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STATICS

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A TREATISE ON STATICS

CONTAINING THE

FUNDAMENTAL PRINCIPLES OF ELECTROSTATICS AND ELASTICITY

BY

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IN THE ROYAL INDIAN ENGINEERING COLLEGE, COOPER'S HILL

SECOND EDITION

CORRECTED AND ENLARGED

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PREFACE

TO THE SECOND EDITION.

THE present edition of this work is, I venture to think, a considerable improvement on the previous one. With such a large number of examples, not only were misprints, but mistakes, more or less trifling, almost inevitable; but, owing chiefly to the kindness of correspondents, very few of these can remain in this edition.

My acknowledgments are, in the first place, due to Mr. Robert Graham, of Kingstown, who supplied me with a long list of corrections and some good suggestions. Mr. J. C. Malet kindly called my attention to some mistakes which I had overlooked in the chapter on Attractions. In some places where a better choice of language was possible for the elucidation of the subject, I have adopted alterations kindly pointed out by Colonel Chesney.

In the earlier part of the book the examples have been subjected to a rearrangement, the order of relative difficulty being better kept in view; and some of them which were of the purely mathematical and fantastic character have been expunged.

Besides alterations of the above description, four others deserve special mention.

Firstly, the proof of the parallelogram of forces has been based entirely on the Newtonian definition of force, and has

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therefore been made to follow from the composition of velocities.

Secondly, the principal propositions of Graphic Statics (so far as coplanar forces are concerned) have been introduced. The subject is a small one and very simple, and I believe that in the few pages in which I have treated it (see end of Chapter V) the student will find enough to enable him to read with ease a more elaborate and formal treatise on graphic methods.

Thirdly, the portion dealing with Electrostatics has been so enlarged as to contain several propositions of importance which had been omitted in the previous edition.

Fourthly, and chiefly, a Chapter on Strains and Stresses has been introduced. So far as English works on Statics, in general, are concerned, this is an innovation, and a very important one. In view of the enormous development of Mathematical Physics, and the wonderful inventions depending on the small strains and vibrations of natural solids, which have been made within the last few years, the study of the equilibrium and motion of bodies *as they are*, and not *as they exist in abstraction*, is surely a subject of which it is impossible to exaggerate the importance. We may well ask whether in this country too much valuable time is not spent in the discussion of neat mathematical unrealities—in the calculation of the behaviour of impossible bodies under impossible conditions. A certain amount of this is of course necessary for the study of the fundamental principles of Dynamics; but the equilibrium and motion of natural solids ought to occupy the attention of every student of Physics after he has acquired a sound and firm knowledge of the fundamental propositions concerning the action of Force. Yet Applied Mechanics, as a sequel to, and corrective of, Rational Dynamics, is a subject the study of which is confined almost exclusively to scientific students of Engineering.

I am very far indeed from asserting or implying that the few pages on Strains and Stresses in this work supply adequately this deficiency in our general scientific education. They are addressed to students who have attained considerable proficiency in pure mathematics, and have a reference much more to the Theories of Light, Magnetism, and Electricity than to ordinary Applied Mechanics. For students of lower attainments a short treatise dealing first with plane elasticity and proceeding thence to strains in three dimensions would be extremely desirable.

In dealing with the theory of Strains and Stresses and with the subject of Electrostatics, I have had the benefit of the invaluable advice and criticism of Mr. Fitzgerald, whose assistance was always given with the utmost zeal. In two Chapters of his *Elements of Dynamic* the late Professor Clifford gave a discussion of 'Strain-Steps' and 'Strain-Velocities' marked by all the elegance and simplicity of treatment which characterised everything he wrote. From these chapters I have derived considerable assistance; but their (quaternion) method is, of course, different from that which I have adopted.

For the view of the theory of Friction presented to the student in this work, I am almost wholly indebted to Mr. Jellett, whose method of treating the rational theory of Friction, both in his Lectures and in his Treatise on the subject, has invested it with a completeness and precision which it had not previously attained. Our knowledge of the laws of Friction has been recently extended by the experiments of Professor Osborne Reynolds on *rolling friction* (*Phil. Trans.*, vol. 166, pt. 1), and by experiments made on an extensive scale on the London Chatham and Dover Railway by Captain Douglas Galton (*Proceedings of the Institution of Mechanical Engineers*, June and October, 1878).

A reference to these experiments will be useful to the student.

I have again to thank Mr. Eagles for his very useful and painstaking assistance in correcting the press and verifying results.

Mr. Reilly's references to sources of information have been, as before, of very great value to me; and I have to thank Professor Wolstenholme for continuing his permission to draw from the inexhaustible store accumulated in his *Book of Mathematical Problems.*

COOPER'S HILL, *December*, 1879.

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ERRATUM.

In page 179, line 3, for W read $\frac{W}{2}$.

NOTE.

In example 39, p. 159, omit everything after the words 'horizontal line through A,' and insert instead the words 'show that as the point C varies, the position of the beam being always the same, the magnitudes and lines of action of the pressure on the axis will be represented by lines drawn from A to a certain right line parallel to AB; and if the position of the beam varies, while AC is always equal to AB, find the curve whose radii vectores will represent the pressure on the axis.'

STATICS.

CHAPTER I.

THE COMPOSITION AND RESOLUTION OF FORCES ACTING IN ONE PLANE AT A POINT.

ARTICLE 1.] **Definition of Force.** *Force is an action exerted upon a body in order to change its state either of rest or of moving uniformly forward in a right line.*

This is the definition of Force given by Newton (see Principia, Book I, Def. IV).

2.] **Divisions of the Science.** The Science which treats of the action of Force on bodies is called *Dynamics*. Of this science there are two branches: the first treats of the laws to which forces are subject when they keep bodies at rest, and this branch is called *Statics;* the second treats of the laws to which the motions of bodies are subject when these motions are produced by given forces, and this branch is called *Kinetics*.

3.] **Matter.** Matter is something which exists in space, and attests its presence by such observed qualities as extension, resistance, and impenetrability.

A limited portion of matter is called a *Body*, and the quantity of matter contained in a body is called its *Mass*. A very small portion of matter is called a *Particle*. *

4.] **Velocity.** Suppose a point to move along a right line in such a way that it always takes the same time, t, to travel over the same length, s, of the line, in whatever points of the line the extremities of this length are situated. Then we readily say that the point's 'rate of moving' is the same all through, and this rate we measure by the quotient $\frac{s}{t}\cdot$ The rate of moving we call the *velocity* of the moving point. But if the time of moving over the length s is not the same all through but depends on the

* A particle is a [illegible] carries with it the [illegible] a point does [illegible]

points of the line between which it is measured, the velocity, or rate of moving, is clearly not uniform. Nevertheless we recognise the fact that at each of its positions the moving point has a particular rate of going. How is this rate to be estimated? Like all rates, it must be measured by a differential coefficient. Thus, if P and Q are two extremely close positions, and if O is any fixed point on the line of motion, the distance between O and P being called s and the distance OQ being called $s + \Delta s$, and if the point has taken the infinitesimal time Δt to get from P to Q, we shall be very near the truth in assuming that its rate of moving has remained uniform in the passage from P to Q, and the velocity in this interval will, as above, be the quotient $\frac{\Delta s}{\Delta t}$. The smaller the interval PQ (and therefore the smaller Δs and Δt) the more nearly true is the assumption of uniformity of the rate of moving from P to Q. Hence if we could find the value of the ratio $\frac{\Delta s}{\Delta t}$ when both Δs and Δt are indefinitely diminished, we should have the exact rate of moving at P. But the limit of this ratio is the differential coefficient $\frac{ds}{dt}$, which is easily found by the rules of the Differential Calculus.

We have thus not only a conception of different rates of moving, but also a method of estimating these rates numerically at different points of the path.

5.] **Criterion of the Action of Force.** Instead of the motion of a mere mathematical point, let us consider the motion of a material particle. How can we tell whether this moving particle is acted on by *force* or not? The answer is—unless the particle is completely at rest, or failing this, *moving with a uniform velocity in a right line,* it is acted on by some force. Observe the two distinct characters which must be possessed by the motion of a particle which is not acted on by force—the velocity must be constant in magnitude and the path must be a right line.

6.] **Measure of Force.** Suppose a particle to move along a right line in such a way that in any interval of time, t, there is the same addition made to its velocity, between whatever epochs of time the interval t is reckoned. Then the velocity is obviously increased at the same rate at every point of the path, and the particle is said to be continuously acted on by a *uniform force* in the line of motion. The rate at which this increase of velocity takes place is taken as the measure of the force acting on the particle; that is, if the same particle moves along a right line in such a way that its velocity is increased at a constant rate which is double the previous rate, it will be continuously acted upon in the second motion by a force which is double the previous force.

If the rate of increase (or in other words, the *acceleration*) of the particle's motion is not uniform, the force acting on it is not uniform, and its magnitude at any point of the particle's path is estimated by the rate of increase of the velocity of the particle at this point.

Since the velocity of one and the same particle is capable of having all possible rates of increase, all forces may be compared with each other by means of their effects on a single particle.

7.] **Ways in which Force is produced.** One of the simplest ways in which a force can be made to act on a particle consists in attaching a string to the particle and pulling this string so as to cause the particle to move. If no other force acts on the particle, and if the string is always pulled in the same right line, the particle will continue to move in this right line; and the *rate*, per unit of time, at which its velocity is being increased at any point of its path is a measure of the magnitude of the force with which the string pulls it; so that if for any finite time we observed its velocity to remain constant, we should know that during this time the string ceased to be pulled, and that no force acted on the particle in this particular interval.

There are other ways in which forces act on particles, but the manner in which they act is not in every case known to us. For example, if the particle consists of a small piece of soft iron and we hold it near the pole of a magnet we shall see it rushing with continually increased velocity towards the magnet, and it is therefore by definition acted on by some force towards the magnet. This force can be measured, as before, at every point of the particle's path by the rate, per unit of time, at which it produces an increase of velocity in the particle; nevertheless it is quite uncertain how this force is produced—whether it is an action at a distance or a stress in some intervening medium.

But whatever its cause may be, we can measure it numerically by its effect—viz., rate of increase of velocity produced in a material particle.

Again, since the velocities of planets towards the sun and of meteoric stones towards the earth are perpetually accelerated, the planets are acted upon by forces towards the sun, and the meteors by forces towards the earth. These forces are called *forces of attraction;* but the nature or precise mode of operation of this attraction is a matter on which no certainty exists.

8.] **Linear Representation of Forces.** Consider a single material particle. Every velocity which it can have possesses three characteristics—it must have a certain numerical magnitude, it must take place in a certain right line, and it must take place in a certain *sense* (from right to left or from left to right) along this line; or, in other words, it must have *magnitude*, *line of action*, and *sense*.

Now every velocity can be regarded as produced in the particle by the uniform action of a force for a definite time. Hence forces are also characterised by magnitude, line of action, and sense.

Two forces acting on a particle are therefore compared by specifying the two lines and senses in which they would cause it to move if each acted separately, and also the magnitudes of the velocities which they would thus generate in it if they both acted for the same time on it.

Hence any force may be completely represented by a right line drawn in the direction and sense in which it would cause a material particle to move, the length of this line representing, on any scale, the rate per unit of time at which the force would generate velocity in the particle. And all other forces may be compared with this force as to magnitude, direction, and sense by drawing right lines in the several directions in which they would produce motion, and taking the lengths of these lines to represent, on the same scale as before, the rates at which they would severally generate velocity in *one and the same particle.*

Forces may also be compared with each other by means of their effects on *different particles.* For, let n perfectly equal particles be placed side by side in a row (fig. 1), and let each of them be acted upon uniformly for the same time by a force which at the end of this time generates the same velocity,

f, in each of them. Now if instead of being n separate particles they were all glued together so as to form a body of n times the mass of each particle, and if each of them is still acted on by the same force as before, this body will, at the end of the time considered, have the same velocity as each separate particle had, and will be acted upon by n times the force which generated this velocity in the particle. Comparing a single particle, then, with the body whose mass is n times the mass of this particle we see that to produce the same velocity in two bodies by forces acting on them for the same time, the magnitudes of the forces must be proportional to the masses to which they are applied.

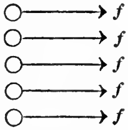

Fig. 1.

And hence, generally, if we define *momentum* as the product of mass and velocity—

The magnitude of a force is proportional to the rate per unit of time at which it generates momentum.

The greater the mass on which the force acts, the less the rate at which it increases the velocity of this mass; and the less the mass, the greater the rate of increase of velocity; the product of the two being always the same for the same force, *whatever be the masses to which it is applied.*

So that if P is a force which generates velocity at the rate $\frac{dv}{dt}$ in a body of mass m, and if P' is a force which generates velocity at the rate $\frac{dv'}{dt}$ (per unit of time) in a body of mass m', we have

$$\frac{P}{P'} = \frac{\frac{d}{dt}(mv)}{\frac{d}{dt}(m'v')}.$$

9.] **Composition of Velocities.** We propose to show how a particle may be moving with two velocities in two different directions at the same time. Let a board be placed on a horizontal table; let a rectilinear groove, OA (fig. 2), be cut in this board, and let a particle be placed at O in the groove. Suppose, for definiteness, that the unit of time is one second. Let the particle be moved along the groove with a uniform velocity represented by OA, and at

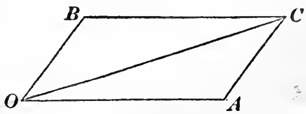

Fig. 2.

the same time let the board (i.e. every point in the board) be moved along a groove cut in the table with a uniform velocity represented in magnitude and direction by OB. Over what point in the table will the particle be found at the end of one second? Before the motions begin, complete the parallelogram $OACB$.

At the end of a second the particle must be found in the groove at the point A, and also at the end of the same time the point A of the groove must be found at the point of the table vertically under C. Hence this latter point is the position of the particle at the end of a second.

Let the foot of a perpendicular dropped from the particle on the table be called *the position of the particle referred to the table.* How do we know that the position of the particle referred to the table has described the right line OC (or rather a line in the table vertically under OC)? In this way—if we demanded the position of the particle referred to the table at the end of any fraction or multiple of a second, we should find that the distance which it has travelled along OA is to the distance which the groove has travelled in the direction OB as OA is to AC, and therefore the positions of the particle referred to the table trace out a right line vertically under OC.

Consequently the two simultaneous velocities OA and OB which were impressed on the particle have combined to give it a single velocity represented in magnitude and direction by OC.

The velocity OC is called the *resultant* of the velocities OA and OB, and these latter are called *components* of the velocity OC. Hence we arrive at the proposition which is the foundation of Dynamics:—

If a point, O, move with two coexistent velocities represented in magnitudes, directions, and senses by two right lines, OA and OB, it will have a resultant velocity represented in magnitude, direction, and sense by the diagonal, drawn through O, of the parallelogram determined by the lines OA and OB.

This proposition is called by the name of *The Parallelogram of Velocities.*

10.] **Composition of Forces.** From the Parallelogram of Velocities, the *Parallelogram of Forces* follows at once. Since two simultaneous velocities, OA and OB, of a particle result in a single velocity, OC, and since these three velocities may be supposed to be produced by the separate action of three forces all acting for the same time, it follows that the effect produced on a particle by the combined action, for the same time, of two forces may be produced by the action, for the same time, of a single force which is therefore called the *resultant* of the other two forces.

And these forces will be represented in magnitudes, lines of action, and senses by the lines OA, OB, and OC (Art. 8); hence—

If two forces be represented in magnitudes, lines of action, and senses by two right lines OA and OB, their resultant is represented in magnitude, line of action, and sense by the diagonal, OC, of the parallelogram $OACB$ determined by these lines.

This is the proposition of the Parallelogram of Forces.

Cor. The resultant of two forces acting along the same right line and in the same sense is equal to their sum; and if they act in different senses, the resultant is equal to their difference.

11.] **Equilibrium of Three Forces.** In fig. 2 produce CO through O to C' so that $CO = OC'$. Now imagine that, when the particle is started along the groove and the board along the table, the table itself is moved in a groove cut in the floor in the direction OC' with a velocity represented by OC'. In this case it is evident that the position of the particle with reference to the floor is fixed; that is, the particle is at rest with regard to fixed space (the floor being supposed fixed).

Consequently if three forces represented by the lines OA, OB, and OC' act together on the particle, no motion will ensue. In this case *each force is equal and opposite to the resultant of the other two;* for it is obvious that OA is equal and opposite to the diagonal, through O, of the parallelogram determined by OB and OC'; and that OB is equal and opposite to the diagonal of the parallelogram determined by OA and OC'.

12.] **Statical point of view.** The primary conception of *force* is that of a cause of motion in a body or in a material particle, and the magnitude of any force is estimated by the rate at which it generates momentum (Art. 8). Nevertheless in Statics it is only the *tendency* which forces have to produce motion that is considered. Forces in this branch of Dynamics are considered as acting in such ways as to counteract each other's tendency to produce motion, or as producing a state of equilibrium in the bodies to which they are applied; but the magnitude of each force is estimated with reference to the amount of momentum which it would *actually* generate if it were completely unfettered by the action of other forces.

Forces in Statics are usually expressed as multiples of the *weight* of some standard body arbitrarily chosen. Thus a force is said to be a force of 10 kilogrammes if it is *just* capable of lifting vertically a body whose weight is equal to that of the mass of water which at a temperature of 4°C. fills a volume of 10 cubic decimetres. But even here the Newtonian definition of force, as a cause of change of motion, is not discarded but merely kept in the background. For the weight which is called a kilogramme is merely a force which generates momentum at a certain rate in a body of certain mass; and the vertical force which is *just* able to raise a body from the ground is a force which could generate momentum in the body at the same rate as its weight and in the opposite sense. For practical purposes this measurement of forces as multiples of a weight is used by engineers and others; but in the very important branch of Dynamics which treats of Electricity and Magnetism an *absolute measure* of force is resorted to—i.e. a measure which is one and the same all over the earth, and indeed all through the universe. The mass of a body is something which cannot conceivably change, whether the body is taken to different parts of the earth or to different parts of the universe; and the force which, acting uniformly on this mass for a certain time (say one sidereal second), will at the end of this time have caused it to move with a certain velocity (say one centimetre per second), must be one and the same wherever the experiment is tried. The mass selected to define the unit force is a mass equal to that of the water which, at its temperature of maximum density, fills one cubic centimetre; and this *absolute unit of force* is called a *Dyne*. Compared with even such a small force as the weight of a gramme, the dyne is exceedingly small; but in many problems of Electricity and Magnetism where the forces at play are very small, the dyne as a unit force is convenient enough.

13.] **Force must act upon Matter.** Although the Newtonian definition and measure of force render it clear that whenever force acts it must act on something material, it is not impossible that beginners may lose sight of this fact and suppose that a force could, for example, act on a mathematical *point*. We may without error speak of forces as acting *at* a point, but not *on* it, if their lines of action pass through the point. Thus, in fig. 2, two forces acting along the lines OA and OB may be spoken of as two forces acting *at* the point O; but their action would be physically impossible unless it took place on some material body, such as a particle placed at O. *Wherever force is exhibited, there is evidence of the existence of matter, both acting and acted upon.*

14.] **Proper Representation of Forces.** In representing the resultant of two forces which act together at a point, O, the student should be careful to draw the two forces acting *from* the point. Thus, if of the two forces, P and Q, one, P, is represented as acting *from* O, and the other *towards* O, we must produce the line QO to Q', so that $OQ' = OQ$; completing, then, the parallelogram $OPRQ'$, its diagonal, OR, will represent in magnitude and direction the resultant of P and Q. The marking of lines representing forces with arrow-heads will serve to exhibit the *senses* of the forces in every case.

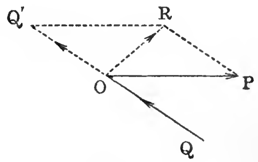

Fig. 3.

15.] **Resolution of Forces.** Having proved the principle of the Composition of Forces, the principle of the Resolution of Forces at once follows. If two forces, P and Q, are equivalent to a single force $OO' = R$ (fig. 4), it is evident that the single force R acting along OO' can be replaced by the two forces P and Q, represented in magnitude and direction by two adjacent sides of a parallelogram of which OO' is the diagonal. Since an infinite number of parallelograms, of each of which OO' is the diagonal, can be constructed, the force R can be resolved in an infinite number of ways into two other forces. These forces are called the *components* of R.

16.] **Theorem.** It being given that the direction of the resultant of every two forces is that of the diagonal of their parallelogram, its magnitude must be represented by this diagonal; and conversely,

Let it be granted that the resultant of P and Q acts in the diagonal, OO' (fig. 4), of the parallelogram determined by P and Q. Measure backwards through O a line, OR, the length of which represents the magnitude, R, of the resultant. A system of forces acting at O, represented in magnitude and direction by P, Q, and R, will evidently be in equilibrium. Each force is, therefore, equal and opposite to the resultant of the other two. If, then, we consider P as equal and opposite to the resultant of Q and R, OP', the production of OP, must be the diagonal of the parallelogram determined by Q and R. Now, since $OQP'R$ is a parallelogram, $OR = P'Q$; and since $OP'QO'$ is a parallelogram, $P'Q = OO'$; therefore $OR = OO'$.—Q. E. D.

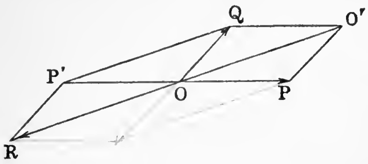

Fig. 4.

Again, for the converse proposition, let it be granted that $OR = OO'$, while OO' and OR are not necessarily in one right line; and let OP' be diagonal of the parallelogram, $OQP'R$, determined by OQ and OR; then OP is equal in magnitude to OP', since the resultant of Q and R has a magnitude equal to OP'.

Comparing the triangles OQO' and OQP' we have $OO' = QP'$, $QO' = OP'$, and OQ common to both; therefore the angle QOO' = the angle OQP', therefore QP' is parallel to OO'; but QP' is also parallel to OR, therefore OR and OO' are in one right line. Therefore, &c., Q. E. D.

17.] **Relations between Three Forces in Equilibrium.** When three forces maintain a particle in equilibrium, each force is equal in magnitude to the resultant of the other two, and acts in the sense exactly opposite to this resultant. Thus, in fig. 4, each of the lines, OP, OQ, and OR, which represent in magnitude and direction the forces P, Q, R, is equal and opposite to the diagonal of the parallelogram determined by the two remaining lines.

This enables us to express the relative magnitudes of three forces in equilibrium by means of the three angles between them. For (fig. 4) the forces P, Q, R are equal in magnitude to the lines OP, PO', $O'O$, respectively. Now, since the sides of a plane triangle are to each other as the sines of the opposite angles, we have

$$OP : PO' : O'O = \sin PO'O : \sin O'OP : \sin OPO'.$$

Denote by $\hat{PQ}$, $\hat{QR}$, $\hat{RP}$, the angles between the directions of the forces P and Q, Q and R, R and P, respectively. Then, evidently,

$$\sin PO'O = \sin QOO' = \sin QOR = \sin \hat{QR};$$

$\sin O'OP = \sin ROP = \sin \hat{RP}$; $\sin OPO' = \sin POQ = \sin \hat{PQ}$.
Hence we have the fundamental relations

$$P : Q : R = \sin \hat{QR} : \sin \hat{RP} : \sin \hat{PQ}.$$

It may, perhaps, assist the beginner to mark the angle opposite to each force by the corresponding small letter (fig. 5); and then the ratios between the forces may easily be remembered in the form

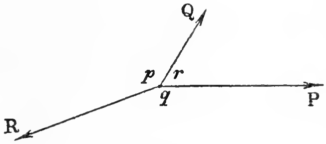

Fig. 5.

$$P : Q : R = \sin p : \sin q : \sin r. \qquad (a)$$

Since the sides of the triangle OPO' (fig. 4) are connected by the equation

$$OO'^2 = OP^2 - 2OP \, . \, PO' \cos OPO' + PO'^2,$$

we have evidently

$$R^2 = P^2 + 2PQ \cos \hat{PQ} + Q^2,$$

an equation which gives the *magnitude* of the resultant of two forces in terms of the magnitudes of the two forces and the angle between their directions, the forces being represented by two lines, both drawn *from* the point at which they act, as in Art. 14. If $\hat{PQ} = 0$, the above equation gives $R = P + Q$, or the resultant of two coincident forces is equal to the sum of the forces. If $\hat{PQ} = \pi$, $R = P - Q$; or, the resultant of two forces which act at a point in exactly opposite senses is equal to the difference of the forces.

18.] **Theorem.** *If any one set of forces* (P, Q, R) *acting in three given directions is in equilibrium, all other sets acting in equilibrium in the same directions are merely multiples of the set* (P, Q, R).

For, let the given directions make angles p, q, r with each other in pairs, and let the sets (P, Q, R) and (P', Q', R') acting in these directions be separate systems in equilibrium. Then we have

$$P : Q : R = \sin p : \sin q : \sin r$$

and

$$P' : Q' : R' = \sin p : \sin q : \sin r.$$

Therefore, $P : Q : R = P' : Q' : R'$, or $\dfrac{P'}{P} = \dfrac{Q'}{Q} = \dfrac{R'}{R}$. Hence the forces P', Q', R' are separately proportional to P, Q, R, and therefore the former set is not essentially distinct from the latter. This theorem is equivalent to the statement—*when we have determined any one set of forces in equilibrium in three given directions, we have determined all such sets.*

Thus, if we know (see Example 1, p. 16) that three forces acting along the bisectors of the sides of a triangle drawn from the opposite angles, and proportional to the lengths of these bisectors, are in equilibrium, we know that this is the *only* set in equilibrium in these directions.

19.] **Principle of the Transmissibility of Force.** When a force acts on a particle, the force will produce the same effect if it be supposed applied at *any* point along a string connected with the particle, the string lying in the line of action of the force. Thus, if a force of P grammes (fig. 6) act on a particle, O, in the direction OA, P may be supposed to act at A or B at the end of a string attached to O. Imagine the particle O to be connected with an indefinitely thin rigid membrane, abc; then any force P acting on O may be supposed to be directly applied at *any* point of the membrane in the line of action of P.

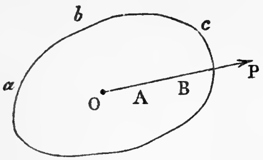

Fig. 6.

This axiom is known as the *principle of the transmissibility of force;* it is one of the fundamental principles of Rational Statics, and in most treatises on the subject, it constitutes the basis of the investigation of the conditions of equilibrium. It is essentially necessary to observe that it holds good only for a *rigid* body—that is, a body whose parts, under all circumstances, must maintain constant distances from each other. Thus, if we suppose such a body *about to be* acted on by any set of forces given in magnitudes and directions, we can say, *before the forces are actually applied* at certain points in the body, that the effect will be the same if these forces are applied at *any other points* in their respective lines of action. On the contrary, if the body is deformable, we can make no such assertion. Take, for example, a set of parallel rulers, $ABCD$ (fig. 7), of which the ruler CD is fixed, and suppose a force F to act on the ruler AB at the point a. If, *previous to the action of the force*, it were allowable to transfer its point of application to b, on the fixed ruler CD, it is clear that the system would remain at rest. But we know that the force F, applied at a, will cause the ruler AB to move until the braces AD and CB are parallel to the direction of F. However, after *the deformable body has taken up a position of equilibrium under the action of the forces*, each force may be transferred to any point in its line of action, just as in the case of an indeformable body.

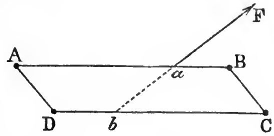

Fig. 7.

Several other very obvious instances of the inapplicability of this principle will doubtless present themselves to the student.

It is essential to observe at the outset that in nature there are no such things as rigid bodies. For a great many practical matters there are bodies which may be treated as if they were rigid or indeformable; but the fact that the particles of solid bodies like iron can be thrown into vibration by the application of even small impulses—as is evidenced by the production of sound from bells and gongs—proves that these bodies are not absolutely rigid.

Bodies which most nearly approximate to the notion of rigidity are called *Natural Solids*.

Examples.

1. Find the magnitude of the resultant of two forces of 10 kilogrammes and 8 kilogrammes which act at an angle of 105°.

Ans. $R = 2\sqrt{41-10(\sqrt{6}-\sqrt{2})} = 11{\cdot}06$ kilogrammes.

2. Two forces, P and Q, of which P is given, act at an angle of 60°; given the magnitude of their resultant, R, find the magnitude of Q.

$$\textit{Ans.}\quad Q = \frac{\sqrt{4R^2-3P^2}-P}{2}.$$

From this it appears that R cannot be less than $\frac{\sqrt{3}}{2}\cdot P$; explain this result by a figure.

3. Two forces, P and Q, inclined at an angle of 120°, have a resultant, R; when they are inclined at an angle of 60°, the resultant becomes n times as great as before; show that

$$P = \frac{R}{2\sqrt{2}}(\sqrt{3n^2-1}+\sqrt{3-n^2})$$

$$\text{and } Q = \frac{R}{2\sqrt{2}}(\sqrt{3n^2-1}-\sqrt{3-n^2}).$$

4. If two forces, acting at a given angle, be each multiplied by the same number, show that their resultant is also multiplied by this number and unchanged in direction.

5. Two forces act at an angle ω; each force becomes n times as great as before, and the angle between the forces is reduced to $\frac{\omega}{2}$; each of these latter forces again becomes n times as great as before, and the angle between them reduced to $\frac{\omega}{4}$. It is observed, that in all these cases the magnitude of the resultant is unaltered. Show that

$$\omega = 4\cos^{-1}\left(\frac{\sqrt{9+4n^2}-1}{4}\right).$$

6. Two chords, OA and OB, of a circle represent in magnitude and direction two forces acting at the point O; show that if their resultant passes through the centre of the circle, either the chords are equal or they contain a right angle.

7. Find the components of a force, P, along two directions making angles of 30° and 45° with P on opposite sides.

Ans. $\frac{2P}{1+\sqrt{3}}$, and $\frac{P\sqrt{2}}{1+\sqrt{3}}$.

8. Show that a force represented in magnitude and direction by the diameter of a circle may be resolved into two rectangular components represented by any two rectangular chords of the circle drawn from the extremity of the diameter.

9. Two rectangular forces, P and $P\sqrt{3}$, act on a particle lying on the ground. If P makes an angle of 30° with the horizon, show that the particle will have no horizontal motion.

10. Three forces equal to P, $P+Q$, and $P-Q$, act on a particle in directions mutually including an angle $\frac{2\pi}{3}$; find the magnitude and direction of their resultant.

20.] **Theorem.** The following theorem is of wide application in the composition of forces—

If two forces acting at a point, O, are represented in magnitudes and directions by OB and $n \cdot OA$, their resultant is represented in magnitude and direction by $(n+1)\ OG$, the point G being taken on AB so that $BG = n \cdot AG$.

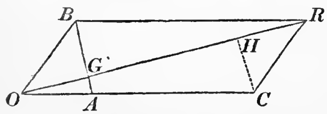

Fig. 8.

For, produce OA to C so that $OC = n \cdot OA$. Then the two forces acting at O are represented by OC and OB. Complete the parallelogram $OCRB$. Then the diagonal OR is the resultant force.

From C draw CH parallel to AB. Then the triangles CHR and BGO are equal in all respects, therefore $HR = OG$. Now since $OC = n.OA$, it follows that $OH = n.OG$, therefore $OR = (n+1)\ OG$, which proves the proposition for the magnitude of the resultant.

Again, $\dfrac{CH}{AG} = \dfrac{CO}{OA} = n$, therefore $CH = n.AG$, and since $CH = BG$, we have $BG = n.AG$.

As a particular case, the resultant of two forces represented by OA and OB passes through the middle point of AB, and is equal to twice the line joining O to this point.

If the two forces are equal to $n.OA$ and $m.OB$, the resultant passes through the point G determined so that $\dfrac{BG}{AG} = \dfrac{n}{m}$, and is represented on the same scale by $(m+n).OG$.

For, diminishing the scale to which the forces are drawn in the ratio of $m:1$, the two forces will be represented by OB and $\dfrac{n}{m}.OA$. It then follows, by what precedes, that the resultant acts through a point G, such that $BG = \dfrac{n}{m}.AG$, and is equal in magnitude to $\left(\dfrac{n}{m}+1\right).OG$. If, now, we revert to the original scale, this must be multiplied by m, and we have for the resultant $(n+m).OG$.—Q. E. D.

21.] **Graphic Representation of the Resultant.** If several forces, P_1, P_2, ... act together at a point, their resultant is found thus:—Take the resultant of P_1 and P_2; compounding this resultant with P_3, we get a new force which is the resultant of P_1, P_2, and P_3; compounding this force with P_4, we get the resultant of P_1, P_2, P_3, and P_4; and carrying on this process until all the forces have been used, we obtain in magnitude and direction the resultant of the whole system.

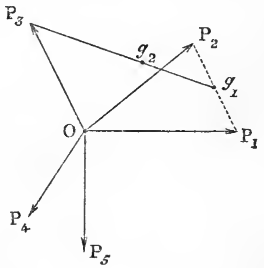

Fig. 9.

Let g_1 be the middle point of the line P_1P_2, which joins the extremities of the first two forces. Then the resultant of P_1 and P_2 is represented in magnitude and direction by $2.Og_1$. Compounding the force $2.Og_1$ with P_3, we get a resultant represented in magnitude and direction by $3 . Og_2$ (Art. 20), where g_2 is a point on $g_1 P_3$ such that $P_3 g_2 = 2 . g_1 g_2$. Again, the resultant of $3 . Og_2$ and P_4 is $4 . Og_3$, where g_3 is the point on $P_4 g_2$ such that $P_4 g_3 = 3 . g_2 g_3$. If there are n forces acting on O, and if G is the last point determined as above, the resultant is represented in magnitude and direction by $n . OG$.

Def. The point G, thus determined, is called the *Centroid* of the points $P_1, P_2, \ldots P_n$.

Cor. 1. If the point O, at which the given forces act, is the centroid of the extremities of the forces $P_1, P_2, \ldots P_n$, the resultant force vanishes, and the point O is in equilibrium.

Cor. 2. The more advanced student will perceive that if at the points $P_1, P_2, \ldots P_n$ there be placed equal particles, each of mass m, and if each of these particles attracts or repels the particle O with a force proportional to m and to the distances $OP_1, OP_2, \ldots OP_n$, respectively, the resultant attraction or repulsion on O will be $nm . OG$, or $M . OG$, where $M =$ the sum of the masses and G is their centre of mass.

Cor. 3. If the attracting or repelling particles form a continuous body, of mass M, and the law of attraction or repulsion is that of the direct distance, the resultant attraction or repulsion will be $M . OG$, acting in the line OG, where G is the centre of mass of the body.

This important result is, therefore, seen to be a simple consequence of the theorem in this Article concerning the resultant of a number of forces acting on a particle—a theorem which was first given by Leibnitz.

Examples.

1. Find a point inside a triangle such that, if it be acted on by forces represented by the lines joining it to the vertices, it will be in equilibrium.

Ans. The intersection of the bisectors of the sides drawn from the opposite angles.

2. $P_1, P_2, \ldots P_n$ are points which divide the circumference of a circle into n equal parts. If a particle, Q, lying on the circumference, be acted upon by forces represented by $QP_1, QP_2, \ldots QP_n$, show that the magnitude of the resultant is constant wherever Q is taken on the circumference. *Ans.* It is $n . QO$, O being the centre of the circle.

3. A particle placed at O is acted on by forces represented in magnitudes and directions by the lines, $OA_1, OA_2, \ldots OA_n$, which join O to any fixed points, $A_1, A_2, \ldots A_n$; where must O be placed so that the magnitude of the resultant force may be constant?

Ans. If the resultant is represented by a line of length R, O may be placed anywhere on a sphere of radius $\frac{R}{n}$ described round the centroid of the fixed points as centre.

4. Two forces are represented by two semi-conjugate diameters of an ellipse; prove that their resultant is a maximum when the diameters are equal and so taken as to include an acute angle; and that their resultant is a minimum when they are equal and include an obtuse angle.

5. $ABCD$ is a quadrilateral of which A and C are opposite vertices. Two forces acting at A are represented in magnitudes and directions by the sides AB and AD; and two forces acting at C are represented in magnitudes and directions by the sides CB and CD. Prove that the resultant force is represented in magnitude and direction by four times the line joining the middle points of the diagonals of the quadrilateral.

6. O is any point in the plane of a triangle, ABC, and D, E, F are the middle points of the sides. Show that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF. (Wolstenholme, *Book of Mathematical Problems.*)

7. If O be the centre of the circumscribed circle of a triangle, ABC, and L the intersection of perpendiculars from the angles on the sides, prove that the resultant of forces represented by LA, LB, and LC will be represented in magnitude and direction by $2\ LO$. (Wolstenholme, *ibid.*)

If G is the centroid of the triangle, the resultant is $3 . LG$ (Art. 21); but this, by a well-known theorem in Geometry, is $2 . LO$.

22.] **Graphic Representation of the Resultant.** There is another mode of exhibiting the resultant of a number of forces acting on a particle.

When two forces, OA and OB (fig. 2, p. 6) act at O, their resultant is the diagonal of the parallelogram $OACB$; or, again, it may be considered as the third side of the triangle determined by OA and AC, the latter line being drawn from the extremity of the force OA parallel to the other force, OB.

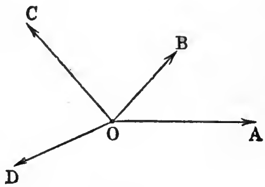

Fig. 10.

Let any number of forces, OA, OB, OC, OD (fig. 10), act at O. Then drawing oa (fig. 11) parallel and equal (or proportional) to OA, and from the extremity a drawing ab parallel and equal (or proportional, on the same scale) to OB, the resultant of the forces OA and OB is represented by ob, the third side of the triangle oab. (Of course the resultant acts at O, and is parallel to ob). Again, drawing bc parallel and equal (or proportional) to OC, the resultant of ob and bc is oc. Compounding this with cd, which represents OD in the above manner, we get the resultant of the whole system represented in magnitude and direction by od, the last side of the polygon $oabcd$.

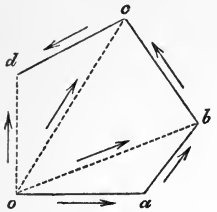

Fig. 11.

Hence to represent the resultant of any number of forces acting at a point, O—

Take any point, o, and draw the sides of a polygon successively parallel and equal (or proportional) to the forces acting at O; then the last side, or that which is required to close up the polygon, represents in magnitude and direction the resultant of the system.

COR. 1. If the last vertex, d, of the polygon of forces closed up into o, the side od would vanish, or the resultant force would vanish; that is, the system of forces would be in equilibrium. Hence—

If the sides of a closed polygon marked with arrows, which all go round the polygon IN THE SAME SENSE, *represent in magnitude and direction the forces which act together on a particle, these forces form a system in equilibrium.*

COR. 2. When only three forces act, the preceding Cor. shows that they will be in equilibrium if they are parallel and proportional to the sides of a triangle which are marked with arrows all going round the triangle in the same sense.

This proposition is known as the *Triangle of Forces*.

oplace
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23.] LAPLACE'S PROOF OF THE PARALLELOGRAM OF FORCES. Among purely statical proofs of this fundamental proposition, i.e. proofs which do not depend on the consideration of velocity, Laplace's appears to be the most elegant, and as, moreover, it does not involve the principle of transmissibility, it is thought desirable to include it in the present treatise.

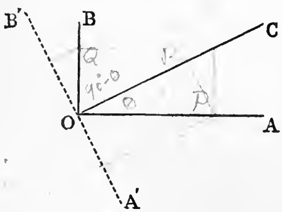


Fig. 12.

Let two rectangular forces, P and Q, represented by the lines OA and OB (fig. 12) act at O, and let R be the unknown magnitude, and OC the unknown direction, of their resultant. It is evident that if P and Q give a resultant equal to R acting in OC, nP and nQ will give a resultant equal to nR acting also in OC, because taking multiples of the forces is the same thing as merely altering the scale of magnitude to which they are referred. Conversely, whatever n may be, nR may be replaced by nP, making an angle θ ($=COA$) and nQ, making an angle $\frac{\pi}{2}-\theta$ ($=COB$) with the direction of R. Let n be taken $=\frac{P}{R}$ and draw $A'OB'$ perpendicular to OC. Then, since

R may be replaced by P in OA and Q in OB,

P " " " $\frac{P^2}{R}$ in OC " $\frac{PQ}{R}$ in OA';

Q may be replaced by $\frac{PQ}{R}$ in OB' and $\frac{Q^2}{R}$ in OC.

Hence the forces P and Q are equivalent to a force $=\frac{P^2}{R}+\frac{Q^2}{R}$ in OC, a force $\frac{PQ}{R}$ in OA', and a force $\frac{PQ}{R}$ in OB'. But these last are equal and opposite, and therefore they destroy each other. Hence P and Q are equivalent to a single force $=\frac{P^2+Q^2}{R}$ acting in the direction of their resultant; therefore

$$R=\frac{P^2+Q^2}{R},$$

or

$$R=\sqrt{P^2+Q^2}. \qquad (1)$$

Thus we have found the *magnitude* of the resultant of any two rectangular forces. We now proceed to find its direction.

If P and Q are equal, their resultant bisects the angle between them, and (1) therefore shows that it is represented in magnitude and direction by the diagonal of their parallelogram.

Let three forces, at right angles to each other, OA, OB, and OC (fig. 13) each equal to P, act on a particle O; complete the cube as in the figure. By what precedes, the resultant of OB and OC is OF; combining this with OA, we see that the direction of the resultant lies in the plane FOA. Similarly, it can be proved to lie in the plane COD; hence its direction is OO', the intersection of these planes, or the diagonal of the cube. Now from (1) $OF=P\sqrt{2}$, and the resultant of the three forces is the same as the resultant of $P\sqrt{2}$ along OF and P along OA.

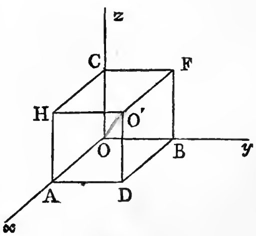


Fig. 13.

By (1) the magnitude of the resultant is $P\sqrt{3}$, and since

$OO' = P\sqrt{3}$, we have proved that the diagonal, OO', of the parallelogram FOA represents in magnitude and direction the resultant of two forces P and $P\sqrt{2}$.

Suppose now that $OA = P$, $OB = P\sqrt{2}$, and $OC = P$, and complete the parallelopiped. We have just proved that the resultant of OB $(=P\sqrt{2})$ and OC $(=P)$ is the diagonal OF $(=P\sqrt{3})$; and since the resultant of the three forces must lie in the planes COD and FOA, it must act in the diagonal OO'. But this resultant is the resultant of $P\sqrt{3}$ along OF and P along OA, and by (1) its magnitude is $P\sqrt{4}$, which is the magnitude of OO', the diagonal of the parallelogram FOA.

By keeping OA and OC each equal to P, and giving OB the values P, $P\sqrt{2}$, $P\sqrt{3}, \ldots P\sqrt{m}$, successively, we prove in this way that the parallelogram law holds for P and $P\sqrt{m}$; hence, multiplying the forces by $\sqrt{n}$, the law holds for $P\sqrt{n}$ and $P\sqrt{mn}$; or, replacing mn by k, the law holds for $P\sqrt{n}$ and $P\sqrt{k}$, where n and k are any two integers. But the numbers n and k can be varied in such a way that $\sqrt{\frac{k}{n}}$ shall be equal to any given quantity. Hence the parallelogram law holds for two rectangular forces which bear to each other any given ratio.

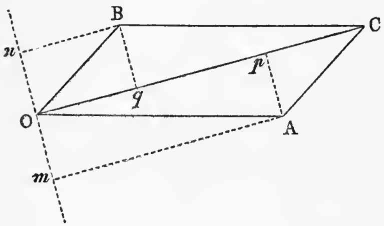


Fig. 14.

From this the proposition follows easily for oblique forces.

Let OA and OB (fig. 14) represent two oblique forces, P and Q; complete the parallelogram, draw the line mn through O perpendicular to the diagonal OC, and let fall the perpendiculars Ap, Am, Bq, and Bn, on OC and mn. By what we have proved, the force OB $(= Q)$ can be replaced by Oq and On, and OA $(= P)$ can be replaced by Op and Om. But Om is evidently equal and opposite to On, therefore OC is the line of action of the resultant, and its magnitude $= Op + Oq$, which $= OC$. This proof will be found at greater length in the first chapter of Moigno's *Leçons de Mécanique Analytique*.

CHAPTER II.

GENERAL CONDITIONS OF THE EQUILIBRIUM OF A PARTICLE UNDER THE ACTION OF FORCES IN ONE PLANE.

24.] **Absolute Condition of Equilibrium.** One condition is necessary and sufficient for the equilibrium of a particle—and that condition is, that *the magnitude of the resultant force acting upon it shall be zero.* In the case of a body (as distinguished from a mere particle) the student will afterwards see that this single condition is not sufficient. The vanishing of the Resultant may be called the *absolute* condition of the equilibrium of a particle.

25.] **Several Forces.** When several forces act upon a particle, the condition of its equilibrium may be expressed as in Cor. 1, p. 16; or as in Cor. 1, p. 18. But, in practice, these representations would frequently be found clumsy, and we obtain simpler results by using the principle of the Resolution of Forces than those given by the principle of Composition. It is to be observed that forces acting on a particle are to be considered as *forces whose lines of action all pass through one common point.*

26.] **Resolution of Forces in given Directions.** It has been proved that a force can be resolved into two others along any two directions in the same plane. Simplicity is gained by taking these two directions at right angles to each other. Thus, let Ox and Oy be any two lines at right angles to each other, and P any force acting at O in the plane Oxy. Then, completing the parallelogram $OXPY$, we find the components, OX and OY, of the force P along the axes Ox and Oy. Let OX and OY be denoted simply by X and Y. It is, then, evident that

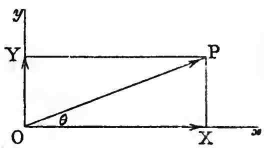


Fig. 15.

$$X = P \cos \theta,$$
$$Y = P \sin \theta,$$

where θ is the angle which the direction of P makes with Ox.

In strictness, when we speak of the component of a given force along a certain line, it is necessary to mention the other line along which the other component acts. For example, the force P may have an infinite number of components along the same right line Ox. If the line associated with Ox be Om, and if the parallelogram $OMPM'$ be completed, the component of P along Ox will be OM, the other component being OM'. If, again, the resolution of P be effected along Ox and On, and the parallelogram $ONPN'$ be drawn, the component of P along Ox will be ON; and it is evident that if ω be the angle between the axes along which P is resolved, the component along Ox will be $P \cdot \dfrac{\sin(\omega - \theta)}{\sin \omega}$.

