A TREATISE ON SOME

NEW GEOMETRICAL METHODS,

CONTAINING ESSAYS ON

THE GEOMETRICAL PROPERTIES OF ELLIPTIC INTEGRALS,
ROTATORY MOTION,
THE HIGHER GEOMETRY,
AND CONICS DERIVED FROM THE CONE,
WITH
AN APPENDIX TO THE FIRST VOLUME.

Nova methodus, nova seges.

IN TWO VOLUMES.—VOL. II.

BY

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INTRODUCTION

TO THE SECOND VOLUME.

AFTER the lapse of nearly four years, in the face of many hindrances, untoward events, and difficulties, I have succeeded in bringing through the press this second and concluding volume of my mathematical and physical researches.

It is proper to mention that the volume will be found to contain four distinct treatises:—(α) on Elliptic Integrals, (β) on Rotatory Motion, (γ) on the Higher Geometry, and (δ) on Conic Sections, followed by an Appendix to the first volume.

An outline of the following researches on the Geometrical Properties of Elliptic Integrals was published in the *Philosophical Transactions* of the ROYAL SOCIETY for 1852, p. 311, followed by a Supplement printed in the volume for 1854, p. 53. Ample time and unbroken leisure have enabled me to recast and enlarge those essays. Though the work was onerous, it was also, I may say, a labour of love, lightened by the discovery, sometimes unexpected, of new truths of great geometrical beauty.

Amongst these researches not the least important is the discovery of three curves of double curvature whose rectification may be effected by elliptic integrals of the first and third orders. These are the geometrical types of those transcendental expressions due to Legendre and Lagrange. The algebraical relations discovered by these illustrious geometers are the exponents of the geometrical properties of those curves. Those versed in the subject will not need to be told how the simplicity of these relations contrasts with the abortive attempts of the most illustrious mathematicians to devise, on a plane, curves whose quadrature or rectification might represent those expressions. I do not here propose to give an analysis of the work; but, for the sake of the few who care to inquire into those matters, I would call attention to Chapter VIII. on *conjugate amplitudes,* and to Chapter X. on *derivative* hyperconic sections.

In the course of these investigations this important truth is clearly established, that the theory of those celebrated functions constitutes a general trigonometry for those curves in which surfaces of the second order intersect. Of this general trigonometry circular and parabolic trigonometry are the extreme cases. In the former the modulus is zero, in the latter unity. Thus an unbroken analogy runs throughout the whole, and the several cases are linked together under the great mathematical law of continuity.

As a test of the utility of those researches in physics, I have applied them, in the following essay, to the discussion of the celebrated problem, to determine the rotation of a rigid body, in free motion, round a fixed point; and I have shown how the position of such a body at the end of any given epoch may be made to depend on the evaluation of those algebraical expressions or their equivalents, the arcs of hyperconic sections.

The investigations on rotatory motion given in this volume were made, the greater portion of them, very many years ago. Some of them appeared from time to time in those periodical publications whose pages are open to discussions on subjects of this nature.

In this treatise a complete investigation has been attempted of the laws of the motion of a rigid body revolving round a fixed point, and free from the action of accelerating forces—an investigation based on the properties of surfaces of the second order, of the curves in which these surfaces intersect, and on the theory of elliptic integrals. The results which have been obtained are exact and not approximate, general and unrestricted by any imposed hypothesis.

I have carefully abstained from introducing any methods which, to one moderately versed in the first principles of the integral calculus, might not fairly be assumed as known. There is but one exception. In a few cases, where the method was peculiarly applicable, I have ventured to make use of *tangential coordinates*, the theory of which is fully developed in the first volume of this work. The reader may, however, if he chooses, omit those applications, without in any way breaking the continuity of the subject.

I have not been led away by mathematical pedantry to attempt to render this essay purely algebraical, by rejecting geometrical conceptions and the aids thence derived to simplicity and clearness, knowing that, very often, the elegance of the analysis is owing to the distinctness of the graphical conception, and that, though the forms of the reasoning may be different, the subject is identically the same.

The problem of the rotation of a rigid body round a fixed point is one that has engaged the attention of the most eminent mathematicians of Europe. More than a century has passed away since D'Alembert first, and Euler soon after, investigated the analytical conditions of such a motion, and formed those differential equations, on the integration of which the determination of the motion ultimately depends. In their investigations, which were purely algebraical, they used to a great extent the principles of the transformation of coordinates—a method of research, it must be conceded, which, though unexceptionable on the ground of mathematical rigour, is generally found to lead through operose and cumbrous processes to complicated results.

Some years afterwards, Lagrange took up the subject, and developed the theory in formulæ of great symmetry and generality. Combining the principle of D'Alembert with that of virtual velocities, he deduced the differential equations of motion, containing six quantities to be determined. By means of these equations, the three components of the angular velocities round the principal axes, which determine the position of the instantaneous axis of rotation in the body, and the three other angular quantities which define the position of the body itself in space, at any epoch, may be expressed in terms of the time. But these quantities, however they may be determined, furnish us, as it has been justly observed, with no conception of the motion during during the time. They prove to us that the body, after the lapse of a certain time, must be in a certain position; but we are not shown how it gets there. We may, by means of calculations, more or less long and complicated, ascertain the bearings of the body at any required instant; we cannot, so to speak, accompany it during its motion. It is determined *per saltum,* and not continuously; we are wholly unable to keep it in view and follow it, as it were, with our eyes during the whole progress of rotation.

To furnish a clear idea of the rotatory motion of a body round a fixed point, and free from the action of accelerating or other external forces, but in motion from the influence of one or more primitive impulses, was the object of a memoir, presented many years ago to the Institute, by that eminent mathematician, M. Poinsot. In this memoir, the motion of a body round a fixed point, and free from the action of accelerating forces, is reduced to the motion of a certain ellipsoid, whose centre is fixed, and which rolls, without sliding, on a plane fixed in space. The axes of this ellipsoid are assumed proportional to the inverse square roots of the moments of inertia round the principal axes of the body, passing through the fixed point, and coinciding in direction with them. He states as his final result, that the time and the other ultimate quantities must be determined by the aid of elliptic integrals. He does not, however, give any account of the processes by which he arrived at his results; and few of the attempts which have since been made to supply that omission have been very successful.

Some time afterwards the late Professor M'Cullagh, of Dublin, turned his attention to this problem, which, owing to the recent researches of Poinsot, had then attracted considerable notice. He adopted an ellipsoid, the reciprocal of that chosen by the latter geometer, and deduced those results which had long before been arrived at by the more operose methods of Euler and Lagrange. His method of investigation, however, was peculiarly his own; but, so far as the author is aware, he never published his method of research.

The idea of substituting, as a means of investigation, an ideal ellipsoid, having certain relations with the actually revolving body, claims as its author the illustrious Legendre. Although he conducts his own investigations on principles altogether different, he yet seemed to be, in his *Traité des Fonctions Elliptiques,* well aware of the use which might be made of this happy conception.

Several years ago, when engaged in applying the new analytical method, founded on my peculiar system of *tangential coordinates,* I was led to views somewhat similar to those of Legendre, from remarking the close analogy or rather identity which exists between the formulæ for finding the position of the principal axes of a body, and those for determining the symmetrical diameters of an ellipsoid. I still further observed, that the expression for the length of a perpendicular from the centre on a tangent plane to an ellipsoid, in terms of the cosines of the angles which it makes with the axes, is precisely the same in form with that which gives the value of the moment of inertia round a line passing through the origin. Guided by this analogy, I was led to assume an ellipsoid, the squares of whose axes should be *directly* proportional to the moments of inertia round the coinciding principal axes of the body.

At first sight the inverse ellipsoid, assumed by Poinsot, may seem to possess some advantages over the direct ellipsoid, at least so far as such an ellipsoid may be said to approximate in form to the natural body. For example, if we consider the case of the rotation of a solid homogeneous ellipsoid round its centre, the ideal or mathematical ellipsoid of Poinsot will bear a resemblance to the figure actually in motion. In the direct ellipsoid of moments, which is made the instrument of investigation in the following pages, this resemblance does not exist; for the coinciding axes of the material and mathematical ellipsoids are such that the sum of their squares is constant. Should the revolving figure be an oblate spheroid, its mathematical representative will be a prolate spheroid. The reader must bear this diversity of figure in mind, in applying the conclusions of theory to an actually revolving ellipsoid. Although it may seem a matter of little moment which of the ellipsoids we choose as the geometrical substitute for the revolving body, it is not so in reality when we come to treat of the properties of the integrals which determine the motion. These integrals depend on the properties of those curves of double flexion in which cones of the second degree are intersected by concentric spheres. By means of the properties of these curves, a complete solution may be obtained, even in the most general case, to which only an approximation has hitherto been made. The solution of a mathematical problem may only then be said to be complete, when in the final result the calculation of the sought quantities may be made to depend on some known elementary quantity or quantities, such as a certain straight line, an arc of a circle, &c. So in this problem, the elliptic transcendents, to the determination of which the calculation of the motion is ultimately reduced, are shown to represent arcs of spherical conic sections, whose elements depend on the nature of the body and on the magnitude and position of the impressed moment. In all the solutions of this problem which have hitherto appeared, the investigations are brought to a close when the expressions, either for the time or other sought quantity, are reduced so as to include the square roots of quadrinomials involving the independent variable to the fourth power. In this treatise the investigations are continued beyond that point, and the quadrinomials have been reduced, as shown in the preceding treatise, to arcs of hyperconic sections.

An elliptic integral of the first order being shown to be only a particular case of elliptic integrals of the third order, as the circle is a species of ellipse, it follows that the analogies between integrals of the first and third orders will be more numerous and intimate than between the second and either of the others. Such is in fact the case. Elliptic integrals of the first and third orders constantly occur in combination. In the discussions of the following pages, for example, integrals of the first and third orders present themselves in various combinations, while an integral of the second order does not once occur in the essay.

The application of the theory of elliptic functions to the discussion of the problem of a rigid body revolving round a fixed point, has led to the following remarkable theorem:

The length of the spiral between two of its successive apsides, described in absolute space, on the surface of a fixed concentric sphere, by the instantaneous axis of rotation, is equal to a quadrant of the spherical ellipse described by the same axis on an equal sphere, moving with the body.

The ordinary equations of motion being established, the author proceeds to show that if the direct ellipsoid of moments be constructed, the rotatory motion of a body, acted on solely by primitive impulses, may be represented by this ellipsoid moving round its centre, in such a way that its surface shall always pass through a point fixed in space. This point, so fixed, is the extremity of the axis of the plane of the impressed couple, or of the plane known to mathematicians as the invariable plane of the motion.

But a still clearer idea of the motion of such a body may be formed by the aid of another theorem, which shows that the whole motion of a revolving body may be represented by a cone which rolls, without sliding, on a fixed plane passing through its vertex, while this plane revolves with a uniform motion round its own axis. This, perhaps, is the simplest conception we can form of a revolving body. Now the principal axes of this cone, and its focal lines, depend on the constitution and form of the body, or, in other words, are functions of the moments of inertia round the principal axes; while the initial position of the plane of the impressed couple in the body will determine the tangent plane to this cone. But when the two focal lines of a cone, and a tangent plane to it, are given, the cone may as completely be determined as a conic section when its foci and a tangent to it are given. Nothing more simple than this conception: a cone rigidly connected with the body, the position of whose focal lines, and whose principal vertical angles, depend on the form and constitution of the body, revolves without sliding on a plane, while the plane itself revolves uniformly round its own axis. We may also observe, that when the plane of the impressed couple passes through one of the focals of the rolling cone the motion is *sui generis*; it no longer may be represented by a rolling cone. The cone degenerates into a plane segment of a circle, which swings round one or other of the cyclic axes of the ellipsoid of moments, these cyclic axes being the boundaries of the circular segment.

Although it may be, and doubtless is, very satisfactory in this way to be enabled to place before our eyes, so to speak, the very actual motion of the revolving body, yet it is not on such grounds solely that the following essay has been published. Were the theory of no other use than to give strength and clearness to vague and obscure notions on this confessedly most difficult subject, enough had been already accomplished by the celebrated geometer whose name is so deservedly associated with this subject. It is as a method of investigation that it must rest its claims to the notice of mathematicians—as a means of giving simple and elegant interpretations of those definite integrals, on the evaluation of which the dynamical state of a body at any epoch can alone be ascertained. If the author has to any degree succeeded in accomplishing this, it is because he has drawn largely upon the properties of lines and surfaces of the second order, and of those curve lines in which these surfaces intersect. If he has been enabled to advance any thing new, it is solely owing to the somewhat unfrequented path he has pursued. That it was antecedently probable such might lead to undiscovered truths, no one conversant with the applications of mathematical conceptions to the discussions of those sciences will deny. To introduce auxiliary surfaces into the discussions and investigations of physical science is an idea no less luminous than it has been successful. The properties of such surfaces often aid our conceptions or facilitate our calculations in a very remarkable manner. M. Dupin, for example, reduces the problem of the equilibrium of a floating body to that of a solid resting on a horizontal plane, the solid being the envelope of all the planes which retrench from the floating body a given volume. We have a still more striking instance in the wave-theory of light. Therein we find the surface of elasticity the equimomental surface in the theory of rotation. Few indeed there are among mathematicians who require to be informed how the wave-surface of Fresnel, and its reciprocal polar, the surface of wave-slowness of Sir William R. Hamilton, have served to clear our conceptions on a subject as yet scarcely understood, to realize and embody an indistinct and shadowy theory. Nay, more, the geometrical properties of the surface of *wave-slowness* in the hands of Sir W. Rowan Hamilton have led to the anticipation of the theory of conical refraction. They have enabled us to invert the natural order of induction and to place theory in advance of experiment. Were further illustration needed, one might refer with confidence to the treatise of Maclaurin on the figure of the earth, to the researches of Clairaut on the same subject, and to the investigations of Poisson, Chasles, and Ivory on the attraction of ellipsoids. A theorem in surfaces of the second order, on which he has bestowed his name, enabled Ivory to evade the difficulties of the problem on which he was engaged. So true is the fine anticipation of Bacon:—"For as Physicall knowledge daily growes up, and new Actioms of nature are disclosed; there will be a necessity of new Mathematique inventions"*.

The author has taken occasion, during the progress of the essay, to derive those partial solutions on particular hypotheses, which are given in the usual text-books on this portion of dynamical science. To the reader familiar with those solutions it will, doubtless, be satisfactory to see them follow, as simple conclusions, from principles more widely general. These partial solutions serve, as it were, to test the truth and accuracy of the principles on which the entire theory is based. To those who read the subject as a portion of academical study, it will, no doubt, prove interesting to discover an additional link connecting the deductions of pure thought with the laws of matter and motion. They will not fail to observe the analogy, that as the properties of the sections of a cone by a plane have elucidated the motions of translation of the planets in their orbits, so likewise the theory of the rotation of those bodies, round their axes, may be founded on the properties of the sections of a cone by a sphere.

* *Of the Advancement of Learning*, book iii. chap. 6.

As introductory to the treatise on conics, I have given an essay on what may be called the higher geometry on a plane. This embraces the theory of transversals, invented and developed by Carnot, and the principles of harmonic and anharmonic ratio, a powerful instrument in the able hands of Chasles. The properties of triangles with reference to inscribed and circumscribed circles, the properties of orthocentres and of orthocentral triangles, the remarkable theory of the nine-point circle, and of the excentral triangles connected with it are also fully developed. In this old and seemingly worn-out subject the reader will yet find something new.

The substance of the following essay was read before the Royal Irish Academy, nearly forty years ago (March 1837)*. It has lain by me unpublished ever since. I have been strongly recommended to add it to this volume by a friend of mathematical attainments of a very high order to whom I had shown this essay (Mr. W. J. C. Miller, Mathematical Editor of the '*Educational Times*,' and Registrar of the General Medical Council, to whom I am much indebted for his judicious advice and suggestions in this portion of the volume, and also for the care and accuracy which he has bestowed on the correction of the press). The shortness and simplicity of the demonstrations encouraged me to submit those propositions to geometers, few of them requiring any more knowledge than that of the simplest propositions of Euclid.

It may be objected to the method developed in the following pages that all the properties of the conic sections are derived almost exclusively from those of the right cone. In reply to this objection, it may be observed that the object is not to investigate the properties of cones or other surfaces of the second order, but those only of plane curves; that the right cone is used as a simpler and more powerful instrument of discovery than the oblique cone; and that any argument for deriving those properties from this latter

* The Secretary communicated the substance of a paper "On the Conic Sections," by James Booth, Esq.

The methods hitherto adopted in deducing the central and focal properties of the conic sections from arbitrary definitions having appeared to the author defective in geometrical elegance, he has endeavoured in this paper to derive them from a new definition.

If two spheres be inscribed in a right cone touching the plane of a conic section, the points of contact are called *foci*.

The property from which the definition of a focus here given is derived, though known for some time, has not been hitherto applied further than to show that this point is identical with the focus as usually defined.

By the help of the above definition, and of the simplest elementary principles, the central and focal properties already known have been deduced, generally in one or two steps, and several new theorems have been likewise discovered in the development of the method.—Extract from the *Proceedings of the Royal Irish Academy*, March 16, 1837.

surface would be equally applicable in favour of deducing them from any other suitable surface of the second order. Besides, any conic section being given on a plane, a right cone of which it may be considered a section, can always be constructed. The mere extension to the oblique cone is too trivial, when compared with the number of other surfaces of the second order having like properties, to merit any special attention*.

The right cone with a circular base is selected in preference to any other surface, because the properties of its plane sections, hence called conic sections, may be derived with more clearness, brevity, and simplicity, than those of like sections in any other surface. It must be borne in mind that the surface is used simply as a means or instrument to obtain the properties of its plane sections; and these can be deduced from the right circular cone with greater facility than from any other†.

The prolix diffuseness of most of the treatises on this subject, the interminable series of proportionals which cumber every page, and the tediousness of the demonstrations follow from the fact that, as soon as the cone had afforded one or two principal properties of its sections, these have been selected as definitions of the sections, and the attempt is made, often with much ingenuity, to base a wide and general system of these curves on the apex of one narrow definition‡.

* La construction que nous venons de donner des foyers des coniques, prises dans le cône oblique, ne se prête pas à la démonstration des propriétés de ces points, et n'est pas propre même à indiquer à priori leur existence dans les coniques. Il reste donc à rechercher comment, par la considération des coniques dans le cône, on peut être conduit à la découverte de leurs foyers.—CHASLES, *Aperçu*, p. 286.

† Les Anciens avaient considéré les sections coniques dans le cône, mais seulement pour en concevoir la génération et en démontrer quelques propriétés principales, et faire servir ensuite ces propriétés primitives à la recherche, et à la démonstration de toutes les autres: de sorte qu'ils formaient leur théorie des coniques sans connaître la nature ni aucune propriété du cône, et indépendamment de celles du cercle qui lui sert de base.—CHASLES, *Aperçu*, p. 119.

‡ Nous dirons, en passant, qu'outre la méthode des Anciens et celle adoptée par De la Hire, nous en concevons une troisième qui n'a point été employée, et qui eût été propre pourtant, si nous ne nous abusons, à simplifier extrêmement les démonstrations, et à mettre dans tout leur jour les principes et la véritable origine des diverses propriétés des coniques: sous ce rapport, on ne peut se dissimuler que la méthode des Anciens n'offrait qu'obscurité.

Cette méthode eût consisté à étudier les propriétés du cône lui-même, et à les formuler, indépendamment et abstraction faite de celles des coniques; et celles-ci se seraient déduites des premières avec une facilité et une généralité ravissantes. On le concevra sans peine, car partout où les Anciens employaient trois démonstrations différentes pour démontrer la même propriété dans les trois sections coniques, ellipse, hyperbole et parabole, parce qu'ils s'appuyaient sur les caractères particuliers à chacune de ces courbes, une seule démonstration suffira pour démontrer, dans le cône même la propriété analogue, d'où celles des trois coniques doivent se déduire comme de leur vraie et commune origine.

De cette manière, on eût vu prendre naissance dans le cône à plusieurs pro-

Thus if we were to assume the determining ratio, so simply established in the following treatise, as the basis of a system of conic sections, we should follow that adopted by Boscovich, Walker, Sir John Leslie, and others, in their several treatises on this subject. The numerous books compiled for the use of the Universities start from the same definition. De la Hire suggested as a fundamental definition of a system of conics the constancy of the sum or difference of the focal vectors to any point on the conic.

But a much more fertile property was derived by Dr. Hugh Hamilton, author of a treatise of conic sections published in 1758, and very celebrated in its day. He shows that if two fixed lines be drawn, and two other intersecting lines parallel to them, but variable in position and cutting the cone, the ratio of the rectangles under their segments is constant, and independent of their position, subject only to the condition that they remain parallel to the two fixed lines given in position. This is perhaps the most general property of the cone with reference to the properties of its several plane sections. But Dr. Hamilton's anxiety to abandon the cone and to arrive as speedily as possible at those theorems which relate to the foci, directrices, and centres, led him into a course of investigation but little calculated to exhibit the peculiar advantages of the basis he had chosen*.

The definition of a focus, on which this treatise chiefly rests, is derived from a beautiful theorem discovered a few years since by MM. Quetelet and Dandelin, first published in 1822.

It follows indeed so obviously from prop. 37, lib. ii. of Hamilton's Conic Sections, that one is at a loss to understand how this acute and original geometer failed to discover it. The wonder is how he missed stumbling over it, as it lay so obviously in his way; and none of his readers has since supplied the omission.

Although largely to augment the number of general and remarkable properties of those curves which have been brought to light by the continuous labours of accomplished geometers in successive ages may be considered very arduous, (as I wrote in 1837), yet it is hoped that several new theorems, especially those on the curvature of these sections, derived from the properties of the cone, will not be found elsewhere.

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priétés des coniques, telles que celle des *foyers,* qu'il semble qu'Apollonius ait devinée; et que ce géomètre, ni aucun de ceux qui l'ont suivi, n'ont rattachée ni aux propriétés du cercle, ni à celles du cône; de sorte que l'origine première de ces points singuliers, celle qui ne participe que de la nature du cône ou la courbe prend naissance, est restée ignorée.—CHASLES, *Aperçu,* p. 121.

* Quoniam Apollonius omnia ferè conicorum demonstrata conatus est in planum redigere, antiquioribus insignior: neglectâ conorum descriptione, et aliunde quærens argumenta, cogitur persæpe obscurius et indirectè demonstrare id, quod contemplando solidæ figuræ sectionem apertius et brevius demonstratur.—*D. Francisci Maurolici opera Mathematica,* p. 280. See CHASLES, *Aperçu,* p. 120.

The properties of conic sections may be divided into two distinct classes, the angular and the metrical. The former will be found chiefly to depend on the *focal* properties of the sections developed from the definition of the foci as the points of contact of the plane of the section with spheres inscribed in the cone, while the latter will be more easily established by the methods of harmonic lines and planes. The definition of a centre is founded on the properties of harmonic pencils. Thus the two classes of properties are quite distinct. The shortness and simplicity of the demonstrations prove that these two principles, the definitions of the foci and the centres of these curves, afford the true key to their investigation.

In most modern treatises on this subject, the three sections are treated independently, as if they had no common genesis, and the demonstrations rest, not on geometrical constructions, but on endless rows of tedious and repulsive proportionals. In the following pages an attempt is made to derive the cardinal properties of those celebrated curves from their common origin, the cone, independently of any arbitrary definition. Some of those properties, and these amongst the most important, which are commonly established by the tedious processes of a disguised algebra, come out at once clear and self-evident from mere inspection. When those leading theorems are once established for conics in general, it becomes a matter of the utmost facility to apply them to the investigation and discussion of theorems and problems of a less general character on a plane.

There is also to be observed in some of those treatises a puerile affectation of geometrical rigour, in rejecting the use of such abbreviations as sin, cos, tan, so generally used in mathematical works to denote certain constantly occurring ratios. One is at a loss to understand how the force of a demonstration is augmented by using instead of sin A the circumlocution "In the right-angled triangle ABC the ratio of the perpendicular BC to the hypotenuse BA." This notation, borrowed from trigonometry, wherever it is adopted, gives a singular clearness and brevity to the demonstrations. And again, it is difficult to imagine in what respect it is less rigorous to say *a* than the straight line AB.

The reader's attention is specially directed to Chapter XXIX., in which the radius of curvature of conics is derived directly from the right cone, without the help either of the Differential Calculus, or of Infinitesimals or of any other such device. I am not aware that any attempt has ever been made to obtain the curvature of a conic directly from the cone whereof it forms a section.

There cannot be a more powerful help to develop that faculty of the mind which may be called geometrical imagination, that power to place clearly before the mind's eye the several positions which planes, lines, and surfaces assume as they intersect in space, than the contemplation of those curves considered as the intersections of planes and surfaces. In no science is this power of clear and steady conception so necessary as in Astronomy and Mechanics.

It is worthy of remark that solid geometry as it is called, or a reference to space of three dimensions, facilitates very often, and that too in a striking manner, the proofs of theorems concerning figures on a plane. A signal example of this will be seen in the simple proofs of the principal properties of conics established by the help of the right cone.

The object which the author has proposed to himself in the following pages is not so much to use a single method in the solution of a cloud of problems and theorems, many of them remarkable only for their intricacy, but to apply a variety of methods to the discussion of a class of selected properties, and to show that while some questions yield with ease to one method, they are almost insoluble by another.

Thus in some instances several demonstrations will be found for the same theorem. It is of far greater importance, and will give a wider grasp of the subject, to contrast and compare different methods when applied to the investigation of the same theorem. The student will then perceive that every method has something inherent to recommend it, and that the method which in one case will give a simple and easy demonstration, will afford obscure and complicated results in other cases apparently not more difficult.

For this reason I have been more solicitous to develop a variety of methods than to follow out some one selected principle into all its details. It is no doubt a test of ingenuity and mathematical ability to be able to build up an imposing structure of mathematical demonstration based upon one fundamental principle alone. But this apparent simplicity is found often to lead to long calculations and complicated results in the development of the principle assumed.

To the well-informed reader it will be evident that the modern methods of geometrical investigation which in recent times have been applied to the development of geometry have to a great extent superseded the old. In the geometry of the Greeks, the demonstrations were partial, often requiring a separate proof for every modification of figure. Some one property (as in the conic sections for example) was made the basis of a superstructure erected with infinite ingenuity and matchless skill, but often tedious, complicated, and involved, owing to the narrowness and remoteness of the definition.

It has been well observed by a very profound mathematician and elegant writer, that when a subject is contemplated from a true point of view it may be explained in a few words to a passenger in the street*. As disjointed limbs and broken fragments (confused

* Nous ajouterons avec un des géomètres modernes qui ont le plus médité sur

images) when viewed from the focus of a conical mirror range themselves in symmetrical order and assume definite forms, so it is with the truths of science; confused, isolated, and indistinct they remain until their true stand-point of view be taken.

The aim and scope of the modern geometry widely transcend the limits which ancient science imposed on itself, while the traditional reverence in which those old methods were held was long an obstacle to the development of physical and mathematical knowledge*. We have no just reason, however, to be surprised at this superstitious veneration for the great works and mighty genius of antiquity. Strange indeed had it been otherwise. It is sometimes said that we do not retain that traditional reverence for antiquity, that veneration for great names, which distinguished the promoters of intellectual advancement at the birth of modern civilization—that we no longer feel that exclusive admiration for the literature and science of Greece and Rome, which, three or four centuries ago, was a marked characteristic of every one who professed to cultivate them. Now this veneration for ancient wisdom is founded on a fallacious analogy. The young naturally confide in the experience and knowledge of the old; and as the old have preceded them in point of time, we are led by the seeming analogy to look upon the early life of the world as its old age instead of its youth. Lord Bacon, in his *Advancement of Learning*, says, "certainly our times are the ancient times when the world is now ancient, and not those which we count ancient, *ordine retrogrado*, by a computation backward from our own times." Again, an exaggerated admiration of antiquity, and a sort of longing regret for times passed away, are by no means hopeful signs of a present healthy progress. It has sometimes been remarked of those who can boast a long line of ancestors, and yet have degenerated

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la philosophie des mathématiques, "qu'on ne peut se flatter d'avoir le dernier mot d'une théorie, tant qu'on ne peut pas l'expliquer en peu de paroles à un passant dans la rue."

Et en effet, les vérités grandes et primitives, dont toutes les autres derivent, et qui sont les vraies bases de la science, ont toujours pour attribut caractérisque la simplicité et l'intuition.—CHASLES, *Aperçu*, p. 115.

* Si présentement on me demande mon opinion sur la géométrie pure, je demanderai à mon tour de faire une distinction s'agit-il de la géométrie d'Archimède, d'Euclide, d'Apollonius, et de tous ceux d'entre les modernes qui, comme Viviani, Halley, Viète et Fermat, ont marché sur leurs traces? J'avouerai franchement, quelque opinion que l'on puisse en prendre de moi, que je n'en suis pas enthousiaste. Que si, au contraire, on veut parler de cette géométrie qui, née, pour ainsi dire, des méditations de l'illustre Monge, a fait de si immenses progrès entre les mains de ses nombreux disciples, on me trouvera toujours disposé à lui rendre le plus éclatant hommage, et à reconnaître qu'elle nous a fait découvrir en vingt années plus de propriétés de l'étendue qu'on n'en avait pu découvrir dans les vingt siècles qui les avaient précédées.—*Annales de Mathématique*, tom. viii. p. 159.

in the descent, that they were satisfied to base their claims to consideration, not on the grounds of personal merit, but on the greatness of those who had gone before them. The same is as true of nations as of individuals. Diodorus and Plutarch, by their extravagant eulogies of the extinct republics and legendary heroes of antiquity, tried to console themselves for the degeneracy of the times in which they wrote. By their enthusiastic admiration of forms of government that had been abolished, they indirectly censured the enormities of the grinding despotisms under which they could scarcely call even their lives their own; and the language in which they lauded the liberties they had lost was the surest index of the slavery under which they groaned. The same tone of saddened retrospection breathes through the fine preface of Livy's immortal history.

But, independently of these considerations, there is a legitimate cause and weighty reason for this profound admiration of antiquity. Let us in imagination go back to the year 1500 of our era, or thereabouts; let us imagine a man somewhere in the south of Europe, or in one of the Greek cities of the lesser Asia, within sight of that purple sea, beyond whose sunny shores civilization had never yet been able to advance. Let us further suppose him to be profoundly versed in all human learning, and acquainted with every cardinal event in man's history. What are the reflections that would naturally arise in the mind of so accomplished and philosophical a spectator taking a comprehensive view of the annals of mankind, and of the progress of civilization from its earliest recorded dawn down to his own time?

He would have seen all human knowledge either absolutely stationary or actually retrograding. He would have seen that the mathematical science of his own day had not made a single step in advance during the long period of 1700 years, from the state in which it was left by Archimedes and Euclid and Apollonius; for the Roman civilization throughout its long duration never produced even a fifth-rate mathematician. He would have seen that since the days of Hippocrates and Galen the science of medicine had degenerated into a mere empirical art; that there was no body of laws worthy of the name but the Roman codes; that alchemy flourished, for chemistry was not yet; that astrology had displaced the little astronomy that was known; that there was absolutely no such thing as physical science; that the multitudinous facts of natural history had yet to be observed and noted, excepting those only investigated by Aristotle, that most profound and accurate physicist; that in poetry, oratory, architecture, and the kindred arts of painting and sculpture, the ancients transcended rivalry or even successful imitation; in short, that the whole sum of human knowledge, scant as it was, had continued without augmentation or accession during fifteen long centuries of man's eventful history; that the acutest wits and the most subtle intellects were forced to move round and round in the same dull mill-circle, and thresh the straw that had been threshed a thousand times before; that the profoundest thinkers failed to make even the shallowest discovery either in science or in art; that the most learned men occupied themselves, century after century, in piling up pyramids of commentaries on those wondrous men Aristotle and Plato, who, like the Pillars of Hercules in the old mythology, separated the clear, the definite, the settled, and the known from the dark, the vague, the boundless, and the obscure,—when, moreover, our supposed inquirer, continuing his survey, would have learned that whole regions of the earth's surface were passing clean out of the knowledge of civilized man, that the ideas which learned professors and adventurous travellers formed about countries not far remote were vague and contradictory, that less was known four centuries ago about the geography of the world and the relative magnitudes and positions of the several regions thereof than in the times of Scylax, Herodotus, Strabo, Ptolemy, or even Agatharchides, that the knowledge of many fine inventions and curious processes in the arts had actually perished (and has never to this day been rediscovered)—when, in addition to this, looking to the political aspects of the world, he would have seen the very fairest and most hallowed regions of the earth's surface overrun by the wild fanatics of Arabia, or trodden down by the savage hordes of Turkestan, who with unbroken front were advancing like the ocean tide rushing up an estuary, to overwhelm under one undistinguishing flood every monument and every institution that survived of the ancient civilization (even now who shall truly say that the liberties of the west and the civilization of our own time, beginning to show symptoms of early decline and marks of premature decay, are entirely beyond the reach of the ever advancing wave of Russian despotism, urged onwards by the barbarous hordes of the deserts of Eastern Asia?)—and when, lastly, to such an ideal spectator, reviewing the history of man's progress upon earth, that great renovating institution the Church, would have been presented to his view, not as the living, breathing incarnation of the Gospel, giving health and vigour to the nations of antiquity worn out and effete, but like Niobe of old petrified into stone, and becoming herself a huge stumblingblock in the way of progress, a rock of offence to those who saw not that her corruptions and errors were, in some measure at least, due to the evil days through which she had had to pass.

Nor from such a retrospect could our spectator have drawn, with regard to the future, other than the most desponding anticipations. No man could foresee that as the night is darkest before the dawn, so out of this dense moral night and deep darkness of the human understanding a new order of things was soon to arise, and the light of a higher and better civilization to gladden mankind. It is no wonder then that men, looking back through the vista of a lengthened period of time, and seeing that every thing that was worth preserving in literature, science, and art—whether it be poetry, oratory, or the drama—whether it be architecture, sculpture, or painting, was the creation of comparatively a small number of gifted minds and the birth of a few remote centuries, it is no wonder that men in those days held the deep conviction that nearly every thing that could be known was already discovered. In fact they had a special name for it. They called it the "omne scibile." They called it not "omnis Scientia," but "omne scibile," not merely every thing that was known, but every thing that could be known. It is not strange, then, that a feeling of admiration apparently akin to hero-worship should have been felt for those who at a bound had reached the limits and touched the very outer verge of knowledge attainable by man.

It is generally assumed, as an assertion not admitting of dispute, that the origin of the present methods of physical investigation is due to Bacon, and that an outline of those methods may be traced throughout his works, more especially in the 'Novum Organum,' the 'Instauratio Magna,' and the 'De Augmentis Scientiarum.' It requires some hardihood to call in question such an established opinion; yet, to one who, free from prejudices and preconceived notions, shall carefully read those works, it will be abundantly evident that Bacon's great merit lay in giving form and pressure to the accepted modes of thought of his own time. His chief object seems to have been to denounce authority, to set at naught antiquity, to undervalue ancient philosophers and their theories, to prove that no natural knowledge could be established by their methods of procedure, and that the ancient syllogism was an impotent instrument of investigation. Now this was the very spirit of Bacon's age. Human authority had already been denounced in Ecclesiastical affairs; and the fruit of this was the Reformation. The authority of Aristotle and the old Greek philosophers was questioned; and a general *scepsis scientifica* was the result. In politics this denial of human supremacy led to the great rebellion of 1641. Bacon deserves the credit of realizing the spirit of his own times, which was intensely sceptical. He first showed that all advance in the natural sciences must be based on original and independent inquiry, without reference to the theories of the old philosophy.

A very brief examination of Bacon's works would completely establish this view. In the 84th aphorism of the first book of the 'Novum Organon' he says "Reverence for antiquity has retarded mankind, and thrown as it were a spell over them, and the authority of men who were held to be great in philosophy. It is a mark of feebleness to yield every thing to ancient authors, and to deny his supremacy to time; for truth is the daughter of time, not of authority." He adds that "the present time is to be considered as the ripe maturity of the world, with all our accumulated facts and experiences, and not antiquity, which may rather be called the childhood of mankind." In fact the whole tone and spirit of the book is a powerful protest against the influence of authority in matters of science.

It is often said that Bacon was opposed to the construction of philosophical hypotheses. This is true in one sense, but not in another. There are what may be called *provisional*, as well as *established* theories. When Newton saw the historical apple fall to the ground, and conjectured whether the moon might not itself be a big apple, he made his calculations, assuming the law of gravitation as his hypothesis. But when he found that, owing to an erroneous estimate of the mass of the earth, then accepted by astronomers as correct, his calculations did not confirm his theory, he abandoned his hypothesis. Now this is an instance of a *provisional* hypothesis. When, some years afterwards, Newton obtained a more correct value of the mass of the earth, he resumed his calculations, established his theory, and thus turned his *provisional* into an *established* hypothesis, which, for countless ages yet to come, is likely to respond to the mechanism of the heavens.

Bacon agrees with Cousin that the syllogism does not investigate first principles. This, however, nowise invalidates the use of logic. It is not the business of logic to investigate first principles. In the longest and most subtle demonstration there can be found nothing in the conclusion that was not previously involved in the principles assumed as the basis of the proof. In most physical inquiries—if we except Mathematical Astronomy and, perhaps, Optics—there are but very few steps in the process of physical induction.

Bacon, however, was much more successful in the work of destruction than in that of reconstruction. He could pull down; but he could not build up. The specimens of philosophical induction which he gives in the second book of the 'Novum Organon' are most of them puerile, if not silly, and frequently contradict his own principles. He equally fails in laying down the true goal and just object to be kept in view in the cultivation of natural knowledge. He holds up no higher standard than gross material utility. He proposes to make men comfortable in their persons and dwellings. This is a low standard; it falls far below that of the old Greeks. But some allowance must be made for him. He lived in a cold ungenial clime, very different from the bright and sunny lands of Attica. In the great object of his works—the subversion of the authority of the ancient philosophers, and the uprooting of all reverence for antiquity—he has thoroughly succeeded; and he succeeded because he embodied the spirit of his age and cleared the ground for those who were to follow.

The word science has in these latter days been divorced from its original meaning, geometry and the creations of the pure intellect. It is now appropriated to observations in natural history and to experiments in chemistry. These subjects of research are no doubt very interesting and valuable; but they are not science in the original and best sense of the word. Yet without a knowledge of mathematics it is impossible to make any real advance in the discoveries of physical science. Take the case of that great science Physical Astronomy, of which Sir J. Herschel says, "admission to its sanctuary and to the privileges and feelings of a votary is only to be gained by one means—sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range."

But, notwithstanding the concurrent testimony of the greatest men of every age, it is in the mouths of many a very common objection which leads them to ask, "What possible use can there be in mathematics? how few are they to whom they can be of the least utility in after life!" So it might with equal plausibility be asked why practise running, leaping, or wrestling? seeing that very few become professed athletes. But just as athletic exercises develop the muscles, improve the health, and invigorate the body, so severe studies strengthen the understanding, form habits of thinking, and deepen the grooves of thought, even though the subjects of those studies be in the course of time wholly forgotten. Like those old quarries we read of in Pentelicus or Paros, though the blocks of marble, the material of the breathing bust or god-like statue have gone, never more to return, yet the ruts of the wheels that bore them, the grooves in the face of the rock along which the guiding gear and cordage ran, are as fresh and as sharp as if they had left off working only yesterday.

And nowhere is this low utilitarian sentiment more loudly expressed than amongst those who have acquired such attainments as they possess at our national Universities. Those persons pick up just as much learning or science as may suit their purpose and help them forward on the path of life they have selected. In fact, learning and science are valued just as acquaintance with book-keeping by double entry is valued, as a means to an end, and that end by no means the noblest. To secure their approbation, research must have a bearing on some useful practical money-making object. This is in accord with the spirit of the age, a spirit of pretence and vanity and sham*.

At this state of things we ought not to feel any surprise. Our Universities are no longer calm retreats for the encouragement of patient and continuous thought expended on the development of branches of science which do not promise an immediate ready-money return; they are now almost wholly engaged in conducting the elementary education of the upper and middle classes of this country. And hence it is that some of those who have most widely extended the boundaries of knowledge are men who early abandoned their college retreats, or have never been inside the portals of a University college at all. Men, such as Thomas Simpson, and Boole, and Davies, and Horner and others, not to speak of those whom, as still alive, it might be invidious to mention, have had the genial current of their souls frozen by a chill penury, or were relegated to a dull oblivion, or at least to a passing obscurity, by combinations of cliques, nowhere more general or more potent than in the mathematical world. It would be a curious but perhaps a bootless inquiry to discuss why, from the days of Apollonius of Perga, called the great geometer, to our own, a characteristic failing of mathematicians has always been envy.

The education of our own day tends to produce a dead level of mediocrity. There will be few to note for crass ignorance, and scarcely any to admire for profound learning. The age is so fast that it cannot stop to think; it cannot pause to ponder. Nay, more, it cannot with common propriety express its own wants and wishes; for the "pure well of English undefiled" is rapidly turning into a puddle of slang. If ridicule be a test of truth, as the author of the *Characteristics* asserts it to be, we ought by this time to have reached the very extreme limit of correct opinion. For every thing, now-a-days, is treated in a spirit of mockery, levity, or contemptuous indifference. That this happy result has not yet been obtained is a proof of the fallacy of LORD SHAFTESBURY's great discovery in ethics. There will be, as in all human affairs, a reaction and a change; and men will once again follow the more excellent way.

Attempts are perseveringly made to remove the Elements of Euclid from the high position which it has held for more than two thousand years, of being unquestionably the best introduction to geometry. It is assailed on the ground that it is too tedious, too rigorous in its demonstrations, that it wants order, and is deficient in symmetry. It is asserted that it is time such old-world notions

* At apud plerosque tantum abest, ut homines id sibi proponant, ut scientiarum et artium massa augmentum obtineat; ut ex eâ, quæ præsto est, massâ nil amplius sumant aut quærant, quam quantum ad usum professorium, aut lucrum, aut existimationem, aut hujusmodi compendia convertere possint.—BACON, *Nov. Org.* lib. i. Aph. 81.

and methods were exploded, and that what we want now, is some easy, handy compilation, on a level with the comprehension of most people, which would commend itself by its practical utility in meeting the passing needs of daily life; and if such a short cut to geometry be not rigorous in its demonstrations, what possible difference could it make to any one whether the proofs were real or only seeming?

But Euclid is not likely to be dethroned for some little time longer. Not very long ago a Committee was appointed by a new geometrical Society to draw up a syllabus of the elements of geometry to supersede the tedious and repulsive work of Euclid. The Committee, which consisted of six members, was requested to draw up a joint report on the subject. But, *Quot homines tot sententiæ*, six different reports were sent in!! no two members so far agreeing in their views as to unite in drawing up a joint report.

It is also, we are told, likely that the study of Greek in this country will soon be given up, if not altogether, at least in a great measure. This is a prospect even still darker; for it implies a decline in the cultivation of the finest language that has ever yet been spoken on the earth, and a consequent degradation of the standard of that learning by which a nation is ennobled.

It hardly needs to be said that I publish these volumes not only without the expectation of reimbursement, but with the certainty of heavy pecuniary loss. I can appeal to no University syndicate to share my burden. It is perhaps right that for this act of indiscretion I should make an apology to the public, whose one sole test of literary and scientific excellence is *Will it pay?* That old-world notion of working for work's sake is now utterly exploded, not alone among the ignorant and the vulgar, in whom it might be forgiven, but even amongst those who stand highest in the ranks of science in our own day. How often do we hear such researches stigmatized as unprofitable and vain! Yet the great masters of wisdom in every age have otherwise taught; and I have followed their teaching, not deterred by the conviction that abstract science has become obsolete and stale. Many of those discoveries, the fruit of a long and desultory life, I would not willingly let die. Popularity as an author or reputation as a discoverer in science is to me a matter of supreme indifference. Neither is it an object with me of any importance to make money by the publication of my discoveries, as I am fortunately placed above those needs which sometimes press so heavily on many of the most illustrious cultivators of literature and science.

J. B.

Stone Vicarage,
New Year's Day, 1877.

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		For	*Substitute*
29.	22 a.	*point*	*points*
31.	1 a.	x	$x_{\prime}$
	19 b.	Place 2 before the square root.	
32.	1 a.	BC−AC	$\mathrm{BC}-\mathrm{AC}_{\prime}$
69.	15 a.	$x=$	$z=$
114.	1 b.	$a^2\xi^2+b^2v^2$	$a^2v^2+b^2\xi^2$
115.	2 a.	$(ay)^{\frac{2}{3}}+(bx)^{\frac{2}{3}}$	$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}$
	(g).	$a^2\xi^2+b^2v^2$	$a^2v^2+b^2\xi^2$
	(h).	$\mathrm{A}=\frac{a^2-b^2}{b}, \mathrm{B}=\frac{a^2-b^2}{a}$	$\mathrm{A}=\frac{a^2-b^2}{a}, \mathrm{B}=\frac{a^2-b^2}{b}$
141.	15 b.	$\frac{\mathrm{dF}}{\mathrm{d}y}$	$\frac{\mathrm{dF}}{\mathrm{d}y}y$
154.	9 a.	$a^2\xi^2+b^2v^2$	$a^2v^2+b^2\xi^2$
287.	(f)	$(p+q)^2$	$(p^2+q^2)^2$

ERRATA IN THE SECOND VOLUME.

		For	*Substitute*
14.	5 a.	$\cos^2\alpha$	$\cos^2\beta$
37.	(70).	$1+j$	$(1+j)$
40.	5 a.	$\sqrt{-1\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}$	$\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}$
41.	(90).	$(1+)j\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}$	$(1+j)\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}$
41.	(90).	$\sqrt{1-\left(\frac{1-j}{1+j}\right)\sin^2\phi}$	$\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}$
43.	10 b.	$\sqrt{1-\cos^2\gamma\sin\mu}$	$\sqrt{1-\cos^2\gamma\sin^2\mu}$
81.	(248).	$[n+m-2mn^2]$	$[n+m-2mn]^2$
96.	27 a.	[9]	[7]
206.	(a).	$\frac{\mathrm{d}\Psi}{t}$	$\frac{\mathrm{d}\Psi}{\mathrm{d}t}$

ON THE

GEOMETRICAL PROPERTIES

OF

ELLIPTIC INTEGRALS.

INTRODUCTION.

In publishing the following researches on the geometrical types of elliptic integrals, I may be permitted briefly to advert to what had already been effected in this department of geometrical research. Legendre, to whom this important branch of mathematical science owes so much, devised a plane curve whose rectification might be effected by an elliptic integral of the first order. Since that time many other geometers have followed his example, in contriving similar curves, to represent, either by their quadrature or rectification, elliptic functions. Of those who have been most successful in devising curves which should possess the required properties, may be mentioned M. Gudermann, M. Verhulst of Brussels, and M. Serret of Paris. These geometers, however, have succeeded in deriving from those curves scarcely any of the properties of elliptic integrals, even the most elementary. This barrenness in results was doubtless owing to the very artificial character of the genesis of those curves, devised, as they were, solely to satisfy one condition only of the general problem*.

In 1841 a step was taken in the right direction. MM. Catalan and Gudermann, in the journals of Liouville and Crelle, showed how the arcs of spherical conic sections might be represented by elliptic integrals of the third order and *circular* form. They did not, however, extend their investigations to the case of elliptic in-

* Legendre a cherché à représenter en général, la fonction dig. (c, ϕ) par un arc de courbe ; mais ses tentatives ne nous ont pas semblé heureuses, car il n'est parvenu à résoudre complétement le problème, qu'en employant une courbe transcendante, dans laquelle l'amplitude ϕ et l'arcs ont entre eux une relation géométrique encore plus difficile à saisir que dans la lemniscate.—Verhulst, *Traité des Fonctions Elliptiques*, p. 295.

tegrals of the third order and *logarithmic* form; nor even to that of the first order. These cases still remained without any analogous geometrical representative, a blemish to the theory.

It will be shown in the following pages that the elliptic integral of the first order, which is merely a particular case of the circular form of the elliptic integral of the third order, represents a spherical conic section whose principal arcs have a certain relation to each other, and that the true geometrical representative of an elliptic integral of the third order and logarithmic form, is the curve of intersection of a right elliptic cylinder by a paraboloid of revolution having its axis coincident with that of the cylinder. The geometrical representative of the peculiar form when the parameter is negative and greater than 1, is shown to be a curve which I call the *Logarithmic hyperbola,* and which may be thus generated. If a right cylinder standing on a plane hyperbola as a base, be substituted for the elliptic cylinder, the curve of intersection may be named the *logarithmic hyperbola.* It will have four infinite branches, whose asymptotes will be the infinite arcs of two equal plane parabolas. This curve, and not the spherical ellipse, is the true analogue of the common hyperbola.

The main object of the following treatise is to prove, that *Elliptic Integrals of every order, the parameter taking any value whatever between positive and negative infinity, represent the intersections of surfaces of the second order.*

To these curves may be given the appropriate name of *Hyperconic sections.*

These surfaces divide themselves into two classes, of which the sphere and the paraboloid of revolution are the respective types; from the one arise the circular functions, from the other the logarithmic and exponential. The circular integral of the third order is derived from the sphere, while the logarithmic function of the same order is founded on the paraboloid of revolution.

Although in the following pages I have, for the sake of simplicity, derived the properties of those curves, or of the integrals which represent them, from the intersections of these normal surfaces (the sphere and the paraboloid) with certain cylindrical surfaces, yet the intersections so produced may be considered as the intersections of these normal surfaces with various other surfaces of the second order. Let $U=0$ be the equation of the sphere or paraboloid, and $V=0$ the equation of the cylinder. The simultaneous equations $U=0$, $V=0$ give the equations of the curve of intersection. Let f be any abstract number whatever; then $U+fV=0$ is the equation of another surface of the second order passing through the curve of intersection. Let $U=0$ be the equation of a sphere, for example. Accordingly as we assign suitable values to the number f, we may make the equation $U+fV=0$ represent any central surface of the second order. But we cannot, by any substitution or rational transformation, make the equation $U+fV=0$ represent a non-central surface instead of a central one, or *vice versâ*.

Although a remarkable relation exists between the areas and lengths of some of these hyperconics, such as the circle and the spherical ellipse, yet more distinctly to show the analogy which pervades all those curves, I have not had recourse in any case to the method of "elliptic quadratures," as it is termed*. We cannot admit such a violation of the law of geometrical continuity as to suppose that while a function in one state represents a curve line, in another, immediately succeeding, it must express an area. Such can only be taken as a conventional explanation, until the real one, characterized by the simplicity of truth, shall present itself.

In the course of these investigations, it will be shown that the formulæ for the comparison of elliptic integrals, which are given by Legendre and other writers on this subject, follow simply as geometrical inferences from the fundamental properties of these curves, and that the ordinary conic sections are merely particular cases of those more general curves above referred to under the name of hyperconic sections.

It will doubtless appear not a little singular that the principal properties of those functions, their classification, their transformations, the comparison of integrals of the third order with conjugate or reciprocal parameters, were all investigated and developed before geometers had any idea of the true geometrical origin of those functions. It is as if the formulæ of trigonometry had been derived from an algebraical definition, before the geometrical conception of the circle had been admitted. As circular trigonometry may be defined the development of the functions of circular arcs, whether described on a plane or on the surface of a sphere, and parabolic trigonometry † as the development of the relations which exist between the arcs of a parabola, so this higher trigonometry, or the theory of elliptic integrals, may best be interpreted as the development of the relations which exist between the arcs of hyperconic sections.

* En considérant les fonctions elliptiques comme des secteurs, dont l'angle est précisément égal à l'amplitude ϕ, nous avons eu l'avantage de justifier la dénomination d'amplitude appliquée à l'angle ϕ; et même celle de *fonctions elliptiques*, en général, puisque les courbes algébriques par lesquelles nous avons représenté ces transcendantes, se construisent avec facilité au moyen des rayons vecteurs d'une ou de deux ellipses données.—VERHULST, '*Traité des Fonctions Elliptiques*,' p. 295.

M. Verhulst has represented the three kinds of elliptic integrals by means of sectorial areas of certain curves. It is manifest, however, that it is incomparably easier to do this than to represent these transcendents by means of the arcs of curves.—R. L. ELLIS, *Report on the recent progress of Analysis*, p. 73.

† See Vol. I. page 313.

Indeed it may with truth be asserted that nearly all the principal functions, on which the resources of analysis have chiefly been exhausted, whether they be circular, logarithmic, exponential or elliptic, arise out of the solution of this one general problem, to determine the length of an arc of a hyperconic section.

It may be said, we cannot by this method derive any properties of elliptic integrals which may not algebraically be deduced from the fundamental expressions appropriately assumed. But surely no one will assert that the properties of curve lines should be algebraically developed without any reference to their geometrical types.

We might, from algebraical expressions suitably chosen, derive every known property of curve lines, without having in any instance a conception of the geometrical types which they represent. The theory of elliptic integrals was developed by a method the inverse of that pursued in establishing the formulæ of common trigonometry. In the latter case, the geometrical type was given—the circle—to determine the algebraical relations of its arcs. In the theory of elliptic integrals, the relations of the arcs of unknown curves are given, to determine the curves themselves. This is briefly the object of the present paper.

The true geometrical basis of this theory would doubtless long since have been developed, had not geometers sought to discover the types of those functions among plane curves. They were beguiled into this course by observing, that in one case—that of the second order—the representative curve is obviously a plane ellipse. Hence they were led by a seeming analogy to search for the types of the other integrals among plane curves also.

I have attempted thus to place on its true geometrical basis a somewhat abstruse department of analysis, and to clear up the elementary notions from which it may, with the utmost simplicity, be developed. It is only in the maturity of a science that the relations which bind together its cardinal ideas become simplified. An author, who has himself contributed much to the progress of mathematical science, well observes,—" qu'il est bien rare qu'une théorie sorte sous sa forme la plus simple des mains de son premier auteur. Nous pensons qu'on sert peut-être plus encore la science en simplifiant, de la sorte, des théories déjà connues, qu'en l'enrichissant de théories nouvelles, et c'est là un sujet auquel on ne saurait s'appliquer avec trop de soin."—Gergonne, '*Annales des Mathématiques*,' tom. xix. p. 338.

It may be asked, of what use is the theory of elliptic integrals? This is a very natural inquiry in an age when every intellectual acquisition, when every exercise of the understanding is tested by its gross material utility. Yet it may suffice to say in reply, that this theory will be found of use in many geometrical and physical inquiries. These functions not only exhibit the rectification and quadrature of conic and hyperconic sections, but they subserve the theories of the common and conical pendulums and of the elastic curve. In Astronomy, the elements of the orbits of the planets, the attraction of ellipsoids, and the problem of the rotation of a solid body round a fixed point, receive their final and complete solutions by the help of these integrals. M. Lame has proved how questions which involve the distribution of heat and the nature of isothermal surfaces may be reduced to the same functions.

In a subsequent portion of this volume, it will be shown that the complete mathematical solution of that celebrated problem the rotation of a solid body, has been for the first time obtained by the aid of those functions in their state of complete development.

CHAPTER I.

1.] The theory of Elliptic Integrals is founded on the development of the quadrinomial integral,

$$\int \frac{\mathrm{f}(x)\,\mathrm{d}x}{\sqrt{\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+x^4}};$$

in which A, B, C, and D are constants, while $\mathrm{f}(x)$ denotes a rational function of x.

It has been shown by Legendre, and, after him, by Verhulst, Hymers, and others, that by the help of some ingenious transformations the above integral may be reduced to one or other of the following fundamental forms,

$$\int \frac{\mathrm{d}\varphi}{\sqrt{1-c^2\sin^2\varphi}}, \quad \int \mathrm{d}\varphi\,\sqrt{1-c^2\sin^2\varphi},$$

and

$$\int \frac{\mathrm{d}\varphi}{[1\pm p\sin^2\varphi]\,\sqrt{1-c^2\sin^2\varphi}},$$

or, as they have been denoted by Legendre,

$$\mathrm{F}c(\varphi),\ \mathrm{E}c(\varphi),\ \text{and}\ \Pi c(p,\varphi).$$

I have ventured to make some alterations in the established notation of elliptic integrals. I have written i for the modulus, instead of c, and j for its complement instead of b; so that $i^2+j^2=1$.

The symbol c, used by writers on this subject to designate the modulus, was adopted by analogy from the formula for the rectification of a plane elliptic arc by an integral of the second order.

Although in the circular forms of the third order it still signifies a certain ellipticity, yet it has no longer the same signification in the usual form of the first order, or in the logarithmic form of the third.

Instead of the usual symbol, $\Delta = \sqrt{1-c^2\sin^2\phi} = \sqrt{1-i^2\sin^2\phi}$, $\sqrt{\mathrm{I}}$ has been substituted when i is the modulus. Should it become necessary to designate the amplitude ϕ, the expression may be written $\sqrt{\mathrm{I}_\phi}$, while $\sqrt{\mathrm{I}_\prime}$ may denote a function whose modulus is $i_\prime$.

For the elliptic integrals of the first and second orders, $\int\frac{d\phi}{\sqrt{\mathrm{I}}}$ and $\int d\phi\sqrt{\mathrm{I}}$ have been substituted. Hence $\int\frac{d\phi}{\sqrt{\mathrm{I}}}$ represents $\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}$, and $\int d\psi\sqrt{\mathrm{I}_\prime}$ may be put for $\int d\psi\sqrt{1-i_\prime^2\sin^2\psi}$. The surface of revolution may be named the *generating surface*, while the intersecting surface is always a right cylindrical surface. The parameter, of which p is the general symbol, we shall suppose to vary from positive to negative infinity, and to pass through all intermediate states of magnitude.

The nature of the representative curve will depend on the value assigned to the parameter p in the expression

$$\mathrm{K}\int\frac{d\phi}{[\pm p\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}.$$

The modulus i we shall assume to be invariable and less than 1. In this progress from $+\infty$ to $-\infty$, the parameter passes through thirteen distinct values, each of which will cause a variation in the species or properties of the hyperconic section, the representative curve of the given elliptic integral.

In the following Table we may observe that the generating surface in passing from a sphere to a paraboloid, in its course of transition becomes a plane.

It is somewhat remarkable that the common form of the elliptic integral of the first order does not appear in the Table, although it is implicitly contained in cases II. and VIII.; for the circular form of the third order, when the parameter is equal to the modulus i, may be reduced to the first. The reason why the first form of elliptic integral does not appear in the Table is this: in the thirteen cases given, the origin is placed at the centre, or symmetrically with respect to the represented curve. When the elliptic integral of the first order is given in the usual form, without a parameter, it represents a spherical parabola, but the origin is non-symmetrical, that is, the origin is placed at a focus.

Instead of p, the general symbol for the parameter, we may substitute for it particular values, such as l, m, or n, as the case may require. The quantities l, m, n, i, and j are connected by the following equations :—

$$\left.\begin{array}{l} i^2+j^2=1,\ lm=i^2,\ \text{and}\ m-n+mn=i^2,\ \text{in the circular form,} \\ i^2+j^2=1,\ ln=i^2,\ \text{and}\ m+n-mn=i^2,\ \text{in the logarithmic form,} \end{array}\right\} \quad (1)$$

m and n may be called *conjugate parameters*; while l and m, or l and n may be termed *reciprocal parameters*.

For $(1-m\sin^2\varphi)$ we may put M, and N for $(1+n\sin^2\varphi)$.

These thirteen cases are exhibited in the following Table :—

Case.	Sign.	Parameter.	Generating surface.	Cylindrical surface.	Hyperconic section.
I.	$+$	$p=n=\infty$.	Sphere.	Elliptic cylinder.	Circular sections of elliptic cylinder.
II.	$+$	$p=n=i$, or $m=n$.	Sphere.	Elliptic cylinder.	Spherical parabola.
III.	$+$	$p=n>0$.	Sphere.	Elliptic cylinder.	Spherical ellipse.
IV.	$\pm$	$p=n=0$.	Plane.	Elliptic cylinder.	Plane ellipse.
V.	$-$	$p=m=1-j$, or $m=n$.	Paraboloid indefinitely attenuated.	Circular cylinder.	Circular logarithmic ellipse.
VI.	$-$	$p=m$, or $p=n<i^2$.	Paraboloid.	Elliptic cylinder.	Logarithmic ellipse.
VII.	$-$	$p=m=i^2$.	Plane.	Elliptic cylinder.	Plane ellipse.
VIII.	$-$	$p=m=i$.	Sphere.	Elliptic cylinder.	Spherical parabola.
IX.	$-$	$p=m>i^2$ $p=m<1$.	Sphere.	Elliptic cylinder.	Spherical ellipse.
X.	$-$	$p=l=1$.	Plane.	Hyperbolic cylinder.	Plane hyperbola.
XI.	$-$	$p=l>1$.	Paraboloid.	Hyperbolic cylinder.	Logarithmic hyperbola.
XII.	$-$	$p=l=1+j$, or $m=n$.	Paraboloid.	Hyperbolic cylinder.	Equiparametral logarithmic hyperbola.
XIII.	$-$	$p=l=\infty$.	Paraboloid.	Vertical plane.	Parabola.

Cases I., IV., VII., X., XIII. give the formulæ for the rectification of the ordinary conic sections, the generating surface in these cases being a plane. When the generating surface is a sphere, we get the spherical hyperconic sections; when a paraboloid, the logarithmic hyperconic sections result.

ON THE SPHERICAL ELLIPSE.

2.] A spherical ellipse may be defined as the curve of intersection of a cone of the second degree with a concentric sphere.

In the spherical ellipse there are two points analogous to the foci of the plane ellipse, such *that the sum of the arcs of the great circles, drawn from these points to any point on the curve, is constant.* Let α and β be the principal semiangles of the cone; 2α and 2β are therefore the *principal arcs* of the spherical ellipse. Let two straight lines be drawn from the vertex of the cone, in the plane of the angle of 2α, making with the internal axis of the cone equal angles ϵ, such that

$$\cos\epsilon = \frac{\cos\alpha}{\cos\beta}. \qquad \qquad (2)$$

These lines are usually called *focals*, or the *focal lines* of the cone. The points in which they meet the surface of the sphere are termed the *foci* of the spherical ellipse.

Every umbilical surface of the second order has two concentric circular sections, whose planes, in the case of cones, pass through the greater of the external axes. Perpendiculars drawn to the planes of these sections, passing through the vertex (*they may be called the* CYCLIC AXES *of the cone*), *make with the internal axis of the cone in the plane of* 2β (*the plane passing through the internal and the lesser external axis*) equal angles η, such that

$$\cos\eta = \frac{\sin\beta}{\sin\alpha}. \qquad \qquad (3)$$

Let a series of planes be drawn through the vertex, and perpendicular to the successive sides of the cone. This series of planes will envelop a second cone, which is usually called the *supplemental cone* to the former. The cones are so related, that the planes of the circular sections of the one are perpendicular to the focal lines of the other, and conversely.

The equation of the spherical ellipse may be found as follows, from simple geometrical considerations.

Let 2α and 2β be the greatest and least vertical angles of the cone; the origin of coordinates being placed at the common centre of the sphere and cone. Let the internal axis of the cone meet the surface of the sphere in the point Z, which may be taken as the pole. Let ρ be an arc of a great circle drawn from the point Z to any point Q on the curve, ψ being the angle which the plane of this circle makes with the plane of 2α. We shall then have for the polar equation of the spherical ellipse,

$$\frac{1}{\tan^2\rho} = \frac{\cos^2\psi}{\tan^2\alpha} + \frac{\sin^2\psi}{\tan^2\beta}.$$

To show this, through the point Z let a tangent plane be drawn to the sphere. This plane will intersect the cone in an ellipse. This ellipse may be called the *plane base* of the cone, while the portion of the surface of the sphere within the cone may be termed the *spherical base* of the cone. The plane of the great circle passing through Z and Q will cut the plane base of the cone in the radius vector R; and if we write A and B for the semiaxes of this ellipse, whose plane touches the sphere, we shall have for the common polar equation of this ellipse, the centre being the pole,

$$\frac{1}{R^2}=\frac{\cos^2\psi}{A^2}+\frac{\sin^2\psi}{B^2}.$$

Now, the radius of the sphere being k, and ρ, α, β the angles subtended at the centre by R, A, B, we shall clearly have

$$R=k\tan\rho,\quad A=k\tan\alpha,\quad B=k\tan\beta;$$

whence
$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}. \quad .\ .\ .\ .\ .\ . \quad (4)$$

We may write this equation in the form

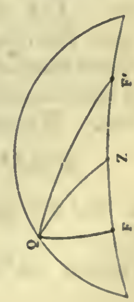
$$\frac{1-\sin^2\rho}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}(1-\sin^2\alpha)+\frac{\sin^2\psi}{\sin^2\beta}(1-\sin^2\beta);$$

or reducing,
$$\frac{1}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}+\frac{\sin^2\psi}{\sin^2\beta}. \quad .\ .\ .\ .\ .\ . \quad (5)$$

This is the equation of the spherical ellipse under another form, which may be obtained independently by orthogonally projecting the spherical ellipse on the plane of the external axes; or by taking the spherical ellipse as the symmetrical intersection of a right elliptic cylinder with the sphere.

3.] *If in the major principal arc 2α of a spherical ellipse, we assume two points equidistant from the centre, the distance ϵ being determined by the condition* $\cos\epsilon=\frac{\cos\alpha}{\cos\beta}$, *as in* (2), *the sum of the arcs of the great circles drawn from these points—the foci—to any point on the spherical ellipse is constant, and equal to the principal arc 2α.*

Fig. 1.



Let θ and θ' denote the arcs drawn from the points F, F′ to a point Q upon the curve, $QZ=\rho$, and the angle $QZF=\psi$, $FZ=F'Z=\epsilon$.

Then, as FZQ, F′ZQ are spherical triangles, we get

$$\cos\psi=\frac{\cos\theta-\cos\epsilon\cos\rho}{\sin\epsilon\sin\rho}, \quad .\ .\ .\ . \quad (a)$$

$$-\cos\psi=\frac{\cos\theta'-\cos\epsilon\cos\rho}{\sin\epsilon\sin\rho}, \quad .\ .\ .\ . \quad (b)$$

$\cos\epsilon=\frac{\cos a}{\cos\beta}$ (c), and the equation of the curve given in [(2)]

$$\cot^2\rho=\cot^2 a\cos^2\psi+\cot^2\beta\sin^2\psi \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Between (a), (b), (c), (d), we must eliminate ρ, ψ, and ϵ. Adding together (a) and (b), also subtracting (b) from (a), we get

$\cos\theta+\cos\theta'=2\cos\rho\cos\epsilon$; and $\cos\theta-\cos\theta'=2\sin\rho\sin\epsilon\cos\psi$;

from (d), $1=\cot^2 a\tan^2\rho\cos^2\psi+\tan^2\rho\cot^2\beta-\tan^2\rho\cot^2\beta\cos^2\psi$;

or $\left(\frac{\cos^2\beta-\cos^2 a}{\sin^2 a\sin^2\beta}\right)\sin^2\rho\cos^2\psi=\cot^2\beta-\frac{\cos^2\rho}{\sin^2\beta}$; substituting for $\sin\rho\cos\psi$, its value deduced by subtracting (b) from (a), we find
$\cos^2 a\ (\cos\theta-\cos\theta')^2+\sin^2 a\ (\cos\theta+\cos\theta')^2=\sin^2 2a$,
or $\cos^2\theta+\cos^2\theta'-2\cos\theta\cos\theta'(\cos^2 a-\sin^2 a)=1-\cos^2 2a$;
whence $\cos^2 2a-2\cos\theta\cos\theta'\cos 2a=1-\cos^2\theta-\cos^2\theta'$.

Completing the square and reducing, we obtain

$$\cos 2a=\cos\theta\cos\theta'\mp\sin\theta\sin\theta'=\cos(\theta\pm\theta') \text{ or}$$

$$2a=\theta\pm\theta' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

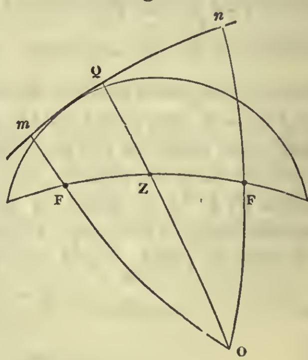
The positive sign to be taken when the curve is the spherical ellipse.

4.] *The product of the sines of the perpendicular arcs let fall from the foci of a spherical ellipse on the arc of a great circle touching it, is constant.*

Let ϖ and ϖ' be the perpendicular arcs let fall from the foci on the tangent arc of a great circle; we shall have

$$\sin\varpi\sin\varpi'=\sin(a+\epsilon)\sin(a-\epsilon).$$

Fig. 2.



Let ϖ, ϖ', ϖ'', be the perpendicular arcs, let fall from the centre, and the two foci F and $F_{\prime}$, on the tangent arc mn. These three arcs will meet in the point o, the pole of the arc mn. Let p be the perpendicular from the centre on the straight line which touches the plane elliptic base; of this straight line, mn is the projection. We shall therefore have

$$p^2=A^2\cos^2\lambda+B^2\sin^2\lambda,$$

or $\tan^2\varpi=\tan^2 a\cos^2\lambda+\tan^2\beta\sin^2\lambda$,

whence $\cos^2\varpi=\frac{\cos^2 a}{1-\sin^2\epsilon\sin^2\lambda}$.

Now FZQ$=\lambda$, whence in the spherical triangle FZO,

as
$$\mathrm{FO}=\frac{\pi}{2}-\varpi', \ \mathrm{ZO}=\frac{\pi}{2}-\varpi,$$

we shall have
$$\cos\lambda=\frac{\sin\varpi'-\cos\epsilon\sin\varpi}{\sin\epsilon\cos\varpi}.$$

In the other spherical triangle F′ZO, we shall also have

$$-\cos\lambda=\frac{\sin\varpi''-\cos\epsilon\sin\varpi}{\sin\epsilon\cos\varpi}.$$

Adding first, and then subtracting these equations, one from the other, we shall find

$$\sin\varpi'+\sin\varpi''=2\cos\epsilon\sin\varpi,$$
$$\sin\varpi'-\sin\varpi''=2\sin\epsilon\cos\varpi\cos\lambda.$$

Squaring these equations, and subtracting the latter from the former, we shall obtain

$$\sin\varpi'\sin\varpi''=\cos^2\epsilon-\cos^2\varpi\,(1-\sin^2\epsilon\sin^2\lambda).$$

Substituting for $\cos\varpi$ its value given above, and reducing,

$$\sin\varpi'\sin\varpi''=\sin(\alpha+\epsilon)\sin(\alpha-\epsilon)\,.\quad.\quad.\quad.\quad.\qquad(6)$$

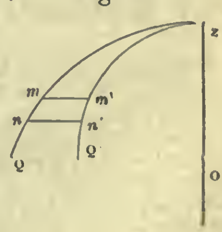
5.] The area of any portion of a spherical surface bounded by a closed curve, may be determined by the formula,

$$\text{area}=\int_0^{2\pi}d\psi\int_0^{\rho}d\sigma\,[\sin\sigma],$$

where σ is the arc of a great circle intercepted between the fixed point Z taken within the curve as pole (fig. 3), and any variable point m assumed within the bounding curve on the surface of the sphere, ρ being the spherical radius vector of the curve measured from the pole Z, and passing through m, while ψ is the angle which the plane of this great circle, passing through the points Z, m, makes with the fixed plane of a great circle passing through Z.

Let O be the centre of the sphere, Z the pole, m the assumed point, ZQ the great circle passing through them. Through Z let a great circle OZQ′ be drawn, indefinitely near to the former, $d\psi$ being the angle between the planes. Through m let a plane be drawn perpendicular to the axis OZ, meeting the great circle OZQ′ in m'. Through n, a point on ZQ indefinitely near to m, a parallel plane being drawn, it will meet the great circle OZQ′ in a point n', indefinitely near to m'. Now it is manifest from this construction that the

Fig. 3.



whole spherical area to be determined is the sum of all the indefinitely small trapezia, such as $m\,n\,m'\,n'$, into which in this manner it may be divided. To compute the value of this elementary trapezium, we have $mm' = \sin\sigma\, d\psi$, $m\,n = d\sigma$. As the pole Z is within the curve, the limits of σ are 0 and ρ; and as the surface is assumed to extend all round Z, the limits of ψ are 0 and 2π.

Whence $$\text{area} = \int_0^{2\pi} d\psi \int_0^{\rho} d\sigma\,[\sin\sigma] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

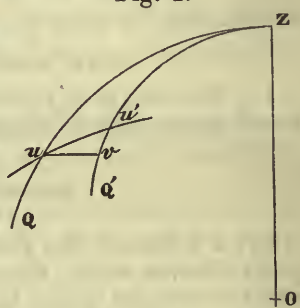
Integrating this equation between the limits 0 and ρ, we find

$$\text{area} = \int_0^{2\pi} d\psi\,[1-\cos\rho] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)^*$$

The second integration can be accomplished only when we know the relation between ρ and ψ, or the equation of the bounding curve.

6.] To find an expression for the length of a curve described on the surface of a sphere, whose radius is 1.

Fig. 4.



Let u and u' be two consecutive points on the curve, ZQ, ZQ$'$ the arcs of two great circles passing through them inclined to each other at the indefinitely small angle $d\psi$. Through u let a plane be drawn perpendicular to OZ, and meeting the great circle ZQ$'$ in v.

Then ultimately uvu' may be taken as a right-angled triangle, whence

$$\overline{uu'}^2 = \overline{uv}^2 + \overline{u'v}^2.$$

Now $uu' = d\sigma$, $uv = \sin\rho\, d\psi$, $u'v = d\rho$, whence

$$d\sigma = [d\rho^2 + \sin^2\rho\, d\psi^2]^{\frac{1}{2}}.$$

Integrating this expression between the limits $\rho_{,}$ and $\rho_{,,}$, or ψ and 0, accordingly as we take ρ or ψ for the independent variable, we get

$$\sigma = \int_{\rho''}^{\rho_{,}} d\rho\left[1+\sin^2\rho\left(\frac{d\psi}{d\rho}\right)^2\right]^{\frac{1}{2}}; \text{ or } \sigma = \int_0^{\psi} d\psi\left[\left(\frac{d\rho}{d\psi}\right)^2 + \sin^2\rho\right]^{\frac{1}{2}} \quad (7)$$

* Equation (b) may be established by the help of the simplest elementary principles. We know that the surface of the segment of a sphere comprised between a tangent plane and a parallel secant plane is equal to the circumference of a great circle multiplied into the distance between these planes. This distance is $1-\cos\rho$; ρ being the arc of a great circle, measured from the point of contact of the tangent plane to the parallel secant plane. If through the diameter perpendicular to these planes we draw two great circles, inclined one to the other at the angle $d\psi$, the surface of the sperical wedge thus formed will be $d\psi\,(1-\cos\rho)$.

7.] To apply these expressions to find the length of an arc of a spherical ellipse.

In this case it will be found simpler to integrate the differential expression for an arc of a curve, taking ρ instead of ψ as the independent variable. We may derive from (5) the following expressions,

$$\left.\begin{aligned}\sin^2\psi&=\frac{\sin^2\beta}{\sin^2\rho}\left\{\frac{\sin^2\alpha-\sin^2\rho}{\sin^2\alpha-\sin^2\beta}\right\}\\ \cos^2\psi&=\frac{\sin^2\alpha}{\sin^2\rho}\left\{\frac{\sin^2\rho-\sin^2\beta}{\sin^2\alpha-\sin^2\beta}\right\}\end{aligned}\right\}\quad .\;.\;.\;.\;.\quad \text{(a)}$$

Differentiating the former with respect to ψ and ρ, and eliminating $\sin\psi$, $\cos\psi$, using for this purpose the relations established in (a), we shall find

$$\frac{d\psi}{d\rho}=\frac{-\sin\alpha\sin\beta\cos\rho}{\sin\rho\sqrt{\sin^2\alpha-\sin^2\rho}\sqrt{\sin^2\rho-\sin^2\beta}}.\quad .\;.\;.\;.\quad \text{(b)}$$

Substituting this value of $\frac{d\psi}{d\rho}$ in the general expression for the arc in the last section, the resulting equation will become

$$\sigma=\int d\rho\left[\frac{\sin\rho\sqrt{\cos^2\rho-\cos^2\alpha\cos^2\beta}}{\sqrt{(\sin^2\alpha-\sin^2\rho)(\sin^2\rho-\sin^2\beta)}}\right],\quad .\;.\;.\quad \text{(c)}$$

an elliptic integral which may be reduced to the usual form by the following transformation: assume

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\varphi+\sin^2\beta\sin^2\varphi}{\tan^2\alpha\cos^2\varphi+\tan^2\beta\sin^2\varphi}.\quad .\;.\;.\;.\quad \text{(d)}$$

The limits of integration are 0 and $\frac{\pi}{2}$. Differentiating this expression, and introducing into (c) the relations assumed in (d), we shall obtain for the arc the following expression:—

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\varphi}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\right)\sin^2\varphi}}\right].\quad (8)$$

Let e be the eccentricity of the plane base of the cone, whose semi-axes are A and B, as in sec [2],

$$e^2=\frac{A^2-B^2}{A^2}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\cos^2\beta},$$

we may derive from (2) and (3)

$$\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\text{ and }\sin^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta};$$

or grouping these results together,

$$e^2=\frac{\tan^2 a-\tan^2\beta}{\tan^2 a}=\frac{\sin^2 a-\sin^2\beta}{\sin^2 a\cos^2\beta}=m,$$

$$\sin^2\eta=\frac{\sin^2 a-\sin^2\beta}{\sin^2 a}=i^2,\ \sin^2\epsilon=\frac{\sin^2 a-\sin^2\beta}{\cos^2 a}=n. \quad . \quad (9)$$

These quantities m, n, and i^2 fulfil the equation of condition assumed in (1)

$$m-n+mn=i^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

If we introduce these values into (8), the transformed equation will become

$$\sigma=\frac{\tan\beta}{\tan a}\sin\beta\int\left[\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\eta\sin^2\phi}}\right], \quad . \quad (10)$$

an elliptic integral of the third order and circular form, since e^2 is greater than $\sin^2\eta$, and less than 1.

This is case IX. in the Table, page 7.

This is the simplest form to which the rectification of an arc of a spherical ellipse can be reduced. The parameter of the elliptic integral is the square of the eccentricity of the plane elliptic base; and the modulus is the sine of half the angle between the planes of the circular sections of the cone.

If we write m for e^2, i for $\sin\eta$, and express the coefficient $\frac{\tan\beta}{\tan a}\sin\beta$ in terms of m and i, the expression (10) may be transformed into

$$\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\left[\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]. \quad . \quad (11)$$

It is easily shown that the coefficient $\frac{\tan\beta}{\tan a}\sin\beta$ of the elliptic integral in (10) or its equivalent $\left(\frac{1-m}{m}\right)\sqrt{mn}$ is the square root of the *criterion of sphericity*,

$$\kappa=(1-m)\left(1-\frac{i^2}{m}\right).$$

For if we substitute in this expression for i its value given in (1) $m-n+mn=i^2$, we shall find

$$\sqrt{\kappa}=\frac{\tan\beta}{\tan a}\sin\beta=\left(\frac{1-m}{m}\right)\sqrt{mn}. \quad . \quad . \quad . \quad . \quad (f)$$

As $\sqrt{\kappa}$ is manifestly real, the elliptic integral is of the circular form.

8.] To find the area of a spherical ellipse.

Resuming equations (4) and (5) of the spherical ellipse,

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}, \text{ and } \frac{1}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}+\frac{\sin^2\psi}{\sin^2\beta},$$

dividing the former by the latter, and reducing, we shall find

$$\cos\rho=\cos\alpha\frac{\sqrt{1+\frac{\tan^2\alpha}{\tan^2\beta}\tan^2\psi}}{\sqrt{1+\frac{\sin^2\alpha}{\sin^2\beta}\tan^2\psi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Substituting this value of $\cos\rho$ in the general expression for the spherical area (b) sec. [5], we obtain the result

$$\text{area}=\psi-\cos\alpha\int d\psi\left[\frac{1+\frac{\tan^2\alpha}{\tan^2\beta}\tan^2\psi}{1+\frac{\sin^2\alpha}{\sin^2\beta}\tan^2\psi}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad \text{(b)}$$

To integrate this equation, let us assume

$$\tan\psi=\frac{\tan\beta}{\tan\alpha}\tan\phi\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

and we shall find, on making the necessary transformations in the preceding expressions, the area $=\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\times$

$$\int\left[\frac{d\phi}{\left\{1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right\}\sqrt{1-\left(\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta}\right)\sin^2\phi}}\right] \quad \text{(12)}$$

Let A and B be the semiaxes of the plane elliptic base of the cone, and e its eccentricity, then we shall obviously have

$$e^2=\frac{A^2-B^2}{A^2}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

and ϵ being the angle between the spherical focus and centre,

$$\cos\epsilon=\frac{\cos\alpha}{\cos\beta} \text{ as in sec [2], whence } \sin^2\epsilon=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta}. \quad \text{(e)}$$

Introducing these relations into (12), we shall obtain the formula

$$\text{area}=\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\left[\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\epsilon\sin^2\phi}}\right]. \quad \text{(13)}$$

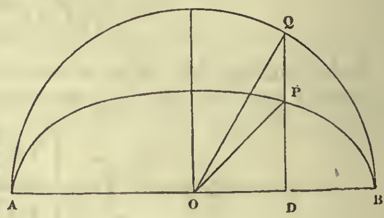
This is an elliptic function of the *third order* and *circular* form, since e^2 is less than 1, and greater than $\sin^2\epsilon$.

This seems to be the simplest form that can be found for the quadrature of the spherical ellipse, the *parameter* and squared *modulus* of the elliptic transcendent being the squares of the eccentricities of the plane and spherical ellipses respectively.

We shall show hereafter that there is a class of spherical ellipses whose quadrature may be effected by elliptic functions of the *first* order.

To determine the geometrical signification of the angle of reduction φ, in the above transformation.

Fig. 5.



On the major axis of the plane elliptic base of the cone, let a semicircle be described. Let OP be drawn, making the angle ψ with the major axis OB. Let the ordinate through P be produced to meet the circle in Q, join OQ;

$$\text{then } \frac{\tan\psi}{\tan \mathrm{QOB}}=\frac{\mathrm{PD}}{\mathrm{QD}}=\frac{\mathrm{B}}{\mathrm{A}}=\frac{\tan\beta}{\tan\alpha}; \text{ but } \frac{\tan\psi}{\tan\varphi}=\frac{\tan\beta}{\tan\alpha}; \text{ see (10)}$$

whence $\mathrm{QOB}=\varphi$, or φ is the *eccentric anomaly* of the point P.

Now, when $\psi=0$, $\varphi=0$, and when $\psi=\frac{\pi}{2}$, $\varphi=\frac{\pi}{2}$, whence φ and ψ coincide at these limits. Writing S for the area of the quadrant of the spherical ellipse; as the surface evidently consists of four symmetrical quadrants, the area or length of one quadrant will manifestly be one fourth of the area or length of the whole; whence

$$\text{area}=\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int_0^{\frac{\pi}{2}}\left[\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\epsilon\sin^2\varphi}}\right]. \quad (14)$$

9.] Let $2\alpha'$ and $2\beta'$ be the principal arcs of the supplemental cone, α' being in the plane of β, and β' in that of α. Let Σ' be the length of a quadrant of the spherical ellipse the intersection of this cone with the concentric sphere. Then we may deduce from (10)

$$\Sigma'=\frac{\tan\beta'}{\tan\alpha'}\sin\beta'\int_0^{\frac{\pi}{2}}\left[\frac{d\varphi}{\{1-e'^2\sin^2\varphi\}\sqrt{1-\sin^2\eta'\sin^2\varphi}}\right]. \quad . \quad (a)$$

Now, as the cones are assumed to be supplemental,

$$\alpha+\beta'=\frac{\pi}{2},\quad \beta+\alpha'=\frac{\pi}{2},\quad \text{whence}\quad \sin\alpha'=\cos\beta,\quad \sin\beta'=\cos\alpha,$$

$\cos \alpha' = \sin \beta$, $\cos \beta' = \sin \alpha$; therefore $\dfrac{\tan \beta'}{\tan \alpha'} = \dfrac{\tan \beta}{\tan \alpha}$, $e'^2 = e^2$, and $\sin \eta' = \sin \epsilon$. . . . . (b)

Introducing these transformations into the last formula

$$\Sigma' = \frac{\tan \beta}{\tan \alpha} \cos \alpha \int_0^{\frac{\pi}{2}} \left[\frac{d\varphi}{\{1 - e^2 \sin^2 \varphi\} \sqrt{1 - \sin^2 \epsilon \sin^2 \varphi}} \right]. \quad (15)$$

Now, if we turn to the expression found for the area of a spherical ellipse, given in (13), we shall find that it consists of two parts—a circular arc, and an elliptic integral identically the same with the one just investigated, when taken between the limits 0 and $\frac{\pi}{2}$. We thus arrive at the very remarkable result, that the rectification of a spherical ellipse depends on the quadrature of the supplemental ellipse, and reciprocally.

If we add together (13) and (15),

$$S + \Sigma' = \frac{\pi}{2}; \quad . \ . \ . \ . \ . \ . \ . \ . \quad (16)$$

or taking the whole surface 4S of the spherical conic, and the circumference $4\Sigma'$ of the supplemental conic, introducing, moreover, k the radius of the sphere, we obtain the remarkable theorem

$$4S + 4k\Sigma' = 2k^2\pi. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

Now $4k\Sigma'$ is twice the lateral surface of the supplemental cone, and 4S is the surface of the spherical ellipse. We may therefore infer that

The spherical base of any cone, together with twice the lateral surface of the supplemental cone, is equal to the surface of the hemisphere.

Let 4S′ denote the spherical base of the supplemental cone, and L the lateral surface of the original cone: from the preceding equations we obtain

$$2S + L' = k^2\pi, \; 2S' + L = k^2\pi.$$

Adding these equations,

$$\left.\begin{array}{l} 4(S + S') + 2(L + L') = 4k^2\pi. \\ \text{Subtracting one from the other,} \\ 4(S - S') = 2(L - L'); \end{array}\right\} \quad . \ . \ . \ . \ . \quad (18)$$

or, if any two cones, supplemental one to the other, are cut by a concentric sphere,

The sum of their spherical bases, together with twice the sum of their lateral surfaces, is equal to the surface of the sphere.

And, *The difference of their bases is equal to twice the difference of their lateral surfaces.*

Again, let a cone whose principal angles are supplemental be cut by a concentric sphere,

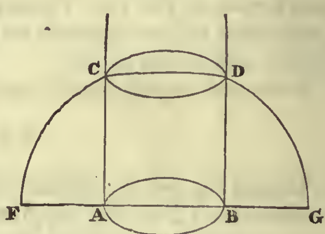
The area of the spherical base, together with twice the lateral surface, is equal to the surface of the hemisphere.

10.] We may, by the method of projective coordinates, derive an expression for the arc of a spherical ellipse.

In this case we shall consider the spherical ellipse as the curve of intersection of a right elliptic cylinder by a sphere having its centre on the axis of the cylinder.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } x^2+y^2+z^2=k^2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Fig. 6.



be the equations of the cylinder and sphere, ABCD and FGCD; then, $d\sigma$ being the element of an arc on the surface of a sphere whose radius is 1, $kd\sigma$ will be the element of the corresponding arc on the surface of the sphere whose radius is k.

Hence $$k\frac{d\sigma}{d\lambda_{\prime}}=\sqrt{\left(\frac{dx}{d\lambda_{\prime}}\right)^2+\left(\frac{dy}{d\lambda_{\prime}}\right)^2+\left(\frac{dz}{d\lambda_{\prime}}\right)^2}, \quad . \quad . \quad . \quad . \quad (20)$$

x, y and z being functions of the independent variable $\lambda_{\prime}$.

Assume

$$\left.\begin{aligned} x^2&=\frac{a^4\cos^2\lambda_{\prime}}{a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}}, \quad y^2=\frac{b^4\sin^2\lambda_{\prime}}{a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}} \\ z^2&=\frac{a^2(k^2-a^2)\cos^2\lambda_{\prime}+b^2(k^2-b^2)\sin^2\lambda_{\prime}}{a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}}. \end{aligned}\right\} \quad . \quad . \quad (21)$$

Differentiating these expressions,

$$\left.\begin{aligned} &\left(\frac{dx}{d\lambda_{\prime}}\right)^2=\frac{a^4b^4\sin^2\lambda_{\prime}}{[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]^3}, \quad \left(\frac{dy}{d\lambda_{\prime}}\right)^2=\frac{a^4b^4\cos^2\lambda_{\prime}}{[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]^3}; \\ &\text{and as } xdz+ydy+zdz=0, \\ &\left(\frac{dz}{d\lambda_{\prime}}\right)^2=\frac{a^4b^4(a^2-b^2)^2\sin^2\lambda_{\prime}\cos^2\lambda_{\prime}}{[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]^3[a^2(k^2-a^2)\cos^2\lambda_{\prime}+b^2(k^2-b^2)\sin^2\lambda_{\prime}]}. \end{aligned}\right\} (22)$$

Substituting these expressions in (20), we find

$$\left(\frac{d\sigma}{d\lambda_{\prime}}\right)^2=\frac{a^4b^4[a^2(k^2-a^2)\cos^2\lambda_{\prime}+b^2(k^2-b^2)\sin^2\lambda_{\prime}+(a^2-b^2)^2\sin^2\lambda_{\prime}\cos^2\lambda_{\prime}]}{k^2[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]^3[a^2(k^2-a^2)\cos^2\lambda_{\prime}+b^2(k^2-b^2)\sin^2\lambda_{\prime}]}. \quad (23$$

The numerator of this expression may be resolved into the factors

$$[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]\ [(k^2-a^2)\cos^2\lambda_{\prime}+(k^2-b^2)\sin^2\lambda_{\prime}],$$

and the equation may now be written

$$\frac{d\sigma}{d\lambda_{\prime}}=\frac{a^2b^2\sqrt{(k^2-a^2)\cos^2\lambda_{\prime}+(k^2-b^2)\sin^2\lambda_{\prime}}}{k[a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}]\sqrt{a^2(k^2-a^2)\cos^2\lambda_{\prime}+b^2(k^2-b^2)\sin^2\lambda_{\prime}}}. \quad (24)$$

Assume $$\tan^2\varphi_{\prime}=\left(\frac{k^2-b^2}{k^2-a^2}\right)\tan^2\lambda_{\prime}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence $$\frac{d\lambda_{\prime}}{d\varphi_{\prime}}=\frac{\sqrt{(k^2-a^2)(k^2-b^2)}}{(k^2-a^2)\sin^2\varphi_{\prime}+(k^2-b^2)\cos^2\varphi_{\prime}}.$$

(24) may now be transformed into

$$\frac{d\sigma}{d\varphi_{\prime}}=\frac{d\sigma}{d\lambda_{\prime}}\frac{d\lambda_{\prime}}{d\varphi_{\prime}}=\frac{a^2b^2\sqrt{(k^2-a^2)(k^2-b^2)}}{k[a^2(k^2-b^2)\cos^2\varphi_{\prime}+b^2(k^2-a^2)\sin^2\varphi_{\prime}]\sqrt{a^2\cos^2\varphi_{\prime}+b^2\sin^2\varphi_{\prime}}}. \quad (2$$

If we imagine a concentric cone to pass through the mutual intersection of the cylinder and the sphere, we shall have

$$\left.\begin{aligned} a=k\sin\alpha,\quad b=k\sin\beta,\\ \sin^2\eta=\frac{a^2-b^2}{a^2},\quad e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{k^2(a^2-b^2)}{a^2(k^2-b^2)}\end{aligned}\right\} . \quad . \quad . \quad (27)$$

Whence (26) may be transformed into

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\phi_{\prime}}{[1-e^2\sin^2\varphi_{\prime}]\sqrt{1-\sin^2\eta\sin^2\varphi_{\prime}}}\right], \quad . \quad (28)$$

an expression identically the same with (10).

The angle $\varphi_{\prime}$ in this expression is identical with φ in (10).

For $$x^2+y^2=\frac{a^4\cos^2\lambda_{\prime}+b^4\sin^2\lambda_{\prime}}{a^2\cos^2\lambda_{\prime}+b^2\sin^2\lambda_{\prime}}=\frac{a^4+b^4\tan^2\lambda_{\prime}}{a^2+b^2\tan^2\lambda_{\prime}};$$

eliminating $\tan\lambda_{\prime}$ by (25),

$$x^2+y^2=\frac{a^4(k^2-b^2)\cos^2\varphi_{\prime}+b^4(k^2-a^2)\sin^2\varphi_{\prime}}{a^2(k^2-b^2)\cos^2\varphi_{\prime}+b^2(k^2-a^2)\sin^2\varphi_{\prime}}.$$

Now $a^2=k^2\sin\alpha$, $b^2=k^2\sin^2\beta$, $k^2-a^2=k^2\cos^2\alpha$, $k^2-b^2=k^2\cos^2\beta$, and $x^2+y^2=k^2\cos^2\rho$.

Reducing, we get

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\varphi_{\prime}+\sin^2\beta\sin^2\varphi_{\prime}}{\tan^2\alpha\cos^2\varphi_{\prime}+\tan^2\beta\sin^2\varphi_{\prime}}. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Comparing this expression with (d) sec. [7], it follows that $\varphi=\varphi_{\prime}$. (30)

In the foregoing expressions (11) sec.[7] and (28) for the rectification of an arc of a spherical ellipse, the elliptic integrals are of the third order and circular form, with *negative* parameters. We shall now proceed to show that the same arc may be expressed by an elliptic integral of the third order and circular form, having a *positive* parameter.

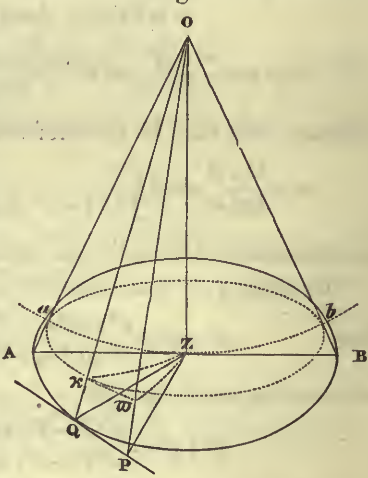
11.] It is shown in the first volume of this work, at page 184, that if p, the perpendicular let fall from a fixed point as pole on a tangent to the curve, makes the angle λ with a fixed straight line drawn through the pole, t being the intercept of the tangent between the point of contact and the foot of the perpendicular, we shall have

$$s=\int p\,\mathrm{d}\lambda \pm \frac{\mathrm{d}p}{\mathrm{d}\lambda}, \text{ and } t=\pm\frac{\mathrm{d}p}{\mathrm{d}\lambda}; \quad . \quad . \quad . \quad . \quad . \quad (31)$$

the upper sign to be taken when the radius of curvature is greater than p, the lower sign to be used when it is less than p.

To investigate an analogous formula for the rectification of a spherical curve, the intersection of a cone of any order with a concentric sphere.

Fig. 7.



Let a point Z be assumed on the surface of the sphere as pole, and through this point a tangent plane ZAQB, or (Θ), to the sphere being drawn, the cone whose vertex is at O, the centre of the sphere, and which passes through the given spherical curve, will cut this tangent plane (Θ) in a plane curve AQB, whose rectification may be effected, when possible, by the preceding expression. Now a tangent plane OQP, or (T), may be conceived as drawn touching the cone, and cutting the tangent plane (Θ) in a straight line QP or t, which will be a tangent to the plane curve in (Θ). It will also cut the sphere in an arc of a great circle ($\kappa\varpi$) which will touch the spherical curve in κ. Let the distance QO of the point of contact of the line t with the plane curve from the centre of the sphere be R. Through the centre of the sphere let a plane OZP, or (Π), be drawn at right angles to the straight line t. Now this plane, as it is perpendicular to t, must be perpendicular to the planes (Θ) and (T) which pass through t. As the plane (Π) is perpendicular to the plane (Θ), it must pass through (Z) the point of contact of this plane with the sphere, and cut the plane of the curve AQB in a straight line ZP, or p, which passes through the pole, the point of contact of (Θ) with the sphere. This line p being in (Π), must be perpendicular to t. The plane (Π) will also cut the sphere in an arc of a great circle $Z\varpi=\varpi$, perpendicular to $\kappa\varpi$, the tangent arc to the spherical curve; for these arcs must be at right angles to each other, since the planes in which they lie, (Π) and (T), are at right angles. Let P be the distance OP of the point in which the plane (Π) cuts the straight line t, from the centre of the sphere; r the distance QZ of the pole of the plane curve to the point in which t touches it, τ being the angle which t subtends at the centre of the sphere, and k its radius;

$$\left.\begin{array}{l} R^2=k^2+r^2,\ P^2=k^2+p^2,\ t^2=r^2-p^2=R^2-P^2, \\ p=k\sin\varpi,\quad t=P\tan\tau, \end{array}\right\} \quad . \quad . \quad (32)$$

τ is the angle between OQ and OP.

Let ds be the element of an arc of the plane curve between any two consecutive positions of R, indefinitely near to each other; $k\mathrm{d}\sigma$ the corresponding element of the spherical curve between the same consecutive positions of R. Then the areas of the elementary triangles on the surface of the cone, between these consecutive positions of R, having their vertices at the centre of the sphere, and for bases the elements of the arcs of the plane and spherical curves respectively, are as their bases multiplied by their altitudes. Let S and S′ be these areas; then

$$S : S' : : P\frac{ds}{d\lambda} : k^2\frac{d\sigma}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But the areas of triangles are also as the products of their sides into the sines of the contained angles, *i. e.* in this case as the squares of the sides, or

$$S : S' : : R^2 : k^2, \quad . \quad . \quad (b) \qquad \text{or } \frac{d\sigma}{d\lambda}=\frac{P}{R^2}\frac{ds}{d\lambda}; \quad . \quad . \quad . \quad (c)$$

putting for ds its value given in (31), $\dfrac{d\sigma}{d\lambda}=\dfrac{P}{R^2}\left\{\dfrac{d^2p}{d\lambda^2}+p\right\}. \quad . \quad (d)$

Now $p=P\sin\varpi$, $P^2=R^2-t^2$, and $P^2=k^2+p^2$;

whence $$P\frac{dP}{d\lambda}=p\frac{dp}{d\lambda},\ \text{and } t=-\frac{dp}{d\lambda}.$$

Substituting these values in (d),

$$\frac{d\sigma}{d\lambda}=\sin\varpi+\frac{1}{R^2}\left\{P\frac{d^2p}{d\lambda^2}-\frac{dP}{d\lambda}\frac{dp}{d\lambda}\right\}. \quad . \quad . \quad . \quad . \quad (e)$$

We shall now proceed to show that the last term of this equation is the differential of the arc, with respect to λ, subtended at the centre of the sphere.

This arc being τ, $\tan\tau=\frac{t}{\mathrm{P}}$, $\cos\tau=\frac{\mathrm{P}}{\mathrm{R}}$.

Therefore $$\frac{d\tau}{d\lambda}=\frac{1}{\mathrm{R}^2}\left\{\mathrm{P}\frac{dt}{d\lambda}-t\frac{d\mathrm{P}}{d\lambda}\right\}, \quad \text{(f)}$$

or as $$t=-\frac{dp}{d\lambda},\quad \frac{d\tau}{d\lambda}=-\frac{1}{\mathrm{R}^2}\left\{\mathrm{P}\frac{d^2p}{d\lambda^2}-\frac{dp}{d\lambda}\frac{d\mathrm{P}}{d\lambda}\right\} \quad \text{(g)}$$

Adding this equation to (e), we get for the final result,

$$\left.\begin{array}{l} \sigma=\int d\lambda\sin\varpi-\tau. \\ \text{If } t=\frac{dp}{d\lambda}, \text{ the formula becomes } \sigma=\int d\lambda\sin\varpi+\tau. \end{array}\right\} \quad (33)$$

12.] This formula serves a twofold purpose; for it will also enable us to give the quadrature of the supplemental figure on the surface of the sphere. Let ρ' be that radius vector of the supplemental figure on the surface of the sphere which is the prolongation of ϖ; $\rho'+\varpi=\frac{\pi}{2}$, and therefore $\sin\varpi=\cos\rho'$; λ remains the same in both curves; whence

$$\int\sin\varpi d\lambda=\int\cos\rho' d\lambda. \quad \text{(h)}$$

But it was shown in (b) sec. [5] that the expression for the area of a spherical curve is

$$\text{area}=\int(1-\cos\rho')\,d\lambda=\lambda-\int\sin\varpi d\lambda. \quad \text{(i)}$$

Thus the proposition established in sec. [9] as to the reciprocal relations between the rectification and quadrature of supplemental spherical conics of the second order, is shown to hold with respect to supplemental conics of any order described on the surface of a sphere.

Throughout these pages, to avoid circumlocution and needless repetitions, we shall designate as the *pro*-jected tangent, or briefly as the *protangent*, that portion of a tangent to a curve, whether it be a straight line, a circle, or a parabola, between its point of contact, and a perpendicular from a fixed point let fall upon it, whether this perpendicular be a straight line, or a circular or a parabolic arc. This definition is the more necessary, as the protangent will continually occur in the following investigations. The term is not inappropriate, as the *pro*-tangent is the *projection* of the radius vector on the tangent.

13.] To apply the formula (33) to the rectification of the spherical ellipse.

Let, as before, A and B be the semiaxes of the plane elliptic base of the cone, r the central radius vector drawn to the point of contact of the tangent t, p the perpendicular from the centre on this tangent, t the intercept of the tangent to the plane ellipse between the point of contact and the foot of the perpendicular, λ the angle between p and A. Let $\alpha, \beta, \rho, \varpi, \tau$ be the angles subtended at the centre of the sphere, whose radius is 1, by the lines A, B, r, p, t, we shall consequently have

$$\left.\begin{aligned} \mathrm{A}=k\tan\alpha,\ \mathrm{B}=k\tan\beta,\ r=k\tan\rho,\ p=k\tan\varpi, \\ \text{and } t=\sqrt{k^2+p^2}\tan\tau=\mathrm{P}\tan\tau. \end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

Now in the plane ellipse

$$p^2=\mathrm{A}^2\cos^2\lambda+\mathrm{B}^2\sin^2\lambda,\ \text{and } t^2=\frac{(\mathrm{A}^2-\mathrm{B}^2)^2\sin^2\lambda\cos^2\lambda}{p^2};$$

therefore in the spherical ellipse

$$\tan^2\varpi=\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda; \quad . \quad . \quad . \quad (35)$$

whence $$\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda.$$

Dividing the former by the latter,

$$\sin^2\varpi=\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}. \quad . \quad . \quad . \quad (36)$$

Introducing this value of $\sin\varpi$ into (33), the general form for spherical rectification, the resulting equation will become

$$\sigma=\int d\lambda\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}\right]^{\frac{1}{2}}-\tau. \quad . \quad . \quad (37)$$

14.] To reduce this expression to the usual form of an elliptic integral.

Assume $$\tan\chi=\cos\epsilon\tan\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

It must first be shown that this amplitude χ is equal to the amplitude φ in (d) sec. [7], and therefore to $\varphi_{\prime}$ in (25), as was established in sec. [10].

In an ellipse, if ψ and λ are the angles which a central radius vector, and a perpendicular from the centre, on the tangent drawn through its extremity, make with the major axis, we know that $\tan\psi=\frac{\mathrm{B}^2}{\mathrm{A}^2}\tan\lambda=\frac{\tan^2\beta}{\tan^2\alpha}\tan\lambda$. Introducing this value of $\tan\psi$ into (5) sec. [2] and reducing,

$$\cos^2\rho=\cos^2\alpha\cos^2\beta\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\tan^2\alpha\cos^2\beta\cos^2\lambda+\tan^2\beta\cos^2\alpha\sin^2\lambda}\right].$$

Comparing this value of $\cos^2\rho$ with that assumed for $\cos^2\rho$ in (d) sec. [7], namely,

$$\cos^2\rho = \frac{\sin^2 a \cos^2\varphi + \sin^2\beta \sin^2\varphi}{\tan^2 a \cos^2\varphi + \tan^2\beta \sin^2\varphi},$$

we get, after some reductions, $\tan\varphi = \cos\epsilon \tan\lambda$. . . . . (39)

But in (38) we assumed $\tan\chi = \cos\epsilon \tan\lambda$. Hence the amplitudes φ, φ', and χ in (d) sec. [7], (25), and (38) are equal. We may accordingly write φ instead of χ or $\varphi_{\prime}$. Substituting the value of $\tan\lambda$ derived from the equation, $\tan\varphi = \cos\epsilon \tan\lambda$, in (38), the integral in (37) becomes

$$\int \frac{\cos a \cos\beta [\sin^2 a - (\sin^2 a - \sin^2\beta)\sin^2\varphi] d\varphi}{[\cos^2 a + (\sin^2 a - \sin^2\beta)\sin^2\varphi]\sqrt{\sin^2 a \cos^2\varphi + \sin^2\beta \sin^2\varphi}}.$$

Now $\cos\epsilon = \dfrac{\cos a}{\cos\beta}$, $\tan^2\epsilon = \dfrac{\sin^2 a - \sin^2\beta}{\cos^2 a}$, $\sin^2\eta = \dfrac{\sin^2 a - \sin^2\beta}{\sin^2 a}$. (40)

Making the substitutions suggested by these relations, and reducing, we shall find

$$\left.\begin{aligned} \sigma = \frac{\cos\beta}{\cos a \sin a} \int \left[\frac{d\varphi}{[1 + \tan^2\epsilon \sin^2\varphi]\sqrt{1 - \sin^2\eta \sin^2\varphi}} \right] \\ - \frac{\cos a \cos\beta}{\sin a} \int \frac{d\varphi}{\sqrt{1 - \sin^2\eta \sin^2\varphi}} - \tau \end{aligned}\right\}, \quad (41)$$

an elliptic integral of the third order, with a *positive* parameter, and therefore of the *circular* form.

This is case IX. in the Table, page 7.

Writing n for $\tan^2\epsilon$, i for $\sin\eta$, and expressing $\sin a$, $\cos a$, $\sin\beta$, $\cos\beta$ in terms of n and i, (41) becomes

$$\left.\begin{aligned} \sigma = \left(\frac{1+n}{n}\right)\sqrt{mn} \int \left[\frac{d\varphi}{[1 + n \sin^2\varphi]\sqrt{1 - i^2 \sin^2\varphi}} \right] \\ - \frac{i^2}{\sqrt{mn}} \int \frac{d\varphi}{\sqrt{1 - i^2 \sin^2\varphi}} - \tau \end{aligned}\right\} . \quad (42)$$

If we put $k_{\prime}$ for the *criterion of sphericity*, as in sec. [7], with respect to n the *positive parameter*, or $\kappa_{\prime} = (1+n)\left(n + \dfrac{i^2}{n}\right)$, or $\sqrt{\kappa_{\prime}} = \left(\dfrac{1+n}{n}\right)\sqrt{mn}$, it may easily be shown that $\sqrt{\kappa_{\prime}} = \dfrac{\cos\beta}{\cos a \sin a}$, the coefficient of the preceding integral. Hence also, $\kappa\kappa_{\prime} = j^4$.

15.] To express the *protangent* τ in terms of λ and φ. We found in sec. [11]

$$\tan^2\tau=\frac{t^2}{\mathrm{P}^2}=\frac{t^2p^2}{\mathrm{P}^2p^2}=\frac{(\mathrm{A}^2-\mathrm{B}^2)^2\sin^2\lambda\cos^2\lambda}{[k^2+a^2\cos^2\lambda+b^2\sin^2\lambda]\,[a^2\cos^2\lambda+b^2\sin^2\lambda]}.$$

Now

$$\mathrm{A}=k\tan\alpha,\ \mathrm{B}=k\tan\beta,\ e^2=\frac{\mathrm{A}^2-\mathrm{B}^2}{\mathrm{A}^2},\ \text{and}\ \sin^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta},$$

whence
$$\tan\tau=\frac{e^2\sin\alpha\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}\,\sqrt{1-\sin^2\epsilon\sin^2\lambda}}. \quad . \quad . \quad (43)$$

To express $\tan\tau$ *in terms of the amplitude* φ.

Assume the relation established in (d) sec. [7] or (25) or (38) or (39), $\tan\varphi=\cos\epsilon\tan\lambda$. Introducing this condition into (43), we obtain

$$\tan\tau=\frac{e\tan\epsilon\sin\varphi\cos\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}}; \quad . \quad . \quad . \quad . \quad . \quad (44)$$

or as
$$\sqrt{m}=e,\ \ \sqrt{n}=\tan\epsilon,\ \ i=\sin\eta,$$

the last equation becomes
$$\tan\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}. \quad . \quad . \quad . \quad (45)$$

Hence (42) may now be written

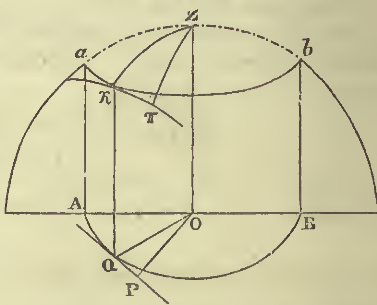
$$\left.\begin{aligned}\sigma=&\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{[1+n\sin^2\varphi]\,\sqrt{1-i^2\sin^2\varphi}}\right]\\ &-\frac{i^2}{\sqrt{mn}}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right]\end{aligned}\right\}. \quad (46)$$

Now this formula and (11) represent the same arc of the spherical ellipse; they may therefore be equated together. Accordingly

$$\left.\begin{aligned}&\left(\frac{1+n}{n}\right)\int\left[\frac{d\varphi}{[1+n\sin^2\varphi]\,\sqrt{1-i^2\sin^2\varphi}}\right]\\ &-\left(\frac{1-m}{m}\right)\int\left[\frac{d\varphi}{[1-m\sin^2\varphi]\,\sqrt{1-i^2\sin^2\varphi}}\right]\\ =&\frac{i^2}{mn}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right]\end{aligned}\right\}. \quad (47)$$

This is the well-known theorem established by LEGENDRE, *Traité des Fonctions Elliptiques,* tom. i. p. 72, for the comparison of elliptic integrals of the circular form, with positive and negative parameters respectively. These circular forms arise from treating the element of the spherical conic either as the hypotenuse of an infinitesimal right-angled triangle, or as an element of a circular arc, having the same curvature. When we adopt the former principle, we obtain for the arc an elliptic integral of the third order, circular form, and negative parameter. When we select the latter, we get a circular form of the same order, with a positive parameter. Equating these expressions for the same arc of the curve, the resulting relation is Legendre's theorem. We thus see how an elliptic integral with a *positive* parameter may be made to depend on another with a negative parameter less than 1 and greater than i^2.

Fig. 8.



16.] We must not confound the angle λ in the preceding article with the angle $\lambda_{,}$ in Art. [10]. We shall investigate the relation between them. Through ZO, the axis of the cylinder, let a plane be drawn making the angle ψ with the plane ZOAa. Let this plane cut the spherical ellipse in the point κ, and the plane ellipse the orthogonal projection of the latter in the point Q. Through κ draw an arc of a great circle $\kappa\pi$ touching the curve, and through Q draw a right line touching the plane ellipse. From Z let fall the perpendicular arc Zπ on the tangent arc of the circle, making the angle λ with the arc Za. From O let fall on the tangent to the plane ellipse at Q the perpendicular OP making the angle $\lambda_{,}$ with OA.

Then $$\tan\lambda = \frac{\tan^2 a}{\tan^2\beta}\tan\psi, \text{ and } \tan\lambda_{,} = \frac{\sin^2 a}{\sin^2\beta}\tan\psi.$$

Hence we derive $$\frac{\tan\lambda_{,}}{\tan\lambda} = \cos^2\epsilon.$$

Consequently $$\tan\lambda \,.\, \tan\lambda_{,} = \cos^2\epsilon\,\tan^2\lambda.$$

But we have shown in (39) that
$$\tan^2\phi = \cos^2\epsilon\,\tan^2\lambda,$$
whence $$\tan^2\phi = \tan\lambda\,\tan\lambda_{,}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

or *the tangent of the amplitude* ϕ *is a mean proportional between the tangents of the normal angles which a point of contact* κ *on the spherical ellipse, and its projection* Q *on the plane ellipse the base of the cylinder produce.*

17.] We may obtain, under another form, the rectification of the spherical ellipse.

Assume the equations of the right cylinder and generating sphere as given in (19),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } x^2+y^2+z^2=k^2.$$

Make $$x=a\sin\theta,\ \ y=b\cos\theta;$$

hence $$z^2=k^2-a^2\sin^2\theta-b^2\cos^2\theta; \quad \ldots \quad (49)$$

and therefore $$k\frac{d\sigma'}{d\theta}=\left[\frac{a^2(k^2-b^2)\cos^2\theta+b^2(k^2-a^2)\sin^2\theta}{(k^2-b^2)\cos^2\theta+(k^2-a^2)\sin^2\theta}\right]^{\frac{1}{2}}. \quad (50)$$

Now

$$a^2(k^2-b^2)=k^4\sin^2\alpha\cos^2\beta,\ \ b^2(k^2-a^2)=k^4\sin^2\beta\cos^2\alpha,$$

$$k^2-b^2=k^2\cos^2\beta,\ \ k^2-a^2=k^2\cos^2\alpha.$$

Substituting these values in (50), and integrating,

$$\sigma'=\int d\theta\left[\frac{\tan^2\alpha\cos^2\theta+\tan^2\beta\sin^2\theta}{\sec^2\alpha\cos^2\theta+\sec^2\beta\sin^2\theta}\right]^{\frac{1}{2}}. \quad \ldots \quad (51)$$

If we now compare this formula with (37) and make $\theta=\lambda$, we shall have $$\sigma'-\sigma=\tau. \quad \ldots \quad (52)$$

Hence we may represent the difference between two arcs of a spherical ellipse, measured from the vertices of the major and minor arcs of the curve, by an arc τ of a great circle which touches the spherical ellipse.

18.] We may thus, by the help of the foregoing theorems, show that when any elliptic integral of the third order and circular form is given, whether the parameter be positive or negative, we may always obtain the elements of the spherical ellipse, of whose arc the given function is the representative.

Let the parameter be negative.

As $$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=m, \text{ and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2,$$

we shall have $$\tan^2\alpha=\frac{m-i^2}{i^2(1-m)},\ \tan^2\beta=\frac{m-i^2}{i^2}. \quad \ldots \quad (53)$$

In order that these values of $\tan\alpha$, $\tan\beta$ may be real, we must have $m>i^2$ and $m<1$.

Let the parameter be positive.

Now $$\tan^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}=n, \text{ and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2;$$

hence $$\tan^2\alpha=\frac{n}{i^2},\ \tan^2\beta=\frac{n}{i^2}\frac{(1-i^2)}{1+n}. \quad \ldots \quad (54)$$

There is in this case no restriction on the magnitude of n.

19.] To determine the value of the expression

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right],$$

when n is infinite.

As $\quad m-n+mn=i^2,\;$ or $(1-m)(1+n)=1-i^2=j^2,$

when n is infinite, $m=1$.

Resuming the expression given in (47),

$$\left.\begin{aligned}\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right]\\ -\frac{i^2}{\sqrt{mn}}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}\right]\end{aligned}\right\},$$

we find that when n is infinite, a is a right angle.

For $\quad n=\tan^2\epsilon=\dfrac{\sin^2 a-\sin^2\beta}{\cos^2 a}=\infty$; therefore $a=\dfrac{\pi}{2}$.

Now ψ being the angle between the spherical radius vector drawn to the extremity of the arc, and the major principal arc, we have

$$\tan\psi=\frac{\tan^2\beta}{\tan^2 a}\tan\lambda,\quad\text{and }\tan\varphi=\frac{\cos a}{\cos\beta}\tan\lambda,$$

or

$$\tan\psi=\frac{\tan\beta}{\tan a}\,\frac{\sin\beta}{\sin a}\tan\varphi.$$

Hence ψ is indefinitely less than φ, when n is infinite, or when a is a right angle. In this case therefore $\sigma=0$, and we get, when n is infinite, and φ not 0,

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}\right]=\frac{\pi}{2}. \qquad (55)$$

We might have derived this theorem directly from (46), by the transformation

$$\sqrt{n}\sin\varphi=\tan\omega.$$

This is case I. in the Table, p. 7.

CHAPTER II.

ON THE SPHERICAL PARABOLA.

20.] It remains now to exhibit a class of spherical conic sections whose rectification may be effected by elliptic integrals of the *first* order.

The curve which is the gnomonic projection of a plane parabola on the surface of a sphere, the focus being the pole, may be rectified by an elliptic integral of the first order.

Let a sphere be described touching the plane of the parabola at its focus. *The spherical curve which is the intersection of the sphere with a cone, whose vertex is at its centre, and whose base is the parabola, may be called the spherical parabola.*

To find the polar equation of this curve.

The polar equation of the parabola, the focus being the pole, is $r=\frac{2g}{1+\cos\omega}$, $4g$ being the parameter of the parabola. Let γ be the angle which g subtends at the centre of the sphere, and ρ the angle subtended by r, then

$$\tan\rho=\frac{2\tan\gamma}{1+\cos\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Let p be the perpendicular from the focus on a tangent to the parabola, μ the angle which this perpendicular makes with the axis of the parabola; $p=\frac{g}{\cos\mu}$. Whence in the spherical curve, as $p=k\tan\varpi$, $g=k\tan\gamma$,

$$\tan\varpi=\frac{\tan\gamma}{\cos\mu}; \text{ or } \sin\varpi=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}. \quad . \quad . \quad . \quad (57)^*$$

* The expression for a perpendicular arc from the focus of any spherical ellipse on a tangent arc to it may be found as follows:—

The spherical triangle, fig. 2, sec. [4], F O F′, in which $\mathrm{OFF'}=\mu$, $\mathrm{OF}=\frac{\pi}{2}-\varpi'$,

$$\mathrm{OF'}=\frac{\pi}{2}-\varpi'', \text{ gives } \cos\mu=\frac{\sin\varpi''-\cos 2\epsilon\sin\varpi'}{\sin 2\epsilon\cos\varpi'};$$

from (6) we have $\sin\varpi'\sin\varpi''=\sin(a+\epsilon)\sin(a-\epsilon)$; eliminating $\sin\varpi''$ between these equations, we obtain, after some reductions,

$$\sin^2\varpi_{\prime}=\frac{\sin^2(2\epsilon)\cos^2\mu+2\sin(a+\epsilon)\sin(a-\epsilon)\cos(2\epsilon)\pm\sin(2\epsilon)\cos\mu\sqrt{\sin^2(2a)-\sin^2(2\epsilon)\sin^2\mu}}{2[1-\sin^2(2\epsilon)\sin^2\mu]}.$$

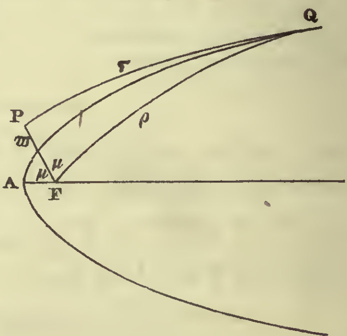
When the curve is the spherical parabola, $a+\epsilon=\frac{\pi}{2}$, $a-\epsilon=\gamma$, and the preceding expression becomes $\sin\varpi'=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$ or $\sin\varpi'=1$ as we take the sign $-$ or $+$.

The locus of the foot of this perpendicular is a great circle touching the spherical parabola at its vertex. Draw the tangent circle at A, and produce the perpendicular ϖ' until it meets this tangent circle in D. Write δ for this produced perpendicular arc. Hence in the right-angled spherical triangle D O A, $\cos\mu=\tan\gamma\cot\delta$, or $\tan\delta=\frac{\tan\gamma}{\cos\mu}$. But $\tan\varpi'=\frac{\tan\gamma}{\cos\mu}$. Whence $\varpi'=\delta$. The second value of ϖ', when the circle is drawn touching the spherical parabola at the other vertex B, is $\frac{\pi}{2}$, as shown above. This is manifestly the true value of ϖ', since the focus F is the pole of the great circle touching the curve at B.

Introduce this expression into the general form for spherical rectification, $\sigma=\int \sin\varpi \mathrm{d}\mu+\tau$, given in (31); we use the positive sign with τ, since $t=\frac{\mathrm{d}p}{\mathrm{d}\mu}$; and as τ, ϖ, and μ are the sides and an angle of a right-angled spherical triangle, since $2\mu=\omega$, we get, by Napier's rules, $\tan\tau=\sin\varpi\tan\mu$, whence, by substitution,

$$\sigma=\sin\gamma\int\frac{\mathrm{d}\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right]. \tag{58}$$

Fig. 9.



When the sphere becomes indefinitely great, the spherical parabola approaches in its contour indefinitely near to the plane parabola; k being the radius of the sphere,

$$\sin\gamma=\tan\gamma=\frac{g}{k},$$

since γ in this case is indefinitely small, whence $\cos^2\gamma=1$. In this manner, since $s=k\sigma$, (58) may be transformed into

$$s=g\int\frac{\mathrm{d}\mu}{\cos\mu}+g\frac{\sin\mu}{\cos^2\mu},$$

the well-known formula for the rectification of a plane parabola. When, on the other hand, the sphere becomes indefinitely small compared with the parabola, γ approximates to a right angle, and (58) becomes

$$s=\mu+\tan^{-1}(\tan\mu)=2\mu,$$

as it should be, since 2μ is the angle which the radius vector ρ makes with the axis.

We shall find the notice of these extreme cases useful.

21.] Although we have called this curve the spherical parabola, as indicating its mode of generation, it is in fact a closed curve, like all other curves which are the intersections of cones of the second degree with concentric spheres. It is a spherical ellipse; and we shall now proceed to determine its principal arcs.

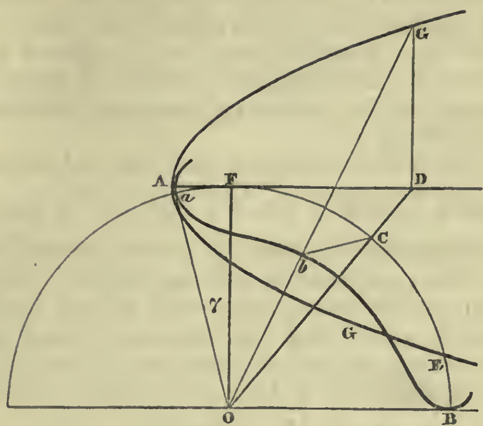
Let ADG be a parabola, F its focus, O being the centre of the sphere which touches the plane of the parabola at F, and being also the vertex of the obtuse-angled cone, of which the parabola ADG is a section parallel to the side of the cone OB. Let the angle AOF or the arc Fa be γ, a and β being the principal semiangles of the cone;

$$2a=\frac{\pi}{2}+\gamma=\mathrm{AOB},$$

whence

$$\tan^2 a=\frac{1+\sin\gamma}{1-\sin\gamma}.$$

Fig. 10.



To determine the angle β, or the arc Cb. Bisect the vertical angle AOB of the cone by the line OD, and draw DG an ordinate of the parabola. Then $\tan^2\beta=\left(\frac{\text{DG}}{\text{OD}}\right)^2$. As AOD is an isosceles triangle, $\text{AD}=\text{AO}=\frac{\text{OF}}{\cos\gamma}$; and

$$\overline{\text{OD}}^2=\frac{\overline{\text{OF}}^2}{\sin^2 a}=\frac{\overline{\text{OF}}^2}{\sin^2\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)}=\frac{2\overline{\text{OF}}^2}{1+\sin\gamma}.$$

We have also, as DG is an ordinate of the parabola,

$$\overline{\text{DG}}^2=4\text{AF}\times\text{AD}=4\text{OF}.\tan\gamma\times\frac{\text{OF}}{\cos\gamma}=4\frac{\overline{\text{OF}}^2.\sin\gamma}{\cos^2\gamma}.$$

Hence, substituting, $\tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}$.

We may therefore announce the following important theorem :—

The spherical ellipse, whose principal arcs are given by the equations

$$\tan^2 a=\frac{1+\sin\gamma}{1-\sin\gamma},\quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}, \quad . \quad . \quad . \quad (59)$$

γ being any arbitrary angle, may be rectified by an elliptic function of the first order.

Write x for $\tan a$, y for $\tan\beta$, and eliminate $\sin\gamma$ from the preceding equations,

$$\tan^2 a-\tan^2\beta=x^2-y^2=1, \quad . \quad . \quad . \quad . \quad (59^*)$$

the equation of an equilateral hyperbola. We thus obtain the following theorem :—

Any spherical conic section, the tangents of whose principal semiarcs are the ordinates of an equilateral hyperbola whose transverse semiaxis is 1, *may be rectified by an elliptic function of the first order. The quadrature of a spherical conic may be effected by an elliptic function of the first order, when the cotangents of the principal semiangles of the cone are the ordinates of an equilateral hyperbola whose transverse semiaxis is* 1.

22.] When we take the complete function, and integrate between the limits 0 and $\frac{\pi}{2}$, we get, not the length of a quadrant of the spherical parabola, as we do when we take the centre as origin, but the length of two quadrants or half the ellipse. We derive also this other remarkable result, that when μ is a right angle, the spherical triangle whose sides are the radius vector, the perpendicular arc on the tangent, and the intercept of the tangent arc between the point of contact and the foot of the perpendicular, is a quadrantal equilateral triangle. For when $\mu=\frac{\pi}{2}$,

$$\rho=\frac{\pi}{2}, \quad \varpi=\frac{\pi}{2}, \quad \tau=\frac{\pi}{2}.$$

It may also easily be shown, that the arc of a great circle which touches the spherical parabola, intercepted between the perpendicular arcs let fall upon it from the foci, is in every position constant, and equal to a quadrant*.

Hence the spherical parabola is the envelope of a quadrantal arc of a great circle, which always has its extremities on two fixed great circles of the sphere, the angle between the planes of these circles being $=\frac{\pi}{2}+\gamma$.

If we take the spherical conic supplemental to the given spherical parabola, the foci of this latter are the extremities of the minor principal arc of the former, and the cyclic arcs of the former are tangents to the latter at the extremities of its major principal arc.

Resuming the equations given in (59), which express the tangents of the principal semiarcs of the spherical parabola in terms of $\sin\gamma$, namely

$$\tan^2 \alpha=\frac{1+\sin\gamma}{1-\sin\gamma}, \quad \tan^2 \beta=\frac{2\sin\gamma}{1-\sin\gamma},$$

* As $\sin\varpi'=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$, and $\sin\varpi'\sin\varpi''=\sin\gamma$, see (6), we must have

$\sin\varpi''=\sqrt{1-\cos^2\gamma\sin^2\mu}$. Hence, as $\varpi'=\frac{\pi}{2}-\text{FO}$, and $\varpi''=\frac{\pi}{2}-\text{F'O}$,

$\cos\text{FO}\,.\,\cos\text{FO}'=\sin\gamma=\cos\text{FF}'$: or the angle FOF' is a right angle. (Fig. 2.)

writing i for $\cos\gamma$, and j for $\sin\gamma$, we get

$$\left.\begin{array}{ll} & \tan^2\epsilon=\frac{1-j}{1+j},\ \ e^2=\frac{1-j}{1+j}\ \sin^2\eta=\left(\frac{1-j}{1+j}\right)^2, \\ \text{whence} & \tan^2\epsilon=e^2=\sin\eta=\cos^2\beta. \\ \text{Again, since} & 2\epsilon+\gamma=\frac{\pi}{2},\ \sin 2\epsilon=\cos\gamma=i, \text{and} \cos 2\epsilon=j. \\ \text{Now} & n=\tan^2\epsilon,\ \ m=e^2;\ \ \text{hence } n=m=\frac{1-j}{1+j}. \end{array}\right\} \quad (60)$$

It is proper to remark that, in the case of the spherical parabola, i is not the modulus, but $\frac{1-j}{1+j}$.

23.] We shall now proceed to the rectification of an arc of the spherical parabola, the centre being the pole. By this method we shall obtain certain geometrical results which have hitherto appeared as mere analytical expressions. In (8) or (28) we found for an arc of a spherical ellipse measured from the major principal arc, the following expression, the centre being the pole,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{(1-e^2\sin^2\phi)\sqrt{1-\sin^2\eta\sin^2\phi}},$$

or, substituting the values of the constants given by the preceding equations,

$$\sigma=\frac{2j}{1+j}\int\frac{d\phi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}. \quad . \quad (61)$$

But when the focus is the pole, we found for the arc the following expression in (58),

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right].$$

Equating these values of σ, we obtain the resulting equation,

$$\left.\begin{array}{l} \frac{2j}{1+j}\int\frac{d\phi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}} \\ =j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right] \end{array}\right\}. \quad . \quad (62)$$

24.] We shall now show that the amplitudes ϕ and μ in the preceding formula are connected by the equation

$$\tan(\phi-\mu)=j\tan\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

a relation long ago established by Lagrange.

Let ϖ and ϖ' be the perpendicular arcs from the centre and focus of the spherical parabola on a tangent arc to the curve. Let λ and μ be the angles which these perpendicular arcs make with the major principal arc. The distance between the centre and focus of the spherical parabola, with the complements of these perpendiculars, constitute the sides of a spherical triangle. We shall therefore have

$$\sin^2\lambda = \sin^2\mu \frac{\sec^2\varpi}{\sec^2\varpi'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Now $\sec^2\varpi = \sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda$, as in (35); or writing for $\sec\alpha$, $\sec\beta$ their particular values in the spherical parabola, given in (59),

$$\sec^2\varpi = \frac{2}{1-\sin\gamma} - \sin^2\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Again, as $\tan\varpi' = \dfrac{\tan\gamma}{\cos\mu}$, $\sec^2\varpi' = \dfrac{\tan^2\gamma + \cos^2\mu}{\cos^2\mu}$;

reducing (64), the result is

$$\tan^2\lambda = \frac{2(1+\sin\gamma)}{(\cot\mu - \sin\gamma\tan\mu)^2}. \quad . \quad . \quad . \quad . \quad . \quad (66)$$

In the case of the spherical parabola,

$\cos^2\epsilon = \dfrac{1+\sin\gamma}{2}$, whence (66) becomes

$$\cos\epsilon\tan\lambda = \frac{1+\sin\gamma}{\cot\mu - \sin\gamma\tan\mu}, \text{ or } \cos\epsilon\tan\lambda = \frac{\tan\mu + \sin\gamma\tan\mu}{1-\sin\gamma\tan\mu\,.\tan\mu}. \quad (67)$$

The second member of this equation is manifestly the expression for the tangent of the sum of two arcs μ and ν, if we make $\tan\nu = \sin\gamma\tan\mu$.

Hence $\cos\epsilon\tan\lambda = \tan(\mu+\nu)$.

In (25), or (38) or (39), we assumed $\tan\phi = \cos\epsilon\tan\lambda$.

Hence $\phi = \mu + \nu$, or $\tan(\phi-\mu) = \tan\nu = \sin\gamma\tan\mu$.

A simple geometrical interpretation of Lagrange's theorem,

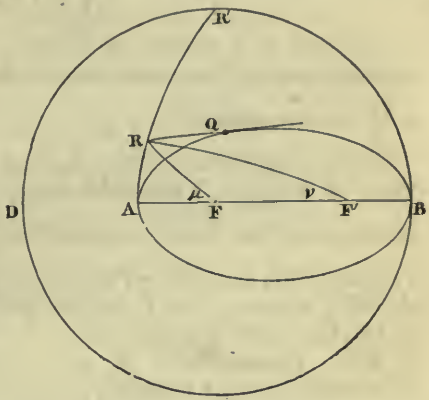
$$\tan(\phi-\mu) = \sin\gamma\tan\mu$$

may be given by the aid of the spherical parabola.

Let DR′B be the great circle, the base of the hemisphere, whose pole is F (fig. 11). Let BQA be a spherical parabola, touching the great circle at B, and having one of its foci at F the pole of the hemisphere whose base is the circle DR′B. Let RQ be an arc of a great circle, a tangent to the curve at Q. From F let fall upon it the perpendicular arc FR. The point R is in the great circle AR which touches the curve at its vertex A (see note to p. 29). The pole of this circle is the second focus F′; for $AF' = FB = \frac{\pi}{2}$. Let the arcs RF, RF′ make the angles μ and ν with the transverse arc

AB. Hence AR$=\nu$. In the spherical triangle FAR, right-angled at A, we have $\sin \text{AF}=\tan\nu \cot\mu$. Now, as AF$=\gamma$, $\sin \text{AF} =\sin\gamma=j$; and if $\phi=\mu+\nu$, $\nu=\phi-\mu$, or, reducing,

Fig. 11.



$$\tan(\phi-\mu)=j\tan\mu;$$

whence we may infer that while *the original amplitude is the angle μ at the focus* F, *the derived amplitude ϕ is the sum of the angles μ and ν at the foci* F *and* F′, *or the amplitude ϕ is the sum of the arcs of two great circles, touching the spherical parabola at the extremities of the principal major arc of the curve, intercepted between those points of contact and the perpendicular arc* FR *let fall from the focus* F *on the tangent arc* RQ *to the curve.*

Hence while the original amplitude μ is equal to an arc of the tangent circle at B, made by RF produced to meet this circle BR′, the derived amplitude ϕ is equal to the sum of two arcs of the tangent circles drawn at A and B, and given by the same construction.

When the function is complete, or $\mu=\frac{\pi}{2}$, R will coincide with R′ the pole of the great circle AB, whence ν is also$=\frac{\pi}{2}$; and as $\phi=\mu+\nu$, $\phi=\pi$. This shows that when the function is complete, or the amplitude is a right angle, the amplitude of the derived function will be two right angles.

When the spherical parabola approximates to a great circle of the sphere, the second focus F′ will approach to F the immovable focus. The arc RF′ will therefore approach to coincidence with the arc RF, or the angle ν will approximate to μ, so that $\phi=\mu+\nu =2\mu$ nearly.

This is the geometrical explanation of the analytical fact observed in this theory, that when the modulus diminishes, or the spherical parabola approximates to a great circle of the sphere, the ratio of any two successive amplitudes approximates to that of two to one.

When the greater principal arc of the spherical parabola is a right angle and a half, $\sin\gamma=\frac{1}{\sqrt{2}}$, and, if C be its circumference,

$C=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}+\pi$. But two quadrants $2s$, or the loop of a lemniscate, are$=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}$. Hence $2s=C-\pi$.

Or the *loop of a lemniscate is equal to the difference between the circumference of the spherical parabola whose greater principal arc is* $\frac{3\pi}{2}$, and a semicircle.

When a quadrant of the spherical parabola is taken, or when the point of contact Q coincides with the extremity of the principal minor arc of the curve, we shall have $\phi=\frac{\pi}{2}$.

Since in this case $RQ=PQ$, $FV=F'V$; therefore $\mu=OFV=OF'V$, or $RF'V=\mu+\nu$. As V is the pole of RP, and F' is the pole of AR, the point R is the pole of VF'. Hence RF'V is a right angle; but $\mu+\nu=RF'V$, whence $\phi=\frac{\pi}{2}$. As

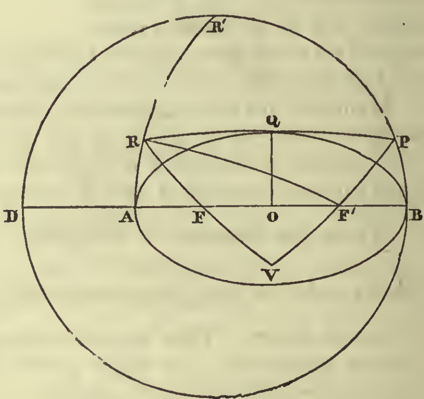
$$\tan(\phi-\mu)=j\tan\mu,$$

when $\phi=\frac{\pi}{2}$, $\tan\mu=\frac{1}{\sqrt{j}}$. If in the expression

$$\tan\tau=\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$$

given in (58), we substitute this value of $\tan\mu$, we shall get $\tan\tau=1$, or $\tau=\frac{\pi}{4}$.

Fig. 12.



Since FVF' (fig. 12) is an isosceles spherical triangle, and $\cos FF'=\cos 2\epsilon=j$, and $\tan^2 F'FV=\tan^2\mu=\frac{1}{j}$, $\cos FF'\tan^2 F'FV=1$, or the angle V is a right angle, or PR is a quadrant.

As two quadrants of a spherical parabola are together double of one, we shall have, writing the integral $\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}$ in the abbreviated form $\int\frac{d\mu}{\sqrt{I}}$,

$$j\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}+\frac{\pi}{2}=2j\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}+2.\frac{\pi}{4}, \text{ or } \int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=2\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}. \quad (68)$$

Now, when i is nearly 1, $\int \frac{d\mu}{\sqrt{I}} = \int \frac{d\mu}{\cos\mu} = \log(\sec\mu + \tan\mu)$.

Taking this expression between the limits $\mu=0$, and $\mu=\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}$,

we shall have, since $\sin\mu=\frac{1}{\sqrt{1+j}}$, $\cos\mu=\frac{\sqrt{j}}{\sqrt{1+j}}$, and neglecting j and its powers when added to 1, j being very small,

$$\sec\mu+\tan\mu=\frac{2}{\sqrt{j}},\ \text{whence}\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{2}{\sqrt{j}}\right).$$

Therefore (68) gives
$$\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{4}{j}\right)^{*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

25.] To show that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}},$$

the amplitudes ϕ and μ being connected as before, by the equation $\tan(\phi-\mu)=j\tan\mu$. Since, as in 67,

$$\tan\phi=\frac{1+\sin\gamma}{\cot\mu-\sin\gamma\tan\mu}=\frac{1+j}{\cot\mu-j\tan\mu},$$

differentiating this expression with respect to ϕ and μ,

$$\frac{1+j}{\sin^2\phi}\frac{d\phi}{d\mu}=\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu\sin^2\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (70)$$

We have also
$$\tan^2\phi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{(\cos^2\mu-j\sin^2\mu)^2}. \quad . \quad . \quad . \quad . \quad . \qquad (71)$$

Whence, after some reductions,

$$\sin^2\phi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{1-i^2\sin^2\mu}. \quad . \quad . \quad . \quad . \quad . \qquad (72)$$

Multiplying this expression by $\left(\frac{1-j}{1+j}\right)^2$, and reducing,

$$\frac{1}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}=\frac{\sqrt{1-i^2\sin^2\mu}}{\cos^2\mu+j\sin^2\mu}. \quad . \quad . \qquad (73)$$

Multiplying together the left-hand members of the equations (70),

* . . . . . . "résultat fort remarquable, déjà signalé par Legendre; mais nous ignorons comment il y est parvenu."—VERHULST, *Traité Elémentaire des Fonctions Elliptiques*, p. 158.

(72) and (73), and also the right-hand members together, we shall get, after some obvious reductions, and integrating,

$$\int \frac{d\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2 \sin^2\phi}} = (1+j)\int \frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}. \quad . \quad . \quad (74)$$

This is the well-known relation between two elliptic integrals of the first order whose moduli are i and $\frac{1-j}{1+j}$, or, in the common notation, whose moduli are c and $\frac{1-b}{1+b}$.

26.] Let τ be the arc whose tangent is $\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$,

then $$\tan 2\tau = \frac{2j\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu - i^2\sin^4\mu}; \quad . \quad . \quad . \quad (75)$$

and combining (71) and (73), we shall find

$$\frac{\tan\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}} = \frac{(1+j)\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu - j^2\sin^4\mu}. \quad (76)$$

Dividing (75) by (76), the result becomes

$$\tan 2\tau = \frac{\frac{2j}{1+j}\tan\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}. \quad . \quad . \quad . \quad . \quad (77)$$

We are thus enabled to express τ, the portion of the tangent arc between the point of contact and the foot of the perpendicular arc on it from the focus, in terms of ϕ instead of μ.

If we introduce this value of τ into (62) and combine with it the relations established in (74), the resulting equation will become

$$\left.\begin{aligned} 2\int \frac{d\phi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}} &= \int \frac{d\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}} \\ &+ \left(\frac{1+j}{2j}\right)\tan^{-1}\left[\frac{\frac{2j}{1+j}\tan\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}\right] \end{aligned}\right\}. \quad (78)$$

Adopting for the moment the ordinary notation of elliptic integrals,

$$m=-c=\frac{1-j}{1+j}, \text{ whence } 1+c=\frac{2j}{1+j}.$$

Introducing this notation, the last formula will become

$$2\Pi_c(-c,\,\phi)=\mathrm{F}_c(\phi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\phi}{\sqrt{1-i^2\sin^2\phi}}\right]. \quad . \quad . \quad (79)$$

In the 'Traité des Fonctions Elliptiques,' tom. i. p. 68, we meet with the formula

$$\Pi_c(n,\,\phi)+\Pi_c\left(\frac{c^2}{n},\,\phi\right)=\mathrm{F}_c(\phi)+\frac{1}{\sqrt{a}}\tan^{-1}\left[\frac{\sqrt{a}\tan\phi}{\sqrt{1-c^2\sin^2\phi}}\right]. \quad (80)$$

Now, when $n=-c$, this formula becomes

$$2\Pi_c(-c,\,\phi)=\mathrm{F}_c(\phi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\phi}{\sqrt{1-c^2\sin^2\phi}}\right], \quad . \quad . \quad (81)$$

whence (79) and (80) are identical.

27.] Let us now proceed to rectify the spherical parabola by the formula for rectification given in (47), the centre being the pole. For this purpose, resuming the formula for rectification established in (41), and deducing the values of the parameter, modulus, and coefficients in that expression from the given relations,

$$\tan^2 a=\frac{1+\sin\gamma}{1-\sin\gamma}=\frac{1+j}{1-j},\ \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}=\frac{2j}{1-j}, \quad . \quad (82)$$

we get

$$\left.\begin{array}{ll}
\text{The parameter,} & \tan^2\epsilon=\dfrac{1-j}{1+j}.\\[2ex]
\text{The modulus,} & \sin\eta=\dfrac{1-j}{1+j}.\\[2ex]
\text{The coefficient} & \dfrac{\cos\beta}{\sin a\cos a}=\dfrac{2}{1+j}.\\[2ex]
\text{The coefficient} & \dfrac{\cos a\cos\beta}{\sin a}=\dfrac{1-j}{1+j}.\\[2ex]
\text{and} & e\tan\epsilon=\dfrac{1-j}{1+j}.
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (83)$$

Making these substitutions in (41), the resulting equation will become

$$\left.\begin{array}{l}
\sigma=\dfrac{2}{(1+j)}\displaystyle\int\frac{d\phi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}\\[3ex]
\quad-\dfrac{(1-j)}{(1+j)}\displaystyle\int\frac{d\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}-\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\phi\cos\phi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\phi}}\right]
\end{array}\right\}. \quad (84)$$

But from (58), the focus being the pole, we derive

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]. \quad . \quad . \quad (85)$$

In (74) we showed that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\varphi}{\sqrt{-1\left(\frac{1-j}{1+i}\right)^2\sin^2\varphi}}.$$

Introducing this relation into the last formula, and equating together the equivalent expressions for the arcs in (84) and (85), we get for the resulting equation,

$$\left.\begin{array}{l}2\int\frac{d\varphi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}=\int\frac{d\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}\\ +(1+j)\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\varphi\cos\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}\right]+(1+j)\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]\end{array}\right\}. (86)$$

We shall now proceed to show that the common formula for the comparison of elliptic integrals having the same modulus and amplitude but reciprocal parameters is, in this particular case, identical with the geometrical theorem just established.

The formula is, in the ordinary notation,

$$2\Pi_c(c,\varphi)=F_c(\varphi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\varphi}{\sqrt{1-c^2\sin^2\varphi}}\right]. \quad . \quad (87)$$

We must accordingly show that, c being $\tan^2\epsilon$, and therefore $\frac{1}{1+c}=\frac{1+j}{2}$

$$\left.\begin{array}{l}(1+j)\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\varphi\cos\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}\right]+(1+j)\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]\\ =\frac{(1+j)}{2}\tan^{-1}\left[\frac{(1+\tan^2\epsilon)\tan\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}\right]\end{array}\right\}. \quad (88)$$

If we write τ, τ', and ϑ for these angles respectively, we have to show that

$$2(\tau+\tau')=\vartheta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

$\tau+\tau'$ is the arc of the great circle, which touches the spherical parabola, intercepted between the perpendicular arcs let fall from the centre and focus upon it.

We must, in the first place, by the help of Lagrange's equation between the amplitudes, established on geometrical principles in sec. [24], reduce these angles to a single variable. μ is taken as the independent variable instead of φ, as the trigonometrical function of φ in terms of μ is in the first power only.

We have, therefore,

$$\left.\begin{aligned} \tan\vartheta &= \frac{2\tan\varphi}{(1+j)\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}} \\ \tan\tau &= \frac{\left(\frac{1-j}{1+j}\right)\sin\varphi\cos\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)\sin^2\varphi}} \\ \tan\tau' &= \frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}} \end{aligned}\right\} \quad . \quad . \quad . \quad (90).$$

The equation between the amplitudes φ and μ,

$$\tan(\varphi-\mu)=j\tan\mu, \text{ gives}$$

$$\tan\varphi=\frac{(1+j)\sin\mu\cos\mu}{\cos^2\mu-j\sin^2\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

Eliminating φ by the help of this equation, from the value of $\tan\tau$ given in the preceding group,

$$\tan\tau=\frac{(1-j)\sin\mu\cos\mu}{\sqrt{1-i^2\sin^2\mu}}\times\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu-j\sin^2\mu}.$$

Using this transformation and reducing,

$$\tan(\tau+\tau')=\tan\mu\sqrt{1-i^2\sin^2\mu}, \quad . \quad . \quad . \quad . \quad (92)$$

a simple expression for the length of the tangent arc to the spherical parabola between the perpendicular arcs let fall from the centre and focus upon it.

From the last equation we may derive

$$\tan 2(\tau+\tau')=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}. \quad . \quad . \quad (93)$$

Using the preceding transformations, we may show that

$$\tan\vartheta=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}.$$

Hence $\qquad \vartheta=2(\tau+\tau'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$

Therefore (86) becomes

$$\left.\begin{aligned} &2\int\frac{d\varphi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}} \\ &-\int\frac{d\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}=(1+j)\,\frac{\vartheta}{2}=(1+j)\,(\tau+\tau') \end{aligned}\right\}. \quad (95)$$

We have thus shown that in the particular case of the general formula for comparing elliptic functions of the third order with reciprocal parameters, when the parameter is *positive* and equal to the *modulus,* the circular arc in the formula of comparison (87) is equal to twice the arc of the great circle touching the curve and intercepted between the perpendicular arcs let fall from the centre and focus upon it.

If we take the parameter with a *negative* sign, the circular arc τ in (62) will represent the tangent arc between the point of contact and the foot of the focal perpendicular.

The spherical parabola, like any other spherical ellipse, may be considered as the intersection of an elliptic cylinder with a sphere whose centre is on the axis of the cylinder.

Let a and b be the semiaxes of the base of the cylinder, and k the radius of the sphere, α and β being the principal semiarcs of the spherical parabola,

$$\tan^2\alpha=\frac{a^2}{k^2-a^2},\quad \tan^2\beta=\frac{b^2}{k^2-b^2};$$

but in (59*) we found $\tan^2\alpha-\tan^2\beta=1$; hence, substituting, we shall have, i being the eccentricity of the base of the elliptic cylinder,

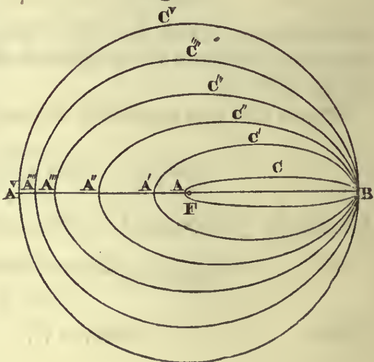
$$k^2=a^2(1+i). \quad \ldots\ldots\ldots\ldots \quad (96)$$

28.] The foregoing investigations furnish us with the geometrical interpretation of the transformations of Lagrange. Let the successive amplitudes φ, ψ, χ of the derived functions be connected by the equations

$$\left.\begin{aligned} &\tan(\varphi-\mu)=j\tan\mu, \\ &\tan(\psi-\varphi)=j_{\prime}\tan\varphi, \\ &\tan(\chi-\psi)=j_{\prime\prime}\tan\psi, \\ &\quad \&c., \qquad \&c., \end{aligned}\right\}. \quad (97)$$

We may imagine a series of confocal parabolas having a common

Fig. 13.



axis, described on a plane in contact with a sphere at their common focus. These parabolas will generate a series of confocal spherical parabolas on the surface of the sphere, BCA, BC′A′, BC″A″, BC‴A‴, which will all mutually touch at the vertex B remote from the common focus F. Let the distances between the common focus F and the vertices of the plane parabolas subtend, at the centre of the sphere, angles $\gamma, \gamma', \gamma''$, &c., whose cosines i, $i_{\prime}$, $i_{\prime\prime}$, &c. are connected by the equations

$$i_{\prime}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}},\ i_{\prime\prime}=\frac{1-\sqrt{1-i_{\prime}^2}}{1+\sqrt{1-i_{\prime}^2}},\ i_{\prime\prime\prime}=\frac{1-\sqrt{1-i_{\prime\prime}^2}}{1-\sqrt{1+i_{\prime\prime}^2}} \ldots \text{\&c.,} \qquad (98)$$

it is plain that $\gamma=\text{FA}$, $\gamma'=\text{FA}'$, $\gamma''=\text{FA}''$, $\gamma'''=\text{FA}'''$, &c.

We may repeat this construction successively, until the parameter of the last of the applied tangent plane parabolas shall become so indefinitely small, compared with the radius of the sphere, that it may ultimately be taken to coincide with its projection. We shall in this way reduce, at least geometrically, the calculation of an elliptic integral of the first order to the rectification of an arc of a parabola—that is, to a logarithm, as in sec. [20]. If, on the contrary, the moduli i, $i_{\prime}$, $i_{\prime\prime}$, &c. proceed in a descending series, the angles γ, $\gamma_{\prime}$, $\gamma_{\prime\prime}$ continually increase, the magnitudes of the confocal applied parabolas increase, till at length their parameters become so large, compared with the radius of the sphere, that their central projections pass into great circles of the sphere. The evaluation of the elliptic integral will therefore ultimately be reduced to the rectification of a circular arc. These are the well-known results of the modular transformation of Lagrange.

The formula established in (58) for the rectification of the spherical parabola, gives

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin\mu}}\right],$$

or, writing i for $\cos\gamma$, j for $\sin\gamma$, and $\sqrt{\text{I}}$ for $\sqrt{1-i^2\sin^2\mu}$,

$$\sigma-\tau=j\int\frac{d\mu}{\sqrt{\text{I}}},$$

σ' and τ' being the corresponding quantities for the next derived spherical parabola,

$$\sigma'-\tau'=j_{\prime}\int\frac{d\varphi}{\sqrt{\text{I}_{\prime}}}.$$

Now $j_{\prime}=\dfrac{2\sqrt{j}}{1+j}$, and $\displaystyle\int\frac{d\mu}{\sqrt{\text{I}}}=\frac{1}{1+j}\int\frac{d\varphi}{\sqrt{\text{I}_{\prime}}}$, as in (98) and (74),

whence
$$2(\sigma-\tau)=\sqrt{j}(\sigma'-\tau'). \quad \ldots\ldots \quad (99)$$

Thus a simple ratio exists between the arcs, diminished by the protangents, of two consecutive confocal spherical parabolas.

When the functions are complete, μ is taken between 0 and $\frac{\pi}{2}$; ϕ therefore, as in sec. [24], must be taken between 0 and π; but when the amplitude is taken between 0 and π the function is doubled. Moreover, when the functions are complete, the point Q coincides with B; so that in this case the complete function represents not one, but two quadrants of the spherical parabola, the focus being the pole. Hence as $\tau = \frac{\pi}{2}$, $\tau' = \pi$. It must be remembered that σ denotes two quadrants of the spherical parabola as shown in sec. [24].

Whence putting C, C′, C″, C‴, &c. for the circumferences of the successive confocal spherical parabolas, derived by the preceding law, we may write

$$\left.\begin{aligned} C - \pi &= \sqrt{j}\,(C_{\prime} - \pi) \\ C_{\prime} - \pi &= \sqrt{j_{\prime}}\,(C_{\prime\prime} - \pi) \\ C_{\prime\prime} - \pi &= \sqrt{j_{\prime\prime}}\,(C_{\prime\prime\prime} - \pi) \\ C_{\prime\prime\prime} - \pi &= \sqrt{j_{\prime\prime\prime}}\,(C_{\text{IV}} - \pi) \\ C_{\text{IV}} - \pi &= \sqrt{j_{\text{IV}}}\,(C_{\text{V}} - \pi) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Multiplying successively by the square roots of j, $j_{\prime}$, $j_{\prime\prime}$, $j_{\prime\prime\prime}$, &c., adding, and stopping at the fifth derived parabola,

$$C - \pi = \sqrt{j j_{\prime} j_{\prime\prime} j_{\prime\prime\prime} j_{\text{IV}}\ \&c.}\ (C_{\text{V}} - \pi).$$

Let this coefficient be $\sqrt{J}$, and we shall have

$$C - \pi = \sqrt{J}\,(C_{\text{V}} - \pi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Now we may extend this series, until the last of the derived spherical parabolas shall differ as little as we please from a great circle of the sphere. Let the circumference of this last derived spherical parabola be C_{ν}. Then $C_{\nu} = 2\pi$, and (101) becomes

$$C = \pi(1 + \sqrt{J}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Hence, calculating the quantity J, we may express the circumference of a spherical parabola by the circumference of a circle.

When all the spherical parabolas are nearly great circles of the sphere, $i, i_{\prime}, i_{\prime\prime}, i_{\prime\prime\prime} = 0$, nearly; and $j, j_{\prime}, j_{\prime\prime} j_{\prime\prime\prime} = 1$, nearly. Whence $J = 1$, nearly; or

$$C = 2\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

When the spherical parabolas are indefinitely diminished, $i, i_{\prime}, i_{\prime\prime} = 1$, nearly, and $j, j_{\prime}, j_{\prime\prime}, j_{\prime\prime\prime} = 0$, nearly, therefore $J = 0$ nearly; or

$$C = \pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Hence the circumferences of all spherical parabolas are greater than two and less than four quadrants of a great circle of the sphere.

29.] Denoting the angles at the centre of the sphere, subtended by the halves of the semiparameters of the applied confocal parabolas, by $\gamma, \gamma', \gamma''$, &c., we shall have $\cos\gamma=i$, $\cos\gamma'=i_{\prime}$, $\cos\gamma''=i_{\prime\prime}$, $\cos\gamma'''=i_{\prime\prime\prime}$, and $\sin\gamma=j$, $\sin\gamma'=j_{\prime}$, $\sin\gamma''=j_{\prime\prime}$, $\sin\gamma'''=j_{\prime\prime\prime}$. We may, using successively the equation $i_{\prime}=\dfrac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, determine in terms of j the successive values of $i_{\prime}, i_{\prime\prime}, i_{\prime\prime\prime}$, and of $j_{\prime}, j_{\prime\prime}, j_{\prime\prime\prime}$, &c., as follows:—

$$\left.\begin{aligned}
&i_{\prime}=\frac{1-j}{1+j},\quad i_{\prime\prime}=\left[\frac{1-j^{\frac12}}{1+j^{\frac12}}\right]^2,\quad i_{\prime\prime\prime}=\left[\frac{(1+j)^{\frac12}-2^{\frac12}j^{\frac14}}{(1+j)^{\frac12}+2^{\frac12}j^{\frac14}}\right]^2,\\
&i_{\text{iv}}=\left[\frac{1+j^{\frac12}-2^{\frac12}2^{\frac14}(1+j)^{\frac14}j^{\frac18}}{1+j^{\frac12}+2^{\frac12}2^{\frac14}(1+j)^{\frac14}j^{\frac18}}\right]^2,\\
&i_{\text{v}}=\left[\frac{(1+j)^{\frac12}+2^{\frac12}j^{\frac14}-2^{\frac12}2^{\frac14}2^{\frac18}(1+j^{\frac12})^{\frac12}(1+j)^{\frac18}j^{\frac1{16}}}{(1+j)^{\frac12}+2^{\frac12}j^{\frac14}+2^{\frac12}2^{\frac14}2^{\frac18}(1+j^{\frac12})\ (1+j)^{\frac18}j^{\frac1{16}}}\right]^2 \text{ \&c.}
\end{aligned}\right\} . \quad (105)$$

Hence we may derive the successive values of $j_{\prime}, j_{\prime\prime}, j_{\prime\prime\prime}$ in terms of j. For

$$\left.\begin{aligned}
&j_{\prime}^2=\frac{2^2 j}{(1+j)^2},\quad j_{\prime\prime}^2=\frac{2^2 2 j^{\frac12}(1+j)}{(1+j^{\frac12})^4},\quad j_{\prime\prime\prime}^2=\frac{2^2 2^1 2^{\frac12} j^{\frac14}(1+j)^{\frac12}(1+j^{\frac12})^2}{[(1+j)^{\frac12}+2^{\frac12}j^{\frac14}]^4},\\
&j_{\text{iv}}^2=\frac{2^2 2^1 2^{\frac12} 2^{\frac14}(1+j^{\frac12})(1+j)^{\frac14}j^{\frac18}[(1+j)^{\frac12}+2^{\frac12}j^{\frac14}]^2}{[(1+j^{\frac12})+2^{\frac12}2^{\frac14}(1+j)^{\frac14}j^{\frac18}]^4},\\
&j_{\text{v}}^2=\frac{(2^2 2^1 2^{\frac12} 2^{\frac14} 2^{\frac18})(1+j^{\frac12})^{\frac12}(1+j^{\frac18}j^{\frac1{16}}[(1+j)^{\frac12}+2^{\frac12}j^{\frac14}][(1+j^{\frac12})+2^{\frac12}2^{\frac14}(1+j)^{\frac14}j^{\frac18}]^2}{[(1+j)^{\frac12}+2^{\frac12}j^{\frac14}+2^{\frac12}2^{\frac14}2^{\frac18}(1+j^{\frac12})^{\frac12}(1+j)^{\frac18}j^{\frac1{16}}]^4}
\end{aligned}\right\} . \quad (106)$$

We may express the coefficient J, or the continued product of $j, j_{\prime}, j_{\prime\prime}, j_{\prime\prime\prime}$, &c., in terms of j, the complement of the original modulus. Including in our approximation the fifth derived modulus, we get

$$\mathrm{J}=\frac{(2)^1.(2)^{1+\frac12}.(2)^{1+\frac12+\frac14}.(2)^{1+\frac12+\frac14+\frac18}.(2)^{1+\frac12+\frac14+\frac18+\frac1{16}}(j\,j^{\frac12}j^{\frac14}j^{\frac18}j^{\frac1{16}})}{(1+j)^{\frac1{16}}(1+j^{\frac12})^{\frac14}[(1+j)^{\frac12}+2^{\frac12}j^{\frac14}]^{\frac12}[(1+j^{\frac12})+2^{\frac12}2^{\frac14}(1+j)^{\frac14}j^{\frac18}][(1+j)^{\frac12}+2^{\frac12}j^{\frac14}+2^{\frac12}2^{\frac14}2^{\frac18}(1+j^{\frac12})^{\frac12}(1-j)^{\frac18}j^{\frac1{16}}]^2} . \quad (107)$$

As an elliptic integral of the first order may be multiplied, or divided into any number of equal parts, as shown in every treatise on this subject, so its representative, an arc of the spherical parabola, like that of the circle, may be multiplied, or divided into any number of equal parts.

30.] It may not be out of place here to show, although the investigation more properly belongs to another part of the subject, that the arc of a spherical parabola may be represented as the sum of two elliptic integrals of the third order, having imaginary parameters; or in other words, that every elliptic integral of the *first* order may be exhibited as the sum of two elliptic integrals of the third order, having *imaginary reciprocal* parameters.

Parameters, whose product is equal to the square of the modulus, may be called *reciprocal parameters*.

Assume the expression given in (58) for an arc of the spherical parabola, the focus being the pole, and μ the angle which the perpendicular arc from the focus, on the tangent arc of a great circle to the curve, makes with the principal transverse arc,

$$\sigma = \sin\gamma \int \frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}.$$

Let $\cos\gamma = i$, $\sin\gamma = j$, and, to preserve uniformity in the notation, write ϕ for μ. Then differentiating the preceding equation, it becomes after some reductions,

$$\frac{d\sigma}{d\phi} = \frac{j[1-i^2\sin^2\phi+\cos^2\phi+j^2\sin^2\phi]}{[\cos^2\phi-i^2\sin^2\phi\cos^2\phi+j^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad . \quad \text{(a)}$$

Now the numerator is equivalent to $2j(1-i^2\sin^2\phi)$, and the first factor of the denominator may be written in the form

$$1-2i^2\sin^2\phi+i^2\sin^4\phi.$$

But $i^2 = i^2(i^2+j^2)$, hence this last expression may be put under the form $1-2i^2\sin^2\phi+i^4\sin^4\phi+i^2j^2\sin^4\phi$. This expression is the sum of two squares. Resolving this sum into its constituent factors, we get

$$\frac{d\sigma}{d\phi} = \frac{2j(1-i^2\sin^2\phi)}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad \text{(b)}$$

Now this product may be resolved into the sum of two terms. Let

$$\left.\begin{aligned}\frac{d\sigma}{d\phi} &= \frac{P}{[1-i(i+j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} \\ &+ \frac{Q}{[1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\end{aligned}\right\}, \quad . \quad . \quad . \quad \text{(c)}$$

or, reducing these expressions to a common denominator,

$$\frac{d\sigma}{d\phi} = \frac{(P+Q)-(P+Q)i^2\sin^2\phi+\sqrt{-1}(P-Q)ij\sin^2\phi}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}, \quad \text{(d)}$$

and comparing this expression with that given in (b), we shall see that

$$P+Q=2j, \ P-Q=0; \text{ therefore } P=j, \ Q=j. \quad . \quad . \quad \text{(e)}$$

Integrating (c), we get

$$\left.\begin{aligned}\sigma=j\int\frac{d\varphi}{[1-i(i+j\sqrt{-1})\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\\+j\int\frac{d\varphi}{[1-i(i-j\sqrt{-1})\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\end{aligned}\right\}\quad .\ .\ .\quad (108)$$

If we replace i by $\cos\gamma$, and j by $\sin\gamma$, the parameters become $\cos\gamma\,(\cos\gamma+\sqrt{-1}\,\sin\gamma)$ and $\cos\gamma\,(\cos\gamma-\sqrt{-1}\sin\gamma)$, whose product is $\cos^2\gamma$, the square of the modulus. They are therefore reciprocal; and putting m for $\cos\gamma\,(\cos\gamma+\sqrt{-1}\,\sin\gamma)$ and $-n$ for $\cos\gamma\,(\cos\gamma-\sqrt{-1}\,\sin\gamma)$, we shall find that these values of m and n satisfy the equation of *circular* conjugation, $m-n+mn=i^2$. It follows therefore that when the rectification of the spherical parabola is effected, the centre being the origin, the representative elliptic integral is of the *third* order and circular form; the parameters m and n are equal to each other, and to the modulus, and are therefore reciprocal. But when the focus of the spherical parabola is assumed as the origin, the rectification of this curve may be effected by an elliptic integral of the *first* order, and this integral may also be exhibited as an integral of the *third* order and circular form, but with imaginary parameters, which are also reciprocal.

CHAPTER III.

ON SPHERICAL CONIC SECTIONS WITH RECIPROCAL PARAMETERS.

31.] Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of an ellipse, the base of an elliptic cylinder. Let two spheres be described, having their centres at the centre of this elliptic base, and intersecting the cylinder in two spherical conic sections. These sections will have reciprocal parameters, if k, $k_{\prime}$, the radii of the spheres, are connected by the equation

$$(k^2-a^2)\,(k_{\prime}^2-a^2)=a^4i^2,\quad .\ .\ .\ .\ .\quad (109)$$

i^2 being, as before, equal to $\frac{a^2-b^2}{a^2}$.

When k and $k_{\prime}$ are equal, we get $k^2=a^2(1+i)$. This value of k agrees with that found for k in (96); or, in other words, when the two spheres coincide, the section of the elliptic cylinder by the sphere is a spherical parabola. Hence also, *a spherical parabola always lies between two spherical conic sections having reciprocal parameters.*

Let e^2 and e'^2 be the parameters of those sections of the cylinder made by the spheres. Then, as shown in (9),

$$e^2=\frac{\sin^2 a-\sin^2\beta}{\sin^2 a\cos^2\beta}=\frac{(a^2-b^2)k^2}{a^2(k^2-b^2)}=\frac{k^2i^2}{k^2-a^2+a^2i^2};$$

but the equation of condition (109) gives

$$k^2-a^2=\frac{a^4i^2}{k_i^2-a^2}; \text{ hence } e^2=\frac{k^2(k_i^2-a^2)}{a^2k_i^2}.$$

In the same manner the spherical conic whose radius is k_i gives

$$e_i^2=\frac{k_i^2(k^2-a^2)}{a^2k^2}; \text{ therefore } e^2e_i^2=\frac{(k^2-a^2)(k_i^2-a^2)}{a^4}=i^2, \quad (110)$$

or e^2 and e_i^2 are reciprocal parameters.

To compute in this case the value of the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ in the expression given in (10) for rectification,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}.$$

Since $\qquad \tan^2\beta=\frac{b^2}{k^2-b^2}, \ \tan^2\alpha=\frac{a^2}{k^2-a^2}, \ \sin\beta=\frac{b}{k};$

we obtain by substitution, $\frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)}{a^2k^2(k^2-a^2+a^2i^2)};$

but the equation of condition (109) gives

$$k^2-a^2=\frac{a^4i^2}{k_i^2-a^2}; \text{ hence } \frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)(k_i^2-a^2)}{a^4i^2k^2k_i^2}=\frac{b^4}{k^2k_i^2}.$$

As this expression is symmetrical, we shall have for the spherical conic section whose radius is k_i;

$$\frac{\tan\beta_i}{\tan\alpha_i}\sin\beta_i=\frac{b^2}{kk_i}. \quad \ldots\ldots \quad (111)$$

Hence $$\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{\tan\beta_i}{\tan\alpha_i}\sin\beta_i; \quad \ldots \quad (112)$$

or the coefficients of the elliptic integrals which determine the arcs of two spherical conic sections having reciprocal parameters are equal.

Let κ be the criterion of sphericity; then, as

$$\kappa=(1-m)\left(1-\frac{i^2}{m}\right)=(1-e^2)(1-e_i^2)=\frac{b^4}{k^2k_i^2},$$

$$\kappa=\kappa_i. \quad \ldots\ldots \quad (113)$$

32.] To determine the values of the angles λ and λ' which correspond to the same angle ϕ in the expressions for the arcs of spherical conic sections having reciprocal parameters.

Since $$\cos^2\epsilon=\frac{\cos^2\alpha}{\cos^2\beta}=\frac{k^2-a^2}{k^2-b^2}=\frac{k^2-a^2}{k^2-a^2+a^2i^2},$$

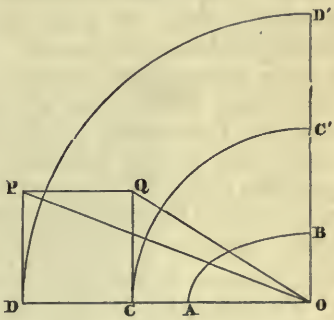
introducing the equation of condition $(k^2-a^2)(k_i^2-a^2)=a^4 i^2$, we get $\cos\epsilon=\frac{a}{k_i}$; but $\tan\phi=\cos\epsilon \tan\lambda$, as in (39); hence

$\tan\lambda=\frac{k_i}{a}\tan\phi$, and $\tan\lambda_i=\frac{k}{a}\tan\phi$;

therefore $$k\tan\lambda=k_i\tan\lambda_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

or the tangent of the angle λ which the perpendicular arc from the centre of the spherical conic, on the arc of a great circle touching it, makes with the principal major arc, is inversely as the radius of the sphere.

Fig. 14.



A simple geometrical construction will give the magnitude of those angles λ and λ_i. Let the ellipse OAB be the base of the cylinder; OCC′, ODD′ being the bases of the hemispheres whose intersections with the cylinder give the spherical conic sections with reciprocal parameters. Erect the tangents DP, CQ, each equal to $\frac{kk_i}{a}\tan\phi$, and join PO, QO. The angles AOP, AOQ are λ and λ_i.

When $DP=CQ=0$, $\lambda=\lambda_i=0$; when $DP=CQ=\infty$, $\lambda=\lambda_i=\frac{\pi}{2}$. The condition (109) shows that when $k=a$, $k_i=\infty$. Now as $k_i\tan\lambda_i=a\tan\lambda$ is finite always so long as λ is not absolutely $=\frac{\pi}{2}$, in order that its equal $k_i\tan\lambda_i$ may be finite also, we must have λ_i always equal to 0 for every finite value of $\tan\lambda$.

33.] *The tangent of the principal arc of a spherical parabola is a mean proportional between the tangents of the principal arcs of two spherical conics with reciprocal parameters,* the three curves being the sections of the same elliptic cylinder by three concentric spheres.

Since
$$\tan^2 a=\frac{a^2}{k^2-a^2},\ \tan^2 a_i=\frac{a^2}{k_i^2-a^2},\ \tan^2 a\tan^2 a_i=\frac{a^4}{(k^2-a^2)(k_i^2-a^2)}.$$

Introducing the equation of condition $(k^2-a^2)(k_i^2-a^2)=a^4 i^2$ (109), we get
$$\tan a\ \tan a_i=\frac{1}{i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

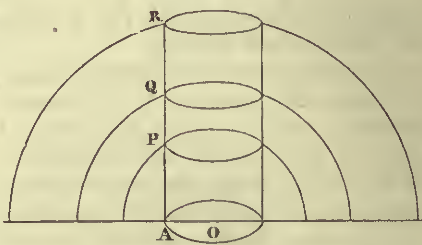
Let $k_{\prime\prime}$ be the radius of the sphere whose intersection with the cylinder gives the spherical parabola; then $k_{\prime\prime}^2=a^2(1+i)$. See (96).

$$\text{Hence} \qquad k_{\prime\prime}^2-a^2=a^2 i\,; \text{ and } \tan^2 a_{\prime\prime}=\frac{a^2}{k_{\prime\prime}^2-a^2}=\frac{1}{i}\,;$$

$$\text{therefore} \qquad \tan a \tan a_{\prime}=\tan^2 a_{\prime\prime}. \quad \ldots \ldots \quad (116)$$

The altitudes of the vertices of the three principal major arcs of the two spherical conics with reciprocal parameters, and of the spherical parabola, above the plane of the elliptic base of the cylinder, are in geometrical progression. Let AQ be the altitude of the vertex of the major arc of the spherical parabola; AP, AR the corresponding altitudes of the vertices of the major arcs of the spherical ellipses which have reciprocal parameters.

Fig. 15.



Then $\text{AP}=\sqrt{k^2-a^2}$, $\text{AR}=\sqrt{k_{\prime}^2-a^2}$, $\text{AQ}=\sqrt{k_{\prime\prime}^2-a^2}=a\sqrt{i}$. The equation of condition gives, as in (109), $\text{AP}\times\text{AR}=\overline{\text{AQ}}^2$.

We shall give, further on, an expression for the sum of the arcs of two spherical conic sections having the same amplitude, but reciprocal parameters.

34.] The projections of supplemental spherical ellipses on the plane of XY are confocal plane ellipses.

For $\qquad \sin\eta=\sin\epsilon'$, $\sin\eta'=\sin\epsilon$. See sec. [9].

$$\text{Hence} \qquad \frac{a^2-b^2}{a^2}=\frac{a_{\prime}^2-b_{\prime}^2}{k^2-b_{\prime}^2},\ \frac{a_{\prime}^2-b_{\prime}^2}{a_{\prime}^2}=\frac{a^2-b^2}{k^2-b^2}.$$

This gives as the resulting value,

$$k^2=a^2+b_{\prime}^2=a_{\prime}^2+b^2, \text{ or } a^2-b^2=a_{\prime}^2-b_{\prime}^2.$$

Two cones, supplemental to each other, are cut by a plane at right angles to their common internal axis. The sections are concentric similar ellipses, having the major and the minor axes of the one coinciding with the minor and major axes of the other.

$$\text{For} \qquad \frac{\tan^2 a-\tan^2\beta}{\tan^2 a}=e^2,$$

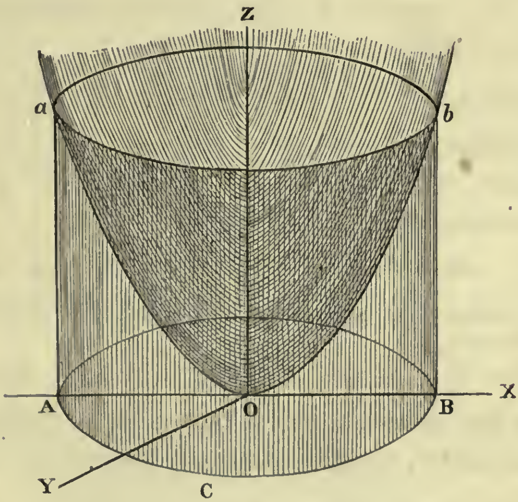
$$\text{and} \quad e_{\prime}^2=\frac{\tan^2 a_{\prime}-\tan^2\beta_{\prime}}{\tan^2 a_{\prime}}=\frac{\cot^2\beta-\cot^2 a}{\cot^2\beta}=\frac{\tan^2 a-\tan^2\beta}{\tan^2 a}, \text{ or } e'=e.$$

CHAPTER IV.

ON THE LOGARITHMIC ELLIPSE.

35.] The logarithmic ellipse may be defined as the curve of symmetrical intersection of a paraboloid of revolution with an elliptic cylinder. This section of the cylinder by the paraboloid is analogous to the section of the cone by the concentric sphere in sec. [7]; for this cylinder may be viewed as a cone having its vertex at the centre of the paraboloid, *i. e.* at an infinite distance.

Fig. 16.



Let the axes of the paraboloid and cylinder coincide with the axis of Z, the vertex of the paraboloid being supposed to touch the plane of XY at the origin O.

It may be proper to note that every tangent plane to the elliptic cylinder will cut the paraboloid in a parabola, just as every tangent plane to a cone will cut a concentric sphere in a great circle.

Let k be the semiparameter of the paraboloid Oab, and let a and b be the semiaxes of the base of the elliptic cylinder ACB; then the equations of these surfaces, and consequently of the curve in which they intersect, are

$$x^2+y^2=2kz, \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad . \quad . \quad . \quad (117)$$

Let $d\Sigma$ be an element of the required curve,

then
$$\frac{d\Sigma}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2+\left(\frac{dz}{d\theta}\right)^2}, \quad . \quad . \quad . \quad . \quad (118)$$

x, y, and z being dependent variables on a fourth independent variable θ.

Assume
$$x=a\cos\theta,\ y=b\sin\theta,\ \text{then } a^2\cos^2\theta+b^2\sin^2\theta=2kz. \quad (119)$$

Differentiating and substituting,
$$\left(\frac{d\Sigma}{d\theta}\right)^2=a^2\sin^2\theta+b^2\cos^2\theta+\frac{(a^2-b^2)^2}{k^2}\sin^2\theta\cos^2\theta. \quad (120)$$

To reduce this expression to a form suited for integration, it may be written,
$$k^2\left(\frac{d\Sigma}{d\theta}\right)^2=b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta. \quad (121)$$

This expression may be reduced as follows:

Let $\mathrm{P}=b^2k^2$, $\mathrm{Q}=(a^2-b^2)[k^2+a^2-b^2]$, $\mathrm{R}=-(a^2-b^2)^2$; (122)

and the preceding equation will become
$$k\Sigma=\int d\theta\sqrt{\mathrm{P}+\mathrm{Q}\sin^2\theta+\mathrm{R}\sin^4\theta}. \quad . \quad . \quad . \quad . \quad (123)$$

Let this trinomial be put under the form of a product of two quadratic factors,
$$(\mathrm{A}+\mathrm{B}\sin^2\theta)(\mathrm{C}-\mathrm{B}\sin^2\theta)=\mathrm{AC}+\mathrm{B}(\mathrm{C}-\mathrm{A})\sin^2\theta-\mathrm{B}^2\sin^4\theta. \quad (124)$$

Comparing this expression with the preceding in (121), we get
$$\mathrm{AC}=b^2k^2,\ \mathrm{C}-\mathrm{A}=k^2+a^2-b^2,\ \mathrm{B}=a^2-b^2. \quad . \quad . \quad (125)$$

To integrate (123): assume $\tan^2\varphi=\frac{\mathrm{A}+\mathrm{B}}{\mathrm{A}}\tan^2\theta$. . . (126)

The limits of integration of the complete functions will continue as before. Making the substitutions indicated by the preceding transformations, the integral will now become
$$\frac{\sqrt{\mathrm{C}(\mathrm{A}+\mathrm{B})}}{\mathrm{AC}}k\Sigma=\int\frac{d\varphi\left[1-\frac{\mathrm{B}}{\mathrm{C}}\left(\frac{\mathrm{A}+\mathrm{C}}{\mathrm{A}+\mathrm{B}}\right)\sin^2\varphi\right]}{\left[1-\frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}}\sin^2\varphi\right]^2\sqrt{1-\frac{\mathrm{B}}{\mathrm{C}}\left(\frac{\mathrm{A}+\mathrm{C}}{\mathrm{A}+\mathrm{B}}\right)\sin^2\varphi}}. \quad (127)$$

Let
$$\left.\begin{array}{l}\frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}}=n,\ \frac{\mathrm{B}}{\mathrm{C}}\frac{(\mathrm{A}+\mathrm{C})}{(\mathrm{A}+\mathrm{B})}=i^2,\ \frac{\mathrm{B}}{\mathrm{C}}=m,\\ \mathrm{N}=1-n\sin^2\varphi,\ \mathrm{I}=1-i^2\sin^2\varphi.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (128)$$

These values of A, B, C satisfy the equation $m+n-mn=i^2$, as assumed in (1). As $k^2=\mathrm{C}-\mathrm{A}-\mathrm{B}$, $\mathrm{C}>\mathrm{A}+\mathrm{B}$, or $n>m$, the preceding expression may now be written

$$\frac{[2n-i^2-n^2]}{\sqrt{n}\,(i^2-n)\,(1-n)}\,\frac{\Sigma}{k}=(1-n)\int\frac{d\phi\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad . \quad . \quad . \quad (129)$$

It will presently be shown that A and C must always have the same sign, whence $i^2>n$.

As $i^2=\dfrac{1+\frac{\mathrm{A}}{\mathrm{C}}}{1+\frac{\mathrm{A}}{\mathrm{B}}}$, and as C is always greater than B, $i^2<1$. From (125) we may derive

$$\frac{a^2}{k^2}=\frac{(\mathrm{A}+\mathrm{B})(\mathrm{C}-\mathrm{B})}{(\mathrm{C}-\mathrm{A}-\mathrm{B})^2},\quad \frac{b^2}{k^2}=\frac{\mathrm{AC}}{(\mathrm{C}-\mathrm{A}-\mathrm{B})^2},\quad \text{and } k^2=(\mathrm{C}-\mathrm{A}-\mathrm{B}).$$

Now, that the values of a and b may be real, we must have $\mathrm{C}>\mathrm{B}$, while A and C must be of the same sign; but as B is essentially positive, C, and therefore A, must be positive.

Since $$\frac{\mathrm{B}}{\mathrm{A}+\mathrm{B}}=n,\quad \text{and } \frac{\mathrm{A}+\mathrm{C}}{\mathrm{C}}=\frac{i^2}{n},\ \text{as in (128)},$$

we may eliminate A, B, C from the values of the semiaxes of the base of the elliptic cylinder, and express a, b, and k in terms of i and n. We may thus obtain

$$\frac{a^2}{k^2}=\frac{n(1-i^2)(i^2-n)}{[2n-i^2-n^2]^2},\quad \frac{b^2}{k^2}=\frac{n(i^2-n)(1-n)^2}{[2n-i^2-n^2]^2};$$

or more simply in terms of n and m,

$$\frac{a^2}{k^2}=\frac{mn(1-m)}{(n-m)^2},\quad \frac{b^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}. \quad . \quad . \quad . \quad . \quad . \quad (130)$$

In order that these values of a and b may be real, we must have n positive, $i^2>n$, and $1>i^2$.

This is case VI. in the Table, p. 7.

If we put c for the eccentricity of the plane elliptic base of the cylinder, we shall have after some obvious reductions,

$$(1-i^2)(1-c^2)=(1-n)^2,\ \text{or } c^2=\frac{n-m}{1-m}. \quad . \quad . \quad . \quad (131)$$

Now this simple equation between n, m, and c enables us with great ease to determine the eccentricity c of the base of the elliptic cylinder whose section with the paraboloid gives the logarithmic ellipse, when we know the parameters m and n, or the modulus i, of the given elliptic integral.

36.] To integrate the expression given in (127), we must assume

$$\Phi_n = \frac{\sin\phi \cos\phi \sqrt{1-i^2\sin^2\phi}}{[1-n\sin^2\phi]}. \quad . \quad . \quad . \quad . \quad (132)$$

Differentiate this expression with respect to ϕ, and we shall have

$$\frac{d\Phi_n}{d\phi} = \frac{1-2(1+i^2)\sin^2\phi + 3i^2\sin^4\phi}{[1-n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} + \frac{2n(\sin^2\phi - \sin^4\phi)(1-i^2\sin^2\phi)}{[1-n\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}. \quad (a)$$

Let $\qquad 1-n\sin^2\phi = \mathrm{N}, \; 1-i^2\sin^2\phi = \mathrm{I}$, as before.

Separating the numerators of the preceding expression into their component parts, and attaching to each their respective denominators, we shall have

$$\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}} = \frac{1}{\mathrm{N}\sqrt{\mathrm{I}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and

$$-\frac{2(1+i^2)\sin^2\phi}{\mathrm{N}\sqrt{\mathrm{I}}} = \frac{2(1+i^2)}{n}\;\frac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}} = \frac{2(1+i^2)}{n\sqrt{\mathrm{I}}} - \frac{2(1+i^2)}{n\mathrm{N}\sqrt{\mathrm{I}}}. (c)$$

The next term gives

$$\frac{3i^2\sin^4\phi}{\mathrm{N}\sqrt{\mathrm{I}}} = -\frac{3i^2}{n}\;\frac{(1-n\sin^2\phi-1)\sin^2\phi}{\mathrm{N}\sqrt{\mathrm{I}}} = -\frac{3i^2}{n}\;\frac{\sin^2\phi}{\sqrt{\mathrm{I}}} + \frac{3i^2\sin^2\phi}{n\mathrm{N}\sqrt{\mathrm{I}}}. \quad . \quad (d)$$

Now these two terms may be still further resolved; for

$$-\frac{3i^2}{n}\;\frac{\sin^2\phi}{\sqrt{\mathrm{I}}} = \frac{3}{n}\;\frac{(1-i^2\sin^2\phi-1)}{\sqrt{\mathrm{I}}} = \frac{3\sqrt{\mathrm{I}}}{n} - \frac{3}{n\sqrt{\mathrm{I}}},$$

and $\qquad \dfrac{3i^2\sin^2\phi}{n\mathrm{N}\sqrt{\mathrm{I}}} = -\dfrac{3i^2}{n^2}\;\dfrac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}} = -\dfrac{3i^2}{n^2\sqrt{\mathrm{I}}} + \dfrac{3i^2}{n^2\mathrm{N}\sqrt{\mathrm{I}}},$

whence (d) becomes

$$\frac{3i^2\sin^4\phi}{\mathrm{N}\sqrt{\mathrm{I}}} = \frac{3\sqrt{\mathrm{I}}}{n} - \frac{3}{n\sqrt{\mathrm{I}}} - \frac{3i^2}{n^2\sqrt{\mathrm{I}}} + \frac{3i^2}{n^2\mathrm{N}\sqrt{\mathrm{I}}}. \quad . \quad . \quad (e)$$

Combining the expressions in (b), (c), (d) or (e), the first term of the second member of (a) may be written

$$\left.\begin{aligned} &\frac{[1-2(1+i^2)\sin^2\phi+3i^2\sin^4\phi]}{[1-n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} = \frac{3\sqrt{\mathrm{I}}}{n} \\ &+\left[\frac{2}{n}(1+i^2) - \frac{3i^2}{n^2} - \frac{3}{n}\right]\frac{1}{\sqrt{\mathrm{I}}} + \left[1 - \frac{2}{n}(1+i^2) + \frac{3i^2}{n^2}\right]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}} \end{aligned}\right\}. \quad (f)$$

The second term, $\dfrac{2n(\sin^2\varphi-\sin^4\varphi)\sqrt{\mathrm{I}}}{(1-n\sin^2\varphi)^2}$, of (a) may be thus developed,

$$\frac{2n\sin^2\varphi\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2n}{n}\frac{(1-n\sin^2\varphi-1)\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2\mathrm{I}}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{2\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \text{(g)}$$

and these two latter expressions may be written

$$-\frac{2\mathrm{I}}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2(1-i^2\sin^2\varphi)}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2i^2}{n}\frac{(1-n\sin^2\varphi-1)}{\mathrm{N}\sqrt{\mathrm{I}}}$$

$$=-\frac{2i^2}{n}\frac{1}{\sqrt{\mathrm{I}}}+\frac{2i^2}{n}\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2}{\mathrm{N}\sqrt{\mathrm{I}}};$$

whence (g) becomes

$$\frac{2n\sin^2\varphi\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2i^2}{n\sqrt{\mathrm{I}}}-2\left(1-\frac{i^2}{n}\right)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{2\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad \text{(h)}$$

The term $-\dfrac{2n\sin^4\varphi\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}$ may be written

$$\left.\begin{aligned}-\frac{2n\mathrm{I}\sin^4\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}&=-\frac{2\mathrm{I}}{n}\left[\frac{1-2n\sin^2\varphi+n^2\sin^4\varphi-2+2n\sin^2\varphi+1}{\mathrm{N}^2\sqrt{\mathrm{I}}}\right]\\&=-\frac{2\mathrm{I}}{n\,.\sqrt{\mathrm{I}}}+\frac{4\mathrm{I}}{n\,.\,\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n\,.\,\mathrm{N}^2\sqrt{\mathrm{I}}}\end{aligned}\right\}. \quad \text{(k)}$$

Now
$$\frac{2\mathrm{I}}{n\sqrt{\mathrm{I}}}=\frac{2}{n}\sqrt{\mathrm{I}},$$

and
$$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4(1-i^2\sin^2\varphi)}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4}{n\mathrm{N}\sqrt{\mathrm{I}}}+\frac{4i^2}{n^2}\frac{(1-n\sin^2\varphi-1)}{\mathrm{N}\sqrt{\mathrm{I}}};$$

whence
$$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-4\frac{(i^2-n)}{n^2}\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad . \quad . \quad . \quad . \quad \text{(m)}$$

Combining (k) with (m), we shall have

$$-\frac{2n\mathrm{I}\sin^4\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\sqrt{\mathrm{I}}}{n}+\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-\frac{4}{n^2}(i^2-n)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \text{(n)}$$

adding (n) to (h),

$$\left.\begin{aligned}\frac{2n(\sin^2\varphi-\sin^4\varphi)\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}&=-\frac{2\sqrt{\mathrm{I}}}{n}+\left(\frac{4i^2}{n^2}-\frac{2i^2}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}\\&+\left[\frac{2i^2}{n}-2+\frac{4}{n}-\frac{4i^2}{n^2}\right]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1}{n}-1\right)\frac{1}{\mathrm{N}^2\sqrt{\mathrm{I}}}\end{aligned}\right\}; \quad \text{(p)}$$

adding (f) and (p) together, we shall obtain as the final result,

$$\frac{d\Phi_n}{d\varphi}=\frac{\sqrt{\mathrm{I}}}{n}+\frac{1}{n}\left(\frac{i^2-n}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}+\frac{1}{n^2}[2n-n^2-i^2]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1-n}{n}\right)\frac{\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \text{(q)}$$

or multiplying by n, putting for i^2 its value $n+m-mn$, transposing and integrating,

$$\left.\begin{aligned}2(1-n)\int\frac{\mathrm{I}d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}&=-n\Phi_n+\int d\varphi\sqrt{\mathrm{I}}\\&+\frac{m}{n}(1-n)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+(n-m)\frac{(1-n)}{n}\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(r)}$$

But we have shown in (129) that

$$\frac{2(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}=2(1-n)\int\frac{\mathrm{I}d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}, \quad . \quad . \quad . \quad . \quad . \quad \text{(s)}$$

whence

$$\left.\begin{aligned}\frac{2(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}&=-n\Phi_n+\int d\varphi\sqrt{\mathrm{I}}\\&+\frac{m}{n}(1-n)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\frac{(n-m)(1-n)}{n}\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\end{aligned}\right\} \quad . \quad . \quad (133)$$

Hence an arc of a logarithmic ellipse may be expressed by a straight line $k\Phi_n$, and in terms of elliptic integrals of the first, second, and third orders, the latter being of the logarithmic form.

The expression $\int\frac{\mathrm{I}d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}$ may be reduced to

$$\frac{i^2}{n}\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{m}{n}(1-n)\int\frac{d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}};$$

and therefore combining this expression with (r),

$$\left.\begin{aligned}2\kappa\int\frac{d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}&=n\Phi+\frac{(1-n)}{n}[m+n+2m(1-n)]\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}\\&-\frac{m}{n}(1-n)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\int d\varphi\sqrt{\mathrm{I}}\end{aligned}\right\} . \, (134)$$

37.] When the elliptic cylinder and the paraboloid are given, we may determine the parameter, modulus, and constants of the functions which represent the curve of intersection of these surfaces, in the terms of the constants a, b, and k.

The modulus, parameter, coefficients, and criterion of sphericity may be expressed as linear products of constants having simple relations with those of the given surfaces.

Resuming the equations given in (125),

$$AC=b^2k^2,\ C-A=k^2+a^2-b^2,\ B=a^2-b^2,$$

we find $$(A+C)^2=(k^2+a^2-b^2)^2+4b^2k^2.$$

Assume $$4p^2=k^2+(a+b)^2,\ 4q^2=k^2+(a-b)^2, \qquad . \quad (135)$$

we shall then have the following equations :—

$$\left.\begin{array}{ll} A+C=4pq\,; & B=(a+b)(a-b) \\ A+B=(a+p-q)(a+q-p)\,; & C-B=(p+q+a)(p+q-a) \\ A=(b+p-q)(b+q-p)\,; & C=(p+q+b)(p+q-b) \\ ab=(p+q)(p-q)\,; & k^2+a^2+b^2=2(p^2+q^2). \end{array}\right\} . \quad (136)$$

Substituting these values in (129), we obtain as the resulting expressions

$$\left.\begin{array}{l} i^2=\dfrac{4(a+b)(a-b)pq}{(p+q+b)(p+q-b)(a+p-q)(a+q-p)} \\ j^2=(1-i^2)=\dfrac{(p+q+a)(p+q-a)(b+p-q)(b+q-p)}{(p+q+b)(p+q-b)(a+p-q)(a+q-p)} \\ n=\dfrac{(a+b)(a-b)}{(a+p-q)(a+q-p)},\quad m=\dfrac{(a+b)(a-b)}{(p+q+b)(p+q-b)} \end{array}\right\} . \quad . \quad . \quad (137)$$

These values of m, n, and i^2 satisfy the second equation of condition in (1),

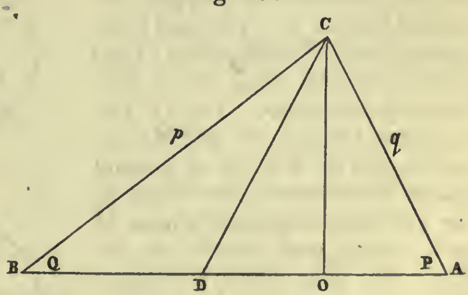
$$m+n-mn=i^2\,;$$

and if we denote by κ the criterion of sphericity,

$$\kappa=\frac{-b^4}{a^2(p+q)^2}\left(\frac{p+q+a}{p+q+b}\right)^2\left(\frac{p+q-a}{p+q-b}\right)^2, \quad . \quad . \quad . \quad (138)$$

we may express the parameters and modulus of the elliptic integral of the third order and logarithmic form by a geometrical construction of remarkable simplicity when the intersecting surfaces are given, or when a, b, and k are given.

Fig. 17.



Take $BA=a$, $BD=b$, and from O the point of bisection of AD, erect the perpendicular $OC=\dfrac{k}{2}$.

Then (135) gives $p=BC$, $q=AC$; and putting P and Q for the angles BAC and ABC, $a+b=2p\cos Q$, $a-b=2q\cos P$. As p, q, b are the sides of the triangle BCD, and the angle BCD$=$P$-$Q,

$$\cos^2\left(\frac{P-Q}{2}\right)=\frac{(b+p+q)(p+q-b)}{4pq}.$$

Again, as a, p, q are the sides of the triangle ABC,

$$\cos^2\left(\frac{P+Q}{2}\right)=\frac{(a+p-q)(a+q-p)}{4pq}.$$

Substituting these values in (137), we get

$$\left.\begin{aligned} n&=\frac{\cos P\cos Q}{\cos^2\left(\frac{P+Q}{2}\right)}, \quad m=\frac{\cos P\cos Q}{\cos^2\left(\frac{P-Q}{2}\right)}, \\ i&=\frac{2[\cos P\cos Q]^{\frac{1}{2}}}{\cos Q+\cos P}, \text{ and } j=\frac{\cos Q-\cos P}{\cos Q+\cos P} \end{aligned}\right\}; \quad . \quad (139)$$

and if c be the eccentricity of the elliptic base of the cylinder,

$$c^2=\frac{\sin 2P.\sin 2Q}{\sin^2(P+Q)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

These are expressions remarkable for their simplicity.

We also find for the criterion of sphericity κ,

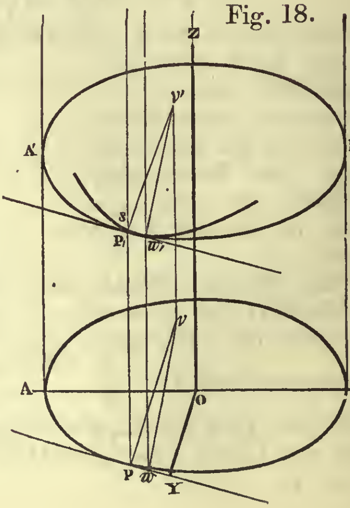
$$\kappa=-\left[\frac{\sin^2\left(\frac{P-Q}{2}\right)}{\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)}\right]^2. \quad . \quad . \quad . \quad (141)$$

As $\frac{k}{2}$ is the altitude of a triangle whose sides are a, p, q.

$$a^2k^2=(a+p+q)(p+q-a)(a+q-p)(a+p-q).$$

38.] In the preceding investigations, the element of the curve has been taken as a side of a limiting rectilinear polygon inscribed within it. We may however effect the rectification of the curve, starting from other elementary principles. Let APB be the plane base of the elliptic cylinder, and let a series of normal planes PP$'vv'$ $\varpi\varpi'vv'$ be drawn to the cylinder, indefinitely near to each other, and parallel to its axis. We may conceive of every element Pϖ of this plane ellipse between the normal planes as the projection of the corresponding

Fig. 18.



element $s\varpi'$ of the logarithmic ellipse. Let τ be the inclination of the element $d\Sigma$ of the logarithmic ellipse to the corresponding element ds of the plane ellipse. We shall have, $d\lambda$ being the elementary angle between the planes $PP'\nu\nu'$ and $\varpi\varpi'\nu\nu'$,

$$\frac{d\Sigma}{d\lambda}=\sec\tau\frac{ds}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

Now (31) gives $\dfrac{ds}{d\lambda}=p+\dfrac{d^2p}{d\lambda^2}$;

and therefore $$\Sigma=\int\frac{p}{\cos\tau}\,d\lambda+\int\frac{d^2p}{d\lambda^2}\sec\tau\,.\,d\lambda. \quad . \quad . \quad . \quad (143)$$

In the plane ellipse $p^2=a^2\cos^2\lambda+b^2\sin^2\lambda$, whence

$$\frac{d^2p}{d\lambda^2}=-\frac{(a^2-b^2)(a^2\cos^4\lambda-b^2\sin^4\lambda)}{(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad (144)$$

We have now to express $\cos\tau$ in terms of λ.

From (119) combined with (120) we may derive

$$\sec^2\tau=\frac{d\Sigma^2}{dx^2+dy^2}=\frac{b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta}{k^2(a^2\sin^2\theta+b\cos^2\theta)}. \quad (145)$$

Eliminating $\frac{y}{x}$ between the equations $\tan\lambda=\frac{a^2}{b^2}\frac{y}{x}$, and $\frac{y}{x}=\frac{b}{a}\tan\theta$, we shall have

$$\tan\lambda=\frac{a}{b}\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

If we eliminate $\tan\theta$ by the help of this equation from (145), we shall obtain

$$\cos^2\tau=\frac{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}{a^2k^2+(a^2-b^2)[a^2-b^2-k^2]\sin^2\lambda-(a^2-b^2)^2\sin^4\lambda}. \quad (147)$$

Substituting this value of $\cos\tau$ in (143), and writing P', Q', R' for the coefficients of powers of $\sin\lambda$, the resulting equation will become

$$\left.\begin{aligned} k\Sigma=&\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}\\ &-(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{k(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}\cos\tau}\end{aligned}\right\} \quad . \quad . \quad . \quad (148)$$

As the first of these integrals is precisely similar in form to the integral in (123), we may in the same manner reduce the expression into factors. Accordingly let

$$P'+Q'\sin^2\lambda+R'\sin^4\lambda=(\alpha+\beta\sin^2\lambda)(\gamma-\beta\sin^2\lambda). \quad . \quad . \quad (149)$$

Writing α, β, γ instead of A, B, C, and following step by step the investigation in sec. [35], we shall have, as in (126) and (128), ψ, m, and $i_{\prime}$ being the amplitude, parameter and modulus,

$$\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda,\ m=\frac{\beta}{\alpha+\beta},\ i_{\prime}^2=\frac{\beta}{\gamma}\left(\frac{\alpha+\gamma}{\alpha+\beta}\right). \quad . \quad (150)$$

As $\qquad \alpha\gamma=a^2k^2,\ \beta=a^2-b^2,$ and $\gamma-\alpha=a^2-b^2-k^2, \quad . \quad (151)$

we shall have the following relations between the constants α, β, γ, m, $i_{\prime}$, and A, B, C, n, i, in (150) and (128),

$$\left.\begin{array}{l}\beta=\mathrm{B},\ \alpha=\mathrm{C}-\mathrm{B},\ \gamma=\mathrm{A}+\mathrm{B},\ \alpha+\gamma=\mathrm{A}+\mathrm{C},\\ \gamma-\beta=\mathrm{A},\ \alpha+\beta=\mathrm{C},\ \gamma-\alpha-\beta+\mathrm{C}-\mathrm{A}-\mathrm{B}=0,\\ i_{\prime}^2=\dfrac{\beta(\alpha+\gamma)}{\gamma(\alpha+\beta)}=\dfrac{\mathrm{B}(\mathrm{A}+\mathrm{C})}{(\mathrm{A}+\mathrm{B})\mathrm{C}}=i^2, \text{ or } i_{\prime}=i,\ \ m=\dfrac{\beta}{\alpha+\beta}=\dfrac{\mathrm{B}}{\mathrm{C}}.\end{array}\right\}. \quad (152)$$

Hence the moduli are the same in the two forms of integration, and the parameters m and n will be found to be connected by the equation

$$m+n-mn=i^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

m and n are therefore *conjugate* parameters, as they fulfil the condition assumed in (1).

The amplitudes ϕ and ψ are equal; for in (126) we assumed $\tan^2\phi=\dfrac{\mathrm{A}+\mathrm{B}}{\mathrm{A}}\tan^2\theta$, and in (150) $\tan^2\psi=\dfrac{\alpha+\beta}{\alpha}\tan^2\lambda$; but $\tan\lambda=\dfrac{a}{b}\tan\theta$, as in (146), whence $\tan^2\psi=\dfrac{a^2}{b^2}\dfrac{(\alpha+\beta)\mathrm{A}}{(\mathrm{A}+\mathrm{B})\alpha}\tan^2\phi$.

In (152) we have found $\alpha+\beta=\mathrm{C}$, and $\mathrm{A}+\mathrm{B}=\gamma$, whence

$$\tan^2\psi=\frac{a^2}{b^2}\frac{\mathrm{AC}}{\alpha\gamma}\tan^2\phi. \quad \text{But } \mathrm{AC}=b^2k^2, \text{ and } \alpha\gamma=a^2k^2,$$

as shown in (125) and (151), whence

$$\psi=\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

We shall now proceed to find the value of the second integral in (148).

From (147) we may derive

$$\tan^2\tau=\frac{(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}. \quad . \quad . \quad . \quad . \quad (155)$$

Differentiating this expression, reducing, dividing by $\cos\tau$, and integrating, we shall finally obtain

$$k\int\frac{d\tau}{\cos^3\tau}=(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{\cos\tau(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}; \quad . \quad (156)$$

(148) may now be written

$$k\Sigma=\int d\lambda\ \sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau}. \quad . \quad (157)$$

If we measure the arc of the logarithmic ellipse from the minor principal axis, or from the parabolic arc which is projected into b, instead of placing the origin at the vertex of the major axis as in (119), we must put

$$x=a\sin\vartheta,\quad y=b\cos\vartheta\,; \quad . \quad . \quad . \quad . \quad . \quad (158)$$

and following the steps indicated in that article, we shall obtain

$$k\Sigma_{\prime}=\int d\vartheta\ \sqrt{P'+Q'\sin^2\vartheta+R'\sin^4\vartheta}. \quad . \quad . \quad . \quad (159)$$

If we now make $\vartheta=\lambda$, and subtract the two latter equations one from the other, the resulting equation will be

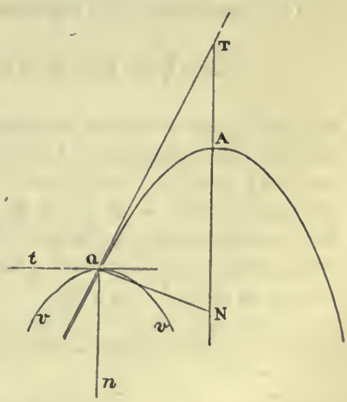
$$\Sigma_{\prime}-\Sigma=k\int\frac{d\tau}{\cos^3\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

But this integral is, we know, the expression for an arc of a common parabola whose semiparameter is k, measured from the vertex of the curve to a point on it where its tangent makes the angle τ with the ordinate.

Thus the difference between two elliptic arcs measured from the vertices of the curve, which in the plane ellipse may, as we know, be expressed by a straight line, and in the spherical ellipse by an arc of a circle (as shown in sec. [15]), will in the logarithmic ellipse be expressed by an arc of a parabola. As a parabolic arc can be rectified only by a logarithm, we may hence see the propriety of the term *logarithmic*, by which this function is designated.

39. If from the vertex A of a paraboloid, an arc of a parabola be drawn, at right angles to a parabolic section of the paraboloid, it will meet this parabolic section at its vertex. Let the arc AQ be drawn at right angles to the parabolic section Qv of the paraboloid, the point Q is the vertex of the parabola Qv.

Fig. 19.



Draw QT and Qt tangents to the arcs QA and Qv. Then QT and Qt are at right angles, since the arcs AQ, Qv are at right angles. As QT is a tangent to a principal section passing through the axis of the paraboloid, it will meet this axis in a point T; and as Qt is a tangent to the surface of the paraboloid, it will be perpendicular to QN the normal to the surface. Now as Qt is perpendicular to QT and to QN, it is perpendicular to the plane QTN which passes through them, and therefore to every line in this plane, and therefore to the axis AN, or to any line parallel to it, as the diameter Qn. Hence, as the tangent Qt to the parabola Qv is perpendicular to the diameter Qn, Q is the vertex of the parabola.

Hence, in the logarithmic ellipse, one extremity of the protangent arc is always the vertex of the parabola which touches the logarithmic ellipse at its other extremity.

This is a very important theorem, as the protangents are arcs of equal parabolas, all measured from the vertices of the parabolas. Hence also the length of the protangent arc depends solely on its normal angle.

As an arc of a circle may be expressed by the notation $s=\sin^{-1}\left(\frac{y}{k}\right)$, y being the ordinate and k the radius, so in like manner an arc of a parabola may be designated by the form $s=\tan^{-1}\left(\frac{y}{k}\right)$, y being the ordinate, and k the semiparameter. To distinguish the parabolic arc from the circular arc, the former may be written $s=\tau\alpha\nu^{-1}\left(\frac{y}{k}\right)$. Again, as we say, in the case of the circle, the angle ω and the arc $k\omega$, ω being the angle contained between the normals to the curve at the extremities of the arc, so in the parabola, we may write ω for the angle between the normals, and $(k.\omega)$ for the corresponding parabolic arc. In the case of the parabola the arc is always supposed to be measured from the vertex; in the circle the arc may be measured from any point, as every point is a vertex.

40.] Resuming the equation (157),

$$k\Sigma=\int d\lambda\,\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau},$$

we shall now proceed to develop the first integral of the second side of this equation. As the integral is precisely the same in form as (123), and the amplitude $\psi=\phi$, as also the modulus $i_{,}=i$, we may substitute α, β, γ for A, B, C, m for n, Φ_m for Φ_n, retaining the modulus and amplitude, which continue unchanged, as we have established in (152) and (154); or substituting for α, β, γ their values in m and i, we get

$$\left.\begin{aligned}\frac{2[i^2+m^2-2m]}{\sqrt{m(i^2-m)(1-m)}}\frac{\Sigma}{k}=-m\Phi_m-\frac{[i^2+m^2-2m]}{m}\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\\+\frac{[i^2-m]}{m}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\int d\phi\,\sqrt{1-i^2\sin^2\phi}-\frac{2[i^2+m^2-2m]}{\sqrt{m(i^2-m)(1-m)}}\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad (161)$$

If we eliminate i from the coefficients of this equation, putting M for $(1-m\sin^2\phi)$, and N for $(1-n\sin^2\phi)$, as also $\sqrt{\mathrm{I}}$ for $\sqrt{1-i^2\sin^2\phi}$, (133) may be written

$$\left.\begin{aligned}\frac{2(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}=-n\Phi_n+\frac{(1-n)(n-m)}{n}\int\frac{\mathrm{d}\phi}{\mathrm{N}\sqrt{\mathrm{I}}}\\ +\frac{m}{n}(1-n)\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\int\mathrm{d}\phi\sqrt{\mathrm{I}}\end{aligned}\right\}, \quad . \quad (162)$$

and (161) will be transformed into

$$\left.\begin{aligned}\frac{2(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}=-m\Phi_m-\frac{(1-m)(n-m)}{m}\int\frac{\mathrm{d}\phi}{\mathrm{M}\sqrt{\mathrm{I}}}\\ +\frac{n}{m}(1-m)\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\int\mathrm{d}\phi\sqrt{\mathrm{I}}-\frac{2(n-m)}{\sqrt{mn}}\int\frac{\mathrm{d}\tau}{\cos^3\tau}\end{aligned}\right\}. \quad . \quad . \quad (163)$$

If we compare together (162) and (163), which are expressions for the same arc of the logarithmic ellipse, and make the obvious reductions, putting for Φ_n and Φ_m their values $\dfrac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{N}}$ and $\dfrac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{M}}$, we shall get the following as the resulting equation of comparison,

$$\left.\begin{aligned}\left(\frac{1-n}{n}\right)\int\frac{\mathrm{d}\phi}{\mathrm{N}\sqrt{\mathrm{I}}}+\left(\frac{1-m}{m}\right)\int\frac{\mathrm{d}\phi}{\mathrm{M}\sqrt{\mathrm{I}}}\\ =\frac{i^2}{mn}\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}-\frac{2}{\sqrt{mn}}\int\frac{\mathrm{d}\tau}{\cos^3\tau}+\frac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{MN}}\end{aligned}\right\}. \quad . \quad (164)$$

From (155) we may deduce

$$\sin\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{\mathrm{I}}}; \quad . \quad . \quad . \quad . \quad . \quad (165)$$

we shall therefore have

$$\tan\tau\sec\tau=\frac{\sqrt{mn}\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{MN}}. \quad . \quad . \quad . \quad . \quad (166)$$

It may easily be shown that $\tan\tau\sec\tau$ represents the portion of a tangent to a parabola intercepted between the point of contact and the perpendicular from the focus.

Hence $$\tan\tau\sec\tau=2\int\frac{\mathrm{d}\tau}{\cos^3\tau}-\int\frac{\mathrm{d}\tau}{\cos\tau}. \quad . \quad . \quad . \quad . \quad (167)$$

Combining (164), (166) and (167), and using the ordinary notation of elliptic integrals,

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\,\phi)+\left(\frac{1-m}{m}\right)\Pi_c(m,\,\phi)=\frac{c^2}{mn}\,\mathrm{F}_c(\phi)-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau};$$

or as

$$\frac{d\tau}{\cos\tau}=\frac{d\sin\tau}{1-\sin^2\tau},\quad \frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}=\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]d\phi}{1-\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]^2}, \quad (168)$$

we have therefore

$$\left.\begin{aligned}&\left(\frac{1-n}{n}\right)\Pi_c(n,\,\phi)+\left(\frac{1-m}{m}\right)\Pi_c(m,\,\phi)\\ &=\frac{c^2}{mn}\,\mathrm{F}_c(\phi)-\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-c^2\sin^2\phi}}\right]d\phi}{1-\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-c^2\sin^2\phi}}\right]^2}\end{aligned}\right\}. \quad . \quad (169)$$

This is the expression given by LEGENDRE, *Traité des Fonctions Elliptiques,* tom. i. p. 68. Written in the notation adopted in this paper, the formula would be

$$\left(\frac{1-n}{n}\right)\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}+\left(\frac{1-m}{m}\right)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{\mathrm{I}}}-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}. \quad (170)$$

41.] We may express a and b, the semiaxes of the elliptic base of the cylinder, in terms of m and n, the conjugate parameters of the elliptic integrals in the preceding equations. From the equation of condition $m+n-mn=i^2$, and the expressions given in (130), we may eliminate i^2, and obtain

$$\frac{a^2}{k^2}=\frac{mn\,(1-m)}{(n-m)^2},\quad \frac{b^2}{k^2}=\frac{mn\,(1-n)}{(n-m)^2}. \quad . \quad . \quad . \quad (171)$$

Therefore $\displaystyle \frac{b}{a}=\sqrt{\frac{1-n}{1-m}}=\frac{\sqrt{(1-n)\,(1-m)}}{(1-m)}=\frac{\sqrt{1-i^2}}{1-m}=\frac{j}{1-m}.$

Hence the ratio of the axes of the elliptic base of the cylinder is a function of the modulus and parameter.

The ratio of the corresponding quantities in the case of the spherical ellipse may be derived from the equation

$$\frac{a^2-b^2}{a^2}=i^2;\ \text{or}\ \frac{b}{a}=\sqrt{1-i^2}=j.$$

This ratio is therefore independent of the parameter. There is, then, an important difference in the two cases. In the one case, the ratio of the axes is independent of the parameter, and will continue invariable while the parameter passes through every stage of magnitude. But in the logarithmic ellipse the vertical cylinder will change its base with the change of the parameter. We shall see the importance of this remark presently.

These ratios are :—

In the sphere, $\frac{b}{a}=j$; in the paraboloid, $\frac{b}{a}=\frac{j}{1-m}$. . . (172)

42.] Resuming equation (157) and developing it by a process similar to that applied to (127), we get

$$\Sigma=\frac{\alpha\gamma}{k\sqrt{\gamma(\alpha+\beta)}}\int\frac{[1-i^2\sin^2\phi]\,d\phi}{\left[1+\frac{\beta}{\alpha+\beta}\sin^2\phi\right]^2\sqrt{1-i^2\sin^2\phi}}-k\int\frac{d\tau}{\cos^3\tau}. \quad (173)$$

Now (151) and (152) give

$$\frac{\beta}{\alpha+\beta}=m,\ \alpha\gamma=a^2k^2,\ \sqrt{\gamma(\alpha+\beta)}=\frac{k^2\sqrt{mn}}{(n-m)},\ \text{and}\ a^2=\frac{k^2mn(1-m)}{(n-m)^2}.$$

Making these substitutions, we get

$$\Sigma=a\sqrt{(1-m)}\int\frac{[1-i^2\sin^2\phi]\,d\phi}{[1-m\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}-k\int\frac{d\tau}{\cos^3\tau}, \quad (174)$$

Now let $m=0$, then (165) gives $\tau=0$, and we shall have

$$\Sigma=a\int d\phi\sqrt{1-i^2\sin^2\phi}.$$

This is the common expression for the rectification of a plane ellipse whose greater semiaxis is a, and eccentricity i. This is case IV. of the Table, p. 7.

We cannot arrive at this limiting expression by making $e^2=m=0$ in (53); for this supposition would render $i=0$, which, throughout these investigations, is assumed to be invariable.

43.] If, as in the case of the spherical parabola, we make $n=m$, or $n=1-\sqrt{1-i^2}$, the values of $\frac{a}{k}$ and $\frac{b}{k}$ become infinite. What, then, is the meaning of the elliptic integral of the logarithmic form of the third order when $n=m$, or $n=1-\sqrt{1-i^2}$? In the circular form of the third order, when $m=n$, $n=\frac{1-j}{1+j}$, and the spherical ellipse becomes the spherical parabola, which, as we know, may be rectified by an elliptic integral of the first order.

Not only do the ratios $\frac{a}{k}$, $\frac{b}{k}$ become infinite, but they become equal; for $\frac{b^2}{a^2}=\frac{1-n}{1-m}=1$ when $m=n$. What, then, does the integral in this case signify? It does not become imaginary or change its species.

Resuming the equation established in (133),

$$\frac{2[2n-i^2-n^2]}{\sqrt{n(1-n)(i^2-n)}}\frac{\Sigma}{k}=-n\Phi_n$$
$$+\frac{[2n-i^2-n^2]}{n}\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\,\sqrt{I}.$$

If we now introduce the relation given in (130)

$$\frac{a}{k}=\frac{\sqrt{n(i^2-n)(1-i^2)}}{2n-i^2-n^2},$$

we shall have by substitution

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a}=-n\Phi_n$$
$$+\left(\frac{2n-i^2-n^2}{n}\right)\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\,\sqrt{I}. \quad (175)$$

If we now suppose $m=n$, or $n=1-\sqrt{1-i^2}$, or $2n-i^2-n^2=0$, the last equation will become

$$2\sqrt{j}\,\frac{\Sigma}{a}=j\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\,\sqrt{I}-n\Phi_n, \quad . \quad . \quad . \quad (176)$$

In this case $$\Phi_n=\frac{\tan\varphi\,\sqrt{I}}{1+j\tan^2\varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

This is the expression for the length of an arc of a logarithmic ellipse, the intersection of a cylinder, now become circular, with a paraboloid whose semiparameter $k=0$; therefore, the dimensions of the paraboloid being indefinitely diminished in magnitude, this intersection of a finite circular cylinder by a paraboloid indefinitely attenuated must take place at an infinite altitude. We naturally should suppose that the section of a cylinder which indefinitely approaches in its limit to a circular cylinder by a paraboloid of revolution, would be a circle; yet the fact is not so. The intersection of these surfaces, instead of being a circle, is a logarithmic ellipse, whose rectification may be effected by an elliptic integral of the second order, as we shall now proceed to show.

In the first place let us conceive the paraboloid as of definite magnitude, and the cylinder to be elliptical, its semiaxes as before being a and b. Then, as a and b are the ordinates of a parabola, at the points where the elliptic cylinder meets the paraboloid, at its greatest and least distances from the axis of the surfaces, we shall have

$$a^2=2kz', \ b^2=2kz''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

Hence $a^2-b^2=2k\,(z'-z'')$. Let $z'-z''=h$, then h is the thickness or height of that portion of the cylinder within which the logarithmic ellipse is contained.

Now (171) gives $\quad a^2-b^2=\dfrac{k^2mn}{n-m}$; therefore $2h=\dfrac{kmn}{n-m}$;

and we have also $\quad a=\dfrac{k\sqrt{mn(1-m)}}{n-m}$; hence $h=\dfrac{a}{2}\dfrac{\sqrt{mn}}{\sqrt{1-m}}$.

Now when $n=m$, $a=b$, $k=0$, while we get for h

$$h=\frac{a}{2}\frac{n}{\sqrt{1-n}}=\frac{a}{2}\frac{1-j}{\sqrt{j}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

We thus arrive at this most remarkable result, that though the cylinder changes from elliptic to circular, while the parameter of the paraboloid approximates to its limiting value 0, yet the thickness of the zone (that is, h) does not also indefinitely diminish, but assumes the limiting value given above.

Now if we cut this circular cylinder, the radius of whose base is a, by a plane making with the plane of the circular section, or with the plane of XY, an angle whose tangent is $\frac{h}{a}$, the semiaxes $\mathfrak{A}$ and $\mathfrak{B}$ of this plane section will manifestly be

$$\mathfrak{B}=a, \text{ and } \mathfrak{A}=\sqrt{a^2+h^2} \text{ or } \mathfrak{A}=\frac{a(2-n)}{2\sqrt{1-n}}. \quad . \quad . \quad . \quad (180)$$

If we denote the eccentricity of this plane ellipse by $i_{\prime}$,

$$i_{\prime}=\frac{n}{2-n}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}=\frac{1-j}{1+j}. \quad \text{Hence } j=\frac{1-i_{\prime}}{1+i_{\prime}}. \quad . \quad (181)$$

It is shown by Legendre and other writers on this subject that, if c and $c_{\prime}$ are two moduli connected by the equation

$$c_{\prime}=\frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}=\frac{1-b}{1+b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

and φ and ψ two angles related as in (63), writing φ for μ, so that

$$\tan(\psi-\varphi)=b\tan\varphi, \quad \ldots\ldots \quad (183)$$

we shall have

$$(1+c_{\prime})\mathrm{E}_c(\varphi)=\mathrm{E}_{c_\prime}(\psi)+c_{\prime}\sin\psi-\tfrac{1}{2}b_{\prime}^2\mathrm{F}_{c_\prime}(\psi). \quad \ldots \quad (184)$$

An independent demonstration of this theorem will be given in sec. [44].

Now $\qquad 1+c_{\prime}=\dfrac{2}{1+b}, \quad b_{\prime}^2=1-c_{\prime}^2=\dfrac{4b}{(1+b)^2}$; hence

$$\mathrm{E}_c(\varphi)=\frac{(1+b)}{2}\mathrm{E}_{c_\prime}(\psi)+\frac{(1-b)}{2}\sin\psi-\frac{b}{1+b}\mathrm{F}_{c_\prime}(\psi);$$

and, using the common notation for the present, (74) gives

$b\mathrm{F}_c(\varphi)=\dfrac{b}{1+b}\mathrm{F}_{c_\prime}(\psi)$. Adding these equations, we get

$$\mathrm{E}_c(\varphi)+b\mathrm{F}_c(\varphi)=\frac{1+b}{2}\mathrm{E}_{c_\prime}(\psi)+\frac{(1-b)}{2}\sin\psi, \quad \ldots \quad (185)$$

or, using the notation adopted in this work,

$$\frac{(1+j)}{2}\int d\psi\,\sqrt{\mathrm{I}_{\prime}}+\frac{n}{2}\sin\psi-\left[\int d\varphi\,\sqrt{\mathrm{I}}+j\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]=0, \quad . \quad (186)$$

since $\qquad n=1-b=1-j.$

Substituting the value of the first member of this equation in (176), the resulting equation will be

$$2\sqrt{j}\,\frac{\Sigma}{a}=\frac{(1+j)}{2}\int d\psi\,\sqrt{\mathrm{I}_{\prime}}+\frac{n}{2}\sin\psi-n\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}. \quad (187)$$

Having put for Φ_n its value in this case, namely

$$\Phi_n=\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi},$$

we must now combine the last two members of this equation. Adding, they become

$$\frac{n}{2}\left\{\sin\psi-\frac{2\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}\right\}. \quad \ldots\ldots \quad (188)$$

From this expression we must eliminate the functions of φ.

Now (73) gives $\qquad \sqrt{\mathrm{I}}=\dfrac{\cos^2\varphi+j\sin^2\varphi}{\sqrt{\mathrm{I}'}}, \quad \ldots\ldots \quad (189)$

writing φ for μ.

Substituting this value of $\sqrt{\mathrm{I}}$ in the preceding expression, for which we put t, we get

$$t=\frac{n}{2}\left\{\sin\psi-\frac{2\sin\phi\cos\phi}{\sqrt{\mathrm{I}_{,}}}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (190)$$

From this equation we must eliminate $\sin\phi$, $\cos\phi$.

If we solve the preceding equation (189), we shall obtain as the resulting expressions

$$\left.\begin{array}{l} 2\sin^2\phi=1-\sqrt{\mathrm{I}_{,}}\cos\psi+i_{,}\sin^2\psi \\ 2\cos^2\phi=1+\sqrt{\mathrm{I}_{,}}\cos\psi-i_{,}\sin^2\psi \end{array}\right\}. \quad . \quad . \quad . \quad (191)$$

Multiplying these equations together, and recollecting that $\mathrm{I}_{,}=1-i_{,}^2\sin^2\psi$, we shall find

$$4\cos^2\phi\sin^2\phi=\sin^2\psi[\mathrm{I}_{,}+2\sqrt{\mathrm{I}_{,}}i_{,}\cos\psi+i_{,}^2\cos^2\psi]. \quad (192)$$

Now the second member of this equation is a perfect square,

whence $\qquad 2\sin\phi\cos\phi=\sin\psi[\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi]. \quad . \quad . \quad . \quad (193)$

Substituting this value of $2\sin\phi\cos\phi$ in (190), we get

$$t=\frac{n}{2}\sin\psi\left[1-\frac{\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi}{\sqrt{\mathrm{I}_{,}}}\right]=-\frac{n}{2}\frac{i_{,}\sin\psi\cos\psi}{\sqrt{\mathrm{I}_{,}}}. \quad (194)$$

As $\qquad n=1-j$ and $i_{,}=\dfrac{1-j}{1+j}$, $n=\dfrac{2i_{,}}{1+i_{,}}$;

equation (187) may now be written

$$2\Sigma=\frac{a}{2}\left(\frac{1+j}{\sqrt{j}}\right)\int d\psi\sqrt{\mathrm{I}_{,}}-\frac{ai_{,}^2\sin\psi\cos\psi}{\sqrt{j}(1+i_{,})\sqrt{\mathrm{I}_{,}}}. \quad . \quad . \quad (195)$$

Now, as $\qquad \mathfrak{A}=\dfrac{a}{2}\dfrac{(2-n)}{\sqrt{1-n}}=\dfrac{a}{2}\dfrac{(1+j)}{\sqrt{j}}$ and $1+i_{,}=\dfrac{2}{1+j}$,

we get ultimately

$$2\Sigma=\mathfrak{A}\int d\psi\sqrt{\mathrm{I}_{,}}-\mathfrak{A}\frac{i_{,}^2\sin\psi\cos\psi}{\sqrt{1-i_{,}^2\sin^2\psi}}. \quad . \quad . \quad (196)$$

The second term of the last member of this equation is evidently the common expression for the protangent to a plane ellipse between the point of contact and the foot of a perpendicular on it from the centre; while $\mathfrak{A}\int d\psi\sqrt{\mathrm{I}}$, or $\mathfrak{A}\int d\psi\sqrt{1-i_{,}^2\sin^2\psi}$, is the expression for the arc of a plane ellipse whose semi- transverse axis is $\mathfrak{A}$, and eccentricity $i_{,}$.

When the function is complete, $\phi=\dfrac{\pi}{2}$ and $\psi=\pi$. See (183).

Hence, as $\int_0^{\pi} d\psi\, \sqrt{\mathrm{I}_{\prime}} = 2\int_0^{\frac{\pi}{2}} d\psi\, \sqrt{\mathrm{I}_{\prime}}$,

$$\Sigma = \mathfrak{A}\int_0^{\frac{\pi}{2}} d\psi\, \sqrt{\mathrm{I}_{\prime}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (197)$$

Σ therefore, in this case, is equal to a quadrant of the plane ellipse whose principal semiaxis $\mathfrak{A}$, and eccentricity $i_{\prime}$ are given by the equations

$$\mathfrak{A} = \sqrt{a^2+h^2}, \text{ and } i_{\prime} = \frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}. \quad . \quad . \quad . \qquad (198)$$

To distinguish this variety of the curve, we may call it the *circular logarithmic ellipse,* as it is a section of a circular cylinder. Accordingly, in the two forms of the third order, when the conjugate parameters are equal, or $m=n$, the representative curves of these forms become the spherical parabola and the circular logarithmic ellipse.

This is Case V. in the Table, p. 7. The results of the preceding investigation will reappear in the demonstration of the theorem, that quadrants of the spherical or logarithmic ellipse may be expressed by the help of integrals of the first and second orders.

44.] It is not difficult to show that this particular case of the logarithmic form, when the parameters m and n are equal, also represents the curve of intersection of a circular cylinder by a paraboloid whose principal sections are unequal.

Let $$x^2+y^2=a^2 \text{ and } \frac{x^2}{k}+\frac{y^2}{k'}=2z \quad . \quad . \quad . \quad . \qquad (199)$$

be the equations of the circular cylinder and of the elliptic paraboloid.

Assume $x=a\cos\theta$, $y=a\sin\theta$; then $2z=a^2\left\{\frac{\cos^2\theta}{k}+\frac{\sin^2\theta}{k'}\right\}$, (200)

$$\text{and } \frac{dx}{d\theta}=-a\sin\theta, \frac{dy}{d\theta}=a\cos\theta, \frac{dz}{d\theta}=a^2\left(\frac{1}{k'}-\frac{1}{k}\right)\sin\theta\cos\theta. \qquad (201)$$

Hence $$\frac{d\Sigma}{d\theta}=a\left[1+a^2\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta\cos^2\theta\right]^{\frac{1}{2}}. \quad . \quad . \quad . \qquad (202)$$

Now we may reduce this expression by two different methods to the form of an elliptic integral.

By the first method, eliminating $\cos^2\theta$, this expression becomes

$$\frac{d\Sigma^2}{d\theta^2}=a^2+a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta-a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^4\theta. \quad . \qquad (203)$$

We may, as in (124), reduce this expression to the form of a product of two quadratic factors,

$$(A+B\sin^2\theta)(C-B\sin^2\theta)=AC+B(C-A)\sin^2\theta-B^2\sin^4\theta. \quad (204)$$

Comparing this expression with the preceding,

$$\left.\begin{array}{c} AC=a^2,\ B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right),\ C-A=a^2\left(\frac{1}{k'}-\frac{1}{k}\right) \\ \text{or } C=A+B, \text{ and } AC=A^2+AB=a^2 \end{array}\right\} \quad . \quad . \quad (205)$$

Let us now, as in (126), assume

$$\tan^2\varphi=\frac{A+B}{A}\tan^2\theta, \quad . \quad . \quad . \quad . \quad . \quad (206)$$

and, following the steps there indicated, we shall have

$$\Sigma=A\int\frac{d\varphi\left[1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi}}, \quad . \quad (207)$$

an expression of the same form as (127).

Let $$\frac{B}{A+B}=n,\ \frac{B(2A+B)}{(A+B)^2}=i^2; \quad . \quad . \quad . \quad . \quad (208)$$

$$\left.\begin{array}{ll} \text{therefore} & 1-n=\frac{A}{A+B}, \text{ and } 1-i^2=\frac{A^2}{(A+B)^2} \\ \text{Hence} & 1-n=\sqrt{1-i^2}, \text{ or } n=m \end{array}\right\} \quad . \quad . \quad . \quad (209)$$

If we develop this integral by the method indicated in sec. [36], the coefficient $\frac{2n-i^2-n^2}{n}$ of the integral $\int\frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}$, in the result, will be 0, and the reduced integral will become, since

$$\frac{B}{A+B}=n,\ B=\frac{nA}{1-n}, \text{ and } B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right), \quad . \quad . \quad (210)$$

$$\Sigma=\frac{A}{2(1-n)}\left[\int d\varphi\sqrt{I}+(1-n)\int\frac{d\varphi}{\sqrt{I}}-n\Phi\right]. \quad . \quad (211)$$

Let z' and z'' be the altitudes of the points above the plane of XY, in which the principal sections of the elliptic paraboloid meet the circular cylinder. Then $z''-z'$ is the height or thickness of the zone of the cylinder on which the curve is traced.

Now $a^2=2kz',\ a^2=2k'z''$; whence $z''-z'=\frac{a^2}{2}\left(\frac{1}{k'}-\frac{1}{k}\right)$.

Let this altitude or thickness of the zone be put h, and we shall have

$$\Sigma = \frac{h}{n}\left[\int d\varphi\,\sqrt{\mathrm{I}} + (1-n)\int\frac{d\varphi}{\sqrt{\mathrm{I}}} - n\Phi\right]. \quad . \quad . \quad (212)$$

Hence the arc of this species of logarithmic ellipse may be expressed by integrals of the first and second orders.

It is not a little remarkable that whether the integrals of the third order be circular or logarithmic, or, looking to their geometrical origin, spherical or parabolic, when the conjugate parameters are equal, or $m=n$, we may express the arcs of the hyperconic sections thus represented, in terms of integrals of the first and second orders only, the integral of the third order being in this case eliminated.

If we now resume equation (202) and make

$$2\theta = \frac{\pi}{2} + \chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

$\sin 2\theta = 2\sin\theta\cos\theta = \cos\chi$, and $2d\theta = d\chi$. Therefore (202) will become

$$\frac{4d\Sigma^2}{d\chi^2} = a^2 + \frac{a^4}{4}\left(\frac{1}{k'} - \frac{1}{k}\right)^2\cos^2\chi; \quad . \quad . \quad . \quad (214)$$

hence, as $h = \frac{a^2}{2}\left(\frac{1}{k'} - \frac{1}{k}\right)$, we shall have

$$2\Sigma = \sqrt{a^2+h^2}\int d\chi\sqrt{1 - \frac{h}{\sqrt{a^2+h^2}}\sin^2\chi}. \quad . \quad . \quad (215)$$

This is the common form for the rectification of a plane ellipse, whose principal semiaxes are $\sqrt{a^2+h^2}$ and a. Let $i_{,}$ be the eccentricity of this plane ellipse,

$$i_{,} = \frac{h}{\sqrt{a^2+h^2}} = \frac{\mathrm{B}}{2\mathrm{A}+\mathrm{B}} = \frac{n}{2-n} = \frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}, \quad . \quad (216)$$

and the relation between φ and χ is given by the equations

$$2\theta = \frac{\pi}{2} + \chi, \quad \tan^2\theta = \frac{\mathrm{A}}{\mathrm{A}+\mathrm{B}}\tan^2\varphi, \quad \text{or } \tan\theta = \sqrt{1-n}\tan\varphi.$$

Hence

$$\frac{1+\sin\chi}{1-\sin\chi} = (1-n)\tan^2\theta, \text{ or } \sec\chi + \tan\chi = \sqrt{j}\tan\varphi. \quad (217)$$

When $\chi=0$, $\tan\varphi = \frac{1}{\sqrt{j}}$; when $\chi = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$; when $\chi = -\frac{\pi}{2}$, $\varphi = 0$.

Hence χ is measured from the perpendicular on the tangent to the ellipse, at the point which divides the elliptic quadrant into two segments whose difference is equal to $a-b$, as will be shown further on; while ϕ is measured from the semi-transverse axis a. Thus while χ varies from $-\frac{\pi}{2}$ (that is, from the position at right angles to this perpendicular, and *below* it) to 0 (that is, to the perpendicular itself), ϕ varies from 0 to $\tan^{-1}\frac{1}{\sqrt{j}}$; and while χ varies from 0 to $\frac{\pi}{2}$, ϕ varies from $\tan^{-1}\frac{1}{\sqrt{j}}$ to $\frac{\pi}{2}$. Thus while χ passes over two right angles, ϕ passes over one right angle.

We may now equate the two expressions (211) and (215); and the resulting equation will be

$$\int d\chi\,\sqrt{1-i_i^2\sin^2\chi}=\frac{2h}{\sqrt{a^2+h^2}}\left[\frac{1}{n}\int d\phi\,\sqrt{\text{I}}+\frac{(1-n)}{n}\int\frac{d\phi}{\sqrt{\text{I}}}-\Phi\right],$$

or
$$\frac{(1+j)}{2}\int d\chi\,\sqrt{\text{I}_{i_i}}+(1-j)\Phi-\left[\int d\phi\,\sqrt{\text{I}}+j\int\frac{d\phi}{\sqrt{\text{I}}}\right]=0. \quad (218)$$

Thus we may express an elliptic integral of the first order by means of two elliptic integrals of the second order. Hence we obtain the geometrical origin of the well-known theorem, given in (184).

When the functions are complete, since

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_i^2\sin^2\chi}=2\int_0^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_i^2\sin^2\chi},\text{ we get}$$

$$\int_0^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_i^2\sin^2\chi}=(1+j)\left[\int_0^{\frac{\pi}{2}}d\phi\,\sqrt{\text{I}}+(1-n)\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\text{I}}}\right], \quad (219)$$

which agrees with (186).

44*.] From the foregoing investigations it will follow that, if there are two moduli so related that

$$i_i=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}=\frac{1-j}{1+j}, \quad \ldots\ldots \quad (a)$$

and two amplitudes such that

$$\sec\chi+\tan\chi=\sqrt{j}\tan\phi, \quad \ldots\ldots \quad (b)$$

we may express an elliptic integral of the first order by the help of two elliptic integrals of the second order, whose moduli are i and i_i and whose amplitudes are ϕ and χ.

A like relation is established by Lagrange's theorem (186) in which the moduli are the same, but the amplitudes are given by the equation

$$\tan(\psi-\varphi)=j\tan\varphi. \quad \text{(c)}$$

Lagrange's theorem as given in (186) is

$$\left(\frac{1+j}{2}\right)\int d\psi\ \sqrt{_{\psi}\mathrm{I}_{,}}+\frac{n}{2}\sin\psi-\left[\int d\varphi\ \sqrt{\mathrm{I}}+j\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]=0. \quad \text{(d)}$$

While the theorem established in (218) is

$$\left(\frac{1+j}{2}\right)\int d\chi\ \sqrt{_{\chi}\mathrm{I}_{,}}+(1-j)\,\Phi-\left[\int d\varphi\ \sqrt{\mathrm{I}}+j\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]=0. \quad \text{(e)}$$

It may be proper to show that these theorems (d) and (e), though apparently diverse, are identical.

These equations will be identical if we can prove that

$$\left(\frac{1+j}{2}\right)\int d\psi\ \sqrt{_{\psi}\mathrm{I}_{,}}+\frac{n}{2}\sin\psi=\left(\frac{1+j}{2}\right)\int d\chi\ \sqrt{_{\chi}\mathrm{I}_{,}}+(1-j)\Phi. \quad \text{(f)}$$

To show this, we must eliminate φ between the equations

$$\tan(\psi-\varphi)=j\tan\varphi, \text{ and } \sec\chi+\tan\chi=\sqrt{j}\tan\varphi.$$

Eliminating φ and reducing,

$$\tan^2\psi\tan^2\chi=\frac{(1+j)^2}{4j}. \quad \text{(g)}$$

Hence

$$\sin\psi=\frac{\cos\chi}{[1-i_{,}^2\sin^2\chi]^{\frac{1}{2}}}, \quad \text{(h)}$$

and

$$\Phi=\frac{\sin\varphi\cos\varphi\ \sqrt{_{\varphi}\mathrm{I}}}{[1-n\sin^2\varphi]}=\frac{\tan\varphi\ \sqrt{_{\varphi}\mathrm{I}}}{1+j\tan^2\varphi}, \text{ since } n=1-j;$$

consequently

$$2\Phi=\frac{\cos\chi-\left(\frac{1-j}{1+j}\right)\sin\chi\cos\chi}{[1-i_{,}^2\sin^2\chi]^{\frac{1}{2}}}; \quad \text{(i)}$$

therefore

$$\sin\psi-2\Phi=\frac{\left(\frac{1-j}{1+j}\right)\sin\chi\cos\chi}{[-i_{,}^2\sin^2\chi]^{\frac{1}{2}}}.$$

We have also

$$d\psi\ \sqrt{1-i_{,}^2\sin^2\psi}=\frac{4j}{(1+j)^2}\ \frac{d\chi}{[1-i_{,}^2\sin^2\chi]^{\frac{3}{2}}}. \quad \text{(j)}$$

But $$\frac{4j}{(1+j)^2}\int\frac{d\chi}{[1-i_{\prime}^2\sin^2\chi]^{\frac{3}{2}}}=\int d\chi\,\sqrt{1-i_{\prime}^2\sin^2\chi}$$
$$-\left(\frac{1-j}{1+j}\right)^2\frac{\sin\chi\cos\chi}{[1-i_{\prime}^2\sin^2\chi]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (k)$$

as will be shown further on; consequently

$$\int d\psi\,\sqrt{1-i_{\prime}^2\sin^2\psi}=\int d\chi\,\sqrt{1-i_{\prime}^2\sin^2\chi}-\left(\frac{1-j}{1+j}\right)^2\frac{\sin\chi\cos\chi}{[1-i_{\prime}^2\sin^2\chi]^{\frac{1}{2}}}. \quad (l)$$

Substituting these values in (j), the equations are manifestly identical.

We may thus by the help of Lagrange's formula, as given in (d), or by the new expression enunciated in (e), express an elliptic integral of the *first* order by the help of *two* elliptic integrals of the *second* order; but we are unable to reverse the process, and exhibit an elliptic integral of the *second* order, as a function of two elliptic integrals of the *first* order. The problem has been tried, but in vain.

If we multiply (218) by a, bearing in mind that $a^2-b^2=a^2i^2$ and $b=aj$, we shall have, since $n=1-j$,

$$\left(\frac{a+b}{2}\right)\int d\chi\,\sqrt{\mathrm{I}_{j_{\prime}}}-a\int d\phi\,\sqrt{\mathrm{I}}=b\int\frac{d\phi}{\sqrt{\mathrm{I}}}-(a-b)\,\Phi\,; \quad . \quad . \quad (m)$$

but when the functions are complete, since

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_{\prime}^2\sin^2\chi}=2\int_0^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_{\prime}^2\sin^2\chi},$$

we shall have

$$(a+b)\int_0^{\frac{\pi}{2}}d\chi\,\sqrt{1-i_{\prime}^2\sin^2\chi}-a\int_0^{\frac{\pi}{2}}d\phi\,\sqrt{1-i^2\sin^2\phi}=b\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}; \quad (n)$$

$(a+b)$ and $2\sqrt{ab}$ are the semiaxes of the ellipse whose amplitude is χ and modulus $i_{\prime}$. Hence we may derive the following theorem :—

The difference between the quadrants of two ellipses whose semiaxes are a, b, and $(a+b)$, $2\sqrt{ab}$ is equal to a complete elliptic integral of the first *order whose modulus is i*; or, *The difference between the quadrants of two ellipses whose semiaxes are a, b and $(a+b)$, $2\sqrt{ab}$ is equal to half the difference between the circumference of a spherical parabola and a semicircle, both described on a sphere whose radius is a.*

It may be worth while to mention that $a+b$ is the length of the tangent drawn to the ellipse whose semiaxes are a, b, and intercepted between the axes; while the point of contact is the *critical* point, or the point where, as Fagnano has shown, the constituent arcs of the quadrant of the ellipse differ by $a-b$.

$\sqrt{ab}$ is the perpendicular from the centre on this tangent.

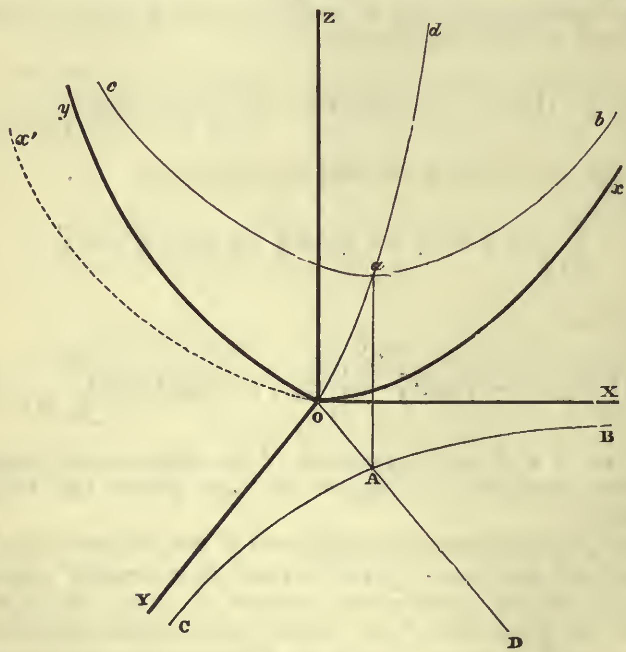
CHAPTER V.

ON THE LOGARITHMIC HYPERBOLA.

45.] The logarithmic hyperbola may be defined as the curve of symmetrical intersection of a paraboloid of revolution with a right cylinder standing on a plane hyperbola as a base.

Let Oxx' be a paraboloid of revolution, whose vertex is at O, and whose axis is OZ. Let ACB be an hyperbola in the plane of

Fig. 20.



XY, whose vertex is at A, and whose axis is the straight line OAD. Let the planes ZOX, ZOD, ZOY cut the paraboloid in the plane parabolas Ox, Od, Oy, and let cab be the curve on the surface of the paraboloid whose orthogonal projection on the plane of xy is the plane hyperbola ABC. Then acb is the logarithmic hyperbola.

Vertical planes erected on the asymptotes of the hyperbola in the plane of XY will pass through the axis OZ, and will cut the paraboloid in two parabolas passing through the vertex O, which will be asymptotic curves to the logarithmic hyperbola. These curves will be found to have properties analogous to those of the plane hyperbola and its asymptotes.

Let
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ and } x^2+y^2=2kz \quad . \quad . \quad . \quad . \quad (220)$$

be the equations of the hyperbolic cylinder and of the paraboloid of revolution, and consequently of the curve in which they intersect; let Υ be an arc of this curve,

then
$$\Upsilon=\int \mathrm{d}\lambda \left[\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad (221)$$

x, y, z being functions of a fourth independent variable λ.

Assume
$$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda},\ y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}*. \quad (222)$$

It is manifest that these assumptions are compatible with the first of equation (220); and the second of that group gives

$$x^2+y^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}=2kz.$$

Differentiating (222), we get

$$\left.\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4\sin^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\ \left(\frac{\mathrm{d}y}{\mathrm{d}\lambda}\right)^2=\frac{a^4b^4\cos^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\\ \left(\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^2=\frac{(a^2+b^2)^2\,a^4b^4\sin^2\lambda\cos^2\lambda}{k^2\,(a^2\cos^2\lambda-b^2\sin^2\lambda)^4}.\end{aligned}\right\} \quad (223)$$

* We might, by the help of the imaginary transformation $\sin\theta=\sqrt{-1}\tan\theta'$, pass at once from the elliptic cylinder to the hyperbolic cylinder. Let $\tan\theta'=u$, and the resulting equation will be of the form

$$\frac{\mathrm{d}\Upsilon}{\mathrm{d}u}=\frac{\alpha+\beta u^2+\gamma u^4}{\sqrt{\mathrm{A}+\mathrm{B}u^2+\mathrm{C}u^4+\mathrm{D}u^6}},$$

an expression which, on trial, it would be found very difficult to reduce. The difficulty is eluded by making the transformation pointed out and adopted in the text.

Hence

$$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{[a^2k^2+(a^2+b^2)(a^2+b^2-k^2)\sin^2\lambda-(a^2+b^2)^2\sin^4\lambda]^{\frac{1}{2}}}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^2}. \quad (224)$$

Let this radical be put $=\sqrt{\mathrm{R}}$.

Assume
$$\left.\begin{aligned}\mathrm{R}&=(\mathrm{A}+\mathrm{B}\sin^2\lambda)(\mathrm{C}-\mathrm{B}\sin^2\lambda)\\&=\mathrm{AC}+\mathrm{B}(\mathrm{C}-\mathrm{A})\sin^2\lambda-\mathrm{B}^2\sin^4\lambda;\end{aligned}\right\} \quad . \quad . \quad . \quad (225)$$

hence $\mathrm{AC}=a^2k^2,\ \mathrm{B}=a^2+b^2,\ \mathrm{C}-\mathrm{A}=a^2+b^2-k^2,$
and therefore $k^2=\mathrm{A}+\mathrm{B}-\mathrm{C}.$ (226)

Let us now assume $\sin\phi$ such that

$$\sin^2\lambda=\frac{\mathrm{AC}\sin^2\phi}{\mathrm{AB}+\mathrm{BC}\cos^2\phi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

then $$\mathrm{A}+\mathrm{B}\sin^2\lambda=\frac{\mathrm{A}(\mathrm{A}+\mathrm{C})}{\mathrm{A}+\mathrm{C}\cos^2\phi},\quad \mathrm{C}-\mathrm{B}\sin^2\lambda=\frac{\mathrm{C}(\mathrm{A}+\mathrm{C})\cos^2\phi}{\mathrm{A}+\mathrm{C}\cos^2\phi},$$

and $$a^2\cos^2\lambda-b^2\sin^2\lambda=a^2-\frac{(a^2+b^2)\,\mathrm{AC}\sin^2\phi}{\mathrm{B}(\mathrm{A}+\mathrm{C}\cos^2\phi)};$$

or as $$a^2+b^2=\mathrm{B},\ \mathrm{AC}=a^2k^2,\ \mathrm{C}+k^2=\mathrm{A}+\mathrm{B},$$

there results $$a^2\cos^2\lambda-b^2\sin^2\lambda=\frac{a^2(\mathrm{A}+\mathrm{C})}{\mathrm{A}+\mathrm{C}\cos^2\phi}\left[1-\frac{\mathrm{A}+\mathrm{B}}{\mathrm{A}+\mathrm{C}}\sin^2\phi\right].$$

Hence $$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{\sqrt{\mathrm{AC}}\,.\,[\mathrm{A}+\mathrm{C}\cos^2\phi]\cos\phi}{a^4(\mathrm{A}+\mathrm{C})[1-l\sin^2\phi]^2}. \quad . \quad . \quad . \quad (228)$$

Making $$l=\frac{\mathrm{A}+\mathrm{B}}{\mathrm{A}+\mathrm{C}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

differentiating the equation $$\sin^2\lambda=\frac{\mathrm{AC}\sin^2\phi}{\mathrm{AB}+\mathrm{BC}\cos^2\phi}, \quad . \quad . \quad (230)$$

we find $$\frac{d\lambda}{d\phi}=\frac{ak\sqrt{\mathrm{A}+\mathrm{C}}\cos\phi}{\sqrt{\mathrm{B}}[\mathrm{A}+\mathrm{C}\cos^2\phi]\sqrt{1-\frac{\mathrm{C}}{\mathrm{B}}\left(\frac{\mathrm{A}+\mathrm{B}}{\mathrm{A}+\mathrm{C}}\right)\sin^2\phi}}; \quad . \quad (231)$$

or as $$\frac{d\Upsilon}{d\phi}=\frac{d\Upsilon}{d\lambda}\frac{d\lambda}{d\phi},\ \text{making } i^2=\frac{\mathrm{C}(\mathrm{A}+\mathrm{B})}{\mathrm{B}(\mathrm{A}+\mathrm{C})}, \quad . \quad . \quad . \quad (232)$$

we get, finally, $$\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}\int\frac{\cos^2\phi\,d\phi}{[1-l\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}. \quad (233)$$

46.] We may develop another formula for the rectification of an arc of the logarithmic hyperbola.

Assuming the principles established in sec. [38], we may put

$$\Upsilon = -\int p \sec \upsilon d\lambda - \int \frac{d^2p}{d\lambda^2} \sec \upsilon d\lambda. \quad . \quad . \quad . \quad . \quad . \quad (234)$$

In this formula p is the perpendicular from the axis of the hyperbolic cylinder let fall on a tangent plane to it, passing through the element of the curve, and υ is the angle which a tangent to this element makes with the plane of the base. υ in this equation is analogous to τ in the last section.

In the above expression the negative sign is used, as the curve and the angle λ are incremented in opposite directions.

Now $p^2 = a^2 \cos^2 \lambda - b^2 \sin^2 \lambda$, and $\tan \upsilon = \dfrac{\dfrac{dz}{d\lambda}}{\sqrt{\left(\dfrac{dx}{d\lambda}\right)^2 + \left(\dfrac{dy}{d\lambda}\right)^2}}$.

We must substitute for these differentials, their values given in (223), and introduce the value of ϕ assumed in (227), whence

$$\sec^2 \upsilon = \frac{(A+C)^2 AC \cos^2\phi}{k^2[A + C \cos^2\phi]^2 (a^2 \cos^2\lambda - b^2 \sin^2\lambda)}; \quad . \quad (235)$$

therefore
$$p \sec \upsilon = \frac{\sqrt{AC}\,(A+C) \cos\phi}{k[A + C \cos^2\phi]}. \quad . \quad . \quad . \quad . \quad . \quad (236)$$

But (231) gives $\dfrac{d\lambda}{d\phi} = \dfrac{\sqrt{A+C}\,.\,ak \cos\phi}{\sqrt{B}[A + C\cos^2\phi]\,\sqrt{1-i^2\sin^2\phi}}$,

whence

$$p \sec \upsilon d\lambda = \frac{a^2 k \cos^2\phi d\phi}{\sqrt{B(A+C)}\left[1 - \dfrac{C}{A+C}\sin^2\phi\right]^2 \sqrt{1-i^2\sin^2\phi}}. \quad (237)$$

We must now determine the value of the second integral in (234), namely

$$\int \frac{d^2p}{d\lambda^2} \sec \upsilon d\lambda.$$

Since
$$p^2 = a^2 \cos^2\lambda - b^2 \sin^2\lambda,$$

$$\frac{d^2p}{d\lambda^2} \sec \upsilon d\lambda = -\frac{(a^2+b^2)[a^2 \cos^4\lambda + b^2 \sin^4\lambda] \sec \upsilon d\lambda}{(a^2 \cos^2\lambda - b^2 \sin^2)^{\frac{3}{2}}}. \quad (238)$$

Now we may derive from (223)

$$\tan v = \frac{(a^2+b^2)\sin\lambda\cos\lambda}{k(a^2\cos^2\lambda - b^2\sin^2\lambda)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (239)$$

Differentiating this expression, then multiplying by sec v, and integrating, we obtain

$$k\int\frac{dv}{\cos^3 v} = (a^2+b^2)\int\frac{[a^2\cos^4\lambda + b^2\sin^4\lambda]\sec v d\lambda}{(a^2\cos^2\lambda - b^2\sin^2\lambda)^{\frac{3}{2}}}. \quad . \quad . \quad (240)$$

Comparing this expression with (238), and introducing into (234) the values found in (237) and (240), we obtain

$$\frac{\Upsilon}{k} = \int\frac{dv}{\cos^3 v} - \frac{a^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\phi d\phi}{[1-m\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}. \quad (241)$$

Making $$m = \frac{C}{A+C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

since $l = \dfrac{A+B}{A+C}$, and $i^2 = \dfrac{C}{B}\left(\dfrac{A+B}{A+C}\right)$, assume $n = \dfrac{i^2}{l} = \dfrac{C}{B}$, . (243)

and we shall have m and n connected by the equation of condition, defined in (1),

$$m+n-mn = i^2.$$

The three parameters l, m, n, and the modulus i are connected by the equations

$$ln = i^2, \quad m+n-mn = i^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (244)$$

l and n are *reciprocal* parameters, the reader will recollect, while m and n are *conjugate* parameters.

By the help of these equations, any one of the quantities l, m, n, i^2 may be eliminated, and an equation established between the three remaining quantities.

47.] It was shown in (226), that $C-A = a^2+b^2-k^2$, $B = a^2+b^2$, $k^2 = A+B-C$, and $a^2k^2 = AC$, whence

$$\frac{a^2}{k^2} = \frac{AC}{(A+B-C)^2}, \quad \frac{b^2}{k^2} = \frac{(A+B)(B-C)}{(A+B-C)^2}. \quad . \quad . \quad . \quad (245)$$

In order that these values of a and b may be real, we must have $B>C$, and A of the same sign with C, both positive; otherwise $\sqrt{R}$ in (225) would be imaginary. As $l = \dfrac{A+B}{A+C}$, $l>1$; here the parameter l is greater than 1, while m and n are each less than 1.

We may express the semiaxes of the hyperbola, the base of the hyperbolic cylinder, in terms of the modulus i and the parameter l; for by the equations immediately preceding we may eliminate A, B, and C in (243). We thus find

$$\frac{a^2}{k^2}=\frac{li^2(l-1)(1-i^2)}{[l^2+i^2-2li^2]^2},\ \frac{b^2}{k^2}=\frac{l(l-1)(l-i^2)^2}{[l^2+i^2-2li^2]^2}; \quad . \quad . \quad (246)$$

therefore

$$\frac{\mathrm{B}}{k^2}=\frac{a^2+b^2}{k^2}=\frac{l(l-1)}{l^2+i^2-2li^2}, \text{ and } (\mathrm{A}+\mathrm{C})=\frac{n\mathrm{B}}{m}. \quad . \quad . \quad . \quad (247)$$

We may express the semiaxes in terms of the conjugate parameters m and n,

$$\frac{a^2}{k^2}=\frac{n^2m\,(1-m)}{[n+m-2mn]^2},\ \frac{b^2}{k^2}=\frac{m\,(1-n)\,(n+m-mn)}{[n+m-2mn^2]}; \quad . \quad (248)$$

hence

$$\frac{\mathrm{B}}{k^2}=\frac{a^2+b^2}{k^2}=\frac{m}{[m+n-2mn]}, \text{ and } \sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}=\frac{k^2\sqrt{mn}}{[m+n-2mn]}; \ (249)$$

or we may express a and b more simply in terms of l and m. Eliminating n and i^2, we get

$$\frac{a^2}{k^2}=\frac{m\,(1-m)}{(l-m)^2},\ \frac{b^2}{k^2}=\frac{l(l-1)}{(l-m)^2}. \quad . \quad . \quad . \quad . \quad . \quad (250)$$

Let $c_{\prime}$ be the eccentricity of the hyperbolic base of the cylinder, the following equation between $c_{\prime}$, i and l, analogous to (131), will follow from (246),

$$(c_{\prime}^2-1)\,i^2j^2=(l-i^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

Hence when i and l are given, $c_{\prime}$ may easily be found.

48.] If we equate together the values found for Υ, the arc of the logarithmic hyperbola, in (233) and (241), we shall have

$$\left.\begin{aligned} b^2\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}} \\ +a^2\int\frac{\cos^2\varphi d\varphi}{[1-m\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}=\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}\int\frac{dv}{\cos^3 v}\end{aligned}\right\}. \ (252)$$

For brevity, put

$$\mathrm{L}=1-l\sin^2\varphi, \mathrm{M}=1-m\sin^2\varphi, \mathrm{N}=1-n\sin^2\varphi, \mathrm{I}=1-i^2\sin^2\varphi. \ (253)$$

The preceding equation may now be written

$$b^2\int\frac{\cos^2\varphi d\varphi}{\mathrm{L}^2\sqrt{\mathrm{I}}}+a^2\int\frac{\cos^2\varphi d\varphi}{\mathrm{M}^2\sqrt{\mathrm{I}}}=\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}\int\frac{dv}{\cos^3 v}; \quad (254)$$

or if we substitute for the coefficients of this equation their values given in (246), we shall have

$$\left.\begin{aligned}(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}\\ =\frac{[l^2+i^2-2li^2]\sqrt{l-i^2}}{\sqrt{l(l-1)}}\int\frac{dv}{\cos^3 v}\end{aligned}\right\}. \quad . \quad (255)$$

Let

$$\left.\begin{aligned}f=l^2+i^2-2li^2,\ f_l=m^2+i^2-2mi^2,\\ \Phi_l=\frac{\sin\varphi\cos\varphi\sqrt{I}}{L},\ \Phi_m=\frac{\sin\varphi\cos\varphi\sqrt{I}}{M}\end{aligned}\right\}. \quad . \quad (256)$$

Now the process given in sec. [36] will enable us to develop the integrals

$$\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}\ \text{and}\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}},\ \text{as follows :—}$$

$$\left.\begin{aligned}2(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}=l(l-i^2)\Phi_l-(l-i^2)\int d\varphi\sqrt{I}\\ +\frac{(l-i^2)^2}{l}\int\frac{d\varphi}{\sqrt{I}}+\frac{f}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}\end{aligned}\right\}; \quad . \quad (257)$$

and

$$\left.\begin{aligned}2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=-\frac{m(1-i^2)i^2}{(i^2-m)}\Phi_m+\frac{i^2(1-i^2)}{i^2-m}\int d\varphi\sqrt{I}\\ +\frac{i^2(1-i^2)}{m}\int\frac{d\varphi}{\sqrt{I}}-\frac{f_l i^2(1-i^2)}{m(i^2-m)}\int\frac{d\varphi}{M\sqrt{I}}\end{aligned}\right\}. \quad (258)$$

The equations of condition $ln=i^2$ and $m+n-mn=i^2$ give

$$\frac{i^2(1-i^2)}{i^2-m}=l-i^2\ \text{and}\ \frac{(l-i^2)^2}{l}+\frac{i^2(1-i^2)}{m}=\frac{(l-i^2)f}{l(l-1)}. \quad . \quad (259)$$

We have also, since

$$m=\frac{i^2(l-1)}{l-i^2},\ l(l-i^2)\Phi_l-\frac{mi^2(1-i^2)}{(i^2-m)}\Phi_m=\frac{f\sin\varphi\cos\varphi\sqrt{I}}{LM}. \quad (260)$$

Making these substitutions, adding together (257) and (258), the coefficient of $\int d\varphi\sqrt{I}$ vanishes, and we shall have

$$\begin{aligned}2(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\frac{f\sin\varphi\cos\varphi\sqrt{I}}{LM}\\ +\frac{f(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{f}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{f_l(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}};\end{aligned}$$

but (255) gives

$$(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=f\sqrt{\frac{(l-i^2)}{l(l-1)}}\int\frac{dv}{\cos^3 v}.$$

Combining this equation with the preceding,

$$\left.\begin{aligned}\frac{f}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{f_l(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}}+\frac{f(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}&\\+\frac{f\sin\varphi\cos\varphi\sqrt{I}}{LM}=2f\sqrt{\frac{l-1}{l(l-1)}}\int\frac{dv}{\cos^3 v}&\end{aligned}\right\}. \quad . \quad (261)$$

Now $\qquad f_l=m^2+i^2-2mi^2=\dfrac{fi^2(1-i^2)}{(l-i^2)^2};$

and as $\qquad m=\dfrac{i^2(l-1)}{l-i^2},\ \dfrac{f_l(l-i^2)}{m}=\dfrac{f(1-i^2)}{(l-1)}.$

In the last equation, substituting this value of f_l, and then dividing by f, we obtain

$$\left.\begin{aligned}\frac{(l-i^2)}{l}\int\frac{d\varphi}{L\sqrt{I}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\varphi}{M\sqrt{I}}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}&\\+\frac{\sin\varphi\cos\varphi\sqrt{I}}{LM}=2\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{dv}{\cos^3 v}&\end{aligned}\right\}. \quad . \quad (262)$$

Now $\qquad 2\displaystyle\int\frac{dv}{\cos^3 v}=\tan v\sec v+\int\frac{dv}{\cos v}$ and $\cos^2 v=\dfrac{LM}{\cos^2\varphi}$, $\quad . \quad (263)$

as may be shown by combining (226) with (235).

Hence $\qquad \sin v=\sqrt{\dfrac{l(l-1)}{l-i^2}}\tan\varphi\sqrt{I}, \quad . \quad . \quad . \quad . \quad (264)$

and therefore $\tan v\sec v=\sqrt{\dfrac{l(l-1)}{l-i^2}}\dfrac{\sin\varphi\cos\varphi\sqrt{I}}{LM}. \quad . \quad . \quad (265)$

Substituting this value in the preceding equation, we find

$$\left.\begin{aligned}\left(\frac{l-i^2}{l}\right)\int\frac{d\varphi}{L\sqrt{I}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\varphi}{M\sqrt{I}}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}&\\=\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{dv}{\cos v}&\end{aligned}\right\}. \quad . \quad (266)$$

In (170) we showed that, m and n being conjugate parameters connected by the equation $m+n-mn=i^2$,

$$\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{N\sqrt{I}}+\frac{(1-m)}{m}\int\frac{d\varphi}{M\sqrt{I}}=\frac{i^2}{mn}\int\frac{d\varphi}{\sqrt{I}}-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}.$$

Now $$\left(\frac{1-n}{n}\right)=\frac{l}{i^2}\left(\frac{l-i^2}{l}\right),\ \left(\frac{1-m}{m}\right)=\frac{l}{i^2}\left(\frac{1-i^2}{l-1}\right),$$

$$\frac{i^2}{mn}=\frac{l}{i^2}\left(\frac{l-i^2}{l-1}\right),\text{ and }\frac{1}{\sqrt{mn}}=\frac{l}{i^2}\sqrt{\frac{l-i^2}{l(l-1)}}.$$

Substituting these values in the preceding equation, and dividing by $\frac{l}{i^2}$, we obtain

$$\left.\begin{aligned}&\left(\frac{l-i^2}{l}\right)\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{1-i^2}{l-1}\right)\int\frac{d\varphi}{M\sqrt{I}}\\&\quad=\left(\frac{l-i^2}{l-1}\right)\int\frac{d\varphi}{\sqrt{I}}-\sqrt{\frac{l-1}{l(l-1)}}\int\frac{d\tau}{\cos\tau}\end{aligned}\right\}\quad . \quad . \quad . \quad (267)$$

If we add this equation to (266), the coefficient of the integral $\int\frac{d\varphi}{M\sqrt{I}}$ will vanish, and the resulting equation will become

$$\int\frac{d\varphi}{L\sqrt{I}}+\int\frac{d\varphi}{N\sqrt{I}}=\int\frac{d\varphi}{\sqrt{I}}+\frac{\sqrt{l}}{\sqrt{(l-1)(l-i^2)}}\left[\int\frac{dv}{\cos v}-\int\frac{d\tau}{\cos\tau}\right].\quad(268)$$

We shall now proceed to show that $\int\frac{dv}{\cos v}-\int\frac{d\tau}{\cos\tau}$ may be put under the form $\int\frac{dv'}{\cos v'}$, if we make the assumption

$$\sin v'=\frac{\sqrt{\kappa'}\tan\varphi}{\sqrt{I}},\quad . \quad . \quad . \quad . \quad . \quad . \quad (269)$$

κ' being equal to $(1-n)\left(\frac{i^2}{n}-1\right)=\frac{(l-i^2)(l-1)}{l}$.

Now $$\cos^2 v=\frac{(1-m\sin^2\varphi)(1-l\sin^2\varphi)}{\cos^2\varphi},\text{ as in (263);}$$

hence

$$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{dv}{\cos v}=\int\frac{d\varphi}{\sqrt{I}}\left[\frac{[1-i^2\sin^2\varphi-i^2\sin^2\varphi\cos^2\varphi]}{LM}\right].\quad(270)$$

But we derive from (165) and (166) the value

$$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\tau}{\cos\tau}=\int\frac{d\phi}{\sqrt{I}}\frac{[n\cos^2\phi-n\sin^2\phi+ni^2\sin^4\phi]}{MN}, \quad (271)$$

or, subtracting,

$$\left.\begin{aligned}\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]&=\int\frac{\sqrt{I}}{M}\left[\frac{1}{L}+\frac{n\sin^2\phi}{N}\right]d\phi\\ &-\int\frac{\cos^2\phi}{M\sqrt{I}}\left[\frac{i^2\sin^2\phi}{L}+\frac{n}{N}\right]d\phi\end{aligned}\right\}. \quad (272)$$

These two latter integrals may be combined into the single integral

$$\int\frac{[1-i^2\sin^2\phi-n\cos^2\phi][1-i^2\sin^4\phi]d\phi}{LMN\sqrt{I}}. \quad . \quad (273)$$

Now, as $m+n-mn=i^2$, the first factor of the numerator becomes $(1-n)(1-m\sin^2\phi)=(1-n)M$; and therefore

$$\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]=\left(\frac{l-i^2}{l}\right)\int\frac{[1-i^2\sin^4\phi]}{LN\sqrt{I}}. \quad (274)$$

Substituting the first member of this equation for the last term in (268), we find

$$\int\frac{d\phi}{L\sqrt{I}}+\int\frac{d\phi}{N\sqrt{I}}-\int\frac{d\phi}{\sqrt{I}}=\int\frac{[1-i^2\sin^4\phi]}{LN\sqrt{I}}. \quad . \quad . \quad (275)$$

Now, since we have assumed in (269)

$$\left.\begin{aligned}&\sin\upsilon'=\frac{\sqrt{\kappa'}\tan\phi}{\sqrt{I}}, \quad \cos^2\upsilon'=\frac{LN}{I\cos^2\phi};\\ &\text{hence } \frac{d\upsilon'}{\cos\upsilon'}=\frac{\sqrt{\kappa'}[1-i^2\sin^4\phi]d\phi}{LN\sqrt{I}}\end{aligned}\right\}; \quad . \quad . \quad (276)$$

and consequently

$$\int\frac{d\phi}{L\sqrt{I}}+\int\frac{d\phi}{N\sqrt{I}}=\frac{d\phi}{\sqrt{I}}+\frac{1}{\sqrt{\kappa'}}\int\frac{d\upsilon'}{\cos\upsilon'}. \quad . \quad . \quad (277)$$

This formula is usually written

$$\left.\begin{aligned}\int\frac{d\phi}{\left[1-\frac{c^2}{n}\sin^2\phi\right]\sqrt{1-c^2\sin^2\phi}}&+\int\frac{d\phi}{[1-n\sin^2\phi]\sqrt{1-c^2\sin^2\phi}}\\ &=F_c(\phi)+\frac{1}{\sqrt{\alpha}}\int\frac{\frac{d}{d\phi}\left(\frac{\sqrt{\alpha}\tan\phi}{\Delta}\right)d\phi}{1-\left(\frac{\sqrt{\alpha}\tan\phi}{\Delta}\right)^2}\end{aligned}\right\}. \quad (278)$$

We have thus shown that from the comparison of two expressions for the same arc of the logarithmic hyperbola, we may derive the well-known equation which connects two elliptic integrals of the third order, and of the logarithmic form, whose parameters are reciprocal*.

Hence also it follows that if υ, τ, and υ' are the angles which normals to a parabola make with the axis, and if these angles, which may be called *conjugate amplitudes*, are connected by the equations

$$\left.\begin{array}{ll} \cos^2\upsilon=\dfrac{\mathrm{ML}}{\cos^2\phi}, & \sin\upsilon=\sqrt{\dfrac{m}{n}}\tan\phi\,\sqrt{\mathrm{I}}, \\ \cos^2\tau=\dfrac{\mathrm{MN}}{\mathrm{I}}, & \sin\tau=\dfrac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{\mathrm{I}}}, \\ \cos^2\upsilon'=\dfrac{\mathrm{LN}}{\mathrm{I}\cos^2\phi}, & \sin\upsilon'=\sqrt{\dfrac{m}{n}}(1-n)\dfrac{\tan\phi}{\sqrt{\mathrm{I}}} \end{array}\right\} \quad . \quad . \quad (279)$$

we shall have

$$\int\frac{d\upsilon}{\cos\upsilon}=\int\frac{d\upsilon'}{\cos\upsilon'}+\int\frac{d\tau}{\cos\tau}. \quad . \quad . \quad . \quad . \quad . \quad (280)\dagger$$

49.] *The difference between an arc of a logarithmic hyperbola, and the corresponding arc of the tangent parabola, may be expressed by the arcs of a plane, a spherical, and a logarithmic ellipse.*

Resuming the equation (241),

$$\int\frac{d\upsilon}{\cos^3\upsilon}-\frac{\Upsilon}{k}=\frac{a^2}{\sqrt{\mathrm{B(A+C)}}}\int\frac{\cos^2\phi d\phi}{\mathrm{M}^2\sqrt{\mathrm{I}}},$$

and combining (248) with (249), we may easily show that

$$\frac{a^2}{\sqrt{\mathrm{B(A+C)}}}=\frac{n(1-m)\sqrt{mn}}{m+n-2nm}; \quad . \quad . \quad . \quad . \quad (281)$$

and from (258) we may deduce that

$$2n(1-m)\int\frac{\cos^2\phi d\phi}{\mathrm{M}^2\sqrt{\mathrm{I}}}=\frac{n}{m}(1-m)\int\frac{d\phi}{\sqrt{\mathrm{I}}}+\int d\phi\sqrt{\mathrm{I}}-m\Phi_m$$
$$-\left(\frac{1-m}{m}\right)(m+n-2mn)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}.$$

* We might by the aid of the imaginary transformation $\sin\phi=\sqrt{-1}\tan\psi$ have passed from this theorem, connecting integrals with reciprocal parameters, to the corresponding theorem in the circular form. It seems better to give an independent proof of this theorem by the method of differentiating under the sign of integration, as we shall do further on. Although these theorems have algebraically the same form, their geometrical significations are entirely different. In the logarithmic form, the theorem results from the comparison of two expressions for the *same* arc of the logarithmic hyperbola. But in the circular form, the theorem represents the sum of the arcs of two different spherical conic sections described on the same cylinder by two concentric spheres, or on the same sphere by two cylinders having their axes coincident.

† These values of υ, τ, and υ' satisfy the equation of condition which connects the conjugate amplitudes in parabolic trigonometry, $\tan\omega=\tan\phi\sec\chi+\tan\chi\sec\phi$. We must replace ω, ϕ, χ by υ, υ', and τ. See vol. i. p. 313, (a).

Let $$\mathrm{G}=\frac{n}{m}(1-m)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\int d\varphi\sqrt{\mathrm{I}}-m\Phi_m. \quad . \quad . \quad (282)$$

Substituting this value of $\int\frac{\cos^2\varphi d\varphi}{\mathrm{M}^2\sqrt{\mathrm{I}}}$ in the preceding equation, we find, after some obvious reductions,

$$2\int\frac{dv}{\cos^3 v}-\frac{2\Upsilon}{k}=\frac{\sqrt{mn}\,\mathrm{G}}{m+n-2mn}-\frac{n(1-m)}{\sqrt{mn}}\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}.$$

Now, $a_{\prime}$ and $b_{\prime}$ being the semiaxes of the base of an elliptic cylinder whose curve of section with the paraboloid is a logarithmic ellipse, let, as in (171),

$$\frac{a_{\prime}^2}{k^2}=\frac{mn(1-m)}{(n-m)^2},\quad \frac{b_{\prime}^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}; \quad . \quad . \quad . \quad (283)$$

and if we put Σ for an arc of this logarithmic ellipse, we shall have, as in (163),

$$\frac{2\Sigma}{k}=\frac{\sqrt{mn}}{n-m}\mathrm{G}-\frac{n(1-m)}{\sqrt{mn}}\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}-2\int\frac{d\tau}{\cos^3\tau}.$$

Subtracting this equation from the preceding, we shall finally obtain

$$\Upsilon=k\int\frac{dv}{\cos^3 v}-k\int\frac{d\tau}{\cos^3\tau}-\Sigma+\frac{\sqrt{mn}(1-n)mk\,\mathrm{G}}{(n-m)(m+n-2mn)}. \quad . \quad (284)$$

We may express the arc Υ by the help of one parabolic arc only, if we introduce the equation established in (160),

$$\Sigma_{\prime}=\Sigma+k\int\frac{d\tau}{\cos^3\tau};\text{ hence}$$

$$\left.\begin{aligned}\Upsilon&=k\int\frac{dv}{\cos^3 v}-\Sigma_{\prime}\\&+\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}\left[\frac{n}{m}(1-m)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\int d\varphi\sqrt{\mathrm{I}}-m\Phi\right]\end{aligned}\right\}.(285)$$

replacing G by its value in (282).

When $\sin\varphi=\frac{1}{\sqrt{l}}$, $v=\frac{\pi}{2}$, and the arc of the logarithmic hyperbola becomes infinite, the arc of the parabola also becomes infinite and an asymptote to the logarithmic hyperbola; the difference, however, between these infinite quantities is finite, and equal to $\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}\mathrm{G}-\Sigma_{\prime}$, integrated between the limits $\varphi=0$, and $\varphi=\sin^{-1}l^{-\frac{1}{2}}$.

It is needless here to dwell on the analogy which this property bears to the finite difference between the infinite arc of the common hyperbola and its asymptote. When $n=m$, the above expression becomes illusory. We shall, however, in the next article find a remarkable value for the arc of the logarithmic hyperbola when $m=n$.

We may express the above formula somewhat more simply.

As in (248) $\frac{b}{k}=\frac{i\sqrt{m(1-n)}}{m+n-2mn}$, and

$$\frac{b_\prime}{k}=\frac{\sqrt{mn(1-n)}}{n-m}, \quad \frac{bb_\prime}{k^2}=\frac{i}{\sqrt{m}}\frac{\sqrt{mn}(1-n)m}{(n-m)(n+m-2mn)}.$$

The equation given in (285) now becomes

$$\Upsilon=k\int\frac{dv}{\cos^3 v}-\Sigma_\prime+\frac{\sqrt{m}}{i}\frac{bb_\prime}{k}\mathrm{G}. \quad . \quad . \quad . \quad . \quad (286)$$

The ratio between the axes of the original hyperbolic cylinder and of the derived elliptic cylinder may easily be determined; for

$$\frac{b^2}{a^2}=\frac{i^2(1-m)}{n^2(1-n)}, \text{ (a)} \quad \text{and } \frac{b_\prime^2}{a_\prime^2}=\frac{1-m}{1-n}. \text{ (b)}$$

Let $c_\prime$ be the eccentricity of the hyperbolic base, and c that of the elliptic base, then

$$n^2(c_\prime^2-1)=i^2(1-c^2).$$

Comparing (a) with (*b*),

$$\sqrt{n}\frac{a_\prime}{a}=\sqrt{l}\frac{b_\prime}{b}=1+\frac{2m(1-n)}{(n-m)}.$$

This equation gives at once the ratio between the axes of the hyperbolic and elliptic cylinders.

50.] *On the rectification of the logarithmic hyperbola when the conjugate parameters are equal,* or $m=n$.

We have shown in sec. [43] that, when $m=n$, the arc of the logarithmic ellipse is equivalent to an arc of a plane ellipse; so, when $m=n$, the arc of a logarithmic hyperbola may be represented by a straight line, an arc of a parabola, and an arc of a plane hyperbola.

In (262), if we make $m=n$, or $l=1+j$, $n=1-j$, we shall have, writing N for M,

$$2j\int\frac{d\phi}{\mathrm{L}\sqrt{\mathrm{I}}}-2j\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2\sin\phi\cos\phi\sqrt{\mathrm{I}}}{\mathrm{LN}}-2\int\frac{d\phi}{\sqrt{\mathrm{I}}}+4\int\frac{dv}{\cos^3 v}; \text{ (a)}$$

and in (170), if we make $m=n$, and $M=N$,

$$2(1-n)\int\frac{d\phi}{N\sqrt{I}}=(2-n)\int\frac{d\phi}{\sqrt{I}}-2\int\frac{d\tau}{\cos^3\tau}+\frac{n\sin\phi\cos\phi\sqrt{I}}{N^2}. \quad . \quad (b)$$

Adding these equations together, as $1-n=j$, we get

$$\left.\begin{aligned}2j\int\frac{d\phi}{L\sqrt{I}}=-(1-j)\int\frac{d\phi}{\sqrt{I}}+4\int\frac{d\upsilon}{\cos^3\upsilon}-2\int\frac{d\tau}{\cos^3\tau}\\+\frac{\sin\phi\cos\phi\sqrt{I}}{N}\left[\frac{n}{N}-\frac{2}{L}\right]\end{aligned}\right\}, \quad . \quad (c)$$

while the arc of the logarithmic hyperbola, as in (233), is

$$\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

In this case, the coefficient $\dfrac{b^2}{\sqrt{B(A+C)}}=\dfrac{l}{2}$, as may be shown by putting, in the general value for this expression given in (249), $m=n$; hence

$$\frac{2\Upsilon}{k}=l\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Now (257) gives

$$2(l-i^2)\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}=l\Phi_l-\int d\phi\sqrt{I}+\left(\frac{l-i^2}{l}\right)\int\frac{d\phi}{\sqrt{I}}+\frac{f}{l}\int\frac{d\phi}{L\sqrt{I}}; \quad (f)$$

and the general value of f being $l^2+i^2-2li^2$, as in (256), $f=2l(1-n)^2$, $l=2-n$, and $l-i^2=l(1-n)$, since $ln=i^2$.

The last equation may now be written, combining (e) with it,

$$\frac{4\Upsilon}{k}=\frac{l}{1-n}\Phi_l-\frac{1}{1-n}\int d\phi\sqrt{I}+\int\frac{d\phi}{\sqrt{I}}+2j\int\frac{d\phi}{L\sqrt{I}}. \quad (287)$$

Adding this equation to (c),

$$\left.\begin{aligned}\frac{4\Upsilon}{k}=4\int\frac{d\upsilon}{\cos^3\upsilon}-2\int\frac{d\tau}{\cos^3\tau}+j\int\frac{d\phi}{\sqrt{I}}-\frac{1}{j}\int d\phi\sqrt{I}+\frac{l}{j}\Phi_l\\+\frac{\sin\phi\cos\phi}{N}\left[\frac{n}{N}-\frac{2}{L}\right]\end{aligned}\right\}. \quad (288)$$

Now $$\frac{l\Phi_l}{j}=\frac{(1+j)\sin\phi\cos\phi\sqrt{I}}{jL}=\frac{\tan\phi\sqrt{I}}{j}+\frac{\tan\phi\sqrt{I}}{L}.$$

Combining this value of Φ_l with the preceding equation, we shall find

$$\left.\begin{aligned}\frac{4\Upsilon}{k}=4\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-2\int\frac{\mathrm{d}\tau}{\cos^3\tau}+\frac{1}{j}\left[\tan\varphi\sqrt{\mathrm{I}}-\int\mathrm{d}\varphi\sqrt{\mathrm{I}}+j^2\int\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\\+\tan\varphi\sqrt{\mathrm{I}}\left[\frac{n\cos^2\varphi}{\mathrm{N}^2}-\frac{2\cos^2\varphi}{\mathrm{LN}}+\frac{1}{\mathrm{L}}\right]\end{aligned}\right\};\quad(289)$$

and this latter term, in this case, may be reduced to $-\dfrac{j\tan\varphi\sqrt{\mathrm{I}}}{\mathrm{N}^2}$.

But, a and b being the semiaxes of the hyperbolic cylinder, (248) gives $\dfrac{ab}{k^2}=\dfrac{mnij}{(m+n-2mn)^2}$, or in this case, as $m=n$, $\dfrac{2\sqrt{ab}}{\sqrt{ij}}=\dfrac{k}{j}$.

Now $\sqrt{\dfrac{ab}{ij}}$ is the distance from the centre to the focus of an hyperbola the squares of whose semiaxes are $\dfrac{i}{j}ab$ and $\dfrac{j}{i}ab$; hence

$$\frac{k}{2j}\left[\tan\varphi\sqrt{\mathrm{I}}-\int\mathrm{d}\varphi\sqrt{\mathrm{I}}+j^2\int\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]$$

represents an arc of an hyperbola the squares of whose semiaxes are $\dfrac{i}{j}ab$ and $\dfrac{j}{i}ab$, as will be shown in sec. [52].

Introduce this value of $\dfrac{k}{j}$, and divide by 2,

$$\left.\begin{aligned}2\Upsilon=2k\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}-k\int\frac{\mathrm{d}\tau}{\cos^3\tau}\\+\sqrt{\frac{ab}{ij}}\left[\tan\varphi\sqrt{\mathrm{I}}-\int\mathrm{d}\varphi\sqrt{\mathrm{I}}+j^2\int\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]-\frac{kj\tan\varphi\sqrt{\mathrm{I}}}{2\mathrm{N}^2}\end{aligned}\right\}.\quad(290)$$

Now, when this equation is integrated between the limits $\varphi=0$ and $\varphi=\sin^{-1}\sqrt{\dfrac{1}{l}}$, or, taking the corresponding values, between $\tau=0$ and $\tau=\sin^{-1}\left(\dfrac{1-j}{1+j}\right)$, or between $\upsilon=0$ and $\upsilon=\dfrac{\pi}{2}$, Υ is infinite, and the arc of the asymptotic parabola $k\displaystyle\int\frac{\mathrm{d}\upsilon}{\cos^3\upsilon}$ is also infinite; but twice the difference Δ between these infinite quantities is finite. Let $\sin^2\varphi_l=\dfrac{1}{l}$, $\sin\tau_l=\dfrac{1-j}{1+j}$; then

$$\Delta=\frac{k(1+j)^2}{8j}+k\int_0^{\tau_l}\frac{\mathrm{d}\tau}{\cos^3\tau}-\sqrt{\frac{ab}{ij}}\left[1-\int_0^{\varphi_l}\mathrm{d}\varphi\sqrt{\mathrm{I}}+j^2\int_0^{\varphi_l}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right].\quad(291)$$

Hence the difference between an infinite arc of the equilateral logarithmic hyperbola, and the corresponding infinite arc of its asymptotic parabola, is equal to a straight line + an arc of a plane parabola — an arc of a plane hyperbola.

When the parameters m and n are equal, the logarithmic hyperbola may by analogy be called equilateral, seeing that though the squares of the axes of the hyperbolic base of the cylinder are not equal, they differ by a constant quantity.

Resuming (250),

$$\frac{b^2}{k^2}=\frac{l(l-1)}{(l-m)^2}, \text{ and } \frac{a^2}{k^2}=\frac{m(1-m)}{(l-m)^2}.$$

But when $m=n$, $l=1+j$, $m=1-j$, substituting these values in the preceding expressions,

$$2(b^2-a^2)=k^2.$$

51.] On the logarithmic hyperbola when $l=\infty$. Case XIII., p. 7.

Resume (233), or $\dfrac{\Upsilon}{k}=\dfrac{b^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}\displaystyle\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}$.

Now, as $ln=i^2$, and as i is finite, while $l=\infty$, $n=0$.

The equation of condition $m+n-mn=i^2$, gives therefore $m=i^2$. Equations (248) and (249) give $a=0$, $b=k$.

And as $\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}=\dfrac{\mathrm{B}\sqrt{n}}{\sqrt{m}}$, we get

$$\frac{b^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}=\frac{k^2\sqrt{m}\,\sqrt{l}}{k^2\sqrt{n}\,\sqrt{l}}=\sqrt{l}, \text{ since } m=i^2=nl;$$

hence
$$\frac{\Upsilon}{k}=\sqrt{l}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad . \quad . \quad . \quad \text{(a)}$$

Let $l\sin^2\varphi=\sin^2\psi$; therefore

$$\sqrt{l}\cos\varphi d\varphi=\cos\psi d\psi,\ [1-l\sin^2\varphi]^2=\cos^4\psi,$$

$$\sqrt{1-i^2\sin^2\varphi}=\sqrt{1-\frac{i^2}{l}\sin^2\psi}=\sqrt{1-n\sin^2\psi},$$

$$\text{and } \cos\varphi=\sqrt{1-\frac{\sin^2\psi}{l}}.$$

Making these substitutions in the preceding equation, we get

$$\frac{\Upsilon}{k}=\frac{\sqrt{l}}{\sqrt{l}}\int\frac{d\psi}{\cos^3\psi}\,\frac{\sqrt{1-\frac{1}{l}\sin^2\psi}}{\sqrt{1-n\sin^2\psi}}.$$ When $l=\infty$, $\frac{1}{l}=0$, $n=0$;

hence $$\Upsilon = k\int\frac{d\psi}{\cos^3\psi}, \quad \ldots \ldots \quad (292)$$

or the logarithmic hyperbola in this case becomes a common parabola.

As $a=0$, $b=k$, the hyperbolic cylinder becomes a vertical plane, at right angles to the transverse axis.

Hence, comparing this result with that in sec. [19], we find that, when the parameters are either $+\infty$ or $-\infty$, the corresponding hyperconic section is a plane principal section of the generating surface, *i. e.* either a circle or a parabola.

52.] By giving a double rectification of the common hyperbola, we shall the more readily discover the striking analogy which exists between this curve and the logarithmic hyperbola.

Let Y be an arc of a common hyperbola, whose equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Assume $$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}, \quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}. \quad \ldots \quad (a)$$

Differentiating these expressions, and substituting, we get

$$\frac{dY}{d\lambda}=\frac{b^2}{a\left[1-\frac{a^2+b^2}{a^2}\sin^2\lambda\right]^{\frac{3}{2}}}.$$

Assume $$\sin^2\phi=\frac{a^2+b^2}{a^2}\sin^2\lambda, \text{ and let } i^2=\frac{a^2}{a^2+b^2}. \quad \ldots \quad (b)$$

Finding from this equation the value of $\frac{d\lambda}{d\phi}$, as $\frac{dY}{d\phi}=\frac{dY}{d\lambda}\cdot\frac{d\lambda}{d\phi}$, we shall finally obtain, since $\frac{b^2}{\sqrt{a^2+b^2}}=\frac{a(1-i^2)}{i}$,

$$\frac{Y}{a}=\frac{(1-i^2)}{i}\int\frac{d\phi}{[1-\sin^2\phi]\sqrt{1-i^2\sin\phi^2}}. \quad \ldots \quad (c)$$

Sec. [38] gives $$Y=\int p\,d\lambda+\int\frac{d^2p}{d\lambda^2}, \text{ or } Y=-\int p\,d\lambda-\frac{dp}{d\lambda}. \quad \ldots \quad (d)$$

Now, as $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, $\frac{dp}{d\lambda}=-\frac{(a^2+b^2)\sin\lambda\cos\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{1}{2}}}$,

and as $$\sin^2\phi=\frac{a^2}{a^2+b^2}\sin^2\lambda, \quad \ldots \ldots \quad (e)$$

$$\frac{d\phi}{d\lambda}=\frac{\sqrt{a^2+b^2}\sqrt{1-i^2\sin^2\phi}}{a\cos\phi}; \quad \ldots \ldots \quad (f)$$

hence $$\frac{dp}{d\lambda} = -\sqrt{a^2+b^2} \tan\varphi \sqrt{1-i^2\sin^2\varphi}\,; \qquad \text{(g)}$$

and as $p = a\cos\varphi$,

$$p\,d\lambda = \frac{a^2\cos^2\varphi\, d\varphi}{\sqrt{(a^2+b^2)}\sqrt{1-i^2\sin^2\varphi}} = \frac{a}{i}\,\frac{\{1+i^2-i^2\sin^2\varphi-1\}}{\sqrt{1-i^2\sin\varphi}}; \qquad \text{(h)}$$

whence, finally,

$$\frac{i}{a}\,\mathrm{Y} = \tan\varphi\sqrt{\mathrm{I}} - \int d\varphi\sqrt{\mathrm{I}} + (1-i^2)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}. \qquad \text{(i)}$$

The integral $$j^2\int\frac{d\varphi}{[1-i^2\sin^2\varphi]^{\frac{3}{2}}} = \int d\varphi\sqrt{\mathrm{I}} - \frac{i^2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}}, \qquad \text{(k)}$$

as may be shown by putting

$$\Phi = \frac{\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}. \qquad \text{(l)}$$

Differentiating this expression and multiplying by i^2,

$$\frac{i^2 d\Phi}{d\varphi} = \frac{i^2 - 2i^2\sin^2\varphi + i^4\sin^4\varphi}{[1-i^2\sin^2\varphi]^{\frac{3}{2}}}. \qquad \text{(m)}$$

This expression may be put in the form $\dfrac{[1-i^2\sin^2\varphi]^2-(1-i^2)}{[1-i^2\sin^2\varphi]^{\frac{3}{2}}}$,

integrating

$$j^2\int\frac{d\varphi}{[1-i^2\sin^2\varphi]^{\frac{3}{2}}} = \int d\varphi\sqrt{1-i^2\sin^2\varphi} - \frac{i^2\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}.$$

This is the integral referred to in sec. [44*].

Adding the integral (k) to (i),

$$\frac{i}{a}\,\mathrm{Y} + (1-i^2)\int\frac{d\varphi}{\mathrm{I}\sqrt{\mathrm{I}}} = (1-i^2)\int\frac{d\varphi}{\sqrt{\mathrm{I}}} + \tan\varphi\sqrt{\mathrm{I}} - \frac{i^2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}}; \qquad \text{(n)}$$

but $$\tan\varphi\sqrt{\mathrm{I}} - \frac{i^2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}} = \frac{(1-i^2)\tan\varphi}{\sqrt{\mathrm{I}}}.$$

Hence, dividing by $(1-i^2)$,

$$\frac{i\mathrm{Y}}{a(1-i^2)} + \int\frac{d\varphi}{\mathrm{I}\sqrt{\mathrm{I}}} = \frac{\tan\varphi}{\sqrt{\mathrm{I}}} + \int\frac{d\varphi}{\sqrt{\mathrm{I}}}; \qquad \text{(o)}$$

and (c) gives $$\frac{i\mathrm{Y}}{a(1-i^2)} = \int\frac{d\varphi}{[1-\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Eliminating Y from these equations, we obtain

$$\left.\begin{aligned}\int\frac{d\varphi}{[1-\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}+\int\frac{d\varphi}{[1-i^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\\ =\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\frac{\tan\varphi}{\sqrt{1-i^2\sin^2\varphi}}\end{aligned}\right\}. \quad (293)$$

The parameters are reciprocal in this equation, being 1 and i^2.

Now this is the extreme case of the formula for the comparison of elliptic integrals of the third order and logarithmic form. We perceive that this formula results from the comparison of two expressions for the same arc of a common hyperbola. We may also see that it is the limiting case of the general formula for the comparison of elliptic integrals of the third order having reciprocal parameters—a formula which in like manner has been deduced from the comparison of two expressions for the same arc of the logarithmic hyperbola. It is also evident that $j^2\frac{\tan\varphi}{\sqrt{\mathrm{I}}}$ being the difference between $\tan\varphi.\sqrt{\mathrm{I}}$ and $\frac{i^2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}}$, it is the difference between tangents, one drawn to the hyperbola, the other to the plane ellipse; for $\tan\varphi\sqrt{\mathrm{I}}$ denotes the portion of a tangent to an hyperbola between the point of contact and the perpendicular on it from the centre, and $\frac{i^2\cos\varphi\sin\varphi}{\sqrt{\mathrm{I}}}$ denotes a similar quantity in an ellipse. This difference is precisely analogous to the expression that occurs in (284) $\int\frac{dv}{\cos^3 v}-\int\frac{d\tau}{\cos^3\tau}$, which denotes the difference between two parabolic arcs, one drawn a tangent to the logarithmic hyperbola, the other a tangent to the logarithmic ellipse.

Hence a hyperbolic arc may be expressed by two elliptic arcs. (Landen's theorem.)

For, eliminating the integral of the first order between (i) and (218), we get, putting $\sqrt{a^2+b^2}=f$,

$$\mathrm{Y}-f\tan\varphi\sqrt{\mathrm{I}}=\sqrt{\frac{1+j}{1-j}}\left[\frac{b}{2}\int d\chi\sqrt{_\chi\mathrm{I}_{,}}-a\int d\varphi\sqrt{\mathrm{I}}\right]+\sqrt{\frac{1-j}{1+j}}\,b\Phi.$$

The difference Δ between the infinite arc of the hyperbola and its asymptote is found by integrating the above expressions between 0 and $\frac{\pi}{2}$. Φ becomes $=0$; and the difference is given by the equation

$$\Delta=\sqrt{\frac{1+j}{1-j}}\left[b\int_0^{\frac{\pi}{2}}d\chi\sqrt{_\chi\mathrm{I}_{,}}-a\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right].$$

CHAPTER VI.

ON THE VALUES OF COMPLETE ELLIPTIC INTEGRALS OF THE THIRD ORDER.

53.] We have hitherto investigated the properties and lengths of elliptic curves, on the supposition that the generating surface, whether sphere or paraboloid, was invariable, and that the lengths of the curves were made up by the summation of the elements produced by the successive values given to the amplitude ϕ between certain limits; 0 and $\frac{\pi}{2}$, suppose, if the arcs are to be quadrants. Thus the length of the quadrant is obtained by adding together the successive increments that result from the continuous additions, indefinitely small, which are made to the amplitude. We may, however, proceed on another principle to effect the rectification of those curves. If, to fix our ideas, we want to determine the length of a quadrant of the spherical ellipse, we may imagine the vertical cylinder, which we shall suppose invariable, to be successively intersected by a series of all possible concentric spheres. Every quadrant will differ in length from the one immediately preceding it in the series, by an infinitesimal quantity; and if we take the least of these quadrants, and add to it the successive elements by which one quadrant differs from the next immediately preceding, we shall thus obtain the length of a quadrant of the required spherical ellipse, equal to the least quadrant which can be described on the elliptic cylinder, plus the sum of all the elements between the least quadrant and the required one. Thus, for example, the least quadrant which can be drawn on an elliptic vertical cylinder, is its section by a horizontal plane, or a quadrant of the plane ellipse, whose semiaxes are a and b. In this case the radius of the sphere is infinite. The least sphere is that whose radius is a, and which cuts the cylinder in its circular sections. Hence the greatest spherical elliptic quadrant is the quadrant of the circle whose radius is a. All the spherical elliptic quadrants will therefore be comprised between the quadrants of an ellipse, and of a circle whose radius is a. Any quadrant, therefore, of a given spherical ellipse is equal to a quadrant of a plane ellipse plus a certain increment, or to a quadrant of a circle minus a certain decrement. The same reasoning will hold as well when we take any other limits besides 0 and $\frac{\pi}{2}$. These considerations naturally lead to the process of differentiation under the sign of integration, because we cannot express, under a finite known form, the arc of a spherical or logarithmic ellipse, and then differentiate the expression so found, with respect to a quantity which hitherto has been taken as a constant.

We may conceive the generation of successive curves of this kind to take place in another manner. Let us imagine an invariable sphere to be cut by a succession of concentric or coaxal right cylinders indefinitely near to each other, and generated after a given law. These cylinders will cut the sphere in a series of spherical ellipses, any one of which will differ from the one immediately preceding by an indefinitely small quantity. If we sum all these indefinitely small quantities, or, in other words, integrate the expression so found, we shall discover the finite difference between any two curves of the series separated by a finite interval. One of the limits being a known curve, the other may thus be determined.

To apply this reasoning.

In the following investigations we shall assume the generating sphere to be invariable, and the modulus i with the amplitude ϕ to be constant. The intersecting cylinder we shall suppose to vary from curve to curve on the surface of the sphere. But i is constant, and $i^2=\frac{a^2-b^2}{a^2}$, sec (27). Now, a and b being the semiaxes of the base of the cylinder, it follows that the bases of all the varying cylinders are concentric and similar ellipses. Hence in the elliptic integral of the third order, which represents the spherical ellipse, the parameter e^2 or m and the criterion of sphericity $\sqrt{\kappa}$ will vary.

In [9] we found for a quadrant of a spherical conic section, which we may denote by $\breve{\sigma}$, the expression

$$\breve{\sigma}=\sqrt{\kappa}\int_0^{\frac{\pi}{2}}\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}.$$

Let k be the radius of the sphere.

Since $e^2=\frac{k^2i^2}{k^2-j^2a^2}$, e will vary, as being a function of a the transverse semiaxis of the variable cylinder. We have also

$$e^2\kappa=(1-e^2)(e^2-i^2). \quad . \quad . \quad . \quad . \quad . \quad (294)$$

Hence $$\frac{d\kappa}{de}=-2e\left(1-\frac{i^2}{e^4}\right);$$

and if, as before, we write M for $1-m\sin^2\phi$, or $1-e^2\sin^2\phi$, we shall have

$$\breve{\sigma}=\sqrt{\kappa}\int_0^{\frac{\pi}{2}}\frac{d\phi}{M\sqrt{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (295)$$

Differentiating this expression on the hypothesis that i and φ are constant, while e is variable, we shall have

$$\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}e}=\frac{1}{2}\frac{1}{\sqrt{\kappa}}\frac{\mathrm{d}\kappa}{\mathrm{d}e}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}+\frac{\sqrt{\kappa}}{e}2\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}^2\sqrt{\mathrm{I}}}-\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}\right].$$

Multiplying this equation by $\frac{\sqrt{\kappa}}{e}$, and recollecting that $\frac{\mathrm{d}\kappa}{\mathrm{d}e}=-2e\left(1-\frac{i^2}{e^4}\right)$, we shall have

$$\frac{\sqrt{\kappa}}{e}\frac{\mathrm{d}\breve{\sigma}}{\mathrm{d}e}=-\left(1-\frac{i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}+\frac{2\kappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}^2\sqrt{\mathrm{I}}}-\frac{2\kappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}. \quad (296)$$

But (134) gives, writing M^2 for N^2, e^2 for m, and i^2 for $m-n+mn$,

$$\left.\begin{aligned}\frac{2\kappa}{e^2}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}^2\sqrt{\mathrm{I}}}&=\left[\frac{2}{e^2}(1+i^2)-1-\frac{3i^2}{e^4}\right]\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}\\&\quad-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\end{aligned}\right\} \quad (297)$$

Introducing this value into the preceding equation, the coefficient of $\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}$ will vanish, and we shall have

$$\frac{\sqrt{\kappa}}{e}\frac{\mathrm{d}\sigma}{\mathrm{d}e}=-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}. \quad (298)$$

Dividing by $\frac{\sqrt{\kappa}}{e}$, and integrating on the hypothesis that φ and i are constant,

$$\breve{\sigma}=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}e}{e\sqrt{\kappa}}-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\int\frac{\mathrm{d}e\,(e^2-i^2)}{e^3\sqrt{\kappa}}+\text{constant};$$

or, as in (294) $\quad e\sqrt{\kappa}=\sqrt{(1-e^2)(e^2-i^2)}$, we shall have

$$\left.\begin{aligned}\breve{\sigma}&=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}e}{\sqrt{(1-e^2)(e^2-i^2)}}\\&\quad-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\int\frac{\mathrm{d}e}{e^2}\sqrt{\frac{e^2-i^2}{1-e^2}}+\text{constant}\end{aligned}\right\} \quad (299)$$

We must recollect that the *complete* integrals within the brackets are functions, not of φ, but of i^2, 0, and $\frac{\pi}{2}$. They are therefore constants.

It is not a little remarkable that the *coefficients* of the *complete* elliptic integrals are themselves also elliptic integrals of the first and second orders. To show this, assume

$$e^2 = \cos^2\theta + i^2 \sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{300}$$

Therefore $1-e^2 = j^2\sin^2\theta$, and $e^2-i^2=j^2\cos^2\theta$; we have also $ede = -j^2 \sin\theta \cos\theta d\theta$.
Hence, if $1-j^2\sin^2\theta = \mathrm{J}$,

$$\int \frac{\mathrm{d}e}{\sqrt{(e^2-i^2)(1-e^2)}} = -\int \frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}} = -\int \frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}, \tag{301}$$

and
$$\sqrt{\kappa} = \frac{j^2 \sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}. \quad . \quad . \quad . \quad . \quad . \tag{302}$$

In the same manner we may show that

$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{\mathrm{d}e}{e^2} = -\int\frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}} + i^2\int\frac{\mathrm{d}\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}}; \tag{303}$$

but
$$i^2\int\frac{\mathrm{d}\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}} = \int \mathrm{d}\theta\,\sqrt{1-j^2\sin^2\theta} - j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}. \tag{304}$$

Hence
$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{\mathrm{d}e}{e^2} = \int \mathrm{d}\theta\,\sqrt{\mathrm{J}} - \int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}} - j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}. \tag{305}$$

Substituting these values in (299), we obtain

$$\left.\begin{aligned}\breve{\sigma} = \left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}} - \int \mathrm{d}\theta\,\sqrt{\mathrm{J}} + j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}\right] \\ -\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\,\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}} + \text{constant}\end{aligned}\right\}. \tag{306}$$

To determine this constant. We must not suppose $i=0$, in this case, as is generally done, to determine the constant. This would be to violate the supposition on which we have all along proceeded, namely, that the variable cylinders are all similar, and therefore that i must be constant. We must determine the constant from other considerations.

Since $e^2 = \frac{i^2k^2}{k^2-j^2a^2}$, when $a=0$, $e^2=i^2$. But as in (300)

$e^2=\cos^2\theta+i^2\sin^2\theta$, therefore $\theta=\frac{\pi}{2}$. As a, the major semiaxis of the base of the cylinder, is supposed to vanish, the curve diminishes to a point, and therefore $\breve{\sigma}=0$.

When $a=k$, $e^2=1$, and $\theta=0$. We have in this case $\breve{\sigma}=\frac{\pi}{2}$; for the sections of a sphere by an elliptic cylinder, whose greater axis is equal to the diameter of the sphere, are two semicircles of a great circle of the sphere. Hence, when $\theta=0$, $\breve{\sigma}=\frac{\pi}{2}$, $\sin\theta=0$, $\int d\phi\,\sqrt{\mathrm{J}}=0$, $\int\frac{d\theta}{\sqrt{\mathrm{J}}}=0$; therefore the constant is equal to $\breve{\sigma}$ when $\theta=0$. But when $\theta=0$, $\breve{\sigma}=\frac{\pi}{2}$, or the constant is equal to $\frac{\pi}{2}$.

The formula now becomes

$$\left.\begin{aligned}\breve{\sigma}=\frac{\pi}{2}-&\left[\int_0^{\frac{\pi}{2}}d\phi\,\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{\mathrm{J}}}\\ &+\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{d\theta}{\sqrt{\mathrm{J}}}-\int d\theta\,\sqrt{\mathrm{J}}+j^2\frac{\sin\theta\cos\theta}{\sqrt{\mathrm{J}}}\right]\end{aligned}\right\}. \qquad (307)$$

When $\theta=\frac{\pi}{2}$, $e=i$, and $\breve{\sigma}=0$, as the variable cylinder is in this case diminished to a straight line; therefore the preceding formula will become

$$\left.\begin{aligned}\frac{\pi}{2}=&\left[\int_0^{\frac{\pi}{2}}d\phi\,\sqrt{\mathrm{I}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\mathrm{J}}}\right]\\ &+\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}d\theta\,\sqrt{\mathrm{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\mathrm{J}}}\right]\end{aligned}\right\}, \qquad (308)$$

or, using the ordinary notation of elliptic integrals,

$$\frac{\pi}{2}=\mathrm{E}_i\mathrm{F}_j+\mathrm{E}_j\mathrm{F}_i-\mathrm{F}_i\mathrm{F}_j. \quad .\ .\ .\ .\ .\ (308^*)$$

Hence we obtain the true geometrical meaning of this curious formula of verification discovered by Legendre. In its general form (307) represents the *difference* between the quadrants of a great circle and of a spherical ellipse. When the spherical ellipse vanishes to a point, this expression must represent, as in (308), the quadrant of a circle.

54.] If we now apply the preceding investigations to the curve described on the same sphere by the reciprocal cylinder, or by the cylinder which gives a function having a reciprocal parameter as defined in sec. [31], we shall find by substitution in (299)

$$\left.\begin{aligned}\breve{\sigma}_{\prime}=&\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\,\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}e'}{\sqrt{(e'^2-i^2)(1-e'^2)}}\\&-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\int\sqrt{\frac{e'^2-i^2}{1-e'^2}}\frac{\mathrm{d}e'}{e'^2}+\text{constant}\end{aligned}\right\}\quad .\ .\ (309)$$

But by the conditions of the question, as in (110),

$$ee'=i,\ e'^2=\frac{i^2}{1-j^2\sin^2\theta};$$

therefore
$$\int\frac{\mathrm{d}e'}{\sqrt{(e'^2-i^2)(1-e'^2)}}=\int\frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}},\quad .\ .\ (310)$$

and
$$\int\frac{\mathrm{d}e'}{e'^2}\sqrt{\frac{e'^2-i^2}{1-e'^2}}=\int\frac{j^2\sin^2\theta\,\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}}$$
$$=\int\frac{\mathrm{d}\theta}{\sqrt{1-j^2\sin^2\theta}}-\int\mathrm{d}\theta\,\sqrt{1-j^2\sin^2\theta}.$$

Substituting these values of the integrals in (309),

$$\breve{\sigma}_{\prime}=\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\,\sqrt{\mathrm{I}}\right]\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}-\int\mathrm{d}\theta\,\sqrt{\mathrm{J}}\right]+\mathrm{C}.\ (311)$$

We shall now show that the constant $\mathrm{C}=0$.

When $\theta=0$, $e=1$, and therefore $e'=i$. Since $e'=i$, and $\breve{\sigma}$ is a quadrant of the vanishing spherical ellipse whose principal arcs $\alpha=0$, $\beta=0$, we shall have $\breve{\sigma}=0$. Hence also $\int\mathrm{d}\theta\,\sqrt{\mathrm{J}}=0$, $\int\frac{\mathrm{d}\theta}{\mathrm{J}}=0$; therefore the constant is 0. When $\theta=\frac{\pi}{2}$, $e'=1$, and (309) becomes

$$\left.\begin{aligned}\frac{\pi}{2}=&\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\,\sqrt{\mathrm{I}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}\right]\\&+\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\sqrt{\mathrm{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{\mathrm{J}}}\right]\end{aligned}\right\},\ (312)$$

or, in the common notation, $\frac{\pi}{2}=E_iF_j+E_jF_i-F_iF_j$,

a formula already established in (308).

If we add together (307) and (312), we shall have, since $\sqrt{\kappa}=\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}$,

$$\breve{\sigma}+\breve{\sigma}_{\prime}=\frac{\pi}{2}+\sqrt{\kappa}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]. \quad . \quad . \quad . \quad . \qquad (313)$$

Now, as in (11) $\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}$,

$$\sigma'=\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\phi}{[1-m_{\prime}\sin^2\phi]\sqrt{1-i^2\sin^2\phi}},$$

in which mm' or $e^2e'^2=i^2$.