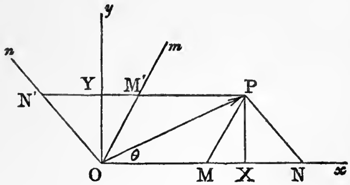


Fig. 16.

ON : OP :: Sin OPN : sin ONP
i.e. $ON = \dfrac{P \sin(\omega - \theta)}{\sin \omega}$

In what follows, unless the contrary is expressed, by the component of a force along any line we shall understand the *rectangular component*; that is, the resolution is supposed to be made along this line and the line perpendicular to it. It must be remembered, then, that—

The component of a force, P, along a right line is $P.\cos$ (angle between right line and direction of P).

27.] **Equations of Equilibrium, or Analytical Conditions.** If several forces, P_1, P_2, P_3, ..., act at O, each of them may be replaced by its two components, one along Ox, and the other along Oy, which is perpendicular to Ox (fig. 17). Thus, the components of P_1 are $P_1 \cos\theta_1$, and $P_1 \sin\theta_1$; those of P_2 are $P_2 \cos\theta_2$, and $P_2 \sin\theta_2$, and these latter are measured in exactly the same senses as the components of P_1; that is to say, $P_2 \cos\theta_2$ is the component of P_2 along Ox in the sense Ox. The component of P_2 in the figure is actually in the sense *opposite* to Ox, that is, in the sense $O, -x$; still,

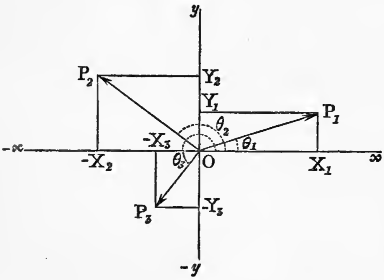


Fig. 17.

the component in the sense Ox is $P_2 \cos \theta_2$, for $\cos \theta_2$ is negative. If the senses Ox and Oy are regarded as the positive senses, any components which act in the opposite senses, $O, -x$ and $O, -y$, would subtract from the positive components, and must be considered negative. It will be seen that the negative sign of every component will be perfectly represented and accounted for by the general expressions, $P \cos \theta$ and $P \sin \theta$, for the two components. Thus, the figure shows that both components of P_3 are negative, and accordingly both of the expressions $P_3 \cos \theta_3$ and $P_3 \sin \theta_3$ are negative, since θ_3 is $> 180°$.

In order that the expressions $P \cos \theta$ and $P \sin \theta$ may always represent components in the positive senses Ox and Oy, the angle θ must be measured from Ox towards the line of action of the force in a fixed sense—that opposite to watch-hand rotation being generally chosen.

With this understanding, then, we may say that the components of P_1, P_2, P_3 in the direction Ox are $P_1 \cos \theta_1$, $P_2 \cos \theta_2$, and $P_3 \cos \theta_3$, and those in the direction Oy are $P_1 \sin \theta_1$, $P_2 \sin \theta_2$, and $P_3 \sin \theta_3$.

Replacing each of the forces, $P_1, P_2, P_3, \ldots$, by its components, we have

$$P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + \ldots, \text{ or } \Sigma P \cos \theta \text{ along } Ox,$$

and

$$P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 + \ldots, \text{ or } \Sigma P \sin \theta \text{ along } Oy.$$

If the component, $P \cos \theta$, of a force, P, along Ox, be denoted by X, and that along Oy by Y, the whole system of forces is equivalent to the two single forces,

$$X_1 + X_2 + X_3 + \ldots, \text{ or } \Sigma X \text{ along } Ox,$$

and

$$Y_1 + Y_2 + Y_3 + \ldots, \text{ or } \Sigma Y \text{ along } Oy.$$

Now, since (Art. 23, p. 20) the resultant of two forces, P and Q, at right angles is $\sqrt{P^2 + Q^2}$, the resultant, R, of the system of forces $P_1, P_2, \ldots$, is given by the equation

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}. \tag{1}$$

For the equilibrium of O it is necessary and sufficient that $R = 0$. Hence

$$(\Sigma X)^2 + (\Sigma Y)^2 = 0. \tag{2}$$

Now this equation cannot be satisfied, so long as ΣX and ΣY are real quantities, unless

$$\Sigma X = 0 \text{ and } \Sigma Y = 0. \tag{3}$$

These, then, are the two necessary and sufficient conditions for the equilibrium of the particle, and they are equivalent to the *single* condition $R = 0$. (See Art. 24).

The equations (3) are equivalent to the following statement:—

For the equilibrium of a particle acted on by any number of forces in one plane, it is necessary and sufficient that *the algebraic sum of the rectangular components of the forces, along each of two right lines at right angles to each other in the plane of the forces, should vanish.* Since the directions Ox and Oy, along which the forces are resolved, may be any whatever in their plane, we may evidently vary the above statement thus—*the algebraic sum of the rectangular components of the forces along every right line in their plane is zero.*

It is merely for uniformity of notation that we have measured θ_1, θ_2, θ_3, ... (fig. 17) all in the same sense — that opposite to watch-hand rotation. In resolving forces along a line, Ox, it is simpler in practice to use the acute angles made by the forces with the line, and to indicate negative components by the sign *minus*.

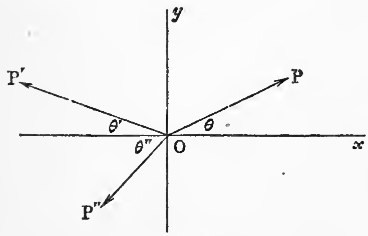


Fig. 18.

Thus, if (fig. 18) the forces P, P', P'' make acute angles θ, θ', θ'', with Ox, the sum of the components of the forces along Ox is

$$P\cos\theta - P'\cos\theta' - P''\cos\theta'',$$

and that along Oy is

$$P\sin\theta + P'\sin\theta' - P''\sin\theta''.$$

The rectangular component of a force along a line is sometimes called the *effective* component along this line.

Cor. A force has no effective component in a direction at right angles to itself.

28.] **Direction of the Resultant.** The direction of the resultant of any number of forces acting in one plane on a particle, O, is known when its components, ΣX and ΣY, along any two directions, Ox and Oy, are known. For, if Ox and Oy are rectangular, and α be the angle which the resultant, R, makes with Ox, we have, evidently (fig. 19),

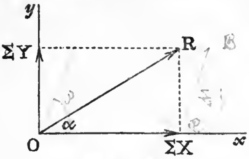


Fig. 19.

$$\tan \alpha = \frac{\Sigma Y}{\Sigma X}; \tag{4}$$

and if Ox and Oy include an angle ω,

$$\frac{\sin \alpha}{\sin (\omega - \alpha)} = \frac{\Sigma Y}{\Sigma X}.$$

29.] **Tension of a String.** When a string is employed to connect two or more particles which are acted on by given forces, the fibres of the string become subject to a certain pull, stress, or *tension*, which, if increased beyond a certain limit, will cause the string to break. This tension is a force *which at any point of the string may be conceived as acting in either of two opposite senses, or in both of these senses at once*, according to the nature of the question under discussion. Let us consider, as a simple example, the case of a string, AB (fig. 20), whose weight we may neglect, fixed at the extremity A, and attached at B to a weight W. If, now, we imagine the string to be cut at any point p, and the lower portion, pB, to be removed, it is clear that the remaining portion, pA, will not be in the same state of stress as before unless we apply at the section p a force equal to W, and acting downwards. Again, let the string be cut a little above p, at q, and suppose the portion qA removed. Then the small portion, pq, will not remain in its place unless an upward force equal to W is applied at the section q. The small portion of the string included between p and q is then kept at rest by two equal and opposite forces, each equal to W. Thus, then, if we consider any portion, pq, as isolated from the rest of the string, we must represent it as subject to two equal tensions directly opposed to each other. If we considered the action of the upper portion, pA, on the lower, pB, we should represent pB as acted on by an upward force applied at p; and if we consider the

Fig. 20.

action of the lower on the upper, we must represent pA as acted on by a downward force applied at the section of separation of pA and pB. Thus, the action at B of the string on the body W is an upward force, or tension, equal to W; while the action of W on the string consists of an equal force in the opposite direction.

30.] **String passing over Smooth Pegs or Surfaces.** When a string whose weight we neglect passes over a smooth peg, or over any number of smooth surfaces, we shall assume for the present that the stress of its fibres, or its tension, is the same at all of its points. Should it, however, be *knotted* at any of its points to the other strings, we must regard its continuity as broken, and the tension will not be the same in the two portions which start from a knot. Thus, if the string pass over two smooth surfaces, A and B (fig. 21), and if it is pulled at one extremity by a force P, it must be pulled at the other extremity with an equal force; but if, after leaving the surface A, it is knotted at C to another string which is pulled with a force equal to R, the tensions in the portions between C and A and between C and B are no longer the same, and their relative magnitudes must be determined by equation (a) of Chap. I, Art. 17.

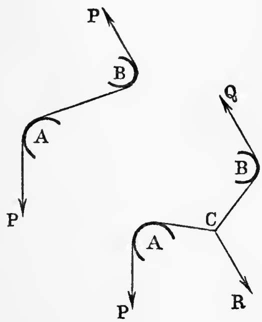


Fig. 21.

31.] **Equilibrium of a System of Particles.** When several particles are connected together and form a system, each particle being acted upon by special forces in addition to the forces produced upon it by its connexion (by strings or rods) with the other particles, we can consider the equilibrium of any *one* particle apart from all the others, *provided that we take account of all the forces which are produced on it by its connexion with the others, in addition to the special forces acting upon it.*

Thus, in No. 8 of the following examples, we may write down equations for the equilibrium of the particle N as if it were entirely disconnected with the other points, A, P, M, B, if we represent it as acted on by the force, W, and by the tensions, T_2 and T_3, of the strings by which it is connected with the system.

EXAMPLES.

1. At the point, O, of intersection of diagonals of a square (fig. 22), let two forces of 8 grammes, and 12 grammes, act along the diagonals, and two forces of 10 grammes, and 2 grammes, act perpendicularly to two sides; required the magnitude and direction of their resultant.

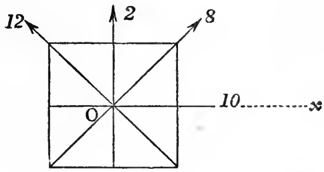


Fig. 22.

Resolving the forces along Ox, the line of action of one of them, the component of the force 10 is 10, that of the force 8 is 8 cos 45°, that of 2 is zero, and that of 12 is $-12 \cos 45°$. Hence

$$\Sigma X = 10 + \frac{8}{\sqrt{2}} - \frac{12}{\sqrt{2}} = 10 - 2\sqrt{2}.$$

$$\text{Similarly,} \quad \Sigma Y = \frac{8}{\sqrt{2}} + 2 + \frac{12}{\sqrt{2}} = 2 + 10\sqrt{2}.$$

Therefore $R = \sqrt{(10 - 2\sqrt{2})^2 + (2 + 10\sqrt{2})^2} = \sqrt{312}.$

Again, if a be the angle made by R with Ox,

$$\tan a = \frac{2 + 10\sqrt{2}}{10 - 2\sqrt{2}} = \frac{1 + 5\sqrt{2}}{5 - \sqrt{2}} = 2\tfrac{1}{4} \text{ (nearly)}.$$

2. Three forces, P, Q, R, act on a particle: find the magnitude of their resultant.

Let the angles opposite P, Q, and R be denoted by p, q, r (fig. 5, p. 11). Then resolving all the forces along the direction of P, we get for their combined component in this direction $P + Q \cos r + R \cos q$. Resolving them perpendicularly to P, the component $= Q \sin r - R \sin q$. Hence the square of the resultant $= (P + Q \cos r + R \cos Q)^2 + (Q \sin r - R \sin q)^2$. Remembering that $p + q + r = 2\pi$, this is easily seen to be

$$P^2 + Q^2 + R^2 + 2PQ.\cos r + 2QR.\cos p + 2RP.\cos q.$$

3. Verify in the last question that if the three forces are in equilibrium, the expression given for the resultant vanishes.

When the forces are in equilibrium,

$$P : Q : R = \sin p : \sin q : \sin r.$$

Hence the expression for the square of the resultant is proportional to

$$\sin^2 p + \sin^2 q + \sin^2 r + 2 \sin p \sin q \cos r + 2 \sin q \sin r \cos p + 2 \sin r \sin p \cos q.$$

The last two terms $=$

$$2 \sin r \sin (p+q) = -2 \sin^2 r, \qquad \because \quad p+q = 2\pi - r.$$

Therefore the above expression is

$$\sin^2 p + \sin^2 q - \sin^2 (p+q) + 2 \cos (p+q) \sin p \sin q = \sin^2 p + \sin^2 q - 1 + \cos (p+q) \cos (p-q), \quad \because \; 2 \sin p \sin q = \cos (p-q) - \cos (p+q).$$

Now, $\cos(p+q)\cos(p-q) = 1-\sin^2 p-\sin^2 q$,

therefore the square of the resultant $= 0$.

4. A heavy particle, O (fig. 23), whose weight is W, is held in equilibrium by three forces (in addition to its weight)—$\frac{W}{n}$ acting horizontally, F acting in a direction making an angle i with the horizon, and R at right angles to F; find the magnitudes of F and R in terms of the given force W.

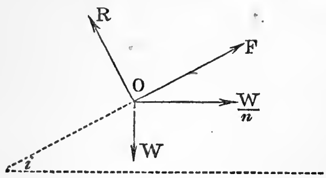


Fig. 23.

Resolve all the forces along the directions of F and R successively. These directions are chosen rather than any others, because, since R is at right angles to F, it will give no component along F, and, for the same reason, F will give no component along R.

The component along OF is $F+\frac{W}{n}\cos i-W\sin i$.

For equilibrium it is necessary (Art. 27, equations (3)) that this component shall be zero. Hence

$$F+\frac{W}{n}\cos i-W\sin i=0,$$

$$\therefore\quad F=W\left(\sin i-\frac{1}{n}\cos i\right).$$

Again, the sum of the components along OR is

$$R-W\cos i-\frac{W}{n}\sin i;$$

and this must also be zero. Hence

$$R=W\left(\cos i+\frac{1}{n}\sin i\right).$$

The same values would, of course, be found if we had selected any two other directions for the resolution. Thus, if we resolve all the forces vertically, or in the direction OW, we get

$$W-F\sin i-R\cos i=0;$$

and resolving horizontally, or in the direction of $\frac{W}{n}$, we get

$$\frac{W}{n}+F\cos i-R\sin i=0.$$

Solving these last two equations for R and F, we get the same values as before.

The advantage of a judicious selection of directions for the resolution of the forces is now apparent. By resolving at right angles to one of the unknown forces, we obtained an equation free from that force; whereas when the directions were selected at random, both of the unknown forces entered into each of our equations, and to find these forces it was then necessary to solve the equations.

Having selected one direction for resolution, it is not necessary that the second should be selected at right angles to it; for the student has seen (p. 24) that when a particle is in equilibrium, the sum of the components of the forces along *any direction whatever* must be zero. Hence we might, in the present case, have resolved vertically and along the direction OF, and the equations thus obtained would have given the same results as before.

5. One end of a string is attached to a fixed point, A (fig. 24); the string, after passing over a smooth peg, B, sustains a given weight, P, at its other extremity, and to a given point, C, in the string is knotted a particle of given weight, W. Find the position of equilibrium of the system.

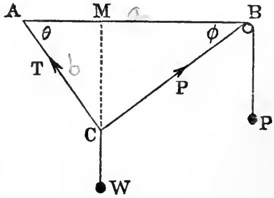


Fig. 24.

Before setting about the solution of statical problems of this kind, the student will clear the ground before him, and greatly simplify his labour by asking himself the following questions:—

(*a*) What lines are there in the figure whose lengths are already given?

(*b*) What forces are there whose magnitudes are already given, and what are the forces whose magnitudes are as yet unknown?

(*c*) What variable or variables in the figure would, if it or they were known, determine the required position of equilibrium?

Now, in the present case (*a*), the linear magnitudes which are given are the lines AB and AC. The entire length of the string is of no consequence, since it is clear that, once equilibrium is established, P might be suspended from a point at any distance whatever from B. The forces (*b*) acting at the point C are the weight, W, a tension in the string CA, and another tension in the string CB. Of these, W is given, and so is the tension in CB, which must, since the peg is smooth, be equal to P (see Art. 30); but there is, as yet, nothing determined about the magnitude of T, the tension in CA. And (*c*) the angle, θ, of inclination of the string CA to the horizon would, if known, at once determine the position of equilibrium. For, if θ is known, we draw AC of the given length: then, joining C to B, the position of the system is completely known. The angle, ϕ, of inclination of BC to the horizon, would do equally well; and it is evident that, since either angle suffices, each must be capable of being expressed in terms of the other, and the given magnitudes in the question.

Let $AB = a$, $AC = b$. Then, for the equilibrium of the point C we have, by equation (a), p. 11,

$$\frac{P}{W} = \frac{\cos\theta}{\sin(\theta+\phi)}.$$

To this equation must be joined the relation between θ and ϕ given by the geometry of the figure. We have, evidently,

$$AC \,.\, \sin ACB = AB \,.\, \sin \phi,$$

or

$$b \sin (\theta + \phi) = a \sin \phi. \quad (1)$$

Equation (1) gives

$$\frac{a \sin \phi}{b \cos \theta} = \frac{W}{P},$$

or

$$\sin \phi = \frac{bW}{aP} \cos \theta,$$

$$\therefore \quad \cos \phi = \frac{\sqrt{a^2 P^2 - b^2 W^2 \cos^2 \theta}}{aP}.$$

Expanding $\sin (\theta + \phi)$ in (2), and substituting these values of $\sin \phi$ and $\cos \phi$, and reducing, we have the equation

$$\cos^3 \theta - \frac{P^2 a^2 + W^2 (a^2 + b^2)}{2\, ab\, W^2} \cdot \cos^2 \theta + \frac{P^2 a}{2 W^2 b} = 0.$$

The student will do well to observe that the coefficients of this equation are ratios of magnitudes of the same kind. Thus, force and linear magnitude are quantities of essentially different kinds. It is true, indeed, that the magnitude of a force may be conventionally represented by the length of a line, but it is only in comparison with other forces that any one force can be so represented, and the scale of representation is arbitrary. Hence $\cos \theta$, which is a mere number, if it is expressed in terms of force, must be expressed as the *ratio of one force to another*; and if it is expressed in terms of linear magnitude, it must be as the *ratio of one line to another*. If, for example, the coefficient of $\cos^3 \theta$ in (3) being unity, the last term had been $\frac{Pa^3}{W^2 b}$, we should have known at once that the result was wrong. For the numerator and denominator of this expression are not of the *same degree* in force; neither are they of the same degree in linear magnitude. Such a term as $\frac{Pa^3}{W^2 b}$ denotes the product of an area, $\frac{a}{b}$, by the reciprocal of a force, $\frac{P}{W^2}$.

Similar remarks as to the *homogeneity* of our results will be of frequent occurrence in the sequel. By attention to considerations of this kind the student will often be able to detect an error in his work.

6. If, in the last example, the weight W, instead of being knotted to the string at C, is suspended from a smooth ring which is at liberty to slide along the string ACB, find the position of equilibrium.

In this case, the string $PBCA$, which passes over a smooth surface at B, and through the smooth ring, will have its tension constant at each of its points (Art. 25), and therefore equal to P. Hence, putting $T = P$, and resolving forces vertically for the equilibrium of C, we have

$$W - 2P \sin \theta = 0,$$

or

$$\sin \theta = \frac{W}{2P}.$$

7. A string, whose weight is neglected, passes over three smooth pegs, A, B, C, which are in the same horizontal line. From the extremities of the string are suspended two weights, P and P'; and to two given points in it are knotted two weights, W and W', the first suspended between A and B, and the second between B and C. Find the position of equilibrium.

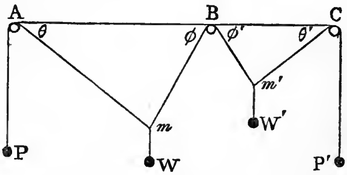


Fig. 25.

In this problem the given quantities are the suspended weights, P, W, P', and W', the distances AB and BC, and the length of the portion mBm' of the string (fig. 25).

Evidently the quantities which we wish to determine are the inclinations, θ, ϕ, ... , of the portions of the string to the horizon.

Let $AB = a$, $BC = a'$, and the length of $mBm' = k$. Consider the equilibrium of the point m. Since the string PAm passes over a smooth peg at A, the tension in it $= P$ throughout. If $T =$ tension in mBm', we have for the equilibrium of m,

$$\frac{P}{W} = \frac{\cos \phi}{\sin (\theta + \phi)}, \tag{1}$$

$$\frac{T}{W} = \frac{\cos \theta}{\sin (\theta + \phi)}.$$

Again, for the equilibrium of m',

$$\frac{P'}{W'} = \frac{\cos \phi'}{\sin (\theta' + \phi')}, \tag{2}$$

$$\frac{T}{W'} = \frac{\cos \theta'}{\sin (\theta' + \phi')}.$$

Equating the two values of T, we have

$$\frac{W \cos \theta}{\sin (\theta + \phi)} = \frac{W' \cos \theta'}{\sin (\theta' + \phi')}. \tag{3}$$

These are all the equations that can be obtained from statical considerations. One more equation is required to determine the four unknown quantities, θ, ϕ, θ', and ϕ'. This is obtained by expressing that the length of $mBm' = k$. Evidently

$$Bm = \frac{a \sin \theta}{\sin (\theta + \phi)}, \text{ and } Bm' = \frac{a' \sin \theta'}{\sin (\theta' + \phi')};$$

$$\therefore \quad \frac{a \sin \theta}{\sin (\theta + \phi)} + \frac{a' \sin \theta'}{\sin (\theta' + \phi')} = k. \tag{4}$$

These four equations determine θ, ϕ, θ', ϕ', and therefore the position of equilibrium.

8. A string, $BMNPA$, whose weight is neglected, is suspended from two fixed points, A and B; and from given points, M, N, P, ..., in the string, are suspended a series of equal particles whose weight is W. Find the inclinations, θ_1, θ_2, θ_3, ..., of the successive portions of the string to the horizon.

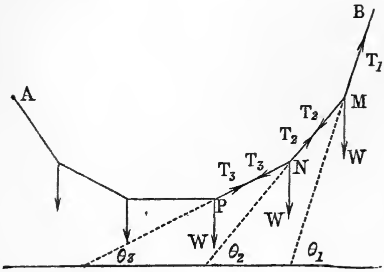


Fig. 26.

Consider the equilibrium of the particle M. It is acted on by three forces, viz., W acting vertically, T_1, the tension of the string MB, and T_2 the tension of MN.

Resolving these forces vertically,

$$W + T_2 \sin \theta_2 - T_1 \sin \theta_1 = 0; \tag{1}$$

and, resolving horizontally,

$$T_1 \cos \theta_1 - T_2 \cos \theta_2 = 0.$$

For the equilibrium of N, resolving horizontally,

$$T_2 \cos \theta_2 - T_3 \cos \theta_3 = 0.$$

Hence

$$T_1 \cos \theta_1 = T_2 \cos \theta_2 = T_3 \cos \theta_3 = \dots;$$

or in other words, *the horizontal components of the tensions in the different portions of the string are constant.* Let this constant be denoted by T; then

$$T_1 = \frac{T}{\cos \theta_1}, \quad T_2 = \frac{T}{\cos \theta_2}, \text{ \&c.}$$

Substituting these values in (1), we have

$$\tan \theta_1 = \tan \theta_2 + \frac{W}{T}.$$

Similarly,

$$\tan \theta_2 = \tan \theta_3 + \frac{W}{T},$$

$$\tan \theta_3 = \tan \theta_4 + \frac{W}{T},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence *the tangents of the successive inclinations form a series in Arithmetical Progression.* In the figure

$$\theta_4 = 0, \quad \therefore \ \tan\theta_3 = \frac{W}{T}, \ \tan\theta_2 = \frac{2W}{T}, \ \tan\theta_1 = \frac{3W}{T}.$$

If the suspended weights are not equal, it is still true that the horizontal components of the tensions are all equal.

The figure formed by the string $BMNPA$ is called the *Funicular Polygon.*

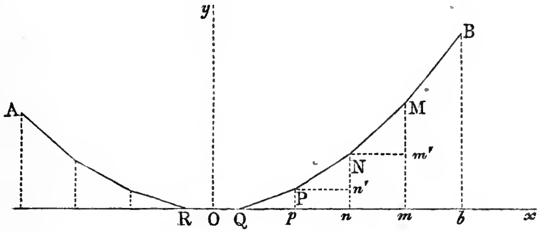


Fig. 27.

9. To construct the Funicular Polygon, when the horizontal projections, RQ, Qp, pn, nm, mb, ..., of the successive portions of the chain are all of constant length, a.

Let $Pp = c$; then, since (last example) the tangent of the inclination of $PN = 2$. tangent of inclination of PQ, it follows that, Pn' being horizontal, $Nn' = 2Pp = 2c$. Also tan of inclination of MN $= 3$ tan of inclination of PQ; therefore $Mm' = 3c$.

Hence, taking the middle point, O, of the horizontal portion, RQ, as origin, and the horizontal and vertical lines through it as axes of x and y, the co-ordinates of P are $(\frac{3}{2}a, c)$; those of N are $(\frac{5}{2}a, c+2c)$; those of M are $(\frac{7}{2}a, c+2c+3c)$; and those of the n^{th} vertex from Q are evidently

$$x = \frac{2n+1}{2} \cdot a, \qquad y = \frac{n(n+1)}{2} \cdot c.$$

The value of the ordinate, y, of any vertex at once enables us to determine this vertex.

If we eliminate n from the two equations for x and y, we get an equation which is satisfied by all the vertices indifferently. This equation denotes, therefore, a curve passing through all the vertices of the polygon. Eliminating n, we get

$$x^2 = \frac{2a^2}{c} \cdot y + \frac{a^2}{4}.$$

This denotes a parabola whose axis is the vertical line Oy. The vertex of the parabola is vertically below O at a distance $= \frac{c}{8}$.

The smaller the distance RQ, Qp, pn, ..., the more nearly does the Funicular Polygon coincide with the parabolic curve.

10. To represent graphically the forces in the general case of the Funicular Polygon.

For convenience, let the vertices of the string or chain be denoted by the numbers 1, 2, 3,..., and let the forces P_2, P_3, ... act at the vertices. Let also the tension in the portion of the string (1, 2) be denoted by T_{12}, &c.

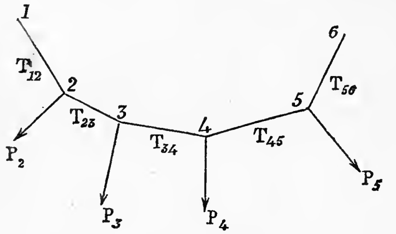


Fig. 28.

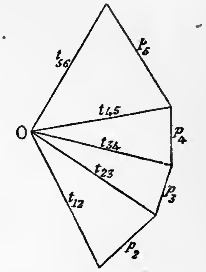


Fig. 29.

Now, take any point, O, and from it draw the line t_{12} parallel to the string (1, 2), and proportional to the tension T_{12}. From the extremity of t_{12} draw the line, p_2, parallel and proportional to the force P_2. It follows, then, that since the forces T_{12}, T_{23}, and P_2 form a system in equilibrium at the point (2), the third side, t_{23}, of the triangle t_{12}, p_2, t_{23} is parallel to T_{23}, and proportional to it (Cor. 2, p. 18). In the same way, drawing p_3 parallel and proportional to P_3, the side t_{34} is parallel and proportional to T_{34}; and continuing this construction, the tensions in the successive portions of the string are all represented by the lines t_{12}, t_{23}, t_{34}, ... in the new figure (fig. 29).

The figure (fig. 29) which represents by its lines, both in magnitude and in direction, all the forces of the system in fig. 28, is called by Professor J. Clerk Maxwell, a 'Force Diagram' of the system. (*Transactions of the Royal Society of Edinburgh*, vol. xxvi.)

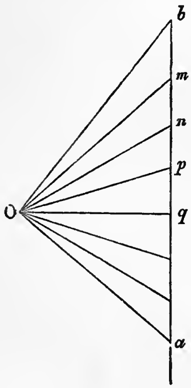


Fig. 30.

When, as in example 8, all the applied forces, P_2, P_3, ... are parallel, the Force Diagram of the system consists of a triangle with lines drawn from the vertex to different points in the base. Thus, taking any point, O (fig. 30), and drawing ob parallel to MB (fig. 27), and proportional to the tension in it; and then drawing bm vertical and proportional to the weight suspended at M, it follows that om will be parallel to MN, and proportional to the tension in it. Similarly for the rest of the figure. If all the suspended weights are equal, the lines bm, mn, np, pq,... are all equal, and fig. 30 at once shows that the tangents of the successive inclinations of the parts of the chain are in Arithmetical Progression. This figure also exhibits the constancy of the horizontal components of the tensions ob, om, on, ... these components being all equal to oq.

11. Suspension Bridge. The number of vertices of the polygon being very great, and the suspended weights all equal, the parabola which passes through all the vertices virtually coincides with the chain forming the polygon, and gives the figure of the Suspension Bridge. In this bridge the weights suspended from the successive portions of the chain are the weights of equal portions of the flooring. The weight of the chain itself and the weights of the sustaining bars are negligible in comparison with the weight of the flooring and the load which it carries.

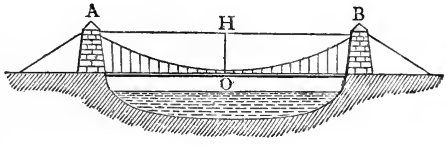


Fig. 31.

Fig. 30 may be taken to represent the Force Diagram of the Suspension Bridge, the vertical line *ab*, representing the weight of the flooring, being divided into as many equal parts as there are divisions of the chain. If these parts are sufficiently numerous, the lines *ob*, *om*, *on*, &c., are parallel to tangents to successive points of the chain. Let the span, AB, of the bridge $= 2a$, and let the height $OH = h$. Then, the equation of the parabola referred to horizontal and vertical axes of y and x, respectively, through O (fig. 31) is

$$y^2 = 4mx,$$

m being a constant; and the tangent of the inclination to the vertical of any portion

$$= \frac{dy}{dx} = \frac{2m}{y} = \frac{y}{2x}.$$

Hence the tangent at the point of support, B, makes with the horizon an angle whose tangent is $\frac{2h}{a}$.

Therefore, oq (fig. 30) being parallel to the tangent at the lowest point of the bridge, and ob parallel to the tangent at the point B,

$$\tan boq = \frac{2h}{a}.$$

Hence, since bq represents half the weight of the bridge, and ob the terminal tension of the chain at B,

$$\text{Terminal tension} = \frac{W}{2\sin boq} = W\frac{\sqrt{a^2+4h^2}}{4h},$$

W being the weight of the flooring.

Also, the vertical tension at $B = \frac{1}{2}W$, and the constant

$$\text{Horizontal tension} = W\frac{a}{4h}.$$

12. The entire load of a suspension bridge is 160,000 kilograms, the span is 64 metres, and the height is 5 metres; find the tension at the points of support, and also the tension at the lowest point.

Ans. Terminal tension = 268,208 kilograms.
Horizontal tension = 256,000 „

13. If the vertical bars which support the roadway of a suspension bridge are not at equal horizontal distances, prove that the vertices of the polygon formed by the chain will still lie on a parabola, provided that each vertical bar supports half of the adjacent portions of the roadway.

This follows from the fact that the cotangent of the inclination of any chord of a parabola to the axis is proportional to the sum of the ordinates of the extremities of the chord.

14. If R is the resultant of any number of forces, $P_1, P_2, P_3, \ldots$, acting in one plane on a particle, prove that

$$R^2 = \Sigma P^2 + 2\Sigma P_1 P_2 \cos(P_1 \hat{,} P_2),$$

where $P_1 \hat{,} P_2$ means the angle between P_1 and P_2.

(This result is true for non-coplanar forces).

15. If a particle is in equilibrium under the action of any forces, prove that the sum of the *oblique* components of the forces along any right line is zero.

If ΣX and ΣY denote the sums of the components along two lines inclined at an angle $= \omega$, the square of the resultant is equal to

$$(\Sigma X)^2 + 2(\Sigma X)(\Sigma Y)\cos\omega + (\Sigma Y)^2;$$

and this $\qquad \equiv (\Sigma X + \Sigma Y)^2 \cos^2\frac{\omega}{2} + (\Sigma X - \Sigma Y)^2 \sin^2\frac{\omega}{2}.$

Hence the result follows as in equations (3), p. 24. It is otherwise evident, since the resultant is the third side of a triangle, two of whose sides are ΣX and ΣY.

16. If in example 7 the weights W and W', instead of being knotted to two given points in the string, are attached to two smooth rings which are capable of sliding freely along the string, determine the condition and position of equilibrium.

Here, since the string passes freely over and under smooth surfaces, the tension is constant throughout its length. Now, the tension in Am is P, and that in $Cm' = P'$. Hence

$$P = P'.$$

For the equilibrium of m, we have, resolving vertically,

$$W = 2P\sin\theta; \quad \therefore \sin\theta = \frac{W}{2P};$$

and for the equilibrium of m',

$$W' = 2P\sin\theta'; \quad \therefore \sin\theta' = \frac{W'}{2P}.$$

17. A heavy particle is attached to one end of a string, the other end of which is fixed. Find the horizontal force which must be applied to the particle in order that the string may deviate by a given angle from the vertical, and find also the tension of the string.

Ans. If $F =$ the horizontal force required, $T =$ tension of string, $W =$ weight of particle, and $\theta =$ angle of string's deviation,

$$F = W\tan\theta, \quad T = W\sec\theta.$$

18. A string ACB (fig. 24, example 5) has its extremities tied to two fixed points, A and B; to a given point, C, in the string is knotted a given weight, W. Find the tensions in the portions CA and CB.

Ans. Since AC and BC are given, the angles CAB and CBA are also given. If these angles are denoted by θ and θ', and if T and T' are the tensions in CA and CB,

$$T = \frac{W\cos\theta'}{\sin(\theta+\theta')}, \quad T' = \frac{W\cos\theta}{\sin(\theta+\theta')}.$$

19. If (same figure) the extremities A and B are fixed, and the weight W is that of a smooth heavy ring at C, which is capable of sliding freely along the string, find the horizontal force which must be applied to the ring C in order that the system may take a given position of equilibrium.

Ans. If the angles CAB and CBA are θ and θ', and F = the required force,

$$F = W\tan\frac{\theta-\theta'}{2}.$$

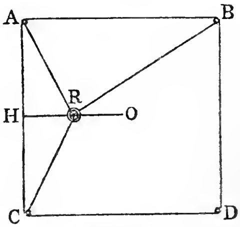


Fig. 32.

20. $ABCD$ (fig. 32) is a system of pegs forming a square in a vertical plane; a string attached to A and B passes through a heavy smooth ring, R, while another string is attached to C and R. The ring is kept in equilibrium half way between H, the middle point of CA, and O, the centre of the square; find the tensions in the strings ARB and CR.

Ans. If W = weight of ring, T = tension in ARB, and T' = tension in CR,

$$T = W.\frac{13\sqrt{5}+5\sqrt{13}}{32}, \quad T' = W.\frac{\sqrt{5}+5\sqrt{13}}{16}.$$

21. In the last example if the tensions in the two strings are equal, find the point at which the ring must be placed on OH.

Ans. If $\dfrac{OR}{OH} = x$, x is determined by the equation

$$3x^4 - 3x^2 - 10x + 6 = 0.$$

This equation has only two real roots, one between 0 and 1, and the other between 1 and 2.

22. A string whose weight is neglected passes over three smooth pegs, A, B, C (fig. 33), in a vertical plane, and sustains two equal weights, W, from its extremities. Find the pressures on the pegs; and find also the magnitudes of the angles α, β, and γ when the system of pegs is least likely to break, the pegs being all equally strong.

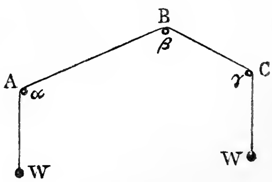


Fig. 33.

Ans. If P, Q, and R be the pressures on the pegs A, B, and C, respectively, $P = 2W\cos\frac{\alpha}{2}$, $Q = 2W\cos\frac{\beta}{2}$, $R = 2W\cos\frac{\gamma}{2}$; and since the sum of α, β, and γ is given ($= 2\pi$), it follows that in the best arrangement $\alpha = \beta = \gamma = \frac{2}{3}\pi$. For, unless each of the angles $= \frac{2}{3}\pi$, some one of the pressures must be $> 2W\cos\frac{\pi}{3}$, or W; and if the pegs are of equal strength, it is best under these conditions, to have the pressures on them all equal.

23. If the string passes over any number of equally strong smooth pegs in the same vertical plane, find the best arrangement.

Ans. If there are n pegs, each of the angles, $\alpha, \beta, \gamma, \delta, \ldots$ must be $= \dfrac{(n-1)\pi}{n}$.

24. In example 14 calculate the pressures on the pegs A, B, C.

Ans. The squares of the pressures are respectively

$P(2P+W)$, $\frac{1}{2}\{4P^2 + WW' - \sqrt{(4P^2 - W^2)(4P^2 - W'^2)}$, $P(2P+W')$.

25. If the strengths of the pegs, A, B, C, in example 20, are proportional to l, m, n, find the best arrangement of the system.

Ans. The angle α is given by the equation

$$2mnx^3 + (l^2 + m^2 + n^2)x^2 - l^2 = 0,$$

in which $x = \cos\frac{\alpha}{2}$. The angles β and γ are at once found from α.

26. Let $A_0 A_1 \ldots A_5$ (fig. 34) be any funicular polygon, with weights P_1, P_2, P_3, P_4 suspended at its vertices A_1, A_2, A_3, A_4, respectively; draw any line, $a_0\, a_5$, meeting the verticals through A_0, $A_1, \ldots$ in the points a_0, d_1, $d_2, \ldots$, and let $A_0 A_5$ meet these verticals in $A_0, D_1, D_2, \ldots$. Now construct a new polygon, $a_0 a_1 a_2 \ldots a_5$, by taking $d_1 a_1 = \frac{1}{n} D_1 A_1$; $d_2 a_2 = \frac{1}{n} D_2 A_2$; and so on, n being any number.

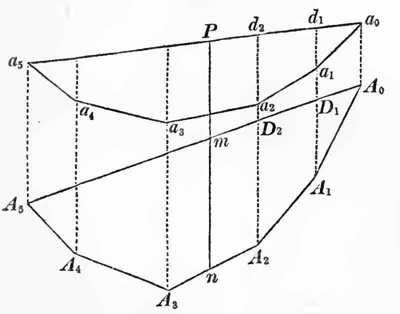


Fig. 34.

Prove that the new polygon, whose fixed ends are a_0 and a_5 will be kept in equilibrium by the set of forces P_1, P_2, P_3, P_4 applied at its vertices a , a_2, a_3, a_4.

Although this may be readily proved geometrically by principles of Graphic Statics, the student will do well to establish it by the method of example 8. He will easily prove that, if α and β are the inclinations of $A_0 A_5$ and $a_0\, a_5$ to the horizon, θ_{01}, $\theta_{02}, \ldots$ the inclinations of the sides $A_0 A_1, A_1 A_2, \ldots$, and $\phi_{01}, \phi_{12}, \ldots$ those of $a_0 a_1, a_1 a_2, \ldots$ to the horizon, we shall have

$$\tan\phi_{01} - \tan\beta = \frac{1}{n}(\tan\theta_{01} - \tan\alpha);$$

$$\tan\phi_{12} - \tan\beta = \frac{1}{n}(\tan\theta_{12} - \tan\alpha), \text{ \&c.}$$

But if T denotes the constant horizontal tension in a funicular polygon, the conditions of its equilibrium are

$$\tan\theta_{01} - \tan\theta_{12} = \frac{P_1}{T}; \quad \tan\theta_{12} - \tan\theta_{23} = \frac{P_2}{T}; \quad \text{\&c.}$$

These conditions are satisfied in the polygon $a_0 a_1, \ldots a_5$ on the supposition that the horizontal tension in it $= nT$; and it is axiomatic that if internal forces can preserve equilibrium, they will.

Of course all the ordinates (and not merely those through the vertices) of the derived polygon are proportional to the corresponding ordinates of the original.

27. Show that the last example enables us to construct for a given parallel system of forces a funicular polygon which shall pass through three given points.

(A solution of this problem for any system of forces will be given in a subsequent chapter).

28. Given the base, NS (fig. 35), of a triangle NPS, and also the sum of the cosines of the base angles, SNP and NSP; let the curve locus of P be constructed. Prove that if a particle be placed at any point of the curve and acted on by two forces, one repulsive from N and equal to $\frac{\mu}{NP^2}$, and the other attractive towards S and equal to $\frac{\mu}{SP^2}$, the resultant force is, at every position of the particle, directed along the tangent to the curve.

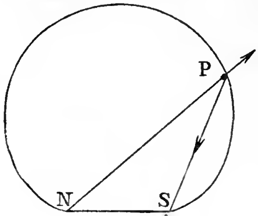


Fig. 35.

N. B.—This curve is called the 'Magnetic Curve,' being one of those in which small iron filings would arrange themselves under the influence of a fixed magnet whose poles are N and S.

It is to be observed that each little piece of iron is a magnet, having two poles at its extremities, and that it must therefore set at the point, P, where it is placed, in the direction of the resultant force on either of its poles.

29. Prove that the line of action of the resultant force of a magnet on a magnetic pole at P divides NS externally in the ratio $NP^3 : SP^3$.

30. Iron filings are sprinkled over a sheet of paper on which a magnet rests; prove that all those filings which dip towards the same point on the line of the magnet lie on a circle (neglecting their mutual actions).

CHAPTER III.

THE EQUILIBRIUM OF A PARTICLE ON PLANE CURVES.

SECTION I.

Smooth Curves.

32.] **Smooth Surface.** When a body is placed in contact with a surface, it is evident that, in addition to the given forces acting on the body, there is a certain force produced by the surface—the force, namely, which the surface exerts to prevent the body from passing through it. This force is called the *Reaction* of the surface. Now, the surface being supposed to be rigid, there is evidently no limit to the *magnitude* of the force with which it is capable of reacting; but the *direction* of the force depends on the nature of the surface itself. If the surface be perfectly smooth, it can react on any body in contact with it only in the direction of the *normal* to the surface at the point where the body is in contact with it. Thus (fig. 36), if a body, M, acted on by any given system of forces, be in contact at a point O with a *smooth* surface, AB, the force which this surface exerts on the body takes the direction, ON, of the normal to the surface at the point of contact, O, and its magnitude will be such as to destroy the effect of all the other forces acting upon M. To the magnitude of the reaction, R, there is no limit; so that if each of the other forces acting on M were increased 100 times, for example, the surface would react with a force equal to 100 R; but the *direction* of R is strictly limited to that of the normal. We may therefore state that—

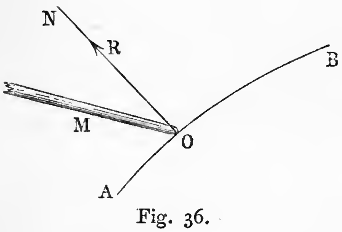


Fig. 36.

When two smooth bodies are in contact, their mutual reaction is normal to the surface of contact.

33.] **Example.** If P (fig. 37) is a heavy particle whose weight is W, placed on a smooth spherical surface whose vertical diameter is AB, what is the position of equilibrium?

Here the forces acting on P are only two in number—namely, its weight, W, and R, the reaction of the smooth surface. Now, this reaction takes place in the direction of the normal, PO, to the sphere at P; and since the particle is in equilibrium under the action of only two forces, these must be equal in magnitude, and act in opposite senses along the same right line. Hence, since W acts vertically, PO must be a vertical line; that is, P must be placed at A, the lowest point of the sphere, or outside the surface at B, the highest point.

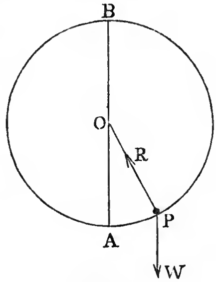


Fig. 37.

Whatever be the smooth surface on which the particle is placed, it is evident that the points on it at which the particle will rest are points the normals at which are vertical lines. And, generally—

A particle will rest at those points of a smooth surface at which the normal coincides with the direction of the resultant of all the forces acting on the particle.

34.] **Normal to a Curve.** The normal to a curve at a given point is not, like the normal to a surface at a given point, a definite line, but is *any line whatever in the plane perpendicular to the tangent at the point.*

Hence, for the equilibrium of a particle placed inside a smooth tube of any form, the resultant force on the particle need not act in a given right line, but must act in a given plane—namely, the plane which is normal to the tube at the point where the particle is placed. Thus, for example, let AB (fig. 38) be a smooth tube of any form, and let P be a particle placed inside it. If we imagine a string attached to P, coming out of the tube through an opening at P, which is not sufficiently large to allow P to come out, it is evident that we may pull at P with any force however great in the plane normal to the tube, and in all directions round P

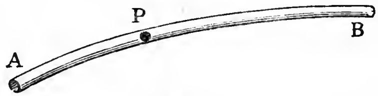


Fig. 38.

and the equilibrium of the particle will not be disturbed. But if we incline the string ever so little to the normal plane at P, motion will ensue along the tube.

35.] **Plane Curve.** In the present chapter we shall consider only *plane curves*, i.e., curves which lie altogether in one plane.

Moreover, when a particle is placed on a curve, and acted on by given forces, we shall suppose that all the forces act in the plane of the curve.

Now, it is evident that the only effect which a curve produces on a particle placed upon it is a normal reaction of some definite magnitude. If, then, we produce upon the particle, by any other means, a force identical with this reaction, we may dispense with the curve altogether. This being so, if we call the reaction of the curve R, we may suppose the particle acted upon by all the given forces, and also by a new force equal to R, this latter acting in the direction of the normal to the curve. Thus, the case is the same as that treated in the last chapter—namely, the equilibrium of a particle acted upon by any number of forces in one plane; and in writing down the equations of equilibrium, we shall merely have to include the new force R among all the others.

EXAMPLES.