Whence, as $\left(\frac{1-m}{m}\right)\sqrt{mn}=\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}=\sqrt{\kappa}$, as we have shown in (113),

$$\left.\begin{array}{l}\sqrt{\kappa}\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} \\ \quad+\sqrt{\kappa}\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\phi}{\left[1-\frac{i^2}{m}\sin^2\phi\right]\sqrt{1-i^2\sin^2\phi}}=\sqrt{\kappa}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\frac{\pi}{2}\end{array}\right\}, \quad (314)$$

The reader will observe how very different are the geometrical origins of two algebraical formulæ apparently similar. In the logarithmic form of the elliptic integral, the formula for the comparison of elliptic integrals, with reciprocal parameters (one of which is greater, while the other is less than 1), resulted from putting in equation two algebraical expressions for the *same* arc of the *one* logarithmic hyperbola. See sec. [48]. In the preceding case, that of the spherical ellipse, the analogous formula expresses the sum of the arcs of two inverse spherical ellipses, whose amplitudes are the same.

We shall use the term *inverse spherical ellipses* to denote curves whose representative elliptic integrals have *reciprocal* parameters. The terms *reciprocal* and *supplemental* have long since been appropriated to curves otherwise related.

Let α and β, $\alpha_{\prime}$ and $\beta_{\prime}$ denote the principal semiarcs of two such curves. Since the modulus i is the same in both integrals, the orthogonal projections of these curves, on the base of the hemisphere, are similar ellipses. (9) gives

$$e^2=i^2\sec^2\beta,\ e_{\prime}^2=i^2\sec^2\beta_{\prime};\ \text{and we assume } e^2e_{\prime}^2=i^2.$$

Hence $\qquad \sec\beta \sec\beta_{\prime} i = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (315)$

Again, as $\qquad \tan^2\alpha(1-e^2) = \tan^2\beta = \sec^2\beta - 1,$

and $\qquad \tan^2\alpha_{\prime}(1-e_{\prime}^2) = \tan^2\beta_{\prime} = \sec^2\beta_{\prime} - 1,$

multiplying these expressions together, and introducing the relation established in (315),

$$\tan^2\alpha \tan^2\alpha_{\prime} i^2 = \frac{i^2\sec^2\beta\sec^2\beta_{\prime} - i^2(\sec^2\beta + \sec^2\beta_{\prime}) + i^2}{1 + i^2 - i^2(\sec^2\beta + \sec^2\beta_{\prime})} = 1. \quad (316)$$

Hence the principal arcs of the inverse spherical ellipses are connected by the symmetrical relations

$$\tan\alpha\tan\alpha_{\prime} i = 1, \text{ and } \sec\beta\sec\beta_{\prime} i = 1. \quad . \quad . \quad . \quad (317)$$

When the inverse curves coincide, $\alpha = \alpha_{\prime}$, $\beta = \beta_{\prime}$, and the last equations may be reduced to $\tan^2\alpha - \tan^2\beta = 1$. Now we have shown in (59) that when the principal arcs of a spherical hyperconic section are so related, the curve is the spherical parabola, or *when the curve becomes its own inverse it is the spherical parabola.*

We have shown in (9) that $i^2 = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha} = 1 - \dfrac{\sin^2\beta}{\sin^2\alpha}$; but (3) gives $\cos\eta = \dfrac{\sin\beta}{\sin\alpha}$, 2η being the angle between the cyclic arcs of the spherical ellipse. Hence $i = \sin\eta$, but i is constant. Therefore *all inverse spherical ellipses have the same cyclic arcs.*

Resuming equation (314), and making the assumption that the two inverse spherical ellipses coalesce and become identical, the resulting curve is the spherical parabola. In this case $m = n = i$, and (314) may now be written

$$2\sqrt{\kappa}\int\frac{d\varphi}{[1 - m\sin^2\varphi]\sqrt{1 - m^2\sin^2\varphi}} = \sqrt{\kappa}\int\frac{d\varphi}{\sqrt{1 - m^2\sin^2\varphi}} + \frac{\pi}{2}. \quad (318)$$

But as $\qquad \sqrt{\kappa} = \left(\dfrac{1-m}{m}\right)\sqrt{mn} = 1 - m,$

and $m = \dfrac{1-j}{1+j}$, see (60), we shall have $\sqrt{\kappa} = \dfrac{2j}{1+j}$, and the foregoing equation becomes

$$\frac{2j}{1+j}\int\frac{d\varphi}{\left[1 - \left(\frac{1-j}{1+j}\right)\sin^2\varphi\right]\sqrt{1 - \left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}$$

$$= \frac{j}{1+j}\int\frac{d\varphi}{\sqrt{1 - \left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}} + \frac{\pi}{4}.$$

But (62) gives

$$\frac{2j}{1+j}\int\frac{d\varphi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}$$

$$=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right].$$

Consequently

$$\left.\begin{aligned}&\frac{j}{1+j}\int\frac{d\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}+\frac{\pi}{4}\\ &=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\frac{[j\tan\mu]}{\sqrt{1-i^2\tan^2\mu}}\end{aligned}\right\}. \qquad (319)$$

Now it is shown in (68) that when the second side of this equation is integrated between 0 and $\mu_{\prime}$, $\tan\mu_{\prime}$ being $=\frac{1}{\sqrt{j}}$, the quadrant of the spherical parabola becomes $j\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\frac{\pi}{4}$, since $\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$ is equal to 1 when $\tan\mu=\frac{1}{\sqrt{j}}$.

Hence the first side of this equation represents a quadrant of a spherical parabola, or

$$\breve{\sigma}=\frac{j}{1+j}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}+\frac{\pi}{4};$$

and this expression is identical with (313), since $\sqrt{\kappa}=\frac{2j}{1+j}$ when

$$\breve{\sigma}=\breve{\sigma}_{\prime} \text{ or } \breve{\sigma}=\frac{j}{1+j}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\varphi}}+\frac{\pi}{4},$$

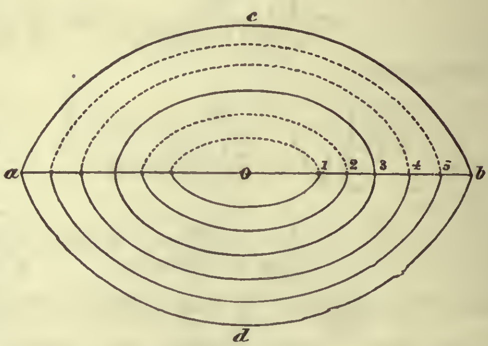
an expression derived from principles quite remote from those established in the earlier portions of this book. These coincidences may be taken as satisfactory tests of the accuracy of some rather complicated investigations, based on principles both obscure and remote.

55.] That portion of the surface of a sphere which lies between the cyclic circles may be called the *cyclic area*.

The spherical parabola divides the cyclic area into two regions.

In the one, between the pole and the spherical parabola, lie all the inverse curves, whose parameters range from i^2 to i. In the other, between the spherical parabola and the cyclic circles, lie all the conjugate inverse curves, whose parameters range from i to 1.

Fig. 21.

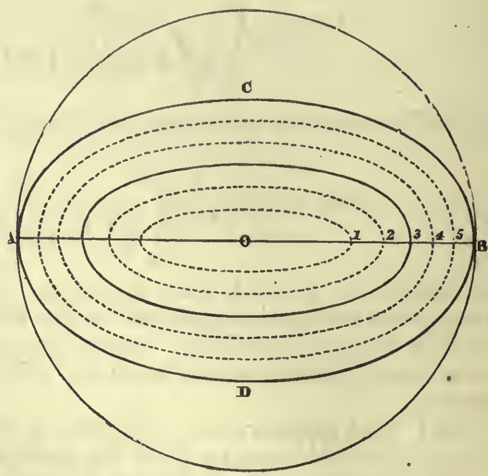


Let acb, adb be the cyclic circles, the intersection of the sphere by an elliptic cylinder whose transverse axis is equal to the diameter of the sphere, and whose minor axis is $2j$. Let a series of concyclic spherical ellipses be described within this cyclic area, whose semitransverse arcs are $O1$, $O2$, $O4$, $O5$, and let $O3$ be the spherical parabola of the series. For every curve, $O1$ or $O2$, inside the spherical parabola, there may be found another *outside* it, $O5$ or $O4$, such that their principal arcs are connected by the equations

$$\tan\alpha\tan\alpha_{\prime} i=1, \quad \sec\beta\sec\beta_{\prime} i=1.$$

The algebraic expressions for the arcs of these curves having the same amplitude give elliptic integrals with *reciprocal* parameters.

Fig. 22.

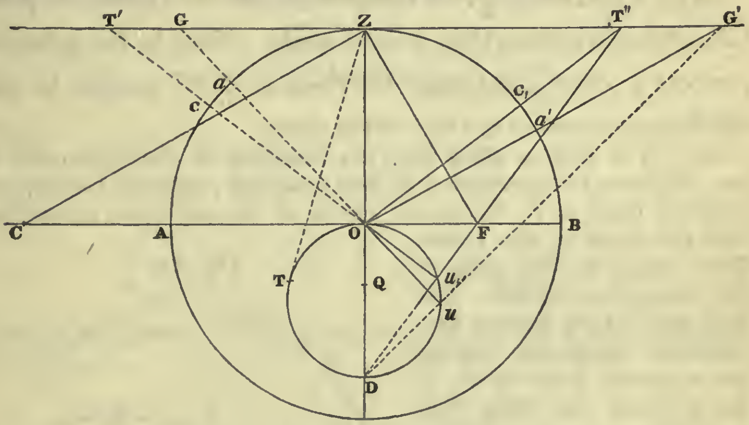


The concyclic spherical ellipses will be orthogonally projected on the base of the hemisphere into as many concentric and similar plane ellipses, whose semiaxes are $O1$, $O2$, $O4$, $O5$. The cyclic area will be projected into the plane ellipse A B C D, and the spherical parabola into the area of the plane ellipse, whose transverse semiaxis is $\dfrac{k}{\sqrt{1+i}}$. Let E be the area of the plane ellipse the projection of the cyclic area, and Π the area of the plane ellipse the projection of the spherical parabola. Then $E=\pi jk^2$, and $\Pi=\frac{\pi jk^2}{1+i}$, whence $\frac{E-\Pi}{\Pi}=i$; or the ellipse the projection of the spherical parabola divides the area of the ellipse the projection of the cyclic area into two portions, such that the outer is to the inner as $i:1$. The semiaxes of E are k and kj, while the semiaxes of Π are $\frac{k}{\sqrt{1+i}}$ and $k\sqrt{(1-i)}$, where $i=\sin\eta$, 2η being as in (9), the cyclic angle.

The importance of this curve, the spherical parabola, in the discussion of the geometrical theory of elliptic integrals is obvious.

We may determine the principal arcs of two inverse spherical ellipses by a simple geometrical construction. Let AZB be a ver-

Fig. 23.



tical section of the hemisphere on which the curves are to be described. Let F be the focus of the elliptic base of the maximum cylinder, whose principal transverse axis is accordingly equal to the diameter of the sphere. Join OZ, FZ, and draw ZC at right angles to ZF, meeting the line AO in C. Produce ZO until OD=AC, and on OD as diameter describe a circle. We are required, given one principal arc Za, to determine the corresponding principal arc Za' of the inverse hyperconic. Draw the tangent ZG. Through a draw the line G$\mathrm{O}u$. Through D draw the line $\mathrm{D}u\mathrm{G}'$. Join OG$'$, it will cut the sphere in a', the vertex of the principal arc Za'. Let $OZ=k$, then $ZG=k\tan\alpha$; and as CZF is a right-angled triangle, $CO=ZD=\frac{k^2}{\sqrt{k^2-B^2}}=\frac{k}{i}$, k and B being the semiaxes of the maximum cylinder. As all the bases of the cylinders are similar, $\frac{k^2-B^2}{k^2}=\frac{a^2-b^2}{a^2}=i^2$.

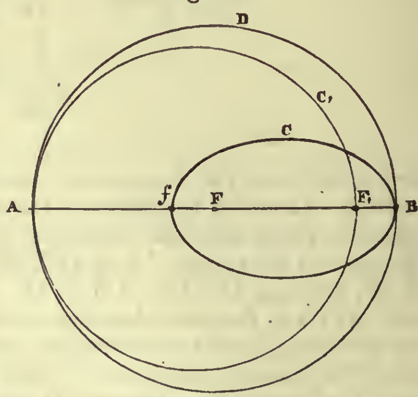
Now as ZOG and ZDG′ are similar triangles, ZG : ZO :: ZD : ZG′, or $k \tan \alpha : k :: \frac{k}{i} : ZG'$, or $ZG'=\frac{k}{i \tan \alpha}$. But $ZG'=k \tan \alpha_{\prime}$; hence $\tan \alpha \tan \alpha_{\prime} i=1$, or the arcs α and $\alpha_{\prime}$ are connected by the equation established in (317).

When we require to know which of these successive curves on this sphere is the spherical parabola, the same construction will enable us to determine it. Draw ZT, a tangent to the circle on OD, take ZT′=ZT″=ZT, and join T′ and T″ with O cutting the sphere in c and c'. $Zc=Zc'$ is the principal semi-transverse arc of the spherical parabola; for $\overline{ZT_{\prime}}^2=k^2 \tan^2 \alpha = OZ\,.\,ZD=\frac{k^2}{i}$, or $\tan^2 \alpha=\frac{1}{i}$.

As $ZT'>ZO$, $cZc'>\frac{\pi}{2}$; or the principal arc of a spherical parabola is always greater than a right angle. Since in the spherical parabola $\gamma+2\epsilon=\frac{\pi}{2}$, the angle $COT'=\epsilon$, or COT′ is equal to half the distance between the foci of the curve.

56.] It is easy to show that the integrals of the first order in sec. [53] may be represented by two spherical parabolas having one common focus at F, the nearer vertex of the one curve coinciding with the focus of the other. Thus, let F be the pole of the hemisphere ABD. Let BCf and $AC_{\prime}F_{\prime}$ denote two spherical parabolas having one common focus at F, $F_{\prime}$ and f being the other foci. Let $Ff=\gamma$, and therefore $FF_{\prime}=\frac{\pi}{2}-\gamma$. Hence the modular angles of the two curves are γ and $\frac{\pi}{2}-\gamma$; and if we make $\cos\gamma=i$, $\cos\left(\frac{\pi}{2}-\gamma\right)=j$.

Fig. 24.



Thus, while the arc of the one is given by the integral $j\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}$, the arc of the other depends on the integral $i\int\frac{d\varphi}{\sqrt{1-j^2\sin^2\varphi}}$.

57.] *On the value of the complete elliptic integral of the third order and logarithmic form.*

Let $$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{[1-n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}. \quad . \quad . \qquad (320)$$

Assume κ the criterion of sphericity $=(1-n)\left(\frac{i^2}{n}-1\right)$, . . . (321)

then $$\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}\right] = \frac{1}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N^2\sqrt{I}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}.$$

Multiply by 2κ, then

$$2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}\right] = \frac{2\kappa}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N^2\sqrt{I}} - \frac{2\kappa}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}. \qquad (322)$$

But (134) gives, making the necessary substitutions as in (297),

$$\left.\begin{aligned} \frac{2\kappa}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N^2\sqrt{I}} &= \left[1-\frac{2}{n}(1+i^2)+\frac{3i^2}{n^2}\right]\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}} \\ &-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{I}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} d\varphi\,\sqrt{I}\int_0^{\frac{\pi}{2}} d\varphi\,\sqrt{I} \end{aligned}\right\}, \quad . \qquad (323)$$

and $$\frac{2\kappa}{n}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}} = \left[\frac{2i^2}{n^2}-\frac{2}{n}-\frac{2i^2}{n}+2\right]\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}.$$

Introducing the substitutions suggested by the two latter equations into (322),

$$\left.\begin{aligned} 2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}\right] &= \left(\frac{i^2}{n^2}-1\right)\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}} \\ &-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{I}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} d\varphi\,\sqrt{I} \end{aligned}\right\} \quad . \quad . \quad . \qquad (324)$$

Now $\frac{d\kappa}{dn} = -\left(\frac{i^2}{n^2}-1\right)$, whence

$$\left.\begin{aligned} 2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}\right] &+ \left[\int_0^{\frac{\pi}{2}} \frac{d\varphi}{N\sqrt{I}}\right]\frac{d\kappa}{dn} \\ &= -\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{I}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} d\varphi\,\sqrt{I} \end{aligned}\right\} \quad . \quad . \quad . \qquad (325)$$

If we divide this equation by $2\sqrt{\kappa}$, the first member will be the differential of $\sqrt{\kappa}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]$. Integrating this equation,

$$2\sqrt{\kappa}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}=-\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\int\frac{(i^2-n)\,dn}{n^2\sqrt{\kappa}}-\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}\right]\int\frac{dn}{n\sqrt{\kappa}}. \qquad (326)$$

Assume

$$n=i^2\sin^2\theta,\text{ then }\kappa=\frac{1-i^2\sin^2\theta}{\tan^2\theta},\ dn=2i^2\sin\theta\cos\theta d\theta. \qquad (327)$$

Hence $$\int\frac{(i^2-n)dn}{n^2\sqrt{\kappa}}=2\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}. \quad . \quad . \quad . \qquad (328)$$

We must now integrate this expression,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}}-\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}},\\
\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}}&=-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}+\int\frac{i^2\cos^2\theta d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}\\
\int\frac{i^2\cos^2\theta d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}&=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}\\
-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}&=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};
\end{aligned}\right\} \qquad (329)$$

adding these equations,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};\\
\therefore-\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\cot\theta\sqrt{1-i^2\sin^2\theta}+\int d\theta\sqrt{1-i^2\sin^2\theta}\ (299).
\end{aligned}\right\} \qquad (330)$$

We have next to compute the value of the integral $\int\frac{dn}{n\sqrt{\kappa}}$.

Now $$\int\frac{dn}{n\sqrt{\kappa}}=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}=\int\frac{d\theta}{\sqrt{I_\theta}}.$$

Substituting these values of the integrals in (326),

$$\left.\begin{aligned}
\sqrt{\kappa}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}=\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]&\left[\cot\theta\sqrt{_\theta I}+\int d\theta\sqrt{_\theta I}\right]\\
&-\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{_\theta I}}
\end{aligned}\right\} \quad . \quad . \qquad (331)$$

If we now substitute this value of $\int_0^{\frac{\pi}{2}} \frac{d\phi}{N\sqrt{I}}$ in the equation given in (175) for a quadrant of the logarithmic ellipse, namely

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a} = \frac{[2n-i^2-n^2]}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}} + \frac{(i^2-n)}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}} + \int_0^{\frac{\pi}{2}} d\phi\sqrt{I},$$

since $\frac{2n-i^2-n^2}{n} = (1-i^2\sin^2\theta) - \cot^2\theta$, we shall obtain the resulting equation,

$$\left.\begin{aligned} \frac{2\sqrt{1-i^2}}{\sqrt{(I_\theta)}}\frac{\Sigma}{a} = \left[\int_0^{\frac{\pi}{2}} d\phi\sqrt{I}\right] + (I_\theta)\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right] \\ + H\left[\frac{\sqrt{(I_\theta)}}{\cot\theta} - \frac{\cot\theta}{\sqrt{(I_\theta)}}\right] + \text{constant} \end{aligned}\right\}, \quad . \quad (332)$$

writing H for

$$\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\int d\theta\sqrt{(I_\theta)} - \left[\int_0^{\frac{\pi}{2}} d\phi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{(I_\theta)}}, \quad . \quad (333)$$

or in the ordinary notation,

$$H = F_i E_i(\theta) - E_i F_i(\theta).$$

When we require to determine the constant, we must not suppose $\theta=0$; for this would render $n=0$, and so change the nature of the curve. Neither should we be justified in making $i=0$ (as some writers do); for this would be to violate the original supposition (and all the conclusions derived from it), namely that i is constant and less than 1. Moreover, since $m+n-mn=i^2=0$, on this hypothesis, $m+n=mn$; or m and n would each be greater than 1, which is inconsistent with the possible values of those quantities.

We have now to determine the value of the constant. In these investigations we have all along supposed $n>m$. The least value n can have is $n=m$. Were we to suppose n to be less than m, it would be nothing more than to write m for n, since m and n are connected by the equation $m+n-mn=i^2$. Hence, if m is not equal to n, one of them must be the greater, and this one we agree to call n, writing m for the lesser. To determine the constant, let us assume $n=m$.

Now $n=i^2\sin^2\theta$, as in (327), and n, when equal to m, is $= 1 - \sqrt{1-i^2} = 1-j$, $(I_\theta) = 1 - i^2\sin^2\theta = j$, $\cot^2\theta = j$, and $\tan\theta = \left(\frac{1}{j}\right)^{\frac{1}{2}}$. Hence the coefficient of H in the last equation,

$\left(\frac{\sqrt{(\mathrm{I}_\theta)}}{\cot\theta}-\frac{\cot\theta}{\sqrt{(\mathrm{I}_\theta)}}\right)$, becomes 0, since in this case $\cot\theta=\sqrt{j}$; and as $n=m$, the curve is the circular logarithmic ellipse. See sec. [43].

The last equation now becomes

$$2\sqrt{j}\,\frac{\Sigma}{a}=\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\,\sqrt{\mathrm{I}}+j\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}}}+\text{constant.} \quad . \quad . \quad . \quad (334)$$

Now, if we turn to (176), we shall find this, without the constant, to be the expression for the quadrant of a circular logarithmic ellipse, or the curve in which a circular cylinder, the radius of whose base is a, intersects at an infinite distance a paraboloid indefinitely attenuated. Hence the constant is 0; and (332) without the constant represents a quadrant of the logarithmic ellipse expressed by elliptic functions of the first and second orders.

CHAPTER VII.

ON THE LOGARITHMIC PARABOLA.

58.] The logarithmic parabola may be defined as the curve of intersection of a parabolic cylinder and a paraboloid of revolution—the vertex of this surface being supposed to touch at its focus the plane of the parabola, the base of the parabolic cylinder.

Let the equation of the paraboloid be

$$x^2+y^2=2kz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and $y^2=4h^2+4hx$ that of the parabolic base of the cylinder, the origin being at the focus. k is the semiparameter of the paraboloid, and h is one fourth of the parameter of the base.

Therefore $$x^2+y^2=(2h+x)^2=2kz; \quad . \quad . \quad . \quad . \quad . \quad (b)$$

hence, x being the independent variable,

$$\frac{\mathrm{d}z^2}{\mathrm{d}x^2}=\frac{(2h+x)^2}{k^2},\ \frac{\mathrm{d}y^2}{\mathrm{d}x^2}=\frac{h}{h+x}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

therefore

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}x}=\frac{(2h+x)[k^2+(h+x)(2h+x)]}{\{k^2(h+x)(2h+x)[k^2+(h+x)(2h+x)]\}^{\frac{1}{2}}}. \quad . \quad . \quad (d)$$

Now the expression under the radical being a quadrinomial in x, must be reducible to the usual form of an elliptic integral. We must choose a suitable transformation. Let

$$\tan^2\tau=\frac{dz^2}{dx^2+dy^2}=\frac{(2h+x)(h+x)}{k^2}, \quad . \quad . \quad . \quad . \quad \text{(e)}$$

deriving this value from (c). Substituting this value in (d) and reducing, we obtain the simple expression

$$\frac{d\Sigma}{dx}=\frac{2h+x}{k\sin\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

τ is evidently the inclination to the plane of XY, of a tangent drawn to the curve.

We must now eliminate x. Since

$$k^2\tan^2\tau=2h^2+3hx+x^2,$$

adding and subtracting $2h^2-hx$, we shall have

$$k^2\tan^2\tau=(2h+x)^2-h(2h+x).$$

Completing the square by adding $\frac{h^2}{4}$, and taking the square root,

$$2(2h+x)=h+\sqrt{(4k^2\tan^2\tau+h^2)}. \quad . \quad . \quad . \quad \text{(g)}$$

The positive sign only must be taken; for when $x=-h$, $\tan\tau=0$. Substituting this value of $2h+x$ in the expression for the arc,

$$\frac{d\Sigma}{dx}=\frac{h+\sqrt{(4k^2\tan^2\tau+h^2)}}{2k\sin\tau}. \quad . \quad . \quad . \quad . \quad \text{(h)}$$

If now we differentiate (e), we shall obtain

$$\frac{dx}{d\tau}=\frac{2k^2\sin\tau}{\cos^3\tau\sqrt{(4k^2\tan^2\tau+h^2)}}. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Multiplying the last equation by this expression,

$$\frac{d\Sigma}{d\tau}=\frac{d\Sigma}{dx}\frac{dx}{d\tau}=\frac{hk}{\cos^3\tau\sqrt{(4k^2\tan^2\tau+h^2)}}+\frac{k}{\cos^3\tau},$$

or
$$\Sigma=k\int\frac{d\tau}{\cos^3\tau}+hk\int\frac{d\tau}{\cos^3\tau\sqrt{(h^2+4k^2\tan^2\tau)}}, \quad . \quad . \quad \text{(k)}$$

59.] There are now three cases (α), (β), (γ) to be considered:

$$2k=h,\ 2k<h,\ 2k>h.$$

Case (α). Let $h=2k$, and the last equation will become

$$\Sigma=k\int\frac{d\tau}{\cos^3\tau}+k\int\frac{d\tau}{\cos^2\tau}=k\int\frac{d\tau}{\cos^3\tau}+k\tan\tau. \quad . \quad . \quad \text{(a)}$$

Now $k\tan\tau$ is the ordinate of a parabola, and $k\int\frac{d\tau}{\cos^3\tau}$ is the length of an arc of this parabola from the vertex to a point where a tangent to it makes the angle τ with the ordinate. Hence, if we assume on the logarithmic parabola a point M, and through this point draw a plane touching the parabolic cylinder, this plane will be vertical, and will cut the vertical paraboloid in a parabola whose semiparameter will be k. This parabola will touch the logarithmic parabola at the point M. Hence in this case the length of the logarithmic parabola to the point M will be equal to the arc of the plane parabola from its vertex to the point M, plus the ordinate of this parabola at the point M.

Case (β). Let $h>2k$.

The general expression may be written

$$\Sigma=k\int\frac{d\tau}{\cos^2\tau\left[1-\left(\frac{h^2-4k^2}{h^2}\right)\sin^2\tau\right]^{\frac{1}{2}}}+k\int\frac{d\tau}{\cos^3\tau}. \quad . \quad . \quad \text{(b)}$$

Let
$$\frac{h^2-4k^2}{h^2}=i^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

and the last equation becomes

$$\Sigma=k\int\frac{d\tau}{\cos^2\tau\sqrt{(1-i^2\sin^2\tau)}}+k\int\frac{d\tau}{\cos^3\tau}. \quad . \quad . \quad . \quad \text{(d)}$$

Now, Y being the arc of an hyperbola, a the transverse axis, and $i^2=\frac{a^2}{a^2+b^2}$, it was shown in (c) sec. [52] that

$$\frac{iY}{a(1-i^2)}=\int\frac{d\tau}{\cos^2\tau\sqrt{(1-i^2\sin^2\tau)}}; \quad . \quad . \quad . \quad . \quad \text{(e)}$$

hence, if $k=\frac{a(1-i^2)}{i}$, we shall have

Logarithmic parabola = plane hyperbola + plane parabola. (f)

The semiaxes a, b of this hyperbola may easily be determined by the equations

$$k=\frac{a(1-i^2)}{i},\ i^2=\frac{a^2}{a^2+b^2};\ \text{or } a^2=\frac{h^2(h^2-4k^2)}{16k^2},\ b=\frac{h}{2}. \quad . \quad \text{(g)}$$

We may eliminate the arc of the hyperbola and introduce instead elliptic integrals of the first and second orders.

Let $\sqrt{I}=1-i^2\sin^2\tau$, then as in (d)

$$\frac{\Sigma}{k}=\int\frac{d\tau}{\cos^2\tau\sqrt{I}}+\frac{d\tau}{\cos^3\tau},$$

and the formula (293), for comparing elliptic integrals with reciprocal parameters, gives

$$\int\frac{d\tau}{\cos^2\tau\sqrt{I}}+\int\frac{d\tau}{I\sqrt{I}}=\int\frac{d\tau}{\sqrt{I}}+\frac{\tan\tau}{\sqrt{I}}. \quad . \quad . \quad . \quad . \qquad (h)$$

We have also, as in (l) sec. [52]

$$-\int\frac{d\tau}{I\sqrt{I}}=\frac{-1}{1-i^2}\int d\tau\sqrt{I}+\frac{i^2}{1-i^2}\,\frac{\sin\tau\cos\tau}{\sqrt{I}}.$$

Adding and reducing,

$$\Sigma=k\left[\int\frac{d\tau}{\sqrt{I}}+\int\frac{d\tau}{\cos^3\tau}\right]+\frac{h^2}{4k}\left[\tan\tau\sqrt{I}-\int d\tau\sqrt{I}\right]. \quad . \qquad (i)$$

Case (γ). Let $2k>h$.

To integrate in this case, we must transform the second member of the equation (h) sec. [58]. Assume

$$2k\tan\tau=h\tan\upsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (j)$$

Then if we make $\dfrac{4k^2-h^2}{4k^2}=j^2$, we shall have

$$2\,(2h+x)=h+h\sec\upsilon, \text{ and } \frac{dx}{d\upsilon}=\frac{h}{2}\,\frac{\sin\upsilon}{\cos^2\upsilon}.$$

But
$$\sin^2\tau=\frac{h^2\sin^2\upsilon}{4k^2\,(1-j^2\sin^2\upsilon)},$$

hence
$$\frac{d\Sigma'}{d\upsilon}=\frac{h}{2}\,\frac{\sqrt{(1-j^2\sin^2\upsilon)}}{\cos^2\upsilon}+\frac{h}{2}\,\frac{\sqrt{(1-j^2\sin^2\upsilon)}}{\cos^3\upsilon}. \quad . \quad . \qquad (k)$$

Now, since
$$\frac{k\,d\tau}{\cos^2\tau}=\frac{h}{2}\,\frac{d\upsilon}{\cos^2\upsilon},$$

and
$$\cos\tau=\frac{\cos\upsilon}{\sqrt{(1-j^2\sin^2\upsilon)}}=\frac{\cos\upsilon}{\sqrt{J}},$$

writing J for $(1-j^2\sin^2\upsilon)$, we shall have

$$k\int\frac{d\tau}{\cos^3\tau}=\frac{h}{2}\int\frac{d\upsilon\sqrt{(1-j^2\sin^2\upsilon)}}{\cos^3\upsilon},$$

or

$$\left.\begin{aligned}\Sigma'=\frac{hj^2}{2}\int\frac{d\upsilon}{\sqrt{(1-j^2\sin^2\upsilon)}}\\ +\frac{h}{2}\,(1-j^2)\int\frac{d\upsilon}{\cos^2\upsilon\sqrt{(1-j^2\sin^2\upsilon)}}+k\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\} \quad . \quad . \quad . \qquad (l)$$

Now the second term of the right-hand member of this equation is the expression for an arc of an hyperbola the distance between whose foci is h. Hence

$$\Sigma' = \frac{hj^2}{2}\int \frac{dv}{\sqrt{(1-j^2\sin^2 v)}} + \mathrm{Y} + \Pi, \quad . \quad . \quad . \quad . \quad . \quad \text{(m)}$$

Π being an arc of the parabola.

We may eliminate the function of the first order and represent in this case the arc of the logarithmic parabola by the arcs of an ellipse, an hyperbola, and a parabola.

Let Y be the arc of an hyperbola whose semitransverse axis is $\frac{1}{j}$, and putting E and Π for the elliptic and parabolic arcs,

$$\Sigma' = \frac{h}{2}\frac{j^2}{(1-j^2)}\left[\mathrm{Y}_{(v)} + \mathrm{E}_j(v) - \tan v\sqrt{\mathrm{J}}\right] + \Pi(\tau), \quad . \quad . \quad \text{(n)}$$

or, as the equation may be written,

$$\Sigma' = \frac{h}{2}\left[\int\frac{dv}{\sqrt{\mathrm{J}}} - \int dv\sqrt{\mathrm{J}} + \tan v\sqrt{\mathrm{J}}\right] + k\int\frac{d\tau}{\cos^3\tau}. \quad . \quad . \quad \text{(o)}$$

ON THE CURVE OF SYMMETRICAL INTERSECTION OF AN ELLIPTIC PARABOLOID BY A SPHERE.

60.] *The curve of symmetrical intersection of a sphere by a paraboloid, whose principal sections are unequal, may be rectified by an elliptic integral of the third order and circular form.*

Let $$x^2+y^2+z^2=2rz \text{ and } \frac{x^2}{k}+\frac{y^2}{k_1}=2z \quad . \quad . \quad . \quad . \quad \text{(a)}$$

be the equations of the sphere and paraboloid, in contact at the vertex of the latter. Then, finding the values of dx, dy, and dz,

$$\left(\frac{ds}{dz}\right)^2 = \frac{(r^2-kk_1)z-2r(r-k)(r-k_1)}{z[z-2(r-k)][2(r-k_1)-z]}. \quad . \quad . \quad . \quad \text{(b)}$$

Assume $$z=2(r-k)\cos^2\theta+2(r-k_1)\sin^2\theta. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Introducing the new variable θ and its functions,

$$\frac{ds}{d\theta} = 2\frac{\sqrt{k_1(r-k)^2+k(r-k_1)^2\tan^2\theta}}{\sqrt{(r-k)+(r-k_1)\tan^2\theta}}. \quad . \quad . \quad . \quad \text{(d)}$$

Assume $$k(r-k_1)^2\tan^2\theta = k_1(r-k)^2\tan^2\varphi\,; \quad . \quad . \quad . \quad . \quad \text{(e)}$$

then, introducing the variable φ and its functions,

$$\frac{ds}{d\theta}=\frac{\sqrt{kk_1}\,\sqrt{(r-k)(r-k_1)}}{\sqrt{k(r-k_1)\cos^2\varphi+k_1(r-k)\sin^2\varphi}}$$

$$=\frac{\sqrt{k_1(r-k)}}{\sqrt{1-\frac{r}{k}\left(\frac{k-k_1}{r-k_1}\right)\sin^2\varphi}},$$

and $$\frac{d\theta}{d\varphi}=\frac{\sqrt{k_1}(r-k)}{\sqrt{k}(r-k_1)\left[1-\frac{(k-k_1)}{k}\frac{(r^2-kk_1)}{(r-k_1)^2}\sin^2\varphi\right]}. \quad . \quad . \quad (f)$$

Multiplying together the foregoing values of $\frac{ds}{d\theta}$ and $\frac{d\theta}{d\varphi}$, and integrating,

$$s=\frac{k_1(r-k)^{\frac{3}{2}}}{\sqrt{k}(r-k_1)}\int\frac{d\varphi}{[1-m\sin^2\varphi]\,\sqrt{1-i^2\sin^2\varphi}}, \quad . \quad . \quad (g)$$

if we write m for

$$\frac{k-k_1}{k}\left(\frac{r^2-kk_1}{(r-k_1)^2}\right), \text{ and } i^2 \text{ for } \frac{k-k_1}{r-k_1}\frac{r}{k}. \quad . \quad . \quad . \quad . \quad . \quad (h)$$

Now, as $i^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}$, and $m=e^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\cos^2\beta}$, as in (9),

we get from these equations

$$\tan^2\alpha=\frac{k(r-k_1)}{r(r-k)}, \quad \tan^2\beta=\frac{k_1(r-k)}{r(r-k_1)}, \quad . \quad . \quad . \quad . \quad (i)$$

whence $$\sqrt{r^2-kk_1}\,\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{k_1(r-k)^{\frac{3}{2}}}{\sqrt{k}(r-k_1)}.$$

Making these substitutions, (g) will become

$$s=\sqrt{r^2-kk_1}\,\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\,\sqrt{1-\sin^2\eta\sin^2\varphi}}. \quad (j)$$

Now, as we have shown in (10), this expression denotes an arc of the spherical ellipse whose principal angles are given by the equations (i), and whose radius is $\sqrt{r^2-kk_1}$. Hence, if a sphere be described whose radius is not r, but $\sqrt{r^2-kk_1}$, the length of the curve, the intersection of the sphere (r) with the paraboloid (kk_1) will be equivalent to that of a spherical ellipse described on the sphere whose radius is $\sqrt{r^2-kk_1}$.

When $r=k$, k being greater than k_1, (d) becomes

$$\frac{ds}{d\theta}=2\sqrt{k(k-k_1)} \text{ or } s=2\sqrt{k(k-k_1)}\,\theta=2r\sqrt{\frac{k-k_1}{k}}.$$

Hence s is an arc of a circle. That such ought to be the case is manifest; for in this case the sphere intersects the paraboloid in its circular sections, and $\sqrt{\frac{k-k_1}{k}}$ is the cosine of the angle which the plane of the circular section of the paraboloid makes with its axis.

It is obvious that the square of the radius of the sphere must be greater than the product of the semiparameters of the principal sections of the paraboloid; otherwise the surface of the sphere would fall within that of the paraboloid and their intersections would become imaginary.

CHAPTER VIII.

ON CONJUGATE AMPLITUDES, AND CONJUGATE ARCS OF HYPERCONIC SECTIONS.

61.] Conjugate arcs of hyperconic sections may be defined, as arcs whose amplitudes ϕ, χ, ω are connected by the equation

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{1-i^2\sin^2\omega}. \quad . \quad (335)$$

This is a fundamental theorem in the theory of elliptic integrals, and may be called *the equation of conjugate amplitudes*. It holds equally in the three orders of elliptic integrals.

The angles ϕ, χ, ω may be called *conjugate amplitudes*.

When the hyperconic section is a circle, $i=0$, and

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi,$$

whence $\omega=\phi+\chi$, or the conjugate amplitudes are $\phi+\chi$, ϕ, and χ. The development of this expression is the foundation of circular trigonometry.

$$\left.\begin{aligned}\text{When } \omega=\frac{\pi}{2},\ \sin\chi&=\frac{\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \text{ and}\\ \sin\phi&=\frac{\cos\chi}{\sqrt{1-i^2\sin^2\chi}}.\end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (a)$$

When the hyperconic section is a parabola, $i=1$, and (335) may be reduced to

$$\tan\omega=\tan\phi\sec\chi+\tan\chi\sec\phi. \quad . \ . \ . \ . \quad (b)$$

If we make the imaginary transformations

$$\tan\omega=\sqrt{-1}\sin\omega',\ \tan\phi=\sqrt{-1}\sin\phi',\ \tan\chi=\sqrt{-1}\sin\chi',$$
$$\sec\phi=\cos\phi',\ \sec\chi=\cos\chi',$$

the preceding formula will become, on substituting these values, and dividing by $\sqrt{-1}$,

$$\sin\omega'=\sin\phi'\cos\chi'+\sin\chi'\cos\phi',$$

the well-known trigonometrical expression for the sine of the sum of two circular arcs.

Hence, by the aid of imaginary transformations, we may interchangeably permute the formulæ of the trigonometry of the circle with those of the trigonometry of the parabola. In the trigonometry of the circle, $\omega=\phi+\chi$; and in the trigonometry of the parabola ω is such a function of the angles ϕ and χ as will render $\tan[(\phi,\chi)]=\tan\phi\sec\chi+\tan\chi\sec\phi$. We must adopt some appropriate notation to represent this function. Let the function (ϕ,χ) be written $\phi\perp\chi$, so that $\tan(\phi\perp\chi)=\tan\phi\sec\chi+\tan\chi\sec\phi$. This must be taken as the *definition* of the function $\phi\perp\chi$.

The theory of parabolic trigonometry, which more properly belongs to this part of the subject, has been fully developed in the first volume of this work (see page 313).

If we take (335), square it, and add $(\cos\omega\cos\chi)^2$ to each side to complete the squares, and reduce, we shall have

$$\cos\phi=\cos\omega\cos\chi+\sin\omega\sin\chi\sqrt{1-i^2\sin^2\phi}. \quad . \quad . \quad \text{(c)}$$

In like manner

$$\cos\chi=\cos\phi\cos\omega+\sin\phi\sin\omega\sqrt{1-i^2\sin^2\chi}; \quad . \quad . \quad \text{(d)}$$

since (335) shows that when $\phi=0$, $\cos\omega=\cos\chi$, it follows that in (c) and (d) the radical must be affected with the positive sign.

62.] Let us assume the equation given in (335) between the conjugate amplitudes,

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{1-i^2\sin^2\omega}.$$

Differentiating this equation on the assumption that ω is constant,

$$\left.\begin{aligned}&[\sin\phi\cos\chi+\cos\phi\sin\chi\sqrt{\text{I}_\omega}]\,d\phi\\ &\quad+[\cos\phi\sin\chi+\sin\phi\cos\chi\sqrt{\text{I}_\omega}]\,d\chi=0\end{aligned}\right\}, \quad . \quad (336)$$

writing $\sqrt{\text{I}_\omega}$ for $\sqrt{1-i^2\sin^2\omega}$.

But
$$\sqrt{\text{I}_\omega}=\frac{\cos\phi\cos\chi-\cos\omega}{\sin\phi\cos\chi}; \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

substituting this value of $\sqrt{I_\omega}$ in the preceding equation,

$$[\sin\phi\cos\chi+\cos\phi\sin\chi\sqrt{I_\omega}] \text{ becomes } \sin\omega\sqrt{I_\chi},$$

and

$$\cos\phi\sin\chi+\sin\phi\cos\chi\sqrt{I_\omega} \text{ becomes } \sin\omega\sqrt{I_\phi};$$

consequently

$$\sin\omega\sqrt{I_\chi}\,d\phi+\sin\omega\sqrt{I_\phi}\,d\chi=0; \quad \ldots \quad \text{(b)}$$

or dividing by $\sin\omega\sqrt{I_\phi}\sqrt{I_\chi}$, we shall obtain

$$\frac{d\phi}{\sqrt{I_\phi}}+\frac{d\chi}{\sqrt{I_\chi}}=0. \quad \ldots \quad \text{(c)}$$

Integrating this expression,

$$\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}=C. \quad \ldots \quad \text{(d)}$$

Now, when $\chi=0$, $\phi=\omega$. Hence $C=\int\frac{d\omega}{\sqrt{I_\omega}}$;

and the resulting expression becomes

$$\int\frac{d\phi}{\sqrt{I_\omega}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}=0. \quad \ldots \quad (337)$$

This is the fundamental equation that connects conjugate elliptic integrals of the first order, if their conjugate amplitudes are connected by the algebraical equation

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{1-i^2\sin^2\omega}.$$

63.] The equation (335) between the conjugate amplitudes,

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{1-i^2\sin^2\omega},$$

which gives the foregoing relation between conjugate elliptic integrals of the first order, naturally leads to the assumption of such a form as the following,

$$\int d\phi\sqrt{1-i^2\sin^2\phi}+\int d\chi\sqrt{1-i^2\sin^2\chi}-\int d\omega\sqrt{1-i^2\sin^2\omega},$$

as equal to some function of ϕ, χ, and ω; or as ω may be assumed to be independent, and χ a function of ϕ by virtue of (335), we may assume, using the notation adopted in this work

$$\int d\phi\sqrt{I_\phi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}=f(\phi), \quad \ldots \quad (338)$$

and proceed to determine $f(\phi)$.

Differentiating this expression,

$$d\phi\,\sqrt{I_\phi}+d\chi\,\sqrt{I_\chi}=d\,.\,f(\phi); \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

but
$$\sqrt{I_\phi}=\frac{\cos\phi-\cos\omega\cos\chi}{\sin\omega\sin\chi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

$$\sqrt{I_\chi}=\frac{\cos\chi-\cos\omega\cos\phi}{\sin\omega\sin\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{b}_{\prime})$$

or, reducing to a common denominator,

$$\frac{d\phi(2\sin\phi\cos\phi-2\sin\phi\cos\omega\cos\chi)+d\chi(2\sin\chi\cos\chi-2\sin\chi\cos\omega\cos\phi)}{2\sin\phi\sin\chi\sin\omega}.$$

Now $2\sin\phi\cos\phi d\phi=d\sin^2\phi$, and $2\sin\chi\cos\chi d\chi=d\sin^2\chi$, while

$$-2\,d\phi\sin\phi\cos\chi\cos\omega-2\,d\chi\sin\chi\cos\phi\cos\omega=2d\,.\,(\cos\phi\cos\chi\cos\omega).$$

Hence

$$d\phi\,\sqrt{I_\phi}+d\chi\,\sqrt{I_\chi}=\frac{d[\sin^2\phi+\sin^2\chi+2\cos\phi\cos\chi\cos\omega]}{2\sin\phi\sin\chi\sin\omega}. \quad . \quad . \quad \text{(c)}$$

But if we square and reduce the conjugate equation (335) we shall have

$$\sin^2\phi+\sin^2\chi+2\cos\phi\cos\chi\cos\omega=1+\cos^2\omega+i^2\sin^2\phi\sin^2\chi\sin^2\omega;$$

hence

$$d[\sin^2\phi+\sin^2\chi+2\cos\phi\cos\chi\cos\omega]=i^2d(\sin\phi\sin\chi\sin\omega)^2, \quad . \quad \text{(d)}$$

or
$$d\,f(\phi)=i^2d(\sin\phi\sin\chi\sin\omega). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

Substituting this value of $f(\phi)$ and integrating

$$\int d\phi\,\sqrt{I_\phi}+\int d\chi\,\sqrt{I_\chi}=C+i^2\sin\phi\sin\chi\sin\omega.$$

To determine the constant C. When $\chi=0$, $\phi=\omega$, and $f(\phi)=0$; and therefore $C=\int d\omega\,\sqrt{I_\omega}$.

Hence finally

$$\int d\phi\,\sqrt{I_\phi}+\int d\chi\,\sqrt{I_\chi}-\int d\omega\,\sqrt{I_\omega}=i^2\sin\phi\sin\chi\sin\omega. \quad . \quad \text{(339)}$$

64.] To prove that

$$\left.\begin{aligned}&\int\frac{d\phi}{(1+n\sin^2\phi)\sqrt{I_\phi}}+\int\frac{d\chi}{(1+n\sin^2\chi)\sqrt{I_\chi}}\\&=\int\frac{d\omega}{(1+n\sin^2\omega)\sqrt{I_\omega}}+\frac{1}{\sqrt{\kappa}}\tan^{-1}\left[\frac{n\sqrt{\kappa}\sin\phi\sin\chi\sin\omega}{1+n-n\cos\phi\cos\chi\cos\omega}\right]\end{aligned}\right\},\text{(340)}$$

or putting $U=\sin\phi\sin\chi\sin\omega$, and $V=\cos\phi\cos\chi\cos\omega$, . (a) and using the notation hitherto adopted in this book,

$$\int\frac{d\phi}{N_\phi\sqrt{I_\phi}}+\int\frac{d\chi}{N_\chi\sqrt{I_\chi}}-\int\frac{d\omega}{N_\omega\sqrt{I_\omega}}=\frac{1}{\sqrt{\kappa}}\tan^{-1}\left[\frac{n\sqrt{\kappa}U}{1+n-nV}\right]. \quad (341)$$

Differentiating the foregoing expression on the supposition that ω is independent of ϕ and χ, and that χ is a function of ϕ, as in (335), we shall have

$$\frac{d\phi}{\sqrt{I_\phi}(1+n\sin^2\phi)}+\frac{d\chi}{\sqrt{I_\chi}(1+n\sin^2\chi)}=df(\phi), \quad . \quad (342)$$

assuming as before $f(\phi)$ for the unknown function of ϕ.

But as $\frac{d\phi}{\sqrt{I_\phi}}=-\frac{d\chi}{\sqrt{I_\chi}}$, as shown in sec. [62], the last expression may be written

$$\frac{d\phi}{\sqrt{I_\phi}}\left[\frac{1}{1+n\sin^2\phi}-\frac{1}{1+n\sin^2\chi}\right]=df(\phi),$$

or $$\frac{nd\phi}{\sqrt{I_\phi}}\left[\frac{\sin^2\chi-\sin^2\phi}{1+n\sin^2\phi+n\sin^2\chi+n^2\sin^2\phi\sin^2\chi}\right]=df(\phi).$$

or $$\frac{n}{i^2}\frac{d\phi}{\sqrt{I_\phi}}\left[\frac{I_\phi-I_\chi}{1+n\sin^2\phi+n\sin^2\chi+n^2\sin^2\phi\sin^2\chi}\right]=df(\phi).$$

Now $$\frac{I_\phi-I_\chi}{\sqrt{I_\phi}}=\sqrt{I_\phi}-\frac{\sqrt{I_\chi}}{\sqrt{I_\phi}}\cdot\sqrt{I_\chi},$$

and $\frac{\sqrt{I_\chi}}{\sqrt{I_\phi}}=-\frac{d\chi}{d\phi}$, as in sec. [62], substituting in the preceding expression

$$\frac{n}{i^2}\frac{[\sqrt{I_\phi}\,d\phi+\sqrt{I_\chi}\,d\chi]}{[1+n\sin^2\phi+n\sin^2\chi+n^2\sin^2\phi\sin^2\chi]}=df(\phi). \quad . \quad (343)$$

But it has been shown in sec. [63] that

$$\sqrt{I_\phi}\,d\phi+\sqrt{I_\chi}\,d\chi=i^2 d\,(\sin\phi\sin\chi\sin\omega)\,;$$

hence $$\frac{nd\,(\sin\phi\sin\chi\sin\omega)}{[1+n\sin^2\phi+\sin^2\chi+n^2\sin^2\phi\sin^2\chi]}=df(\phi). \quad . \quad (344)$$

For brevity let this denominator be put D, and let

$$\sin\phi\sin\chi\sin\omega=U, \text{ as in (a)},$$

then the preceding expression becomes $\frac{ndU}{D}=df(\phi)$. . . (345)

We must develop this expression. (335) gives

$$\cos^2\phi\cos^2\chi=\cos^2\omega+2\cos\omega\sin\phi\sin\chi\sqrt{I}+\sin^2\phi\sin^2\chi I,$$

and $$\cos^2\phi\cos^2\chi=1-\sin^2\phi-\sin^2\chi+\sin^2\phi\sin^2\chi.$$

By the help of this relation we may eliminate $\sin^2\phi+\sin^2\chi$ from the denominator D, and we shall get finally,

$$D=1+n\sin^2\omega-2n\cos\omega\sin\phi\sin\chi\sqrt{I_\omega}+\sin^2\phi\sin^2\chi[n^2+ni^2\sin\omega],$$

$$\text{or } D=1+n\sin^2\omega-2n\cos\omega\frac{U}{\sin\omega}\sqrt{I_\omega}+\frac{U^2}{\sin^2\omega}[n^2+ni^2\sin^2\omega], \quad (346)$$

writing as before U for $\sin\phi\sin\chi\sin\omega$.

The equation now becomes

$$\left.\begin{aligned}\frac{\sqrt{\kappa}\,n\sin^2\omega dU}{\sin^2\omega(1+n\sin^2\omega)-2n\sin\omega\cos\omega\sqrt{I_\omega}U+[n^2+ni^2\sin^2\omega]U^2}\\ =\sqrt{\kappa}\,df(\phi)\end{aligned}\right\}, \quad (347)$$

having multiplied this expression by $\sqrt{\kappa}$, of which we shall presently see the need.

65.] Assume the equation,

$$\tan\Theta=\frac{AU}{B+CU}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (348)$$

in which A, B, C are undetermined constants.

Differentiating this expression,

$$d\Theta=\frac{ABdU}{B^2+2CBU+(A^2+C^2)U^2}; \quad . \quad . \quad . \quad (349)$$

comparing the denominator of this expression with that of the preceding formula, and also the numerators, so that the coefficients of U and U^2 may be the same in both expressions, we shall then have

$$\left.\begin{aligned}
&B^2=\sin^2\omega(1+n\sin^2\omega),\ CB=-n\sin\omega\cos\omega\sqrt{I_\omega},\\
&\text{and}\quad A^2+C^2=n^2+ni^2\sin^2\omega.\\
&\text{Hence}\ B=\sin\omega\sqrt{1+n\sin^2\omega},\ C=-\frac{n\cos\omega\sqrt{I_\omega}}{\sqrt{1+n\sin^2\omega}},\\
&\text{and}\quad A=n\sin\omega\left(\frac{1+n}{n}\right)\frac{\sqrt{mn}}{\sqrt{1+n\sin^2\omega}}.\\
&\text{But it has been shown in (42) that}\\
&\qquad\sqrt{\kappa}=\left(\frac{1+n}{n}\right)\sqrt{mn};\\
&\text{hence}\quad A=\frac{n\sin\omega\sqrt{\kappa}}{\sqrt{1+n\sin^2\omega}},\ \text{and}\ AB=n\sqrt{\kappa}\sin^2\omega.
\end{aligned}\right\} \quad . \quad (350)$$

Having thus found for A, B, C values which satisfy the equation (347), and render the differential expressions (347) and (349) identical, their integrals must be identical. But the integral of (347) is $\sqrt{\kappa}\,\mathrm{f}(\phi)$, and the integral of (349) is $\Theta=\tan^{-1}\left[\frac{\mathrm{AU}}{\mathrm{B}+\mathrm{CU}}\right]$.

Hence $$\sqrt{\kappa}\,\mathrm{f}(\phi)=\tan^{-1}\left[\frac{\mathrm{AU}}{\mathrm{B}+\mathrm{CU}}\right]; \quad . \quad . \quad . \quad . \quad . \quad (351)$$

or substituting for A, B, C, U their values as given in (350), we shall have, putting for $\sqrt{\mathrm{I}_\omega}$ in the constant C its value $\frac{\cos\phi\cos\chi-\cos\omega}{\sin\phi\sin\chi}$ as derived from (335),

$$\sqrt{\kappa}\,\mathrm{f}(\phi)=\tan^{-1}\left[\frac{n\sqrt{\kappa}\sin\phi\sin\chi\sin\omega}{1+n-n\cos\phi\cos\chi\cos\omega}\right],$$

or finally, putting for $\sqrt{\kappa}$ its value $\left(\frac{1+n}{n}\right)\sqrt{mn}$,

$$\sqrt{\kappa}\,\mathrm{f}(\phi)=\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1-\frac{n}{1+n}\cos\phi\cos\chi\cos\omega}\right] . \quad . \quad . \quad (352)$$

66.] Hence if we assume the conjugate amplitudes ϕ, χ, ω as defined by the equation (335), and take the sums of the conjugate integrals of the first, second, and third orders, we shall find them connected by the following equations:—

$$\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}_\phi}}+\int\frac{\mathrm{d}\chi}{\sqrt{\mathrm{I}_\chi}}-\int\frac{\mathrm{d}\omega}{\sqrt{\mathrm{I}}}=0;$$

$$\int\mathrm{d}\phi\sqrt{\mathrm{I}_\phi}+\int\mathrm{d}\chi\sqrt{\mathrm{I}_\chi}-\int\mathrm{d}\omega\sqrt{\mathrm{I}_\omega}=i^2\sin\phi\sin\chi\sin\omega;$$

$$\int\frac{\mathrm{d}\phi}{\mathrm{N}_\phi\sqrt{\mathrm{I}_\phi}}+\int\frac{\mathrm{d}\chi}{\mathrm{N}_\chi\sqrt{\mathrm{I}_\chi}}-\int\frac{\mathrm{d}\omega}{\mathrm{N}_\omega\sqrt{\mathrm{I}_\omega}}=\frac{1}{\sqrt{\kappa}}\tan^{-1}\left[\frac{n\sqrt{\kappa}\sin\phi\sin\chi\sin\omega}{1+n-n\cos\phi\cos\chi\cos\omega}\right].$$

When κ is negative, $\mathrm{f}(\phi)$ is no longer a circular function, but a logarithm, or, in other words, the circular arc becomes the arc of a parabola, since the elliptic integral of the third order and logarithmic form represents the sections of a paraboloid.

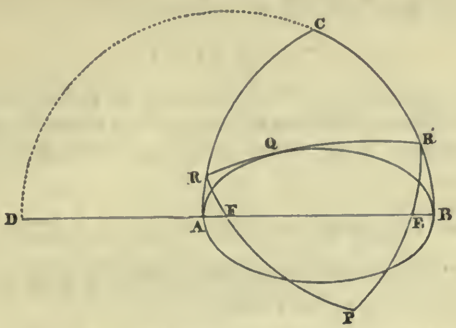
On Conjugate Arcs of a Spherical Parabola.

67.] The well-known relations between elliptic integrals of the first order, whose amplitudes are conjugate, develop some very elegant geometrical theorems.

Thus, in fig. 25, since the arc $\mathrm{AQ}=j\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}_\phi}}+\mathrm{QR}$, and the arc $\mathrm{BQ}=j\int\frac{\mathrm{d}\chi}{\sqrt{\mathrm{I}_\chi}}+\mathrm{QR}'$ (see sec. [20]), the arcs

$$\mathrm{AQ}+\mathrm{BQ}=j\left[\int\frac{\mathrm{d}\phi}{\sqrt{\mathrm{I}_\phi}}+\int\frac{\mathrm{d}\chi}{\sqrt{\mathrm{I}_\chi}}\right]+\mathrm{QR}+\mathrm{QR}'. \quad . \quad . \quad (a)$$

Fig. 25.



Now $AQ+BQ$ = two quadrants of the spherical parabola, and $QR+QR'=\frac{\pi}{2}$; whence half the circumference, or

$$AQB=j\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}\right]+\frac{\pi}{2}.$$

In sec. [22] it has been shown that the complete integral represents the semicircumference, whence

$$AQB=j\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}+\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Comparing these equations (a) and (b) together, we get

$$\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}=\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}.$$

Now, as the triangle RR'P is a quadrantal right-angled triangle, see sec. [24], the relation between the angles AFR, BER', or ϕ and χ, is easily discovered. Since FPE is a spherical triangle right-angled at P, and $FE=2\epsilon=\frac{\pi}{2}-\gamma$, we get $j\tan\phi\tan\chi=1$, since $\sin\gamma=j$. When $AQ=BQ$, $\phi=\chi$, and $\tan\phi=\frac{1}{\sqrt{j}}$.

The locus of the point P is a spherical ellipse, supplemental to the former, having the extremities of its principal minor arc in the foci F, E of the former.

68.] Let σ, $\sigma_{\prime}$, $\sigma_{\prime\prime}$ be three arcs of a spherical parabola, corresponding to the conjugate amplitudes ϕ, χ, ω. Then, successively substituting these amplitudes in (58), the resulting equation becomes

$$\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=j\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}\right]+\tau+\tau_{\prime}-\tau_{\prime\prime}.$$

But as the amplitudes ϕ, χ, ω are conjugate, the sum of these integrals of the first order is $=0$, whence

$$\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=\tau+\tau_{\prime}-\tau_{\prime\prime}. \quad (353)$$

Or, *when the amplitudes of three arcs of a spherical parabola are conjugate amplitudes, the sum of the arcs is equal to the sum of the protangent circular arcs.* The word sum is used in its algebraical sense.

On Conjugate Arcs of a Spherical Ellipse.

69.] If, in (42), we substitute successively ϕ, χ, ω, and add the resulting equations, we shall have

$$\left.\begin{aligned}\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=\left(\frac{1+n}{n}\right)\sqrt{mn}\left[\int\frac{d\phi}{N_{\phi}\sqrt{I_{\phi}}}+\int\frac{d\chi}{N_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{N_{\omega}\sqrt{I_{\omega}}}\right]\\ -\frac{i^2}{\sqrt{mn}}\left[\int\frac{d\phi}{\sqrt{I_{\phi}}}+\int\frac{d\chi}{\sqrt{I_{\chi}}}-\int\frac{d\omega}{\sqrt{I_{\omega}}}\right]-\tau-\tau_{\prime}+\tau_{\prime\prime}\end{aligned}\right\}. \quad (354)$$

Now the conjugate relation between ϕ, χ, and ω renders the sum of the integrals of the first order $=0$, and the sum of the integrals of the third order equal to a circular arc Θ, which is given by the following equation, as shown in (352),

$$\tan\Theta=\frac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1-\frac{n}{1+n}\cos\phi\cos\chi\cos\omega}. \quad (355)$$

Hence $$\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=\Theta-\tau-\tau_{\prime}+\tau_{\prime\prime}. \quad (356)$$

Or, *when the amplitudes are conjugate, the sum of three arcs of a spherical ellipse may be expressed as the sum of four circular arcs.*

When one of the amplitudes ω is a right angle, $\sigma_{\prime\prime}$ becomes a quadrant of the spherical ellipse$=\breve{\sigma}$. $\tau_{\prime\prime}=0$, and $\Theta=\tau=\tau_{\prime}$, as we shall show presently, whence

$$(\breve{\sigma}-\sigma_{\prime})-\sigma=\tau, \text{ which agrees with (52).}$$

Or *the difference between two arcs of a spherical ellipse, measured from the vertices of the curve, may be expressed by a circular arc.*

In (45) we found $\tan\tau=\dfrac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}$, $\tan\tau_{\prime}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}$.

Now, when $\omega=\frac{\pi}{2}$, (a) sec. [61]

$$\text{gives }\sin\chi=\frac{\cos\phi}{\sqrt{1-i^2\sin^2\phi}},\ \sin\phi=\frac{\cos\chi}{\sqrt{1-i^2\sin^2\chi}};$$

whence $\sqrt{mn}\sin\phi\sin\chi=\dfrac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}$,

or $\Theta=\tau=\tau_{\prime}$ when $\tau_{\prime\prime}=0$ or $\omega=\dfrac{\pi}{2}$.

70.] When we take the negative parameter m instead of the positive n, (11) gives

$$\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=\left(\frac{1-m}{m}\right)\sqrt{mn}\left[\int\frac{d\phi}{M_\phi\sqrt{I_\phi}}+\int\frac{d\chi}{M_\chi\sqrt{I_\chi}}-\int\frac{d\omega}{M_\omega\sqrt{I_\omega}}\right]. \quad (357)$$

Now the sum of these arcs is equal to a circular arc $-\Theta_{\prime}$, which may be determined by the expression

$$\tan\Theta_{\prime}=\frac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1+\dfrac{m}{1-m}\cos\phi\cos\chi\cos\omega}, \quad . \quad . \quad . \quad (358)$$

as in (352); whence $\sigma+\sigma_{\prime}-\sigma_{\prime\prime}=-\Theta_{\prime}$. . . . . . . (359)

A little consideration will show that $\Theta_{\prime}$ must be taken with the negative sign; for if we compute the values of $\tan\Theta$ and $\tan\Theta_{\prime}$ from (355) and (358), we shall find

$$\tan\Theta-\tan\Theta_{\prime}=\frac{UV\sqrt{mn}\,(m+n)}{j^2+i^2V+mn(1-V)},$$

a symmetrical expression which remains essentially positive, however we may transpose $\tan\Theta$ and $\tan\Theta_{\prime}$.

Hence $-\tan\Theta_{\prime}=-\tan(-\Theta_{\prime})=\tan\Theta_{\prime}$, or $\Theta_{\prime}$ must be taken with the negative sign.

If we compare together (356) and (359), we shall have the following simple relation between the five circular arcs Θ, $\Theta_{\prime}$, τ, $\tau_{\prime}$, $\tau_{\prime\prime}$,

$$\Theta+\Theta_{\prime}=\tau+\tau_{\prime}-\tau_{\prime\prime}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (360)$$

We may give an independent proof of this remarkable theorem.

The primary theorem (335) $\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{I_\omega}$

gives $$\frac{\sin\omega\cos\omega}{\sqrt{I_\omega}}=\frac{\sin\phi\sin\chi\sin\omega\cos\omega}{\cos\phi\cos\chi-\cos\omega},$$

and $\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2\cos\phi\cos\chi\cos\omega-i^2\sin^2\phi\sin^2\chi\sin^2\omega$.

Let $\sin\phi\sin\chi\sin\omega=U,\quad \cos\phi\cos\chi\cos\omega=V$. . (361)

Now $$\tan\tau_{\prime\prime}=\frac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}=-\frac{\sqrt{mn}\,U\cos^2\omega}{\cos^2\omega-V},$$

whence $$\tan\tau=\frac{\sqrt{mn}\,U\cos^2\phi}{\cos^2\phi-V},\quad \tan\tau=\frac{\sqrt{mn}\,U\cos^2\chi}{\cos^2\chi-V};$$

and $\tan(\tau+\tau_{,}-\tau_{,,})=\dfrac{\tan\tau+\tan\tau_{,}-\tan\tau_{,,}+\tan\tau\tan\tau_{,}\tan\tau_{,,}}{1+\tan\tau_{,,}\tan\tau_{,}+\tan\tau\tan_{,,}-\tan\tau_{,}\tan\tau}$,

whence $\tan(\tau+\tau_{,}-\tau_{,,})=$

$$\frac{\sqrt{mn}\,\mathrm{U}\left[\dfrac{\cos^2\phi}{\cos^2\phi-\mathrm{V}}+\dfrac{\cos^2\chi}{\cos^2\chi-\mathrm{V}}+\dfrac{\cos^2\omega}{\cos^2\omega-\mathrm{V}}-\dfrac{mn\mathrm{U}^2\cos^2\phi\cos^2\chi\cos^2\omega}{(\cos^2\phi-\mathrm{V})(\cos^2\chi-\mathrm{V})(\cos^2\omega-\mathrm{V})}\right]}{1-mn\mathrm{U}^2\left[\dfrac{\cos^2\chi\cos^2\omega}{(\cos^2\chi-\mathrm{V})(\cos^2\omega-\mathrm{V})}+\dfrac{\cos^2\omega\cos^2\phi}{(\cos^2\omega-\mathrm{V})(\cos^2\phi-\mathrm{V})}+\dfrac{\cos^2\phi\cos^2\chi}{(\cos^2\phi-\mathrm{V})(\cos^2\chi-\mathrm{V})}\right]}.$$

If we reduce this expression, we shall have, on introducing the relations

$$\left.\begin{aligned}&\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2\mathrm{V}-i^2\mathrm{U}^2\\ \text{and}\quad&\cos^2\omega\cos^2\chi+\cos^2\phi\cos^2\omega+\cos^2\chi\cos^2\phi=\mathrm{V}^2+2\mathrm{V}+j^2\mathrm{U}^2,\end{aligned}\right\}, \quad (362)$$

$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{\sqrt{mn}\,\mathrm{U}[2j^2+(i^2+mn)\mathrm{V}]}{j^2+(i^2+mn)\mathrm{V}-mn(\mathrm{V}^2+j^2\mathrm{U}^2)}. \quad . \quad (363)$$

If we now combine the values of $\tan\Theta$ and $\tan\Theta_{,}$, given in (355) and (358), we shall have

$$\tan(\Theta+\Theta_{,})=\frac{\sqrt{mn}\,\mathrm{U}[2j^2+(i^2+mn)\mathrm{V}]}{j^2+(i^2+mn)\mathrm{V}-mn(\mathrm{V}^2+j^2\mathrm{U}^2)}\,; \quad . \quad . \quad (364)$$

whence $\qquad \Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,}$,

as is evident from an inspection of the preceding formulæ.

On Conjugate Arcs of a Logarithmic Ellipse.

71.] In (162) substitute χ and ω successively for ϕ. Let

$$\left.\begin{aligned}\sqrt{\kappa}=\left(\frac{1-n}{n}\right)\sqrt{mn},\quad \Phi=\frac{\sin\phi\cos\phi\sqrt{\mathrm{I}_\phi}}{1-n\sin^2\phi},\\ \mathrm{X}=\frac{\sin\chi\cos\chi\sqrt{\mathrm{I}_\chi}}{1-n\sin^2\chi},\quad \Omega=\frac{\sin\omega\cos\omega\sqrt{\mathrm{I}_\omega}}{1-n\sin^2\omega},\end{aligned}\right\}, \quad . \quad (365)$$

we shall have, adding the three resulting equations together, and dividing by $\dfrac{n-m}{\sqrt{mn}}$,

$$\left.\begin{aligned}\frac{2}{k}[\Sigma_\omega-\Sigma_\chi-\Sigma_\phi]=\frac{\sqrt{mn}}{n-m}\Big[n\Phi+n\mathrm{X}-n\Omega-\Big(\int d\phi\sqrt{\mathrm{I}_\phi}+\int d\chi\sqrt{\mathrm{I}_\chi}-\int d\omega\sqrt{\mathrm{I}_\omega}\Big)\Big]\\ -\frac{m(1-n)}{n(n-m)}\sqrt{mn}\left[\int\frac{d\phi}{\sqrt{\mathrm{I}_\phi}}+\int\frac{d\chi}{\sqrt{\mathrm{I}_\chi}}-\int\frac{d\omega}{\sqrt{\mathrm{I}_\omega}}\right]\\ -\sqrt{\kappa}\left[\int\frac{d\phi}{\mathrm{N}_\phi\sqrt{\mathrm{I}_\phi}}+\int\frac{d\chi}{\mathrm{N}_\chi\sqrt{\mathrm{I}_\chi}}-\int\frac{d\omega}{\mathrm{N}_\omega\sqrt{\mathrm{I}_\omega}}\right]\end{aligned}\right\}. (366)$$

Now, as ϕ, χ, and ω are conjugate amplitudes,

$$\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}=0,$$

and $\int d\phi\,\sqrt{I_\phi}+\int d\chi\,\sqrt{I_\chi}-\int d\omega\,\sqrt{I_\omega}=i^2\sin\phi\,\sin\chi\,\sin\omega$. See (339).

Whence

$$\left.\begin{aligned}\frac{2}{k}\left[\Sigma_\omega-\Sigma_\chi-\Sigma_\phi\right]=\frac{\sqrt{mn}}{n-m}\left[n\Phi+nX-n\Omega-i^2\sin\phi\,\sin\chi\,\sin\omega\right]\\ -\sqrt{\kappa}\left[\int\frac{d\phi}{N_\phi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\right]\end{aligned}\right\}\quad(367)$$

We have now to compute the sum of $n\Phi+nX-n\Omega$.

Since $$\sqrt{I_\omega}=\frac{\cos\phi\,\cos\chi-\cos\omega}{\sin\phi\,\sin\chi},$$

$$\frac{\sin\omega\,\cos\omega\,\sqrt{I_\omega}}{1-n\sin^2\omega}=\Omega=-\frac{\sin^2\omega\,(\cos^2\omega-V)}{UN_\omega},$$

if we make, as before, $\cos\phi\,\cos\chi\,\cos\omega=V$, and $\sin\phi\,\sin\chi\,\sin\omega=U$. Finding like expressions for Φ and X, we shall have

$$\left.\begin{aligned}n\Phi+nX-n\Omega=\frac{n}{U}\left[\frac{\sin^2\phi\cos^2\phi}{N_\phi}+\frac{\sin^2\chi\cos^2\chi}{N_\chi}-\frac{\sin^2\omega\cos^2\omega}{N_\omega}\right]\\ -\frac{V}{U}\left[\frac{n\sin^2\omega}{N_\omega}+\frac{n\sin^2\chi}{N_\chi}+\frac{n\sin^2\phi}{N_\phi}\right]\end{aligned}\right\}\quad(368)$$

Now $$\frac{n\sin^2\phi\cos^2\phi}{UN_\phi}=\frac{\cos^2\phi\,(1+n\sin^2\phi-1)}{UN_\phi}=\frac{\cos^2\phi}{UN_\phi}-\frac{\cos^2\phi}{U},$$

and $$\frac{\cos^2\phi}{UN^\phi}=\frac{1+n-n\sin^2\phi-1}{nUN_\phi}=\frac{1}{nU}-\frac{(1-n)}{nUN_\phi};$$

whence $$\frac{n\sin^2\phi\cos^2\phi}{UN_\phi}=\frac{1}{nU}-\frac{\cos^2\phi}{U}-\frac{(1-n)}{nUN},$$

and $$-\frac{Vn\sin^2\phi}{UN_\phi}=\frac{V}{U}-\frac{V}{UN_\phi}.$$

Finding similar expressions for the functions of ω and χ, and recollecting that, as in (362), $\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2V-i^2U^2$, we shall have, making $W=1-n+nV$,

$$nU(n\Phi+nX-n\Omega)=3-n+nV+ni^2U^2-W\left[\frac{1}{N_\phi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}\right].$$

Now $\int d\phi\sqrt{I_\phi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}=i^2U$, as in (339), whence

$$\left.\begin{aligned}nU\left[n\Phi+nX-n\Omega-\left(\int d\phi\sqrt{I_\phi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}\right)\right]\\=2-W\left[\frac{1}{N_\phi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}-1\right]\end{aligned}\right\}. \quad (369)$$

We shall find, after some complicated calculations,

$$N_\phi N_\chi N_\omega=W^2-n^2\kappa U^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (370)$$

and $N_\chi N_\omega+N_\omega N_\phi+N_\phi N_\chi=W^2+2W-n(1-n)(i^2+m)U^2$. (371)

Substituting the values hence derived, the whole expression becomes divisible by nU^2, and we shall obtain, finally, the following expression,

$$\left.\begin{aligned}\frac{\sqrt{mn}}{n-m}\left[n\Phi+nX-n\Omega-i^2U\right]\\=\frac{n\sqrt{\kappa}WU}{W^2-n^2\kappa U^2}+\frac{2mn^2\sqrt{\kappa}UV}{(n-m)(W^2-n^2\kappa U^2)}\end{aligned}\right\}. \quad . \quad . \quad (372)$$

It will be shown that

$$\left.\begin{aligned}-\sqrt{\kappa}\left[\int\frac{d\phi}{N_\phi\sqrt{I_\phi}}+\int\frac{d\chi}{N_\chi\sqrt{I_\chi}}-\int\frac{d\phi}{N_\omega\sqrt{I_\omega}}\right]\\=\frac{1}{2}\log\left[\frac{1-n+nV+n\sqrt{\kappa}U}{1-n+nV-n\sqrt{\kappa}U}\right]\end{aligned}\right\}, \quad . \quad (373)$$

or writing, as before, W for $1-n+nV$, and multiplying numerator and denominator by the numerator,

$$-\sqrt{\kappa}\left[\int\frac{d\phi}{N_\phi\sqrt{I_\phi}}+\int\frac{d\chi}{N_\chi\sqrt{I_\chi}}-\int\frac{d\omega}{N_\omega\sqrt{I_\omega}}\right]=\log\left[\frac{W+n\sqrt{\kappa}U}{\sqrt{W^2-n^2\kappa U^2}}\right].$$

When κ becomes negative or $\sqrt{\kappa}$ imaginary, we may pass from the *circular* to the *logarithmic* form of the third order by the usual imaginary transformation. When κ is negative (352) gives

$$-\sqrt{\kappa}\,f(\phi)=\sqrt{-1}\,\Theta, \text{ where } \tan\Theta=\frac{n\sqrt{\kappa}U}{W}.$$

It is a well-known theorem (see vol. i. p. 335) that

$$\sqrt{-1}\,\Theta=\tfrac{1}{2}\log\left[\frac{1+\sqrt{-1}\tan\Theta}{1-\sqrt{-1}\tan\Theta}\right]. \quad . \quad . \quad . \quad (374)$$

Now instead of $\sqrt{-1}\tan\Theta$ we must write $\sin\xi=\frac{n\sqrt{\kappa}U}{W}$, and the preceding equation becomes

$$-\sqrt{\kappa}\,f(\varphi)=\log\left[\frac{W+n\sqrt{\kappa}U}{\sqrt{W^2-n^2\kappa U^2}}\right]=\log\left[\frac{1+\sin\xi}{\sqrt{1-\sin^2\xi}}\right]; \quad (375)$$

and this logarithm becomes $\log(\sec\xi+\tan\xi)$, which is, we know, the integral of $\int\frac{d\xi}{\cos\xi}$. We shall therefore have

$$\frac{n\sqrt{\kappa}WU}{W^2-n^2\kappa U^2}=\sec\xi\tan\xi; \text{ and as } 2\int\frac{d\xi}{\cos^3\xi}=\sec\xi\tan\xi+\int\frac{d\xi}{\cos\xi},$$

the result, dividing by 2, becomes

$$\Sigma_\omega-\Sigma_\varphi-\Sigma_\chi=k\int\frac{d\xi}{\cos^3\xi}+\frac{kmn^2\sqrt{\kappa}UV}{(n-m)(W^2-n^2\kappa U^2)}. \quad . \quad (376)$$

Hence the sum of three arcs of a logarithmic ellipse may be expressed by an arc of a parabola and a straight line.

When one of the arcs Σ_ω is a quadrant, $V=0$, and the equation becomes

$$[\Sigma_{\frac{\pi}{2}}-\Sigma_\chi]-\Sigma_\varphi=k\int\frac{d\xi}{\cos^3\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (377)$$

which coincides with (160).

If we apply to (163) the same process, step by step, and make $\sin\zeta=\frac{m\sqrt{\kappa_{\prime}}U}{W_{\prime}}$, in which $W_{\prime}=1-m+mV$, we shall find

$$\left.\begin{aligned}\Sigma_\omega-\Sigma_\chi-\Sigma_\varphi=-k\int\frac{d\zeta}{\cos^3\zeta}+\frac{km^2n\sqrt{\kappa_{\prime}}UV}{(n-m)(W_{\prime}^2-m^2\kappa_{\prime}U^2)}\\ +k\int\frac{d\tau}{\cos^3\tau}+k\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}-k\int\frac{d\tau_{\prime\prime}}{\cos^3\tau_{\prime\prime}}\end{aligned}\right\} \quad . \quad . \quad (378)$$

If we subtract this equation from (376), we shall have

$$\left.\begin{aligned}\int\frac{d\xi}{\cos^3\xi}+\int\frac{d\zeta}{\cos^3\zeta}=\int\frac{d\tau}{\cos^3\tau}+\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}-\int\frac{d\tau_{\prime\prime}}{\cos^3\tau_{\prime\prime}}\\ +\frac{mn}{n-m}UV\left[\frac{m\sqrt{\kappa_{\prime}}}{W_{\prime}^2-m^2\kappa_{\prime}U^2}-\frac{n\sqrt{\kappa}}{W^2-n^2\kappa U^2}\right]\end{aligned}\right\} \quad . \quad . \quad (379)$$

Now this last term is divisible by $(n-m)$, and may be reduced to the expression

$$\frac{mn\sqrt{mn}\mathrm{UV}[\mathrm{V}^2+j^2mn\mathrm{U}^2-j^2(1-\mathrm{V})^2]}{[\mathrm{W}^2-n^2\kappa\mathrm{U}^2][\mathrm{W}_{\prime}^2-m^2\kappa_{\prime}\mathrm{U}^2]}. \quad . \quad . \quad . \quad . \tag{380}$$

If in (170), which gives the relation between elliptic integrals of the third order, we substitute successively the conjugate amplitudes φ, χ, and ω, and add the equations thence resulting, we shall have

$$\int\frac{d\xi}{\cos\xi}+\int\frac{d\zeta}{\cos\zeta}=\int\frac{d\tau}{\cos\tau}+\int\frac{d\tau_{\prime}}{\cos\tau_{\prime}}-\int\frac{d\tau_{\prime\prime}}{\cos\tau_{\prime\prime}}, \quad . \quad . \tag{381}$$

in which

$$\left.\begin{array}{l}
\sin\xi=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\dfrac{n}{1-n}\cos\varphi\cos\chi\cos\omega}, \\[2ex]
\sin\zeta=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\dfrac{m}{1-m}\cos\varphi\cos\chi\cos\omega}, \quad \sin\tau=\dfrac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \\[2ex]
\sin\tau_{\prime}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}, \quad \sin\tau_{\prime\prime}=\dfrac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}.
\end{array}\right\} \quad . \tag{382}$$

If, in these equations, we change n into $-n$, and therefore $\sin\xi$ into $\sqrt{-1}\tan\Theta$, $\sin\zeta$ into $\sqrt{-1}\tan\Theta_{\prime}$,

$\sin\tau$ into $\sqrt{-1}\tan\tau$, $\sin\tau_{\prime}$ into $\sqrt{-1}\tan\tau_{\prime}$,

and $\sin\tau_{\prime\prime}$ into $\sqrt{-1}\tan\tau_{\prime\prime}$,

the preceding equations will become

$$\left.\begin{array}{l}
\tan\Theta=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1-\dfrac{n}{1+n}\cos\varphi\cos\chi\cos\omega}, \tan\Theta_{\prime}=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\dfrac{m}{1-m}\cos\varphi\cos\chi\cos\omega}, \\[2ex]
\tan\tau=\dfrac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \tan\tau_{\prime}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}, \tan\tau_{\prime\prime}=\dfrac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}
\end{array}\right\}, \tag{383}$$

and $\Theta+\Theta_{\prime}=\tau+\tau_{\prime}-\tau_{\prime\prime}$, as in (360), values which coincide with those found in sec. [69] for the circular form. Or we may pass from the logarithmic to the circular form, or from the paraboloid to the sphere, or inversely, by the imaginary transformations above referred to*.

* Un examen plus approfondi des fonctions de troisième espèce, nous fera connaître que ces deux classes sont essentiellement irréductibles entre elles.—VERHULST, *Traité des Fonctions Elliptiques*, p. 78.

CHAPTER IX.

ON THE MAXIMUM PROTANGENT ARCS OF HYPERCONIC SECTIONS.

72.] Since the protangents vanish at the summits of these curves, there must be some intermediate position at which they attain their maximum. When the curve has but one summit, as is the case in the parabola, the hyperbola, the logarithmic parabola, and the logarithmic hyperbola, there evidently can be no maximum*.

In the plane ellipse, the protangent $t=\dfrac{ai^2\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}$. If we differentiate this expression with respect to ϕ, and make the differential coefficient $\dfrac{dt}{d\phi}=0$, we shall get

$$\tan\phi=\frac{1}{\sqrt{j}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (384)$$

Substituting this value of $\tan\phi$ in the preceding expression,

$$t=a-b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (385)$$

In this case, the arcs drawn from the vertices of the curve, and which are compared together, have a common extremity, or they together constitute the quadrant, as may be thus shown.

The coordinates x, y of the arc measured from the vertex of the minor axis are $x=a\sin\vartheta$, $y=b\cos\vartheta$; therefore $\dfrac{y}{x}=\dfrac{b}{a}\cot\vartheta=j\cot\vartheta$, since $ja=b$. If we now make $\cot\vartheta=\sqrt{j}$, $\dfrac{y}{x}=j^{\frac{3}{2}}$. Again, as $\tan\lambda=\dfrac{a^2}{b^2}\dfrac{y'}{x'}$, $\dfrac{y'}{x'}=j^2\tan\lambda$; or making $\lambda=\vartheta$, or $\tan\lambda=\dfrac{1}{\sqrt{j}}$, hence $\dfrac{y'}{x'}=j^{\frac{3}{2}}$, or $\dfrac{y'}{x'}=\dfrac{y}{x}$. Therefore the arcs have a common extremity. We have also $\tan^2\lambda=\dfrac{a}{b}$. This property of the plane ellipse, called Fagnani's theorem, may be found in any elementary treatise on elliptic functions.

* The investigation of these particular values of those portions of the tangent arcs to the curves, which lie between the points of contact and the perpendicular arcs from the origin upon them—or, as they have been termed in this paper, protangent arcs—is of importance, because, as we shall show in the next chapter, in the different series of derived hyperconic sections, the maximum protangent arc of any curve in the series becomes a parameter in the integral of the curve immediately succeeding.

On the Maximum Protangent Arc in a Spherical Hyperconic Section.

73.] If we assume the expression found for this arc τ in (45), where i represents $\sin\eta$, 2η being the angle between the cyclic planes of the cone,

$$\tan\tau = \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \quad . \quad . \quad . \quad . \quad . \quad (386)$$

τ being the angle which the linear protangent t to the elliptic base subtends at the centre of the sphere. Differentiate this expression, as in the last article, and make $\frac{d\tau}{d\phi}=0$, we shall find, as before,

$$\tan\phi = \frac{1}{\sqrt{j}} = \sqrt{\frac{\sin\alpha}{\sin\beta}}. \quad . \quad . \quad . \quad . \quad . \quad (387)$$

If we substitute this value of $\tan\phi$ in the preceding expression, we shall obtain

$$\tan\bar{\tau} = \tan\alpha\sec\beta - \tan\beta\sec\alpha, \quad . \quad . \quad . \quad (388)$$

writing $\bar{\tau}$ to denote the maximum protangent.

Now if we turn to sec. [68], we shall there find that this equation connects the amplitudes of three conjugate arcs of a plane parabola. Or if $\bar{\tau}$, β, and α are made the three normal angles of a plane parabola, and $(k.\bar{\tau})$, $(k.\beta)$, $(k.\alpha)$ the three corresponding arcs of the parabola, we shall have

$$(k.\alpha) - (k.\beta) - (k.\tau) = k\tan\alpha\tan\beta\tan\tau.$$

If in (386) we substitute for $\sin\phi$ and $\cos\phi$ their values $\frac{1}{\sqrt{1+j}}$ and $\frac{\sqrt{j}}{\sqrt{1+j}}$, the expression will become

$$\tan\bar{\tau} = \frac{\sqrt{mn}}{(1+j)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (389)$$

We shall see the importance of this value of $\bar{\tau}$ in the next chapter.

In the spherical parabola, as $m=n=i$, $\tan^2\bar{\tau} = \frac{1-j}{1+j} = i_{\prime}$.

Precisely in the same manner as in the plane ellipse, we may show that when $\tan\tau$ has the preceding value, the arcs drawn from the vertices of the curve have a common extremity. This will be shown by proving that the vector arcs, drawn from the centre of the curve to the extremities of the compared arcs, have the same inclination to the principal arc 2α. Now, ψ and ψ' being these inclinations, as in sec. [14], we find

$$\tan^2\lambda = \frac{\tan^4\alpha}{\tan^4\beta}\tan^2\psi;$$

and (39) shows that $\tan\phi=\cos\epsilon\tan\lambda$. Hence, reducing,

$$\tan^2\psi=\frac{\tan^2\beta}{\tan^2\alpha}\frac{\sin^2\beta}{\sin^2\alpha}\tan^2\phi. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

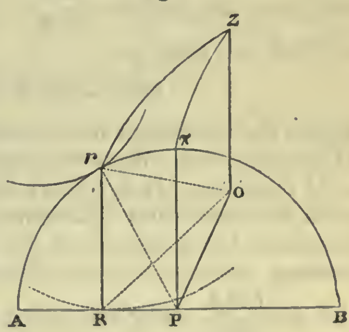
Again, (49) shows, when we measure the arc from the minor principal arc, that $\cot\theta=\frac{a}{b}\frac{y}{x}$, or $\cot\theta=\frac{\sin\alpha}{\sin\beta}\tan\psi'$. Now, in order that we may compare these arcs together, we must have $\theta=\lambda$. Hence

$$\tan^2\psi'=\frac{\tan^2\beta}{\tan^2\alpha}\cdot\frac{1}{\tan^2\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

When we substitute for ϕ any particular value, (a) and (b) will give the corresponding values of $\tan\psi$ and $\tan\psi'$; but when we make $\tan^2\phi=\frac{\sin\alpha}{\sin\beta}=\frac{1}{j}$, the values of ψ and ψ' become equal, or the compared arcs together constitute the quadrant.

74.] To determine the inclination, to the horizontal plane, of the tangent drawn to any point of the spherical ellipse. The spherical ellipse being taken as the curve of intersection of a cylinder by a sphere as in sec. [10], through a side Rr of the cylinder let a plane be drawn, it will cut the sphere in a *small* circle, which will touch the spherical ellipse in the point r, and will cut the base of the hemisphere in the straight line RP, which touches the base of the cylinder at the point R. Let O be the centre of the sphere and Z the centre of the spherical hyperconic. Through the line OZ let a plane be drawn at right angles to the plane of the small circle R$r\pi$P, it will cut the sphere in the arc of a great circle Zπ at right angles to the arc $r\pi$; and as the three planes, namely the horizontal plane, the plane of the small circle R$r\pi$P, and the plane of the great circle ZOPπ, are mutually at right angles, the straight lines in which they intersect PR Pπ, PO are mutually at right angles; therefore P is the foot of the perpendicular drawn from the centre O of the base of the cylinder, to the tangent RP which touches the curve. P is also the centre of the small circle A$r\pi$, since AB is a chord of the sphere. Hence Aπ is a quadrant, and therefore $r\pi$ or ν is the inclination of the element of the spherical ellipse at r to the base of the hemisphere. Now ZO is the radius of the sphere, and Pr that of the small circle. RPO is a right angle; and there-

Fig. 26.



fore $\overline{OR}^2=\overline{OP}^2+\overline{PR}^2$. Hence $\overline{Rr}^2=\overline{Or}^2-\overline{OR}^2$. Now for the moment putting A and B for the semiaxes of the base of the cylinder, $\overline{OP}^2=A^2\cos^2\lambda_{,}+B^2\sin^2\lambda_{,}$, and

$$\overline{PR}^2=\frac{(A^2-B^2)^2\sin^2\lambda_{,}\cos^2\lambda_{,}}{A^2\cos^2\lambda_{,}+B^2\sin^2\lambda_{,}}; \text{ whence } \overline{OR}^2=\frac{A^4\cos^2\lambda_{,}+B^4\sin^2\lambda_{,}}{A^2\cos^2\lambda_{,}+B^2\sin^2\lambda_{,}}, \quad \text{(a)}$$

and therefore $\overline{Rr}^2=\overline{Or}^2-\dfrac{A^4\cos^2\lambda_{,}+B^4\sin^2\lambda_{,}}{A^2\cos^2\lambda_{,}+B^2\sin^2\lambda_{,}}$.

Let $\qquad Or=1,\ A=\sin\alpha,\ B=\sin\beta, \quad . \quad . \quad . \quad . \quad . \quad$ (b)

and as $\tan^2\nu=\dfrac{\overline{RP}^2}{\overline{Rr}^2}$, $\tan^2\nu=\dfrac{(\sin^2\alpha-\sin^2\beta)^2\sin^2\lambda_{,}\cos^2\lambda_{,}}{\sin^2\alpha\cos^2\alpha\cos^2\lambda_{,}+\sin^2\beta\cos^2\beta\sin^2\lambda_{,}}$.

Let, as in (25), $\tan\lambda_{,}=\cos\epsilon\tan\phi=\dfrac{\cos\alpha}{\cos\beta}\tan\phi$. Substituting, we get the expression

$$\tan\nu=\frac{\sin\epsilon\sin\eta\sin\phi\cos\phi}{\sqrt{(1-\sin^2\epsilon\sin^2\phi)(1-\sin^2\eta\sin^2\phi)}}. \quad . \quad (390)$$

In supplemental spherical ellipses, since $\sin\eta$ and $\sin\epsilon$, see sec. [9], are respectively equal to $\sin\epsilon'$ and $\sin\eta'$, we infer therefore that in supplemental spherical ellipses the inclinations to the plane of XY of the tangents to the curves are the same when the amplitudes ϕ are the same.

If we now differentiate this expression, and make $\dfrac{d\nu}{d\phi}=0$, we shall find that $\tan^2\phi=\dfrac{\tan\alpha}{\tan\beta}$. If we substitute this value of $\tan\phi$ in (390), we shall get

$$\tan\nu=\tan(\alpha-\beta), \text{ or } \nu=\alpha-\beta. \quad . \quad . \quad . \quad (391)$$

Hence the maximum inclination to the plane of XY of the tangent to the spherical ellipse is equal to the difference between its principal semiarcs. It is remarkable that the point of the curve which gives the maximum difference between the arcs, which together constitute the quadrant of the spherical ellipse, is not the point of greatest inclination; for this latter point is found by making $\tan^2\phi=\dfrac{\tan\alpha}{\tan\beta}$, while the point of maximum difference is obtained by putting $\tan^2\phi=\dfrac{\sin\alpha}{\sin\beta}$. This is the more worthy of notice, as we shall find the two points—the point of maximum division, and the point of greatest inclination—to coincide in the logarithmic ellipse.

If we take the two plane ellipses which are the projections of the spherical ellipse, one being the perspective, and the other the orthogonal projection, and seek on these plane ellipses their points of maximum division, we shall find that the angles, which the perpendiculars on the tangents, through these points of maximum division of those plane curves, make with the principal arc, are the values which must be assigned to the amplitude ϕ, to determine the point where the tangent to the curve has the greatest inclination to the plane of XY, and the point which divides the quadrant into two parts such that their difference shall be a maximum. This is plain; for the semiaxes of one ellipse are $k\tan\alpha$, $k\tan\beta$; while the semiaxes of the other are $k\sin\alpha$ and $k\sin\beta$. And these angles are given by the equations

$$\tan^2\lambda=\frac{\tan\alpha}{\tan\beta}; \text{ and } \tan^2\lambda_{\prime}=\frac{\sin\alpha}{\sin\beta}.$$

On the Maximum Protangent Arc in a Logarithmic Ellipse.

75.] We must follow the steps previously indicated, and differentiate the expression found in (165),

$$\sin\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

τ here denotes the inclination of the element of the curve to its orthogonal projection on the ellipse, the base of the cylinder, which intersects the paraboloid in the logarithmic ellipse, see sec. [38]. τ is also the normal angle of the tangent parabolic arc to the logarithmic ellipse, whose plane touches the vertical cylinder. This expression will be a maximum when the parabolic arc is a maximum. Put the differential coefficient $\frac{d\tau}{d\phi}=0$. This gives, as before, $\tan\phi=\frac{1}{\sqrt{j}}$. Substituting this expression in (a), we get

$$\sin\bar{\tau}=\frac{\sqrt{mn}}{(1+j)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (392)$$

We shall find the importance of this expression in the next chapter.

From (392) we derive $\qquad \tan^2\bar{\tau}=\frac{mn}{(1+j)^2-mn}.$

Now $\qquad (1+j)^2=2+2j-i^2=2+2j-m-n+mn.$ Hence, as

$$j=\sqrt{(1-m)(1-n)},\ (1+j)^2-mn=[\sqrt{1-m}+\sqrt{1-n}]^2;$$

whence we get $\tan\bar{\tau}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}}$. Multiply this equation, numerator and denominator, by $\sqrt{1-m}-\sqrt{1-n}$, and the last expression will become

$$\tan\bar{\tau}=\frac{\sqrt{mn}\,\sqrt{1-m}}{n-m}-\frac{\sqrt{mn}\,\sqrt{1-n}}{n-m}.$$

In (171) we found for the semiaxes of the cylinder, whose intersection with the paraboloid is the logarithmic ellipse,

$$\frac{a}{k}=\frac{\sqrt{mn}\,\sqrt{1-m}}{n-m},\quad \frac{b}{k}=\frac{\sqrt{mn}\,\sqrt{1-n}}{n-m},\ \text{or}\ \tan\bar{\tau}=\left(\frac{a-b}{k}\right). \tag{393}$$

This gives a simple expression for the tangent of the maximum parabolic arc, analogous to (385) and (391). We have only to take in the parabola, whose semiparameter is k, an arc whose ordinate is $a-b$, to determine the maximum protangent parabolic arc.

The value $\tan\phi=\frac{1}{\sqrt{j}}$, which fixes the position and magnitude of the maximum protangent arc to the logarithmic ellipse, renders $\tan^2\lambda=\frac{a}{b}$. For (150) gives $\tan^2\phi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda$. But (152) gives $\frac{\alpha+\beta}{\alpha}=\frac{C}{C-B}$, and $\frac{C}{C-B}=\frac{1}{1-m}$; hence $\tan^2\phi=\frac{\tan^2\lambda}{1-m}$.
If we now make

$$\tan^2\phi=\frac{1}{j}=\frac{1}{\sqrt{(1-n)(1-m)}},\ \tan^2\lambda=\sqrt{\frac{1-m}{1-n}}=\frac{a}{b},$$

as we may infer from (171). Now, substituting this value of $\tan^2\lambda$ in (155), we shall get

$$\tan\tau=\frac{a-b}{k}.$$

Comparing this expression with (393), we find that the maximum protangent arc is equal to the maximum inclination.

Again, if we differentiate the values of x, y, z given in (158), the coordinates of the extremity of the arc measured from the minor axis, and substitute them in the general expression for the tangent of the inclination of any curve to the plane of XY, namely $\frac{dz}{\sqrt{dx^2+dy^2}}$, and make $\vartheta=\lambda$, as before, putting for $\tan^2\lambda=\tan^2\vartheta$ the value $\frac{a}{b}$, we shall get $\frac{dz}{\sqrt{dx^2+dy^2}}=\frac{a-b}{k}$. Hence the arcs have a common extremity, since they have the same inclination to the plane of XY. As $\frac{a}{b}=\tan^2\lambda$ is the value of $\tan^2\lambda$ which gives the maximum protangent $=a-b$ in the plane ellipse the base of the cylinder, it follows that the point of maximum division on the logarithmic ellipse is orthogonally projected into the point of maximum division on the plane ellipse; and the corresponding protangent in the latter $a-b$ is the ordinate of the parabolic arc which expresses the difference between the corresponding arcs of the former. Thus, while the arcs which together constitute the quadrant on the plane ellipse differ by the difference of the semi-axes $a-b$, the corresponding arcs of the logarithmic ellipse will differ by an arc of a parabola whose ordinate is $a-b$.

76.] When the amplitude ϕ is given by the equation $\tan\phi=\frac{1}{\sqrt{j}}$, or when the protangent is a maximum, the corresponding arc of the spherical ellipse, or of the logarithmic ellipse, may be expressed by functions of the first and second orders only. This may be shown as follows. When $\tan\phi=\frac{1}{\sqrt{j}}$, the arcs σ and $\sigma_{,}$ of the spherical ellipse, or the arcs Σ and $\Sigma_{,}$ of the logarithmic ellipse, together make up the quadrants Q or $Q_{,}$; see sections [73] and [75]. Hence $\sigma+\sigma_{,}=Q$, or $\Sigma+\Sigma_{,}=Q_{,}$. But we have also $\sigma_{,}-\sigma=\tau$, as in (52), and $\Sigma_{,}-\Sigma=\tau$, as in (160). Therefore $2\sigma=Q-\tau$, $2\sigma_{,}=Q+\tau$, $2\Sigma_{,}=Q_{,}+\tau$, $2\Sigma=Q_{,}-\tau$. Or σ and $\sigma_{,}$, or Σ and $\Sigma_{,}$ may be expressed as simple functions of the quadrant and τ. Now the quadrant, as we have shown in the last section, may be expressed by functions of the first and second orders only, while τ is an arc either of a circle or of a parabola. Hence an elliptic integral of the third order, whose amplitude $\phi=\tan^{-1}\left(\frac{1}{\sqrt{j}}\right)$ may be expressed by functions of the first and second orders only*.

CHAPTER X.

ON DERIVATIVE HYPERCONIC SECTIONS.

77.] We shall now proceed to show that, when a hyperconic section is given, whether it be spherical or paraboloidal, we may from it derive a series of curves whose moduli and parameters shall decrease or increase according to a certain law; so that ulti-

* Tout kappa dont l'amplitude a pour tangente trigonométrique $\frac{1}{\sqrt{b}}$, [or, as it is written in this work, $\left(\frac{1}{\sqrt{j}}\right)$], peut s'exprimer par des fonctions d'une espèce inférieure.—VERHULST, *Traité des Fonctions Elliptiques*, p. 99.

mately the rectification of these curves may be reduced to the calculation of circular or parabolic arcs, or, in other words, to circular functions or logarithms. We shall also show that all these derived curves, together with the original curve, may be traced on the *same* generating surface, *i. e.* on the same sphere or paraboloid.

In sec. [44]* we have shown that the rectification of a plane ellipse whose semiaxes are a and b, may be reduced to the rectification of another plane ellipse whose semiaxes $a_{\prime}$, $b_{\prime}$ are given by the equations $a_{\prime}=a+b$, $b_{\prime}=2\sqrt{ab}$, of which the eccentricity is less than that of the former. $a+b$ is that portion of the tangent, drawn through the point of maximum division, which lies between the axes; and $\sqrt{ab}$ is the perpendicular from the centre on it.

We have shown in (63) and (74), that if φ and ψ are connected by the equation $\tan(\psi-\varphi)=j\tan\varphi$, while i and $i_{\prime}$ are so related that

$$i_{\prime}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}=\frac{1-j}{1+j},$$

we shall have

$$\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{1-i_{\prime}^2\sin^2\psi}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}.$$

Let us now introduce this transformation into the elliptic integral of the third order, *circular* form, and *negative* parameter. In (191) we found

$$2\sin^2\varphi=1+i_{\prime}\sin^2\psi-\cos\psi\sqrt{\mathrm{I}_{\prime}}.$$

Now $$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Or replacing φ by its equivalent functions in ψ, and recollecting that $m-n+mn=i^2$, since m and n are conjugate parameters of the circular form, we shall find

$$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{\prime})\int\frac{d\psi}{[2-m-mi_{\prime}\sin^2\psi+m\cos\psi\sqrt{\mathrm{I}_{\prime}}]\sqrt{\mathrm{I}_{\prime}}}. \quad (394)$$

We may eliminate the radical $m\cos\psi\sqrt{\mathrm{I}_{\prime}}$ from the denominator of this expression by treating it as the sum of two terms.

Multiplying and dividing the function by their difference, since $1+i_{\prime}=\frac{2}{1+j}$,

$$4(1-m)\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{\prime})\int\frac{d\psi[2-m-mi_{\prime}\sin^2\psi-m\cos\psi\sqrt{\mathrm{I}_{\prime}}]}{\left[1+\frac{mn}{(1+j)^2}\sin^2\psi\right]\sqrt{\mathrm{I}_{\prime}}}. \quad (395)$$

It is truly remarkable that whether the parameter of the original function we start from be positive or negative, the parameter of the first derived integral will always be positive. Indeed it is necessary that this should be the case, because the parameters of the derived functions, increasing or diminishing as they do, must at length pass from between the limits 1 and i^2. Should they do so, the integral would be no longer of the circular form, but of the logarithmic. Now we cannot pass from one of these forms to the other by any but an imaginary transformation. This objection does not hold when the parameter is positive, because the limits of the positive parameter are 0 and ∞. It is, too, worthy of remark that the first derived parameter is always the same, whether we transform from positive or negative parameters. Write

$$n_{,}=\frac{mn}{(1+j)^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (396)$$

$n_{,}$ is the first derived parameter.

We may transform (395) into

$$4(1-m)\int\frac{d\phi}{M\sqrt{I}}=(1+i_{,})\int\frac{d\psi\left[2-m-\frac{mi_{,}}{n_{,}}(1+n_{,}\sin^2\psi-1)-m\cos\psi\sqrt{I_{,}}\right]}{[1+n_{,}\sin^2\psi]\sqrt{I_{,}}}.$$

Now $$\frac{mi_{,}}{n_{,}}=\frac{i^2}{n}, \text{ and } 2-m+\frac{mi_{,}}{n_{,}}=\frac{m+n}{n}. \quad . \quad . \quad . \quad (397)$$

Hence

$$\left.\begin{array}{l} 2\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}=\frac{(m+n)\sqrt{n_{,}}}{mn\,.\sqrt{mn}}\int\frac{d\psi}{[1+n_{,}\sin^2\psi]\sqrt{I_{,}}} \\ \qquad -\frac{(1+i_{,})}{2}\frac{i^2}{mn}\int\frac{d\psi}{\sqrt{I_{,}}}-\frac{(1+i_{,})}{2\sqrt{n_{,}}}\tan^{-1}(\sqrt{n_{,}}\sin\psi) \end{array}\right\}. \quad (398)$$

We shall now show that

$$\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\sqrt{n_{,}}\sin\psi. \quad . \quad . \quad . \quad . \quad (399)$$

If we revert to (189), (191), and (193), we there find

$$2\sin\phi\cos\phi=\sin\psi[\sqrt{I_{,}}+i_{,}\cos\psi],$$

and $$2\sqrt{I}=(1+j)[\sqrt{I_{,}}+i_{,}\cos\psi].$$

(396) gives $\sqrt{mn}=\sqrt{n_{,}}(1+j)$; therefore $\dfrac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}=\sqrt{n_{,}}\sin\psi$.

If we replace $\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{I_{\prime}}}$ in the preceding equation by its value $\int\frac{d\phi}{\sqrt{I}}$, and put $N_{\prime}$ for $1+n_{\prime}\sin^2\psi$,

$$\left.\begin{aligned}2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}&=\frac{(m+n)}{mn}\sqrt{\frac{n_{\prime}}{mn}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}\\&-\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}-\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]\end{aligned}\right\}\ldots \quad (400)$$

Now the common formula for comparing circular integrals with conjugate parameters is, we know, see (47),

$$\begin{aligned}\left(\frac{1+n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}-\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}\\=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right].\end{aligned}$$

Adding these equations we obtain this new formula,

$$\left.\begin{aligned}\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}\\=\left(\frac{m+n}{mn}\right)\sqrt{n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}\end{aligned}\right\}\ldots \quad (401)$$

By the help of this important formula we may establish a simple relation between the sum of the original conjugate functions of the third order and the first derived function of the same order.

78.] If σ be the arc of a spherical ellipse, it is shown in (46) that

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]. \quad (401^*)$$

and in (11) that $\quad \sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}.$

Adding these equations together, and introducing the relation just now established,

$$2\sigma=\frac{(m+n)}{mn}\sqrt{n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]. \quad (402)$$

Now, as $m-n=i^2-mn$, $(m+n)^2=i^4-2i^2mn+m^2n^2+4mn$.

We have also $mn = n_{\prime}(1+j)^2$, $i_{\prime} = \frac{1-j}{1+j}$, $i^2 = (1-j)(1+j)$, and $2(2-i^2) = 2(1+j^2) = \frac{4(1+i_{\prime}^2)}{(1+i_{\prime})^2}$; hence

$$m+n = (1+j)^2(1+n_{\prime})\sqrt{m_{\prime}}, \quad . \quad . \quad . \quad . \quad (403)$$

and therefore
$$\left(\frac{m+n}{mn}\right)\sqrt{n_{\prime}} = \left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}. \quad . \quad . \quad . \quad (404)$$

It is worthy of especial remark that this coefficient of $\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}$ in (401) is precisely the same in form as the coefficient of $\int \frac{d\phi}{N\sqrt{I}}$.

The preceding equation (402) may now be written

$$2\sigma = \left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}} - \frac{i^2}{\sqrt{mn}}\int \frac{d\phi}{\sqrt{I}} - \tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]. \quad (405)$$

Let $\sigma_{\prime}$, $n_{\prime}$, $i_{\prime}$, ψ be analogous quantities for the derived spherical ellipse $\sigma_{\prime}$; substituting their values in (401*),

$$\sigma_{\prime} = \left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}} - \frac{i_{\prime}^2}{\sqrt{m_{\prime}n_{\prime}}}\int \frac{d\psi}{\sqrt{I_{\prime}}} - \tan^{-1}\left[\frac{\sqrt{m_{\prime}n_{\prime}}\sin\psi\cos\psi}{\sqrt{I_{\prime}}}\right]. (406)$$

Let q, $q_{\prime}$, $q_{\prime\prime}$, $q_{\prime\prime\prime}$, &c. denote $\frac{i^2}{\sqrt{mn}}$, $\frac{i_{\prime}^2}{\sqrt{m_{\prime}n_{\prime}}}$, $\frac{i_{\prime\prime}^2}{\sqrt{m_{\prime\prime}n_{\prime\prime}}}$, &c., and put r, $r_{\prime}$, $r_{\prime\prime}$, $r_{\prime\prime\prime}$, &c. for $(1+j)$, $(1+j)(1+j_{\prime})$, $(1+j)(1+j_{\prime})(1+j_{\prime\prime})$, $(1+j)(1+j_{\prime})(1+j_{\prime\prime})(1+j_{\prime\prime\prime})$, &c. Let also Φ, Ψ, $\Psi_{\prime}$, $\Psi_{\prime\prime}$, &c. denote the arcs, whose tangents are

$$\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \frac{\sqrt{m_{\prime}n_{\prime}}\sin\psi\cos\psi}{\sqrt{1-i_{\prime}^2\sin^2\psi}}, \frac{\sqrt{m_{\prime\prime}n_{\prime\prime}}\sin\psi_{\prime}\cos\psi_{\prime}}{\sqrt{1-i_{\prime\prime}^2\sin^2\psi_{\prime}}}, \text{ \&c.}$$

Making these substitutions, and writing Q, $Q_{\prime}$, $Q_{\prime\prime}$, &c. for the coefficients of

$\int \frac{d\phi}{N\sqrt{I}}$, $\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}$, $\int \frac{d\psi_{\prime}}{N_{\prime\prime}\sqrt{I_{\prime\prime}}}$, (405) and (406) become

$$2\sigma = Q_{\prime}\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}} - q\int \frac{d\phi}{\sqrt{I}} - \Phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\sigma_{\prime} = Q_{\prime}\int \frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}} - qr\int \frac{d\phi}{\sqrt{I}} - \Psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a_{\prime})$$

Taking the derivatives of these expressions, we may write

$$2\sigma_{\prime}=\mathrm{Q}_{\prime\prime}\int\frac{d\psi_{\prime}}{\mathrm{N}_{\prime\prime}\sqrt{\mathrm{I}_{\prime\prime}}}-q_{\prime}r\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\Psi. \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{b})$$

$$\sigma_{\prime\prime}=\mathrm{Q}_{\prime\prime}\int\frac{d\psi_{\prime}}{\mathrm{N}_{\prime\prime}\sqrt{\mathrm{I}_{\prime\prime}}}-q_{\prime\prime}r_{\prime}\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\Psi_{\prime}. \quad . \quad . \quad . \quad . \quad (\mathrm{b})$$

$$2\sigma_{\prime\prime}=\mathrm{Q}_{\prime\prime\prime}\int\frac{d\psi_{\prime\prime}}{\mathrm{N}_{\prime\prime\prime}\sqrt{\mathrm{I}_{\prime\prime\prime}}}-q_{\prime\prime}r_{\prime}\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\Psi_{\prime}. \quad . \quad . \quad . \quad . \quad (\mathrm{c})$$

$$\sigma_{\prime\prime\prime}=\mathrm{Q}_{\prime\prime\prime}\int\frac{d\psi_{\prime\prime}}{\mathrm{N}_{\prime\prime\prime}\sqrt{\mathrm{I}_{\prime\prime\prime}}}-q_{\prime\prime\prime}r_{\prime\prime}\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\Psi_{\prime\prime}. \quad . \quad . \quad . \quad . \quad (\mathrm{c}_{\prime})$$

Subtract ($a_{\prime}$) from (a), ($b_{\prime}$) from (b), and ($c_{\prime}$) from (c), the integrals of the third order disappear, and we shall have

$$\left.\begin{aligned}
2\sigma-\sigma_{\prime}&=(q_{\prime}r-q)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\Psi-\Phi,\\
2\sigma_{\prime}-\sigma_{\prime\prime}&=(q_{\prime\prime}r_{\prime}-q_{\prime}r)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\Psi_{\prime}-\Psi,\\
2\sigma_{\prime\prime}-\sigma_{\prime\prime\prime}&=(q_{\prime\prime\prime}r_{\prime\prime}-q_{\prime\prime}r_{\prime})\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\Psi_{\prime\prime}-\Psi_{\prime},\\
2\sigma_{\prime\prime\prime}-\sigma_{\prime\prime\prime\prime}&=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}-q_{\prime\prime\prime}r_{\prime\prime})\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\Psi_{\prime\prime\prime}-\Psi_{\prime\prime}.
\end{aligned}\right\} \quad . \quad (407)$$

If we add these equations together,

$$\sigma+\sigma_{\prime}+\sigma_{\prime\prime}+\sigma_{\prime\prime\prime}+(\sigma-\sigma_{\prime\prime\prime\prime})=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}-q)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\Psi_{\prime\prime\prime}-\Phi. \quad (408)$$

If we multiply the first of (407) by 2^3, the second by 2^2, the third by 2, and the fourth by 2^0, and add the results,

$$\left.\begin{aligned}
2^4\sigma-\sigma_{\prime\prime\prime\prime}=(q_{\prime\prime\prime\prime}r_{\prime\prime\prime}+q_{\prime\prime\prime}r_{\prime\prime}+2q_{\prime\prime}r_{\prime}+4q_{\prime}r-8q)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\\
+(\Psi_{\prime\prime\prime}+\Psi_{\prime\prime}+2\Psi_{\prime}+4\Psi-8\Phi)
\end{aligned}\right\}, \quad (409)$$

an integral which enables us to approximate with ease to the value of the integral of the third order and circular form, in terms of an integral of the first order.

We have shown in sec. [28] how the integral of the first order may be reduced.

The above expressions may be reduced to simpler forms when the functions are complete. In this case $\Phi=0$, $\Psi=0$, $\Psi_{,}=0$, $\Psi_{,,}=0$, &c.; and when σ is a quadrant, $\sigma_{,}$ will be two quadrants, $\sigma_{,,}$ will be four quadrants, $\sigma_{,,,}$ will be eight quadrants, and so on. The preceding expression may now be written, denoting a quadrant by the symbol $\breve{\sigma}$,

$$16(\breve{\sigma}-\breve{\sigma}_{,,,,})=(q_{,,,}r_{,,,}+q_{,,}r_{,,}+2q_{,}r_{,}+4q_{,}r-8q)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}. \qquad (410)$$

In (396) we found for the parameter of the derived integral of the third order the expression $n_{,}=\dfrac{mn}{(1+j)^2}$. On referring to the geometrical representatives of these expressions, we find for the focal distance $\epsilon_{,}$ of this derived curve the expression $n_{,}=\tan^2\epsilon_{,}=\dfrac{mn}{(1+j)^2}$; but if we turn to (389) we shall see that this is the expression for the maximum protangent to the original spherical ellipse, which is given by the equation $\tan^2\tau=\dfrac{mn}{(1+j)^2}$. We thus arrive at this curious relation between the curves successively derived, that the *maximum protangent of any one of the spherical ellipses becomes the focal distance of the one immediately succeeding in the series.*

79.] Given m, n, and i, we may determine $m_{,}$, $n_{,}$, and $i_{,}$, for $i_{,}=\dfrac{1-j}{1+j}$, $n_{,}=\dfrac{mn}{(1+j)^2}$. Substituting these values of $i_{,}$ and $n_{,}$ in the equation which connects the parameters, $m_{,}-n_{,}+m_{,}n_{,}=i_{,}^2$,

$$m_{,}=\left[\frac{\sqrt{1+n}-\sqrt{1-m}}{\sqrt{1+n}+\sqrt{1-m}}\right]^2. \quad . \quad . \quad . \quad . \quad (411)$$

Hence, given m, n, and i, we can easily compute the values of $m_{,}$, $n_{,}$, and $i_{,}$, and then of $m_{,,}$, $n_{,,}$, and $i_{,,}$, and so on as far as we please.

Given the semiaxes a and b of the elliptic cylinder whose intersection with the sphere is the original spherical ellipse, to determine the semiaxes $a_{,}$ and $b_{,}$ of the cylinder whose intersection with the sphere shall be the first derived spherical ellipse.

We may derive from (53) and (54) the values of a and b in terms of m, n, and i, or, eliminating i, in terms of m and n only; for

$$\sin^2\alpha=\frac{a^2}{k^2}=\frac{n}{m(1+n)}, \quad \sin^2\beta=\frac{b^2}{k^2}=\frac{n(1-m)}{m}.$$

Hence
$$\frac{a_{,}^2}{k^2}=\frac{n_{,}}{m_{,}(1+n_{,})}, \quad \frac{b_{,}^2}{k^2}=\frac{n_{,}(1-m_{,})}{m}.$$

Or substituting the values of $m_{\prime}$ and $n_{\prime}$ in terms of m and n, and therefore of a and b,

$$a_{\prime}=\frac{a+b}{1+\frac{ab}{k^2}},\ b_{\prime}=\frac{2\sqrt{ab}}{1+\frac{ab}{k^2}}. \quad \ldots \quad (412)$$

When the radius of the sphere is infinite, or the derived curve is a plane ellipse, $a_{\prime}=a+b$, $b_{\prime}=2\sqrt{ab}$, as in sec. [77].

When $m=n=i$, $m_{\prime}=n_{\prime}=i_{\prime}$; or when the given curve is a spherical parabola, the derived curve will also be a spherical parabola. Hence all the curves of the series will be spherical parabolas.

If we take the corresponding integral of the third order with a reciprocal parameter l, such that $lm=i^2$, and deduce by the foregoing process the first derived function of the third order, we shall find the parameter $l_{\prime}$ of this function to be positive also, and reciprocal to $n_{\prime}$, so that $l_{\prime}n_{\prime}=i_{\prime}^2$.

Hence, if we deduce a series of derived functions from two primitive functions of the third order and circular form, having either *positive* or *negative* reciprocal parameters, the parameters of all the derived functions $l_{\prime}$, $l_{\prime\prime}$, $l_{\prime\prime\prime}$, $n_{\prime}$, $n_{\prime\prime}$, $n_{\prime\prime\prime}$ will be *positive*, and reciprocal in pairs, so that $l_{\prime}n_{\prime}=i_{\prime}^2$, $l_{\prime\prime}n_{\prime\prime}=i_{\prime\prime}^2$, $l_{\prime\prime\prime}n_{\prime\prime\prime}=i_{\prime\prime\prime}^2$, &c.

80.] We may apply the same method of proceeding to the logarithmic ellipse, or to the logarithmic integral of the third order,

$$\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}},\ \text{in which } i^2>m.$$

If on this function we perform the operations effected on the similar integral in (394), we shall have, after like reductions,

$$\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{(1+i_{\prime})}{4(1-m)}\int\frac{d\psi[2-m-mi_{\prime}\sin^2\psi-m\cos\psi\sqrt{\mathrm{I}_{\prime}}]}{[1-m_{\prime}\sin^2\psi]\sqrt{\mathrm{I}_{\prime}}}. \quad (413)$$

We must recollect that

$$\left.\begin{aligned}\mathrm{M}=1-m\sin^2\phi,\ \mathrm{M}_{\prime}=1-m_{\prime}\sin^2\psi,\ \mathrm{I}=1-i^2\sin^2\phi,\\ \mathrm{I}_{\prime}=1-i_{\prime}^2\sin^2\psi,\ \text{and}\ m_{\prime}=\frac{mn}{(1+j)^2}\end{aligned}\right\}. \quad (414)$$

We may reduce this expression.

The numerator may be put under the form

$$2-m+\frac{mi_{\prime}}{m_{\prime}}\{1-m_{\prime}\sin^2\psi-1\}-m\cos\psi\sqrt{\mathrm{I}_{\prime}}.$$

Now $$2-m-\frac{mi_{\prime}}{m_{\prime}}=\frac{(n-m)}{n},\ \text{and}\ \frac{mi_{\prime}}{m_{\prime}}=\frac{i^2}{m}.$$

We have also $\frac{\sqrt{i_{\prime}}}{i}=\frac{1}{1+j}$.

Hence, making the necessary transformations,

$$2\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{\prime}}}{i}\int\frac{d\psi}{M_{\prime}\sqrt{I_{\prime}}}+\frac{i\sqrt{i_{\prime}}}{mn}\int\frac{d\psi}{\sqrt{_{\prime}I}}-\frac{\sqrt{i_{\prime}}}{i}\int\frac{\cos\psi d\psi}{M_{\prime}}.$$

If into this expression we introduce the relation given in (74),

$$\int\frac{d\phi}{\sqrt{I}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{I_{\prime}}},$$

writing ϕ for μ, and ψ for ϕ, we shall have

$$2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{\prime}}}{i}\int\frac{d\psi}{M_{\prime}\sqrt{I_{\prime}}}+\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}-\frac{\sqrt{i_{\prime}}}{i}\int\frac{\cos\psi d\psi}{M_{\prime}}. \quad (415)$$

Now in (399) it has been shown that $\sqrt{m_{\prime}}\sin\psi=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}$;

and as $\sqrt{mn}=\sqrt{m_{\prime}}(1+j)$, the last term of the preceding equation may be written

$$\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]d\phi}{1-\frac{mn\sin^2\phi\cos^2\phi}{1-i^2\sin^2\phi}}.$$

Substituting this value in the preceding equation, and comparing it with (169) or (170), we shall find

$$\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}=\frac{(n-m)}{nm}\frac{\sqrt{i_{\prime}}}{i}\int\frac{d\psi}{M_{\prime}\sqrt{I_{\prime}}}. \quad (416)$$

This equation is analogous to (401). By the help of it and the last equation we can always express

$$\int\frac{d\phi}{M\sqrt{I}} \text{ or } \int\frac{d\phi}{N\sqrt{I}} \text{ in terms of } \int\frac{d\psi}{M_{\prime}\sqrt{I_{\prime}}}.$$

Since $m_{\prime}=\frac{mn}{(1+j)^2}$ is symmetrical with respect to n and m, we should have obtained the same value for the derived parameter had it been deduced from $\int\frac{d\phi}{N\sqrt{I}}$ instead of $\int\frac{d\phi}{M\sqrt{I}}$. Since $i_{\prime}=\frac{1-j}{1+j}$

and $m_{\prime}=\frac{mn}{(1+j)^2}$, $n_{\prime}=\frac{(1-j)^2-mn}{(1+j)^2-mn}=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2$. (417)

81.] We may express $m_{\prime}$ and $n_{\prime}$ simply, in terms of a and b, the semiaxes of the base of the elliptic cylinder, whose curve of section with the paraboloid is the logarithmic ellipse.

In (171) we have found the values of m and n in terms of a, b, and k, namely,

$$\frac{a}{k}=\frac{\sqrt{mn(1-m)}}{n-m},\quad \frac{b}{k}=\frac{\sqrt{mn(1-n)}}{n-m} \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Hence $\frac{a-b}{a+b}=\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}$, or, assuming the value of $n_{\prime}$ in (417), $n_{\prime}=\left(\frac{a-b}{a+b}\right)^2$. Now

$$n-m=(1-m)-(1-n)=(\sqrt{1-m}+\sqrt{1-n})(\sqrt{1-m}-\sqrt{1-n}).$$

Or as $\quad m_{\prime}=\frac{mn}{(1+j)^2},\ 1-m_{\prime}=\frac{(1+j)^2-mn}{(1+j)^2}=\frac{(\sqrt{1-m}+\sqrt{1-n})^2}{(1+j)^2},$

and (a) gives
$$\frac{a-b}{k}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}},$$

therefore
$$\frac{1-m_{\prime}}{m_{\prime}}=\frac{(\sqrt{1-m}-\sqrt{1-n})^2}{mn}.$$

Hence, reducing,
$$m_{\prime}=\frac{(a-b)^2}{k^2+(a-b)^2}.$$

If we now compare together these expressions for $m_{\prime}$ and $n_{\prime}$, namely,

$$m_{\prime}=\frac{(a-b)^2}{k^2+(a-b)^2},\quad n_{\prime}=\left(\frac{a-b}{a+b}\right)^2, \quad . \quad . \quad . \quad (418)$$

we shall find that $n_{\prime}>m_{\prime}$, so long as $k>2\sqrt{ab}$, that when $k=2\sqrt{ab}$, $n_{\prime}=m_{\prime}$, and that when $k<2\sqrt{ab}$, $n_{\prime}<m_{\prime}$.

To determine the axes of the base of the cylinder whose intersection with the paraboloid gives the derived logarithmic ellipse.

Since $\frac{a_{\prime}^2}{k^2}=\frac{m_{\prime}n_{\prime}(1-m_{\prime})}{(n_{\prime}-m_{\prime})^2}$, $\frac{b_{\prime}^2}{k^2}=\frac{m_{\prime}n_{\prime}(1-n_{\prime})}{(n_{\prime}-m_{\prime})^2}$, as we may infer from (171), we shall have, substituting the preceding values of $m_{\prime}$ and $n_{\prime}$,

$$\left.\begin{aligned} &\frac{a_{\prime}^2}{k^2}=\frac{(a+b)^2k^2}{[k^2-4ab]^2},\quad \frac{b_{\prime}^2}{k^2}=\frac{4ab[k^2+(a-b)^2]}{[k^2-4ab]^2},\\ &\qquad\text{and }\ i_{\prime}^2=\left(\frac{a-b}{a+b}\right)^2\left[\frac{k^2+(a+b)^2}{k^2+(a-b)^2}\right]. \end{aligned}\right\} \quad . \quad . \quad (419)$$

When $k=\infty$, or when the paraboloid is a plane, $a_{\prime}=(a+b)$, $b_{\prime}=2\sqrt{ab}$, which are the values of the semiaxes of a plane ellipse whose eccentricity is $\dfrac{a-b}{a+b}=\dfrac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, as we should have anticipated; for these are the values found in sec. [77] and sec. [79] for the axes of the derived plane ellipse.

When $\quad m=n=1-j,\ m_{\prime}=\dfrac{mn}{(1+j)^2}=\left(\dfrac{1-j}{1+j}\right)^2=i_{\prime}^2$, and $n_{\prime}=0$.

Hence, when the original logarithmic ellipse is of the circular form, the first derived ellipse is a plane ellipse.

When $k^2=4ab$, (418) shows that $m_{\prime}=n_{\prime}$, or $\dfrac{a_{\prime}}{k}=\dfrac{b_{\prime}}{k}=\infty$, as in sec. [43]; but $m_{\prime}=n_{\prime}$, is equivalent to $n=m(\sqrt{1+j}+\sqrt{j})^2$.

Whenever therefore this relation exists between the parameters and modulus of the original integral, the first derived integral will represent the circular logarithmic ellipse, which may be integrated by functions of the first and second orders. Accordingly whenever the above relation exists between the parameters, the integral of the third order may be reduced to others of the first and second orders.

If in the second, third, or any other of the derived logarithmic ellipses we can make the parameters equal, this derived ellipse will be of the circular form, and its rectification may be effected by integrals of the first and second orders only; accordingly the rectification of all the logarithmic ellipses which precede it in the scale may be effected by integrals of the first and second orders only.

We may repeat the remark made in sec. [79]. The derived functions of two integrals of the logarithmic form with reciprocal parameters, have themselves reciprocal parameters.

82.] If we now add together (162) and (163), we shall have

$$\left.\begin{aligned}\frac{4(n-m)}{\sqrt{mn}}\frac{\Sigma}{k}&=-[n\Phi_n+m\Phi_m]+\left[\frac{i^2}{m}+\frac{i^2}{n}-2\right]\int\frac{d\phi}{\sqrt{\mathrm{I}}}\\&\quad-(n-m)\left[\frac{(1-m)}{m}\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]\\&\quad+2\int d\phi\sqrt{\mathrm{I}}-2\frac{(n-m)}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \tag{420}$$

We must now reduce this equation into functions of ψ instead of ϕ, ψ and ϕ being connected, as before, by the fundamental equation

$$\tan(\psi-\phi)=j\tan\phi.$$

The elements of these transformations are given in page 69, namely

$$2\sin^2\phi = 1 + i_{\prime}\sin^2\psi - \cos\psi\,\sqrt{\mathrm{I}_{\prime}}, \text{ and } \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}} = \sqrt{n_{\prime}}\sin\psi.$$

From this last equation we may derive

$$(1-n\sin^2\phi)(1-m\sin^2\phi) = \mathrm{I}(1-m_{\prime}\sin^2\psi).$$

Now, as $\Phi_n = \dfrac{\sin\phi\cos\phi\,\sqrt{\mathrm{I}}}{1-n\sin^2\phi}$, we shall have

$$n\Phi_n = \frac{\sqrt{m_{\prime}}\sin\psi}{2\sqrt{mn}}\left[\frac{2n-2nm\sin^2\phi}{1-m_{\prime}\sin^2\psi}\right], \quad . \quad . \quad . \quad (421)$$

or, putting for $\sin^2\phi$ its value,

$$n\Phi_n = \frac{\sqrt{m_{\prime}}\sin\psi}{2\sqrt{mn}}\,\frac{[2n - nm - nmi_{\prime}\sin^2\psi + mn\cos\psi\,\sqrt{\mathrm{I}_{\prime}}]}{[1-m_{\prime}\sin^2\psi]}. \quad (422)$$

In the same manner, we may find

$$m\Phi_m = \frac{\sqrt{m_{\prime}}\sin\psi}{2\sqrt{mn}}\,\frac{[2m - mn - mni_{\prime}\sin^2\psi + mn\cos\psi\,\sqrt{\mathrm{I}_{\prime}}]}{[1-m_{\prime}\sin^2\psi]}. \quad (423)$$

Adding these equations together, and recollecting that $m+n-mn=i^2$, we shall get

$$n\Phi_n + m\Phi_m = \frac{\sqrt{m_{\prime}}i^2\sin\psi}{\sqrt{mn}} + \frac{\sqrt{m_{\prime}}\,\sqrt{mn}\cos\psi\sin\psi\,\sqrt{\mathrm{I}_{\prime}}}{[1-m_{\prime}\sin^2\psi]}. \quad (424)$$

Now, as

$$i^2 = (1+j)(1-j), \text{ and } \sqrt{mn} = \sqrt{m_{\prime}}(1+j),$$

$$\{n\Phi_n + m\Phi_m\} = (1-j)\sin\psi + (1+j)\frac{m_{\prime}\sin\psi\cos\psi\,\sqrt{\mathrm{I}_{\prime}}}{(1-m_{\prime}\sin^2\psi)}. \quad (425)$$

In (186) we found

$$2\int d\phi\,\sqrt{\mathrm{I}} = (1+j)\int d\psi\,\sqrt{\mathrm{I}_{\prime}} - \frac{2j}{1+j}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}} + (1-j)\sin\psi. \quad (426)$$

Subtracting this expression from the preceding, the terms involving $\sin\psi$ will disappear.

We must now compute the sum of the coefficients of $\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}$ in (420) and (426). Since

$\displaystyle\int\frac{d\phi}{\sqrt{\mathrm{I}}} = \frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}$, this coefficient becomes $\displaystyle\frac{(1+i_{\prime})}{2}\left[\frac{i^2}{m} + \frac{i^2}{n} - 2(1+j)\right]$.

Or as $m+n=i^2+mn$, this coefficient may be written

$$\left[\frac{i^4}{mn}+i^2-2(1+j)\right]\frac{(1+i_{\prime})}{2}.$$

Or as $mn=m_{\prime}(1+j)^2$, it becomes finally, $\dfrac{2}{1+i_{\prime}}\left[\dfrac{i_{\prime}^2}{m_{\prime}}-1\right]$. . (427)

$$\text{Hence } \left[\frac{i^2}{m}+\frac{i^2}{n}-2(1+j)\right]\left(\frac{1+i_{\prime}}{2}\right)\int\frac{d\psi}{\sqrt{I_{\prime}}}=\frac{2}{1+i_{\prime}}\left[\frac{i_{\prime}^2}{m_{\prime}}-1\right]\int\frac{d\psi}{\sqrt{I_{\prime}}}; \quad (428)$$

$$\left.\begin{aligned}\text{and}\quad &(n-m)\left[\left(\frac{1-m}{m}\right)\int\frac{d\varphi}{M\sqrt{I}}-\frac{(1-n)}{n}\int\frac{d\varphi}{N\sqrt{I}}\right]\\ &=\frac{(n-m)}{\sqrt{mn}}\,\frac{(n-m)}{\sqrt{mn}}\,\frac{1}{(1+j)}\int\frac{d\psi}{[1-m_{\prime}\sin^2\psi]\sqrt{I_{\prime}}}\end{aligned}\right\}. \quad (429)$$

Now, as $\quad n+m=i^2+mn,\ (n+m)^2=i^4+2mni^2+m^2n^2.$

Hence $\quad (n-m)^2=i^4+2mni^2+m^2n^2-4mn\,;$

and as $\quad i^4=(1+j)^2(1-j)^2,\ mn=m_{\prime}(1+j)^2$, substituting

$(n-m)^2=(1+j)^2(1-j)^2+2m_{\prime}(1+j)^3(1-j)+m_{\prime}^2(1+j)^4-4m_{\prime}(1+j)^2,$

therefore $(n-m)^2=(1+j)^4\left[\left(\dfrac{1-j}{1+j}\right)^2+2m_{\prime}\left(\dfrac{1-j}{1+j}\right)+m_{\prime}^2-\dfrac{4m_{\prime}}{(1+j)^2}\right];$

and as $\dfrac{4}{(1+j)^2}=(1+i_{\prime})^2$, the expression will finally become

$n-m=(1+j)^2(1-m_{\prime})\sqrt{n_{\prime}}$; hence $\dfrac{n-m}{\sqrt{mn}}\,\dfrac{i_{\prime}}{\sqrt{i}}=\left(\dfrac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}$. (430)

If we now add together (420), (425), (426), (428), and (429), we shall have, dividing by $\dfrac{(n-m)}{\sqrt{mn}}$, putting $\Psi_{m_{\prime}}$ for $\dfrac{\sin\psi\cos\psi\sqrt{I_{\prime}}}{(1-m_{\prime}\sin^2\psi)}$,

$$\left.\begin{aligned}\frac{4\Sigma}{k}=&-\frac{m_{\prime}\sqrt{m_{\prime}}}{(1-m_{\prime})\sqrt{n_{\prime}}}\Psi_{m_{\prime}}+\frac{\sqrt{m_{\prime}}}{(1-m_{\prime})\sqrt{n_{\prime}}}\int d\psi\sqrt{I_{\prime}}\\ &-\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{M_{\prime}\sqrt{I_{\prime}}}+\sqrt{\frac{n_{\prime}}{m_{\prime}}}\int\frac{d\psi}{\sqrt{I_{\prime}}}-2\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad (431)$$

Let us now take the logarithmic ellipse whose equation contains $m_{\prime}, n_{\prime}, i_{\prime}, \psi$ instead of m, n, i, and φ, we shall have from (163),

$$\left.\begin{aligned}\frac{2\Sigma'}{k'}=&-\frac{m_{\prime}\sqrt{m_{\prime}n_{\prime}}}{n_{\prime}-m_{\prime}}\Psi_{m_{\prime}}-\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}\\ &+\frac{\sqrt{m_{\prime}n_{\prime}}}{n_{\prime}-m_{\prime}}\int d\psi\sqrt{I_{\prime}}+\frac{n_{\prime}}{m_{\prime}}\left(\frac{1-m_{\prime}}{n_{\prime}-m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{\sqrt{I_{\prime}}}-2\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}\end{aligned}\right\}. \quad (432)$$

If we now subtract these equations one from the other, combining together like integrals, the integral of the third order will vanish and we shall have

$$\left.\begin{aligned}\frac{2\Sigma_{\prime}}{k}-\frac{4\Sigma}{k}=\frac{m_{\prime}(1-n_{\prime})\ \sqrt{m_{\prime}n_{\prime}}}{n_{\prime}(n_{\prime}-m_{\prime})(1-m_{\prime})}\left[\int d\psi\ \sqrt{\mathrm{I}_{\prime}}+\frac{n_{\prime}}{m_{\prime}}(1-m_{\prime})\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}-m_{\prime}\Psi_{m_{\prime}}\right]\\ +2\int\frac{d\tau}{\cos^3\tau}-2\int\frac{d\tau_{\prime}}{\cos^3\tau_{\prime}}\end{aligned}\right\}.(433)$$

Hence, as we may express an arc of a plane ellipse by an arc of a derived ellipse, an integral of the first order, and a straight line—a known theorem—so we may extend this analogy and express an arc of a logarithmic ellipse by an arc of a derived logarithmic ellipse, by functions of the first and second orders, by an arc of a parabola and by a straight line. The relations between the moduli and amplitudes are the same in both cases,

$$i_{\prime}=\frac{1-j}{1+j},\ \text{and}\ \tan(\psi-\phi)=j\tan\phi.$$

Let $m_{\prime\prime}$, $n_{\prime\prime}$, $i_{\prime\prime}$, $\psi_{\prime}$ be derived from $m_{\prime}$, n , $i_{\prime}$, ψ, by the same law as these latter are derived from m, n, i, ϕ, namely,

$$i_{\prime}=\frac{1-j}{1+j},\ \tan(\psi-\phi)=j\tan\phi,\ m=\frac{mn}{(1+j)^2},$$

$$n_{\prime}=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2,$$

and derive an arc of a third logarithmic ellipse, we shall have, putting A, B, C, D for the coefficients of the integrals, and Π for the parabolic arc,

$$\frac{2\Sigma_{\prime}}{k}-\frac{4\Sigma}{k}=\mathrm{A}\int d\psi\ \sqrt{\mathrm{I}_{\prime}}+\mathrm{B}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}-\mathrm{C}\Psi+\mathrm{D}\Pi,$$

$$\frac{2\Sigma_{\prime\prime}}{k}-\frac{4\Sigma_{\prime}}{k}=\mathrm{A}_{\prime}\int d\psi_{\prime}\ \sqrt{\mathrm{I}_{\prime\prime}}+\mathrm{B}_{\prime}\int\frac{d\psi_{\prime\prime}}{\sqrt{\mathrm{I}_{\prime\prime}}}-\mathrm{C}_{\prime}\Psi_{\prime}+\mathrm{D}_{\prime}\Pi_{\prime}.$$

Multiply the first of these equations by 2 and add them, $\Sigma_{\prime}$ will be eliminated. In this way we may successively eliminate $\Sigma_{\prime}$, $\Sigma_{\prime\prime}$, $\Sigma_{\prime\prime\prime}$, until ultimately we shall have

$$\frac{2\Sigma_{\nu}}{k}-2^{\nu+1}\frac{\Sigma}{k}=\nu\mathrm{E}+\nu\mathrm{F}+\nu\overline{\Psi}-\nu\overline{\Pi},$$

ν being the number of operations, and denoting by F and E, the sum of the integrals of the first and second orders, by $\overline{\Psi}$ the sum of the straight lines, and by $\overline{\Pi}$ the sum of the parabolic arcs.

If in (401) and (416) we substitute the coefficients of the derived integrals as transformed in (404) and (430), the relation between the original and the derived integrals of the third order will be, for the circular form or the spherical ellipse,

$$\left.\begin{aligned}&\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\\&\quad+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\\&\quad=\left(\frac{1+n_{\prime}}{n_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{(1+n_{\prime}\sin^2\psi)\sqrt{1-i_{\prime}^2\sin^2\psi}}\end{aligned}\right\}, \tag{434}$$

and for the logarithmic form or logarithmic ellipse,

$$\left.\begin{aligned}&\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\\&\quad-\left(\frac{1-n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1-n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\\&\quad=\left(\frac{1-m_{\prime}}{m_{\prime}}\right)\sqrt{m_{\prime}n_{\prime}}\int\frac{d\psi}{(1-m_{\prime}\sin^2\psi)\sqrt{1-{}_{\prime}i^2\sin^2\psi}}\end{aligned}\right\}, \tag{435}$$

83.] The preceding investigations lead us to consider a new classification of elliptic integrals, which, in a geometrical point of view, would seem to be more natural than the one at present in use. As the first order is merely a particular case of the circular form of the third, its geometrical type (the spherical parabola) being a particular species of spherical conic, while the two forms which are classed under the third order are irreducible one to the other, representing, as they do, curves of different species, it would seem a more appropriate division to found their classification on their geometrical types, the *plane*, the *logarithmic*, and the *spherical* ellipses, which those integrals represent. Thus that which is now the second would stand the first, the logarithmic form of the third order would hold the second place, while the circular form of the third order, of which the present first order is a particular case, would occupy the third rank. However, as the present division has been sanctioned by time, and by the great names of the founders of this department of mathematical science, Legendre, Jacobi, Abel, and others, it would be presumptuous to propose to change it. Besides, in a point of view purely analytical (the view of the inventors) the present division of these integrals may be held to be the most appropriate; for example, it naturally presents itself in the computation of tables of the numerical values of those integrals.

Hitherto we have considered the elliptic integral or its equivalent, the arc of the hyperconic section σ, as a function of its amplitude ϕ, or assumed as it were, the amplitude ϕ as the independent variable. But we may reverse this course and consider the amplitude as a function of the arc σ of the hyperconic section. A notation has been devised by which the amplitude ϕ may be expressed as a function of the integral or its equivalent σ. When the modulus of the elliptic integral is 0, the integral becomes $\int_0^x \frac{dx}{\sqrt{1-x^2}}$ or $\sin^{-1}x$. Now this is a function very little used as compared with $\sin x$; so that $\sin x$ is always considered the *direct* function, and $\sin^{-1}x$ or the arc the *inverse* function. The reason of this is, as I have elsewhere shown, that our acquaintance with circular functions is not derived from the integral calculus, while our knowledge of the properties of the arcs of hyperconic sections can in no other way be obtained. It will render our language more precise, if we apply the term elliptic integral to those expressions in which the amplitude is the independent variable, and elliptic functions to these expressions in which the arc is the independent variable.

In this way, writing $\sin\phi = \sin \text{amp.}\ \sigma$, we might develop a great system of trigonometry for the hyperconic sections. In this general system when the modulus $i=0$, we pass into circular trigonometry, and when the modulus $i=1$, we may develop an equally extensive system of parabolic trigonometry as given in the first volume of this work, p. 313. In truth that essay ought to have been incorporated in this treatise, in which passing over elliptic functions, we confine our researches to the geometrical properties of elliptic integrals. To enter on the wide field of elliptic functions, or as it may be called the trigonometry of the hyperconic sections, would lead us very far beyond the limits we have prescribed to ourselves; and it has, moreover, been amply treated by Legendre, Jacobi, Abel, and other great continental mathematicians.

There are several plane curves whose lengths we may express by elliptic integrals of the third order. For example, the length of the elliptic lemniscate, or the locus of the intersections of central perpendiculars on tangents to an ellipse, is equal to that of a spherical ellipse which is supplemental to itself, or the sum of whose principal arcs is equal to π, as shown in vol. i. p. 196. We cannot represent elliptic integrals of the third order generally by the arcs of curves whose equations in their simplest forms contain only two constants. Thus let a and b be the constants. We shall have two equations between the constants, the parameter, and the modulus of the function, $i = \mathrm{f}(a, b)$, $n = \mathrm{f}_{\prime}(a, b)$. Assume a as invariable, and eliminate b, we shall have one resulting equation between i, n, and a, or $F(a, i, n)=0$; or n depends on i.

When there are three independent constants, as in the preceding investigations, a, b, and k, we shall have $i=f(a, b, k)$, $n=f'(a, b, k)$. Eliminating successively b and k, we shall have two resulting equations, instead of one, $F(a, k, i, n)=0$, and $F'(a, b, i, n)=0$; or i and n depend on two equations, and may therefore be independent.

The general fundamental expressions for the rectification of curve lines, whether of single or double flexion, show that the arc of a curve may in general be represented as the sum of two quantities, an integrated and a non-integrated part; or, as the proposition may be more briefly put, an arc of a curve may be expressed as the sum of an integral and a residual. Thus the arc of a plane ellipse is equal to an integral and a residual, which latter is a straight line. An arc of a parabola is the sum of an integral and a residual, which latter is also a straight line. An arc of a spherical ellipse is the sum of an integral and a residual, the latter being an arc of a circle, while an arc of a logarithmic ellipse is made up of two portions, one a sum of integrals, the other (the residual) being an arc of a common parabola. It appears therefore to be an expenditure of skill in a wrong direction to devise curves whose arcs should differ from the corresponding arcs of hyperconic sections by the above-named residuals. Thus geometers have sought to discover plane curves whose arcs should be represented by elliptic integrals of the first order, without any residual quantity—the common lemniscate for example, when the modulus has a particular value. It is possible that such may be found. In the same way, an exponential curve may be devised whose arc shall be represented by the integral $k\int\frac{d\theta}{\cos\theta}$, instead of taking it with the residual quantity $k\tan\theta\sec\theta$ as the expression for an arc of a common parabola. Thus geometers have been led to look for the types of elliptic integrals among the higher orders of plane curves, overlooking the analogy which points to the intersection of surfaces of the second order as the natural geometrical types of those integrals.

It has thus been shown that the curves of intersection of *concentric* surfaces of the second order may in all cases be rectified by elliptic integrals. When the intersecting surfaces are not concentric, the rectification of the curve of intersection may be reduced to the integration of an expression which may be called an hyperelliptic integral.

The general expression for the length of an arc of this curve will be an integral of the form

$$s=\int dx\sqrt{\frac{\alpha x^4+\beta x^3+\gamma x^2+\delta x+\epsilon}{ax^4+bx^3+cx^2+ex+f}}.$$

When the surfaces are symmetrically placed and have a common plane of contact, the above expression may be reduced to

$$s=\int \mathrm{d}x \sqrt{\frac{\alpha x^3+\beta x^2+\gamma x+\delta}{a x^3+b x^2+c x+e}}.$$

This form may be reduced to an elliptic integral.

When, moreover, the surfaces are concentric and symmetrically placed, the preceding expression may still further be simplified to

$$s=\int \mathrm{d}x \sqrt{\frac{\alpha x^2+\beta x+\gamma}{a x^2+b x+c}},$$

which is the general form for elliptic integrals.

We can perceive therefore that the solution of the general problem, to determine the length of the curve in which two surfaces of the second order may intersect, investigated under its most general form, far transcends the present powers of analysis. It is only when one of the surfaces becomes a plane, or when they are concentric and symmetrically placed, that the problem under these restricted conditions admits of a complete solution.

We may hence also surmise how vast are the discoveries which still remain to be explored in the wide regions of the integral calculus. We see how questions which arise from the investigation of problems based on the most elementary geometrical forms (surfaces of the second order) baffle the utmost powers of a refined analysis, with all the aids of modern improvements. It is not a little curious, that nearly all the branches of modern analysis, such as plane and spherical trigonometry, the doctrine of logarithms and exponentials, with the theory of elliptic integrals, may all be derived from the investigation of one geometrical problem—to determine the length of an arc of the intersecting curve of two surfaces of the second order.

In the logarithmic hyperconic sections, we may develop properties analogous to those found in the spherical and plane sections, if we substitute parabolic arcs for arcs of great circles in the one, and for straight lines in the other. Here follow a few of those theorems.

1. From any point on a parabolic section of the paraboloid let two parabolas be drawn touching the logarithmic ellipse or the logarithmic hyperbola, the parabolic arcs joining the points of contact will all pass through one point on the surface of the paraboloid.

2. If a hexagon, whose sides are parabolic arcs, be inscribed in a logarithmic ellipse or logarithmic hyperbola, the opposite parabolic arcs will meet two by two on a parabola.

3. If a hexagon, whose sides are parabolas, be circumscribed to

a logarithmic ellipse, the parabolic arcs joining the opposite vertices will pass through a fixed point on the surface of the paraboloid.

4. If through the centre of a logarithmic ellipse or logarithmic hyperbola two parabolic arcs are drawn at right angles to each other, meeting the curve in two points, and parabolic arcs be drawn touching the curve in these points, they will meet on another logarithmic ellipse or logarithmic hyperbola.

5. If a circle whose radius is a be described on the surface of the paraboloid, and therefore touching the logarithmic ellipse or the logarithmic hyperbola at the extremities of its major axis, and from the extremities of any diameter two parabolic arcs be drawn to any third point on the circle, if one of these parabolic arcs touches the logarithmic ellipse or the logarithmic hyperbola, the other will pass through a fixed point on the surface of the paraboloid.

6. If on the paraboloid we describe a circle whose radius is $\sqrt{a^2+b^2}$, and if from the extremities of any diameter of this circle we draw parabolic arcs touching the logarithmic ellipse or the logarithmic hyperbola, these tangent parabolic arcs will meet on the circle.

These theorems will suffice. There would be little difficulty in extending the list. In fact nearly all the projective properties of straight lines and conic sections on a plane may be transformed into analogous properties of great circles and spherical conic sections on the surface of a sphere, and of parabolic arcs and logarithmic sections on the surface of a paraboloid.

CHAPTER XI.

ON THE QUADRATURE OF THE LOGARITHMIC ELLIPSE AND OF THE LOGARITHMIC HYPERBOLA.

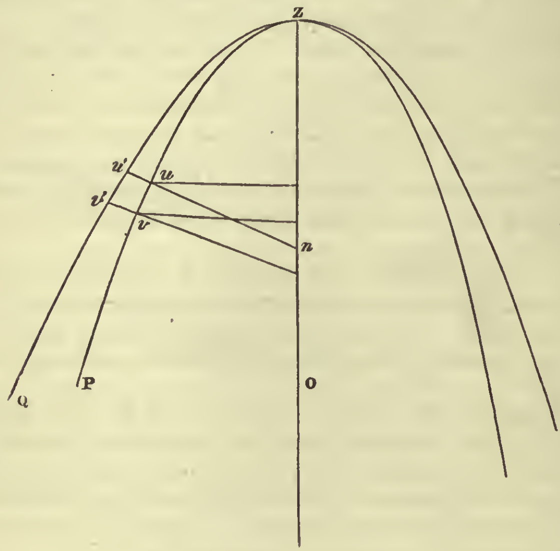
84.] The properties of the Logarithmic Ellipse and the Logarithmic Hyperbola have the same analogy to the paraboloid of revolution that spherical conics have to a sphere, or which conic sections bear to a plane. To determine the areas of these curves, or rather the portions of surface of the paraboloid bounded by them, is a problem not undeserving of investigation.

The *Logarithmic ellipse* has been defined in Chapter IV. as the curve of intersection of a paraboloid of revolution with an elliptic cylinder whose axis coincides with that of the paraboloid.

The *Logarithmic hyperbola*, in like manner, has been defined in Chapter V. as the curve of intersection of a paraboloid of revolution with a cylinder whose base is an hyperbola, and whose axis coincides with that of the paraboloid.

Through the vertex Z of the paraboloid let two parabolas be drawn indefinitely near to each other, ZP, ZQ, and let two planes

Fig. 27.



indefinitely near to each other at right angles to the axis OZ cut the parabolas in the points u, u', v, v'.

The little trapezoid $uvu'v'$ is the element of the surface; and if the normal un makes the angle μ with the axis OZ, $d\psi$ being the elementary angle between the planes, $uu' = k \tan \mu d\psi$, k being the semiparameter of the generating parabola.

Now $uv = ds = k \dfrac{d\mu}{\cos^3 \mu}$. Hence the elementary trapezoid $uvu'v' = \dfrac{k^2 \sin \mu d\mu d\psi}{\cos^4 \mu}$.

Integrating this expression, $\text{area} = k^2 \int d\psi \int \dfrac{\sin \mu}{\cos^4 \mu} d\mu$; . (436)

or performing the integration with respect to μ,

$$\text{area} = \frac{k^2}{3} \int d\psi \sec^3 \mu + \text{constant}.$$

Now when the area is 0, $\sec\mu=1$, and therefore

$$\text{constant}=-\frac{k^2}{3}\int d\psi. \quad \text{Whence}$$

$$\text{area}=\frac{k^2}{3}\int d\psi(\sec^3\mu-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (437)$$

This is the general expression for the surface of a paraboloid between two principal planes, and bounded by a curve.

When this curve is the logarithmic ellipse, let the area be put [LE].

We must now express ψ and μ as functions of another variable, θ.

Let $x=a\cos\theta$, $y=b\sin\theta$, the base of the cylinder being the ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. ψ is the angle which $\sqrt{x^2+y^2}$ makes with the axis a.

Now $$\tan\psi=\frac{y}{x}=\frac{b}{a}\tan\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (438)$$

and $$d\psi=\frac{ab\,d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (439)$$

But $$\tan^2\mu=\frac{r^2}{k^2}=\frac{a^2\cos^2\theta+b^2\sin^2\theta}{k^2};$$

therefore $$\sec^2\mu=\frac{(k^2+a^2)\cos^2\theta+(k^2+b^2)\sin^2\theta}{k^2}. \quad . \quad (440)$$

Hence substituting these values in (437), we get for the area

$$[LE]=\left.\begin{aligned}&\frac{k^2}{3}\frac{ab}{k^3}\int\frac{d\theta[(k^2+a^2)\cos^2\theta+(k^2+b^2)\sin^2\theta]^{\frac{3}{2}}}{[a^2\cos^2\theta+b^2\sin^2\theta]}\\&\qquad-\frac{k^2}{3}ab\int\frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\end{aligned}\right\}. \quad (441)$$

Let $$\frac{a^2-b^2}{a^2+k^2}=i^2, \quad \frac{a^2-b^2}{a^2}=e^2, \quad . \quad . \quad . \quad . \quad . \quad (442)$$

i being the modulus and e^2 the parameter, as in (15).

The above expression may be written

$$\left.\begin{aligned}3\mathrm{d}\,[\mathrm{LE}] &= \frac{ab}{k}\left[\frac{k^4\mathrm{d}\theta}{(a^2\cos^2\theta+b^2\sin\theta)\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}\right]\\ &+\frac{2ab}{k}\,\frac{k^2\mathrm{d}\theta}{\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}\\ &+\frac{ab}{k}\,\frac{(a^2\cos^2\theta+b^2\sin^2\theta)}{\sqrt{(k^2+a^2)-(a^2-b^2)\sin^2\theta}}-k^2\,\frac{\mathrm{d}\left(\frac{b}{a}\tan\theta\right)}{1+\left(\frac{b^2}{a}\tan\theta\right)^2}\end{aligned}\right\}. \quad (443)$$

Therefore, integrating the preceding expression,

$$\left.\begin{aligned}3[\mathrm{LE}] &= \frac{bk^3}{a\sqrt{k^2+a^2}}\int\frac{\mathrm{d}\theta}{[1-e^2\sin^2\theta]\sqrt{1-i^2\sin^2\theta}}\\ &-\frac{2abk}{\sqrt{k^2+a^2}}\int\frac{\mathrm{d}\theta}{\sqrt{1-i^2\sin^2\theta}}\\ &+\frac{ab}{k}\sqrt{a^2+k^2}\int\mathrm{d}\theta\,\sqrt{1-i^2\sin^2\theta}-k^2\tan^{-1}\left(\frac{b}{a}\tan\theta\right)\end{aligned}\right\}. \quad (444)$$

Hence the area of the logarithmic ellipse, or rather the area of the paraboloid bounded by the logarithmic ellipse, may be expressed as a sum of elliptic integrals of the first, second, and third orders, with a circular arc.

Since $\dfrac{a^2-b^2}{a^2} > \dfrac{a^2-b^2}{a^2+k^2}$, $e^2 > i^2$, or the function of the third order is of the circular form. Assume a spherical conic section such that

$$\tan\alpha=\frac{a}{k},\quad \tan\beta=\frac{b}{k},\quad i^2=\frac{a^2-b^2}{a^2+k^2},$$

therefore $\dfrac{\tan\beta}{\tan\alpha}\cos\alpha=\dfrac{bk}{a\sqrt{a^2+k^2}}$, $\sin^2\epsilon=\dfrac{a^2-b^2}{a^2+k^2}$, $e^2=\dfrac{a^2-b^2}{a^2}$.

Combining the first and last terms of the preceding equation, they become

$$-k^2\left[\tan^{-1}\left(\frac{b}{a}\tan\theta\right)-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{\mathrm{d}\theta}{[1-e^2\sin^2\theta]\sqrt{1-\sin^2\epsilon\sin^2\theta}}\right].$$

Now this is the expression for the surface of a segment of a spherical ellipse whose principal angles are 2α and 2β, as shown in sec. [8]. Let this be S.

In the next place, $k\sqrt{a^2+k^2}\int d\theta\sqrt{1-i^2\sin^2\theta}$

is a portion of the elliptic cylinder whose altitude is k, and the semiaxes of whose base are $\sqrt{a^2+k^2}$ and $\sqrt{b^2+k^2}$. Let this be C,

and
$$\frac{abk}{\sqrt{a^2+k^2}}\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}$$

is an expression for an arc of the spherical parabola whose focal distance is one half the focal distance of the former. Let this be denoted by P.

Hence, if we denote the entire surface round Z by [LE],

$$3[\mathrm{LE}]=4h\mathrm{C}+\frac{8abk}{\sqrt{b^2+k^2}}\mathrm{P}-4k^2\,\mathrm{S}. \quad . \quad . \quad (445)$$

Or the area of the logarithmic ellipse may be expressed as a sum of the arcs of a plane ellipse, of a spherical ellipse, and of a spherical parabola, multiplied by constant linear coefficients.

85.] To find the area of the logarithmic hyperbola.

The general expression for the area, as in (437), is $\frac{k^2}{3}\int(\sec^3\mu-1)d\psi$.

Now, the equation of the base of the hyperbolic cylinder being $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, let $\quad x=a\sec\theta,\ y=b\tan\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (446)$

then
$$\tan\psi=\frac{y}{x}=\frac{b}{a}\sin\theta,$$

and
$$\frac{d\psi}{\cos^2\psi}=\frac{b}{a}\cos\theta d\theta,\ \cos^2\psi=\frac{a^2}{a^2+b^2\sin^2\theta};$$

hence
$$d\psi=\frac{ab\cos\theta d\theta}{a^2+b^2\sin^2\theta}.$$

Since
$$\tan^2\mu=\frac{r^2}{k^2}=\frac{a^2+b^2\sin^2\theta}{k^2\cos^2\theta},$$

$$\sec^2\mu=\frac{a^2+k^2\cos^2\theta+b^2\sin^2\theta}{k^2\cos^2\theta};$$

$$\therefore\ \sec^3\mu=\frac{[k^2\cos^2\theta+a^2+b^2\sin^2\theta]^{\frac{3}{2}}}{k^3\cos^3\theta}.$$

Let [LY] denote the area of the logarithmic hyperbola, then

$$3[\mathrm{LY}]=k^2\int\frac{[k^2\cos^2\theta+a^2+b^2\sin^2\theta]^{\frac{3}{2}}\,ab\cos\theta d\theta}{k^3\cos^3\theta[a^2+b^2\sin^2\theta]}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right). \quad (447)$$

Let V be put for $k^2\cos^2\theta+a^2+b^2\sin^2\theta$, ... (448)

and the last equation will become

$$3[\mathrm{LY}]=\int\frac{abk^2\cos^2\theta\,\mathrm{d}\theta}{[a^2+b^2\sin^2\theta]\sqrt{\mathrm{V}}}+\int\frac{2abk\,\mathrm{d}\theta}{\sqrt{\mathrm{V}}}$$
$$+\frac{ab}{k}\int\frac{[a^2+b^2\sin^2\theta]}{\cos^2\theta\sqrt{\mathrm{V}}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right);$$

and this may be written in the form

$$\left.\begin{aligned}3[\mathrm{LY}]=\frac{ak^3(a^2+b^2)}{b}\int\frac{\mathrm{d}\theta}{(a^2+b^2\sin^2\theta)\sqrt{\mathrm{V}}}\\+\left[2abk-\frac{ak^3}{b}-\frac{ab^3}{k}\right]\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{V}}}\\+\frac{ab}{k}(a^2+b^2)\int\frac{\mathrm{d}\theta}{\cos^2\theta\sqrt{\mathrm{V}}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right)\end{aligned}\right\}. \qquad (449)$$

Let $\frac{b^2}{a^2}=\tan^2\epsilon=n,\ \frac{k^2-b^2}{k^2+a^2}=i^2,$

and the preceding equation may be written

$$\left.\begin{aligned}3[\mathrm{LY}]=\frac{(k^3(a^2+b^2)}{ab\sqrt{a^2+k^2}}\int\frac{\mathrm{d}\theta}{[1+n\sin^2\theta]\sqrt{1-i^2\sin^2\theta}}\\+\frac{ab(a^2+b^2)}{k\sqrt{a^2+k^2}}\int\frac{\mathrm{d}\theta}{\cos^2\theta\sqrt{1-i^2\sin^2\theta}}\\-\frac{a}{bk}\frac{(k^2-b^2)^2}{\sqrt{a^2+k^2}}\int\frac{\mathrm{d}\theta}{\sqrt{1-i^2\sin^2\theta}}-k^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right)\end{aligned}\right\}. \qquad (450)$$

Since $n=\frac{b^2}{a^2},\ \frac{1+n}{n}=\frac{a^2+b^2}{b^2},$

and as $(1-m)(1+n)=1-i^2,\ m=\frac{k^2}{a^2+k^2}$, and (47) gives

$$\left(\frac{1+n}{n}\right)\int\frac{\mathrm{d}\theta}{\mathrm{N}\sqrt{\mathrm{I}}}-\left(\frac{1-m}{m}\right)\int\frac{\mathrm{d}\theta}{\mathrm{M}\sqrt{\mathrm{I}}}$$
$$=\frac{i^2}{mn}\int\frac{\mathrm{d}\theta}{\sqrt{\mathrm{I}}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left(\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right),$$

hence

$$\left.\begin{aligned}\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\theta}{N\sqrt{I}}=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\theta}{M\sqrt{I}}\\ +\frac{i^2}{\sqrt{mn}}\int\frac{d\theta}{\sqrt{I}}+\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{I}}\right]\end{aligned}\right\}\quad . \quad . \qquad (451)$$

But
$$\left(\frac{1+n}{n}\right)\sqrt{mn}=\frac{k(a^2+b^2)}{ab\sqrt{a^2+k^2}}.$$

Hence

$$\left.\begin{aligned}\frac{3[\mathrm{LY}]}{k^2}=\frac{ab}{k\sqrt{k^2+a^2}}\int\frac{d\theta}{\left[1-\frac{k^2}{a^2+b^2}\sin^2\theta\right]\sqrt{I}}\\ +\frac{ab(k^2-b^2)}{k^3\sqrt{k^2+a^2}}\int\frac{d\theta}{\sqrt{I}}+\frac{ab(a^2+b^2)}{k^3\sqrt{k^2+a^2}}\int\frac{d\theta}{\cos^2\theta\sqrt{I}}\\ +\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{I}}\right]-\tan^{-1}\left[\frac{b}{a}\sin\theta\right]\end{aligned}\right\}. \qquad (452)$$

Now, if Y be an arc of the plane hyperbola of which $\sqrt{k^2-b^2}$ is the transverse axis, and i the reciprocal of the eccentricity, we shall have

$$\frac{ab}{k^3}\mathrm{Y}=\frac{ab(a^2+b^2)}{k^3\sqrt{a^2+k^2}}\int\frac{d\theta}{\cos^2\theta\sqrt{I}}. \quad . \quad . \quad . \quad . \qquad (453)$$

And if we take the spherical ellipse whose principal semiangles, α and β, are given by the equations

$$\cos\alpha=\frac{b}{k},\ \cos\beta=\frac{b}{k}\sqrt{\frac{k^2+a^2}{k^2+b^2}},$$

we shall have $\quad \sin^2\epsilon=\dfrac{k^2-b^2}{k^2+a^2},\ e^2=\dfrac{k^2}{k^2+a^2}$

and $\quad \dfrac{\tan\beta}{\tan\alpha}\cos\alpha=\dfrac{ab}{k\sqrt{k^2+a^2}},\ $ also $\psi=\tan^{-1}\left(\dfrac{b}{a}\sin\theta\right)$.

Hence the sum of the first and last terms may be written

$$\left[\psi-\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{d\theta}{[1-e^2\sin^2\theta]\sqrt{1-\sin^2\epsilon\sin^2\theta}}\right];$$

and this expression is S, the value of the area of the spherical ellipse $(\alpha\beta)$, as shown in (13).

Now, let $\mathfrak{A}$ be the transverse axis of the auxiliary hyperbola;

$$\mathfrak{A}=\sqrt{k^2-b^2},\ \text{and}\ \mathfrak{B}=\sqrt{a^2+b^2}.$$

Hence the coefficient of $\int \frac{d\theta}{\sqrt{\mathrm{I}}}$ may be written $\frac{ab}{k^3}\frac{\mathfrak{A}^2}{\mathfrak{B}}j$, and the equation (452) finally assumes the form

$$3k[\mathrm{LY}] = ab\left[\mathrm{Y} + \frac{\mathfrak{A}^2}{\mathfrak{B}}j\int\frac{d\theta}{\sqrt{\mathrm{I}}}\right] - k^3\mathrm{S} + k^3\tan^{-1}\left[\frac{\sqrt{mn}\sin\theta\cos\theta}{\sqrt{\mathrm{I}}}\right]. \quad (454)$$

Or the area of the logarithmic hyperbola may be expressed as a sum of the arcs of a common hyperbola, of a spherical ellipse, of a spherical parabola, and of a circular arc, multiplied by constant coefficients.

There is one particular case in which the area of the logarithmic hyperbola may be represented by a very simple expression. Let $k=b$; then, if we turn to (448), $\mathrm{V}=a^2+b^2$, and $\mathrm{I}=1$, since $i=0$. Hence (452) may be changed into

$$3[\mathrm{LY}] = a\sqrt{a^2+b^2}\tan\theta + b^2\tan^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\tan\theta\right)$$

$$+ b^2\tan^{-1}\left(\frac{b^2}{a\sqrt{a^2+b^2}}\sin\theta\cos\theta\right) - b^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right);$$

and this expression may be reduced to

$$\left.\begin{aligned}3[\mathrm{LY}] = a\sqrt{a^2+b^2}\tan\theta + b^2\tan^{-1}\left(\frac{\sqrt{a^2+b^2}}{a}\tan\theta\right)\\ - b^2\tan^{-1}\left(\frac{b}{a}\sin\theta\right)\end{aligned}\right\}, \quad (455)$$

a value entirely independent of elliptic integrals, and which may be represented by a straight line and the difference of two circular arcs.

CHAPTER XII.

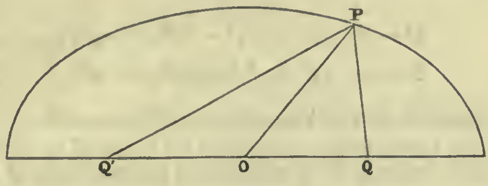
ON THE RECTIFICATION OF LEMNISCATES.

86.] There is a particular class of plane curves, of which the lemniscate of Bernoulli is an example, to which the principles established in the foregoing pages may be applied with much elegance.

Definition.—This entire class of curves may be defined by the following property. The square of the rectangle under the radii vectores drawn from the foci to any point on the curve is equal to a constant, plus or minus the square of the semidiameter passing through this point multiplied by a constant quantity.

Let Q, Q′ be the foci, and O the centre, ρ, $\rho_{\prime}$ r the lines drawn

Fig. 28.



from these points to any point on the curve. Let $OQ=OQ'=c$, and let f be a variable constant.

Then by the definition

$$\rho^2\rho_{\prime}^2=c^4\pm f^2r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But $$\rho^2\rho_{\prime}^2=(x^2+y^2)^2+c^4+2c^2y^2-2c^2x^2,$$

and $$r^2=x^2+y^2;$$

hence $$(x^2+y^2)^2=(f^2+2c^2)x^2+(f^2-2c^2)y^2. \quad . \quad . \quad (456)$$

This is the general equation of the curve, which assumes different forms as we assign varying values to f and c. Some examples may be given.

(α) Let $c=0$, or $f=\infty$, the equation is that of a circle.

(β) Let $f^2>2c^2$, and make $f^2+2c^2=a^2$, $f^2-2c^2=b^2$, the equation will become $\quad (x^2+y^2)^2=a^2x^2+b^2y^2.$

This is the equation of a curve which may be called the elliptic lemniscate. It is the locus, as is well known, of the intersection of central perpendiculars with tangents to an ellipse; and its rectification has been fully investigated in vol. i. p. 196.

(γ) Let $f^2=2c^2$. The equation becomes $(x^2+y^2)^2=4c^2x^2$, or the equation is that of two equal circles in external contact.

(δ) Let $f^2<2c^2$. The equation becomes

$$(x^2+y^2)^2=(2c^2+f^2)x^2-(2c^2-f^2)y^2, \text{ and } a^2>b^2.$$

(ϵ) Let $f^2=0$. The equation becomes $(x^2+y^2)^2=2c^2(x^2-y^2)$, or the equation is that of the lemniscate of Bernoulli.

(ζ) Let f^2, passing through 0, be taken with a negative sign. The equation in this case becomes

$$(x^2+y^2)^2=(2c^2-f^2)x^2-(2c^2+f^2)y^2, \text{ and } b^2>a^2.$$

In one case only does the equation of the lemniscate in its general form coincide with that of Cassini's ellipse, namely when $f=0$ and $h=c$, h^2 being the product of the radii vectores from the foci.

The definition of Cassini's ellipse being "a curve such that the product of the radii vectores drawn from two fixed points, the foci, to a third point on the curve, shall be constant and equal to h^2," its equation will obviously be, $2c$ being the distance between the foci,

$$h^4-c^4=(x^2+y^2)^2-2c^2(x^2-y^2)\,; \quad . \quad . \quad . \quad . \quad \text{(b)}$$

when

$$h=c, \quad (x^2+y^2)^2=2c^2(x^2-y^2). \quad . \quad . \quad . \quad . \quad \text{(c)}$$

This is the equation of the lemniscate of Bernoulli.

These elliptic lemniscates may also be defined as the orthogonal projections of the curves of symmetrical intersection of a paraboloid of revolution with cones of the second degree, having their centres at the vertex of the paraboloid. Let α and β be the principal semiangles of one of the cones. Its equation is

$$\cot^2\alpha \,.\, x^2 \pm \cot^2\beta \,.\, y^2 = z^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Make $\tan\alpha=\dfrac{2k}{a}$, $\tan\beta=\dfrac{2k}{b}$, and the equation of the cone becomes

$$a^2x^2 \pm b^2y^2 = 4k^2z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

Let the equation of the paraboloid be x^2+y^2+2kz.

Eliminating z, the equation of the projection of the curve of intersection will become

$$(x^2+y^2)^2=a^2x^2 \pm b^2y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (457)$$

When the section is an ellipse, the equation of this curve is, as in (β),

$$(x^2+y^2)^2=a^2x^2+b^2y^2.$$

On the Hyperbolic Lemniscate.

87.] The equation of the lemniscate in this case is

$$(x^2+y^2)^2=a^2x^2-b^2y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (458)$$

Following the steps indicated in sec. [86], we shall find

$$\frac{ds^2}{d\lambda^2}=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the limits of λ are 0 and $\tan^{-1}\dfrac{a}{b}$.

Assume

$$\sin^2\lambda=\frac{a^4\sin^2\phi}{a^2b^2+a^4\sin^2\phi+b^4\cos^2\phi}. \quad . \quad . \quad . \quad \text{(b)}$$

The limits of ϕ, corresponding to $\lambda=0$ and $\lambda=\tan^{-1}\dfrac{a}{b}$, are

$$\phi=0, \text{ and } \phi=\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Substituting this value of $\sin^2\lambda$ in the preceding equation, we shall find

$$\frac{ds}{d\lambda}=\frac{a}{\cos\varphi}. \quad \text{(d)}$$

From (b) we may derive

$$\frac{d\lambda}{d\varphi}=\frac{a^2b(a^2+b^2)\cos\varphi}{[a^2b^2+a^4\sin^2\varphi+b^4\cos^2\varphi]\sqrt{a^2+b^2\cos^2\varphi}}. \quad \text{(c)}$$

Multiplying the two latter equations together and reducing, we get

$$s=\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\varphi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\varphi\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\varphi}}. \quad (459)$$

When $a=b$, or when the lemniscate is that of Bernoulli, there results the well-known expression

$$s=\frac{a}{\sqrt{2}}\int\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}.$$

When $a>b$ the integral is of the third order and *circular* form; but when $a<b$ the integral is of the third order and *logarithmic* form. That it is of the logarithmic form may thus be shown.

Let $\qquad \dfrac{b^2-a^2}{b^2}=m,$ and $i^2=\dfrac{b^2}{a^2+b^2}.$

Hence $\qquad i^2-m=\dfrac{a^4}{b^2(a^2+b^2)}, \quad (460)$

or i^2 is greater than m; but we know that the form is logarithmic when the square of the modulus is greater than the parameter, when it is affected with a negative sign.

This is a remarkable result. All analysts know the impossibility of transforming the circular form into the logarithmic, or *vice versâ*, by any other than an imaginary transformation; the utmost efforts of the most accomplished analysts have been exhausted in the attempt; yet in this particular case their geometrical connexion is very close. The modulus and the parameter are connected by the equation

$$\frac{1}{i^2}\mp m=2; \quad (461)$$

the upper sign to be taken in the circular form, the lower in the logarithmic.

There are two distinct cases to be considered—when a is greater than b, and when a is less than b.

Case I. $a>b$.

Let a plane ellipse be constructed whose principal semiaxes A and B are given by the equations

$$A^2=a^2+b^2, \quad B^2=a^2, \quad \ldots \ldots \quad (f)$$

and let a sphere be described from the centre of this ellipse with a radius

$$=\frac{a^2}{\sqrt{a^2-b^2}}=\frac{B^2}{\sqrt{2B^2-A^2}}=R.$$

Then we can find, as follows, the length of an arc of the spherical ellipse, the intersection of the sphere whose radius is R with the cylinder standing on the ellipse whose semiaxes are A and B.

$$\left.\begin{array}{ll}
\text{Since} & \sin^2\alpha=\frac{A^2}{R^2}=\frac{a^4-b^4}{a^4}, \ \cos^2\alpha=\frac{b^4}{a^4}, \\
\text{and} & \sin^2\beta=\frac{B^2}{R^2}=\frac{a^2-b^2}{a^2}, \ \cos^2\beta=\frac{b^2}{a^2}, \\
\text{therefore} & \frac{R\cos\beta}{\cos\alpha\sin\alpha}=\frac{a^5}{b(a^2-b^2)\sqrt{a^2+b^2}}. \\
\text{We have also} & \\
& \frac{R\cos\beta\cos\alpha}{\sin\alpha}=\frac{ab^3}{(a^2-b^2)\sqrt{a^2+b^2}}, \\
& \tan^2\epsilon=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\alpha}=\frac{a^2-b^2}{b^2}, \\
& i^2=\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=\frac{b^2}{a^2+b^2}.
\end{array}\right\} \quad \ldots \quad (g)$$

Substituting these values in (46) the expression for an arc of a spherical ellipse with a *positive* parameter, and writing $\bar{s}$ for the arc, we get

$$\left.\begin{array}{l}
\frac{a^2-b^2}{a^2}\bar{s}=\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\phi}{\left[1+\left(\frac{a^2-b^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\phi}} \\
\qquad -\frac{b^3}{a\sqrt{a^2+b^2}}\int\frac{d\phi}{\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\phi}}-(a^2-b^2)^{\frac{1}{2}}\tau
\end{array}\right\}. \quad (462)$$

Comparing this with (459), we find

$$s-\left(\frac{a^2-b^2}{a^2}\right)\bar{s}=\frac{b^3}{a\sqrt{a^2+b^2}}\int\frac{d\phi}{\sqrt{1-\left(\frac{b^2}{a^2+b^2}\right)\sin^2\phi}}+(a^2-b^2)^{\frac{1}{2}}\tau, \quad (463)$$

or the *difference* between an arc of a hyperbolic lemniscate and an arc of a spherical ellipse may be expressed by an integral of the first order, together with a circular arc. When $a=b$, the radius of the sphere is infinite, the sphere becomes a plane, so that it is not possible to express an arc of a spherical ellipse by the common lemniscate.

Case II. Let $b>a$.

In this case the arc of the hyperbolic lemniscate may be expressed by an arc of a logarithmic ellipse of a *particular species*, or one whose parameter and modulus are connected by the relation given in (461).

Resuming the expression in (459) for the arc of the hyperbolic lemniscate,

$$s=\frac{a^3}{b\sqrt{a^2+b^2}}\int\frac{d\phi}{\left[1-\left(\frac{b^2-a^2}{b^2}\right)\sin^2\phi\right]\sqrt{1-\frac{b^2}{b^2+a^2}\sin^2\phi}}. \quad . \quad \text{(h)}$$

Let $\quad \frac{b^2-a^2}{b^2}=m, \quad \frac{b^2}{b^2+a^2}=i^2;$

then, as $\quad m+n-mn=i^2, \quad n=\frac{a^2}{a^2+b^2}$ . . . . . (i)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be the semiaxes of the base of the elliptic cylinder, k the parameter of the paraboloid whose intersection with the cylinder gives the logarithmic ellipse. Assume for the principal semi- major axis of the elliptic base

$$\mathfrak{A}=\sqrt{a^2+b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(j)}$$

In (171) we found the following relations between $\mathfrak{A}, \mathfrak{B}, k, m, n,$

$$\frac{\mathfrak{A}^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}, \quad \frac{\mathfrak{B}^2}{k^2}=\frac{mn(1-m)}{(n-m)^2},$$

and as we assume $\mathfrak{A}=\sqrt{a^2+b^2}$, we get, substituting for m and n their values in terms of a and b, the semiaxes of the hyperbola

$$\mathfrak{B}=\frac{b^2}{a}, \text{ and } k=\frac{a^2b^2+a^4-b^4}{a^2\sqrt{b^2-a^2}}. \quad . \quad . \quad . \quad . \quad \text{(k)}$$

In (163) we found for the equation of the logarithmic ellipse measured from the *minor* axis, and multiplied by the undetermined factor Q,

$$\left.\begin{aligned} 2Q\Sigma = &-\left(\frac{1-m}{m}\right)\sqrt{mn}kQ\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}} \\ &+\frac{kQ\sqrt{mn}}{n-m}\left[\int d\varphi\sqrt{I}+\left(\frac{i^2-m}{m}\right)\int\frac{d\varphi}{\sqrt{I}}-m\Phi\right] \end{aligned}\right\} . \quad (464)$$

If in this equation we substitute for m, n, and k their values as given in (i) and (k), and equate the coefficient $\left(\frac{1-m}{m}\right)\sqrt{mn}kQ$ with the coefficient $\frac{a^3}{b\sqrt{a^2+b^2}}$ of the expression for the lemniscate in (459), we shall find

$$Q=\frac{a^2(b^2-a^2)}{a^2b^2+a^4-b^4};$$

hence the last equation, substituting this value of Q, will become

$$\left.\begin{aligned} \frac{2a^2(b^2-a^2)}{a^2b^2+a^4-b^4}\Sigma+s=&\frac{ab(b^2-a^2)\sqrt{a^2+b^2}}{a^2b^2+a^4-b^4}\int d\varphi\sqrt{I} \\ &+\frac{a^5b}{[a^2b^2+a^4-b^4]\sqrt{a^2+b^2}}\int\frac{d\varphi}{\sqrt{I}}-\frac{a(b^2-a^2)^2\sqrt{a^2+b^2}}{b(a^2b^2+a^4-b^4)}\Phi \end{aligned}\right\}; \quad (465)$$

or the *sum* of an arc of a hyperbolic lemniscate and of an arc of a logarithmic ellipse may be expressed as a sum of integrals of the first and second orders with a circular arc.

When $b=a$, the above expression will become

$$s=\frac{a}{\sqrt{2}}\int\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}.$$

In this case the parameter of the paraboloid becomes infinite, and therefore the paraboloid a plane, just as the sphere became a plane in the last case; so that we cannot express integrals of the third order, whether circular or logarithmic, by an arc of a common lemniscate.

Although the lemniscates may be rectified by elliptic integrals of the third order, as well circular as logarithmic, yet these curves cannot be accepted as general representatives of integrals of the third order, because, in the functions which represent those curves, the parameters and the moduli are connected by an invariable relation, as in (461). Thus the elliptic lemniscate, whatever be the ratio of the axes of the generating plane ellipse, can be represented only by a particular species of spherical ellipse, that whose principal arcs are supplemental.

THE THEORY

OF

ELLIPTIC INTEGRALS,

AND THE

PROPERTIES OF SURFACES OF THE SECOND ORDER,

APPLIED TO THE INVESTIGATION OF

THE MOTION OF A RIGID BODY

ROUND A FIXED POINT.

"Quant aux sciences des phénomènes naturels, nous ne doutons point que les surfaces du second degré ne doivent s'y présenter aussi dans un grand nombre de questions, et y jouer un rôle aussi important que celui des sections coniques dans le système planétaire."
—Chasles, *Aperçu Historique*, p. 251.

CHAPTER XIII.

88.] We shall now proceed to apply the principles developed in the foregoing pages to the investigation of a physical problem of much celebrity and great interest in Astronomy—the motion of rotation of a rigid body round a fixed point. The discovery of the geometrical properties of elliptic integrals may be applied with singular felicity to the illustration of the complicated motions of the several axes of this body, the spirals, curves, and cones described by them during its rotation round the fixed point. Let this point be taken as the origin of three rectangular coordinates, their direction being arbitrary as well with respect to the body as to absolute space. Let us, moreover, make the supposition that the body is not subject to the action of accelerating forces, but in a state of motion originated by a single impulse, or by any number of single impulses, which may be combined into one. This may be considered as the normal state of the rotation of a body; because if it should besides be subjected to accelerating forces, such new forces will introduce variations into the arbitrary constants of the problem. It has, moreover, the advantage of admitting a complete solution; we are not compelled to have recourse to approximations. It will be shown that the curves which the final integrals represent are spherical conic sections—curves which may as easily be determined, from the principles laid down in the preceding chapters, by means of the constants which enter into the integrals, and the amplitudes of those functions, as the arc of a circle may be ascertained when we know its radius and the angle which the arc subtends at the centre. Hitherto there has not been any attempt made, at least so far as the author is aware, to carry the solution further than to show that as the final integrals involve the square roots of quadrinomial expressions with respect to the independent variable, they might be reduced to the usual forms of elliptic functions. But these integrals have not been interpreted so as to give a graphic representation of the motion, by means of the properties of those functions.

Assuming the usual definition of the moment of inertia of a body with respect to a certain straight line (that it is the sum of all the constituent elements of the body, each multiplied into the square of its distance from this axis), we shall briefly give the usual method of finding it.

Let the given axis make the angles λ, μ, ν, with the axes of coordinates, R being the distance of one of the elements dm from the origin, and θ the angle which this line makes with the axis. The distance, therefore, of the particle dm from the axis is $R \sin \theta$; and the moment of inertia round this axis is the sum or integral of all the terms, such as $R^2\sin^2\theta dm$, which the body affords. Writing H for the moment of inertia round this axis,

$$H=\int dm[R\sin\theta]^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (466)$$

the integral being extended to the whole mass of the body. H is therefore a quantity of five dimensions.

To transform this integral into another, which shall contain the rectangular coordinates xyz of the particle dm. We have

$$R\cos\theta=x\cos\lambda+y\cos\mu+z\cos\nu;$$

deriving the value of $\sin\theta$ from this expression, and substituting it in (466), we get

$$\left.\begin{aligned}H=\cos^2\lambda\int dm(y^2+z^2)+\cos^2\mu\int dm(x^2+z^2)+\cos^2\nu\int dm(x^2+y^2)\\ -2\cos\mu\cos\nu\int dm yz-2\cos\lambda\cos\nu\int dm xz-2\cos\lambda\cos\mu\int dm xy\end{aligned}\right\}. \quad (467)$$

Now these six integrals depend solely on the assumed position of the coordinate planes with respect to the body, and not on the position of the axis of moments, which is determined by the angles λ, μ, ν. These integrals, referred to the same system of coordinates, will therefore be the same for every assumed axis. Let them be computed and designated as follows—

$$\left.\begin{aligned}&\int dm(y^2+z^2)=L,\ \int dm(x^2+z^2)=M,\ \int dm(x^2+y^2)=N,\\ &\int dm yz=U,\ \int dm xz=V,\ \int dm xy=W.\end{aligned}\right\}. \quad (468)$$

The value of H may now be written,

$$\left.\begin{aligned}H=L\cos^2\lambda+M\cos^2\mu+N\cos^2\nu-2U\cos\mu\cos\nu\\ -2V\cos\lambda\cos\nu-2W\cos\lambda\cos\mu\end{aligned}\right\}. \quad (469)$$

We may reduce this expression to represent a straight line drawn from the origin to some curved surface, by the following transformations:

$$\left.\begin{aligned}\text{let }H=nP^2,\ L=nA,\ M=nA_{\prime},\ N=nA_{\prime\prime},\\ U=nB,\ V=nB_{\prime},\ W=nB_{\prime\prime}.\end{aligned}\right\}. \quad . \quad (470)$$

Substitute these values, and divide by the cubical constant n, equation (469) becomes

$$\left.\begin{aligned}A\cos^2\lambda+A_{\prime}\cos^2\mu+A_{\prime\prime}\cos^2\nu-2B\cos\mu\cos\nu\\ -2B_{\prime}\cos\lambda\cos\nu-2B_{\prime\prime}\cos\lambda\cos\mu=P^2\end{aligned}\right\}. \quad . \quad (471)$$

Now this, as may easily be shown, is the expression for the length of a perpendicular let fall from the centre of a surface of the second order on a tangent plane to this surface. As the coefficients L, M, N are necessarily finite and positive, the coefficients of the surface A, $A_{,}$, $A_{,,}$, which have a given ratio to the former, must also be finite and positive. The surface is therefore an ellipsoid. That the above expression represents such a perpendicular may be shown as follows.

89.] The tangential equation of a surface of the second order (see vol. i. p. 66), the origin being at the centre, is

$$A\xi^2+A_{,}v^2+A_{,,}\zeta^2+2B\zeta v+2B_{,}\xi\zeta+2B_{,,}\xi v=1. \quad . \quad (472)$$

In this equation ξ, v, ζ denote the reciprocals of the portions of the axes of coordinates between the origin and the variable tangent plane, supposed to envelop the surface in every successive possible position. The squared reciprocal of the perpendicular from the centre on the tangent plane is $\xi^2+v^2+\zeta^2$. If λ, μ, ν denote the angles which this perpendicular $P_{,}$ makes with the axes of coordinates, $\cos\lambda=P_{,}\xi$, $\cos\mu=P_{,}v$, $\cos\nu=P_{,}\zeta$. Substituting these values of ξ, v, ζ in the preceding equation, and multiplying by $P_{,}^2$, we find

$$\left.\begin{array}{r} A\cos^2\lambda+A_{,}\cos^2\mu+A_{,,}\cos^2\nu+2B\cos\mu\cos\nu \\ +2B_{,}\cos\lambda\cos\nu+2B_{,,}\cos\lambda\cos\mu=P_{,}^2 \end{array}\right\} . \quad . \quad (473)$$

an equation which coincides with (471); hence $P_{,}=P$.

If we divide (469) by P^2, and introduce the quantities ξ, v, ζ by the help of the equations $\cos\lambda=P\xi$, $\cos\mu=Pv$, $\cos\nu=P\zeta$, $H=nP^2$, we shall find

$$L\xi^2+Mv^2+N\zeta^2-2U\zeta v-2V\xi\zeta-2W\xi v=n. \quad . \quad (474)$$

It is shown in the first volume of this work, p. 63, that, if x, y, z denote the projective coordinates of the point of contact of the tangent plane to the surface,

$$\left.\begin{array}{l} nx=L\xi-V\zeta-Wv \\ ny=Mv-W\xi-U\zeta \\ nz=N\zeta-Uv-V\xi \end{array}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (475)$$

Let $x_{,}y_{,}z_{,}$ denote the coordinates of the foot of the perpendicular P on the tangent plane; then as $P\cos\lambda=x_{,}$, and $P\xi=\cos\lambda$, $x_{,}=P^2\xi$; in like manner, $y_{,}=P^2v$, $z_{,}=P^2\zeta$: whence

$$\left.\begin{array}{l} n(x_{,}-x)=(L-nP^2)\xi-V\zeta-Wv \\ n(y_{,}-y)=(M-nP^2)v-W\xi-U\zeta \\ n(z_{,}-z)=(N-nP^2)\zeta-Uv-V\xi \end{array}\right\} . \quad . \quad . \quad (476)$$

Now, writing T for the distance measured along the tangent plane between the foot of the perpendicular upon it from the centre, and the point of contact of this tangent plane, $x_{,}-x$, $y_{,}-y$, $z_{,}-z$ are the projections of T upon the three coordinate axes. It is also evident that $(x_, y_, z_,)$, (xyz), and $(0, 0, 0)$ are the projective coordinates of the three angles of the right-angled triangle whose vertex is at the centre and whose base is T.

It may easily be shown, and we may therefore assume, that the orthogonal projections of the area of this triangle upon the coordinate planes of xy, yz, and xz are

$$\left.\begin{array}{r}[y(x_,-x)-x(y_,-y)],\ [z(y_,-y)-y(z_,-z)],\\ \text{and }\ [x(z_,-z)-z(x_,-x)]\end{array}\right\} . \quad (477)$$

respectively.

If we substitute in these expressions the values of the projective coordinates, which may be deduced from (476), writing Δ for the area of this triangle, and $\Delta l_,$, $\Delta m_,$, $\Delta n_,$ for its projections on the coordinate planes of yx, xz, and xy, ($l_,$, $m_,$, $n_,$ being the direction cosines which a normal to the plane of Δ makes with the axes of x, y, z respectively), we shall have

$$\left.\begin{array}{l}\mathrm{n}\Delta l_, = \mathrm{P}^2[(\mathrm{M}-\mathrm{N})\zeta\upsilon-(\mathrm{W}\zeta-\mathrm{V}\upsilon)\xi-\mathrm{U}(\zeta^2-\upsilon^2)]\\ \mathrm{n}\Delta m_, = \mathrm{P}^2[(\mathrm{N}-\mathrm{L})\xi\zeta-(\mathrm{U}\xi-\mathrm{W}\zeta)\upsilon-\mathrm{V}(\xi^2-\zeta^2)]\\ \mathrm{n}\Delta n_, = \mathrm{P}^2[(\mathrm{L}-\mathrm{M})\xi\upsilon-(\mathrm{V}\upsilon-\mathrm{U}\xi)\zeta-\mathrm{W}(\upsilon^2-\xi^2)]\end{array}\right\} . \quad (478)$$

We shall discover the dynamical illustrations of these expressions further on.

90.] To determine the axes of figure of the ellipsoid.

It is manifest, whenever the distance T between the foot of the perpendicular from the centre on the tangent plane, and the point of contact of this tangent plane with the surface, vanishes, that the radius through the point of contact becomes also a perpendicular to the tangent plane, and therefore one of the axes of the surface. When $\mathrm{T}=0$, its projections on the coordinate axes vanish, or $x_,-x=0$, $y_,-y=0$, $z_,-z=0$; (476) then becomes, putting n, as we evidently may do, equal to 1,

$$\left.\begin{array}{l}(\mathrm{L}-\mathrm{P}^2)\xi-\mathrm{V}\zeta-\mathrm{W}\upsilon=0\\ (\mathrm{M}-\mathrm{P}^2)\upsilon-\mathrm{W}\xi-\mathrm{U}\zeta=0\\ (\mathrm{N}-\mathrm{P}^2)\zeta-\mathrm{U}\upsilon-\mathrm{V}\xi=0\end{array}\right\} . \quad . \quad . \quad . \quad . \quad (479)$$

From these equations eliminating the quantities ξ, υ, ζ, we get the following cubic equation in P^2,

$$\left.\begin{array}{r}(\mathrm{L}-\mathrm{P}^2)(\mathrm{M}-\mathrm{P}^2)(\mathrm{N}-\mathrm{P}^2)-\mathrm{U}^2(\mathrm{L}-\mathrm{P}^2)-\mathrm{V}^2(\mathrm{M}-\mathrm{P}^2)\\ -\mathrm{W}^2(\mathrm{N}-\mathrm{P}^2)-2\mathrm{UVW}=0\end{array}\right\} . \quad (480)$$

The roots of this equation are the three semiaxes squared of the ellipsoid.

We need not here stop to show that the three roots of this cubic equation are real, as the proposition has already been established in various ways, see vol. i. sec. [84]. The following is a group of symmetrical formulæ for determining the position of any one of these axes in space when its magnitude is determined.

Let $P_{\prime}^2$ be one of the roots of the cubic equation, or the square of one of the semiaxes; let $L-P_{\prime}^2=Q$, $M-P_{\prime}^2=Q_{\prime}$, $N-P_{\prime}^2=Q_{\prime\prime}$; also let λ, μ, ν be the angles which this axis $P_{\prime}^2$ makes with the axes of coordinates; then $\cos\lambda=P_{\prime}\xi$, $\cos\mu=P_{\prime}\upsilon$, $\cos\nu=P_{\prime}\zeta$.

This equation may also be written

$$QQ_{\prime}Q_{\prime\prime}-QU^2-Q_{\prime}V^2-Q_{\prime\prime}W^2-2UVW=0. \quad . \quad . \quad (481)$$

Resuming equations (479), and introducing the given value $P_{\prime}^2$ of P^2,

$$\left.\begin{array}{l} Q\xi-V\zeta-W\upsilon=0, \\ Q_{\prime}\upsilon-W\xi-U\zeta=0, \\ Q_{\prime\prime}\zeta-U\upsilon-V\xi=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (482)$$

Combining the first of these equations with the second, and eliminating υ,

$$\frac{\zeta}{\xi}=\frac{QQ_{\prime}-W^2}{VQ_{\prime}+UW};$$

combining the second with the third, and again eliminating υ,

$$\frac{\zeta}{\xi}=\frac{VQ_{\prime}+UW}{Q_{\prime}Q_{\prime\prime}-U^2};$$

multiplying the two latter,

$$\frac{\zeta^2}{\xi^2}=\frac{\cos^2\nu}{\cos^2\lambda}=\frac{Q_{\prime}Q-W^2}{Q_{\prime}Q_{\prime\prime}-U^2}.$$

In like manner $\quad \dfrac{\upsilon^2}{\xi^2}=\dfrac{\cos^2\mu}{\cos^2\lambda}=\dfrac{Q_{\prime\prime}Q-V^2}{Q_{\prime\prime}Q_{\prime}-U^2},$

whence, adding,

$$\cos^2\lambda=\frac{Q_{\prime\prime}Q_{\prime}-U^2}{(Q_{\prime\prime}Q_{\prime}-U^2)+(QQ_{\prime\prime}-V^2)+(Q_{\prime}Q-W^2)}; \quad . \quad (483)$$

and like expressions for $\cos^2\mu$ and $\cos^2\nu$ may be found. See vol. i. p. 73.

We may express these formulæ in a more compact notation as follows:

If we take the first derivative of (480), we shall find it to consist of three members. Substituting for P^2 one of its values, $P_{\prime}^2$ suppose, the resulting expression may be written

$\Upsilon+\Phi+\Omega$, and the last formula becomes

$$\cos^2\lambda=\frac{\Upsilon}{\Upsilon+\Phi+\Omega}; \text{ also } \cos^2\mu=\frac{\Phi}{\Upsilon+\Phi+\Omega}, \cos^2\nu=\frac{\Omega}{\Upsilon+\Phi+\Omega}. \quad (484)$$

91.] In every revolving body there exists an instantaneous axis of rotation, or a line of particles which remain at rest during an instant. Let C be the position of a point in the revolving body at any given time, C′ the position of the point during the next instant. Let the arc CC′ be ds. At the extremities of this arc ds let normal planes be drawn to the curve. If these planes are parallel, the motion is one of rotation round an axis infinitely distant, or the motion is one of translation. If the planes are not parallel, let them meet; the straight line in which they intersect is the axis of rotation during the indefinitely small time in which the arc CC′ or ds has been described.

This line, the intersection of the normal planes, must pass through the fixed point, if there be one in the body; otherwise there would exist in the body a fixed point and a fixed straight line not passing through the point, which would retain the body in a state of rest, contrary to the supposition.

Again, there cannot be, during the same instant, two or more axes of rotation in the body; for two fixed lines are equivalent to three fixed points, which would retain the body in a state of rest.

The same considerations will show that the instantaneous axis of rotation could not possibly be a curve.

The angular velocity of a body is defined to be the arc of a circle whose radius is 1, described in the element of the time, and whose centre is on the axis of rotation.

92.] To determine equations of the instantaneous axis of rotation.

The fixed point being taken as origin, let $x'y'z'$ be the coordinates of the point C, $(x'+\mathrm{d}x')$, $(y'+\mathrm{d}y')$, $(z'+\mathrm{d}z')$ of the point C′. The equation of the normal plane passing through C is

$$x\mathrm{d}x'+y\mathrm{d}y'+z\mathrm{d}z'=x'\mathrm{d}x'+y'\mathrm{d}y'+z'\mathrm{d}z'=0, \quad . \quad . \quad . \quad \text{(a)}$$

since the plane must pass through the origin; hence as

$$x'\mathrm{d}x'+y'\mathrm{d}y'+z'\mathrm{d}z'=0,$$

the point C must move on the surface of a sphere. The equation of the normal plane passing through C′ is

$$x\mathrm{d}^2x'+y\mathrm{d}^2y'+z\mathrm{d}^2z'=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

The equation of the osculating plane passing through the arc ds being

$$\mathrm{A}(x-x')+\mathrm{B}(y-y')+\mathrm{C}(x-z')=0, \quad . \quad . \quad . \quad . \quad \text{(c)}$$

we may determine the constants from the consideration that the osculating plane is perpendicular to each of the normal planes. The osculating plane is therefore perpendicular to the intersection of these planes—that is, to the instantaneous axis of rotation.

Let λ, μ, ν be the angles which this line makes with the axes of coordinates, then $\frac{\cos\lambda}{\cos\nu}=\frac{A}{C}$, $\frac{\cos\mu}{\cos\nu}=\frac{B}{C}$; and the equations of this straight line become

$$Az-Cx=0,\ Bx-Ay=0,\ Cy-Bz=0. \quad . \quad . \quad . \quad (d)$$

Let ω be the angular velocity round the instantaneous axis of rotation;

$$\text{then } \omega=\frac{ds}{R},\ \text{R being the radius of curvature.}$$

Make $r=\omega\cos\nu$, and as

$$\cos\nu=\frac{C}{\sqrt{A^2+B^2+C^2}},\quad r=\frac{ds}{R}\frac{C}{\sqrt{A^2+B^2+C^2}}.$$

Now R (as is shown in treatises on the geometry of three dimensions*) is equal to $\frac{ds^3}{\sqrt{A^2+B^2+C^2}}$;

whence $r=\frac{C}{ds^2}$. In like manner, let $p=\omega\cos\lambda$, $q=\omega\cos\mu$;

$$\text{then } p=\frac{A}{ds^2},\quad q=\frac{B}{ds^2}.$$

Substituting in (d) these values of A, B, C, we get

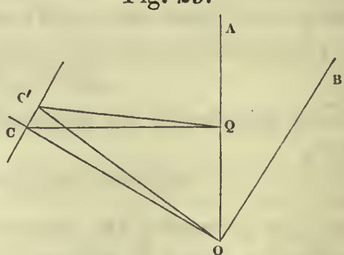
$$pz-rx=0,\ qx-py=0,\ ry-qz=0. \quad . \quad . \quad . \quad (485)$$

These are the equations of the instantaneous axis of rotation, as we shall show presently from dynamical considerations.

93.] The angular velocity round the instantaneous axis being ω, the angular velocity round any other axis which makes the angle θ with the former is $\omega\cos\theta$.

Let OA be the instantaneous axis of rotation, OB an axis which makes the angle θ with the former. Through O let a plane be drawn perpendicular to OB. In this plane assume any point C, with the centre O and radius OC let a sphere be described, and through C let a plane be drawn perpendicular to OA and meeting this line in Q. The point C will move, in consequence of this rotation, on the circumference of the circle the intersection of the sphere by this plane, and therefore on the surface of the sphere itself. Hence the tangent CC′ is perpendicular as well

Fig. 29.



* Leroy, *Analyse appliquée à la Géométrie des Trois Dimensions*, p. 295.

to the line CO as to CQ. Let the angle $CQC'=\omega$, the angle $COC'=\omega'$; then $CC'=CQ.\omega=OC.\omega'$, and $CQ=OC\cos\theta$; hence

$$\omega'=\omega\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (486)$$

Now, as the angular velocities of every other element of the body, round the axes OA, OB, are ω and ω' respectively during this instant, it is plain that the angular velocity of every particle of the body round these axes is connected by the relation

$$\omega'=\omega\cos\theta;$$

hence p, q, r in the last section are the angular velocities round the axes of x, y, z.

94.] Let as before Ox, Oy, Oz be any three rectangular coordinates passing through the fixed point O, and X, Y, Z the velocities of the particle dm of the body resolved along these axes, x, y, z being the coordinates of the particle dm. These velocities being translated to the origin are there equilibrated by the resistance of the fixed point O; while they generate the moments $(Yx-Xy)dm$, $(Zy-Yz)dm$, $(Xz-Zx)dm$ in the planes of xy, yz, zx respectively.

We may conventionally assume that the rotations from x to y, from y to z, and from z to x, shall be taken as positive, and the rotations in any of the opposite directions as negative. Let ω be the angular velocity round the instantaneous axis of rotation, λ, μ, ν the angles this axis makes with the axes of coordinates, p, q, r the components of the angular velocities along the axes of xyz, so that

$$p=\omega\cos\lambda, \quad q=\omega\cos\mu, \quad r=\omega\cos\nu. \quad . \quad . \quad (487)$$

The velocity of the particle dm parallel to the plane of xy is $r\sqrt{x^2+y^2}$; and this resolved along the axes of x and y is $-r\sqrt{x^2+y^2}.\dfrac{y}{\sqrt{x^2+y^2}}$ and $r\sqrt{x^2+y^2}.\dfrac{x}{\sqrt{x^2+y^2}}$, or $-yr$ and xr, in accordance with the conventional agreement as to the signs of rotation in the coordinate planes; whence

the velocities parallel to the axes of x and y are $-yr$ and xr,
" " " of y and z are $-zp$ and yp,
" " " of z and x are $-xq$ and zq,

whence $X=zq-yr$, $Y=xr-zp$, $Z=yp-xq$;

and these velocities translated to the origin generate the moments

$$\left.\begin{aligned} Yx-Xy&=(xr-zp)x-(zq-yr)y, \text{ in the plane of } xy,\\ Zy-Yz&=(yp-xq)y-(xr-zp)z, \text{ in the plane of } yz,\\ Xz-Zx&=(zq-yr)z-(yp-xq)x, \text{ in the plane of } xz.\end{aligned}\right\} . \quad (488)$$

We may determine the position of that group of particles (if any) in the body which at the given instant are at rest, by making $X=0$, $Y=0$, $Z=0$. These conditions are satisfied by making $xr-zp=0$, $zq-yr=0$, $yp-xq=0$.

These, it is hardly necessary to observe, are the equations of a straight line passing through the origin, equations which we have already found in (485) from geometrical considerations.

95.] If we extend to the whole mass the velocities found for the single particle dm in the preceding section, we must integrate the expressions for these velocities. Introducing the notation adopted in (468), we find, multiplying the last equation by dm and integrating,

$$\left.\begin{aligned}\int(Zy-Yz)\,\mathrm{dm}&=\mathrm{L}p-\mathrm{V}r-\mathrm{W}q,\\ \int(Xz-Zx)\,\mathrm{dm}&=\mathrm{M}q-\mathrm{W}p-\mathrm{U}r,\\ \int(Yx-Xy)\,\mathrm{dm}&=\mathrm{N}r-\mathrm{U}q-\mathrm{V}p.\end{aligned}\right\} \quad . \quad . \quad . \quad (489)$$

Now, as the impressed couple or the resultant of all the impressed couples must, by the principle of D'Alembert, be equivalent to the effective moments, if we make this impressed couple K, and l, m, n the direction-cosines of its axis k,

$$\left.\begin{aligned}\mathrm{K}l&=\mathrm{L}p-\mathrm{V}r-\mathrm{W}q,\\ \mathrm{K}m&=\mathrm{M}q-\mathrm{W}p-\mathrm{U}r,\\ \mathrm{K}n&=\mathrm{N}r-\mathrm{U}q-\mathrm{V}p.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (490)$$

When the principal axes are the axes of coordinates, $U=0$, $V=0$, $W=0$, and we get the well-known equations

$$\mathrm{K}l=\mathrm{L}p,\ \mathrm{K}m=\mathrm{M}q,\ \mathrm{K}n=\mathrm{N}r. \quad . \quad . \quad . \quad (491)$$

Hence the components of the angular velocity round the principal axes are equal to the components of the impressed couple at right angles to these axes, divided by the moments of inertia about them, or

$$p=\frac{\mathrm{K}l}{\mathrm{L}},\ q=\frac{\mathrm{K}m}{\mathrm{M}},\ r=\frac{\mathrm{K}n}{\mathrm{N}}. \quad . \quad . \quad . \quad . \quad (491^*)$$

96.] If we compare together the formulæ given in (475) and (490), we shall make the second members identical by assuming

$$p=f\xi,\ q=f\upsilon,\ r=f\zeta,\ f \text{ being a linear quantity}; \quad . \quad (492)$$

whence $\omega^2=f^2(\xi^2+\upsilon^2+\zeta^2)=\frac{f^2}{\mathrm{P}^2}$; or the angular velocity is inversely proportional to the perpendicular on the tangent plane, which may be called *the instantaneous plane of rotation.*

Resuming the equations (475) and (490), introducing also the relations established in (492), we obtain

$$\mathrm{K}l=\mathrm{L}p-\mathrm{V}r-\mathrm{W}q=f(\mathrm{L}\xi-\mathrm{V}\zeta-\mathrm{W}v)=f\mathrm{n}x, \text{ or}$$

$$\mathrm{K}l=f\mathrm{n}x; \text{ in like manner } \mathrm{K}m=f\mathrm{n}y,\ \mathrm{K}n=f\mathrm{n}z, \text{ whence}$$

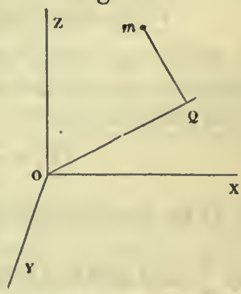
$$\mathrm{K}^2=f^2\mathrm{n}^2(x^2+y^2+z^2)=f^2\mathrm{n}^2k^2. \quad . \quad . \quad . \quad . \quad (493)$$

Now x, y, z are the coordinates of the point of contact of the tangent plane; whence we infer that k, *the semidiameter drawn from the centre to the point of contact of the instantaneous plane of rotation, is constant during the motion.*

From the relations of (492), it also follows that if through the fixed point we draw any three rectangular axes in the body, the angular velocities round these axes are always inversely proportional to the segments of those axes cut off by the instantaneous plane of rotation; or, in other words, the symbols ξ, v, ζ, the tangential coordinates of the instantaneous plane of rotation, will denote the components of angular *slowness* round those axes.

97.] Resulting from the rotation of the body, there arises a new class of forces, which in general tend to alter the position of the axes of rotation of the body. They are known as the centrifugal forces. When translated to the origin they generate a couple, whose magnitude and position we are now to determine.

Fig. 30.



Let OQ be the instantaneous axis of rotation, $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$ the cosines of the angles it makes with the axes. x, y, z are the coordinates of the particle dm. The centrifugal force which acts on this particle dm is equal to the square of the velocity divided by the radius—that is, $=\frac{\omega^2\overline{\mathrm{Q}m}^2}{\mathrm{Q}m}=\omega^2.\mathrm{Q}m$; and this force, as it acts in the direction of $\mathrm{Q}m$, may be resolved into the forces $\omega^2(x-x_i)$, $\omega^2(y-y_i)$, $\omega^2(z-z_i)$, respectively parallel to the axes of x, y, and z. $x_i\, y_i\, z_i$ are the coordinates of the point Q. Now

$$\mathrm{OQ}=\mathrm{O}m\cos m\mathrm{OQ}=\frac{px+qy+rz}{\omega}; \text{ we also have}$$

$$x_i=\mathrm{OQ}\cos\mathrm{QOX}=\mathrm{OQ}\,\frac{p}{\omega}=\frac{(px+qy+rz)}{\omega^2}p, \text{ or}$$

$$\omega^2x_i=(px+qy+rz)p; \text{ but } \omega^2x=x(p^2+q^2+r^2), \text{ whence}$$

$$\left.\begin{aligned}\omega^2(x-x_i)&=q(qx-py)+r(rx-pz)=\mathrm{X}',\\ \omega^2(y-y_i)&=r(ry-qz)+p(py-qx)=\mathrm{Y}',\\ \omega^2(z-z_i)&=p(pz-rx)+q(qz-ry)=\mathrm{Z}'.\end{aligned}\right\} \quad . \quad . \quad . \quad (494)$$

From these equations we obtain

$$Y'x - X'y = pq(y^2 - x^2) + yx(p^2 - q^2) + rz(py - qz),$$

or, extending this expression to the whole mass,

$$\int (Y'x - X'y)\,\mathrm{dm} = pq\int[(y^2+z^2) - (x^2+z^2)]\,\mathrm{dm}$$
$$+ (p^2 - q^2)\int yx\,\mathrm{dm} + pr\int zy\,\mathrm{dm} - qr\int zx\,\mathrm{dm}.$$

Writing analogous formulæ for the other axes, making $Gl_{\prime}$, $Gm_{\prime}$, $Gn_{\prime}$ equal respectively to $\int \mathrm{dm}(Z'y - Y'z)$, $\int \mathrm{dm}(X'z - Z'x)$, $\int \mathrm{dm}(Y'x - X'y)$, and using the notation established in (468), we get

$$\left.\begin{aligned} Gl_{\prime} &= (M-N)qr + (Vq - Wr)p + U(q^2 - r^2), \\ Gm_{\prime} &= (N-L)pr + (Wr - Up)q + V(r^2 - p^2), \\ Gn_{\prime} &= (L-M)pq + (Up - Vq)r + W(p^2 - q^2). \end{aligned}\right\} \quad . \quad . \quad (495)$$

When the principal axes coincide with the axes of coordinates, $U=0$, $V=0$, $W=0$, and the formulæ become

$$Gl_{\prime} = (M-N)qr,\ \ Gm_{\prime} = (N-L)pr,\ \ Gn_{\prime} = (L-M)pq. \quad (496)$$

When one of the axes of coordinates, that of z suppose, coincides with the instantaneous axis of rotation, we have $p=0$, $q=0$, $r=\omega$, and (495) becomes

$$Gl_{\prime} = -U\omega^2,\ \ Gm_{\prime} = V\omega^2,\ \ Gn_{\prime} = 0. \quad . \quad . \quad . \quad (496^*)$$

If we multiply the first of (495) by $\frac{p}{\omega}$, the second by $\frac{q}{\omega}$, the third by $\frac{r}{\omega}$, and add the results, the sum will be zero, or

$$\frac{G}{\omega}[l_{\prime}p + m_{\prime}q + n_{\prime}r] = 0; \quad . \quad . \quad . \quad . \quad (497)$$

whence it follows that *the plane of the centrifugal couple always passes through the instantaneous axis of rotation.*

Multiply together line by line the groups in (490) and (495), and add the results; the sum will be cipher, or

$$KG[ll_{\prime} + mm_{\prime} + nn_{\prime}] = 0. \quad . \quad . \quad . \quad . \quad (498)$$

Whence we may infer that the *planes of the impressed and centrifugal couples are always at right angles to each other.*

98.] If we compare (478) with (495), we shall find the second members identical, if we assume, as in (492),

$$p = f\xi,\ \ q = fv,\ \ r = f\zeta;\ \text{ whence } f^2 = P^2\omega^2,$$

and therefore $$G = \Delta n\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (499)$$

We may hence infer that *the triangle whose sides are the semidiameter to the point of contact of the tangent plane, and the perpendicular on this tangent plane from the centre, coincides in position with the plane of the centrifugal couple. The centrifugal couple is also equal to the* centrifugal triangle *multiplied by the mass and the square of the angular velocity,* as shown in the preceding formula.

The reader will not fail to have observed the ease and simplicity with which the properties of the ellipsoid, treated generally, without reference to the principal axes, by the method of tangential coordinates, may be used to illustrate and establish the corresponding states of a body in motion round a fixed point. The subsequent investigations might in most cases have been discussed with the same generality and facility; but as the principle of this new analytical geometry, the method of tangential coordinates, as developed in the first volume of this work, is probably as yet but little known, it may be more satisfactory to conduct these investigations on principles universally admitted. To simplify the results, we shall adopt a particular system of coordinates which will render the formulæ much more manageable. If we choose the principal axes of the body as axes of coordinates, $U=0$, $V=0$, $W=0$, and our investigations will therefore be very much simplified.

Let $a>b>c$ be the three semiaxes of the ellipsoid in the order of magnitude, L, M, N the moments of inertia about the coinciding principal axes of the body. We may assume, as in (470), the squares of the semiaxes of the ellipsoid proportional to the moments of inertia round these axes, so that

$$a^2\mathrm{n}=\mathrm{L},\ b^2\mathrm{n}=\mathrm{M},\ c^2\mathrm{n}=\mathrm{N}, \quad . \quad . \quad . \quad . \quad (500)$$

n being a constant depending on the mass and constitution of the body.

This ellipsoid we shall call the *ellipsoid of moments.*

Introducing these transformations and simplifications, (469), (490), and (495) become,

$$\mathrm{H}=\mathrm{n}[a^2\cos^2\lambda+b^2\cos^2\mu+c^2\cos^2\nu], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (501)$$

$$\mathrm{K}l=\mathrm{n}a^2p,\ \mathrm{K}m=\mathrm{n}b^2q,\ \mathrm{K}n=\mathrm{n}c^2r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (502)$$

$$\mathrm{G}l_{\prime}=\mathrm{n}(b^2-c^2)qr,\ \mathrm{G}m_{\prime}=\mathrm{n}(c^2-a^2)pr,\ \mathrm{G}n_{\prime}=\mathrm{n}(a^2-b^2)pq. \quad (503)$$

In formula (501) it is evident that the part within the brackets is the expression for the square of a perpendicular from the centre on a tangent plane to the ellipsoid. Let this perpendicular be P, and (501) will become

$$\mathrm{H}=\mathrm{n}\mathrm{P}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (504)$$

Hence it follows that the moment of inertia of any rigid body round a given axis is the mass of the body multiplied into the square of the coinciding perpendicular from the centre on a tangent plane to the ellipsoid of moments.

Square the terms of (502), add them, and multiply by ω^2, we shall obtain the result

$$\mathrm{K}^2\omega^2=\mathrm{n}^2[a^4p^2+b^4q^2+c^4r^2](p^2+q^2+r^2);$$

also, as $\omega\cos\lambda=p$, $\omega\cos\mu=q$, $\omega\cos\nu=r$,

$$\mathrm{H}^2\omega^4=\mathrm{n}^2[a^2p^2+b^2q^2+c^2r^2]^2,$$

whence we shall obtain

$$\mathrm{G}^2=\mathrm{K}^2\omega^2-\mathrm{H}^2\omega^4, \quad \ldots\ldots \quad (505)$$

an important formula, which gives the value of the centrifugal couple in terms of the impressed couple, the moment of inertia, and the angular velocity round the instantaneous axis of rotation.

99.] Assume the impressed couple $\mathrm{K}=\mathrm{n}fk$, k being the semi-diameter of the ellipsoid perpendicular to the plane of K. The product fk is of course constant; it will be shown presently that f and k are each constant.

As the axes of coordinates are the principal axes,

$$p=\frac{\mathrm{K}l}{\mathrm{L}},\ q=\frac{\mathrm{K}m}{\mathrm{M}},\ r=\frac{\mathrm{K}n}{\mathrm{N}}. \quad \text{See (491).}$$

Let x, y, z be the coordinates of the vertex of k, then

$$\left.\begin{array}{l} l=\frac{x}{k},\ m=\frac{y}{k},\ n=\frac{z}{k},\ \mathrm{L}=\mathrm{n}a^2,\ \mathrm{M}=\mathrm{n}b^2,\ \mathrm{N}=\mathrm{n}c^2, \\ \text{and } \mathrm{K}=\mathrm{n}fk;\ \text{whence } p=\frac{fx}{a^2},\ q=\frac{fy}{b^2},\ r=\frac{fz}{c^2}. \end{array}\right\} \quad . \quad . \quad (506)$$

Squaring these values and adding,

$$(p^2+q^2+r^2)=\omega^2=f^2\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]=\frac{f^2}{\mathrm{P}^2}. \quad . \quad . \quad (507)$$

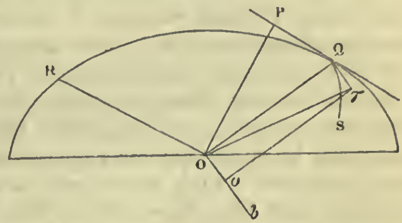
The cosines of the angles which this perpendicular makes with the axes are $\frac{\mathrm{P}x}{a^2}, \frac{\mathrm{P}y}{b^2}, \frac{\mathrm{P}z}{c^2}$, while the cosines of the angles which the instantaneous axis of rotation makes with the same axes are $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$; but $p=\frac{fx}{a^2}$ and $\omega=\frac{f}{\mathrm{P}}$, whence $\frac{p}{\omega}=\frac{\mathrm{P}x}{a^2}$; similarly $\frac{q}{\omega}=\frac{\mathrm{P}y}{b^2}$, $\frac{r}{\omega}=\frac{\mathrm{P}z}{c^2}$; we may therefore infer that

The instantaneous axis of rotation coincides with the perpendicular from the centre on the instantaneous tangent plane drawn through the vertex of k the axis of the impressed couple. The angular velocity round this axis is inversely proportional to this perpendicular.

100.] *During the whole period of rotation, the semidiameter k of the ellipsoid, perpendicular to the plane of the impressed couple, is constant.*

Through any point Q on the surface of an ellipsoid let a tangent plane be drawn, and through the centre a plane parallel to it. Let a concentric sphere be described through the point Q, intersecting the surface of the ellipsoid in the curve of double curvature Qs. To this curve, let a tangent $Q\tau$ be drawn at the point Q; and through this tangent let a diametral plane be drawn intersecting in the straight line Ob the diametral plane ROb parallel to the tangent plane through Q.

Fig. 31.



Hence it follows that QO, Ob are the semiaxes of the plane section QOb of the surface. Let $OQ=k$, $Ob=u$. Let fall from O a perpendicular OP on the tangent plane $QP\tau$. This line will also be perpendicular to the parallel diametral plane ObR, and therefore to every line in this plane, and therefore to the line Ob. Now the tangent line $Q\tau$, as it is on the tangent plane to the ellipsoid, and passes through the point Q, must be a tangent to the plane section of the ellipsoid passing through it; and as it is besides a tangent to a curve drawn upon the surface of the sphere, it must be at right angles to the radius of the sphere OQ; hence $OQ\tau$ is a right angle, and therefore OQ must be a semiaxis of the section $OQ\tau$, because, when a tangent to a conic section is perpendicular to the diameter passing through the point of contact, this diameter must be an axis of the section. Now, as the parallel planes $QP\tau$, ORb are cut by the plane $QO\tau$, Ob is parallel to $Q\tau$ and consequently at right angles to OQ. Hence OQ, Ob are the semiaxes of the section $OQ\tau$.

Since Ob is perpendicular to OP as well as to OQ, it is perpendicular to the plane of OPQ, which passes through OP, OQ—that is, to the plane of the centrifugal couple; whence we are led to infer that the semiaxes k and u of the diametral section of the ellipsoid, whose plane passes through the tangent to the curve of double curvature in which the ellipsoid and sphere intersect, coincide with the axes of the impressed and centrifugal couples K and and G respectively.

Assume a point v on the line Ob, so that Ov may be to k as the centrifugal couple G is to the impressed couple K. The diagonal $O\tau$ of this instantaneous rectangle will represent, as well in magnitude as in direction, the axis of the resultant couple at the end of the first instant. During this instant, accordingly, the vertex of the axis of the impressed couple will have travelled on the surface of the ellipsoid, as also on the surface of the concentric sphere whose radius is k. It follows therefore that, at the end of the first instant, the vertex of the axis of the resultant couple will be found on the curve of double curvature in which the ellipsoid and sphere intersect. The same proof will hold for the second and for every succeeding instant, whence k continues always invariable. Now the impressed couple K was assumed in sec. [99] equal to $\mathrm{n}fk$; but as n and k are each constant, f must likewise be constant.

If, to fix our ideas, we take the plane of K horizontal, and k therefore vertical, we may infer that the rotatory motion of the body will be such that its representative ellipsoid will bring all its semidiameters which are equal to k successively into a vertical position, and therefore the *surface of the representative ellipsoid will always pass through a fixed point in space.*

Hence the motion of rotation of a rigid body round a fixed point may easily be conceived by the help of the ellipsoid of moments.

Let us imagine *the centre of this ellipsoid to be fixed, that its surface always passes through a fixed point in space, and that tangent planes are always drawn to the ellipsoid through this fixed point. The perpendiculars from the centre on these successive tangent planes will represent in magnitude and position the instantaneous axis of rotation.*

101.] It was shown in (507) that the angular velocity ω was equal to $\frac{f}{\mathrm{P}}$; and as f is constant, the angular velocity round the instantaneous axis of rotation varies inversely as P (the perpendicular let fall from the centre on the instantaneous plane of rotation).

Hence it follows that *the square of the angular velocity round the instantaneous axis of rotation is always proportional to the area of the diametral section of the ellipsoid perpendicular to this axis.*

The angular velocity κ round the axis of the impressed couple is constant during the motion.

Let θ be the angle between k and P. Then $\cos\theta=\frac{\mathrm{P}}{k}$; now $\kappa=\omega\cos\theta$, as shown in (486), and $\omega=\frac{f}{\mathrm{P}}$, whence $\kappa=\frac{f}{\mathrm{P}}\frac{\mathrm{P}}{k}=\frac{f}{k}$; but f and k are each constant, or $\kappa=\frac{f}{k}=\text{constant}$. . . . (508)

The magnitude of the centrifugal couple G *varies as the tangent of the angle between the axis of the impressed moment and the instantaneous axis of rotation.*

Resume the equation given in (505), $\mathrm{G}^2=\mathrm{K}^2\omega^2-\mathrm{H}^2\omega^4$. Write for K, H, and ω their values as given in sec. [99], (501), and

(507)—namely, $K = nfk$, $H = nP^2$, and $\omega = \frac{f}{P}$. We have also

$\tan\theta = \frac{\sqrt{k^2 - P^2}}{P}$, and $\kappa = \frac{f}{k}$, whence

$$G = K\kappa \tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (509)$$

It will be evident on inspection, that the indefinitely small portion Ov of the line Ob parallel to the tangent drawn at Q, to the section of the ellipsoid whose semiaxes are k and u, and which is equal to $Q\tau$, may be taken as the element of the arc of the spherical curve traced out by the vertex of k during the element of the time dt. Writing $\frac{ds}{dt}$ for this element Ov, and referring to sec. [100], we have the ratio $Ov : k :: G : K$,

$$\text{or } Ov = \frac{ds}{dt} = \frac{Gk}{K}, \text{ but } G = K\kappa \tan\theta, \text{ and } f = \kappa k.$$

Whence $$\frac{ds}{dt} = f\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (510)$$

Now $\frac{ds}{dt}$ is the velocity with which the curve of double curvature passes through Q, the fixed point in space. We may thence infer that *the velocity with which the pole of the impressed couple passes along this curve, or the velocity with which the curve passes through the fixed pole, varies as the tangent of the angle θ between the axis of the impressed couple and the instantaneous axis of rotation.*

102.] To find the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, or of the velocities of the pole of the impressed couple in the direction of the principal axes of the body.

We have $\frac{dz}{ds} = \frac{\frac{dz}{dt}}{\frac{ds}{dt}}$, and $\frac{ds}{dt} = f\tan\theta$, whence $\frac{dz}{dt} = \frac{dz}{ds} f\tan\theta$, and $\frac{ds^2}{dz^2} = 1 + \frac{dx^2}{dz^2} + \frac{dy^2}{dz^2}$. Now (xyz) is a point on the surface of the ellipsoid of moments, as also on that of a concentric sphere whose radius is k. The equations of these surfaces are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ and } x^2 + y^2 + z^2 = k^2. \quad . \quad . \quad (511)$$

Eliminating y and x successively, and then differentiating, we find

$$\frac{dx}{dz}=\frac{z}{x}\frac{a^2}{c^2}\left(\frac{b^2-c^2}{a^2-b^2}\right),\ \frac{dy}{dz}=\frac{z}{y}\frac{b^2}{c^2}\left(\frac{a^2-c^2}{a^2-b^2}\right). \quad . \quad . \quad . \quad (512)$$

Whence $\dfrac{ds^2}{dz^2}=\dfrac{a^4(b^2-c^2)^2y^2z^2+b^4(c^2-a^2)^2x^2z^2+c^4(a^2-b^2)^2x^2y^2}{c^4(a^2-b^2)^2x^2y^2}.$

Now $\tan^2\theta=\dfrac{k^2-P^2}{P^2}=k^2\left[\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}\right]-\left[\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right]^2$

and $k^2=x^2+y^2+z^2$; hence

$$\tan^2\theta=\frac{a^4(b^2-c^2)^2y^2z^2+b^4(c^2-a^2)^2z^2x^2+c^4(a^2-b^2)^2x^2y^2}{a^4b^4c^4}. \quad . \quad . \quad (513)$$

or eliminating x and y by (511),

$$\tan^2\theta=\frac{(a^2-c^2)(b^2-c^2)k^2z^2-c^4(a^2-k^2)(b^2-k^2)}{a^2b^2c^4}. \quad . \quad . \quad (514)$$

Making the substitutions suggested by these equations, we shall obtain

$$\frac{dx}{dt}=\frac{f(b^2-c^2)yz}{b^2c^2},\ \frac{dy}{dt}=\frac{f(c^2-a^2)xz}{a^2c^2},\ \frac{dz}{dt}=\frac{f(a^2-b^2)xy}{a^2b^2}. \quad (515)^*$$

103.] *The axis of rotation due to the centrifugal forces lies in the plane of the impressed couple.*

Let ω' be the angular velocity round the axis of rotation due to the centrifugal couple, and $p_{,}, q_{,}, r_{,}$ its components round the prin-

* When the axis of the impressed moment very nearly coincides with one of the principal axes (that of c suppose), the differential equations of motion may easily be deduced.

In this case as x and y are each very small, their product xy may be neglected; now $p=\dfrac{fx}{a^2}$, $q=\dfrac{fy}{b^2}$, $r=\dfrac{fz}{c^2}$, and $\dfrac{dr}{dt}=\dfrac{f}{c^2}\dfrac{dz}{dt}=\dfrac{f^2(a^2-b^2)}{a^2b^2c^2}xy=0$. Hence r is constant, equal to n suppose. We also have

$$\frac{dp}{dt}=\frac{f}{a^2}\frac{dx}{dt}=\frac{f^2(b^2-c^2)yz}{a^2b^2c^2};\ \text{but } y=\frac{b^2q}{f},\ z=\frac{c^2n}{f},$$

whence $\dfrac{dp}{dt}=\dfrac{f^2}{a^2}\dfrac{(b^2-c^2)}{b^2c^2}\dfrac{b^2c^2}{f^2}nq=\dfrac{b^2-c^2}{a^2}nq$, or writing $A=na^2$, $B=nb^2$, $C=nc^2$;

$$A\frac{dp}{dt}+(C-B)nq=0. \quad \text{Similarly}$$

$$B\frac{dq}{dt}+(C-A)np=0.$$

These are the equations deduced by Poisson for this particular case. (*Traité de Mécanique*, tom. ii. p. 159.)

cipal axes. Then, as the angular velocity round any principal axis is equal to the couple which produces the motion resolved at right angles to this axis, and divided by the corresponding moment of inertia,

$$p_{\prime}=\frac{\mathrm{G}\frac{\mathrm{d}x}{\mathrm{d}s}}{\mathrm{L}};\ \text{now } \mathrm{G}=\mathrm{K}\kappa\tan\theta,\ \mathrm{K}=\mathrm{n}fk,\ \mathrm{L}=\mathrm{n}a^2,$$

$$\text{and } \frac{\mathrm{d}x}{\mathrm{d}s}=\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}}=\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{f\tan\theta},\ \text{whence } p_{\prime}=f\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{a^2}.$$

Making corresponding substitutions for $q_{\prime}$ and $r_{\prime}$, we shall have

$$p_{\prime}=f\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{a^2},\ q_{\prime}=f\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{b^2},\ r_{\prime}=f\frac{\frac{\mathrm{d}z}{\mathrm{d}t}}{c^2}. \quad \ldots\ldots \quad (516)$$

Now the cosines of the angles which this axis of rotation makes with the axes of coordinates are $\frac{p_{\prime}}{\omega_{\prime}}, \frac{q_{\prime}}{\omega_{\prime}}, \frac{r_{\prime}}{\omega_{\prime}}$; and the cosines of the angles which the axis k makes with the same axes are $\frac{x}{k}, \frac{y}{k}, \frac{z}{k}$. If we denote the angle between the axis k of the impressed couple and $\mathrm{P}_{\prime}$ the instantaneous axis of rotation due to the centrifugal couple by $k\hat{\mathrm{O}}\mathrm{P}_{\prime}$,

$$\cos k\hat{\mathrm{O}}\mathrm{P}_{\prime}=\frac{1}{k\omega_{\prime}}\,(p_{\prime}x+q_{\prime}y+r_{\prime}z)=\frac{f}{k\omega_{\prime}}\left(\frac{\frac{x\mathrm{d}x}{\mathrm{d}t}}{a^2}+\frac{\frac{y\mathrm{d}y}{\mathrm{d}t}}{b^2}+\frac{\frac{z\mathrm{d}z}{\mathrm{d}t}}{c^2}\right)=0, \quad (517)$$

since the part within the brackets is the differential of the equation of the ellipsoid.

We may infer, therefore, *that not only is the axis of the centrifugal couple contained in the plane of the impressed couple, but the axis round which the centrifugal couple would give the body a tendency to revolve lies in the same plane also**.

* To determine the angular velocity when $\mathrm{L}=\mathrm{M}$, or, using Poisson's notation, when $\mathrm{A}=\mathrm{B}$.

As $r=\frac{fz}{c^2}$, $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{f}{c^2}\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{f^2(a^2-b^2)}{a^2b^2c^2}xy=0$, since $a^2=b^2$. Hence r is constant $=n$.

Now $\omega^2=p^2+q^2+r^2=n^2+\frac{f^2}{a^4}(x^2+y^2)$. Let $\frac{x^2+y^2}{k^2}=\sin^2\epsilon$; then $\omega^2=n^2+\frac{f^2k^2}{a^4}\sin^2\epsilon$. We have $\mathrm{K}=\mathrm{n}fk$, $\mathrm{A}=\mathrm{n}a^2$; whence $\omega^2=n^2+\frac{\mathrm{K}^2}{\mathrm{A}^2}\sin^2\epsilon$. The expression given by Poisson, *Traité de Mécanique*, p. 159.

104.] Through the vertex of u the axis of the centrifugal couple, let a tangent plane to the ellipsoid be drawn. The perpendicular from the centre on this tangent plane, is the instantaneous axis of rotation due to the centrifugal couple.

Let $x_{\prime}\, y_{\prime}\, z_{\prime}$ be the coordinates of the vertex of u; $l_{\prime}, m_{\prime}, n_{\prime}$, the cosines of the angles it makes with the axes; $\lambda_{\prime}, \mu_{\prime}, \nu_{\prime}$, the angles which $\mathrm{P}_{\prime}$ the instantaneous axis of rotation due to the centrifugal couple makes with the same axes. Then, as u is perpendicular as well to k as to $\mathrm{P}_{\prime}$,

$$\frac{l_{\prime}x}{k}+\frac{m_{\prime}y}{k}+\frac{n_{\prime}z}{k}=0, \quad \mathrm{P}_{\prime}\left[\frac{l_{\prime}x}{a^2}+\frac{m_{\prime}y}{b^2}+\frac{n_{\prime}x}{c^2}\right]=0. \quad . \quad . \quad \text{(a)}$$

Eliminating from these equations $m_{\prime}$ and $l_{\prime}$ successively,

$$\frac{l_{\prime}}{n_{\prime}}=\frac{a^2 z}{c^2 x}\left(\frac{b^2-c^2}{a^2-b^2}\right). \quad . \quad \text{(b)} \qquad \frac{m_{\prime}}{n_{\prime}}=\frac{b^2 z}{c^2 y}\left(\frac{a^2-c^2}{a^2-b^2}\right). \quad . \quad . \quad . \quad \text{(c)}$$

Now
$$\frac{\cos\lambda_{\prime}}{\cos\nu_{\prime}}=\frac{\dfrac{\mathrm{P}_{\prime}x_{\prime}}{a^2}}{\dfrac{\mathrm{P}_{\prime}z_{\prime}}{c^2}}=\frac{c^2x_{\prime}}{a^2z_{\prime}}, \quad \frac{\cos\mu_{\prime}}{\cos\nu_{\prime}}=\frac{c^2y_{\prime}}{b^2z_{\prime}},$$

and
$$\frac{l_{\prime}}{n_{\prime}}=\frac{\dfrac{x_{\prime}}{u}}{\dfrac{z_{\prime}}{u}}=\frac{x_{\prime}}{z_{\prime}}, \quad \frac{m_{\prime}}{n_{\prime}}=\frac{y_{\prime}}{z_{\prime}}; \text{ whence}$$

$\dfrac{\cos\lambda_{\prime}}{\cos\nu_{\prime}}=\dfrac{c^2}{a^2}\dfrac{l_{\prime}}{n_{\prime}}, \ \dfrac{\cos\mu_{\prime}}{\cos\nu_{\prime}}=\dfrac{c^2}{b^2}\dfrac{m_{\prime}}{n_{\prime}}$. Substituting for $\dfrac{l_{\prime}}{n_{\prime}}, \dfrac{m_{\prime}}{n_{\prime}}$ their values given in the preceding equations, and reducing, we find

$$\cos^2\nu_{\prime}=\frac{(a^2-b^2)^2x^2y^2}{(a^2-b^2)^2x^2y^2+(b^2-c^2)^2y^2z^2+(c^2-a^2)^2z^2x^2}. \quad \text{(518)}$$

We may find analogous expressions for $\cos\lambda_{\prime}$ and $\cos\mu_{\prime}$.

Introducing the terms $\dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}y}{\mathrm{d}t}, \dfrac{\mathrm{d}z}{\mathrm{d}t}$, by the help of (515),

$$\cos^2\nu_{\prime}=\frac{a^4b^4\left(\dfrac{\mathrm{d}z}{\mathrm{d}t}\right)^2}{a^4b^4\left(\dfrac{\mathrm{d}z}{\mathrm{d}t}\right)^2+b^4c^4\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2+c^4a^4\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2}. \quad . \quad . \quad \text{(519)}$$

Now the cosine of the angle which the axis due to G makes with the axis of z is $\frac{r_{\prime}}{\omega_{\prime}}$; writing for $r_{\prime}$ and $\omega_{\prime}$ their values as given in (516),

$$\frac{r_{\prime}^2}{\omega_{\prime}^2}=\frac{a^4b^4\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}{a^4b^4\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2+b^4c^4\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2+a^4c^4\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}. \quad . \quad . \quad . \quad (520)$$

Whence, comparing (519) with (520),

$$\frac{r_{\prime}}{\omega_{\prime}}=\cos\nu_{\prime}; \text{ in like manner } \frac{p_{\prime}}{\omega_{\prime}}=\cos\lambda_{\prime},\ \frac{q_{\prime}}{\omega_{\prime}}=\cos\mu_{\prime},$$

or, *The perpendicular let fall from the centre on the tangent plane, drawn through the vertex of the axis of the centrifugal couple, coincides with the instantaneous axis of rotation due to this couple.*

The perpendicular $\mathrm{P}_{\prime}$ is therefore in the plane of the impressed couple.

105.] To find the component of the angular velocity $\omega_{\prime}$ due to the centrifugal couple resolved along the instantaneous axis of rotation.

Let δ be the angle between the axes of the rotations due to the impressed and centrifugal couples. Then

$$\cos\delta=\frac{pp_{\prime}+qq_{\prime}+rr_{\prime}}{\omega\omega_{\prime}};$$

or substituting the values of $\omega, p, q, r, \omega_{\prime}, p_{\prime}, q_{\prime}, r_{\prime}$, as given in (506) and (516), we shall have

$$\omega_{\prime}\cos\delta=\mathrm{P}f\left[\frac{x}{a^4}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{y}{b^4}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{z}{c^4}\frac{\mathrm{d}z}{\mathrm{d}t}\right].$$

Now the part within the brackets is the differential of $\frac{1}{2\mathrm{P}^2}$, whence $\omega_{\prime}\cos\delta=-\frac{f}{\mathrm{P}^2}\frac{\mathrm{dP}}{\mathrm{d}t}=f\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mathrm{P}}\right)$; but as $\omega=\frac{f}{\mathrm{P}}$,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=f\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mathrm{P}}\right), \text{ whence } \frac{\mathrm{d}\omega}{\mathrm{d}t}=\omega_{\prime}\cos\delta. \quad . \quad . \quad (521)$$

Or, *The increment of the angular velocity round the instantaneous axis of rotation, is due to the component of the angular velocity arising from the centrifugal couple, and resolved along the axis.*

106.] To investigate expressions for the lengths of u and $\mathrm{P}_{\prime}$.

As u makes angles with the coordinate axes whose cosines are $\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}$, since u is parallel to the tangent to the common intersection of the ellipsoid and sphere, and is besides a semidiameter of the surface,

$$\frac{1}{u^2}=\frac{\left(\frac{dx}{ds}\right)^2}{a^2}+\frac{\left(\frac{dy}{ds}\right)^2}{b^2}+\frac{\left(\frac{dz}{ds}\right)^2}{c^2}.$$

Now $\frac{dx}{ds}=\frac{dx}{dt}\frac{dt}{ds}$, $\frac{dy}{ds}=\frac{dy}{dt}\frac{dt}{ds}$, $\frac{dz}{ds}=\frac{dz}{dt}\frac{dt}{ds}$, and $\frac{ds}{dt}=f\tan\theta$, as in (511).

Whence $$\frac{1}{u^2}=\frac{b^2c^2\left(\frac{dx}{dt}\right)^2+a^2c^2\left(\frac{dy}{dt}\right)^2+a^2b^2\left(\frac{dz}{dt}\right)^2}{a^2b^2c^2f^2\tan^2\theta}. \quad . \quad . \quad . \quad (522)$$

Again, as $P_{\prime}^2=a^2\cos^2\lambda_{\prime}+b^2\cos^2\mu_{\prime}+c^2\cos^2\nu_{\prime}$, we shall have, putting for $\cos\lambda_{\prime}$, $\cos\mu_{\prime}$, $\cos\nu_{\prime}$ their values as given in (518),

$$P_{\prime}^2=\frac{a^2b^2c^2\left[a^2b^2\left(\frac{dz}{dt}\right)^2+a^2c^2\left(\frac{dy}{dt}\right)^2+b^2c^2\left(\frac{dx}{dt}\right)^2\right]}{a^4b^4\left(\frac{dz}{dt}\right)^2+b^4c^4\left(\frac{dx}{dt}\right)^2+a^4c^4\left(\frac{dy}{dt}\right)^2}. \quad . \quad . \quad (523)$$

If we combine (511), (522), and (523), we shall find

$$\frac{\left(\frac{ds}{dt}\right)^2}{P_{\prime}^2u^2}=\frac{\left(\frac{dx}{dt}\right)^2}{a^4}+\frac{\left(\frac{dy}{dt}\right)^2}{b^4}+\frac{\left(\frac{dz}{dt}\right)^2}{c^4}; \quad . \quad . \quad . \quad (523^*)$$

but $p_{\prime}=f\frac{\left(\frac{dx}{dt}\right)}{a^2}$, $q_{\prime}=f\frac{\left(\frac{dy}{dt}\right)}{b^2}$, $r_{\prime}=f\frac{\left(\frac{dz}{dt}\right)}{c^2}$, as shown in (516).

Whence $$\frac{\left(\frac{ds}{dt}\right)^2}{P_{\prime}^2u^2}=\frac{\omega_{\prime}^2}{f^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (524)$$

And as $\frac{ds}{dt}=\frac{Gk}{K}$ [see (501)] and $\omega=\frac{f}{P}$, we shall have

$$\frac{\omega_{\prime}}{\omega}=\frac{GPk}{KP_{\prime}u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (525)$$

To investigate an expression for the angle ρ, between the axes of rotation due to the impressed and centrifugal couples.

The cosines of the angles which the axes of rotation make with the axes of coordinates are

$$\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}, \frac{p_{\prime}}{\omega_{\prime}}, \frac{q_{\prime}}{\omega_{\prime}}, \frac{r_{\prime}}{\omega_{\prime}}, \text{ whence } \cos\rho = \frac{pp_{\prime}+qq_{\prime}+rr_{\prime}}{\omega\omega_{\prime}}.$$

$$\text{Now } p = \frac{fx}{a^2}, \text{ and } p_{\prime} = \frac{f\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}{a^2} = \frac{f^2(b^2-c^2)yz}{a^2b^2c^2};$$

whence $pp_{\prime} = \dfrac{f^3xyz}{a^2b^2c^2}\left(\dfrac{b^2-c^2}{a^2}\right)$. Finding like expressions for

$$qq_{\prime} \text{ and } rr_{\prime},\ \omega\omega_{\prime}\cos\rho = \frac{f^3xyz}{a^2b^2c^2}\left[\frac{a^2-b^2}{c^2}+\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}\right];$$

but
$$\left[\frac{a^2-b^2}{c^2}+\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}\right] = \frac{(a^2-b^2)(b^2-c^2)(a^2-c^2)}{a^2b^2c^2};$$

whence
$$\omega\omega_{\prime}\cos\rho = \frac{f^3xyz(a^2-b^2)(b^2-c^2)(a^2-c^2)}{a^4b^4c^4}. \quad . \quad (526)$$

The values of ω and $\omega_{\prime}$ are given in (507) and (524).

This formula shows that *whenever any two of the axes of the ellipsoid of moments are equal, or whenever the axis of the impressed couple happens to lie in one of the principal planes of the ellipsoid, the angle between the axes of rotation due to the impressed and centrifugal couples is a right angle.*

CHAPTER XIV.

ON THE CONES DESCRIBED BY THE SEVERAL AXES DURING THE MOTION OF THE BODY.

To determine the cones described by the axes of the impressed and centrifugal couples, as also by the axes of rotation due to those couples—in other words, to investigate the loci of *k*, P, *u*, and $\mathrm{P}_{\prime}$ referred to the principal axes of the body during the motion, will be the object of the present chapter.

107.] To find the locus of *k*, the axis of the impressed couple.

The equation of the cone whose vertex is at the centre, and which passes through the curve in which the ellipsoid of moments, and the invariable sphere whose radius is k, intersect, may easily be investigated, as k passes through the intersection of the ellipsoid and sphere—

$$\left(\frac{1}{a^2}-\frac{1}{k^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{k^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{k^2}\right)z^2=0, \quad . \quad . \quad (527)$$

the equation of a cone of the second degree, whose axes coincide with those of the ellipsoid.

This cone and the spherical conic section which constitutes its base will repeatedly present themselves in the course of the following pages; it may therefore be proper to denote them by some appropriate name.

As the side of this cone is constant, being the axis of the impressed couple, it may with propriety be named the *invariable cone*; and the spherical conic may be termed the *invariable spherical ellipse*.

108.] To investigate the nature of the surface described by P the instantaneous axis of rotation.

λ, μ, ν, being the angles which P makes with the axes,

$$\frac{\cos\lambda}{\cos\nu}=\frac{c^2x}{a^2z}, \quad \frac{\cos\mu}{\cos\nu}=\frac{c^2y}{b^2z}.$$

We have also the equations of the ellipsoid and sphere,

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $x^2+y^2+z^2=k^2$. Eliminating x, y, z, we get

$$a^4\left(\frac{1}{a^2}-\frac{1}{k^2}\right)\cos^2\lambda+b^4\left(\frac{1}{b^2}-\frac{1}{k^2}\right)\cos^2\mu+c^4\left(\frac{1}{c^2}-\frac{1}{k^2}\right)\cos^2\nu=0.$$

Let $x\,y\,z$ be the coordinates of any point on the surface of the cone at the distance R from the origin; then $\cos\lambda=\frac{x}{\mathrm{R}}$, $\cos\mu=\frac{y}{\mathrm{R}}$, $\cos\nu=\frac{z}{\mathrm{R}}$, and the equation of the cone becomes

$$a^4\left(\frac{1}{a^2}-\frac{1}{k^2}\right)x^2+b^4\left(\frac{1}{b^2}-\frac{1}{k^2}\right)y^2+c^4\left(\frac{1}{c^2}-\frac{1}{k^2}\right)z^2=0, \quad . \quad (528)$$

the equation of a cone which is also of the second degree.

As this cone too will frequently recur, we may name it the *cone of rotation*.

109.] To determine the equation of the cone described by the axis u of the centrifugal couple.

Let $x' y' z'$ be the coordinates of a point on the axis u of the centrifugal couple; then

$$\frac{x'}{z'}=\frac{\frac{dx}{ds}}{\frac{dz}{ds}}=\frac{a^2}{c^2}\left(\frac{b^2-c^2}{a^2-b^2}\right)\frac{z}{x},\quad \frac{y'}{z'}=\frac{\frac{dy}{ds}}{\frac{dz}{ds}}=\frac{b^2}{c^2}\left(\frac{a^2-c^2}{a^2-b^2}\right)\frac{z}{y}.\quad \text{See (512).}$$

From these equations and the equations of the ellipsoid and sphere, eliminating x, y, z, we find, omitting the traits as no longer necessary, the following equation of the fourth degree*,

$$\left.\begin{array}{l} a^2(a^2-k^2)(b^2-c^2)^2y^2z^2+b^2(b^2-k^2)(a^2-c^2)^2x^2z^2 \\ \qquad +c^2(c^2-k^2)(a^2-b^2)^2x^2y^2=0 \end{array}\right\}. \qquad (529)$$

110.] To determine the equation of the cone described in the body by $P_{\prime}$ the axis of rotation due to the centrifugal couple.

The axis $P_{\prime}$ makes with the axes of coordinates the angles $\lambda_{\prime}, \mu_{\prime}, \nu_{\prime}$. Let $x_{\prime}\, y_{\prime}\, z_{\prime}$ be the coordinates of a point on the surface of this cone; then

$$\frac{x'}{z'}=\frac{\cos\lambda_{\prime}}{\cos\nu_{\prime}}=\frac{z}{x}\left(\frac{b^2-c^2}{a^2-b^2}\right),\quad \frac{y_{\prime}}{z_{\prime}}=\frac{\cos\mu_{\prime}}{\cos\nu_{\prime}}=\frac{z}{y}\left(\frac{a^2-c^2}{a^2-b^2}\right).\quad \text{See (518).}$$

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* It may not be out of place to show that the equations of the *invariable cone*, and of the *cone of rotation* given in sec. [107] and sec. [108] are equivalent to the equations of the same cones given by Poisson in his *Traité de Mécanique* (tom. ii. pp. 151, 152). To show this, assume the equation of the *vis viva* given at page 140 of the same volume, $h=Ap^2+Bq^2+Cr^2$. Now

$$A=na^2,\ p=\frac{fx}{a^2},\ \text{whence}\ Ap^2=nf^2\frac{x^2}{a^2};$$

finding similar values for Bq^2 and Cr^2, we obtain

$$h=Ap^2+Bq^2+Cr^2=nf^2\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=nf^2;$$

we also have $\qquad A=na^2,\ B=nb^2,\ C=nc^2.$

$$k'=K=nfk;\ \text{hence}\ k'^2=n.nf^2k^2=\frac{A}{a^2}.h.k^2,\ \text{or}\ \frac{a^2}{k^2}=\frac{Ah}{k'^2}.$$

But the coefficient

$$\left(\frac{1}{a^2}-\frac{1}{k^2}\right)\ \text{may be written}\ \frac{1}{a^2}\left(1-\frac{a^2}{k^2}\right)=\frac{n}{A}\left(1-\frac{Ah}{k'^2}\right)=\frac{n}{k'^2}\left(\frac{k'^2-Ah}{A}\right);$$

making similar substitutions for the other coefficients and dividing by $\frac{n}{k'^2}$, we get

$$\left(\frac{k'^2-Ah}{A}\right)x^2+\left(\frac{k'^2-Bh}{B}\right)y^2+\left(\frac{k'^2-Ch}{C}\right)z^2=0.$$

In the same way (528) may be transformed into

$$(k'^2-Ah)x^2+(k'^2-Bh)y^2+(k'^2-Ch)z^2=0.$$

These are the equations given by Poisson.

Eliminating x, y, z from these equations, as also from those of the ellipsoid and sphere,

$$\left.\begin{aligned}(a^2-k^2)(b^2-c^2)^2b^2c^2y^2z^2+(b^2-k^2)(a^2-c^2)^2a^2c^2x^2z^2\\ +(c^2-k^2)(a^2-b^2)^2a^2b^2x^2y^2=0\end{aligned}\right\}, \quad (530)$$

which is also an equation of the fourth degree.

111.] The circular sections of the invariable cone coincide in position with the circular sections of the ellipsoid.

It is a property of surfaces of the second order*, that if in two such surfaces referred to the same or parallel axes the coefficients of the squares of the corresponding variables differ all by the same quantity, the circular sections of any two such surfaces are parallel.

Now the coefficients of the squares of the variables in the equation of the ellipsoid are $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, and the coefficients of

* Let $\quad Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z=1$

be the equation of a surface of the second degree, referred to rectangular axes. Let the surface now be referred to a new system of rectangular coordinates, such that the plane of $x'y'$ shall be parallel to one of the umbilical tangent planes, or to one system of circular sections of the surface. If in this transformed equation we make $z'=0$, we shall obtain the equation of a circle referred to rectangular axes, if the roots are real. The equation being that of a circle, we thence derive two conditions—the equality of the coefficients of the squares of the variables, and the evanescence of the coefficient of the rectangle $x'y'$. Let θ be the angle between the axes of z and z'. If we take the intersection of the plane of xy with the plane of one of the circular sections as the axis of x', ψ being the angle between the axis of x and x', we shall have, by the known transformations of coordinates, and putting $z'=0$,

$$x=\cos\psi x'+\cos\theta\sin\psi y', \quad y=-\sin\psi x'+\cos\theta\cos\psi y', \quad z=-\sin\theta y'.$$

Substituting these values of x, y, z in the given equation, the resulting equation in x' and y' is that of the conic section in which the plane of $x'y'$ intersects the given surface. As this section must be a circle, we get the two conditions

$$\begin{aligned}[(A-A'')\cos^2\psi+(A'-A'')\sin^2\psi-2B''\sin\psi\cos\psi]\tan^2\theta+2[B\cos\psi+B'\sin\psi]\tan\theta\\ =4B''\sin\psi\cos\psi-(A-A')(\cos^2\psi-\sin^2\psi)\end{aligned}$$

and

$$\tan\theta=\frac{B''(\cos^2\psi-\sin^2\psi)+(A-A')\sin\psi\cos\psi}{B'\cos\psi-B\sin\psi}.$$

From these equations eliminating $\tan\theta$, we should obtain a resulting equation of condition in ψ, whose coefficients would be functions of $(A-A')$, $(A-A'')$, $(A'-A'')$, B, B′, B″.

As the coefficients of the squares of the variables do not enter the coefficients of the resulting equation, but the differences of those coefficients only, it follows that two surfaces of the second order whose equations are of the form

$$\begin{aligned}&Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy \text{ \&c.}=1,\\ &(A+h)x^2+(A'+h)y^2+(A''+h)z^2+2Byz+2B'xz+2B''xy \text{ \&c.}=1,\end{aligned}$$

will have the planes of their circular sections parallel.

the equation of the cone are $\frac{1}{a^2}-\frac{1}{k^2}$, $\frac{1}{b^2}-\frac{1}{k^2}$, $\frac{1}{c^2}-\frac{1}{k^2}$, of which the constant difference is $\frac{1}{k^2}$.

112.] There are some general properties of rotatory motion, such as the principles of the *conservation of areas*, the *conservation of living forces*, &c., which may with much simplicity be here established.

Resuming the equation (466) and multiplying by ω^2, we get

$$\mathrm{H}\omega^2=\int \mathrm{dm}[\mathrm{R}\omega\sin\theta]^2,$$

the integral being extended to the whole mass of the body. Now $\mathrm{R}\omega\sin\theta$ is the velocity of the particle dm. The above integral therefore denotes the sum of all the elementary particles of the body multiplied each into the square of its velocity. This is termed the *vis viva* of the body.

In (504) it was shown that $\mathrm{H}=\mathrm{nP}^2$, and $\omega=\frac{f}{\mathrm{P}}$; whence $\mathrm{H}\omega^2=\mathrm{n}f^2$, or the *vis viva* of the body is constant, since n and f are constant.

Let the *vis viva* of the body be denoted by F, we shall have

$$\mathrm{F}=\text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (531)$$

Multiply the tangential equation of the ellipsoid of moments given in (474) by f^2, then

$$\mathrm{n}f^2=\mathrm{L}f^2\xi^2+\mathrm{M}f^2v^2+\mathrm{N}f^2\zeta^2-2\mathrm{U}f^2v\zeta-2\mathrm{V}f^2\xi\zeta-2\mathrm{W}f^2\xi v.$$

In (492) it was shown that $p=f\xi$, $q=fv$, $r=f\zeta$, whence

$$\mathrm{F}=\mathrm{L}p^2+\mathrm{M}q^2+\mathrm{N}r^2-2\mathrm{U}qr-2\mathrm{V}pr-2pq, \quad . \quad . \quad (532)$$

which is the equation of the *vis viva* in its most general form. When we take the principal axes as axes of coordinates,

$$\mathrm{U}=0,\ \mathrm{V}=0,\ \mathrm{W}=0, \text{ or } \mathrm{F}=\mathrm{L}p^2+\mathrm{M}q^2+\mathrm{N}r^2, \quad . \quad (533)$$

the form in which the equation of the *vis viva* is usually exhibited.

If we square the equations given in (490), and add the results,

$$\left.\begin{array}{r}(\mathrm{L}^2+\mathrm{V}^2+\mathrm{W}^2)p^2+(\mathrm{M}^2+\mathrm{W}^2+\mathrm{U}^2)q^2+(\mathrm{N}^2+\mathrm{U}^2+\mathrm{V}^2)r^2\\ -[\mathrm{U}(\mathrm{M}+\mathrm{N})-\mathrm{VW}]qr-2[\mathrm{V}(\mathrm{N}+\mathrm{L})-\mathrm{WU}]pr\\ -2[\mathrm{W}(\mathrm{M}+\mathrm{L})-\mathrm{UV}]pq=\mathrm{K}^2\end{array}\right\}. \quad (534)$$

In this equation is contained the principle of *the conservation of areas*; for $(\mathrm{K}l=\mathrm{L}p-\mathrm{V}r-\mathrm{W}q)$, see (490), is the sum of the areas described on the plane of yz, multiplied into the particles which describe those areas. Now these areas are projected on the plane of the impressed couple, by multiplying this expression by the cosine of the angle between the planes—that is, by l or its equal $\frac{\mathrm{L}p-\mathrm{V}r-\mathrm{W}q}{\mathrm{K}}$; and therefore $\frac{(\mathrm{L}p-\mathrm{V}r-\mathrm{W}q)^2}{\mathrm{K}}$ denotes the sum of the particles of the body multiplied into the areas described by these particles on the plane of yz, and then projected on the plane of the impressed couple. Finding analogous expressions for the two other coordinate planes, we get for the sum of all the particles of the body multiplied into the areas which they describe on the plane of the impressed couple,

$$\frac{(\mathrm{L}p-\mathrm{V}r-\mathrm{W}q)^2}{\mathrm{K}}+\frac{(\mathrm{M}q-\mathrm{W}p-\mathrm{U}r)^2}{\mathrm{K}}+\frac{(\mathrm{N}r-\mathrm{U}q-\mathrm{V}p)^2}{\mathrm{K}};$$

but the sum of these expressions must, we know, be equal to K, whence we obtain the formula given above.

When the axes of coordinates are the principal axes, V=0, U=0, W=0, and we get the well-known equation,

$$\mathrm{K}^2=\mathrm{L}^2p^2+\mathrm{M}^2q^2+\mathrm{N}^2r^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (535)$$

We may, in a very simple manner, establish the equations which embody the principles of the *vis viva*, and the *conservation of areas*, without using the method of tangential coordinates, when we restrict our choice of coordinates to the principal axes of the body; for

$$\mathrm{L}=\mathrm{n}a^2,\ p=\frac{fx}{a^2}, \text{ as shown in (500) and (506).}$$

Finding like values for the other analogous quantities and adding,

$$\mathrm{L}p^2+\mathrm{M}q^2+\mathrm{N}r^2=\mathrm{n}f^2\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=\mathrm{n}f^2=\mathrm{F}. \quad . \qquad (536)$$

Again, $\mathrm{L}^2p^2+\mathrm{M}^2q^2+\mathrm{N}r^2=\mathrm{n}^2f^2(x^2+y^2+z^2)=\mathrm{n}^2f^2k^2=\mathrm{K}^2.$ (537)

Let $p_{\prime}, q_{\prime}, r_{\prime}$, denote the angular velocities round the principal axes, the components of the angular velocities due to the centrifugal couple; then

$$\mathrm{L}^2p_{\prime}^2+\mathrm{M}^2q_{\prime}^2+\mathrm{N}^2r_{\prime}^2=\mathrm{K}^2\kappa^2\tan^2\theta. \quad . \quad . \quad . \quad . \qquad (538)$$

We have $\mathrm{L}=\mathrm{n}a^2,\ p_{\prime}=f\frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}{a^2}$. Writing similar expressions for the other analogous quantities,

$$\mathrm{L}^2p_{\prime}^2+\mathrm{M}^2q_{\prime}^2+\mathrm{N}^2r_{\prime}^2=\mathrm{n}^2f^2\left[\frac{\mathrm{d}x^2}{\mathrm{d}t^2}+\frac{\mathrm{d}y^2}{\mathrm{d}t^2}+\frac{\mathrm{d}z^2}{\mathrm{d}t^2}\right]=\mathrm{n}^2f^2\frac{\mathrm{d}s^2}{\mathrm{d}t^2}.$$

Now $\qquad \dfrac{ds}{dt}=f\tan\theta$, see (501), and $\dfrac{f}{k}=\kappa$, as in (508);

whence
$$L^2p_{\prime}^2+M^2q_{\prime}^2+N^2r_{\prime}^2=n^2f^2f^2\tan^2\theta$$
$$=n^2f^2k^2.\frac{f^2}{k^2}\tan^2\theta=K^2\kappa^2\tan^2\theta.$$

We may also show that,
$$Lp_{\prime}^2+Mq_{\prime}^2+Nr_{\prime}^2=F\frac{k^2}{u^2}.\kappa^2\tan^2\theta. \qquad (539)$$

113.] Using the principles established in the foregoing pages, the reader will find little difficulty in verifying the following theorems:—
$$p_{\prime}l+q_{\prime}m+r_{\prime}n=0, \qquad (a)$$
$$Lp_{\prime}l+Mq_{\prime}m+Nr_{\prime}n=0, \qquad (b)$$
$$\frac{p_{\prime}l}{L}+\frac{q_{\prime}m}{M}+\frac{r_{\prime}n}{N}=\frac{1}{2k}\frac{d}{dt}\omega^2. \qquad (c)$$

The sum of the squares of the distances of the vertices of the three semiaxes of the ellipsoid of moments from the plane of the impressed couple, divided by the corresponding moments of inertia, is constant during the motion.

Let $x_{\prime}$ be the distance of the vertex of a from the plane of the impressed couple. Then $x_{\prime}=al$, and $l=\dfrac{x}{k}$; hence $x_{\prime}=\dfrac{ax}{k}$ and $L=na^2$, or $\dfrac{x_{\prime}^2}{L}=\dfrac{x^2}{nk^2}$, whence
$$\frac{x_{\prime}^2}{L}+\frac{y_{\prime}^2}{M}+\frac{z_{\prime}^2}{N}=\frac{1}{nk^2}(x^2+y^2+z^2)=\frac{1}{n}. \qquad (d)$$

The sum of the squares of the distances of the vertices of the three semiaxes of the ellipsoid from the plane of the impressed couple, divided by the squares of the corresponding moments of inertia, is constant during the motion.

As before $x_{\prime}^2=\dfrac{a^2x^2}{k^2}$, $L^2=n^2a^4$; therefore $\dfrac{x_{\prime}^2}{L^2}=\dfrac{1}{n^2k^2}\left(\dfrac{x^2}{a^2}\right)$, whence
$$\left(\frac{x_{\prime}}{L}\right)^2+\left(\frac{y_{\prime}}{M}\right)^2+\left(\frac{z_{\prime}}{N}\right)^2=\frac{1}{n^2k^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\frac{1}{n^2k^2}. \qquad (e)$$

Let tangent planes be drawn to the vertices of a, b, c, the three semiaxes of the ellipsoid, cutting off from the axis of the plane of the impressed couple three segments. The sum of the squares of the reciprocals of those segments will be constant during the motion. Denoting these reciprocals by ξ, v, ζ, we shall have $\xi^2+v^2+\zeta^2=\frac{1}{k^2}$, during the motion; for $a\xi=l=\frac{x}{k}$, or $k\xi=\frac{x}{a}$; hence

$$\xi^2+v^2+\zeta^2=\frac{1}{k^2}\left(\frac{x^2}{k^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\frac{1}{k^2}. \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Again, $a^2\xi^2+b^2v^2+c^2\zeta^2=(l^2+m^2+n^2)=1$.

ξ, v, ζ, the reciprocals of the segments cut off from the axis of the plane of the impressed couple by three tangent planes drawn through the vertices of the axes of the surface, may be the segments of the axes of coordinates cut off by *any* tangent plane to the ellipsoid.

If through the vertex of k, which is a point fixed in space, a plane be drawn parallel to the plane of the impressed couple, this fixed plane will cut off segments from the axes of the ellipsoid during the motion, the sum of the squares of the reciprocals of which is constant.

Writing ξ, v, ζ for these reciprocals, we have

$$k\xi=l,\ kv=m,\ k\zeta=n;\ \text{hence}\ \ \xi^2+v^2+\zeta^2=\frac{1}{k^2}. \quad . \quad . \quad \text{(g)}$$

CHAPTER XV.

INVESTIGATION OF THE POSITION OF THE BODY AT THE END OF A GIVEN TIME.

114.] We must now proceed to the investigation of formulæ by whose aid we may be enabled to determine the position of the body at the end of a given epoch. For this purpose we shall obtain two distinct classes of formulæ, to determine not only relatively to certain fixed lines within the body (the principal axes suppose) the position of certain other lines, but also absolutely the position of these lines themselves in space. This double investigation is necessary, because the locus of a point will vary accordingly as we choose the axes of coordinates fixed in space, or varying in position according to some given law. For example, the instantaneous axis of rotation describes on a sphere concentric with the body, and moving along with it, a spherical conic, while it describes on a concentric sphere fixed in space a spiral which undulates continually between two small parallel circles of the sphere.

Again, under certain conditions the same straight line may describe in the body a plane, or on the moving sphere a great circle, while it describes in absolute space a sort of spiral cone, or on the surface of the fixed sphere a spiral approaching very nearly to the loxodromic or rumb line.

We have hitherto assumed k as lying between the mean and least semiaxes of the ellipsoid, or $a^2>b^2>k^2>c^2$. Should we require to consider the case when k lies between the greatest and mean semiaxes of the ellipsoid, the formulæ will be most easily modified so as to embrace this hypothesis also, by taking in that case c as the greatest semiaxis, and b the mean semiaxis as before, or $a^2<b^2<k^2<c^2$. While on the former supposition the binomials a^2-b^2, a^2-c^2, b^2-c^2, a^2-k^2, b^2-k^2, k^2-c^2, are all positive, on the latter they will all be negative. Now, in the formulæ which we shall have to deal with in the remaining portion of this subject, these binomials occur generally in pairs, connected either by multiplication or division. It will result, therefore, that no effective change of sign will generally take place, whether we suppose k to lie between the greatest and mean semiaxes, or between the mean and the least. The case where k is equal to the mean axis will require a separate investigation. When the body is a solid of revolution we cannot take N equal to L or M, or c equal to a or b, because we suppose c to be the greatest or the least of the three semiaxes. The only hypothesis, not inconsistent with previous assumptions, is L=M, or $a=b$; and this is the assumption generally made when the case of a solid of revolution is considered.

Resuming one of the equations (515),

$$\frac{dt}{dz}=\frac{a^2b^2}{f(a^2-b^2)xy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

If we agree to take $\frac{dt}{dz}$ with the positive sign when $a>b$, we must attach the negative sign when $a<b$.

To integrate this equation, we must express x and y in terms of z. This we can easily do by eliminating x and y alternately from the equations of the ellipsoid of moments and the concentric sphere. We hence find

$$x=\frac{a\sqrt{(b^2-c^2)z^2-c^2(b^2-k^2)}}{c\sqrt{a^2-b^2}},\ y=\frac{b\sqrt{c^2(a-k^2)-(a^2-c^2)z^2}}{c\sqrt{a^2-b^2}}. \text{(b)}$$

Making these substitutions in (a), the last equation becomes

$$\frac{dt}{dz}=\frac{abc^2}{f\sqrt{[(b^2-c^2)z^2-c^2(b^2-k^2)][c^2(a^2-k^2)-(a^2-c^2)z^2]}}.\text{*} \quad (540)$$

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* If we assume the relations established in the note at page 186,

$$\text{A}=\text{n}a^2,\ \text{B}=\text{n}b^2,\ \text{C}=\text{n}c^2,\ h=\text{n}f^2,\ k'=\text{n}fk,\ r=\frac{fz}{c^2},\ \frac{dz}{dr}=\frac{\text{G}^2}{f},\ \frac{dz}{dr}=\frac{c^2}{f},$$

To facilitate the integration of this equation, assume

$$z^2=\frac{c^2(a^2-k^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi}. \qquad (541)$$

Substituting the value of z derived from this equation in (540),

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)^2=\frac{a^2b^2c^2}{f^2[(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi]}, \quad (541^*)$$

or integrating, we obtain the following elliptic integral of the first order,

$$t=\frac{\pm abc}{f\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{\mathrm{d}\varphi}{\sqrt{1-\left[\frac{(a^2-b^2)(k^2-c^2)}{(b^2-c^2)(a^2-k^2)}\right]\sin^2\varphi}}. \qquad (542)$$

115.] *The modulus of this function is the sine of the semifocal angle of the invariable cone.*

Resuming the equation of this cone given in (527), and writing α and β for its principal semiangles,

$$\tan^2\alpha=\frac{b^2(k^2-c^2)}{c^2(b^2-k^2)},\ \tan^2\beta=\frac{a^2(k^2-c^2)}{c^2(a^2-k^2)}. \quad . \quad . \quad . \quad (a)$$

Now, ϵ being the semifocal angle of this cone, $\cos\epsilon=\frac{\cos\alpha}{\cos\beta}$ as in (2), or $\sin^2\epsilon=\frac{\cos^2\beta-\cos^2\alpha}{\cos^2\beta}=\frac{(a^2-b^2)(k^2-c^2)}{(b^2-c^2)(a^2-k^2)}$; . . (b)

hence $\cos^2\epsilon=\frac{(a^2-c^2)(b^2-k^2)}{(b^2-c^2)(a^2-k^2)}$, and $\sec\alpha\cos\epsilon=\frac{k\sqrt{(a^2-c^2)}}{c\sqrt{(a^2-k^2)}}$; . (c)

Consequently the coefficient of the elliptic integral in (542),

$\frac{abc}{f\sqrt{(a^2-k^2)(b^2-c^2)}}$, may now be written $\frac{abc^2\sec\alpha\cos\epsilon}{k^2\kappa\sqrt{(a^2-c^2)(b^2-c^2)}}$. (d)

In (508) it was shown that $f=k\kappa$. Introducing this relation into the preceding coefficient, and making

$$j=\frac{\kappa\cos\alpha\, k^2\sqrt{(a^2-c^2)(b^2-c^2)}}{abc^2}, \quad . \quad . \quad . \quad . \quad (543)$$

---

and by the help of these relations eliminate from (540) the quantities a, b, c, f, z, k, we shall obtain the resulting equation

$$\mathrm{d}t=\frac{\pm\sqrt{\mathrm{AB}}.\mathrm{C}\mathrm{d}r}{[k'^2-\mathrm{B}h+(\mathrm{B}-\mathrm{C})\mathrm{C}r^2]^{\frac{1}{2}}[\mathrm{A}h-k'^2+(\mathrm{C}-\mathrm{A})\mathrm{C}r^2]^{\frac{1}{2}}},$$

the expression which Poisson arrives at, *Traité de Mécanique*, tom. ii. p. 140.

(542) may now be written

$$jt=\cos\epsilon\int\frac{d\phi}{\sqrt{1-\sin^2\epsilon\sin^2\phi}}. \quad . \quad . \quad . \quad . \quad (544)$$

In (58) it was shown that the arc $\bar{\sigma}$ of a spherical parabola whose principal arcs $\bar{\alpha}$ and $\bar{\beta}$ are given by the equations

$$\tan^2\bar{\alpha}=\frac{1+\sin\gamma}{1-\sin\gamma},\quad \tan^2\bar{\beta}=\frac{2\sin\gamma}{1-\sin\gamma},$$

may be represented by an integral of the first order, or

$$\bar{\sigma}=\sin\gamma\int\frac{d\phi}{\sqrt{1-\cos^2\gamma\sin^2\phi}}+\tan^{-1}\left[\frac{\sin\gamma\tan\phi}{\sqrt{1-\cos^2\gamma\sin^2\phi}}\right]; \quad (e)$$

writing ς for the circular arc, we get the simple formula

$$jt=\bar{\sigma}-\varsigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (545)$$

In this case, $\tan^2\bar{\alpha}=\dfrac{1+\cos\epsilon}{1-\cos\epsilon}=\cot^2\tfrac{1}{2}\epsilon$, or $2\bar{\alpha}+\epsilon=\pi$. . . . (f)

$2\bar{\alpha}$ and ϵ are therefore supplemental.

When ϵ vanishes, $\bar{\alpha}=\dfrac{\pi}{2}$, $\bar{\beta}=\dfrac{\pi}{2}$, or the spherical parabola becomes a great circle of the sphere.

When the moment N of the body is very nearly equal to L or M, c^2 must very nearly be equal to a^2 or b^2, and the coefficient j becomes indefinitely small.

116.] It may easily be shown that the amplitude ϕ of the elliptic integral assumed in (541) is the eccentric anomaly of the vertex of k, the axis of the impressed couple. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the semiaxes of the plane ellipse, the intersection of the invariable cone with a plane which touches the sphere whose radius is k, which is drawn at right angles to the axis c of the ellipsoid, the internal axis of this cone.

Let the plane which passes through the axis c and k cut the plane of the ellipse in the semidiameter R, making the angle ψ with the axis $\mathfrak{a}$ of the ellipse. Then, as $\mathfrak{a}=k\tan\alpha$, $\mathfrak{b}=k\tan\beta$, and, ρ being the angle which k makes with the axis of z, $\mathrm{R}=k\tan\rho$, we shall have

$$\cos^2\rho=\frac{\dfrac{1}{\tan^2\alpha}+\dfrac{\tan^2\psi}{\tan^2\beta}}{\dfrac{1}{\sin^2\alpha}+\dfrac{\tan^2\psi}{\sin^2\beta}}, \text{ as shown in sec. [8].}$$

Let ϕ' be the eccentric anomaly, then $\tan\phi'=\dfrac{\mathfrak{a}}{\mathfrak{b}}\tan\psi$, . (a)

or $\tan\psi=\dfrac{\tan\beta}{\tan\alpha}\tan\phi'$, and $\cos^2\rho=\dfrac{\cos^2\alpha}{1-\sin^2\epsilon\sin^2\phi'}$. . . . (b)

In (541) we assumed

$$\frac{z^2}{k^2}=\frac{\frac{c^2}{k^2}\left(\frac{b^2-k^2}{b^2-c^2}\right)}{1-\left[\frac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}\right]\sin^2\phi}; \text{ but } \frac{z}{k}=\cos\rho,\ \frac{c^2(b^2-k^2)}{k^2(b^2-c^2)}=\cos^2\alpha;$$

and $\frac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}=\sin^2\epsilon$. Comparing this expression with (b) we find $\phi=\phi'$.

Or ϕ is the eccentric anomaly of the vertex of k.

117.] Resuming the equation established in (544), we may invert the formula, $jt=\cos\epsilon\int\frac{d\phi}{\sqrt{1-\sin^2\epsilon\sin^2\phi}}$, and express the amplitude ϕ in terms of the function jt. Accordingly let ϕ be a function of jt, or $\phi=(jt)$*, the parenthesis denoting a function of jt. Substituting this value in the value assumed for z in (541), we find the following values of x, y, z—

$$\left.\begin{aligned}
x^2&=\frac{a^2(b^2-k^2)(k^2-c^2)\sin^2(jt)}{(a^2-k^2)(b^2-c^2)\cos^2(jt)+(b^2-k^2)(a^2-c^2)\sin^2(jt)},\\
y^2&=\frac{b^2(a^2-k^2)(k^2-c^2)\cos^2(jt)}{(a^2-k^2)(b^2-c^2)\cos^2(jt)+(b^2-k^2)(a^2-c^2)\sin^2(jt)},\\
z^2&=\frac{c^2(a^2-k^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)\cos^2(jt)+(b^2-k^2)(a^2-c^2)\sin^2(jt)}.
\end{aligned}\right\}\quad(546)$$

---

* That the assumption here made is allowable, may be shown as follows.

Let $(1-i^2\sin^2\phi)^{-\frac{1}{2}}$ be developed in a series of cosines of multiple arcs; for the successive integral powers of $\sin^2\phi$ may be so developed. Accordingly let

$$\frac{j}{\sqrt{1-i^2\sin^2\phi}}=A+2B\cos 2\phi+4C\cos 4\phi+6D\cos 6\phi \text{ &c.}$$

Integrating these equivalent expressions, and putting t for $\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}$, we get $jt=A\phi+B\sin 2\phi+C\sin 4\phi+D\sin 6\phi\ \ldots\ldots$ &c. now

$$\sin 2\phi=2\phi-\frac{(2\phi)^3}{1\,2\,3}+\frac{(2\phi)^5}{1\,2\,3\,4\,5}\text{ &c.}$$

$$\sin 4\phi=4\phi-\frac{(4\phi)^3}{1\,2\,3}+\frac{(4\phi)^5}{1\,2\,3\,4\,5}\text{ &c.}$$

$$\sin 6\phi=6\phi-\frac{(6\phi)^3}{1\,2\,3}\text{ &c.}$$

Substituting these values of the sines of the multiple arcs of ϕ in the preceding equation,

$$jt=\alpha\phi+\beta\phi^3+\gamma\phi^5+\text{&c.};$$

or, by the inverse method of series,

$$\phi=\alpha_{,}[jt]+\beta_{,}[jt]^3+\gamma_{,}[jt]^5\text{ &c.};$$

or ϕ may be taken as a function of jt, or we may put $\phi=(jt)$, as in the text.

We may also express x, y, z in terms of the time and of the constants of the invariable cone. Transforming the expressions given in the preceding formulæ, we find

$$\left.\begin{aligned}\frac{x^2}{k^2}&=\frac{\tan^2\beta\sin^2(jt)}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)},\\ \frac{y^2}{k^2}&=\frac{\tan^2\alpha\cos^2(jt)}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)},\\ \frac{z^2}{k^2}&=\frac{1}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}.\end{aligned}\right\} \quad . \quad . \quad (547)$$

From either of these groups of equations we may find the coordinates $x\,y\,z$ of the vertex of k the axis of the impressed couple, in terms of the time. We can thus determine the particular diameter of the ellipsoid which happens to coincide with the axis of the impressed couple at the end of the time t. And if we suppose the ellipsoid brought into this position, we shall have the inclination of the equator of the body to the plane of the impressed couple. This, however, is not sufficient to determine completely the position of the body. The body might take any position round this line as an axis, $x\,y\,z$ remaining unchanged. We must therefore determine the position of some other fixed line or plane in the body. One of the most obvious is the intersection of the plane of the equator of the body or of the plane of $x\,y$ with the plane of the impressed couple. The position of this line being ascertained at any epoch, the position of the body will be completely determined.

118.] To determine the value of ω the angular velocity at the end of any given time.

Since $\omega^2=\frac{f^2}{\mathrm{P}^2}=f^2\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right]$, substituting for $x\,y\,z$ their values given in terms of the time in (547), we find

$$\omega^2=k^2f^2\frac{\left[\frac{1}{c^4}+\frac{\tan^2\alpha}{b^4}\cos^2(jt)+\frac{\tan^2\beta}{a^4}\sin^2(jt)\right]}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}. \quad . \quad (548)$$

This formula may be simplified as follows.

It was shown in sec. [108] that the instantaneous axis of rotation describes a cone of the second degree, whose equation is

$$a^4\left(\frac{1}{a^2}-\frac{1}{k^2}\right)x^2+b^4\left(\frac{1}{b^2}-\frac{1}{k^2}\right)y^2+c^4\left(\frac{1}{c^2}-\frac{1}{k^2}\right)z^2=0.$$

Let α' and β' be the principal angles of this cone. It may easily be shown that

$$\tan^2\alpha'=\frac{c^2(k^2-c^2)}{b^2(b^2-k^2)},\quad \tan^2\beta'=\frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}\,; \quad . \quad . \quad (\mathrm{a})$$

whence $\tan\alpha' = \frac{c^2}{b^2}\tan\alpha, \quad \tan\beta' = \frac{c^2}{a^2}\tan\beta. \quad . \quad . \quad . \quad .$ (b)

Introducing into the value of ω these functions, we get

$$\omega^2 = \frac{f^2k^2}{c^4}\left[\frac{\sec^2\alpha'\cos^2(jt) + \sec^2\beta'\sin^2(jt)}{\sec^2\alpha\cos^2(jt) + \sec^2\beta\sin^2(jt)}\right]^*. \quad . \quad (549)$$

---

* Let the axis of the impressed couple very nearly coincide with one of the principal axes (that of c suppose); then k is very nearly equal to c, or to z, and the angular velocity round the axis of z, being given by the equation $r = \frac{fz}{c^2}$, as in (506), $r = \frac{f}{c}$, a constant quantity which may be put equal to n, or $\kappa = n$.

In this case the invariable cone becoming indefinitely attenuated, $\sec\alpha = 1$, $\sin\epsilon = 0$, and $k = c$ nearly; so that the formula given in sec. [114]

$$t = \frac{c^2}{kf}\frac{\sec\alpha}{\sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2b^2}}} \times \cos\epsilon \int \frac{d\phi}{\sqrt{1 - \sin^2\epsilon\sin^2\phi}}$$

may now be written, $nt = \frac{\phi}{\sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2b^2}}}$. To use the notation adopted by Poisson in the *Traité de Mécanique*, let A, B, C denote the moments of inertia round the principal axes; then $A = na^2$, $B = nb^2$, $C = nc^2$,

$$\text{whence } \sqrt{\frac{(a^2-c^2)(b^2-c^2)}{a^2b^2}} = \sqrt{\frac{(A-C)(B-C)}{AB}} = \delta,$$

or $n\delta t = \phi$, whence $j = n\delta$.

In (546) we found $x^2 = \frac{a^2(b^2-k^2)(k^2-c^2)\sin^2 jt}{(a^2-k^2)(b^2-c^2)\cos^2 jt + (b^2-k^2)(a^2-c^2)\sin^2 jt}$.

Since k^2 is equal to c^2 nearly, let $k^2 = c^2 + \nu^2$, in which ν is a quantity indefinitely small; the above formula may now be written

$$x^2 = \frac{\nu^2a^2[b^2-c^2-\nu^2]\sin^2 n\delta t}{(a^2-c^2)(b^2-c^2) - \nu^2[(b^2-c^2)\cos^2 n\delta t + (a^2-c^2)\sin^2 n\delta t]},$$

or, neglecting ν^2 when added to finite quantities,

$$x^2 = \frac{\nu^2a^2(b^2-c^2)\sin^2 n\delta t}{(a^2-c^2)(b^2-c^2)}.$$

Taking the square root and reducing,

$$\frac{fx}{a^2} = \frac{\nu f \mathrm{n}\sqrt{b^2(b^2-c^2)}\,.\sin n\delta t}{\sqrt{\mathrm{n}^2a^2b^2(a^2-c^2)(b^2-c^2)}}.$$

Now assume $$\frac{\nu f}{\sqrt{\mathrm{n}^2a^2b^2(a^2-c^2)(b^2-c^2)}} = \alpha,$$

whence $\frac{fx}{a^2} = \alpha\sqrt{B(B-C)}\sin(n\delta t + \gamma)$. γ is added, since x and t may be supposed not to vanish together. In like manner, $\frac{fy}{b^2} = \alpha\sqrt{A(A-C)}\cos(n\delta t + \gamma)$.

We may also express the components p, q, r of the angular velocity in terms of the time—

$$\left.\begin{aligned}
p^2&=\frac{f^2k^2}{a^4}\left[\frac{\tan^2\beta\sin^2(jt)}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}\right],\\
q^2&=\frac{f^2k^2}{b^4}\left[\frac{\tan^2\alpha\cos^2(jt)}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}\right],\\
r^2&=\frac{f^2k^2}{c^4}\left[\frac{1}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}\right].
\end{aligned}\right\}\quad . \quad (550)$$

The angles which the instantaneous axis of rotation makes with the principal axes, are given by the equations

$$\frac{\cos\lambda}{\cos\nu}=\frac{c^2}{a^2}\frac{x}{z}=\frac{c^2}{a^2}\tan\beta\sin(jt),\quad \frac{\cos\mu}{\cos\nu}=\frac{c^2y}{b^2z}=\frac{c^2}{b^2}\tan\alpha\cos(jt),$$

or, as $\tan\alpha'=\frac{c^2}{b^2}\tan\alpha$, $\tan\beta'=\frac{c^2}{a^2}\tan\beta$, as in (b),

$$\frac{\cos\lambda}{\cos\nu}=\tan\beta'\sin(jt),\quad \frac{\cos\mu}{\cos\nu}=\tan\alpha'\cos(jt),\quad . \quad . \quad (c)$$

$$\left.\begin{aligned}
\cos^2\lambda&=\frac{\tan^2\beta'\sin^2(jt)}{\sec^2\alpha'\cos^2(jt)+\sec^2\beta'\sin^2(jt)},\\
\cos^2\mu&=\frac{\tan^2\alpha'\cos^2(jt)}{\sec^2\alpha'\cos^2(jt)+\sec^2\beta'\sin^2(jt)},\\
\cos^2\nu&=\frac{1}{\sec^2\alpha\cos^2(jt)+\sec^2\beta\sin^2(jt)}.
\end{aligned}\right\}\quad . \quad . \quad (551)$$

These equations give us the position of the instantaneous axis of rotation with reference to the principal axes, in terms of the time.

119.] We must now, in order completely to determine the position of the body at the end of the time t, investigate a formula which will enable us to ascertain the position of some other line in the body at the end of the given epoch. We may take the straight line

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In (506) it was shown that $p=\frac{fx}{a^2}$, $q=\frac{fy}{b^2}$; whence

$$p=\alpha\sqrt{B(B-C)}\sin(n\delta t+\gamma),\quad q=\alpha\sqrt{A(A-C)}\cos(n\delta t+\gamma).$$

These are the formulæ established by Poisson, on this particular hypothesis, by methods wholly dissimilar. (*Traité de Mécanique*, tom. ii. p. 154.)

When k is absolutely equal to c, $\nu=0$, and therefore $\alpha=0$, or $p=0$, $q=0$, whatever be the value of t. Since $K=fkn$, $F=f^2n$, we get

$$a^2=\frac{K^2-FN}{LM(L-N)(M-N)},\text{ or, using Poisson's notation, } \alpha^2=\frac{k^2-hC}{AB(A-C)(B-C)}.$$

in which the equator of the body (the plane of xy suppose) and the plane of the impressed couple intersect.

The angular velocity of the body round the axis k being uniform and equal to κ, the angle described on the plane of the impressed moment in the element of the time dt will be κdt, or the angle κt in the time t, measured from a given line in this plane, its intersection with the plane of the equator of the body, or the plane of the axes a, b. But this line, which may be called the *line of the nodes,* will itself have an angular motion on the plane of the impressed moment during the time; this angle may be denoted by ψ, whence the whole elementary angle will be

$$\frac{d\psi}{dt}+\kappa. \quad \text{Let this angle be } \frac{d\vartheta}{dt}, \text{ then } \frac{d\psi}{t}+\kappa=\frac{d\vartheta}{dt}. \quad . \quad \text{(a)}$$

Now this elementary angle is the projection, on the plane of the impressed moment, of the angle on the plane of $a\,b$, over which the projection of the axis k on the plane of $a\,b$ passes in the time dt. Let ρ be the angle between these planes, or the angle between k and the axis of z. Then $\cos\rho=\frac{z}{k}$, and the angle of which $\frac{d\vartheta}{dt}$ is the projection is $\frac{k}{z}\frac{d\vartheta}{dt}$. Hence the area described on the plane of $a\,b$ by the projection of k upon it is $\frac{1}{2}(x^2+y^2)\,\frac{k}{z}\frac{d\vartheta}{dt}$. This area may also be represented by the expression $\frac{1}{2}\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)$. Equating these expressions for the same elementary area,

$$(x^2+y^2)\,\frac{k}{z}\frac{d\vartheta}{dt}=\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Now $\frac{dx}{dt}=\frac{f(b^2-c^2)yz}{b^2c^2}$, $\frac{dy}{dt}=\frac{f(c^2-a^2)xz}{a^2c^2}$, as in (515).

Whence

$$y\frac{dx}{dt}-x\frac{dy}{dt}=\frac{fz}{a^2b^2c^2}\{a^2b^2y^2+a^2b^2x^2-a^2c^2y^2-b^2c^2x^2\}. \quad . \quad . \quad \text{(c)}$$

The equations of the ellipsoid and sphere give

$$b^2c^2x^2+a^2c^2y^2=a^2b^2c^2-a^2b^2z^2, \quad a^2b^2y^2+a^2b^2x^2=a^2b^2k^2-a^2b^2z^2.$$

$$\text{Consequently } y\frac{dx}{dt}-x\frac{dy}{dt}=fz\left(\frac{k^2-c^2}{z^2}\right). \quad . \quad . \quad . \quad \text{(d)}$$

And as $x^2+y^2=k^2-z^2$, $\frac{f}{k}=\kappa$, we at length obtain

$$\frac{d\vartheta}{dt}=\kappa\left(\frac{k^2-c^2}{c^2}\right)\left(\frac{z^2}{k^2-z^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

whence

$$-\frac{d\psi}{dt}=\kappa\left[1-\left(\frac{k^2-c^2}{c^2}\right)\left(\frac{z^2}{k^2-z^2}\right)\right]. \quad . \quad . \quad . \quad (552)$$

To integrate this equation, assume as in (541)

$$\frac{z^2}{c^2}=\frac{(a^2-k^2)(b^2-k^2)}{(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi}, \quad . \quad \text{(f)}$$

whence

$$\frac{k^2-c^2}{c^2}\cdot\frac{z^2}{k^2-z^2}=\frac{(a^2-k^2)(b^2-k^2)}{b^2(a^2-k^2)-k^2(a^2-b^2)\sin^2\varphi}; \quad . \quad . \quad \text{(g)}$$

and writing for dt its value as given in (541*), we obtain by integration the elliptic integral

$$-\psi=\kappa t\mp\frac{ac(b^2-k^2)}{kb\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{d\varphi}{\left[1-\frac{k^2(a^2-b^2)}{b^2(a^2-k^2)}\sin^2\varphi\right]\sqrt{1-\sin^2\epsilon\sin^2\varphi}}. \quad \text{(h)}$$

Now, e being the eccentricity of the plane base of the cone the locus of the axis of the impressed couple, (a) sec. [115] gives

$$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{k^2(a^2-b^2)}{b^2(a^2-k^2)}. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

We find also

$$\frac{ac(b^2-k^2)}{bk\sqrt{(a^2-k^2)(b^2-c^2)}}=\pm\frac{\tan\beta}{\tan}\cos\alpha, \quad . \quad . \quad . \quad \text{(j)}$$

taking the negative sign when $b>k$.

Introducing these transformations, the last equation (h) becomes

$$-\psi=\kappa t\mp\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\epsilon\sin^2\varphi}}. \quad (553)$$

If we now turn to the formula given in (15), we shall there find that this elliptic integral is the algebraical expression for an arc of the spherical ellipse, supplemental to the one whose principal arcs are α and β, supplemental in this case, therefore, to the invariable spherical conic. Writing σ for this arc, we get the simple relation

$$-\psi=\kappa t\mp\sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (554)$$

We may hence infer that the line of the nodes, or the intersection of the plane of the equator ab with the plane of the impressed couple, describes an angle which is made up of two parts: one of these parts is a circular arc increasing uniformly with the time; the other, σ, is an arc of the spherical ellipse which is the base of the cone supplemental to the invariable cone. Now, as the axis of the impressed couple is always a side of the invariable cone, the plane of the impressed couple will always be a tangent plane to the supplemental cone; and it may easily be shown that the line of contact of the plane of the impressed couple with this cone is always at right angles to the line of the nodes.

It follows, therefore, that the line of the nodes is *retrograde*, and in the time t will describe the angle $\kappa t \mp \sigma$.

The angle $-\psi$ equal to $\kappa t \mp \sigma$, we may imagine to be thus described. Let this supplemental cone be conceived to roll on the plane of the impressed couple with such a velocity that the axis of the conjugate tangent plane may describe the invariable cone with the velocity given in (510). Let, moreover, the invariable plane be conceived to revolve uniformly round its axis. We shall then have a perfect idea of the rotatory motion of a body revolving round a fixed point, free from the action of accelerating forces. In this manner it is shown that the most general motion of a body round a fixed point may be reduced to that of a cone which rolls without sliding with a certain variable velocity on a plane whose axis is fixed, while this plane rotates round its axis with a certain uniform velocity.

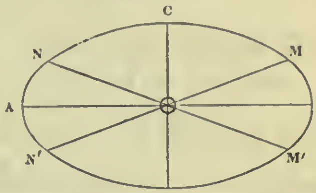
This cone is always given, and may be determined as follows:—

The circular sections of the invariable cone coincide with the circular sections of the ellipsoid of moments (see sec. [111]), whence the cyclic axes of the ellipsoid, or the diameters perpendicular to the planes of those sections, will be the focal lines of the supplemental cone. As the invariable plane is always a tangent plane to this cone, we have elements sufficient given to determine it; for when the two focals of a cone and a tangent plane to it are given, we may determine it, just as we may a conic section when its foci and a tangent to it are given.

120.] From these considerations it follows that we may altogether dispense with the ellipsoid of moments, and say that if two straight lines are drawn through the fixed point of the body, in the plane of the greatest and least moments, making equal angles with the axis of greatest moment, whose cosines shall be equal to the square root of the expression $\frac{L(M-N)}{M(L-N)}$, and a cone be conceived having these lines as focals, and touching, moreover, the plane of the impressed couple, the entire motion of this body will consist in the rotation of this cone on the invariable plane, with a variable velocity, while the plane revolves round its own axis with a uniform velocity.

Let ACB be the mean plane section of the ellipsoid, or that which passes through the axes $2a$, $2c$; ON, ON$'$ the cyclic axes; then, if the plane of the impressed couple coincides with any of the principal planes, the cones round the cyclic axes as focals become planes also, and the axis of rotation coincides with one of the axes of the figure.

Fig. 32.



Again, if the plane of the impressed couple intersects the mean plane between N and C, it will envelope the cone whose focals are ON, ON$'$, and whose internal axis is therefore OA. But if it intersect between A and N, it will envelope the cone whose focals are ON, OM, and whose internal axis is OC. Whence the range in the former case (which may be taken as the measure of the stability of rotation round the axis whose moment is the greatest) is to the range in the latter case (which may also be assumed as the representative of the stability of rotation round that axis whose moment of inertia is the least) as the supplement of the angle between the cyclic axes of the ellipsoid is to the angle between these axes.

It is also evident that the sign of the spherical elliptic arc will depend on the sign of the binomial (b^2-k^2) in (j) sec. [119]. The signs of κt and σ being contrary when $b<k$, they will be the same when $b>k$. We may therefore infer that the direction in which the angle σ shall be described will depend upon the position of the axis k in the body—whether it lies within the region between the planes of the circular sections of the ellipsoid, or without.

From the theorem established in sec. [4] we may infer that the product of the sines of the angles, which the cyclic axes of the body make with the plane of the impressed couple, is constant during the motion; for the cyclic axes of the ellipsoid of moments are the focals of the cone supplemental to the invariable cone.

121.] To determine the angle between the instantaneous axis of rotation and the line of the nodes.

Let this angle be $\delta_{/}$. The cosines of the angles which the axis of the impressed couple makes with the axes of coordinates being as before l, m, n, let the cosines of the angles which the line of the nodes makes with the same axes be $l_{//}$, $m_{//}$, $n_{//}$; λ, μ, ν, are the angles which the instantaneous axis of rotation makes with the same axes.

Then $$\cos\delta_{/}=l_{//}\cos\lambda+m_{//}\cos\mu+n_{//}\cos\nu. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

As the line of the nodes lies in the plane of the impressed couple, and is therefore at right angles to its axis k,

$$0=l_{//}l+m_{//}m+n_{//}n; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

and as it is perpendicular to the axis of Z, see sec. [119],

$$0=l_{\prime\prime}\cos\frac{\pi}{2}+m_{\prime\prime}\cos\frac{\pi}{2}+n_{\prime\prime}\cos 0, \text{ or } 0=n_{\prime\prime}; \quad . \quad . \quad \text{(c)}$$

hence (a) and (b) become

$$\cos\delta_{\prime}=l_{\prime\prime}\cos\lambda+m_{\prime\prime}\cos\mu, \quad l_{\prime\prime}l+m_{\prime\prime}m=0; \text{ and } l_{\prime\prime}^2+m_{\prime\prime}^2=1.$$

These equations give $m_{\prime\prime}=\frac{l}{\sqrt{l^2+m^2}}$, $l_{\prime\prime}=\frac{-m}{\sqrt{l^2+m^2}}$,
whence
$\cos\delta_{\prime}=\frac{l\cos\mu-m\cos\lambda}{\sqrt{l^2+m^2}}$; now $l=\frac{x}{k}$, $m=\frac{y}{k}$, $\cos\lambda=\frac{\mathrm{P}x}{a^2}$, $\cos\mu=\frac{\mathrm{P}y}{b^2}$, . (d)

$$\text{or } \cos\delta_{\prime}=\frac{\mathrm{P}(a^2-b^2)xy}{a^2b^2\sqrt{x^2+y^2}}. \quad . \quad . \quad . \quad . \quad . \quad (555)$$

When two of the moments of inertia are equal (L=M, suppose), $a=b$, and $\cos\delta_{\prime}=0$, or $\delta_{\prime}=90°$. Whence we may infer that *when the body is a solid of revolution, the angle between the instantaneous axis of rotation and the line of the nodes is always a right angle.*

The angle $\delta_{\prime}$ is also a right angle whenever the axis of the impressed couple lies in one of the planes of the principal sections of the ellipsoid; for then $x=0$, or $y=0$.

122.] To determine the angle between the line of the nodes and the axis u of the centrifugal couple.

Let χ be the angle which the axis u of the centrifugal couple makes with a fixed line, ψ the angle which the line of the nodes makes with the same fixed line; then as the line of the nodes and u are in the plane of the impressed couple, see (498), the angle to be determined is $(\chi-\psi)$.

Now the cosines of the angles which u makes with the axes are $\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}, \frac{\mathrm{d}z}{\mathrm{d}s}$; whence $\cos(\chi-\psi)=l_{\prime\prime}\frac{\mathrm{d}x}{\mathrm{d}s}+m_{\prime\prime}\frac{\mathrm{d}y}{\mathrm{d}s}+n_{\prime\prime}\frac{\mathrm{d}z}{\mathrm{d}s}$.

The values of $l_{\prime\prime}$, $m_{\prime\prime}$, $n_{\prime\prime}$ were found in the last section to be

$$l_{\prime\prime}=\frac{m}{\sqrt{l^2+m^2}}, \quad m_{\prime\prime}=\frac{-l}{\sqrt{l^2+m^2}}, \quad n_{\prime\prime}=0; \quad l=\frac{x}{k}, \quad m=\frac{y}{k}.$$

We may hence deduce

$$\cos(\chi-\psi)=\frac{k}{\sqrt{k^2-z^2}}\left[\frac{y}{k}\frac{\mathrm{d}x}{\mathrm{d}s}-\frac{x}{k}\frac{\mathrm{d}y}{\mathrm{d}s}\right]; \quad . \quad . \quad . \quad \text{(a)}$$

but
$$\frac{\mathrm{d}x}{\mathrm{d}s}=\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}, \quad \frac{\mathrm{d}y}{\mathrm{d}s}=\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s},$$

and $\frac{\mathrm{d}x}{\mathrm{d}t}=f\,\frac{(b^2-c^2)}{b^2c^2}yz$, $\frac{\mathrm{d}y}{\mathrm{d}t}=f\,\frac{(c^2-a^2)}{a^2c^2}xz$, as in (515).

Whence

$$\frac{ds}{dt}\cos(\chi-\psi)=\frac{fz}{\sqrt{k^2-z^2}}\left[\frac{a^2b^2y^2+a^2b^2x^2-a^2c^2y^2-b^2c^2x^2}{a^2b^2c^2}\right].$$

The part within the brackets is $\frac{(k^2-c^2)}{c^2}$; and $\frac{ds}{dt}=f\tan\theta$, see (510);

whence $$\cos(\chi-\psi)=\frac{z}{\sqrt{k^2-z^2}}\left(\frac{k^2-c^2}{c^2}\right)\cot\theta. \quad . \quad . \quad . \quad \text{(b)}$$

ρ being the angle between the axes c and k, $\cos\rho=\frac{z}{k}$. Introducing this value of z into (514) and the trigonometrical functions of α and β the principal semiangles of the invariable cone, as given in (a), sec. [115],

$$\tan\theta=\left(\frac{k^2-c^2}{c^2}\right)\sqrt{\frac{\cos^2\rho-\cos^2\alpha\cos^2\beta}{\sin^2\alpha\sin^2\beta}}, \quad . \quad . \quad \text{(c)}$$

whence $$\cos^2(\chi-\psi)=\frac{\sin^2\alpha\sin^2\beta}{\sin^2\rho-\cos^2 a\cos^2\beta\tan^2\rho},$$

and $$\tan^2(\chi-\psi)=\frac{(\sin^2\alpha-\sin^2\rho)(\sin^2\rho-\sin^2\beta)}{\sin^2\alpha\sin^2\beta\cos^2\rho}. \quad . \quad (556)$$

This formula leads us to infer that when $\alpha=\beta$, $\chi-\psi$ is always 0, or $\chi=\psi$; whence the axis of the centrifugal couple, when the solid is one of revolution, always coincides with the line of the nodes.

Again, when $\rho=\alpha$, or $\rho=\beta$, $\chi=\psi$; that is, whenever the axis of the impressed couple lies in one of the principal planes of the solid, *the axis of the centrifugal couple coincides with the line of the nodes.*

CHAPTER XVI.

123.] In the preceding sections formulæ are given which enable us to determine the position of the axis of rotation, and of the axis of the plane of the impressed couple, with reference to fixed lines taken within the body. It still, however, remains to determine the positions not only of those lines, but of the fixed lines within the body, relatively to absolute space. True, we may by transformations of coordinates, and by the choice of other variables, obtain solutions from the formulæ already established, by methods which, however, are tedious, complex, and not a little obscure. It will be found not only the most direct, but by far the most elegant method of procedure, to conduct the investigation independently, and start from first principles.

As the body must now be referred to fixed lines in space, it is no less obvious than natural that we should assume the plane of the impressed couple as one of the coordinate planes. Let this plane be taken as that of $x\,y$, its axis that of z. Moreover let the plane of the greatest and least principal axes of the ellipsoid of moments coincide with the plane of $x\,z$, at the beginning of the time t. The instantaneous axis of rotation will be in the same plane at the same epoch, and will make with the vertical axis k an angle whose tangent is given by the equation

$$\tan^2\Theta=\frac{(a^2-k^2)(k^2-c^2)}{a^2c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (557)$$

This may easily be shown; for the perpendicular from the centre on a tangent through the vertex of k, a semidiameter of an ellipse whose semiaxes are a and c, makes with k an angle whose tangent is given by the last formula.

In like manner, for the principal section whose semiaxes are b and c, we get

$$\tan^2\Theta_{\prime}=\frac{(b^2-k^2)(k^2-c^2)}{b^2c^2}. \quad . \quad . \quad . \quad . \quad . \quad (557^*)$$

Θ and $\Theta_{\prime}$ are the maximum and minimum values of θ, the angle between the axis of the impressed couple and the instantaneous axis of rotation.

124.] We now proceed to establish the following proposition:—

The area described by the axis u of the centrifugal couple, on the plane of the impressed couple, varies as the time.

The following relations were established in (524), (510), (507), (508)—

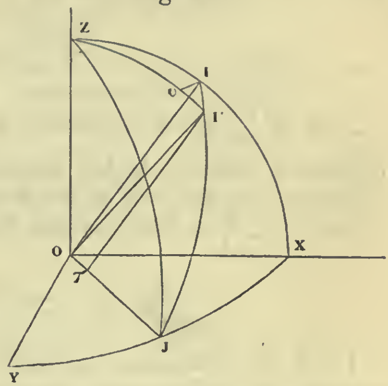
$$\frac{\frac{\mathrm{d}s}{\mathrm{d}t}}{\mathrm{P}_{\prime}u}=\frac{\omega_{\prime}}{f}, \quad \frac{\mathrm{d}s}{\mathrm{d}t}=f\tan\theta, \quad \omega=\frac{f}{\mathrm{P}}, \quad \text{and} \quad \kappa=\frac{f}{k};$$

whence

$$\frac{\omega_{\prime}}{\omega}=\frac{\mathrm{P}k}{\mathrm{P}_{\prime}u}\,\kappa\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (558)$$

Let O be the centre of a sphere whose radius is 1, concentric with the ellipsoid of moments, Z the point in which the axis of the plane of the impressed couple meets it, and OI the direction of the instantaneous axis of rotation at the end of time t. Let the plane which passes through these lines OZ, OI, or the plane of the centrifugal couple coincide with the plane of $x\,z$ at the same instant. Then the axis of Y will at that instant be the axis of the centrifugal couple; and the perpendicular from the centre on the tangent plane to the ellipsoid, at the point where the axis of Y intersects it, will be the axis of rotation due to the centrifugal couple, see sec. [104]. Let the direction of this perpendicular be OJ. Through OIJ let a plane be drawn. If, along OI, OJ the instantaneous axes of rotation, we assume lengths OI, $O\tau$, proportional to the angular velocities ω, ω' round these axes, the diagonal OI′, of the parallelogram constructed with those lines as sides, will represent in direction the instantaneous axis of rotation at the end of the time $t+dt$.

Fig. 33.



Let OI, $O\tau$ taken in this proportion, be the sides of the parallelogram; the diagonal OI′ will be the contemporaneous position of this axis of rotation.

Let the angle $ZOI=\theta$, $YOJ=\theta'$; also let δ be the angle between the planes of IOJ and ZOX. Then, as the instantaneous axis of rotation due to the centrifugal couple lies always in the plane of the impressed couple, see sec. [103], the line OJ is in the plane of xy, and the angle $JOX=\frac{\pi}{2}-\theta'$. Let χ be the angle which the vector arc θ makes with a fixed great circle of the sphere passing through Z. The instantaneous axis having moved into the position OI′, the arc ZI will have moved into the position ZI′, or through the angle $d\chi$, in the time dt. Let Iv be an arc of a great circle perpendicular to ZI′, and as II′v is an infinitesimal right-angled triangle we shall have $II'\sin\delta=Iv=\frac{d\chi}{dt}\sin\theta$. Again, as IJX is a spherical triangle, right-angled at X; sin IJ : sin JX : : 1 : sin δ,

$$\text{or } \sin IJ=\frac{\cos\theta'}{\sin\delta}.$$

We are also given by the construction,

$$\frac{\omega'}{\omega}=\frac{\sin II'}{\sin IJ}=\frac{II'\sin\delta}{\cos\theta'}=\frac{d\chi}{dt}\frac{\sin\theta}{\cos\theta'};$$

and (525) gives
$$\frac{\omega'}{\omega}=\frac{Pk}{P'u}\kappa\tan\theta.$$

Equating these values of $\frac{\omega'}{\omega}$, and introducing the relations $P=k\cos\theta$, $P'=u\cos\theta'$, we get

$$u^2\frac{d\chi}{dt}=\kappa k^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (559)$$

Now $u^2\frac{d\chi}{dt}$ is the elementary area described on the plane of the impressed moment by the semidiameter u of the ellipsoid which coincides with the axis of the centrifugal couple; whence the area described by this semidiameter is proportional to the time, or

$$\int u^2\frac{d\chi}{dt}\,dt=\kappa t k^2+\text{constant}. \quad . \quad . \quad . \quad . \quad (560)$$

125.] To determine the position of the instantaneous axis of rotation in absolute space, at the end of any given time.

If along the axes of rotation due to the impressed and centrifugal couples, we take two lines to represent the angular velocities due to those couples, the diagonal of the parallelogram, constructed with these lines as sides, will represent the instantaneous position of the axis of rotation.

Now, if we turn to the figure at p. 213, we shall see that

$$\sin II' : \sin IJ :: \omega' : \omega, \text{ and ultimately } \frac{d\sigma}{dt}=\sin II'; \text{ whence}$$

$$\frac{d\sigma}{dt}=\frac{\omega'}{\omega}\sin IJ; \quad \text{or} \quad \left(\frac{d\sigma}{dt}\right)^2=\frac{\omega'^2}{\omega^2}-\frac{\omega'^2}{\omega^2}\cos^2 IJ. \quad . \quad . \quad (a)$$

The general formula for the element of an arc measured on the surface of a sphere is

$$\left(\frac{d\sigma}{dt}\right)^2=\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\chi}{dt}\right)^2, \text{ whence}$$

$$\left(\frac{d\theta}{dt}\right)^2=\frac{\omega'^2}{\omega^2}-\sin^2\theta\left(\frac{d\chi}{dt}\right)^2-\frac{\omega'^2}{\omega^2}\cos^2 IJ. \quad . \quad . \quad (b)$$

We must now reduce this formula.

In (559) it was shown that $\frac{d\chi}{dt}=\frac{\kappa k^2}{u^2}$, and in (525) that $\frac{\omega'}{\omega}=\frac{\kappa k^2}{P'u}\sin\theta$. Making the substitutions suggested by these transformations, we shall find

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = k^4\kappa^2\sin^2\theta\left[\frac{1}{\mathrm{P}'^2u^2} - \frac{1}{u^4}\right] - \frac{\omega'^2}{\omega^2}\cos^2\mathrm{IJ}. \quad . \quad . \qquad (c)$$

We shall now proceed to reduce the first term of the second member of this formula. To facilitate the calculations, let

$$\mathrm{Q} = k^4\kappa^2\sin^2\theta\left[\frac{1}{\mathrm{P}_{\prime}^{2}u^2} - \frac{1}{u^4}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

Multiplying by $\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^4$, we shall have

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^4\mathrm{Q} = k^4\kappa^2\sin^2\theta\left[\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{\mathrm{P}_{\prime}^{2}u^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 - \left(\frac{\frac{\mathrm{d}s}{\mathrm{d}t}}{u}\right)^4\right]. \quad . \quad . \qquad (e)$$

s, it must be borne in mind, is the arc of the invariable conic; and zyx are the coordinates of the vertex of k referred to the principal planes of the ellipsoid.