1. A heavy particle is placed on a smooth inclined plane, AB (fig. 39), and is sustained by a force, F, which acts along AB in the vertical plane which is at right angles to AB; find F, and also the pressure on the inclined plane.

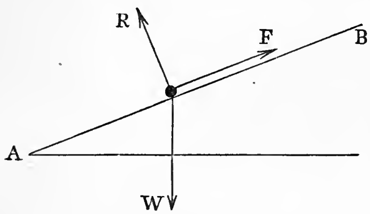


Fig. 39.

The only effect of the inclined plane is to produce a normal reaction, R, on the particle. Hence, if we introduce this force, we may imagine the plane removed. Let W be the weight of the particle, and i the inclination of the plane to the horizon.

Resolving the forces along AB, we have

$$F - W \sin i = 0, \text{ or } F = W \sin i;$$

and, resolving perpendicularly to AB,

$$R - W \cos i = 0, \text{ or } R = W \cos i.$$

If, for example, the weight of the particle is 4 grammes and the inclination of the plane 30°, there will be a normal pressure of $2\sqrt{3}$ grammes on the plane, and the force F will be 2 grammes.

2. In the previous example, if F act horizontally, find its magnitude, and also that of R.

Resolving along AB, and perpendicularly to it, we have, successively,

$$F\cos i - W\sin i = 0, \quad \text{or} \quad F = W\tan i;$$

and $$F\sin i + W\cos i - R = 0, \quad \therefore \quad R = \frac{W}{\cos i};$$

R is therefore in this case greater than it was before, as is sufficiently evident *a priori*.

3. If the particle is sustained by a force, F, making a given angle, θ, with the inclined plane, find the magnitude of this force, and of the pressure, all the forces acting in the same vertical plane.

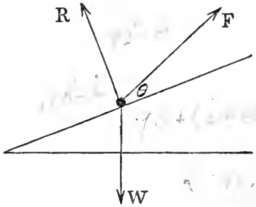


Fig. 40.

Resolving along the plane, (fig. 40),

$$F\cos\theta - W\sin i = 0, \quad \text{or } F = \frac{W\sin i}{\cos\theta};$$

and resolving perpendicularly to the plane,

$$R + F\sin\theta - W\cos i = 0, \therefore R = W\frac{\cos(i+\theta)}{\cos\theta}.$$

The student will, of course, observe that these values of F and R could have been at once obtained, without resolution, by the equation (a), p. 11.

4. A heavy particle, whose weight is W, is sustained on a smooth inclined plane, by three forces applied to it, each equal to $\frac{W}{3}$; one acts vertically, another horizontally, and the third along the plane (fig. 41); find the inclination of the plane.

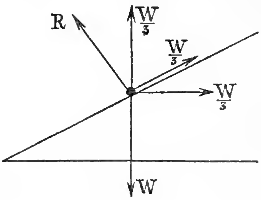


Fig. 41.

Since we do not want R, the pressure on the plane, we shall resolve forces at right angles to R, that is, along the plane. Hence

$$\frac{W}{3}\sin i + \frac{W}{3} + \frac{W}{3}\cos i - W\sin i = 0,$$

or $$2\sin i = 1 + \cos i, \quad \therefore \quad 2\sin\frac{i}{2}\cos\frac{i}{2} = \cos^2\frac{i}{2}. \qquad (1)$$

If we reject the factor $\cos\frac{i}{2}$ for the present, we have

$$\tan\frac{i}{2} = \frac{1}{2},$$

which determines the inclination.

The student should observe that we have expelled the factor $\cos\frac{i}{2}$ from equation (1), and this amounts to rejecting the solution

$$\cos\frac{i}{2} = 0.$$

Now in this, as well as in many physical and geometrical problems, such a solution ought not to be rejected, unless it is shown to be irrelevant to the question. So long as our equations are *perfect* interpretations of the physical or geometrical conditions of the problem, no factor can furnish an irrelevant solution. It is only when an equation expresses more or less than is implied in the given conditions that irrelevant factors can present themselves. Instances of these factors frequently occur in the operations of Algebra and Analytic Geometry—as, for example, when we rationalize an equation by the process of squaring. If, before this process, the square root of a quantity was affected with a *minus* sign, this sign will be indifferent in the rationalized result, and this latter, consequently, expresses *more* than was contained in the original equation. Hence it may happen that the result will furnish us not only with what is relevant, but, in addition, with what is wholly irrelevant.

In the present instance the equation $\cos\frac{i}{2} = 0$ would give the inclination of the plane $= 180°$, and the figure would then become fig. 42, in which the particle is placed underneath the plane in such a way that equilibrium is manifestly impossible.

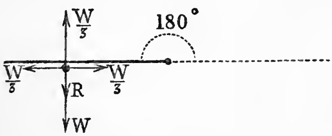


Fig. 42.

Hence it appears as if the equation $\cos\frac{i}{2} = 0$ were wholly without meaning.

A little reflection, however, will show that it is quite relevant. For equation (1) is merely the analytical expression of the physical condition that the component of the acting forces *along the plane* shall be zero. Now it is not enough *for equilibrium* that the component along some one line shall be zero; for this, the component along some other line must vanish as well. Hence our result does not express the complete condition of the particle's equilibrium, but merely *a part* of that condition; and each of the equations

$$\tan\frac{i}{2} = \frac{1}{2}, \quad \text{and} \quad \cos\frac{i}{2} = 0,$$

expresses perfectly all the physical conditions contained in (1). For when the inclination is 180°, the force $\frac{W}{3}$ which acted along the inclined plane becomes a horizontal force opposite to the given horizontal force $\frac{W}{3}$; and the vertical $\frac{W}{3}$ furnishes no component along the plane.

The magnitude of R is $\frac{2}{3}W$.

5. A heavy particle, P (fig. 43), is placed inside a smooth parabolic tube whose axis is vertical, and is acted upon by a horizontal force, F,

equal μ PM, PM being the ordinate of the point P; find the position of equilibrium,

Here the forces acting are W, the weight of the particle, R, the normal reaction of the tube, and F. We shall obtain an equation between F and W, without R, by resolving along the tangent at P. If $\theta =$ angle between the tangent at P and the vertical,
$W \cos\theta = F \sin\theta = \mu y . \sin\theta$, where $y = PM$.
Hence, for the position of equilibrium, retaining the factor $\cos\theta$,

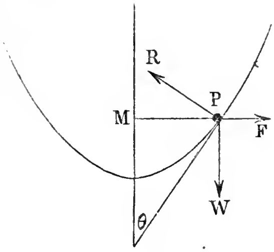


Fig. 43.

$$\cos\theta\,(W - \mu y \tan\theta) = 0.$$

But if the equation of the parabola is $y^2 = 4mx$, $\tan\theta = \dfrac{2m}{y}$. Hence the equation is

$$\cos\theta\,(W - 2\mu m) = 0. \tag{1}$$

This equation of equilibrium can be satisfied in two ways. Firstly we can have

$$\cos\theta = 0, \tag{2}$$

or $\theta = \dfrac{\pi}{2}$, which gives the vertex of the tube as the position of equilibrium. This position is *a priori* evident, since the particle would at the vertex be acted upon only by its weight and the reaction of the tube, the force F here being $= 0$.

Secondly, the equation will be satisfied if

$$W - 2\mu m = 0. \tag{3}$$

Now, this is simply a relation between the constants of the problem, and gives no value of θ—that is, no definite position of equilibrium. In fact, if the equation (3) is satisfied, (1) will be satisfied, no matter what θ may be. In physical language, then, the result is as follows: if $\mu = \dfrac{W}{2m}$, the particle will rest in all positions; and if this relation does not hold, the vertex is the only position.

It is well for the student to observe that μ is here the quotient of a force by a line, the force being expressed in the same units as those of W, and the line in the same units as those of PM. For since we have put $F = \mu\, PM$, if Q is a force in the same units as those of W, and l a line in the same units as those of PM, it is clear that the proper representation of F would be something of the form $Q\dfrac{PM}{l}$; therefore $\mu = \dfrac{Q}{l}$.

6. A heavy particle, resting on a smooth inclined plane, is attached to a string which, passing over a smooth pulley, sustains another heavy particle: find the conditions and position of equilibrium.

Let W be the weight of the particle on the plane, P that of the hanging particle, and θ the inclination of the string to the inclined plane in the position of equilibrium.

For the equilibrium of the particle on the plane, we have, resolving along the plane (since the tension of the string $= P$),

$$W \sin i = P \cos \theta;$$

$$\therefore \quad \cos \theta = \frac{W \sin i}{P}.$$

In order that there may be a position of equilibrium, this value of $\cos \theta$ must be < 1, therefore $W \sin i$ must be $< P$.

Explain the result when $P = W$.

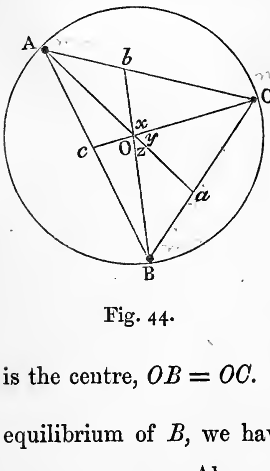


Fig. 44.

7. Three particles, whose masses are m_1, m_2, m_3, are placed at three points, A, B, C (fig. 44), inside a smooth circular tube; they attract or repel each other with forces directly proportional to their masses and their distances; find the position of equilibrium of the system.

Consider the equilibrium of m_1 at A. It is acted upon by two forces equal to $m_2 AB$ and $m_3\, AC$, in the directions AB and AC. The resultant of these must be normal to the tube at A. But (Cor. 2, p. 16) the resultant acts towards a, the centre of mass of m_2 and m_3, and if O is the centre, $OB = OC$. Hence $\dfrac{\sin y}{\sin z} = \dfrac{m_2}{m_3}$; and, by considering the equilibrium of B, we have $\dfrac{\sin x}{\sin z} = \dfrac{m_1}{m_3}$. Therefore $\sin x : \sin y : \sin z = m_1 : m_2 : m_3$. Also $x + y + z = \pi$; therefore x, y, and z are the angles of a triangle whose sides are proportional to m_1, m_2, and m_3. These angles being known from some such equations as $\cos x = \dfrac{m_2^2 + m_3^2 - m_1^2}{2 m_2 m_3}$, &c., the relative positions of the particles are at once determined. The centre, O, of the tube is the centre of mass of the particles.

8. Two smooth heavy rings, A and C (fig. 45), slide on two rods which are inclined to the horizon at angles i and i'; a string connecting A and C passes through a smooth heavy ring, B. Find the condition of equilibrium.

Let the weights of A, B, C, be P, W, P', respectively, and let R and R' be the reactions of the rods on A and C. Construct the force-diagram of the system by drawing om from an arbitrary origin, O, parallel and proportional to R', and mn parallel and proportional to P'; then on will be parallel to BC and proportional to the tension in it. Drawing again np parallel and proportional to W, op will be parallel to BA, and represent its tension. Finally, if pq represents P, oq will represent R. Since the tension in ABC is constant, $on = op$; therefore a perpendicular from O on mq bisects np. The

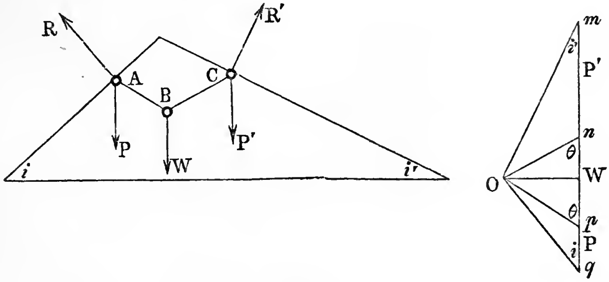


Fig. 45.

length of this perpendicular is on the one hand $(mn + \frac{1}{2}np) \tan i'$, and on the other $(pq + \frac{1}{2}np) \tan i$. Hence, equating these, we have

$$(P' + \tfrac{1}{2}W) \tan i' = (P + \tfrac{1}{2}W) \tan i.$$

This is a relation between the constants of the problem, and it therefore constitutes a condition that equilibrium should be at all possible. If this condition is fulfilled, the position of equilibrium can be obtained by finding the angle, θ, which the string BC makes with the vertical. Evidently, from the force-diagram

$$\tan\theta = \frac{W + 2P'}{W} \tan i'.$$

9. Two heavy rings, whose weights are P and P' (fig. 46), rest on the circumference of smooth vertical circle, and are connected by a weightless string on which a heavy ring, whose weight is Q, slides freely. Find the position of equilibrium.

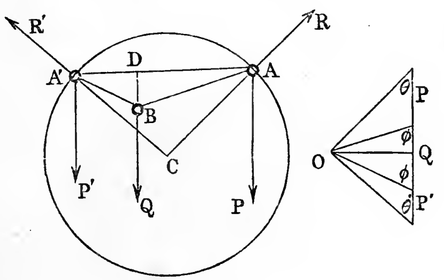


Fig. 46.

Construct the force-diagram. Let θ and θ' be the inclinations of the radii CA and CA' to the vertical, and let ϕ be the inclination of the portions of the string AB and BA' to the vertical.

The force-diagram then gives the statical equations

$$\left(\frac{Q}{2}+P\right)\tan\theta=\left(\frac{Q}{2}+P'\right)\tan\theta', \tag{1}$$

$$\left(\frac{Q}{2}+P\right)\tan\theta=\frac{Q}{2}\tan\phi. \tag{2}$$

To these must be added the geometrical equation which connects the length, l, of the string, with the radius, a, of the circle.

Since the horizontal projections of the broken lines ACA' and ABA' are the same, we have

$$a(\sin\theta+\sin\theta')=l\sin\phi. \tag{3}$$

Equations (1), (2), and (3) are sufficient to determine the unknown angles θ, θ', and ϕ.

10. A body, whose weight is 10 kilogrammes, is supported on a smooth inclined plane by a force of 2 kilogrammes acting along the plane and a horizontal force of 5 kilogrammes; find the inclination of the plane.

Ans. $\sin^{-1}\left(\frac{3}{5}\right)$.

11. A heavy body is sustained on a smooth inclined plane (inclination i) by a force P acting along the plane, and a horizontal force, Q. The inclination being halved, and the forces P and Q each halved, the body is still observed to rest; find the ratio of P to Q.

Ans. $\frac{P}{Q}=2\cos^2\frac{i}{4}$.

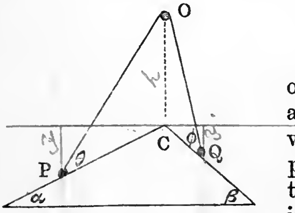


Fig. 47.

12. Two weights, P and Q (fig. 47), rest on a smooth double-inclined plane, and are attached to the extremities of a string which passes over a smooth peg, O, at a point vertically over the intersection of the planes, the peg and the weights being in a vertical plane. Find the position of equilibrium.

Ans. If $l=$ the length of the string, and $CO=h$, the position of equilibrium is defined by the equations

$$P\frac{\sin\alpha}{\cos\theta}=Q\frac{\sin\beta}{\cos\phi},$$

$$\frac{\cos\alpha}{\sin\theta}+\frac{\cos\beta}{\sin\phi}=\frac{l}{h}.$$

13. Two weights, P and Q, connected by a string, rest on the convex side of a smooth vertical circle. Find the position of equilibrium, and show that the heavier weight will be higher up on the circle than the lighter.

Ans. If the radius of the circle drawn to P make an angle θ with the vertical diameter, $l=$ length of the string, and $a=$ radius of the circle, the position of equilibrium is defined by the equation

$$P \sin \theta = Q \sin \left(\frac{l}{a} - \theta\right),$$

θ being circular measure.

14. Show, by considering the equilibrium of P and Q (in the last example) as one system, that their centre of gravity lies in the vertical radius of the circle.

15. Two rods are fixed in the same vertical plane at inclinations α and β to the horizon; two rings, whose weights are P and Q, are connected by a string of given length and placed one on each rod; find the position of equilibrium.

Ans. If P is placed on the rod of inclination α, the inclination, θ, of the string to the vertical is given by the equation

$$(P+Q) \cot \theta = P \cot \beta - Q \cot \alpha.$$

16. Two heavy rings, P and Q, connected directly by a string of given length, rest on a smooth circular wire fixed in a vertical plane; find the position of equilibrium.

Ans. If 2α is the angle subtended at the centre of the circle by the string, the inclination, θ, of the string to the vertical is given by the equation

$$(P+Q) \cot \theta = (P-Q) \tan \alpha.$$

17. Two heavy rings, P and Q, connected directly by an elastic string whose tension is proportional to its length*, rest on a smooth circular wire fixed in a vertical plane; find the position of equilibrium.

Ans. If C is the magnitude of the tension of the string when the string is stretched to the length of the radius of the wire, construct a triangle whose base and two sides are respectively proportional to $\frac{PQ}{C}$, P, Q. Then the base angles of this triangle are those made with the vertical by the radii of the wire drawn to the rings.

18. Two weights rest on the convex side of a parabola whose axis is vertical, and are connected by a string which passes over a smooth peg at the focus; show that equilibrium is impossible unless the weights are equal.

19. Two weights, P and Q (fig. 48), rest on the concave side of a parabola whose axis is horizontal, and are connected by a string which passes over a smooth peg at the focus F. Find the position of equilibrium.

Ans. Let l = length of the string; θ = the angle which FP makes with the axis; $4m$ = the latus rectum of the parabola; then

$$\cot \frac{\theta}{2} = \frac{P}{\sqrt{P^2+Q^2}} \sqrt{\frac{l-2m}{m}}.$$

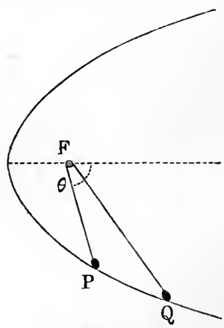


Fig. 48.

* The student will afterwards see that this would be the case if the natural length of the string were so small as to be negligible in the problem.

20. A particle is placed on the convex side of a smooth ellipse, and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the centre. Find the position of equilibrium.

Ans. If $r =$ the distance of the particle from the centre of the curve; $b =$ semi-axis minor; and $n = \frac{F-F'}{F''}$; then

$$r = \frac{b}{\sqrt{1-n^2}}.$$

21. A heavy particle, P, is placed on the concave side of a smooth vertical circle whose lowest point is A and highest point B. If the particle is acted upon by two forces, in the directions AP and BP, equal to μBP, and μAP, respectively, find the position of equilibrium.

Ans. Let $W =$ the weight of the particle; $\theta =$ the angle made with the vertical by the radius to P; $a =$ the radius of the circle; then

$$\tan\theta = \frac{2\mu a}{W}.$$

22. A particle, P, is acted upon by two forces towards two fixed points, S and H, these forces being $\frac{\mu}{SP}$ and $\frac{\mu}{HP}$, respectively; prove that P will rest at all points inside a smooth tube in the form of a curve whose equation is $SP.PH = k^2$, k being a constant.

23. A particle, P, is placed inside a smooth circular tube, and acted upon by two forces towards the extremities, A and B, of a fixed diameter, AB; the forces are respectively proportional to PA and PB: prove that the particle will rest in all positions.

24. Two weights, P and Q, connected by a string rest on the convex side of a smooth cycloid. Find the position of equilibrium.

Ans. If $l =$ the length of the string, and $a =$ radius of generating circle, the position of equilibrium is defined by the equation

$$\sin\frac{\theta}{2} = \frac{Q}{P+Q}\cdot\frac{l}{4a},$$

where θ is the angle between the vertical and the radius to the point on the generating circle which corresponds to P.

25. Two weights, P and Q, rest on the convex side of a smooth vertical circle, and are connected by a string which passes over a smooth peg vertically over the centre of the circle; find the position of equilibrium.

Ans. Let $h =$ the distance between the peg, B, and the centre of the circle; θ and $\phi =$ the angles made with the vertical by the radii to P and Q, respectively; α and $\beta =$ the angles made with the tangents to the circle at P and Q by the portions PB and QB of the string; $l =$ length of the string; then

$$P\frac{\sin\theta}{\cos\alpha} = Q\frac{\sin\phi}{\cos\beta},$$

$$h\left(\frac{\sin\theta}{\cos\alpha} + \frac{\sin\phi}{\cos\beta}\right) = l,$$

$$h\cos(\theta+\alpha) = a\cos\alpha,$$

$$h\cos(\phi+\beta) = a\cos\beta.$$

Section II.

Rough Curves.

36.] **Friction.** The curves and surfaces which we have hitherto considered were supposed to be incapable of offering resistance in any other than a normal direction. Such curves and surfaces, however, exist only in the abstractions of Rational Statics, and are not to be found in nature. Every surface is capable of destroying a certain amount of force in its tangent plane; or, in other words, every surface in nature possesses a certain degree of roughness, in virtue of which it resists the sliding of other surfaces upon it.

Now, there are two ways in which a surface may resist a sliding motion. Firstly, it may possess small inequalities which act as *fixed obstacles* to sliding; and, secondly, there may exist an *adhesion* between its substance and that of another body in contact with it. In virtue of inequalities, the two surfaces get interlocked, and an effort to cause one to slide on the other causes a strain in each of the surfaces, the force which resists this sliding being called *Friction*. Rankine (*Applied Mechanics*, p. 209) distinguishes adhesion from friction on the ground that adhesion between two surfaces is independent of the force by which they are pressed together, and is analogous to *shearing stress*, i.e., to the force (called cohesion) which resists an attempt to divide a solid by causing one part of it to slide on another.

At the same time he holds (*Mechanical Text-Book*, p. 153) that friction is a kind of shearing stress, and this view gives probably the most real and vivid conception of its nature.

37.] **Laws of Friction.** Experiments made by Coulomb and Morin have established the following laws of friction :—

1°. The tangential force necessary to establish the *beginning* of a sliding motion is a constant fraction of the normal pressure between the two surfaces in contact.

2°. With a given normal pressure, the tangential force necessary to establish the beginning of a sliding motion is independent of the extent of the surface of contact.

Subsequent experiments have, however, considerably modified the first of these laws, and shown that it can be regarded only as an approximation to the truth. If R be the normal pressure between the bodies, F the force of friction, and μ the constant ratio of the latter to the former *when slipping is about to ensue*, we have

$$F = \mu R. \qquad (a)$$

The fraction μ in this equation is called *the coefficient of friction*, and if the first law were true, μ would be strictly constant for the same pair of bodies, whatever the magnitude of the normal pressure between them might be. This, however, is not the case. For great differences of normal pressure there are considerable differences in the value of μ. When the normal pressure is nearly equal to that which would crush either of the surfaces in contact, the force of friction increases more rapidly than the normal pressure. Equation (a) is nevertheless very nearly true when the differences of normal pressure are not very great, and in what follows we shall assume this to be the case.

38.] **Causes which Modify the Coefficient of Friction.** Friction being a force called into play by the mutual action of two bodies in contact, μ depends on the particular *pair* of bodies in contact, and is not a quantity pertaining to any *one* body by itself. Moreover, it varies for the same two bodies according as the state of each body varies. Thus, it is not the same for iron and dry oak, as for iron and the same piece of oak with a moistened surface. Neither, again, is it the same for two pieces of wood when their fibres are parallel as when they are perpendicular. In fact, when great accuracy is required, a special experiment should be made to ascertain the coefficient of friction between two bodies which in any case are to act upon and sustain each other. Tables of the coefficient of friction between bodies in specified states are to be found in most practical treatises on Statics.

39.] **Independence of the Extent of the Surface of Contact.** The second law of Friction may at first sight appear strange;

but a little reflection will remove objections against its truth. If the total normal pressure between two bodies be R, and the surface of contact S, the pressure per unit of area (which is called the *intensity of pressure*) is $\frac{R}{S}\cdot$ If now, while the normal pressure remains the same as before, the surface of contact is doubled, the pressure per unit of area is only $\frac{R}{2S}$, which is just half as great as before. Hence, though the area over which friction acts is doubled, the intensity of pressure is halved; and it is consistent with common sense that the friction per unit of area should be halved also. Thus, on the whole, the same total tangential force is required to set up sliding in both cases.

40.] **Actual Magnitudes of Coefficients of Friction.** It is well that the student should have some idea of the actual magnitudes of coefficients of friction between bodies. For this purpose he should look at a table of these coefficients. Practically there is no observed coefficient much greater than 1. In Rankine's table the coefficient for damp clay on damp clay is given as 1, and that for shingle on gravel is at the most 1.11. Most of the ordinary coefficients are less than $\frac{1}{2}$.

41.] **Other Coefficients of Friction.** It is found by experiment that the friction which resists the *beginning* of sliding is greater than that which resists its *continuance*. Again, the resistance which is opposed to the rolling of one surface on another is distinguished by the special name of *Rolling Friction*, but it would more properly be called *Resistance to Rolling*. At present we shall limit ourselves to the consideration of the friction of the beginning of motion which is expressed by the equation

$$F = \mu R.$$

42.] **Reaction of a Rough Curve or Surface.** Let AB (fig. 49) be a rough curve or surface; P the position of a particle on it; and suppose the forces acting on P to be confined to the plane of the paper. Let R_1 = the normal resistance of the surface, acting in the normal, PN, and F = the force of friction, acting along the tangent, PT.

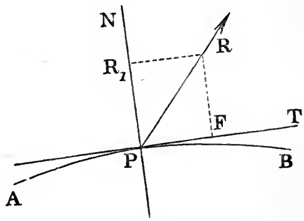


Fig. 49.

The resultant of R_1 and F is a force which we shall call the *Total Resistance* of the surface. It is represented in magnitude and direction by the line $PR = R$, which is the diagonal of the parallelogram determined by R_1 and F. We have seen that the total resistance of a smooth surface is normal; but this limitation does not apply to a rough surface. The angle, ϕ, between R and the normal is given by the equation

$$\tan\phi = \frac{F}{R_1}.$$

Hence, ϕ will be a maximum when the force of friction bears the greatest ratio to the normal pressure. But this greatest ratio is what we have called the coefficient of friction, μ; and this ratio is attained when the particle is just on the point of slipping along the surface. Therefore *the greatest angle by which the Total Resistance of a rough curve or surface can deviate from the normal is the angle whose tangent is the coefficient of friction for the bodies in contact; and this deviation is attained when slipping is about to commence.*

43.] **Angle of Friction.** The angle between the normal and the total resistance of a rough surface *when slipping is about to take place* is called the *Angle of Friction**. We shall throughout denote it by λ; and if μ is the coefficient of friction,

$$\tan\lambda = \mu.$$

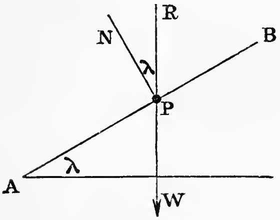


Fig 50.

44.] **Experimental Determination of μ.** Let P be the position of a heavy particle, whose weight is W, on a rough plane, AB, whose inclination is gradually increased until P is on the point of slipping down. Consider the equilibrium of P in these circumstances. It is acted upon by *two* forces, namely, its weight, W, and the total resistance, R, of the plane. For equilibrium these forces must be equal and act in opposite senses. Hence R acts in a vertical line; and since slipping is about to take place, the angle between R and the normal, PN, to the plane must (Art. 42) be equal to λ, the angle of friction. But the angle between the

* Sometimes called the *Angle of Repose.*

vertical and PN is also equal to the inclination of the plane to the horizon. Hence *the inclination of a rough plane on which a particle, acted upon solely by its own weight, is just about to slip, is the Angle of Friction.*

This result might have been proved by the resolution of forces. Thus, if R_1 be the normal pressure, the force of friction acting up the plane is μR_1, since slipping is about to begin. Hence, resolving forces horizontally for the equilibrium of P,

$$R_1 \sin i - \mu R_1 \cos i = 0,$$

i being the inclination; or $\tan i = \mu$, therefore $i = \lambda$.

Morin determined the coefficient of friction between two substances by placing one on a fixed horizontal plane made of the other, and then measuring the least horizontal force which should be applied to the body resting on the plane to cause it to slide. The ratio of this force to the weight of the body is the required coefficient of friction.

45.] **Limitation of the Total Resistance.** As in the case of the resistance of a smooth curve or surface, there is no limit to the *magnitude* of the total resistance of a rough curve or surface —for the surfaces with which we are at present concerned are supposed to be capable of resisting penetration to any extent— the only limitation to which the total resistance is subject being one of *direction*, and this limitation is thus expressed :—

The Total Resistance of a rough curve or surface, though unrestricted in magnitude, can never make with the normal an angle greater than the angle of friction corresponding to the two bodies in contact.

Within this limit, the total resistance can assume any magnitude and direction, so that we at once deduce the following important principle :—

If the Total Resistance can maintain equilibrium, it will do so.

Thus, let P (fig. 51) be a heavy particle placed upon a rough plane whose inclination is less than λ, the angle of friction. Then it is clear that, to keep P at rest, the total resistance, R, has only to be equal and opposite to W, the weight of P.

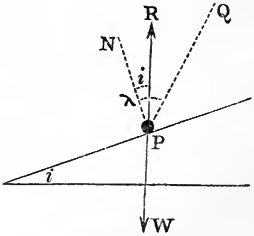


Fig. 51.

But drawing PQ, making the angle of friction, λ, with the normal, PN, we see that the direction of R falls within the prescribed limit; and therefore the equilibrium will subsist, no matter how great W may be, for there is no limit as to the *magnitude* of R.

46.] **Limiting Equilibrium.** A particle acted upon by any forces and placed upon a rough surface is said to be in *limiting equilibrium* when it is in such a position that the total resistance of the surface makes the angle of friction with the normal. In such a position if any slight change should occur in the circumstances of the particle, in virtue of which the total resistance would be compelled to make a greater angle with the normal, equilibrium could subsist no longer; for the total resistance can never be inclined to the normal at an angle greater than the angle of friction. Or we may put the matter thus. In every case the equilibrium of a particle restricted to a rough curve or surface is broken only by some circumstance which compels the total resistance to make with the normal an angle greater than the angle of friction. The manner in which this is supposed to happen depends on the particular problem. For example, let us enquire into the circumstances of the equilibrium of a heavy particle, whose weight is W, on a rough curve, AB (fig. 52), whose plane is vertical, the particle being acted upon by a horizontal force, F.

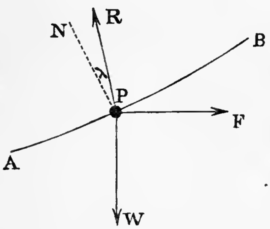


Fig. 52.

The problem proposed for solution may be any one of the three following:—

(*a*) Determine the least horizontal force that will sustain a particle, of weight W, at a given point, P, of a given rough curve, AB.

(*b*) Determine the point at which a particle, of weight W, will be just sustained by a given horizontal force, F, on a given rough curve, AB.

(*c*) Determine the least coefficient of friction that will allow a particle, of weight W, to rest at a given point, P, of a curve, AB, the particle being acted on by a given horizontal force, F.

If PN be the normal at P, and PR be drawn making the angle of friction, λ, with it, PR will be the direction of the total resistance, since, by supposition, the particle is about to slip down. All three problems are solved by the equation

$$\frac{W}{F} = \cot(\theta - \lambda),$$

θ being the inclination of the tangent at P to the horizon. But the manner in which equilibrium is supposed to be broken is not the same in each of them. If, in the first case, $F < W \tan(\theta - \lambda)$, in the second, $\theta > \lambda + \tan^{-1}\left(\frac{F}{W}\right)$, and in the third, $\lambda < \theta - \tan^{-1}\left(\frac{F}{W}\right)$, the particle will not rest at P. Thus the equilibrium may be broken by—

(*a*) a slight change in some of the acting forces;

(*b*) a slight change in the position of the particle; or,

(*c*) a slight change in the nature of the supporting surface, i.e., a diminution of its roughness.

If the particle is in limiting equilibrium (i.e., if the total resistance makes the angle of friction with the normal to the supporting surface) it is evident that equilibrium will always be broken if the third of these changes occurs; but it may not be broken by either of the others. Take, for example, a heavy particle placed on an inclined plane whose inclination to the horizon is the angle of friction. It is evident that any change may be made, either in its weight or in its position on the plane, and equilibrium will still subsist; for in neither case is the total resistance (equal and opposite to W) compelled to make with the normal an angle $> \lambda$.

In every case of equilibrium it is to be observed that the *Force of Friction* (Art. 37) *acts in the sense opposite to that in which motion would ensue if the bodies in contact became gradually smoother.*

47.] **Friction in non-limiting equilibrium.** The beginner is very prone to assume that, if μ is the coefficient of friction between two bodies, *in every case* in which one of these bodies rests against the other the force of friction is μR, where R is the normal pressure between them. That this is not so he will easily see by considering the case in which a heavy piece of metal rests on a horizontal plane of wood the coefficient of friction between the metal and the wood being, say, $\frac{2}{3}$, and no forces, other than its weight and the resistance of the plane, acting on the body. So far from the force of friction being $\frac{2}{3}$ of the normal pressure, the force of friction is zero, and will come into existence only when some horizontal force is applied to the body. The force of friction will always be equal to this horizontal force and will attain the value $\frac{3}{4}R$ only when slipping is about to take place.

The changes both in magnitude and in direction which the Total Resistance between two rough surfaces in contact undergoes while equilibrium changes from a state bordering on motion in one direction to a state bordering on motion in the opposite direction may be very simply illustrated by solving the following problem:—

A heavy body of weight W is held on a rough inclined plane of inclination i by a horizontal force P; the force P being varied gradually from the value required just to sustain the body to the value required just to drag it up the plane, it is required to represent graphically the different magnitudes and directions of the Total Resistance corresponding to the successive values of P.

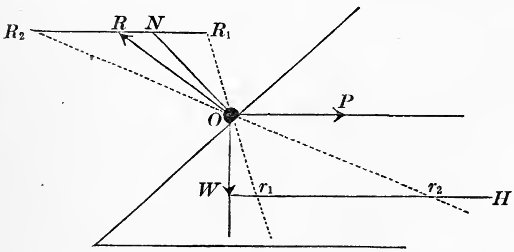


Fig. 53.

Let O (fig. 53) be the position of the body, and measure off a vertical line OW to represent the magnitude of W.

Then, for different values of P, the resultant of W and P will be represented by lines drawn from O and terminating on the horizontal line WH. The Total Resistance of the plane on the body is, of course, equal and opposite to the resultant of P and W, and it will therefore be represented by a line drawn from O to a horizontal line, R_1R_2, drawn at the same distance above O as the line WH is below it.

Let ON be the normal to the plane at O, and draw the lines OR_1 and OR_2 making the angle, λ, of friction with the normal at opposite sides of it. Let these lines be produced to meet the line WH in the points r_1 and r_2.

Then for equilibrium the resultant of P and W must be represented by some line intermediate between Or_1 and Or_2.

When the resultant of P and W is Or_1, the Total Resistance of the plane is OR_1, and since this makes the angle of friction with the normal, the body is on the point of slipping down. When the resultant of P and W is Or_2, the Total Resistance is OR_2, and the body is on the point of slipping up.

The values of P which will just sustain the body and just drag it up are, respectively,

$$W \tan(i-\lambda) \text{ and } W \tan(i+\lambda),$$

as appears at once from the figure or by calculation.

If P has a value between these limits, the Total Resistance, OR, will be intermediate between OR_1 and OR_2, and the equilibrium will not be limiting, i.e., the body will not be on the point of slipping either up or down; and the force of friction, which is the component of R along the plane, will not be μ times the normal pressure, except in the two states bordering on motion.

If P has the value $W \tan i$, which is intermediate between its extreme values, the Total Resistance will be normal to the plane, and in this state there will be *no force of friction* exerted between the plane and the body.

48.] **Passive Resistances.** The force of friction between a body and a rough surface belongs to a class of forces called *Passive Resistances*, i.e., forces which come into existence only on account of the action of other forces and which always endeavour to destroy the effect of these other forces. To this class, indeed, belongs also the normal pressure between any two bodies, and also the resistance of air or any other fluid to a body moving through it.

And it is an axiom with regard to all passive resistances that if they can preserve equilibrium they will.

Examples.

1. A heavy particle is placed on a rough plane inclined to the horizon at an angle less than the angle of friction; find the limits of the direction of the force required to drag it down.

Let PN (fig. 54) be the normal to the inclined plane, and let PQ be drawn, making the angle $NPQ = \lambda$, the angle of friction. Now, the necessary and sufficient condition that equilibrium should exist is, that the resultant of the weight, W, and the force applied, F, should fall within the angle NPQ. Hence, producing NP and QP to n and q, we see that no force applied to P within the angle nPq

will disturb the equilibrium. F must, therefore, be applied within the angle NPq, and act from P towards the left of the figure.

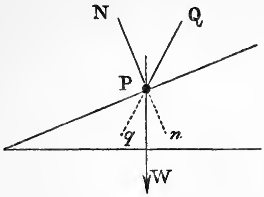


Fig. 54.

2. Two heavy particles, whose weights are P and Q, rest in limiting equilibrium on a rough double-inclined plane, and are connected by a string which passes over a smooth peg at a point, A (fig. 55), vertically over the intersection, B, of the two planes. Find the position of equilibrium.

Let the inclinations of the planes be a and β; let the length of the string be l, and $AB = h$; and let the portions of the string make angles θ and ϕ with the planes.

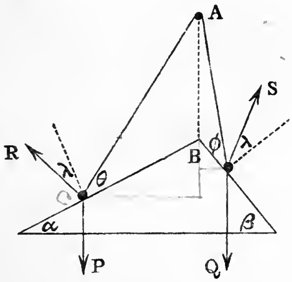


Fig. 55.

Suppose that P is on the point of ascending, and Q of descending. Then, since the motion of each body is about to ensue, the total resistances, R and S, must each make the angle of friction with the corresponding normal; and since the weight P is about to move upwards, R must act towards the left of the normal, while, since Q is about to move downwards, S must act to the right of the corresponding normal.

If T is the tension of the string, we have for the equilibrium of P,

$$T = P\frac{\sin(a+\lambda)}{\cos(\theta-\lambda)}.$$

Again, for the equilibrium of Q,

$$T = Q\cdot\frac{\sin(\beta-\lambda)}{\cos(\phi+\lambda)}.$$

Hence, equating the values of T,

$$P\cdot\frac{\sin(a+\lambda)}{\cos(\theta-\lambda)} = Q\cdot\frac{\sin(\beta-\lambda)}{\cos(\phi+\lambda)}. \quad (1)$$

This is the only statical equation connecting the given quantities. We obtain a geometrical equation by expressing that AB and the length of the string are given. This is, evidently,

$h : AC :: \sin\theta : \cos\alpha$
$\therefore h\left(\frac{\cos\alpha}{\sin\theta} + \&c\right) = l.$

$$h\left(\frac{\cos a}{\sin\theta} + \frac{\cos\beta}{\sin\phi}\right) = l. \quad (2)$$

Equations (1) and (2) determine the values of θ and ϕ, and constitute the solution of the problem.

Other Solution. Instead of considering the total resistances, R and S, we may consider two normal resistances, R_1 and S_1, and two forces of friction, μR_1 and μS_1, acting respectively down the plane a and up the plane β. In this case, considering the equilibrium of P, and resolving forces along and perpendicular to the plane a, we have

$$\left.\begin{aligned} P\sin a+\mu R_1 &= T\cos\theta, \\ P\cos a &= R_1+T\sin\theta; \end{aligned}\right\} \qquad (A)$$

and for the equilibrium of Q,

$$\left.\begin{aligned} Q\sin\beta &= \mu S_1+T\cos\phi, \\ Q\cos\beta &= S_1+T\sin\phi. \end{aligned}\right\} \qquad (B)$$

Eliminating R_1, S_1, and T from the systems (A) and (B), we arrive at the same statical equation as before.

The method of considering total resistances instead of their normal and tangential components is almost always more simple than the separate consideration of the latter forces.

3. If in the last question P is given, what are the limits of Q consistent with equilibrium?

If Q be so large that it is about to drag P up, its value, Q_1, will be given by equation (1),

$$Q_1 = P.\frac{\sin(a+\lambda)\cos(\phi+\lambda)}{\sin(\beta-\lambda)\cos(\theta-\lambda)};$$

and if Q be so small that P is about to descend, its value, Q_2, will be

$$Q_2 = P.\frac{\sin(a-\lambda)\cos(\phi-\lambda)}{\sin(\beta+\lambda)\cos(\theta+\lambda)},$$ by changing signs

the angles θ and ϕ being connected by equation (2).

4. A heavy ring is placed on a rough vertical circle; find the limits of its position consistent with equilibrium.

Ans. Draw two diameters making the angle of friction with the vertical diameter. The ring will rest anywhere on the circumference between the two upper extremities, or between the two lower extremities, of these diameters.

5. A heavy body whose weight is 20 kilogrammes is just sustained on a rough inclined plane by a horizontal force of 2 kilogrammes, and a force of 10 kilogrammes along the plane; the coefficient of friction is $\frac{2}{5}$; find the inclination of the plane. *Ans.* $2\tan^{-1}\left(\frac{25}{48}\right)$.

6. A heavy particle is placed on a rough plane whose inclination to the horizon is $\sin^{-1}\left(\frac{3}{5}\right)$, and is connected by a string passing over a smooth pulley with a particle of equal weight, which hangs freely. Supposing that motion is on the point of ensuing up the plane, find the inclination of the string to the plane, the coefficient of friction being $\frac{1}{2}$.

Ans. By resolving forces along the inclined plane, we have, if $\theta =$ inclination of the string to the plane,

$$\frac{1}{2}\sin\theta = 1-\cos\theta, \quad \text{or} \quad \frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin^2\frac{\theta}{2},$$

one solution of which is $\theta = 0$, and the other is $\tan \frac{\theta}{2} = \frac{1}{2}$.

7. In the second solution of the last question, exhibit the position of the string, and explain the result.

8. A heavy particle acted upon by a force equal in magnitude to its weight is just about to ascend a rough inclined plane under the influence of this force; find the inclination of the force to the inclined plane.

Ans. If θ is the required inclination, λ = angle of friction, and i = inclination of the plane,

$$\theta = \frac{\pi}{2} - i, \quad \text{and} \quad \theta = 2\lambda + i - \frac{\pi}{2}$$

are possible solutions. (θ is here supposed to be measured from the *upper* side of the inclined plane. If $\frac{\pi}{2} > 2\lambda + i$, the applied force will act towards the *under* side).

9. In the first solution of the last question, what is the magnitude of the pressure on the plane?

Ans. Zero. Explain this.

10. Prove that the horizontal force which will just sustain a heavy particle on a rough inclined plane will sustain the particle on the same plane supposed smooth, if the inclination is diminished by the angle of friction.

11. What is the least coefficient of friction that will allow of a heavy body's being just kept from sliding down an inclined plane of given inclination, the body (whose weight is W) being *sustained* by a given horizontal force, P?

Ans. $\dfrac{W \tan i - P}{W + P \tan i}$.

Explain *à priori*, why we get a negative value for the coefficient of friction unless $W \tan i > P$.

12. It is observed that a body whose weight is known to be W can be just sustained on a rough inclined plane by a horizontal force P, and that it can also be just sustained on the same plane by a force Q up the plane; express the angle of friction in terms of these known forces.

Ans. Angle of friction $= \cos^{-1} \dfrac{PW}{Q\sqrt{P^2 + W^2}}$.

13. It is observed that a force, Q_1, acting up a rough inclined plane will just sustain on it a body of weight W, and that a force, Q_2, acting up the plane will just drag the same body up; find the angle of friction.

Ans. Angle of friction $= \sin^{-1} \dfrac{Q_2 - Q_1}{2\sqrt{W^2 - Q_1 Q_2}}$.

14. A body is held on a rough inclined plane ($i > \lambda$) by a force which acts up the plane; this force being varied gradually from the value required just to sustain the body to the value just required to drag it up, it is required to represent graphically the different magnitudes and directions of the Total Resistance.

15. In example 8, p. 47, if the rings A and C are equally rough, find the condition that there may be a limiting equilibrium in which each is about to slip down.

Ans. If λ is the angle of friction, the required condition is

$$(P'+\frac{W}{2})\tan(i'-\lambda) = (P+\frac{W}{2})\tan(i-\lambda).$$

In this case the lines Om and Oq must be drawn making angles $i'-\lambda$, and $i-\lambda$, respectively, with the line mq.

16. In the same example, if one of the rings, C, is in a position of limiting equilibrium, find the direction of the string, the position of the other ring, A, and the direction of the total resistance at it.

Ans. The position of the string is determined by the equation

$$\frac{W}{2}\cdot\tan\theta = (\frac{W}{2}+P')\tan(i'\pm\lambda),$$

the $+$ or $-$ sign being used according as C is about to slip up or down. When θ is known, the position of A is known; and the direction of the total resistance at A is found from the equation

$$(\frac{W}{2}+P)\tan Oqm = (\frac{W}{2}+P')\tan(i'\pm\lambda).$$

17. A heavy body is to be dragged up a rough inclined plane: find the direction of the least force requisite.

Ans. The force must make the angle of friction with the inclined plane. This follows at once either by resolution of forces, or by drawing the force-diagram. Viewed in the latter way, the problem is this:—Given one force (the weight) in magnitude and direction, and the direction of another (the total resistance), when is the resultant a minimum? Evidently when it is at right angles to the total resistance.

N. B. This result is often expressed thus:—The best angle of traction up a rough inclined plane is the angle of friction.

18. Two weights, P and Q, connected by a string, whose weight is neglected rest on a rough vertical circle, the string being supposed not to be anywhere in contact with the circle; find the limits of the position of equilibrium.

Ans. If θ be the angle made by the radius to P with the vertical, $l =$ the length of the string, and $a =$ the radius of the circle, θ may have any value between θ_1 and θ_2, these being given by the equations

$$\tan\theta_1 = \frac{Q\sin\left(\frac{l}{a}+\lambda\right) + P\sin\lambda}{Q\cos\left(\frac{l}{a}+\lambda\right) + P\cos\lambda},$$

$$\tan\theta_2 = \frac{Q\sin\left(\frac{l}{a}-\lambda\right) - P\sin\lambda}{Q\cos\left(\frac{l}{a}-\lambda\right) + P\cos\lambda},$$

λ being the angle of friction.

19. Two heavy bodies rest, at points P and Q, on any rough curve in a vertical plane, and are connected by a string, which is nowhere in contact with the curve; show that in the *limiting* positions of equilibrium the total resistances at P and Q intersect on the circle passing through P, Q, and the point of intersection of the normals at P and Q.

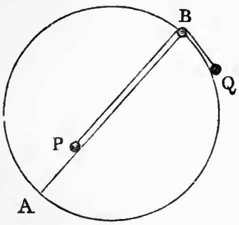


Fig. 56.