Now, if we turn to sec. [106] and sec. [107], we shall there find

$$\left.\begin{aligned}
\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{\mathrm{P}_{\prime}^{2}u^2} &= \frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}{a^4} + \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}{b^4} + \frac{\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}{c^4},\\
\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{u^2} &= \frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}{a^2} + \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}{b^2} + \frac{\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}{c^2},\\
\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2.
\end{aligned}\right] \quad . \quad . \quad . \quad . \qquad (f)$$

Introducing the substitutions suggested by these transformations, we shall obtain

$$\frac{a^4b^4c^4\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^4\mathrm{Q}}{k^4\kappa^2\sin^2\theta} = \left[\begin{aligned}
&b^4c^4\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\left[\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - 2\frac{a^2}{b^2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]\\
&+a^4c^4\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - 2\frac{b^2}{c^2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\\
&+a^4b^4\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - 2\frac{c^2}{a^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right].
\end{aligned}\right] \qquad (g)$$

Making the obvious reduction in this equation,

$$\left.\begin{aligned}\frac{\left(\frac{ds}{dt}\right)^4 Q}{\kappa^2 k^4 \sin^2\theta} &= \left(\frac{1}{c^2}-\frac{1}{b^2}\right)^2\left(\frac{dy}{dt}\right)^2\left(\frac{dz}{dt}\right)^2 \\ &+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)^2\left(\frac{dz}{dt}\right)^2\left(\frac{dx}{dt}\right)^2+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2\left(\frac{dx}{dt}\right)^2\left(\frac{dy}{dt}\right)^2.\end{aligned}\right] \quad (h)$$

We have also, see (515),

$$\left(\frac{dx}{dt}\right)^2 = f^2\frac{(b^2-c^2)^2}{b^4c^4}y^2z^2, \quad \left(\frac{dy}{dt}\right)^2 = f^2\frac{(a^2-c^2)^2}{a^4c^4}x^2z^2,$$

$$\left(\frac{dz}{dt}\right)^2 = f^2\frac{(a^2-b^2)^2}{a^4b^4}x^2y^2; \text{ or, reducing,}$$

$$\left(\frac{1}{c^2}-\frac{1}{b^2}\right)^2\left(\frac{dy}{dt}\right)^2\left(\frac{dz}{dt}\right)^2 = \frac{f^4(b^2-c^2)^2(a^2-c^2)^2(a^2-b^2)^2x^4y^2z^2}{a^8b^8c^8}.$$

Finding similar values for the other symmetrical expressions, substituting, introducing the relation $x^2+y^2+z^2=k^2$, and writing for $\frac{ds}{dt}$ its value $f\tan\theta$, we shall finally obtain

$$Q = \left[\frac{\kappa k^3 \sin\theta\,(a^2-b^2)\,(a^2-c^2)\,(b^2-c^2)xyz}{a^4b^4c^4\tan^2\theta}\right]^2. \quad . \quad . \quad (i)$$

We have now to compute the term $\frac{\omega'^2}{\omega^2}\cos^2 \mathrm{I\,J}$.

In sec. [106] it was shown that the angle between the axes of rotation due to the impressed and centrifugal couples, was given by the formula

$$\cos \mathrm{I\,J} = \frac{pp_{\prime}+qq_{\prime}+rr_{\prime}}{\omega\omega_{\prime}};$$

$$\text{whence } \frac{\omega_{\prime}^2}{\omega'^2}\cos^2\mathrm{I\,J} = \left(\frac{pp_{\prime}+qq_{\prime}+rr_{\prime}}{\omega^2}\right)^2.$$

In (506) and (516) it was shown that

$$p=\frac{fx}{a^2},\ p_{\prime}=\frac{f^2(b^2-c^2)yz}{a^2b^2c^2}, \quad \text{or } pp_{\prime}=\frac{f^3xyz}{a^2b^2c^2}\left(\frac{b^2-c^2}{a^2}\right).$$

Finding analogous expressions for $qq_{\prime}$ and $rr_{\prime}$,

$$pp_{\prime}+qq_{\prime}+rr_{\prime} = \frac{f^3xyz}{a^2b^2c^2}\left[\frac{b^2-c^2}{a^2}+\frac{c^2-a^2}{b^2}+\frac{a^2-b^2}{c^2}\right]. \quad . \quad . \quad (j)$$

Now $\omega = \frac{f}{\mathrm{P}} = \frac{f}{k \cos \theta}$, as in (508); and

$$\frac{b^2-c^2}{a^2} + \frac{c^2-a^2}{b^2} + \frac{a^2-b^2}{c^2} = \frac{(b^2-c^2)(a^2-c^2)(a^2-b^2)}{a^2b^2c^2}; \text{ whence}$$

$$\frac{\omega'^2}{\omega^2} \cos^2 \mathrm{IJ} = \frac{f^2k^4 \cos^4 \theta (a^2-b^2)^2 (a^2-c^2)^2 (b^2-c^2)^2 x^2y^2z^2}{a^8b^8c^8}. \quad \text{(k)}$$

Multiplying this expression, numerator and denominator, by $\tan^4 \theta$, writing κk for f, and in the expression

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \mathrm{Q} - \frac{\omega'^2}{\omega^2} \cos^2 \mathrm{IJ}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(l)}$$

substituting for the terms of the second member the values found in the preceding equations, reducing, and taking the square root,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\kappa k^3 \sin \theta \cos \theta (a^2-b^2)(b^2-c^2)(a^2-c^2)xyz}{a^4b^4c^4 \tan^2 \theta}. \quad . \quad (561)$$

We have now to express x, y, z in terms of θ.

Combining the simultaneous equations of the ellipsoid of moments, of the concentric sphere, and of the perpendicular from the centre on the tangent plane to the ellipsoid, namely

$$\left.\begin{array}{l} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1, \qquad x^2+y^2+z^2 = k^2, \\ \text{and } \dfrac{k^2}{\mathrm{P}^2} = 1 + \tan^2 \theta = k^2 \left[\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}\right], \end{array}\right\} \quad . \quad . \quad . \quad \text{(m)}$$

we obtain from these equations,

$$\left.\begin{array}{l} \dfrac{x^2}{a^4} = \dfrac{[b^2c^2 \tan^2 \theta - (b^2-k^2)(k^2-c^2)]}{k^2(a^2-c^2)(a^2-b^2)}, \\ \dfrac{y^2}{b^4} = \dfrac{[(a^2-k^2)(k^2-c^2) - a^2c^2 \tan^2 \theta]}{k^2(a^2-b^2)(b^2-c^2)}, \\ \dfrac{z^2}{c^4} = \dfrac{[a^2b^2 \tan^2 \theta + (a^2-k^2)(b^2-k^2)]}{k^2(a^2-c^2)(b^2-c^2)}. \end{array}\right\} \quad . \quad . \quad (562)$$

Substituting these values of x, y, z in (561), the resulting equation will become

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{a^2b^2c^2 \sin \theta \sec^3 \theta}{\kappa \sqrt{[a^2b^2 \tan^2 \theta + (a^2-k^2)(b^2-k^2)][b^2c^2 \tan^2 \theta - (b^2-k^2)(k^2-c^2)][(a^2-k^2)(k^2-c^2) - a^2c^2 \tan^2 \theta]}}. \quad (563)$$

This is an elliptic integral of the first order, which may be reduced to the usual form by assuming

$$a^2b^2c^2\tan^2\theta=(k^2-c^2)\left[b^2(a^2-k^2)\cos^2\lambda+a^2(b^2-k^2)\sin^2\lambda\right]. \quad (564)$$

Before we proceed further, we shall give the geometrical interpretation of this assumption.

Let a cone be conceived whose internal axis shall coincide with the axis of the plane of the impressed couple, or with the axis of z, and whose principal arcs shall be the greatest and least elongations of the instantaneous axis of rotation from the axis of the impressed couple. This cone will generate on the surface of the sphere a spherical conic, the tangents of whose principal arcs ($2\alpha''$, $2\beta''$) are given as in (557) by the equations,

$$\tan^2\alpha''=\frac{(a^2-k^2)(k^2-c^2)}{a^2c^2}, \quad \tan^2\beta''=\frac{(b^2-k^2)(k^2-c^2)}{b^2c^2}. \quad . \quad \text{(n)}$$

This cone may be named the *cone of nutation.*

Now, if from the centre of this curve the vector arc θ is drawn to a point on it, λ is the angle which the perpendicular arc from the centre on the tangent arc through the vertex of θ, makes with the principal arc α''.

To simplify the notation, let

$$\left.\begin{aligned} \mathrm{X}&=b^2c^2\tan^2\theta-(b^2-k^2)(k^2-c^2), \\ \mathrm{Y}&=(a^2-k^2)(k^2-c^2)-a^2c^2\tan^2\theta, \\ \mathrm{Z}&=a^2b^2\tan^2\theta+(a^2-k^2)(b^2-k^2), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (565)$$

and the equation (563) will become

$$\frac{dt}{d\theta}=\frac{a^2b^2c^2\tan^2\theta}{\kappa\sin\theta\cos\theta\sqrt{\mathrm{X.Y.Z.}}}. \quad . \quad . \quad . \quad . \quad (566)$$

If we differentiate (564), and make the transformations resulting from that assumption, we shall get the following relations:—

$$\left.\begin{aligned} a^2\mathrm{X}&=k^2(a^2-b^2)(k^2-c^2)\cos^2\lambda; \\ b^2\mathrm{Y}&=k^2(a^2-b^2)(k^2-c^2)\sin^2\lambda; \text{ and} \\ c^2\mathrm{Z}&=k^2(a^2-k^2)(b^2-c^2)\cos^2\lambda+k^2(b^2-k^2)(a^2-c^2)\sin^2\lambda. \end{aligned}\right\} . \quad (566^*)$$

By the help of these transformations, equation (566) takes the form

$$t=\frac{\mp abc}{f\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{d\lambda}{\sqrt{1-\left[\frac{(a^2-b^2)(k^2-c^2)}{(b^2-c^2)(a^2-k^2)}\right]\sin^2\lambda}}, \quad (567)$$

which is precisely the same elliptic integral we found in (542), differing from it only in the amplitude λ and the sign. When $b>a$ the positive sign must be taken. We shall show presently that ϕ and λ have opposite signs.

This formula may be thus written, as in (544),

$$t=\frac{\mp abc^2 \sec\alpha \cos\epsilon}{k^2\kappa\sqrt{(a^2-c^2)(b^2-c^2)}}\int\frac{\mathrm{d}\lambda}{\sqrt{1-\sin^2\epsilon\sin^2\lambda}}. \quad . \quad (568)$$

When the integrals are complete they are identical, as they manifestly should be, because the maximum and minimum values of θ, the greatest and least elongations of the instantaneous axis of rotation from the axis of the plane of the impressed couple, should be given by the same formula, whatever system of axes we choose—since this value must be independent of the position of any axes chosen at will, being a function of the constitution of the body, and of the magnitude and position of the impressed couple.

126.] To determine the angle χ, which θ the vector arc, drawn from the vertex of k, to the pole of the instantaneous axis of rotation, makes with a fixed plane passing through k the axis of the impressed couple.

Resuming the equation $\frac{\mathrm{d}\chi}{\mathrm{d}t}=\frac{\kappa k^2}{u^2}$, established in (560), we have now to express u^2 in terms of λ.

If we turn to (522), we shall there find

$$\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}{u^2}=\frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}{a^2}+\frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}{b^2}+\frac{\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}{c^2}, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

or $$\frac{a^2b^2c^2}{u^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=b^2c^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+a^2c^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+a^2b^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2.$$

Eliminating $\frac{\mathrm{d}z}{\mathrm{d}t}$ by the relation $\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$,

$$\frac{a^2b^2c^2}{u^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=a^2b^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2-b^2(a^2-c^2)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-a^2(b^2-c^2)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2.$$

Now $$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=\frac{f^2(b^2-c^2)^2z^2y^2}{b^4c^4}, \qquad \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=\frac{f^2(a^2-c^2)^2x^2z^2}{a^4c^4},$$

as shown in (515).

Having made these substitutions, we shall find

$$\frac{a^2b^2c^2}{u^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2=a^2b^2\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2-\frac{(a^2-c^2)(b^2-c^2)z^2}{a^2b^2c^4}f^2(a^2b^2y^2+a^2b^2x^2-a^2c^2y^2-b^2c^2x^2).$$

Eliminating x^2 and y^2 by the equations of the ellipsoid and sphere, introducing also the relations $\frac{ds}{dt}=f\tan\theta$ and

$$a^2b^2c^4\tan^2\theta=(a^2-c^2)(b^2-c^2)k^2z^2-c^4(a^2-k^2)(b^2-k^2),$$

as given in (514), we get

$$\frac{k^2}{u^2}=\frac{a^2b^2c^2\tan^2\theta+(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2b^2c^2\tan^2\theta}. \quad . \quad . \quad (569)$$

In this equation substituting the value of $\tan\theta$, given in terms of λ in (564), we obtain

$$\frac{1}{u^2}=\frac{(a^2-k^2)\cos^2\lambda+(b^2-k^2)\sin^2\lambda}{b^2(a^2-k^2)\cos^2\lambda+a^2(b^2-k^2)\sin^2\lambda}. \quad . \quad . \quad . \quad (570)$$

Now this may easily be reduced to the form

$$\frac{1}{u^2}=\frac{\frac{1}{b^2}-\frac{(a^2-b^2)}{b^2(a^2-k^2)}\sin^2\lambda}{1-\frac{k^2}{b^2}\left(\frac{a^2-b^2}{a^2-k^2}\right)\sin^2\lambda}. \quad . \quad . \quad . \quad . \quad (571)$$

But it has been already shown in (i) sec. [119] that

$$\frac{k^2}{b^2}\frac{(a^2-b^2)}{(a^2-k^2)}=e^2,$$

e being the eccentricity of the plane elliptic base of the invariable cone.

Whence $$\frac{k^2}{u^2}=1-\left(\frac{b^2-k^2}{b^2}\right)\left[\frac{1}{1-e^2\sin^2\lambda}\right]. \quad . \quad . \quad . \quad . \quad (572)$$

Introducing this value of $\frac{k^2}{u^2}$ into the equation $\chi=\kappa\int\frac{k^2}{u^2}\,dt$, writing for dt its value as given in (567), and integrating, we shall obtain the final result,

$$\chi=-\kappa t\pm\frac{ac}{bk}\frac{(b^2-k^2)}{\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{d\lambda}{[1-e^2\sin^2\lambda]\,\sqrt{1-\sin^2\epsilon\sin^2\lambda}}. \quad (573)$$

The positive sign to be taken when $b>k$.

This elliptic integral differs from (553) only in the amplitude.

When the integrals (553) and (573) are complete, the values of ψ and χ become identical, as they manifestly ought to be, because in sec. [122] it was shown that the line of the nodes coincides with the axis of the centrifugal couple whenever the instantaneous axis of rotation lies in one of the principal planes of the ellipsoid.

127.] If we eliminate z and $\tan\theta$ between (514), (541), and (564), we shall get the following relation between ϕ and λ,

$$\tan\phi \tan\lambda = \sec\epsilon; \quad . \quad . \quad . \quad . \quad . \quad . \quad (574)$$

hence $\dfrac{d\phi}{d\lambda} = -\dfrac{\sin 2\phi}{\sin 2\lambda}$; or ϕ and λ have opposite signs.

But these angles differ in their origin by a right angle, since ϕ is measured from the plane of bc, while λ is measured from that of ac; subtracting ϕ from $\frac{\pi}{2}$ to make their origins coincide, then

$$\tan\phi = \cos\epsilon \tan\lambda; \quad . \quad . \quad . \quad . \quad . \quad (575)$$

this formula coincides with that given in (39).

Now, when the ellipsoid is a figure of revolution (a equal to b, suppose), the invariable cone becomes a right cone of revolution, whence the angles between its focals vanish, or $\epsilon=0$. Therefore ϕ is always equal to λ; that is, the amplitudes of the functions are identical throughout their whole extent, as plainly they ought to be, because in this case the line of the nodes always coincides with the axis of the centrifugal couple.

$$\text{when } \phi=0,\ \lambda=0; \text{ and when } \phi=\frac{\pi}{2},\ \lambda=\frac{\pi}{2}.$$

We may repeat here what has been said in sec. [119], that the expression

$$\frac{ac(b^2-k^2)}{bk\sqrt{(a^2-k^2)(b^2-c^2)}}\int\frac{d\lambda}{[1-e^2\sin^2\lambda]\sqrt{1-\sin^2\epsilon\sin^2\lambda}}$$

may be transformed into this other,

$$\frac{\tan\beta}{\tan\alpha}\cos\alpha\int\frac{d\lambda}{[1-e^2\sin^2\lambda]\sqrt{1-\sin^2\epsilon\sin^2\lambda}},$$

which represents, as has been shown in sec. [8], an arc of the spherical conic, supplemental to the invariable spherical ellipse.

The relation between χ and λ is given by the following elliptic integral,

$$\left.\begin{aligned}[(a^2-k^2)(b^2-c^2)]^{\frac{1}{2}}\chi = \frac{ac(b^2-k^2)}{bk}\int\frac{d\lambda}{[1-e^2\sin^2\lambda]\sqrt{1-\sin^2\epsilon\sin^2\lambda}}\\ -\frac{abc}{k}\int\frac{d\lambda}{\sqrt{1-\sin^2\epsilon\sin^2\lambda}}.\end{aligned}\right\}. \quad (576)$$

128.] We may now determine the angular velocity round the instantaneous axis of rotation, and the nutation of this axis, in formulæ of great simplicity.

Since in (568) the time is given in terms of λ, we may reverse the formula and obtain λ a function of t'. (See note, p. 202). t' in this equation is no longer the same numerical quantity as t in sec. [117]; for while all the constants in (542) and (568) are the same, the amplitudes φ and λ are different. Accordingly let

$$j_{\prime} : j :: t' : t; \text{ hence } j_{\prime}t = jt'. \quad . \quad . \quad . \quad . \quad (a)$$

Let $$\lambda = (jt'), \text{ or } \lambda = (j_{\prime}t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Then in (186) writing for $\tan^2\theta$ its value $\frac{k^2}{P^2}-1$, we get

$$\frac{1}{P^2} = \frac{(a^2+c^2-k^2)}{a^2c^2}\cos^2\lambda + \frac{(b^2+c^2-k^2)}{b^2c^2}\sin^2\lambda. \quad . \quad . \quad (c)$$

Let $P_{\prime}$ and $P_{\prime\prime}$ be the greatest and least values of P; then

$$\frac{1}{P^2} = \frac{\sin^2\lambda}{P_{\prime}^2} + \frac{\cos^2\lambda}{P_{\prime\prime}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

or P is a semidiameter of a plane ellipse whose principal semiaxes are $P_{\prime}$ and $P_{\prime\prime}$.

If Ω and $\Omega_{\prime}$ are put for the greatest and least angular velocities,

$$\Omega = \frac{f}{P_{\prime\prime}}, \quad \Omega' = \frac{f}{P_{\prime}};$$

we hence get for the angular velocity the very simple expression

$$\omega^2 = \Omega^2\cos^2(j_{\prime}t) + \Omega_{\prime}^2\sin^2(j_{\prime}t); \quad . \quad . \quad . \quad (577)$$

or the angular velocity varies as the perpendicular on a tangent to a plane ellipse whose principal semiaxes are proportional to Ω and Ω'.

In the same way writing Θ and Θ' for the greatest and least values of θ, the nutation of the instantaneous axis of rotation from the axis of the plane of the impressed couple, we obtain

$$\tan^2\theta = \tan^2\Theta\cos^2(j_{\prime}t) + \tan^2\Theta'\sin^2(j_{\prime}t). \quad . \quad (578)$$

This formula may easily be obtained, if we multiply (d) by k^2, subtract 1 from the first number, and $\cos^2\lambda + \sin^2\lambda$ from the second.

CHAPTER XVII.

ON THE SPIRAL DESCRIBED ON A FIXED CONCENTRIC SPHERE BY THE INSTANTANEOUS AXIS OF ROTATION OF THE BODY.

129.] If it were possible to eliminate λ from the equations (564) and (576), we should have a direct equation between θ and χ, the polar spherical coordinates of the curve. We cannot do this; but still we may perceive that as the equations involve the angle χ simply and no trigonometrical function of it, while θ is a periodic function involving sines and cosines of arcs which increase with the time, the curve must be some sort of spiral described on the surface of the fixed sphere. But although this direct elimination is in the general case extremely difficult, perhaps impossible to effect, we may however be enabled successfully to investigate some of the more important properties of this spiral in the general case, and to give its polar equation in a particular case of rotatory motion.

The spiral, analogous to the herpoloid of Poinsot, has two asymptotic circles on the surface of the sphere.

The angle τ which the vector arc θ of a spherical curve, drawn from the origin to any point on the curve, makes with a tangent at that point, is given by the equation

$$\tan\tau=\sin\theta\,\frac{d\chi}{d\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (579)$$

This is evident, because the sides of the elementary right-angled triangle on the surface of the sphere are the element of the arc, the differential of the vector arc θ, and the distance $\sin\theta d\chi$ at that point between two consecutive meridians.

We may transform this equation into

$$\tan\tau=\sin\theta\,\frac{d\chi}{dt}\cdot\frac{dt}{d\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Now in (559) it was shown that $\frac{d\chi}{dt}=\kappa\frac{k^2}{u^2}$, and in (569) that

$$\frac{k^2}{u^2}=\frac{a^2b^2c^2\tan^2\theta+(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2b^2c^2\tan^2\theta},$$

while (563) gives $\frac{dt}{d\theta}=\frac{a^2b^2c^2\tan^2\theta}{\kappa\sin\theta\cos\theta\sqrt{XYZ}}$; whence

$$\tan\tau=\frac{a^2b^2c^2\tan^2\theta+(a^2-k^2)(b^2-k^2)(c^2-k^2)}{\cos\theta\sqrt{XYZ}}. \quad . \quad (580)$$

Now, whatever supposition we make with respect to the magnitude of k, some one of the factors X, Y, Z, in (565), must be essentially positive, and cannot become cipher. In this case Z is essentially positive. Making $X=0$, and $Y=0$, successively, we get

$$\tan^2\theta=\frac{(b^2-k^2)(k^2-c^2)}{b^2c^2} \text{ and } \tan^2\theta=\frac{(a^2-k^2)(k^2-c^2)}{a^2c^2};$$

but when $X=0$, or $Y=0$, $\tan\tau=\infty$, or τ is a right angle; hence, when θ has either of these values, the spiral touches one or other of the circles whose spherical radii are the values of $\tan\theta$ given above.

If we make θ greater or less than the limiting values just given, either X or Y will become negative, and the value of $\tan\theta$ therefore imaginary. We may hence infer that the spiral on the surface of the sphere is confined between two planes parallel to the plane of the impressed couple, and that it always undulates between two parallel small circles of the sphere, having its apsides alternately upon them.

Let $P_{\prime}$ and $P_{\prime\prime}$ be the greatest and least values of P, the perpendicular from the centre of the ellipsoid of moments on the instantaneous tangent plane. The area of the spherical belt or zone, within which the undulations of the spiral are contained, is equal to $2\pi k(P_{\prime}-P_{\prime\prime})$.

130.] It was shown in sec. [108] that the instantaneous axis of rotation referred to the principal axes of the body generates a cone of the second degree. We shall now proceed to establish the following remarkable theorem.

The length of the spiral between any two successive apsides is constant, and equal to a quadrant of the spherical ellipse generated by the cone of rotation.

Let σ be the arc of this spiral,

$$\text{then } \left(\frac{d\sigma}{dt}\right)^2=\left(\frac{d\theta}{dt}\right)^2+\sin^2\theta\left(\frac{d\chi}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad (a)$$

(566) (559) and (569) give us $\left(\frac{d\theta}{dt}\right)^2=\frac{\kappa^2\sin^2\theta\cos^2\theta\,(X.Y.Z)}{a^4b^4c^4\tan^4\theta};$

also
$$u^2\frac{d\chi}{dt}=\kappa k^2,$$

and

$$\sin^2\theta\left(\frac{d\chi}{dt}\right)^2=\frac{\kappa^2\sin^2\theta[a^2b^2c^2\tan^2\theta+(a^2-k^2)(b^2-k^2)(c^2-k^2)]^2}{a^4b^4c^4\tan^4\theta}.$$

Making the requisite substitutions in the general formula for the spherical arc, we shall find

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2=\frac{\kappa^2\sin^2\theta\cos^2\theta\,(\mathrm{X.Y.Z})+\kappa^2\sin^2\theta\,[a^2b^2c^2\tan^2\theta+(a^2-k^2)(b^2-k^2)(c^2-k^2)]^2}{a^4b^4c^4\tan^4\theta}. \quad \text{(b)}$$

In (565) we found

$$\mathrm{X}=b^2c^2\tan^2\theta-(b^2-k^2)(k^2-c^2),$$
$$\mathrm{Y}=(a^2-k^2)(k^2-c^2)-a^2c^2\tan^2\theta,$$
$$\mathrm{Z}=a^2b^2\tan^2\theta+(a^2-k^2)(b^2-k^2).$$

Substituting these values of X, Y, Z in the preceding formula, squaring the second member, and adding, we shall find, after some rather complicated reductions,

$$\frac{a^2b^2c^2}{k^4\kappa^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2=(a^2+b^2+c^2-2k^2)\sin^2\theta\cos^2\theta+(a^2+b^2+c^2-k^2)\left[\frac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2b^2c^2}\right]\cos^4\theta. \quad \text{(c)}$$

We must now reduce this formula to a form suited for integration. In (564) we made the assumption,

$$a^2b^2c^2\tan^2\theta=(k^2-c^2)\,[b^2(a^2-k^2)\cos^2\lambda+a^2(b^2-k^2)\sin^2\lambda].$$

Let us continue this assumption: reducing we find

$$\sin^2\theta=\frac{(k^2-c^2)}{k^2}\left[\frac{b^2(a^2-k^2)\cos^2\lambda+a^2(b^2-k^2)\sin^2\lambda}{b^2(a^2+c^2-k^2)\cos^2\lambda+a^2(b^2+c^2-k^2)\sin^2\lambda}\right], \quad \text{(d)}$$

and

$$\cos^2\theta=\frac{a^2b^2c^2}{k^2[b^2(a^2+c^2-k^2)\cos^2\lambda+a^2(b^2+c^2-k^2)\sin^2\lambda]}. \quad \text{(e)}$$

Substituting and reducing

$$\frac{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2}{\kappa^2k^2(k^2-c^2)}=\frac{(a^2-k^2)(a^2+c^2-k^2)\cos^2\lambda+(b^2-k^2)(b^2+c^2-k^2)\sin^2\lambda}{[b^2(a^2+c^2-k^2)\cos^2\lambda+a^2(b^2+c^2-k^2)\sin^2\lambda]^2}, \quad \text{(f)}$$

$\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ denotes the velocity of the pole of the instantaneous axis of rotation along the spiral which it describes. We thus have the velocity of this point given in terms of λ. We shall return to this expression.

To change the independent variable from t to λ.

Multiply the last equation by the equivalent expression given in (567), namely

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2=\frac{a^2b^2c^2}{\kappa^2k^2[(a^2-k^2)(b^2-c^2)\cos^2\lambda+(b^2-k^2)(a^2-c^2)\sin^2\lambda},$$

and we shall have

$$\left(\frac{d\sigma}{d\lambda}\right)^2=\frac{a^2b^2c^2(k^2-c^2)[(a^2-k^2)(a^2+c^2-k^2)\cos^2\lambda+(b^2-k^2)(b^2+c^2-k^2)\sin^2\lambda]}{[b^2(a^2+c^2-k^2)\cos^2\lambda+a^2(b^2+c^2-k^2)\sin^2\lambda]^2[(a^2-k^2)(b^2-c^2)\cos^2\lambda+(b^2-k^2)(a^2-c^2)\sin^2\lambda]}. \quad (581)$$

We shall now proceed to show that this expression may be reduced to an elliptic integral of the third order and *circular* form. To simplify the calculations, write

$$\left.\begin{array}{l} A=(a^2-k^2)(a^2+c^2-k^2),\ C=b^2(a^2+c^2-k^2),\ U=(a^2-k^2)(b^2-c^2), \\ B=(b^2-k^2)(b^2+c^2-k^2),\ D=a^2(b^2+c^2-k^2),\ V=(b^2-k^2)(a^2-c^2). \end{array}\right\} \quad (582)$$

Making these substitutions, dividing by $a^2b^2c^2$, and taking the square root, we shall obtain

$$\left(\frac{d\sigma}{d\lambda}\right)=\left[\frac{A\cos^2\lambda+B\sin^2\lambda}{C\cos^2\lambda+D\sin^2\lambda}\right]\frac{abc\sqrt{k^2-c^2}}{\sqrt{(A\cos^2\lambda+B\sin^2\lambda)(U\cos^2\lambda+V\sin^2\lambda)}}. \quad (583)$$

To integrate this equation, assume

$$V\tan^2\lambda=U\tan^2\Phi. \quad . \quad . \quad . \quad . \quad . \quad (584)$$

Introducing the changes arising from this transformation, the last equation may be reduced to

$$\left.\begin{array}{l} \dfrac{d\sigma}{d\Phi}=-\dfrac{(AV-BU)}{(DU-CV)\sqrt{AV}}\left[\dfrac{abc\sqrt{k^2-c^2}}{\sqrt{1-\left(\dfrac{AV-BU}{AV}\right)\sin^2\Phi}}\right] \\ +\dfrac{U(AD-CB)}{C(DU-CV)\sqrt{AV}}\left[\dfrac{abc\sqrt{k^2-c^2}}{\left[1+\left(\dfrac{DU-CV}{CV}\right)\sin^2\Phi\right]\sqrt{1-\left(\dfrac{AV-BU}{AV}\right)\sin^2\Phi}}\right] \end{array}\right\}. \quad (585)$$

We have now to compute the values of the coefficients, modulus, and parameter of this expression.

From the relations established in (582), we get, writing E and F, for the first and second coefficients,

$$\left.\begin{array}{l} E=\dfrac{U(AD-CB)abc\sqrt{k^2-c^2}}{C(DU-CV)\sqrt{AV}}=\dfrac{a(b^2-c^2)(b^2+c^2-k^2)\sqrt{a^2-k^2}}{bc\sqrt{(b^2-k^2)(k^2-c^2)(a^2-c^2)(a^2+c^2-k^2)}}, \\ F=\dfrac{(AV-BU)abc\sqrt{k^2-c^2}}{(DU-CV)\sqrt{AV}}=\dfrac{ab}{c}\sqrt{\dfrac{(a^2-k^2)(b^2-k^2)}{(k^2-c^2)(a^2-c^2)(a^2+c^2-k^2)}}, \\ \text{the parameter}=\dfrac{DU-CV}{CV}=\dfrac{c^2(a^2-b^2)(k^2-c^2)(a^2+b^2-k^2)}{b^2(a^2-c^2)(b^2-k^2)(a^2+c^2-k^2)}, \\ \text{the square of the modulus}=\dfrac{AV-BU}{AV}=\dfrac{(a^2-b^2)(a^2+b^2-k^2)}{(a^2-c^2)(a^2+c^2-k^2)}. \end{array}\right\} \quad (586)$$

Let us now take the cone described by the instantaneous axis of rotation, with reference to the principal axes of the body. The equation is given in (528), namely,

$$a^2(a^2-k^2)x^2+b^2(b^2-k^2)y^2+c^2(c^2-k^2)z^2=0\,;$$

and we shall find, writing as before α' and β' for the principal arcs of the spherical ellipse the intersection of this cone with a concentric sphere, that

$$\left.\begin{aligned}
\tan^2\alpha' &= \frac{c^2(k^2-c^2)}{b^2(b^2-k^2)}, & \tan^2\beta' &= \frac{c^2(k^2-c^2)}{a^2(a^2-k^2)},\\
\cos^2\alpha' &= \frac{b^2(b^2-k^2)}{(b^2-c^2)(b^2+c^2-k^2)}, & \cos^2\beta' &= \frac{a^2(a^2-k^2)}{(a^2-c^2)(a^2+c^2-k^2)},\\
\sin^2\alpha' &= \frac{c^2(k^2-c^2)}{(b^2-c^2)(b^2+c^2-k^2)}, & \sin^2\beta' &= \frac{c^2(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)}.
\end{aligned}\right\}\quad(587)$$

If we write $2\epsilon'$ for the angle between the focals of this cone, we know from (e) sec. [8] that its value, in terms of the principal arcs of the spherical ellipse, is given by the equation

$$\tan^2\epsilon'=\frac{\cos^2\beta'-\cos^2\alpha'}{\cos^2\alpha'}.$$

Substituting the particular values of these functions just given, we obtain

$$\tan^2\epsilon'=\frac{c^2(a^2-b^2)(k^2-c^2)(a^2+b^2-k^2)}{b^2(a^2-c^2)(b^2-k^2)(a^2+c^2-k^2)}.$$

Hence $\tan^2\epsilon'$ is the parameter.

Let $2\eta'$ be the angle between the circular sections of the same cone. It was found in (9) that $\sin^2\eta'=\dfrac{\sin^2\alpha'-\sin^2\beta'}{\sin^2\alpha'}$,

$$\text{whence}\quad \sin^2\eta'=\frac{(a^2-b^2)(a^2+b^2-k^2)}{(a^2-c^2)(a^2+c^2-k^2)},$$

or $\sin\eta'$ is the modulus.

Let us compute the value of the first coefficient E.

Making the necessary substitutions, we obtain the resulting expressions,.

$$\mathrm{E}=\frac{a(b^2-c^2)(b^2+c^2-k^2)\sqrt{a^2-k^2}}{bc\sqrt{(a^2-c^2)(b^2-k^2)(k^2-c^2)(a^2+c^2-k^2)}}=\frac{\cos\beta'}{\cos\alpha'\sin\alpha'}.$$

In like manner we find for the second coefficient F,

$$\mathrm{F}=\frac{ab}{c}\sqrt{\frac{(a^2-k^2)(b^2-k^2)}{(a^2-c^2)(k^2-c^2)(a^2+c^2-k^2)}}=\frac{\cos\alpha'\cos\beta'}{\sin\alpha'}.$$

Making all the substitutions just indicated, (585) may be transformed into

$$\left.\begin{array}{r}\text{Arc of spiral} = \dfrac{\cos\beta'}{\cos\alpha'\sin\alpha'}\displaystyle\int\frac{d\Phi}{[1+\tan^2\epsilon'\sin^2\Phi]\sqrt{1-\sin^2\eta'\sin^2\Phi}} \\ -\dfrac{\cos\alpha'\cos\beta'}{\sin\alpha'}\displaystyle\int\frac{d\Phi}{\sqrt{1-\sin^2\eta'\sin^2\Phi}}\end{array}\right\}. \quad (588)$$

When the body is one of revolution or $a=b$, $\alpha'=\beta'$ and the preceding expression becomes, Arc of spiral $=\sin\alpha'\,\Phi$, an arc of a circle, since $\epsilon'=0$ and $\eta'=0$.

It may be shown by comparing (10) with (41) that if there are two circular elliptic integrals of the third order with positive and negative parameters, having the same modulus and amplitude, the parameters being respectively the square of the tangent of the semi- focal angle, and the square of the eccentricity of the plane elliptic base of the cone, the expressions are connected by the following equation :—

$$\left.\begin{array}{r}\dfrac{\cos\beta}{\cos\alpha\sin\alpha}\displaystyle\int\frac{d\varphi}{[1+\tan^2\epsilon\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}} \\ -\dfrac{\cos\alpha\cos\beta}{\sin\alpha}\displaystyle\int\frac{d\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}} \\ =\dfrac{\tan\beta}{\tan\alpha}\sin\beta\displaystyle\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}} \\ +\tan^{-1}\left[\dfrac{e\tan\epsilon\sin\varphi\cos\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}}\right]\end{array}\right\}. \quad 589)$$

If now we introduce this relation into the preceding equation (588), we shall obtain for the final result,

$$\left.\begin{array}{r}\text{Arc of spiral} = \dfrac{\tan\beta'}{\tan\alpha'}\sin\beta'\displaystyle\int\frac{d\Phi}{[1-e'^2\sin^2\Phi]\sqrt{1-\sin^2\eta'\sin^2\Phi}} \\ +\tan^{-1}\left[\dfrac{e'\tan\epsilon'\sin\Phi\cos\Phi}{\sqrt{1-\sin^2\eta'\sin^2\Phi}}\right]\end{array}\right\}. \quad (590)$$

In sec. [7] it was established that the elliptic integral

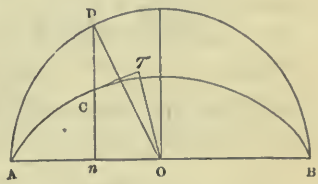
$$\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}}$$

is the value of an arc of the spherical ellipse, the principal angles of whose generating cone are 2α and 2β, the angle between whose circular sections is 2η, and the eccentricity of whose plane elliptic base is e. And it is shown in (44) that

$$\tan^{-1}\left[\frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}\right]$$

is the arc of a great circle touching the spherical conic, intercepted between the point of contact and the foot of the perpendicular arc from the centre on the tangent arc.

Fig. 34.



Make the angle AOD $=\phi$, draw the arc Dn a secondary to AB, and through C draw the tangent arc Cτ.

The length of the spiral = spherical elliptic arc AC + circular arc Cτ.

The length of the spiral between any two successive apsides is found by taking Φ between the limits 0 and $\frac{\pi}{2}$. At these limits tangent vanishes, and the expression becomes the length of a quadrant of the ellipse; hence we obtain this remarkable proposition:—

The length of the spiral, described on a fixed concentric sphere, between any two of its successive apsides, is equal to a quadrant of the spherical ellipse, described by the pole of the instantaneous axis of rotation, on an equal concentric sphere which moves with the body.

If we turn to the relation assumed in (584) between λ and Φ for the purpose of facilitating the integrations, and substitute for U and V their values in the equation

$$V\tan^2\lambda=U\tan^2\Phi,$$

we shall find $\tan^2\Phi=\frac{V}{U}\tan^2\lambda$, or $\tan^2\Phi=\frac{(b^2-k^2)(a^2-c^2)}{(a^2-k^2)(b^2-c^2)}\tan^2\lambda$,

or $\tan\Phi=\cos\epsilon\tan\lambda$. This result is identical with the expression found in (39).

But λ and the amplitude ϕ used in the investigations in this and the foregoing chapter, are connected by the relation established in (575),

$$\tan\phi=\cos\epsilon\tan\lambda.$$

Hence $$\phi=\Phi. \qquad (591)$$

131.] Let $\epsilon, \epsilon', \epsilon''$ be the semi-focal angles of the *invariable cone*, of the *cone of rotation*, and of the *cone of nutation* respectively.

Then

$$\cos^2\epsilon = \frac{\cos^2\alpha}{\cos^2\beta} = \frac{(b^2-k^2)(a^2-c^2)}{(a^2-k^2)(b^2-c^2)} \text{ as in (c) sec. [115];}$$

$$\cos^2\epsilon' = \frac{\cos^2\alpha'}{\cos^2\beta'} = \frac{b^2(b^2-k^2)(a^2-c^2)(a^2+c^2-k^2)}{a^2(a^2-k^2)(b^2-c^2)(b^2+c^2-k^2)} \text{ as in (587);}$$

$$\cos^2\epsilon'' = \frac{\cos^2\alpha''}{\cos^2\beta''} = \frac{a^2(b^2+c^2-k^2)}{b^2(a^2+c^2-k^2)} \text{ from (n) sec. [125].}$$

Whence $$\cos\epsilon = \cos\epsilon' \cos\epsilon''. \qquad (592)$$

Let e'' be the eccentricity of the plane base of the cone of nutation. From (n) sec. [125] we may derive

$$e''^2 = \frac{\tan^2\alpha'' - \tan^2\beta''}{\tan^2\alpha''} = \frac{k^2(a^2-b^2)}{b^2(a^2-k^2)}.$$

But it was shown in (i) sec. [119], that $e^2 = \frac{k^2(a^2-b^2)}{b^2(a^2-k^2)}$; whence $e = e''$, or the plane elliptic base of the cone of nutation is similar to that of the invariable cone.

132.] When the revolving body is very nearly a sphere, as in the case of the planetary bodies, a, b, c are very nearly equal. In this case, the ellipse of rotation is indefinitely greater than the ellipse of nutation, as may thus be shown:

$$\tan^2\alpha' = \frac{c^2(k^2-c^2)}{b^2(b^2-k^2)}, \qquad \tan^2\beta' = \frac{c^2(k^2-c^2)}{a^2(a^2-k^2)},$$

$$\tan^2\alpha'' = \frac{(a^2-k^2)(k^2-c^2)}{a^2c^2}, \qquad \tan\beta'' = \frac{(b^2-k^2)(k^2-c^2)}{b^2c^2}; \text{ whence}$$

$$\frac{\tan\alpha''}{\tan\alpha'} = \frac{b^2}{c^2}\sqrt{\frac{(a^2-k^2)(b^2-k^2)}{a^2b^2}}, \quad \frac{\tan\beta''}{\tan\beta'} = \frac{a^2}{c^2}\sqrt{\frac{(a^2-k^2)(b^2-k^2)}{a^2b^2}}. \qquad (593)$$

Now, when a, b, c are very nearly equal, k also must nearly be equal to one of these quantities; whence as k approaches in magnitude to one of the axes, the above ratio becomes indefinitely small.

As the length of one undulation of the spiral depends solely on the magnitude of the principal arcs of the ellipse of rotation, and is independent of that of nutation; it is evident that when the body approaches in shape to a sphere, several revolutions of the body must occur between one extreme position of the axis of rotation and the one immediately following.

When the body is very nearly a sphere, we may approximate to this number. In this case the ellipses are very nearly circles, and the number of revolutions n will be the ratio of their circumferences, or

$$n=\frac{\text{circumference of circle of rotation}}{\text{circumference of circle of nutation}}=\frac{\sin\alpha'}{\sin\alpha''}=\frac{\tan\alpha'}{\tan\alpha''}=\frac{\mathrm{N}}{\mathrm{L}-\mathrm{N}};$$

or, in the usual notation, $n=\dfrac{\mathrm{C}}{\mathrm{A}-\mathrm{C}}$ nearly, since $a=b=k=c$ nearly.

133.] *On the velocity of the pole of the instantaneous axis of rotation along the spiral.*

The velocity V along the spiral is the value of the expression $\dfrac{d\sigma}{dt}$. This value has been found, (f) sec. [130], to be, in terms of λ,

$$\mathrm{V}^2=\frac{\kappa^2k^2(k^2-c^2)\left[(a^2-k^2)(a^2+c^2-k^2)\cos^2\lambda+(b^2-k^2)(b^2+c^2-k^2)\sin^2\lambda\right]}{\left[b^2(a^2+c^2-k^2)\cos^2\lambda+a^2(b^2+c^2-k^2)\sin^2\lambda\right]^2}. \quad (594)$$

We shall now proceed to find the maximum and minimum values of V by the ordinary process of differentiation. For this purpose differentiating equation (c) of sec. [130] and putting the differential of $\left(\dfrac{d\sigma}{dt}\right)$ equal to 0, we shall obtain

$$0=\frac{d\theta}{dt}\cdot\sin\theta\cos\theta\left[\mathrm{Q}(\sin^2\theta-\cos^2\theta)-2\mathrm{W}\cos^2\theta\right], \quad (595)$$

writing Q for $a^2+b^2+c^2-2k^2$,

and W for $(a^2+b^2+c^2-k^2)\left[\dfrac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2b^2c^2}\right]$.

In this equation there are four factors, any one of which, equated to cipher, would satisfy the equation; either $\dfrac{d\theta}{dt}=0$, $\sin\theta=0$, $\cos\theta=0$, or $\mathrm{Q}(\sin^2\theta-\cos^2\theta)-2\mathrm{W}\cos^2\theta=0$.

We shall now proceed to show that they are all inadmissible except the first.

We cannot have $\sin\theta=0$, or $\cos\theta=0$; or $\theta=0$, or $\theta=\dfrac{\pi}{2}$; because the magnitude of the angle θ is confined within certain limits, given by the equations (557); neither can we have $\mathrm{Q}(\sin^2\theta-\cos^2\theta)-2\mathrm{W}\cos^2\theta=0$; for if we assume the truth of this supposition, we shall find, writing $\theta_{\prime}$ for θ,

$$\tan^2\theta_{\prime}=\frac{\mathrm{Q}-2\mathrm{W}}{\mathrm{Q}}, \text{ or } \sec^2\theta_{\prime}=\frac{2(\mathrm{Q}-\mathrm{W})}{\mathrm{Q}}. \quad . \quad . \quad . \quad (a)$$

We must now compute the value of this expression.

Since $Q=a^2+b^2+c^2-2k^2$, and

$$W=(a^2+b^2+c^2-k^2)\left[\frac{(a^2-k^2)(b^2-k^2)(c^2-k^2)}{a^2b^2c^2}\right],$$

we get, after some reductions,

$$\left.\begin{aligned}\frac{a^2b^2c^2}{k^2}(Q-W)=a^2b^2c^2+(b^2c^2+a^2c^2+a^2b^2)(a^2+b^2+c^2)\\ -k^2(b^2c^2+a^2c^2+a^2b^2)-k^2\{a^4+b^4+c^4+2b^2c^2+2a^2c^2+2a^2b^2\}\\ +2k^4(a^2+b^2+c^2)-k^6.\end{aligned}\right\}\quad\text{(b)}$$

Now this expression may be reduced to the symmetrical form

$$(b^2+c^2-k^2)(a^2+c^2-k^2)(a^2+b^2-k^2); \quad . \quad . \quad . \quad \text{(c)}$$

$$\text{whence } \sec^2\theta_{\prime}=\frac{2k^2(b^2+c^2-k^2)(a^2+c^2-k^2)(a^2+b^2-k^2)}{a^2b^2c^2(a^2+b^2+c^2-2k^2)}. \quad . \quad \text{(d)}$$

The greatest value of $\sec\theta$, which the conditions of the problem admit, is given by the equation (557),

$$\sec^2\Theta=\frac{k^2(a^2+c^2-k^2)}{a^2c^2}.$$

Let the ratio of these secants be n, we shall find that n is always greater than 1: put $\sec\theta_{\prime}=n\sec\Theta$,

$$\text{or}\quad \frac{\sec^2\theta_{\prime}}{\sec^2\Theta}=n^2=\frac{2(b^2+c^2-k^2)(a^2+b^2-k^2)}{b^2(a^2+b^2+c^2-2k^2)},$$

$$\text{or}\quad n^2=2-\frac{2(a^2-k^2)(k^2-c^2)}{b^2(a^2+b^2+c^2-2k^2)}.$$

As the extreme limits of k are a and c, let $k^2=a^2-\alpha^2$, $k^2=c^2+\gamma^2$, α and γ being positive quantities, which are small when compared with the axes. This expression may now be written

$$n^2=2-\frac{2\alpha^2\gamma^2}{b^2(b^2+\alpha^2-\gamma^2)};$$

or n is equal to $\sqrt{2}$ nearly, since the second term may be neglected. We have therefore

$$\sec\theta=\sqrt{2}\sec\Theta,$$

a value of θ which cannot be admitted, since Θ is the maximum value of θ.

The only remaining factor is $\frac{d\theta}{dt}$; differentiating (564) and making $\frac{d\theta}{dt}=0$, we get $-k^2(a^2-b^2)\sin 2\lambda=0$, an equation which is satisfied by $\lambda=0$ or $\lambda=\frac{\pi}{2}$; but these values of λ give $\theta=\Theta$, and $\theta=\Theta'$; or, *the maximum and minimum velocities of the pole of the instantaneous axis of rotation along the spiral are at its greatest or least distances from the centre of the spiral,* as we might indeed have anticipated.

Taking the second differential of this expression,

$$-k^2(a^2-b^2)\cos 2\lambda,$$

this is negative when $\lambda=0$, and positive when $\lambda=\frac{\pi}{2}$. Therefore the velocity is a maximum when $\lambda=0$, and a minimum when $\lambda=\frac{\pi}{2}$. Or the velocity is least at the inner, and greatest at the outer apside.

CHAPTER XVIII.

134.] We shall now proceed to determine the curves traced out by the poles of the principal axes of the body, during the motion, on an immovable concentric sphere. We shall first investigate the curve traced out by the axis c of the ellipsoid, or the C spiral, as for the sake of brevity it may be named.

Let ρ be the angle between the pole of the impressed couple and the pole of the axis c. Then the usual formula gives us

$$\left(\frac{d\sigma}{dt}\right)^2=\left(\frac{d\rho}{dt}\right)^2+\sin^2\rho\left(\frac{d\psi}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad (a)$$

Now, ρ being the angle between k and the axis c of the ellipsoid, $\cos\rho=\frac{z}{k}$, $\sin\rho=\frac{\sqrt{k^2-z^2}}{k}$, $\tan\rho=\frac{\sqrt{k^2-z^2}}{z}$; hence $\left(\frac{d\rho}{dz}\right)^2=\frac{1}{k^2-z^2}$.

In (540) it was shown that $\left(\frac{dz}{dt}\right)^2=\frac{\kappa^2k^2XY}{a^2b^2c^4}$,

where $X=[(b^2-c^2)z^2-c^2(b^2-k^2)]$, $Y=[c^2(a^2-k^2)-z^2(a^2-c^2)]$.

In (552) we found $\frac{d\psi}{dt}=\frac{\kappa k^2}{c^2}\left(\frac{z^2-c^2}{k^2-z^2}\right)$.

Before we proceed further, it is proper to show that the curve has two asymptotic circles; for, τ being the inclination of the vector arc to the curve at the point of contact,

$$\tan\tau = \frac{\sin\rho\,\dfrac{d\psi}{dt}}{\dfrac{d\rho}{dt}} = \frac{ab(z^2-c^2)}{\sqrt{XY}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

When $X=0$, or $Y=0$, we shall have $\tan\tau=\infty$, or $\tau=\frac{\pi}{2}$. . (c)

The radii of the asymptotic circles may be found by making $X=0$ and $Y=0$,

$$\left.\begin{array}{l}\text{or } (a^2-k^2)c^2-(a^2-c^2)z^2=0,\\ \text{and } (b^2-c^2)z^2-(b^2-k^2)c^2=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Resuming our equations, and making the suggested substitutions in (a),

$$\frac{a^2b^2c^4}{\kappa^2k^2}\left(\frac{d\sigma}{dt}\right)^2 = \frac{XY+a^2b^2(c^2-z^2)^2}{(k^2-z^2)}. \quad . \quad . \quad . \quad . \quad \text{(e)}$$

This expression, by the help of the preceding relations, becomes

$$\frac{a^2b^2c^2}{\kappa^2k^2}\left(\frac{d\sigma}{dt}\right)^2 = c^2(a^2+b^2-k^2)-(a^2+b^2-c^2)z^2. \quad . \quad . \quad (596)$$

135.] Let distances a', b', c' be assumed along the axes of the ellipsoid a, b, c, and inversely proportional to these axes, so that $aa'=bb'=cc'=h^2$. Let v, v', v'' be the velocities of the extremities of these lines respectively. Whence $c'\left(\frac{d\sigma}{dt}\right)$ will be the velocity of the extremity of c',

$$\text{or } v''=c'\left(\frac{d\sigma}{dt}\right)=\frac{h^2}{c}\left(\frac{d\sigma}{dt}\right); \text{ hence } \left(\frac{d\sigma}{dt}\right)^2=\frac{c^2}{h^4}v''^2.$$

Substituting this value in the last equation, and multiplying by a^2b^2, we find

$$\frac{a^4b^4c^4}{\kappa^2h^6}\cdot v''^2 = a^2b^2c^2(a^2+b^2-h^2)+2a^2b^2c^2z^2-(a^2+b^2+c^2)a^2b^2z^2.$$

Writing analogous expressions for the other axes, and introducing the relations given by the equations of the ellipsoid and sphere, we shall find, on adding those expressions,

$$v^2+v'^2+v''^2=\frac{\kappa^2h^6}{a^2b^2c^2}(a^2+b^2+c^2-h^2). \quad . \quad . \quad . \quad \text{(a)}$$

We have therefore this theorem :—

If straight lines are taken along the three principal axes of the body from the centre, and inversely proportional to the square roots of the moments of inertia round these axes, the sum of the squares of the velocities of their extremities is constant during the motion.

Let segments equal to R measured from the centre be assumed on the three principal axes of the body, the sum of the areas described by the projections of these lines on the plane of the impressed couple varies as the time.

Let S_c be the area described by the projection of a portion of the axis of c equal to R on the plane of the impressed couple; then the projection of R on this plane is $R \sin \rho$, and the differential of the area

$$\frac{dS_c}{dt} = \tfrac{1}{2} R^2 \sin^2 \rho \frac{d\psi}{dt}. \quad \text{(b)}$$

Now $$\sin^2 \rho = \frac{k^2 - z^2}{k^2},$$

and $$\frac{d\psi}{dt} = \kappa \left[1 - \left(\frac{k^2 - c^2}{c^2}\right)\left(\frac{z^2}{k^2 - z^2}\right)\right], \text{ as in (552);}$$

whence $$\frac{dS_c}{dt} = \tfrac{1}{2} \kappa R^2 \left[1 - \frac{z^2}{c^2}\right]. \quad \text{(e)}$$

In like manner $$\frac{dS_a}{dt} = \tfrac{1}{2} \kappa R^2 \left[1 - \frac{x^2}{a^2}\right], \quad \frac{dS_b}{dt} = \tfrac{1}{2} \kappa R^2 \left[1 - \frac{y^2}{b^2}\right];$$

whence $$\frac{dS_a}{dt} + \frac{dS_b}{dt} + \frac{dS_c}{dt} = \tfrac{1}{2} R^2 \kappa \left[3 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right] = R^2 \kappa,$$

or $$S_a + S_b + S_c = R^2 \kappa t + \text{constant.} \quad \text{(d)}$$

Should the lengths R, instead of being equal, be proportional to the square roots of the moments of inertia round the corresponding axes, the sum of the areas described by the projections of those lines, on the plane of the impressed couple, is still proportional to the time.

Let $R^2 = \frac{N}{W} = \frac{nc^2}{W}$. W being a constant. Then (b) in the last article may be changed into the following, $\frac{dS_c}{dt} = \kappa \frac{n}{W} (c^2 - z^2)$.

Whence $$S_a + S_b + S_c = \frac{n}{W} \kappa (a^2 + b^2 + c^2 - k^2) t + \text{constant.} \quad \text{(c)}$$

136.] Let us now resume the general equation, and proceed to find the lengths of the spirals traced by the principal axes during the motion. The equation for the C spiral is, as in (596),

$$\frac{a^2b^2c^2}{\kappa^2k^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2=c^2(a^2+b^2-k^2)-(a^2+b^2-c^2)z^2. \quad . \quad . \quad \text{(a)}$$

Assume, as in (541),

$$z^2=\frac{(a^2-k^2)(b^2-k^2)c^2}{(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi},$$

and substitute this value of z in (a); we shall then have

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2=\frac{\kappa^2k^2(k^2-c^2)[a^2(a^2-k^2)\cos^2\varphi+b^2(b^2-k^2)\sin^2\varphi]}{a^2b^2[(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi]} \quad . \quad \text{(b)}$$

and

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)^2=\frac{a^2b^2c^2}{\kappa^2k^2[(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi]} \text{ as in (541*)}, \quad \text{(c)}$$

whence

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right)^2=\frac{c^2(k^2-c^2)[a^2(a^2-k^2)\cos^2\varphi+b^2(b^2-k^2)\sin^2\varphi]}{[(a^2-k^2)(b^2-c^2)\cos^2\varphi+(b^2-k^2)(a^2-c^2)\sin^2\varphi]^2}. \quad \text{(597)}$$

Let

$$\left.\begin{aligned}\mathrm{A}&=a^2(a^2-k^2), & \mathrm{C}&=(a^2-k^2)(b^2-c^2),\\ \mathrm{B}&=b^2(b^2-k^2), & \mathrm{D}&=(b^2-k^2)(a^2-c^2);\end{aligned}\right\}. \quad . \quad . \quad \text{(d)}$$

then

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right)=\frac{\mathrm{A}\cos^2\varphi+\mathrm{B}\sin^2\varphi}{\mathrm{C}\cos^2\varphi+\mathrm{D}\sin^2\varphi}\times\frac{c\sqrt{k^2-c^2}}{\sqrt{\mathrm{A}\cos^2\varphi+\mathrm{B}\sin^2\varphi}}; \quad . \quad \text{(e)}$$

and this expression may be transformed into

$$\left.\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}\right)=\frac{\mathrm{BC}-\mathrm{AD}}{\mathrm{C}(\mathrm{C}-\mathrm{D})\sqrt{\mathrm{A}}}&\frac{c\sqrt{k^2-c^2}}{\left[1-\left(\frac{\mathrm{C}-\mathrm{D}}{\mathrm{C}}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}}\right)\sin^2\varphi}}\\ &+\frac{\mathrm{A}-\mathrm{B}}{(\mathrm{C}-\mathrm{D})\sqrt{\mathrm{A}}}\frac{c\sqrt{k^2-c^2}}{\sqrt{1-\left(\frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}}\right)\sin^2\varphi}}\end{aligned}\right\}. \quad \text{(598)}$$

Equations (d) give us

$$\left.\begin{aligned}\frac{\mathrm{BC}-\mathrm{AD}}{\mathrm{C}(\mathrm{C}-\mathrm{D})}&=\frac{-(a^2+b^2-c^2)(b^2-k^2)}{(b^2-c^2)(k^2-c^2)}, & \frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}-\mathrm{D}}&=\frac{a^2+b^2-k^2}{k^2-c^2},\\ \frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}}&=\frac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)}, & \frac{\mathrm{C}-\mathrm{D}}{\mathrm{C}}&=\frac{(a^2-b^2)(k^2-c^2)}{(b^2-c^2)(a^2-k^2)}.\end{aligned}\right\} \quad . \quad . \quad \text{(f)}$$

Now $e'^2 = \dfrac{\tan^2\alpha' - \tan^2\beta'}{\tan^2\alpha'}$, as in sec. [7]. Substituting the values of $\tan\alpha'$, $\tan\beta'$ given in (587), we get

$$e'^2 = \frac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)} = \frac{A-B}{A};$$ hence e' is the modulus.

In (b) sec. [115] it was shown that

$$\sin^2\epsilon = \frac{(a^2-b^2)(k^2-c^2)}{(b^2-c^2)(a^2-k^2)} = \frac{C-D}{C},$$

whence $\sin^2\epsilon$ is the parameter. Making these substitutions, and integrating, we obtain the resulting equation,

$$\left.\begin{aligned} \sigma = &\sqrt{\frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}}\left(\frac{a^2+b^2-k^2}{k^2-c^2}\right)\int\frac{d\varphi}{\sqrt{1-e'^2\sin^2\varphi}} \\ &-\sqrt{\frac{c^2(k^2-c^2)}{a^2(a^2-k^2)}}\left(\frac{(a^2+b^2-c^2)(b^2-k^2)}{(k^2-c^2)(b^2-c^2)}\right)\int\frac{d\varphi}{[1-\sin^2\epsilon\sin^2\varphi]\sqrt{1-e'^2\sin^2\varphi}} \end{aligned}\right\} . \quad (599)$$

As $\sin^2\epsilon$ is less than e'^2, this elliptic integral is of the third order and *logarithmic* form. That it is so, may be shown by constructing the expression $(1+n)\left(1+\dfrac{i^2}{n}\right)$; or in this case, in which $n = -\sin^2\epsilon$ and $i^2 = e'^2$,

$$\cos^2\epsilon\left(1 - \frac{e'^2}{\sin^2\epsilon}\right) = \cot^2\epsilon\,(\sin^2\epsilon - e'^2);$$

whence the *criterion* of *sphericity* becomes, as in (138),

$$-\frac{(b^2-k^2)^2(a^2-c^2)(a^2+b^2-c^2)}{a^2(a^2-k^2)(k^2-c^2)(b^2-c^2)} \quad . \quad . \quad . \quad . \quad . \quad (g)$$

This is a quantity essentially negative, whatever be the value we assign to k between its limits a and c. Hence the polar spiral described during the motion by the least principal axis, may be rectified by an elliptic integral of the *third* order and *logarithmic* form.

When the ellipsoid is one of revolution, the elliptic integral may be reduced from the third order to a circular arc. In this case $a = b$, since $\sin\epsilon = 0$, $e' = 0$.

Adding together the coefficients of the integrals, now become identical, we get

$$\sigma = \frac{a^2}{a^2-c^2}\tan\alpha'\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (600)$$

137.] Multiply equation (599) by the expression

$$\frac{(a^2-b^2)\sqrt{a^2+b^2-c^2}}{abc},$$

which depends solely on the moments of inertia of the body. Let i be written for this factor; then (599) will become

$$\left.\begin{aligned} i\sigma = & \frac{(a^2-b^2)(a^2+b^2-k^2)\sqrt{a^2+b^2-c^2}}{a^2b\sqrt{(a^2-k^2)(k^2-c^2)}}\int\frac{d\varphi}{\sqrt{1-e'^2\sin^2\varphi}} \\ & -\frac{(a^2-b^2)(b^2-k^2)(a^2+b^2-c^2)^{\frac{3}{2}}}{a^2b(b^2-c^2)\sqrt{(a^2-k^2)(k^2-c^2)}}\int\frac{d\varphi}{[1-\sin^2\epsilon\sin^2\varphi]\sqrt{1-e'^2\sin^2\varphi}} \end{aligned}\right\}. \quad (601)$$

Now ϵ is the focal angle of the invariable cone, and e' is the eccentricity of the plane base of the cone of rotation. Let there be a cone which shall have the same focal lines as the invariable cone, and a plane elliptic base similar to that of the cone of rotation. Then, $\alpha_{\prime}$ and $\beta_{\prime}$ being the principal angles of such a cone, we shall have, see (19),

$$\frac{\tan^2\alpha_{\prime}-\tan^2\beta_{\prime}}{\tan^2\alpha_{\prime}}=e'^2, \text{ and } \frac{\sin^2\alpha_{\prime}-\sin^2\beta_{\prime}}{\cos^2\beta_{\prime}}=\sin^2\epsilon, \quad . \quad (a)$$

$$\text{or } \tan^2\alpha_{\prime}=\frac{a^2(k^2-c^2)}{(b^2-k^2)(a^2+b^2-c^2)}, \quad \tan^2\beta_{\prime}=\frac{b^2(k^2-c^2)}{(a^2-k^2)(a^2+b^2-c^2)}; \quad (b)$$

whence
$$\cos^2\alpha_{\prime}=\frac{(a^2+b^2-c^2)(b^2-k^2)}{(a^2+b^2-k^2)(b^2-c^2)}, \quad . \quad . \quad . \quad . \quad (c)$$

$$e_{\prime}^2=e'^2=\frac{(a^2-b^2)(a^2+b^2-k^2)}{a^2(a^2-k^2)}, \quad \sin^2\epsilon_{\prime}=\sin^2\epsilon=\frac{(a^2-b^2)(k^2-c^2)}{(a^2-k^2)(b^2-c^2)}. \quad (d)$$

By the help of these relations, if we construct the expression $\frac{e'^2}{\tan\beta_{\prime}}$ we shall find it to be equal to the coefficient of the elliptic integral of the *first* order in the equation (601). In like manner if we construct the expression $\frac{e_{\prime}^2}{\tan\beta_{\prime}}\cos^2\alpha$, we shall obtain the coefficient of the elliptic integral of the *third* order in the same equation. Accordingly (601) may be written,

$$\left.\begin{aligned} j\sigma = & \frac{e_{\prime}^2}{\tan\beta_{\prime}}\int\frac{d\varphi}{\sqrt{1-e_{\prime}^2\sin^2\varphi}} \\ & -\frac{e_{\prime}^2}{\tan\beta_{\prime}}\cos^2\alpha_{\prime}\int\frac{d\varphi}{[1-\sin^2\epsilon_{\prime}\sin^2\varphi]\sqrt{1-e_{\prime}^2\sin^2\varphi}} \end{aligned}\right\}. \quad (602)$$

138.] When the parameter of the elliptic integral of the third order is negative and less than the square of the modulus, the function no longer represents any spherical curve of the second order. It is possible, however, to construct a spherical curve whose rectification may be effected by an elliptic integral of the third order, and *logarithmic* form.

Let us conceive a spherical curve which shall cut all its spherical vectors in angles whose cosines shall have a given ratio to the sines of double the angles which the equal central vectors of a certain spherical ellipse make with the major arc. Let τ be this angle, and ρ the distance of the point from the centre of the curve. In the spherical ellipse, of which the principal arcs are α and β, let this vector ρ make with the major arc the angle ψ. Then, by the law of the generation of the curve,

$$\cos\tau = j\sin\psi\cos\psi. \quad \text{(a)}$$

Now, as the spherical radii of the ellipse which are equal to α and β respectively, make with the major arc angles 0 and $\frac{\pi}{2}$, at these distances $\cos\tau=0$, and the curve has therefore apsides at these distances from the centre.

To find the length of the curve, we must compare the values of $\cos\tau$. $\cos\tau=j\sin\psi\cos\psi$ (this relation may be taken as the definition of the curve); and $\cos\tau=\left(\frac{d\rho}{d\sigma}\right)$; $j\left(\frac{d\sigma}{d\rho}\right)^2=\frac{1}{\sin^2\psi\cos^2\psi}$; . . . (b)

while the equation of the spherical ellipse gives

$$\cot^2\rho=\cot^2\alpha\cos^2\psi+\cot^2\beta\sin^2\psi. \quad \text{(c)}$$

Let ϕ be the *eccentric anomaly*, as in (c) sec. [8]; then

$$\tan\psi=\frac{\tan\beta}{\tan\alpha}\tan\phi; \quad \text{(d)*}$$

whence

$$\left.\begin{aligned}\sin^2\psi&=\frac{\tan^2\beta\sin^2\phi}{\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi},\\ \cos^2\psi&=\frac{\tan^2\alpha\cos^2\phi}{\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi}.\end{aligned}\right\} \quad \text{(e)}$$

Substituting these values of $\sin\psi$, $\cos\psi$ in (b), we find

$$j^2\left(\frac{d\sigma}{d\rho}\right)^2=\frac{[\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi^2]}{\tan^2\alpha\tan^2\beta\sin^2\phi\cos^2\phi}. \quad \text{(f)}$$

* The eccentric anomaly ϕ in (c) sec. [8] is not the same angle as ϕ in (d) sec. [7].

Again, as $\tan^2\rho=\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi, \quad . \quad . \quad . \quad (g)$

$$\text{differentiating,}\ \left(\frac{d\rho}{d\phi}\right)^2=\frac{(\tan^2\alpha-\tan^2\beta)^2\sin^2\phi\cos^2\phi}{\tan^2\rho\sec^4\rho},$$

whence, as $j\left(\frac{d\sigma}{d\phi}\right)=j\left(\frac{d\sigma}{d\rho}\right)\cdot\left(\frac{d\rho}{d\phi}\right)$,

$$j^2\left(\frac{d\sigma}{d\phi}\right)^2=\frac{(\tan^2\alpha-\tan^2\beta)^2}{\tan^2\alpha\tan^2\beta}\frac{[\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi]}{[\sec^2\alpha\cos^2\phi+\sec^2\beta\sin^2\phi]^2}. \quad (h)$$

$$\text{Now}\qquad \frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=e^2,\ \frac{\sec^2\alpha-\sec^2\beta}{\sec^2\alpha}=\sin^2\epsilon; \quad . \quad . \quad (i)$$

making these substitutions, reducing and taking the square root, the transformed equation becomes

$$\left.\begin{aligned} j\sigma=&\frac{e^2}{\tan\beta}\int\frac{d\phi}{\sqrt{1-e^2\sin^2\phi}}\\ &-\frac{e^2\cos^2\alpha}{\tan\beta}\int\frac{d\phi}{[1-\sin^2\epsilon\sin^2\phi]\sqrt{1-e^2\sin^2\phi}}\end{aligned}\right\} \quad . \quad . \quad (603)$$

As $e^2>\sin^2\epsilon$, this is an elliptic integral of the third order and *logarithmic* form.

Now, if we compare this formula with (602), we shall find them identical,—whence we may infer that the length of the spiral described by the pole of the greatest or the least axis of the ellipsoid on a fixed sphere (*the semidiameter k being the next in the order of magnitude to such greatest or least axis*) will be equal to the length of the curve there defined as generated on the surface of a sphere according to a given law.

139.] On the spiral described by the pole of the greater principal axis, or the A spiral.

In the general equation (596) substitute x for z, and interchange a and c; we shall then have

$$\frac{a^2b^2c^2}{\kappa^2k^2}\left(\frac{d\sigma'}{dt}\right)^2=a^2(c^2+b^2-k^2)-(c^2+b^2-a^2)x^2. \quad . \quad . \quad (a)$$

In (546) we found

$$x^2=\frac{a^2(b^2-k^2)(k^2-c^2)\sin^2\phi}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi}.$$

Substituting this value of x^2 in the preceding equation, and introducing the value of $\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)$ given in (541*), we shall obtain the resulting equation

$$\frac{\left(\frac{\mathrm{d}\sigma'}{\mathrm{d}t}\right)}{a\sqrt{a^2-k^2}}=\frac{\sqrt{(b^2-c^2)(c^2+b^2-k^2)\cos^2\phi+b^2(b^2-k^2)\sin^2\phi}}{(a^2-k^2)(b^2-c^2)\cos^2\phi+(b^2-k^2)(a^2-c^2)\sin^2\phi}. \quad \text{(b)}$$

This expression may be reduced in the same way as (597), omitting the steps for the sake of brevity. The resulting equation will be found as follows :—

$$\left.\begin{array}{l}\sigma_{\prime}=\dfrac{c^2}{a^2-b^2}\left[\dfrac{a^2(a^2-k^2)}{(b^2-c^2)(b^2+c^2-k^2)}\right]^{\frac{1}{2}}\displaystyle\int\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}}\\ -\dfrac{a^2(a^2-k^2)(b^2-k^2)}{(a^2-b^2)(a^2-k^2)}\left[\dfrac{(b^2+c^2-a^2)}{(b^2-c^2)(b^2+c^2-k^2)}\right]^{\frac{1}{2}}\displaystyle\int\frac{\mathrm{d}\phi}{[1-\sin^2\epsilon\sin^2\phi]\sqrt{1-\sin^2\alpha'\sin^2\phi}}\end{array}\right\},(6$$

an elliptic integral which is also of the third order and *logarithmic* form.

The parameter is the square of the sine of the semifocal angle of the invariable cone, while the modulus is the sine of the major principal arc of the cone of rotation.

When $a=b$, $\sin\epsilon=0$, and the above expression assumes the form,

$$\sigma'=\left(\frac{a^2+c^2-k^2}{a^2-k^2}\right)\cos\alpha'\int\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}}. \quad . \quad . \quad (605)$$

In (58) it was shown that $\cos\alpha'\int\frac{\mathrm{d}\phi}{\sqrt{1-\sin^2\alpha'\sin^2\phi}}$ is the algebraical representative of an arc of the spherical parabola whose major principal arc $\alpha_{\prime}$ is given by the equation

$$\tan^2\alpha_{\prime}=\frac{1+\cos\alpha'}{1-\cos\alpha'}=\frac{1}{\tan^2\frac{\alpha'}{2}};\ \text{whence}\ \alpha_{\prime}+\frac{\alpha'}{2}=\frac{\pi}{2},$$

or α' and $2\alpha_{\prime}$ are supplemental.

140.] On the spiral described by the mean axis b of the ellipsoid, or the *mean* or B spiral.

In the general equation (596), interchanging b and c, also y and z, we obtain the result

$$\frac{a^2b^2c^2}{\kappa^2k^2}\left(\frac{\mathrm{d}\sigma''}{\mathrm{d}t}\right)^2=b^2(a^2+c^2-k^2)-(a^2+c^2-b^2)y^2. \quad . \quad . \quad \text{(a)}$$

For y^2 substitute its value given in (546),

$$y^2 = \frac{b^2(a^2-k^2)(k-c^2)\cos^2\varphi}{(a^2-k^2)(b^2-c^2)\cos^2\varphi + (b^2-k^2)(a^2-c^2)\sin^2\varphi}.$$

Introducing the value of $\left(\frac{dt}{d\varphi}\right)$ found in (541*), we shall obtain

$$\frac{1}{b\sqrt{b^2-k^2}}\left(\frac{d\sigma''}{d\varphi}\right) = \frac{\sqrt{a^2(a^2-k^2)\cos^2\varphi + (a^2-c^2)(a^2+c^2-k^2)\sin^2\varphi}}{(a^2-k^2)(b^2-c^2)\cos^2\varphi + (b^2-k^2)(a^2-c^2)\sin^2\varphi}. \quad \text{(b)}$$

Let

$$\left.\begin{array}{ll} A = a^2(a^2-k^2), & C = (a^2-k^2)(b^2-c^2), \\ B = (a^2-c^2)(a^2+c^2-k^2), & D = (b^2-k^2)(a^2-c^2), \end{array}\right\} . \quad \text{(c)}$$

and the preceding equation may be written

$$\frac{1}{b\sqrt{b^2-k^2}}\left(\frac{d\sigma''}{d\varphi}\right) = \frac{A\cos^2\varphi + B\sin^2\varphi}{C\cos^2\varphi + D\sin^2\varphi}\,\frac{1}{\sqrt{A\cos^2\varphi + B\sin^2\varphi}}; \quad \text{(d)}$$

or, as $B > A$, this equation may be transformed into

$$\left.\begin{array}{l} \dfrac{\sqrt{B}}{b\sqrt{b^2-k^2}}\left(\dfrac{d\sigma''}{d\varphi}\right) = \dfrac{BC-AD}{D(C-D)}\displaystyle\int\frac{d\varphi}{\left[1+\left(\frac{C-D}{D}\right)\cos^2\varphi\right]\sqrt{1-\left(\frac{B-A}{B}\right)\cos^2\varphi}} \\ \qquad\qquad - \dfrac{(B-A)}{C-D}\displaystyle\int\frac{d\varphi}{\sqrt{1-\left(\frac{B-A}{B}\right)\cos^2\varphi}} \end{array}\right\}. \quad (606)$$

If we now compute the value of the coefficients in this equation by the help of (c), we shall find, 2ϵ being the focal angle of the invariable cone, as shown in (b) sec. [115],

$$\frac{C-D}{D} = \frac{(a^2-b^2)(k^2-c^2)}{(a^2-c^2)(b^2-k^2)} = \tan^2\epsilon. \quad . \quad . \quad . \quad . \quad \text{(e)}$$

$\frac{B-A}{B} = \frac{c^2(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)} = \sin^2\beta'$, β' being the lesser principal angle of the cone of rotation as in (587). We have also

$$\frac{BC-AD}{D(C-D)} = \frac{(a^2-k^2)}{(b^2-k^2)}\,\frac{(a^2+c^2-b^2)}{(a^2-b^2)},$$

and

$$\sqrt{B} = \sqrt{(a^2-c^2)(a^2+c^2-k^2)}.$$

Making those substitutions, (606) becomes

$$\left.\begin{aligned}\sigma'' = &\frac{b(a^2-k^2)(a^2+c^2-b^2)}{(a^2-b^2)\sqrt{(a^2-c^2)(b^2-k^2)(a^2+c^2-k^2)}}\int\frac{d\phi}{[1+\tan^2\epsilon\cos^2\phi]\sqrt{1-\sin^2\beta'\cos^2\phi}}\\ &-\frac{c^2 b\sqrt{b^2-k^2}}{(a^2-b^2)\sqrt{(a^2-c^2)(a^2+c^2-k^2)}}\int\frac{d\phi}{\sqrt{1-\sin^2\beta'\cos^2\phi}}\end{aligned}\right\}. \text{(6C}$$

As the parameter $\tan^2\epsilon$ is positive, the elliptic integral of the third order is of the *circular* form.

When $a=b$, $\tan\epsilon=0$ and the elliptic integral of the third order in the preceding equation is reduced to the first. Adding the above expressions together, and reducing,

$$\sigma'' = \left(\frac{a^2+c^2-k^2}{a^2-k^2}\right)\cos\alpha'\int\frac{d\phi}{\sqrt{1-\sin^2\alpha'\cos^2\phi}}. \quad . \quad . \quad \text{(f)}$$

This expression agrees with the one found for the greater spiral, differing from it only in the amplitude, which is complementary.

We shall now proceed to eliminate from the preceding equation the integral of the first order.

Multiply this equation by the factor $\sqrt{\dfrac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}}$.

Let as before $\alpha_{\prime}$ and $\beta_{\prime}$ be the principal semiangles of a cone whose focals shall coincide with those of the invariable cone, and the planes of whose circular sections shall make the angles β' with the internal axis; then, assuming the equations established in sec. [8] and (e), we shall have

$$\frac{\tan^2\alpha_{\prime}-\tan^2\beta_{\prime}}{\sec^2\beta_{\prime}} = \tan^2\epsilon_{\prime} = \tan^2\epsilon = \frac{(a^2-b^2)(k^2-c^2)}{(a^2-c^2)(b^2-k^2)},$$

and

$$\frac{\sin^2\alpha_{\prime}-\sin^2\beta_{\prime}}{\sin^2\alpha_{\prime}} = \sin^2\eta_{\prime} = \sin^2\beta' = \frac{c^2(k^2-c^2)}{(a^2-c^2)(a^2+c^2-k^2)},$$

as in (587); whence, making the substitutions indicated,

$$\tan^2\alpha_{\prime} = \frac{(a^2-b^2)(a^2+c^2-k^2)}{c^2(b^2-k^2)}, \quad \tan^2\beta_{\prime} = \frac{a^2(a^2-b^2)}{c^2(b^2-c^2)}, \quad . \quad \text{(g)}$$

by the help of these equations we may show that

$$\frac{\cos\beta_{\prime}}{\cos\alpha_{\prime}\sin\alpha_{\prime}} = \frac{(a^2-k^2)\sqrt{(b^2-c^2)(a^2+c^2-b^2)}}{\sqrt{(a^2-c^2)(a^2-b^2)(b^2-k^2)(a^2+c^2-k^2)}}, \quad . \quad \text{(h)}$$

and

$$\frac{\cos\beta_{\prime}\cos\alpha_{\prime}}{\sin\alpha_{\prime}} = \frac{c^2\sqrt{(b^2-k^2)(b^2-c^2)}}{\sqrt{(a^2-c^2)(a^2-b^2)(a^2+c^2-k^2)(a^2+c^2-b^2)}}. \quad \text{(i)}$$

Whence (606) may now be written

$$\left.\begin{aligned}\left[\frac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}\right]^{\frac{1}{2}}\sigma'' = \frac{\cos\beta_{\prime}}{\cos\alpha_{\prime}\sin\alpha_{\prime}}\int\frac{d\phi}{[1+\tan^2\epsilon\cos^2\phi]\;\sqrt{1-\sin^2\eta_{\prime}\cos^2\phi}} \\ -\frac{\cos\beta_{\prime}\cos\alpha_{\prime}}{\sin\alpha_{\prime}}\int\frac{d\phi}{\sqrt{1-\sin^2\eta_{\prime}\cos^2\phi}}\end{aligned}\right\}. \quad (608)$$

If now we turn to (41) and (47), in which elliptic integrals are compared, having the same amplitude, but positive and negative parameters respectively, we shall find them identical with the preceding equation, which may now therefore be written

$$\left.\begin{aligned}\left[\frac{(a^2-b^2)(b^2-c^2)}{b^2(a^2+c^2-b^2)}\right]^{\frac{1}{2}}\sigma'' = \frac{\tan\beta_{\prime}}{\tan\alpha_{\prime}}\sin\beta_{\prime}\int\frac{d\phi}{[1-e_{\prime}^2\cos^2\phi]\;\sqrt{1-\sin^2\eta_{\prime}\cos^2\phi}} \\ +\tan^{-1}\left[\frac{e_{\prime}\tan\epsilon_{\prime}\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta_{\prime}\cos^2\phi}}\right]\end{aligned}\right\}. \quad (609)$$

If we take the complete function, the circular arc vanishes. We may therefore conclude that *the length of the mean or* B *spiral, or of the spiral described by the pole of the mean axis b of the ellipsoid, between any two of its asymptotic positions, is equal to a quadrant of a spherical ellipse.* The cone of which this spherical ellipse is the base, may with ease be determined. It must have the same focal lines as the invariable cone; and its minor principal arc is the angle between the cyclic diameters of the ellipsoid; for the cyclic semidiameter whose square is $a^2+c^2-b^2$ makes with the axis c an angle the square of whose tangent is

$$\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)}. \quad \text{In (g) we found } \tan^2\beta_{\prime}=\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)},$$

or $2\beta_{\prime}$ is the angle between the cyclic diameters of the ellipsoid.

We have thus investigated the equations of the spirals described on a fixed concentric sphere by the three principal axes of a body, which we have named the *greater, mean,* and *lesser,* or the A, B, and C spirals. It is not a little remarkable that the rectification of the greater and lesser spirals must be effected by elliptic integrals of the third order and *logarithmic* form, while the rectification of the mean spiral depends on an elliptic integral of the third order and *circular* form. It will moreover be evident, on referring to the preceding sections, that the elliptic integrals which express the lengths of the spirals described by the instantaneous axis of rotation and the mean principal axis of the body have the same amplitude, and are each of the *circular* form; while the integrals which determine the spirals described by the greatest and the least principal axes of the body also have the *same* amplitude, which is complementary to the former, and are of the *logarithmic* form.

141.] We may determine the maximum and minimum velocities with which the poles of the principal axes of the body describe their respective spirals on the fixed concentric sphere. Resuming the equation of the spirals traced by the principal axes,

$$\frac{a^2b^2c^2}{\kappa^2k^2}\left(\frac{d\sigma}{dt}\right)^2=c^2(a^2+b^2-k^2)-(a^2+b^2-c^2)z^2;$$

differentiating and putting the differential equal to cipher, we get $\frac{dz}{dt}=0.$

It was shown in (515) that $\frac{dz}{dt}=\frac{f(a^2-b^2)xy}{a^2b^2}.$

This is $=0$ whenever the position of the axis k renders $x=0$ or $y=0$; and as k is at its greatest or least distance from the axis c of the ellipsoid whenever it lies in one of the principal planes, the velocity of the pole of c on the spiral is the greatest or the least whenever the axis c is at its greatest or least distance from the axis k.

The same proof may be applied to determine the extreme velocities of the poles of a and b.

CHAPTER XIX.

142.] There are two particular cases of the general problem which require separate investigations—when the plane of the impressed couple is at right angles to, or coincides with, the plane of one of the circular sections of the ellipsoid of moments.

We shall first take the case when the plane of the impressed couple is at right angles to the plane of one of the circular sections of the ellipsoid, or $k=b$. If we introduce this value of k into the equation of the invariable cone in (527), we shall obtain the following equation:—

$$c^2(a^2-b^2)x^2+a^2(c^2-b^2)z^2=0. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

This expression is the equation of the two plane circular sections of the ellipsoid which intersect in the mean axis b. If, then, to fix our ideas, we conceive the plane of the impressed couple to be horizontal, one of the circular sections of the ellipsoid will be vertical during the motion.

To determine in this case the locus of the instantaneous axis of rotation in the body. If we write b for k in the equation of the cone of rotation (528), we get

$$a^2(a^2-b^2)x^2+c^2(c^2-b^2)z^2=0, \quad . \quad . \quad . \quad . \quad \text{(b)}$$

the equation of two plane sections of the ellipsoid passing through the mean axis, and perpendicular to the umbilical diameters of the ellipsoid.

We may perceive therefore that the axis of the impressed couple, and the instantaneous axis of rotation, describe planes in the body during the motion.

To find the greatest elongation of the axis of rotation from the axis k. This is nothing more than to find the angle which a perpendicular from the centre, on a tangent passing through the vertex of k or b, makes with it, in an ellipse whose semiaxes are a and c. Now, h being the conjugate diameter to k or b, and P the perpendicular on the tangent,

$$h^2+b^2=a^2+c^2, \text{ and } \mathrm{P}h=ac. \text{ Let this angle be } \vartheta.$$

Then $$\tan^2\vartheta=\frac{b^2-\mathrm{P}^2}{\mathrm{P}^2}=\frac{(a^2-b^2)(b^2-c^2)}{a^2c^2}=w^2. \quad . \quad . \quad . \quad \text{(c)}$$

To determine the time.

In the general equation (540) let $k=b$, and we shall find

$$\frac{\mathrm{d}t}{\mathrm{d}z}=\frac{abc^2}{fz\sqrt{b^2-c^2}\sqrt{c^2(a^2-b^2)-(a^2-c^2)z^2}}. \quad . \quad . \quad \text{(d)}$$

Assume $$(a^2-c^2)z^2=c^2(a^2-b^2)\sin^2\varphi, \quad . \quad . \quad . \quad . \quad \text{(e)}$$

in which φ is the angle between k and the mean axis of the ellipsoid, measured on a circular section of the surface. By this transformation, equation (a) may be changed into

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{ac}{\kappa\sqrt{a^2-b^2)(b^2-c^2)}}\left(\frac{1}{\sin\varphi}\right). \quad . \quad . \quad . \quad . \quad \text{(f)}$$

It was shown in (c) that $\tan\vartheta$ is the maximum value of $\tan\theta$.

Hence $$\tan\theta=\sqrt{\frac{(a^2-b^2)(b^2-c^2)}{a^2c^2}}=w.$$

Let $j=\kappa w$; the preceding equation, when integrated, will become, putting C for the constant,

$$jt=\log\tan\frac{\varphi}{2}+\mathrm{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

To determine the value of this constant. Let δ be the initial distance of the pole of k from the axis b, at the beginning of the motion; then $0=\log\tan\frac{\delta}{2}+\mathrm{C}$. Subtracting we shall have

$$jt=\log\left[\frac{\tan\frac{\varphi}{2}}{\tan\frac{\delta}{2}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(h)}$$

Let $\tan\frac{\delta}{2}=m$, and the last equation may be written

$$\tan\frac{\varphi}{2}=m\mathrm{e}^{jt}, \text{ or, as } j=\kappa w,\ \tan\frac{\varphi}{2}=m\mathrm{e}^{\kappa\ t}, \quad . \qquad (610)$$

e being the base of the Neperian logarithms.

When δ is absolutely equal to 0, m also is equal to 0, and φ is 0, however large the value we may assign to the time t. But when δ is only very small, m will be a very small quantity, and therefore t must be very large before its magnitude can have any appreciable effect on the magnitude of φ. Hence the pole of k will diverge slowly from the mean axis b. When the initial distance δ is supposed to be considerable, then m is no longer an indefinitely small quantity, and a small increase in t will produce a considerable effect in the magnitude of φ.

Again, let the axis of the impressed couple, by the motion of the semicircular section passing through it, be approximated to indefinitely, by the prolongation of the principal axis b, within a very small angle δ'.

Let T be the future time at which the prolongation of the axis b shall arrive within a certain small angle δ' of k. Then $\varphi=\pi-\delta'$, and $j\mathrm{T}=\log\tan\left(\frac{\pi}{2}-\frac{\delta'}{2}\right)+\mathrm{C}$. As the initial distance of b from k must be supposed as before to be δ,

$$0=\log\tan\left(\frac{\delta}{2}\right)+\mathrm{C}, \quad \text{whence } -j\mathrm{T}=\log\left[\tan\left(\frac{\delta}{2}\right)\tan\left(\frac{\delta'}{2}\right)\right].$$

Let $m=\tan\left(\frac{\delta}{2}\right)$ as before; then

$$m\tan\frac{\delta'}{2}=\mathrm{e}^{-j\mathrm{T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (611)$$

In this equation T will be infinite on two suppositions, either $m=0$, or $\tan\frac{\delta'}{2}=0$. The former shows that T will be infinite if b never departs *from* coincidence with the axis of the impressed couple. From the second we may infer that b never can, having once departed *from* coincidence with k, again coincide *with* it.

We may therefore infer that the motion of k in the body will be as follows. When the coincidence of k with the mean axis is disturbed, and the disturbance takes place along one or other of the circular sections of the ellipsoid, the axis b at first diverges very slowly from k, then with greater rapidity until this velocity reaches a maximum state. The velocity then decreases, so that b, with a motion continually retarded, approaches indefinitely near to, without ever absolutely reaching, the axis of the impressed couple.

143.] To find the value of θ the angle between the axis of rotation and the axis of the plane of the impressed couple.

In (514) writing b for k, and $c^2(a^2-b^2)\sin^2\varphi$ for $(a^2-c^2)z^2$, we obtain $\tan\theta=w\sin\varphi$. Hence θ varies from its inferior limit to ϑ as φ varies from δ to $\frac{\pi}{2}$.

It was shown in (510) that the velocity of the pole of the plane of the impressed couple along the invariable conic was $f\tan\theta$. Writing V for this velocity, $\mathrm{V}=b\kappa w\sin\varphi$. . . . . . (611*)

As $\tan\theta=w\sin\varphi$, $\omega=\frac{f}{\mathrm{P}}$, $\tan^2\theta=\frac{k^2-\mathrm{P}^2}{\mathrm{P}^2}$, ω being the angular velocity, whence $\omega^2=\kappa^2[1+w^2\sin^2\varphi]$, or $\omega=\kappa\sec\theta$. . . (612)

To determine the angle ψ which the line of the nodes makes with a fixed line in the plane of the impressed couple.

Resuming the equation (552), putting b for k, and

$c^2(a^2-b^2)\sin^2\varphi$ for $z^2(a^2-c^2)$, as in (e) sec. [142], we get

$$\frac{z^2}{b^2-z^2}=\frac{\tan^2\eta\sin^2\varphi}{1+\tan^2\eta\cos^2\varphi}.\quad\text{Writing }\tan^2\eta\text{ for }\frac{c^2(a^2-b^2)}{a^2(b^2-c^2)},$$

which represents the tangent of half the dihedral angle between the circular sections of the ellipsoid, or half the angle between the cyclic axes. We also have

$$\frac{dt}{d\varphi}=\frac{1}{\kappa w\sin\varphi},\text{ as in (f) sec. [142].}$$

Making these substitutions in the equation (552),

$$\psi=-\kappa t+\kappa\left(\frac{k^2-c^2}{c^2}\right)\int\frac{z^2dt}{k^2-z^2},\text{ we find}$$

$$-\psi=\kappa t+\tan^{-1}[\tan\eta\cos\varphi]+\text{constant}.\qquad(613)$$

To determine this constant.

At the beginning of the motion let the axis of the plane of the impressed couple very nearly coincide with the mean axis of the ellipsoid. Then ϕ is very small, and $\cos\phi$ very nearly equal to 1: we thus get $0=\tan^{-1}(\tan\eta)+C$, or $C=-\eta$; hence

$$-\psi=+\kappa t+\tan^{-1}(\tan\eta\cos\phi)-\eta. \quad . \quad . \quad (614)$$

The limits of ϕ are 0 and π, between which limits the pole of the impressed couple lies during the motion. Now when $\phi=0$, $\cos\phi=1$, and $\tan^{-1}(\tan\eta\cos\phi)=\eta$. When $\phi=\pi$, $\cos\phi=-1$, and $\tan^{-1}\{\tan\eta\times-1\}=-\eta$. Whence

$$-\psi=\kappa T+2\eta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (614^*)$$

writing T for the period in which the semicircle is described by k.

Thus we perceive that the infinite angle ψ is made up of two parts, one of which increases as the time, while the other continually approximates to a fixed limit 2η, 2η being the angle between the cyclic axes of the surface.

The geometrical interpretation of this formula it is not difficult to discover. In sec. [119] it was shown that the angle ψ was made up of two parts, one of which κt increases as the time, while the other may be represented by an arc of the spherical ellipse, generated by the cone supplemental to the invariable cone. As the circular sections of this latter coincide in direction with the circular sections of the ellipsoid, the cyclic axes of this latter surface will coincide with the focals of the supplemental cone. Hence, as before mentioned, the whole motion of the body may be represented by conceiving this supplemental cone to roll without sliding on the plane of the impressed couple, while this plane revolves uniformly round its axis. But when the plane, as in this case, passes through one of the cyclic axes of the ellipsoid, this supplemental cone degenerates into a plane sector of a circle, the angle between whose bounding diameters is 2η. Now, when the plane of the impressed couple is slightly disturbed from coincidence with the plane of this circular sector (for when k coincides with b, the plane of the impressed couple coincides with the principal plane ac, which contains the cyclic axes), it will revolve round a straight line (one of the cyclic axes bounding the circular sector) instead of rolling upon a conical surface; and this straight line (the cyclic axis of the ellipsoid, or the focal of the rolling cone) becomes, in the ultimate state of this cone, the *edge* of the circular sector. The plane of ac, being disturbed from a state of coincidence with the plane of the impressed couple, will revolve round one of the cyclic axes until it approximates indefinitely on *its other side* to this plane.

Now if, instead of the cone, we imagine the sector of the circle to revolve upon the plane, the line of contact with the plane will no longer *advance continuously* upon this plane, but *per saltum*, starting forward through an angle 2η at each half-revolution; so that if we imagine a number of semirevolutions to occur, the line of contact of this sector with the plane would advance through the angles 2η, 4η, &c. From the nature of this motion, however, we can have but half a revolution, and even that only as a limit. It follows, therefore, that when half the semicircle is completed, or when the axis of the plane of the impressed couple comes into the plane of ac, an angle η must at once be added to the angle ψ, or that the line of the nodes starts forward through the angle η.

144.] We shall now investigate the nature of the spiral described by the pole of the instantaneous axis of rotation in the case when $k=b$.

The spherical polar coordinates of this spiral are θ and χ.

They are connected as follows:—

In general $\chi=\kappa k^2\int_0 \frac{dt}{u^2}$, as shown in (560): put b for k in the equation (571) which determines u, and we shall have $u=b$; hence

$$\chi=\kappa t. \qquad \text{(a)}$$

This equation shows that the motion of the radius vector arc θ is uniform, being proportional to the time.

It was shown in (610) that $\tan\frac{\phi}{2}=m e^{\kappa w t}$: writing χ for κt, we get $\tan\frac{\phi}{2}=m e^{w\chi}$, and $\tan\theta=w\sin\phi$. . . . . (b)

These are the equations of the spiral. We must eliminate ϕ from these equations.

As $\sin\phi=2\sin\frac{\phi}{2}\cos\frac{\phi}{2}=2\tan\frac{\phi}{2}\cos^2\frac{\phi}{2}=\dfrac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$, we get

$$\tan\theta=\frac{2mw\,e^{w\chi}}{1+m^2e^{2w\chi}}, \text{ or } \tan\theta=\frac{2w}{(m\,e^{w\chi})+(m\,e^{w\chi})^{-1}}, \qquad (615)$$

a relation between the variables θ and χ, consequently the equation of the spiral.

145.] The *rhumb line* may be defined as the curve described on the surface of a sphere which cuts all the meridians in a given angle. Let this constant angle be the complement of ϑ, then its cotangent is w, ϕ and χ being the polar spherical coordinates of the curve; therefore

$$w\sin\phi=\frac{d\phi}{d\chi}. \qquad \text{(a)}$$

This is the equation of the rhumb line.

Taking the integral of this equation, $\log \tan \frac{\phi}{2} = w\chi + C$.

Let the value of ϕ be δ when $\chi = 0$. Then $\log \tan \frac{\delta}{2} = C$,

and $\tan \frac{\delta}{2} = m$; hence

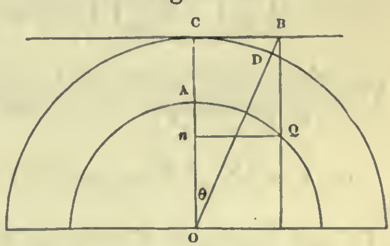
$$w\chi = \log \left[\frac{\tan \frac{\phi}{2}}{\tan \frac{\delta}{2}} \right], \text{ or } \tan \frac{\phi}{2} = m e^{w\chi}. \quad . \quad . \quad (616)$$

This is the usual equation of the rhumb line, and is identical with (610). Hence the polar spiral is a sort of curtated rhumb line. If a rhumb line be described on the surface of the sphere, its ordinate angle being $\left(\frac{\pi}{2} - \vartheta\right)$, and if we shorten its spherical central vectors ϕ in the constant ratio given by the equation $\tan \theta = \tan \vartheta \sin \phi$, the extremity of θ will describe the polar spiral.

Another construction exhibiting the relation between these spirals may be given.

Let a concentric sphere be described, whose radius $OA = \tan \vartheta = w$. On this sphere let a rhumb line be constructed, having its pole at A in the axis of z. Let this rhumb line be orthogonally projected on the tangent plane to the sphere whose radius is 1, parallel to the plane of xy. Now, if this plane curve be considered as the gnomonic projection (*i. e.* the eye being supposed at the centre) of a spherical curve described on the surface of the *outer* sphere, this latter curve will be the polar spiral, or Q and D are corresponding points.

Fig. 35.



This we may thus show. In this construction we always have $\tan \theta = \tan \vartheta \sin \phi$.

Now $CB = Qn$, $CB = \tan \theta$, and $Qn = \tan \vartheta \sin \phi$. Q and D are therefore the corresponding points of the rhumb line and of the polar spiral, whose vector arcs are $CD = \theta$, $AQ = \phi$.

It is evident that the polar spiral has an asymptotic circle, whose radius is $\sin \vartheta$. In the vicinity of the pole, the polar spiral approximates indefinitely to the rhumb line.

146.] To find the length of this spiral from the pole to the asymptotic circle.

$$\left(\frac{d\sigma}{d\phi}\right)^2=\left(\frac{d\theta}{d\phi}\right)^2+\sin^2\theta\left(\frac{d\chi}{d\phi}\right)^2=\left(\frac{d\theta}{d\phi}\right)^2+\sin^2\theta\left(\frac{d\chi}{dt}\right)^2\left(\frac{dt}{d\phi}\right)^2,$$

$$\tan\theta=w\sin\phi,\quad \frac{d\theta}{d\phi}=\frac{w\cos\phi}{1+w^2\sin^2\phi},\quad \frac{d\chi}{dt}=\kappa,\quad \frac{dt}{d\phi}=\frac{1}{\kappa w\sin\phi},$$

and $\sin^2\theta=\dfrac{w^2\sin^2\phi}{1+w^2\sin^2\phi}$.

Introducing these relations, we get

$$\frac{d\sigma}{d\phi}=\frac{\sqrt{1+w^2}}{1+w^2\sin^2\phi};$$

dividing by $\cos^2\phi$, and integrating, we shall find

$$\sigma=\tan^{-1}(\sqrt{1+w^2}\tan\phi). \quad . \quad . \quad . \quad . \quad (617)$$

When $\phi=0$, $\sigma=0$, and when $\phi=\frac{\pi}{2}$, $\sigma=\frac{\pi}{2}$.

We thus find that the length of the polar spiral between the pole and the asymptotic circle is equal to a quadrant of a great circle of a sphere,—a result in strict accordance with the more general theorem established in sec. [130].

When $\sqrt{1+w^2}\tan\phi=1$, or $\tan\phi=\cos\vartheta$, $\sigma=\tan^{-1}(1)$ or $\sigma=\frac{\pi}{4}$.

147.] To determine the velocity of the pole along the spiral.

As $V^2=\left(\frac{d\sigma}{dt}\right)^2=\left(\frac{d\sigma}{d\phi}\right)^2\left(\frac{d\phi}{dt}\right)^2=\dfrac{(1+w^2)\kappa^2w^2\sin^2\phi}{(1+w^2\sin^2\phi)^2}$,

$$V^2=\frac{\kappa^2(1-w^2)w^2\sin^2\phi}{\{1+w^2\sin^2\phi\}^2}=\frac{\kappa^2(1+w^2)\tan^2\theta}{\sec^4\theta}=\kappa^2(1+w^2)\sin^2\theta\cos^2\theta;$$

or $V=\dfrac{\kappa\sqrt{1+w^2}}{2}\sin 2\theta$, or $V=\frac{1}{2}\dfrac{\kappa\sin 2\theta}{\cos\vartheta}$, since $w=\tan\vartheta$.

It may be shown that when k coincides with the greatest or the least principal axes of the body, the spirals described by the two other axes are equivalent to circular arcs. But when k coincides with b the mean axis, the lengths of the spirals described by the greatest and the least principal axes are given by logarithms. Omitting the investigations (which, though somewhat complicated, the reader, assuming the principles established in the foregoing pages, may supply), the final result will be found as follows—

$$p\sigma = q \log \tan\frac{\psi}{2} + \log(1 + q\sec\psi),\ p \text{ and } q \text{ being constants.}$$

148.] When the plane of the impressed moment coincides with the plane of one of the circular sections of the ellipsoid of moments, the elliptic integrals which determine the motion may be reduced from the *third* order to the *first*.

In this case $2k$ is the cyclic axis of the ellipsoid, or the diameter *perpendicular* to the plane of one of its circular sections.

Accordingly $\frac{1}{k^2} = \frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2}$. Substitute this value of k in (i) and (j) sec. [119], and (553). Reducing, we shall have $\psi + \kappa t =$

$$\pm\left[\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}\right]\int\frac{d\varphi}{\left[1-\frac{c^2}{a^2}\left(\frac{a^2-b^2}{b^2-c^2}\right)\sin^2\varphi\right]\sqrt{1-\frac{c^4}{a^4}\left(\frac{a^2-b^2}{b^2-c^2}\right)^2\sin^2\varphi}}. \quad (618)$$

This integral, as the parameter is equal to the modulus, may be reduced to the first order as follows :—

Let γ as in sec. [20] be the *parametral angle* of the spherical parabola. Assume $\frac{1-\sin\gamma}{1+\sin\gamma} = \frac{c^2(a^2-b^2)}{a^2(b^2-c^2)} = \tan^2\eta$, η being half the angle between the circular sections of the ellipsoid. Whence

$$\sin\gamma = \frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\frac{1}{c^2}-\frac{1}{a^2}},\quad \cos^2\gamma = 4\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)^2}. \quad (a)$$

The preceding equation may now be written

$$\psi + \kappa t = \frac{2\sin\gamma}{1+\sin\gamma}\int\frac{d\varphi}{\left[1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)\sin^2\varphi\right]\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\varphi}}, \quad (619)$$

or, as it may be more succinctly written,

$$\psi + \kappa t = (1-\tan^2\eta)\int\frac{d\varphi}{[1-\tan^2\eta\sin^2\varphi]\ \sqrt{1-\tan^4\eta\sin^2\varphi}}.$$

If we compare (619) with (62), we shall find that the second member is equivalent to the following elliptic integral of the first order,

$$\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}} + \tan^{-1}\left[-\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right], \quad (b)$$

the amplitudes ϕ and μ being connected by Lagrange's formula, $\tan(\phi-\mu)=\sin\gamma\tan\mu$, as in (63), or, as it may in this case be written,

$$\tan\phi=\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\sin^2\mu}{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\cos 2\mu}. \quad . \quad . \quad . \quad \text{(c)}$$

Should we require to reduce the integrals of the third and first order of the same amplitude, equation (58) will enable us with ease to do so, by assuming the theorem established in that equation,

$$\left.\begin{aligned}\psi+\kappa t&=\frac{\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\\ &+\tfrac{1}{2}\tan^{-1}\left[\frac{\left(\frac{2-\sin\gamma}{1+\sin\gamma}\right)\tan\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}\right]\end{aligned}\right\} . \quad . \quad (620)$$

Hence ψ depends on an integral of the *first* order,—the theorem it was proposed to establish.

Again, if we substitute the foregoing value of k in (542), which connects the time with the amplitude ϕ, on which immediately depends the position of the axis k in the body at the end of the given time, we shall have

$$\kappa t=\frac{1}{f^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}\int\frac{d\phi}{\sqrt{1-\frac{c^4}{a^4}\left(\frac{a^2-b^2}{b^2-c^2}\right)^2\sin^2\phi}};$$

$$\text{and as } \frac{2\sin\gamma}{1+\sin\gamma}=\frac{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)}$$

$$\frac{\left[\left(\frac{1}{c^2}-\frac{1}{b^2}\right)-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\right]}{2\left[\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}\right]}\kappa t=\frac{\sin\gamma}{1+\sin\gamma}\int\frac{d\phi}{\sqrt{1-\left(\frac{1-\sin\gamma}{1+\sin\gamma}\right)^2\sin^2\phi}}. \quad (621)$$

But this elliptic integral, as shown in sec. [25], is the expression for an arc of a spherical parabola whose *parametral angle* is γ, the centre being the pole. In this case the two elliptic functions which determine the motion are represented by arcs of the *same* spherical parabola.

We may eliminate the latter integral by the equation established in sec. [25], and the last equation will now become

$$\kappa t = 2\frac{\left[\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}\right]}{\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}\int\frac{d\mu}{\sqrt{1-\frac{4a^2c^2(a^2-b^2)(b^2-c^2)}{b^4(a^2-c^2)^2}\sin^2\mu}}.$$

The moduli are two successive terms of Lagrange's modular scale.

149.] Thus have we shown in the foregoing investigations how the properties of elliptic integrals applied to the theory of the motion of a rigid body round a fixed point have led us to a complete solution of this celebrated problem, a solution which has enabled us to place before our eyes, so to speak, the very actual motion of the revolving body. Yet it is not on such grounds solely that this treatise has been published. Were the investigations of no other use than to give strength and clearness to vague and obscure notions on this confessedly most difficult subject, enough had been already accomplished by the celebrated geometer whose name is so deservedly associated with this theory. It is as a method of investigation that it must rest its claims to the notice of mathematicians, as a means of giving simple and elegant interpretations of those definite integrals on the evaluation of which the dynamical state of a body at any epoch can alone be ascertained. If the author has to any degree succeeded in accomplishing this, it is because he has drawn largely upon the properties of lines and surfaces of the second order, and of those curve lines in which these surfaces intersect. If he has been enabled to advance any thing new, it is owing solely to the somewhat unfrequented path he has pursued. That it was antecedently probable such might lead to undiscovered truths, no one conversant with the applications of mathematical conceptions to the discussions of those sciences will deny. To introduce auxiliary surfaces into the discussions and investigations of physical science is an idea as luminous as it has been successful.

A TREATISE

ON

THE HIGHER GEOMETRY,

AND ON

CONICS

CONSIDERED AS SECTIONS OF A RIGHT CIRCULAR CONE.

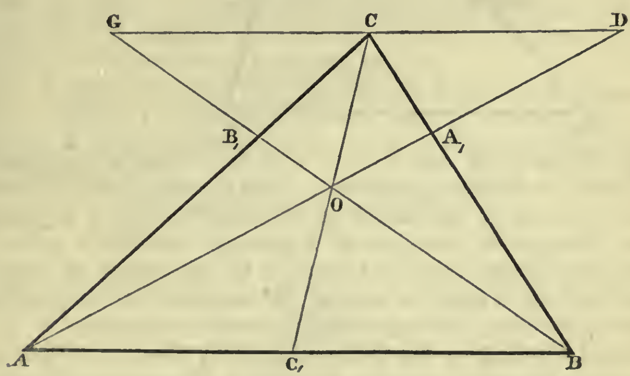
"Les sections coniques offrent une source intarissable de propriétés, et l'on ne peut dire sans témérité que cette matière est épuisée."—QUETELET, *Correspondance Mathématique et Physique,* tom. i. p. 162.

CHAPTER XX.

ON TRANSVERSALS.

150.] *If through any point* O *in the plane of a triangle* ABC *transversals are drawn from the vertices* A, B, C, *and meet the opposite sides in the points* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$, *the continued products of the alternate segments of the sides are equal.*

Fig. 1.



Through one of the vertices C let a parallel DG to the opposite side AB be drawn, and let the transversals $AA_{\prime}$, $BB_{\prime}$ meet it in the points D, G. Then, by similar triangles, we have

$$AC_{\prime} : BC_{\prime} = DC : CG,$$
$$BA_{\prime} : CA_{\prime} = AB : DC,$$
$$CB_{\prime} : AB_{\prime} = CG : AB.$$

Compounding these proportions together, we shall have

$$AC_{\prime} . BA_{\prime} . CB_{\prime} = AB_{\prime} . BC_{\prime} . CA_{\prime}.$$

This product is a maximum when O is the *centroid* (that is, the centre of gravity) of the triangle.

151.] *If the sides of a triangle* ABC *are cut by a transversal* $C_{\prime}A_{\prime}B_{\prime}$, *it divides the sides into segments such that the continued product of these alternate segments are equal.*

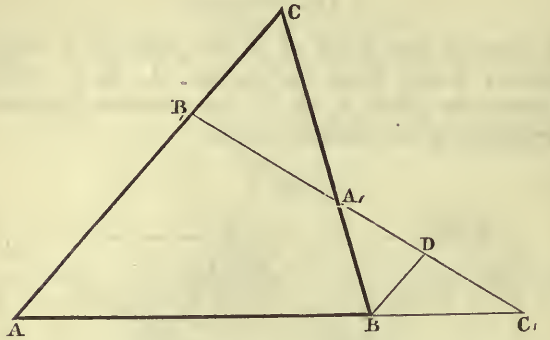
Through one of the vertices B draw the straight line BD parallel to the opposite side AC. Then, by similar triangles, we have

$$AB_{\prime} : AC_{\prime} = BD : BC_{\prime}, \quad CA_{\prime} : BA_{\prime} = CB_{\prime} : BD, \quad BC_{\prime} : CB_{\prime} = BC_{\prime} : CB_{\prime}.$$

Compounding these proportions together, we shall have

$$AB_{\prime}\,.\,BC_{\prime}\,.\,CA_{\prime}=AC_{\prime}\,.\,BA_{\prime}\,.\,CB_{\prime}.$$

Fig. 2.



These propositions are of very wide and important application. Thus, from the former proposition (150) it immediately follows (α) that the bisectors of the angles of a triangle meet in a point, (β) that the bisectors of the sides of a triangle meet in a point*, (γ) that the lines drawn from the vertices to the points of contact of the inscribed circle meet in a point, (δ) and that the perpendiculars from the vertices on the opposite sides meet in a point.

If an odd number of the points $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ (that is, either one or three) are on the sides between the angles, transversals drawn from the vertices will meet in a point; but if an even number (that is, either two or none) are so found, then the three points A, B, C will range in a straight line.

152.] *Through a point* O *in the plane of a triangle* ABC, *let straight lines be drawn from the vertices* A, B, C, *meeting the opposite sides in the points* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$; *then we shall have*

$$\frac{AO}{AA_{\prime}}+\frac{BO}{BB_{\prime}}+\frac{CO}{CC_{\prime}}=2, \quad \text{(a)} \qquad \frac{OA_{\prime}}{AA_{\prime}}+\frac{OB_{\prime}}{BB_{\prime}}+\frac{OC_{\prime}}{CC_{\prime}}=1. \quad \text{(b)}$$

We have manifestly, see fig. 1,

$$\frac{AO+OA_{\prime}}{AA_{\prime}}+\frac{BO+OB_{\prime}}{BB_{\prime}}+\frac{CO+OC_{\prime}}{CC_{\prime}}=3. \quad . \quad . \quad . \quad \text{(c)}$$

But this may be written

$$\frac{AO}{AA_{\prime}}+\frac{BO}{BB_{\prime}}+\frac{CO}{CC_{\prime}}+\frac{OA_{\prime}}{AA_{\prime}}+\frac{OB_{\prime}}{BB_{\prime}}+\frac{OC_{\prime}}{CC_{\prime}}=3. \quad . \quad . \quad \text{(d)}$$

* These bisectors (β) are called by French geometers *median* lines, a term which we shall adopt and make use of hereafter.

Let Δ denote the area of the triangle, and $\delta_{\prime}$, $\delta_{\prime\prime}$, $\delta_{\prime\prime\prime}$ the areas of the component triangles whose vertices are at O, and whose bases are BC, AC, and AB. Then $\Delta=\delta_{\prime}+\delta_{\prime\prime}+\delta_{\prime\prime\prime}$.

But $$\frac{OA_{\prime}}{AA_{\prime}}=\frac{\delta_{\prime}}{\Delta},\quad \frac{OB_{\prime}}{BB_{\prime}}=\frac{\delta_{\prime\prime}}{\Delta},\quad \frac{OC_{\prime}}{CC_{\prime}}=\frac{\delta_{\prime\prime\prime}}{\Delta}.$$

Hence we have $$\frac{OA_{\prime}}{AA_{\prime}}+\frac{OB_{\prime}}{BB_{\prime}}+\frac{OC_{\prime}}{CC_{\prime}}=1. \quad \ldots\ldots \quad (e)$$

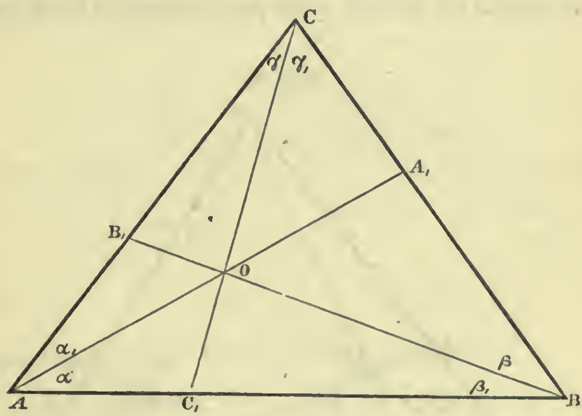
and therefore $$\frac{AO}{AA_{\prime}}+\frac{BO}{BB_{\prime}}+\frac{CO}{CC_{\prime}}=2. \quad \ldots\ldots \quad (f)$$

When the point O is assumed outside the triangle, the above proportions still hold good, but *one* of the component triangles must then be taken with a negative sign.

153.] *In any triangle* ABC, *if lines be drawn from the vertices through a point* O *to the opposite sides, making with the sides at the vertices* A, B, C *the angles* $\alpha, \alpha_{\prime}$; $\beta, \beta_{\prime}$; $\gamma, \gamma_{\prime}$, *then the following relation will hold good*:—

$$\sin\alpha\sin\beta\sin\gamma=\sin\alpha_{\prime}\sin\beta_{\prime}\sin\gamma_{\prime}. \quad \ldots\ldots \quad (a)$$

Fig. 3.



The triangles $BAA_{\prime}$ $CAA_{\prime}$ are as their bases $BA_{\prime}$, $CA_{\prime}$. But twice the triangle $BAA_{\prime}=BA\,.\,AA_{\prime}\sin\alpha$, and twice the triangle $CAA_{\prime}=CA\,.\,AA_{\prime}\sin\alpha_{\prime}$, or $BA\,.\,AA_{\prime}\sin\alpha : CA\,.\,AA_{\prime}\sin\alpha_{\prime}=BA_{\prime} : CA_{\prime}$.

Hence we have $$\frac{\sin\alpha}{\sin\alpha_{\prime}}=\frac{CA}{BA}\cdot\frac{BA_{\prime}}{CA_{\prime}}. \quad \ldots\ldots \quad (b)$$

In like manner $$\frac{\sin\beta}{\sin\beta_{\prime}}=\frac{BA}{BC}\cdot\frac{CB_{\prime}}{AB_{\prime}}, \text{ and } \frac{\sin\gamma}{\sin\gamma_{\prime}}=\frac{BC}{CA}\cdot\frac{AC_{\prime}}{BC_{\prime}}.$$

Multiplying these expressions together, we obtain

$$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin\alpha_{,}\sin\beta_{,}\sin\gamma_{,}}=\frac{BA_{,}\,.\,CB_{,}\,.\,AC_{,}}{CA_{,}\,.\,AB_{,}\,.\,BC_{,}}=1, \text{ by sec. [150]}. \qquad (c)$$

154.] *From the vertices* A, B, C *of a triangle, pairs of lines are drawn, making with the adjacent sides equal angles* α, α; β, β; γ, γ. *If the first set of three lines pass through a point* O, *the second set will also meet in a point* Q.

Let the angles which the lines AO, BO, CO (fig. 4) make at the vertices A, B, C with the adjacent sides be α, $A-\alpha$; β, $B-\beta$; γ, $C-\gamma$; then the angles which the second set of lines makes at the same vertices will be $A-\alpha$, α; $B-\beta$, β; $C-\gamma$, γ. Now, since the first set of lines pass through a fixed point O, we shall have, by sec. [153],

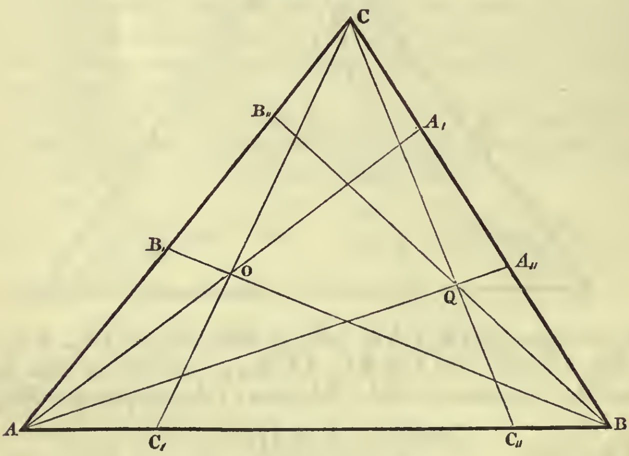
$$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)}=1; \quad . \quad . \quad . \quad . \qquad (a)$$

and we must therefore have

$$\frac{\sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)}{\sin\alpha\sin\beta\sin\gamma}=1; \quad . \quad . \quad . \quad . \qquad (b)$$

hence the second set of lines must pass through a fixed point Q.

Fig. 4.

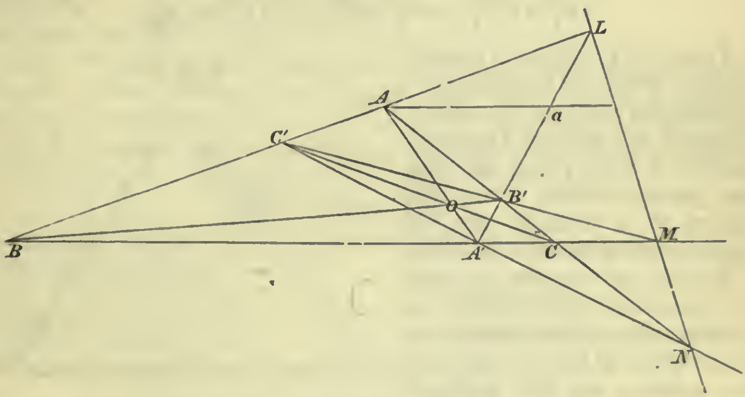


Hence it obviously follows that since the perpendicular drawn on the opposite side from any vertex of a triangle, and the diameter of the circumscribing circle passing through this vertex, make equal angles with the adjacent sides, if one set of these lines pass through a point, the other set will also pass through a point. But the diameters of the circumscribing circle drawn through the three angles of the triangle concur in a point, the centre; hence also the three perpendiculars meet in a point. This point, which is of constant occurrence in the higher geometry, has been called by some geometers the *orthocentre*. We shall henceforth adopt this term. The triangle formed by joining the feet of these perpendiculars may be appropriately called the *orthocentric triangle*.

155.] *Three lines meeting in a point* O *are drawn from the vertices of a triangle* ABC, *and produced to meet the opposite sides in the points* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$. *The sides of the triangle* $A_{\prime}B_{\prime}C_{\prime}$ *will meet the sides of the triangle* ABC *in three points which range in a straight line.*

Through A draw the straight line Aa parallel to the side BC of the triangle. Let the corresponding sides of the two triangles ABC, $A_{\prime}B_{\prime}C_{\prime}$ meet in the points L, M, N. Then L, M, N will range in a straight line.

Fig. 5.



In the two triangles LAa and LB$A_{\prime}$, we have

$$LA : LB = Aa : BA_{\prime}, \text{ and } Aa : CA_{\prime} = AB_{\prime} : CB_{\prime}.$$

Hence
$$\frac{LA}{LB} = \frac{CA_{\prime} . AB_{\prime}}{BA_{\prime} . CB_{\prime}}.$$

In like manner $\dfrac{MB}{MC} = \dfrac{BC_{\prime} . AB_{\prime}}{AC_{\prime} . CB_{\prime}}$ and $\dfrac{N}{NA} = \dfrac{CA_{\prime} . BC_{\prime}}{BA_{\prime} . AC_{\prime}}$.

Multiplying these expressions together, we obtain

$$\frac{LA . MB . NC}{LB . MC . NA} = \left[\frac{CA_{\prime} . AB_{\prime} . BC_{\prime}}{BA_{\prime} . CB_{\prime} . AC_{\prime}}\right]^2.$$

But as the three lines meet in a point, the expression between the brackets is equal to unity; hence $\frac{LA \cdot MB \cdot NC}{LB \cdot MC \cdot NA}=1$, and therefore, by sec. [151], L, M, N range in a straight line.

When the point in which the three lines concur lies outside the triangle, a slight modification of the same proof will apply.

156.] *If from any point* P *in the plane of a triangle* ABC, *perpendiculars are drawn to meet the opposite sides in the points* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$; *then we shall have*

$$\overline{AB_{\prime}}^2+\overline{CA_{\prime}}^2+\overline{BC_{\prime}}^2=\overline{AC_{\prime}}^2+\overline{BA_{\prime}}^2+\overline{CB_{\prime}}^2.$$

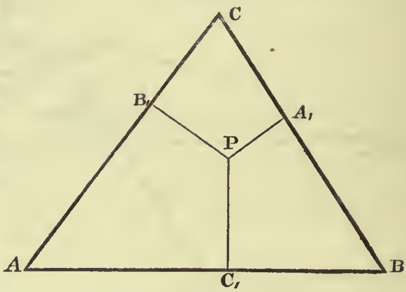
For

$$\overline{AB_{\prime}}^2+\overline{PB_{\prime}}^2=\overline{AC_{\prime}}^2+\overline{PC_{\prime}}^2,$$
$$\overline{CA_{\prime}}^2+\overline{PA_{\prime}}^2=\overline{CB_{\prime}}^2+\overline{PB_{\prime}}^2,$$
$$\overline{BC_{\prime}}^2+\overline{PC_{\prime}}^2=\overline{BA_{\prime}}^2+\overline{PA_{\prime}}^2.$$

Adding these equations together, the squares on the perpendiculars cancel each other, and we shall have

$$\overline{AB_{\prime}}^2+\overline{CA_{\prime}}^2+\overline{BC_{\prime}}^2=\overline{AC_{\prime}}^2+\overline{BA_{\prime}}^2+\overline{CB_{\prime}}^2.$$

Fig. 6.



From this proposition we may at once infer, (α) that the perpendiculars through the middle points of the sides of a triangle meet in a point, and (β) that the perpendiculars from the angles of a triangle on the opposite sides meet in a point.

For as the square on each perpendicular is the difference between the squares on the adjacent sides of the triangle and the squares on the adjacent segments of the opposite side, the proposition becomes manifest.

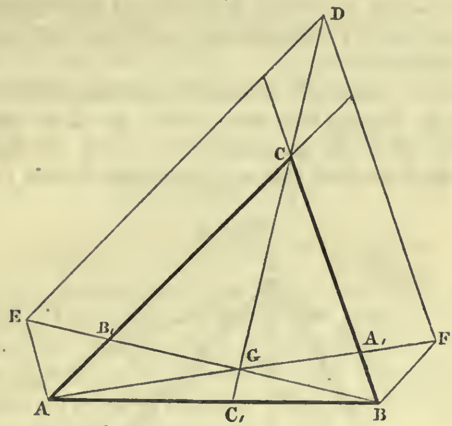
157.] *From the ends* A, B *of the base of a triangle* ABC, *lines* AE, BF, *of arbitrary equal lengths, are drawn parallel to the opposite sides of the triangle; and through* E, F *lines* ED, FD *are drawn parallel to the adjacent sides of the triangle and meeting in* D. *Then if* AF, BE *cut the opposite sides in* $A_{\prime}$ *and* $B_{\prime}$, *these lines will intersect in the line* DC.

By similar triangles, we have

$$AE : CB = AB_{\prime} : CB_{\prime} \text{ and } AC : BF = CA_{\prime} : BA_{\prime},$$

but $AE=BF$; hence $AC : CB = AB_{\prime} \cdot CA_{\prime} : CB_{\prime} \cdot BA_{\prime}$.

Fig. 7.



But since the line DC bisects the angle C,

we have $\quad AC : CB = AC_{,} : BC_{,}$,

therefore $\quad AB_{,} . CA_{,} . BC_{,} = AC_{,} . CB_{,} . BA_{,}$;

and therefore the three lines AF, BE, CD meet in a point*.

158.] *From any point* O *within or without the angle* BAC *a transversal is drawn cutting the sides of the angle in the points* B *and* C; *the sum of the reciprocals of the areas of the triangles* AOB *and* AOC *is constant, and independent of the position of the transversal.*

For the sum or difference of the triangles, we find

$$AOC \pm AOB = ABC;$$

or dividing by the product of the areas of the triangles AOC and AOB, we shall have

$$\frac{1}{AOB} \pm \frac{1}{AOC} = \frac{ABC}{AOB . AOC} = \frac{2AB . AC . \sin A}{(AO . AB . \sin BAO)(AO . AC \sin CAO)}$$

$$= \frac{2 \sin A}{\overline{AO}^2 . \sin BAO . \sin CAO},$$

* The following extension of this theorem is due to Mr. W. J. C. Miller, Vice-Principal of Huddersfield College:—From the ends of the base of a triangle straight lines are drawn—in the same or in a different direction—parallel to the opposite sides, and proportional in length to the *adjacent* sides; then (1) the straight lines joining the ends of these parallels with the remote ends of the base, intersect each other on one of two straight lines which pass through the vertex of the triangle, and divide the base internally and externally in the duplicate ratio of the adjacent sides; (2) if the vertical angle is a right angle, the internal locus is perpendicular to the base; and (3) if the parallels are proportional in length to the *opposite* sides, the locus of the intersections will be a line from the vertex bisecting the base, or else parallel to the base.

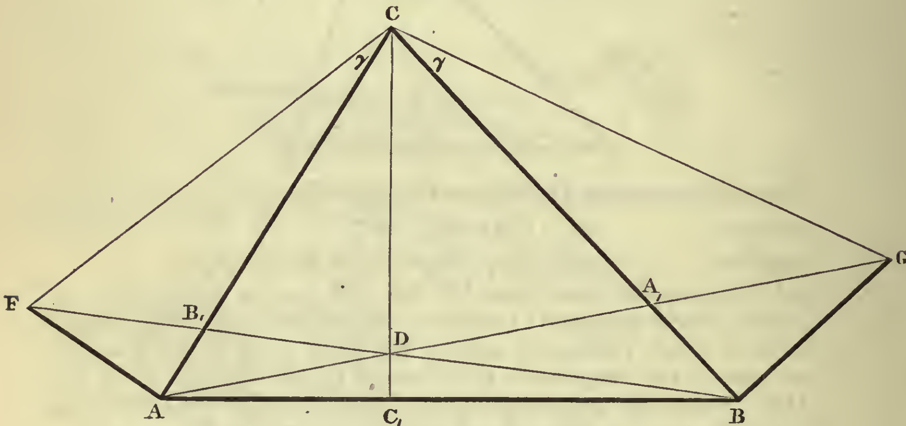
Proofs of these theorems will be found in pages 18, 19 of vol. xxii. of the mathematical 'Reprints from the Educational Times,' edited by Mr. Miller.

which is a constant quantity independent of the positions of the points B and C—that is, of the direction of the transversal.

159.] The method of transversals may also be applied to prove the following theorem :—

At the ends of the base of a triangle perpendiculars to the adjacent sides are drawn, having the same ratio to these sides; the lines joining the ends of these perpendiculars with the opposite corners of the triangle will meet on the perpendicular drawn from the vertex to the base.

Fig. 8.



Let AF, BG be perpendicular to AC and BC, having the same ratio $2m$ to the sides AC, BC. Join FC and GC. Let $CF=2n.AC$, $CG=2n.BC$, the angle $ACF=BCG=\gamma$. Put $AC=b$ and $BC=a$, then the area of the triangle FCB is $nab\sin(C+\gamma)$, and the area of the triangle AFB is $mcb\cos A$. But these areas are in the proportion of $CB_{\prime}$ to $AB_{\prime}$; hence

$$\frac{CB_{\prime}}{AB_{\prime}}=\frac{na\sin(C+\gamma)}{mc\cos A}\,;\ \text{and similarly}\ \frac{CA_{\prime}}{BA_{\prime}}=\frac{nb\sin(C+\gamma)}{mc\cos B}.$$

Therefore $$\frac{CB_{\prime}.BA_{\prime}}{AB_{\prime}.CA_{\prime}}=\frac{a\cos B}{b\cos A}.$$

But $AC_{\prime}=b\cos A$, and $BC_{\prime}=a\cos B$; consequently

$$CB_{\prime}.BA_{\prime}.AC_{\prime}=AB_{\prime}.CA_{\prime}.BC_{\prime}.$$

If the ratio be one of equality and the vertical angle C be a right angle, it follows that the transversals will meet on the perpendicular in the diagram of Euclid's proof of the Pythagorean theorem (Euc. I. 47).

In the same way we may establish the following theorems :—

1. *If on the sides of a triangle similar rectangles be drawn and the adjacent extremities of these rectangles be joined, the perpendi-*

culars from the three vertices of the triangle on these lines will meet in a point.

2. *If on the sides of a triangle similar isosceles triangles be drawn, the lines joining the vertices of these triangles with the opposite vertices of the given triangle will meet in a point.*

CHAPTER XXI.

ON HARMONIC AND ANHARMONIC RATIOS.

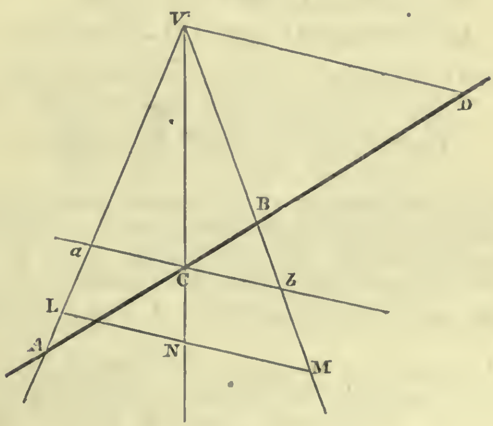
160.] The principles developed in these methods will be found of wide application, and most powerful instruments of investigation. Let any straight line LM (fig. 9) be bisected in N; and from any point V let straight lines be drawn through the points L, M, N; and let VD be drawn parallel to LM. The four lines VL, VM, VN, VD form what is called an *harmonic pencil.*

If any straight line—called a *transversal*—be drawn across this pencil, it will be divided so as to have AC : CB = AD : BD.

Draw aCb parallel to LM or VD; then, since $aC = Cb$, we have

$$\frac{VD}{Ca} = \frac{AD}{AC}, \text{ and } \frac{VD}{Cb} = \frac{BD}{BC}.$$

Fig. 9.



But
$$\frac{VD}{Ca} = \frac{VD}{Cb},$$

consequently
$$\frac{AD}{AC} = \frac{BD}{BC}, \text{ or AD : BD = AC : BC.} \quad . \quad . \quad . \quad (a)$$

The points C and D are called *Harmonic points* with reference to the line AB.

When the transversal is drawn parallel to one of the rays of the harmonic pencil, its segments between the remaining three rays of the pencil are equal. This is evident from the preceding figure. On this property may be based the development of the properties of the centres of the conic sections.

161.] Let the line AB, harmonically divided in C and D, be bisected in O.

Fig. 10.

Then we have $\quad AC : AC+CB=AD : AD+DB;$

but $\quad AC+CB=2AO, \text{ and } AD+DB=2DO;$

therefore $$AC : AO=AD : DO, \quad \ldots\ldots \quad (b)$$

and $$AO : DO=BC : BD. \quad \ldots\ldots \quad (c)$$

If we take the original ratio $AC : CB=AD : DB$,

and apply the principle of the composition and division of ratios, we shall have

$$AC+CB : AC-CB=AD+DB : AD-DB.$$

But $\quad AC+CB=2AO,\ AC-CB=2CO,$

$\quad AD+DB=2DO, \text{ and } AD-DB=2AO,$

or $$AO : CO=DO : AO, \text{ or } \overline{AO}^2=CO\,.\,DO; \quad \ldots \quad (d)$$

also $$\frac{CO}{DO}=\frac{CO\,.\,DO}{\overline{DO}^2}=\frac{\overline{AO}^2}{\overline{DO}^2}=\frac{\overline{BC}^2}{\overline{BD}^2}=\frac{\overline{AC}^2}{\overline{AD}^2}, \quad \ldots \quad (e)$$

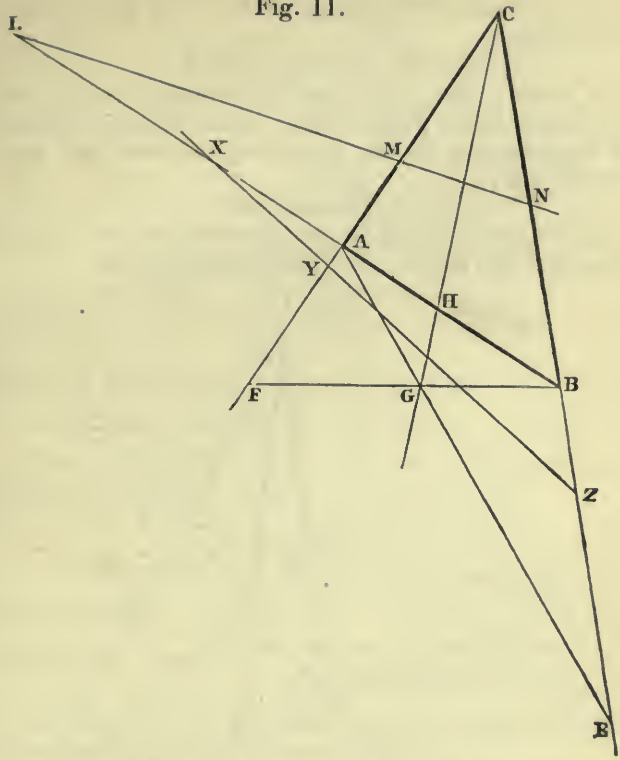
or *the ratio of the distances of the middle point of the given line from the harmonic points of division is equal to the square of the ratio of the distances of the same middle point to the ends of the whole line* AD—*and also equal to the squares of the ratios of the distances from the ends of the given line* A, B *to the harmonic points of division* C, D.

162.] *Let* L, M, N *be the points in which a transversal meets the sides of the triangle* ABC; *then the lines drawn from the vertices of this triangle to the harmonic points of* L, M, N *will meet in a point* G.

Let H, F, E be the harmonic points of L, M, N; then we have

$$\frac{MA}{MC}=\frac{FA}{FC}, \quad \frac{LB}{LA}=\frac{HB}{HA}, \quad \frac{EC}{EB}=\frac{NC}{NB}.$$

Fig. 11.



Multiplying these equations together, we shall have

$$\frac{MA \cdot LB \cdot EC}{MC \cdot LA \cdot EB} = \frac{FA \cdot HB \cdot NC}{FC \cdot HA \cdot NB}.$$

But as the points L, M, N are on a transversal, the first side of the equation is, by sec. [151], equal to unity, and therefore also the second side; hence the straight lines AE, BF, CH must pass through the same point G.

The point G may be called the *pole* of the *transversal* LMN with respect to the triangle ABC.

Let the segments LH, FM, and EN be bisected in the points X, Y, Z; then the points X, Y, Z will range in a straight line.

For, by sec. [161], $\frac{XA}{XB} = \frac{\overline{LA}^2}{\overline{LB}^2}$, $\frac{YC}{YA} = \frac{\overline{MC}^2}{\overline{MA}^2}$, $\frac{ZB}{ZC} = \frac{\overline{NB}^2}{\overline{NC}^2}$;

consequently $$\frac{XA \cdot ZB \cdot YC}{XB \cdot ZC \cdot YA} = \left[\frac{LA \cdot MC \cdot NB}{LB \cdot MA \cdot NC}\right]^2.$$

But the latter member of this equation is, by sec. [151], equal to unity; therefore

$$XA . ZB . YC = XB . ZC . YA;$$

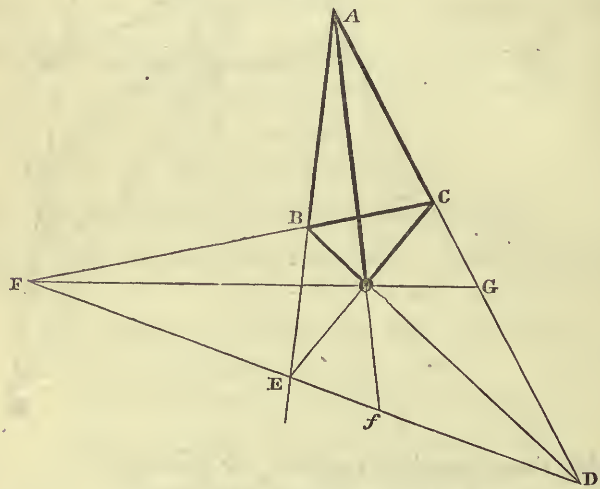
hence the three points X, Y, Z lie in the same straight line.

163.] *In a complete quadrilateral* ABOCED *any diagonal* ED *is divided harmonically by the two other diagonals* AO *and* BC *in the points* F *and f.*

DEFINITION.

A complete quadrilateral is that in which all the sides are produced to meet two by two, as ABOCED in fig. 12.

Fig. 12.



For as FC is a transversal to the triangle AED, we shall have

$$FE . DC . AB = FD . CA . BE, \text{ see sec. [151]}, \quad . \quad . \quad \text{(a)}$$

and as O is a point in the triangle AED, through which pass the three lines Af, CE, BD, we shall have, see sec. [150],

$$DC . AB . Ef = CA . BE . Df; \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

dividing the preceding equation by this latter, we shall have

$$\frac{FE}{Ef} = \frac{FD}{Df}, \text{ or } FE : FD = Ef : Df. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

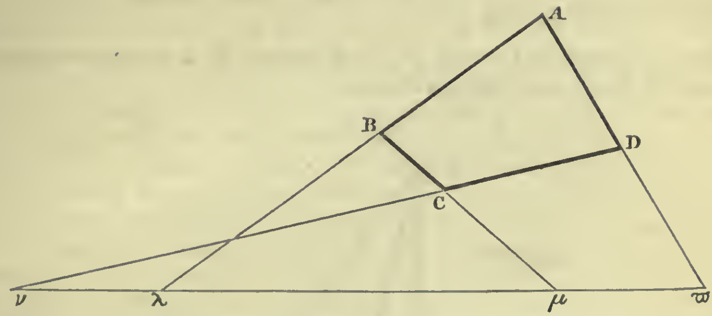
Hence the diagonal DE is *harmonically divided* in F and f.

The line Af may be called the *harmonic conjugate* of the point F; and FG is, similarly, the *harmonic conjugate* of the point A.

164.] *If a quadrilateral* ABCD *be cut by a transversal in the points* λ, μ, ν, ϖ, *the continued product of the alternate segments will be equal, or*

$$A\lambda . B\mu . C\nu . D\varpi = A\varpi . B\lambda . C\mu . D\nu. \quad . \quad . \quad . \quad \text{(d)}$$

Fig. 13.



By comparing the partial triangles, we have

$$A\lambda : \lambda\varpi = \sin\varpi : \sin A, \quad C\nu : \mu\nu = \sin\mu : \sin C,$$
$$B\mu : \lambda\mu = \sin\lambda : \sin B, \quad D\varpi : \varpi\nu = \sin\nu : \sin D.$$

Therefore, compounding these proportions, we have

$$A\lambda \cdot C\nu \cdot B\mu \cdot D\varpi = \frac{\lambda\varpi \cdot \mu\nu \cdot \lambda\mu \cdot \varpi\nu \cdot \sin\lambda \cdot \sin\mu \cdot \sin\nu \cdot \sin\varpi}{\sin A \cdot \sin B \cdot \sin C \cdot \sin D}.$$

In like manner

$$A\varpi \cdot B\lambda \cdot C\mu \cdot D\nu = \frac{\lambda\mu \cdot \mu\nu \cdot \varpi\nu \cdot \lambda\varpi \cdot \sin\lambda \cdot \sin\mu \cdot \sin\nu \cdot \sin\varpi}{\sin A \cdot \sin B \cdot \sin C \cdot \sin D}.$$

Hence the truth of the proposition, which may be extended to a polygon of any number of sides as follows:—

When it is proved that in a triangle cut by a transversal the products of the alternate segments of the sides are equal, we may extend the proposition to the case of the quadrilateral or to any other linear polygon.

On one of the sides of the given triangle let another triangle be constructed, whose sides shall be cut by the given transversal.

Let α, β, γ be the ratios of the alternate segments of the sides of the triangle; then $\alpha\beta\gamma=1$. Let δ and ϵ be the ratios of the segments of the sides of the newly applied triangle, then $\frac{\delta\epsilon}{\gamma}=1$; consequently $\alpha\beta\delta\epsilon=1$.

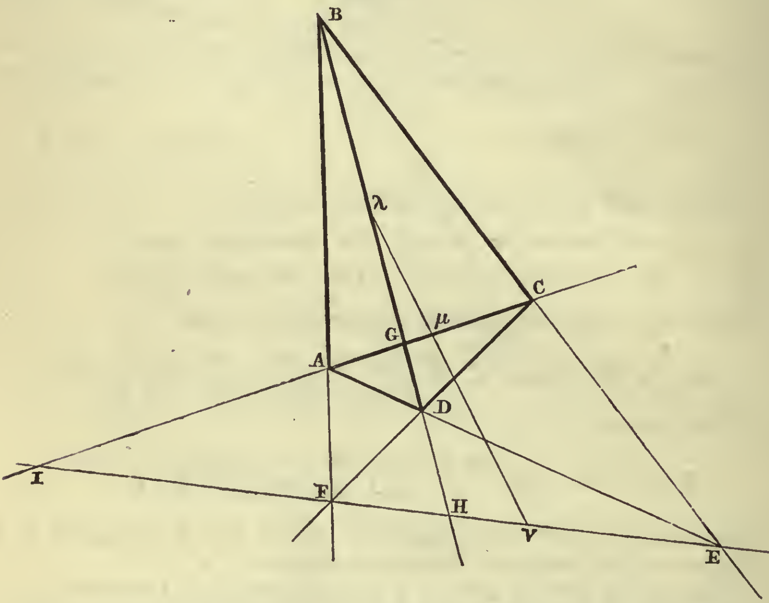
In like manner the theorem may be extended to a polygon of any number of sides.

It is obvious that in going round the triangle ABC we proceed from A to B, from B to C, and from C to A. In the same way in going round the quadrilateral we proceed from A to B, from B to C, from C to D, and from D to A; while in going round the applied triangle we proceed from C to D, from D to A, and from A to C.

Hence, if the ratio of the segments of CA be γ, the ratio of the segments of AC will be $\frac{1}{\gamma}$.

165.] *The middle points of the diagonals of a complete quadrilateral lie in the same straight line.*

Fig. 14.



Let λ, μ, ν be the middle points of these diagonals; they lie in a straight line. Since the line BD is harmonically divided in the points G, H, and bisected at λ, we shall have

$$\frac{\lambda G}{\lambda H}=\frac{\overline{BG}^2}{\overline{BH}^2},$$ see sec. [161]. We shall also have

$$\frac{\mu I}{\mu G}=\frac{\overline{CI}^2}{\overline{CG}^2} \text{ and } \frac{\nu H}{\nu I}=\frac{\overline{FH}^2}{\overline{FI}^2}=\frac{\overline{EH}^2}{\overline{EI}^2}.$$

Multiplying these expressions together, we shall have

$$\frac{\lambda G\,.\,\mu I\,.\,\nu H}{\lambda H\,.\,\mu G\,.\,\nu I}=\left\{\frac{BG\,.\,CI\,.\,EH}{BH\,.\,CG\,.\,EI}\right\}^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

But the three diagonals constitute a triangle GHI, of which the line BCE is a transversal. Consequently BG.CI.EH = BH.CG.EI, and therefore $\lambda G\,.\,\mu I\,.\,\nu H=\lambda H\,.\,\mu G\,.\,\nu I$, or λ, μ, and ν range along a straight line.

ON ANHARMONIC RATIO.

166.] This theory, the invention of M. Chasles (unquestionably the greatest geometer of this age, and, perhaps, equal to the best in any age), is an extension of the principle of harmonic ratio.

The theorem on which this powerful instrument of investigation is founded may be traced to the mathematical collections of Pappus*. This simple relation has been made the basis of a general system of conics by M. Chasles. Before his day it lay barren of results, until he developed its properties and gave it the name of *anharmonic ratio,* from its analogy to harmonic ratio, a particular case of the more general relation. There is one signal peculiarity of this method. If we take any theorem and its *dual,* as for example Pascal's and Brianchon's hexagons, the one inscribed in, and the other circumscribed to, a conic section, or any other like dual property, and if the one admits of investigation by Cartesian or *projective* coordinates, the dual must be treated by *tangential* coordinates, as discussed in the first volume of this work. But the anharmonic method is alike applicable to each, as we shall show further on. Another element of the great power of the anharmonic method is that its properties are projective.

From a point V let four fixed lines be drawn, meeting a fifth straight line variable in position, in the points A, B, C, D. Let these lines be put $VA=a$, $VB=b$, $VC=c$, $VD=d$, and let the sines of the angles between a and c be α, between c and b be γ, between b and d be β, and between d and a be δ. Let the sine of the angles between a and b be $(\alpha+\gamma)$, and that between c and d be $(\beta+\gamma)$, and let p be the perpendicular from the point V on the range.

Now twice the area of the triangle AVC is

$$AC\,.\,p=ac\alpha;\ \text{therefore}\ AC=\frac{ac\alpha}{p}.$$

In like manner

$$\left.\begin{aligned} CB=\frac{cb\gamma}{p},\quad BD=\frac{bd\beta}{p},\quad AD=\frac{ad\delta}{p},\\ AB=\frac{ab(\alpha+\gamma)}{p},\quad CD=\frac{cd(\beta+\gamma)}{p}.\end{aligned}\right\}\quad .\ .\ .\ (a)$$

Now these six segments of the range may be combined in the three following distinct groups—and no more—so that the variable rays and the common perpendicular p may be eliminated by division.

$$\left.\begin{aligned} \frac{CA}{CB}\div\frac{DA}{DB}&=\frac{ca\,.\,db}{cb\,.\,da}\,\frac{\alpha\beta}{\gamma\delta},\\ \frac{AC}{AB}\div\frac{DC}{DB}&=\frac{ac\,.\,db}{ab\,.\,dc}\,\frac{\alpha\beta}{(\alpha+\gamma)(\beta+\gamma)},\\ \frac{AB}{AD}\div\frac{CB}{CD}&=\frac{ab\,.\,cd}{ad\,.\,cb}\,\frac{(\alpha+\gamma)(\beta+\gamma)}{\gamma\delta}.\end{aligned}\right\}\quad .\ .\ .\ .\ (b)$$

* See Commandine's translation, Prop. 129, Lib. vii.

Dividing by a, b, c, d, the three anharmonic ratios become

$$\text{(I.)} \quad \frac{CA}{CB} \div \frac{DA}{DB} = \frac{\alpha\beta}{\gamma\delta}, \qquad \text{(II.)} \quad \frac{AC}{AB} \div \frac{DC}{DB} = \frac{\alpha\beta}{(\alpha+\gamma)(\beta+\gamma)},$$

and

$$\text{(III.)} \quad \frac{AB}{AD} \div \frac{CB}{CD} = \frac{(\alpha+\gamma)(\beta+\gamma)}{\gamma\delta}.$$

The first of these forms may be easily recollected, as it is the form of an harmonic pencil. The second has the same arrangement of the rays in the numerator as the first, $ca \, . \, db$, while the only arrangement possible for the denominator is $ab \, . \, dc$. The third form is the result of dividing the denominator of the second by that of the first.

There are in fact six different forms, which may be reduced to three.

It is not possible to write the four letters a, b, c, d two by two in more than three ways; namely $ab \, . \, cd$, $ac \, . \, bd$, $ad \, . \, cb$; hence there can be but three anharmonic arrangements of the segments of the range.

A peculiar notation may be devised to indicate briefly the several ratios of the anharmonic range.

Let V be the vertex of the pencil, and A, B, C, D the four points; then the ratio

$$\left.\begin{array}{l}
\dfrac{CA}{CB} \div \dfrac{DA}{DB} \text{ may be written } V\left(\dfrac{A}{B}\right)(C \div D), \\[2ex]
\dfrac{AC}{AB} \div \dfrac{DC}{DB} \text{ may be written } V\left(\dfrac{C}{B}\right)(A \div D), \\[2ex]
\dfrac{AB}{AD} \div \dfrac{CB}{CD} \text{ may be written } V\left(\dfrac{B}{D}\right)(A \div C).
\end{array}\right\}$$

The following relations may be easily established.

$$\left.\begin{array}{l}
V\left(\dfrac{A}{B}\right)(C \div D) = V\left(\dfrac{C}{D}\right)(A \div B), \\[2ex]
V\left(\dfrac{C}{B}\right)(A \div D) = V\left(\dfrac{A}{D}\right)(C \div B), \\[2ex]
V\left(\dfrac{B}{D}\right)(A \div C) = V\left(\dfrac{A}{C}\right)(B \div D).
\end{array}\right\} \quad . \quad . \quad . \quad (c)$$

Hence these six forms may be reduced to three.

If the given pencil of rays be cut by any other transversal, the ratios of the segments of this latter range will be the same as those of the former; for the sines of the radial angles remain unchanged.

If four points A, B, C, D be taken on a range, and through any point in space four rays be drawn through these four points, the anharmonic ratio of this pencil will be the same as that of the four points on the range.

The anharmonic ratio of any four points ranged along a straight line in one figure is equal to the anharmonic ratio of the corresponding pencil on the *reciprocal polar* of the original figure.

167.] Should the rays a, b, c, d meet the circumference of a circle in four fixed points, while the vertex V of the pencil moves along the circumference, the anharmonic ratios of these successive pencils will continue unchanged, because the sines of the radial angles (that is, of the angles between the rays) continue unchanged.

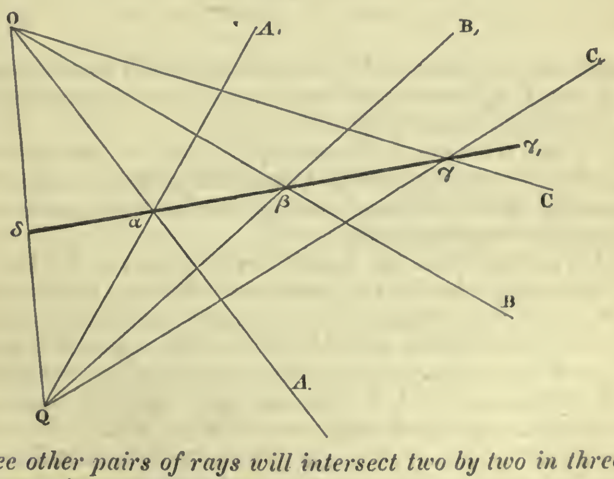
When the pencil is turned through a right angle, the anharmonic ratios continue unchanged, because the sines of the radial angles are still the same.

If four fixed tangents drawn to a circle be intersected by a fifth tangent variable in position, the anharmonic ratio of the segments of this tangent made by the fixed tangents will be constant and independent of its position.

It may easily be shown that if two fixed tangents are drawn to a circle, the segment of a third variable tangent intercepted between them subtends a constant angle at the centre, equal to half the external angle of the two fixed tangents. Hence the variable segments of the tangent range to the circle subtend fixed angles at the centre; and consequently their anharmonic ratio is constant.

168.] *If two equal anharmonic pencils have a common ray or axis,*

Fig. 15.



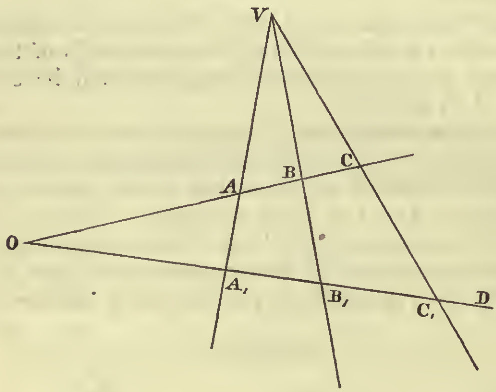
the three other pairs of rays will intersect two by two in three points which range in a straight line.

Let the two equal anharmonic pencils OQ, OA, OB, OC and QO, $QA_{,}$, $QB_{,}$, $QC_{,}$ have a common ray or axis OQ, the remaining three rays will intersect in three points α, β, γ, which range in a straight line.

Join α and β, and produce $\alpha\beta$ to δ in the common ray OQ, and let it meet the fourth rays OC, $QC_{,}$ in the points γ and $\gamma_{,}$; then these points must coincide, since the anharmonic ratio of $\delta\alpha\beta\gamma$ is equal to the anharmonic ratio of $\delta\alpha\beta\gamma_{,}$.

When the anharmonic ratios of two straight lines which meet in a point are equal, the straight lines which join the corresponding points two by two will all three meet in a point. Let OABC and $OA_{,}B_{,}C_{,}$ be two equal anharmonic ranges. Join $AA_{,}$, $BB_{,}$, and let

Fig. 16.