20. Two heavy particles, P and Q (fig. 56) rest, one on a rough diameter, AB, of a rough vertical circle, and the other on the convex side of the circle, the particles being connected by a string which passes over a smooth peg at the upper extremity, B, of the diameter. Find the position of equilibrium, the string being supposed to be nowhere in contact with any rough surface, and the coefficients of friction for P and Q being different.

Ans. If $a =$ the inclination of AB to the vertical, $\theta =$ inclination of the radius drawn to Q to the vertical, $\mu =$ coefficient of friction between P and AB, $\mu' =$ coefficient of friction between Q and the circle, the limiting positions of equilibrium are given by the equations

$$Q(\sin\theta_1 + \mu'\cos\theta_1) = P(\cos a - \mu\sin a),$$

$$Q(\sin\theta_2 - \mu'\cos\theta_2) = P(\cos a + \mu\sin a).$$

21. AB is the vertical diameter of a rough circular tube, of which C is the centre; P is a heavy particle placed inside the tube, and attached to three strings which, passing through a narrow slit in the inner side of the tube, pass over smooth pegs fastened at A, B, and C. Find the position of equilibrium.

Ans. If the weight of the particle $= W$, and the weights suspended over the pegs, A, C, and $B = P_1$, P_2, and P_3, respectively, the angle θ, which CP makes with the vertical when the particle is about to slip down, is given by the equation

$$W\sin(\theta+\lambda) + P_1\cos\left(\frac{\theta}{2}+\lambda\right) - P_2\sin\lambda - P_3\sin\left(\frac{\theta}{2}+\lambda\right) = 0;$$

and by changing the sign of λ in this equation we obtain the position in which P is about to slip up. Anywhere between these positions the particle will rest in non-limiting equilibrium.

22. Two heavy particles, P and Q (fig. 57), rest on two rough circular arcs which have a common vertical tangent at O; P and Q are connected by a string which passes over a smooth pulley at O; find the positions of limiting equilibrium.

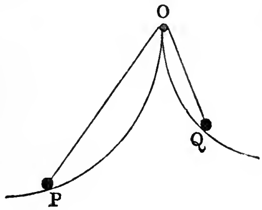


Fig. 57.

Ans. Let θ and ϕ be the angles subtended by the arcs OP and OQ at the centres of the corresponding circles, a and b the radii of the circles, λ and ϵ the angles of friction for P and Q, respectively, and l the length of the string; then, if P is about to slip down, the equations

$$P\frac{\cos(\theta+\lambda)}{\cos(\frac{\theta}{2}+\lambda)} = Q\frac{\cos(\phi-\epsilon)}{\cos(\frac{\phi}{2}-\epsilon)},$$

and
$$a\sin\frac{\theta}{2}+b\sin\frac{\phi}{2}=\frac{l}{2},$$

determine the position of equilibrium. Changing the signs of λ and ϵ, we obtain the position in which Q is about to slip down.

23. A particle rests on a rough curve whose equation is $f(x, y)=0$, and is acted on by forces the sums of whose components along the axes of x and y are X and Y; prove that the particle will rest at all points on the curve at which

$$\frac{X\frac{df}{dx}+Y\frac{df}{dy}}{\sqrt{X^2+Y^2}\,.\sqrt{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2}} > \cos\lambda.$$

24. Two rings whose weights are P and Q are moveable on a rough rod inclined to the horizon at an angle i; these rings are connected by a string of given length which passes through and supports a smooth heavy ring W; find the greatest distance between P and Q.

Ans. If θ is the inclination of either portion of the string to the vertical, the greatest distance between the rings is obtained by giving $\tan\theta$ the greater of the values

$$\frac{W+2Q}{W}\tan(\lambda-i), \qquad \frac{W+2P}{W}\text{an}(\lambda+i),$$

Q being the upper ring.

CHAPTER IV.

THE PRINCIPLE OF VIRTUAL WORK.

SECTION I.

A Single Particle.

49.] **Orthogonal Projection.** Let Ox and AB (fig. 58) be any two right lines inclined at an angle θ. If from the extremities, A and B, of the right line AB, two perpendiculars, Aa and Bb, be let fall on Ox, the line ab is called the *orthogonal projection* of AB on Ox. If the lines Aa and Bb had been each drawn parallel to a given line, which is not perpendicular to Ox, ab would be an *oblique* projection of AB.

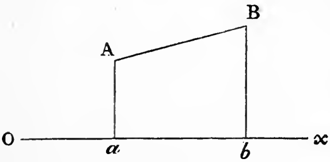


Fig. 58.

In the case of orthogonal projection it is evident that $ab = AB\cos\theta$.

50.] **Projection of a Broken Line.** Let $ABCD$ (fig. 59) be a zig-zag or broken line. Then it is evident that the projection (orthogonal or oblique) of the line AD, joining the first and last

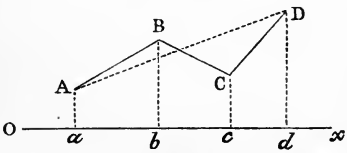


Fig. 59.

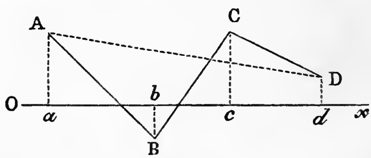


Fig. 60.

points, A and D, is equal to the sum of the projections of the separate lines, AB, BC, and CD, on any line Ox.

This is also true when the line Ox, on which the projection takes place, cuts any or all of the lines AB, BC,... between the vertices, $A, B, C, \ldots$, of the polygon formed by them, as in fig. 60.

If the sides of a closed polygon taken in order be marked with arrows pointing from each vertex to the next one, and if their projections be marked with arrows flying in the same directions, then, lines measured from left to right being considered positive, and lines from right to left negative, we may evidently state this result as follows :—

The sum of the projections of the sides of a closed polygon on any right line, allowance being made for positive and negative projections, is zero.

51.] **Virtual* Displacement. Virtual Work.** If a point at O (fig. 61) be conceived as displaced to A, OA may be called the *virtual displacement* of the point.

Let OP be the direction of a force, P, and let AN be drawn perpendicular to it; then ON is the projection of the virtual displacement along OP, and the product of the force, P, by the projection, ON, of the virtual displacement is called the *virtual work* of the force. We shall therefore say that—

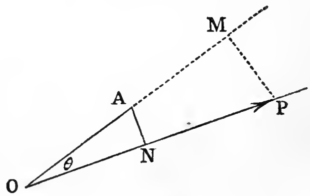


Fig. 61.

The Virtual Work *of a force is the product of the force and the projection along its direction of the Virtual Displacement of its point of application.*

If θ be the angle between the force and the virtual displacement,

$$\text{The Virtual Work} = P.\,ON = P.\,OA\cos\theta = P\cos\theta\,.\,OA.$$

Now $P\cos\theta$ is the projection of the force along the direction of displacement, and is equal to OM, if PM is perpendicular to OA. Hence we may also define the virtual work of a force as follows :—

The virtual work of a force is the product of the virtual displacement of its point of application and the projection (or component) of the force in the direction of this displacement.

This latter definition is for some purposes more convenient than the former. It is to be observed that the projection of a line, AB (fig. 58), of given length remains unaltered in magnitude when AB is moved parallel to itself into any position.

* Imaginary, supposed or hypothetically made

52.] **Theorem.** *The virtual work of a force is equal to the sum of the virtual works of its components, rectangular or oblique.*

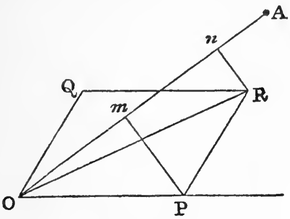


Fig. 62.

Let a force R, represented by OR (fig. 62), act at O, and let its components be P and Q, represented by OP and OQ. Let OA be the virtual displacement of O, and let its projections on R, P, and Q, be r, p, and q, respectively. Then the virtual works of these forces are $R.r$, $P.p$, $Q.q$. Draw Pm and Rn perpendicular to OA. Then On is the projection of R in the direction of the displacement, and by the end of Art. 51,

$$R.r = OA \times On.$$

Similarly $P.p = OA \times Om$, and $Q.q = OA \times mn$. Hence

$$P.p + Q.q = OA(Om + mn) = OA \times On = R.r.\text{—Q. E. D.}$$

53.] **Theorem.** *The sum of the virtual works of any number of forces acting at a point is equal to the virtual work of the resultant.*

This may be proved by taking the forces two-and-two; and using the last Theorem, or by making use of the polygon of forces (see fig. 11, p. 18). The sum of the virtual works of the forces is equal to the virtual displacement multiplied by the sum of the projections along it of the sides of the polygon parallel to the forces (Art. 51). But (Art. 50) the sum of these projections is equal to the projection of the remaining side of the polygon, and this side represents the resultant. Therefore, &c.

It follows, then, that—

When a system of forces acting at a point is in equilibrium, the sum of the virtual works of the forces = 0.

For such a system will be represented by a closed polygon, and (Art. 50) the sum of the projections of the sides of the polygon along any right line = 0.

54.] **Convention of Signs.** If the virtual displacement, OA (fig. 63), project on the line of the force P in the sense *opposite* to that in which P acts, the projection ON is to be considered negative, and the virtual work is negative. In this case P will also project on the line of displacement in the sense opposite to OA.

In fig. 64 the virtual displacement, OA, is such as to give positive projections, Or and Op, along the forces R and P, and a negative projection, Oq, along Q. And if in this case the

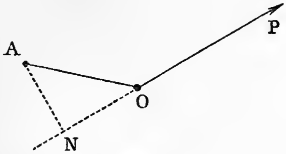


Fig. 63.

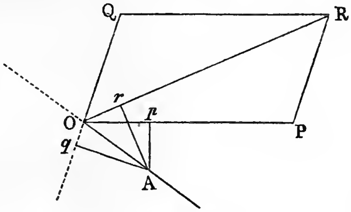


Fig. 64.

lengths of Or, Op, and Oq are denoted by r, p, and q, the equation of virtual work will be $R \,.\, r = P \,.\, p - Q \,.\, q$.

55.] **Nature of the Displacement.** It must be carefully observed that the displacement of the particle on which the forces act is both VIRTUAL and perfectly ARBITRARY. In the motion of a particle, treated of in Kinetics, the displacement is often taken to be that which the particle *actually* undergoes; but in the statical problem of the equilibrium of forces, the relation between them, expressed in an equation of virtual work, holds, whatever the displacement may be—that is, it holds whether the displacement be an actual or merely an imagined one. Since with regard to the equilibrium of forces a state of absolute rest and a state of uniform motion in a right line are not essentially different, we shall see that the most useful applications of the Principle of Work are made in the case of machines moving uniformly. The second characteristic of the displacement—namely, its *arbitrariness*—is most important, as will presently appear.

56.] **General Equation of Virtual Work.** Let several forces, P_1, P_2, ... (fig. 65), act in equilibrium on a particle, O, and let OA be any conceived, or *virtual*, displacement of O. Letting fall perpendiculars, Ap_1, Ap_2, ..., on the forces, the projections Op_2, Op_3, and Op_4, are all positive, while Op_1 and Op_5 are negative (Art. 54). Hence the equation of virtual work is

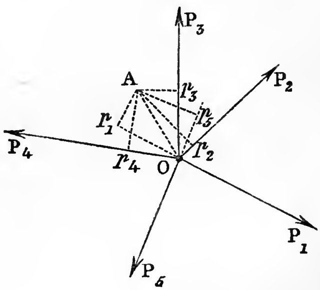


Fig. 65.

$$-P_1 \,.\, Op_1 + P_2 \,.\, Op_2 + P_3 \,.\, Op_3 + P_4 \,.\, Op_4 - P_5 \,.\, Op_5 = 0.$$

If the projections of the displacement be denoted by $p_1, p_2, \dots$, and if these quantities are supposed to carry their proper signs with them, this equation becomes, the number of forces being any whatever,

$$P_1 \cdot p_1 + P_2 \cdot p_2 + P_3 \cdot p_3 + \dots = 0, \tag{1}$$

or

$$\Sigma (P \cdot p) = 0. \tag{2}$$

57.] **General Displacement of a Particle.** The most general displacement of a single particle is a simple motion of translation from the point, O, which it occupies, to another point, A. It is true that in Molecular Dynamics, very small portions of matter are conceived as capable not only of translations but also of rotations about axes through themselves. Indeed every portion of matter, since it must possess extension in space, must be capable of both kinds of displacement; but the second kind does not belong to our present purpose.

58.] **Deduction of the Equations of Equilibrium from the Equation of Virtual Work.** Through O draw any two axes, Ox and Oy, rectangular or oblique, and let a and β be the projections of the virtual displacement, OA, along these axes. Replace the force P_1 by its components, X_1 and Y_1, along Ox and Oy. Then (Art. 52)

$$P_1 \cdot p_1 = aX_1 + \beta Y_1.$$

Similarly,

$$P_2 \cdot p_2 = aX_2 + \beta Y_2,$$

$$P_3 \cdot p_3 = aX_3 + \beta Y_3.$$

. . . .

. . . .

Hence equation (1) of Art. 56 becomes

$$a(X_1 + X_2 + X_3 + \dots) + \beta(Y_1 + Y_2 + Y_3 + \dots) = 0,$$

or

$$a\Sigma X + \beta \Sigma Y = 0. \tag{1}$$

Now a and β are perfectly independent of each other. For the displacement OA may be chosen so as to keep a constant while varying β at pleasure, or *vice versâ*. Suppose, then, that β' and a are the projections of a new virtual displacement, and we shall have

$$a\Sigma X + \beta' \Sigma Y = 0. \tag{2}$$

Subtracting (2) from (1), we have

$$(\beta - \beta') \Sigma Y = 0.$$

Now $\beta - \beta'$ is not $= 0$, therefore ΣY must $= 0$; and in the same way $\Sigma X = 0$. Hence we arrive at the equations of resolution of forces

$$\Sigma X = 0, \Sigma Y = 0,$$

which were deduced in Chap. II.*

59.] **Elementary Virtual Work.** In the general equation of virtual work, for forces acting in equilibrium on a *single* particle, namely,

$$P_1 . p_1 + P_2 . p_2 + P_3 . p_3 + \ldots = 0, \quad \text{or} \quad \Sigma P . p = 0,$$

no limitation has been placed upon the magnitude of the virtual displacement. This equation is true, independently of its magnitude; but it is generally more convenient to assume the virtual displacement to be infinitesimal, even in the case of the equilibrium of a single particle, and it is absolutely necessary to do so (as will presently be seen) in treating of the equilibrium of a connected system of particles.

If the virtual displacement is infinitesimal, its projections, $p_1, p_2, \ldots$, on the several forces acting upon the particle are all infinitesimal. We shall, therefore, denote these small projections in future by $\delta p_1, \delta p_2, \ldots$, and the equation of elementary virtual work will be

$$P_1 . \delta p_1 + P_2 . \delta p_2 + P_3 . \delta p_3 + \ldots = 0,$$

or

$$\Sigma P \delta p = 0.$$

60.] **Case in which the Virtual Work of a Force vanishes.** If a force P act at a point O, and if the virtual displacement OA is at right angles to the direction of P, it is clear that δp, the projection of OA on the direction of P, is equal to zero. Hence, *when the virtual displacement is at right angles to the direction of the force, the virtual work of the force* = 0, *and the force will not enter into the equation of virtual work.* Such a virtual displacement is always a convenient one to choose when we desire to get rid of some unknown force which acts upon a particle or a system. For example, let a particle, O, of weight W, be sustained on a smooth inclined plane by a force, P, making an angle

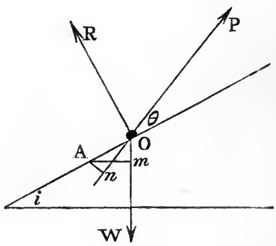


Fig. 66.

* These equations are, of course, implied in the proof of the principle of virtual work (Art. 53).

θ with the plane. If we wish to find the magnitude of P in terms of W, without bringing the unknown reaction, R, into our equation, we conceive O as receiving a virtual displacement, OA (the magnitude of which is, in the present case, unlimited), at right angles to R, that is, along the plane. Drawing Am and An perpendicular to W and P, respectively, the equation of virtual work is

$$W.Om - P.On = 0.$$

But $Om = OA.\sin i$, and $On = OA.\cos\theta$; therefore

$$W\sin i - P\cos\theta = 0.$$

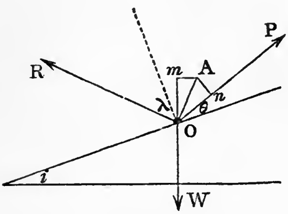


Fig. 67.

As a second example, let us suppose that the plane is rough, and that the particle is on the point of being dragged up the plane. The normal resistance will then be replaced by the total resistance, R, inclined to the normal at an angle $= \lambda$, the angle of friction. Let the virtual displacement, OA (fig. 67), now take place perpendicularly to R. Then the equation of virtual work is

$$W.Om - P.On = 0.$$

But $Om = OA.\sin(i+\lambda)$, and $On = OA.\cos(\lambda - \theta)$; therefore

$$W.\sin(i+\lambda) = P\cos(\lambda - \theta).$$

As a third example, let us find the horizontal force which is necessary to keep a heavy particle in a given position inside a smooth circular tube (fig. 68).

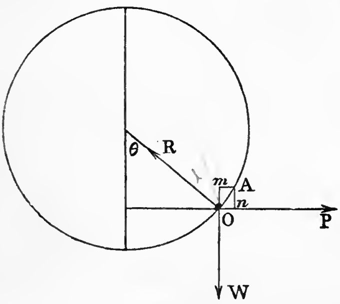


Fig. 68.

Let the virtual displacement, OA, be an indefinitely small one $= ds$, along the tube. Then since ds is infinitesimal, the projection of OA on R will be zero. Also $Om = ds.\sin\theta$, and $On = ds.\cos\theta$; therefore the equation of virtual work is

$$-Wds.\sin\theta + Pds.\cos\theta = 0,$$

or $$P = W\tan\theta.$$

If the tube is rough, and the particle in limiting equilibrium, instead of the normal reaction we must draw the total resistance, making the angle λ with the normal at the right or left hand side, according as P is the force which *just sustains* the particle, or the force which will *just drag* it up the tube, and take the virtual displacement, not along the tube, but at right angles to the total resistance. In this case we obtain

$$P = W \tan(\theta \mp \lambda).$$

61.] **Condition of Equilibrium of a Particle as determined by the Principle of Virtual Work.** It will now be sufficiently clear that—

For the equilibrium of a free particle acted on by any forces in one plane it is necessary and sufficient that the virtual work of the system of forces for every arbitrary displacement whatsoever should vanish.

First, it is *necessary* that the virtual work should vanish for every displacement. For the sum of the virtual works of the forces is equal to the virtual work of their resultant, and if this sum did not vanish, the resultant force could not vanish, and therefore the particle could not be in equilibrium.

Secondly, it is *sufficient* that this sum should vanish for every displacement. This sum is equal to the virtual work of the resultant, and if this vanishes for all possible displacements, the resultant force itself must be zero, and therefore the particle is at rest. For, if possible, let there be a resultant R, which is not zero. Then, since the virtual displacement is quite arbitrary, we may choose it so that it gives a projection $= \delta r$ (which is not $= 0$) on the direction of R. Now, since the virtual work of the system vanishes, we have $R\delta r = 0$. But since δr is not $= 0$, R must be $= 0$, and the particle is, therefore, at rest.

62.] **Normals to Curves.** The equation of virtual work furnishes a ready method of drawing normals to certain curves. For example, to draw a normal at any point, O, of an ellipse (fig. 69): let a particle be placed at O inside a smooth elliptic tube whose foci are F and F', and let it be kept in equilibrium by two forces, P and P', directed towards the foci. Let $OF = r$, $OF' = r'$. Then by the property of the ellipse,

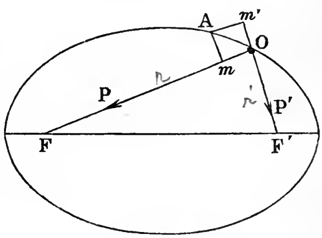


Fig. 69.

$$r + r' = \text{a constant.} = 2A$$

Hence, proceeding to a close point, A, we have

$$\delta r + \delta r' = 0. \tag{1}$$

Now the resultant of P and P' is normal to the curve, and is destroyed by the normal reaction. Drawing Am and Am' perpendicular to P and P', the equation of virtual work is

$$P.Om - P'.Om' = 0.$$

But $Om = -\delta r$, and $Om' = \delta r'$; therefore this equation becomes

$$P.\delta r + P'.\delta r' = 0. \tag{2}$$

Equation (1) gives $\delta r' = -\delta r$; therefore, substituting in (2), we have

$$P = P',$$

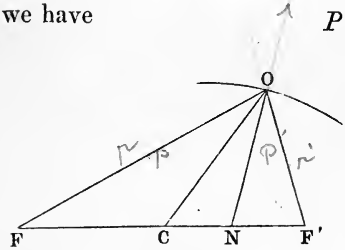


Fig. 70.

or the forces towards the foci must be equal. But the resultant of two equal forces bisects the angle between them.

Hence the normal at any point of an ellipse bisects the angle between the focal radii drawn to the point.

Again, the ovals of Cassini are given by the equation

$$rr' = k^2,$$

r and r' being the distances of a point, O, on the curve (fig. 70), from two fixed points, F and F'. If two forces, P and P', act at O towards F and F', their resultant being normal to the curve, we have for a small virtual displacement along the curve

$$P\delta r + P'\delta r' = 0. \tag{1}$$

But, differentiating the equation of the curve,

$$r'\delta r + r\delta r' = 0. \tag{2}$$

Hence from (1) and (2)

$$\frac{P}{P'} = \frac{r'}{r}.$$

Now, if C is the middle point of FF', we have

$$\frac{r'}{r} = \frac{\sin F}{\sin F'} = \frac{\sin COF}{\sin COF'}. \quad *$$

Therefore
$$\frac{P}{P'} = \frac{\sin COF}{\sin COF'}.$$

* OC : FC :: sin F : sin COF.
OC : CF'(=FC) :: sin F' : sin COF'
$\therefore \frac{\sin F}{\sin F'} = \frac{\sin COF}{\sin COF'}$

But if ON be the direction of the resultant,

$$\frac{P}{P'} = \frac{\sin NOF'}{\sin NOF}.$$ by Art. 17.

Hence $NOF' = COF$; and the normal is, therefore, constructed by joining the point O, on the curve, to the middle point of the line joining the foci, F and F', and then drawing the right line ON so that $\angle NOF' = \angle COF$. The line ON is the normal at O.

Examples.

1. If the equation of a curve is expressed in the form $\frac{r}{r'} = k$, k being a constant, and r, r' the distances of any point on the curve from two fixed points, A, B, show that the normal to the curve divides AB externally in the ratio $k^2 : 1$, and that the curve is therefore a circle.

2. Prove that the normal to the curve $\frac{1}{r^n} + \frac{1}{r'^n} = \frac{1}{a^n}$ divides AB in the ratio $\left(\frac{r}{r'}\right)^{n+2}$.

3. Give a simple construction for the normal to a Cartesian oval, whose equation is $lr + mr' = a$.

4. The equation of the magnetic curve is $\cos\omega + \cos\omega' = k$ (example 28, p. 39). If N and S are the poles, prove that the normal at a point P is constructed by measuring, on lines perpendicular to PN and PS, lengths proportional to PS^2 and PN^2, respectively, and proceeding as in last Article.

5. The equation of any curve being $f(r, r') = 0$, prove that if the normal is constructed by measuring constant lengths, Pa and Pb, from a point P on the curve, along the lines PA and PB, the curve must belong to the Cartesian ovals.

[This follows at once from the integral of the equation $\frac{df}{dr} = k\frac{df}{dr'}$; for this integral gives $f = \phi(kr + r')$; therefore all such curves give $kr + r' = \text{const.}$]

6. Show that for curves given by the equation $f(\omega, \omega') = 0$, a construction similar to that in the last example (except that the constant lengths are measured on perpendiculars to PA and PB) will hold only when the equation is

$$\tan^n \frac{\omega}{2} \tan^m \frac{\omega'}{2} = k.$$

[This follows from the integral of the equation

$$\frac{1}{r}\frac{df}{d\omega} = \frac{k}{r'}\frac{df}{d\omega'}, \quad \text{or} \quad \sin\omega\frac{df}{d\omega} = k\sin\omega'\frac{df}{d\omega'},$$

for the method of obtaining which integral see Boole's *Differential Equations*, p. 328].

7. Apply the result in the last example to construct the normal to an ellipse at any point.

[The equation of the ellipse is $\tan\frac{\omega}{2}\cdot\tan\frac{\omega'}{2} = k$.]

The general theorem* of which these are particular cases is the following:—Let the equation of any curve be expressed in the form

$$f(r_1, r_2, r_3, \ldots r_n) = 0,$$

where $r_1, r_2, r_3 \ldots r_n$, denote the distances of any point, P, (fig. 71), on the curve, from a number of fixed points, $A_1, A_2, A_3, \ldots A_n$; then, if on $PA_1, PA_2, PA_3, \ldots PA_n$, we measure off lengths $Pa_1, Pa_2, Pa_3, \ldots Pa_n$ proportional to

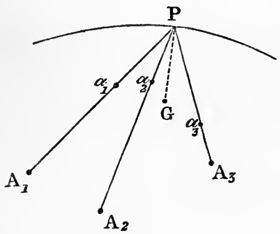


Fig. 71.

$$\frac{df}{dr_1}, \frac{df}{dr_2}, \frac{df}{dr_3}, \ldots \frac{df}{dr_n},$$

and find G, the centroid of the points $a_1, a_2, a_3, \ldots a_n$, PG will be the normal to the curve at P [f is used for shortness instead of $f(r_1, r_2, r_3, \ldots r_n)$].

The proof of this theorem is exceedingly simple from a statical point of view. Suppose a number of forces, $P_1, P_2, P_3, \ldots P_n$, to act at P along the lines $PA_1, PA_2, PA_3, \ldots PA_n$; then these forces will have a resultant normal to the curve if

$$P_1\,\delta r_1 + P_2\,\delta r_2 + P_3\,\delta r_3 \ldots + P_n\,\delta r_n = 0.$$

But $$\frac{df}{dr_1}\delta r_1 + \frac{df}{dr_2}\delta r_2 + \frac{df}{dr_3}\delta r_3 \ldots + \frac{df}{dr_n}\delta r_n = 0;$$

hence if $$P_1 : P_2 : P_3 : \ldots P_n = \frac{df}{dr_1} : \frac{df}{dr_2} : \frac{df}{dr_3} : \ldots \frac{df}{dr_n},$$

the resultant acts in the direction of the normal. The rest easily follows by Leibnitz's graphic method of representing the resultant of any number of concurrent forces (see p. 15).

This theorem may be extended to curves given by equations of the form

$$f(\omega_1, \omega_2, \omega_3, \ldots \omega_n) = 0,$$

* This theorem is, I believe, due to Tschirnhausen. The student will find another proof of this and the following theorem in Williamson's *Differential Calculus*, Art. 193, third edition.

where $\omega_1, \omega_2, \omega_3, \dots \omega_n$ are the angles which PA_1, PA_2, PA_3, $\dots PA_n$ make with a fixed line.

Let forces $Q_1, Q_2, Q_3, \dots Q_n$, act at P perpendicularly to the lines $PA_1, PA_2, PA_3, \dots PA_n$. Then the virtual work of Q_1 for a displacement along the curve is evidently $Q_1 r_1 \delta\omega_1$. Hence the forces will have a resultant normal to the curve if

$$Q_1 r_1 \delta\omega_1 + Q_2 r_2 \delta\omega_2 + Q_3 r_3 \delta\omega_3 \dots + Q_n r_n \delta\omega_n = 0.$$

But $$\frac{df}{d\omega_1}\delta\omega_1 + \frac{df}{d\omega_2}\delta\omega_2 + \frac{df}{d\omega_3}\delta\omega_3 \dots + \frac{df}{d\omega_n}\delta\omega_n = 0;$$

therefore the resultant will be normal if

$$Q_1 : Q_2 : Q_3 : \dots Q_n = \frac{1}{r_1}\frac{df}{d\omega_1} : \frac{1}{r_2}\frac{df}{d\omega_2} : \frac{1}{r_3}\frac{df}{d\omega_3} : \dots \frac{1}{r_n}\frac{df}{d\omega_n}.$$

Consequently, the rule is—measure off lengths, Pb_1, Pb_2, &c., proportional to $\frac{1}{r_1}\frac{df}{d\omega_1}$, $\frac{1}{r_2}\frac{df}{d\omega_2}$, &c., on lines drawn at P perpendicularly to PA_1, PA_2, &c., in the directions in which the angles ω_1, ω_2, &c., increase; find the centroid of the points, b_1, b_2, &c.; then the line joining this point to P is the normal to the curve.

Section II.

A System of two Particles.

63.] Projection of a Displaced Line of Constant Length. Let a line, AB (fig. 72), be a right line which is displaced into any close position, $A'B'$, its length remaining constant. Let $\delta\theta$ be the small angle between AB and $A'B'$, and let ab be the projection of $A'B'$ on its original position. Then Aa, the projection of the displacement AA', is equal to Bb, the projection of the displacement BB', if infinitesimals of a higher order than the first are neglected.

Fig. 72.

For, $ab = A'B'.\cos(\delta\theta) = A'B'\left(1 - \frac{(\delta\theta)^2}{1.2} + \dots\right)$.

Hence the difference between ab and $A'B'$ (or AB) is of the order of $(\delta\theta)^2$; and therefore, rejecting $(\delta\theta)^2$, we have

$$AB = ab,$$

$$\therefore \quad Aa = Bb.$$

The result may be thus stated:—the difference between AB and ab is infinitesimal compared with the greatest displacement in the figure.

64.] **Projection of a Displaced String of Constant Length.** Let APB be a string which passes over a peg at P, and, the length of the string remaining the same, let the extremities, A and B, be slightly displaced to A' and B'. Let Aa and Bb be the projections of the displacements AA' and BB' on the original portions of the string (fig. 73). Then $Aa = Bb$.

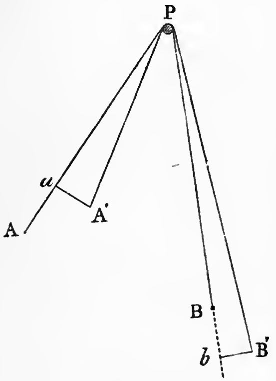


Fig. 73.

For $Pa = PA'.\cos aPA' = PA'$, as in the last Article.

Also $Pb=PB'$. Hence, since $PA' + PB' = PA+PB$, $Pa+Pb = PA+PB$, therefore $Aa = Bb$.

If in the last Article $l =$ the length of AB, and in the present, $l =$ length of the string, both results are expressed in the equation $l =$ a constant.

$$\delta l = 0.$$

65.] **Virtual Work of the Tension of an Inelastic String.** In fig. 73 suppose the peg to be smooth. Let A and B be two particles which are acted on by any forces which keep the system in equilibrium in the position indicated by the figure. Then if we consider the equilibrium of A alone, we may replace the string by a force $= T$ (the tension) acting in AP. Considering then a virtual displacement AA', the tension would furnish the term

$$T.Aa, \text{ or } -T.\delta r,$$

to the equation of virtual work, the length PA being denoted by r. Similarly, considering the equilibrium of B, the tension would furnish to its equation of virtual work, for the virtual displacement BB', the term

$$-T.Bb, \text{ or } -T.\delta r',$$

r' denoting the length of PB.

Taking the two equations together, the term contributed by the tension will be, by addition,

$$-T(\delta r+\delta r'), \text{ or } -T.\delta l,$$

which $= 0$, since the particles A and B are imagined to be simultaneously displaced in such a manner that the length of the connecting string is constant. Hence—

For any small virtual displacement in which the length of a string is unaltered, the virtual work of its tension $= 0$.

In the same way, if, in fig. 72, the rod AB, connecting two particles A and B, be subject to a tension, T, in the direction of its length, the virtual work of this tension for the displacement $A'B'$ will be

$$T.(Aa - Bb), \text{ or } T.\delta AB,$$

which $= 0$, because the length of AB is constant.

Hence—*The virtual work of the tension of a rod connecting two points whose mutual distance is unaltered in the virtual displacement is zero.*

66.] Typical Expression for the Virtual Work of a Force. Example.—We have seen (Art. 59) that if a force, P, act on a particle, O, whose virtual displacement, OA, has a projection $= \delta p$ on the line of action of *P in the sense in which P acts*, the virtual work of P is $P.\delta p$.

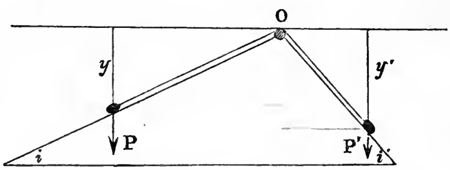


Fig. 74.

Generally, *if p denote the co-ordinate, referred to some fixed axis, of the point of application of a force, P, the virtual work of the force is $P.\delta p$, δp being supposed to be a positive increment, and the co-ordinate being measured in the sense in which P acts.*

As an example, let us determine the relation between two weights, P and P' (fig. 74), which rest on two smooth inclined planes, of inclinations i and i'. Let y and y' denote the co-ordinates of the weights, referred to a horizontal plane through O. Then the equation of virtual work for the system, the displacement being supposed to be along the planes, is

$$P.\delta y + P'.\delta y' = 0. \qquad (1)$$

[Here it will be observed that the normal reactions do not enter, because the virtual displacements take place at right angles to them (see Art. 60); and the tension does not enter, since the virtual displacement does not alter the length of the string (see Art. 65)].

To this must be added the geometrical equation connecting y and y'. If l be the length of the string, we have, clearly,

$$\frac{y}{\sin i} + \frac{y'}{\sin i'} = l.$$

Differentiating this equation, we have

$$\frac{\delta y}{\sin i} + \frac{\delta y'}{\sin i'} = 0. \tag{2}$$

Hence, from (1) and (2),

$$P \sin i = P' \sin i',$$

an equation which is, of course, otherwise evident.

If the weights are connected, as in example 12, p. 48, we have still the equation of virtual work,

$$P\delta y + Q\delta y' = 0, \tag{3}$$

y and y' denoting the vertical distances of P and Q in the figure of that example from a horizontal plane through C.

The geometrical equation connecting y and y' is, evidently,

$$\sqrt{y^2 \operatorname{cosec}^2 a + 2hy + h^2} + \sqrt{y'^2 \operatorname{cosec}^2 \beta + 2hy' + h^2} = l. \tag{4}$$

Differentiating (4), we have

$$\frac{y \operatorname{cosec}^2 a + h}{\sqrt{y^2 \operatorname{cosec}^2 a + 2hy + h^2}} \cdot \delta y + \frac{y' \operatorname{cosec}^2 \beta + h}{\sqrt{y'^2 \operatorname{cosec}^2 \beta + 2hy + h^2}} \cdot \delta y' = 0. \tag{5}$$

Hence, from (3) and (5), we obtain

$$P \cdot \frac{\sqrt{y^2 \operatorname{cosec}^2 a + 2hy + h^2}}{y \operatorname{cosec}^2 a + h} = Q \cdot \frac{\sqrt{y'^2 \operatorname{cosec}^2 \beta + 2hy' + h^2}}{y' \operatorname{cosec}^2 \beta + h}. \tag{6}$$

Equations (4) and (6) are sufficient to determine y and y', on which the position of equilibrium depends.

67.] **Geometrical Forces.** When a particle is compelled to satisfy some geometrical condition—as, for instance, to rest on a given smooth surface, or to preserve a constant distance from some other particle—this condition is equivalent to the action of a certain force on the particle. If the particle is compelled to rest under given forces on a smooth inclined plane, we have seen that this condition may be removed if we produce, by any means, a force exactly equal to the normal reaction of the plane on the particle. In the same way, the connexion of the particle with another by means of a rigid rod may be severed if we produce on the particle the force which is actually impressed upon it by the rod.

Forces proceeding from geometrical connexions are called *Geometrical Forces*, and if these forces are actually produced on the particle by other means, the conditions may be violated, and the particle considered absolutely free from constraint.

68.] **Choice of Virtual Displacements.** When two or more particles constituting a system are connected by rods or strings, and constrained to rest on given smooth curves or surfaces, there is an advantage, when seeking for the position of equilibrium, in choosing *such virtual displacements as do not violate any of these conditions;* because, as we have seen, the tensions of the connecting rods or strings, and the reactions of the smooth curves or surfaces, will, for such virtual displacements, contribute nothing to the equation of virtual work of the system. Thus we get rid at once of all such unknown forces. Of course, any geometrical condition may be violated in a virtual displacement at the expense of bringing into the equation of virtual work the corresponding geometrical force.

For example, if a particle, O (fig. 75), is placed on a smooth plane whose inclination is i, and we wish to find the horizontal force, P, which will sustain it, the best displacement to choose is one along the plane, i. e., one which does not violate the geometrical condition, because, if this is chosen, the unknown reaction, R, will not appear in the equation of virtual work. But we shall still get a valid equation if we choose a virtual displacement, OA, which does violate the condition. This equation is

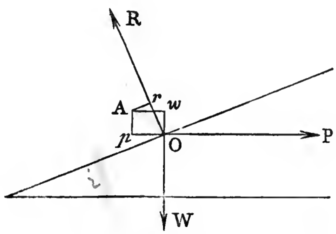


Fig. 75.

$$R \,.\, Or - P \,.\, Op - W \,.\, Ow = 0,$$

Or, Op, and Ow being the projections of OA on the directions of R, P, and W, respectively.

On the other hand, if we wish to determine R, without determining P, the best virtual displacement to choose is one at right angles to P, i.e., a vertical displacement which does violate the geometrical condition.

In the typical expression, $P\delta p$, for the virtual work of a force, the letter δ has been used to signify that the small displacement is any whatever; but it is usual in the Differential Calculus to denote small increments of the co-ordinates of a point on a curve or surface by the letter d. Hence in the following examples we shall denote small displacements on the curves considered by this letter.

Examples.

1. Two heavy particles, P and P' (fig. 76), rest on the concave side of a smooth vertical circle, and are connected by a string passing over a smooth peg, A, at the extremity of the vertical diameter. If the particles are acted upon by two horizontal forces, F and F', proportional to the distances, PQ and $P'Q'$, of the particles from the vertical diameter, find the position of equilibrium by the principle of virtual work.

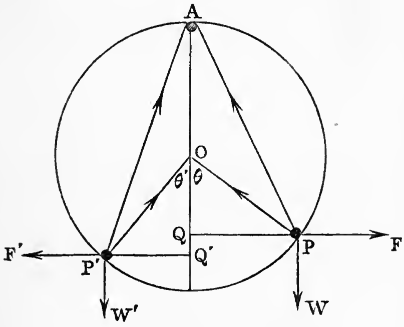


Fig. 76.

Let θ and θ' be the angles which the radii to P and P' make with the vertical; let the weights of the particles be W and W'; the radius of the circle $= a$, the length of the string $= l$, and the forces F and $F' = \mu \,.\, PQ$ and $\mu' \,.\, P'Q'$, respectively. Finally, let the distances PQ and $P'Q'$ be x and x', and let the vertical distances of P and P' below the horizontal diameter of the circle be y and y'.

Then, choosing virtual displacements of P and P' along the circle in such a manner that the length of the connecting string remains unaltered, we have

$$Wdy + W'dy' + Fdx + F'dx' = 0,$$

or

$$Wdy + W'dy' + \mu x \,.\, dx + \mu' x' \,.\, dx' = 0. \tag{1}$$

Now $y = a\cos\theta,\quad y' = a\cos\theta',\quad x = a\sin\theta,\quad x' = a\sin\theta'.$

Hence (1) becomes

$$(W - \mu a\cos\theta)\sin\theta\, d\theta + (W' - \mu' a\cos\theta')\sin\theta' d\theta' = 0. \tag{2}$$

Again, $\qquad AP = 2a\cos\frac{\theta}{2},\quad AP' = 2a\cos\frac{\theta'}{2}.$

Hence the geometrical equation is

$$\cos\frac{\theta}{2} + \cos\frac{\theta'}{2} = \frac{l}{2a}. \tag{3}$$

Differentiating this, we have

$$\sin\frac{\theta}{2}\cdot d\theta + \sin\frac{\theta'}{2}\cdot d\theta' = 0. \tag{4}$$

From (2) and (4), we have, therefore,

$$(W - \mu a\cos\theta)\cos\frac{\theta}{2} = (W' - \mu' a\cos\theta')\cos\frac{\theta'}{2}. \tag{5}$$

The solution of the problem is contained in equations (3) and (5).

2. Two heavy particles, P and P', rest on two smooth curves in a vertical plane, and are connected by an inextensible string which passes over a smooth peg, A (fig. 77), in the same plane. Prove that in the position of equilibrium, the centre of gravity of the particles is at the greatest or least height above the horizon that it can occupy consistently with the given conditions.

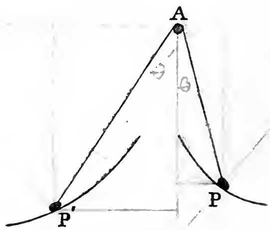


Fig. 77.

Let y and y' denote the vertical distances of P and P' from a horizontal line through A (or through any other fixed point). Then, the displacement being made consistently with the geometrical conditions, we have

$$W dy + W' dy' = 0, \tag{1}$$

W and W' being the weights of P and P'.

Now, the depth of the centre of gravity is

$$\bar{y} = \frac{Wy + W'y'}{W + W'}. \tag{2}$$

Hence, differentiating (2),

$$(W + W')\,d\bar{y} = W dy + W' dy' = 0; \tag{3}$$

and $\bar{y}$ is therefore a maximum or minimum.

If equation (3) holds in all positions of the particles, they will rest in all positions, and their centre of gravity is at a constant height.

3. If the normals at P and P' meet the vertical line through A in n and n', prove that in the position of equilibrium

$$W\frac{AP}{An} = W'\frac{AP'}{An'}.$$

4. If the particle P hang freely, find the curve on which P' will rest in all positions of the system.

Ans. A conic having A for focus.

5. If P and P' rest in all positions, and if the curve on which P' rests is given, find that on which P rests.

Ans. Let the horizontal line through A be taken as axis of x, $l =$ the length of the string, $y' = f(AP')$ be the equation of the given curve, and $Wy + W'y' = k$; then the equation of the other curve will be

$$Wy = k - W'f(l-r), \text{ or } r = \phi(y),$$

where $r = AP$.

6. A particle is attracted towards two fixed points by two constant forces: find the curve on which it will rest in all positions.

Ans. A Cartesian oval.

7. A particle is acted upon by forces emanating from a given number of fixed points and proportional, respectively, to the distances of the particle from the fixed points; find (by Virtual Work) the surface on which the particle will rest in all positions.

Ans. A sphere. [See also p. 16.]

[The student is recommended to solve some of the examples on pp. 49 and 50 by the Principle of Virtual Work.]

3rd. $T : W :: AP : An$

$T : W' :: AP' : An'$

$\therefore W \frac{AP}{An} = W' \frac{AP'}{An}$.

CHAPTER V.

COMPOSITION AND RESOLUTION OF FORCES ACTING IN ONE PLANE ON A RIGID BODY.

69.] **Resultant of Two Parallel Forces.** Let two parallel forces, P and Q (fig. 78), act at points A and B, in the same direction, on a rigid body. It is required to find the resultant of the forces P and Q.

At A and B introduce two equal and opposite forces, F. The introduction of these forces will not disturb the action of P and Q, since, the body being indeformable (see p. 12), the force F at A may be supposed to be transferred to B, at which point it would be directly opposed to the other force, F. Compound P and F at A into a single force, R, and compound Q and F at B into a single force, S. Then let R and S be supposed to act at O, the point of intersection of their lines of action. At this point let them be resolved into their components,

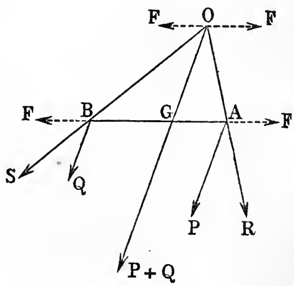


Fig. 78.

P, F, and Q, F, respectively. The forces F at O destroy each other, and the components P and Q are superposed in a right line, OG, parallel to their lines of action at A and B. The *magnitude* of the resultant is, therefore, $P+Q$. To find the point, G, in which its line of action intersects AB, let the extremities of P and R (acting at A) be joined. Then the triangle PAR is evidently similar to the triangle GOA; therefore $\frac{P}{F} = \frac{OG}{GA}$. Similarly, $\frac{Q}{F} = \frac{OG}{GB}$; therefore, by division, $\frac{P}{Q} = \frac{GB}{GA}$. Hence—

The resultant of two parallel forces acting in the same direction at the extremities of a given line divides this line internally into two segments in such a way that each segment is inversely proportional to the force acting at its extremity.

Suppose, now, that the parallel forces, P and Q, act in opposite directions. At A and B (fig. 79), let two equal and opposite forces, F, be introduced, as before; and let R, the resultant of P and F, and S, the resultant of Q and F, be transferred to O, their point of intersection. If at O the forces R and S are decomposed into their original components, it is clear that the system will reduce to a force, P, acting in the direction GO, parallel to the direction of P and Q, and a force, Q, acting in the direction OG. Hence the resultant is a force $= P - Q$ acting in the line GO. To determine the point G, we have, from the similar triangles, PAR and OGA,

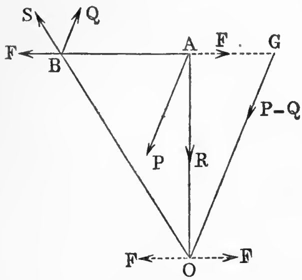


Fig. 79.

$$\frac{P}{F} = \frac{OG}{GA}; \text{ also we have } \frac{Q}{F} = \frac{OG}{GB}; \text{ therefore } \frac{P}{Q} = \frac{GB}{GA}.$$

Hence—

The resultant of two parallel forces acting in opposite directions at the extremities of a given line cuts this line externally into two segments, in such a way that each segment is inversely proportional to the force acting at its extremity.

DEF.—The *segments* of a right line, AB, made by a point G in it or its production, are the distances, GA and GB, of the point G, from the extremities A and B of the given line, whether G is on AB, or on AB produced.

In both cases we have the equation

$$P \times GA = Q \times GB.$$

Hence we have, evidently, the theorem—

If from a point on the resultant of two parallel forces a right line be drawn meeting the forces, whether perpendicularly or not, the products obtained by multiplying each force by its distance from the resultant, measured along the arbitrary line, are equal.

70.] **Composition of Parallel Forces deduced directly**

from that of Concurrent Forces. Let two forces, P and Q (fig. 80), act, in inclined directions, at two points, A and B, of a rigid body. Let O be the point in which their lines of action meet, and measure off Om and On equal to P and Q respectively. Then, completing the parallelogram $Omrn$, the diagonal, Or, represents the resultant of P and Q in magnitude and direction. Let G be the point in which Or meets AB. Then we have

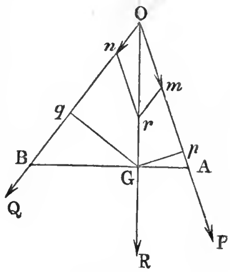


Fig. 80.

$$\frac{P}{Q} = \frac{Om}{mr} = \frac{\sin rOn}{\sin rOm}.$$

From G let fall perpendiculars, Gp and Gq, on P and Q. Then

$\sin rOn = \frac{Gq}{GO}$, and $\sin rOm = \frac{Gp}{GO}$; therefore

$$\frac{P}{Q} = \frac{Gq}{Gp}. \tag{1}$$

Again, if R is the resultant of P and Q, we have

$$\frac{R}{P} = \frac{Or}{Om} = \frac{\sin nOm}{\sin nOr}, = \frac{\sin P'OQ}{\sin nOR'}$$

or,
$$\frac{R}{P} = \frac{\text{perp. from } B \text{ on } P}{\text{perp. from } B \text{ on } R}. \tag{2}$$

Now, if P and Q are parallel, R becomes parallel to P and Q, and we shall evidently have $\frac{Gq}{Gp} = \frac{GB}{GA}$; hence (1) gives for parallel forces

$$\frac{P}{Q} = \frac{GB}{GA};$$

and (2) gives, since R is parallel to P and Q,

$$\frac{R}{P} = \frac{BA}{BG} = \frac{BG + GA}{BG} = 1 + \frac{Q}{P},$$

$$\therefore \quad R = P + Q.$$

A similar demonstration holds when P and Q act in opposite directions.

71.] **Construction for the Resultant of two Parallel Forces.** If the lines AP and BQ (figs. 81 and 82), represent in magnitudes and lines of action two parallel forces, the student will easily prove the following construction for the resultant :—

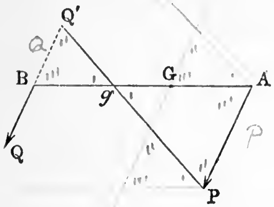


Fig. 81.

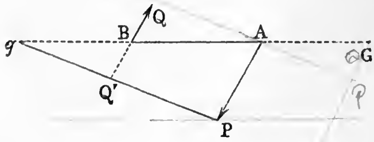


Fig. 82.

Draw BQ' equal and opposite to BQ, and draw PQ', meeting AB in g. Then measure off $AG = Bg$. G is a point on the resultant. Through G draw an indefinite right line parallel to P and Q, and from A and P draw parallels to PQ' and AB, respectively. These lines will intercept on the line through G a length $= P + Q =$ resultant.

72.] **Moment of a Force with respect to a Point.** Let a force, P (fig. 83), act on a rigid body in the plane of the paper, and let an axis perpendicular to this plane pass through the body at any point, O. It is clear, then, that the effect of the force will be to turn the body round this axis, (the axis being supposed to be fixed,) and the rotatory effect will depend on two things—firstly, the magnitude of the force, P, and, secondly, the perpendicular distance, p, of P from O. If P passes through O, it is evident that no rotation of the body round O can take place, whatever be the magnitude of P; while if P vanishes, no rotation will take place, however great p may be. Hence we may regard the product

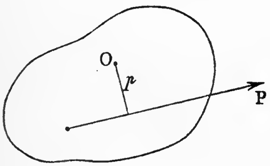


Fig. 83.

$$P \cdot p$$

as a representation of the power of the force to produce rotation about O; and to this product the special name *Moment* has, for convenience of reference, been given by writers on Statics.

When all the forces under consideration act in one plane, we may speak of the *point*, O, in which the axis of *Moments* meets this plane, instead of the axis itself. We shall therefore define the *Moment*, with respect to a point, of a force acting on a body to be *the product of the force and the perpendicular let fall on its line of action from the point.*

The unit of force being a pound and the unit of length a foot, the unit of Moment will obviously be a *foot-pound.*

73.] **Moments of Different Signs.** *If two forces tend to produce rotations of a body in opposite senses round a point, their moments with respect to this point are affected with opposite signs.* Thus (fig. 84), the force P tends to turn the body round O in a sense opposite to that of watch-hand rotation, while Q tends to turn it in the opposite sense. If, then, the former rotation is considered positive, the algebraic sum of the moments of P and Q round O is

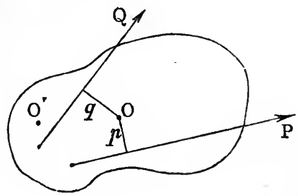


Fig. 84.

$$P \cdot p - Q \cdot q,$$

p and q being the perpendiculars from O on P and Q.

Round the point O' both forces would produce rotation in the same sense, and therefore the algebraic sum of their moments with respect to this point is

$$P \cdot p' + Q \cdot q',$$

p' and q' being the perpendiculars from O' on P and Q, respectively.

In future we shall speak simply of the *sum* of the moments, instead of the *algebraic sum* of the moments, of forces with respect to a point, as we shall suppose the moment of each force to be affected with its proper sign, in accordance with the rule given at the beginning of this Article.

74.] **Case of Two Equal and Opposite Parallel Forces.** If the forces P and Q in Art. 69, fig. 79, are equal, the equation

$$P \times GA = Q \times GB$$

gives $GA = GB$, or $\frac{GA}{GB} = 1$, an equation which is true only when G is at infinity on AB. Also the resultant of the forces, being equal to their difference, is equal to zero. Two equal and opposite parallel forces acting on a rigid body constitute what is called a *Couple.*

Theorem I. *Two equal and opposite parallel forces have a* *constant moment with respect to all points in their plane.*—Let O (fig. 85), be any point in the plane of two equal and opposite parallel forces, P, and let fall the perpendiculars Om and On on their lines of action. Then, if O is inside the lines of action of the forces, these forces tend to produce rotation round O in the same sense, and therefore the sum of their moments is equal to

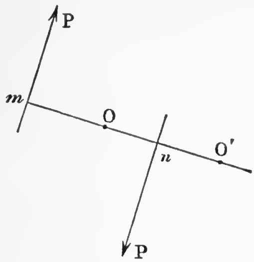


Fig. 85.

$$P\,(Om + On), \quad \text{or} \quad P \times mn.$$

If the point chosen is O', the sum of the moments is evidently

$$P\,(O'm - O'n), \quad \text{or} \quad P \times mn,$$

which is the same as before.

The perpendicular distance between the two forces of a couple is called the *Arm* of the couple.

The *Moment* of a couple is the product of the arm and one of the forces.

The *Axis* of a couple is a right line drawn anywhere perpendicular to the plane of the couple, and in a particular sense, its length being proportional to the moment of the couple. The sense of the axis is determined thus:—imagine a watch placed in the plane in which several couples act. Then let the axes of those couples which tend to produce rotation in the direction opposed to that of the rotation of the hands be drawn upwards through the face of the watch, and the axes of those which tend to produce the contrary rotation be drawn downwards.

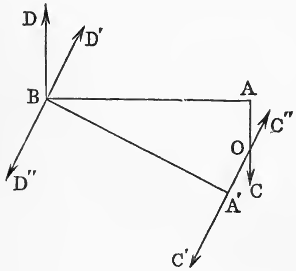


Fig. 86.

Theorem II. *The effect of a couple on a rigid body is not altered if the arm be turned through any angle round one extremity.*

Let AC and BD (fig. 86) be a couple whose arm is AB, and let the arm turn round B into the position BA'. At A' introduce two equal and opposite forces, $A'C'$ and $A'C''$, each of which is equal to one of the forces, P, of the given couple, and perpendicular to BA'. At B introduce two equal and opposite forces, BD' and BD'', perpendicular to BA', each force being equal to AC

or P. The effect of the given couple is, of course, unaltered by the introduction of these forces. Now the forces BD and BD'' may be replaced by their resultant, $2P\cos\dfrac{DBD''}{2}$, or $2P\sin\dfrac{ABA'}{2}$, which acts in the bisector, BO, of the angle DBD''; and the forces AC and $A'C''$ may be replaced by their resultant, $2P\cos\dfrac{COC''}{2}$, or $2P\sin\dfrac{ABA'}{2}$, which also acts in the line BO in a sense opposed to the previous resultant. Hence the forces BD, BD'', AC, and $A'C''$, are a null system. There remain, then, the forces BD' and $A'C'$ which form a couple whose arm is BA'. Hence the couple of forces P acting at A and B may be replaced by a couple of forces P acting at the extremities of an arm of length equal to AB having one extremity common with AB.

THEOREM III. *The effect of a couple on a rigid body is not altered if the arm is moved parallel to itself anywhere in the plane of the couple.*

Let two forces, AC and BD, each equal to P (fig. 87), act with arm AB, and draw $A'B'$ equal and parallel to AB in the plane of the couple. At A' and B' introduce, perpendicularly to $A'B'$, four forces $A'C'$, $A'C''$, $B'D'$, and $B'D''$, each equal to P. This does not alter the effect of the given couple.

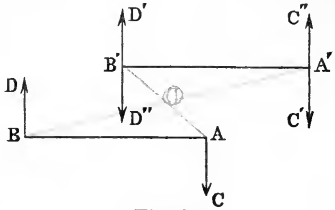


Fig. 87.

Now since AB and $A'B'$ are equal and parallel, the lines AB' and BA', being the diagonals of the parallelogram $ABB'A'$, bisect each other in the point O, suppose. Replace the forces BD and $A'C''$ by their resultant, $2P$, which acts at O parallel to BD; and replace the forces AC and $B'D''$ by their resultant, $2P$ which also acts at O in a sense opposite to the previous resultant. These two resultants destroy each other, and therefore the forces BD, AC, $B'D''$, and $A'C''$, constituting a null system, may be removed. There remain the forces, $A'C'$ and $B'D'$, which constitute a couple whose arm is $A'B'$. Therefore, &c.

THEOREM IV. *The effect of a couple on a rigid body is not altered if the couple is changed into another having the same moment, the arms of the couples being in the same line and having a common extremity.*

Let the given couple be AC and BD (fig. 88), each equal to P. Produce BA to A' so that $\frac{BA}{BA'} = \frac{Q}{P}$, and at A' and B introduce equal and opposite forces $A'C'$ and $A'C''$, BD' and BD'', the magnitude of each of these forces being Q. Now the forces AC and $A'C''$ give a resultant $= P - Q$ at B (Art. 69) in the direction BD''; and this force added to BD'' gives a force $= P$ which destroys BD. Hence there remain the forces $A'C'$ and BD', which form a couple whose moment is equal to that of AC and BD, since (by construction)

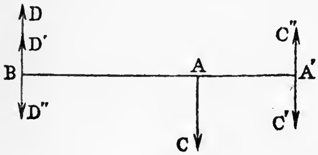


Fig. 88.

$$P \,.\, BA = Q \,.\, BA'.$$

Therefore, &c.

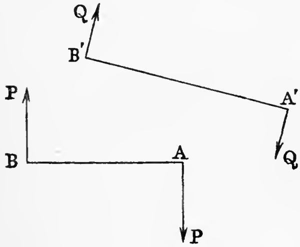


Fig. 89.

Theorem V. *A couple acting on a rigid body may be replaced by any other couple in the same plane if the moments of the couples are the same in magnitude and sign.*

Let P, P and Q, Q (fig 89), be two couples in the same plane, having the same moment, and tending to produce rotation in the same sense; then P, P may be transformed into Q, Q. For, we can first turn the arm AB round B until it is parallel to $B'A'$ (Theorem II); then we can lengthen it until it becomes equal to $B'A'$, changing, at the same time, the forces P into forces Q (Theorem IV); and finally, we can move it into the position $B'A'$ (Theorem III).

The sign of the moment of a couple is indicated by the sense in which the axis is drawn, as has been already explained (p. 90). Axes drawn upwards through the face of the watch are then considered positive, and axes drawn downwards are negative.

From the foregoing Theorems it is clear that the addition of co-planar couples is effected by adding their axes, regard being had to the signs of the axes.

Theorem VI. *A force and a couple acting in the same plane on a rigid body are equivalent to a single force.*

Let the force be F and the couple (P, a)—that is, P is the magnitude of each force in the couple whose arm is a. Then (Theorem IV) the couple $(P, a) =$ the couple $\left(F, \frac{aP}{F}\right)$. Let this latter be moved until one of its forces acts in the same line as the given force F, but in the opposite sense. The given force F will then be destroyed, and there will remain a force F acting in the same direction as the given one and at a perpendicular distance $= \frac{aP}{F}$ from it.

This Theorem is equivalent to the statement—*A force and a couple acting in the same plane cannot produce equilibrium.*

THEOREM VII. *A force acting on a rigid body at any point A may be replaced by an equal force acting in the same direction at any other point B together with a couple whose moment is the moment of the original force about B.*

This important proposition is easily demonstrated.

THEOREM VIII. *The resultant of any member of coplanar couples* num *is a couple whose moment is equal to the sum (with the proper signs) of the moments of the given couples.*

For, let the component couples have moments $L, M, N, \ldots$, and let each of them be changed into a couple, having the same right line AB (whose length is x) for arm. Then (Theorem IV), the couple L will give rise to a force $\frac{L}{x}$ at A, and an equal force in opposite sense at B. Hence at A we shall have the force $\frac{L+M+N+\ldots}{x}$ and an equal and opposite force at B. Thus we have a couple whose moment is the product of this force by the arm x; i.e., its moment is $L+M+N+\ldots$, or the sum of the given moments.

75.] **Geometrical Representation of the Moment of a Force with respect to a Point.** Let the line AB (fig. 90) represent a force in magnitude and direction, and let it be required to represent its moment with respect to a point O. If $p =$ the perpendicular from O on AB, the moment is $AB \times p$. Now this is double the area of the triangle AOB. Hence *the moment of a force with respect to a point is geometrically represented by double the area of the triangle whose base is the line representing*

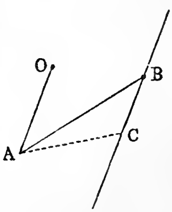


Fig. 90.

the force in magnitude and line of action, and whose vertex is the given point.

Draw AO, and from the other extremity, B, of the given force draw an indefinite right line, BC, parallel to AO. Join A to any point, C, of this line. Then the area of the triangle AOB = the area of the triangle AOC, since these triangles have the same base and are between the same parallels. Consequently the moment of a force represented by AB about O = the moment of a force represented by AC about O, wherever C be taken on the indefinite line through B.

76.] **Varignon's Theorem of Moments.** *The sum of the moments of two forces with respect to any point in their plane is equal to the moment of their resultant with respect to the point.*

Let AP and AQ (fig. 91) represent two forces whose resultant is AR, and let O be the point about which moments are taken. Draw AO, and draw PC and QD parallel to it.

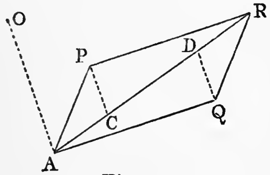


Fig. 91.

By the last Article the moment of AP about O = the moment of AC about O, and the moment of AQ = the moment of AD; therefore the sum of the moments of AP and AQ about O = the sum of the moments of AC and AD about O = the moment of the sum of AC and AD (since AC and AD are forces acting in the same line); but, by equal triangles AC is evidently = DR; therefore the sum of the moments = the moment of AR = the moment of the resultant. Q. E. D.

The student will find no difficulty in considering the case in which O is between AP and AQ, observing that in this case their moments are opposed, and that in the new figure AR will be equal to $AD \backsim AC$.

Of course it follows that the sum of the moments (with their proper signs) of any number of co-planar forces with respect to any point in their plane is equal to the moment of their resultant with respect to this point; for the forces may be replaced in pairs by their resultants, &c. It also follows that the sum of the moments of the forces about any point on the line of action of the resultant is equal to zero.

77.] **Varignon's Theorem of Moments for Parallel Forces.** *The sum of the moments of two parallel forces about* *any point is equal to the moment of their resultant about the point.*

Let the forces be P and Q (fig. 92), and let O be the point about which moments are to be taken. From O let fall perpendiculars OA, OB, and OG on the lines of action of P, Q, and their resultant, R, and let the forces be applied at the points A, B, and G, respectively.

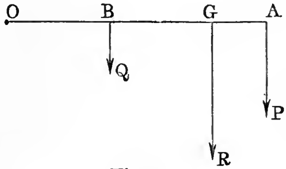


Fig. 92.

Then, moment of

$$P \text{ about } O = P . OA = P(OG + GA);$$

and moment of

$$Q \text{ about } O = Q . OB = Q(OG - GB);$$

therefore, by addition, the sum of the moments $= (P+Q) . OG + P . GA - Q . GB$. But $P . GA = Q . GB$; therefore the sum of the moments $= (P+Q) . OG = R . OG$.

A similiar proof holds when P and Q act in opposite directions, and also when O is between the lines of action of P and Q.

It follows that *the sum of the moments (with their proper signs) of any number of co-planar parallel forces with respect to a point in their plane is equal to the moment of their resultant with respect to the point.*

78.] **Centre of Parallel Forces. Theorem.** *If any number of parallel forces, P_1, P_2, P_3, ... P_n, act in one plane at points A_1, A_2, A_3, ... A_n, their resultant passes through a fixed point if all the forces are turned in the same sense round their points of application through an arbitrary but common angle.*

The point, g_1 (fig. 93), of application of the resultant of P_1 and P_2 has been determined (Art. 69) by dividing the line $A_1 A_2$ so that

$$\frac{A_1 g_1}{A_2 g_1} = \frac{P_2}{P_1},$$

on the supposition that the forces P_1 and P_2 are parallel, but no assumption has been made as to their common direction. Hence g_1 will be a point on their resultant in whatever direction they act, and the force at this point is $P_1 + P_2$. The point of application of the resultant of

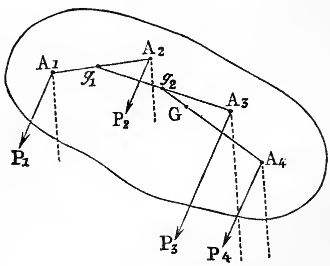


Fig. 93.

P_1, P_2, and P_3, is determined by joining g_1 to A_3, and dividing it in g_2, so that

$$\frac{g_1 g_2}{A_3 g_2} = \frac{\text{force at } A_3}{\text{force at } g_1} = \frac{P_3}{P_1+P_2},$$

and the force at g_2 is $P_1+P_2+P_3$. Similarly, the point of application of the resultant of P_1, P_2, P_3, and P_4 is a point, G, on $g_2 A_4$, such that

$$\frac{g_2 G}{A_4 G} = \frac{P_4}{P_1+P_2+P_3},$$

and the force at $G = P_1+P_2+P_3+P_4$.

We thus see that the point, G, of application of the resultant of the system is determined by dividing the lines $g_1 A_3$, $g_2 A_4$, ... in certain ratios which depend simply on the *magnitudes*, and not on the *directions*, of the forces at A_1, A_2, A_3, ... The theorem is, therefore, evident.

Of course no one point on the line of action of a force which acts on an indeformable body has a special right to be called *the* point of application of the force; nevertheless, we shall speak of the point, G, as *the* point of application of the resultant force, since, as we have seen, it is a point through which the resultant of forces equal to P_1, P_2, ... always passes, whatever be the common direction of these forces.

The theorem of this article is true also in the case in which neither the parallel forces nor their fixed points of application lie in the same plane.

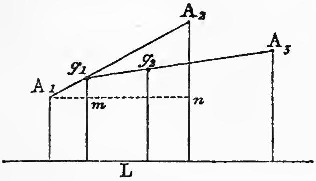


Fig. 94.

79.] **Centre of Mean Position.** Let there be any number of points, A_1, A_2, A_3, ... (fig. 94), in one plane, and let the line, $A_1 A_2$, be divided at g_1, so that

$$\frac{g_1 A_2}{g_1 A_1} = \frac{m_1}{m_2};$$

let $g_1 A_3$ be divided at g_2, so that

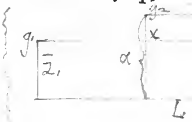
$$\frac{g_2 A_3}{g_2 g_1} = \frac{m_1+m_2}{m_3};$$

let $g_2 A_4$ be divided at g_3, so that

$$\frac{g_3 A_4}{g_3 g_2} = \frac{m_1+m_2+m_3}{m_4};$$

and so on, until by a final construction we arrive at a point, G.

It is required to express the distance of G from an arbitrary line, L, in the plane of the points in terms of the distances, z_1, z_2, z_3, ... of A_1, A_2, A_3, ... from this line*.



Draw A_1mn parallel to L. Then

$$\frac{g_1m}{A_2n}=\frac{A_1g_1}{A_1A_2}=\frac{m_2}{m_1+m_2},$$

$$\therefore\ g_1m=\frac{m_2}{m_1+m_2}\cdot A_2n=\frac{m_2}{m_1+m_2}(z_2-z_1).$$

$\frac{g_1g_2}{g_2A_3}=\frac{g_2x}{A_3y}=\frac{m_3}{m_1+\dots}$

But the distance of g_1 from L is equal to

$\therefore\ \frac{\alpha-\bar{z}_1}{z_3-\alpha}=\frac{m_3}{m_1+m_2}$

$$z_1+g_1m=z_1+\frac{m_2}{m_1+m_2}(z_2-z_1)=\frac{m_1z_1+m_2z_2}{m_1+m_2}.$$

$\therefore\ \alpha=\frac{(m_1+m_2)\bar{z}_1+m\dots}{m_1+m_2+m\dots}=\frac{m_1z_1+m_2z_2\dots}{m_1+m\dots}$

Calling this distance $\bar{z}_1$, we have the distance of g_2 from L equal to

$$\frac{(m_1+m_2)\bar{z}_1+m_3z_3}{m_1+m_2+m_3}=\frac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3},$$

since g_1A_3 is divided at g_2 in the ratio $\frac{m_3}{m_1+m_2}$. Continuing the application of this method, we have evidently

$$\bar{z}=\frac{m_1z_1+m_2z_2+m_3z_3+\dots m_nz_n}{m_1+m_2+m_3+\dots+m_n}, \qquad (1)$$

$\bar{z}$ being the distance of G from L.

This equation is generally written in the form

$$\bar{z}=\frac{\Sigma mz}{\Sigma m}, \qquad (2)$$

in which Σ denotes a summation.

The point G thus arrived at is called *The Centre of Mean Position of the given points for the system of multiples* m_1, m_2, m_3, ... m_n.

The points A_1, A_2, A_3, ... remaining the same, and the system of multiples being altered to p_1, p_2, p_3, ... the point G arrived at would, of course, be different. The distance of the new point would be

$$\frac{\Sigma pz}{\Sigma p}.$$

In particular, the distance, z, of the centre of parallel forces from any plane is given by the equation

$$\bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

---

* All this holds if the points A_1, A_2, ... are not in the same plane and L represents any plane from which their distances are measured.

Examples.

1. The centre of mean position of three points, A, B, C, for a system of equal multiples, is the intersection of the bisectors of the sides of the triangle ABC drawn from the opposite angles.

2. The centre of mean position of three points, A, B, C, for a system of multiples $\sin 2A$, $\sin 2B$, $\sin 2C$, is the centre of the circle circumscribed about the triangle ABC.

3. The sides of the triangle being a, b, c, the centre of mean position of A, B, C, for the system of multiples a, b, c, is the centre of the inscribed circle.

4. For the system of multiples $\tan A$, $\tan B$, $\tan C$, the centre of mean position is the intersection of perpendiculars.

The construction given in this Article for the Centre of Mean Position of the points $A_1, A_2, A_3, \ldots$ is of course the same when the points do not all lie in one plane. In the latter case it is easily seen that if $z_1, z_2, z_3, \ldots$ denote the distances of the points from an arbitrary *plane*, the distance, $\bar{z}$, of the centre of mean position from this plane, for the system of multiples $m_1, m_2, m_3, \ldots$, is given by the equation

$$\bar{z} = \frac{\Sigma mz}{\Sigma m}.$$

Centre of Mean Position is a generic term which comprises under it particular points which must be specially noticed. One, *the Centre of Parallel Forces*, has been already mentioned. Another is the *Centre of Mass*, called also the *Centre of Inertia*. If at the points considered, A_1, A_2, $A_3, \ldots$ there be placed material particles whose masses are respectively $m_1, m_2, m_3, \ldots$ and we find the centre of mean position of these points for the system of multiples $m_1, m_2, m_3, \ldots$ we shall arrive at the *Centre of Mass* of this system of particles. Nothing is here assumed about the closeness of the points $A_1, A_2, A_3, \ldots$, or the particles placed at them, and the process of arriving at the point G will be unaltered if these particles constitute a continuous body. Hence the *Centre of Mass* of any body is *the Centre of Mean Position of all the points within it for a system of multiples proportional to the masses of the particles placed at these points respectively.*

A body whose points do not suffer any relative changes of position will therefore continue to possess the same centre of mass no matter into what part of the universe the body may be taken. A different arrangement of its particles, would, of course, in general alter its centre of mass. The centre of mass of a rigid body is, then, something which it possesses absolutely, or apart from all contingency of position in space or relation to other bodies.

The distance of this point from any plane is given by the equation last written, in which the sign Σ is to be replaced by the integral sign $\int$, and the element of mass at a distance z from the plane denoted by dm. Thus

$$\bar{z} = \frac{\int z dm}{\int dm}.$$

Again, if at the points $A_1, A_2, A_3, \ldots$ there be placed particles whose *weights* are $w_1, w_2, w_3, \ldots$ these weights constituting a system of parallel forces, the centre of these parallel forces is called the *Centre of Gravity* of the given particles.

The effect of altering the position of the body in the most general manner possible is merely to turn the forces, $w_1, w_2, w_3, \ldots$ round their fixed points of application, $A_1, A_2, \ldots$ through the same angle, and by the last article we see that the resultant of the weights of the particles will, in all positions of the body, pass through a fixed point, G, in the body. The resultant of all the elementary weights is equal to their sum, and is called the *weight of the body*. We may, therefore, define the centre of gravity of a body thus—*The centre of gravity of a body is that unique point in it through which passes, in all possible positions of the body, the resultant of the system of parallel forces formed by the weights of the indefinitely great number of indefinitely small particles into which the body can be divided.*

The centre of gravity of a body is, then, the centre of the particular set of parallel forces which act on its various elements in virtue of the attraction of the Earth. The existence of such a point depends on the parallelism of the forces produced by the Earth on the elements of the body, and this parallelism, again, depends on the minuteness of the volume of the body in comparison with that of the Earth. If the body were carried to the surface of the Sun, or any other such large attracting mass, the individual weights of its elementary portions, and therefore its total weight, would be greater than they are at the Earth's surface, but the position of the centre of gravity in the body would remain the same. On the other hand, if the dimensions of the body were comparable with those of the attracting mass, the forces of attraction on its elementary portions would not be a parallel system, and the resultant attraction would not, in general, pass through any fixed point in the body independently of the relative positions of the two masses. The term *weight of a body* is used to signify the resultant attraction produced on the body by the Earth, or other planet, on whose surface the body exists, and it is therefore, unlike mass, a mere contingent property of the body; and the centre of gravity is *essentially* distinguished from the centre of mass; although, since weight and mass are always proportional, when the first point exists, it coincides with the second.

In considering the equilibrium of a rigid heavy body we represent its weight as a single force acting vertically through its centre of gravity.

80.] **Conditions of Equilibrium of a Rigid Body acted on by Forces in One Plane.** 1. Let the forces be parallel. Take any point, O, and draw through it a right line, Oy, parallel to the forces (fig. 95). At O introduce two forces, P_1' and P_1'', each equal to P_1, these new forces being directly opposed to each other along Oy. Now, P_1' and P_1'' form a couple whose moment is $P_1 . p_1$, if p_1 is the perpendicular from O on the line $A_1 P_1$. Introducing, in the same way, two forces, P_2' and P_2'', equal to P_2, directly opposite to each other along Oy, we have P_2 at A_2 replaced by a force P_2'' acting at O along Oy' and a couple whose moment is $-P_2 . p_2$, p_2 being the perpendicular from O on the line $A_2 P_2$. The $-$ sign is attached to this couple because the couple (P_2', P_2) tends to produce rotation in a sense opposite to that in which the couple (P_1'', P_2) tends to produce rotation.

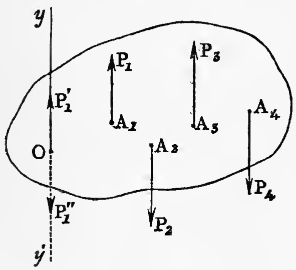


Fig. 95.

Proceeding in this way with all the forces in the above figure, we have the whole system of forces at A_1, A_2, A_3, A_4, ... equivalent to a single force,

$$P_1 - P_2 + P_3 - P_4 + \dots,$$

acting at O in the direction Oy, and a couple,

$$P_1 . p_1 - P_2 . p_2 + P_3 . p_3 - P_4 . p_4 + \dots,$$

tending to turn the body round O in a sense opposite to that of watch-hand rotation.

In general, denoting the resultant force by R, and the moment of the resultant couple by G, we have

$$R = \Sigma P, \tag{1}$$

$$G = \Sigma(P.p). \tag{2}$$

Now, by Theorem VI, of Art. 74, a couple and a force in the same plane are equivalent to a single force, and cannot, therefore, conjointly produce equilibrium. Hence, for equilibrium, the force and the couple must vanish; or

$$\Sigma P = 0, \tag{3}$$

and

$$\Sigma(P.p) = 0; \tag{4}$$

that is to say, for the equilibrium* of a system of coplanar parallel forces acting on a body—

(*a*) *The sum of forces must* = 0, and

(*b*) *The sum of the moments of the forces about every point in their plane must* = 0.

2. Let the forces act in any directions.

Take any point whatever, O, (fig. 96), in the plane of the forces. At O introduce two opposite forces, P_1' and P_1'', each equal and parallel to P_1. Let P_1 and P_1'' be considered as forming a couple. Then P_1 at A_1 is equivalent to P_1 acting at O, and a couple whose moment $= P_1 . p_1$. Replace P_2 at A_2 in the same way by P_2'' (or P_2) acting at O, and a couple (P_2, P_2') whose moment is $-P_2 . p_2$. Thus the whole system of forces will be replaced by forces, $P_1, P_2, P_3, P_4, \ldots$, acting at O, and a number of couples whose moments are $P_1 . p_1, -P_2 . p_2, P_3 . p_3, -P_4 . p_4, \ldots$ (the forces acting as in the above figure). The forces acting at O will have a single resultant, R, and the couples will form a

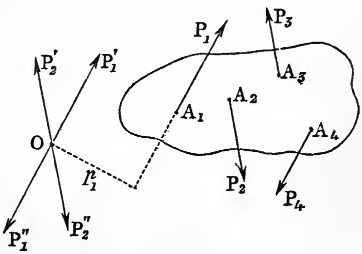


Fig. 96.

* The attention of the student is particularly directed to the remark at the end of this chapter.

single couple whose moment, G, is (Theor. VIII, Art. 74) the sum of the moments of the couples. For equilibrium it is necessary that each of these should vanish. Hence, for the equilibrium * of a body acted on by coplanar forces—

(*a*) *The resultant which the forces would have if they all acted together at a point, each in the direction in which it acts on the given body, must* $= 0$; and

(*b*) *The sum of the moments of the forces round every point in their plane must* $= 0$.

The first of these conditions asserts that there must be no *force* in any direction; and the second that there must be no *moment* round any point. Thus, the conditions of equilibrium of a rigid body embrace the condition (*a*) of the equilibrium of a particle (Art. 24, p. 21); and (*b*) a condition distinctive of the susceptibility of a body of finite extension to receive a motion of rotation.

It is to be observed, then, that a system of coplanar forces acting on a body can be reduced to a single resultant force, R, acting at any arbitrary point, O, in the plane of the forces, and a couple, G, also in this plane; and that whatever point, O, is chosen, the force R is constant in magnitude and direction, while the magnitude of the couple G varies with the point chosen. The force R is called the *Resultant of Translation*.

81.] **Analytical Conditions of Equilibrium.** Through any point, O, draw two rectangular lines, Ox and Oy, and resolve the force, P_1, acting at A_1, into two components, X_1 and Y_1, parallel to Ox and Oy. Now (Art. 76) the moment of P_1 about O is equal to the sum of the moments of X_1 and Y_1 about O. But if rotation opposite to that of a watch-hand is considered positive, the moment of Y_1 about O is $Y_1 . x_1$; and the moment of X_1 is $-Y_1 . y_1$, where x_1 and y_1 are the co-ordinates of A_1 referred to the axes Ox and Oy. Hence the moment of P_1 about O is

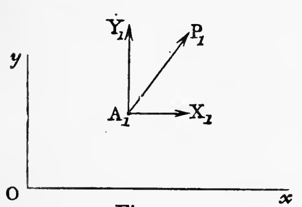


Fig. 97.

$$Y_1 x_1 - X_1 y_1.$$

* See remark at the end of this chapter.

Adding together the moments of P_1, P_2, ..., we get the total moment

$$G = \Sigma\,(Yx - Xy). \tag{1}$$

If the sum of the components of the forces along Ox is denoted by ΣX, and the sum of the components along Oy by ΣY, the resultant of the forces acting at O (fig. 96) is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2. \tag{2}$$

Now, since for equilibrium we must have $R = 0$, and $G = 0$, the conditions, analytically expressed, are

$$\Sigma X = 0,\ \Sigma Y = 0, \tag{3}$$

$$\Sigma\,(Yx - Xy) = 0. \tag{4}$$

These equations are the expressions of the conditions of Art. 80.

82.] **Equation of the Resultant.** We have seen (Art. 80), that a system of coplanar forces is equivalent to a single force, R, acting at any arbitrary origin, together with a couple, G. The direction and magnitude of the resultant force, R, will be the same whatever origin may be chosen, but the couple will vary with the origin. Now, *supposing that the resultant of the forces does not vanish*, the couple and the force R can (Theorem VI, Art. 74) be replaced by a single force equal to R; and the sum of the moments of the forces about any point on its line of action is equal to zero (Art. 76).

Let (α, β) be the co-ordinates of any point referred to rectangular axes through an arbitrary origin, O (fig. 97). Then the moment of the force, P_1, about this point, is evidently

$$Y_1(x_1 - \alpha) - X_1(y_1 - \beta), \text{ or } Y_1 x_1 - X_1 y_1 - \alpha Y_1 + \beta X_1.$$

Taking the sum of the moments of all the forces about the point, we have

$$G' = G - \alpha \Sigma Y + \beta \Sigma X, \tag{1}$$

G' being the sum of the moments about the point (α, β).

Since, for any point on the resultant $G' = 0$, the equation of its line of action is

$$\alpha \Sigma Y - \beta \Sigma X = G.$$

Equation (1) gives at once the following result—*The sum of the moments of a system of coplanar forces about any point, O, is equal to the sum of their moments about any other point, O′, plus* *the moment about O of their resultant of translation supposed acting at O′.*

83.] **Force Polygon and Funicular Polygon.** Let there be any system of forces, P_1, P_2, P_3, P_4, P_5, (fig. 98) acting in one

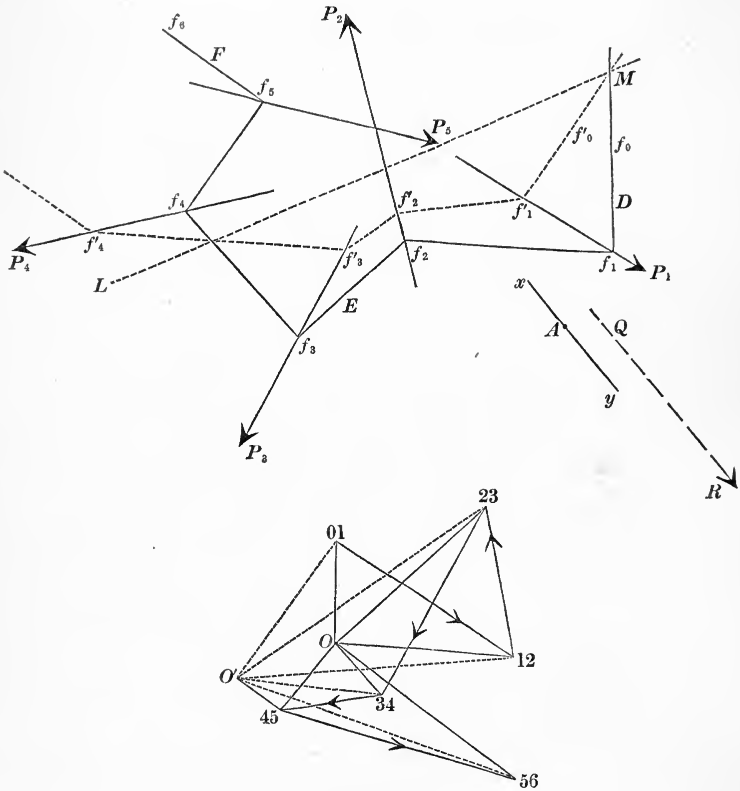


Fig. 98.

plane on a body. Starting with any point, 01, draw lines, (01, 12), (12, 23), (23, 34), (34, 45), (45, 56), parallel to the lines of action of the forces and respectively proportional to them. The figure formed by these lines, (01, 12), (12, 23), ..., is called the *Force Polygon* of the given system of forces. Now take any point, O, and from it draw lines, O 01, O 12, O 23, ..., to the vertices of the force polygon. From any point, f_1, on the line of action of P_1 draw two lines, $f_1 f_0$ and $f_1 f_2$, parallel to the lines

$O\,01$ and $O\,12$; from the point f_2 in which $f_1 f_2$ meets P_2, draw $f_2 f_3$ parallel to $O\,23$ and meeting P_3 in f_3; from f_3 draw $f_3 f_4$ parallel to $O\,34$; and so on.

The system of lines $f_0\ f_1\ f_2\ f_3\ f_4\ f_5\ f_6$ parallel to the radii drawn to the vertices of the force polygon from any point, O, is called a *Funicular Polygon* of the given system of forces.

The point O the radii from which to the vertices of the force polygon determine the funicular is called the *Pole* corresponding to the funicular.

Let any other pole, O', be chosen, and from an arbitrary point, f_1', on P_1, let $f_1' f_0'$ and $f_1' f_2'$ be drawn parallel to $O'\,01$ and $O'\,12$, respectively; and let a new funicular, $f_0' f_1' \cdots f_6'$, be constructed.

Then the sides (such as $f_2 f_3$ and $f_2' f_3'$) of these polygons which reach between the lines of action of the same two forces are called corresponding sides.

Since the point f_1 may be taken anywhere on P_1 it is clear that for a given pole, O, we may construct an infinite number of funiculars of the system, but the corresponding sides of them are of course parallel. If the force at each vertex of a funicular of the system is resolved into two components directed along the two sides of the funicular which meet at this vertex, the components at the extremities of each side of the funicular are equal and opposite. For, suppose P_3 resolved into two components in $f_3 f_2$ and $f_3 f_4$; then these components are represented by the lines $23\,O$ and $O\,34$; also if P_2 is resolved into components in $f_2 f_3$ and $f_2 f_1$, these will be represented by $O\,23$ and $12\,O$, respectively; thus the components in the side $f_2 f_3$ are equal and opposite.

84.] **Theorem.** *The corresponding sides of any two funiculars of a given system of forces intersect on a right line, which is parallel to that joining the poles of the two funiculars.*

At the points f_2 and f_2' let two equal forces (each P_2) be applied in opposite senses along the line $f_2 f_2'$; suppose them to act away from both of these points, as P_2 is represented in fig. 98. Considered as acting on a rigid body, these forces are in equilibrium. Now let P_2 at f_2 be resolved into its components along $f_2 f_1$ and $f_2 f_3$. These components will be represented in magnitudes and senses by $O\,12$ and $23\,O$, respectively. Similarly, resolve P_2 at f_2' along $f_2' f_1'$ and $f_2' f_3'$; and these components will be represented by 12 O' and O' 23. These four components are therefore in equilibrium. Take the sum of their moments about the point of intersection of the lines $f_2 f_3$ and $f_2' f_3'$. Then, since this sum is zero, it follows that the resultant of the two components (O12 and 12 O') in the lines $f_2 f_1$ and $f_1' f_2'$ must pass through the point of intersection of $f_2 f_3$ and $f_2' f_3'$; but it also passes through the point of intersection of $f_2 f_1$ and $f_2' f_1'$; therefore its line of action is the line joining these two intersections. Now this line of action is parallel to the line OO'; for, two forces represented by O12 and 12 O' give a resultant represented by OO' in magnitude and sense.

Hence the corresponding sides $f_1 f_2$ and $f_1' f_2'$, $f_2 f_3$ and $f_2' f_3'$ intersect on a line parallel to OO'; similarly the sides $f_2 f_3$ and $f_2' f_3'$, $f_3 f_4$ and $f_3' f_4'$ intersect on a line parallel to OO', which, of course, must be the same line as before. This line is LM in the figure.

85.] **Problem.** *Given one funicular of a given system of coplanar forces, to construct all funiculars of the system.*

Let the given funicular be $f_0 f_1 f_2 f_3 \dots$. Draw any line LM in the plane of the forces; produce the sides, $f_0 f_1$, $f_1 f_2, \dots$, of the given funicular to meet LM; from the point of intersection of LM and $f_0 f_1$ draw the arbitrary line $f_0' f_1'$, which meets P_1 in f_1'; join f_1' to the point of intersection of LM and $f_1 f_2$; this joining line will meet P_2 in f_2', which is the second vertex of the new funicular; join f_2' to the point of intersection of LM and $f_2 f_3$; this will give f_3'; and so on. Hence a new funicular is formed, and since the lines LM and $f_0' f_1'$ were drawn at random, an infinite number of funiculars of the system can be described in this way.

86.] **Problem.** *To construct the resultant of a given system of coplanar forces.*

On any scale construct a force polygon 01, 12, 23, ... of the given system; then the line of action of the resultant must be parallel to the side (01, 56) which closes the force polygon. Take any pole, O, and construct a funicular $f_0 f_1 f_2 \dots$, of the system. Then the resultant must pass through the point of intersection of the extreme sides, $f_0 f_1$ and $f_5 f_6$, of the funicular. For, by resolving each force into components along the two sides of the funicular which start from the vertex at which the force may be supposed to act, these components will be mutually destroyed, with the exception of those in the extreme sides, $f_0 f_1$ and $f_5 f_6$. Hence the whole system of forces is equivalent to two forces acting in these sides, and represented in magnitudes on the scale adopted by the lines $O01$ and $O56$. The line of action of the resultant therefore passes through the intersection of the extreme sides and is parallel to the line joining 01 to 56, and the magnitude is represented by the length of this joining line, its sense being of course from 01 to 56.

Cor. 1. Whatever be the path described by the pole, the point of intersection of the extreme sides of the funicular describes a fixed right line. This is the line of action of the resultant of the given system of forces.

Cor. 2. The point of intersection of any two sides of a funicular describes a fixed right line, when the pole varies in any manner. Thus the sides $f_1 f_2$ and $f_4 f_5$ will always intersect on the line of action of the resultant of the forces P_2, P_3, P_4.

87.] **Graphic Conditions of Equilibrium.** When a system of coplanar forces acting on a rigid body is in equilibrium, the forces when compounded two and two must finally reduce to two equal forces of opposite senses acting in the same right line. Since the resultant is proportional to the line required to close the force polygon, this line must be zero; hence the force polygon of the system must close up of itself. Again, since the system is finally reducible to two forces acting in the first and last sides, $f_0 f_1$ and $f_5 f_6$, of any funicular, these sides must coincide; or, in other words, the funicular must be closed.

Hence the conditions of equilibrium are—

1. The Force Polygon of the system must be closed.

2. Any Funicular Polygon of the system must be closed.

Cor. 1. If any one funicular of the system is closed, every funicular of the system is closed.

Cor. 2. If the system is equivalent to a couple, the force polygon is closed, and the first and last sides of all funiculars are parallel.

88.] **Problem.** *To represent the moment of a force about a point.*

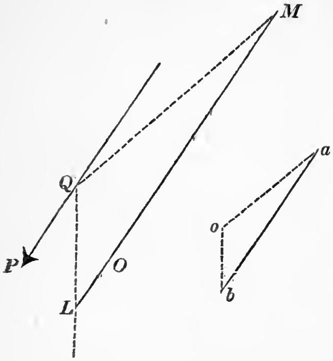


Fig. 99.

Let it be required to represent the magnitude of the moment of a force P about a point O (fig. 99). Draw ab parallel to P and representing it on any scale.

Let o be a point taken at a unit distance from ab; draw oa and ob. Assume any point, Q, on the line of action of P, and draw QM and QL parallel to oa and ob, respectively. From O draw a line, LM, parallel to P. Then the length LM represents the moment of P about O. For, the triangles oab and QML are similiar; therefore if p is the length of the perpendicular from Q on LM, we have $\frac{LM}{p} = \frac{ab}{1}$, therefore $LM = P.p$, since ab represents P.

Hence LM is the moment on the scale adopted.

If the pole o is at a distance k units from ab, we shall have $P.p = LM \times k$.

89.] **Problem.** *To represent the sum of the moments of any system of coplanar forces about a point.*

Let A (fig. 98) be the point about which the sum of the moments of the forces is required.

The sum of their moments = the moment of their resultant about the point. Let this resultant be constructed by Art. 86, and let the moment of the resultant be constructed by last Art. Now the resultant is represented by the line joining 01 to 56 (fig. 98), and if O is a pole assumed at any distance, k, from this line, we are to draw from any point on the resultant, two lines parallel to $O\,01$ and $O\,56$, and through A a line parallel to the resultant, R.

Now the extreme sides, $f_0 f_1$ and $f_5 f_6$, of the funicular intersect in a point on R, and are parallel to the lines $O\,01$ and $O\,56$. Hence *the intercept made by the extreme sides of the funicular on a line drawn through the given point A parallel to the resultant will represent the sum of the moments of the forces about the point.*

This intercept multiplied by k will be the sum of moments.