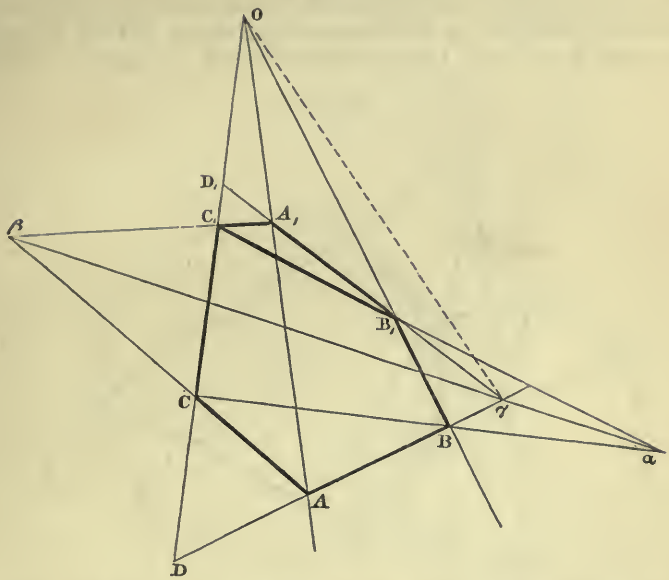
them meet in V. Then if $VC_{,}$ be drawn, it will pass through C; for if it cut $OA_{,}$ in some other point D, the anharmonic range $OA_{,}B_{,}C_{,}$ would be equal to $OA_{,}B_{,}D$.

169.] *If two triangles* ABC *and* $A_{,}B_{,}C_{,}$ (fig. 17) *have their corresponding vertices on three straight lines which meet in a point* O, *the corresponding sides will meet two by two in three points* α, β, γ *which range in a straight line.*

Join $O\gamma$; then since the pencil $O\gamma BAC$ is cut by the transversal $DAB\gamma$, and also by the transversal $D_{,}A_{,}B_{,}\gamma$, the anharmonic ratios of these two straight lines or ranges are equal; and as the pencils CD, CA, CB, $C\gamma$ and $C_{,}D_{,}$, $C_{,}A_{,}$, $C_{,}B_{,}$, $C_{,}\gamma$ have a common ray $CC_{,}$, and their anharmonic ratios are equal, the three remaining pairs of rays CA, $C_{,}A_{,}$, CB, $C_{,}B_{,}$, and AB, $A_{,}B_{,}$ will meet in the three points β, α, γ, which range in a straight line.

The triangles ABC, $A_{,}B_{,}C_{,}$ are called by Poncelet *homologous triangles*; the common point in which the three directrix lines meet,

Fig. 17.



the *centre of homology*; and the straight line in which each pair of sides meet, the *homologous axis*.

170.] *Let two homologous triangles* ABC and $A_{\prime}B_{\prime}C_{\prime}$ (fig. 17) *have their sides* AB, $A_{\prime}B_{\prime}$ *meeting in* γ, *their sides* BC, $B_{\prime}C_{\prime}$ *meeting in* α, *and their sides* AC, $A_{\prime}C_{\prime}$ *meeting in* β; *then, if* α, β, γ *range along a straight line, the lines joining the points* $AA_{\prime}$, $BB_{\prime}$, $CC_{\prime}$ *will meet in a point.*

As the pencil OCABγ is cut by the two ranges γBAD and $\gamma B_{\prime}A_{\prime}D_{\prime}$, their anharmonic ratios are equal, and they have besides an homologous point γ, therefore the lines joining the homologous points $AA_{\prime}$, $BB_{\prime}$, $CC_{\prime}$ meet in a point.

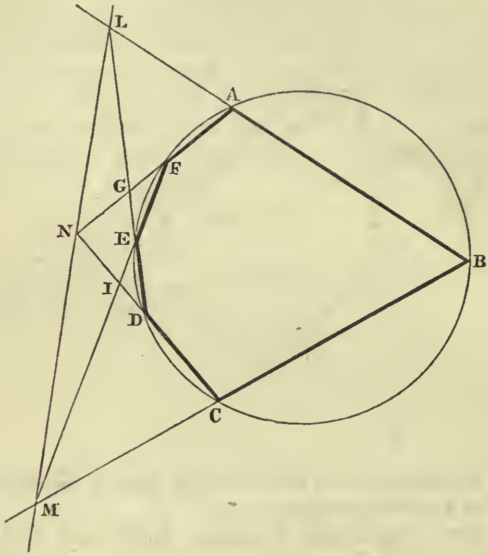
It is rather remarkable that when the two triangles are in the same plane, some such demonstration as that above given is required, but when the triangles lie in different planes the proposition becomes self-evident, the triangles constituting the bases of the same pyramid, and their sides will manifestly meet in the line in which the plane bases intersect—that is, in a straight line.

171.] *If the opposite sides of a hexagon inscribed in a circle be produced, they will meet two by two in the same straight line.*

Let B and E be the ends (fig. 18) of one of the diagonals of the hexagon BAFEDC, A and C the angles adjoining to B, and F and D the angles adjoining A and C.

Then, as these points lie on the circumference of a circle, the anharmonic ratios A(BFED), C(BDEF) will be equal. See sec. [167]. And as the pencil A(BFED) is cut by the transversal LGED, and the pencil C(BDEF) is cut by the transversal MIEF, the anharmonic ratios of these two transversals will be equal. Moreover

Fig. 18.

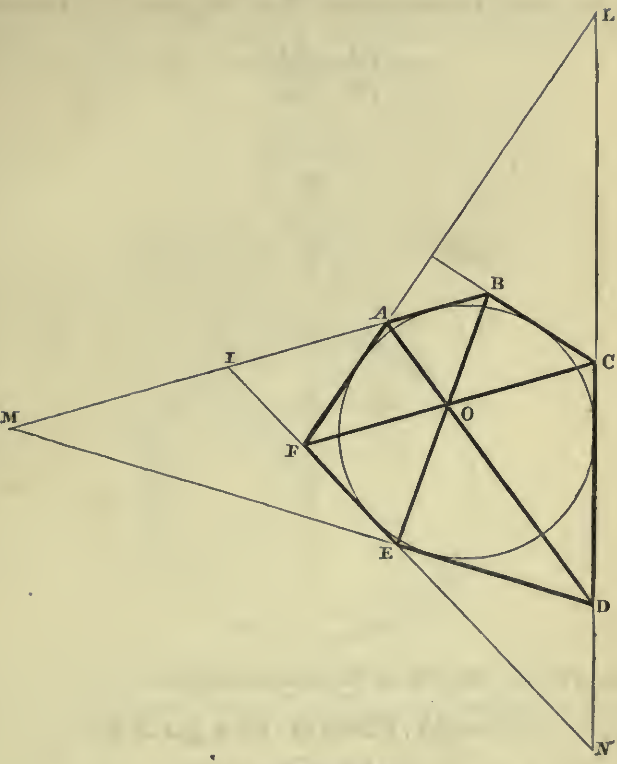


they have a common or homologous point E; hence the lines joining the other homologous points will all three meet in the same point, or the lines joining the points L and M, G and F, I and D, will meet in the same point N. Hence L, M, N are in the same straight line.

172.] *The diagonals of a hexagon circumscribed to a circle meet in a point.*

Since the four tangents CB, AF, FE, ED (fig. 19) meet the two tangents AB and CD in the points B and C, A and L, I and N, and in the points M and D, and as the anharmonic ratios of these two ranges BAIM and CLND are equal, the anharmonic pencils which pass through them will be equal. Therefore the anharmonic ratio of the pencil E(BAIM) will be equal to the anharmonic ratio of the pencil F(CLDN); and as these pencils have a common ray EF, the remaining three rays of each pencil will meet two by two in three points which range in a straight line: that is, EB and FC will meet in O, EA and FL will meet in A, while EM and FD will meet in D. Hence the point O must be on the line AD, or the three diagonals meet in the same point O.

Fig. 19.



CHAPTER XXII.

Definition.

173.] If from the centre of a circle a perpendicular be drawn on any straight line in its plane, and if in this perpendicular a point be taken so that the rectangle contained by its distance from the centre and the perpendicular shall be equal to the square on the radius, the point so found and the straight line are called *pole* and *polar* with respect to the circle*.

Lemma I.

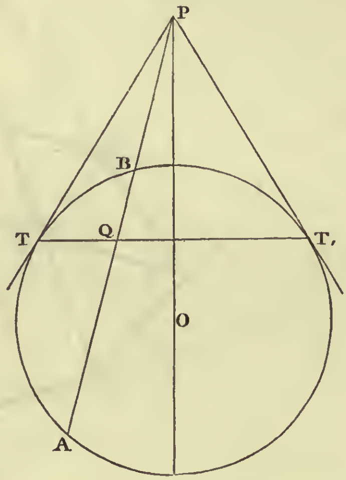
If a chord be drawn in a circle, and any point taken in this chord, the polar of this point will divide the chord into segments which will have to each other the same ratio as the segments of this chord made by the pole.

* Like most of our terms in this important branch of geometry, we owe the very convenient terms pole and polar to French geometers—the former to Servois, the latter to Gergonne.

Let P be the pole taken in the chord AB, and let this chord be cut in the point Q by the polar $TT_{\prime}$ of the point P. Then we shall have

$$\frac{PA}{PB} = \frac{QA}{QB}.$$

Fig. 20.



Since PT is a tangent to the circle, we have

$$\overline{PT}^2 = PA \,.\, PB = PQ \,.\, PB + AQ \,.\, PB;$$

and as $PTT_{\prime}$ is an isosceles triangle,

therefore $\overline{PT}^2 = QT.QT_{\prime} + \overline{PQ}^2 = QA \,.\, QB + PQ \,.\, QB + PQ \,.\, PB,$

$$= PA \,.\, QB + PQ \,.\, PB.$$

Equating these two values of $\overline{PT}^2$, and taking away the common rectangle PQ . PB, we shall have QA . PB = PA . QB;

therefore $$\frac{PA}{PB} = \frac{QA}{QB}.$$

Since $PA - PQ = QA$ and $PQ - PB = QB$, we have

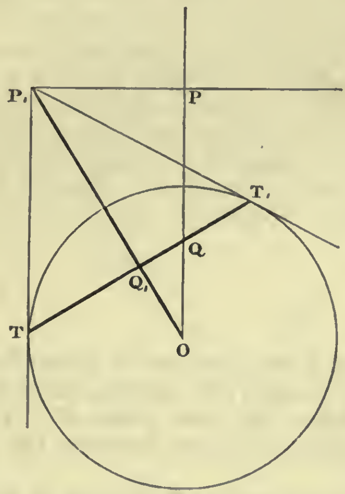
$$\frac{PA-PQ}{PQ-PB} = \frac{QA}{QB} = \frac{PA}{PB};$$

hence PA, PQ, PB are in harmonical proportion, since the first is to the third as the difference between the first and the second is to the difference between the second and the third.

LEMMA II.

Let a point and a straight line be assumed as pole and polar with reference to a circle. The polar of any point taken in this straight line will pass through the point assumed as pole.

Fig. 21.



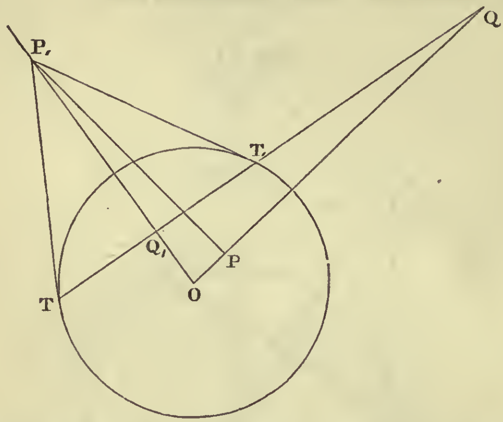
(α). Let the pole Q be taken within the circle. Join OQ, and produce it to P, so that $OQ \cdot OP = R^2$; then, by the definition of pole and polar, the polar of Q will pass through P and be at right angles to OQ.

Through Q draw a chord $TT_{\prime}$, and tangents $TP_{\prime}$, $T_{\prime}P_{\prime}$ meeting in $P_{\prime}$; and join $PP_{\prime}$. Then, as $OQ \cdot OP = OQ_{\prime} \cdot OP_{\prime}$ (since each rectangle is equal to R^2), the triangles $OQ_{\prime}Q$ and $OPP_{\prime}$ are similar, and the angle $OQ_{\prime}Q$ is equal to the angle $OPP_{\prime}$; but $OQ_{\prime}Q$ is a right angle, therefore $OPP_{\prime}$ is a right angle, or the line $PP_{\prime}$ is the polar of the point Q.

(β). Let the polar $PP_{\prime}$ cut the circle. Then, if OP be the perpendicular on $PP_{\prime}$ the distance of the pole of $PP_{\prime}$ from the centre is $\frac{R^2}{OP}$.

From any point $P_{\prime}$ in the polar $PP_{\prime}$ let tangents $P_{\prime}T$ and $P_{\prime}T_{\prime}$ be drawn to the circle, the line $TT_{\prime}$, the polar of $P_{\prime}$, will pass through the pole of $PP_{\prime}$. Let $TT_{\prime}$ meet the perpendicular OP in the point Q; then, as triangles $P_{\prime}OP$ and $QOQ_{\prime}$ are similar, $P_{\prime}O \cdot OQ = QO \cdot OP$; but $P_{\prime}O \cdot OQ_{\prime} = R^2$; therefore $QO \cdot OP = R^2$ or $QO = \frac{R^2}{OP}$; therefore the point Q in which the secant $TT_{,}$ cuts the perpendicular OP, coincides with the pole of the polar $PP_{,}$.

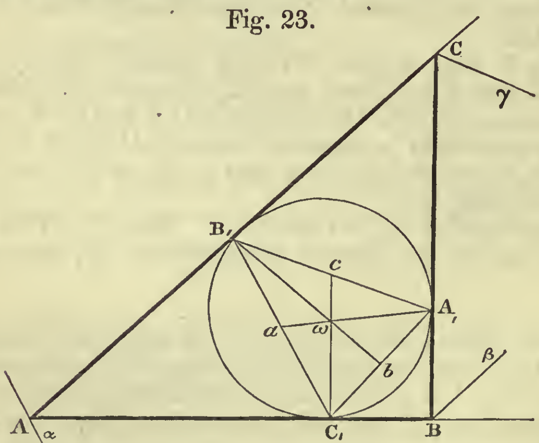
Fig. 22.



It is evident that if we substitute a sphere for the circle, and a plane for the polar straight line, we may infer that if any point be assumed in the plane, the polar plane of this point, taken with reference to a sphere, will pass through the pole of the polar plane.

174.] *If the external angles of a triangle be bisected, the bisectors will meet the opposite sides of the triangle in three points* α, β, γ *which range in a straight line.*

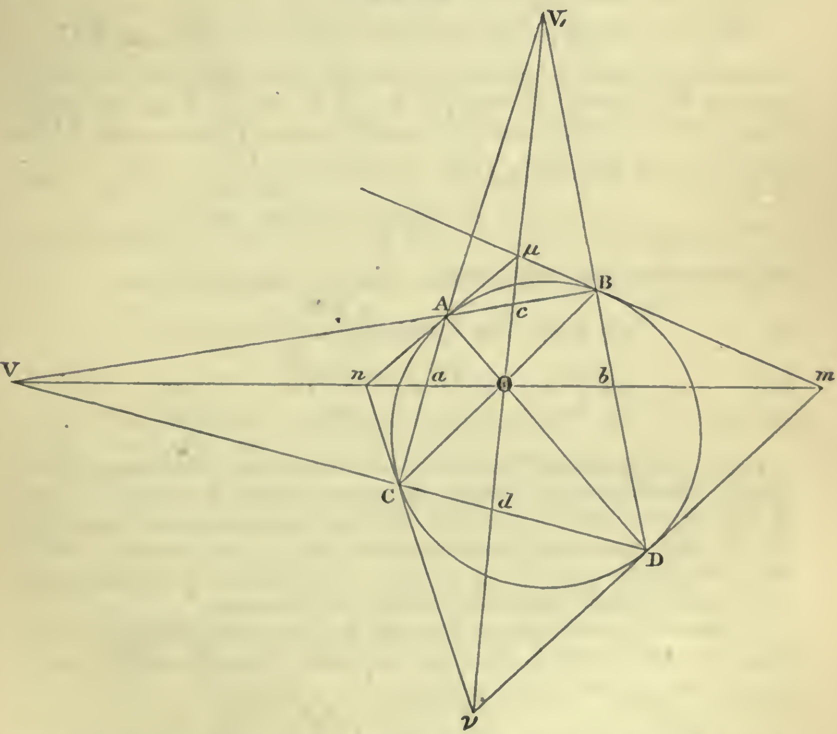
Fig. 23.



Let a circle be inscribed in the triangle, and let the points of contact $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ be joined. Let the inscribed circle be taken as a *polarizing* circle. Then, as the bisector of the external angle at C is the polar of the point c (the middle point of the line $A_{\prime}B_{\prime}$), and as the side AB is the polar of the point $C_{\prime}$, the point γ, in which the side AB meets the bisector of the vertical angle at C, is the pole of the line $C_{\prime}c$. In the same way it may be shown that a and b are the poles of the two other bisectors, while $A_{\prime}a$ and $B_{\prime}b$ are the polars of the points in which these bisectors meet the opposite sides. But the lines drawn from the angles of a triangle to the middle points of the opposite sides meet in a point, the centre of gravity or *centroid* of the triangle. Consequently the centroid of the triangle $A_{\prime}B_{\prime}C_{\prime}$ is the pole of the straight line $\alpha\beta\gamma$, and the perpendicular from the centre of the circle on this line will pass through the centroid.

175.] *If the opposite sides of a quadrilateral inscribed in a circle be produced to meet in* V, $V_{\prime}$ (fig. 24), *and the diagonals* AD, BC *be*

Fig. 24.



drawn to meet in O, *and tangents to the circle be drawn at the points* A, B, C, D, *these tangents will meet two by two on the lines* VO, $V_{\prime}$O *in the points* μ, ν, m, n, *so that the points* $V_{\prime}, \mu, O, \nu$ *and* V, m, O, n *will lie on the straight lines* VO *and* $V_{\prime}O$.

Since ABCD is a quadrilateral, the line AB is harmonically divided in c and V, and the line ab in O and V, and CD in d and V. See sec. [163]. And again, as ABCD is a quadrilateral inscribed in a circle, the polar of V will divide harmonically the chords AB and CD in c and d; therefore the line cd is the polar of V, and this line will therefore pass through the poles μ, ν of AB and CD. Hence the points $V_{\prime}, \mu, O, \nu$ are in the same straight line.

In the same way it may be shown that the points V, n, O, m are in the same straight line.

Without using poles and polars the proposition may be proved as follows by the method of transversals:—

If we can show that the straight lines $V_{\prime}\mu$, $A\mu$, $B\mu$ make angles with the sides of the triangle $V_{\prime}AB$, such that the product of the sines of the alternate angles may be equal, these lines must meet in one point μ, see sec. [153]—that is, if

$$\sin \mu A V_{\prime} . \sin \mu V_{\prime} B . \sin \mu BA = \sin \mu V_{\prime} A . \sin V_{\prime} B \mu . \sin BA\mu.$$

Now $\sin \mu BA = \sin BA\mu$, since $A\mu$ and $B\mu$ are tangents to the circle; also $\sin AV_{\prime}O : \sin BV_{\prime}O = V_{\prime}B . Ac : V_{\prime}A . cB$; and as the angle $V_{\prime}A\mu$ is equal to the angle ABO, and the angle $V_{\prime}B\mu$ equal to BAO, $\sin V_{\prime}A\mu : \sin V_{\prime}B\mu = \sin ABO : \sin BAO = \dfrac{P_{\prime}}{OB . Bc} : \dfrac{P}{OA . Ac}$; but since the angle $V_{\prime}AO = V_{\prime}BO$, $\dfrac{V_{\prime}A . AO}{P_{\prime}} = \dfrac{V_{\prime}B . BO}{P}$, P and $P_{\prime}$ being the perpendiculars drawn from A and B on the line $OV_{\prime}$;

or $$\sin V_{\prime}A\mu : \sin V_{\prime}B\mu = \frac{V_{\prime}B}{Bc} : \frac{V_{\prime}A}{Ac},$$

and $$\sin BV_{\prime}c : \sin AV_{\prime}c = V_{\prime}A . Bc : V_{\prime}B . Ac.$$

Hence $$\sin V_{\prime}A\mu . \sin BV_{\prime}c = \sin V_{\prime}B\mu . \sin AV_{\prime}c.$$

176.] *If a quadrilateral be inscribed in a circle, then* (α) *the square on the* outer *diagonal of the complete quadrilateral is equal to the sum of the squares on the tangents drawn from its ends to the circle,* (β) *the diagonal itself is equal to the sum of the tangents drawn from its middle point, and* (γ) *the circle drawn on this diagonal as diameter will cut the given circle at right angles.*

(α) Since P is the pole of EG (fig. 25), the *outer* diagonal of the complete quadrilateral, therefore $On . OE = Om . OG = R^2$.

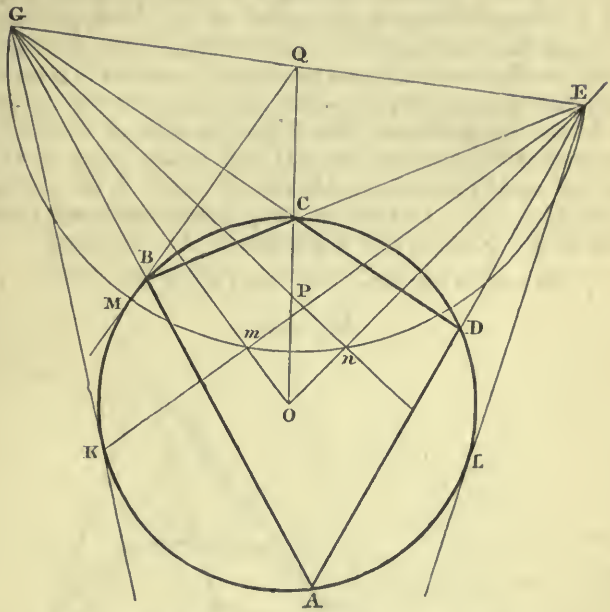
But $\overline{EG}^2 = \overline{EO}^2 + \overline{GO}^2 - 2GO . Om$.

Now $\overline{EO}^2 = \overline{EL}^2 + R^2$, $\overline{GO}^2 = \overline{GK}^2 + R^2$, and $2GO . Om = 2R^2$.

Hence we have $\overline{EG}^2 = \overline{EL}^2 + GK^2$.

(β) Let $\overline{QM}$ be a tangent drawn to the circle from the middle point Q of EG.

Fig. 25.



Then, $\quad 2(\overline{GO}^2+\overline{EO}^2)=4\overline{OQ}^2+4\overline{EQ}^2,\ \overline{GO}^2=\overline{GK}^2+R^2,$

$$EO^2=\overline{EL}^2+R^2,\ OQ^2=\overline{QM}^2+R^2,\ 4\overline{EQ}^2=\overline{EG}^2;$$

therefore $\qquad EQ=GQ=QM.$

(γ) Since QM is a tangent to one circle and a radius of the other, the circles must cut orthogonally.

It may also be shown that the squares of the *inner* diagonals are to each other as the distances of their middle points from the middle point Q of the *outer* diagonal.

177.] *The line joining the middle points of the diagonals of a quadrilateral circumscribing a circle passes through the centre*.*

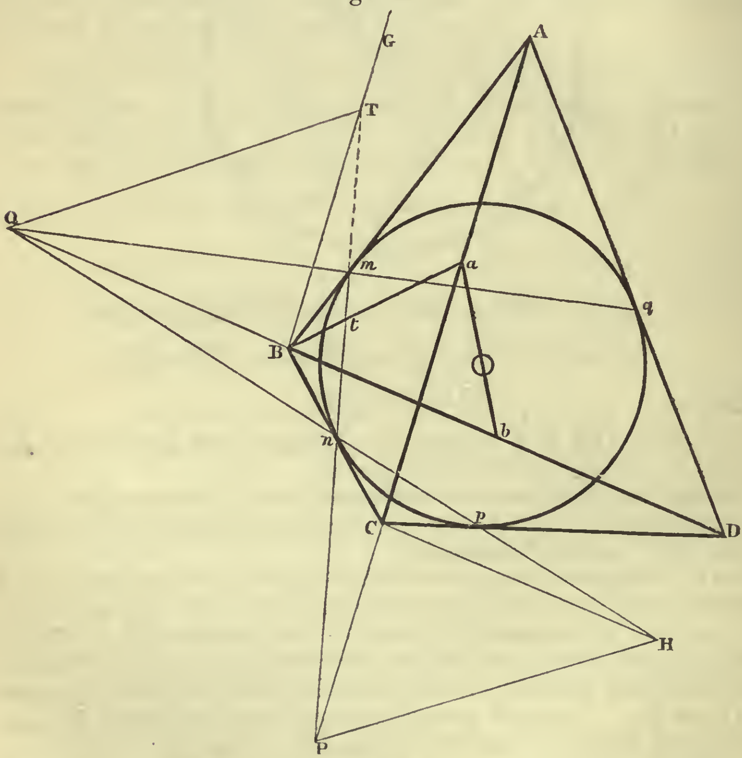
Let a and b be the middle points of the diagonals AC, BD of the quadrilateral ABCD (fig. 26) circumscribing the circle. Through B and C draw straight lines BG and CH parallel to the diagonals AC and BD. Through m and n, the points of contact of the quadrilateral, draw the chord mn meeting BG in T, and the line Ba in t.

* Of this theorem,—which is due to Newton,—a proof by the method of tangential coordinates will be found in the first volume of this work, p. 40.

Now since BG, BA, Ba, BC is an harmonic pencil, since AC is bisected in a and is parallel to BG, the transversal mn is harmonically divided in t and T; therefore the polar of T passes through t; and as moreover T is a point in the chord mn, the polar of T will pass through B; therefore Ba is the polar of T. Now producing the chords pn, qm they will meet in a point Q, the polar of which point, as it is on the line pn, will pass through C; and as it is on the line qm it will pass through A; therefore AC is the polar of the point Q; and as it has been shown that T is the pole of the line Ba, it will follow that the straight line QT will be the polar of the point in which AC and Bt intersect—that is, the point a, the middle point of the diagonal AC. In the same way it may be shown that PH is the polar of b. Now as BG is parallel to AP, we have

$$\mathrm{T}n : n\mathrm{P} = \mathrm{B}n : n\mathrm{C}, \text{ and } \mathrm{Q}n : n\mathrm{H} = \mathrm{B}n : n\mathrm{C};$$

Fig. 26.

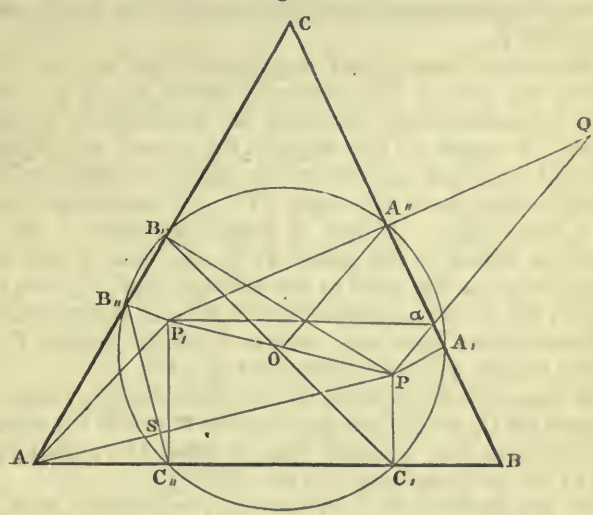


Therefore Tn : Qn = nP : nH. Hence the triangles QTn and HnP are similar; therefore QT is parallel to PH. But these lines are the polars of the points a and b; the point in which they meet will therefore be the pole of the line ab. But as they meet at infinity, the line ab must pass through the centre of the circle.

The same proof will hold when the curve is a conic.

178.] *If from any point* P (fig. 27) *perpendiculars* $PA_{,}$, $PB_{,}$, $PC_{,}$ *are drawn on the sides of a triangle* ABC, *a circle through the three points* $A_{,}$, $B_{,}$, $C_{,}$ *will cut the sides of the triangle in three other points* $A_{,,}$, $B_{,,}$, $C_{,,}$ *such that if perpendiculars to the sides of the triangle be drawn through these points, they will also meet in a point* P.

Fig. 27.



Since $AC_{,} \cdot AC_{,,} = AB_{,} \cdot AB_{,,}$, and $AC = AC_{,} + C_{,}C_{,,}$, while $AB_{,} = AB_{,,} + B_{,}B_{,,}$, we shall have

$$\overline{AC_{,,}}^2 + AC_{,,} \cdot C_{,}C_{,,} = \overline{AB_{,,}}^2 + AB_{,} \cdot B_{,}B_{,,},$$

and $$\overline{AC_{,}}^2 - AC_{,} \cdot C_{,}C_{,,} = \overline{AB_{,}}^2 - AB_{,} \cdot B_{,}B_{,,};$$

adding these two expressions, we shall have

$$\overline{AC_{,}}^2 + \overline{AC_{,,}}^2 - \overline{C_{,}C_{,,}}^2 = \overline{AB_{,}}^2 + \overline{AB_{,,}}^2 - \overline{B_{,}B_{,,}}^2;$$

so also $$\overline{BA_{,}}^2 + \overline{BA_{,,}}^2 - \overline{A_{,}A_{,,}}^2 = \overline{BC_{,}}^2 + \overline{BC_{,,}}^2 - \overline{C_{,}C_{,,}}^2,$$

and $$\overline{CB_{,}}^2 + \overline{CB_{,,}}^2 - \overline{B_{,}B_{,,}}^2 = \overline{CA_{,}}^2 + \overline{CA_{,,}}^2 - \overline{A_{,}A_{,,}}^2;$$

adding these equals, the squares on the intervals between the feet of the perpendiculars mutually cancel, and we shall have

$$\overline{AC_{\prime}}^2+\overline{BA_{\prime}}^2+CB_{\prime}^2+\overline{AC_{\prime\prime}}^2+\overline{BA_{\prime\prime}}^2+\overline{CB_{\prime\prime}}^2$$
$$=\overline{AB_{\prime}}^2+\overline{BC_{\prime}}^2+\overline{CA_{\prime}}^2+\overline{AB_{\prime\prime}}^2+\overline{BC_{\prime\prime}}^2+\overline{CA_{\prime\prime}}^2.$$

But since $PA_{\prime}$, $PB_{\prime}$, $PC_{\prime}$ are perpendiculars to the sides of the triangle, we shall have, see sec. [156],

$$\overline{AC_{\prime}}^2+\overline{BA_{\prime}}^2+\overline{CB_{\prime}}^2=\overline{AB_{\prime}}^2+\overline{BC_{\prime}}^2+\overline{CA_{\prime}}^2,$$

therefore $$\overline{AC_{\prime\prime}}^2+\overline{BA_{\prime\prime}}^2+\overline{CB_{\prime\prime}}^2=\overline{AB_{\prime\prime}}^2+\overline{BC_{\prime\prime}}^2+\overline{CA_{\prime\prime}}^2;$$

hence the perpendiculars through the points $A_{\prime\prime}$, $B_{\prime\prime}$, $C_{\prime\prime}$ meet in a common point $P_{\prime}$.

The line drawn from A to P is perpendicular to the line $B_{\prime\prime}C_{\prime\prime}$, which joins the feet of the perpendiculars $P_{\prime}B_{\prime\prime}$, $P_{\prime}C_{\prime\prime}$. For since $AC_{\prime}PB_{\prime}$ is a quadrilateral that may be inscribed in a circle, the angle $APC_{\prime}$ is equal to the angle $AB_{\prime}C_{\prime}$; and as $C_{\prime}B_{\prime}B_{\prime\prime}C_{\prime\prime}$ is a quadrilateral inscribed in a circle, the angle $AC_{\prime\prime}B_{\prime\prime}$ is equal to the angle $AB_{\prime}C_{\prime}$—that is, to the angle $APC_{\prime}$. Consequently the angle $ASC_{\prime\prime}$ is a right angle. Hence, if from the angles of the triangle ABC lines be drawn to the points P, $P_{\prime}$, the lines drawn to P will be perpendicular to the sides of the triangle $A_{\prime\prime}B_{\prime\prime}C_{\prime\prime}$, and the lines drawn to $P_{\prime}$ will be perpendicular to the sides of the triangle $A_{\prime}B\,C_{\prime}$.

The lines drawn from any vertex A to the points P, $P_{\prime}$ will make equal angles with the sides AB and AC*.

For the angle $P_{\prime}AB_{\prime\prime}$ is equal to the angle $P_{\prime}C_{\prime\prime}B_{\prime\prime}$ which is equal to the angle $PB_{\prime}C_{\prime}$, which has been proved equal to the angle PAC.

179.] The foregoing theorem may be proved in a simpler way by the help of the property given in sec. [134].

Let the perpendiculars $B_{\prime\prime}P_{\prime}$ and $C_{\prime\prime}P_{\prime}$ be erected at $B_{\prime\prime}$ and $C_{\prime\prime}$ to meet in $P_{\prime}$. Then, as $AB_{\prime\prime}P_{\prime}C_{\prime\prime}$ is a quadrilateral that may be inscribed in a circle, the angle $B_{\prime\prime}AP_{\prime}$ is equal to the angle $B_{\prime\prime}C_{\prime\prime}P_{\prime}$; and as $C_{\prime}B_{\prime}B_{\prime\prime}C_{\prime\prime}$ is a like quadrilateral, the angle $AC_{\prime\prime}B_{\prime\prime}$ is equal

* Hence the points P, $P_{\prime}$ are the foci of an ellipse inscribed in the triangle ABC, of which O is the centre, and the major axis the diameter of the circle.

Produce $P_{\prime}A_{\prime\prime}$ to Q until $A_{\prime\prime}Q$ is equal $A_{\prime\prime}P$. Join PQ cutting the side of the triangle in α. Join $P_{\prime}P\alpha$. Then as $P_{\prime}\alpha$ is equal to $Q\alpha$ and $P\alpha=P\alpha$, $P_{\prime}\alpha+P\alpha=PQ=2OA_{\prime\prime}$, or $P_{\prime}\alpha+P\alpha$ is constant, being equal to the diameter of the circle.

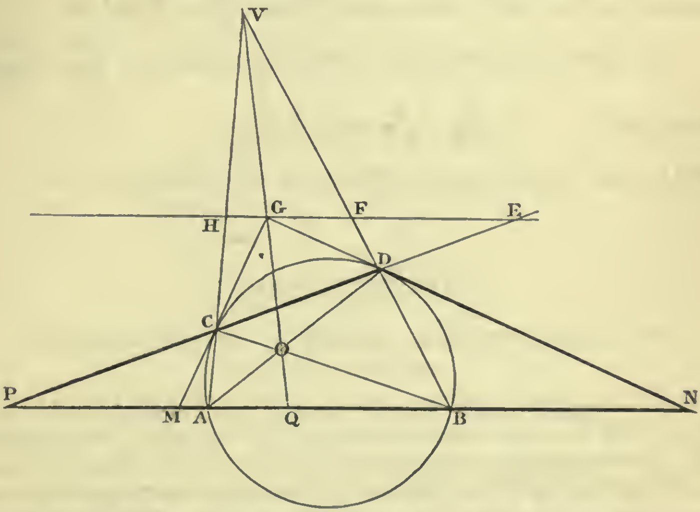
We may hence infer that if three tangents to an ellipse be given, and one of its foci, we can at once construct the ellipse. From the focus draw perpendiculars on the three tangents, the circle that passes through the feet of the perpendiculars will cut the tangents in three other points, through which if perpendiculars be drawn, they will meet in the second focus. The major axis of this ellipse will be the diameter 2R of the circle; and the eccentricity will be $\frac{PP_{\prime}}{2R}$.

to the angle $AB_{\prime}C_{\prime}$; but $AC_{\prime\prime}P_{\prime}$ and $AB_{\prime}P$ are right angles; hence the angle $B_{\prime\prime}C_{\prime\prime}P_{\prime}$ is equal to the angle $C_{\prime}B_{\prime}P$—that is, to the angle $C_{\prime}AP$, since $AB_{\prime}PC$ is also a quadrilateral that may be inscribed in a circle. Hence the angle $B_{\prime\prime}AP_{\prime}$ is equal to the angle $CAP_{\prime}$. But in sec. [154] it is shown that if two sets of lines be drawn from the angles of a triangle making equal angles with the adjacent sides, and if one set meet in a point, so likewise the other set will also meet in a point.

180.] *If through a given point* P (fig. 28) *two secants* PAB, PCD *be drawn to a circle, the first fixed, the second movable, and if from the points of intersection of this latter with the circle tangents be drawn meeting the fixed secant in the points* M, N, *we shall have* *

$$\frac{1}{PM}+\frac{1}{PN}=\frac{1}{PA}+\frac{1}{PB}.$$

Fig. 28.



Through C and D let tangents be drawn meeting in G, and cutting the fixed secant in the points M, N. Join AD, BC meeting in O, and AC, BD meeting in V. Then VO will pass through G, the intersection of the tangents at C and D, and will cut the line AB

* This theorem is taken from Maclaurin's *Tractatus de linearum curvarum proprietatibus generalibus*, p. 11, a treatise of rare originality and beauty. The theorem in the text, which is proved for algebraical curves of all orders by a simple application of an elementary principle of the differential calculus, Maclaurin makes the foundation of a system of geometry of curve lines of singular elegance.

in a point Q. Let the line GE be drawn through G parallel to AB meeting the lines VA, VB in H and F. Then by similar triangles

$$PM : PC = EG : CE \text{ and } PC : PA = CE : EH.$$

Compounding these ratios, PM : PA = EG : EH.

Hence $\frac{1}{PM} = \frac{1}{PA} \cdot \frac{EH}{EG}$. In like manner we obtain $\frac{1}{PN} = \frac{1}{PB} \cdot \frac{EF}{EG}$.

But EH = EG + GH and EF = EG − FG ;

therefore $$\frac{1}{PM} = \frac{1}{PA}\left[\frac{EG+GH}{EG}\right] = \frac{1}{PA}\left[1+\frac{GH}{EG}\right],$$

and $$\frac{1}{PN} = \frac{1}{PB}\left[\frac{EG-FG}{EG}\right] = \frac{1}{PB}\left[1-\frac{FG}{EG}\right];$$

therefore $$\frac{1}{PM} + \frac{1}{PN} = \frac{1}{PA} + \frac{1}{PB} + \frac{1}{EG}\left[\frac{GH}{PA} - \frac{FG}{PB}\right].$$

But as the line AB is harmonically divided in P and Q,

$$PA : PB = AQ : BQ = GH : FG \text{; and therefore } \frac{GH}{PA} = \frac{FG}{PB};$$

consequently $$\frac{1}{PM} + \frac{1}{PN} = \frac{1}{PA} + \frac{1}{PB}.$$

This proof, without any modification, will hold for conics.

CHAPTER XXIII.

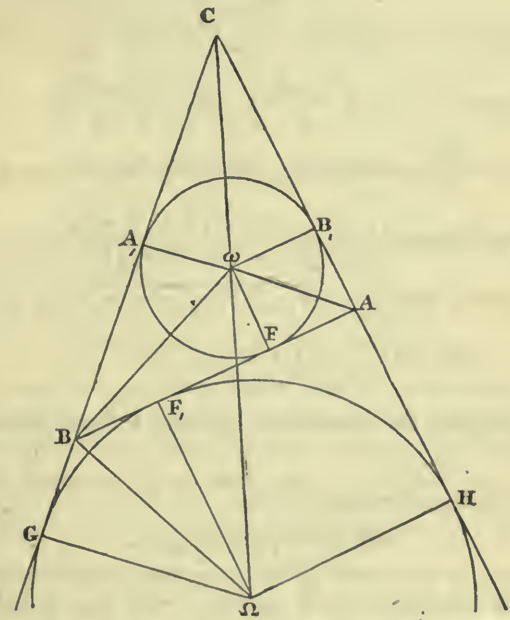
ON CIRCLES INSCRIBED, EXSCRIBED, AND CIRCUMSCRIBED TO A TRIANGLE.

When a triangle is given, sixteen circles may be described in connexion with it :—one circumscribed to the triangle ; one inscribed in it ; three touching, each a side and the other two sides produced ; six passing through the centres of the circles of contact and the vertices of the given triangle taken two by two ; four through the centres of the inscribed and exscribed circles taken three by three ; and, lastly, a sixteenth circle passing through the feet of the perpendiculars drawn from the vertices of the triangle on the opposite sides. This may be called the *orthocentric circle,* as it circumscribes the orthocentric triangle. It is also known as the *nine-point circle.* The other circles will be named as definitions are required.

The four circles which touch the sides of this triangle may with propriety and brevity be named *the circles of contact* ; and their centres may be called *the centres of contact.*

181.] Let $r, r_{\prime}, r_{\prime\prime}, r_{\prime\prime\prime}$, R be the radii of the inscribed, exscribed*, and circumscribed circles of the triangle ABC, and let $\omega, \Omega, \Omega_{\prime}, \Omega_{\prime\prime}$, O be the centres of these circles, while Θ is the centre of the circle, whose radius is ρ, inscribed in the orthocentric triangle. Let the inscribed circle touch the sides of the triangle in the points $B_{\prime}, A_{\prime}$, F, and the exscribed circle touch the same sides in the points G, H, $F_{\prime}$, and as $BG=BF_{\prime}$, and $AH=AF_{\prime}$, $BG+AH=BA=c$, if a, b, c be the sides of the triangle opposite to the angles A, B, C. Hence $CG+CH$ is equal to the perimeter of the triangle, or as $CG=CH$, CG or CH is half the perimeter of the triangle; let this semiperimeter be denoted by s. And as $CA_{\prime}=CB_{\prime}$, $GA_{\prime}=HB_{\prime}$; and as $GA_{\prime}=BF_{\prime}+BF$ and $HB_{\prime}=AF+AF_{\prime}$, therefore $BF_{\prime}+BF=AF+AF$, or $BF=AF$; hence $BA=GA_{\prime}=HB_{\prime}=c$. Therefore $BG=s-a$, $BA=s-b$, and $CA_{\prime}=s-c$.

Fig. 29.



Let Δ be the area of the triangle, then it is well known that

$$\Delta=sr, \;=\sqrt{s(s-a)(s-b)(s-c)}, \;=\frac{abc}{4R}. \quad . \quad . \quad (a)$$

* Not *escribed*, as it is usually written, but *exscribed*, in accordance with the analogy of the pronunciation of other like words, such as *exscind*, *exsection*, *exsert*, *exsiccate*, &c.

We have also $r_{\prime}=\dfrac{sr}{s-a}$, $r_{\prime\prime}=\dfrac{sr}{s-b}$, $r_{\prime\prime\prime}=\dfrac{sr}{s-c}$. . . . . (b)

Therefore $rr_{\prime}r_{\prime\prime}r_{\prime\prime\prime}=\dfrac{s^4r^4}{s(s-a)(s-b)(s-c)}=\Delta^2$. . . . (c)

Taking the reciprocals of (b), we shall have

$$\frac{1}{r}=\frac{1}{r_{\prime}}+\frac{1}{r_{\prime\prime}}+\frac{1}{r_{\prime\prime\prime}}, \quad . \ (d) \qquad \text{whence } r=\frac{r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}}{r_{\prime}r_{\prime\prime}+r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime}r_{\prime\prime\prime}}. \ (e)$$

But as $rr_{\prime}r_{\prime\prime}r_{\prime\prime\prime}=\Delta^2$ and $sr=\Delta$, $s^2=r_{\prime}r_{\prime\prime}+r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime}r_{\prime\prime\prime}$. . . (f)

Since $s-a=\dfrac{sr}{r_{\prime}}$ and $s-b=\dfrac{sr}{r_{\prime\prime}}$,

therefore $2s-a-b=c=sr\left(\dfrac{1}{r_{\prime}}+\dfrac{1}{r_{\prime\prime}}\right)=\dfrac{srr_{\prime\prime\prime}(r_{\prime}+r_{\prime\prime})}{r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}}$.

But $r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}=s^2r$; therefore $c=r_{\prime\prime\prime}\left(\dfrac{r_{\prime}+r_{\prime\prime}}{s}\right)$.

In like manner $a=r_{\prime}\left(\dfrac{r_{\prime\prime}+r_{\prime\prime\prime}}{s}\right)$, $b=r_{\prime\prime}\left(\dfrac{r_{\prime}+r_{\prime\prime\prime}}{s}\right)$; . . . (g)

and since $4\mathrm{R}=\dfrac{abc}{sr}$, substituting the foregoing values of a, b, c,

we shall have $4\mathrm{R}=\dfrac{r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}(r_{\prime\prime}+r_{\prime\prime\prime})(r_{\prime\prime\prime}+r_{\prime})(r_{\prime}+r_{\prime\prime})}{sr[r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime}r_{\prime\prime}+r_{\prime}r_{\prime\prime\prime}]^{\frac{3}{2}}}$. . . . (h)

But $r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}=s^2r$, and $s=\sqrt{r_{\prime}r_{\prime\prime}+r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime\prime\prime}r_{\prime}}$, as in (f);

hence $4\mathrm{R}=\dfrac{(r_{\prime}+r_{\prime\prime})(r_{\prime\prime}+r_{\prime\prime\prime})(r_{\prime\prime\prime}+r_{\prime})}{r_{\prime}r_{\prime\prime}+r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime\prime\prime}r_{\prime}}$. . . . . (i)

Now if we develop the numerator and add to both sides

$r=\dfrac{r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}}{r_{\prime}r_{\prime\prime}+r_{\prime\prime}r_{\prime\prime\prime}+r_{\prime\prime\prime}r_{\prime}}$, as given in (e), we shall have

$$4\mathrm{R}+r=r_{\prime}+r_{\prime\prime}+r_{\prime\prime\prime}, \quad . \ . \ . \ . \ . \ . \ . \ . \ (j)$$

Thus *the sum of the radii of the exscribed circles is equal to the radius of the inscribed circle together with four times the radius of the circumscribed circle.*

Since $s-a=\dfrac{sr}{r_{\prime}}$, and $s-b=\dfrac{sr}{r_{\prime\prime}}$, we have

$$(s-a)(s-b)=\frac{s^2r^2}{r_{\prime}r_{\prime\prime}} \text{ or } s^2-(a+b)s+ab=\frac{s^2r^2r_{\prime\prime\prime}}{r_{\prime}r_{\prime\prime}r_{\prime\prime\prime}}.$$

Finding like expressions for the other sides, and adding,

we obtain $3s^2-4s^2+bc+ac+ab=\dfrac{s^2r^2(r_{,}+r_{,,}+r_{,,,})}{r_{,}r_{,,}r_{,,,}}$.

But $r_{,}r_{,,}r_{,,,}=s^2r$; consequently $bc+ac+ab=s^2+r(r_{,}+r_{,,}+r_{,,,})$.

But $$r_{,}+r_{,,}+r_{,,,}=4\mathrm{R}+r;$$

hence $$bc+ac+ab=s^2+4\mathrm{R}r+r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (k)$$

and therefore $$a^2+b^2+c^2=2s^2-8\mathrm{R}r-2r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (l)$$

These useful theorems may be more simply established by successively eliminating $(bc+ca+ab)$ and $(a^2+b^2+c^2)$ between the formulæ $2s=a+b+c$ and $sr^2=(s-a)(s-b)(s-c)$.

182.] Since $\dfrac{1}{r_{,}}+\dfrac{1}{r_{,,}}+\dfrac{1}{r_{,,,}}=\dfrac{1}{r}$, see (d) sec. [181],

squaring $$\frac{1}{r_{,}^2}+\frac{1}{r_{,,}^2}+\frac{1}{r_{,,,}^2}=\frac{1}{r^2}-2\left[\frac{1}{r_{,}r_{,,}}+\frac{1}{r_{,,}r_{,,,}}+\frac{1}{r_{,,,}r_{,}}\right],$$

or $$\frac{1}{r_{,}^2}+\frac{1}{r_{,,}^2}+\frac{1}{r_{,,,}^2}+\frac{1}{r^2}=\frac{2}{r^2}-\frac{2(r_{,}+r_{,,}+r_{,,,})r}{r_{,}r_{,,}r_{,,,}r}.$$

Now $$r_{,}+r_{,,}+r_{,,,}=4\mathrm{R}+r \text{ and } r_{,}r_{,,}r_{,,,}r=s^2r^2;$$

therefore $$\frac{1}{r_{,}^2}+\frac{1}{r_{,,}^2}+\frac{1}{r_{,,,}^2}+\frac{1}{r^2}=\frac{2s^2-8\mathrm{R}r-2r^2}{s^2r^2}. \quad . \quad . \quad . \quad (a)$$

But it has been shown, in (l) in the last section, that

$$a^2+b^2+c^2=2s^2-8\mathrm{R}r-2r^2;$$

and therefore $$\frac{1}{r_{,}^2}+\frac{1}{r_{,,}^2}+\frac{1}{r_{,,,}^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{\Delta^2}. \quad . \quad . \quad . \quad . \quad (b)$$

Thus *the sum of the squares of the reciprocals of the radii of the four inscribed and exscribed circles to a triangle is equal to the sum of the squares of the three sides divided by the square of the area of the triangle.*

183.] Let $h_{,}$, $h_{,,}$, $h_{,,,}$ denote the perpendiculars from the vertices of a triangle on the opposite sides, then $ah_{,}$, $bh_{,,}$, $ch_{,,,}$ are each equal to $2sr$; and as $\dfrac{1}{r_{,}}=\dfrac{s-a}{sr}$, $\dfrac{1}{r_{,,}}=\dfrac{s-b}{sr}$,

we have $$\frac{1}{r_{,}}+\frac{1}{r_{,,}}=\frac{c}{sr}=\frac{2c}{h_{,,,}c}, \text{ or } h_{,,,}\left(\frac{1}{r_{,}}+\frac{1}{r_{,,}}\right)=2.$$

In like manner $h_{,}\left(\dfrac{1}{r_{,,}}+\dfrac{1}{r_{,,,}}\right)=2$, and $h\left(\dfrac{1}{r_{,}}+\dfrac{1}{r_{,,,}}\right)=2$;

consequently $$\frac{(h_{,}+h_{,,})}{r_{,,,}}+\frac{(h_{,,}+h_{,,,})}{r_{,}}+\frac{(h_{,,,}+h_{,})}{r_{,,}}=6. \quad . \quad . \quad . \quad (a)$$

184.] *The sum of the squares of the sides of a triangle is equal to twice the sum of the products of each height multiplied by the distance between the corresponding angle and the* orthocentre.

Let h be the altitude corresponding to the angle A; then the distance from the vertex A to the orthocentre is $2\text{R}\cos\text{A}$, and the product by h is $2\text{R}h\cos\text{A}=4\Delta\text{R}\frac{\cos\text{A}}{a}$, putting Δ for the area of the triangle; and this may be written $2\Delta\text{R}\frac{(b^2+c^2-a^2)}{abc}$. Finding like expressions for the other angles, and bearing in mind that $abc=4\text{R}\Delta$, we get

$$a^2+b^2+c^2=2\text{R}[h\cos\text{A}+h_{\prime}\cos\text{B}+h_{\prime\prime}\cos\text{C}].$$

In any plane triangle we shall have the relation

$$\frac{a\cos\text{A}+b\cos\text{B}+c\cos\text{C}}{2s}=\frac{r}{\text{R}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

For if $p, p_{\prime}, p_{\prime\prime}$ denote the perpendiculars from the centre of the circumscribed circle on the sides of the triangle a, b, c, we have

$$\cos\text{A}=\frac{p}{\text{R}},\ \cos\text{B}=\frac{p_{\prime}}{\text{R}},\ \cos\text{C}=\frac{p_{\prime\prime}}{\text{R}}. \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence
$$\frac{ap+bp_{\prime}+cp_{\prime\prime}}{2s\text{R}}=\frac{2\Delta}{2s\text{R}}=\frac{\Delta r}{\text{R}sr}=\frac{r}{\text{R}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

The sum of the ratios of each perpendicular from the centre of the circumscribing circle on a side of the triangle to the perpendicular from the opposite angle on the same side is equal to unity.

For
$$\frac{p}{h}=\frac{\text{area COB}}{\text{area CAB}};\ \text{therefore}\ \frac{p}{h}+\frac{p_{\prime}}{h_{\prime}}+\frac{p_{\prime\prime}}{h_{\prime}}=1. \quad . \quad . \quad \text{(d)}$$

The sum of the reciprocals of the perpendiculars from the angles of a triangle on the opposite sides is equal to the reciprocal of the radius of the inscribed circle.

Let ω be the centre of this circle, then

$$\frac{r}{h}=\frac{\text{area B}\omega\text{C}}{\text{area BAC}};$$

finding like expressions for the other terms, and adding, we shall have

$$\frac{1}{h}+\frac{1}{h_{\prime}}+\frac{1}{h_{\prime\prime}}=\frac{1}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

If we turn to fig. 29 (p. 289) it will easily be seen that

$$\frac{\Omega\omega}{\text{C}\omega}+\frac{\Omega_{\prime}\omega}{\text{B}\omega}+\frac{\Omega_{\prime\prime}\omega}{\text{A}\omega}=2.$$

For
$$\frac{\Omega\omega}{C\omega}=\frac{c}{s},\ \frac{\Omega_{,}\omega}{B\omega}=\frac{b}{s},\ \frac{\Omega_{,,}\omega}{A\omega}=\frac{a}{s};$$

and the sum of these ratios is obviously 2.

ON THE TRIGONOMETRICAL RELATIONS OF THE ANGLES OF A TRIANGLE.

185.] In the following propositions the terms sin, cos, tan are used as brief and familiar expressions to denote certain ratios of lines connected with a triangle and its inscribed and circumscribed circles.

Since
$$\cot\frac{A}{2}=\frac{s-a}{r},\ \text{and}\ \cot\frac{A}{2}=\frac{s}{r_{,}},\ .\ .\ .\ .\ .\quad \text{(a)}$$

finding like expressions for the other angles, and adding, we have
$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}=\frac{s}{r},$$

and
$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s}{r_{,}}+\frac{s}{r_{,,}}+\frac{s}{r_{,,,}};$$

hence dividing these equations by s, we obtain
$$\frac{1}{r}=\frac{1}{r_{,}}+\frac{1}{r_{,,}}+\frac{1}{r_{,,,}}.\ .\ .\ .\ .\ .\ .\ .\quad \text{(b)}$$

Multiplying together the cotangents in (a),

we have
$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{(s-a)(s-b)(s-c)}{r^3},$$

and
$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s^3}{r_{,}r_{,,}r_{,,,}}.$$

Hence
$$\frac{s^3r}{r\,r_{,}r_{,,}r_{,,,}}=\frac{s(s-a)(s-b)(s-c)}{sr^3}=\frac{s^2r^2}{sr^3}=\frac{s}{r},$$

or
$$sr=\sqrt{rr_{,}r_{,,}r_{,,,}}.\ .\ .\ .\ .\ .\ .\ .\quad \text{(c)}$$

Hence *the square root of the continued product of the four radii of the inscribed and exscribed circles is equal to the area of the triangle.*

To prove the following relations:—

(α)
$$\sin A+\sin B+\sin C=\frac{s}{R}.$$

Since
$$\sin A=\frac{a}{2R},\ \sin B=\frac{b}{2R},\ \sin C=\frac{c}{2R},$$

adding these expressions,

$$\sin A + \sin B + \sin C = \frac{s}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Multiplying these values,

$$(\beta) \qquad \sin A \sin B \sin C = \frac{rs}{2R^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

If we square (α) and subtract the values of the squares of the sines, we shall have

$$(\gamma) \quad \sin A \sin B + \sin B \sin C + \sin A \sin C = \frac{4s^2 - (a^2+b^2+c^2)}{8R^2}.$$

But $a^2+b^2+c^2 = 2s^2 - 2r^2 - 8Rr$.

Hence $$\sin A \sin B + \sin B \sin C + \sin A \sin C = \frac{s^2+r^2+4Rr}{4R^2}. \quad . \quad . \quad (f)$$

$$(\delta) \qquad \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Hence $$4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = \frac{4s(s-a)(s-b)(s-c)}{sabc} = \frac{r}{R}. \quad . \quad . \quad (g)$$

$$(\epsilon) \qquad 1 + \cos A = \frac{2s(s-a)}{bc};$$

finding like expressions for $\cos B$ and $\cos C$, adding, we shall have

$$\cos A + \cos B + \cos C = \frac{2s^2(a+b+c) - 2s(a^2+b^2+c^2) - 3abc}{abc};$$

but $\quad a^2+b^2+c^2 = 2s^2 - 2r^2 - 8Rr$, as in (1) sec. [181];

hence $$\cos A + \cos B + \cos C = 1 + \frac{r}{R}. \quad . \quad . \quad . \quad . \quad . \quad (h)$$

If $p, p_{\prime}, p_{\prime\prime}$ denote the perpendiculars from the centre of the circumscribed circle on the sides,

$$p = R\cos A, \quad p_{\prime} = R\cos B, \quad p_{\prime\prime} = R\cos C.$$

Hence $$p + p_{\prime} + p_{\prime\prime} = R + r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

186.] To prove that $\quad a\cot A + b\cot B + c\cot C = 2(R+r)$. $\quad$ (a)

$$a\cot A = \frac{a\cos A}{\sin A} = \frac{2aR\cos A}{2R\sin A} = \frac{2ap}{a} = 2p.$$

Hence $\quad a\cot A + b\cot B + c\cot C = 2(p + p_{\prime} + p_{\prime\prime})$.

But $\quad p + p_{\prime} + p_{\prime\prime} = R + r$, as shown in (i), last section.

If we square the expression (h) in sec. [185], and put for $\cos^2 A$ its value $1-\frac{a^2}{4R^2}$, and like expressions for $\cos^2 B$, $\cos^2 C$, we shall have

$$2(\cos A\cos B+\cos B\cos C+\cos A\cos C)=\left(1+\frac{r}{R}\right)^2-3+\frac{a^2+b^2+c^2}{4R^2};$$

putting for $a^2+b^2+c^2$ its value, and reducing,

$$\cos B\cos C+\cos A\cos C+\cos A\cos B=\frac{r^2+s^2-4R^2}{4R^2}. \quad . \quad (b)$$

187.] To show that

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C. \quad . \quad (a)$$

Since $A+B+C=\pi$, $\cos C=-\cos(A+B)$;

therefore
$$\cos^2 C=\cos^2 A\cos^2 B-2\cos A\cos B\sin A\sin B$$
$$+1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B,$$

putting for $\sin^2 A\sin^2 B$ in the developed form

its equivalent $1-\cos^2 A+\cos^2 B+\cos^2 A\cos^2 B$.

Hence the expression $\cos^2 A+\cos^2 B+\cos^2 C$ now becomes

$$1+2\cos A\cos B(\cos A\cos B-\sin A\sin B)=1-2\cos A\cos B\cos C.$$

We have also, as shown in (ϵ) section [185],

$$(1+\cos A)(1+\cos B)(1+\cos C)=\frac{s^2}{2R^2}. \quad . \quad . \quad (b)$$

If we multiply together the expressions

$(1+\cos A)$, $(1+\cos B)$, $(1+\cos C)$, we shall have

$$(1+\cos A)(1+\cos B)(1+\cos C)=1+\cos A+\cos B+\cos C$$
$$+\cos A\cos B+\cos B\cos C+\cos A\cos C+\cos A\cos B\cos C=\frac{s^2}{2R^2}.$$

Substituting for $(\cos A+\cos B+\cos C)$ and

$\cos A\cos B+\cos B\cos C+\cos A\cos C$ their values

as given in (h) and (i) in section [185], we shall find

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}. \quad . \quad . \quad . \quad . \quad (j)$$

Since $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

$$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{4s\,.\,sr}{4Rrs}=\frac{s}{R}; \quad . \quad . \quad . \quad . \quad (k)$$

comparing this expression with (d), sec. [185], we find

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

188.] To show that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Let $a_{\prime}$ and $a_{\prime\prime}$ be the segments of the side a, made by the perpendicular h drawn to it from the vertex A, then

$$\cot B\cot C=\frac{a_{\prime}a_{\prime\prime}}{h^2}=\frac{h\varpi}{h^2},$$

ϖ being the perpendicular from the orthocentre on the side a.

Hence $\cot B\cot C=\frac{\varpi}{h}=\frac{\varpi a}{ha}$. Let δ, $\delta_{\prime}$, $\delta_{\prime\prime}$ be the component triangles of the original triangle, then

$$\varpi a=2\delta \text{ and } ha=2\Delta;$$

hence $$\cot B\cot C+\cot C\cot A+\cot A\cot B=\frac{\delta+\delta_{\prime}+\delta_{\prime\prime}}{\Delta}=1.$$

Multiplying these expressions by tan A tan B tan C,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C. \quad . \quad . \quad (a)$$

Since $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{(s-b)(s-c)(s-a)}{s\sqrt{s(s-a)(s-b)(s-c)}}=\frac{r}{s}. \quad . \quad (b)$$

Again, as

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{(s-b)(s-c)}{sr},$$

therefore $$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Hence also

$$\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1; \quad . \quad . \quad (d)$$

for $$\tan\frac{B}{2}=\frac{r}{s-b}, \ \tan\frac{C}{2}=\frac{r}{s-c}; \text{ therefore}$$

$$\tan\frac{B}{2}\tan\frac{C}{2}=\frac{s-a}{s}.$$ Hence results the theorem.

189.] Since $ha=2\Delta$, $h=\dfrac{2\Delta}{a}$, and therefore

$$h+h_{,}+h_{,,}=2\Delta\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\Delta\frac{(bc+ac+ab)}{abc}=\frac{s^2+r^2+4Rr}{2R}. \quad \text{(a)}$$

To show that $$\varpi+\varpi_{,}+\varpi_{,,}=\frac{s^2+r^2-4R^2}{2R}. \quad . \quad . \quad . \quad \text{(b)}$$

Let $a_{,}$, $a_{,,}$ be the segments of the side a made by the perpendicular h from the vertex, ϖ the corresponding perpendicular from the orthocentre; then

$$\cos B\cos C=\frac{a_{,}a_{,,}}{bc}=\frac{h\varpi}{bc}=\frac{2\Delta\varpi}{abc}.$$

(b) sec. [186] gives $\cos B\cos C+\cos A\cos C+\cos A\cos B=\dfrac{r^2+s-4R^2}{4R^2}$.

Hence $\varpi+\varpi_{,}+\varpi_{,,}=\dfrac{s^2+r^2-4R^2}{2R}$, as above;

and therefore $$(h+h_{,}+h_{,})-(\varpi+\varpi_{,}+\varpi_{,,})=2(R+r). \quad . \quad . \quad \text{(c)}$$

But this quantity denotes the sum of the lines drawn from the orthocentre to the vertices of the triangle; and as it may be shown that the sum of these distances is equal to twice the sum of the perpendiculars on the sides of the triangle, these perpendiculars being written

p, $p_{,}$, $p_{,,}$, we shall have, as in (i) sec. [185],

$$p+p_{,}+p_{,,}=R+r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

190.] *In any triangle the sum of the reciprocals of the sides of the six inscribed and exscribed squares is equal to twice the reciprocal of the radius of the inscribed circle.*

Let a be the base and h the height of the triangle, and x the side of the square inscribed, then $x=\dfrac{ah}{a+h}$.

Let $x_{,}$ be the side of the square *exscribed*, then $x_{,}=\dfrac{ah}{a-h}$.

Hence $\dfrac{1}{x}+\dfrac{1}{x_{,}}=\dfrac{2}{h}$. Let y, $y_{,}$ and z, $z_{,}$ be the sides of the squares on the other two sides of the triangle, and we shall have

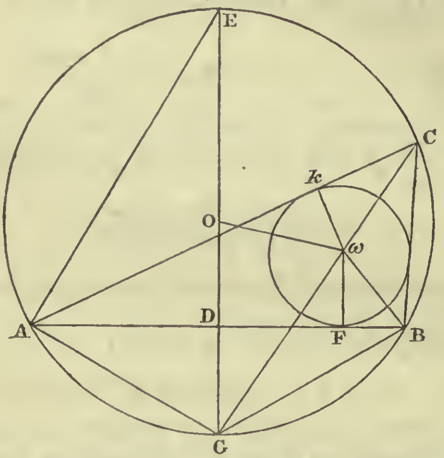
$$\frac{1}{x}+\frac{1}{x_{,}}+\frac{1}{y}+\frac{1}{y_{,}}+\frac{1}{z}+\frac{1}{z}=2\left(\frac{1}{h}+\frac{1}{h_{,}}+\frac{1}{h_{,,}}\right)=\frac{2}{r},$$

as shown in (e) sec. [184].

ON TRIANGLES INSCRIBED IN ONE CIRCLE AND CIRCUMSCRIBED TO ANOTHER.

191. Let the triangle ABC be inscribed in the circle AEBG and circumscribed to the circle $F\omega k$; we proceed to find an expression for the distance between O and ω the centres of these circles. Let D be this distance, and let R and r be the radii of the circles; then manifestly $(R+D)(R-D)=C\omega \,.\, G\omega$ or $D^2=R^2-C\omega \,.\, G\omega$.

Fig. 30.



Through G draw the diameter GOE; join AG and AE. Since the triangles $Ck\omega$ and AGE are similar, $GE \,.\, \omega k = C\omega \,.\, GA$; but $GE = 2R$, and $\omega k = r$, while $GA = GB = G\omega$. Consequently $2Rr = C\omega \,.\, G\omega$, and therefore

$$D^2=R^2-2Rr. \qquad \text{(a)}$$

The value of D is independent of the sides of the triangle. Hence, if two circles be described so that the interval between their centres shall be equal to $\sqrt{R^2-2Rr}$, any triangle inscribed in the one may be circumscribed to the other*.

* Another proof of this theorem may be given. Let two tangents to the inscribed circle be drawn from the points A and B meeting in C. If C be on the circumference the proposition is established. But if not let another circle be described passing through the points A, B, C, Let $R_{\prime}$ be the radius of this circle, its centre will be on the line GE, suppose at $O_{\prime}$, and let $D_{\prime}$ be the distance from $O_{\prime}$ to ω. Let $OO_{\prime}=\mu$, and let $OD=k$; then

$$R_{\prime}^2=R^2+\mu^2+2\mu k, \quad D_{\prime}^2=D^2+\mu^2-2\mu(r-k).$$

But $D_{\prime}^2=R_{\prime}^2-2R_{\prime}r$; or, substituting the value of $R_{\prime}$,

$$D^2+\mu^2-2\mu(r-k)=R^2+\mu^2+2\mu k-2r\sqrt{R^2+\mu^2+2\mu k};$$

and $$D^2=R^2-2Rr; \text{ consequently } R+\mu=\sqrt{R^2+\mu^2+2\mu k},$$

which is impossible unless $\mu=0$; or the two centres of the circumscribing circles must coincide; and as they pass through the same points A and B, they must be identical.

192.] Let $r_{\prime}$ be the radius of one of the outer circles of contact; then, making the necessary transformations, it may be shown that

$$D_{\prime}^2 = R^2 + 2Rr_{\prime}. \quad \ldots \ldots \ldots \quad (a)$$

If we take like expressions for the other two sides we shall have, adding them together,

$$D^2 + D_{\prime}^2 + D_{\prime\prime}^2 + D_{\prime\prime\prime}^2 = 4R^2 + 2R(r_{\prime} + r_{\prime\prime} + r_{\prime\prime\prime} - r).$$

But $r_{\prime} + r_{\prime\prime} + r_{\prime\prime\prime} - r = 4R$, as shown in (j) sec. [181]; hence

$$D^2 + D_{\prime}^2 + D_{\prime\prime}^2 + D_{\prime\prime\prime}^2 = 12R^2; \quad \ldots \ldots \quad (b)$$

or *the sum of the squares of the distances from the centre of the circumscribed circle to the centres of the four circles of contact is equal to twelve times the square of the radius of the circumscribed circle.* It may easily be shown that G, the middle point of the arc AB, is the centre of the circle which passes through A, B, the ends of the base AB, and through the centres ω and Ω of the inscribed and exscribed circles.

193.] *If a triangle circumscribe one circle and be inscribed in another circle, the circles will have a common pole and polar.*

Let d be the distance from O the centre of the circumscribing circle to the common polar, let δ be the distance between the centre of the inscribed circle and the common pole, and, as before, let D be the distance between the centres of the circles whose radii are R and r.

Then obviously we shall have

$$(D + \delta)d = R^2, \text{ and } (d - D)\delta = r^2.$$

Eliminating δ, we shall find for d, the distance of O from the common polar,

$$d = \frac{(R + r)(R - r) + R(R - 2r) \pm r\sqrt{4Rr + r^2}}{2D}.$$

CHAPTER XXIV.

ON THE ORTHOCENTRIC TRIANGLE.

194.] The *orthocentric* triangle has been defined in sec. [154] as the triangle formed by joining the feet of the perpendiculars drawn from the vertices of a triangle to the opposite sides; and these perpendiculars, as it has been shown, meet in the *orthocentre*.

The circle which circumscribes this triangle may be called the *orthocentric* circle. It has also by Poncelet been named the *nine-point circle,* from a property which will be established further on.

Let A, B, C *be the angles of the given triangle,* a, b, c *the opposite sides, and* R *the radius of the circumscribing circle.*

The sides of the orthocentric triangle are $a\cos A$, $b\cos B$, $c\cos C$.

Let $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ be the vertices of the orthocentric triangle opposite the vertices A, B, C of the given triangle, then the sides of the triangle $A_{\prime}BC_{\prime}$ are $a\cos B$, $c\cos B$, and $\overline{A_{\prime}C_{\prime}}$. Hence

$$\overline{A_{\prime}C_{\prime}}^2=a^2\cos^2 B+c^2\cos^2 B-2ac\cos^3 B,$$

or $$\overline{A_{\prime}C_{\prime}}^2=\cos^2 B\left[a^2+c^2-2ac\cos B\right].$$

But the part put within brackets is equal to $\overline{AC}^2$ or b^2;

hence $$\overline{A_{\prime}C_{\prime}}^2=\overline{AC}^2\cos^2 B, \text{ or } b_{\prime}=b\cos B. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We have also $2R=\dfrac{a}{\sin A}$, a well-known theorem.

But the sides of the orthocentric triangle are $a\cos A$, $b\cos B$, $c\cos C$; and if $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ be the angles of the orthocentric triangle opposite the sides $a\cos A$, $b\cos B$, $c\cos C$, we shall have

$$A+2A=\pi, \text{ or } \sin A_{\prime}=\sin 2A=2\sin A\cos A. \quad . \quad . \quad (b)$$

Hence, if $R_{\prime}$ be the radius of the circle circumscribing the orthocentric triangle, we have

$$2R_{\prime}=\frac{a\cos A}{\sin A_{\prime}}=\frac{a\cos A}{2\sin A\cos A}=\frac{a}{2\sin A}.$$

Hence $2R_{\prime}=R$, or the diameter of the circle circumscribing the original triangle is twice that of the circle circumscribing the orthocentric triangle.

195.] *To determine the area of the orthocentric triangle.*

In general the area of a triangle Δ is determined by the equation $abc=4R\Delta$, Δ being the area of the triangle.

In the orthocentric triangle the sides are $a\cos A$, $b\cos B$, $c\cos C$, and $2R_{\prime}=R$; hence

$$abc\cos A\cos B\cos C=4R_{\prime}\Delta_{\prime}.$$

But $$abc=4R\Delta, \text{ and } 2R_{\prime}=R;$$

hence $$\cos A\cos B\cos C=\frac{\Delta_{\prime}}{2\Delta}, \text{ or } \frac{\Delta_{\prime}}{\Delta}=2\cos A\cos B\cos C. \quad . \quad (c)$$

If perpendiculars be drawn from the vertices of a triangle to the sides of its orthocentric triangle, they will pass through the centre of the circle circumscribing the given triangle.

As the perpendiculars drawn from the vertices of the given triangle ABC on its opposite sides bisect the angles of the orthocentric triangle, the perpendiculars drawn from any two vertices of the given triangle, A and B suppose, to the sides of the orthocentric triangle will make equal angles with the side C. Hence by sec. [154] these lines will meet in a point; and as these three lines are equal, they must meet in the centre of the circle ABC.

Hence, as the perpendiculars drawn from the vertices of the triangle ABC to the opposite sides determine by their intersection the orthocentre, so the perpendiculars drawn from the same vertices to the sides of the orthocentric triangle determine by their intersection the centre of the circumscribing circle.

196.] Since the perpendiculars drawn from Θ, the orthocentre, to the sides of the original triangle bisect the angles of the orthocentric triangle, Θ is the centre of the circle inscribed in it.

To find the value of the radius ρ of the circle inscribed in the orthocentric triangle.

Let $s_{\prime}$ be half the sum of the sides of the orthocentric triangle and $\Delta_{\prime}$ its area, then $\Delta_{\prime}=s_{\prime}\rho$. But, as in sec. [194],

$$2s_{\prime}=a\cos A+b\cos B+c\cos C;\ \text{hence } 2s_{\prime}=\frac{1}{R}(ap+bp_{\prime}+cp_{\prime\prime}), \quad \text{(a)}$$

$p, p_{\prime}, p_{\prime\prime}$ being the perpendiculars drawn from the centre O on the sides of the triangle.

$$\text{But } ap+bp_{\prime}+cp_{\prime\prime}=2\Delta;\ \text{hence } \rho=\frac{\Delta_{\prime}}{s_{\prime}},\ \text{and } s_{\prime}=\frac{\Delta}{R}; \quad .\ .\ \text{(b)}$$

$$\text{therefore } \rho=\frac{R\Delta_{\prime}}{\Delta}.\ \text{But } \frac{\Delta_{\prime}}{\Delta}=2\cos A\cos B\cos C, \text{as in (e) sec. [195];}$$

$$\text{hence } \rho=2R\cos A\cos B\cos C. \ \text{(c)} \quad \text{We have also } \frac{\rho}{R}=\frac{\Delta_{\prime}}{\Delta}, \ \text{(d)}$$

or the areas of the orthocentric and original triangles are to each other as the radii of the circles inscribed in the former and circumscribed to the latter.

Hence, as (b) gives $\Delta=Rs_{\prime}$, it follows that *the area of a triangle is equal to the radius of the circumscribed circle, multiplied into the semiperimeter of its orthocentric triangle.*

We have thus an additional theorem for finding the area of a triangle. This simple expression may be added to those given in sec. [181].

197.] To show that

$$2R+\rho=2R^2\left[\frac{\sin^3 A}{a}+\frac{\sin^3 B}{b}+\frac{\sin^3 C}{c}\right].$$

The area of the original triangle is the sum of the areas of the orthocentric triangle and the three component triangles on its sides; and twice the area of one of these triangles is $bc\cos^2 A\sin A$. Hence

$$bc\cos^2 A\sin A+ac\cos^2 B\sin B+ab\cos^2 C\sin C+2\Delta_{\prime}=2\Delta,$$

or $$bc \sin A + ac \sin B + ab \sin C$$
$$- abc\left[\frac{\sin^3 A}{a} + \frac{\sin^3 B}{b} + \frac{\sin^3 C}{c}\right] = 2\Delta - 2\Delta_{,}.$$

But $$bc \sin A = ac \sin B = ab \sin C = 2\Delta;$$

hence $$4\Delta + 2\Delta_{,} = 4R\Delta\left[\frac{\sin^3 A}{a} + \frac{\sin^3 B}{b} + \frac{\sin^3 C}{c}\right],$$

or, as $R\Delta_{,} = \Delta\rho$,
$$2R + \rho = 2R^2\left[\frac{\sin^3 A}{a} + \frac{\sin^3 B}{b} + \frac{\sin^3 C}{c}\right]. \quad . \quad . \quad . \quad (a)$$

198.] *To determine an expression for the square of the distance between the centre of the circumscribed circle and the orthocentre, or an expression for* $\overline{O\Theta}^2$.

If we take the triangle whose vertices are O, Θ, and one of the vertices, A suppose, of the given triangle, the sides of this new triangle will be OΘ, R, and 2R cos A, while the angle at A will be C − B. Hence obviously
$$\overline{O\Theta}^2 = R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos(C - B). \quad . \quad . \quad (a)$$

But $A = \pi - (C + B)$; hence
$$\overline{O\Theta}^2 = R^2[1 - 4 \cos A\{\cos(C + B) + \cos(C - B)\}],$$

or $$\overline{O\Theta}^2 = R^2[1 - 8 \cos A \cos B \cos C]. \quad . \quad . \quad . \quad (b)$$

In (c) sec. [196] it was shown that $\rho = 2R \cos A \cos B \cos C$.

Hence $$\overline{O\Theta}^2 = R^2 - 4R\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Since $\dfrac{\Delta_{,}}{\Delta} = 2 \cos A \cos B \cos C$, see (c), sec. [195],
$$O\Theta^2 = R^2\left[1 - \frac{4\Delta_{,}}{\Delta}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

We have also $$\overline{O\Theta}^2 = 9R^2 - (a^2 + b^2 + c^2). \quad . \quad . \quad . \quad . \quad . \quad (e)$$

199.] *The sum of the squares of the distances of the vertices of a triangle to the orthocentre, diminished by the square of the distance of this point from the centre of the circumscribed circle, is equal to three times the square of the radius of the circumscribing circle.*

Since $\overline{A\Theta}^2 = 4R^2 \cos^2 A$, we shall have
$$\overline{A\Theta}^2 + \overline{B\Theta}^2 + \overline{C\Theta}^2 = 4R^2(\cos^2 A + \cos^2 B + \cos^2 C).$$

But $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$, see sec. [187];

and as $$\overline{O\Theta}^2 = R^2[1 - 8 \cos A \cos B \cos C],$$

subtracting,
$$\overline{A\Theta}^2 + \overline{B\Theta}^2 + \overline{C\Theta}^2 - \overline{O\Theta}^2 = 3R^2. \quad . \quad . \quad . \quad . \quad (a)$$

Since ρ the radius of the circle inscribed in the orthocentric triangle is equal to 2R cos A cos B cos C,

$$\overline{O\Theta}^2 = R(R-4\rho).$$

Hence, adding twice this expression to the above, we shall have

$$\overline{A\Theta}^2+\overline{B\Theta}^2+\overline{C\Theta}^2+\overline{O\Theta}^2=5R^2-8R\rho. \quad . \quad . \quad . \quad (b)$$

200.] *If ρ, $\rho_{\prime}$, $\rho_{\prime\prime}$, $\rho_{\prime\prime\prime}$ denote the radii of the circles inscribed and exscribed to the orthocentric triangle, and if $2\Sigma^2=a^2+b^2+c^2$, we shall have*

$$\rho=\frac{\Sigma^2-4R^2}{2R},\ \rho_{\prime}=\frac{\Sigma^2-a^2}{2R},\ \rho_{\prime\prime}=\frac{\Sigma^2-b^2}{2R},\ \rho_{\prime\prime\prime}=\frac{\Sigma^2-c^2}{2R}. \quad . \quad (a)$$

Since $\rho=2R\cos A\cos B\cos C$, as in sec. [196],

and $2\cos A\cos B\cos C=1-(\cos^2 A+\cos^2 B+\cos^2 C)$,

$$\rho=R[\sin^2 A+\sin^2 B+\sin^2 C-2].$$

But $\sin^2 A=\dfrac{a^2}{4R^2}$; substituting, $\rho=\dfrac{\Sigma^2-4R^2}{2R}$.

Again, since $\rho_{\prime}=\dfrac{bc}{a}\cos A\sin A$,

or $\rho_{\prime}=2\Delta\dfrac{\cos A}{a}$, but $\dfrac{\cos A}{a}=\dfrac{b^2+c^2-a^2}{2abc}$,

hence $\rho_{\prime}=\dfrac{\Delta}{4R\Delta}(b^2+c^2-a^2)=\dfrac{\Sigma^2-a^2}{2R}$;

and like expressions for $\rho_{\prime\prime}$ and $\rho_{\prime\prime\prime}$ may be found.

If we add these expressions

$$\rho+\rho_{\prime}+\rho_{\prime\prime}+\rho_{\prime\prime\prime}=\frac{a^2+b^2+c^2-4R^2}{2R},$$

or

$$\rho+\rho_{\prime}+\rho_{\prime\prime}+\rho_{\prime\prime\prime}=\frac{\Sigma^2-2R^2}{R}. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

201.] Since $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$ and ρ the radius of the circle inscribed in the orthocentric triangle is equal to 2R cos A cos B cos C, see (c) sec. [196], while $\cos^2 A=\dfrac{p^2}{R^2}$, p being the perpendicular from the centre of the circumscribing circle on one of the sides, then we shall have

$$p^2+p_{\prime}^2+p_{\prime\prime}^2=R(R-\rho).$$

202.] *Three times the sum of the squares of the distances of the* *centres of the four circles of contact from the centre of the circumscribing circle is equal to four times the sum of the squares of the sides, and four times the square of the distance of the orthocentre from the centre of the circumscribing circle.*

In sec. [192] it has been shown that

$$\overline{O\omega}^2+\overline{O\Omega}^2+\overline{O\Omega_{I}}^2+\overline{O\Omega_{II}}^2=12R^2,$$

and $\overline{O\Theta}^2=R^2(1-8\cos A\cos B\cos C)$, as in sec. [198];

but $2\cos A\cos B\cos C=\sin^2 A+\sin^2 B+\sin^2 C-2$;

Hence, reducing, $\overline{O\Theta}^2=9R^2-(a^2+b^2+c^2)$. . . . . . . (a)

Substituting this value of $\overline{O\Theta}^2$ the proposition is manifest; that is,

$$3[\overline{O\omega}^2+\overline{O\Omega}^2+\overline{O\Omega_{I}}^2+\overline{O\Omega_{II}}^2]=4(a^2+b^2+c^2)+4\overline{O\Theta}^2. \quad \text{(b)}$$

203.] *The squares of the sides of a triangle added to the squares of the radii of the four exscribed and inscribed circles is equal to sixteen times the square of the radius of the circumscribing circle.*

In (f) sec. [181] it is shown that

$$(a+b+c)^2=4s^2=4(r_{I}r_{II}+r_{II}r_{III}+r_{I}r_{III}), \quad \text{(a)}$$

and $bc+ca+ab=r_{I}r_{II}+r_{II}r_{III}+r_{I}r_{III}+r(4R+r)$. . (b)

But $4R+r=r_{I}+r_{II}+r_{III}$; . . . . (c)

hence, subtracting twice (b) from (a), we get

$$a^2+b^2+c^2=2(r_{I}r_{II}+r_{II}r_{III}+r_{I}r_{III})-2r(r_{I}+r_{II}+r_{III});$$

and as $4R=r_{I}+r_{II}+r_{III}-r$, squaring and subtracting,

$$16R^2=r_{I}^2+r_{II}^2+r_{III}^2+r^2+a^2+b^2+c^2. \quad \text{(d)}$$

204.] *The sum of the squares of the sides of a triangle is equal to twelve times the square of the radius of the circumscribing circle, diminished by four times the sum of the squares of the perpendiculars from its centre on the sides.*

For $a^2+b^2+c^2=4R^2[\sin^2 A+\sin^2 B+\sin^2 C]$;

hence $a^2+b^2+c^2=12R^2-4R^2(\cos^2 A+\cos^2 B+\cos^2 C)$,

or $a^2+b^2+c^2=12R^2-4(p^2+p_{I}^2+p_{II}^2)$.

205] In any triangle $\left(\frac{a}{r_{I}}+\frac{b}{r_{II}}+\frac{c}{r_{III}}\right)\left(\frac{a+b+c}{r_{I}+r_{II}+r_{III}}\right)=4$,

the letters having the usual signification.

Since $\frac{a}{r_{I}}=\frac{a(s-a)}{sr}$, the first factor is $\frac{2s^2-(a^2+b^2+c^2)}{sr}$.

But $2s^2-(a^2+b^2+c^2)=2r(4R+r)$, see (l) sec. [181],

while $r_{\prime}+r_{\prime\prime}+r_{\prime\prime\prime}=4R+r$, and $a+b+c=2s$.

Substituting these values we obtain the result.

206.] *To determine an expression for* $\overline{\Theta\omega}$, *the distance between the centres of the circles inscribed in the original and orthocentric triangles.*

These centres and a vertex A of the original triangle constitute the vertices of a triangle whose sides are r cosec $\frac{1}{2}$A, 2R cos A, and $\overline{\Theta\omega}$, while the vertical angle of this triangle is $\frac{1}{2}$(C$-$B).

$$\text{Hence } \overline{\Theta\omega}^2=4R^2\cos^2 A+r^2\operatorname{cosec}^2\tfrac{1}{2}A-\frac{4Rr\cos A\cos\frac{1}{2}(C-B)}{\sin\frac{1}{2}A}; \quad \text{(a)}$$

but $$\frac{\cos\frac{1}{2}(C-B)}{\sin\frac{1}{2}A}=\frac{c+b}{a},$$

while $$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

Substituting these values in the original equation, we shall have

$$\overline{\Theta\omega}^2=4R^2-8Rr+ab+ac+bc-(a^2+b^2+c^2). \quad . \quad . \quad \text{(b)}$$

But it has been shown in (k) and (l) [sec. 181] that

$$bc+ac+ab=s^2+r^2+4Rr,$$

and $$a^2+b^2+c^2=2s^2-2r^2-8Rr.$$

Hence $$\overline{\Theta\omega}^2=4Rr+4R^2+3r^2-s^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Let Ω, $\Omega_{\prime}$, $\Omega_{\prime\prime}$ denote the centres of the exscribed circles; then, making the necessary substitutions, we shall find

$$\left.\begin{aligned}\overline{\Theta\Omega}^2&=4R^2+8Rr_{\prime}+bc-ac-ab-(a^2+b^2+c^2),\\ \overline{\Theta\Omega}_{\prime}^2&=4R^2+8Rr_{\prime}-bc+ac-ab-(a^2+b^2+c^2),\\ \overline{\Theta\Omega}_{\prime\prime}^2&=4R^2+8Rr_{\prime\prime}-bc-ac+ab-(a^2+b^2+c^2),\end{aligned}\right\} \quad . \quad . \quad \text{(d)}$$

and as in the preceding formula (b)

$$\overline{\Theta\omega}^2=4R^2-8Rr+bc+ac+ab-(a^2+b^2+c^2).$$

Adding these expressions together, and bearing in mind that

$$r_{\prime}+r_{\prime\prime}+r_{\prime\prime\prime}-r=4R, \text{ see sec. [181]},$$

we shall have

$$\overline{\Theta\Omega}^2+\overline{\Theta\Omega}_{\prime}^2+\overline{\Theta\Omega}_{\prime\prime}^2+\overline{\Theta\omega}^2=48R^2-4(a^2+b^2+c^2). \quad . \quad . \quad . \quad \text{(e)}$$

Let the distances of the orthocentre from the vertices of the triangle be $A\Theta$, $B\Theta$, $C\Theta$; then we have

$$\overline{A\Theta}^2=4R^2-a^2,\ \overline{B\Theta}^2=4R^2-b^2,\ \overline{C\Theta}^2=4R^2-c^2;$$

substituting we obtain

$$\overline{\Theta\Omega}^2+\overline{\Theta\Omega_{\prime}}^2+\overline{\Theta\Omega_{\prime\prime}}^2+\overline{\Theta\omega}^2=4(\overline{A\Theta}^2+\overline{B\Theta}^2+\overline{C\Theta}^2). \quad . \quad . \quad (f)$$

Since $\overline{O\Theta}^2=9R^2-(a^2+b^2+c^2)$, see (e) sec. [198],

and $\overline{O\omega}^2+\overline{O\Omega}^2+\overline{O\Omega_{\prime}}^2+\overline{O\Omega_{\prime\prime}}^2=12R^2$, see (b) sec. [192],

therefore

$$\overline{\Theta\Omega}^2+\overline{\Theta\Omega_{\prime}}^2+\overline{\Theta\Omega_{\prime\prime}}^2+\overline{\Theta\omega}^2=\overline{O\omega}^2+\overline{O\Omega}^2+\overline{O\Omega_{\prime}}^2+\overline{O\Omega_{\prime\prime}}^2+4\overline{O\Theta}^2,$$

Hence *the sum of the squares of the distances of the centres of the four circles of contact from the orthocentre, exceeds the squares of the distances of the same points from the centre of the circumscribing circle by four times the square of the distance between the orthocentre and the centre of the circumscribing circle.*

207.] *A perpendicular is drawn from the vertex of a triangle on the opposite side; a line is drawn bisecting the vertical angle and meeting the base; and a circle is inscribed in the triangle. The distances from the middle point of the base to the foot of the perpendicular, to the point of contact of the inscribed circle, and to the point where the bisector meets the base, are in geometrical progression.* For these distances are, as may easily be shown,

$$\frac{c^2-b^2}{2a},\quad \frac{c-b}{2},\quad \frac{a}{2}\frac{(c-b)}{c+b}.$$

When circles are exscribed to and inscribed in any triangle, each side, a suppose, is touched in four points—in two, $F\,F_{\prime}$, within the angle A, and in two external to it. The circles, one inscribed, the other exscribed, which touch the side a within the angle A are on opposite sides of it; and their distance is $(c-b)$ or $\frac{1}{2}(c-b)$ from M the middle of a. The side a is touched by the two remaining exscribed circles on the same side at two points, L and N, outside the angle A, distant from the angles B, C, by $s-a$; and the distance between these two points L and N is $2(s-a)+a=c+b$; and the distance of L and N from the middle point M of a is $\frac{1}{2}(c+b)$

208.] *If α, β, γ be the* median *lines of a triangle whose sides are a, b, c, we shall have the following relations between these lines:—*

$$16(\alpha^4+\beta^4+\gamma^4)=9(a^4+b^4+c^4), \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$16(\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2)=9(b^2c^2+a^2c^2+a^2b^2). \quad . \quad . \quad . \quad (b)$$

By a well known theorem

$$2b^2+2c^2-a^2=4\alpha^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Finding analogous values for β^2 and γ^2, and adding, we obtain

$$4(\alpha^2+\beta^2+\gamma^2)=3(a^2+b^2+c^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

If we square (c) and the other like expressions, and add them, we shall have

$$16(\alpha^4+\beta^4+\gamma^4)=9(a^4+b^4+c^4). \quad . \quad . \quad . \quad . \quad (e)$$

If we square the expression (d) and subtract (e) from it, we shall find

$$16(\beta^2\gamma^2+\alpha^2\gamma^2+\alpha^2\beta^2)=9(b^2c^2+a^2c^2+a^2b^2). \quad . \quad . \quad (f)$$

209.] *If through the points of contact* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ *of the circle inscribed in a given triangle perpendiculars are drawn to meet the corresponding median lines in the points* l, m, n, *we shall have*

$$\frac{1}{A_{\prime}l}+\frac{1}{B_{\prime}m}+\frac{1}{C_{\prime}n}=\frac{2}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The distance between the middle point of the base and the foot of the perpendicular on it from the vertex A is $\dfrac{c^2-b^2}{2a}$.

The distance between the middle point of the base and the point of contact of the inscribed circle is $\dfrac{c-b}{2}$; and their distances are as h and $A_{\prime}l$.

Hence $A_{\prime}l : h=\dfrac{c-b}{2} : \dfrac{c^2-b^2}{2a}=a : c+b$;

therefore
$$\frac{1}{A_{\prime}l}=\frac{c+b}{ah}=\frac{c+b}{2\Delta};$$

consequently
$$\frac{1}{A_{\prime}l}+\frac{1}{B_{\prime}m}+\frac{1}{C_{\prime}n}=\frac{4s}{2\Delta}=\frac{2}{r}.$$

210.] *To determine the distance between the centroid* κ, *and the centre* ω *of the inscribed circle.*

Let us take the triangle of which the vertices are κ, A, ω. Now the sides of this triangle are $\overline{\omega\kappa}$, $\dfrac{r}{\sin\frac{1}{2}A}$ and $\frac{2}{3}\alpha$, α being the median line from A to the opposite side.

Let θ be the angle between the median line α, and the side c, suppose.

Then, as the median line bisects the triangle ABC, we shall have

$$c\sin\theta = b\sin(A-\theta),\ \text{ or, as } 2\alpha = \frac{c+b\cos A}{\cos\theta},$$

$$\sin\theta = \frac{b\sin A}{2\alpha},\ \ \cos\theta = \frac{c+b\cos A}{2\alpha}. \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Let ϵ be the angle between the median line 2α and the bisector of the vertical angle A; then we have

$$\epsilon = (\tfrac{1}{2}A-\theta),\ \text{ and }\ 2\alpha\cos\epsilon = (b+c)\cos\tfrac{1}{2}A.$$

Now
$$\overline{\omega\kappa}^2 = \frac{r^2}{\sin^2\frac{1}{2}A} + \tfrac{4}{9}\alpha^2 - \tfrac{4}{3}\alpha\frac{r}{\sin\frac{1}{2}A}\cos\epsilon \quad . \quad . \quad . \quad . \qquad (b)$$

$$= \frac{bc+ab+ac}{3} - \frac{(a^2+b^2+c^2)}{9} - 4Rr. \quad . \quad . \quad . \qquad (c)$$

But as
$$2s^2 - 2r^2 - 8Rr = a^2+b^2+c^2,$$
and $2s^2+2r^2+8Rr = 2(bc+ac+ab)$, as shown in (l) and (k) sec. [181]

$$4Rr = \frac{bc+ac+ab}{2} - \frac{(a^2+b^2+c^2)}{4} - r^2.$$

Substituting this value of $4Rr$ in the preceding equation, we get

$$\overline{\omega\kappa}^2 = \tfrac{5}{36}(a^2+b^2+c^2) - \tfrac{1}{6}(bc+ac+ab) + r^2. \quad . \quad . \quad . \qquad (d)$$

In the same way, making the necessary substitutions, we shall have

$$\left.\begin{aligned}\overline{\Omega\kappa}^2 &= \tfrac{5}{36}(a^2+b^2+c^2) + \tfrac{1}{6}(ca+ab-bc) + r_{,}^2,\\ \overline{\Omega_{,}\kappa}^2 &= \tfrac{5}{36}(a^2+b^2+c^2) + \tfrac{1}{6}(ab+bc-ac) + r_{,,}^2,\\ \overline{\Omega_{,,}\kappa}^2 &= \tfrac{5}{36}(a^2+b^2+c^2) + \tfrac{1}{6}(bc+ca-ab) + r_{,,,}^2.\end{aligned}\right\} \quad . \quad . \qquad (e)$$

Adding these expressions together, we shall have

$$\overline{\omega\kappa}^2 + \overline{\Omega\kappa}^2 + \overline{\Omega_{,}\kappa}^2 + \overline{\Omega_{,,}\kappa}^2 = \tfrac{5}{9}(a^2+b^2+c^2) + r^2 + r_{,}^2 + r_{,,}^2 + r_{,,,}^2. \quad (f)$$

But it has been shown in (d) sec. [203] that

$$r^2 + r_{,}^2 + r_{,,}^2 + r_{,,,}^2 = 16R^2 - (a^2+b^2+c^2).$$

Hence, eliminating, we obtain

$$\overline{\omega\kappa}^2 + \overline{\Omega\kappa}^2 + \overline{\Omega_{,}\kappa}^2 + \Omega_{,,}\kappa^2 = 16R^2 - \tfrac{4}{9}(a^2+b^2+c^2). \quad . \quad . \quad . \qquad (g)$$

211.] *To determine an expression for the distance between the centroid and the centre of the circumscribing circle.*

Taking the triangle whose vertices are O, κ, and the middle point of the base, the sides of this triangle are $\overline{O\kappa}$, $R\cos A$, and $\frac{1}{3}\alpha$, while the cosine of the angle opposite to $O\kappa$ is $\dfrac{h}{\alpha}$. Hence we have

$$\overline{O\kappa}^2 = \frac{\alpha^2}{9} + R^2 \cos^2 A - \tfrac{2}{3}\alpha R \cos A \frac{h}{\alpha}. \quad . \quad . \quad . \quad . \quad (a)$$

Reducing, we find $\overline{O\kappa}^2 = R^2 - \dfrac{(a^2+b^2+c^2)}{9}$. . . . . (b)

Comparing this expression with that found for the distance of the orthocentre from the centre of the circumscribing circle, we shall have

$$O\Theta = 3O\kappa. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

212.] *The sum of the squares of the twelve lines drawn from the angles of a triangle to the points of contact of the circles of contact in the opposite sides is equal to five times the sum of the squares of the sides of the triangle.*

Let the side BC of the triangle be produced to L and N, so that $BL = s-a$, $CN = s-a$; then it may easily be shown that L and N are the external points of contact, while the distance between F and $F_{\prime}$, the internal points of contact is $c-b$. But $a+s-a+s-a=c+b$. Let $AM=\alpha$, where M is the middle point of the side a (see fig. 29);

then $\quad \overline{AL}^2 + \overline{AN}^2 = 2\alpha^2 + \frac{1}{2}(c+b)^2$, and $\overline{AF}^2 + \overline{AF_{\prime}}^2 = 2\alpha^2 + \frac{1}{2}(c-b)^2$;

therefore $\quad \overline{AL}^2 + \overline{AN}^2 + \overline{AF}^2 + \overline{AF_{\prime}}^2 = 4(\alpha^2 + c^2 + b^2)$.

Making similar constructions on the other sides; the sum of the squares of the twelve lines will be found equal to

$$4(\alpha^2+\beta^2+\gamma^2) + 2(a^2+b^2+c^2).$$

But $\quad 4(\alpha^2+\beta^2+\gamma^2) = 3(a^2+b^2+c^2)$.

Hence the sum of the squares of the twelve lines is equal to

$$5(a^2+b^2+c^2).$$

213.] *The sum of the squares of the twelve lines drawn from the middle points of the sides of a triangle to the centres of the circles of contact, together with the sum of the squares of the sides of the triangle, is equal to twelve times the square of the diameter of the circumscribing circle.*

Let the lines drawn from the middle points of the sides a, b, c to the centre Ω of the exscribed circle opposite the angle A, be α, β, γ, and that to ω the centre of the inscribed circle be δ.

Then $\quad \overline{B\Omega}^2 + \overline{C\Omega}^2 = \alpha^2 + \frac{1}{2}a^2, \ \overline{B\omega}^2 + \overline{C\omega}^2 = 2\delta^2 + \frac{1}{2}a^2,$

$$\overline{A\Omega}^2 + \overline{C\Omega}^2 = 2\beta^2 + \tfrac{1}{2}b^2, \ \overline{A\Omega}^2 + \overline{B\Omega}^2 = 2\gamma^2 + \tfrac{1}{2}c^2; \quad . \quad (a)$$

adding these expressions, and dividing by 2, we have

$$\overline{A\Omega}^2+\overline{B\Omega}^2+\overline{C\Omega}^2+\tfrac{1}{2}(\overline{B\omega}^2+\overline{C\omega}^2)=\alpha^2+\beta^2+\gamma^2+\delta^2+\tfrac{1}{2}a^2+\tfrac{1}{4}(b^2+c^2).$$

But $\quad \overline{A\Omega}^2=r_{,}^2+s^2,\ \overline{B\Omega}^2=r_{,}^2+(s-c)^2,\ \overline{C\Omega}^2=r_{,}^2+(s-b)^2,$

and $\quad \tfrac{1}{2}(\overline{B\omega}^2+\overline{C\omega}^2)=r^2+(s-b)^2+(s-c)^2;$

adding these expressions, we obtain

$$\left.\begin{aligned}&3r_{,}^2+r^2+\tfrac{1}{2}[2s^2+3(s-b)^2+3(s-c)^2]\\&\quad=(\alpha^2+\beta^2+\gamma^2+\delta^2)+\tfrac{1}{2}a^2+\tfrac{1}{4}(b^2+c^2)\end{aligned}\right\}. \quad . \quad . \quad \text{(b)}$$

Writing analogous expressions for the two other centres $\Omega_{,}$ and $\Omega_{,,}$, we shall have

$$\begin{aligned}&3(r_{,}^2+r_{,,}^2+r_{,,,}^3+r^2)+3(a^2+b^2+c^2)=(\alpha^2+\beta^2+\gamma^2+\delta^2)\\&\quad+(\alpha_{,}^2+\beta_{,}^2+\gamma_{,}^2+\delta_{,}^2)+(\alpha_{,,}^2+\beta_{,,}^2+\gamma_{,,}^2+\delta_{,,}^2)+(a^2+b^2+c^2).\end{aligned}$$

But $\quad r_{,}^2+r_{,,}^2+r_{,,,}^2+r^2+a^2+b^2+c^2=16R^2$, see sec. [203].

Hence, substituting, we find,

$$\left.\begin{aligned}48R^2=(\alpha^2+\beta^2+\gamma^2+\delta^2)+\alpha_{,}^2+\beta_{,}^2+\gamma_{,}^2+\delta_{,}^2)\\+(\alpha_{,,}^2+\beta_{,,}^2+\gamma_{,,}^2+\delta_{,,}^2)\end{aligned}\right\}. \quad . \quad . \quad \text{(e)}$$

214.] *The sum of the areas of the four triangles formed by joining three by three, the points of contact of the circles of contact is constant, and equal to twice the area of the given triangle.*

The area of the triangle formed by joining the three interior points of contact must be taken with the negative sign.

In the first place let us take the triangle whose vertex is A and base a, and construct the triangle whose vertices are the points of contact of the exterior circle of contact with the side a, and b, c produced. Then twice the area of this triangle is manifestly

$$s^2\sin A-(s-b)^2\sin C-(s-c)^2\sin B-bc\sin A;$$

and if we make like constructions for the other angles B and C of the given triangle,

$$s^2\sin B-(s-c)^2\sin A-(s-a)^2\sin C-ac\sin B,$$
$$s^2\sin C-(s-a)^2\sin B-(s-b)^2\sin A-ab\sin C,$$

will be twice the areas of the two other triangles.

Let us first combine those terms of which sin A is a factor, or

$$[s^2-(s-c)^2-(s-b)^2-bc]\sin A,$$

which may be reduced by obvious substitution to

$$[4Rr+r^2]\sin A.$$

Making like reductions for the other angles B and C, we get for twice the sum of the areas of the three triangles

$$(4Rr+r^2)(\sin A+\sin B+\sin C).$$

But $\sin A+\sin B+\sin C=\frac{s}{R}$, see (d) sec. [185].

Hence twice the sum of the areas of the three triangles is

$$(4Rr+r^2)\frac{s}{R}=4rs+\frac{r^2s}{R},$$

Now $4rs$ is twice the area of the given triangle, and $\frac{r^2s}{R}$ is twice the area of the triangle whose vertices are the points of internal contact.

215.] *In a triangle* ABC, *let the internal bisectors of the angles* A, B, C *meet the opposite sides in the points* $A_{\prime}$, $B_{\prime}$, $C_{\prime}$, *and let the external bisectors of these angles meet the same sides in the points* $A_{\prime\prime}$, $B_{\prime\prime}$, $C_{\prime\prime}$; *then, if* $a>b>c$, *we shall have*

$$\frac{AA_{\prime\prime}}{A_{\prime}A_{\prime\prime}}\cdot\frac{BB_{\prime\prime}}{B_{\prime}B_{\prime\prime}}\cdot\frac{CC_{\prime\prime}}{C_{\prime}C_{\prime\prime}}=\frac{(b+c)(a+c)(a+b)}{8R^2(a+b+c)}.$$

Now $\qquad c:b=BA_{\prime}:CA_{\prime}$, or $(c+b)=\frac{ac}{BA_{\prime}}$;

but as the angle $A_{\prime}AA_{\prime\prime}$ is a right angle,

$$\cos AA_{\prime\prime}B=\frac{AA_{\prime\prime}}{A_{\prime}A_{\prime\prime}}=\sin AA_{\prime}B.$$

But $\qquad \sin AA_{\prime}B:\sin\tfrac{1}{2}A=c:BA_{\prime}$.

Therefore $\qquad (c+b)=\frac{a\sin AA_{\prime}B}{\sin\frac{1}{2}A}$,

or, putting for $\sin AA_{\prime}B$ its value,

$$\frac{(c+b)}{a}\sin\tfrac{1}{2}A=\frac{AA_{\prime\prime}}{A_{\prime}A_{\prime\prime}}.$$

Finding like expressions for the other two sides, and bearing in mind that

$$4\sin\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}B\,.\,\sin\tfrac{1}{2}C=\frac{r}{R},$$

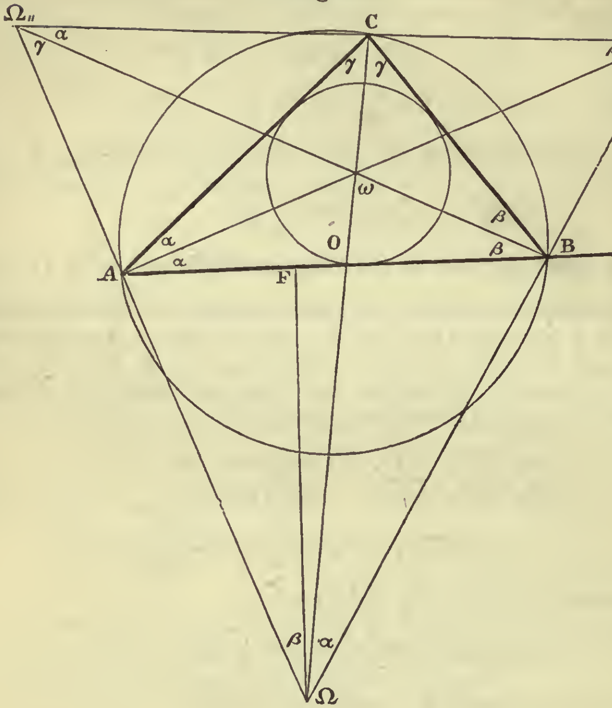
we obtain the theorem.

216.] To find expressions for the sides, angles, and areas of the excentral triangles, $\Omega\Omega_{\prime}\Omega_{\prime\prime}$, $\Omega\omega\Omega_{\prime}$, $\Omega_{\prime}\omega\Omega_{\prime\prime}$, $\Omega_{\prime\prime}\omega\Omega$.

Since (fig. 31) $BF=(s-a)$ is the projection of ΩB, therefore $\Omega B=\frac{(s-a)}{\sin\frac{1}{2}B}$.

In like manner we obtain $\Omega_{\prime}B=\frac{(s-c)}{\sin\frac{1}{2}B}$.

Fig. 31.



Therefore $\Omega B + B\Omega_{\prime} = \Omega\Omega_{\prime} = \dfrac{s-a+s-c}{\sin\frac{1}{2}B}$, or the side

$$\Omega\Omega_{\prime} = \frac{b}{\sin\frac{1}{2}B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let R be the radius of the circle circumscribing the triangle ABC;

$$b = 2R\sin B = 4R\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B.$$

Hence $\quad \Omega\Omega_{\prime} = 4R\cos\frac{1}{2}B. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad$ (b)

In like manner $\Omega\Omega_{\prime\prime} = 4R\cos\frac{1}{2}A$, and $\Omega_{\prime}\Omega_{\prime\prime} = 4R\cos\frac{1}{2}C$.

Hence the semiperimeter S of the excentral triangle is

$$S = 2R(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

The area of this excentral triangle may be found. For this area is equal to $\frac{1}{2}\Omega\Omega_{\prime} . \Omega\Omega_{\prime\prime}\sin\frac{1}{2}(A+B)$; or, substituting for these expressions their values, we have

Area of excentral triangle $= 8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C. \quad . \quad$ (d)

Since $4\cos\frac{1}{2}A\cos\frac{1}{2}B.\cos\frac{1}{2}C=\sin A+\sin B+\sin C$, see sec. [187], this area is equal to $2R^2(\sin A+\sin B+\sin C)$; but

$$\sin A=\frac{a}{2R},\ \sin B=\frac{b}{2R},\ \sin C=\frac{c}{2R}.$$

Hence the area of the excentral triangle is equal to

$$R(a+b+c)=2Rs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

This expression coincides with that found for the area of a triangle in sec. [196]; for ABC is the orthocentric triangle of the excentral triangle $\Omega\Omega_{\prime}\Omega_{\prime\prime}$, the radius of whose circumscribing circle is 2R.

217.] The area of the excentral triangle is

$$8R^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$

and the side opposite the angle A is $4R\cos\frac{1}{2}A$; hence the perpendicular from the vertex A on the opposite side is

$$4R\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The radius $\bar{\rho}$ of the circle inscribed in the excentral triangle may be thus found. Since the radius of the inscribed circle is equal to the area of the triangle divided by its semiperimeter, therefore

$$\frac{1}{\bar{\rho}}=\frac{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}{4R\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

218.] To find the values of $\Omega\omega$, $\Omega_{\prime}\omega$, $\Omega_{\prime\prime}\omega$. Since the projection of $\Omega\omega$ on the side a is equal to c,

therefore $$\Omega\omega=\frac{c}{\cos\frac{1}{2}C}=\frac{2R\sin C}{\cos\frac{1}{2}C}=4R\sin\tfrac{1}{2}C. \quad . \quad . \quad . \quad . \quad (a)$$

In like manner $\Omega_{\prime}\omega=4R\sin\frac{1}{2}A$ and $\Omega_{\prime\prime}\omega=4R\sin\frac{1}{2}B$. . (b)

Hence $$\Omega\Omega_{\prime}\,.\,\Omega_{\prime}\Omega_{\prime\prime}\,.\,\Omega\Omega_{\prime\prime}\,.\,\Omega\omega\,.\,\Omega_{\prime}\omega\,.\,\Omega_{\prime\prime}\omega=64R^3abc. \quad . \quad (c)$$

To find the area of the triangle $\Omega\omega\Omega_{\prime}$. The area of twice this triangle is $\Omega\omega\,.\,\Omega_{\prime}\omega\sin\frac{1}{2}(A+C)$. Hence this area is

$8R^2\sin\frac{1}{2}A\sin\frac{1}{2}C\cos\frac{1}{2}B$, which may be put under the form

$$8R^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}C].$$

Finding like expressions for the two other component triangles of the triangle $\Omega\Omega_{\prime}\Omega_{\prime\prime}$ we shall have for the sum of the three,

$$8R^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C[\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C+\tan\tfrac{1}{2}C\tan\tfrac{1}{2}A$$
$$+\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B].$$

But the sum of the terms within the brackets is equal to 1, as shown in (d) sec. [188].

219.] *The square of the distance between the centres of two of the exscribed circles of a triangle exceeds the square of the sum of their radii by the square of the opposite side of the triangle.*

Let the exscribed circles be taken which are opposite to the angles A and C of the given triangle,

$$\text{then we have } r_{,}=\frac{sr}{s-a} \qquad \text{and } r_{,,}=\frac{sr}{s-c};$$

consequently
$$r_{,}+r_{,,}=\frac{srb}{(s-a)(s-c)}=\frac{s^2r^2b(s-b)}{rs^2r^2},$$

or
$$r_{,}+r_{,,}=\frac{b(s-b)}{r}=b\cot\tfrac{1}{2}\mathrm{B}.$$

Let $\Omega\Omega_{,}$ be the line which joins Ω and $\Omega_{,}$; then the projection of ΩB on the side c is $s-c$, and the projection of B$\Omega_{,}$ on the side a is $(s-a)$; consequently the sum of these projections is b, or

$$\overline{\Omega\Omega_{,}}\sin\tfrac{1}{2}\mathrm{B}=b.$$

Hence
$$\overline{\Omega\Omega_{,}}^2-(r_{,}+r_{,,})^2=b^2(\operatorname{cosec}^2\tfrac{1}{2}\mathrm{B}-\cot^2\tfrac{1}{2}\mathrm{B})=b^2.$$

220.] *The sum of the squares of the tangents drawn from the centres of the four circles of contact of a triangle, to any circle which passes through the centre of the circumscribing circle, is equal to three times the square of the circumscribing diameter.*

Let ω, Ω, $\Omega_{,}$, $\Omega_{,,}$ be the centres of the four circles of contact, and O the centre of the circumscribing circle through which the diameter HD perpendicular to the base BC passes.

Let Q be the centre of the arbitrary circle passing through O; and draw the lines $\mathrm{Q}\omega$, $\mathrm{Q}\Omega$, $\mathrm{Q}\Omega_{,}$, $\mathrm{Q}\Omega_{,,}$, QO, QH, QD, $\mathrm{H}\omega$.

Then
$$\overline{\mathrm{Q}\Omega}^2+\overline{\mathrm{Q}\omega}^2=2\overline{\mathrm{QD}}^2+2\overline{\mathrm{DC}}^2, \text{ since } \mathrm{DC}=\mathrm{D}\omega,$$

and
$$\overline{\mathrm{Q}\Omega_{,}}^2+\overline{\mathrm{Q}\Omega_{,,}}^2=2\overline{\mathrm{QH}}^2+2\overline{\mathrm{HC}}^2.$$

But
$$2\overline{\mathrm{QD}}^2+2\overline{\mathrm{QH}}^2=4r^2+4\mathrm{R}^2,$$

and
$$2\overline{\mathrm{DC}}^2+2\overline{\mathrm{HC}}^2=8\mathrm{R}^2;$$

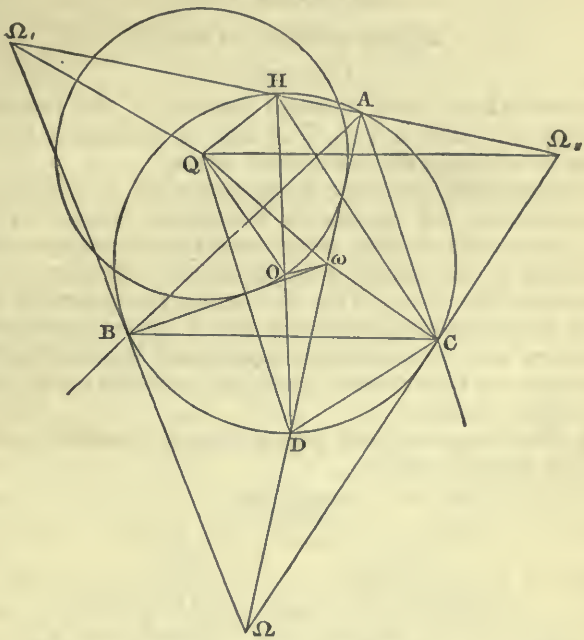
therefore

$$(\overline{\mathrm{Q}\omega}^2-r^2)+(\overline{\mathrm{Q}\Omega}^2-r^2)+(\overline{\mathrm{Q}\Omega_{,}}^2-r^2)+(\overline{\mathrm{Q}\Omega_{,,}}^2-r^2)=12\mathrm{R}^2.$$

But these expressions are the squares of the tangents drawn from the centres of the circles of contact to the circle whose radius is r.

When $r=0$, or the arbitrary circle vanishes to a point, we get the theorem established in sec. [192].

Fig. 32.



221.] *If the sides of the excentral triangle $\Omega\Omega_{\prime}\Omega_{\prime\prime}$ be produced, and circles of contact be drawn touching the sides of this triangle, and the centres of these new circles of contact be joined so as to form a new excentral triangle, and if this process of construction be continued, the successive excentral triangles will approximate to an equilateral triangle.*

Let A, B, C be the angles of the given triangle; $A_{\prime}$, $B_{\prime}$, $C_{\prime}$ the angles of the first derived triangle, $A_{\prime\prime}$, $B_{\prime\prime}$, $C_{\prime\prime}$ the angles of the second derived excentral triangle, and so on; then

$$A_{\prime}=\tfrac{1}{2}(B+C), \quad B_{\prime}=\tfrac{1}{2}(C+A), \quad C_{\prime}=\tfrac{1}{2}(A+B).$$

Therefore

$$B_{\prime}-A_{\prime}=\tfrac{1}{2}(A-B),\ C_{\prime}-B_{\prime}=\tfrac{1}{2}(B-C),\ C_{\prime}-A_{\prime}=\tfrac{1}{2}(A-C).$$

Hence the differences between the angles of the first derived excentral triangle are one half those between the angles of the original triangle.

Again as $A_{\prime\prime}=\tfrac{1}{2}(B_{\prime}+C_{\prime}),\ B_{\prime\prime}=\tfrac{1}{2}(C_{\prime}+{}_{\prime}A),\ C_{\prime\prime}=\tfrac{1}{2}(A_{\prime}+B_{\prime}),$

$$A_{\prime\prime}-B_{\prime\prime}=\tfrac{1}{2}(B_{\prime}-A_{\prime})=\tfrac{1}{4}(A-B).$$

Hence the difference between the angles $A_{\prime\prime}$ and $B_{\prime\prime}$ is one fourth of the difference between the angles A and B. The same is true for the other angles. Hence the successive excentral triangles approximate to an equilateral triangle.

CHAPTER XXV.

ON THE NINE-POINT CIRCLE.

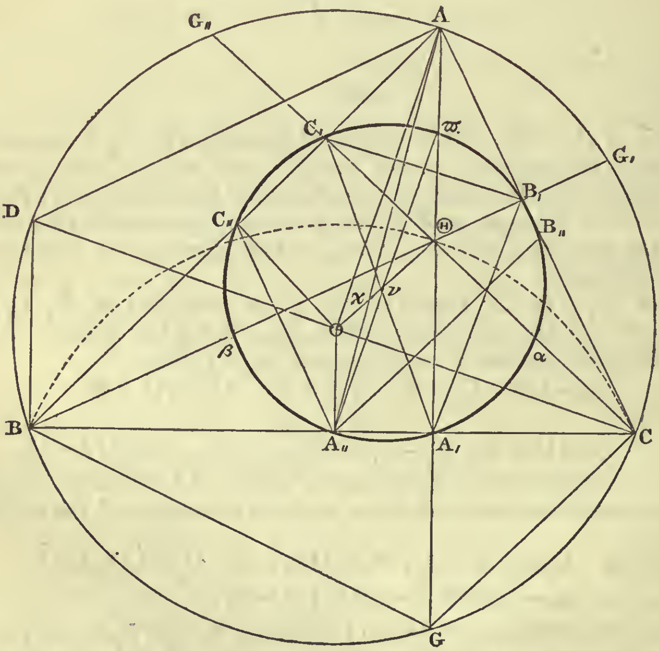
Definition.

The circle which passes through the feet of the perpendiculars drawn from the vertices A, B, C of a given triangle to the opposite sides has been named the *Nine-point circle.*

The properties of the Nine-point circle are unquestionably the most remarkable and elegant in the entire range of plane geometry. Some of the leading properties of this circle were discovered by Poncelet in the early part of the present century. It is a singular fact that the theory of the Nine-point circle escaped the notice not only of the ancient geometers but of modern mathematicians almost to our own time—another proof, were another wanting, how inexhaustible are the truths of geometry, and how many yet remain to be brought to light.

222.] *The nine-point circle passes through the middle points of the sides of the triangle* ABC.

Fig. 33.



Let the nine-point circle which passes through the points $A_{\prime}$, $B_{\prime}$, $C_{\prime}$

cut the sides of the given triangle ABC in the points $A_{\prime\prime}$, $B_{\prime\prime}$, $C_{\prime\prime}$. Join $A_{\prime\prime}C_{\prime\prime}$. As $A_{\prime}C_{\prime}C_{\prime\prime}A_{\prime\prime}$ is a quadrilateral inscribed in the nine-point circle, the angle $BC_{\prime\prime}A_{\prime\prime}$ is equal to the angle $BA_{\prime}C_{\prime}$, which is equal to the angle BAC, since $A_{\prime}C_{\prime}AC$ is also a quadrilateral that may be inscribed in a circle. Hence, as the angle $BC_{\prime\prime}A_{\prime\prime}$ is equal to the angle BAC, $A_{\prime\prime}C_{\prime\prime}$ is parallel to AC a side of the given triangle ABC. In the same way it may be shown that $A_{\prime\prime}B_{\prime\prime}$ is parallel to AB and $B_{\prime\prime}C_{\prime\prime}$ parallel to BC. But when a triangle inscribed in another triangle has its sides parallel to those of the latter, it obviously follows that the vertices of the former will be on the middle points of the latter.

This is a particular case of a far more general theorem which will be given further on.

223.] *The distances of the orthocentre* Θ *from the vertices* A, B, C *of the given triangle are double the distances of the centre of the circumscribing circle from the opposite sides* a, b, c.

From C draw the diameter COD; then CBD is a right angle. Join AD, then CAD is a right angle, and therefore AD is parallel to $BB_{\prime}$ while $A\Theta$ is parallel to BD, each being perpendicular to BC. Therefore $A\Theta BD$ is a parallelogram; and therefore $A\Theta = BD$. But BD is equal to $2A_{\prime\prime}O$, since $BC = 2CA_{\prime\prime}$; hence $A\Theta$ is equal to twice $A_{\prime\prime}O$.

Bisect $A\Theta$ in ϖ, and join $\varpi A_{\prime\prime}$ meeting $O\Theta$ in ν; then, as $OA_{\prime\prime} = \Theta\varpi$, $\varpi\nu$ is equal to $A_{\prime\prime}\nu$, and $O\nu$ is equal to $\Theta\nu$.

Now ν will be the centre of the nine-point circle. For ν is the intersection of the perpendiculars drawn through the middle points of $A_{\prime}A_{\prime\prime}$, $B_{\prime}B_{\prime\prime}$, $C_{\prime}C_{\prime\prime}$ the chords of the nine-point circle.

Since $A\Theta$ is equal to twice $A_{\prime\prime}O$, $A\kappa$ is equal to twice $A_{\prime\prime}\kappa$, or κ is the *centroid* of the triangle ABC.

Hence the line which joins the centre of the circle circumscribing the triangle with its orthocentre passes through the centre of the nine-point circle and the centroid.

Since $A\varpi$ is equal and parallel to $A_{\prime\prime}O$, OA is equal and parallel to $A_{\prime\prime}\varpi$. But OA is the radius of the circumscribed circle, and $A_{\prime\prime}\varpi$ is the diameter of the nine-point circle; hence the radius of the circumscribed circle is equal to the diameter of the nine-point circle.

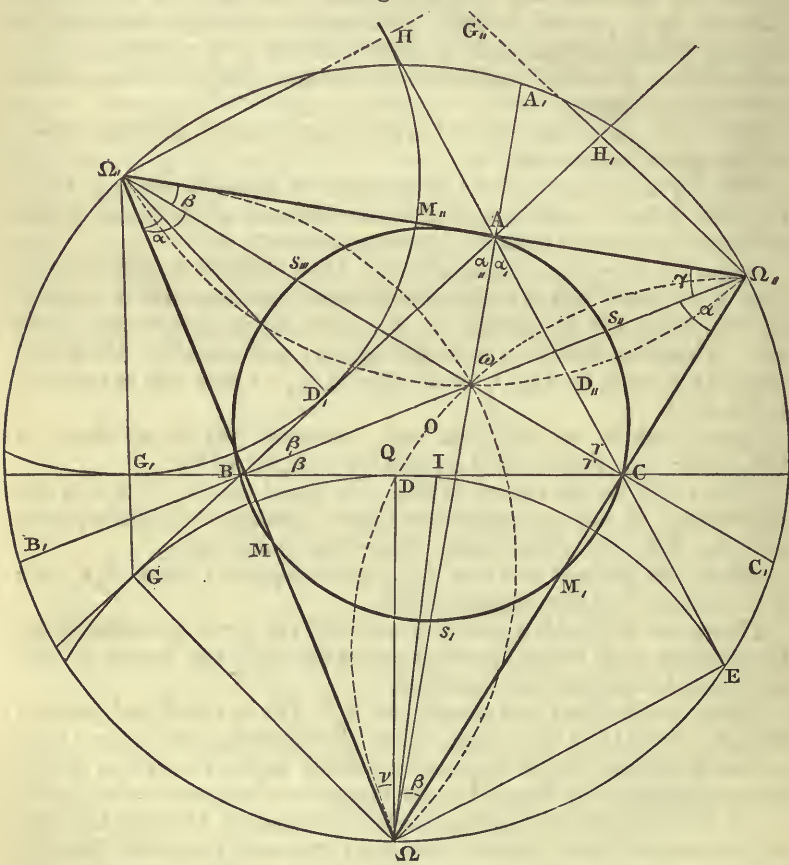
As the orthocentric or nine-point circle passes through the feet of the perpendiculars drawn from the vertices of the given triangle to the opposite sides, through the three middle points of the sides of this triangle, and through the three middle points of the lines which join the orthocentre with the opposite vertices A, B, C of the given triangle, this circle has therefore been called the ***nine-point circle***.

The angle A of the triangle BAC is equal to the angle BDC; and $BC = CD \sin CDB$; hence CD or $2R = \dfrac{BC}{\sin A}$.

ON THE TRIANGLES WHOSE VERTICES ARE, THREE BY THREE, THE FOUR CENTRES OF THE THREE EXSCRIBED AND THE INSCRIBED CIRCLE.

224.] (α) In the given triangle ABC (fig. 34) let a circle be conceived to be inscribed whose centre is ω.

Fig. 34.



Let Ω, $\Omega_{\prime}$, $\Omega_{\prime\prime}$ be the centres of the circles of contact. Join ΩB, B$\Omega_{\prime}$; then, as Bω bisects the internal angle B, and BΩ bisects the external angle B, these bisectors Bω and BΩ meet at right angles, and therefore ΩB and B$\Omega_{\prime}$ are in a straight line.

In the same way it may be shown that $\Omega\Omega_{\prime\prime}$ and $\Omega_{\prime}\Omega_{\prime\prime}$ are in a straight line.

This may be called the *principal excentral triangle*.

(β) There are three other excentral triangles, whose vertices are $\Omega, \Omega_{,}, \omega$, $\Omega_{,}, \Omega_{,,}, \omega$, and $\Omega, \Omega_{,,}, \omega$.

(γ) The sides of these three triangles also pass through the vertices of the given triangle ABC.

(δ) The circles which circumscribe these four triangles are all equal.

It is shown in the last section that the diameter of a circle circumscribing a triangle is equal to a side of the triangle divided by the sine of the opposite angle.

But $\dfrac{\Omega\Omega_{,,}}{\sin \Omega\Omega_{,}\Omega_{,,}} = \dfrac{\Omega\Omega_{,,}}{\sin \Omega\omega\Omega_{,,}}$, since A$\omega$B is the supplement of the angle AΩB$_{,}$.

(ϵ) The triangle ABC is the orthocentric triangle of the excentral triangle $\Omega\Omega_{,}\Omega_{,,}$; and ω, the centre of the circle inscribed in it, is the orthocentre of the triangle $\Omega\Omega_{,}\Omega_{,,}$.

This is evident; for AΩ, B$\Omega_{,,}$, C$\Omega_{,}$ are perpendiculars drawn from the vertices $\Omega\Omega_{,,}\Omega_{,}$ of the excentral triangle to the opposite sides, all passing through the orthocentre ω.

225.] *Any one of the four centres of the circles of contact is the orthocentre of the triangle whose vertices are the other three centres of the circles of contact.*

Thus ω is the orthocentre of the triangle $\Omega\Omega_{,}\Omega_{,,}$, Ω is the orthocentre of the triangle $\Omega_{,}\omega\Omega_{,,}$, $\Omega_{,}$ is the orthocentre of the triangle $\Omega_{,,}\omega\Omega$, and $\Omega_{,,}$ is the orthocentre of the triangle $\Omega\omega\Omega_{,}$.

This is evident from an inspection of the figure.

226.] Since the perpendiculars drawn from the vertices of a triangle on the sides of its orthocentric triangle meet in a point (the centre), it will follow that

If twelve perpendiculars be drawn to the sides of the triangle ABC *from the four centres of the circles of contact, these perpendiculars will meet three by three in four points, and these four points will be the centres of the circles which circumscribe the four excentral triangles.*

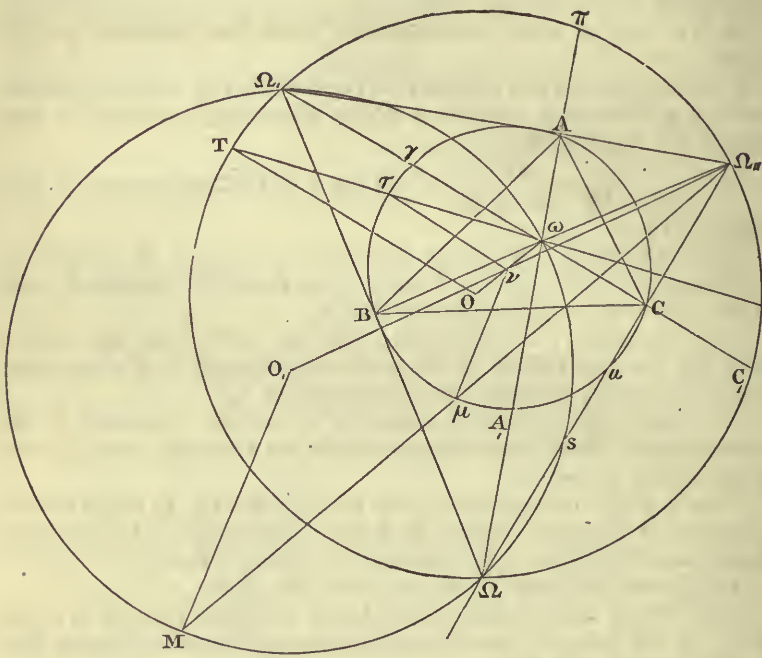
This follows from sec. [195]; for the perpendiculars on the sides of the common orthocentric triangle from the four centres of the circles of contact make equal angles with the sides of the triangles $\Omega\Omega_{,}\Omega_{,,}$, $\Omega\omega\Omega_{,}$, $\Omega_{,}\omega\Omega_{,,}$, and $\Omega_{,,}\omega\Omega$, and therefore the product of their sines taken three by three are equal.

227.] Since the given triangle ABC is the orthocentric triangle of the triangles $\Omega\Omega_{,}\Omega_{,,}$, $\Omega\omega\Omega_{,}$, $\Omega_{,}\omega\Omega_{,,}$, and $\Omega\omega\Omega_{,,}$, the radius of the circle which circumscribes ABC is one half the radius of the circle $\Omega\Omega_{,}\Omega_{,,}$, or its equals $\Omega\omega\Omega_{,}$, $\Omega_{,}\omega\Omega_{,,}$, and $\Omega_{,,}\omega\Omega$.

228.] *The nine-point circle* ABC *bisects all the vectors drawn from the orthocentre to the circumferences of the circles which circumscribe the given triangles* $\Omega\Omega_{,}\Omega_{,,}$, $\Omega\omega\Omega_{,}$, $\Omega_{,}\omega\Omega_{,,}$, and $\Omega\omega\Omega_{,,}$.

Let ω (fig. 35) be the orthocentre of the nine-point circle ABC to the triangle $\Omega\Omega_{,}\Omega_{,,}$. Let ν be the centre of the nine-point circle;

Fig. 35.



therefore ν is the middle point of the line $O\omega$, as shown in the preceding section; and as the radius of the circle which circumscribes the triangle $\Omega\Omega_{,}\Omega_{,,}$ is twice that of the nine-point circle ABC, OT is equal to twice $\nu\tau$; hence OT is parallel to $\nu\tau$; and therefore $\omega\tau = \tau T$; consequently $\omega\gamma = \gamma\Omega_{,}$, $\omega A = A\pi$, $\omega C = CC_{,}$, $\omega A_{,} = A_{,}\Omega$.

If we take the triangle $\Omega\omega\Omega_{,}$ of which $\Omega_{,,}$ is the orthocentre, and $O_{,}$ the centre of the circle circumscribing it, then, as $O_{,}M$ is equal to twice $\nu\mu$ and $\Omega_{,,}O_{,}$ is equal to twice $\Omega_{,,}\nu$, the triangles $\Omega_{,,}\nu\mu$ and $\Omega_{,,}O_{,}M$ are similar. Hence $\Omega_{,,}\mu = M\mu$. Thus the nine-point circle bisects all the vectors drawn from $\Omega_{,,}$ the orthocentre to the circumference of the circle which circumscribes the triangle $\Omega\omega\Omega_{,}$.

229.] *The lines drawn from the orthocentres of the four excentral triangles to the centres of the circles which circumscribe these triangles, all four pass through the centre of the nine-point circle.*

This is evident; for ωO, $\Omega_{,,}O_{,}$, &c. all pass through ν.

230.] *If from the centres Ω, $\Omega_{,}$, $\Omega_{,,}$ of the circles of contact straight lines be drawn to the middle points of the opposite sides of the triangle* ABC, *these lines being produced will meet in a point.*

In fig. 34 let I be the middle point of the side BC. Then the area of the triangle ΩBI is equal to that of the triangle ΩCI. But twice the area of the triangle ΩBI is equal to $\Omega I.\Omega B.\sin B\Omega I$, and twice the area of the triangle ΩCI is equal to $\Omega I.\Omega C.\sin C\Omega I$.

Hence $\Omega I.\Omega B.\sin B\Omega I=\Omega I.\Omega C.\sin C\Omega I$; or, dividing by ΩI, we shall have

$$\frac{\sin B\Omega I}{\sin C\Omega I}=\frac{\Omega C}{\Omega B}=\frac{\cos\beta}{\cos\gamma}.$$

Finding like expressions for the centres $\Omega_{,}$ and $\Omega_{,,}$ we shall have

$$\frac{\sin B\Omega I.\sin C\Omega_{,,}I_{,}.\sin A\Omega_{,}I_{,,}}{\sin C\Omega I.\sin B\Omega_{,}I_{,,}.\sin A\Omega_{,,}I_{,}}=\frac{\cos\beta\cos\gamma\cos\alpha}{\cos\gamma\cos\alpha\cos\beta}=1.$$

But it has been shown in sec. [153] that when three lines are drawn from the vertices of a triangle, making with each side pairs of angles so that the continued product of the three sines of the angles of one triad is equal to the continued product of the three sines of the angles of the alternate triad, these lines will meet in a point.

ON THE RADICAL CIRCLES OF A TRIANGLE.

231.] If on the six lines, as diameters, which join, two by two, the four centres of the circles of contact of a triangle, namely $\omega\Omega$, $\omega\Omega_{,}$, $\omega\Omega_{,,}$, $\Omega\Omega_{,}$, $\Omega_{,}\Omega_{,,}$, $\Omega\Omega_{,,}$, six circles be described, it may be shown that the centres of these circles (see fig. 36) range along the circumference of the circle ABC.

Dividing these diameters into two sets, those which end in the orthocentre ω, and those which end in the centres Ω, $\Omega_{,}$, $\Omega_{,,}$ of the external circles of contact, and which may be called the *inner* and *outer* diameters, the centres of the inner radical circles are on the middle points N, $N_{,}$, $N_{,,}$ of the arcs AB, BC, CA, while the centres of the outer circles are on the points of bisection M, $M_{,}$, $M_{,,}$ of the supplemental arcs of AB, BC, CA; so that the six centres of the radical circles are on the circumference of the circle ABC, and on its three diameters which are perpendicular to the sides of the triangle ABC.

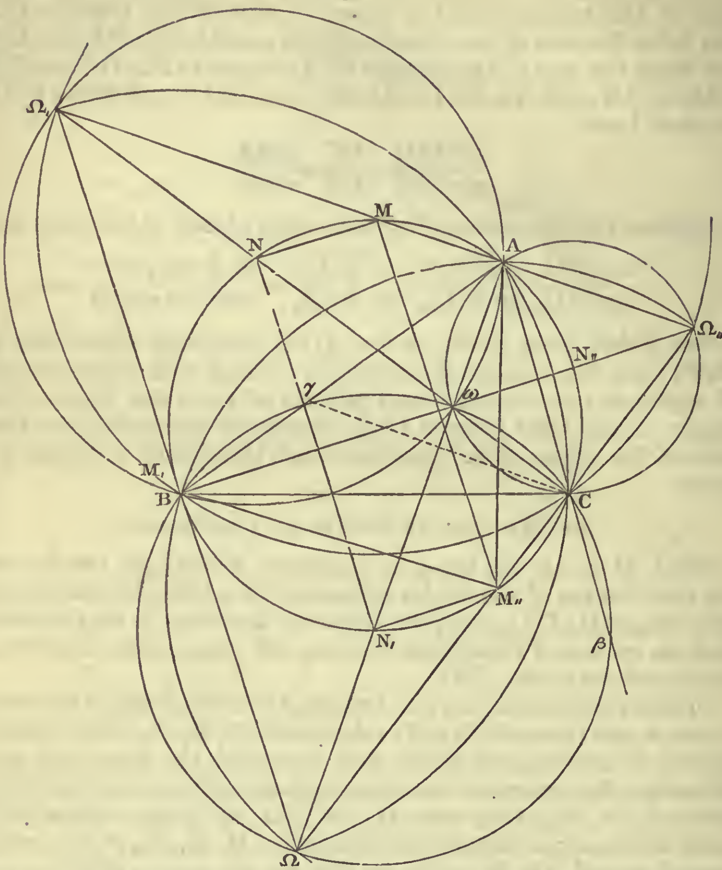
The sides of the triangle are radical axes of each pair of *outer* and *inner* circles, while the orthocentric perpendiculars are radical axes of each pair of *inner* circles.

If from any angle Ω of the excentric triangle tangents be drawn to the circles $\Omega_{,}$BA and $\Omega_{,,}$CA, these tangents will be equal; for their squares are manifestly equal to the rectangle $A\Omega\omega$.

It is evident that ω is the radical centre of the three circles.

Since ω is the orthocentre of the triangle $\Omega\Omega_{,}\Omega_{,,}$, $\omega N_{,}=N\Omega_{,,}$ and $\omega N=N\Omega$; therefore $N_{,}N$ is one half of $\Omega\Omega_{,}$ and parallel to it. In like manner since $\Omega_{,,}$ is the orthocentre of the triangle $\Omega\omega\Omega_{,}$, $\Omega_{,,}M=M\Omega_{,}$ and $\Omega_{,,}M_{,,}=\Omega M_{,,}$; therefore $MM_{,,}$ is one half

Fig. 36.



of $\Omega\Omega_{\prime}$ and parallel to it. Hence $MM_{\prime\prime}=NN_{\prime}$. In like manner $NM=N_{\prime}M_{\prime\prime}$, since each is equal to one half $\Omega_{\prime\prime}\omega$, and $NMM_{\prime\prime}N_{\prime}$ is obviously a rectangle of which the sides are

$$2R\cos\tfrac{1}{2}B \text{ and } 2R\sin\tfrac{1}{2}B.$$

232.] In sec. [216] it has been shown that

$$\Omega\Omega_{\prime\prime}=4R\cos\tfrac{1}{2}A,\ \Omega\Omega_{\prime}=4R\cos\tfrac{1}{2}B,\ \Omega_{\prime}\Omega_{\prime\prime}=4R\cos\tfrac{1}{2}C, \qquad (a)$$

and in sec. [217] that

$$\Omega_{\prime}\omega=4R\sin\tfrac{1}{2}A,\ \Omega_{\prime\prime}\omega=4R\sin\tfrac{1}{2}B,\text{ and } \Omega\omega=4R\sin\tfrac{1}{2}C. \quad (b)$$

If we square these expressions and add them, two by two, we shall have

$$\left.\begin{array}{r}\overline{\Omega\Omega_{\prime\prime}}^2+\overline{\Omega_{\prime}\omega}^2=16R^2,\quad \overline{\Omega\Omega_{\prime}}^2+\overline{\Omega_{\prime\prime}\omega}^2=16R^2,\\ \text{and } \overline{\Omega_{\prime}\Omega_{\prime\prime}}^2+\overline{\Omega\omega}^2=16R^2.\end{array}\right\} \quad . \quad . \quad (c)$$

Therefore *the square of a side of a triangle, and the square of the distance of its orthocentre from the opposite vertex are together equal to the square of the diameter of the circumscribing circle.*

In sec. [216] it has been shown that, if S denote the semiperimeter of the excentral triangle,

$$S=2R(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C).$$

So also, if $S_{,}$ denote half the sum of the three lines drawn from the orthocentre ω to the vertices of the excentral triangle,

$$S_{,}=2R(\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C);$$

such are the geometrical interpretations of these trigonometrical expressions.

If we square the expressions in (a) and (b) and add them, we shall have

$$\overline{\Omega\Omega_{,,}}^2+\overline{\Omega\Omega_{,}}^2+\overline{\Omega_{,}\Omega_{,,}}^2=16R^2(\cos^2\tfrac{1}{2}A+\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C);$$

and this expression becomes by reduction $8R(4R+r)$.

In like manner we have

$$\overline{\Omega_{,}\omega}^2+\overline{\Omega_{,,}\omega}^2+\overline{\Omega\omega}^2=8R(4R-r).$$

These expressions when added give the result obtained in (c).

Hence the sum of the squares of the sides of the excentral triangle is equal to $8R(4R+r)$, and the sum of the squares of the lines drawn from these vertices to ω is equal to $8R(4R-r)$.

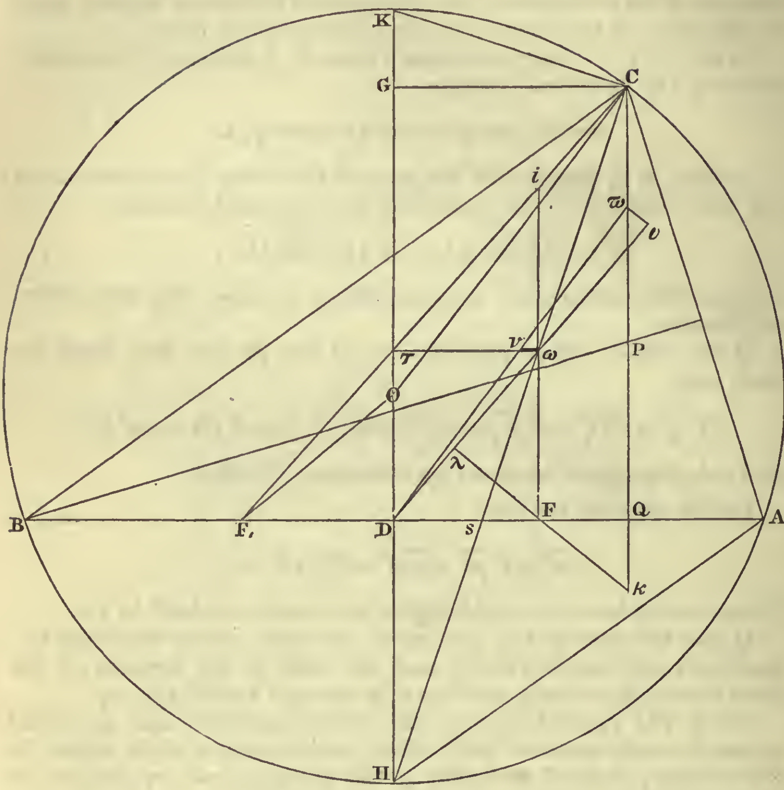
233.] *The radical axes of the circles inscribed and exscribed to any triangle intersect each other, two by two, at right angles, in the middle points of the sides of the triangle, and are parallel to the sides of the principal excentral triangle.*

Let ABC be any triangle, ω, Ω, $\Omega_{,}$, $\Omega_{,,}$ the centres of the inscribed and exscribed circles; then *the twelve circles described about the component triangles of the complete quadrilaterals* $\Omega\Omega_{,}\omega\Omega_{,,}$, $\Omega_{,}\Omega\omega\Omega_{,,}$, and $\Omega_{,,}\Omega_{,}\omega\Omega$ *will intersect four and four in* ABC, *and their centres will lie two and two in six points on the circumscribing circle.*

234.] *The nine-point circle of a triangle touches the inscribed and the three exscribed circles.*

Let O (fig. 37) be the centre of the circle circumscribing the triangle ABC, and let ν be the centre of the nine-point circle which passes through D the middle point of AB, and through ϖ the middle point of PC. Then $D\varpi=R$ the radius of the circumscribed circle. Let ω be the centre of the inscribed circle whose radius is r, and which touches the base AB in the point F. Let Q be the foot of the perpendicular CP on AB. Join $D\omega$, and let fall on it the per-

Fig. 37.



pendicular $\varpi\upsilon$. Let the distance $\nu\omega$ between the centres of the nine-point circle and the inscribed circle be d, and let ϵ be the angle between $D\nu$ and $D\omega$; then, since $\overline{D\omega}^2=\frac{(a-b)^2}{4}+r^2$,

$$d^2=\frac{R^2}{4}+\frac{(a-b)^2}{4}+r^2-2\frac{R}{2}\,\frac{\sqrt{(a-b)^2+4r^2}}{2}\cos\epsilon,$$

or, putting k for $F_i i=\sqrt{(a-b)^2+4r^2}$,

$$4d^2=R^2+(a-b)^2+4r^2-2Rk\cos\epsilon. \quad . \quad . \quad . \quad . \quad (a)$$

As $2DQ=a\cos B-b\cos A=\frac{(a^2-b^2)}{c}$, and $DF=\frac{a-b}{2}$,

$$DQ-DF=FQ=\frac{(a^2-b^2)}{2c}-\frac{(a-b)}{2}=\frac{(a-b)(s-c)}{c}. \quad . \quad (b)$$

On $D\omega$ let fall the perpendicular $F\lambda$ and produce it to meet PQ in κ. Now $Q\kappa = FQ \tan QF\kappa$. But as in (b)

$$FQ = \frac{(a-b)(s-c)}{c} \text{ and } \tan QF\kappa = \frac{(a-b)}{2r}, \text{ substituting}$$

$$Q\kappa = \frac{(a-b)^2(s-c)}{2cr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Let the angle $KCG = KHC = \theta$,

then $KG = KC \sin\theta$, and $KC = 2R \sin\theta$,

or $$KG = 2R \sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

But $$\sin^2\theta = \frac{\overline{\tau\omega}^2}{\overline{H\omega}^2} = \frac{(a-b)^2}{4\overline{HA}^2} = \frac{(a-b)^2}{4.2R.HD},$$

or $$2R \sin^2\theta = \frac{(a-b)^2}{4\frac{1}{2}c \tan \frac{1}{2}C} = \frac{(a-b)^2(s-c)}{2cr}; \quad . \quad . \quad . \quad (e)$$

consequently

$$KG = \frac{(a-b)^2(s-c)}{2cr} \text{ or } Q\kappa = KG. \quad . \quad . \quad . \quad . \quad (f)$$

Hence, as $Q\kappa = KG$, $C\kappa = KD$, and $\varpi\kappa = R$.

This is a new as well as an important property of the circle.

As $D\upsilon$ is the projection of $D\varpi$ or R on the line $D\omega$, and as it is also the projection of $\varpi\kappa$ or R and DF on the same straight line, we shall have

$$R \cos\epsilon = R \sin\delta + \tfrac{1}{2}(a-b)\cos\delta, \quad . \quad . \quad . \quad . \quad . \quad (g)$$

putting δ for the angle ωDF.

Now $\sin\delta = \dfrac{2r}{k}$, $\cos\delta = \dfrac{a-b}{k}$, where $k = \sqrt{(a-b)^2 + 4r^2}$.

Hence $2Rk \cos\epsilon = 4rR + (a-b)^2$.

Substituting this value of $2R\cos\epsilon$ in (a), we shall obtain

$$4d^2 = R^2 + 4r^2 - 4Rr.$$

Reducing, this becomes $d = \frac{1}{2}R - r$. . . . . (h)

235.] Let $d_{\prime}$, $d_{\prime\prime}$, $d_{\prime\prime\prime}$ denote the distances of the centre ν of the nine-point circle from the centres of the exscribed circles; we shall then have by making the necessary transformation of the figure,

$$d_{\prime} = \tfrac{1}{2}R + r_{\prime},\ d_{\prime\prime} = \tfrac{1}{2}R + r_{\prime\prime},\ d_{\prime\prime\prime} = \tfrac{1}{2}R + r_{\prime\prime\prime}; \quad . \quad . \quad . \quad (a)$$

adding these results, we shall have

$$d_{\prime} + d_{\prime\prime} + d_{\prime\prime\prime} + d = 2R + r_{\prime} + r_{\prime\prime} + r_{\prime\prime\prime} - r. \quad . \quad . \quad . \quad (b)$$

Now it has been shown in sec. [192] that if $D, D_{,}, D_{,,}, D_{,,,}$ denote the distances of the centre of the circumscribing circle to the same four points,

$$D^2+D_{,}^2+D_{,,}^2+D_{,,,}^2=4R^2+2R(r_{,}+r_{,,}+r_{,,,}-r) \; ; \quad . \quad (c)$$

consequently

$$D^2+D_{,}^2+D_{,,}^2+D_{,,,}^2=2R(d+d_{,}+d_{,,}+d_{,,,}). \quad . \quad . \quad . \quad (d)$$

Hence the sum of the squares of the distances of the centre of the circle circumscribing a triangle to the centres of the inscribed and exscribed circles divided by the diameter is equal to the sum of the distances of the centre of the nine-point circle to the same four points.

Another proof of this important theorem may be given.

236.] Let ABC be the given triangle as before, circumscribed by the circle whose radius is R, and whose centre is at O. Let F and $F_{,}$ be the points in which the inscribed and exscribed circles touch the base AB or c.

Then $\qquad BF_{,}=s-a, \quad AF=s-b.$

Let ν be the centre of the nine-point circle, and ω that of the inscribed circle; join $CF_{,}$. It may easily be shown that this line $CF_{,}$ or $f_{,}$ will pass through i the extremity of that diameter of the inscribed circle which passes through F its point of contact with AB. Let $D\varpi$ be the diameter of the nine-point circle; then, as OD is equal and parallel to $C\varpi$, OC or R is equal and parallel to $D\varpi$, and as $r : DF=2r : FF_{,}$, $D\varpi$ is parallel to $CF_{,}$ or to $f_{,}$, writing $f_{,}$ for $CF_{,}$. Hence the angle $OCF_{,}$ is equal to the angle $\nu D\omega$. Let this angle as before be ϵ, and let $OF_{,}$ be u; then,

since AOB is an isosceles triangle, $R^2=u^2+(s-a)(s-b)$. . (a)

$$\text{But} \quad u^2=R^2+f_{,}^2-2Rf_{,}\cos\epsilon \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

consequently

$$2f_{,}R\cos\epsilon=f_{,}^2+(s-a)(s-b). \quad . \quad . \quad . \quad . \quad (c)$$

Let δ be the angle which $CF_{,}$ or $f_{,}$ makes with AB the base of the triangle; then, as $CF_{,}$ or $f_{,}$ is parallel to $D\omega$,

$$\cos\delta=\frac{a-b}{k}, \quad \text{and } f_{,}\cos\delta=FF_{,}+FQ. \quad . \quad . \quad . \quad (d)$$

But $\qquad FF_{,}=a-b,$ and $FQ=\dfrac{(a-b)(s-c)}{c}$, see (b) sec. [235];

therefore $f_{,}\cos\delta=\dfrac{s}{c}(a-b)$, and consequently

$$f_{,}=\frac{sk}{c} \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

and therefore, substituting for $f_{\prime}$ its value in (e),

$$2\mathrm{R}k\cos\epsilon=\frac{sk^2}{c}+\frac{c}{s}(s-a)(s-b). \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Now, as before in (a) sec. [234],

$$4d^2=\mathrm{R}^2+(a-b)^2+4r^2-2\mathrm{R}k\cos\epsilon\,;$$

eliminating $\cos\epsilon$ between these equations,

$$4d^2=\mathrm{R}^2+(a-b)^2+4r^2-\frac{sk^2}{c}-\frac{c}{s}(s-a)(s-b).$$

But $$\frac{c}{s}(s-a)(s-b)=cs-c(a+b)+\frac{abc}{s}.$$

Now $$\frac{abc}{s}=4\mathrm{R}r, \text{ and } -c(a+b)=c^2-2sc\,;$$

making these substitutions, the equation becomes

$$4d^2=(\mathrm{R}-2r)^2+(a-b)^2-\frac{s}{c}\left[(a-b)^2+4r^2\right]-c^2+sc,$$

or $$4d^2=(\mathrm{R}-2r)^2+\left[(a-b)^2-c^2\right]-\frac{s}{c}\left[(a-b)^2-c^2\right]-\frac{4sr^2}{c}.$$

Reducing, the final equation becomes as before,

$$d=\tfrac{1}{2}\mathrm{R}-r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

237.] The demonstration of the case when the exscribed circle touches the base of the triangle differs but little from the preceding.

Join O the centre of the circumscribing circle with F the point in which the inscribed circle touches the base; then, as before,

$$\mathrm{R}^2=u^2+(s-a)(s-b).$$

Now as CI meets the circumference of the exscribed circle in the point I the extremity of the diameter $\mathrm{F}_{\prime}\Omega$, and as $\mathrm{I}\Omega=\Omega\mathrm{F}_{\prime}$ and $\mathrm{F}_{\prime}\mathrm{D}=\mathrm{DF}$, the line $\mathrm{D}\Omega$ is parallel to CF or to f, and OC is parallel to $\mathrm{D}\nu$ as before. Let the angle OCF in the triangle OCF be put ϵ, then the angle $\Omega\mathrm{D}\nu$ in the triangle $\Omega\mathrm{D}\nu$ is $\pi-\epsilon$, since the sides of this triangle are parallel to those of the former. Now in the former triangle, putting u for OF, as in the last section,

$$u^2=\mathrm{R}^2+f^2-2\mathrm{R}f\cos\epsilon,$$

and $$\mathrm{R}^2=u^2+(s-a)(s-b).$$

But in the triangle $\Omega\mathrm{D}\nu$, putting $\Omega\nu=d_{\prime}$,

$$4d_{\prime}^2=\mathrm{R}^2+\left[(a-b)^2+4r_{\prime}^2\right]-2\mathrm{R}\left[(a-b)^2+4r_{\prime}^2\right]^{\frac{1}{2}}\cos(\pi-\epsilon)\,; \text{ (a)}$$

or writing $k_{,}$ for $[(a-b)^2+4r_{,}^2]^{\frac{1}{2}}$, and substituting for $\cos\epsilon$ the value found above, we shall have

$$4d_{,}^2=\mathrm{R}^2+k_{,}^2+\frac{(s-c)}{c}k_{,}^2+\frac{(s-a)(s-b)c}{s-c}. \quad . \quad . \quad . \quad \text{(b)}$$

Now $\quad 4r_{,}^2+(a-b)^2=k_{,}^2$, and $\dfrac{(s-a)(s-b)c}{s-c}=\dfrac{abc}{s-c}-sc$,

while $abc=4\mathrm{R}sr$. But $sr=(s-)cr_{,}$; hence $\dfrac{abc}{s-c}=4\mathrm{R}r_{,}$.