90.] **Property of Perspective Triangles.** Two triangles, ABC and $A'B'C'$, are said to be *in perspective* when their vertices can be joined in pairs by three right lines which meet in a point. If the lines joining A to A', B to B', and C to C' meet in a point, A and A' are called *corresponding vertices*, as are also B and B', C and C'; and the sides, AB and $A'B'$, &c., which join corresponding vertices in the triangles are called *corresponding sides*.

The fundamental property of triangles in perspective is that *the points of intersection of corresponding sides lie in one right line.*

To prove this projective property it is sufficient to prove it for the simplest figure into which the two triangles can be projected. Let the line CC' be projected to infinity. Then AA' and BB' will become parallel lines; also the sides AC and BC of the first triangle will become parallel, as will $A'C'$ and $B'C'$ of the second. For the simple figure thus obtained there is no difficulty in proving the proposition.

To construct a triangle whose three sides shall pass each through a given point, and whose three vertices shall each lie on one of three concurrent lines.

Let it be required to construct a triangle whose vertices, A, B, C, shall lie on three concurrent lines, AO, BO, CO, and whose sides shall pass through the points a, b, c, (fig. 100). Suppose it done, and let ABC be the triangle. Take any point, C', on CO, and draw $C'a$ and $C'b$ meeting BO and AO in B' and A' respectively.

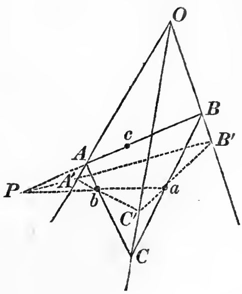


Fig. 100.

Then the triangles ABC and $A'B'C'$ are in perspective, therefore the sides AB and $A'B'$ intersect in P, a point on the line ab. Hence P is known, since it is the intersection of ab with the line $A'B'$ which is constructed by arbitrarily assuming C'. P being known, join it to c, and the vertices A and B are determined, and C follows at once. Q. E. F.

NOTE. In Art. 88 if the unit of force is ϖ and the unit of length λ, the moment of the force P about O will be $LM \times \varpi \times \frac{k}{\lambda}$. For ab will obviously be $\frac{P}{\varpi}\lambda$.

EXAMPLES.

1. A heavy rod, or beam, is supported horizontally on two smooth props at its extremities, and loaded with given weights at given points in its length; find the pressure on the props.

Suppose the line $a_0\, a_5$ (fig. 34, p. 38) to be horizontal and to represent the loaded beam, the loads, P_1, P_2, ... (including its weight among them) being applied at the points, d_1, d_2, ..., and let the pressures at the props a_0 and a_5 be P_0 and P_5. Starting from any point 01 draw a vertical downward line to represent on any scale the force P_1, and let this line terminate at the point 12; from 12 draw a vertical downward line representing P_2 on the same scale, and let this line terminate at the point 23; from this point draw a vertical downward line to the point 34 to represent P_3; from 34 draw a vertical downward line to the point 45 to represent P_4.

Then from 34 we must draw a vertical *upward* line to represent the pressure P_5, and this line will terminate at the point 56, which, however, is at present unknown. The pressure P_0 will, of course, be represented by the upward line between 56 and 01.

To determine 56, assume any pole, O, and join this pole to the points 01, 12, ... Across the lines of action of the forces acting on the beam draw the lines A_0A_1, A_1A_2, ... parallel to the lines $O\,01$, $O\,12$, ..., and draw the closing line, $A_0\,A_5$, of the funicular polygon. Then the line through O parallel to this closing line is that joining O to the required point, 56.

2. A beam is supported horizontally at its extremities on two vertical props and loaded with given weights at given points in its length; it is required to represent the *Bending Moment* at any point of the beam.

Def. When a beam is in equilibrium under the action of any forces, the Bending Moment at any point means the sum (with their proper signs) of the moments about this point of all those forces which act at one side (either side will do) of the point.

Suppose $a_0\, a_5$ (fig. 34, p. 38) to represent the beam, as in last example, and let P be the point about which the leading moment is required. The pressure on the prop a_0 being P_0, the bending moment at P is the sum of the moments of P_0, P_1, and P_2; and if we construct any funicular of the system this moment will, by Art. 89, be the intercept on a vertical line through P made by the extreme sides of the funicular of the forces P_0, P_1, and P_2. But these extreme sides are obviously $A_0\,A_5$ and $A_2\,A_3$. Hence the bending moment at any point P is represented by the vertical ordinate, mn, drawn through P, of any funicular polygon of the system.

Of course, if k is the distance of the pole of the assumed funicular from the vertical line which serves as the force diagram, the bending moment will be $mn \times k \times \frac{\varpi}{\lambda}$. (See Note, p. 109.)

3. To construct for any system of coplanar forces a funicular polygon which shall pass through three assigned points.

Let the given system of forces be P_1, P_2, P_3, P_4, P_5 (fig. 98, p. 104), and let it be required to construct a funicular polygon which shall pass through the points D, E, F.

Consider the triangle formed by the sides, $f_0 f_1$, $f_2 f_3$, and $f_5 f_6$, of the funicular which pass through the three given points.

The vertex formed by the intersection of $f_0 f_1$ and $f_2 f_3$ lies on a given line, R_{12}, (not drawn in figure) which is the resultant of P_1 and P_2 (Cor. 2, Art. 86); the vertex formed by the intersection of $f_2 f_3$ and $f_5 f_6$ lies on a given line, R_{345}, which is the resultant of P_3, P_4, and P_5; and the vertex formed by the intersection of $f_0 f_1$ and $f_5 f_6$ lies on a given line, R_{12345}, which is the resultant of P_1, P_2, P_3, P_4, and P_5.

Moreover the three lines R_{12}, R_{345}, and R_{12345} obviously meet in a point; for the resultant of P_1, ... P_5 may, if we please, be constructed by first finding the resultant of P_1, P_2, and then finding the resultant of P_3, P_4, P_5.

Hence the triangle formed by the sides of the funicular which are to pass through the assigned points is one whose vertices lie on three concurrent lines and whose sides pass each through a fixed point.

Let this triangle be constructed by Art. 90. Then knowing the force diagram of the forces and drawing two lines, 01 O and 23 O say, parallel to the two sides $f_0 f_1$ and $f_2 f_3$, the pole O is known, and thence the whole figure.

4. Construct a funicular polygon which shall pass through three given points, two of which lie on one side of the polygon.

Ans. This side of the polygon is known, and it intersects the side passing through the remaining point in a point lying on a given line. Hence the side passing through the remaining point is known, and hence the pole of the funicular.

5. For a given system of vertical downward forces, P_1, P_2, ... P_{n-1}, equilibrated by two extreme vertical upward forces, P_0, P_n, let any funicular polygon be constructed. Prove that the area of this polygon $=\dfrac{C}{k}$, where C is constant and k the distance of its pole from the vertical line which is the force diagram of the forces.

(The value of C is obtained by multiplying each force of the system by half the product of the distances between its line of action and the lines of action of the extreme forces, and adding all such products together, and multiplying the result by $\dfrac{\lambda}{\varpi}$. See Note, p. 109.)

6. A uniform beam is supported at its extremities on two vertical props; find the bending moment at any point in it.

Ans. If y is the distance of the point from one extremity, the bending moment is $W\dfrac{y(l-y)}{2l}$, where W is the weight of the beam.

7. In the last example, what is the curve of bending moment?

Ans. A parabola passing through the ends of the beam, its vertex lying on the vertical line through the middle of the beam at a distance

$\frac{l}{8}$ from the beam. (The bending moment at any point is the product of W and the vertical distance of the point from the parabola.)

91.] **Astatic Equilibrium.** When any number of forces, P_1, $P_2, \ldots$, acting at points, $A_1, A_2, \ldots$, in a body keep this body in equilibrium, these forces will not, in general, continue to preserve equilibrium when the body is displaced in any manner, each force still retaining its magnitude, direction, and point of application in the body. If for all displacements of the body the forces continue to preserve equilibrium, the body is said to be in *astatic* equilibrium.

The simplest example of astatic equilibrium is furnished by a heavy body suspended by a vertical string attached at its centre of gravity. Here the system of forces consists of the weights of the particles of the body and the tension of the string; and however the body may be displaced about its centre of gravity, all these forces will retain their individual magnitudes, directions, and points of application, and the body will remain at rest.

Again, a system of two equal reversed magnets rigidly connected by an axis through their centres is astatic for displacements round this axis.

When a system of forces applied to a body is not in equilibrium, it happens that in certain cases this system can be astatically equilibrated by a single applied force; i.e., in all displacements which the body can receive, each force acting on it with invariable magnitude, direction, and point of application, it may be possible to equilibrate the system by one force of constant magnitude, direction, and point of application.

It is evident that this is always the case for a system of parallel forces. A single force equal and opposite to their resultant, applied at their centre, will astatically equilibrate them.

Into the general discussion of astatic equilibrium we do not enter *. Suffice it to say that a system of (non-coplanar) forces must in general be astatically equilibrated by *three* forces; and if the forces are all parallel to one plane, by *two*. When (as in the present chapter) the forces are all coplanar we shall prove

* On this subject the student may consult Moigno's *Statique* (Dixième Leçon), a memoir by M. Darboux (*sur L'Équilibre Astatique*), and a paper by the author in the *Proceedings of the London Mathematical Society* (vol. ix).

that for displacements of their points of application in their plane, the system can be astatically equilibrated by a single force.

In this case it is clear that instead of considering the body to which they are applied as displaced, we may consider the body fixed and each force rotated in a fixed sense round its point of application through a constant angle—a motion of translation of the body or points having obviously no effect on the system of forces.

We shall now prove that—*if all the forces in a coplanar system are rotated in the same sense, through the same angle, in their plane, round their points of application, their resultant* (unaltered in magnitude, of course) *passes through a fixed point in the body.*

Let two forces, P and Q, act at two fixed points, A and B, (fig. 101) in the directions OA and OB, O being the point of intersection of their lines of action; and let the forces be turned in the same sense round A and B through the same angle, so that the point of intersection of their new lines of action is O'. Now, since $\angle OAO' = \angle OBO'$, a circle described through A, B, and O will pass through O', and the angle $AO'B$, between P and Q when they are turned round, is equal to the original angle, AOB, between them. Also, the forces being unaltered in magnitude, it follows that the angles which the resultant at O' makes with them are the same as the angles which it makes with P and Q at O. If, then, OC is the direction of the resultant at O, $O'C$ must be the direction of this resultant at O'. Hence, the resultant of P and Q passes through the fixed point C. In exactly the same way it is proved that the resultant of three forces passes through a fixed point when the forces are turned round their fixed points of application through a constant angle; and so on for any number of forces.

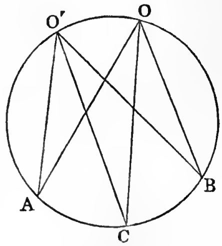


Fig. 101.

This point may be called the *astatic centre* of the system of forces*.

92.] To find the Astatic Centre of a System of Coplanar Forces. Taking an arbitrary origin and arbitrary axes, the

* Of course it is understood throughout this discussion and in the examples at the end of this chapter that the displacements of the body or forces are always supposed to take place in the plane of the forces.

point required lies on the resultant whose equation is (Art. 82)

$$a\Sigma Y-\beta\Sigma X-G=0, \tag{1}$$

(a, β) being the running co-ordinates.

Now, if the force P_1, acting at the point (x_1, y_1) is turned round in the plane of xy through an angle ω, X_1 becomes $P_1 \cos(\theta_1+\omega)$, where θ_1 is the original angle made with the axis of x by P_1, or $X_1 \cos\omega - Y_1 \sin\omega$; Y_1 becomes $X_1 \sin\omega + Y_1 \cos\omega$; and $Y_1x_1 - X_1y_1$ becomes $(Y_1x_1 - X_1y_1)\cos\omega + (X_1x_1 + Y_1y_1)\sin\omega$.

$$\left.\begin{array}{lll}\text{Hence, } \Sigma X & \text{becomes} & \cos\omega.\Sigma X - \sin\omega.\Sigma Y, \\ \quad\Sigma Y & \quad ,, & \sin\omega.\Sigma X + \cos\omega.\Sigma Y, \\ \quad G & \quad ,, & G\cos\omega + \Gamma\sin\omega,\end{array}\right\} \tag{A}$$

where $\Gamma \equiv \Sigma(Xx+Yy)$*.

The equation of the new resultant is, therefore,

$$(a\Sigma Y-\beta\Sigma X-G)\cos\omega+(a\Sigma X+\beta\Sigma Y-\Gamma)\sin\omega=0, \tag{2}$$

and the astatic centre of the system of forces is the intersection of the lines given by equations (1) and (2). This point may evidently be determined by (1) and by the equation

$$a\Sigma X+\beta\Sigma Y-\Gamma=0. \tag{3}$$

Hence for the co-ordinates of the astatic centre we have

$$a=\frac{\Gamma\Sigma X+G\Sigma Y}{R^2},\ \beta=\frac{\Gamma\Sigma Y-G\Sigma X}{R^2}. \tag{4}$$

If the astatic centre were the origin, a and β would be each $=0$, and G would $=0$, since the point is on the resultant (Art. 76). Hence for the centre of the forces we have

$$G=0,\ \Gamma=0. \tag{5}$$

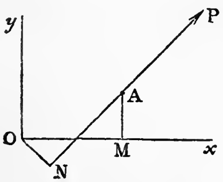


Fig. 102.

If the co-ordinates of A, the point of application of a force, P (fig. 102), with respect to rectangular axes, Ox and Oy, are x and y, the quantity $Xx+Yy$ is equal to $P(x\cos\theta+y\sin\theta)$, θ being the angle which P makes with Ox. Now if OM is x, and AM is y, it is evident that $x\cos\theta+y\sin\theta=AN$, N being the foot of the perpendicular from O on the line of action of P. Denoting AN by q, we have, then, for the Virial

$$\Gamma=\Sigma(Pq).$$

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* This quantity is called by Clausius the *Virial* of the forces.

Hence, *if any number of coplanar forces be turned each round a fixed point of application through an arbitrary but common angle, there exists a point in the plane of the forces such that both the Virial and the sum of the moments of the forces about it, continue to vanish for all displacements.*

It is easy to see that if AN be in the sense in which P acts, the sign of the product Pq will be changed.

The value of Γ with respect to axes through a point (α, β) parallel to Ox and Oy is evidently $\Sigma\{X(x-\alpha)+Y(y-\beta)\}$, or $\Gamma-\alpha\Sigma X-\beta\Sigma Y$. Hence the locus of points for which this quantity $= 0$ is given by equation (3), which denotes a right line passing through the astatic centre, and evidently perpendicular to the resultant.

93.] **Theorem.** *If any number of coplanar forces are in equilibrium, and if the forces be turned, each round a fixed point, in the same sense through any common angle, the new system is equivalent to a couple.*

For, from equations (A) Art. 92, it appears that if $\Sigma X = 0$ and $\Sigma Y = 0$ before the rotation, they will $= 0$ after it; hence the new system has no resultant of translation, and it must, therefore, be a couple. Now, since by hypothesis $G=0$, the axis of the new couple is, by equations (A), equal to

$$\Gamma \sin \omega.$$

We see, then, that the system of forces will remain in equilibrium, whatever be the angle through which they are turned, if

$$\Gamma = 0.$$

94.] **Remark on the Conditions of Equilibrium.** It must be carefully borne in mind that the conditions of equilibrium given in Arts. 80 and 81 are *sufficient* only in the case of indeformable bodies. For, having reduced a system of forces to a resultant of translation, R, acting at an arbitrary point, together with a couple of moment G, the logical conclusion is that—

If $R = 0$ and $G = 0$, those motions of the system which would be produced by R and G respectively are thereby destroyed.

Now by a fundamental principle of Kinetics, which we anticipate, *if $R = 0$ there is no resultant linear momentum of the system in any direction, or in other words its centre of mass is at rest; and if, in addition, $G = 0$, there is no resultant angular momentum about the centre of mass of the system.*

These two things we can conclude from the equations $R = 0$, $G = 0$ for all systems, whether they are gases, liquids, deformable frameworks, natural solids, or rigid bodies.

Now the destruction of resultant linear and angular momentum will, except in the case of rigid bodies, be quite consistent with the existence of motions of parts of the system among themselves, negative momenta cancelling positive. Hence, *whenever a system is capable of altering the relative positions of its parts, the complete equilibrium of the system will require more than the vanishing of the resultants of translation and rotation of the forces applied to it.* In fact, its internal forces will have to be taken into account. In rigid bodies the destruction of the above-mentioned motions will necessitate the destruction of all motion, and the conditions $R = 0$, $G = 0$ are both necessary and sufficient. In these bodies there is no restriction placed on the internal forces, so that they are always capable of assuming such magnitudes and directions as will enable them to destroy the action of the external forces. On the contrary, in deformable bodies, there are restrictions placed on the internal forces so that they are not capable of preserving equilibrium against all systems of external forces. For example, in a freely jointed framework, the action between bar and bar must consist of a single force restricted to passing through the joint. This is the reason why two equal forces applied in opposite senses in the same line to two opposite sides of a set of parallel rulers will not hold them in equilibrium, unless the rulers are placed in a certain configuration; and it is also the reason why two equal and directly opposite forces applied to the ends of a string, elastic or inelastic will not hold it in equilibrium until it has assumed a certain state.

Hence also the necessity for considering the internal forces (pressures) in Hydrostatics.

We shall afterwards enunciate a single principle,* or condition, of equilibrium which will embrace all systems indiscriminately.

These observations are recommended to the most careful consideration of the student.

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* The Principle of Virtual Work.

Examples.

1. If the sums of the moments of any number of coplanar forces round three points which are not in a right line are each $= 0$, the forces are in equilibrium.

2. If the sums of the moments round three points not in a right line are equal, the forces are either in equilibrium or equivalent to a couple.

3. If the sum of the moments of a system of coplanar forces round three given points are l, m, and n, and if the sides and angles of the triangle formed by the points are a, b, c, A, B, C, show that the resultant force is equal to

$$\frac{(l^2a^2 + m^2b^2 + n^2c^2 - 2lmab\cos C - 2mnbc\cos A - 2nlca\cos B)^{\frac{1}{2}}}{2\Delta},$$

where Δ is the area of the triangle ABC.

4. If a system of coplanar forces applied at fixed points is in equilibrium, the co-ordinates of the astatic centre become indeterminate. Explain this.

Ans. In this case the system must be astatically equilibrated by two equal and opposite forces (couple).

5. In the last case show how to find an astatically equilibrating couple for the system.

Ans. Take the astatic centre of any number of the forces, and also the astatic centre of the remaining forces. These will be the points of application of the forces of the required couple (whose moment, of course, varies with the displacement of the body or forces), and the forces of the couple are equal to the resultants of the two partial sets.

6. Three forces are applied at the middle points of the sides of a triangle, ABC, perpendicular to these sides and respectively proportional to them; find a couple which will astatically equilibrate them.

Ans. A couple one of whose forces is applied at the middle point of any one side, AB, and the other applied at the point of intersection of a parallel to AB drawn through C with the perpendicular to AB at its middle point.

7. When a system of coplanar forces in equilibrium continues in equilibrium for all displacements in the plane of the forces, show that the astatic centre of any number of them must be coincident with that of the remainder.

CHAPTER VI.

THE CONDITIONS OF EQUILIBRIUM OF A RIGID BODY UNDER THE ACTION OF FORCES IN ONE PLANE DEDUCED FROM THE PRINCIPLE OF VIRTUAL WORK FOR A SINGLE PARTICLE.

95.] **Theorem.** If a particle in equilibrium under the action of any forces be constrained to maintain a fixed distance from a given fixed point, the force due to the constraint (if any) is directed towards the fixed point.

Let B be the particle, and A the fixed point. Then the string or rigid rod which connects B with A may be removed if we enclose the particle in a smooth circular tube whose centre is A; for evidently the preservation of the constancy of the distance AB receives sufficient expression in this manner. Now, in order that B may be in equilibrium inside the tube, it is necessary that the resultant of the forces acting upon it should be normal to the tube, i.e., directed towards A.

Cor. 1. If A and B be two particles in equilibrium, connected by a rigid rod whose weight is neglected, the reactions of A and B on the rod are two forces equal in magnitude and opposite in direction.

Cor. 2. If any body be in equilibrium under the action of two forces only, these forces must be equal and opposite in the same right line.

Cor. 3. If a particle in equilibrium under the action of any forces is constrained to maintain a fixed distance from each of a number of other particles or points, the forces corresponding to these constraints are directed in the right lines joining the particle to each of the other particles or points.

This is evidently true whether the invariable distances are maintained by straight rigid bars or by crooked bars.

96.] **System of Particles Rigidly Connected.** Let there be any number of particles, $m_1, m_2, m_3, \ldots$ (fig. 103), each acted on by any forces, and connected with the others in such a way that the figure of the system is invariable.

Then, by the last Article, the force proceeding from the connection of m_1 and m_2 is in the line $m_1\, m_2$, which we may

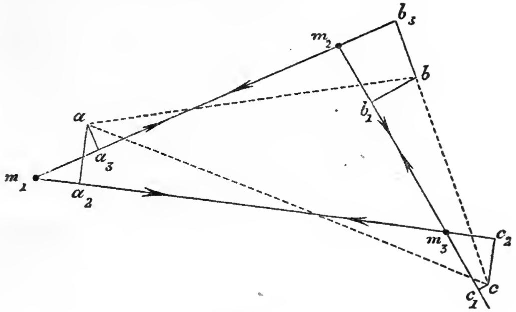

Fig. 103.

imagine to be a rigid bar. Let this force be denoted by T_{12}. Similarly, let the forces in the bars $m_2\, m_3$ and $m_3\, m_1$ be denoted by T_{23} and T_{31}, respectively. These internal forces may tend either to increase the distances between the particles or to diminish them. In the figure we have supposed the latter to be the case, but the result will be the same if the former supposition is made.

Imagine that the system is slightly displaced so as to occupy the position *abc*. Now, it has been already proved (Art. 65, p. 78) that the equation of virtual work for two particles rigidly connected will not involve the force due to the connection; but, for clearness, we reproduce the proof here.

Let fall the perpendiculars aa_2 and aa_3 on the lines $m_1\, m_3$ and $m_1\, m_2$; bb_1 and bb_3, on $m_2\, m_3$ and $m_1\, m_2$; cc_1 and cc_2 on $m_2\, m_3$ and $m_1\, m_3$. Let the sum of the virtual works of the external forces (not including T_{12} and T_{13}) acting on m_1 be denoted by $\Sigma P\,\delta p$, and let $\Sigma\, Q\,\delta q$ and $\Sigma R\,\delta r$ denote similar quantities for m_2 and m_3. Then the equation of virtual work for m_1 is evidently

$$\Sigma P\,\delta p + T_{12}\,.\,m_1\, a_3 + T_{12}\,.\,m_1\, a_2 = 0\,; \tag{1}$$

that for m_2 is

$$\Sigma\, Q\,\delta q - T_{12}\,.\,m_2\, b_3 + T_{23}\,.\,m_2\, b_1 = 0, \tag{2}$$

and that for m_3 is

$$\Sigma R\,\delta r - T_{13}\,.\,m_3\, c_2 - T_{23}\,.\,m_3\, c_1 = 0. \tag{3}$$

Now (Art. 63, p. 77) $m_1\, a_3 = m_2\, b_3$; $m_1\, a_2 = m_3\, c_2$; $m_2\, b_1 = m_3\, c_1$.

Hence, by addition, the internal forces disappear, and the equation of virtual work for the whole system is

$$\Sigma P\delta p + \Sigma Q\delta q + \Sigma R\delta r = 0,$$

or

$$\Sigma (P\delta p + Q\delta q + R\delta r) = 0. \quad (4)$$

The same result is evidently true, whatever be the number of particles forming the system; and it is well to note that we have been enabled to obtain equation (4) connecting the external forces acting on the system, by choosing *a virtual displacement compatible with the geometrical conditions of the system,* that is, in the present case, a virtual displacement which allows the mutual distances of the particles to remain unaltered; or, again, *such a virtual displacement as might be an actual one;* for the system could *actually* occupy the position *abc*.

97.] **Elimination of the Internal Forces of a System.** By the *Internal Forces* of a system it is already sufficiently clear that we mean forces proceeding from the internal connections of the parts of the system among themselves. Such forces are directed from particle to particle, and will contribute nothing to the equation of virtual work of the system, if in the virtual displacement the distance between every two particles remains the same as before.

It is evident that if the virtual displacement violates any geometrical condition of the system, the corresponding internal force will appear in the equation of virtual work. Thus, if in fig. 103, the distance *ab* is not equal to the distance between m_1 and m_2, we shall have by addition the term

$$T_{12} \cdot (m_1 a_3 - m_2 b_3),$$

or

$$T_{12} \cdot \delta (m_1 m_2),$$

where $\delta (m_1 m_2)$ denotes the change or variation of the distance between m_1 and m_2.

And, generally, if any internal force, F, tend to vary any internal function, f, in a system, this force will contribute to the equation of virtual work of the system the term

$$F \cdot \delta f,$$

so that if in the supposed displacement of the system, the function f is actually altered, the force F will appear in the equation, but will not appear if f is unaltered.

98.] **General Equation of Virtual Work for Forces Acting in One Plane on a Rigid Body*.** If the particles $m_1, m_2, m_3, \ldots$ form a continuous body, on which forces $P_1, P_2, P_3, \ldots$ act in one plane at different points $A_1, A_2, A_3, \ldots$ of the system (fig. 104),

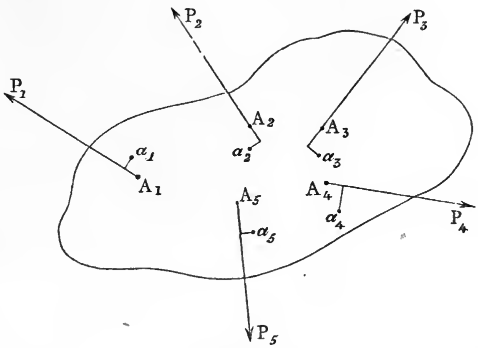

Fig. 104.

the condition necessary and sufficient for the equilibrium of the system is that the sum of the virtual works of the forces is equal to zero for any and every virtual displacement which violates none of the geometrical conditions of the system.

For we have seen (Art. 61, p. 73) that the condition necessary and sufficient for the equilibrium of any one particle of the system is the vanishing of the virtual work of all the forces acting upon it, the internal forces proceeding from its connection with the other particles of the system being, of course, included, as in equations (1), (2), (3) of Art. 96. Expressing thus the conditions for the equilibrium of all particles of the system, and adding the results, there remains for the condition of equilibrium the equation

$$P_1\,\delta p_1 + P_2\,\delta p_2 + P\delta p_3 + \ldots = 0, \tag{1}$$

into which no internal force enters.

Conversely, if the sum of the virtual works of the forces

* We formally confine the discussion for the present to Rigid Bodies, although it is clear from last Article that what follows is applicable to systems such as freely articulated bars which, without being rigid systems, satisfy certain geometrical conditions that are not violated in the virtual displacement; and it is equally clear that these conditions may be violated if we include in our equations the work of internal forces.

vanishes for every virtual displacement, the system is in equilibrium.

For, if it is not, it will take a *determinate* motion, each point of the system describing a certain line in virtue of its connections with the other points. Now, this motion will be in no way interfered with if we introduce new connections which render it the only motion possible for the system. Under the new circumstances it is clear that if we prevent the motion of any one point, we prevent the motion of the system. Suppose the motion of the point A to be stopped by the application of a force, F, in the direction $A'A$, A' being the point to which A moves. Now, equilibrium exists under the action of (a) the given external forces, (β) the newly-introduced geometrical connections, and (γ) the force F; hence the sum of the virtual works of these forces $= 0$ for every displacement. Choose that displacement which the system is supposed actually to undergo when the force F is not applied at A. Now, by the last Article, since none of the geometrical conditions (β) are violated by this displacement, the forces proceeding from them will do no work. Hence the equation of work is

$$\Sigma P \delta p - F . AA' = 0,$$

where $\Sigma P \delta p$ denotes the virtual work of the given acting forces. But, by hypothesis, $\Sigma P \delta p = 0$ for every displacement, and therefore for this one; hence $F . AA' = 0$, i.e., either $AA' = 0$, or $F = 0$, either of which signifies that no motion of the system takes place. Hence the system is in equilibrium.

In fig. 104, $a_1, a_2, a_3, \ldots$ are supposed to be virtual positions of the points of application of the forces $P_1, P_2, P_3, \ldots$.

99.] **Remarks on the Equation of Virtual Work.** Equation (1) of last Article, though *strictly* true in the case of forces acting on a particle, is not so when these forces are applied at points in a body of finite extension, or to a system of particles connected in any manner. In fact, the internal forces of the system have been eliminated from equations (1), (2), and (3) of Art. 96, by assuming that $m_1 a_3 - m_2 b_3 = 0$. Now, we know that this quantity is not strictly equal to zero, but equal to an infinitesimal of the second order, if the angular displacement of the line $m_1 m_2$ is regarded as an infinitesimal of the first order. It is more correct, therefore, to say that for the equilibrium of a body *the virtual work of the applied forces is an infinitesimal of* *the second order, if the greatest displacement in the system is regarded as an infinitesimal of the first order.*

100.] **General Displacement of a Rigid Body in One Plane.** Since the general condition of equilibrium of a rigid body requires the vanishing of the virtual work of the acting forces for *every* virtual displacement which could be an actual one, it is evidently necessary to investigate all the kinds of displacement which such a body could undergo. Now, evidently, the position of a right line is known, if the positions of any two of its points are known; and also the position of any body is known, if the positions of any three * of its points which are not *in directum* are known. Hence, to investigate the displacements to which a rigid body may be subject, it is sufficient to determine the general displacements of a system formed of three points.

In fig. 103 let such a system be $m_1 m_2 m_3$, and let abc be any displacement whatever of this system in its own plane. Then it is clear that if we moved m_1 into the position a, and then got m_2 into the position b, the remaining point, m_3, would take up the position c. This follows from Prop. VII of the first book of Euclid. Now what is necessary to move the line $m_1 m_2$ into the position ab? Two things—

(α) The point m_1 must be moved up to a, by a simple *motion of translation;* and

(β) When this is done, the line $m_1 m_2$ must be *rotated* about a so as to bring m_2 into the position b. This second motion is called *a motion of rotation.*

If we suppose that in the first motion (α) the line $m_1 m_2$ is moved parallel to itself, while m_1 is moved to a, the subsequent motion of rotation which brings m_2 into the position b will be a small one, the position abc being only slightly different from $m_1 m_2 m_3$.

Hence—*If a rigid body receives any displacement parallel to a fixed plane, it may be brought from its old into its new position by (α) a motion of translation which has the same magnitude and direction for all its points, and (β) a motion of rotation which has also the same angular magnitude and sense for all its points.*

Thus, in fig. 105, by the motion of translation common

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* If, as in the present chapter, the displacement is made parallel to one plane, the positions of *two* points will suffice.

to all the points, m_1 is carried to a, while m_2 is carried to b', and m_3 to c', the lines m_1m_2, m_2m_3, and m_1m_3 being carried parallel to themselves to ab', $b'c'$, and ac', respectively. Then, by the motion of rotation ab' is turned round to ab, and c' is made to coincide with c.

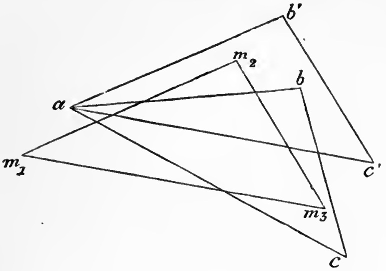

Fig. 105.

101.] **Independence of the Motions of Translation and Rotation.** If we have a system in the position $ab'c'$ (fig. 105), it is clear that no motion of translation will ever bring it into the position abc. The change is effected by a motion of rotation alone. On the other hand, no motion of rotation could bring a system, $m_1m_2m_3$, into the position $ab'c'$. This change is effected by a simple translation common to all the points: hence these motions are quite independent of each other.

102.] **Theorem.** *All the conditions necessary and sufficient for the equilibrium of a rigid body acted on by any forces can be deduced from equations of virtual work corresponding either to a virtual displacement of translation common to all its parts, or to a virtual displacement of rotation common to all its parts.*

For (Art. 98), the condition necessary and sufficient for the equilibrium of the body is the vanishing of the virtual work of the applied forces for *every* virtual displacement; and (Art. 100) every virtual displacement is either one of translation, or one of rotation, or a combination of both. Now (Art. 101), these displacements are independent, and therefore the supposed condition must come either from a virtual displacement of translation alone, or from one of rotation alone.—Q. E. D.

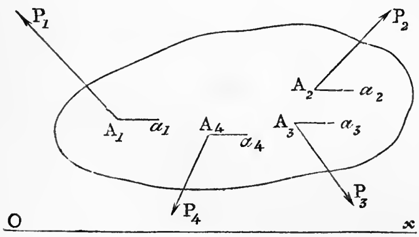

Fig. 106.

103.] **Virtual Work Corresponding to a Virtual Motion of Translation.** Let a rigid body (fig. 106) be in equilibrium under the action of any forces in one plane, P_1, P_2, P_3, ..., and let the body be imagined to receive a motion of translation parallel to an arbitrary line, Ox, whereby the points, A_1, A_2, A_3, ..., of application of the different forces receive virtual displacements, $A_1 a_1$, $A_2 a_2$, $A_3 a_3$, ..., all parallel to Ox, and equal to a. Then (Art. 51, p. 67), the virtual work of the force P_1 is $a \times$ projection of P_1 along Ox. Let the projection of P_1 along Ox be X_1: then the virtual work of P_1 is aX_1. Similarly, if X_2, X_3 ..., be the components of P_2, P_3, ... along Ox, the virtual works of these forces will be aX_2, aX_3, ... Hence the equation of virtual work is

$$a(X_1 + X_2 + X_3 + \ldots) = 0,$$

or

$$a\Sigma X = 0. \tag{1}$$

Consequently, since a is arbitrary, we have

$$\Sigma X = 0. \tag{2}$$

Hence—*For the equilibrium of a rigid body it is necessary that the sum of the components of the acting forces along every arbitrary right line shall be zero.*

This condition is not sufficient, since every virtual displacement of a body is not one of translation alone.

104.] **Virtual Work Corresponding to a Motion of Rotation.** Let several forces, P_1, P_2, P_3, ... (fig. 107), act on a body at points A_1, A_2, A_3, ..., and suppose that the body is rotated through a small angle $= \omega$, round an axis perpendicular to the plane of the forces through an arbitrary point, O. Then the points A_1, A_2, A_3, ... will describe small circular arcs, $A_1 a_1$, $A_2 a_2$, $A_3 a_3$, ... having O as their common centre, and subtending the same angle, ω, at O. Let θ_1 be the angle between OA_1 and the direction of P_1. Then, evidently, the projection of $A_1 a_1$ on the direction of P_1 is $A_1 a_1 . \sin\theta_1$. But $A_1 a_1 = \omega . OA_1$; therefore the virtual work of P_1 is

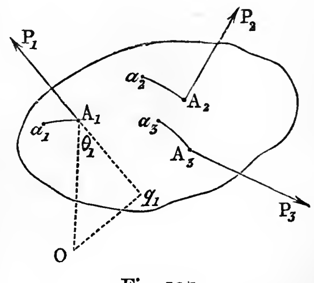

Fig. 107.

$$\omega P_1 . OA_1 \sin\theta_1.$$

If $p_1 =$ the perpendicular, Oq_1, from O on the line of action of P_1, this is evidently

$$\omega P_1 . p_1.$$

Similarly, the virtual work of P_2 is $\omega P_2 . p_2$, and that of P_3 is $-\omega P_3 . p_3$. Hence the equation of virtual work is

$$\omega (P_1 p_1 + P_2 p_2 - P_3 p_3 + \dots) = 0, \tag{1}$$

or

$$\Sigma Pp = 0.$$

But the product of a force, P, and the perpendicular, p, let fall upon it from the point O, is the moment of the force with respect to the point O, or rather with respect to an axis through O perpendicular to the plane of the figure.

Hence, equation (2) asserts that *for equilibrium the sum (with their proper signs) of the moments of the forces with respect to any point in their plane is zero.*

As regards the signs to be given to the moments, $P_1 p_1$, $P_2 p_2$, ... of the forces, we see that—

Those forces which tend to rotate the body in the same sense round the point O give virtual work of the same sign, and therefore have moments of the same sign with respect to O.

Thus, in fig. 107, the forces P_1 and P_2 tend to turn the body round O, in a sense opposite to that of watch-hand rotation, while P_3 tends to turn it in the opposite sense. Hence, in the *Equation of Moments,* as the equation

$$\Sigma Pp = 0$$

is called, $P_1 p_1$ and $P_2 p_2$ have the same sign, and $P_3 p_3$ has an opposite sign.

105.] **Absolute Conditions for the Equilibrium of a Rigid Body Acted on by Forces in One Plane.** It is now clear that, as all possible displacements of a rigid body are exhausted in a motion of translation common to all its parts, and a motion of rotation common to all its parts, all possible conditions of its equilibrium under the action of forces acting in one plane are exhausted in the conditions of Articles 103 and 104, namely—

1. The sum of the components of the acting forces along every arbitrary line in their plane = 0.

2. The sum of the moments of the forces with regard to every arbitrary point in their plane = 0.

These are the conditions which were deduced in the last chapter; and it is clear that since all possible displacements of a deformable system are by no means exhausted in motions of translation and rotation common to all its parts, the equation of virtual work for such a system does not lead to the above conditions as sufficient.

106.] **Analytical Expression for the Displacement of a Rigid Body.** We have seen (Art. 100) that the displacement of a rigid body is known from the displacement of any fixed triangle in it; and that the displacement of such a triangle consists of a motion of translation common to all its parts, and a motion of rotation common to all its parts. The displacement of translation may be that which moves each side of the triangle parallel to itself until the vertex m_1 (fig. 105) comes into the position a; or it may be that which moves each side parallel to itself until the vertex m_2 comes into the position b; or, again, that which moves the system until the vertex m_3 comes into the position c. In the first case the magnitude of the motion of translation is $m_1 a$, in the second, $m_2 b$, and the third, $m_3 c$.

Now these three quantities are all of different magnitudes.

But after any one of these motions of translation has taken place, the motion of rotation is constant, since the angles between the sides of the triangle are invariable. Hence—

If a rigid body occupying the position (A) *is displaced by a motion parallel to one plane into the position* (B), *the body may be brought from the position* (A) *to the position* (B) *by:* (α) *a variable motion of translation common to all its parts, whereby any one point, P, of the body is brought directly from its old to its new position, O; and* (β) *a subsequent motion of rotation round an axis through O perpendicular to the plane of motion, the angular magnitude of the rotation being a constant quantity for all such axes.*

We shall investigate the changes produced in the co-ordinates of a point by given *small* motions of translation and rotation. Let the motion of translation first take place. Then draw any two rectangular axes, Ox and Oy, through O (fig. 108) the new position of a point O_1. Let the motion of translation $O_1 O$, common to all parts of the body, be resolved in two components, a and b, parallel to Ox and Oy.

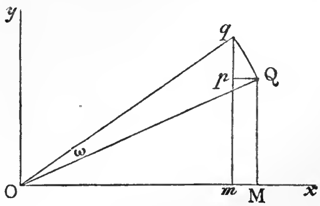

Fig. 108.

Then, if x and y denote the co-ordinates of a point Q_1 in the body with reference to fixed axes drawn through O_1 parallel to Ox and Oy, these quantities will be increased by a and b, respectively, by the motion of translation. To find how much they will be subsequently altered by an angular rotation $= \omega$ round O, let Q describe a small arc of a circle, Qq, round O.

Let fall the perpendiculars QM and qm on Ox, and Qp on qm. It is evident that $OM = x$ and $QM = y$. Then the increase of y produced by the rotation $= qp$, and the increase in $x = -QP$. Now

$$Qp = Qq . \sin QOx = \omega . OQ . \sin QOx = \omega . QM = \omega y;$$

and $\quad qp = Qq . \cos QOx = \omega . OQ . \cos QOx = \omega . OM = \omega x.$

Hence, if δx and δy denote the changes produced in x and y by the two motions combined,

$$\delta x = a - \omega y, \tag{1}$$

$$\delta y = b + \omega x. \tag{2}$$

These are the general analytical expressions for the displacements of a particle in the body. (They can obviously be obtained by differentiating the equations $x = r \cos\theta$, $y = r \sin\theta$, on the supposition that θ alone varies by a quantity $\delta\theta = \omega$, and then adding a and b to the results.)

107.] **Analytical Conditions of Equilibrium.** If any forces, $P_1, P_2, P_3, \ldots$, act on a rigid body in one plane, the condition necessary and sufficient for equilibrium is (Art. 98)

$$P_1 \delta p_1 + P_2 \delta p_2 + P_3 \delta p_2 + \ldots = 0. \tag{1}$$

Let X_1 and Y_1 be components of P_1 along two rectangular axes, Ox and Oy, and let x_1 and y_1 be the co-ordinates of the point at which P_1 acts. Then (Art. 52, p. 68)

$$P_1 \delta p_1 = X_1 \delta x_1 + Y_1 \delta y_1. \tag{2}$$

Making similar substitutions for $P_2 \delta p_2$, $P_3 \delta p_3$, ..., equation (1) becomes

$$X_1 \delta x_1 + Y_1 \delta y_1 + X_2 \delta x_2 + Y_2 \delta y_2 + \ldots = 0, \tag{3}$$

or
$$\Sigma (X \delta x + Y \delta y) = 0. \tag{4}$$

Substituting in (4) the values of δx and δy given in the last Article, we have

$$\Sigma \{X(a - \omega y) + Y(b + \omega x)\} = 0,$$

or
$$a . \Sigma X + b . \Sigma Y + \omega . \Sigma (xY - yX) = 0, \ldots \tag{5}$$

since a, b, and ω are common to all points of the body, and may be taken outside the sign of summation.

Now the displacements a, b, and ω are completely independent of each other, and therefore equation (5) requires that

$$\left.\begin{array}{r}\Sigma X = 0,\ \Sigma Y = 0 \\ \Sigma\,(xY - yX) = 0\end{array}\right\}. \qquad (6)$$

For, choose another virtual displacement in which a and b are the same as before and ω different. Then we have

$$a\Sigma X + b\Sigma Y + \omega'\,\Sigma\,(xY - yX) = 0. \qquad (7)$$

Subtracting (7) from (5),

$$(\omega - \omega')\,\Sigma\,(xY - yX) = 0.$$

But since $\omega - \omega'$ is not $= 0$, this equation requires that

$$\Sigma\,(xY - yX) = 0.$$

Similarly, by making a alone variable, we prove that $\Sigma X = 0$, and by making b alone variable, $\Sigma Y = 0$.

The three equations (6) constitute the *analytical conditions* of equilibrium of the body, and they are the expressions of the two *absolute conditions* of Art. 105.

The first two of the equations (6) are called the *equations of translation*, and the last is called the *equation of moments or rotation.*

108.] **Varignon's Theorem of Moments.** *The moment of the resultant of two forces with respect to any point in their plane is equal to the sum of the moments of the forces with respect to this point.*

Let R (fig. 109) be the resultant of two forces, P and Q, applied at a point A, and let O be any point in their plane. Then the virtual work of R for any displacement of A = the virtual work of P + the virtual work of Q. Let the virtual displacement of A be one of rotation round O, through a small angle $= \omega$. Then, as in Art. 104, the virtual work of R is $\omega \,.\, R \,.\, OA \,.\, \sin OAR$; but this $= \omega \,.\, R \times$ the perpendicular from O on $R = \omega \times$ the moment of R with respect to O. Similarly, the virtual work of $P = \omega \times$ moment of P with respect to O; and virtual work of $Q = \omega \times$ moment of Q with respect to O. Therefore, &c.—Q. E. D.

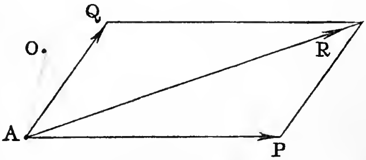

Fig. 109.

In precisely the same way, the moment of the resultant of any number of forces is proved to be equal to the sum of the moments of the forces separately.

109.] **Particular Case in which the Resultant of Translation Vanishes.** When forces applied to a particle have no resultant of translation, their whole effect is null. It is not necessarily so, however, if they are applied to a body of finite dimensions. For example—

If the forces acting upon a rigid body form by their magnitudes and lines of action the sides of a closed polygon taken in order, their resultant of translation vanishes, and they have a constant moment with respect to all points in their plane.

Let forces $P_1, P_2, P_3, \dots$ (fig. 110) act at points $A_1, A_2, A_3, \dots$ in one plane, in a body and let these forces be represented in magnitudes and lines of action by the sides of the polygon formed by their points of application.

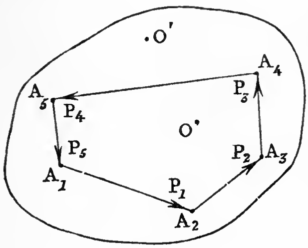

Fig. 110.

Now since (Art. 50) the sum of the projections of the sides of this polygon on any arbitrary line $= 0$, the condition of Art. 103 is fulfilled, and the forces have no resultant of translation.

Let O be any point inside the polygon, and take the sum of the moments of the forces round it. If the perpendiculars from O on the sides A_1A_2, $A_2A_3 \dots$ be $p_1, p_2, \dots$ the sum of the moments will be

$$P_1p_1 + P_2p_2 + P_3p_3 + \dots = G, \text{ suppose.}$$

And since $P_1, P_2, \dots$ are equal to the sides of the polygon, G is evidently $= 2 \,.$ area of polygon. This is a constant for all points inside the polygon.

Now if we take the sum of the moments round any external point, O', we shall have

$$P_1p_1 + P_2p_2 + P_3p_3 - P_4p_4 + P_5p_5,$$

since P_4 turns the body round O' in a sense opposite to that in which the other forces turn it. But this sum is equal to

$$2\,(A_1O'A_2 + A_2O'A_3 + A_3O'A_4 - A_4O'A_5 + A_5O'A_1),$$

and this is again equal to $2 \,.$ area of polygon.

Hence for all points in the plane, the sum of the moments, G, is constant.

110.] **Theorem.** *If a number of forces acting in one plane upon a rigid body have a constant moment with respect to all points in the plane, they can have no resultant force, and must be reducible to a couple.*

For, suppose that they have a resultant $= R$, then if p is the perpendicular let fall on R from any point, O, the sum of the moments of the forces $= R \cdot p$ (Art. 108). Hence by varying the position of O, the sum of the moments varies, which is contrary to hypothesis. They are reducible to two equal, parallel, and opposite forces. For their resultant is zero; hence, compounding them in pairs, they must reduce to two parallel, equal, and opposite forces forming a couple, or to two such forces directly opposite to each other in a right line. But in the latter case the sum of their moments about any point would be zero; therefore if this moment is not zero, the forces must reduce to a couple.

111.] **Problem.** *To find the resultant of two parallel forces, P and Q, acting in the same sense.*

Let AB (fig. 111) be the shortest distance between P and Q, and let the forces be supposed to act at A and B. Also let the reversed resultant, R, act at some point, O, in AB. Since the forces are in equilibrium, their virtual work $= 0$ for every virtual displacement (Art. 98). Choose first a virtual displacement of translation along AB. For this displacement the virtual work of the forces P and $Q = 0$, therefore the virtual work of $R = 0$, therefore R is parallel to P and Q. Again, choose a virtual displacement of rotation about O through an angle $= \omega$. The virtual work of P is then $P \cdot \omega\, OA$, and that of Q is $-Q \cdot \omega\, OB$, while that of R is zero. Hence

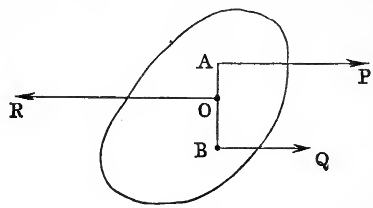

Fig. 111.

$$P \cdot OA - Q \cdot OB = 0, \tag{1}$$

$$\therefore \quad \frac{OA}{OB} = \frac{Q}{P}.$$

Finally, to find the magnitude of R, take a virtual displacement of translation parallel to the forces. This evidently gives

$$R = P + Q. \tag{2}$$

Therefore *the resultant of two parallel forces acting in the same sense is a force parallel to them in the same sense, equal to their sum, and dividing the line joining their points of application in the inverse ratio of the forces.*

Equation (1) asserts that the moments of two parallel forces with respect to any point on their resultant are equal and opposite—a result which is, of course, contained in equation (1) of Art. 104.

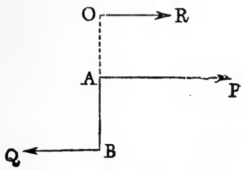

Fig. 112.

If P and Q act in opposite senses (fig. 112), the resultant is obtained in magnitude and direction by simply changing the sign of Q.

Thus (1) becomes

$$\frac{OA}{OB} = \frac{Q}{P}, \tag{3}$$

which shows that O is on the production of AB at the side of the greater force; and (2) gives

$$R = P - Q. \tag{4}$$

In illustration of this chapter some of the examples in the next are solved by the Principle of Virtual Work.

CHAPTER VII.

APPLICATIONS OF THE CONDITIONS OF EQUILIBRIUM OF A BODY.

112.] **Condition of Equilibrium of a Body under the Action of Two Forces in a Plane.** *If two forces maintain a body in equilibrium, they must be equal and opposite in the same right line.*

For, take moments round any point on the line of action of one of them, P. The sum of the moments must (Art. 104) be $= 0$. Hence the other force, Q, must pass through the assumed point. Again, take any other point on P, and take moments round it. The sum must be $= 0$, and Q must, therefore, pass through this point. Hence P and Q act in the same line. Now their sum must $= 0$ (Art. 103). Therefore P and Q are equal and opposite.—Q. E. D.

113.] **Condition of Equilibrium of a Body under the Action of Three Forces in One Plane.** *If three forces maintain a body in equilibrium, their lines of action must meet in a point, or be parallel.*

For, take moments round the point of intersection of two of them, P and Q. The sum must (Art. 103) $= 0$; therefore, either the third force, R, is zero, or it passes through the intersection of P and Q. If R is not $= 0$, it must pass through this point.

The three forces may then be supposed to act at this point, and to keep it at rest. Hence, each force must be equal and opposite to the resultant of the other two; and if the angles between them in pairs be p, q, r, the forces must satisfy the conditions

$$P : Q : R = \sin p : \sin q : \sin r. \qquad (\beta)$$

If two of them are parallel, the third must be parallel to them and equal and directly opposed to their resultant.

EXAMPLES.

1. Three forces, P, Q, R (fig. 113) act at the middle points of the sides of a triangular plate, each force being perpendicular and proportional to the side at which it acts. If the forces all act inwards, or all outwards, they are in equilibrium. For (α) they satisfy the first conditions of equilibrium of three forces, namely, that of meeting in a point (Art. 113); and (β) they are proportional to the sines of the angles between them in pairs, since

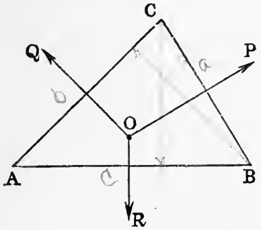

Fig. 113.

$$P : Q : R = a : b : c = \sin A : \sin B : \sin C$$
$$= \sin QOR : \sin ROP : \sin POQ.$$

They, therefore, satisfy both of the conditions of Art. 113.

In exactly the same way it is proved that if three forces act perpendicularly to the sides of a triangle, and be proportional to them, they will be in equilibrium, provided that they pass through any common point, and all act outwards or all inwards.

2. Three forces acting along the perpendiculars of a triangle keep it at rest; find the relations between them.

They satisfy the first condition of equilibrium, namely, that of meeting in a point. Then if the forces perpendicular to the sides a, b, c, be P, Q, R, respectively, the relations (β) of Art. 113 give

$$P : Q : R = \sin A : \sin B : \sin C = a : b : c,$$

as might have been concluded from the remark at the end of the last example.

3. Three forces acting along the bisectors of the angles of a triangle, all either from or towards the angles, keep it at rest; find the relations between them.

The forces evidently satisfy the condition of meeting in a point.

Let P, Q, R, be the forces in the bisectors of A, B, C, respectively. Then the angle between P and Q is easily seen to be $\pi - \dfrac{A+B}{2}$. Hence $P : Q : R = \cos \dfrac{A}{2} : \cos \dfrac{B}{2} : \cos \dfrac{C}{2}$.

4. Three forces acting in the bisectors of the sides of a triangle drawn from the opposite angles maintain equilibrium; find the relations between them.

They satisfy the first condition.

Let the lengths of the bisectors of the sides a, b, c (fig. 114) be β_1, β_2, and β_3, and let p and q be the perpendiculars from C on P and Q.

Take moments round C for the equilibrium of the forces. Then

$$Pp = Qq. \qquad (1)$$

(The moments of P and Q with respect to C have opposite signs, since Q tends to turn the body round C in the sense of watch-hand rotation, while P tends to turn it in the opposite sense).

Again, $\quad p\beta_1 = q\beta_2, \qquad (2)$

each side of this equation being the area of the triangle. Divide the sides of (1) by the corresponding sides of (2).

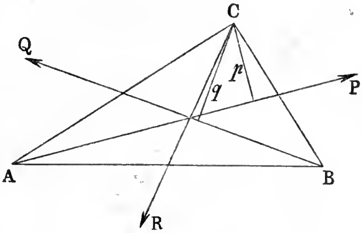

Fig. 114.

Then $$\frac{P}{\beta_1} = \frac{Q}{\beta_2}.$$

Hence $$P : Q : R = \beta_1 : \beta_2 : \beta_3,$$

or the forces are proportional to the bisectors.

5. At the middle points of the sides of any *indeformable* polygon (fig. 115) forces act perpendicularly to the sides, each force being proportional to the side at which it acts. If the forces all act inwards or outwards, they form a system in equilibrium.

For (example 1) the resultant of P_1 and P_2 is a force acting at the middle point of AC, perpendicular and proportional to AC. Again, this force and P_3 may be replaced by a force acting at the middle point of AD, perpendicular and proportional to AD.

Replacing the given forces in this manner, the result follows by example 1.

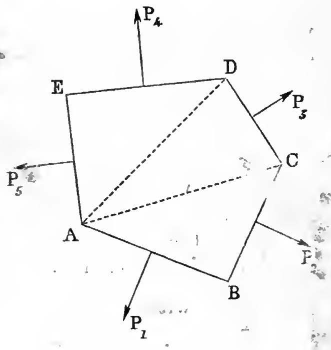

Fig. 115.

6. If from any point perpendiculars be drawn to the sides of a polygon, and forces act along these perpendiculars, either all inwards or all outwards, each force being proportional to the side to which it is perpendicular, the system is in equilibrium.

This follows, exactly as in the last example, by dividing the polygon into triangles, and attending to the remark at the end of example 1.

7. From any point, O, inside (or outside) a triangle, ABC (fig. 116), are let fall perpendiculars, Oa, $O\beta$, $O\gamma$, on the three sides. At the points a, β, γ, are applied forces P, Q, R, each of which is proportional and perpendicular to the side at which it acts. The forces are then all turned round their points of application in the same sense, so as to make equal angles with the perpendiculars Oa, $O\beta$, and $O\gamma$. Show that in this latter case the resultant of the system of forces is

a couple whose moment is proportional to the square root of the area of the triangle $A'B'C'$, enclosed by their lines of action.

(The forces act all outwards or all inwards).

Let the sides of ABC be a, b, c, and let $P = ka$, $Q = kb$, $R = kc$, k being a constant coefficient.

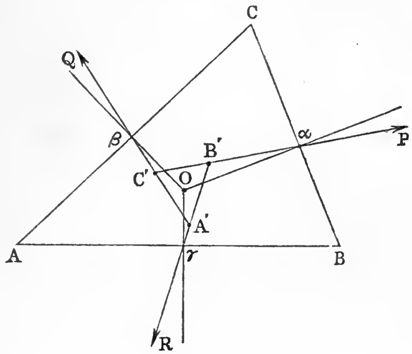

Fig. 116.

Let θ be the angle, OaB', between P and the perpendicular Oa. Then

$$\theta = O\beta C' = O\gamma A'.$$

Replace P by two components, one along BC and the other perpendicular to it. Similarly, replace Q and R. Then the perpendicular components are $ka \cos\theta$, $kb \cos\theta$, and $kc \cos\theta$; and since they meet in a point, O, and are proportional to the sides at which they act, they are in equilibrium (example 1). Hence the forces are equivalent to three, $ka \sin\theta$, $kb \sin\theta$, and $kc \sin\theta$, acting along the sides of ABC in cyclical order, and therefore, by Art. 109, their equivalent is a couple $= 2k\Delta \sin\theta$, Δ denoting the area of the triangle ABC. (See also Art. 93, p. 115.) Now the triangle $A'B'C'$ is similar to ABC. For, since the angles OaB and $O\gamma B$ are right, and the angles OaB' and $O\gamma B'$ are equal, a circle will go round the points $OB'aB\gamma$. Hence $\angle \gamma Oa = \angle \gamma B'a$; therefore their supplements, B and B' are equal. Similarly, $A = A'$, and $C = C'$.

Again, the side $A'B' = AB . \sin\theta$. For in the circle round $\gamma OB'aB$, $\gamma B'$ is a chord making an angle θ with a chord γO, and an angle $\frac{\pi}{2} - \theta$ with the perpendicular chord, γB. Therefore

$$\gamma B' = \gamma O . \cos\theta + \gamma B . \sin\theta. \tag{1}$$

Similarly, in the circle round $\gamma A'O\beta A$, we have

$$\gamma A' = \gamma O . \cos\theta - \gamma A . \sin\theta. \tag{2}$$

Subtracting (2) from (1) we have

$$A'B' = (\gamma B + \gamma A) . \sin\theta = AB . \sin\theta.$$

Now if Δ' be the area of $A'B'C'$,

$$\frac{\Delta'}{\Delta} = \left(\frac{A'B'}{AB}\right)^2 = \sin^2\theta;$$

$$\therefore \quad \sin\theta = \sqrt{\frac{\Delta'}{\Delta}},$$

and therefore the moment of the forces $= 2k\sqrt{\Delta\Delta'}$.

8. If the triangle be replaced by a polygon of any number of sides, prove that the equivalent of the forces is a couple whose moment is proportional to the square root of the area of the (similar) polygon enclosed by their lines of action.

9. A triangular plate, ABC (fig. 117), is acted upon at each angle by forces, along the two sides containing it, represented in magnitudes and lines of action by the distances between the angle and the feet of the perpendiculars let fall from the other two angles on these sides. Find the line of action of the resultant force.

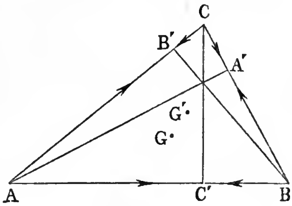

Fig. 117.

Let the perpendiculars let fall on the three sides, a, b, c, from any point, P, on the resultant be x, y, z, respectively, and let A', B', C' be the feet of the perpendiculars. Then the force in AB in the sense AB is $AC'-BC'$, or $b\cos A - a\cos B$. Hence the moment of this force about P is $z\,(b\cos A - a\cos B)$, and since the sum of the moments of all the forces (estimated in cyclical order) round P is $=0$ (Art. 76), we have

$$x(c\cos B - b\cos C) + y(a\cos C - c\cos A) + z(b\cos A - a\cos B) = 0 \ldots \quad (1)$$

Now, one set of values of x, y, and z, which will satisfy this equation, is, evidently, a, b, c. Hence the resultant passes through the point the perpendiculars from which on the sides are proportional to a, b, c. This point is thus found :—Let G be the centre of gravity of the triangle; from A draw a line, AG', which makes $\angle CAG' = \angle BAG$, and from B draw a line, BG', which makes $\angle CBG' = \angle ABG$. These lines intersect in G', the required point.

Again, another set of values of x, y, z, which will satisfy (1), is $\cos A$, $\cos B$, $\cos C$; and the resultant passes through the point whose perpendiculars on the sides are proportional to these quantities. This point is the centre of the circumscribed circle.

Hence the line of action of the resultant is known.

10. Show that the resultant of the system of forces in the last example is

$$\frac{4\Delta}{abc}\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2},$$

where Δ is the area of the triangle.

11. Forces P, Q, R act along the sides of a triangle, ABC, and their resultant passes through the centres of the inscribed and circumscribed circles: prove that

$$\frac{P}{\cos B - \cos C} = \frac{Q}{\cos C - \cos A} = \frac{R}{\cos A - \cos B}$$

(Wolstenholme's Book of Mathematical Problems).

12. A heavy beam, AB (fig. 118), rests against a smooth horizontal plane, CA, and a smooth vertical wall, CB, the lower extremity, A,

being attached to a rope which passes over a smooth pulley at C, and sustains a given weight, P. Find the position of equilibrium, and the pressures on the plane and wall.

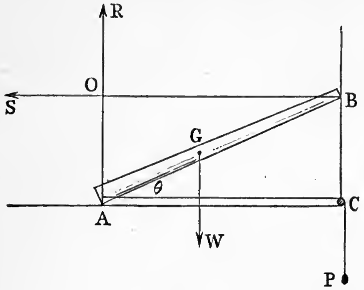

Fig. 118.

Let θ be the inclination of the beam to the horizon in the position of equilibrium; let $W=$ weight of the beam: and let the centre of gravity, G, divide the beam into two portions, $AG=a$, and $BG=b$.

Now, the reactions, R and S, of the wall and plane are normals to these surfaces; and since they are both unknown, we shall obtain an equation for θ which will contain neither of them, by taking moments about O, their point of intersection. Hence, since the force P acts on the beam along AC, and tends to turn it in a sense opposite to that in which W tends to turn it round O, we have

$$P(a+b)\sin\theta - Wa\cos\theta = 0,$$

$$\therefore \quad \tan\theta = \frac{Wa}{P(a+b)}. \tag{1}$$

Again, resolving forces vertically, we have

$$R = W. \tag{2}$$

And resolving horizontally, $\quad S = P. \quad (3)$

Solution by Virtual Work. Imagine a displacement in which the ends A and B remain in contact with the planes. Then the virtual works of R and S are both zero, and the equation of virtual work is (if y is the height of G above the horizontal plane)

$$-W.dy - P.d(AC) = 0. \tag{4}$$

Now $y = a\sin\theta$, $AC = (a+b)\cos\theta$;

$$\therefore \quad dy = a\cos\theta d\theta, \quad d(AC) = -(a+b)\sin\theta d\theta;$$

and (4) gives $\quad Wa\cos\theta = P(a+b)\sin\theta,$

which gives the same value of θ as (1).

13. If the beam rest, as in the last example, against a smooth vertical and a smooth horizontal plane, and a rope be attached firmly to the point C, and to a point in the beam, find the limit to the position of this latter point consistent with equilibrium.

Let fig. 119 represent the beam in any position, and let m be the middle point of the beam. Suppose the rope attached to C, and to a point, n, in the upper half of the beam. Then the forces acting on the beam are W, T (the tension of the rope nC), R, and S. Let p

be the point of intersection of W and T. Now, the resultant of W and T must, for equilibrium, be equal and opposite to the resultant of R and S; hence the resultant of R and S must act in the line Op; but this line is not between the lines of action of T and W, that is, inside the angle WpC; therefore the resultant of R and S cannot be equal and opposite to that of W and T with such a position of the rope, and, therefore, equilibrium is impossible, no matter what the inclination of the beam may be. Hence, in order that equilibrium may be possible, the rope must be attached to some point, such as P, between A and m.

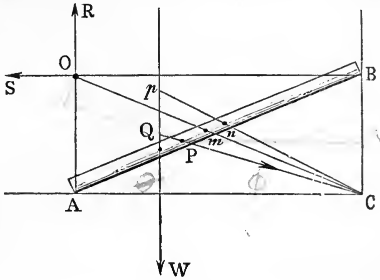

Fig. 119.

14. In the last example, given the point of attachment of the rope, find the tension in it.

It is easy to see that if P, the point of attachment, be given, and also l, the length of the rope, CP, the position of the beam is given. For, if $\theta = \angle BAC$, we have

$$l^2 = BP^2 . \cos^2\theta + AP^2 . \sin^2\theta,$$

an equation which determines θ.

The angle PCA is also known. Denote it by ϕ. To determine T, the tension of the rope, without bringing R and S into our equation, take moments round O, their intersection. Hence, a and b being the segments of the beam made by the centre of gravity, we have

$$Wa \cos\theta = T . OC \sin OCP = T . (a+b) \sin(\theta - \phi),$$

$$\therefore\ T = W . \frac{a \cos\theta}{(a+b)\sin(\theta-\phi)}.$$

To obtain T by the principle of Virtual Work. Choosing a virtual displacement which keeps A and B in contact with the planes, the equation of work is

$$-W dy - T d(PC) = 0, \qquad (1)$$

y denoting the height of G above the horizontal plane.

Now $PC^2 = BP^2 \cos^2\theta + AP^2 \sin^2\theta$, and this equation will also hold in the displaced position. Hence we may differentiate it, and then we obtain

$$PC . d(PC) = -(PB^2 - PA^2) \sin\theta \cos\theta \, d\theta$$
$$= -(a+b)(PB - PA) \sin\theta \cos\theta \, d\theta;$$

or

$$d(PC) = -(a+b)\left(\frac{\cos\phi}{\cos\theta} - \frac{\sin\phi}{\sin\theta}\right) \sin\theta \cos\theta \, d\theta$$
$$= -(a+b) \sin(\theta - \phi) \, d\theta.$$

Also $y = a \sin\theta$, therefore $dy = a \cos\theta d\theta$; and substituting these values of dy and $d(PC)$ in (1), we obtain the same value of T as before.

Note. If $\theta = \phi$, $T = \infty$. In this case the rope is attached to m, the middle point of the beam, and therefore its direction always passes through O, the intersection of R and S. Now, it is easy to see that in this case the conditions of equilibrium are theoretically satisfied, because the resultant of T and W acts along T, whose direction passes through O. But if $\phi > \theta$, no value of T can even theoretically satisfy the conditions (see last example).

15. ABC is any triangle, of which C is the vertex. It is acted on by the forces CA, CB, and AB. Prove that it will be kept in equilibrium by a force equal to $2BC$, acting parallel to BC, at the middle point of AB.

16. In example 12, it is clear that two positions of equilibrium of the beam are a vertical and a horizontal position; explain why these positions are not given by the equation (1) which determines the position of equilibrium.

17. Explain why the proof in example 5 would not hold for a polygon formed of bars freely jointed together and therefore capable of turning about the joints.

114.] **Action of a Hinge or Joint.** Among the internal forces of a system, the action of a joint is one of frequent occurrence. If the joint be smooth, the reaction between two bars or beams connected by it consists of a single force. For, let PQS (fig. 120) represent a section of the joint connecting two beams: then, since their surfaces are in contact, either throughout the whole of the circumference or a part of it, there will be (since the joint is smooth) *normal* reactions at the points of contact, P, Q, ... Now, since all these pass through the centre of the circle, they have a single resultant. Consequently, the action in this case consists of a single force.

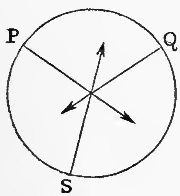

Fig. 120.

But, if the joint be rough, the reactions at the points of contact will not be normal, that is, their lines of action will not meet in a point, and, therefore, they may reduce to a couple, or to a single force. When slipping is about to ensue at the joint, it is easy to see that the total resistances at the points of contact envelop a circle (or rather a cylinder). For, at any point, P, of contact (fig. 121), draw PR, making the angle of friction, λ, with the normal, PC, to the surface of contact. The perpendicular from C, the centre of the joint, is equal

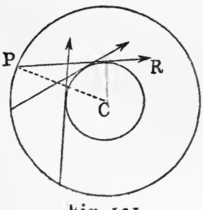

Fig. 121.

to $PC \,.\, \sin \lambda$, and is, therefore, constant. Hence, PR envelops a circle whose radius $= PC \,.\, \sin \lambda$.

If $PC = a$, and ds is the element of the surface of contact at P, it is evident that the sum of the moments of the reactions about C is (R being the reaction per unit of surface)

$$a \sin \lambda \int R ds.$$

As an example, let us consider the equilibrium of two equal beams which are connected by a joint, C, and rests on a perfectly smooth cylinder, in a vertical plane at right angles to the axis of the cylinder.

Firstly, let the joint be rough, and suppose the contact to be complete all over its surface: then it is clear that such a position as that represented in fig. 122 is a possible position of equilibrium if the joint is sufficiently rough. Let fig. 120 represent an enlarged view of the circle which is enveloped by the total resistances at the various points of the surface of contact at the hinge, C. Then, if the total resistances at the lower portion of the joint be considerably greater than those at the upper portion, it is possible that the resultant of the whole set may be a horizontal force, R, acting through a point, P, below the joint.

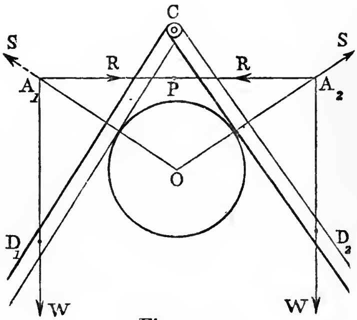

Fig. 122.

In the position of equilibrium of the beams represented in fig. 122 the weight, W, of the beam CD_1, and the normal reaction, S, of the smooth cylinder, meet in a point A_1, through which point the force produced by the action of the other beam must pass. In the same way the action of the beam CD_1 on CD_2 must pass through the point A_2. Hence the resultant action of each beam on the other must be directed in the line $A_1 A_2$; and we have seen that if the contact along the joint extend

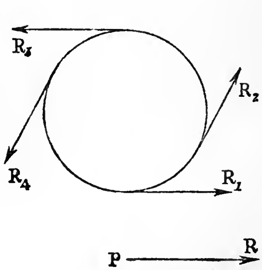

Fig. 123.

over its surface, this is a possible line of action, though it does not intersect the joint.

Secondly, let the joint be rough, and let the contact take place at only one point, N (fig. 124). Suppose the joint to consist of a pin, BN, which forms part of the beam CD_2 (fig. 119), and let this fit loosely into the beam CD_1. It is clear, then, that the action between the beams consists of a single force, R, acting at N, and making the angle of friction, λ, with the radius CN, if slipping is about to take place. As before, this force must pass through the points A_1, A_2.

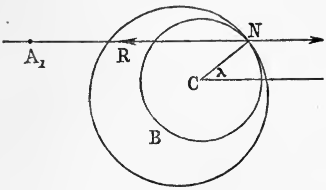

Fig. 124.

In this case, then, the point of contact of the beams is constructed by drawing a radius, CN, of the cylindrical axis constituting the joint, inclined to the horizon (since $A_1 A_2$ is horizontal) at the angle of friction.

Thirdly, let the joint be smooth. In this case the beams must assume such a position that the line $A_1 A_2$ passes through the centre of the joint; and this position is practically the same as that in the last case, because since the dimensions of the joint are negligible compared with those of the beams, the line of resistance RN (fig. 124) may be supposed to pass through the centre, C, of the joint.

A similiar explanation is to be given in the case of two equal beams rigidly connected, and forming one piece, the system resting, as in the previous example, on a smooth cylinder. In this case the beams can take only one position, which must be a position of equilibrium, and the action between them must accommodate itself to the geometrical necessity of the figure. (In the following figure the cylinder is not drawn.) If we consider the equilibrium of one of the beams, CD (fig. 125), by itself, we shall have to supply to it whatever force is actually produced upon it by the other beam. Now, if BC is the section along which the

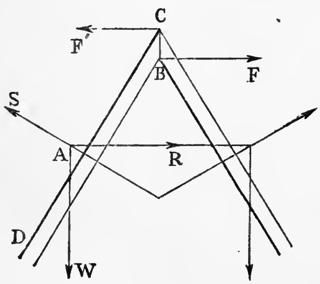

Fig. 125.

system is considered as divided by the removal of the second beam, it is clear that the internal forces in the neighbourhood of B tend to tear the beams apart, if A is below the section BC, while those about C tend to press the beams more closely together. Hence the action of the second beam on CD consists of a number of forces whose horizontal components near B act from left to right, as the force BF, and whose horizontal components near C act from right to left, as the force CF'. If, therefore, the forces near B are greater than those near C, the resultant of the whole system will consist of a horizontal force, AR, acting outside the section CB, so as to pass through the point, A, of intersection of the weight and the normal reaction of the cylinder. In this case, then, the action, over a section BC, between two rigidly connected pieces consists of a force outside the section; which force may, of course, be replaced by one at any point in the section, together with an accompanying couple (see Art. 74).

In all cases in which contact over a finite surface takes place between two bodies, the student must be careful *to examine the nature of the forces exerted between them at the individual points of contact with a view to ascertaining whether the resultant action of one on the other consists of a single force at all; or, if so, whether it can be assumed to act at any point in the surface of contact or must be assumed to act wholly outside it.*

115.] **Geometrico-statical Problems.** In many statical problems which relate to the positions of equilibrium of bodies the result is independent of the magnitude of some given force, and such independence can be perceived *à priori*. Thus, suppose the question to be—What is the limiting inclination to the horizon of a heavy uniform beam which rests against a rough vertical and a rough horizontal plane? In this problem we may, if we please, assume W, the weight of the beam, and $2a$, its length; but it is evident *à priori* that the result cannot involve either of these quantities. For, if the angle which the beam makes with the ground be θ, the position of equilibrium will be defined by some of the trigonometrical functions of θ, such as $\sin\theta$ or $\tan\theta$. Now, the trigonometrical function of an angle are mere numbers, or ratios of quantities *of the same kind*. Hence, if the expression for $\tan\theta$ (suppose) involve *force*, it must involve the *ratio of one force to another force*, and if there is only *one* force given in the problem, we have no other force to combine with it in the form of a ratio or a mere number. Consequently, the weight of the beam can in no way influence its limiting inclination. Precisely similiar remarks hold with regard to the only *linear* magnitude in the question, viz., the length of the beam. There is no other quantity of the same kind with which to compare it. Therefore, we are enabled to state *à priori* that the inclination of the beam to the horizon in its limiting position of equilibrium depends simply on the coefficients of friction for the beam and the two rough planes, or that

$$\theta = f(\mu, \mu'),$$

μ and μ' being these coefficients, and f denoting some (as yet) unknown function.

Again, suppose the question to be—What force applied to one of the handles of a table drawer will pull the drawer out ?* It is evident that the answer must be either—no force, however great, will pull it out, or—any force, however small, will pull it out. And the result will depend simply upon the relation between the coefficient of friction for the drawer and the table, and the ratio of the side of the drawer to the distance between the handles. This is evident, because there is no given force in terms of which the required force could be expressed.

Numerous examples of this class of questions will be given in the sequel. Such problems, then, in which the result is independent of a force magnitude, we shall classify as Geometrico-statical Problems, because, though they involve conceptions concerning the *directions* of forces, they do not involve their *magnitudes*. In all such problems, once the requisite theorems concerning the directions of forces are made use of, the result follows at once from the geometry of the figure; and a solution by the method of resolving forces and taking moments is, in reality, an illogical process.

116.] **Useful Trigonometrical Theorem.** In connexion with the class of geometrico-statical problems, the following theorem in Plane Trigonometry will be found extremely useful :—

If a right line, CP (fig. 126), drawn from the vertex of a triangle, divide the base into two segments m and n, or segments which are to each other in the ratio of m to n,

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* The friction of the bottom is neglected.

$$(m+n)\cot\theta = m\cot a - n\cot\beta, \tag{1}$$

a and β being the angles which CP makes with the sides AC and BC, and θ the angle which CP makes with the base.

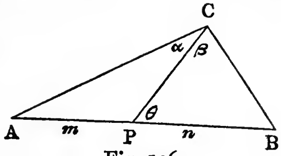

Fig. 126.

For, if $AP = m$, and $BP = n$,

$$CP = m\frac{\sin A}{\sin a} = m\frac{\sin(\theta - a)}{\sin a} = m(\sin\theta\cot a - \cos\theta).$$

Also,

$$CP = n\frac{\sin B}{\sin\beta} = n\frac{\sin(\theta+\beta)}{\sin\beta} = n(\sin\theta\cot\beta + \cos\theta).$$

Hence

$$m(\sin\theta\cot a - \cos\theta) = n(\sin\theta\cot\beta + \cos\theta),$$

from which (1) follows at once.

We have also the equation

$$(m+n)\cot\theta = n\cot A - m\cot B. \tag{2}$$

For,

$$CP = m\frac{\sin A}{\sin a} = m\frac{\sin A}{\sin(\theta - A)} = \frac{m}{\sin\theta\cot A - \cos\theta}.$$

Similarly, $$CP = \frac{n}{\sin\theta\cot B + \cos\theta};$$

therefore, &c.—Q. E. D.

Examples.

1. A heavy beam rests on two smooth inclined planes whose intersection is a horizontal line, the beam lying in a vertical plane perpendicular to this line of intersection; find the position of equilibrium and the pressures on the planes.

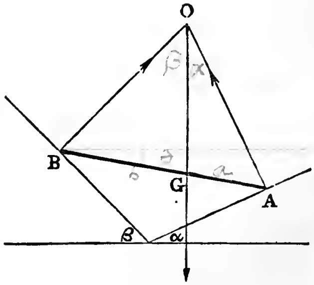

Fig. 127.

Let a and b be the segments, AG and BG, of the beam, made by its centre of gravity, G; θ the inclination of the beam to the horizon, α and β the inclinations of the planes, R and R' the pressures on these planes, respectively, and W the weight of the beam.

Then, since the beam is in equilibrium under the action of only three forces, they must meet in a point, O.

Now the angles GOA and GOB are equal to a and β, respectively, and $BGO = \frac{\pi}{2} - \theta$. Hence

$$(a+b) \cot BGO = a \cot GOA - b \cot GOB,$$

or
$$(a+b) \tan \theta = a \cot a - b \cot \beta, \tag{1}$$

which determines the position of equilibrium.

Again, by the relations between three forces in equilibrium,

$$R = W \frac{\sin \beta}{\sin (a+\beta)}, \tag{2}$$

$$R' = W \frac{\sin a}{\sin (a+\beta)}. \tag{3}$$

Hence, if $\frac{a}{b} = \frac{\tan a}{\tan \beta}$, the beam will rest in a horizontal position.

Suppose that $a \cot a - b \cot \beta$ is positive, and that $(a+b) \tan \beta < a \cot a - b \cot \beta$. Then, *à fortiori* $(a+b) \tan \theta < a \cot a - b \cot \beta$, since θ, the angle made with the horizon by the beam in any such position as AB, is necessarily $< \beta$.

Hence, the only position of equilibrium possible is either one of continued contact with the plane (β), or one of continued contact with the plane (a). Suppose the first, as in fig. 128. To find in this case the point through which the resultant pressure of the plane (β) on the beam acts, draw AO perpendicular to the plane (a); then AO is the line of action of the pressure on this plane.

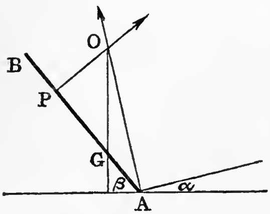

Fig. 128.

Let AO meet the vertical through G in O, and from O draw OP perpendicular to the plane (β). Evidently, P is the point at which the resultant pressure of the plane (β) acts.

But it may now be shown that, with the two inequalities supposed, this position is impossible. For if $AP > a+b$, it will be impossible; that is, if $a \frac{\cos \beta \sin (a+\beta)}{\sin a} > a+b$; or $a \tan \beta (\cot a - \tan \beta) > b$; or $a \cot a - b \cot \beta > a \tan \beta$. But, by supposition $a \cot a - b \cot \beta$ is positive and $> (a+b) \tan \beta$, therefore $AP > AB$, which is manifestly impossible. Hence the only position of equilibrium in this case is one of continuous contact with the plane (a). [We have supposed all through that the end A of the beam is to rest on the plane (a).] The least inclination of the plane (a) which will allow of a position of continuous contact with (β) is found by drawing at B a perpendicular to the plane (β) and joining its point of intersection with the vertical through G with A. The joining line is the normal to the plane of least inclination (a).

2. A uniform heavy beam, AB (fig. 129), rests with one extremity, A, against the internal surface of a smooth fixed hemispherical bowl, while it is supported at some point in its length by the rim of the bowl; find the position of equilibrium.

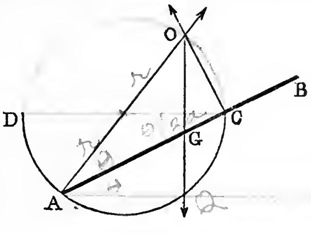

Fig. 129.

It is *à priori* evident that the result must be independent of force, since the weight of the beam is the only force that may be supposed to be given; and it is also evident that the result depends on the only two linear magnitudes which may be supposed to be given—viz., the length of the beam, $2a$, and the radius, r, of the bowl.

Draw the three forces which keep the beam in equilibrium. They are the weight, a reaction at A perpendicular to the *surface of contact*, and therefore perpendicular to the bowl, and a reaction at C which for the same reason is perpendicular to the beam. These must meet in a point, O. Let $\theta =$ the inclination of the beam to the horizon $= \angle ACD$. Let the line OG meet the semicircle DAC in the point Q. Then AQ is a horizontal line. Also $\angle QAG = \angle DCA = \theta$, therefore $\angle OAQ = 2\theta$. Hence $AQ = AO \cos 2\theta$, and also $AQ = AG \cos\theta$; therefore $2r\cos 2\theta = a\cos\theta$, $\cos 2\theta = 2c$

or $$4r\cos^2\theta - a\cos\theta - 2r = 0.$$

This equation gives two values of $\cos\theta$, one of which supposes the hemisphere to be completed into a sphere, the end A of the beam to rest against the *upper* portion of the sphere, and the action of the sphere on A to consist of a *pull*. The student will have no difficulty in representing this position, or in proving that the reaction at $C = W\dfrac{a}{2r}$.

$C : W :: \sin AOG : \sin AOC :: \sin(90^\circ - 2\theta) : \sin(90^\circ - \theta)$
$:: \cos 2\theta : \cos\theta :: a : 2r.$

3. Find the position of equilibrium of a uniform heavy beam, one end of which rests against a smooth vertical plane, and the other against the internal surface of a given fixed smooth sphere.

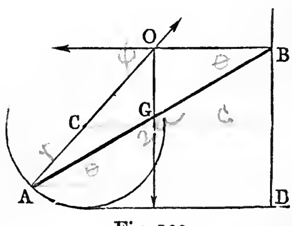

Fig. 130

Let the length of the beam, AB, $= 2a$, $r =$ the radius of the sphere, $c =$ the distance of the centre, C, of the sphere from the vertical wall, DB; also let $\theta =$ the required inclination of the beam to the horizon, and $\phi =$ the inclination of the radius CA to the horizon.

The statics of the problem is exhausted in drawing the figure so that the weight of the beam and the two reactions at A and B shall meet in a point, O. Geometry then gives

$$2\cot OGB = \cot AOG - \cot GOB = \cot AOG,$$

or $$2\tan\theta = \tan\phi. \qquad (1)$$

Again, the perpendicular distance between A and DB is $2a\cos\theta$; but it is also evidently equal to the horizontal projection of CA + the distance of C from BD; that is,

$$2a\cos\theta = r\cos\phi + c. \tag{2}$$

From (1) and (2) a value of θ can be obtained, and hence the position of equilibrium.

If the beam rest on the convex surface, the only change in the equations will be a change of the sign of c in (2).

4. The extremities of a beam rest at two given points against two given smooth curves in the same vertical plane; the beam is to be sustained by a rope attached to its centre of gravity and to a fixed point. Determine the position of this point so that the rope may be the weakest possible.

Let AB (fig. 131) be the beam, G its centre of gravity, O the point of intersection of the normal reactions of the curves A and B; k the length of the perpendicular from O on the line of action of the weight, W, of the beam; p the perpendicular from O on the line, GP, of the rope, and T the tension of the rope.

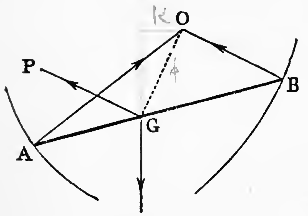

Fig. 131.

Then, taking moments about O,

$$T.p = W.k,$$

or

$$T = W\frac{k}{p}.$$

Hence, since W and k are given, T will be a minimum when p is a maximum. But the maximum value of the perpendicular from O on a right line through G is OG; hence the rope must assume a direction perpendicular to OG.

5. A heavy uniform trap-door, AB (fig. 132), is moveable about a hinge-line represented by A; and to the middle point, B, of the opposite edge is attached a string, BC, the extremity C of the string being fastened to the point occupied by B when the door is horizontal. Given the length of the string, find the magnitude and direction of the pressure on the hinge-line, and the tension of the string.

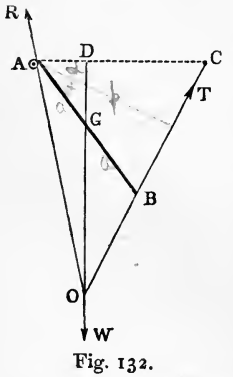

Fig. 132.

Produce the line of the string to meet the line of action of the weight in a point, O. Then, since the door is in equilibrium under the influence of only three forces, they must meet in a point. Hence the pressure on the hinge-line must pass through O, and since the plane of the tension, T, and the weight, W, intersects the hinge-line at A, the pressure, R, must act through A (the hinge being smooth).

To determine T take moments about A. Then, if $p =$ the perpendicular from A on BC, $T.p = W.AD$. (1)

Let the angle $BAC = 2\alpha$, and let $AB = 2a$. Then $p = 2a\cos\alpha$, $AD = a\cos 2\alpha$, therefore

$$T = \tfrac{1}{2}W\frac{\cos 2\alpha}{\cos\alpha}.$$

Again, by the triangle of forces we have

$$R^2 = W^2 + T^2 - 2TW\cos\alpha;$$

and substituting the above value of T, this gives

$$R = \tfrac{1}{2}W\sqrt{4\sin^2\alpha + \sec^2\alpha}.$$

The values of T and R can be at once found in terms of the lengths AB and BC. Denoting the latter by $2l$, we have $\sin\alpha = \frac{l}{2a}$, therefore, &c.

6. If in the last example the string, instead of being attached to C, pass over a smooth pulley at that point, and sustain a given weight, find the position of equilibrium, and the pressure on the hinge-line.

Let P be the suspended weight, and $\theta = \angle CAB$; then the position of equilibrium is defined by the equation

$$\cos^2\frac{\theta}{2} - \frac{P}{W}\cos\frac{\theta}{2} - \frac{1}{2} = 0, \qquad (1)$$

and

$$R^2 = P^2 - 2PW\cos\frac{\theta}{2} + W^2. \qquad (2)$$

Equation (1) gives two positions of equilibrium, and since it shows that one of the values of $\cos\frac{\theta}{2}$ is negative, one position corresponds to a value of θ greater than 180°. Such a position, of course, supposes the door capable of revolving freely about its hinge-line through four right angles.

The student will have no difficulty in representing the position of the door in this case, or in explaining why no linear magnitude enters into the equations.

7. A uniform heavy beam, AB, rests against a smooth peg, P, and against a smooth vertical wall, AD; find the position of equilibrium and the pressures on the wall and peg.

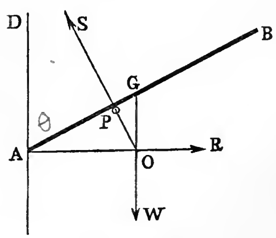

Fig. 133.

This, so far as it relates simply to the position of equilibrium, is another geometrico-statical problem. We have merely to draw AB in such a manner that the vertical through G and the perpendiculars at A and P to the wall and beam shall intersect in a common point, O.

Let $2a =$ the length of the beam, and $c =$ the perpendicular distance of the peg from the wall. Then the position must evidently be expressed as a function of $\frac{c}{a}$. Let θ = the inclination of the beam to the vertical.

Then $AP = \frac{c}{\sin\theta}$, and $AO = \frac{c}{\sin^2\theta}$. But $AO = AG.\sin\theta$; therefore

$$\frac{c}{\sin^2\theta} = a\sin\theta, \qquad \therefore \quad \sin\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}}. \tag{1}$$

Resolving vertically, $\quad S.\sin\theta = W,$

$$\therefore \quad S = W\left(\frac{a}{c}\right)^{\frac{1}{3}}. \tag{2}$$

Resolving horizontally, $\quad S\cos\theta = R,$

$$\therefore \quad R = W\frac{\sqrt{a^{\frac{2}{3}} - c^{\frac{2}{3}}}}{c^{\frac{1}{3}}}. \tag{3}$$

8. A triangular board, BCA (fig. 134), of uniform thickness, rests on two smooth pegs, P and Q, at a given distance from each other, in the same horizontal line. Find its position of equilibrium.

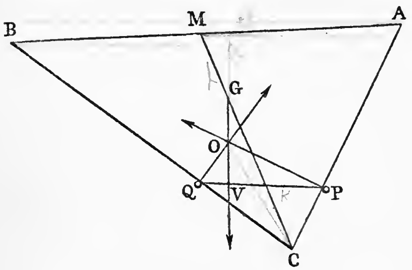

Fig. 134.

The position of equilibrium will evidently be known if the inclination of AB to the horizon is known.

Let this inclination be θ; let the angles of the triangle be denoted by A, B, C; let $a = \angle AMC$, which the bisector, CM, of the base makes with the base; let $CM = l$, and let $PQ = k$.

Then, since no force is given except the weight of the board, θ will depend simply on A, B, C, l, and k, and the problem is geometrical. The reactions of the pegs P and Q are perpendicular to AC and BC, respectively, and they must meet the weight of the board acting through its centre of gravity, G, in a point O. The geometry which gives the solution will express that

$$CO.\sin COV = CG.\sin CGO. \tag{1}$$

Now, $\angle CGO = \frac{\pi}{2} + \theta - a$, and $COV = COQ - VOQ$; but $COQ = QPC$ (since the quadrilateral $QOPC$ is inscribable in a circle) $= A + \theta$; and VOQ evidently $= B - \theta$: therefore $COV = A - B + 2\theta$. Also CO is the diameter of the circle round $QOPC$, a circle in which the chord PQ subtends at the circumference an angle $= C$;

$$\therefore \quad CO = \frac{PQ}{\sin C} = \frac{k}{\sin C}.$$

Then, since $CG = \frac{2}{3}l$, (1) becomes

CO : QC :: sin 90° : sin QOC
QP : QC :: sin C : sin QPC but ∠QOC = ∠QPC
∴ CO : QP :: 1 : sin C ∴ CO = PQ/...

$$k \sin (A - B + 2\theta) = \frac{2}{3} l \sin C . \cos (a - \theta), \tag{2}$$

an equation which determines θ.

9. Two heavy uniform rods, AB and BC (fig. 135), are connected by a smooth joint at B, and, by means of rings at A and C, are also connected with two smooth rods, AD and CD, fixed in a vertical plane. Find the reaction at the joint, the pressures at the rings, and the inclinations of the rods to the vertical in the position of equilibrium.

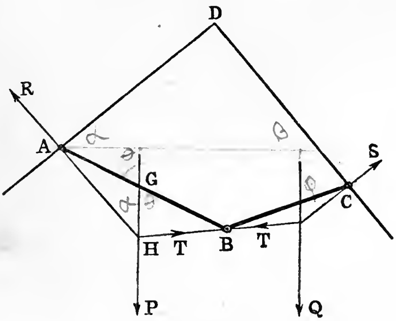

Fig. 135.

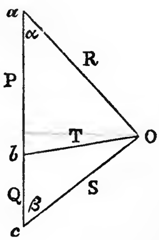

Fig. 136.

Starting from any point, O (fig. 136), draw a force diagram of the system. Let Oa be parallel and proportional to the reaction, R, at A; let ab represent P, the weight of AB: then bO represents T, the reaction at B. In the same way let bc and cO represent Q, the weight of BC, and S the reaction at C. Let a and β be the inclinations of AD and DC to the horizon, θ and ϕ the inclinations of AB and BC to the vertical.

Then we have (from fig. 136)

$$R = (P+Q) \frac{\sin \beta}{\sin (a+\beta)}, \tag{1}$$

$$S = (P+Q) \frac{\sin a}{\sin (a+\beta)}. \tag{2}$$

Also $T^2 = P^2 - 2PR \cos a + R^2$, which, by the substitution of the value of R from (1), becomes

$$T^2 \sin^2 (a+\beta) = P^2 \sin^2 a - 2PQ \sin a \sin \beta \cos (a+\beta) + Q^2 \sin^2 \beta. \tag{3}$$

Again, $\theta = HGB$, and evidently (Art. 116),

$$2 \cot \theta = \cot AHG - \cot GHB$$
$$= \cot a - \cot abO \text{ (fig. 136)}.$$

Now, $\cot abO = \dfrac{P - R \cos a}{R \sin a} = \dfrac{P \cot \beta - Q \cot a}{P+Q}$, by equation (1).