Introducing these values we shall have

$$4d_{,}^2=(\mathrm{R}+2r_{,})^2+(a-b)^2+\left(\frac{s-c}{c}\right)4r_{,}^2+\left(\frac{s-c}{c}\right)(a-b)^2-sc,$$

or $$4d_{,}^2=(\mathrm{R}+2r_{,})^2-\frac{s}{c}\left[c^2-(a-b)^2\right]+4\left(\frac{s-c}{c}\right)r_{,}^2.$$

But $\quad c^2-(a-b)^2=(c+a-b)(c+b-a)$

$$=4(s-a)(s-b), \text{ and } \left(\frac{s-c}{c}\right)4r_{,}^2=\frac{4s\,.\,s-a\,.\,s-b}{c};$$

hence the expression now becomes

$$4d_{,}^2=(\mathrm{R}+2r_{,})^2-\frac{4s}{c}(s-a)(s-b)+\frac{4s}{c}(s-a)(s-b), \quad . \quad \text{(c)}$$

or $$d_{,}=\tfrac{1}{2}\mathrm{R}+r_{,}.$$

The lines $f, f_{,}$ drawn from the vertex C of the triangle to the points of contact F, $\mathrm{F}_{,}$ in which the exscribed and inscribed circles touch the base c of the triangle are of much importance. It will be shown further on that these lines also pass through the extremities of the diameters which pass through the points of contact of the two focal spheres with the plane of the conic section—the foci. These lines may therefore be called the *vertical focals* of the conic section.

Let r and $r_{,}$ be the radii of the inscribed and exscribed circles to the base c.

Let $4r_{,}^2+(a-b)^2=k_{,}^2$, $4r^2+(a-b)^2=k^2$. Then it may easily be shown that

$$f_{,}=\frac{s}{c}k, \text{ and } f=\frac{(s-c)}{c}k_{,}.$$

If we put h and $h_{,}$ for the distances of the vertex C of the triangle to the other extremities the diameters of the inscribed and exscribed circles, we shall have

$$h_{,}=\frac{(s-c)}{c}k, \text{ and } h=\frac{s}{c}k_{,}.$$

Hence also we have $ff_{,}=hh_{,}$, or the area of the triangle $\mathrm{CFF}_{,}$ is equal to the area of the triangle iCI, i and I being the other extremities of the diameters of the inscribed and exscribed circles.

These focal lines f and $f_{\prime}$, passing through F and $F_{\prime}$, the bisector of the vertical angle of the triangle, and the perpendicular from the vertex on the base of the triangle constitute an harmonic pencil.

The distances Fs and $F_{\prime}s$ from the point s the foot of the bisector of the vertical angle are $\frac{s}{c}(a-b)$ and $\frac{s-c}{c}(a-b)$. Hence the bisector of the vertical angle divides the distance between F, $F_{\prime}$ the focal points of the triangle in the ratio of $s:s-c$; that is (as $s:s-c=r_{\prime}:r$), in the ratio of the radii of the exscribed and inscribed circles.

238.] A trigonometrical proof of this theorem may be given.

As in fig. 37, let O be the centre of the circumscribing circle, ν that of the nine-point circle, and ω the centre of the inscribed circle; and let $\nu\omega=d$.

Let the angle νDA be γ, and the angle ωDA be δ; then as Dν is parallel to the diameter 2CO, and the angle COK is equal to the difference between the angles A and B, we shall have $\frac{1}{2}\pi-\gamma=\mathrm{A}-\mathrm{B}$. The radius D$\nu$ of the nine-point circle is equal to $\frac{1}{2}$R; and

$$2\mathrm{D}\omega=[(a-b)^2+4r^2]^{\frac{1}{2}}=k. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let ϵ be the angle between the sides of the triangle Dν and Dω, then $\epsilon=\gamma-\delta$, and

$$4d^2=\mathrm{R}^2+4r^2+(a-b)^2-2\mathrm{R}k\cos(\gamma-\delta). \quad . \quad . \quad (b)$$

But as $\frac{1}{2}\pi-\gamma=\mathrm{A}-\mathrm{B}$, $\cos\gamma=\sin(\mathrm{A}-\mathrm{B})$, and $\sin\gamma=\cos(\mathrm{A}-\mathrm{B})$;

hence $\cos\epsilon=\cos(\mathrm{A}-\mathrm{B})\sin\delta+\sin(\mathrm{A}-\mathrm{B})\cos\delta. \quad . \quad . \quad . \quad . \quad (c)$

Now $\cot\delta=\dfrac{a-b}{2r}$, and $\cot\frac{1}{2}\mathrm{B}-\cot\frac{1}{2}\mathrm{A}=\dfrac{s-b}{r}-\dfrac{s-a}{r}=\dfrac{a-b}{r}$;

therefore $$2\cot\delta=\cot\tfrac{1}{2}\mathrm{B}-\cot\tfrac{1}{2}\mathrm{A}.$$

Hence $$\tan\delta=\frac{2\sin\frac{1}{2}\mathrm{A}\sin\frac{1}{2}\mathrm{B}}{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})}.$$

Multiplying this expression by $2\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})$, we have

$$2\sin^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})\tan\delta=4\sin\tfrac{1}{2}\mathrm{A}\sin\tfrac{1}{2}\mathrm{B}\sin\tfrac{1}{2}(\mathrm{A}-\mathrm{B}),$$

or, reducing,

$$2\sin^2\tfrac{1}{2}(\mathrm{A}-\mathrm{B})\tan\delta=2\sin\mathrm{B}\sin^2\tfrac{1}{2}\mathrm{A}-2\sin\mathrm{A}\sin^2\tfrac{1}{2}\mathrm{B}. \quad (d)$$

Substituting for the squares of these sines their values in terms of the double angles, we have

$$\cos(A-B)\sin\delta+\sin(A-B)\cos\delta=\sin\delta+(\sin A-\sin B)\cos\delta.$$

Now in (c) substituting this latter value for the first, we obtain

$$4d^2=R^2+4r^2+(a-b)^2-2Rk[\sin\delta+(\sin A-\sin B)\cos\delta].$$

But $\sin\delta=\frac{2r}{k}$, $\sin A-\sin B=\frac{a-b}{2R}$, and $\cos\delta=\frac{a-b}{k}$;

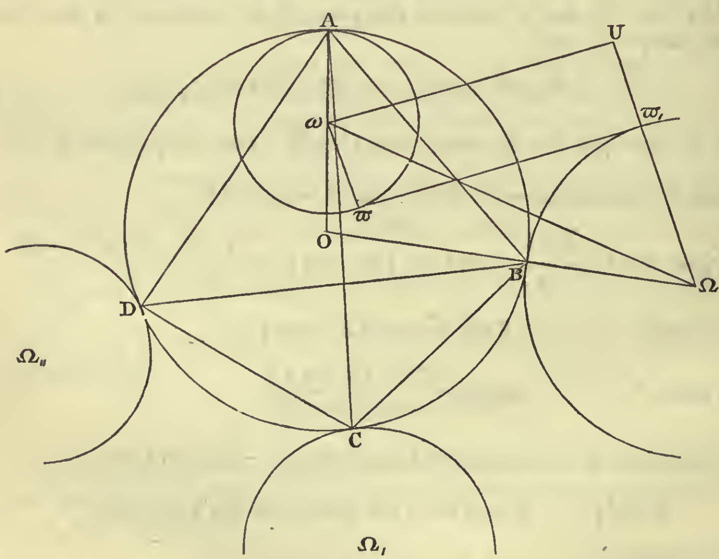
hence, making these substitutions in the preceding equation, we get

$$d=\tfrac{1}{2}R-r. \quad \ldots\ldots\ldots \quad \text{(e)}$$

239.] A proof of this theorem founded on other principles may be appropiately here given.

Four circles whose radii are r, $r_{\prime}$, $r_{\prime\prime}$, $r_{\prime\prime\prime}$ touch a fifth circle, whose radius is R, in four points A, B, C, D, all externally or all internally, or some externally and others internally. To find a general relation between these five circles and their common tangents. Let us assume the particular case of one internal and three external contacts, as in fig. 38. Let O be the centre of the

Fig. 38.



common circle of contact, and let ω, Ω, $\Omega_{\prime}$, $\Omega_{\prime\prime}$, be the centres of the four circles touching the common circle in the points A, B, C, D.

Now in any triangle of which the sides are a, b, c, we shall have, as may easily be shown,

$$4bc\sin^2\tfrac{1}{2}A = a^2 - (b-c)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But in the triangle $O\omega\Omega$ we shall have

$$4O\omega\,.\,O\Omega\,\frac{\overline{AB}^2}{4R^2} = \overline{\Omega\omega}^2 - (A\omega + B\Omega)^2, \quad . \quad . \quad . \quad . \quad (b)$$

since $$\sin^2\tfrac{1}{2}A = \frac{\overline{AB}^2}{4R^2}.$$

Let t be the common transverse tangent $\varpi\varpi_{,}$ to the circles whose centres are ω and Ω, then $t^2 = \overline{\Omega\omega}^2 - (A\omega + B\Omega)^2$; consequently

$$\overline{AB} = \frac{Rt}{\sqrt{rr_{,}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

In like manner $BC = \dfrac{Rt_{,}}{\sqrt{r_{,}r_{,,}}}$, $CD = \dfrac{Rt_{,,}}{\sqrt{r_{,,}r_{,,,}}}$, $DA = \dfrac{Rt_{,,,}}{\sqrt{r_{,,,}r}}$. (d)

Let T and $T_{,}$ be the common tangents to the opposite circles whose centres are ω and Ω as also $\Omega_{,}$ and $\Omega_{,,}$,

$$AC = \frac{RT}{\sqrt{rr_{,,}}} \text{ and } BD = \frac{RT_{,}}{\sqrt{r_{,,,}r_{,}}}.$$

Now as ABCD is a quadrilateral inscribed in a circle, we have, by Ptolemy's theorem,

$$AB\,.\,CD + BC\,.\,AD - AC\,.\,BD = 0, \quad . \quad . \quad . \quad . \quad . \quad (e)$$

substituting the preceding values found for these lines, we obtain

$$tt_{,,} + t_{,}t_{,,,} - TT_{,} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Hence we may infer that when this relation holds between the six common tangents to the four circles, they are all in contact with a fifth circle.

Now let four circles be inscribed in and exscribed to a triangle. Then in this case the four circles have three common tangents, the sides of the triangle, and on each side of the triangle there will be four points of contact, a point of contact with each of the four circles, as shown in sec. [207]. The six tangents coalesce two by two into three tangents. Each side of the triangle will be a direct tangent to two of the circles and an indirect tangent to the other two.

Let γ, $\gamma_{,}$, $\gamma_{,,}$, $\gamma_{,,,}$ be the four points of contact of the side c with the four circles. Then, as

$$\gamma\gamma_{,,,} = a - b, \text{ and } \gamma_{,}\gamma_{,,} = AC_{,} + BC_{,,} - AB = s + s - c = a + b,$$

it will follow that the product of the two tangents in the base AB or $c_{,}$ touching the four circles is $(a-b)(a+b) = a^2 - b^2$.

Therefore the sum of the products of the three sets of coincident tangents taken two by two, is

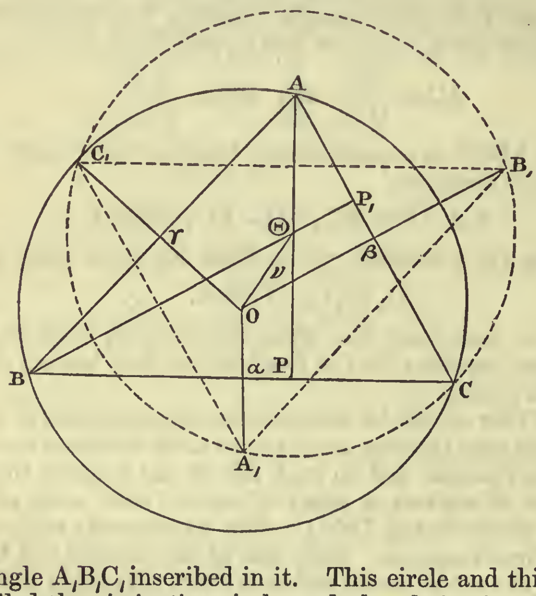
$$a^2-b^2+b^2-c^2+c^2-a^2=0. \quad . \quad . \quad . \quad . \quad . \quad (g)$$

Since this relation holds, the four circles must touch one common circle; and this circle may be easily shown to be the nine-point circle.

240.] As the triangle ABC is the nine-point circle not only to the principal excentral triangle $\Omega\,\Omega_{\prime}\,\Omega_{\prime\prime}$, but also to the other excentral triangles $\Omega\omega\Omega_{\prime}$, $\Omega_{\prime}\omega\Omega_{\prime\prime}$, $\Omega\omega\Omega_{\prime\prime}$, it follows that the nine-point circle will be in contact with the *sixteen* circles which are exscribed to and inscribed in these four triangles. This relation may be still further extended, as we now proceed to show.

Let ABC be a triangle inscribed in a circle. Let $O\alpha$, $O\beta$, $O\gamma$ be the perpendiculars drawn from the centre O on the sides a, b, c, and produced to $A_{\prime}B_{\prime}C_{\prime}$, so that $O\alpha=\alpha A_{\prime}$, $O\beta=\beta B_{\prime}$, and $O\gamma=\gamma C_{\prime}$. Through the points $A_{\prime}\,B_{\prime}\,C_{\prime}$ let a circle be described,

Fig. 39.



and a triangle $A_{\prime}B_{\prime}C_{\prime}$ inscribed in it. This circle and this triangle may be called the *derivative* circle and the *derivative* triangle of the former.

Since α and β are the middle points of CB and CA, $\alpha\beta$ is the half of AB; and as α and β are the middle points of $OA_{\prime}$ and $OB_{\prime}$, $\alpha\beta$ is the half of $A_{\prime}B_{\prime}$. Therefore $A_{\prime}B_{\prime}$ is equal to AB and is also parallel to it. In the same way it may be shown that the other sides of the two triangles are equal and parallel. Hence the circumscribing circles are equal; and while O is the centre of the given circle circumscribing the triangle, Θ its orthocentre is the centre of the derived circle. Therefore the circles interchange their centres and orthocentres. The two triangles have the same nine-point circle, whose centre is at ν the middle point of $O\Theta$.

Hence it follows that this nine-point circle touches the *thirty-two* circles which are circuminscribed* to the excentral triangles of the original triangle and its derivative.

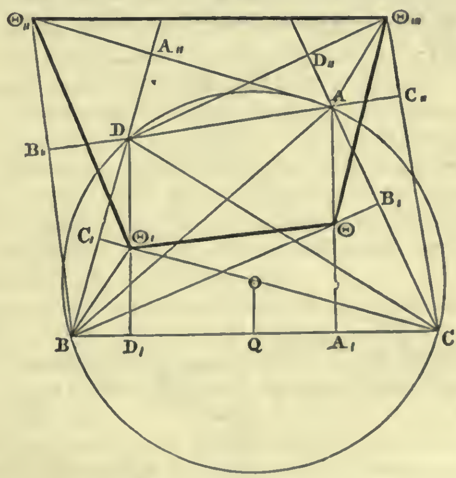
241.] *If a quadrilateral be inscribed in a circle, the orthocentres of its four constituent triangles will range on another circle equal to the former.*

Let ACBD be the quadrilateral, and let $\Theta, \Theta_{\prime}, \Theta_{\prime\prime}, \Theta_{\prime\prime\prime}$ be the orthocentres of the four constituent triangles ABC, DBC, ADB, ACD.

As $A\Theta$ and $D\Theta_{\prime}$ are parallel and equal, since each is double of OQ, therefore $\Theta\Theta_{\prime}$ is equal and parallel to AD. In the same way it may be shown that $\Theta_{\prime}\Theta_{\prime\prime}$ is equal and parallel to AC; so is $\Theta_{\prime\prime}\Theta_{\prime\prime\prime}$ equal and parallel to CB, while $\Theta\Theta_{\prime\prime\prime}$ is equal and parallel to BD.

Hence the two quadrilaterals are equal and alike in every respect, and therefore the circles in which they are inscribed are equal.

Fig. 40.



Since BD is equal and parallel to $\Theta\Theta_{\prime\prime\prime}$, and $D\Theta_{\prime}$ equal and parallel to $A\Theta$, therefore $B\Theta_{\prime}$ is equal and parallel to $A\Theta_{\prime\prime\prime}$, and $AB\Theta_{\prime}\Theta_{\prime\prime\prime}$ is a parallelogram whose diagonals $A\Theta_{\prime}$ and $B\Theta_{\prime\prime\prime}$ bisect

* A short term to denote circles one circumscribed and one inscribed in the same triangle.

each other. Hence the lines joining the corresponding points of the two quadrilaterals all pass through the same point.

242.] Let the derivative circle be taken, and the four derivative triangles inscribed in it. Since the four original triangles are inscribed in the same circle, and have four orthocentres, the derivative group will have only one orthocentre for the four derivative triangles, and these triangles will be circumscribed each by a distinct circle. There will be four nine-point circles, whose centres will be the middle points of the lines joining the common orthocentre with the four centres of the derived circles.

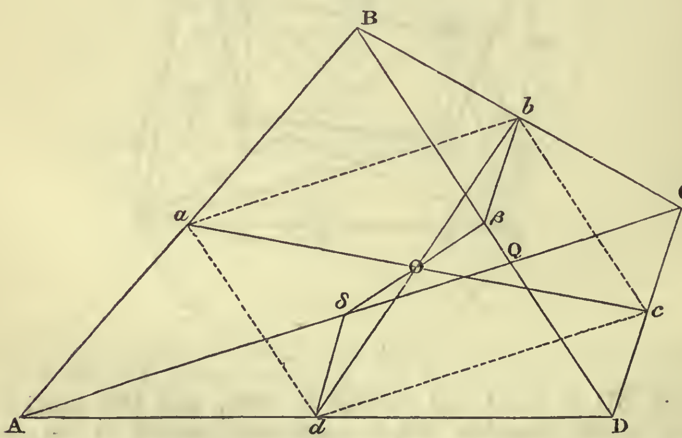
Hence these four nine-point circles will be in contact with the *hundred and twenty-eight circles of contact*, and every vector drawn from this common orthocentre to the circumferences of these one hundred and twenty-eight circles of contact will be bisected by one or other of the four nine-point circles.

CHAPTER XXVI.

ON SOME ELEMENTARY PROPERTIES OF QUADRILATERALS.

243.] (α) *If the middle points of the opposite sides of a quadrilateral be joined, their intersection* O *will lie in the line joining the middle points of the diagonals, and these three lines will mutually bisect each other.*

Fig. 41.



Let a, b, c, d be the middle points of the sides of the quadrilateral ABCD. Then ab is the half of the diagonal AC and parallel to it. Therefore $abcd$ is a parallelogram, and its diagonals ac, bd are therefore bisected in O. Since $b\beta$ and $d\delta$ are each the half of CD and parallel to it, $b\beta = d\delta$, and therefore $\beta O = \delta O$.

(β) *The sum of the squares of any two opposite sides of a quadrilateral, together with twice the square of the line joining their middle points is constant;*

that is $$\overline{AB}^2 + \overline{CD}^2 + 2\overline{ac}^2 = \overline{AD}^2 + \overline{CB}^2 + 2\overline{bd}^2.$$

(γ) Hence also

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 + \overline{AC}^2 + \overline{BD}^2 = 4[\overline{ac}^2 + \overline{bd}^2 + \overline{\beta\delta}^2];$$

that is, *in any tetrahedron the sum of the squares of the six edges is equal to four times the squares of the lines joining the middle points of the opposite edges.*

(δ) We have also $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{\beta\delta}^2$;

that is, *the sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of the two diagonals, together with four times the square of the line joining the middle points of the two diagonals.*

ON QUADRILATERALS INSCRIBED IN ONE CIRCLE AND CIRCUMSCRIBED ABOUT ANOTHER.

244.] In that very celebrated and highly original work, the '*Traité des propriétés projectives*' of Poncelet (pp. 260–283) the very elegant properties of circles inscribed in and circumscribed to the same quadrilateral are treated with much originality. In fact the discovery of those elegant properties is due to Poncelet. The methods of investigation, however, which he has used, have not hitherto been admitted into elementary geometry. As these properties deserve to be better known, and admit of rigorous geometrical demonstration, they should take their place in every treatise of pure geometry. We shall first, by way of preface, state some of those properties of quadrilaterals in connexion with circles which are elementary and have been long known.

(α) *In every quadrilateral inscribed in a circle the sum of the opposite angles is equal to two right angles.*

(β) *In every quadrilateral inscribed in a circle the rectangle under the segments of one of the diagonals is equal to the rectangle under the segments of the other.*

(γ) *In every quadrilateral so inscribed the rectangle under the diagonals is equal to the sum of the rectangles under the two pairs of opposite sides, and the diagonals are to each other as the sums of the rectangles under the sides which terminate in the extremities of these diagonals.*

When, moreover, the diagonals of the inscribed quadrilateral are at right angles we shall have the following properties:—

(δ) *The sum of the squares of the four sides is double the square of the diameter.*

(ϵ) *The sum of the squares of the four segments of the diagonals is equal to the square of the diameter*; and

(ζ) *The sum of the squares of the two diagonals is equal to the square of the diameter diminished by four times the square of the distance between the centre and the point in which the diagonals intersect.*

(η) *If circles be described on the three diagonals of a complete quadrilateral inscribed in a circle, they will have the same radical axis, and the orthocentres of the four component triangles of the complete quadrilateral range on the same straight line.*

And with respect to quadrilaterals circumscribed about a circle, it is easy to show that

(θ) *The sum of two opposite sides is equal to the sum of the two others.*

(ι) *In any quadrilateral circumscribed to a circle, the sum of any two opposite angles is equal to twice the external angle of one of component quadrilaterals into which the given quadrilateral is divided by the two chords.*

When these chords are at right angles the external angles of the component quadrilaterals are right angles; therefore the sum of the opposite angles of the circumscribing quadrilateral is equal to two right angles, or the quadrilateral circumscribing the circle may also be inscribed in a circle.

The proof is very simple, and depends on the equality of the angles which a chord of a circle makes with the tangents at its extremities.

245.] If two quadrilaterals are the one inscribed and the other circumscribed to the same circle, so that the vertices of the inscribed may be on the points of contact of the circumscribed quadrilateral,

(α) *The chords which join the points of contact of the circumscribed quadrilateral will be at right angles.*

(β) *The diagonals of the two quadrilaterals will cut all four in the same point.*

(γ) *The points of concourse of the opposite sides of the two quadrilaterals will range all four on the same straight line*; and

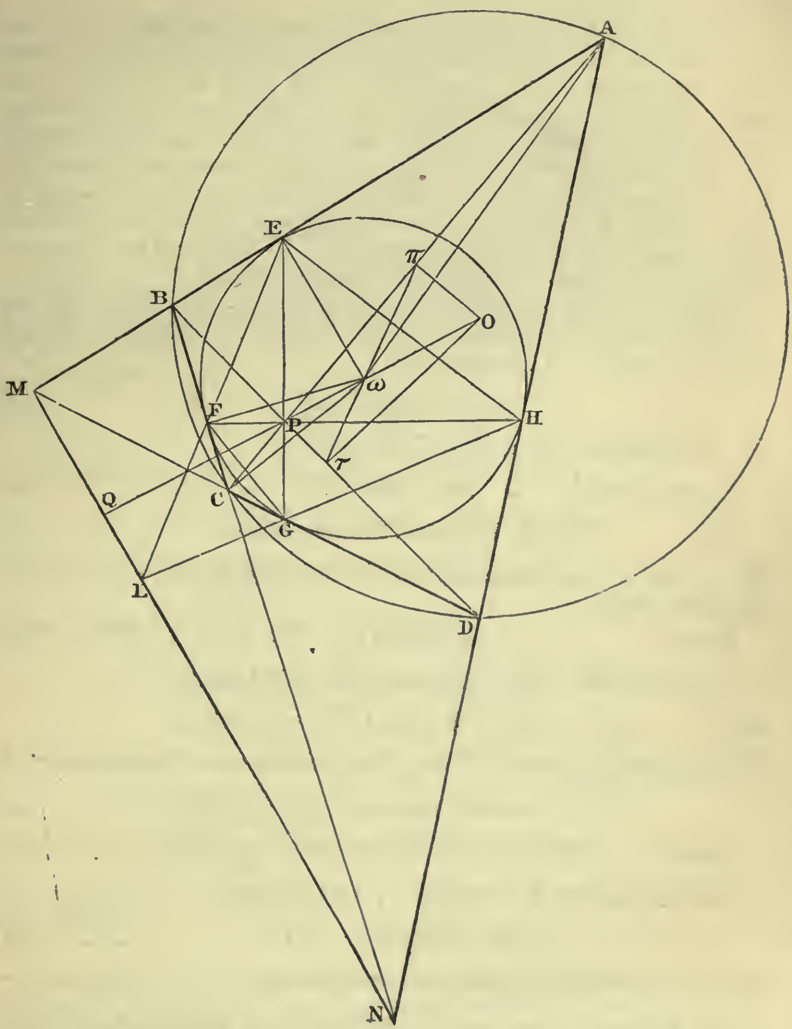
(δ) *The point of intersection of the four diagonals will be the pole of the straight line which contains the points in which the opposite sides of the quadrilaterals intersect.*

(ϵ) *The diagonals* EG *and* FH *of the inscribed quadrilateral meet the intersection of the lines joining the points of contact of the circumscribed quadrilateral; and the angles between the former are bisected by the latter.*

We shall now proceed to establish the foregoing theorems, beginning with the last (ϵ).

As the angle CBD is equal to the angle CAD, and the angle BFP equal to AHP, therefore the triangles BFP and APH are

Fig. 42.



similar; therefore BF or BE : BP as AH or AE : AP. Consequently BE : AE = BP : AP, or in the triangle APB the angle APB is bisected by PE. In the same way it may be shown that the other angles between the diagonals of the inscribed quadrilateral are so bisected.

Hence also the chords of contact EG and FH are at right angles.

246.] *The diagonals of the circumscribed quadrilateral will pass through the pole* P.

Through E and G let tangents be drawn intersecting in M, then M is the pole of EG; in like manner N is the pole of HF; hence MN is the polar of P. Let EF and GH meet in L; then, as the polar of L must pass through P, the point L must be on the line MN; and as L is a point in EF, the polar of L must pass through B; and as L is a point in GH, the polar of L must pass through D; and as L is a point in MN, the polar of L must pass through P. Hence BPD is a straight line, the diagonal of the circumscribed quadrilateral; and it passes through P. In the same way it may be shown that the other diagonal AC passes through P.

247.] Since the angle $E\omega H$ is equal to the angle FCG, the half of $E\omega H$ is equal to the half of FCG; hence the triangles $AE\omega$ and $FC\omega$ are similar. Consequently

$$AE \,.\, FC = \overline{E\omega}^2 = r^2, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

if r be the radius of the inscribed circle.

In like manner $$BF \,.\, DH = r^2. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a_{,})$$

Let $$AE = a,\ BF = b,\ CG = c,\ DH = d, \quad . \; . \; . \; . \quad (b)$$

the radius of the circumscribed circle being R, and r that of the inscribed circle.

Hence $$ac = bd = r^2. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (c)$$

We have also $$AP : AE = \sin AEP : \sin \tfrac{1}{2} BPA,$$

and $$CP : CG = \sin AEP : \sin \tfrac{1}{2} BPA.$$

Let $AP = n \,.\, AE$, and $CP = n \,.\, CG$, writing n for the quotient of

$$\sin AEP \text{ divided by } \sin \tfrac{1}{2} BPA. \quad . \; . \; . \quad (d)$$

Hence $$AC = n(AE + CG) \text{ or } AC = n(a + c). \quad . \; . \; . \quad (e)$$

In like manner $BD = n(b + d)$; and therefore

$$AC : BD = a + c : b + d. \quad . \; . \; . \; . \; . \; . \; . \quad (f)$$

But $AC \,.\, BD = (a + b)(c + d) + (a + d)(b + c)$,

or, by Ptolemy's theorem, $AC \,.\, BD = 4r^2 + (a + c)(b + d)$, . . (g)

since $$ac = bd = r^2.$$

Multiply this expression by $\dfrac{AC}{BD} = \dfrac{a + c}{b + d}$, and we shall have

$$\overline{AC}^2 = (a + c)^2 + 4r^2 \frac{(a + c)}{(b + d)} \text{ and } \overline{BD}^2 = (b + d)^2 + \frac{4r^2(b + d)}{a + c}.$$

Since $AP = na$, and $CP = nc$, we have

$$\frac{AC}{AP} = \frac{a+c}{a}, \text{ and } \overline{AP}^2 = \overline{AC}^2 \left(\frac{a}{a+c}\right)^2, \quad . \quad . \quad . \quad . \quad \text{(h)}$$

$$\left.\begin{array}{l} \overline{AP}^2 = a^2 + \dfrac{4a^2r^2}{(a+c)(b+d)}, \quad \overline{CP}^2 = c^2 + \dfrac{4c^2r^2}{(a+c)(b+d)}, \\ \overline{BP}^2 = b^2 + \dfrac{4b^2r^2}{(a+c)(b+d)}, \quad \overline{DP}^2 = d^2 + \dfrac{4d^2r^2}{(a+c)(b+d)}. \end{array}\right\} . \quad \text{(i)}$$

Let O be the centre of the circle circumscribing the quadrilateral ABCD, ω the centre of the inscribed circle, P the common pole, and let the straight line $O\omega P$ meet the common polar MN in Q; then we shall have

$$\frac{1}{O\omega} + \frac{1}{Q\omega} = \frac{1}{P\omega}.$$

To show this, in the triangle $A\omega P$ we have

$$\overline{P\omega}^2 = \overline{A\omega}^2 + \overline{AP}^2 - 2A\omega \, . \, AP \, . \cos PA\omega,$$

and $$\overline{C\omega}^2 = \overline{CA}^2 + \overline{A\omega}^2 - 2CA \, . \, A\omega \cos PA\omega;$$

hence, eliminating $\cos PA\omega$, we obtain

$$\overline{P\omega}^2 = \overline{A\omega}^2 + \overline{AP}^2 - AP \, . \, AC + \frac{AP}{AC} (\overline{C\omega}^2 - \overline{A\omega}^2).$$

Now $\overline{A\omega}^2 = a^2 + r^2$, $\overline{C\omega}^2 = c^2 + r^2$, $\overline{AP}^2 = a^2 + \dfrac{4a^2r^2}{(a+c)(b+d)}$; hence

$$\overline{P\omega}^2 = r^2 + 2a^2 + \frac{4a^2r^2}{(a+c)(b+d)} - a\left[(a+c) + \frac{4r^2}{(b+d)}\right] - a(a-c),$$

or, reducing, $$r^2 - \overline{P\omega}^2 = \frac{4r^4}{(a+c)(b+d)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(j)}$$

From O the centre of the circumscribing circle draw the perpendiculars $O\pi$ and $O\tau$ on the diagonals AC and BD; then π and τ are the middle points of AC and BD. Hence, by Newton's theorem given in page 283, the line $\pi\tau$ passes through ω the centre of the inscribed circle; and as $OP\pi\tau$ is a quadrilateral that may be inscribed in a circle,

$$O\omega \, . \, P\omega = \omega\pi \, . \, \omega\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(k)}$$

Now as the sum of the squares of the four sides of a quadrilateral is equal to the sum of the squares of the diagonals and four times the square of the distance between the middle points of these diagonals, as shown in (δ) sec. [243],

$$(a+b)^2+(b+c)^2+(c+d)^2+(d+a)^2=\overline{AC}^2+\overline{BD}^2+4\overline{\pi\tau}^2.$$

But $$4\overline{\pi\tau}^2=4\overline{\omega\pi}^2+4\overline{\omega\tau}^2+8\overline{\omega\pi}\times\overline{\omega\tau},$$

and $$8\omega\pi\,.\,\omega\tau=8\mathrm{O}\omega\,.\,\mathrm{P}\omega.$$

Hence $$(a+b)^2+(b+c)^2+(c+d)^2+(d+a)^2$$
$$=2\left[\overline{A\omega}^2+\overline{B\omega}^2+\overline{C\omega}^2+\overline{D\omega}^2\right]+8\mathrm{O}\omega\,.\,\mathrm{P}\omega,$$

since $\overline{AC}^2+4\overline{\omega\pi}^2=2(\overline{A\omega}^2+\overline{C\omega}^2)$, and $\overline{BD}^2+4\overline{\omega\tau}^2=2(\overline{B\omega}^2+\overline{D\omega}^2)$.

Now $$(a+b)^2+(b+c)^2+(c+d)^2+(d+a)^2$$
$$=2(a^2+b^2+c^2+d^2)+2(a+c)(b+d);$$

and as $a^2=\overline{A\omega}^2-r^2$, $b^2=\overline{B\omega}^2-r^2$, $c^2=\overline{C\omega}^2-r^2$, $d^2=\overline{D\omega}^2-r^2$,

$$2\left[\overline{A\omega}^2+\overline{B\omega}^2+\overline{C\omega}^2+\overline{D\omega}^2\right]-8r^2+2(a+c)(b+d)$$
$$=2\left[\overline{A\omega}^2+\overline{B\omega}^2+\overline{C\omega}^2+\overline{D\omega}^2\right]+8\mathrm{O}\omega\,.\,\mathrm{P}\omega,$$

or $$(a+c)(b+d)=4r^2+4\mathrm{O}\omega\,.\,\mathrm{P}\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad (l)$$

But in the preceding paragraph it has been shown that

$$(a+c)(b+d)=\frac{4r^4}{r^2-\overline{\mathrm{P}\omega}^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (m)$$

equating these values of $(a+c)(b+d)$, we obtain the relation

$$\mathrm{O}\omega=\frac{r^2\,.\,\mathrm{P}\omega}{r^2-\overline{\mathrm{P}\omega}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (n)$$

Since $r^2=\mathrm{P}\omega\,.\,\mathrm{Q}\omega$, this expression may be reduced to

$$\frac{1}{\mathrm{O}\omega}+\frac{1}{\mathrm{Q}\omega}=\frac{1}{\mathrm{P}\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (p)$$

a simple relation between the distances of the centre ω of the inscribed circle from O the centre of the circumscribed circle, and from P and Q the pole and polar.

248.] To express R the radius of the circumscribing circle in terms of r the radius of the inscribed circle and p the distance between the centre of this circle and the common pole.

Let q be the distance between the pole P and the polar MN, and D the distance between the centres, then we shall have

$$p(p+q)=r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and $R^2=(D+p)(D+p+q)$, or, since $p+q=\frac{r^2}{p}$,

$$R^2=(D+p)\left(D+\frac{r^2}{p}\right),$$

or

$$R^2=D^2+\frac{D}{p}(r^2+p^2)+r^2. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Substituting in this expression the value of D given in (n), last section,

$$D=\frac{r^2p}{r^2-p^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

we finally obtain

$$R=\frac{r^2\sqrt{2r^2-p^2}}{(r^2-p^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

The least value of R is when the circles are concentric, or $p=0$. In this case $R=\sqrt{2}r$.

From these expressions, namely

$$D=\frac{r^2p}{r^2-p^2} \text{ and } R=\frac{r^2\sqrt{2r^2-p^2}}{r^2-p^2}, \quad . \quad . \quad . \quad . \quad (e)$$

it follows that when r and p are given, D and R are completely determined, or, however the rectangular chords of the inscribed circle may vary in position, the centre and radius of the circumscribed circle are fixed.

If we eliminate p between the preceding expressions,

$$D=[(R^2+r^2)-r\sqrt{4R^2+r^2}]^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (f)$$

249.] Hence it follows that *if through any fixed point in a given circle two rectangular chords be drawn, and at their extremities four tangents be drawn constituting a quadrilateral, this quadrilateral may be inscribed in a circle, and the centre and radius of this circle will be fixed and independent of the directions in which the rectangular chords may be drawn.*

The square of the area of the quadrilateral is equal to

$$(a+b)(b+c)(c+d)(d+a),$$

since half the sum of its sides is $(a+b+c+d)$.

Multiplying out this expression, bearing in mind that $ac=bd=r^2$, and dividing by $abcd$, we obtain the very remarkable symmetrical expression

$$\frac{\overline{\text{area}}^2}{abcd}=(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right). \quad . \quad . \quad . \quad (a)$$

In every quadrilateral which may be inscribed in one circle and circumscribed to another the centres of the two circles and the common point in which the four diagonals intersect are in a straight line.

In every such quadrilateral the distances from the vertices to the point of intersection of the diagonals are proportional to the tangents drawn from these vertices and touching the internal circle.

The diagonals are proportional to the sum of the tangents drawn from their extremities to the interior circle.

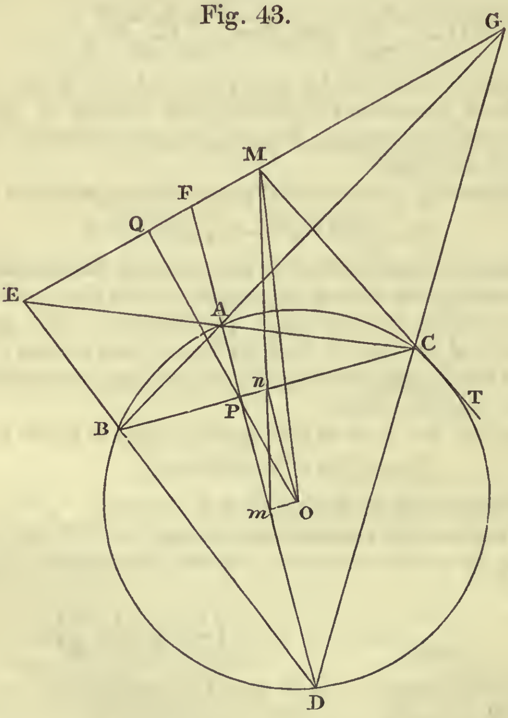
The distance PQ *between the common pole* P *and its polar* MN, *multiplied by the distance between the centres of the inscribed and circumscribing circles, is equal to the square of the radius of the inscribed circle.*

For if q be this distance, it has been shown that $p(p+q)=r^2$ or $q=\dfrac{r^2-p^2}{p}$ and $\mathrm{D}=\dfrac{r^2p}{r^2-p^2}$. Hence $\mathrm{D}q=r^2$.

250.] *If a quadrilateral be inscribed in a circle, the squares of the inner diagonals are to each other as the distances of their middle points from the middle point of the outer diagonal.* See sec. [176].

It has been shown in sec. [165] that the middle points of the three diagonals range in the same straight line.

Fig. 43.



Let ABDC be the inscribed quadrilateral. Let m and n be the middle points of the *inner* diagonals AD and BC. Let M be the middle point of the *outer* diagonal. Then M, m, n are in a straight line. P the intersection of the *inner* diagonals is the pole of the *outer* diagonal EG and O the centre of the circle. Om and On are perpendicular to the diagonals AD, BC, and they bisect them.

Since the line AD is bisected in m, and harmonically divided in P and F, as shown in (d) sec. [161], we have

$$\overline{Dm}^2 = Pm\,.\,Fm. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But $Pm = PO \sin F$, and $Fm : Mm = \sin FMn : \sin F$,

or
$$Fm = \frac{Mm\,.\,\sin FMn}{\sin F}.$$

Therefore
$$\overline{Dm}^2 = PO\,.\,\sin FMm\,.\,Mm. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

In like manner $\overline{Cn}^2 = PO\,.\,\sin FMn\,.\,Mn.$

Therefore
$$\overline{Dm}^2 : \overline{Cn}^2 = Mm : Mn. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

251.] This property will enable us to give a very simple and elegant solution of the following celebrated problem:—Given a circle and the lengths of the three diagonals of a quadrilateral to be inscribed in it, to construct the quadrilateral.

Let $2G$, $2G_{\prime}$, $2G_{\prime\prime}$ be the lengths of the three diagonals, $2G$ being greater than $2G_{\prime}$, and $2G_{\prime\prime}$ the outer diagonal; let $g = mn$.

Since $Mm : Mn = G^2 : G_{\prime}^2$,

$$Mm : Mm - Mn = G^2 : G^2 - G_{\prime}^2.$$

But $Mm - Mn = g$; therefore

$$Mm = \frac{gG^2}{G^2 - G_{\prime}^2} \text{ and } Mn = \frac{gG_{\prime}^2}{G^2 - G_{\prime}^2}. \quad . \quad . \quad . \quad . \quad (d)$$

Let ϵ be the angle O Mm; then in the triangle OmM

We have also
$$\left.\begin{aligned} \overline{Om}^2 &= \overline{OM}^2 + \overline{Mm}^2 - 2OM\,.\,Mm\cos\epsilon. \\ \overline{On}^2 &= \overline{OM}^2 + \overline{Mn}^2 - 2OM\,.\,Mn\cos\epsilon. \end{aligned}\right\} \quad . \quad . \quad . \quad (e)$$

Eliminating $\cos\epsilon$ from these expressions, we get

$$\frac{\overline{OM}^2 + \overline{Mm}^2 - \overline{Om}^2}{Mm} = \frac{\overline{OM}^2 + \overline{Mn}^2 - \overline{On}^2}{Mn}.$$

Now as the tangent drawn from M to the circle is equal to $G_{\prime\prime}$, see sec. [176], and R being the radius of the circle,

$$\overline{OM}^2 = R^2 + G_{\prime\prime}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

But we have found

$$\mathrm{M}m=\frac{g\mathrm{G}^2}{\mathrm{G}^2-\mathrm{G}_{\prime}^2},\ \mathrm{M}n=\frac{g\mathrm{G}_{\prime}^2}{\mathrm{G}^2-\mathrm{G}_{\prime}^2},\ \overline{\mathrm{O}m}^2=\mathrm{R}^2-\mathrm{G}^2,\ \mathrm{O}n=\mathrm{R}^2-\mathrm{G}_{\prime}^2;\quad (g)$$

substituting these values in the preceding equation, we get

$$\frac{g}{\mathrm{G}_{\prime\prime}}+\frac{\mathrm{G}_{\prime}}{\mathrm{G}}-\frac{\mathrm{G}}{\mathrm{G}_{\prime}}=0. \quad (h)$$

This enables us to express the distance (g) between the middle points of the *inner* diagonals in terms of the three diagonals. Hence the three sides of the triangle Omn are given, and this triangle has its vertex at O; and hence the diagonals may be drawn and the quadrilateral constructed.

The circles described on the three diagonals G, $\mathrm{G}_{\prime}$, $\mathrm{G}_{\prime\prime}$ of the quadrilateral as diameters intersect, two by two, in the same two points. Their centres, therefore, range along the same straight line, and have a common radical axis, the common chord. The distance d between the common chord of any two of the circles and the centre of one of them is given by the symmetrical formula

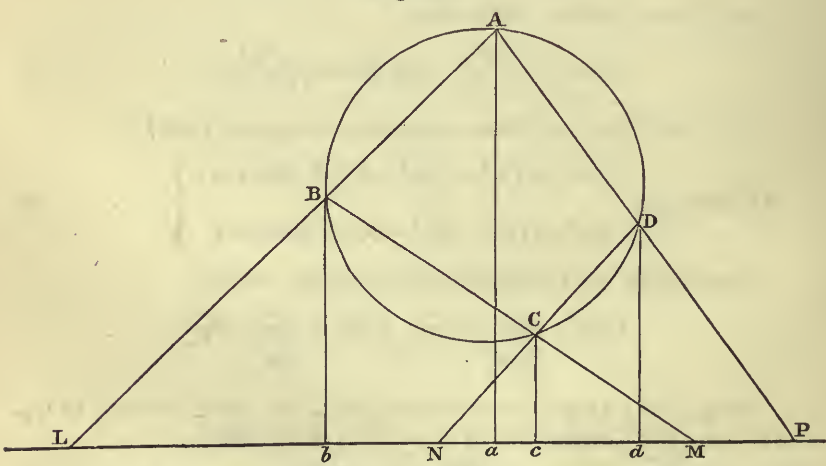
$$2d=\frac{\mathrm{G}^2\mathrm{G}_{\prime}^2+\mathrm{G}^2\mathrm{G}_{\prime\prime}^2-\mathrm{G}_{\prime}^2\mathrm{G}_{\prime\prime}^2}{\mathrm{G}\mathrm{G}_{\prime}\mathrm{G}_{\prime\prime}}. \quad (i)$$

If C be the common chord of the three circles,

$$\mathrm{C}^2=2(\mathrm{G}^2+\mathrm{G}_{\prime}^2+\mathrm{G}_{\prime\prime}^2)-\mathrm{G}^2\mathrm{G}_{\prime}^2\mathrm{G}_{\prime\prime}^2(\mathrm{G}^{-4}+\mathrm{G}_{\prime}^{-4}+\mathrm{G}_{\prime\prime}^{-4}). \quad (j)$$

252.] Let the sides of a quadrilateral inscribed in a circle be cut by a transversal, the continued product of the ratios of the segments of the sides made by the transversal will be equal.

Fig. 44.



Let L, M, N, P be the points in which the transversal is cut by the sides of the quadrilateral; from the points A, B, C, D let perpendiculars to the transversal be drawn. Let these perpendiculars be put a, b, c, d. Then we have

$$\frac{AL}{BL}=\frac{a}{b}, \quad \frac{BM}{CM}=\frac{b}{c}, \quad \frac{CN}{DN}=\frac{c}{d}, \quad \frac{DP}{AP}=\frac{d}{a}.$$

Hence $$\frac{AL.BM.CN.DP}{BL.CM.DN.AP}=\frac{abcd}{bcda}=1.$$ We have also

$$AL.BL.BM.CM.CN.DN.DP.AP=[BL.CM.DN.AP]^2.$$

Let l, m, n, p be the tangents from the points L, M, N, P.

Then $lmnp=BL.CM.DN.AP$, or $lmnp=AL.BM.CN.DP$.

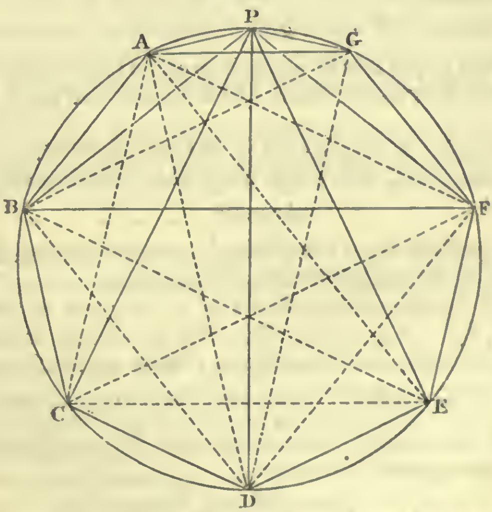
This property may be extended to inscribed regular polygons of any number of sides.

ON THE PROPERTIES OF CHORDS DRAWN FROM A POINT IN THE CIRCUMFERENCE OF A CIRCLE TO THE ANGLES OF AN INSCRIBED REGULAR POLYGON OF AN ODD NUMBER OF SIDES.

253.] When the polygon is an equilateral triangle the properties are obvious and known.

When the polygon is a pentagon. In general let the side of the polygon be put s; let the chord which subtends two adjacent sides of the polygon be t, that which subtends three consecutive sides be u, and that which subtends four sides be z, &c.

Fig. 45.



Let the chords drawn from the point P to the angles A, B, C, D, E, F, G, &c. be a, b, c, d, e, f, g, &c.

Then in the case of the pentagon we have

$$\left.\begin{array}{l} as+cs=bt, \\ es+cs=dt, \end{array}\right\} \left.\begin{array}{l} at+et=cs, \\ bs+ds=ct. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Adding these expressions together, and dividing by $(s+t)$, we have

$$a+c+e=b+d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

We shall have also $\quad c^2=(a+e)(b+d), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$

and $\quad 5c^2=(a+b+d+e)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$

254.] When the regular polygon is a heptagon, then we shall have the following twelve equations:—

$$\left.\begin{array}{l} as+cs=bt, \\ au+gu=ds, \\ bt+ft=du, \\ cs+es=dt, \end{array}\right\} \left.\begin{array}{l} at+gs=bs, \\ gt+as=fs, \\ gs+es=ft, \\ fu+bs=eu, \end{array}\right\} \left.\begin{array}{l} bu+fs=cu, \\ ct+fs=bt+ft, \\ et+bs=du, \\ (a+c+e+g)s=du. \end{array}\right\} . \quad (a)$$

Adding these twelve equations, we shall have

$$(a+c+e+g)(s+t+u)=(b+d+f)(s+t+u),$$

or, dividing by $(s+t+u)$,

$$a+c+e+g=b+d+f; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

or, in other words, the sum of the odd chords drawn from the point P to the alternate vertices of the heptagon will be equal to the sum of the even chords.

We have also $\quad d^3=(a+g)(b+f)(c+e); \quad . \quad . \quad . \quad . \quad . \quad (c)$

that is, *the cube of the middle chord is equal to the continued product of the sums of the first and seventh, of the second and sixth, of the third and fifth.*

When the point P is assumed in the middle of the arc AG, then PD is a diameter 2R, and $a=g$, $b=f$, $c=e$, and therefore

$$abc=R^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

These properties thus established may be extended to regular polygons of $(2n+1)$ sides inscribed in a circle.

Thus let M be the middle chord of a polygon of $(2n+1)$ sides, and let $C_1\,C_2\,C_3\ldots C_{2n-1}$, C_{2n}, C_{2n+1} be the chords drawn from the point P to the angles of the polygon; then we shall have

$$M^n=(C_1+C_{2n+1})(C_2+C_{2n})(C_3+C_{2n-1}) \text{ \&c.}$$

When P is the middle point of the arc, M is a diameter 2R, and the preceding expression becomes

$$R^n=C_1\,C_2\,C_3\ldots \text{\&c.}$$

A TREATISE

ON

CONICS.

CHAPTER XXVII.

Definitions.

255.] Let a straight line be drawn perpendicular to the plane of a circle through its centre, and a point in it assumed, through which a straight line of indefinite length passes, always touching the circumference of the circle; the surface thus generated is called a *cone*, the perpendicular is called the *axis of the cone*, and the fixed point the *vertex*.

The surface thus generated is divided by the vertex of the cone into two portions, which may be called the upper and lower sheets of the cone.

II.

If this surface be cut by a plane, the line in which the cone and the plane intersect is called a *conic section*, or in short a *conic*.

III.

If a sphere be inscribed in this cone touching the plane of the conic section, the point of contact is called a *focus* of the conic.

As there may be in general two spheres so inscribed, one touching the plane of the section above, the other below—or one in each sheet of the cone, both touching the plane of the section on the same side—there are in general two foci in a conic section.

These spheres may be called *focal spheres*.

IV.

The straight line which passes through the foci, and is terminated by the surface of the cone, is called the *major axis.*

V.

The plane drawn through the vertex of the cone and the major axis of the section, cuts the surface of the cone in two straight lines, which together with the major axis constitute a triangle, which may be called the *focal triangle,* since its plane passes through the foci.

VI.

The focal spheres touch the surface of the cone in two circles which may be called the *circles of contact.*

The planes of these circles are manifestly parallel, since they are at right angles to the axis of the cone.

VII.

The straight line in which the plane of a circle of contact cuts the plane of the section is called a *directrix.*

As there are in general two circles of contact, there are in general also two directrices, and they are parallel to each other.

VIII.

A plane drawn through the vertex of the cone parallel to the plane of the section is called the *vertical polar plane;* and the straight line drawn through the vertex of the cone, the polar line of this vertical plane with respect to this cone, is called *the polar axis,* and it meets the plane of the conic section in a point called the *centre.*

IX.

The straight line in which the vertical polar plane cuts the plane of the circle of contact is called the *dirigent.* As there are in general two circles of contact, there are two dirigents, and they are parallel to the directrices.

X.

The dirigent is the polar of the point in which the polar axis of the cone meets the plane of the circle of contact with respect to this circle.

XI.

If a straight line be drawn from the vertex of the cone in the vertical polar plane, the polar plane of this straight line will pass through the polar axis of the cone, and is called a *polar plane of the cone.*

XII.

When the vertical polar plane lies outside the cone the parallel section is called an ellipse; when it touches the side of the cone the parallel section is a parabola; and when the vertical plane cuts the surface of the cone, the parallel section is an hyperbola.

In this latter case the polar axis will lie outside the cone; and if two planes be drawn through this line touching the cone, they will cut the plane of the hyperbola in two straight lines called *asymptotes*; and as the polar axis (the intersection of the tangent planes) cuts the plane of the conic in its centre (see def. VIII.), the asymptotes will meet in the centre of the hyperbola. Moreover, as the polar axis touches the surface of the cone when the conic is a parabola, the two tangent planes drawn through it to the cone coincide and become parallel to the plane of the parabola; consequently the asymptotes of the parabola are two straight lines parallel to the axis of the parabola but at an infinite distance from this axis.

XIII.

The ordinate drawn through the focus of a conic, at right angles to the major axis, is called the *parameter* or *latus rectum.*

XIV.

The *radical plane* of the focal spheres cuts the plane of the conic in a straight line called the *minor axis.*

XV.

Lines drawn from the vertex of the cone to the extremities of the diameters of the focal spheres which are perpendicular to the plane of the conic may be called *vertical focals* of the conic.

ON THE FOCAL PROPERTIES OF CONICS.

256.] *If a sphere be inscribed in a right cone, the curve of contact is a circle.*

Since all tangents drawn from a point to a sphere are equal, the vertex of the cone may be considered as the centre of a sphere whose radii are the sides of the cone intercepted between the vertex and the line of contact with the inscribed sphere. This sphere, therefore, will intersect the inscribed sphere in the line of contact; but two spheres intersect each other in a circle; hence the line of contact is a circle.

257.] *The plane which passes through the vertex of the cone and the two foci, passes also through the axis of the cone, and is at right angles to the plane of the conic.*

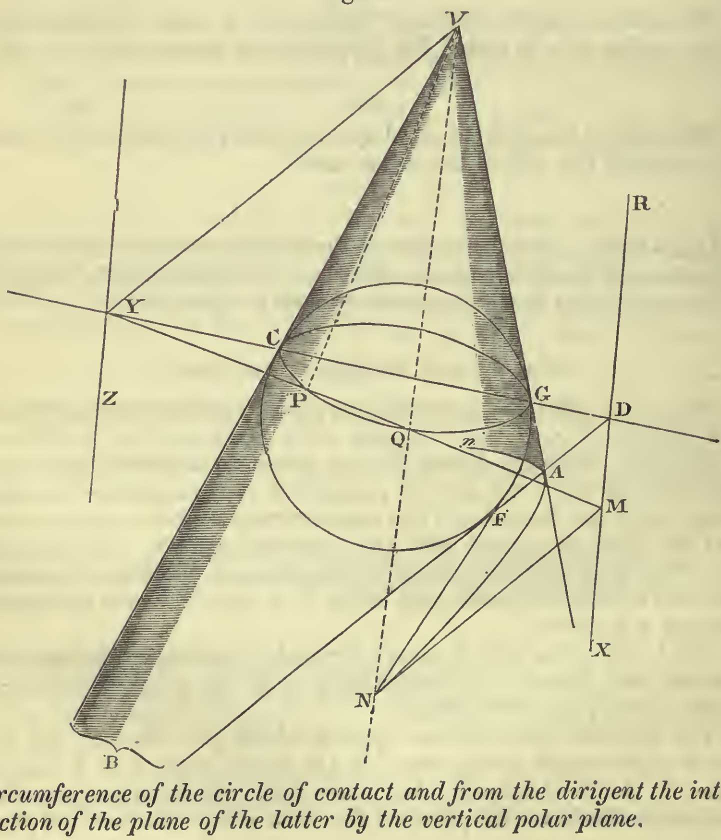
The radii of the inscribed spheres which pass through the foci are at right angles to the plane of the conic, since it is a tangent plane to the focal spheres; but these radii are parallel, since they are perpendicular to the plane of the conic; and therefore the plane which passes through them will pass through the centres of the focal spheres, which are manifestly on the axis of the cone; consequently the plane of the *focal triangle* (see def. v.) will be perpendicular to the plane of the conic.

258.] *The directrix is perpendicular to the major axis of the conic.*

Since the plane of the conic and the plane of the circle of contact are each perpendicular to the plane of the focal triangle, their intersection, the directrix (Euclid, XI.) will be perpendicular to the same plane, and therefore perpendicular to any straight line on it, and therefore to the major axis.

259.] *The distance of any point of a conic from a focus is to its perpendicular distance from the corresponding directrix in a constant ratio—namely, as the distances of the vertex of the cone from the*

Fig. 46.



circumference of the circle of contact and from the dirigent the intersection of the plane of the latter by the vertical polar plane.

The inscribed sphere touches the plane of the conic MDANFn in the point F; and the cone touches the sphere along the circle of contact CGQP.

Draw NM parallel to AF, join NF, NV. Draw the vertical plane VYZ parallel to the plane of the conic, meeting the plane of the circle of contact in the straight line, the dirigent, YZ. In this vertical plane draw the line VY parallel to the major axis AF, to which MN is parallel. Join YQ, and produce it to meet the directrix RX. It must meet the line MN also in the directrix; for as YQ is in the plane of the circle of contact, it can meet the plane of the conic only in their intersection, the directrix RX; but as MN is parallel to VY, a plane may pass through VY, VQN, and NM; hence NM must meet YQ; and as it lies in the plane of the conic, it can only meet it in the directrix RX.

Now as the triangles MQN and VYQ are similar, NQ : NM = VQ : VY. But NQ = NF, since Q and F are points on the same sphere; therefore NF : NM = VQ : VY = VC : VY, since VQ is equal to VC. But VC has a constant ratio to VY independently of the position of the point N; therefore NM has a constant ratio to NF.

This is the theorem which has been made by De la Hire, and by others since his time, the basis of a system of conics in a plane.

Cor. i. When the vertical plane touches the cone, as when the conic is a parabola, VC = VY, consequently NF = NM.

Cor. ii. When the conic is an ellipse, VY is greater than VC, or NM is greater than NF; when VY is equal to VC, NM is equal to NF; when VY is less than VC, or when the vertical plane VY falls *within* the cone, or NM is less than NF, the conic is an hyperbola.

The ratio of VC the side of the cone between the vertex and the circle of contact to the perpendicular VY from the vertex of the cone on the dirigent YZ is called the eccentricity of the conic, and is usually denoted by e.

260.] *If from any point in a conic a line be drawn to the directrix parallel to the straight line the intersection of the vertical plane with the cone, it will be equal to the focal distance of the same point.*

Let VP be the intersection of the vertical plane with the cone; join PQ; and by the same construction and demonstration as the preceding, NF = NM, since VC = VP.

Hence, *If from a point in a conic a line be drawn to the directrix parallel to the axis of a parabola, or to one of the asymptotes of an hyperbola, this straight line will be equal to the focal distance of the same point.*

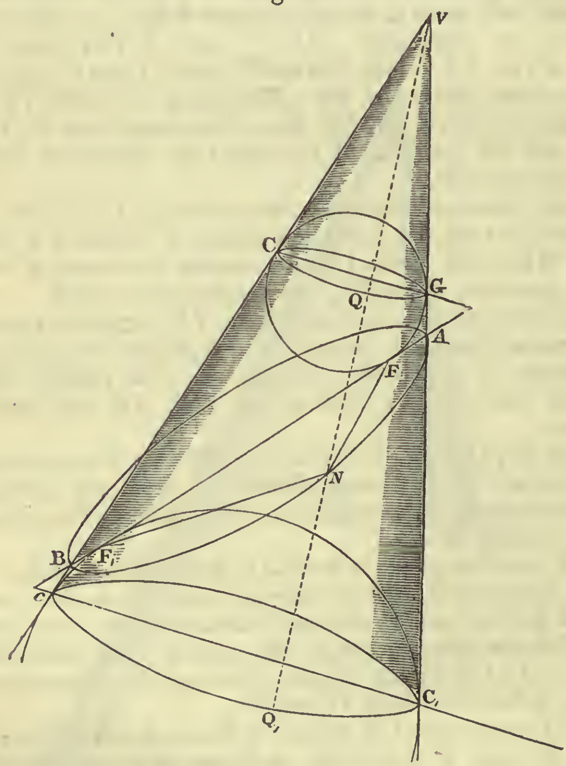
261.] *The major axis of a conic is equal to the segment of a side of the cone intercepted between the circles of contact.*

In the focal triangle ABC the base AB (that is, the major axis of the conic) is equal to the segment of the side CB intercepted between $A_,$ and G, the points of contact of the side CB with the inscribed circles (see fig. 29). But these are great circles of the focal spheres which touch the plane of the conic in its foci F and $F_,$.

262.] *The sum or difference of the focal distances of any point in a conic—the sum, if an ellipse, the difference, if the conic be an hyperbola—is constant, and equal to the portion of the side of the cone intercepted between the circles of contact (that is, to the major axis).*

Let $VQNQ_,$ be a side of the cone touching the focal spheres in the points Q, $Q_,$ and passing through N a point on the conic. Then, as Q and F are points on the same sphere, $NF = NQ$; so also $NF_, = NQ_,$.

Fig. 47.

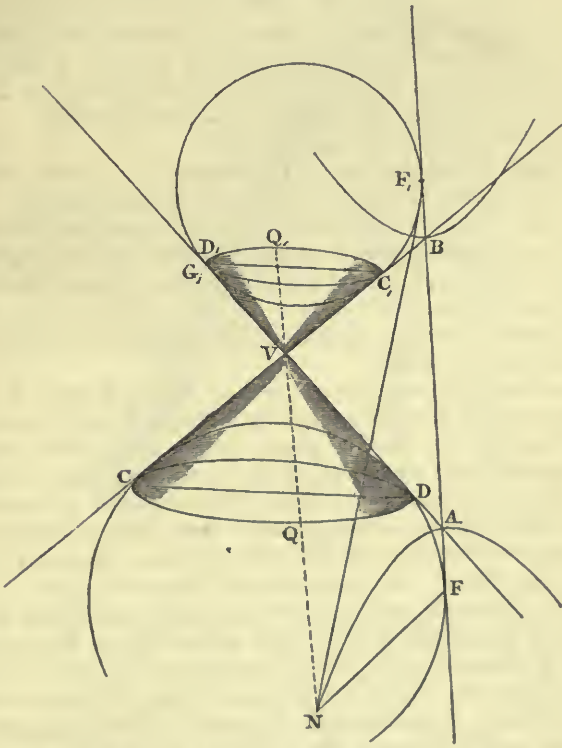


Therefore $NF + NF_,$ is equal to $QQ_,$, the portion of a side of the cone intercepted between the circles of contact.

In the last proposition it was shown that this segment of the side of the cone is equal to the major axis of the conic. Therefore the sum of the focal vectors of an ellipse is equal to its major axis.

In the case of the hyperbola (fig. 48), since NF is equal to NQ and $NF_{\prime}$ equal to $NQ_{\prime}$, therefore $NF_{\prime} - NF$ is equal to $QQ_{\prime}$ the segment of the side VN of the cone intercepted between the circles of contact.

Fig. 48.



Cor. i.] *The distance between the directrices is equal to that portion of the major or transverse axis intercepted between the planes of the circles of contact* (see fig. 47).

The ratio of the major axis of the conic to the distance between the directrices is as $e : 1$.

For, in fig. 46, $NM : NF = VY : VC$; so also with respect to the other directrix $NM_{\prime} : NF_{\prime} = VY : VC$.

Therefore $\quad NM + NM_{\prime} : NF + NF_{\prime} = VY : VC$.

But $NM + NM_{\prime}$ is the distance between the directrices, and $NF + NF_{\prime}$ is equal to AB the major axis; therefore the distance between the directrices is to the major axis as $1 : e$.

Cor. ii.] In the same way it may be shown that the distance between the dirigents is equal to the distance between the directrices.

Cor. iii.] *The distance between the foci is equal to the difference between the sides of the cone terminated in the extremities of the major axis, namely* VB *and* VA.

For $VB-VA=BC-AG=BF-AF=FF_{\prime}$.

In the hyperbola we have $VB+VA=FF_{\prime}$ (see fig. 48).

263.] *The rectangle under the radii of the focal spheres is equal to the rectangle under the focal distances of the vertices of the conic.* Let R and $R_{\prime}$ be the radii of the inscribed spheres, ω and Ω their centres. The triangles $B\omega F$ and $\Omega BF_{\prime}$ are similar, since the angle $\omega BF=B\Omega F$. Therefore $RR_{\prime}=BF_{\prime}.BF$; but $BF_{\prime}=AF$.

Therefore $RR_{\prime}=AF.BF$ (see fig. 29).

264.] *Planes which intersect in a tangent to a conic and pass through the centres of the focal spheres are at right angles.*

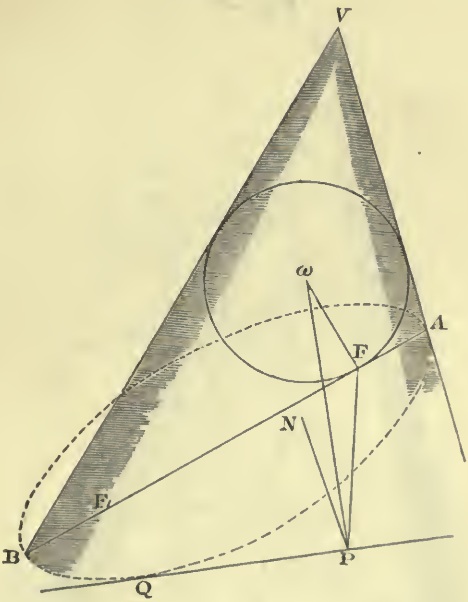
It is evident that the dihedral angle between the plane of the conic and the tangent plane to the cone which contains the tangent to the conic is bisected by the plane passing through the tangent their intersection and the centre of the sphere which touches these two planes; for this point is equidistant from the planes of the section and tangent plane.

In like manner the supplement of this dihedral angle is bisected by the plane which passes through this tangent and the centre of the other focal sphere. Hence planes drawn through the centres of the focal spheres and a tangent to the conic are at right angles.

265.] *If perpendiculars are drawn from the foci of a conic on a tangent to the curve, the rectangle under these perpendiculars is equal to the rectangle under the radii of the focal spheres; that is,* $PP_{\prime}=RR_{\prime}$.

Let FP (fig. 49) be the perpendicular from the focus F on the tangent PQ. Join ωP, and erect the perpendicular PN to the plane of the conic; it is parallel to ωF, and is therefore in the plane $P\omega F$; and as QP is perpendicular to FP by construction, and to PN, it is perpendicular to the plane which passes through them—that is, to the plane $PN\omega F$. Consequently the angle $FP\omega$ is the measure of the dihedral angle between the plane of the conic and the plane which passes through the tangent to it and the centre ω of the focal sphere. Let this angle be ϖ, we shall have, since ωFP is a right angle, $R=P\tan\varpi$; and as the angle between the plane of the conic and the plane passing through the tangent QP and the centre Ω of the other focal sphere is the complement of the former, we shall have $R_{\prime}=P_{\prime}\cot\varpi$, or $RR_{\prime}=PP_{\prime}$.

Fig. 49.

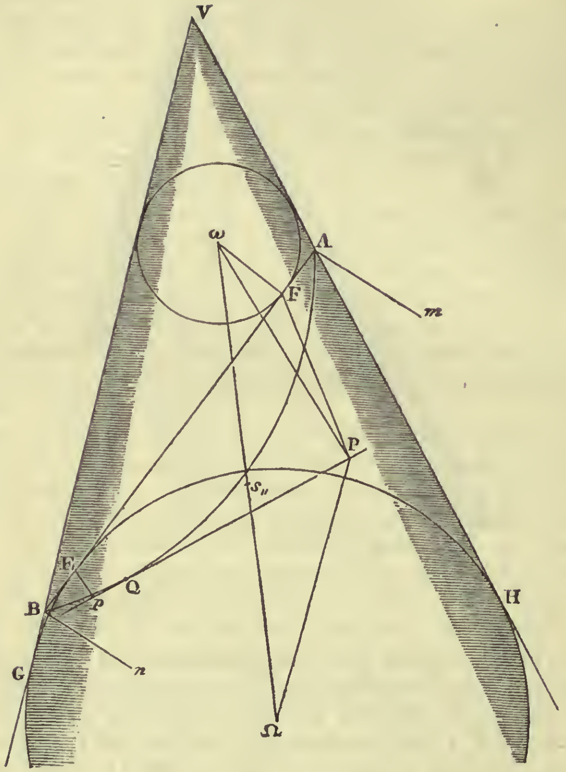


266.] *The locus of the feet of perpendiculars let fall from the foci of a conic on a tangent to the curve is a circle* (see fig. 50).

Since the angle ωPQ is a right angle, see last section, and the planes ωPQ, ΩPQ are at right angles, ωPΩ is also a right angle; therefore the sphere described on $\omega\Omega$, the line which joins the centres of the focal spheres, will pass through the points P, p, and also through the points A and B, since ΩBω and ωAΩ are right angles. But the points P, p, A, B are also in a plane, namely that of the conic. Hence they lie in the intersection of a plane and a sphere—that is, a circle.

Let Am, Bn be the lines in which the plane of the conic intersects the tangent planes along the sides of the cone VA, VB. Now these planes are perpendicular to the focal plane passing through the axis of the cone, which is also a diametral plane of the sphere whose diameter is $\omega\Omega$; therefore these lines are tangents to the sphere; and as they are parallel, the line which joins their points of contact A, B is a diameter of the circle. The locus of the feet of the perpendiculars is therefore a circle whose diameter is the major axis of the conic.

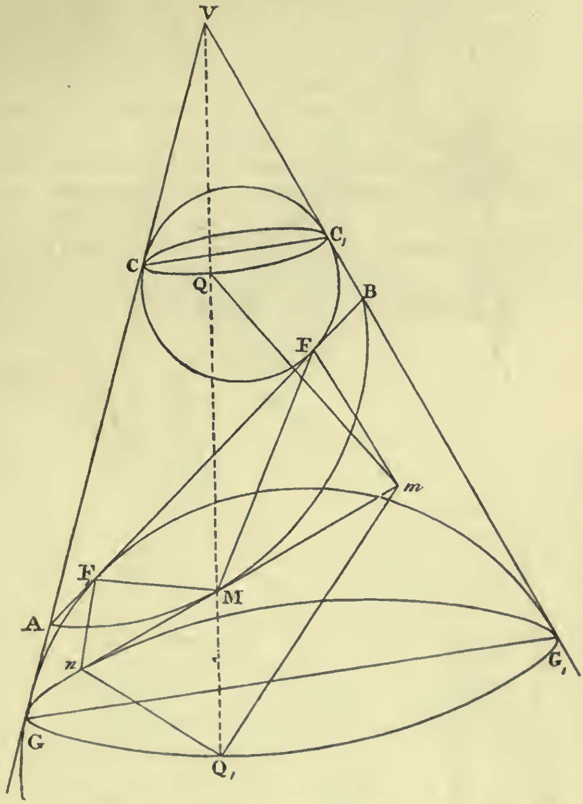
Fig. 50.



267.] *A tangent to a conic makes equal angles with the focal vector and the side of the cone passing through the point of contact* (see fig. 51).

In the tangent $m\mathrm{M}n$, through which the tangent plane $\mathrm{VQ}n\mathrm{M}m\mathrm{Q}_{\prime}$ passes, assume any point m; draw $m\mathrm{F}$, and $m\mathrm{Q}$ to the point Q where the side VM of the eone meets the circle of contact $\mathrm{CQC}_{\prime}$. Then in the triangles $m\mathrm{MQ}$ and $m\mathrm{MF}$ the side $m\mathrm{F}=m\mathrm{Q}$; so also $\mathrm{MF}=\mathrm{MQ}$; and $\mathrm{M}m$ is common to the two triangles; hence these triangles are equal, and therefore the angles $\mathrm{FM}m$ and $\mathrm{QM}m$ are equal. In the same manner the angles $\mathrm{F}_{\prime}\mathrm{M}n$ and $\mathrm{Q}_{\prime}\mathrm{M}n$ are equal.

Fig. 51.



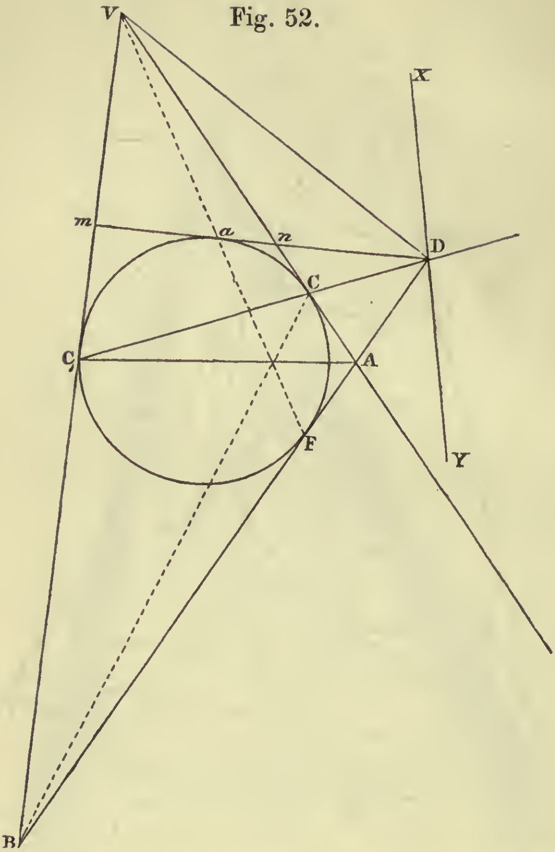
The angles which the focal vectors make with a tangent to the curve at the point of contact are equal.

The angle FMm is equal to the angle QMm; and the angle $F_{\prime}Mn$ is equal to the angle $Q_{\prime}Mn$. But the angle QMm is equal to the angle $Q_{\prime}Mn$ as they are vertically opposite angles, the angles which the side of the cone makes with the tangent to the curve. Hence the angles FMm and $F_{\prime}Mn$ are equal.

268.] *The directrix is the polar of the focus, or the locus of the intersection of every pair of tangents whose chord of contact passes through the focus* (see fig. 52).

Through VF, the line which joins the vertex V of the cone with the focus F, let a plane be drawn, cutting the plane of the conic in the line BAD and the plane of the circle of contact in the points $C_{\prime}CD$. As D is a point in the plane of the conic and in the plane of the circle of contact, D must be on the directrix.

Fig. 52.



Now as the two sides of the cone VA, VB and the line AB in which this plane cuts the plane of the conic constitute a triangle in which the circle $FCC_{,}$ a section of the focal sphere is inscribed, the lines $AC_{,}$, VF, and BC will meet in a point. As in the triangles DAC and DBC the angle at D is common, and the angle DCA is supplemental to the angle $DC_{,}B$,

$$DA : AC = DB : BC_{,}; \text{ but } AC = AF, \text{ and } BC_{,} = BF;$$

therefore $DA : DB = AF : BF$, or VB, VF, VA, VD constitute an harmonic pencil. And as this proof will hold good for any plane drawn through the focus and the vertex of the cone, it is clear that the directrix is the polar of the focus.

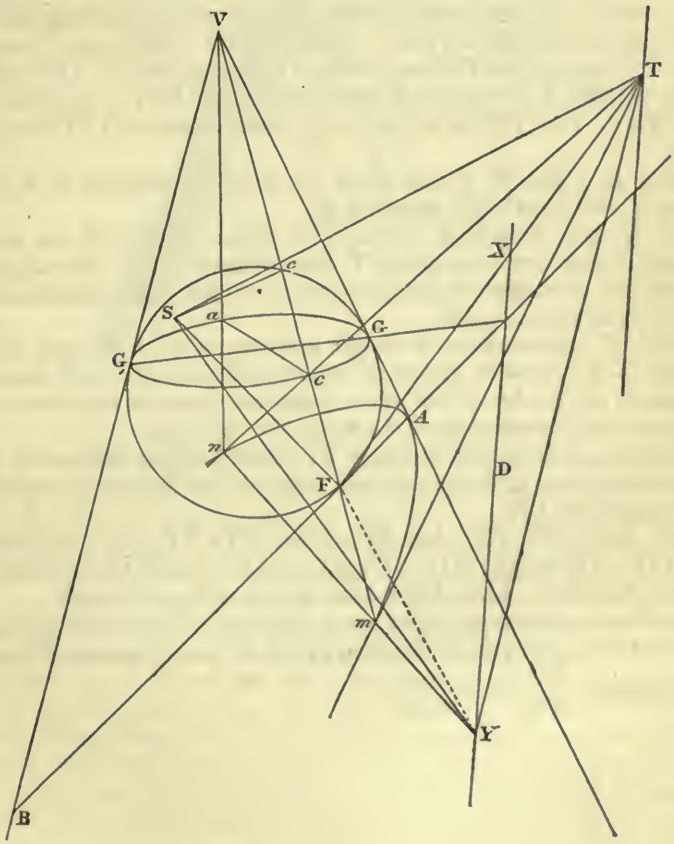
Cor. Join Da and produce it to m. Then mn is harmonically divided in D and a; and therefore $ma : an = m\mathrm{D} : \mathrm{D}n$. VC and $VC_{,}$ are tangents to the circle; therefore Da is also a tangent.

269.] *If any chord of a conic be drawn and produced to meet the directrix, and from the corresponding focus two lines be drawn, one to the intersection of the tangents drawn to the ends of the chord, the other to the intersection of this chord with the directrix, these two lines will be at right angles to each other* (see fig. 53).

Let the chord *mn* meet the directrix DX in the point Y; draw the tangents *m*T, *n*T meeting in T. Draw the tangent planes V*m*T, V*n*T whose lines of contact with the cone meet the circle of contact in the points *a*, *c*. Join *a*, *c*; also T F, F Y; the angle TFY is a right angle.

Join TY, and let a plane be drawn through TY touching the focal sphere in the point S. As TY is in the plane of the conic which touches the focal sphere in F, the line SF is the conjugate

Fig. 53.



polar of the line TY; and as Y is a point in the line TY, the polar plane of the point Y will pass through FS the conjugate polar of TY; and as Fe is the conjugate polar of the directrix in which Y is a point (see fig. 52), the polar plane of Y will also pass through Fe. Hence the plane which passes through FS and Fe is the polar plane of the point Y. This plane will also pass through the point T; for as the point T is in the intersection of the tangent planes to the cone VmT and VnT, the polar plane of T will pass through the chord a c; and as this chord is in the plane of the circle of contact, it must meet the directrix which also lies in the plane of the circle of contact; for otherwise it would be parallel to it, and then the directrix could never meet the secant plane Vmn, contrary to hypothesis. Therefore the chord ac of the circle of contact meets the directrix in the point Y; therefore the polar plane of T passes through Y; therefore T is a point in the plane FSe; and therefore ST is a tangent to the base of the cone whose vertex is at Y, and which touches the focal sphere in the points FSe. Consequently YST is a right angle. Now in the triangles YTF and YTS, YF is equal to YS, TF is equal to TS, and YT is common; therefore the angle YFT is equal to the angle YST. But YST is a right angle; and therefore YFT is a right angle.

270.] *If a line be drawn from the focus to the pole of a focal chord, it will be at right angles to it.*

For, by the preceding proposition, when the chord mn passes through F the focus, the point T, the intersection of the tangents mT, nT will be found on the directrix, and the equal angles mFT, nFT become right angles.

271.] *If a focal chord be drawn perpendicular to the axis, and a tangent to the curve be drawn at its extremity, it will cut off from the tangent to the vertex of the curve a portion equal to the distance of the vertex of the curve from the focus* (see fig. 54).

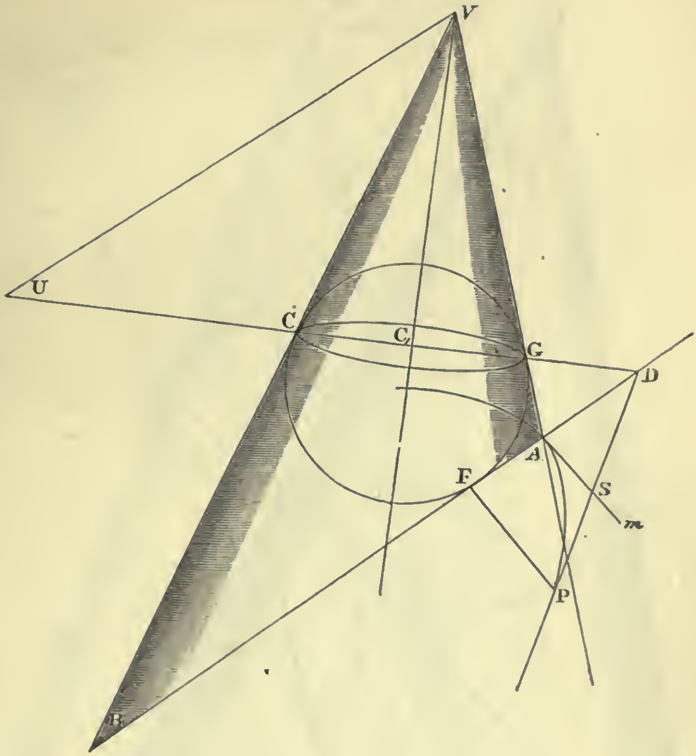
Through F draw the ordinate FP, and through the point P a tangent meeting in S the vertical tangent Am drawn through A; AS is equal to AF.

AS : AD = FP : FD; but FP : FD = VG : VU, as in sec. [259], and VG : VU = AG : AD. Therefore AS is equal to AG = AF.

Therefore AS : AD = AG : AD; hence AS = AG = AF.

Let θ be the semiangle of the cone, and i the inclination of the plane of the conic to the axis of the cone; then $\mathrm{VC}_{,} = \mathrm{VU}\cos i = \mathrm{VG}\cos\theta$; consequently $\dfrac{\mathrm{VG}}{\mathrm{VU}} = e = \dfrac{\cos i}{\cos\theta}$.

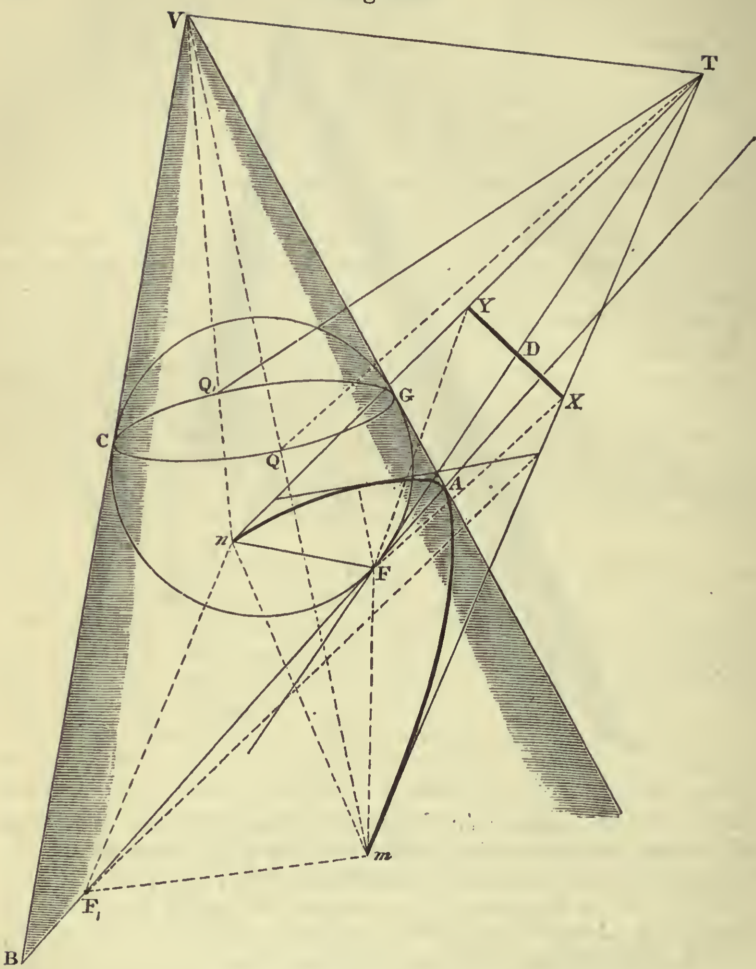
Fig. 54.



272.] *If two tangents be drawn to a conic, the line connecting their point of meeting with a focus bisects the angle contained by the focal vectors drawn from this focus to the points of contact* (see fig. 55).

Let Tm, Tn be tangents to the conic at the points m and n, and meeting in T. Join TF; then TF bisects the angle mFn. Through Tm, Tn let tangent planes to the cone be drawn touching it in the sides Vm, Vn, and therefore touching the sphere in the points Q and $Q_{,}$. Join VT, TQ, and $TQ_{,}$. Then in the triangles TVQ and $TVQ_{,}$ since $VQ=VQ_{,}$, $TQ=TQ_{,}$, and VT common, the angle VQT is equal to the angle $VQ_{,}T$, and their supplements are therefore equal; that is, the angle TQm is equal to the angle $TQ_{,}n$. Now mF is equal to mQ, TQ is equal to TF, and Tm is common; therefore in the triangles TmQ and TmF the angle TQm is equal to the angle TFm. In the same way the angle $TQ_{,}n$ may be proved equal to the angle TFn; consequently the angles TFm and TFn are equal.

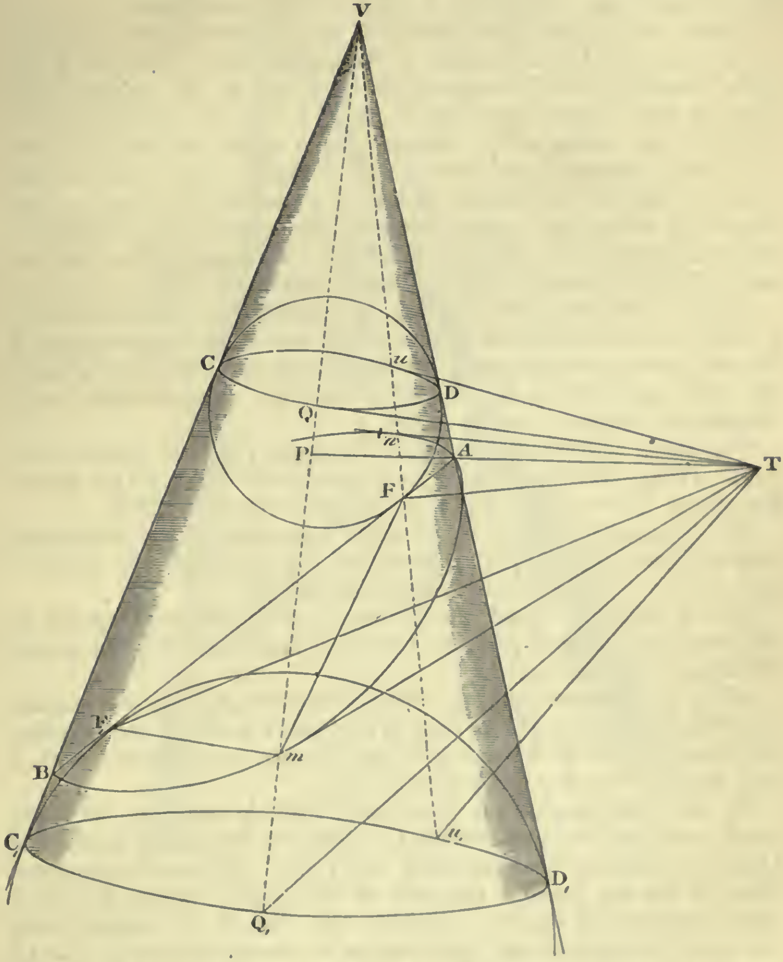
Fig. 55.



273.] *If from the intersections* X, Y, *of the tangents* Tm, Tn *with the directrix* YDX, *focal chords* XF, YF *be drawn, they will make equal angles with the focal chord* FT.

Since X is the pole of mF, mFX is a right angle; and since Y is the pole of Fn, nFY is a right angle. But the angle TFm is equal to the angle TFn; therefore the angle TFX is equal to the angle TFY.

Fig. 56.



274.] *If from the intersection of two tangents to a conic, chords be drawn to the two foci, they will make equal angles with the tangents.*

Let Tm, Tn (fig. 56) be the tangents to the conic, meeting in T. Let the tangent planes VmT, VnT be drawn, touching the cone along the sides VQ$m$$Q_{,}$, V$unu_{,}$, and draw TF, $TF_{,}$. The angles FTn and $F_{,}$Tm are equal. Join Fm, $F_{,}m$.

Now as $TF_{,}$ is equal to $TQ_{,}$ since they are tangents to the same sphere, and $mF_{,}$ for the same reason is equal to $mQ_{,}$, and mT is common, the triangles T$m$$F_{,}$ and T$m$$Q_{,}$ are equal, and therefore the angle $F_{,}$Tm is equal to the angle $Q_{,}$Tm.

In the same way, as TF is equal to TQ, Fm equal to mQ, and Tm common, the angle TmF is equal to the angle TmQ.

Hence the angles $Q_{\prime}$Tm and mTF are together equal to $QTQ_{\prime}$; or, as the angle $Q_{\prime}$Tm is equal to the angle $F_{\prime}$Tm, twice the angle $F_{\prime}$Tm together with the angle $F_{\prime}$TF are equal to the angle $Q_{\prime}$TQ. For the same reason twice the angle FTn with the angle $F_{\prime}$TF are equal to the angle uT$u_{\prime}$. But as $TQ_{\prime}$ is equal to T$u_{\prime}$, TQ equal to Tu, and $QQ_{\prime}$ is equal to $uu_{\prime}$, the triangles $TQ_{\prime}Q$ and T$uu_{\prime}$ are equal; therefore the angle $QTQ_{\prime}$ is equal to the angle uT$u_{\prime}$. Therefore twice the angle $F_{\prime}$Tm with the angle $F_{\prime}$TF are equal to twice the angle FTn with the angle $F_{\prime}$TF; taking away the common angle $F_{\prime}$TF, the angle $F_{\prime}$Tm is equal to the angle FTn.

275.] *If two tangents be drawn to a conic, and from their intersection two lines be drawn to the points where the tangent plane to the cone drawn through one of the tangents touches the focal spheres, the angle contained by the two latter lines will be equal to the angle between the tangents* (see fig. 56)*.

The angle $2F_{\prime}$Tm together with the angle $F_{\prime}$TF are equal to the angle $QTQ_{\prime}$. But the angle $F_{\prime}$Tm is equal to the angle FTn. Hence the angles $F_{\prime}Tm + FTn + F_{\prime}TF$ are together equal to $QTQ_{\prime}$.

But the angle between the tangents is made up of the component angles $F_{\prime}Tm + FTn + F_{\prime}TF$. Therefore the angle between the tangents to the conic is equal to the angle $QTQ_{\prime}$.

It is a matter of indifference through which of the tangents to the conic the tangent plane to the cone be drawn; for the angles $QTQ_{\prime}$ and uT$u_{\prime}$ are equal.

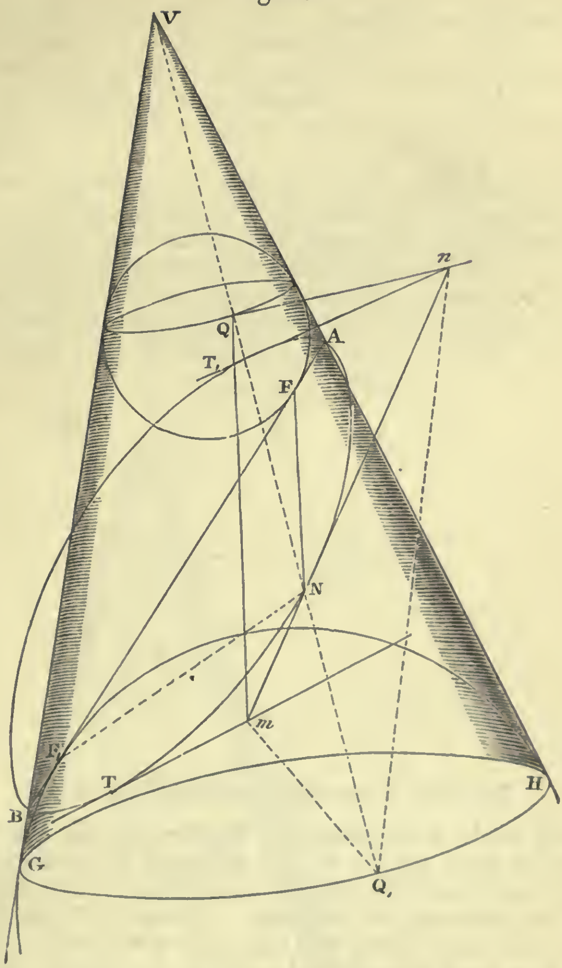
276.] *If a tangent plane be drawn to the cone, meeting two parallel tangents to a section of this cone in the points m and n, and touching the focal spheres in the points* Q *and* $Q_{\prime}$, *the quadrilateral* $QmQ_{\prime}n$ *may be inscribed in a circle* (see fig. 57).

By the last proposition the angle TmN is equal to the angle $QmQ_{\prime}$, and $T_{\prime}n$N is equal to $QnQ_{\prime}$; therefore the angles $QmQ_{\prime}$ and $QnQ_{\prime}$ are together equal to TmN and $T_{\prime}n$N. But as the tangents Tm and $T_{\prime}n$ are parallel, the sum of the angles TmN and $T_{\prime}n$N is equal to two right angles; therefore the sum of the angles $QmQ_{\prime}$ and $QnQ_{\prime}$ is equal to two right angles, or the quadrilateral $QmQ_{\prime}n$ may be inscribed in a circle.

Cor.] Since Nm . Nn is equal to NQ . $NQ_{\prime}$, while NQ is equal to NF, and $NQ_{\prime}$ equal to $NF_{\prime}$, therefore the rectangle under the segments of a tangent between its point of contact and its intersections by two parallel tangents is equal to the rectangle under the focal chords drawn through the point of contact.

* This is perhaps the most important proposition in the theory of conics derived from the cone.

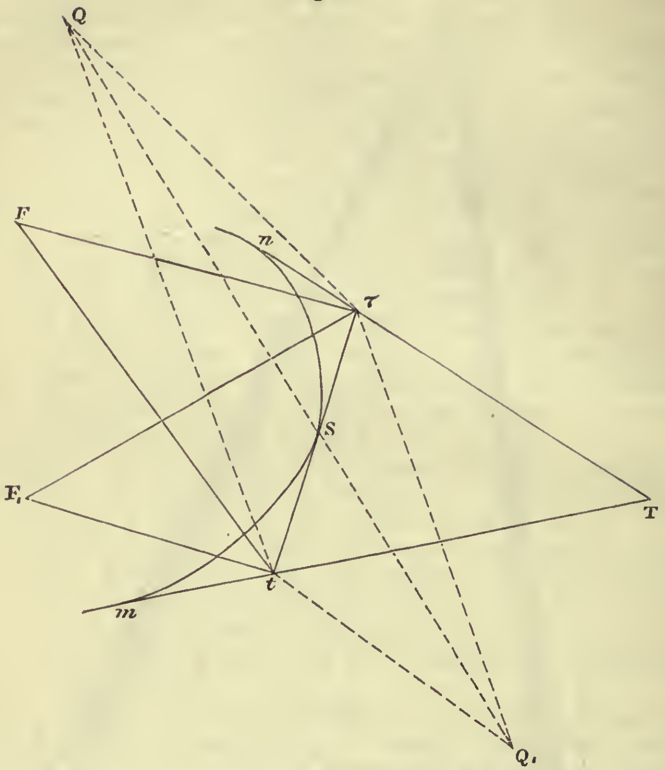
Fig. 57.



277.] *If two fixed tangents be drawn to a conic, and a third tangent variable in position, the segment of this latter tangent between the two former will subtend angles at the foci whose sum is constant and equal to the supplement of the angle contained by the two fixed tangents* (see fig. 58).

Let Tm, Tn be the two fixed tangents touching the conic in the points m, n. Let tSτ be the variable tangent touching the conic in S and cutting the fixed tangents in t and τ. The tangent $t\tau$ will subtend at the foci F, $\text{F}_{\prime}$ angles whose sum is constant, and equal to the supplement of the angle at T.

Fig. 58.



The vertex of the cone is omitted from the figure.

Through $ts\tau$ let a tangent plane $\mathrm{VQ}t\mathrm{S}\tau\mathrm{Q}_{,}$ to the cone be drawn touching the focal spheres in the points Q, $\mathrm{Q}_{,}$, cutting the fixed tangents to the conic Tm, Tn in the points t, τ, and touching the conic in S. Join tQ, τQ, $t\mathrm{Q}_{,}$, $\tau\mathrm{Q}_{,}$. By sec. [275] the angle mtS is equal to the angle $\mathrm{Q}t\mathrm{Q}_{,}$, and the angle $n\tau$S is equal to the angle $\mathrm{Q}\tau\mathrm{Q}_{,}$. Now these two external angles of the triangle T$t\tau$ together with the external angle at T are equal to four right angles; and the four angles of the quadrilateral $\mathrm{Q}t\mathrm{Q}_{,}\tau$ are also equal to four right angles. But two of the angles of this quadrilateral $\mathrm{Q}\tau\mathrm{Q}_{,}$ and $\mathrm{Q}t\mathrm{Q}_{,}$ have been shown to be equal to the external angles of the triangle $t\mathrm{T}\tau$; therefore the remaining two $t\mathrm{Q}\tau$ and $t\mathrm{Q}_{,}\tau$ must be equal to the external angle at T.

Now, in the triangles $t\mathrm{Q}\tau$ and $t\mathrm{F}\tau$, since tQ is equal to tF, and τQ is equal to τF, since the points Q and F are on the same sphere, and $t\tau$ is common, the triangle $t\mathrm{Q}\tau$ is equal to the triangle $t\mathrm{F}\tau$, and therefore the angle $t\mathrm{Q}\tau$ is equal to the angle $t\mathrm{F}\tau$. The same may be shown for the other focus. Hence the angle Q of the quadrilateral is equal to the focal angle at F, and the other angle $Q_{\prime}$ of the quadrilateral is equal to the angle at $F_{\prime}$.

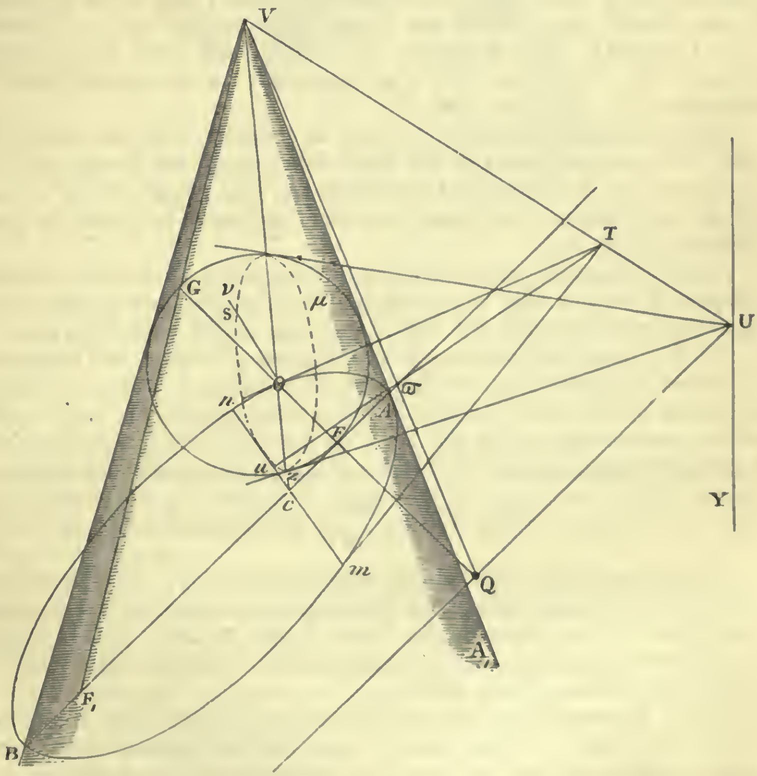
Hence the sum of the angles which $t\tau$ subtends at the foci is equal to the supplement of the angle T.

Cor.] When the fixed tangents are parallel, we get the theorem in sec. [276].

278.] *Two tangents are drawn to a conic; a perpendicular drawn to the chord of these tangents from their point of meeting will cut the major axis in a point which with the two foci and the intersection of the chord with this axis will be four harmonic points.*

Through F the focus of the conic let the diameter of the focal sphere be drawn meeting its surface in G; the focal vertical VG will meet the major axis in the other focus $F_{\prime}$, see def. xv. Draw

Fig. 59.



the plane V*mnc* cutting the focal sphere in a circle $z\mu\nu$, and let this plane cut the focal diameter FG in O. Let T be the intersection of the tangents Tm, Tn. Then U the vertex of the cone $z\mu\nu$U is on VT, since the tangent planes meet in VT. The polar plane of the point O is a plane drawn through U parallel to the plane of the conic. Let this plane meet FG in Q. Join VQ meeting the major axis in ϖ. Through Tϖ let a line be drawn, meeting the chord mn in u; this line will be at right angles to the chord mn. Then, as the plane through U parallel to the plane of the conic is the polar plane of O, QF : FO = QG : GO. Through O let a plane be drawn parallel to the plane of the conic and cutting the sphere in the line Os. This line Os will be parallel to the chord mn.

Let this lesser circle be the base of a cone whose vertex is at Q on the plane through U parallel to the plane of the conic.

Now the line UQ, which joins the vertices of the cones, is the harmonic conjugate of the line Os, in which the bases of the two cones intersect. Hence UQ is at right angles to Os. But UQ is parallel to Tu, since they are in parallel planes; and sO is parallel to the chord mn. Hence mn is at right angles to Tu. Since GO : OF = GQ : QF, therefore (as the vertical focal VG passes through $F_{\prime}$) V$F_{\prime}$, Vc, VF, Vϖ constitute an harmonic pencil; therefore $F_{\prime}c$: Fc = $F_{\prime}\varpi$: Fϖ.

When mn passes through F, c and ϖ coincide with the chord F, and TF is perpendicular to the focal chord as shown in sec. [273].

Hence also it follows that uc bisects the angle $F_{\prime}u$F, which is one of the most general theorems in conics, and may be given in the following form:—

If two rectangular axes are drawn in the plane of a conic, so that the pole of the one may be a point on the other, the lines drawn from their intersection to the foci will make equal angles with these axes.

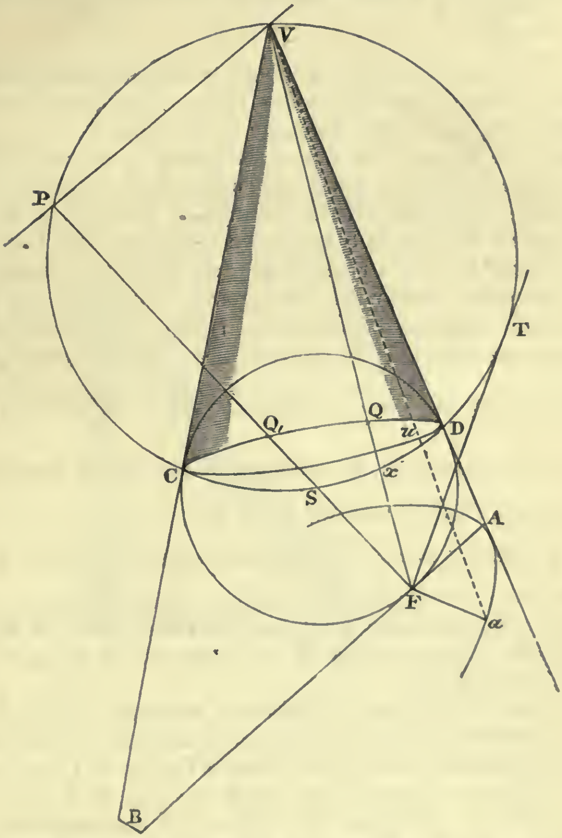
279.] *If any point be assumed in the plane of a conic, and tangents be drawn from this point to the curve, the rectangle under the focal distances of this point is equal to the rectangle under the major axis and a perpendicular from this point on a focal chord drawn through a point of contact divided by the sine of the angle between the tangents,*

or $$\mathrm{TF}_{\prime} \,.\, \mathrm{TF} = \frac{2a\mathrm{P}}{\sin m\mathrm{T}n} \text{ (see fig. 56).}$$

Since TF = TQ and $\mathrm{TF}_{\prime} = \mathrm{TQ}_{\prime}$, therefore TF . $\mathrm{TF}_{\prime}$. sin mTn is the area of the triangle $\mathrm{QTQ}_{\prime}$; but this area is also equal to $\mathrm{QQ}_{\prime}$ ($=2a$) multiplied by the perpendicular drawn from T to $\mathrm{QQ}_{\prime}$. This perpendicular it may easily be shown is equal to the perpendicular from T drawn to the focal chord mF.

Cor.] Hence all the perpendiculars let fall from T on the focal chords are equal. Consequently, if any two points be assumed on a conic section, and two focal chords be drawn through each, the centre of the circle described touching these four chords will be on the intersection of the tangents touching the curve in the two given points.

Fig. 60.



280.] *If a sphere be described about that portion of the cone cut off by the circle of contact, the semiparameter is a third proportional to the side of the cone cut off by the sphere, and the tangent from the focus to this sphere.*

Let $CuDQQ_{\prime}$ be the circle of contact, which is also the common intersection of the focal sphere with the circumscribing sphere VPCDT.

Draw the ordinate F*a* perpendicular to the axis AB; through V and *a* draw a side of the cone V*a* meeting the circle of contact in the point *u*, touching the focal sphere in *u*, and meeting the circumscribed sphere in V and *u*.

Then $Va \, . \, au = \overline{Fa}^2 + FV \, . \, Fx$, since F*a* is perpendicular to F*x*. Now $Va = au + Vu = Fa + VC$; for F*a* and *au* are tangents to the same focal sphere. Therefore $au = Fa$; hence $(Fa + VC) Fa = \overline{Fa}^2 + FV \, . \, Fx$.

But $FV . Fx = \overline{FT}^2$; consequently $VC . Fa = \overline{FT}^2$, or the semiparameter $Fa = \frac{\overline{FT}^2}{VC}$.

281.] *The semiparameter is equal to the perpendicular distance between the plane of the section and the vertical polar plane, multiplied by the tangent of the semiangle of the cone.*

Through the focus F let a perpendicular be drawn to the plane of the conic, meeting the sphere circumscribed to the cone VCD in the point P. This line passes through S the centre of the focal sphere; and as this point S is on the diameter VS of the circumscribed sphere, VPS is a right angle, or VP is the intersection of the vertical polar plane with this sphere. Consequently FP is the perpendicular distance between the planes. Now $\overline{FT}^2 = FS . FP$. But $FS = r$, the radius of the focal sphere; and 2θ being the vertical angle of the cone, $\tan\theta = \frac{r}{VC}$; therefore $\overline{FT}^2 = FP . VC \tan\theta$.

But in the preceding proposition it was shown that $\frac{1}{2}L = \frac{\overline{FT}^2}{VC}$; therefore $\frac{1}{2}L = P\tan\theta$, writing P for FP.

Cor. i.] Since $P\tan\theta = \frac{b^2}{a}$, and the area of the ellipse is πab, the volume of the cone which stands on the ellipse as base is $\frac{1}{3}\pi b^3 \cot\theta$.

Cor. ii.] If a sphere be described with the vertex of the cone as centre, all the plane sections of this cone which touch this sphere have equal parameters.

282.] *Twice the rectangle under the segments of any focal chord is equal to the rectangle under this focal chord and the semiparameter.*

Let the segments of the focal chord mFn (fig. 61) be f and $f_{\prime}$, and let c be the distance from the vertex of the cone to a point C on the circle of contact. Through the vertex V of the cone and the focal chord mFn or $f+f_{\prime}$ let the plane $VamFneV$ pass, intersecting the cone in the triangle Vmn and the focal sphere in the circle $aFeQ$, of which the radius is ρ. From s the centre of the focal sphere draw the perpendicular sx on VF. The plane through sx perpendicular to VF will pass through x the centre of the circle made by the above secant plane whose radius is ρ.

Now $f = mF = ma$, $f_{\prime} = nF = ne$, and $VC = VG_{\prime\prime} = c$.

The following are well known expressions for the area of the triangle Vmn circumscribing the circle $QG_{\prime\prime}Fa$:—

$$[(f+f_{\prime}+l)ff_{\prime}c]^{\frac{1}{2}} = (f+f_{\prime}+c)\rho = \tfrac{1}{2}(f+f_{\prime})p, \quad . \quad . \quad . \quad \text{(a)}$$

p being the perpendicular from V the vertex of the cone on the plane of the conic. But, by similar triangles, $VF : p = Fs_{\prime}$ or $\rho : Fx$.

Hence $p = VF . \frac{Fx}{\rho}$, and x is a point on the circumscribing sphere (see fig. 60).

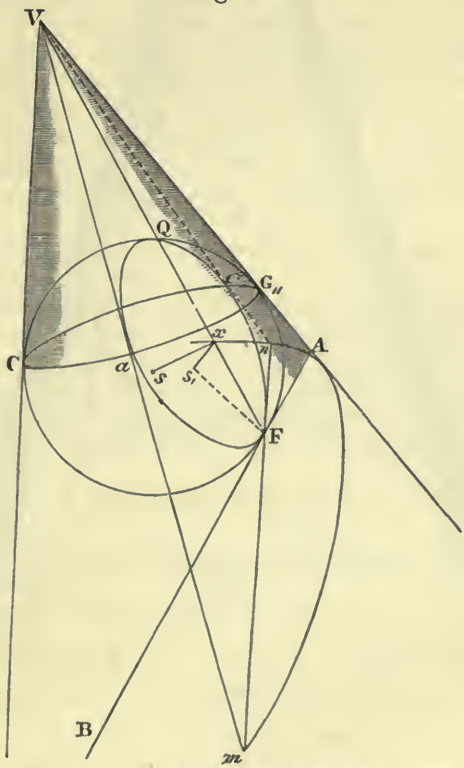
Multiplying together the two latter values for the area, and equating the product with the square of the former, we shall have

$$2[(f+f_{\prime}+c)ff_{\prime}c]=(f+f_{\prime}+c)\rho(f+f_{\prime})p\,;$$

$$\text{therefore }\ 2ff_{\prime}=(f+f_{\prime})\,\frac{\rho p}{c}.$$

Now $p = \mathrm{VF}\,.\,\dfrac{\mathrm{F}x}{\rho}$, while $c = \mathrm{VC}$; and the semiparameter $\frac{1}{2}\mathrm{L}=\dfrac{\mathrm{VF}\,.\,\mathrm{F}x}{\mathrm{VC}}$, as in the last section; consequently $2ff_{\prime}=(f+f_{\prime})\,(\frac{1}{2}\mathrm{L})$.

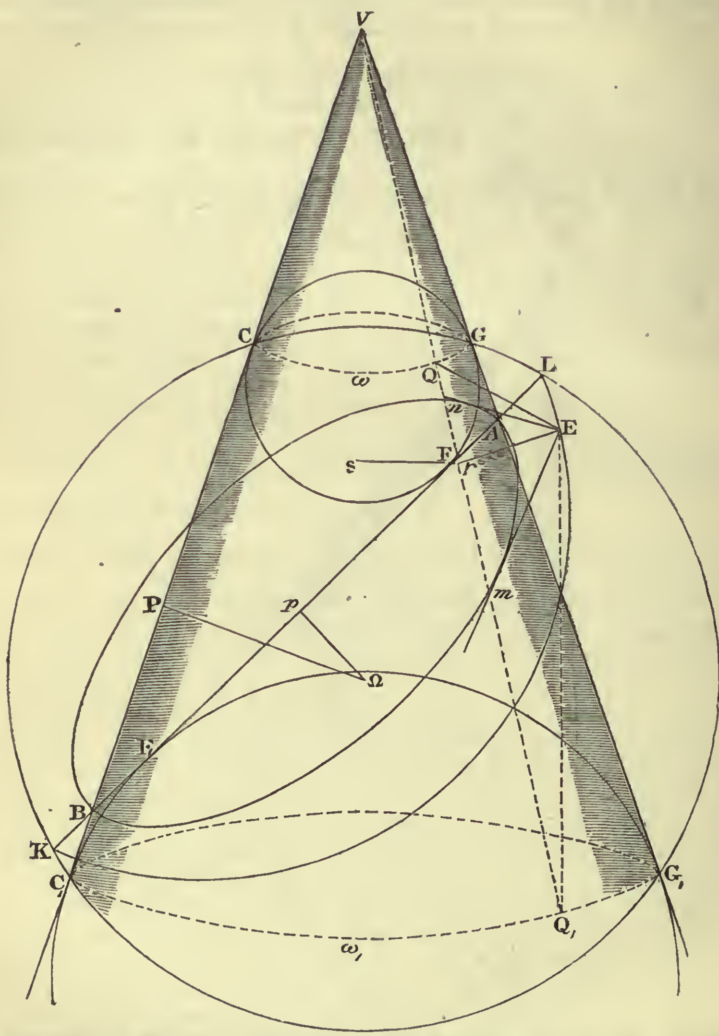
Fig. 61.



283.] *To find the locus of the intersection of pairs of tangents to a conic, meeting at a given angle* (see fig. 62).

Draw any tangent plane $\mathrm{VQ}m\mathrm{Q}_{\prime}\mathrm{E}$ to the cone, and on the line $\mathrm{QQ}_{\prime}$, equal to the major axis of the conic, and in this tangent plane, let a segment of a circle be described capable of containing the given angle. Let a solid be generated by the revolution of this tangent plane to the cone carrying the circular segment with it as described in this plane on the chord $\mathrm{QQ}_{\prime}$.

Fig. 62.



The intersection of this solid (which may be called the *conospheroid*) with the plane of the conic will be the required locus.

In this curve of intersection assume any point E; draw the tangents Em, En to the conic. They will contain the angle mEn. But this angle is equal to the angle QEQ, by the theorem established in sec. [275].

It is evident that this solid will consist of two sheets, the one described by that segment of the generating circle which contains the given angle, and which has $QQ_{,}$ for its chord; the other will be described by the remaining segment of the circle, which contains the supplement of the given angle. It is plain that the two sheets of this conospheroid meet in the two circles of contact of the focal spheres with the cone.

Every plane section of the conospheroid at right angles to the axis of the cone is a circle.

From the point E draw Er at right angles to the side of the cone $VQQ_{,}$, and draw rs at right angles to the line $VQQ_{,}$ until it meets the axis of the cone in s. Then, as Er, rs are each at right angles to the side $VQQ_{,}$ the plane Ers will be perpendicular to the side of the cone $VQQ_{,}$. Therefore the axis of the cone makes a constant angle with this plane srE. E is therefore on the surface of a right cone whose vertex is s and axis sV; and sE is constant, since $\overline{Es}^2=\overline{sr}^2+\overline{Er}^2$, each of which is constant. Consequently, E being on the surface of a right cone, and at a constant distance from the vertex, E must describe a circle at the distance sE from the vertex of this cone.

The projective equation of the conospheroid may be found from the genesis of the surface.

Let θ be the semiangle of the cone, p the perpendicular from the centre of the generating circle on the chord $2a$. Let $2s$ be the sum of the radii of the circles of contact. Let the origin of coordinates be taken on the axis of the cone equidistant from the planes of the circles of contact; let the plane perpendicular to this axis be taken as the plane of xy, and the plane of the focal triangle as the plane of xz.

Then it will not be difficult to show that the projective equation of the conospheroid is

$$x^2+y^2+z^2=s^2+a^2+2p^2-2sz\tan\theta\pm 2p(a^2+p^2-\sec^2\theta z^2)^{\frac{1}{2}}. \quad \text{(a)}$$

The volume V of this surface is

$$V=2\pi r\left[\tfrac{2}{3}r^2+s^2+sr\tan\theta+p^2\pm\tfrac{1}{2}\pi r\cos\theta\right];$$

$V_{,}$ and $V_{,,}$ being the volumes of the two sheets,

$$V_{,}-V_{,,}=2\pi^2 pr^2\cos\theta,$$

an equation of the fourth degree, as it evidently should be;

$$r=\sqrt{a^2+p^2} \text{ is the radius of the generating circle.}$$

Since the expression for the difference of the volumes of the two sheets does not contain $2s$ the sum of the radii of the circles of contact, it will follow that this difference will depend on the *form* but not on the *magnitude* of the cone.

284.] We shall now proceed to find the algebraical equation of the curve which is the locus of the vertex of a constant angle whose sides always touch a conic.

It will add to the simplicity of the investigation, and not detract from its generality, if we assume a right circular cylinder instead of a cone as the dirigent surface.

The equation of the conospheroid as given in (a) is

$$x^2+y^2+z^2=a^2+s^2+2p^2-2sz\tan\theta\pm2p(a^2+p^2-z^2\sec^2\theta)^{\frac{1}{2}};$$

but when the cone becomes a cylinder, 2θ its vertical angle becomes 0, and $s=b$, where b is the radius of the base of the circular cylinder. The equation of the conospheroid now becomes

$$x^2+y^2+z^2=a^2+b^2+2p^2\pm2p(a^2+p^2-z^2)^{\frac{1}{2}}. \quad . \quad . \quad \text{(b)}$$

Let the axis of the conic make the angle ϕ with the base of the cylinder, the axis of Y continuing unchanged. Then we shall have

$$x=x_{\prime}\cos\phi+z_{\prime}\sin\phi, \quad z=x_{\prime}\sin\phi+z_{\prime}\cos\phi.$$

But as we require only the equation of the curve in which these surfaces intersect, we must put $z_{\prime}=0$; and then $x=x_{\prime}\cos\phi$, $z=x_{\prime}\sin\phi$.

Substituting these values in the preceding equation, bearing in mind that $\cos\phi=\frac{a}{b}$, and $\sin\phi=e$, we get, since $\tan^2\alpha=\frac{a^2}{p^2}$,

$$4[a^2y^2+b^2x^2-a^2b^2]=[x^2+y^2-(a^2+b^2)]^2\tan^2\alpha. \quad . \quad \text{(c)}^*$$

285.] When the given conic is a parabola, the locus of the vertex of the constant angle touching the parabola is an hyperbola (fig. 63).

This case may be simply proved by the theorem established in sec. [279].

Let ρ and $\rho_{\prime}$ be the focal distances of any point T outside a conic, then $\rho\rho_{\prime}=\frac{2a\mathrm{P}}{\sin\alpha}$, where α is the angle between the tangents drawn to the conic from the point T.

When the curve is a parabola $\rho_{\prime}=2a=\infty$.

Hence $$\rho=\frac{p}{\sin\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

where p is the perpendicular from T on the chord Fm.

From the point T let tangents Tm, T$m_{\prime}$ be drawn to the parabola, containing the angle α.

Let F be the focus of the parabola, and let the angle FmT be χ. Let T$m=t$, TP$=p$, FT$=\rho$, the angle AFT$=\lambda$, and Tc a perpendicular to the axis of the parabola. We shall have $p^2=\mathrm{T}a\,.\,\mathrm{T}b$.

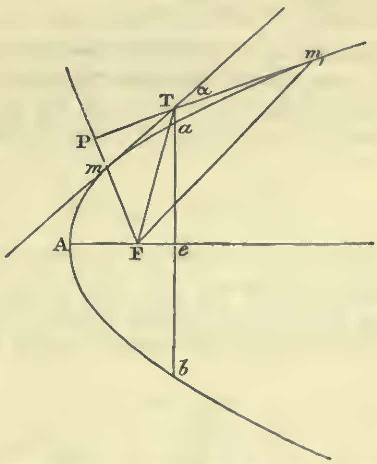
* In the 8th volume of the *Annales de Mathematiques* by GERGONNE the problem to find the locus of the vertex of a given angle is solved by PONCELET. The proof he gives by algebra is complicated and tedious.

DE LA HIRE has also given a solution of this problem. See CHASLES, *Aperçu*, p. 125.

For $t^2 : \mathrm{T}a \,.\, \mathrm{T}b = 1 : \sin^2\chi$; therefore $t^2 \sin^2\chi = \mathrm{T}a \,.\, \mathrm{T}b$. But $p = t\sin\chi$; consequently $p^2 = \mathrm{T}a \,.\, \mathrm{T}b = \overline{\mathrm{T}c}^2 - \overline{ca}^2$. But $\mathrm{T}c = \rho\sin\lambda$; and ca being an ordinate of the parabola whose parameter is $4k$, while $p^2 = \rho^2 \sin^2\alpha$, therefore $\overline{ca}^2 = 4k(k - \rho\cos\lambda)$.

Reducing, we shall find $\rho = \dfrac{2k}{\cos\lambda - \cos\alpha} = \dfrac{2k\sec\alpha}{\sec\alpha \,.\, \cos\lambda - 1}$. (b)

Fig. 63.



If we now compare this expression with the general form of the focal equation of an hyperbola

$$\rho = \frac{\mathrm{A}(e^2 - 1)}{e\cos\lambda - 1},$$

they will be identical if we make $e = \sec\alpha$, and

$$\mathrm{A} = \frac{2k}{\tan\alpha\sin\alpha} = \frac{2k\sec\alpha}{\tan^2\alpha}. \quad \ldots\ldots \quad \text{(c)}$$

The parabola, and the hyperbola which is the locus of the revolving angle, have the same directrix. For the distance of the focus of an hyperbola from its directrix is $\mathrm{A}(e - e^{-1})$; putting for A its value given above, we get for this distance $2k$, the same as in the parabola. α is the angle between the asymptotes of the hyperbola.

286.] When the given angle is a right angle, the generating segments of the tangent circle become semicircles, the two sheets of the conospheroid coalesce, and it becomes a sphere of which the circles of contact are lesser circles.

The radius of this sphere may be thus found. Let $\mathfrak{R}$ be the radius of this sphere; since $\mathrm{GG}_{,} = 2a$, and $\Omega\mathrm{P} = \frac{1}{2}(\mathrm{R} + r)$,

$$\overline{\Omega\mathrm{C}}^2 \text{ or } \mathfrak{R}^2 = a^2 + \tfrac{1}{4}(\mathrm{R} + r)^2 = a^2 + b^2 + \tfrac{1}{4}(\mathrm{R} - r)^2.$$

But $R-r=2a\tan\theta$, 2θ being the vertical angle of the cone.

Therefore $\mathfrak{R}^2=a^2\sec^2\theta+b^2$.

To find the diameter of the circle AEB, since Ω is the centre of the sphere described through the two circles of contact it is the middle point of $\omega\omega_{\prime}$; consequently the circle described round the focal triangle VAB passes through the point Ω.

Now $2\Omega p=R-r=2a\tan\theta$.

Hence, if $\mathfrak{r}$ be the radius of this circle,

$$\mathfrak{r}^2=\overline{\mathfrak{R}}^2-\overline{\Omega p}^2=a^2\sec^2\theta+b^2-a^2\tan^2\theta=a^2+b^2.$$

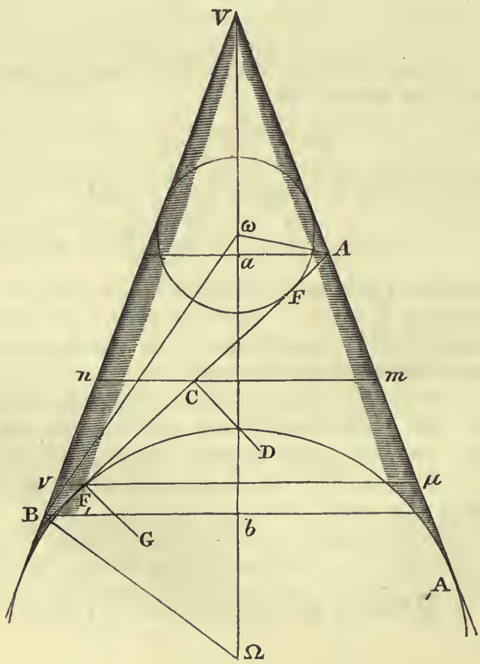
Cor.] When the section is a parabola, the second circle of contact recedes to infinity, the sphere becomes the plane of the circle of contact, CQG therefore the locus becomes the intersection of this plane with the plane of the section, *i. e.* the directrix.

CHAPTER XXVIII.

ON THE CENTRAL PROPERTIES OF CONIC SECTIONS.

287.] *The rectangle under the distances of the vertex of the cone from the centres of the focal spheres is equal to the rectangle under the sides of the cone ending in the vertices of the major axis of the conic.*

Fig. 64.



In fig. 64, since $\Omega B\omega$ is a right angle, the quadrilateral $A\omega B\Omega$ may be inscribed in a circle. Hence the angle $V\Omega B$ is equal to the angle ωAB, which is equal to $VA\omega$. Therefore the triangles $V\Omega B$ and $V\omega A$ are similar; consequently

$$V\Omega \,.\, V\omega = VA \,.\, VB.$$

288.] *Through* C *the middle point of* AB *the major axis of the conic, which is the centre by def.* xiv., *let a plane be drawn at right angles to the axis of the cone. This plane will cut the cone in a circle. The line* CD *in which the planes of this circle and the conic intersect will be the minor axis of this section.*

The square of the common ordinate CD gives $\overline{CD}^2 = Cn \,.\, Cm$.

Now since $BC = CA$, $Cn = Aa = VA \sin\theta$.

In like manner $Cm = Bb = VB \sin\theta$.

Therefore $Cn \,.\, Cm = VA \,.\, VB \sin^2\theta$. But by the preceding theorem $VA \,.\, VB = V\Omega \,.\, V\omega$; and therefore

$$VA \,.\, VB \sin^2\theta = V\Omega \sin\theta \,.\, V\omega \sin\theta = Rr.$$

But $Rr = BF_{\prime} \,.\, AF_{\prime\prime}$ or $AF \,.\, BF$, as in sec. [263].

Therefore the square of half the minor axis is equal to $AF \,.\, BF = Rr$, which is equal to $pp_{\prime}$, as shown in sec. [265].

289.] *The parameter* (*that is, double the ordinate through a focus*) *is* $=\dfrac{2b^2}{a}$.

Through $F_{\prime}$ let a plane be drawn at right angles to the axis of the cone; then, $F_{\prime}G$ being half the ordinate, the intersection of the planes of the circle and conic, $\overline{F_{\prime}G}^2 = F_{\prime}\nu \,.\, F_{\prime}\mu$.

But $F_{\prime}\nu : Cn = BF_{\prime} : BC$ or $F_{\prime}\nu = \dfrac{Cn \,.\, BF_{\prime}}{BC} = \dfrac{Cn \,.\, BF_{\prime}}{a}$.

In like manner $F_{\prime}\mu = \dfrac{Cm \,.\, AF_{\prime}}{a}$. But $AF_{\prime} \,.\, BF_{\prime} = AF \,.\, BF$.

Therefore $F_{\prime}\nu \,.\, F_{\prime}\mu = Cn \,.\, Cm \,.\, \dfrac{AF \,.\, BF}{a^2}$.

Now in the preceding theorem it has been shown that

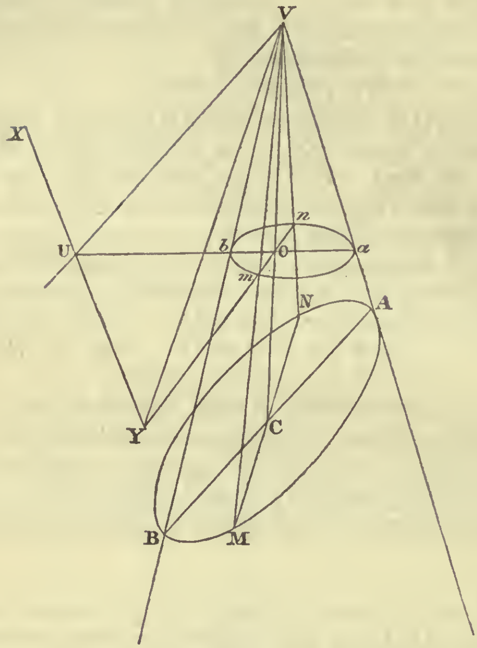
$$Cn \,.\, Cm = b^2, \text{ and } AF \,.\, BF = b^2;$$

hence $\qquad \overline{F_{\prime}G}^2 = F_{\prime}\nu \,.\, F_{\prime}\mu = \dfrac{b^4}{a^2}$; therefore $2F_{\prime}G = \dfrac{2b^2}{a}$.

290.] *The polar axis* VO *of the cone meets the plane of the conic in a point* C, *the centre* (see fig. 65); *and this point bisects all the diameters of the conic.*

Let the *vertical polar* plane, see def. viii., cut the plane of the circle of contact *abmn* in the *dirigent* XY, and let the *polar axis* VOC of this plane meet the plane of the circle of contact in the point O, and the plane of the conic in C. Now as the dirigent XY and the pole O are polar and pole with respect to the circle of contact *abmn*, any plane which passes through the polar axis VO will cut

Fig. 65.



the vertical polar plane in the line VU and the cone in the sides V*a*, V*b*, so that VU, V*b*, VO, V*a* constitute an harmonic pencil in the plane VU*b*O*a* which cuts the plane of the conic in the diameter BCA; and as the plane of the conic is parallel to the vertical polar plane VXY, VU is parallel to AB. Therefore AB is bisected in C.

In the same way, let any other line VY be drawn in the vertical polar plane. Through this line and the polar axis VOC let a plane be drawn cutting the cone in the lines V*m*, V*n*, and the plane of the conic in the diameter MCN. Then as VY, V*m*, VO, V*n* constitute an harmonic pencil, and as MN is parallel to VY, MC = CN.

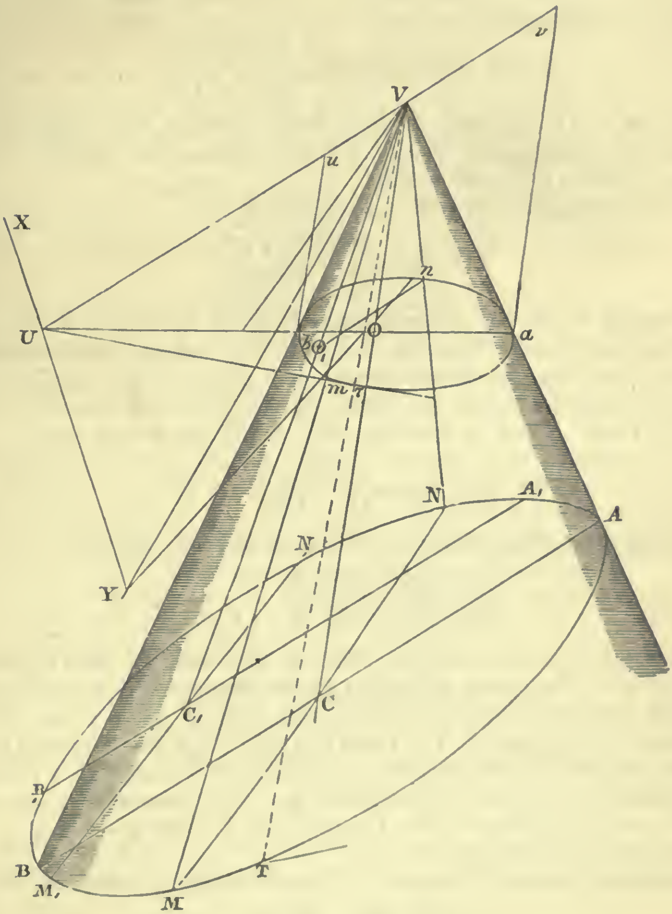
291.] *In any conic the rectangles under segments of parallel chords are proportional to each other.*

Through the polar axis VOC (fig. 66) let two planes be drawn cutting the plane of the conic in the diameters AB, MN, and the vertical polar plane in the lines VU and VY. Through *a* and *b* the points in which the plane VUO cuts the circle of contact draw the lines *av*, *bu* parallel to the polar axis VOC meeting VU in *v* and *u*.

Then we have by similar triangles

$$\mathrm{AC} : \mathrm{V}v = \mathrm{VC} : va \text{ and } \mathrm{V}v : a\mathrm{O} = \mathrm{U}v : \mathrm{U}a. \quad . \quad . \quad \text{(a)}$$

Fig. 66.



Compounding these proportions, we obtain

$$AC : aO = VC \,.\, Uv : va \,.\, Ua. \quad \text{(b)}$$

In like manner $BC \,.\, bO = VC \,.\, Uu : ub \,.\, Ub$, (c)

and therefore

$$AC \,.\, BC : aO \,.\, bO = \overline{VC}^2\, Uv \,.\, Uu : Ua \,.\, Ub \,.\, va \,.\, ub. \quad \text{(d)}$$

* By a suitable alteration in fig. 66 the theorem may as easily be proved when the point in which the chords meet is outside the cone.

Now, by introducing the relations $\frac{Uv}{va}=\frac{Uu}{ub}=\frac{UV}{VO}$, the preceding expression becomes

$$AC \,.\, BC = aO \,.\, bO \,.\, \frac{\overline{VC}^2\,\overline{VU}^2}{\overline{VO}^2\,Ua\,.\,Ub}. \quad . \quad . \quad . \quad . \quad (e)$$

Let $aO\,.\,bO=k^2$, since O is a fixed point in the plane of the circle of contact, and let $Ua\,.\,Ub=t^2$, t being the tangent drawn from U to the circle of contact.

The preceding expression now becomes

$$AC \,.\, BC = \frac{k^2}{t^2}\frac{\overline{VC}^2\,.\,\overline{VU}^2}{\overline{VO}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Through V let any other straight line VY be drawn in the vertical polar plane. Through this line VY and the polar axis VC let another plane be drawn cutting the plane of the conic in the straight line MCN and the plane of the circle of contact in the secant Ymn. Then in the same way it may be shown that

$$MC\,.\,NC = \frac{k^2}{t_{\prime}^2}\frac{\overline{VC}^2\,.\,\overline{VY}^2}{\overline{VO}^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

and therefore, eliminating the common factors, we find

$$\frac{AC\,.\,BC}{MC\,.\,NC} = \frac{\overline{VU}^2\,t_{\prime}^2}{\overline{VY}^2\,.\,t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

Through UV let a tangent plane to the cone be drawn cutting the plane of the circle of contact in the tangent Uτ, and the plane of the conic which is parallel to the vertical polar plane in the tangent to the conic at T. Then the side VT of the cone will make equal angles with the tangent to the cone at T, and with VU which is parallel to it. Let this angle be χ. Then as a side of the cone Vτ is at right angles to the tangent Uτ, UV $\sin\chi=$ Uτ or t. In like manner YV $\sin\chi_{\prime}=t_{\prime}$.

Making these substitutions in the preceding expressions, we get

$$\frac{AC\,.\,BC}{MC\,.\,NC} = \frac{\sin^2\chi_{\prime}}{\sin^2\chi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

It has been shown in sec. [267] that the angle which a side of the cone makes with the tangent to the conic at the point where the side of the cone meets it is equal to the angle which the focal vector makes with the tangent at the same point; and as AC = BC and MC = NC, since C is the centre, we may infer that *any two diameters of a conic are to each other inversely as the sines of the angles which parallel tangents to these diameters make with the focal vectors passing through the points of contact.*

292.] If now through any other point $O_{\prime}$, in the plane of the circle of contact which is *not* the pole of the dirigent XY, a straight line be drawn from the vertex, and meeting the plane of the conic in the point $C_{\prime}$, and if through this line $VC_{\prime}$ and the two lines VU, VY in the vertical polar plane be drawn meeting the plane of the conic in the straight lines $A_{\prime}C_{\prime}B_{\prime}$ and $M_{\prime}C_{\prime}N_{\prime}$, these lines will be parallel to ACB and MCN, since one pair of planes passes through UV, and the other pair through YV, which are each parallel to the plane of the conic. The point $C_{\prime}$, however, will not be the middle point of the chords $A_{\prime}B_{\prime}$, $M_{\prime}N_{\prime}$, since VU, $VB_{\prime}$, $VO_{\prime}$, $VA_{\prime}$ do not constitute an harmonic pencil.

Now repeating the same construction as before, we shall have

$$\frac{A_{\prime}C_{\prime}\,.\,B_{\prime}C_{\prime}}{M_{\prime}C_{\prime}.\,N_{\prime}C_{\prime}}=\frac{\sin^2\chi_{\prime}}{\sin^2\chi};$$

comparing this expression with (i) in the last section, we see that

$$\frac{A_{\prime}C_{\prime}\,.\,B_{\prime}C_{\prime}}{M_{\prime}C_{\prime}.\,N_{\prime}C_{\prime}}=\frac{\overline{AC}^2}{\overline{MC}^2}. \quad \ldots \ldots \quad \text{(a)}$$

Thus may the well known relation between the rectangles under the segments of parallel chords be simply derived from the properties of the right cone, and from this other that if a straight line be drawn parallel to a plane, all the planes drawn through this straight line will cut the plane in parallel straight lines.

293.] Let us assume one of the foregoing rectangles or squares (since AC = BC and MC = NC) as the square of half the major axis a^2, and let the other square be $a_{\prime}^2$; then

$$\frac{a^2}{a_{\prime}^2}=\frac{\sin^2\chi_{\prime}}{\sin^2\chi}, \text{ as before. } \ldots \ldots \quad \text{(b)}$$

Now when the tangent to the conic is drawn parallel to $2a$, the major axis of the conic, $\sin^2\chi=\frac{b^2}{a^2}$, and $\sin^2\chi_{\prime}=\frac{pp_{\prime}}{\rho\rho_{\prime}}$, p, $p_{\prime}$ being the focal perpendiculars on the tangent whose focal angle is $\chi_{\prime}$, and ρ, $\rho_{\prime}$ the focal vectors of the point of contact. Now $pp_{\prime}=b^2$, as shown in sec. [288]. Hence $\sin^2\chi_{\prime}=\frac{b^2}{\rho\rho_{\prime}}$. Substituting for $\sin\chi$, $\sin\chi_{\prime}$ their values, we get

$$a_{\prime}^2=\rho\rho_{\prime}. \quad \ldots \ldots \quad \text{(b)}$$

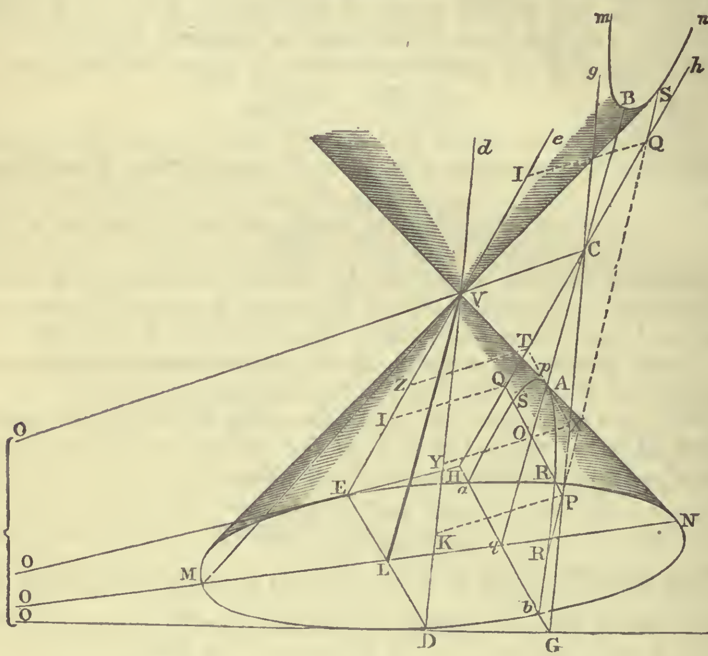
In sec. [276] it has been shown that the rectangle under the segments of a tangent to a conic, intercepted between two parallel tangents to the curve, is equal to the rectangle under the focal vectors of the point of contact. Hence, by the preceding theorem, the rectangle under the segments of the tangents is equal to the square of the parallel semidiameter*.

* This is the theorem which connects the focal and central properties of the conic sections.

ON THE HYPERBOLA AND ITS ASYMPTOTES.

294.] In the preceding sections, the vertical polar plane as defined in def. VIII. is drawn *outside* the cone, while its polar with respect to this cone, the vertical polar axis, is drawn *within* the surface of the cone. We may, however, invert these conditions, and draw the vertical polar axis OV *outside* the cone (as in fig. 67).

Fig. 67.



Through this axis let two tangent planes be drawn to the cone touching it in the sides VD, VE, and cutting the base of the cone in the line DE. These tangent planes may be called Asymptotic Planes. The plane of this triangle VDE will be the vertical polar plane of the axis VO, which meets the tangents DO, EO in the point O.

Let a plane AGH be drawn parallel to the vertical polar plane. This plane will cut the cone in an hyperbola ASaRb. The polar axis OV being produced will meet the plane of the hyperbola in a point C, which, as will be shown, is the centre of the hyperbola; and if the asymptotic tangent planes to the cone drawn through the polar axis OV, and touching the cone along the sides VD, VE, be produced, they will cut the plane of the hyperbola in two straight lines CG, CH meeting in C; and these lines are called the *asymptotes* of the hyperbola.

Since the plane of the hyperbola is parallel to the vertical polar plane VDE, the asymptotic tangent planes to the cone through VD, VE will cut these planes in parallel straight lines VD, CG and VE, CH; or the asymptotes are parallel to the sides of the cone.

Cor. i.] No hyperbola can be cut from a given right cone the angle between whose asymptotes is greater than the vertical angle of the cone.

Cor. ii.] All hyperbolas whose planes are parallel will have the same asymptotic planes; and therefore the angles between their several pairs of asymptotes will be equal.

295.] *Through the vertical polar axis* VO *let a plane be drawn cutting the vertical polar plane in the line* VL, the sides of the cone in the lines VM and VN, and the plane of the hyperbola in the line ACB. Then as VO, VM, VL, VN is an harmonic pencil, and the line ACB is parallel to the line VL in the polar plane VDE, CA=CB, or C the centre bisects all the chords which pass through it.

Since the asymptotes CG, CH are parallel to the sides of cone VD, VE, a line TZ drawn from any point T of an asymptote to the parallel side VZ of the cone and parallel to VC is equal to VC the distance between the vertex of the cone and the centre of the hyperbola, since VCTZ is a parallelogram.

296.] Since the plane of the hyperbola is parallel to the vertical polar plane, the straight lines in which these planes are cut by the asymptotic tangent planes are parallel. As the distance between the plane of the hyperbola and the vertical polar plane is constant, the surface of the cone as it enlarges from the vertex will approach more and more closely to the asymptotes; so also, therefore, will the hyperbola, as it is a curve on the surface of the cone, and whose plane is at a fixed distance from the side of the cone in which it is touched by the asymptotic plane.

297.] *If a straight line meet the hyperbola and its asymptotes, the portions of the line between the curve and the asymptotes are equal.*

Let the secant meet the hyperbola in the points R, S, and the asymptotes in the points Q, P. Through the points Q, P let tangents QI, PK be drawn to the cone parallel to VC, and touching the cone in the points I, K on the sides of the cone VE, VD. Then as VCPK and VCQI are parallelograms, PK is equal to QI, as each is equal to VC. But the rectangle $QS \,.\, QR : PR \,.\, PS = \overline{QI}^2 : \overline{PK}^2$; but $QI = PK$, and therefore $QS \,.\, QR = PR \,.\, PS$ or $QS = PR$.

298.] *If a tangent be drawn to an hyperbola, the portion of it between the asymptotes will be bisected at the point of contact.*

Through A let a tangent TX be drawn, $AT = AX$.

From X and T let tangents TZ, XY be drawn to the cone parallel to VC. They are therefore equal, as each is equal VC. But

$$\overline{AT}^2 : \overline{AX}^2 = \overline{TZ}^2 : \overline{XY}^2;$$

and as TZ = XY, AT = AX.

299.] *The rectangle under the segments of a secant between the asymptotes and a point on the curve is constant, and equal to the square of the parallel tangent between the point of contact and the asymptote.*

Let AT be parallel to the secant QSR; draw tangents to the cone TZ and QI from T and Q, parallel to VC. These tangents are equal, each being equal to VC; hence the rectangle

$$QS \cdot QR : \overline{AT}^2 = \overline{QI}^2 : \overline{TZ}^2.$$

But QI = TZ, as each is equal to VC; therefore $QS \cdot QR = \overline{AT}^2$.

Hence also, *the rectangles under the segments of any parallel secants between the asymptotes and points on the curve are equal.*

300.] While the vertical polar plane and the vertical polar axis are interchanging their positions, the former becomes a tangent plane to the cone, while the polar axis becomes that side of the cone in which it is touched by the vertical polar plane. Hence the plane of the conic which is always parallel to the vertical polar plane, now becomes parallel to a side of the cone; that is, the section is a parabola: and as the centre of the conic is *always* on the polar axis (in this case the side of the cone), the centre of the parabola will be the point in which the side of the cone will meet the plane of the parabola, to which it is parallel—that is, at infinity.

Again, as the vertical polar axis is the line in which the asymptotic planes intersect, and as these tangent planes merge into one when their line of intersection becomes a side of the cone, the asymptotic plane spreads out on either side and meets the plane of the parabola in straight lines parallel to the axis of the conic, but at an infinite distance from it. Hence the parabola partakes of the nature of the hyperbola. It has asymptotes; but they are parallel to its axis at infinity.

CHAPTER XXIX.

ON THE CURVATURE OF THE CONIC SECTIONS DERIVED FROM THE CURVATURE OF THE RIGHT CONE.

Definition.

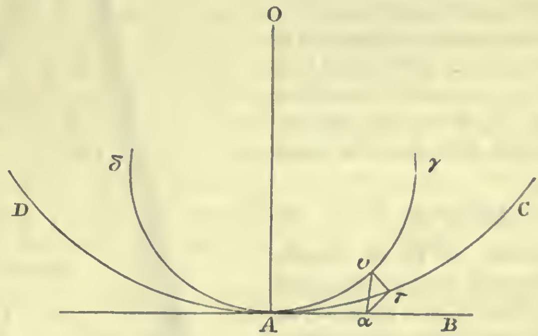
The curvature of a surface at a point A, may be defined as the aggregate of all the curvatures of its sections whose planes pass through the normal to the surface at the point A.

LEMMA.

301.] A tangent being drawn to any curved surface, and $\mathfrak{R}$ being the radius of curvature of a normal section drawn through this tangent, at the point of contact, the radius of curvature of any other plane section drawn through this tangent is $\mathfrak{R}\cos i$, i being the angle between the planes. MEUNIER'S *theorem.*

Let AB be the tangent at the point A, CAOD the normal section, in the plane of the paper, suppose; then the tangent plane to the

Fig. 68.



surface through the tangent AB will be perpendicular to the plane of the paper, and the curved surface on either side of CAD indefinitely near to A is perpendicular to the plane of the curve CAOD. Let $\gamma A\delta$ be a section of the surface made by a plane passing through AB, inclined at an angle i to the plane CAOD. Through α a point assumed on AB indefinitely near to A let the plane $\alpha\tau\upsilon$ be drawn perpendicular to AB, meeting the normal section in τ and the other section in υ. Then $\upsilon\tau\alpha$ is a right angle, and the angle $\upsilon\alpha\tau=i$. Let $\mathfrak{r}$ be the radius of curvature of the section $\gamma A\delta$. Then $\overline{A\alpha}^2=2\mathfrak{R}\,.\,\alpha\tau$, and $\overline{A\alpha}^2=2\mathfrak{r}\,.\,\alpha\upsilon$. But $\alpha\tau=\alpha\upsilon\cos i$; consequently

$$\mathfrak{r}=\mathfrak{R}\cos i. \quad \ldots \ldots \quad \text{(a)}$$

Cor. i.] *Hence if through the circle of curvature of the normal section of the surface, whose plane passes through the tangent* $A\alpha$, *a sphere be described having its centre coincident with that of the circle of curvature of the normal section, a plane passing through the tangent* $A\alpha$ *will cut the surface in a curve and the sphere in a lesser circle, such that the latter will be the circle of curvature of the former at the point* A.

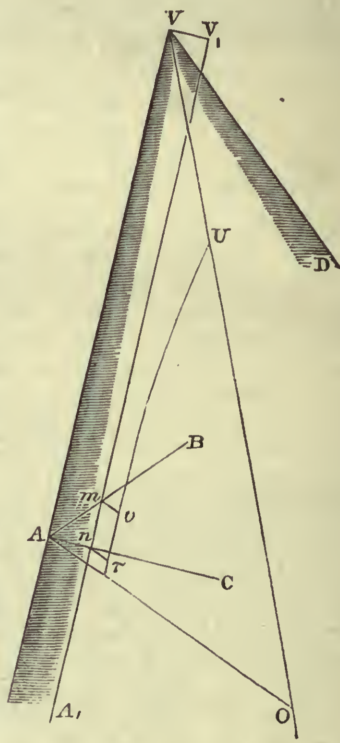
Cor. ii.] *If on the normal to a curved surface, as diameter, a sphere be described passing through the given point* A, *and if the sections of the surface and the sphere made by a plane passing through the* *tangent* AB *have the same curvature, any other plane passing through* AB *will cut the surface and the sphere in sections having the same curvature.*

302.] We shall now proceed to apply this theorem to cones and conics.

If a tangent AB *be drawn to a right cone at a point* A, *and* AC *be drawn in the tangent plane at right angles to the side of the cone* AV, *the radius of curvature of the normal section passing through* AC *is to the radius of curvature of the normal section passing through* AB *at the point* A *as* $\sin^2$ VAB : 1; *or if* $\mathfrak{r}$ *be the radius of curvature through* AC, $\mathfrak{R}$ *the radius of curvature through* AB, *and the angle* VAB be χ,

$$\mathfrak{r}=\mathfrak{R}\sin^2\chi. \quad . \quad . \quad \text{(a)}$$

Fig. 69.



Let a plane AVD be drawn through the axis of the cone, and a tangent plane to the cone along the side AV, and let another plane $A'V'mn$ be drawn parallel and indefinitely near to the former, cutting the tangent plane AVB in the straight line $A_,V_,$ parallel to AV, the tangents AB, AC in the points m and n, and the cone in the hyperbola τ, v, U, of which $A_,V_,$ is one of the asymptotes. Draw $n\tau$, mv parallel to the normal at A, and meeting the hyperbola in τ and v; then $n\tau$, mv are ultimately equal; for in the infinitesimal hyperbola $Uv\tau$, $V_,m \cdot mv = V_,n \cdot n\tau$. But ultimately $V_,m=V_,n$, as each is ultimately equal to VA. Therefore $mv=n\tau$. Now $\overline{Am}^2=2\mathfrak{R}\cdot mv$, and $\overline{An}^2=2\mathfrak{r}\cdot n\tau$, while $\overline{An}^2=\overline{Am}^2\sin^2\chi$; consequently

$$\mathfrak{r}=\mathfrak{R}\sin^2\chi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence *the radii of curvature of all the normal sections of a cone at a given point, and whose planes pass through tangents to the cone at this point, are to each other inversely as the squares of the sines of the angles which these tangents make with the side of the cone passing through the given point.*

303.] *To find the radius of curvature of a conic section at a given* *point on the surface of a cone, whose plane passes through a given tangent to the cone at this point* (fig. 69).

Let A be the given point, AB the given tangent, and VAB $=\chi$.

Let the tangent AC be drawn at right angles to VA. Then if a sphere be described on the normal to the cone touching the tangent plane at A, it will follow from cor. ii. sec. [301] that if any common section of the cone and sphere passing through the tangent AC have the same curvature, every other common section of the sphere and cone passing through the same tangent AC will have the same curvature. Let the sphere now be supposed to be inscribed in the cone, touching the tangent plane at A; it is manifest that the common sections of the cone and sphere passing through the tangent AC parallel to the base of the cone will have the same curvature at A, as the sections in this case are one and the same circle, the "circle of contact" of the sphere with the cone; consequently the great circle of this sphere whose plane passes through the tangent AC is the circle of curvature of the normal section of the cone at A whose plane passes through the tangent AC.

Let VA $=l$, and let the semiangle of the cone be θ, while $\bar{r}$ is the radius of this sphere inscribed in the cone, then $\bar{r}$ is manifestly equal to $l\tan\theta$.

Consequently, if $\mathfrak{R}$ be the radius of curvature of the normal section of the cone through AB, $\mathfrak{R}=\dfrac{\bar{r}}{\sin^2\chi}$ (see sec. [302]), and $\bar{r}=l\tan\theta$. Therefore $\mathfrak{R}=\dfrac{l\tan\theta}{\sin^2\chi}$.

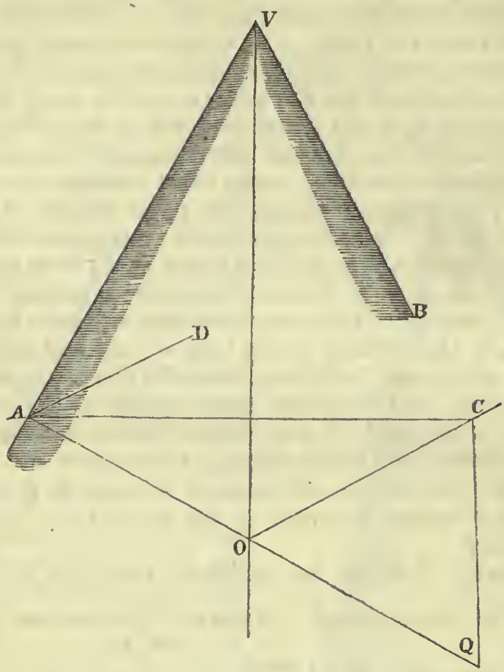
If now a sphere be described touching the tangent plane VACB at A, its radius being $\dfrac{l\tan\theta}{\sin^2\chi}$, every plane passing through the tangent AB will cut the cone in a conic section, and the sphere in a circle, such that the latter will be the circle of curvature of the former at the point A.

304.] *To find the centre of the sphere of curvature for all the sections of the cone whose planes pass through the tangent to the cone* AD (fig. 70).

Let A be a point on the surface of the cone through which the tangent AD is drawn. To the tangent plane VAD draw the normal AO meeting the axis of the cone in O. Through ADO let a plane be drawn, and in this plane make the angle DAC equal to the angle VAD. Through the point O draw the line OC parallel to AD, and meeting the line AC in C. Through the point C draw CQ at right angles to AC, meeting the line AO in Q. AQ is the radius of the sphere of curvature.

Since QCA is a right-angled triangle at C, and OC is at right angles to AQ, the angle CQA $=$ OCA $=$ CAD $=$ VAD $=\chi$. Therefore

Fig. 70.

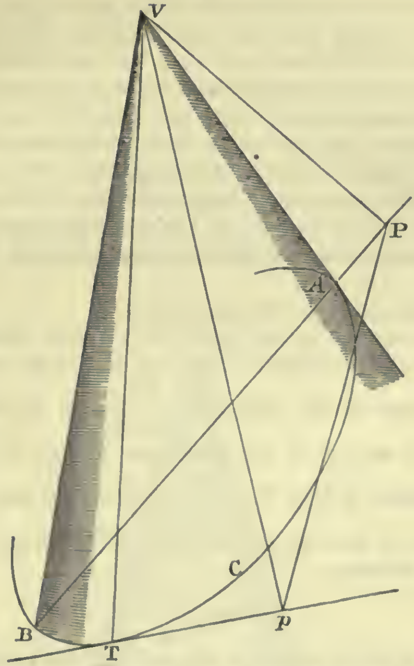


$AC = AQ \sin \chi$, and $AC \sin \chi = AO$. Therefore $AO = AQ \sin^2 \chi$, and $AO = l \tan \theta$. Hence $AQ = \dfrac{l \tan \theta}{\sin^2 \chi}$.

305.] *To find an expression for the radius of curvature of any conic section whose plane passes through the tangent* Tp (fig. 71).

Let ACB be the conic section, Tp the tangent to the cone at the point T in the plane of the conic. From the vertex of the cone draw the perpendicular VP to the plane of the conic, and through VP let a plane be drawn at right angles to the tangent Tp, meeting this tangent in p. Then Vp, Pp are each at right angles to Tp; and therefore the angle VpP is the inclination of the tangent plane to the cone to the plane of the conic. Let $VT = l$, and the angle $VTp = \chi$. The sine of the angle which the plane of the conic makes with the tangent plane is $\dfrac{VP}{Vp} = \dfrac{VP}{l \sin \chi}$. Hence the cosine of the angle which the plane of the conic makes with the normal plane passing through AB is $\dfrac{VP}{l \sin \chi}$. But the radius of curvature of this section is equal to the radius of curvature of the normal section passing through Tp, multiplied by the cosine of the angle between the planes, by

Fig. 71.



Meunier's *theorem*; or the radius of curvature of the conic section at the point T is $\frac{l\tan\theta}{\sin^2\chi}\cdot\frac{\mathrm{VP}}{l\sin\chi}$. Now in sec. [281] it has been shown that $\mathrm{VP}\tan\theta$ is the semiparameter. Hence the radius of curvature is equal to $\frac{(\frac{1}{2}\mathrm{L})}{\sin^3\chi}$. . . . . (a)

Now as χ is also the angle between the tangent and the focal vector at the point A,

$\sin\chi=\frac{p}{\rho}=\frac{p_{\prime}}{\rho_{\prime}}$. But $pp_{\prime}=b^2$, and $\rho\rho_{\prime}=a_{\prime}^2$, p and $p_{\prime}$ being the perpendiculars from the foci on the tangent; therefore $\sin^3\chi=\frac{b^3}{a_{\prime}^3}$, and $\frac{1}{2}\mathrm{L}=\frac{b^2}{a}$. Therefore the radius of curvature $=\frac{a_{\prime}^3}{ab}$. . . . . (b)

Cor.] Hence also the radii of curvature of all conic sections whose planes pass through a given tangent to the cone are, at their points of contact, as their parameters.

In some treatises on conic sections $b_{\prime}$ is put for the semidiameter parallel to the tangent, while $a_{\prime}$ represents the semidiameter through the point of contact; here the notation is reversed.

Definition.

306.] A normal to a conic, at a given point, may be defined as the projection, on the plane of the section, of the radius of the sphere inscribed in the cone, touching the conic at this point.

As the centre of the inscribed sphere is always on the axis of the cone, and as the projection of any point in the axis of the cone on the conic is always on its major axis, therefore the foot of the normal will always be found in the major axis of the conic.

Cor.] The normal is always perpendicular to the tangent to the cone at the given point; for as AO is perpendicular to AB (see fig. 69), its projection on any plane passing through AB will be also perpendicular to AB.

To find an expression for the normal N.

Let N be the normal at the point A, $VA=l$; then the cosine of the angle between the normal plane to the cone passing through AB and the plane of the conic is $\frac{VP}{\sin\chi}$, as shown in the last section; and the radius of the inscribed sphere is $l\tan\theta$; consequently the normal is $l\tan\theta\,.\,\frac{VP}{l\sin\chi}$. Now $VP\tan\theta$ is the semi-parameter $(\frac{1}{2}L)$, as shown in sec. [281]; therefore the expression for the normal becomes

$$N=\frac{(\frac{1}{2}L)}{\sin\chi}.$$

307.] *In any conic section the normal is to the radius of curvature at any given point as the radius of the inscribed sphere is to the radius of the sphere of curvature at that point.*

The radius of curvature of the conic at the given point is $\frac{(\frac{1}{2}L)}{\sin^3\chi}$. The normal at the same point is $\frac{(\frac{1}{2}L)}{\sin\chi}$. The radius of the sphere of curvature is $\frac{l\tan\theta}{\sin^2\chi}$. The radius of the inscribed sphere is $l\tan\theta$. Hence the proposition is manifest*.

* Intelligent students of this subject may have been at a loss to understand why the radius of curvature of a conic section at any point should vary inversely as the cube of the sine of the angle between the tangent and focal vector at that point. These quantities do not appear to have any connexion; there are other quantities with which the radius of curvature would seem to be more nearly allied. But when it is shown that the angle χ is not only the angle between the tangent and the focal vector, but that it is also the angle between a side of the cone and the plane of the normal section of curvature whose radius varies inversely as the square of the sine of this angle, and that the cosine of the angle between the plane of the conic and the plane of this normal circle of curvature varies also inversely as $\sin\chi$, we may thus see how the radius of curvature of the conic section varies as the product of $\frac{1}{\sin^2\chi}$ by $\frac{1}{\sin\chi}$.

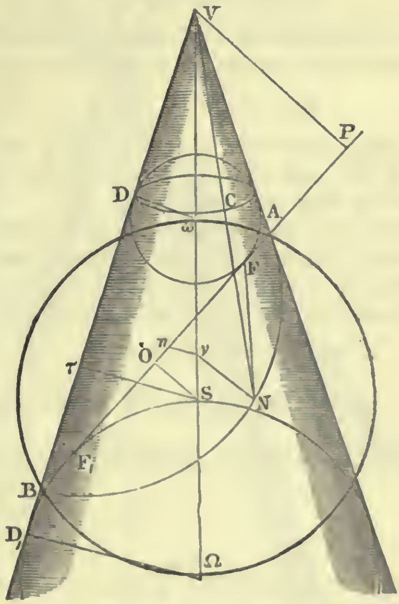
Definition.

The sphere described on the portion of the axis of the cone between the centres of the focal spheres as a diameter, may be called the *central sphere*.

308.] *The distance between the centre of the conic and the foot of the normal is* $(a-\rho)e$, *ρ being the focal vector to the focus* F *from the point* N *to which the normal is drawn*, and e the eccentricity of the conic.

Since the centre S of the *central sphere* is on the axis of the cone, and the centre ν of the normal sphere is also on the same axis, the projections of these two centres on the major axis of the conic will give the centre of this conic and the foot of the normal, as shown in sec. [306].

Fig. 72.



Thus, in fig. 72, let VC$=c$ be a side of the cone between the vertex V and the circle of contact; and as DD$_{,}=2a$, see sec. [261], the distance of V to S, the centre of the central sphere, is $\left(\frac{c+a}{\cos\theta}\right)$, θ being the semiangle of the cone, and i the angle which the axis of the cone makes with the plane of the conic. This line projected on the major axis of the conic, becomes

$$(c+a)\frac{\cos i}{\cos\theta}=(c+a)e=\text{OP}, \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

since $e=\frac{\cos i}{\cos\theta}$, as shown in sec. [271].

In like manner, $c+\rho$ being that portion of the side of the cone to the point N, the distance of the vertex of the cone to the centre ν of the normal sphere will be $\left(\frac{c+\rho}{\cos\theta}\right)$; and this line projected on the major axis of the conic will become

$$\left(\frac{c+\rho}{\cos\theta}\right)\cos i=(c+\rho)e=n\mathrm{P}.$$

Now On is the difference of the projections OP and nP; hence

$$\mathrm{OP}-n\mathrm{P}=\mathrm{O}n=(c+a)e-(c+\rho)e=(a-\rho)e. \quad . \quad . \quad \text{(b)}$$

Cor. i.] The distance between the foot of the normal and the focus is

$$ae-(a-\rho)e=\rho e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Cor. ii.] The distance of the foot of the normal from the other focus is $\rho_i e$; therefore the rectangle under these distances is

$$\rho\rho_i e^2=a_i^2e^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

309.] *The rectangle under the perpendiculars, on the major axis, from the vertex of the cone and the centre of the central sphere is equal to the square of half the minor axis.*

As the major axis (fig. 72) of the conic is a chord of the central sphere whose radius is $a\sec\theta$, the perpendicular on the major axis from this centre will be $a\tan\theta$; and p being the perpendicular from the vertex of the cone on the major axis, the rectangle is

$$a\tan\theta\,.\,p=a\,.\,p\tan\theta.$$

But $$p\tan\theta=\frac{b^2}{a}, \text{ as shown in sec. [281]};$$

therefore the rectangle under the perpendiculars is equal to b^2.

CHAPTER XXX.

ON THE PROPERTIES OF CONFOCAL CONICS DERIVED FROM THE RIGHT CONE.

310.] The consideration of groups of conics that shall have the same centre and foci may be based on an extension of the properties of focal spheres.

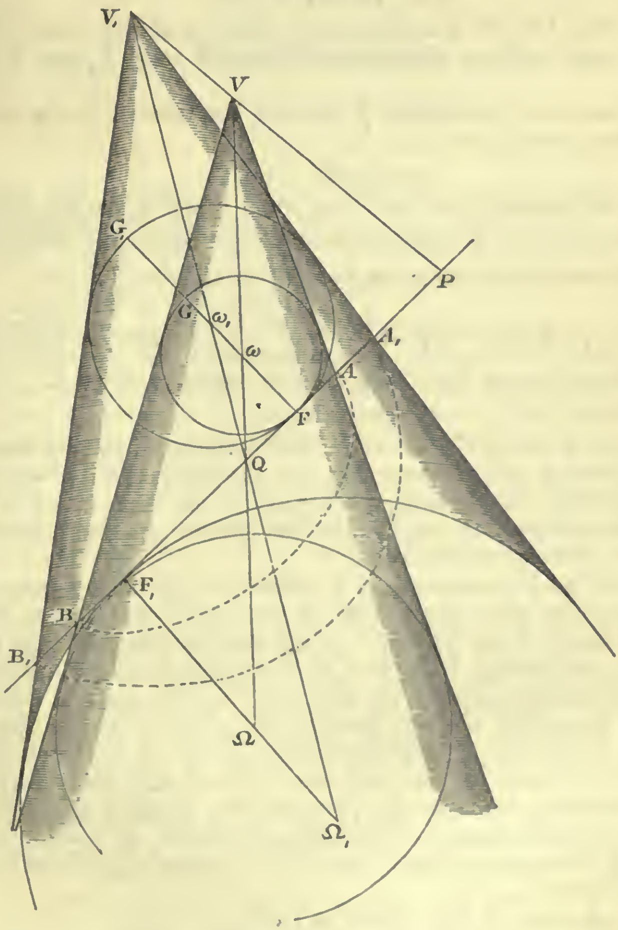
If we conceive the radii of the focal spheres inscribed in the cone to be increased in the same ratio, while the points of contact of the spheres with the plane of the conic continue the same, and if circumscribing cones be drawn to each pair of spheres, whose radii are R, r : nR, nr : $n_{,}$R, $n_{,}r$ and so on, we shall have as many cones circumscribing these spheres, which will cut the plane of the original conic in as many concentric and confocal conics.

Definition.

The point in which the axis of the cone meets the major axis of the conic may be called *the point of axial intersection.*

These cones possess some curious properties.

Fig. 73.



(α) The axes of these cones all pass through a fixed point (the point of axial intersection) on the major axis.

(β) The vertices of all these cones range along the same perpendicular to the plane of the conic.

(γ) The ratio of the distances from the vertex of any one of the cones to the centres of the inscribed focal spheres is constant.

311.] Let planes be drawn through the axes of these cones, they will all cut the major axis in the axial point of intersection Q; and P being the foot of the perpendicular drawn from the vertices of all these cones, we shall have

$$F_{\prime}Q : QF = F_{\prime}P : PF;$$

for $VF_{\prime}$, VQ, VF, VP is an harmonic pencil, as shown in sec. [278].

The angle between the vertical focals VF and $VF_{\prime}$ may be thus found.

The tangent of the angle γ between the vertical focals may be found from the expression

$$\tan\gamma = 2e\tan\theta,$$

θ being the semiangle of the cone, while R and r are the radii of the focal spheres. Let these focal vectors make the angles δ, $\delta_{\prime}$ with the major axis; then $\tan\delta = \frac{ae}{r}$, and $\tan\delta_{\prime} = \frac{ae}{R}$.

Now as $\gamma = \delta - \delta_{\prime}$, $\tan\gamma = \frac{\tan\delta - \tan\delta_{\prime}}{1 + \tan\delta\tan\delta_{\prime}}$, or $\tan\gamma = \frac{ae(R-r)}{a^2e^2 + Rr}$.

But $Rr = b^2$, as in sec. [288], and $(R-r) = 2a\tan\theta$.

Therefore $\tan\gamma = 2e\tan\theta$.

312.] If in sec. [278] the chord mn be supposed to pass through Q, the point of axial intersection, the perpendicular on mn from the intersection of the tangents drawn at the extremities of this chord mn will pass through P the foot of the perpendicular from one of the vertices of the cones.

Hence, if mn be a segment of a common chord to any number of confocal conics, the intersections of every pair of tangents whose common chord is mn will meet in the straight line drawn at right angles to mn through P the foot of the perpendicular to the plane of the section, the locus of the vertices of all the confocal cones.

More generally, if any number of confocal conics have a common chord, and if tangents in pairs be drawn to the conics at the points in which they are met two by two by the common chord, these tangents will meet in pairs on the straight line passing through ϖ at right angles to the common chord. If q be the intersection of the common chord mn with the major axis of the conic, we shall have

$$F_{\prime}q : qF = F_{\prime}\varpi : \varpi F.$$

Hence the position of the point ϖ may be ascertained.

Should the chord mn become a tangent instead of a secant to one of the confocal conics, the pair of tangents coalesce into one tangent meeting on the perpendicular.

313.] Hence we may obtain this other theorem established by a very different method in the first volume, p. 20 :—*If a secant to a conic be a tangent to another confocal conic, and tangents be drawn to the outer conic at the ends of this chord meeting in a point, the line drawn from this point of intersection to the point of contact of the inner confocal section will be perpendicular to this secant.*

CHAPTER XXXI.

ON SIMILAR CONIC SECTIONS.

DEFINITION.

314.] The sections of a cone made by parallel planes may be called *similar conic sections.*

Hence similar conics have the same vertical polar plane and the same polar axis; and therefore all their centres range along the same straight line, the polar axis.

Therefore all circles, parabolas, and equilateral hyperbolas are similar figures; for their vertical polar planes are identical.

Hence all similar hyperbolas have the same asymptotes.

In similar and similarly posited conics all parallel diameters, and homologous lines generally, are in the same ratio, that of the parameters of the conics.

Through the axis VOQ of the cone let a plane be drawn cutting the planes of the parallel conics ABCD and *abnm* in the lines QA, Oa, which lines are themselves parallel; hence (fig. 74)

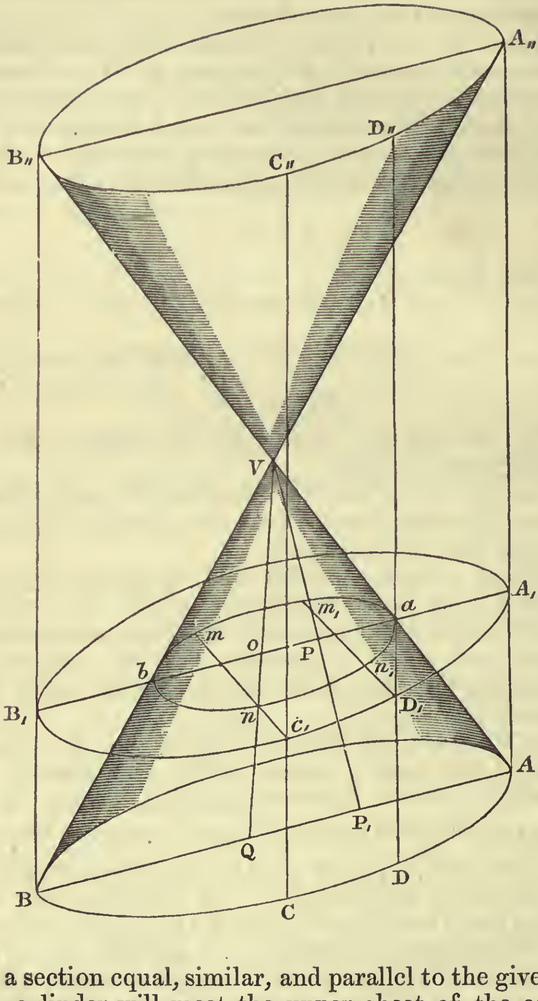
$$Oa : QA = VO : VQ = VP : VP_{,} = VP \cdot \tan\theta : VP_{,} \tan\theta.$$

But $VP\tan\theta$ and $VP_{,}\tan\theta$ are the semiparameters of the two sections, as shown in sec. [281]. In the same way it may be shown of any two homologous lines in the similar sections.

315.] *In two similar concentric and similarly posited conics two parallel chords of one are drawn cutting the other; the rectangle under the segments of the one is equal to the rectangle under the segments of the other.*

Through the opposite cone $VA_{,,}B_{,,}C_{,,}D_{,,}$ let a plane $A_{,,}B_{,,}C_{,,}D_{,,}$ be drawn parallel to the plane ABCD and equidistant from the vertex V. The section of the cone made by this plane will be in every respect equal snd similar to the section ABCD. Now if we conceive a cylinder erected on this base, and having its axis coincident with that of the cone, it will meet the plane of the parallel

Fig. 74.



section in a section equal, similar, and parallel to the given section; hence the cylinder will meet the upper sheet of the cone in the section $A_{\prime\prime}B_{\prime\prime}C_{\prime\prime}D_{\prime\prime}$. Through any point $A_{\prime}$ in the cylinder let a plane be drawn parallel to the given plane ABCD; it will cut the cylinder in a section $A_{\prime}B_{\prime}C_{\prime}D_{\prime}$ equal and similar to ABCD, and the cone in a section *abmn* parallel to the section ABCD, and therefore similar to it. Hence the sections $A_{\prime}B_{\prime}C_{\prime}D_{\prime}$ and *abmn* of the cylinder and the cone are similar and concentric. Through $C_{\prime}$ and $D_{\prime}$, any two points on the surface of the cylinder and in the plane of the section $A_{,}B_{,}C_{,}D_{,}$, let two parallel chords be drawn meeting the section $abmm_{,}nn_{,}$ in the points m, n and the points $m_{,}n_{,}$.

Through $C_{,}$ and $D_{,}$ let two sides of the cylinder be drawn meeting the cone in the points C, $C_{,,}$ and D, $D_{,,}$. Then, as $CC_{,,}$, $DD_{,,}$ and $C_{,}m$, $D_{,}m_{,}$ are parallel secants of the cone,

$$C_{,}C \,.\, C_{,}C_{,,} : C_{,}m \,.\, C_{,}n = D_{,}D \,.\, D_{,}D_{,,} : D_{,}m_{,} \,.\, D_{,}n_{,};$$

but as the three common secant planes of the cylinder and cone are parallel, $C_{,}C = D_{,}D$ and $C_{,}C_{,,} = D_{,}D_{,,}$; therefore $C_{,}m \,.\, C_{,}n = D_{,}m . D_{,}n$. Hence also, if one of the parallel secants of the similar conics becomes a tangent, this tangent will be bisected at the point of contact.

It is manifest that the segments of any chord drawn to meet the similar conics are equal between the sections.

The following properties of right cones and their sections are worthy of notice.

316.] (α) *A tangent to a cone being drawn, there may always be drawn through it two planes cutting the cone in two sections which shall have equal parameters.*

(β) *The conic of maximum parameter which can be drawn through a point on the surface of a right cone is that whose plane is at right angles to the side of the cone passing through the given point, and having its tangent at this point parallel to the circular base of the cone.*

(γ) *Through a given point on the surface of a cone there may be drawn any number of planes cutting the cone in conics having the same parameter; and their planes will all touch a right cone, whose vertex is the given point and whose axis is the side of the original cone passing through the given point.*

(δ) *The locus of the foci of all the parabolas which can be constructed on a given right cone is also a right cone. The locus of the foci of all equal parabolas on the cone is a circle whose plane is parallel to that of the base; and the locus of the foci of all the parabolas whose planes are parallel is a straight line passing through the vertex of the cone. Hence the locus of the foci of all the parabolas that can be drawn on the cone is the combination of the above named loci, a cone.*

CHAPTER XXXII.

ON CONICS IN A PLANE.

317.] Having now, by the help of the right circular cone, established the principal properties of its sections by short and simple demonstrations based on the most elementary principles of the ancient geometry, it is proper to show how these principles may be applied to the development of the properties of conics viewed as curves on a plane. When the cardinal properties of conics have once been established by the help of the right circular cone, there is but little difficulty in applying them to the investigation on a plane of the countless theorems relating to those curves. Geometers have in general ignored their real origin, and have founded their investigations on some arbitrary definition. It is worthy of notice that some of the theorems which are most easily and briefly demonstrated when these curves are taken as sections of a right cone, can be established only by tedious and complicated methods when they are treated as plane curves. Of this the focal properties furnish a striking example. There is, however, a very large class of theorems to whose solutions neither the right cone nor any other cone affords any help. I refer to the properties of *minor foci* and *minor directrices*, which will be found discussed at some length in the first volume of this work (see Vol. I. sec. [288]).

The method of *reciprocal polars* applied to oblate and prolate spheroids is the source of innumerable novel properties of conics. One special application of the method of reciprocal polars to the prolate or elongated ellipsoid of revolution round its major axis (see Vol. I. p. 218) enables us to develop to a very great extent and with much simplicity the properties of surfaces of the second order whose three axes are unequal. In particular it is shown that every such surface has four foci and four directrix planes. Every new method may be a fertile field of new truths.

In sec. [265] it is shown that the rectangle under the focal perpendiculars p, $p_{\prime}$ on a tangent drawn to a central conic is equal to b^2. Let P be the perpendicular from the centre on the same tangent, making the angle λ with the major axis. Then, obviously

$$p+p_{\prime}=2\mathrm{P}. \quad \text{(a)}$$

Let ρ, $\rho_{\prime}$ be the focal vectors to the same point.

Then $\rho+\rho_{\prime}=2a$, as shown in sec. [262].

Let χ be the *focal tangential angle*, then we have

$$2\mathrm{P}=p+p_{\prime}=(\rho+\rho_{\prime})\sin\chi=2a\sin\chi.$$

Hence $$\mathrm{P}=a\sin\chi. \quad \text{(b)}$$

Since $\qquad p=\mathrm{P}+ae\cos\lambda$, and $p_{\prime}=\mathrm{P}-ae\cos\lambda$, . . . (c)

$$b^2=pp_{\prime}=\mathrm{P}^2-a^2e^2\cos^2\lambda.$$

Therefore $\qquad \mathrm{P}^2=a^2(1-e^2\sin^2\lambda)$. . . . . (d)

Comparing this expression with (b) we find

$$\cos\chi=e\sin\lambda, \qquad \text{(e)}$$

a simple relation which connects the focal tangential angle χ with the latitude λ.

318.] Since $\rho+\rho_{\prime}=2a$, squaring,

$\rho^2+2\rho\rho_{\prime}+\rho_{\prime}^2=4a^2$. Now $2\rho\rho_{\prime}=2a_{\prime}^2$, as shown in (b), sec. [293], while $\rho^2+\rho_{\prime}^2=2b_{\prime}^2+2a^2e^2$.

Hence, substituting, $\quad a_{\prime}^2+b_{\prime}^2=a^2+b^2$. . . . . (a)

Since $a_{\prime}^2=\rho\rho_{\prime}$ and $\mathrm{P}^2=a^2\sin^2\chi$, $a_{\prime}^2\mathrm{P}^2=a^2\rho\sin\chi\cdot\rho_{\prime}\sin\chi$,

or $\qquad a_{\prime}^2\mathrm{P}^2=a^2pp_{\prime}=a^2b^2$. . . . . (b)

Hence the areas of parallelograms about the conjugate diameters of a conic are equal.

Let $b_{\prime}^2=x^2+y^2$, then $a^2b_{\prime}^2=a^2x^2+a^2y^2$. But $a^2y^2=a^2b^2-b^2x^2$;

$$\left.\begin{array}{ll}\text{hence} & a^2b_{\prime}^2=a^2b^2+(a^2-b^2)x^2, \text{ or } b_{\prime}^2=b^2+e^2x^2;\\ \text{and in like manner} & a_{\prime}^2=a^2-e^2x^2.\end{array}\right\} \qquad \text{(c)}$$

319.] The following values of the radius of curvature, and chords of curvature passing through the foci and centre, may easily be derived from the expressions in secs. [303], [304], [305], which have themselves been deduced from the properties of the right cone.

In sec. [305] it has been shown that

$$\mathfrak{R}=\frac{b^2}{a\sin^3\chi} \text{ and } \mathrm{N}=\frac{b^2}{a\sin\chi}, \qquad \text{(a)}$$

while $\qquad \mathrm{P}=a\sin\chi$, and $\sin\chi=\dfrac{b}{a_{\prime}}$. . . . . . (b)

From these values we may obtain the following expressions for the radius of curvature and the normal—

$$\left.\begin{array}{c}\mathfrak{R}\mathrm{P}^3=a^2b^2,\ \mathfrak{R}\mathrm{P}=a_{\prime}^2,\ \mathfrak{R}=\dfrac{a_{\prime}^3}{ab},\ \mathrm{N}=\dfrac{b^2}{\mathrm{P}}=\dfrac{ba_{\prime}}{a},\\ a\mathrm{N}=ba_{\prime},\ \mathfrak{R}=\dfrac{a^2\mathrm{N}^3}{b^4},\ \mathfrak{R}\sin^2\chi=\mathrm{N}.\end{array}\right\} \qquad \text{(c)}$$

If ψ and χ be the angles which a tangent to the curve makes with the central and focal vectors,

$$\sin^2\psi=\frac{a^2\sin^4\chi}{a^2\sin^2\chi-b^2\cos^2\chi}. \qquad \text{(d)}$$

Hence C, the chord of curvature through the centre is $\dfrac{2a_{\prime}^2}{b_{\prime}}$, (e)

while the chord of curvature through the focus is $\dfrac{2a_{\prime}^2}{a}$. . . (f)

320.] *If a line be drawn from the focus to the pole of a focal chord, it will be at right angles to this chord,* see sec. [270].

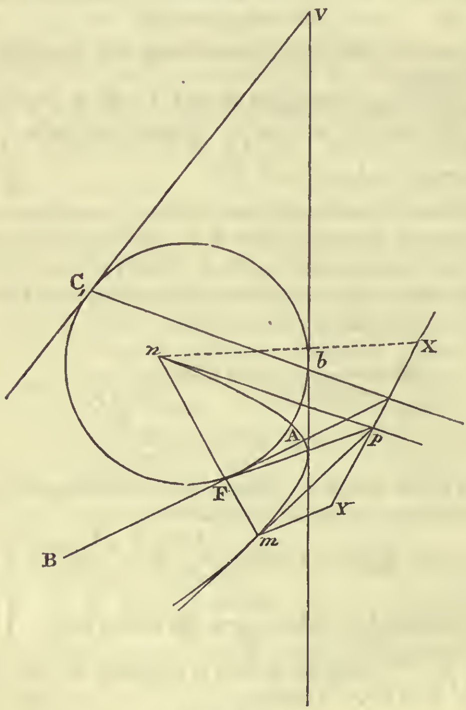
In the parabola mpn is a right angle.

Since $m\mathrm{F}=m\mathrm{Y}$, and the angle $p\mathrm{F}m$ is equal to the angle $m\mathrm{Y}p$, both being right angles, the angle $\mathrm{Y}p\mathrm{F}$ is bisected by the tangent pm. In the same way the angle $\mathrm{X}p\mathrm{F}$ is bisected by the tangent pn; consequently the angle mpn is a right angle.

Cor. i.] We have also $\mathrm{Y}p=\mathrm{X}p=\mathrm{F}p$.

Hence, *if from the ends of a focal chord of a parabola perpendiculars are drawn to the directrix, the pole of the focal chord will bisect the portion of the directrix between the feet of the perpendiculars.*

Fig. 75.



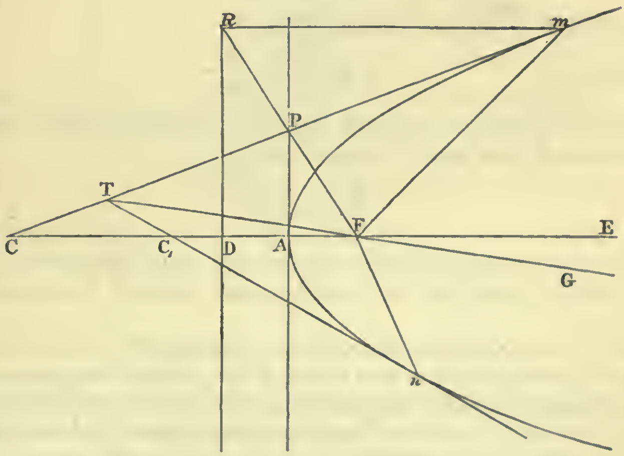
A tangent to a parabola makes equal angles with the focal vector drawn to the point of contact and with the axis of the curve.

This is evident from an inspection of fig. 75.

321.] *The focal vector drawn through the intersection of a pair of tangents to a parabola divides the angle between these tangents into two, which are respectively equal to the alternate angles which the* *tangents make with the focal vectors passing through the points of contact.*

Let the tangents Tm, Tn to the parabola meet in T; let F be

Fig. 76.



the focus of the parabola; and let the tangents meet the axis of the curve in the points C, C$_{,}$.

Then the angle mFE is equal to twice the angle mCF, and the angle nFE is equal to twice the angle nC$_{,}$F; hence, adding, the angle mFn is equal to twice the angle mTn, or half the angle mFn is equal to the angle mTn. Now the line TFG bisects the focal angle mFn; therefore the angle mFG is equal to the angle mTn. But, being external, it is also equal to the sum of the angles FmT and FTm. Therefore the sum of the angles FmT and FTm is equal to the sum of the angles FTn and FTm; consequently the angle FmT is equal to the angle FTn.

Hence, since the angle TFm is equal to the angle TFn, the two triangles TFm and TFn are similar; therefore $m\mathrm{F} : \mathrm{TF} = \mathrm{TF} : n\mathrm{F}$, or

$$m\mathrm{F} \, . \, n\mathrm{F} = \overline{\mathrm{TF}}^2.$$

Hence, *in a parabola, the square of the focal vector drawn to the intersection of a pair of tangents to the curve is equal to the rectangle under the focal vectors drawn to the points of contact of these tangents.*

322.] *The squares of the tangents* Tm, Tn (fig. 76) *drawn to a parabola from any point* T *are in the same ratio as the focal vectors drawn to the points of contact* m, n.

Let FP be a perpendicular drawn from the focus to the tangent Tm, then the area of the triangle TFm is $=\frac{1}{2}\text{FP}\,.\,\text{T}m$. But if ψ be the angle TFm, the area of this triangle is also $\frac{1}{2}\text{FT}\,.\,\text{F}m\,.\,\sin\psi$; therefore

$$\sin\psi=\frac{\text{FP}\,.\,\text{T}m}{\text{FT}\,.\,\text{F}m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

But the angle TFn is also equal to ψ;

therefore $$\sin\psi=\frac{\text{FP}_{\prime}\,.\,\text{T}n}{\text{FT}\,.\,\text{F}n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Equating these values of $\sin\psi$, squaring, putting for $\overline{\text{FP}}^2$ and $\overline{\text{FP}_{\prime}}^2$ their values $k\,.\,\text{F}m$ and $k\,.\,\text{F}n$, we get

$$\frac{\overline{\text{T}m}^2}{\overline{\text{T}n}^2}=\frac{\text{F}m}{\text{F}n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Hence also the chord mn is divided into segments by the line TF, *which are to each other in the duplicate ratio of the tangents* Tm *and* Tn.

323.] *If a parabola be inscribed in a triangle, the circle which circumscribes the triangle passes through the focus of the parabola.*

This theorem follows immediately from that established in sec. [277], in which it is shown that, if a conic be inscribed in a triangle, the sum of the angles subtended at the foci by the base of the triangle is equal to the external vertical angle of the triangle. Now when the conic becomes a parabola, the remote focal angle vanishes, and therefore the angle at the near focus, subtended by the base of the triangle, is equal to the external vertical angle of the triangle; and therefore a circle may be drawn through the vertices of the quadrilateral ACBF*.

Since $\overline{\text{FC}}^2=\text{FA}_{\prime}\,.\,\text{FB}_{\prime}$, $\overline{\text{FB}}^2=\text{FC}_{\prime}\,.\,\text{FA}_{\prime}$, $\overline{\text{FA}}^2=\text{FB}_{\prime}\,.\,\text{FC}_{\prime}$, . (a)
then, as ACBF is a quadrilateral in a circle,

$$\text{CB}\,.\,(\text{FB}_{\prime}\text{FC}_{\prime})^{\frac{1}{2}}+\text{CA}\,.\,(\text{FC}_{\prime}\,.\,\text{FA}_{\prime})^{\frac{1}{2}}=\text{AB}\,.\,(\text{FA}_{\prime}\,.\,\text{FB}_{\prime})^{\frac{1}{2}};$$

consequently by division we obtain finally

$$\frac{\text{CB}}{\sqrt{\text{FA}_{\prime}}}+\frac{\text{CA}}{\sqrt{\text{FB}_{\prime}}}-\frac{\text{AB}}{\sqrt{\text{FC}_{\prime}}}=0. \quad . \quad . \quad . \quad . \quad (b)$$

Hence *the sum of the sides of the circumscribing triangle, each divided by the square root of the focal vector drawn to its point of contact with the parabola, is constant.*

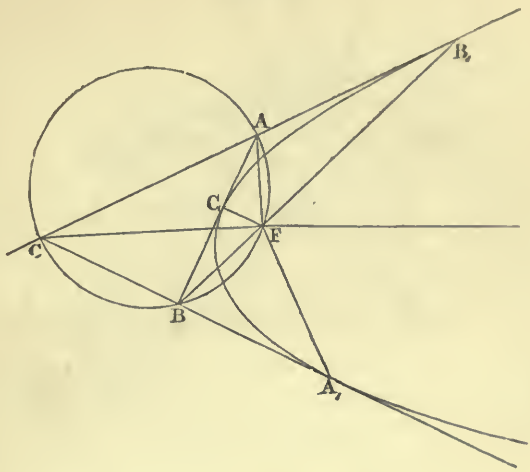
If we multiply together the expressions in (a), we shall have

$$\text{FA}\,.\,\text{FB}\,.\,\text{FC}=\text{FA}_{\prime}\,.\,\text{FB}_{\prime}\,.\,\text{FC}_{\prime}. \quad . \quad . \quad . \quad . \quad (c)$$

Hence, *when a triangle circumscribes a parabola, the product of the focal vectors drawn to the vertices of the triangle is equal to the*

* This theorem is otherwise established, and very simply, in the first volume, see sec. [53], by an application of the method of tangential coordinates.

Fig. 77.



product of the focal vectors drawn to the points of contact of the sides of the triangle with the parabola.

324.] Since the sums of the rectangles under the adjacent sides of a quadrilateral inscribed in a circle are as the diagonals which join the points in which the sides of the rectangles meet, we have

$$AB \, . \, BC \, . \, CA = CA \, . \, FC \, . \, FA + CB \, . \, FC \, . \, FB - AB \, . \, FA \, . \, FB.$$

But $\overline{FA}^2 = FB_{\prime} \, . \, FC_{\prime}$, $\overline{FB}^2 = FA_{\prime} \, . \, FB_{\prime}$, and $\overline{FC}^2 = FA_{\prime} FB_{\prime}$.

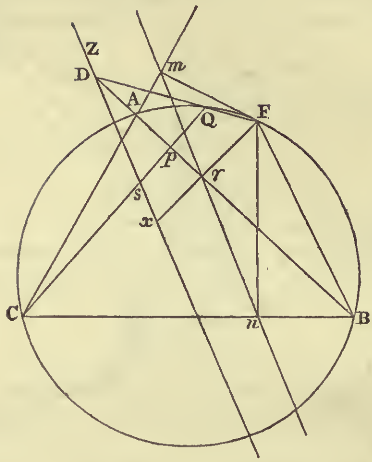
Substituting these values in the preceding equation, we get

$$\frac{AB \, . \, BC \, . \, CA}{[FA_{\prime} \, . \, FB_{\prime} \, . \, FC_{\prime}]^{\frac{1}{2}}} = AC \, . \, \sqrt{FB_{\prime}} + BC \sqrt{FA_{\prime}} - AB \, . \, \sqrt{FC_{\prime}}.$$

325.] *The directrices of all the parabolas inscribed in a triangle pass through the orthocentre of this triangle* (see fig. 78).

From the focus F, on the circumference of the circle, draw the perpendiculars F*m*, F*n*, F*r* on the sides AC, CB, AB of the given triangle. The points *m*, *r*, *n* range along a straight line, which is a tangent to the parabola at its vertex. Produce F*r* until *rx* is equal to F*r*, and through *x* draw *x*Z parallel to *mrn*. *x*Z is the directrix. Produce *r*A to meet the directrix in D. Join DF meeting the circle in Q. Join CQ, FB. Then the angle CQD = the angle FBC, since CQFB is a quadrilateral in a circle. The angle QF*r* = the angle D*xr* = the angle *mr*F = the angle FBC, since F*rn*B is a quadrilateral that may be inscribed in a circle. Therefore the angle DQC is equal to the angle DF*x*, or F*x* is parallel to C*p*.

Fig. 78.



Therefore CpD is a right angle; and therefore $ps = pQ$, or s (a point on the directrix) is the orthocentre.

326.] *The inscription of the maximum parabola in a triangle involves the trisection of an angle* (see fig. 79).

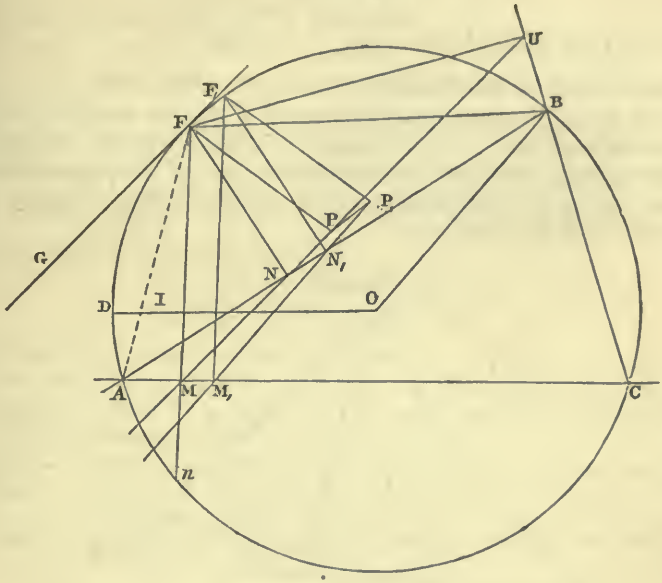
Let ABC be the triangle, and let F be the focus of the maximum parabola. From F draw the perpendiculars FM, FN, FU on the sides of the triangle AC, AB, BC; the line MNU is a tangent to the parabola at its vertex (see preceding theorems). To this tangent MNU draw the perpendicular FP; FP will be one fourth of the parameter of the maximum parabola inscribed in the triangle.

Assume a point $F_{\prime}$ on the circumference of the circle indefinitely near to F, and from this point draw the perpendiculars $F_{\prime}M_{\prime\prime}$, $F_{\prime}N_{\prime}$ to the sides of the triangle AC, AB. The line $M_{\prime}N_{\prime}$ will be a tangent to the parabola whose focus is $F_{\prime}$; draw to this tangent the perpendicular $F_{\prime}P_{\prime}$. $F_{\prime}P_{\prime}$ is one fourth the parameter of the parabola drawn indefinitely near to the former; therefore $F_{\prime}P_{\prime} = FP$; and they are ultimately parallel, therefore FG the tangent to the circle at F is parallel to MNU. But as FAMN is a quadrilateral in a circle, the angle FAB is equal to the angle FMN$=$GFM. Therefore $Fn = FB$. Draw OD parallel to AC, cutting the line Fn in I, then $FI = nI$; therefore FI is equal to the half of FB; and therefore the angle FOI is one half the angle FOB, or the arc BFD is trisected in F.

This question may be taken as a good illustration of the application of the method of infinitesimals to the solution of problems in geometrical maxima and minima.

When the given triangle ABC is isosceles, the angle to be trisected becomes a right angle.

Fig. 79.



327.] By this method of geometrical limits problems which present great difficulty if treated by algebra or the differential calculus, may be solved with great simplicity. For example.

To draw the minimum line through a given point within a given angle (see fig. 80).

Let BAC be the given angle, O the given point, and BOC the minimum straight line. Draw the perpendicular AD from A to BC, and through O draw the line bOc indefinitely near to the line BOC, meeting the sides of the given angle in the points c, b. Then as BOC is the minimum line through O, bOc which is indefinitely near to it, is therefore equal to it. With O as centre draw the circles whose radii are OC, Ob cutting the lines bc and BC in the points m, n. Then as OC$=$Om, and O$b$$=On$, $cm$$=Bn$. Let ω be the infinitesimal angle between the minimum lines. Then B$n$$=$$bn$ cot B, and $bn$$=$OB . ω. Therefore B$n$$=$OB . ω . cot B. In like manner $cm$$=$OC . ω . cot C. Therefore as B$n$$=$$cm$,

$$\text{OC} \tan \text{B} = \text{OB} \tan \text{C} \; ; \text{ hence } \frac{\text{OC}}{\text{OB}} = \frac{\tan \text{C}}{\tan \text{B}}.$$

But $$\tan \text{C} = \frac{\text{AD}}{\text{CD}}, \text{ and } \tan \text{B} = \frac{\text{AD}}{\text{BD}};$$

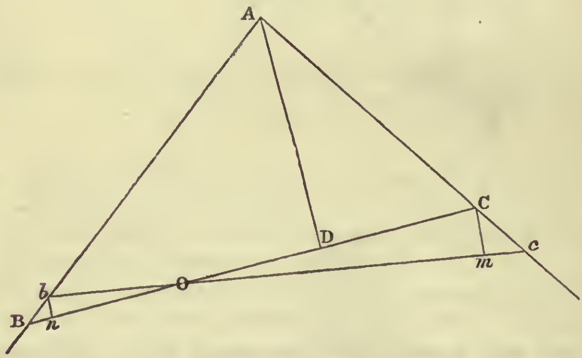
hence $$\frac{OC}{OB}=\frac{BD}{CD}. \quad \text{(a)}$$

This may be reduced to $\frac{CO+OB}{OB}=\frac{CD+DB}{CD}$.

But $CO+OB=CD+DB=BC$; therefore $OB=CD$. . . (b)

Hence the minimum line drawn through a point within an angle may be defined as *the line to which if a perpendicular be drawn from the vertex, the distance between the foot of the perpendicular and one end of the base shall be equal to the distance between the given point and the other end.*

Fig. 80.

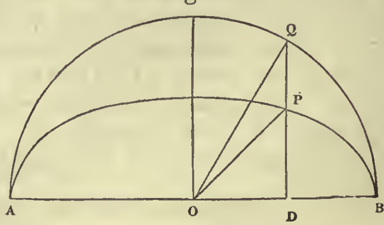


The point D is the intersection of a semicircle, drawn on the line AO, with a curve of the fourth order.

ON THE ECCENTRIC ANOMALY IN AN ELLIPSE.

328.] Let a circle be described on the major axis of an ellipse as diameter, and an ordinate PD to the major axis be produced through the point P to meet the circle in the point Q. The radius of the circle through this point makes the angle μ with the major axis. The angle μ is called the *eccentric anomaly*; and the angle λ which the perpendicular from the centre on the tangent through the point P makes with the major axis is called the *latitude*.

Fig. 81.



Since $x=a\cos\mu$, $y=b\sin\mu$, the semidiameter

$$\overline{\mathrm{OP}}^2 \equiv b_i{}^2=a^2\cos^2\mu+b^2\sin^2\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

In like manner

$$a_i{}^2=a^2\sin^2\mu+b^2\cos^2\mu, \text{ and } \mathrm{P}^2=a^2\cos^2\lambda+b^2\sin^2\lambda. \quad \text{(b)}$$

Thus P and b_i reciprocate their forms. Since $x=a\cos\mu$, and $a^2\xi=x$, $a\xi=\cos\mu$. In like manner $bv=\sin\mu$.

Let d be that semidiameter of the ellipse which coincides with the eccentric radius OQ, and which makes the angle μ with the major axis, then $\frac{1}{d^2}=\frac{\cos^2\mu}{a^2}+\frac{\sin^2\mu}{b^2}$. But P being the perpendicular on the tangent through the point P,

$$\frac{1}{\mathrm{P}^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{\cos^2\mu}{a^2}+\frac{\sin^2\mu}{b^2}; \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

therefore $\mathrm{P}=d$, whence this theorem :—

The perpendicular from the centre of the ellipse on the tangent through the point P *is equal to the semidiameter which coincides with* OQ *the eccentric radius of the circle.*

329.] *To find the relation between the angle of the eccentric anomaly μ, and the focal tangential angle χ.*

Since $\frac{1}{\mathrm{P}^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{\cos^2\mu}{a^2}+\frac{\sin^2\mu}{b^2}$, and $\mathrm{P}=a\sin\chi$, we find

$$\tan\chi=\frac{b}{ae\sin\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Hence

$$\sin^2\chi=\frac{b^2}{a^2\sin^2\mu+b^2\cos^2\mu}. \quad . \quad . \quad . \quad . \quad \text{(e)}$$

To find the relation between λ and μ.

Since $\mathrm{P}^2=a^2\cos^2\lambda+b^2\sin^2\lambda$ and $\frac{1}{\mathrm{P}^2}=\frac{\cos^2\mu}{a^2}+\frac{\sin^2\mu}{b^2}$, we get

$$a\tan\mu=b\tan\lambda; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

therefore λ is greater than μ.

In sec. [305] it has been shown that if $\mathfrak{R}$ be the radius of curvature,

$$\mathfrak{R}=\frac{b^2}{a\sin^3\chi}. \quad \text{(c)} \qquad \text{Hence } \mathfrak{R}=\frac{[a^2\sin^2\mu+b^2\cos^2\mu]^{\frac{3}{2}}}{ab}. \quad \text{(d)}$$

So also

$$\mathfrak{R}=\frac{a^2b^2}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

Since the normal N is equal to $\frac{b^2}{a\sin\chi}$, $a^2\mathrm{N}^2=b^2[a^2\sin^2\mu+b^2\cos^2\mu]$.

Comparing this experiment with the preceding, we get $\mathfrak{R}=\frac{\mathrm{N}^3}{(\frac{1}{2}\mathrm{L})^2}$, or the radius of curvature is equal to the cube of the normal divided by the square of the semiparameter.

CHAPTER XXXIII.

ON ORTHOGONAL PROJECTION.

330.] In orthogonal projection the several points and lines of the original or *projective* figure generate another or *projected* figure on a plane inclined to the former, the locus of the feet of the perpendiculars drawn from every point of the projective to the projected figure. These terms will be found simple and useful in saving much circumlocution. The *projective* figure is cast into its *projected* derivative. Thus in a right circular cylinder, the *projective* ellipse generates the *projected* circle on a horizontal plane.

The principles of orthogonal and divergent projection are often found to be simple yet powerful instruments of investigation, especially where it may be required to pass from the projective properties of a circle to those of a conic. Let an ellipse be conceived as an oblique section of a right cylinder standing on a circle as base. The projective properties of the circle may be at once transferred without demonstration to the ellipse. For example :—

(α) *All the radii of a circle are equal; and therefore all the diameters of an ellipse are bisected in the centre.*

(β) The squares which circumscribe a circle are equal, and the diameters which join the points of contact of the sides of the square are parallel to the sides; hence *all parallelograms about conjugate diameters are equal in area, and the rectangular diameters in a circle are projected into conjugate diameters in an ellipse.*

(γ) *The locus of the angles of a square circumscribed to a circle is a circle whose radius is to that of the former as* $\sqrt{2}:1$. *Hence the locus of the vertices of parallelograms about the conjugate diameters of an ellipse is an ellipse similar to the original ellipse, whose axes are in the ratio of* $\sqrt{2}:1$.

(δ) Since the locus of the intersection of perpendiculars from the centre of a circle on the chords joining the extremities of diameters at right angles to each other is also a circle, *so in an ellipse the locus of the intersections of lines drawn from the centre to the middle points of the chords joining the extremities of conjugate diameters is an ellipse similar to the former, and whose area is to that of the original ellipse as* $1:\sqrt{2}$.

Hence *the area of the original ellipse is a mean proportional between the areas of these loci.*

(ϵ) *As the area of a square circumscribing a circle is the least of all circumscribing quadrilaterals, so the parallelogram about the conjugate diameters of an ellipse is the least of all circumscribing quadri-*

laterals. As the square is the greatest quadrilateral that may be inscribed in a circle, so the area of the parallelogram formed by joining the extremities of conjugate diameters in an ellipse is a maximum.

(ζ) *As the equilateral triangle is the least triangle that can be circumscribed to a circle, so the triangle whose sides are bisected at the points of contact is the least triangle that can be circumscribed to an ellipse.*

(η) *As the equilateral triangle is the greatest triangle that may be inscribed in a circle, so the greatest triangle that may be inscribed in an ellipse is one whose vertex is at the extremity of one conjugate diameter, and whose base is an ordinate to this diameter bisecting it between the centre and the remote vertex.*

Hence all such triangles are equal in area, and their centres of gravity coincide with the centre of the ellipse.

(θ) *In a circle a chord drawn from a point in which two tangents intersect is divided harmonically by this point and the chord of contact; so also in a conic.*

331.] *A perpendicular is drawn from a given point to a given straight line. The point and line are orthogonally projected on a given plane into another point and another straight line; and from the former a perpendicular is drawn to the latter. The ratio of these perpendiculars is independent of the position of the points from which the perpendiculars are drawn* (fig. 82).

Let OA, OB, OC be a set of three rectangular axes in space; let BP be the perpendicular from the given point B on the given line AC; let this line AC be orthogonally projected into the line AO inclined to AC by the angle i; let BO be the perpendicular on this line: then the ratio of BP to BO is independent of the position of B. Let OQ be the perpendicular from O to the plane ABC inclined to OC by the angle θ.

Now the volume of the rectangular pyramid OACB is one sixth of the volume OA . OB . OC. But it is also one sixth of the volume of the triangular base ABC multiplied by OQ.

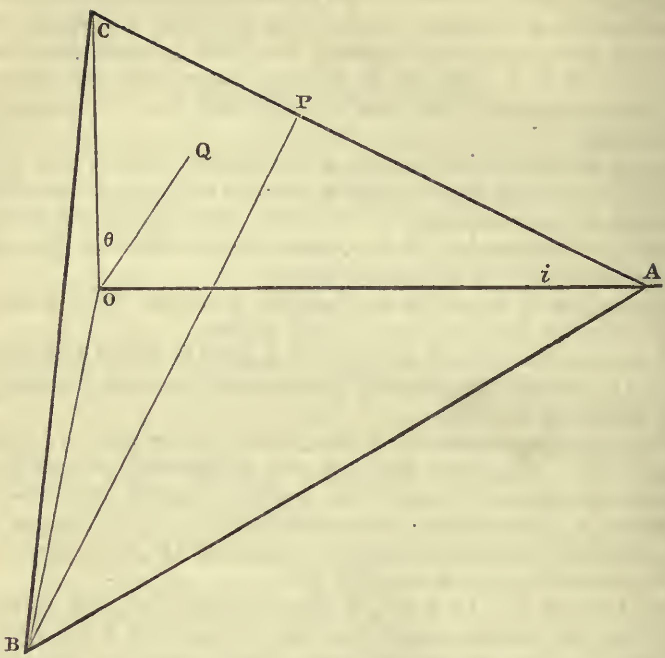
Therefore $$\mathrm{OA}\,.\,\mathrm{OB}\,.\,\mathrm{OC}=\mathrm{OQ}\,.\,\mathrm{AC}\,.\,\mathrm{BP}.$$

But $\mathrm{OQ}=\mathrm{OC}\cos\theta$, and $\mathrm{OA}=\mathrm{AC}\cos i$; hence we obtain

$$\frac{\mathrm{BO}}{\mathrm{BP}}=\frac{\cos\theta}{\cos i},$$ a ratio independent of the position of the point B.

This is a most important theorem. It enables us to pass from the properties of perpendiculars about a circle to the analogous properties of perpendiculars about a conic. By the help of this relation we may give a very simple proof of the following celebrated theorem of PAPPUS, "*Ad quatuor lineas,*" as also of many others.

Fig. 82.



332.] *If from any point* P, *in the circumference of a circle, perpendiculars be drawn to the four sides of an inscribed quadrilateral, the rectangles under each pair of perpendiculars on the opposite sides will be equal*; that is (see fig. 83),

$$PA_{,} . PD_{,} = PB_{,} . PC_{,}.$$

From P let the lines PA, PB, PC, PD be drawn to the four angles of the quadrilateral, and let R be the radius of the circle. Then (Euclid, Book VI. Prop. C) we have

$$PD . PC = 2R . PD_{,} \text{ and } PA . PB = 2R . PA_{,};$$

therefore $PA . PB . PC . PD = 4R^2 PA_{,} . PD_{,}$.

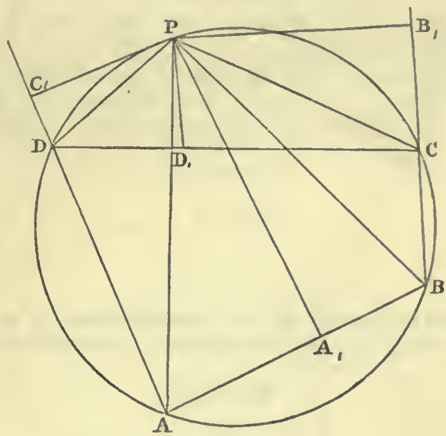
In like manner $PA . PB . PC . PD = 4R^2 PB_{,} PC_{,}$.

Hence $PA_{,} . PD_{,} = PB_{,} PC_{,}$.

If now we orthogonally project the circle into an ellipse, the point P will be projected into a point ϖ on the conic; the perpendiculars PA, $\varpi\alpha$ will have to each other a ratio, the cosine of the inclination of the side AB to its projection $\alpha\beta$, and so for the other sides. Hence the theorem of the "*Ad Quatuor lineas*," viz.:—*If from any point ϖ in the circumference of a conic perpendiculars be drawn to the sides of an inscribed quadrilateral, the rectangles under each pair of perpendiculars on the opposite sides will have a constant ratio.*

It is evident that the inclination of the planes will not enter into the constant ratio, as this relation will be eliminated by division.

Fig. 83.



333.] *If tangents be drawn at the vertices of a triangle inscribed in a circle, and if from any point in the circumference of this circle perpendiculars be drawn to the tangents and to the sides, the product of the perpendiculars on the tangents will be equal to the product of the perpendiculars on the sides.*

Since PBT and $PAQ_{,}$ are similar triangles, we have

$$PB : PT = PA : PQ_{,}.$$

In like manner we have

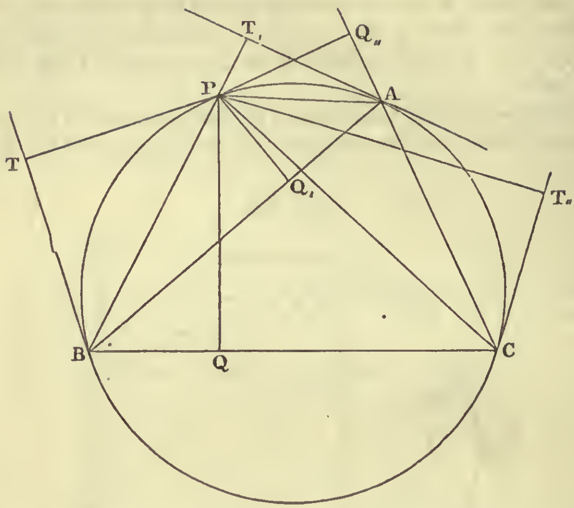
$$PA : PT_{,} = PC : PQ_{,,} \text{ and } PC : PT_{,,} = PB : PQ.$$

Compounding these proportions, we obtain

$$PT \,.\, PT_{,} \,.\, PT_{,,} = PQ \,.\, PQ_{,} \,.\, PQ_{,,}.$$

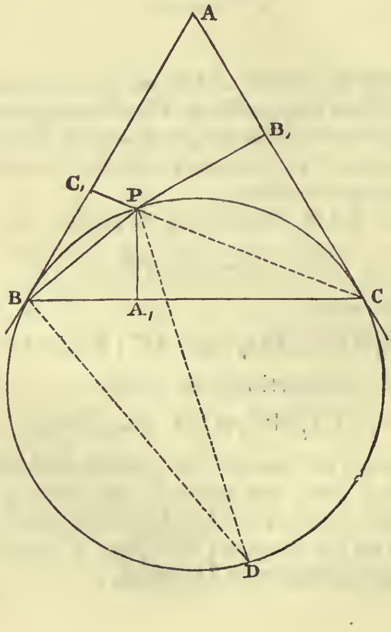
Hence, *if a triangle be inscribed in a conic, and tangents be drawn to its vertices, and if from any point in the conic perpendiculars be drawn to the three tangents and to the three sides, the product of the perpendiculars on the tangents will have a constant ratio to the product of the perpendiculars on the sides.*

Fig. 84.



334.] *If from any point in the circumference of a circle perpendiculars be drawn to a pair of tangents to the circle, the rectangle*

Fig. 85.



under these perpendiculars will be equal to the square of the perpendicular drawn from this point to the common chord.

Let $PA_{\prime}$, $PB_{\prime}$, $PC_{\prime}$ be the three perpendiculars. Then by similar triangles

$$PB_{\prime} : PC = PA_{\prime} : PB, \text{ and } PC_{\prime} : PB = PA_{\prime} : PC.$$

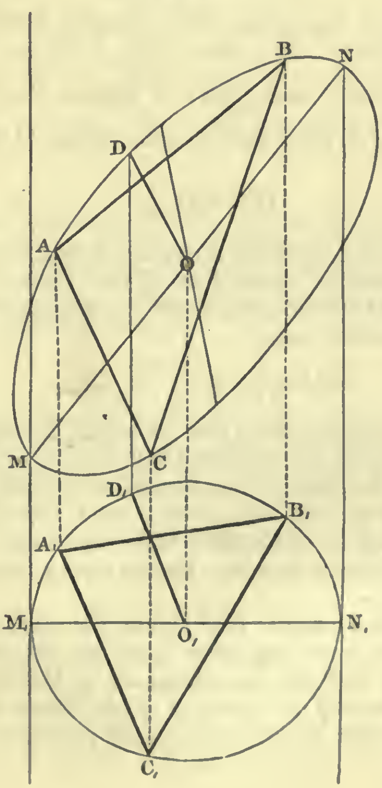
Hence
$$PB_{\prime} \,.\, PC_{\prime} = PA_{\prime}^2.$$

Therefore, if from any point in a conic, perpendiculars be drawn to a pair of tangents and their chord of contact; the rectangle under the perpendiculars on the tangents will have a fixed ratio to the square of the perpendicular on the chord.

335.] *Let a, b, c be the sides of a triangle inscribed in an ellipse of which the semiaxes are* A *and* B, *while the radius of the circle circumscribing the triangle is* R; *let d, $d_{\prime}$, $d_{\prime\prime}$ be the semidiameters parallel to the sides of the triangle a, b, c; then*

$$RAB = dd_{\prime}d_{\prime\prime}. \qquad \text{(a)}$$

Fig. 86.



Let the ellipse be projected into a circle whose radius is B; let the triangle in the ellipse whose sides are a, b, c be projected into another inscribed in the circle whose sides are α, β, γ; let the areas of the projective and projected triangles be S and $S_{,}$, then

$$S=\frac{abc}{4R}, \text{ and } S_{,}=\frac{\alpha\beta\gamma}{4B}. \quad \text{But} \quad S : S_{,}=A : B. \quad . \quad . \quad (b)$$

Now as the lengths of any two parallel lines on a plane have the same ratio to one another as their projections on another plane, and as d is parallel to a, $a : \alpha = d : B$, or

$$a=\frac{d\alpha}{B}. \quad \text{In like manner } b=\frac{d_{,}\beta}{B}, \text{ and } c=\frac{d_{,,}\gamma}{B}.$$

Hence $abc=\frac{\alpha\beta\gamma \,.\, dd_{,}d_{,,}}{B^3}$. But $\frac{abc}{\alpha\beta\gamma}=\frac{RS}{BS_{,}}$, and $\frac{S}{S_{,}}=\frac{A}{B}$; . . (c)

hence $$RAB=d\,.\,d_{,}\,.\,d_{,,}.$$

Let $f, f_{,}, f_{,,}$ be the three focal chords drawn through any focus, and parallel to the sides a, b, c of the triangle ABC; then from sec. [282] and [291] it follows that $d^2=\frac{Af}{2}$. Substituting for $d, d_{,}, d_{,,}$ their values, and writing D for 2R and L for $\frac{2B^2}{A}$, we get

$$D^2L=ff_{,}f_{,,}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

336.] If a circle be described cutting a conic in four points, the vertices of an inscribed quadrilateral, and from a focus six chords be drawn parallel to the four sides and two diagonals of the inscribed quadrilateral, we shall have

$$D^4L^2=f_1\cdot f_2\cdot f_3\cdot f_4\cdot f_5\cdot f_6. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

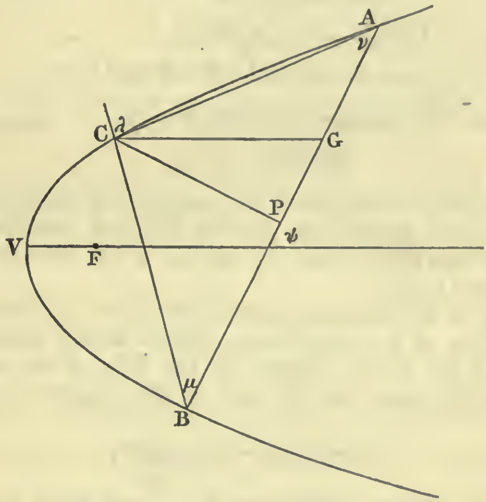
This follows obviously from the preceding theorem; for we may consider the inscribed quadrilateral with its diagonals as equivalent to four triangles, to which the construction in the foregoing theorem being applied, we should have twelve focal chords, three for each triangle. But each focal chord is once repeated; this reduces the number to six different chords. Hence the theorem may be enunciated as follows:—

If a circle meet a conic in four points, the vertices of an inscribed quadrilateral, and from any focus focal chords be drawn parallel to the four sides and the two diagonals of this quadrilateral, we shall have the square of the diameter of the circle multiplied by the parameter of the conic equal to the square root of the product of the six focal chords.

In the preceding theorem the products of the six focal chords, taken two by two, are equal, or

$$f_1 \cdot f_2 = f_3 \cdot f_4,\ f_2 \cdot f_3 = f_1 \cdot f_4,\ f_2 \cdot f_5 = f_4 \cdot f_6, \text{ and } f_2 \cdot f_6 = f_4 \cdot f_5. \quad \text{(b)}$$

Fig. 87.



337.] Without having recourse to orthogonal projection, it may be shown that the product of three focal chords drawn parallel to the sides of an inscribed triangle is equal to the product of the parameter L of the parabola multiplied by the square of the diameter D of the circumscribing circle, or

$$\mathrm{LD}^2 = f_1 f_2 f_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let ABC be the inscribed triangle, V the vertex, and F the focus. Let CG be drawn parallel to the axis, meeting the side AB in G, which makes the angle ψ with the axis; and let $\mathrm{CP}=p$ be the perpendicular on the side AB, and the angle ACG be ω.

Let $\quad \mathrm{AG}=a,\ \mathrm{BG}=b,\ \mathrm{CG}=c,\ \mathrm{AB}=l,\ \mathrm{BC}=m,\ \mathrm{AC}=n.$

Now $\quad \sin^2\nu : \sin^2\omega = c^2 : a^2$ and $\sin^2\omega : 1 = \mathrm{L} : f_1$;

therefore $\quad \sin^2\nu : 1 = c^2\mathrm{L} : f_1 a^2.$

In like manner $\quad \sin^2\mu : 1 = c^2\mathrm{L} : f_2 b^2.$

But $\quad \sin^2\lambda : \sin^2\psi = l^2c^2 : n^2m^2,$

and $\quad \sin^2\psi : 1 = \mathrm{L} : f_3;$

therefore $\quad \sin^2\lambda : 1 = \mathrm{L}l^2c^2 : f_3 n^2 m^2.$

Hence, multiplying these expressions, we obtain

$$\frac{L^3c^6l^2m^2n^2}{\sin^2\lambda\sin^2\mu\sin^2\nu}=f_1f_2f_3\,a^2b^2n^4m^4.$$

But $$D^3=\frac{lmn}{\sin\lambda\sin\mu\sin\nu}, \text{ and } Dp=mn;$$

therefore $$L^3c^6D^2=f_1f_2f_3a^2b^2p^2.$$

But $$p=c\sin\psi, \text{ and } ab\sin^2\psi=Lc.$$

Making these substitutions, we have finally

$$LD^2=f_1\cdot f_2\cdot f_3. \quad \text{(b)}$$

338.] *If a conic described on the surface of a right cone be orthogonally projected on a plane passing through the vertex at right angles to the axis of the cone, the vertex of the cone will be a focus of the projected conic.*

Let a, b, e be the semiaxes and eccentricity of the conic drawn on the surface of the right cone; let θ be the semiangle of the cone; and let l, $l_{\prime}$ be the lengths of the sides of the cone between the vertex of the cone and the ends of the major axis of the given conic.

Then $$4a^2=l^2+l_{\prime}^2-2ll_{\prime}\cos 2\theta; \quad \text{(a)}$$

$2ae=(l-l_{\prime})$, see sec. [262], cor. iii.; $\dfrac{b^2}{a}=p\tan\theta$, see sec. [281],

where p is the perpendicular from the vertex of the cone on the plane of the given conic.

The area of the focal triangle gives $p=\dfrac{ll_{\prime}\sin 2\theta}{2a}=\dfrac{ll_{\prime}\sin\theta\cos\theta}{a}$, or

$$b^2=ll_{\prime}\sin^2\theta. \quad \text{(b)}$$

Then $2a_{\prime}=(l+l_{\prime})\sin\theta$, and $b_{\prime}=b$, since b is parallel to the plane of projection through the vertex. As $b^2=ll_{\prime}\sin^2\theta$, $b_{\prime}^2=ll_{\prime}\sin^2\theta$, or $b_{\prime}^2=l\sin\theta\,.\,l_{\prime}\sin\theta=VA_{\prime}VB_{\prime}$. Therefore V is a focus of the projected curve.

The semiparameter of the projected curve is

$$\frac{b_{\prime}^2}{a_{\prime}}=\frac{2b^2}{(l+l_{\prime})\sin\theta}=\frac{2ll_{\prime}\,.\,\sin\theta}{l+l_{\prime}}, \quad \text{(c)}$$

and as $$e_{\prime}^2=\frac{a_{\prime}^2-b_{\prime}^2}{a_{\prime}^2}, \text{ substituting, } e_{\prime}=\frac{l-l_{\prime}}{l+l_{\prime}}. \quad \text{(d)}$$

339.] *The surface of a right cone bounded by a conic is developed on a plane passing through the vertex of the cone at right angles to its axis; to determine the curve which the conic becomes when the surface of the cone becomes a plane.*

Let the focal equation of the projected conic be

$$\rho_{\prime}=\frac{a_{\prime}(1-e_{\prime}^{2})}{1+e_{\prime}\cos\phi_{\prime}},\ \rho_{\prime}\ \text{being the focal vector.}\quad . \quad . \quad \text{(a)}$$

Through the axis of the cone and the focal vector $\rho_{\prime}$ let a plane be drawn; it will cut the surface of the cone in a side of the cone s so that $\rho_{\prime}=s\,.\sin\theta$. Let $2n\sin\theta=1$; then $2n$ is a constant. Let $d\phi$ and $d\phi_{\prime}$ be the corresponding elementary angles between two successive values of $\rho_{\prime}$ the focal vector of the projected conic, and s the corresponding vector along the side of the cone, so that $\phi_{\prime}=2n\phi$; hence the equation of the projected conic

$$\rho'=\frac{a_{\prime}(1-e_{\prime}^{2})}{1+e\cos\phi_{\prime}}\ \text{becomes}\ s=\frac{\left(\frac{l+l_{\prime}}{2}\right)\left[1-\left(\frac{l-l_{\prime}}{l+l_{\prime}}\right)^{2}\right]}{1+\left(\frac{l-l_{\prime}}{l+l_{\prime}}\right)\cos 2n\phi}.\quad . \quad \text{(b)}$$

But $\cos 2n\phi=\cos^{2}n\phi-\sin^{2}n\phi$; hence this equation now becomes

$$\frac{1}{s}=\frac{\sin^{2}n\phi}{l}+\frac{\cos^{2}n\phi}{l_{\prime}}.\quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

This is a species of spiral curve having two apsides at the distances l and $l_{\prime}$ from the vertex of the cone, when $n\phi=\frac{\pi}{2}$ or when $n\phi=\pi$.

In these cases the vector-angle $\phi=\frac{\pi}{2n}$ or $\phi=\frac{\pi}{n}$.

Hence the curve undulates between two concentric asymptotic circles whose radii are l and $l_{\prime}$.

When the conic is a parabola, $l_{\prime}=\infty$, and the equation of the locus becomes $\frac{1}{s}=\frac{\cos^{2}n\phi}{l}$.

When the conic is an hyperbola the equation of the locus becomes

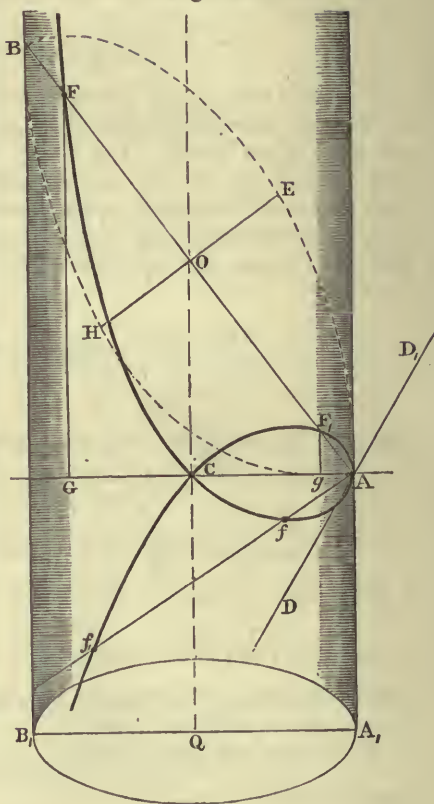
$$\frac{1}{s}=\frac{\cos^{2}n\phi}{l}-\frac{\sin^{2}n\phi}{l_{\prime}}.\quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

If we refer to the ninth section of NEWTON'S *Principia*, we shall see that the formula above given is the equation for movable orbits whose apsides recede.

340.] *If secant planes be drawn through a horizontal tangent to a right circular cylinder, the locus of the foci of the elliptic sections will be the logocyclic curve* (see fig. 88).

Let AD be the horizontal tangent to the right circular cylinder $ABB_{\prime}A_{\prime}$. Let AB be the major axis of the ellipse, and let F, $F_{\prime}$ be its foci. Let a be the radius of the circular base; then it is manifest that a is half the minor axis of the ellipse. Then, as F is a focus, we shall have $AF\,.\,BF=a^{2}$.

Fig. 88.



Let zx be the ordinates of the point F, the axes of coordinates being $AA_{,}$, AC; and let i be the inclination of the secant plane to the circular base of the cylinder.

Then

$$AB = 2a \sec i, \quad AF = \sqrt{x^2+z^2},$$

and

$$BF = 2a \sec i - \sqrt{x^2+z^2},$$

and

$$\sec i = \frac{\sqrt{x^2+z^2}}{x}$$

therefore

$$BF = \frac{2a\sqrt{x^2+z^2}}{x} - \sqrt{x^2+z^2}.$$

Substituting these values for AF and BF, we get

$$(x^2+z^2)(2a-x) = a^2x,$$

the equation of the logocyclic curve, substituting y for z, as shown in sec. [319] of the first volume.

ON DIVERGENT PROJECTION.

341.] In perspective or central or divergent projection (as it may be called), the projecting lines are no longer parallel as in orthogonal projection. They radiate from a single point which may be called the vertex or centre, and so transfer the lines and points of one surface to those of another. In general, as here, the surfaces are planes; one plane figure is projected into another. This sort of projection has been named central projection by Poncelet, the great authority on this subject. This is a simple and powerful method of investigation, so far as the graphical properties of figures are concerned. It is more general in its application than orthogonal projection, in which the vertex or centre of projection is at infinity. For example, in the application of these methods to conics, only the properties of the ellipse may be derived from those of the circle by orthogonal projection, while divergent projection may be applied to all conics.

This method of projection rests on the following simple theorem :—

If a straight line be drawn parallel to a given plane, all planes drawn through this straight line will cut the given plane in parallel straight lines; and if a straight line be drawn meeting the given plane in a point, all planes drawn through this straight line will meet the given plane in the same point.

Of the several ways in which this method may be applied the following appears the simplest.

Through the vertex of a right circular cone let a plane be drawn at right angles to its axis and intersecting one of the plane sections of this cone in a straight line which may be called the *cyclic axis* (while the plane drawn through this axis and the vertex of the cone may be called the *cyclic plane*). Any plane drawn through the axis of the cone will cut the cone in two straight lines, and the cyclic plane in a straight line; and these four lines evidently constitute an harmonic pencil.

The figure whose projective properties it is sought to develop may be drawn on the plane of one of the circular sections of the cone, the vertex of the cone being the centre of projection.

One or two applications of this method must here suffice.

342.] Let two right cones having the same vertex and axis be drawn, they will be cut by a plane at right angles to the common axis in two concentric circles. Let these circles be drawn so that a square inscribed in the one shall be circumscribed to the other; the diagonals of the inscribed square and its chords of contact with the circle inscribed in it will pass through the common centre of the two circles; and if the square be turned about between the two circles, it is obvious that its angles will remain on the outer circle, and its sides remain in contact with the inner circle. If we now project these circles and the square, the circles will become conics and the square a quadrilateral inscribed in one conic and circumscribed to the other. As the opposite sides of the square are parallel, and the chords joining the points of contact are also parallel, the projections of these eight lines will meet two by two in four points along the cyclic axis; and this cyclic axis is the polar of the point in which the common axis of the two cones meets the plane of the two conics. It is also obvious that any number of quadrilaterals may be inscribed in the one conic and circumscribed to the other.

APPENDIX

TO THE FIRST VOLUME,

WITH NOTES AND CORRECTIONS.

343.] At page xii. of the introduction, reference is made to a theorem of Euler's connecting by a simple and invariable relation the numbers denoting the solid angles, faces, and edges of any polyhedron.

A very elegant and simple demonstration of this curious theorem which had so long baffled that illustrious geometer Euler, will be found at page 333 of the XIX.th volume of the *Annales Mathématiques* of Gergonne, based on the relations of a group of reticulated polygons. But the following proof, which some years ago I discovered, will be found still simpler, and requires no knowledge, beyond that of common arithmetical addition, to understand it.

Let the relation $s+f-e=2$ be assumed as established for any one polyhedron, where s denotes the number of solid angles, f the number of faces, and e the number of edges. From this solid let a pyramid be removed whose vertex is one of the solid angles of the polyhedron; let n be the number of plane angles which together constitute the solid angle, the vertex of the retrenched pyramid. Let S, F, and E represent the numbers of the solid angles, faces, and edges of the new polyhedron made by retrenching the aforesaid pyramid. Now, by the removal of the pyramid whose vertex is a solid angle of the first polyhedron, we take away from this figure one solid angle, but we add n solid angles, the number round the base of the retrenched pyramid; so that by the removal of this pyramid we add $n-1$ solid angles to the original polyhedron, or

$$S=s+(n-1).$$

By this operation we add n new edges, which are the sides of the polygon that constitute the polygonal base of the pyramid, or

$E=e+n$; and we evidently add one more face to the original polyhedron by removing the pyramid, or $F=f+1$; consequently

$$S+F-E-2=s+(n-1)+(f+1)-(e+n)-2,$$

or

$$S+F-E-2=s+f-e-2;$$

or the same relation exists between the numbers which represent the solid angles, faces, and edges of the original and derived polyhedrons.

We may now assume any simple polyhedron, a cube suppose, in which the relation $s+f-e-2=0$ is evident, and by the successive removal of pyramid after pyramid thus increase the numbers that denote the solid angles, faces, and edges of the derived polyhedrons, and still find the same invariable relation,

$$s+f-e=2.$$

Cauchy's theorem, which is as follows, may be proved with equal brevity and simplicity. Let m denote a number of polyhedrons, agglutinated together like a mass of crystals, and let S, F, E denote the numbers of the solid angles, faces, and edges of this cluster of polyhedrons, we shall have

$$S+F-E=1+m.$$

This is Cauchy's theorem. When there is but one polyhedron, $m=1$, and we obtain Euler's theorem.

Let s, f, e denote, as before, the numbers of the solid angles, faces, and edges of any polyhedron; then by Euler's theorem we shall have $s+f-e=2$. Now if we conceive one of the polygons which constitute the faces of this polyhedron to have n edges and n solid angles, the removal of this polygon with its n solid angles and n edges will make the closed polyhedron an *open* polyhedron; and we shall have the following relation between the numbers that denote the solid angles, faces, and edges of an *open* polyhedron,

$$s+f-e=1.$$

Let $s_{\prime}, f_{\prime}, e_{\prime}$ denote the numbers of the solid angles, faces, and edges of the *open* polyhedron thus derived, we shall have $s=s_{\prime}+n$, $f=f_{\prime}+1$, and $e=e_{\prime}+n$; substituting these values in Euler's formula, $(s_{\prime}+n)+(f_{\prime}+1)-(e_{\prime}+n)=2$, or $s_{\prime}+f_{\prime}-e_{\prime}=1$.

Let us now conceive a *closed* polyhedron having an *open* polyhedron applied to one of its faces, so as to fit, or, in other words, so that the projecting edges of the open polyhedron may be applied to the solid angles of the closed polyhedron; then we shall have, by Euler's theorem for the closed polyhedron,

$$S+F-E=2,$$

and for the open polyhedron

$$s+f-e=1,$$

as just now shown. But if $S_{,}$, $F_{,}$, and $E_{,}$ denote the numbers of the solid angles, faces, and edges of the compound polyhedron, we shall have

$$S_{,}=S+s, \quad F_{,}=F+f, \quad E_{,}=E+e;$$

consequently

$$S_{,}+F_{,}-E_{,}=2+1,$$

or the difference is one more than in the case of the single polyhedron.

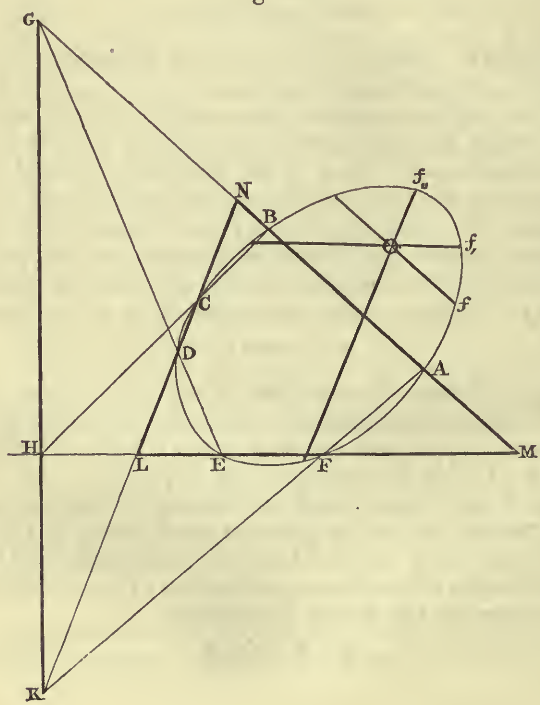
Consequently for every additional *open* polyhedron we attach, the absolute number is increased by unity; or if there be m agglutinated polyhedrons,

$$S+F-E=1+m.$$

[Page xiii.]

344.] *The opposite sides of a hexagon inscribed in a conic meet, two by two, in three points which range along a straight line.*

Fig. 89.



Let A, B, C, D, E, F be the vertices of a hexagon inscribed in a conic, whose opposite sides meet two by two in the three points G, H, K. These points range along a straight line.

Let the alternate sides of the hexagon be taken as forming a triangle LMN, whose sides are cut in the points A, B, C, D, and E, F by a conic and also by the three transversals BC, DE, and FA.

Through a focus of the conic let chords be drawn $f, f_{\prime}, f_{\prime\prime}$ parallel to the sides of the triangle LMN; then, by a well known theorem,

$$\left.\begin{aligned} MA.MB : MF.ME &= f : f_{\prime}, \\ NC.ND : NA.NB &= f_{\prime\prime} : f, \\ LE.LF : LD.LC &= f_{\prime} : f_{\prime\prime}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Multiplying these expressions,

$$MA.MB.NC.ND.LE.LF = MF.ME.NA.NB.LD.LC. \quad (b)$$

Since the triangle LMN is cut by the transversals HB, GE, and KA,

$$\left.\begin{aligned} HL.NC.MB &= HM.NB.LC, \\ GM.ND.LE &= GN.LD.ME, \\ KN.LF.MA &= KL.MF.NA. \end{aligned}\right\} \quad . \quad . \quad . \quad (c)$$

Multiplying together these three sets of proportionals, and dividing the product by the products in (b), we shall have

$$HL.GM.KN = KL.GN.HM; \quad . \quad . \quad . \quad . \quad (d)$$

or the three points G, H, K range along the straight line GHK, which is a transversal to the triangle LMN*.

345.] *A hexagon is circumscribed to a conic; the diagonals which join the opposite vertices meet in a point.*

Through the points A, B, C, D, E, F (see preceding figure) let tangents be drawn to the conic, meeting in the points a, b, c, d, e, f, which therefore constitute a hexagon circumscribed to the conic. Now as a is the pole of the chord AB, the polar of any point in AB will pass through a. But G is a point on AB; therefore the polar of the point G will pass through a. In like manner the polar of any point in DE will pass through d. But G is also a point on DE; therefore the polar of G will pass through d; therefore ad is the polar of the point G. So also be is the polar of the point H, and cf is the polar of the point K. But, as shown above, G, H, K range along a straight line; therefore ad, be, cf, the diagonals of the circumscribed hexagon, meet in a point, the pole of the straight line GHK.

* This solution was given in the 'Ladies' Diary' for 1842 under the initials J. B. B. C.

Page 15. Sec. [24].

346.] More generally, let the *projective* equation of the conic section, referred to rectangular axes, be

$$Ax^2+A_{\prime}y^2+2Bxy+2Cx+2C_{\prime}y-1=0.$$

Therefore by sec. [22] $\xi=\dfrac{Ax+By+C}{1-Cx-C_{\prime}y}$, $v=\dfrac{A_{\prime}y+Bx+C_{\prime}}{1-Cx-C_{\prime}y}$.

Hence $x=\dfrac{(A_{\prime}+C_{\prime}^2)\xi-(B+CC_{\prime})v-(A_{\prime}C-BC_{\prime})}{(AA_{\prime}-B^2)+(AC_{\prime}-BC)v+(A_{\prime}C-BC_{\prime})\xi}$;

also $y=\dfrac{(A+C^2)v-(B+CC_{\prime})\xi-(AC_{\prime}-BC)}{(AA_{\prime}-B^2)+(AC_{\prime}-BC)v+(A_{\prime}C-BC_{\prime})\xi}$.

Substituting these values of x and y in the dual equation $x\xi+yv=1$,

$$(A_{\prime}+C_{\prime}^2)\xi^2+(A+C^2)v^2-2(B+CC_{\prime})\xi v+2(BC-AC_{\prime})v$$
$$+2(BC_{\prime}-A_{\prime}C)\xi+B^2-AA_{\prime}=0.$$

In the *projective* equation of the parabola, $B^2-AA_{\prime}=0$; hence the *tangential* equation of the parabola has no absolute term.

Page 21. Sec. [32].

347.] *If from any point Q, in the plane of a rectangular polygon, perpendiculars are drawn to the sides, if the feet of these perpendiculars be joined two by two, so as to constitute another polygon, and if the area of this latter polygon be constant, the locus of the point Q will be a conic section.*

Let x and y be the projective coordinates of the point Q, and let ξ, v and $\xi_{\prime}$, $v_{\prime}$ be the tangential coordinates of two successive sides of the polygon, and let θ be the angle between them; then, P and $P_{\prime}$ being the first pair of perpendiculars,

$$P=\frac{1-\xi x-vy}{\sqrt{\xi^2+v^2}},\quad P_{\prime}=\frac{1-\xi_{\prime}x-v_{\prime}y}{\sqrt{\xi_{\prime}^2+v_{\prime}^2}},$$

and

$$\sin\theta=\frac{\xi_{\prime}v-\xi v_{\prime}}{\sqrt{(\xi^2+v^2)(\xi_{\prime}^2+v_{\prime}^2)}}\ \text{(see p. 4)}.$$

Hence the area of the first component triangle is

$$PP_{\prime}\sin\theta=\frac{(1-\xi x-vy)(1-\xi_{\prime}x-v_{\prime}y)(\xi_{\prime}v-\xi v_{\prime})}{(\xi^2+v^2)(\xi_{\prime}^2+v_{\prime}^2)}.$$

But ξv and $\xi_{\prime}v_{\prime}$ being constants, we may put

$$A=\frac{(\xi_{\prime}v+\xi v_{\prime})}{(\xi^2+v^2)(\xi_{\prime}^2+v_{\prime}^2)};$$

therefore the first component triangle is equal to

$$\mathrm{A}(1-\xi x-v y)(1-\xi_{\prime} x-v_{\prime} y).$$

In like manner the next component triangle will be equal to

$$\mathrm{A}_{\prime}(1-\xi_{\prime} x-v_{\prime} y)(1-\xi_{\prime\prime} x-v_{\prime\prime} y);$$

and if the sum of these component triangles be assumed as constant and equal to C, we shall have a resulting equation of the form

$$\mathrm{P}x^2+\mathrm{Q}y^2+2\mathrm{R}xy+2\mathrm{S}x+2\mathrm{T}y=\mathrm{C},$$

the projective equation of a conic section—P, Q, R, S, T being functions of the constants ξ, $\xi_{\prime}$, v, $v_{\prime}$, &c.

Page 25. Sec. [36].

348.] *To find the equation to the envelope of equal chords of a given ellipse.*

Proposed by Mr. A. Martin in the *Educational Times*, No. 4519.

Let (x, y) be a point on the ellipse, and (ξ, v) the tangential coordinates of a tangent passing through this point; then, eliminating y between the equations

$$a^2y^2+b^2x^2=ab^2 \quad . \quad . \quad . \quad \text{(a)} \quad \text{and} \quad x\xi+yv=1, \quad . \quad . \quad . \quad \text{(b)}$$

we shall find $\quad (a^2\xi^2+b^2v^2)x^2-2a^2\xi x+a^2(1-b^2v^2)=0. \quad . \quad . \quad$ (c)

Let $x_{\prime}$ and $x_{\prime\prime}$ be the roots of this quadratic equation, we have

$$x_{\prime}+x_{\prime\prime}=\frac{2a^2\xi}{a^2\xi^2+b^2v^2}, \quad \text{and} \quad x_{\prime}x_{\prime\prime}=\frac{a^2(1-b^2v^2)}{a^2\xi^2+b^2v^2};$$

consequently $\quad (x_{\prime}-x_{\prime\prime})^2=\dfrac{4a^2b^2v^2(a^2\xi^2+b^2v^2-1)}{(a^2\xi^2+b^2v^2)^2}. \quad . \quad . \quad . \quad$ (d)

In like manner $\quad (y_{\prime}-y_{\prime\prime})^2=\dfrac{4a^2b^2\xi^2(a^2\xi^2+b^2v^2-1)}{(a^2\xi^2+b^2v^2)^2}. \quad . \quad . \quad . \quad$ (d$_{\prime}$)

Let $2c$ be the chord. Then $(x_{\prime}-x_{\prime\prime})^2+(y_{\prime}-y_{\prime\prime})^2=4c^2, \quad . \quad$ (e)

or $\quad a^2b^2(\xi^2+v^2)[a^2\xi^2+b^2v^2-1]=c^2(a^2\xi^2+b^2v^2)^2. \quad . \quad . \quad$ (f)

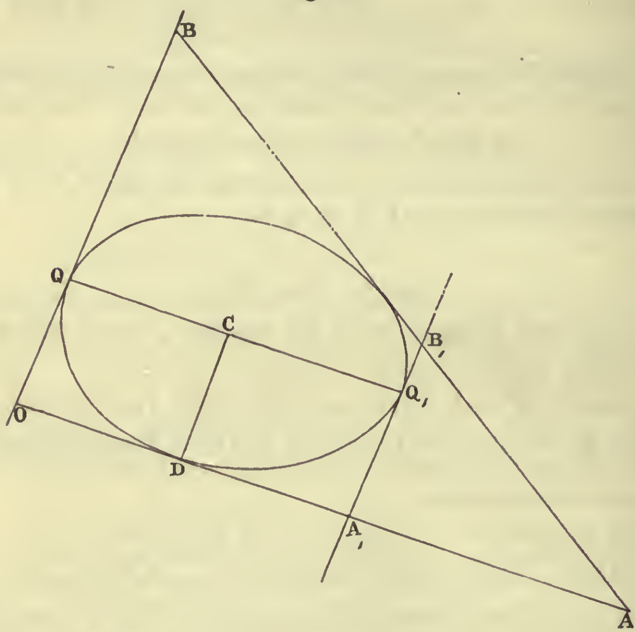
Hence the projective equation of the pedal is

$$a^2b^2(x^2+y^2)[a^2x^2+b^2y^2-(x^2+y^2)^2]=c^2(a^2x^2+b^2y^2)^2. \quad . \quad \text{(g)}$$

Page 40. Sec. [48].

349.] *Two parallel tangents are drawn to a conic, and a third tangent between them, variable in position. This tangent will cut off segments from the parallel tangents between its intersections with* *them and the points of contact, such that the rectangle under these segments will be constant.*

Fig. 90.



Let two tangents to the curve be taken as axes of coordinates, the axis of Y being one of the fixed tangents to the curve, while the axis of X is parallel to the diameter conjugate to the two parallel tangents. Then the tangential equation of the curve referred to these tangents as axes is, as shown in sec. [48],

$$2\beta\xi\upsilon + 2\gamma\xi + 2\gamma_{,}\upsilon = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let the variable tangent cut off from the axes of coordinates $OB = b$, $OA = a$. Then, as this line is a tangent to the curve, $\frac{1}{a}$, $\frac{1}{b}$, are tangential coordinates, and satisfy the equation (a). Hence (a) becomes

$$2\beta + 2\gamma b + 2\gamma_{,}a = ab\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

and as the axis of X is a tangent to the curve, we shall have $\beta\xi + \gamma_{,} = 0$, see sec. [19]; and as in this case $OD = \frac{1}{\xi} = \gamma$, we shall have

$$\beta + \gamma\gamma_{,} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Now $BQ=b-\gamma_{\prime}$; and $B_{\prime}Q_{\prime}$ it may be shown is equal to

$$\frac{ab-2b\gamma-a\gamma_{\prime}}{a}=\frac{2\beta+a\gamma_{\prime}}{a}=-\frac{\gamma_{\prime}}{a}(2\gamma-a),$$

putting for β its value $-\gamma\gamma_{\prime}$.

Hence $BQ\,.\,B_{\prime}Q_{\prime}=-\frac{\gamma_{\prime}}{a}(b-\gamma_{\prime})(2\gamma-a)$; or, multiplying,

$$BQ\,.\,B_{\prime}Q_{\prime}=-\frac{\gamma_{\prime}}{a}[2\beta+2\gamma_{\prime}a+2\gamma b-ab-a\gamma].$$

But $2\beta+2\gamma_{\prime}a+2\gamma b-ab=0$, from (b).

Hence $$BQ\,.\,B_{\prime}Q_{\prime}=\gamma\gamma_{\prime}. \quad \text{(d)}$$

Page 41. Sec. [49].

350.] The tangential equation of the parabola may be obtained from the projective equation of the curve, $y^2=4kx+4k^2$, as follows. The equations of transition give, see sec. [22],

$$\xi=\frac{-1}{2k+x}, \quad v=\frac{y}{2kx+4k^2},$$

and $x\xi+yv=1$. Eliminating x and y we get $k(\xi^2+v^2)+\xi=0$.

Page 43. Sec. [50].

351.] *A parabola is inscribed in a triangle* ABC (fig. 91), *touching the sides of the triangle in the points* $A_{\prime}\,C_{\prime}\,B_{\prime}$. *The rectangle under the sides* CB, CA *of the triangle is equal to the rectangle under the segments of these sides produced until they touch the parabola; or* $CA\,.\,CB=BA_{\prime}\,.\,AB_{\prime}$.

In sec. [55] it is shown that the tangential equation of the parabola, the axes of coordinates being the tangents to the curve, is

$$g\xi v+h\xi+h_{\prime}v=0. \quad \text{(a)}$$

Let $CB=b$, $CA=a$, and, as these values must satisfy this equation, seeing that AB is a tangent, we shall have

$$g+hb+h_{\prime}a=0. \quad \text{(b)}$$

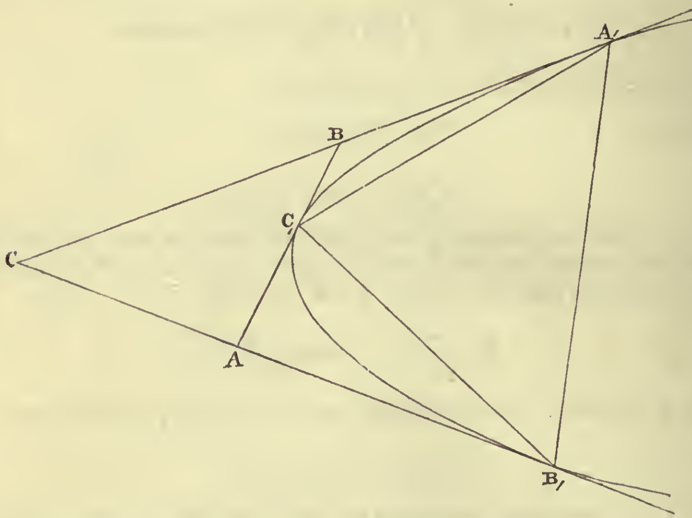
The value of $CA_{\prime}$ the tangent to the curve is found by making $\frac{1}{\xi}=0$ in the given equation, which reduces it to $gv+h=0$, or $CA_{\prime}=\frac{-g}{h}$; in like manner $CB_{\prime}=\frac{-g}{h_{\prime}}$.

Therefore $BA_{\prime}=-\left(\frac{g}{h}+b\right)$, and $AB_{\prime}=-\left(\frac{g}{h_{\prime}}+a\right)$.

Consequently $BA_{\prime}AB_{\prime} = \frac{g}{hh_{\prime}}(g + hb + h_{\prime}a) + ab.$

But the expression between the brackets $=0$ by the tangential equation (b) of the parabola; hence $CB \, . \, CA = BA_{\prime} \, . \, AB_{\prime}$.

Fig. 91.



If we now take the other two angles successively as the origin of coordinates we shall have the same property repeated.

Consequently $a^2b^2c^2 = AB_{\prime} \, . \, BA_{\prime} \, . \, BC_{\prime} \, . \, CB_{\prime} \, . \, CA_{\prime} \, . \, AC_{\prime}$. . (c)

Page 43. Sec. [50].

352.] *A parabola is inscribed in a triangle. The triangle whose vertices are the three points of contact is twice the area of the given triangle* (see fig. in last section).

Let two of the tangents to the parabola be taken as the axes of coordinates. The triangle $A_{\prime}C_{\prime}B_{\prime} = 2ABC$. Let $CB = b$, $CA = a$; since the tangents $CA_{\prime}$ $CB_{\prime}$ are axes of coordinates, the tangential equation of the parabola is

$$g\xi v + h\xi + h_{\prime}v = 0 \equiv V ; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

and the projective coordinates of the point $C_{\prime}$ in which the curve is touched by the tangent AB is found to be, using the *equations of transition* (as in sec. [22]),

$$\left.\begin{aligned} & x=\frac{\frac{dV}{d\xi}}{\frac{dV}{d\xi}\xi+\frac{dV}{dv}v}, \quad y=\frac{\frac{dV}{dv}}{\frac{dV}{d\xi}\xi+\frac{dV}{dv}v}, \\ \text{or} \quad & x=\frac{gv+h}{gv\xi}, \quad y=\frac{g\xi+h_{\prime}}{gv\xi}, \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(b)}$$

or, putting $\frac{1}{a}$ for ξ and $\frac{1}{b}$ for v, as the point C is on the line AB,

$$x=\frac{a}{g}(g+bh), \quad y=\frac{b}{g}(g+ah_{\prime}).$$

Let CA=Y; the value of Y is found by putting $\xi=\infty$ in (a). Hence $\frac{1}{v}$ or $Y=\frac{-g}{h}$, and $X=\frac{-g}{h_{\prime}}$.

Now the triangle $A_{\prime}B_{\prime}C_{\prime}=CA_{\prime}B_{\prime}-CC_{\prime}B-CC_{\prime}A_{\prime}$. . . . (c)

But $CA_{\prime}B_{\prime}=\frac{g^2\alpha}{hh_{\prime}}$, putting α for the sine of the angle of ordination,

$$\left.\begin{aligned} &\text{and } CC_{\prime}B_{\prime}=\alpha Xy=\frac{-g}{h_{\prime}}\left(\frac{bg+abh_{\prime}}{g}\right)=\frac{-b}{h_{\prime}}(g+ah_{\prime}). \\ &\text{In like manner} \qquad \alpha Yx=\frac{-a}{h}(g+bh). \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(d)}$$

Consequently $CA_{\prime}B_{\prime}-CC_{\prime}B_{\prime}-CCA_{\prime}=\frac{\alpha g}{hh_{\prime}}(g+bh+ah_{\prime})+2abc.$

But since $\frac{1}{a}$ and $\frac{1}{b}$ are tangential coordinates, $g+bh+ah_{\prime}=0$; hence the triangle $A_{\prime}B_{\prime}C_{\prime}=2\alpha ab$. But αab is the area of the triangle ABC; therefore the triangle $A_{\prime}B_{\prime}C_{\prime}$ is equal to twice the triangle ABC.

Since

$$\left.\begin{aligned} &\frac{CB}{A_{\prime}B}=\frac{-bh}{(g+bh)} \text{ and } \frac{AC_{\prime}}{C_{\prime}B}=\frac{a-x}{x}=\frac{-bh}{g+bh} \text{ and } \frac{AB_{\prime}}{CA}=\frac{-bh}{(g+bh)}, \\ &\text{it will follow that} \quad \frac{CB}{BA_{\prime}}=\frac{AC_{\prime}}{BC_{\prime}}=\frac{AB_{\prime}}{CA}=\frac{-bh}{g+bh}=\frac{bh}{ah_{\prime}}. \end{aligned}\right\} \quad . \quad \text{(e)}$$

353.] The tangential equation of a parabola referred to two tangents as axes of coordinates, see sec. [55] is

$$g\xi v+h\xi+h_{\prime}v=0\equiv V, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

to determine the projective equation of the curve referred to the same axis.

$$\left.\begin{aligned}\text{Since, as in sec. [22], } \frac{dV}{d\xi}=gv+h,\quad \frac{dV}{dv}=g\xi+h_{,},\\ \text{and } \frac{dV}{d\xi}\xi+\frac{dV}{dv}v=g\xi v,\end{aligned}\right\} \quad . \quad . \quad \text{(b)}$$

$$x=\frac{gv+h}{g\xi v},\quad y=\frac{g\xi+h_{,}}{g\xi v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Hence $$\xi=\frac{g+hy-h_{,}x}{2gx},\quad v=\frac{g-hy+h_{,}x}{2gy}. \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Substituting these values of ξ and v in the tangential equation of the parabola (a), we get

$$h_{,}^2x^2+h^2y^2-2hh_{,}xy+2ghy+2gh_{,}x+g^2=0. \quad . \quad . \quad \text{(e)}$$

Page 47. Sec. [55].

354.] *A quadrilateral is circumscribed to a parabola. Two of its sides are fixed, while two are variable in position. These latter intercept, on the former, segments which are always in a constant ratio.*

Let the tangential equation of the parabola referred to the fixed sides of the quadrilateral as axes of coordinates be

$$g\xi v+h\xi+h_{,}v=0\,;$$

let a and b, $a_{,}$ and $b_{,}$ be the tangential coordinates of the two variable lines; then we shall have $\left(\text{since } \xi=\frac{1}{a},\ v=\frac{1}{b}\right)$

$$g+hb+h_{,}a=0 \text{ and } g+hb_{,}+h_{,}a_{,}=0.$$

Subtract these equations one from the other, the result is $b-b_{,}=\frac{h_{,}}{h}(a_{,}-a)$.

But h and $h_{,}$ are constant quantities depending on the equation of the curve; hence $\frac{b-b_{,}}{a_{,}-a}$ is constant.

Page 65. Sec. [76].

355.] The *projective* equation of a surface of the second order, $f(x, y, z)=0$, referred to three rectangular coordinates in space being

$$Ax^2+A_{,}y^2+A_{,,}z^2+2Byz+2B_{,}xz+2B_{,,}xy+2Cx+2C_{,}y+2C_{,,}z=1,$$

the *tangential* equation of the same surface referred to the same rectangular axes may be found by eliminating x, y, z between the following equations, given in this section.

$$\left.\begin{array}{l} \xi=\dfrac{Ax+B_{,}z+B_{,,}y+C}{1-Cx-C_{,}y-C_{,,}z}; \quad v=\dfrac{A_{,}y+B_{,,}x+Bz+C_{,}}{1-Cx-C_{,}y-C_{,,}z}; \\ \zeta=\dfrac{A_{,,}z+By+B_{,}x+C_{,,}}{1-Cx-C_{,}y-C_{,,}z}; \text{ and } x\xi+yv+z\zeta=1. \end{array}\right\} \quad . \quad . \quad \text{(a)}$$

They may be reduced as follows:—

$$\left.\begin{array}{l} (A\ +C\ \xi)x+(B_{,,}+C_{,}\xi)y+(B_{,}+C_{,,}\xi)z+C\ -\xi=0, \\ (A_{,}\ +C_{,}v)y+(B\ +C_{,,}v)z+(B_{,,}+C\ v)x+C_{,}-v=0, \\ (A_{,,}+C_{,,}\zeta)z+(B_{,}\ +C\ \zeta)x+(B\ +C_{,}\zeta)y+C_{,,}-\zeta=0. \end{array}\right\} \quad . \quad \text{(b)}$$

Let us now assume the three formal linear equations

$$ax+by+cz=d, \quad a_{,}x+b_{,}y+c_{,}z=d_{,}, \quad a_{,,}x+b_{,,}y+c_{,,}z=d_{,,}. \quad \text{(c)}$$

Comparing the coefficients of these expressions with those of the preceding equations (b), we shall have

$$\left.\begin{array}{l|l|l|l} a\ =A\ +C\xi, & b\ =B_{,,}+C_{,}\xi, & c\ =B_{,}\ +C_{,,}\xi, & d\ =\xi-C\ , \\ a_{,}=B_{,,}+Cv, & b_{,}=A_{,}\ +C_{,}v, & c_{,}=B\ +C_{,,}v, & d_{,}=v-C_{,}, \\ a_{,,}=B_{,}\ +C\zeta, & b_{,,}=B\ +C_{,}\zeta, & c_{,,}=A_{,,}+C_{,,}\zeta, & d_{,,}=\zeta-C_{,,}. \end{array}\right\} \quad \text{(d)}$$

If we now solve the group of formal equations (c) for x, y, z, we shall have

$$\left.\begin{array}{l} x=\dfrac{(b_{,,}c_{,}-b_{,}c_{,,})d+(bc_{,,}-b_{,,}c)d_{,}+(b_{,}c-bc_{,})d_{,,}}{a(b_{,,}c_{,}-b_{,}c_{,,})+a_{,}(bc_{,,}-b_{,,}c)+a_{,,}(b_{,}c-bc_{,})}, \\ y=\dfrac{(a_{,}c_{,,}-a_{,,}c_{,})d+(a_{,,}c-ac_{,,})d_{,}+(ac_{,}-a_{,}c)d_{,,}}{a(b_{,,}c_{,}-b_{,}c_{,,})+a_{,}(bc_{,,}-b_{,,}c)+a_{,,}(b_{,}c-bc_{,})}, \\ z=\dfrac{(a_{,,}b_{,}-a_{,}b_{,,})d+(ab_{,,}-a_{,,}b)d_{,}+(a_{,}b-ab_{,})d_{,,}}{a(b_{,,}c_{,}-b_{,}c_{,,})+a_{,}(bc_{,,}-b_{,,}c)+a_{,,}(b_{,}c-bc_{,})}; \end{array}\right\} \quad . \quad \text{(e)}$$

substituting for the nine constants a, b, c, $a_{,}$, $b_{,}$, $c_{,}$, and $a_{,,}$, $b_{,,}$, $c_{,,}$ their values as derived from (d), we shall have, putting Δ for this common denominator,

$$\left.\begin{array}{l} \Delta=AB^2+A_{,}B_{,}^2+A_{,,}B_{,,}^2+AA_{,}A_{,,}-2BB_{,}B_{,,} \\ \quad +[(B^2-A_{,}A_{,,})C+(B_{,,}A_{,,}-BB_{,})C_{,}+(B_{,}A_{,}-B_{,,}B)C_{,,}]\xi \\ \quad +[(B_{,}^2-AA_{,,})C_{,}+(BA-B_{,}B_{,,})C_{,,}+(B_{,,}A_{,,}-BB_{,})C]v \\ \quad +[(B_{,,}^2-AA_{,})C_{,,}+(B_{,}A_{,}-BB_{,,})C+(BA-B_{,}B_{,,})C_{,}]\zeta. \end{array}\right\} \quad \text{(f)}$$

We have also, multiplying by ξ,

$$\left.\begin{aligned}
&(b_{\prime\prime}c-b_{\prime}c_{\prime\prime})\,d\xi\\
&=(B^2-A_{\prime}A_{\prime\prime})\xi^2+(C_{\prime\prime}B-A_{\prime\prime}C_{\prime})\xi^2\upsilon+(BC_{\prime}-A_{\prime}C_{\prime\prime})\xi^2\zeta\\
&\quad-(B^2-A_{\prime}A_{\prime\prime})C\xi-(C_{\prime\prime}B-A_{\prime\prime}C_{\prime})C\xi\upsilon-(BC_{\prime}-A_{\prime}C_{\prime\prime})C\xi\zeta\\
&(bc_{\prime\prime}-b_{\prime\prime}c)\,d_{\prime}\xi\\
&=(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})\xi\upsilon+(A_{\prime\prime}C_{\prime}-BC_{\prime\prime})\xi^2\upsilon+(B_{\prime\prime}C_{\prime\prime}-B_{\prime}C_{\prime})\xi\upsilon\zeta\\
&\quad-(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})C_{\prime}\xi-(A_{\prime\prime}C_{\prime}-BC_{\prime\prime})C_{\prime}\xi^2-(B_{\prime\prime}C_{\prime\prime}-B_{\prime}C_{\prime})C_{\prime}\xi\zeta\\
&(b_{\prime}c-bc_{\prime})\,d_{\prime\prime}\xi\\
&=(A_{\prime}B_{\prime}-B_{\prime\prime}B)\xi\zeta+(A_{\prime}C_{\prime\prime}-BC_{\prime})\xi^2\zeta+(B_{\prime}C_{\prime}-B_{\prime\prime}C_{\prime\prime})\xi\upsilon\zeta\\
&\quad-(A_{\prime}B_{\prime}-B_{\prime\prime}B)C_{\prime\prime}\xi-(A_{\prime}C_{\prime\prime}-BC_{\prime})C_{\prime\prime}\xi^2-(B_{\prime}C_{\prime}-B_{\prime\prime}C_{\prime\prime})C_{\prime\prime}\xi\upsilon.
\end{aligned}\right\}\ (g)$$

If we now add these expressions together we shall have, since the triple products of ξ, υ, ζ vanish,

$$\left.\begin{aligned}
\Delta x\xi=&\,[(B^2-A_{\prime}A_{\prime\prime})+2BC_{\prime}C_{\prime\prime}-A_{\prime\prime}C_{\prime}^2-A_{\prime}C_{\prime\prime}^2]\xi^2\\
&+[(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})+(A_{\prime\prime}C_{\prime}-C_{\prime\prime}B)C+(B_{\prime\prime}C_{\prime\prime}-B_{\prime}C_{\prime})C_{\prime\prime}]\xi\upsilon\\
&+[(A_{\prime}B_{\prime}-BB_{\prime\prime})+(A_{\prime}C_{\prime\prime}-BC_{\prime})C+(B_{\prime}C_{\prime}-B_{\prime\prime}C_{\prime\prime})C_{\prime}]\xi\zeta\\
&+[(A_{\prime}A_{\prime\prime}-B^2)C+(BB_{\prime}-A_{\prime\prime}B_{\prime\prime})C_{\prime}+(B_{\prime\prime}B-A_{\prime}B_{\prime})C_{\prime\prime}]\xi.\\
\text{In like manner}&\\
\Delta y\upsilon=&\,[(B_{\prime}-AA_{\prime\prime})+2B_{\prime}CC_{\prime\prime}-AC_{\prime\prime}^2-A_{\prime\prime}C^2]\upsilon^2\\
&+[(AB-B_{\prime}B_{\prime\prime})+AC_{\prime\prime}-CB_{\prime})C_{\prime}+(BC-B_{\prime\prime}C_{\prime\prime})C]\upsilon\zeta\\
&+[(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})+(A_{\prime\prime}C-B_{\prime}C_{\prime\prime})C_{\prime}+(B_{\prime\prime}C_{\prime\prime}-BC)C_{\prime\prime}]\xi\upsilon\\
&+[(AA_{\prime\prime}-B_{\prime}^2)C_{\prime}+(B_{\prime}B_{\prime\prime}-AB)C_{\prime\prime}+(BB_{\prime}-A_{\prime\prime}B_{\prime\prime})C]\upsilon,\\
\text{and also}&\\
\Delta z\zeta=&\,[(B_{\prime\prime}-AA_{\prime})+2B_{\prime\prime}C_{\prime}C-A_{\prime}C^2-AC_{\prime}^2]\zeta^2\\
&+[(A_{\prime}B_{\prime}-BB_{\prime\prime})+(A_{\prime}C-C_{\prime}B_{\prime\prime})C_{\prime\prime}+(B_{\prime}C_{\prime}-BC)C_{\prime}]\xi\zeta\\
&+[(AB-B_{\prime}B_{\prime\prime})+(AC_{\prime}-B_{\prime\prime}C)C_{\prime\prime}+(BC-B_{\prime}C_{\prime})C]\upsilon\zeta\\
&+[(A_{\prime}A-B_{\prime\prime}^2)C_{\prime\prime}+(BB_{\prime\prime}-A_{\prime}B_{\prime})C+(B_{\prime}B_{\prime\prime}-AB)C_{\prime}]\zeta.
\end{aligned}\right\}\ (h)$$

Bearing in mind that

$$\Delta(x\xi+y\upsilon+z\zeta)=\Delta,$$

we shall have, making the necessary reductions,

$$\left.\begin{aligned}&AB^2+A_{\prime}B_{\prime}^2+A_{\prime\prime}B_{\prime\prime}^2-AA_{\prime}A_{\prime\prime}-2BB_{\prime}B_{\prime\prime}\\&=[(B^2-A_{\prime}A_{\prime\prime})+2BC_{\prime}C_{\prime\prime}-A_{\prime\prime}C_{\prime}^2-A_{\prime}C_{\prime\prime}^2]\xi^2\\&+[(B_{\prime}^2-AA_{\prime\prime})+2B_{\prime}CC_{\prime\prime}-AC_{\prime\prime}^2-A_{\prime\prime}C^2]v^2\\&+[(B_{\prime\prime}^2-AA_{\prime})+2B_{\prime\prime}CC_{\prime}-A_{\prime}C^2-AC_{\prime}^2]\zeta^2\\&+2[(AB-B_{\prime}B_{\prime\prime})+(BC-B_{\prime\prime}C_{\prime\prime})C+(AC_{\prime\prime}-CB_{\prime})C_{\prime}]v\zeta\\&+2[(A_{\prime}B_{\prime}-BB_{\prime\prime})+(B_{\prime}C_{\prime}-BC)C_{\prime}+(A_{\prime}C-C_{\prime}B_{\prime\prime})C_{\prime\prime}]\xi\zeta\\&+2[(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})+(B_{\prime\prime}C_{\prime\prime}-B_{\prime}C_{\prime})C_{\prime\prime}+(A_{\prime\prime}C_{\prime}-BC_{\prime\prime})C]\xi v\\&+2[(A_{\prime}A_{\prime\prime}-B^2)C+(BB_{\prime}-A_{\prime\prime}B_{\prime\prime})C_{\prime}+(B_{\prime\prime}B-A_{\prime}B_{\prime})C_{\prime\prime}]\xi\\&+2[(AA_{\prime\prime}-B_{\prime}^2)C_{\prime}+(B_{\prime}B_{\prime\prime}-AB)C_{\prime\prime}+(BB_{\prime}-A_{\prime\prime}B_{\prime\prime})C]v\\&+2[(AA_{\prime}-B_{\prime\prime}^2)C_{\prime\prime}+(BB_{\prime\prime}-A_{\prime}B_{\prime})C+(B_{\prime}B_{\prime\prime}-AB)C_{\prime}]\zeta.\end{aligned}\right\}\quad\text{(i)}$$

Let X, Y, Z be the projective ordinates of the centre of the surface; then, as shown in sec. [75],

$$X=\frac{(A_{\prime}A_{\prime\prime}-B^2)C+(BB_{\prime}-A_{\prime\prime}B_{\prime\prime})C_{\prime}+(B_{\prime\prime}B-A_{\prime}B_{\prime})C_{\prime\prime}}{AB^2+A_{\prime}B_{\prime}^2+A_{\prime\prime}B_{\prime\prime}^2-AA_{\prime}A_{\prime\prime}-2BB_{\prime}B_{\prime\prime}}, \quad . \quad \text{(j)}$$

and like values for Y and Z may be found.

When the surface is a paraboloid, as in this case the absolute term vanishes, we shall have

$$AB^2+A_{\prime}B_{\prime}^2+A_{\prime\prime}B_{\prime\prime}^2-AA_{\prime}A_{\prime\prime}-2BB_{\prime}B_{\prime\prime}=0. \quad . \quad . \quad \text{(k)}$$

When $C=0$, $C_{\prime}=0$, $C_{\prime\prime}=0$ or when the origin is at the centre; the projective equation of the surface becomes

$$Ax^2+A_{\prime}y^2+A_{\prime\prime}z^2+2Byz+2B_{\prime}xz+2B_{\prime\prime}xy=1,$$

and the tangential equation of the surface referred to the same axes is

$$\left.\begin{aligned}&AB^2+A_{\prime}B_{\prime}^2+A_{\prime\prime}B_{\prime\prime}^2-AA_{\prime}A_{\prime\prime}-2BB_{\prime}B_{\prime\prime}\\&=(B^2-A_{\prime}A_{\prime\prime})\xi^2+(B_{\prime}^2-AA_{\prime\prime})v^2+(B_{\prime\prime}^2-AA_{\prime})\zeta^2\\&+2(AB-B_{\prime}B_{\prime\prime})v\zeta+2(A_{\prime}B_{\prime}-BB_{\prime\prime})\xi\zeta+2(A_{\prime\prime}B_{\prime\prime}-BB_{\prime})\xi v.\end{aligned}\right\}\quad .\quad \text{(l)}$$

Page 133. Sec. [134].

356.] *In a system of confocal ellipses the envelope of the normal that makes with the major axis an angle whose sine is $b(a^2-b^2)^{-\frac{1}{2}}$ is a four-cusped hypocycloid with two opposite cusps at the foci of the system.*

Proposed by Mr. J. L. McKenzie, in the *Educational Times*, No. 4420.

The tangential equation of the evolute of an ellipse (see sec. [156]), since $a^2=b^2+k^2$, may be transformed into

$$b^2(v^2+\xi^2)+k^2v^2=k^4\xi^2v^2; \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

but it is assumed that the sine of the angle which the normal makes with the axis is $\frac{b}{k}$, and as the normal is parallel to the perpendicular from the centre on the tangent, we shall have $\frac{b^2}{k^2}=\frac{\xi^2}{\xi^2+v^2}$, or $b^2(\xi^2+v^2)=k^2\xi^2$. (b) ; eliminating b^2 between (a) and (b), we shall have

$$\xi^2+v^2-k^2\xi^2v^2=0=\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

the tangential equation of the quadrantal hypocycloid, as shown in sec. [134].

The projective equation of this curve may easily be found by the help of the *formulæ* of transition given in sec. [22].

For $$\frac{d\mathrm{V}}{d\xi}=2(k^2v^2-1), \quad \frac{d\mathrm{V}}{dv}=2(k^2\xi^2-1);$$

hence $$x=\frac{k^2v^2-1}{k^2\xi v^2}, \text{ or } k^2(1-\xi x)v^2=1.$$

But (c) gives $(k^2\xi^2-1)v^2=\xi^2$; eliminating v^2 between these equations,

$$\xi^3=\frac{1}{k^2x}; \text{ hence } \frac{1}{\xi^2}=k^{\frac{4}{3}}x^{\frac{2}{3}}. \text{ In like manner } \frac{1}{v^2}=k^{\frac{4}{3}}y^{\frac{2}{3}}.$$

But $\frac{1}{\xi^2}+\frac{1}{v^2}=k^2$; hence we have $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$, the projective equation of the quadrantal hypocycloid.

357.] *Let a, b be two conjugate semidiameters of an ellipse, and $x_{\prime}$, $y_{\prime}$ the coordinates referred to them of a variable point in the curve ; to show that the envelope of a series of ellipses whose semidiameters are coincident in direction with a, b, and in magnitude are mean proportionals between a, $x_{\prime}$ and b, $y_{\prime}$ is* $\left(\frac{x}{a}\right)^{\frac{4}{3}}+\left(\frac{y}{b}\right)^{\frac{4}{3}}=1$.

Proposed by Mr. W. J. C. Miller, in the *Educational Times*, No. 4463 *.

If $ax_{\prime}$ and $by_{\prime}$ be the squares of the semiaxes of the variable ellipse, its tangential equation is $ax_{\prime}\xi^2+by_{\prime}v^2=1$, . . (a)
and the equation of the given ellipse is $a^2y_{\prime}^2+b^2x_{\prime}^2=a^2b^2$, . (b)
Eliminating $y_{\prime}$ between these equations,

$$\frac{x_{\prime}^2}{a^2}+\frac{(1-a\xi^2x_{\prime})^2}{b^4v^4}-1=0=\mathrm{F}. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

To eliminate $x_{\prime}$, $y_{\prime}$ we must manifestly have three equations—(a), (b), and the differential of F with respect to $x_{\prime}$, or

$$\frac{d\mathrm{F}}{dx_{\prime}}=0. \quad . \quad . \quad \text{(d)} \qquad \text{But as } \frac{d\mathrm{F}}{dx_{\prime}}=0, \ \frac{x_{\prime}}{a}=\frac{a^2v^2}{[a^4\xi^4+b^4v^4]}.$$

* This solution embodies an important principle. It shows how the tangential method may be extended to those cases in which the envelope is generated by the successive intersections of curves whose parameters vary according to a given law.

Finding a like expression for $y_{,}$, $\frac{y_{,}}{b} = \frac{b^2 v^2}{[a^4\xi^4 + b^4 v^4]}$.

Substituting these values of $x_{,}$ and $y_{,}$ in (b), the resulting expression is

$$(a\xi)^4 + (bv)^4 = 1. \quad \ldots \ldots \quad \text{(d)}$$

This is the tangential equation of the curve required.

If we require the projective equation of the same curve, we must put

$$V = (a\xi)^4 + (bv)^4 - 1; \ldots \text{(e)} \quad \text{then } \frac{dV}{d\xi} = 4a^4\xi^3, \ \frac{dV}{dv} = 4b^4v^3. \ldots \text{(f)}$$

But $x = \dfrac{\frac{dV}{d\xi}}{\frac{dV}{d\xi}\xi + \frac{dV}{dv}v}$, see sec. [22]. A like value for y is obvious.

Hence $\frac{x}{a} = a^3\xi^3$, or $\left(\frac{x}{a}\right)^{\frac{4}{3}} = a^4\xi^4$.

Finding a like value for y, the projective equation becomes

$$\left(\frac{x}{a}\right)^{\frac{4}{3}} + \left(\frac{y}{b}\right)^{\frac{4}{3}} = 1. \quad \ldots \ldots \quad \text{(g)}$$

Page 112. Sec. [119].

358.] *If at each point of an ellipsoid a distance* $\frac{k^2}{P}$ *be measured along the normal,* P *being the perpendicular from the centre on the tangent plane at that point, the locus of the point so defined is another ellipsoid, the envelope of which for different values of k is the* " surface of centres " *of the original ellipsoid.*

Proposed by Mr. R. F. Scott, B.A., in the *Educational Times*, No. 4466.

Let the tangential equation of the ellipsoid be

$$a^2\xi^2 + b^2v^2 + c^2\zeta^2 = 1, \quad \ldots \ldots \quad \text{(a)}$$

and $\qquad \frac{1}{P} = \sqrt{\xi^2 + v^2 + \zeta^2}$; hence $\frac{k^2}{P} = k^2\sqrt{\xi^2 + v^2 + \zeta^2}$.

Let x, y, z be the projective coordinates of the point on the surface to which the normal is drawn; then $x = a^2\xi$, and the projection of the line $\frac{k^2}{P}$ on the axis of X is $\frac{k^2\xi\sqrt{\xi^2 + v^2 + \zeta^2}}{\sqrt{\xi^2 + v^2 + \zeta^2}} = k^2\xi$; and if $x_{,}$ be the projection of its extremity, we shall have $x_{,} = a^2\xi - k^2\xi$, since $(x - x_{,}) = k^2\xi$. Consequently $\xi = \frac{x_{,}}{a^2 - k^2}$. In like manner

$$v = \frac{y_{,}}{b^2 - k^2}, \text{ and } \zeta = \frac{z_{,}}{c^2 - k^2}. \quad \ldots \ldots \quad \text{(b)}$$

Substituting these values of ξ, v, ζ in (a) we shall have

$$\frac{a^2x_i^2}{(a^2-k^2)^2}+\frac{b^2y_i^2}{(b^2-k^2)^2}+\frac{c^2z_i^2}{(c^2-k^2)^2}=1 \quad . \quad . \quad . \quad (c)$$

for the projective equation of this surface.

Hence by the *formulæ of transition*, p. 68, the tangential equation of this same surface will be

$$(a^2-k^2)^2\frac{\xi^2}{a^2}+(b^2-k^2)\frac{v^2}{b^2}+(c^2-k^2)\frac{\zeta^2}{c^2}-1=0\equiv \mathrm{V}. \quad . \quad (d)$$

We must now eliminate k between this equation and $\frac{d\mathrm{V}}{dk}=0$.

This elimination gives $k^2=\dfrac{\xi^2+v^2+\zeta^2}{\dfrac{\xi^2}{a^2}+\dfrac{v^2}{b^2}+\dfrac{\zeta^2}{c^2}}$. . . . . (e)

Substituting this value of k in the equation $\mathrm{V}=0$ we shall have

$$(\xi^2+v^2+\zeta^2)^2=\left[\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{z^2}\right](a^2\xi^2+b^2v^2+c^2\zeta^2-1), \quad . \quad (f)$$

the tangential equation of the surface of centres as found in sec. [119].

359.] *A given ellipse* F *is one of a system of concentric similar and similarly situated ellipses. A line is drawn touching any other ellipse* H *of the system; and the perpendicular distance of the tangent from the centre is a mean proportional between the semi- major axis of* H, *and the semi- minor axis of* F. *To show that the envelope of the tangent is the first negative pedal of* F, *but turned round a right angle about its centre.*

Proposed by Mr. J. L. McKenzie, in the *Educational Times*, No. 4368.

The tangential equation of the first negative pedal of

$$a^2y^2+b^2x^2-a^2b^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

is

$$a^2v^2+b^2\xi^2=a^2b^2(\xi^2+v^2)^2: \quad . \quad . \quad . \quad . \quad (b)$$

see (β) sec. [163]. The projective equation of the reciprocal polar of (a), a being the radius of the polarizing circle, is

$$a^2x^2+b^2y^2=a^4;$$

and the tangential equation of its first negative pedal is

$$a^2\xi^2+b^2v^2=a^4(\xi^2+v^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Let

$$a^2\xi^2+b^2v^2-1=0\equiv \mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

be the tangential equation of the given ellipse, and let

$$n^2a^2\xi^2+n^2b^2v^2-1=0\equiv \mathrm{W} \quad . \quad . \quad . \quad . \quad (e)$$

be the tangential equation of one of the concentric and similar ellipses. But, by the conditions of the question,

$$nab(\xi^2+v^2)=1\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(f)}$$

eliminating n between this and the preceding equation, we get

$$a^2\xi^2+b^2v^2=a^2b^2(\xi^2+v^2)^2. \quad . \quad . \quad . \quad . \qquad \text{(g)}$$

This equation would coincide with (b) were the axes of coordinates turned through a right angle, or if ξ and v were changed into v and ξ.

If the duplicate ratio of the perpendicular on the tangent to the linear unit be equal to the ratio of linear similarity of V and W, the envelope of this tangent is the first negative pedal of the polar reciprocal of V.

For, by supposition, $\quad n=a^2(\xi^2+v^2)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad$ (h)

eliminating n between (h) and (f) we get

$$a^2\xi^2+b^2v^2=a^2b^2(\xi^2+v^2)^2, \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

which coincides with (g).

360.] Prove that the ellipses

$$a^2y^2+b^2x^2=a^2b^2, \quad a^2x^2\sec^4\varphi+b^2y^2\operatorname{cosec}^4\varphi=a^4e^4 \quad . \qquad \text{(U, V)}$$

are so related that the envelope of (V), for different values of φ, is the evolute of (U), and that a point of contact of (V) with its envelope is the centre of curvature at the point of (U) whose eccentric angle is φ.

Proposed by Mr. R. Tucker, in the *Educational Times*, No. 4240.

Let

$$a^2y^2+b^2x^2-a^2b^2=0\equiv\text{U}, \quad a^2x^2\sec^2\varphi+b^2y^2\operatorname{cosec}^4\varphi-a^4e^4=0\equiv\text{V}.$$

Find the value of φ from the equation $\dfrac{d\text{V}}{d\varphi}=0$; substitute this value in $\text{V}=0$, and we shall have $\text{W}\equiv(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}-(a^2-b^2)^{\frac{2}{3}}=0$; and this is the projective equation of the evolute of $\text{U}=0$.

Again, assuming $x=\dfrac{(a^2-b^2)\cos^3\varphi}{a}$, and $\quad y=\dfrac{(a^2-b^2)}{b}\sin^3\varphi, \quad$ (a)

we shall find that these values of x and y satisfy the equations $\text{V}=0$, $\text{W}=0$. Hence this point is common to the ellipse $\text{V}=0$ and its evolute $\text{W}=0$.

Moreover, if $\bar{x}$ and $\bar{y}$ be the coordinates of a point on the ellipse $\text{U}=0$, of which point φ is the eccentric angle, we shall have

$$\bar{x}=a\cos\varphi, \text{ and } \bar{y}=b\sin\varphi\,; \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

and if we eliminate $\cos\phi$ and $\sin\phi$ between (a) and (b) we shal. have

$$\bar{x}=\frac{a^{\frac{4}{3}}x^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}},\quad \bar{y}=\frac{b^{\frac{4}{3}}y^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Hence x and y are the projective coordinates of the centre of curvature of the point $(\bar{x}, \bar{y})$.

2. The question may be solved as follows by tangential coordinates :—

Let $$a^2\xi^2+b^2v^2-1=0\equiv U', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

and $$\frac{(a^2-b^2)^2}{a^2}\cos^4\phi\xi^2+\frac{(a^2-b^2)^2}{b^2}\sin^4\phi v^2-1=0\equiv V', \quad . \quad \text{(c)}$$

be the tangential equation of the two ellipses. Then, finding the value of $\frac{dV'}{d\phi}=0$, we shall have $\cos^2\phi=\frac{a^2v^2}{a^2v^2+b^2\xi^2}$. Eliminating $\sin\phi$, $\cos\phi$ from $V'=0$, we shall have

$$a^2v^2+b^2\xi^2-(a^2-b^2)^2\xi^2v^2=0\equiv W', \quad . \quad . \quad . \quad \text{(f)}$$

which is the tangential equation of the evolute of $U=0$ (see vol. i. p. 115).

Assume $$\xi=\frac{a\sec\phi}{(a^2-b^2)},\ \text{and}\ v=\frac{b\operatorname{cosec}\phi}{(a^2-b^2)}. \quad . \quad . \quad . \quad . \quad \text{(g)}$$

Now, substituting these assumed values of ξ and v in the equations $V'=0$ and $W'=0$, we shall find that they satisfy these equations; consequently the ellipse $V'=0$, and its evolute $W'=0$, have a common tangent.

Let $\bar{\xi}$ and ξ denote the tangential ordinates along the axis of X, made by two tangents passing through a point on an ellipse, one to the ellipse, the other to the evolute, and let ϕ be the eccentric angle of $U'=0$ at this point; then

$$a\bar{\xi}=\cos\phi,\ b\bar{v}=\sin\phi,\ \text{and}\ a\bar{\xi}\xi=\xi\cos\phi=\frac{a}{a^2-b^2},\ \text{from (a)}. \quad \text{(h)}$$

Hence $(a^2-b^2)\bar{\xi}\xi=1$; consequently the common tangent to $V'=0$ and $W'=0$ passes through the point on $U'=0$, of which the eccentric angle is ϕ.

If we substitute the values of x, y, ξ, v assumed in the equations (a) and (g), we shall find that they satisfy the *dual equation* $x\xi+yv=1$; consequently the common tangent passes through the common point of the two given ellipses.

Page 230. Sec. [254].

361.] From this focal property of a surface of the second order having three unequal axes may be derived this new theorem :—

Let two equal semidiameters k *be drawn in an ellipse whose semi-axes are* a, b. *Assume two points* C *and* D *on the major axis, such that* $\mathrm{CO}=\sqrt{a^2-k^2}$ and $\mathrm{DO}=\frac{a^2}{\sqrt{a^2-k^2}}$, O *being the centre. Through the point* D *let two straight lines be drawn parallel to the equal semidiameters* k. *From any point* Q *on the ellipse let perpendiculars* $\mathrm{P}, \mathrm{P}_{\prime}$ *be drawn to these two lines, and a vector* R *from* Q *to* C; *we shall have* $\frac{\mathrm{P}.\mathrm{P}_{\prime}}{\mathrm{R}^2}=\frac{b^2a^2}{k^2(a^2-b^2)}$ *a constant ratio.*

Cor. i.] When $k=b$, the perpendiculars P, $\mathrm{P}_{\prime}$ coincide and become equal, and the ratio becomes $\frac{1}{e^2}$, the common focal property of the ellipse.

Cor. ii.] When $k=a$, the point D is at infinity, the lines to which the perpendiculars from a point on the curve are drawn become the *minor directrices*, of which the properties are developed in sec. [288].

Page 329. Sec. [354].

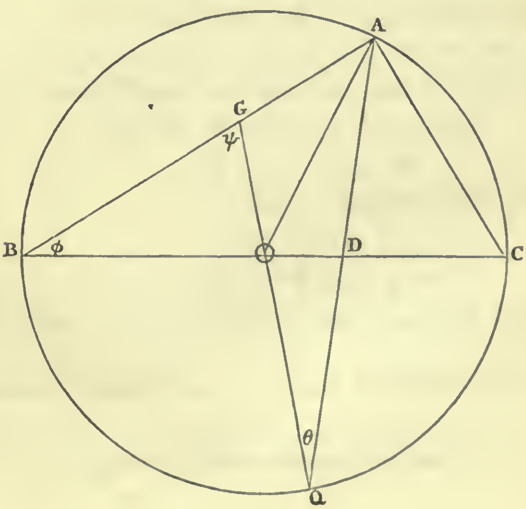
The following are the numerical values of sec ϵ, tan ϵ, e, and e^{-1}.

$\sec\epsilon=1{\cdot}5430806348$ &c., $\tan\epsilon=1{\cdot}1752011936$ &c.,

$\sec\epsilon+\tan\epsilon=e=2{\cdot}7182818284$ &c.,

$\cos\epsilon=e^{-1}=0{\cdot}3678794411$ &c.

Fig. 92.



Page 313. Sec. [343].

Proposed by the Rev. W. Roberts, M.A., in the *Educational Times*, No. 1749.

362.] *In a right-angled triangle* ABC (fig. 92) *a straight line is drawn from the right angle* A *to a point* D *in the line* BC, *whose distance from the middle point of* BC *is one third of the radius of the circumscribing circle. The line* AD *is produced to meet the circle in* Q. *Through* Q *draw the radius* QO *meeting the side* AB *in* G. *Let the angle* ABC *be* ϕ, *and the angle* OGB *be* ψ; *then*

$$(\sec\psi+\tan\psi)=(\sec\phi+\tan\phi)^3. \quad . \quad . \quad . \quad . \quad (a)$$

Let the angle AQO be θ, $AO=r$, and $QD=nr$; hence

$$\left.\begin{aligned} 2\theta&=\psi-\phi,\\ 3n\sin\left(\frac{\psi-\phi}{2}\right)&=\sin(\psi+\phi),\\ \text{and }\ 9n^2&=10+6\cos(\psi+\phi). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (b)$$

Eliminating θ, n, and reducing, we find

$$2(\sec\psi\sec\phi-\tan\psi\tan\phi)^2-(\sec\psi\sec\phi+\tan\psi\tan\phi)=1. \quad (c)$$

Subtracting from the preceding equation the identical expressions $(\sec^2\psi-\tan^2\psi)(\sec^2\phi-\tan^2\phi)=1$, we find

$$\left.\begin{aligned} (\sec^2\psi+\tan^2\psi)(\sec^2\phi+\tan^2\phi)-4\sec\psi\tan\psi\sec\phi\tan\phi\\ =\sec\psi\sec\phi+\tan\psi\tan\phi. \end{aligned}\right\} \quad (d)$$

But in sec. [344], (θ), (η), and (γ), it is shown that

$$\sec^2\psi+\tan^2\psi=\sec(\psi\perp\psi),\ \ 2\sec\psi\tan\psi=\tan(\psi\perp\psi),$$

and $$\sec\psi\sec\phi+\tan\psi\tan\phi=\sec(\psi\perp\phi).$$

Substituting these values in the preceding equation (d), it becomes

$$\sec(\psi\perp\psi)\sec(\phi\perp\phi)-\tan(\psi\perp\psi)\tan(\phi\perp\phi)=\sec(\psi\perp\phi). \ (e)$$

But this formula may be written, as shown in sec. [344],

$$\sec(\psi\perp\psi\top\phi\top\phi)=\sec(\psi\perp\phi),$$

or $$\psi\perp\psi\top\phi\top\phi=\psi\perp\phi.$$

Transposing and changing $\top$ into $\perp$

$$\psi=\phi\perp\phi\perp\phi; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

hence $$\sec\psi+\tan\psi=(\sec\phi+\tan\phi)^3. \quad . \quad . \quad . \quad . \quad (g)$$

Generally, the following relation exists in parabolic trigonometry:—

$$\sec(\phi\perp\phi\ldots\text{to } n \text{ angles})+\tan(\phi\perp\phi\ldots\text{to } n\text{ angles})=(\sec\phi+\tan\phi)^n.$$

THE END OF THE SECOND VOLUME.

A Treatise on some New Geometrical Methods, containing Essays on Tangential Coordinates, Pedal Coordinates, Reciprocal Polars, the Trigonometry of the Parabola, the Geometrical Origin of Logarithms, the Geometrical Properties of Elliptic Integrals, on Rotatory Motion, the Higher Geometry, and Conics. By J. Booth, LL.D., F.R.S., F.R.A.S., &c., Vicar of Stone, Buckinghamshire. (In Two Volumes.) Vol. I. with Photographic Portrait of the Author, 416 pp. and 87 Diagrams. Medium 8vo. Price 18*s.* (June 1873.)

The following reviews and notices of the first volume have appeared :—

From the 'Bulletin des Mathématiques,' Paris, June 1873.

"Le développement du grand principe de la Dualité géométrique est l'idée fondamentale de cet ouvrage. Dans les vingt-deux premiers chapitres, l'auteur établit un système de coordonnées qu'il appelle *coordonnées tangentielles*, le corrélatif du système bien connu des coordonnées cartésiennes, et qu'il base sur une notation algébrique particulière. Il applique sa méthode et à la discussion et à la solution de différents théorèmes et problèmes, en établissant dans chaque cas la corrélation des figures géométriques. Cette théorie est appliquée non-seulement aux courbes et surfaces courbes du second degré, mais à celles des degrés supérieurs.

"L'auteur développe la théorie des polaires réciproques par l'application des relations métriques, et plus particulièrement il déduit les propriétés des surfaces du second ordre, ayant trois axes inégaux, de celles des surfaces de révolution. Il continue ensuite à appliquer ce principe *sans exception* de dualité universelle de la Trigonométrie, et établit, pour la parabole, une trigonométrie analogue à celle du cercle. Il démontre l'origine géométrique des logarithmes et fait voir que, si les nombres naturels sont représentés par les rayons vecteurs d'une courbe qu'il nomme *courbe logocyclique*, les logarithmes correspondants seront représentés par les arcs de paraboles correspondantes. Les principes de la Trigonométrie parabolique servent ensuite à établir de nombreuses relations entre les arcs de la parabole ; et l'auteur a soin de signaler les relations semblables que présentent les arcs de la chaînette et, par suite, les rapports de cette courbe avec la traction.

"Ces quelques mots ne donnent qu'un résumé succinct d'un important ouvrage qui est, ainsi que le déclare avec raison l'auteur, entièrement original."

From the 'Standard' of July 21, 1873.

"The mere title of this book will suffice to show that it treats of the highest and most profound geometrical and mathematical problems, and that, were we to discuss at length the various abstruse questions with which Dr. Booth deals, and to follow him through the new methods of solution of these problems which he proposes, there are but few of our readers who would care to follow us. We notice the appearance of the work, however, because, in the first place, it is a very remarkable addition to mathematical science, and because, in the second place, it suggests a number of questions of general importance, many of which are touched upon by the author himself in his introductory remarks. There is a tendency of the present age to believe that although in the domain of practical science and invention there is still great progress to be made, yet that in the region of abstruse scientific problems there is but slight range open to us, and that, even if there were, it would be altogether useless to investigate it. Unfortunately, too, the spirit of the age is entirely utilitarian. In our universities high mathematics are taught and studied with a view that the learner may obtain high honours, and so reap the substantial benefits of scholarships and fellowships. Men do not study these things for their own sake, nor, having once acquired them for the sake of distinction or pecuniary advantage, do they keep up the knowledge after leaving the University. It is difficult, however, to say that any new scientific problems and discussions whatsoever are useless. The utility may not, indeed, be evident at the time; but, for example, our highest astronomical problems could never have been solved had it not been for the application of mathematical problems hitherto condemned as useless. The world is, indeed, deeply indebted to men like Dr. Booth—deep and original thinkers and students, men who make but little stir in the world, who have nothing in common with the gentlemen who love to place themselves in the front rank, and to sound their own trumpets before the world upon all occasions, but who are content to live quiet and retired, seeking neither fame nor profit, but studying laboriously, and issuing perhaps but one book, conveying to the world the result of a lifetime of unremitting mental toil."

From the 'Cambridge Chronicle,' August 2, 1873.

"It is upwards of thirty years since the Rev. Jas. Booth published his first essay on Tangential Coordinates, since which time he has set himself the task of discovering some method of expressing by common algebra the properties of reciprocal curves and curved surfaces. Having been successful in the discovery of a simple method and compact notation, he now gives the public the result of his prolonged labours and researches in this volume of essays on 'Tangential Coordinates, Pedal Coordinates, Reciprocal Polars, the Trigonometry of the Parabola, the Geometrical origin of Logarithms, the Geometrical properties of Elliptic Integrals, and other kindred subjects,' first explaining in the introduction the considerations which led to the discovery of his method. With the usual modesty of great minds the Rev. Jas. Booth apologizes for thus making public the meditations of the 'better part of a lifetime,' during which he has watched in expectation that some accomplished mathematician would take up these subjects and expand them, producing a treatise from which any student of moderate ability might glean enough to enable him to extend those researches still further. No such mathematical champion having appeared, our learned author has compiled this volume, containing at length results of which he has from time to time frequently given abstracts in the Proceedings of learned societies. It would have been difficult to have found a man better fitted for the task, or one who would bring to bear on the subject more ability, more original and deep thought, or more careful and untiring research; indeed this work is the fruit of a life of laborious study in the deepest and highest branches of mathematical science; and those who deal in abstruse scientific problems will frequently find their path made comparatively easy by the arduous labours of their pioneer, the Rev. Jas. Booth."

From the 'Educational Times,' August 1, 1873.

"This is by far the most interesting of the mathematical works which have for a long time been brought under our notice. Here we find gathered up, and placed before us in a connected form, and with singular clearness and elegance of exposition, the various contributions which Dr. Booth has, from time to time, made to our mathematical literature, along with much new matter, which is both valuable and original. The chief feature of the work is the development of the method of Tangential Coordinates, which now, in some form or other, constitutes a recognized portion of the Modern Geometry. * * * *

"The method of Tangential Coordinates, however, forms but a small portion of the contents of the elegant volume before us. Indeed, we do not remember to have ever met with a mathematical book containing so great a variety of interesting, novel, and important matter. This will be clearly seen from the following brief analysis of the contents of the book. The first twenty-four chapters of the volume treat of the development of the principle of duality, as involved in the system of tangential coordinates, applied to space of two and three dimensions. In the twenty-fifth chapter the principle of duality is established geometrically, and then applied—in what we consider one of the most remarkable and original chapters of the book—to the investigation of the properties of surfaces of the second order having three unequal axes, derived from the corresponding properties of surfaces of revolution. In chapter xxix. metrical methods are applied to the discussion of the great principle of duality with reference to the theory of reciprocal polars. In chapter xxx. the logocyclic curve and the geometrical origin of logarithms are discussed; while in chapter xxxi. the trigonometry of the parabola is fully investigated, and the properties of this new branch of mathematical science applied to the catenary and the tractrix. The last chapter is devoted to the discussion of certain properties of confocal surfaces.

"From this rapid analysis it will be seen that there is much in this volume that cannot fail to meet the tastes of all geometers. In some parts of his work, Dr. Booth professes not to be able to find room for many illustrative examples, as he states that his main object is to lay down the principles of the various methods discussed, as applied to a few particular instances, without following out the investigations into all their details. Yet even in the most sparsely illustrated portions of the work we find a few judicious examples, most aptly chosen, while in those portions wherein the author expresses his fears—which we cannot but think altogether groundless—that examples may be thought to be unduly multiplied, the illustrative exercises are in the highest degree valuable. To the readers of this journal these examples will be especially interesting, inasmuch as many of them have appeared in our mathematical columns, and have there received solutions by methods different, for the most part, from those given by the author in the volume before us. Occasionally a solution is quoted entire from our own columns, with appropriate acknowledgment—an act of justice to ourselves which, we regret to say, is not always rendered— as, amongst other instances, in Mr. Spottiswoode's investigation of the Tangential Equation of the Cardioid (p. 142), and the Editor's method (p. 126) of deriving the projective equations of the bicusped and unicusped hypocycloid from the general tangential equation.

"A noteworthy feature of the volume before us—and it is one which we cannot praise too highly—is the clear and elegant style in which it is written. Usually our mathematical books are little more than mere collections of algebraical symbols, with scarcely two consecutive sentences of English of any kind beyond what is required to connect them, from one end to the other. But Dr. Booth possesses a vigorous and forcible style, and very properly devotes much attention and ample space to the interpretation of the results at which he arrives, and to a lucid exposition of the principles of the methods of which he treats.

"The work treats of subjects of great interest and importance to mathematicians, develops methods of much power and efficacy in geometrical research, is written, as we have already stated, in a remarkably clear and vigorous style, and—what is not by any means one of its least recommendations—is one of the best-printed mathematical books that has ever issued from the English press. The woodcuts, eighty-seven in number, are admirably engraved, and really serve to illustrate the book, a well-drawn diagram being introduced wherever it would be of use in enabling us more easily to follow the demonstrations.

"We cannot but express a hope that some of our own contributors will take up Dr. Booth's methods, and develop and apply them in the mathematical pages of this journal, and its connected volumes of reprints. And we hope, too, that Dr. Booth will find, in the reception which mathematicians will accord to this volume, sufficient encouragement to induce him to carry on soon to its completion the promised *second* volume, wherein he proposes, 'if declining years and failing strength should permit' him, to embody his researches on the geometrical origin and properties of Elliptic Integrals, and to apply them to the investigation of the free motion of a rigid body round a fixed point, together with other collateral inquiries.

"In this country we have no 'Minister of Public Instruction,' or 'Keeper of the Seals,' under whose auspices a costly and unremunerative mathematical work could be brought out without any expense to the author; and it would be a subject of regret if, when an English mathematician takes upon himself some of the duties of the above-mentioned functionaries, so useful to men of science across the Channel, and brings out, at his own cost, a work like the one before us, in every way fit to take its place amongst the best French and German treatises, he should, after all his toil and trouble, be taught by painful experience that, in this country, no mathematical work has any chance of success unless it belongs to the petty and trivial class of cram-books, drawn up for the use of candidates preparing for some one of the innumerable competitive examinations which have become the rage of the day. We hope that the volume Dr. Booth has now given to the world will meet with such a reception as may show the writer that there are still 'a chosen few' who can appreciate a work like that before us, of which it is not too much to say, judging from the instalment we have already received, that it promises to be one of the most valuable contributions to mathematical science which has appeared for many years.

"We have hitherto said nothing about what we regard as one of the most attractive portions of the book, the excellent Introduction, which occupies the first twenty-two pages of the volume. The rest of the work is addressed more exclusively to mathematicians; but this is a part which will not be without interest even to the general reader. We should have been glad, had our space permitted, to lay this introduction *in extenso* before our readers."

From the 'Cambridge Express,' October 25, 1873.

"The work consists of separate essays on tangential coordinates, pedal coordinates, reciprocal polars, the trigonometry of the parabola, the geometrical origin of logarithms, geometrical properties of elliptic integrals, and other kindred subjects. Most of these are old friends that have appeared long since, either as pamphlets or in mathematical journals; but they have all grown in the interval since we last saw them. Thus the essay on tangential coordinates is known to most mathematicians as a tract of 32 pp., published at Dublin in 1840, and entitled 'On the Application of a New Analytic Method to the Theory of Curves and Curved Surfaces,' while here it is presented under its now well-known name of 'Tangential Coordinates,' and occupies, perhaps, over 200 pp. This was one of the earliest of Dr. Booth's works, and is the one by which he is best known; in fact the method is always associated with his name. In the original tract of 1840 Dr. Booth's said that he feared that 'brevity and compression had been too much studied in the following essay;' and here, after an interval of thirty-three years, we have the essay amplified and expanded to a size proportional to the value of the method, and with the addition of the notes and examples which have occurred to its author in a period exceeding the average working length of a lifetime. It would not be easy to give an idea of the contents of the work without transcribing the titles of the different chapters, thirty-three in number. The matter in the book is, of course, not consecutive, as it is formed by reprinting, with additions, Dr. Booth's original papers; but there is a 'golden thread' which runs through and connects all the subjects discussed in the volume.

“ There is prefixed to the volume, by way of introduction, an interesting essay, written in a spirit which here and there recalls Babbage’s ‘ Decline of Science in England.’ Dr. Booth laments the utilitarian spirit of the age in this country, and points out how all knowledge is subordinated to the grand question of money-making. On this point we cannot refrain from making the following extract :—

“ ‘ Will it pay? is the test of all mental labour. It was very different in the schools and agoræ of that nation we are so prone to hold up for admiration as exhibiting models of intellectual greatness hitherto unequalled. Nor is this exclusive devotion to the adaptation of science to money-making so universal in other countries as amongst ourselves. Yet it was not always so. One might appeal to the age of Newton and Locke, the age of deep thinking and profound learning, in proof of this position. The causes of this degradation in the objects of intellectual pursuit are many, and some of them deeply seated. Not the least of these is the influence which the philosophy of Bacon has exerted on the tone and tendency of public opinion in this country. No doubt the author of the ‘ Novum Organon ’ conferred great benefits on mankind by laying down so clearly the true principles of physical investigation. He has marred this philosophy, however, by the motives he presents to us for its cultivation. He who could propound the maxim, worthy of Epicurus, that the true object of science is to make men comfortable, had no very exalted conception of the dignity of man’s understanding.

“ ‘ It is plain from his tone of thought that the philosophical Chancellor had a very clear prænotion, to use his own phraseology, of that emphatically English idea, comfort. There is little doubt that he would have valued more the invention of an efficient kitchen-range, or an ingenious corkscrew, than the ideas of Plato or the discoveries of Archimedes.’

“ What particularly charms us in the above quotation is the estimate of Bacon’s philosophy, which we are afraid is not very far from the truth. It is becoming more apparent to the present age that Bacon’s views are very different to those of the *savant,* and that his philosophy is not in all respects the magnificent structure it was, till recently, heresy to have any doubt about.

“ No one, however, can fail to read with much interest Dr. Booth’s views on the subject; and it must be remembered that they come from him as from one of the most earnest labourers in the field of education. If all the time that Dr. Booth devoted to the formation and improvement of the Society of Arts’ schemes of education had been given to his own pursuits, the volume before us would have been a much larger one.

“ A mathematician who republishes his scattered writings collected in a volume, not only thereby secures whatever posthumous fame is his due, but also confers a benefit on his science. Their accumulation in the same volume places the whole in a much higher rank than would belong to the sum of the parts if separate. It is also to be remembered that in many a country house, cut off from the great journal literature of mathematics, the appearance of a book containing original work (not written for teaching-purposes) is hailed with joy.”