Hence $$\cot \theta = \frac{P(\cot a - \cot \beta) + 2Q \cot a}{2(P+Q)}, \tag{4}$$

and we find a similar expression for $\cot \phi$.

10. A board, $ABC, \ldots$ (fig. 137), in the shape of a regular polygon of n sides, rests at one corner, A, against a smooth vertical wall, AP, the adjacent corner, B, being attached to the wall by a string whose length is equal to the side of the polygon. Find the position of equilibrium.

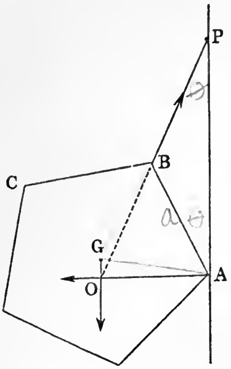

Fig. 137.

Let θ be the inclination, BAP, of the side AB to the vertical; and let O be the point in which the lines of action of the normal pressure at A, the weight of the board, and the tension of the string meet. Then, to determine θ, we have

$$OA = AP \tan\theta,$$

and $$OA = AG\cos OAG = AG\sin GAP,$$

$$\therefore \quad AP\tan\theta = AG\sin GAP.$$

Now, $GAP = GAB + \theta = \frac{\pi}{2} - \frac{\pi}{n} + \theta$; and if $a =$ the side AB, $AP = 2a\cos\theta$; $AG = \frac{a}{2\cos GAB}$; therefore

Since $AGB = \ldots$ $aGA = \ldots$

$$4\sin\theta\sin\frac{\pi}{n} = \cos\left(\frac{\pi}{n} - \theta\right),$$

$2a\cos\theta\tan\theta = \frac{a}{2\sin\frac{\pi}{n}}\cdot\cos\left(\frac{\pi}{n} \ldots\right.$

or

$$* \quad \tan\theta = \frac{1}{3}\cot\frac{\pi}{n}.$$

$\therefore 4\sin\theta\sin\frac{\pi}{n} = \cos\left(\frac{\pi}{n} - \ldots\right.$

This equation determines the position of equilibrium.

The pressure at A is evidently equal to $\frac{W}{3}\cot\frac{\pi}{n}$, W being the weight of the board.

The external angle of the polygon being equal to $\frac{2\pi}{n}$, the inclinations of the successive sides to the vertical are

$$\theta,\ \theta + \frac{2\pi}{n},\ \theta + \frac{4\pi}{n},\ \theta + \frac{6\pi}{n},\ \ldots;$$

and if p_m be the perpendicular distance of the m^{th} vertex from the wall, counting B as the first, we have

$$p_m = a\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \ldots + \sin\left(\theta + \frac{2(m-1)\pi}{n}\right)\right\},$$

or

$$p_m = 2p_1 \frac{\sin\frac{m\pi}{n}}{\sin\frac{2\pi}{n}}\left(2\cos\frac{m-2}{n}\pi - \cos\frac{m\pi}{n}\right).$$

11. A heavy plane body, ABC (fig. 138), of any shape, is suspended from a smooth peg, fixed in a vertical wall, by means of a string of given length, the extremities of which are attached to two fixed

$* \tan\theta = \frac{\cos\frac{\pi}{n}\cos\theta + \sin\frac{\pi}{n}\sin\theta}{4\cos\theta\sin\frac{\pi}{n}} = \frac{1}{4}\cot\frac{\pi}{n} + \frac{1}{4}\tan\theta$

$\frac{3}{4}\tan\theta = \frac{1}{4}\cot\frac{\pi}{n} \quad \therefore \tan\theta = \frac{1}{3}\cot\frac{\pi}{n}$

points, F and F', in the body. Determine the positions of equilibrium.

Let the ellipse $P_1 P_2 P_3$ be described with foci F and F', and axis major equal to the length of the string. The peg will then be somewhere on this ellipse, suppose at P_2. Now, when the body is suspended from the peg, it is kept in equilibrium by its own weight acting vertically through the centre of gravity, and the two tensions in P_2F and P_2F'. But since the peg is smooth, these tensions are equal, and their resultant must bisect the angle FP_2F'; its line of action is, therefore, normal to the ellipse. And if G is the centre of gravity of the body, the resultant tension must pass through G, and be equal and opposite to the weight of the body. Hence the problem is solved by drawing normals from G to the ellipse, and then hanging the figure from the peg in such a manner that any one of these normals is vertical. Now, if G is inside the evolute, four normals can be drawn to the ellipse; but it is easy to see that only three are relevant to the solution if G is inside the lower half of the evolute (as in fig. 135), or only one if G is inside the upper half. For the tangents drawn to the lower half of the evolute belong to the upper half of the ellipse; and in order that the strings should be stretched, it is necessary that the peg should lie somewhere in the upper half of the ellipse. If GP_1, GP_2, and GP_3, are the normals drawn from G, the figure must be placed in a position in which any one of these lines is vertical.

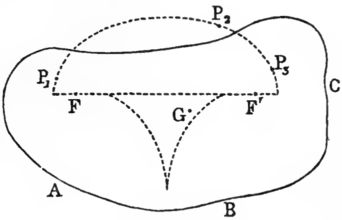

Fig. 138.

12. A beam, whose centre of gravity divides it into two segments a and b, is placed inside a smooth sphere; find the position of equilibrium.

Ans. Let θ be the inclination of the beam to the horizon, and $2a$ the angle subtended by the beam at the centre of the sphere; then

$$\tan\theta = \frac{a-b}{a+b}\tan a.$$

13. A heavy carriage wheel is to be dragged over an obstacle on a horizontal plane by a horizontal force applied to the centre of the wheel; find the magnitude of the required force.

Ans. Let W be the weight and r the radius of the wheel, h the height of the obstacle, and F the requisite force: then

$$F = W\frac{\sqrt{2rh-h^2}}{r-h}.$$

14. If it be attempted to drag the wheel over a smooth obstacle by means of a force whose line of action does not pass through the centre, what happens? Is the result in last example modified if there is friction between the wheel and the obstacle?

15. A heavy uniform beam, moveable in a vertical plane about a smooth hinge fixed at one extremity, is to be sustained in a given position by means of a rope attached to the other extremity; find, geometrically, the least value of the pressure on the hinge, and the corresponding direction of the rope.

Ans. The least pressure on the hinge $= \frac{1}{2} W \sin a$, W being the weight of the beam and a its inclination to the vertical. Also if θ is the angle made by the rope with the vertical when the pressure is least,

$$\cot \theta = 2 \cot a + \tan a.$$

16. A vertical post, loosely fitted into the ground, is exposed to a uniform gale of wind; a rope of given length is to be attached to the post and to the ground; find how the attachment is to be made, in order that the rope may be least likely to break.

Ans. If h is the height of the post and if the length of the rope is $< h\sqrt{2}$, the rope must make an angle of $45°$ with the horizon; but if the length is $> h\sqrt{2}$, the rope must be attached to the top of the post. (See example 4.)

17. A heavy beam rests with one extremity placed at the line of intersection of a smooth horizontal and a smooth inclined plane, the other extremity being attached to a rope which, passing over a smooth pulley at a given point in the inclined plane, sustains a given weight; find the position of equilibrium.

Ans. Let θ be the inclination of the beam, a the inclination of the plane, and ϕ the inclination of the rope, to the horizon; a the distance of the centre of gravity of beam, b the distance of the pulley, from the line of intersection of the planes; and l the length of the beam. Then the position of equilibrium is defined by the equations

$$Wa \cos \theta = Pb \sin (\alpha - \phi),$$
$$b \sin (a - \phi) = l \sin (\theta + \phi).$$

18. A heavy uniform beam, AB, rests with one end, B, against a smooth inclined plane, while the other end, A, is connected with a rope which passes over a pulley and supports a given weight; find the position of equilibrium.

Ans. If a, θ, and ϕ, are the inclinations of the plane, beam, and rope to the horizon, W and P the weight of the beam and the suspended weight, respectively, the position of equilibrium is defined by the equations

$$P \cos (\phi - a) = W \sin a,$$
$$2 \tan \theta = \tan \phi - \cot a.$$

The student will easily explain why no *linear* magnitude enters into the result.

19. A rectangular board, $ABCD$, of uniform thickness, is moveable in a vertical plane about a smooth hinge, P, in the side AD; the side AB is to rest, at a given inclination to the horizon, against a smooth peg, Q: find the position of this peg when the pressure on the hinge is equal to the weight of the board.

Ans. Let O be the point of meeting of the forces which keep the board in equilibrium, and G the centre of gravity of the board. Then QO must bisect the angle POG. Hence from P draw a line, PO, making the same angle with the side AB as AB makes with the vertical; and from the point, O, of intersection of this line with the vertical through G draw a perpendicular, OQ, on AB. This determines Q.

20. A heavy body of any form is moveable round a smooth axis perpendicular to the vertical plane passing through the centre of gravity, and is sustained in a given position by a rope whose weight may be neglected. If the pressure on the axis bears a constant ratio to the weight of the body, prove that the direction of the rope must be a tangent to a conic whose directrix is the vertical line through the centre of gravity, and focus the point in which the axis of suspension cuts the above-mentioned vertical plane.

If, in the last example, QO be the direction of the rope, the ratio $\dfrac{\sin POQ}{\sin QOG}$ is given, and the envelope of QO, as the direction PO varies, is a conic whose focus is P, directrix GO, and eccentricity the given ratio.

21. In example 19, if the hinge is at the corner A, and the position of the peg is given, find the magnitude of the pressure on the hinge.

Ans. Let c = half the length of the diagonal, α = angle between the diagonal and the side AB, x = the distance of peg from A, β = inclination of AB to the vertical; then the pressure on the hinge is

$$W \cdot \frac{\sqrt{x^2 - 2cx \sin\beta \sin(\alpha+\beta) + c^2 \sin^2(\alpha+\beta)}}{x}.$$

22. In the last example, find the position of the peg when the pressure on the hinge is a minimum, and the minimum value.

Ans. At the point in AB vertically under the centre of gravity of the board. The minimum pressure $= W\cos\beta$.

23. A rectangular board of uniform thickness rests in a vertical plane, with two of its adjacent sides in contact with two smooth pegs in the same horizontal line; find the position of equilibrium.

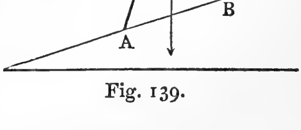

Fig. 139.

Ans. If P and Q (see fig. 134) be the two pegs, CA and CB the sides in contact with P and Q, respectively, α the angle made by the diagonal CD with CB, θ the inclination of this diagonal to the horizon, c half the length of the diagonal, and l the distance PQ, the position of equilibrium is given by the equation

$$c\cos\theta = l\cos 2(\theta - \alpha).$$

24. A triangular board, ABC (fig. 139), of uniform thickness, is placed with its base on a smooth inclined plane, its vertex being connected with a string which passes over a smooth pulley and sustains a weight. Find the conditions of equilibrium.

Ans. Assuming the inclination of the plane to be fixed, the string must take such a direction that the perpendicular let fall on the plane from the point of intersection of the string with the vertical line, Gm through the centre of gravity of the board, falls inside the base. Hence, if Bp be the perpendicular at the extreme point of the base, and if the string cannot cross the surface of the board, all possible directions of the string are included between Cm and Cp. Again, supposing the string to have a direction, Cn, consistent with the possibility of equilibrium, the weight P and the reaction of the plane are thus found: From n let fall a perpendicular on AB, meeting it in a point, q, suppose. Then qn is the line of action of the reaction on the plane: and, resolving along the plane, we have $W \sin i = P \cos \theta$, i being the inclination of the plane, and θ the angle which the string Cn makes with the plane. This equation determines the magnitude of P corresponding to the direction, Cn, of the string. If P is a little greater than the value thus found, the board will begin to slip up, and if P is less than this value, the board will begin to slip down the plane.

25. If, in the last example, the string is parallel to the plane, find the greatest inclination of the plane consistent with equilibrium.

Ans. $\text{Tan}^{-1}(\frac{1}{2} \cot A + \cot B)$.

26. If, in the same example, the string, instead of passing over a pulley and sustaining a weight, is knotted to a fixed peg, how are the previous conditions of equilibrium modified?

Ans. The only condition to be satisfied is that which has reference to the direction of the string. This direction must be somewhere between Cm and Cp.

27. A rectangular board is sustained on a smooth inclined plane by a string attached to its upper corner; the string passes over a smooth pulley and sustains a weight. Find the magnitude of this weight corresponding to a given direction of the string, and find also the pressure on the plane.

Ans. Let i be the inclination of the plane, θ the angle made by the string with the plane, W the weight of the board, P the suspended weight, and R the pressure; then

$$P = W \frac{\sin i}{\cos \theta},$$

$$R = W \frac{\cos(\theta + i)}{\cos \theta}.$$

28. Show that a rectangular board cannot be sustained on a smooth

inclined plane by a string attached to its upper corner, if the inclination of the plane is greater than the angle made by the diagonal of the board with one of the sides perpendicular to the plane.

29. If a rectangular picture be hung from a smooth peg by means of a string, of length $2a$, attached to two points symmetrically placed at a distance $2c$ from each other on the upper side of the frame, show that the only position of equilibrium is one in which this side is horizontal if the adjacent side of the frame is greater than $\dfrac{2c^2}{\sqrt{a^2-c^2}}$.

30. A rod whose centre of gravity is not its middle point is hung from a smooth peg by means of a string attached to its extremities; find the positions of equilibrium.

Ans. There are two positions in which the rod hangs vertically, and there is a third thus defined :—Let F be the extremity of the rod remote from the centre of gravity, k the distance of the centre of gravity from the middle point of the rod, $2a$ the length of the string, and $2c$ the length of the rod; then measure on the string a length FP from F equal to $a\left(1+\dfrac{k}{c}\right)$, and place the point P over the peg. This will define a third position of equilibrium.

31. A smooth hemisphere is fixed on a horizontal plane, with its convex side turned upwards and its base lying in the plane. A heavy uniform beam, AB, rests against the hemisphere, its extremity A being just out of contact with the horizontal plane. Supposing that A is attached to a rope which, passing over a smooth pulley placed vertically over the centre of the hemisphere, sustains a weight, find the position of equilibrium of the beam, and the requisite magnitude of the suspended weight.

Ans. Let W be the weight of the beam, $2a$ its length, P the suspended weight, r the radius of the hemisphere, h the height of the pulley above the plane, θ and ϕ the inclinations of the beam and rope to the horizon; then the position of equilibrium is defined by the equations

$$r \operatorname{cosec} \theta = h \cot \phi, \tag{1}$$

$$r \operatorname{cosec}^2\theta = a(\tan\phi + \cot\theta), \tag{2}$$

which give the single equation for θ,

$$r(r - a\sin\theta\cos\theta) = ah\sin^3\theta. \tag{3}$$

Also $$P = W\frac{\sin\theta}{\cos(\phi-\theta)} = W\frac{a\sin^2\theta\sqrt{r^2+h^2\sin^2\theta}}{r^2}. \tag{4}$$

32. If, in the last example, the position and magnitude of the beam be given, find the locus of the pulley.

Ans. A right line joining A to the point of intersection of the reaction of the hemisphere and W.

33. If, in the same example, the extremity, A, of the beam rest against the plane, state how the nature of the problem is modified, and find the position of equilibrium.

Ans. The suspended weight must be given, instead of being a result of calculation. Equation (1) still holds, but not (2); and the position of equilibrium is defined by the equation

$$Ph^2 \cos^3\phi = War \sin^2\phi.$$

34. If the fixed hemisphere be replaced by a fixed sphere or cylinder resting on the plane, and the extremity of the beam rest on the ground, find the position of equilibrium.

Ans. If h denote the vertical height of the pulley above the point of contact of the sphere or cylinder with the plane, we have

$$r \cot\frac{\theta}{2} = h \cot \phi,$$

$$Pr \left(1 + \cot\frac{\theta}{2} \cot \theta\right) \cos \phi = Wa \cos \theta.$$

35. A heavy regular polygon of any number of sides is attached to a smooth vertical wall by a string which is fastened to the middle point of one of its sides; the plane of the polygon is vertical and perpendicular to the wall, and one end of the side to which the string is attached rests against the wall. For a given position of the polygon, find the requisite direction of the string, and show that in all positions of equilibrium the tension of the string and the pressure on the wall are constant.

Ans. Let A be the vertex of the polygon in contact with the wall, G the centre of gravity, O the point in which the weight and the reaction of the wall meet, and M the middle point of the side to which the string is attached. Then the direction of the string is OM, and, the quadrilateral $GOMA$ being inscribable in a circle, the angle between the string and the vertical is constant and equal to half the angle of the polygon.

36. A square board rests with one corner against a smooth vertical wall, the adjacent corner being attached to the wall by a string whose length is equal to the side of the board; prove geometrically that the distances of the corners from the wall are proportional to 1, 3, and 4.

37. One end, A, of a heavy uniform beam rests against a smooth horizontal plane, and the other end, B, rests against a smooth inclined plane; a rope attached to B passes over a smooth pulley situated in the inclined plane, and sustains a given weight; find the position of equilibrium.

Let θ be the inclination of the beam to the horizon, a the inclination of the inclined plane, W the weight of the beam, and P the suspended weight; then the position of equilibrium is defined by the equation

$$\cos \theta \,(W \sin a - 2P) = 0. \tag{1}$$

Hence we draw two conclusions:—

(*a*) If the given quantities satisfy the equation $W \sin a - 2P = 0$, the beam will rest in all positions.

(*b*) There is one position of equilibrium, namely, that in which the beam is vertical.

This position requires that both planes be conceived as prolonged through their line of intersection.

38. Discuss the second position of equilibrium in the last example, and show that its possibility will depend on the length of the beam, and also on the inequality $W >$ or $< P \operatorname{cosec} a$.

(N. B.—In accounting for this position, the impossible supposition that the reaction of a plane can consist of a *pull* must be rejected.)

39. A uniform beam, AB, moveable in a vertical plane about a smooth horizontal axis fixed at one extremity, A, is attached by means of a rope BC, whose weight is negligible, to a fixed point C in the horizontal line through A, such that $AB = AC$; find the pressure on the axis.

Ans. If $\theta = \angle CAB$, $W =$ weight of beam, the reaction is

$$\tfrac{1}{2} W \sqrt{4 \sin^2 \frac{\theta}{2} + \sec^2 \frac{\theta}{2}}.$$

CHAPTER VIII.

EQUILIBRIUM OF A SYSTEM OF SMOOTH BODIES UNDER THE ACTION OF FORCES IN ONE PLANE.

117.] **Action and Reaction.** If in any system of bodies, connected in any manner, A and B are two bodies in contact between which an action of some kind is exercised; then, whatever be the forces with which the body A acts upon the body B, the very same forces, reversed in directions, will constitute the action of B on A. Let the whole system of forces acting on A, excluding those produced by B, be denoted by (P), and let the forces constituting the action of B on A be denoted by (R); then we may sever the connexion between A and B, provided that we have other means of producing on A the system of forces (R). In the same way, if (Q) denote the whole system of forces acting on B, those constituting the action of A on it being excluded, the body B may be severed from A provided that we have the means of producing a system of forces $(-R)$ on B, $(-R)$ denoting a system of forces obtained by reversing the direction and preserving the magnitude of every force in (R).

For example, the beam CD (fig. 125) may be severed from the other beam along any section, CB, provided that there be introduced on CD either the single force R acting through A, or the complex system of tensile and compressive forces which act at the section CB. This equality of magnitude and oppositeness of direction of the forces existing between two distinct bodies in contact, or between ideally severed portions of the same body, is sometimes spoken of as the principle of *the equality of Action and Reaction;* but it cannot be too strongly impressed on the student that it is by no means the whole of the Newtonian principle called by this name; for Newton specifies several senses in which the terms Action and Reaction can be taken, and in discussing one of them he has explicitly anticipated, in great part, the principle of the Conservation of Energy—as has been pointed out by Thomson and Tait.

118.] **Examples of Internal Action.** The cases which we shall consider in this chapter are those in which the action between two portions of a system ideally severed consists of a single force. The simplest example of such action occurs when a single point of one body rests against the surface of another, the bodies being either rough or smooth. If the bodies are smooth, the action between them consists of a single force which is *normal to the surface of contact* (see p. 40); and if rough, the action is still a single force which is not necessarily normal to this surface. In all cases in which smooth spherical joints or hinges are concerned, the action exercised on bodies connected by them consists of a single force passing through the centre of the joint. When rough joints are used, the action will generally consist of a single force acting somewhere outside the joint; or of a force and a couple acting at the joint; or, possibly, of a couple alone. The tension of a string is also an instance of internal action, and its nature has been already explained in Chapter II.

Again, if we ideally separate into two portions, by an arbitrary surface, a mass of a perfect fluid in equilibrium, the action of one portion on the other over a small area of the ideally separating surface will consist of a single force acting normally on the area. And we may always treat as a separate body any portion whatever of a fluid in equilibrium *, *provided that we produce along the surface of this ideally separated portion all the forces which are actually produced on it by the fluid with which it was surrounded.* It is by such separate consideration of portions of a fluid that we arrive at a knowledge of its internal forces or pressures. For example, if a heavy fluid, whether compressible or incompressible, of uniform or varying density, be contained in a vessel, we can prove that the pressure is the same at all points, P, Q, in the same horizontal plane. For, isolate in imagination a horizontal cylindrical column of the fluid, having small vertical and equal areas at P and Q for extremities, from the rest of the fluid. Then, we may treat the cylinder of fluid PQ as a separate body, provided that, in addition to the external force (gravity) acting on it, we introduce the forces which it actually

* It is usually said that we may, under the above condition, imagine any portion of the fluid *to become solidified*; but this imagined *solidification* is not only wholly unnecessary but misleading to the student.

experienced from the surrounding fluid. Now these forces consist of normal pressures, p and q, on the areas at P and Q, together with normal pressures all over its curved surface, these latter being all at right angles to the axis PQ. If now we resolve horizontally all the forces acting on the cylinder, we get

$$p-q=0, \text{ or } p=q.$$

This demonstration shows, moreover, that in the case of a heavy viscous or imperfect fluid, the pressures are not necessarily equal at all points in the same horizontal plane.

For, in this case, the action of the rest of the fluid on PQ does not necessarily consist of forces normal to its surface, but of oblique forces. Hence the horizontal component of the pressure at P is not equal to the horizontal component at Q; the difference between them is equal to the sum of the horizontal components of the oblique forces.

The importance of keeping such considerations in view may be illustrated by the following example from Hydrostatics.

A conical vessel is filled with water through an aperture at the vertex. From Hydrostatical principles it follows that the pressure on the base of the cone is equal to the weight of a *cylindrical* column of water, standing on the base, and having a height equal to that of the cone; that is, the pressure on the base is much greater than the weight of the water contained in the cone. Now if we imagine the water to become solidified, the curved surface of the cone may be removed, and the pressure on the base will be equal to the weight of the ice, that is, the weight of the water in the cone. An apparent discrepancy is the result. But if we attend to the proviso that in the separate consideration of the equilibrium of any portion of a system, solid or fluid, we must produce upon the isolated portion all the forces which were originally produced upon it by the neighbouring portions of the solid or fluid, the difficulty disappears. In the fluid state the liquid in contact with the curved surface of the cone was pressed normally by a system of varying forces, and the circumstances of the solidified body will not be the same as those of the fluid, unless its surface is pressed in precisely the same way. These pressures have a total vertical component, which must be added to the weight of the block of ice in order that we may obtain the true pressure on the base.

The action between two portions of a perfect fluid ideally separated by a plane surface of any area always consists of a single force which is normal to the area; but the action between two portions of an elastic solid along a plane section is by no means so simple; this latter is not generally reducible to a single force.

119.] **Equilibrium of Several Bodies Forming a System.** It will now be clear that when a system is composed of several bodies in contact with each other, we can consider the whole set as forming a single body in equilibrium under the action of given external forces; or we may consider the separate equilibrium of any one body under the action of given external forces, and the reactions of the other bodies with which it is in contact. A few examples of such systems have already been given; but it is proposed to devote the present chapter more especially to the consideration of such questions.

EXAMPLES.

1. Two uniform beams, connected at a common extremity by a smooth joint, are placed in a vertical plane, their other extremities, which rest on a smooth horizontal plane, being connected by a light rope; find the tension of the rope and the reaction at the joint.

Let AC and CB (fig. 140) be the beams, W and W' their weights, a and a' their inclinations to the horizon, R and R' the reactions of the horizontal plane at A and B, and T the tension of the rope.

If, then, we consider the two beams as forming one system, the mutual reaction at C and the tension of the rope will be *internal* forces of the system, and will therefore disappear from the equations of equilibrium. The forces on this system are simply W, W', R and R'.

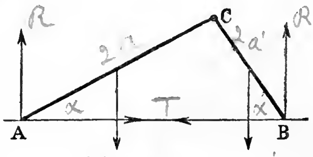

Fig. 140.

Resolving vertically for the equilibrium of the system,

$$R + R' = W + W'. \qquad (1)$$

Again, considering the equilibrium of the beam AC, the forces acting on it are W, R, T, and the unknown reaction at C. This latter will be eliminated by taking moments about C. Thus we get

$$2R\cos a = 2T\sin a + W\cos a,$$

(the length of the beam dividing out), or

$$R = T\tan a + \tfrac{1}{2}W. \qquad (2)$$

Similarly, taking moments about C for the equilibrium of BC,

$$R' = T\tan a' + \tfrac{1}{2}W'. \qquad (3)$$

By adding (2) and (3), and making use of (1), we get

$$T = \frac{W + W'}{2(\tan a + \tan a')}. \qquad (4)$$

Again, let X and Y be the horizontal and vertical components of the reaction at the joint. Then, for the equilibrium of the beam AC,

$$T - X = 0,$$

$$W + Y - R = 0.$$

Hence

$$X = \frac{W + W'}{2(\tan a + \tan a')},$$

$$Y = \frac{W' \tan a - W \tan a'}{2(\tan a + \tan a')}.$$

If we wish to determine T by the principle of virtual work, let y be the height of the middle point of either beam, and we have

$$-(W + W')\,dy - Td(AB) = 0 \qquad (5)$$

for an imagined displacement in which the beams are drawn out, while A and B remain on the ground. If $AC = 2a$, $BC = 2a'$, $y = a \sin a$, $AB = 2a \cos a + 2a' \cos a'$. Therefore

$$dy = a \cos a\,da, \quad d(AB) = -2a \sin a\,da - 2a' \sin a'\,da'$$

$= -2a \dfrac{\sin(a + a')}{\cos a'}\,da$ (since from the equation $a \sin a = a' \sin a'$ we have $a \cos a\,da = a' \cos a'\,da'$).

Substituting these values of dy and $d\,(AB)$ in (5), we get the same value of T as before.

2. Two equal smooth spheres are placed inside a hollow cylinder, open at both ends, which rests on a horizontal plane; find the least weight of the cylinder in order that it may not be upset.

Let figure 141 represent a vertical section of the system through the centres of the spheres. Let P be the weight of the cylinder, a its radius, W and r the weight and radius of each sphere, R and R' the reactions between the cylinder and the spheres whose centres are O and O', respectively.

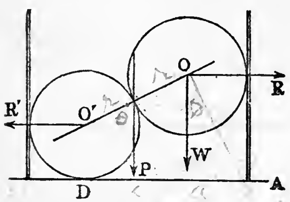

Fig. 141.

Then, the only motion possible for the cylinder is one of tilting over its edge at the point A, in which the vertical plane containing the forces meets it. For, consider the equilibrium of the lower sphere which rests against the ground at D. This sphere is in equilibrium under the influence of R' (reversed in figure), the reaction of the upper sphere, S, acting in the line OO', its weight, W, and the reaction of the ground at D. Now, since three of these forces pass through O', the reaction of the ground, whether the latter is rough or smooth, must also pass through O'. Hence, if θ be the angle which OO' makes with the horizon, we have for the equilibrium of the lower sphere, resolving horizontally,

$$R' = S \cos \theta. \qquad (1)$$

The upper sphere is in equilibrium under the action of R (reversed in figure), W, and S. Hence for its equilibrium we have in the same way,

$$R = S\cos\theta; \tag{2}$$

$$\therefore \quad R = R'. \tag{3}$$

Again, the cylinder is in equilibrium under the action of R, R', P, and the reaction of the ground. Resolving horizontally for its equilibrium, we have the horizontal component of the reaction of the ground $= R - R' = 0$. Hence, even if the ground is rough, there is no tendency to slip, and the only way in which equilibrium can be broken is by turning round A.

Taking moments, then, about A, the point at which the reaction of the ground acts, we have for the equilibrium of the cylinder

$$Pa + R'r = R\,(r + 2r\sin\theta),$$

or

$$Pa = 2Rr\sin\theta. \tag{4}$$

Again, for the equilibrium of the upper sphere, we have

$$\frac{R}{W} = \cot\theta. \tag{5}$$

$R \sin\theta = W\cos\theta$

Substituting this value of R in (4), we have

$$Pa = 2Wr\cos\theta. \tag{6}$$

But evidently

$$\cos\theta = \frac{a-r}{r};$$

therefore, finally,

$$P = 2W\left(1 - \frac{r}{a}\right).$$

3. A heavy beam is moveable in a vertical plane round a smooth hinge fixed at one extremity; a heavy sphere is attached to the hinge by a string; the two bodies rest in contact; find the position of equilibrium and the internal reactions, there being no friction between the bodies.

Let O (fig. 142) be the hinge, OA the string by which the sphere is attached, θ the inclination of the string to the vertical, Cm, ϕ the inclination of the beam to the vertical, W the weight and r the radius of the sphere, l the length of the string, a the distance between O and G, the centre of gravity of the beam, and P its weight.

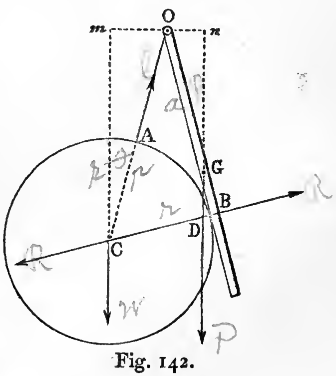

Fig. 142.

Then, considering the sphere and beam as one system, this system is acted on by the given forces W and P, by the tension of the string, and by the resistance of the hinge. The two latter forces will be eliminated by taking moments about O. We have then

$$W.Om = P.On,$$

Om and On being perpendiculars from O on the lines of action of W and P. But $Om = (l+r)\sin\theta$, and $On = a \sin\phi$; therefore

$$W.(l+r)\sin\theta = P.a\sin\phi. \qquad (1)$$

This is the statical equation connecting θ and ϕ; the geometrical equation is

$$\sin COB = \frac{r}{l+r}, \text{ or}$$

$$\sin(\theta+\phi) = \frac{r}{l+r}. \qquad (2)$$

(1) and (2) determine θ and ϕ, and therefore the position of equilibrium. If R is the mutual reaction of the sphere and the beam, we have, by considering the equilibrium of the sphere alone,

$$R = W\frac{\sin\theta}{\cos(\theta+\phi)}. \qquad (3)$$

Again, if the string is attached to the hinge but not to the beam, and if X and Y are the horizontal and vertical components of the pressure of the beam on the hinge, we have for the equilibrium of the beam

$$X = R\cos\phi = W\frac{\sin\theta\cos\phi}{\cos(\theta+\phi)},$$

$$Y = P - R\sin\phi = P - W\frac{\sin\theta\sin\phi}{\cos(\theta+\phi)}.$$

Hence, if S is the resultant of X and Y,

$$S^2 = P^2 - 2PW\frac{\sin\theta\sin\phi}{\cos(\theta+\phi)} + W^2\frac{\sin^2\theta}{\cos^2(\theta+\phi)}. \qquad (4)$$

Evidently S acts in the line OD, which joins the hinge to the point of intersection of P and R.

If the string is attached to the beam, X and Y are the components of the resultant of the tension of the string and the pressure on the hinge.

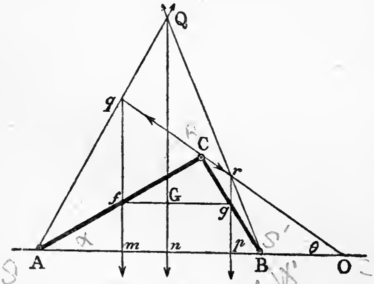

Fig. 143.

4. Two heavy uniform rods are freely jointed at a common extremity, and are connected at their other extremities with two smooth hinges in the same horizontal line. Required the magnitudes and directions of the pressures on the hinges, and the mutual reaction between the rods.

Let AC and CB (fig. 143) be the rods; W and W' their weights, acting through their middle points, f and g; a and a' their inclinations to the horizon; R the mutual reaction at C; S and S' the pressures on the hinges A and B, G the centre of gravity of the system of two rods; and θ the inclination of R to the horizon.

Consider the equilibrium of AC alone. It is acted on by three forces W, R, and S; and since we have drawn the line OC to represent the direction of R, the direction of S must be Aq, q being the point of intersection of W and R. By taking moments about A for the equilibrium of AC, we shall express R in terms of W, a, and θ; and by taking moments about B for the equilibrium of BC, we shall express R in terms of W', a', and θ; equating the two values of R thus obtained, we get a value for $\tan\theta$ which is obtained by dividing the value of Y by that of X in example 1.

Considering the two rods as one system, this system is acted on by the three external forces, S, S', and $W+W'$ acting vertically through G. Hence these must meet in a point, Q.

It is evident that this problem is the same as that in example 1, and that if the reactions S and S' are resolved each into a vertical and a horizontal component, the horizontal components will be equal and opposite (by considering the two rods as one body and resolving horizontally). These horizontal components have each the value of the tension of the rope in example 1, and the vertical components are the values of R and R'. Thus the problem might be completely solved analytically.

*Geometrical Solution**. The direction of the resistance at the joint C can be easily determined as follows:—from A and B draw two lines to any point, D, on the line QG; let AD meet qf in E, and let BD meet rg in H. Then the line EH will meet AB in O, the point through which the line of resistance at C passes. For, the triangles qrQ and EHD are such that the lines, Eq, DQ, Hr, joining corresponding vertices meet in a point (are parallel), therefore, by the well known property of triangles in perspective (which has been given at p. 109), the intersections, A, B, O, of corresponding sides must lie on a right line. Hence O is determined, and therefore OC, the line of resistance.

The direction of R can also be found thus geometrically:—

Since qrO is a transversal cutting the sides of a triangle AQB, we have

$$\frac{AO}{OB}=\frac{Aq}{qQ}\times\frac{Qr}{rB}=\frac{Am}{mn}\times\frac{np}{pB}=\frac{Am}{pB}\times\frac{np}{mn}$$

$$=\frac{Am}{pB}\times\frac{gG}{fG}=\frac{AC\cos a}{BC\cos a'}\cdot\frac{W}{W'}.$$

But $AO=AC\dfrac{\sin(a+\theta)}{\sin\theta}$, and $OB=BC\dfrac{\sin(a'-\theta)}{\sin\theta}$; therefore

$$\frac{\sin(a+\theta)}{\sin(a'-\theta)}=\frac{\cos a}{\cos a'}\cdot\frac{W}{W'},$$

from which we get the same value of $\tan\theta$ as before.

---

* This elegant solution was suggested to me by Mr. Henry Reilly.

5. A sphere and a cone, each resting on a smooth inclined plane, are placed in contact; find the position of equilibrium of the system, and the reactions of the planes.

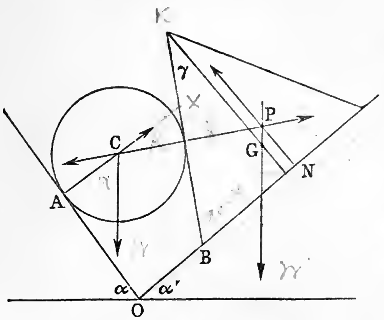

Fig. 144.

Let the sphere rest on the plane OA (fig. 144) whose inclination to the horizon is a, and the cone on OB whose inclination is a'; let W and W' be the weights of the sphere and cone, R the mutual reaction between them, S the reaction of the plane OA on the sphere, T the reaction of OB on the cone, and let γ be the semivertical angle of the cone.

For the equilibrium of the sphere we have

$$R = W\frac{\sin a}{\cos(a + a' - \gamma)}; \qquad (1)$$

$OBX = 90 + \gamma$
$AOB = 180 - (\alpha + \alpha')$
$OAC = 90^\circ$

and for the equilibrium of the cone

$CXB = 360 - [360 - (\alpha + \alpha' - \gamma)]$
$= \alpha + \alpha' - \gamma.$

$$R = W'\frac{\sin a'}{\cos\gamma}. \qquad (2)$$

From (1) and (2) we have

$$W\frac{\sin a}{\cos(a + a' - \gamma)} = W'\frac{\sin a'}{\cos\gamma}, \qquad (3)$$

an equation which, instead of giving a position of equilibrium, gives *a condition to be satisfied* in order that equilibrium may be at all possible.

It is evident that (3) is the only statical equation that can be obtained without involving the unknown reactions. Hence, if it is satisfied, every position in which the bodies are placed is one of equilibrium; and if it is not satisfied, the problem must be radically changed, and one or other of the two bodies must rest in contact with *both* planes. Suppose the cone in contact with both planes.

Here there are only three forces acting on the sphere, and there are four forces acting on the cone, viz., W', R, T, and F, the reaction of the plane OA, which is perpendicular to OA. R must now be determined from the equilibrium of the sphere. Thus

$$R = W\frac{\sin a}{\cos(a + a' - \gamma)}.$$

To determine F, consider the equilibrium of the cone, and resolve along OB. Then

$$F = \left[W' \sin a' - W \frac{\sin a \cos \gamma}{\cos (a + a' - \gamma)}\right] \operatorname{cosec} (a + a').$$

To determine the magnitude of T, resolve the forces on the cone in the direction OA. Then

$$T = (W + W') \frac{\sin a}{\sin (a + a')}.$$

The point N at which T acts is obtained by taking moments about O for the equilibrium of the cone. We thus get

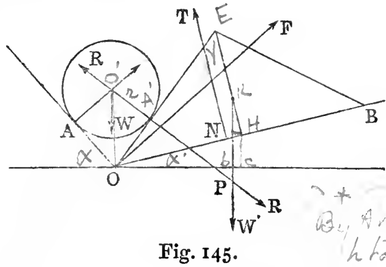

Fig. 145.

* $$T.ON = W'h \left(\tan \gamma \cos a' - \frac{1}{4} \sin a'\right) + Rr \cot \left(\frac{\pi}{4} - \frac{a + a' - \gamma}{2}\right),$$

r being the radius of the sphere, and h the height of the cone.

ON is obtained by substituting in this equation the values of T and R given above, and it is geometrically evident that the point N lies between the foot of the perpendicular from P on OB and the foot of the perpendicular from the intersection of F and W' on OB.

If the sphere is in contact with both planes, the discussion proceeds in a similar manner. R is then determined from the equilibrium of the cone, T acts in the perpendicular from P on OB, and the reactions of the planes on the sphere are easily calculated.

If the weight of the sphere be greater than the value

$$W' \frac{\sin a' . \cos (a + a' - \gamma)}{\sin a \cos \gamma},$$

given by (3), it is sufficiently clear that the sphere will descend to contact with the plane OB; whereas if it is less than this value, the cone will descend.

If the condition (3) is satisfied, the reaction T of the plane OB on the cone is easily found. For, let the directions of W' and R meet in P; then T must act in the perpendicular, PQ, from P on OB, and

$$T = W' . \frac{\cos (a' - \gamma)}{\cos \gamma}.$$

Similarly S may be found.

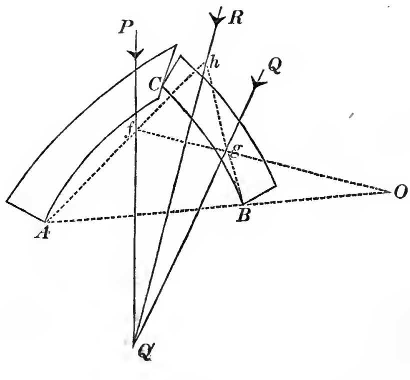

Fig. 146.

6. Two blocks, AC and BC (fig. 146), rest against two fixed

supports at A and B, and against each other at C; each is acted on by a given force (in addition to its weight); find the lines of resistance at A, B, C.

Ans. Let the resultant of the weight of the block AC and the force applied to it be the force P; let the resultant of the weight of BC and the force applied to it be Q; and let the resultant of P and Q be R. Draw the line AB; take any point, h, on R, and draw Ah and Bh, meeting P and Q in f and g, respectively. Then the line fg will intersect AB in O, the point through which the line of resistance at C passes. Draw OC, and let it meet P in F and Q in G. Then AF and BG are the lines of resistance at A and B. (See example 4.)

120.] **System of Jointed Bars.** When a system consists of a number of rods or bars articulated, or connected together by smooth joints, there will be exerted at the extremities of each rod certain forces, or *stresses*, which are produced by the connecting joints, and the calculation of the directions and magnitudes of these stresses forms an important part of Statics as applied to the construction of framework.

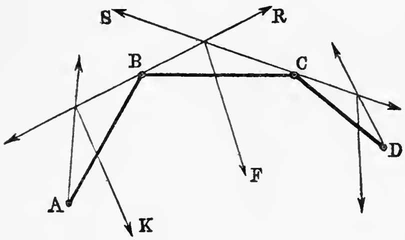

Fig. 147.

The joint connecting any two bars may be either a portion of one of the bars or a hinge-pin distinct from both bars, and the directions of the stresses at the extremities of a bar will depend on the manner in which the external forces are applied. Let us suppose that the joints at B and C (fig. 147), which connect the bar BC with the neighbouring bars, are distinct from BC itself, and that the forces applied to the system act at and *on* the joints. Then the stresses produced at B and C on the bar BC act along this bar. For, the only forces* acting on the bar are the reactions of the joints B and C, and when two forces keep a body in equilibrium, they must be equal and opposite. Hence the stresses must act along BC. Suppose, however, that the forces, still applied at the joints, act on the extremities of the bar BC itself, and let fig. 148 represent the bar apart from the joints. Let the forces applied to it be P and Q. Now the smooth joints must produce reactions which act on the bar through the centres

* The weight of the bar is supposed to be neglected.

of the joints (see p. 140). Hence BC is again kept in equilibrium by forces acting at its extremities, and therefore the resultant of the forces at B must be a force acting in the direction BC or CB, and the resultant of the forces at C must be a force acting in the direction CB or BC. Hence the stresses produced by the joints cannot act along the bar, but must assume some such directions as R and S.

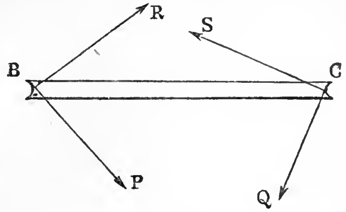

Fig. 148.

Thus, *in any system of articulated bars, when the external forces are applied at the joints, the stresses will be in the directions of the bars only when the external forces act at the joints on pins which are distinct from the bars which they connect.*

121.] **Theorem.** When a system of articulated bars is in equilibrium under the action of external forces applied at given points in the bars, the statical condition of the system may be determined by resolving the force applied to each bar into any two components acting on the joints at its extremities, and then representing each joint as in equilibrium under the action of the components transferred to it together with stresses acting on it along the directions of the bars which it connects.

Let fig. 149 represent one of the bars detached from the

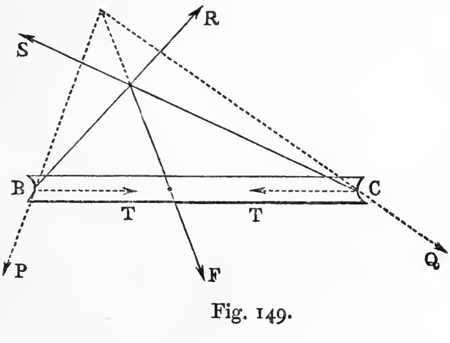

Fig. 149.

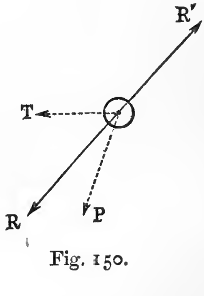

Fig. 150.

joints at its extremities, and let fig. 150 represent the joint which connected the bars AB and BC (fig. 147). If a force F is applied to BC, it is, of course, allowable to break it up into any two components, P and Q, acting on the bar. Let P and

Q act on the bar at its extremities, and let R be the reaction of the joint at B on the bar, and S that of the joint at C. The bar is then kept in equilibrium by the forces P and R at B, and the forces Q and S at C. Hence the resultant of P and R must be a force, T, along the bar; that is to say, if the forces P and R act at any point, they produce a resultant T; or again, if we reverse the directions of R and T (as in fig. 150), the forces P and T are equivalent to R. Now the joint was kept in equilibrium by the equal and opposite reactions, R and R' (fig. 150) of the bars BC and AB. But we have just shown that R is equivalent to the transferred component P of the force F and the stress T, acting along CB. In the same way, R' may be replaced by a component of the force K (fig. 147) acting on AB and a stress acting along AB.

We may, then, replace the external forces, $K, F \ldots$ (fig. 147) which act on the bars by any system of components passing through the centres of the joints, and represent two equal and opposite stresses as acting at the extremities and in the direction of each bar of the system. But it must be remembered that the stresses thus calculated (such as T, fig. 149) are not the *total* stresses at the joints.

The stress in each bar, thus calculated, is the resultant of the total stress at the joint and the component of the force acting on the bar which has been transferred to the joint.

For example, the stress along the bar AB is the resultant of the total stress, R, and the component of K which has been transferred to the joint B.

The external forces, $F, K, \ldots$ may be each broken up into two components passing through the centres of the corresponding joints in an infinite number of ways. In the calculation of stresses in framework it is usual to break each of them up into two parallel forces.

122.] **Method of Separation of the Bars.** Another method, which is not really distinct from the preceding, but which is sometimes convenient in practice, consists in representing the bars as disjointed from each other, and replacing the stresses by two rectangular components at their extremities. A single example will suffice.

Four equal uniform bars, AB, BC, CD, and DE (fig. 151) are connected by smooth joints at B, C, and D, and the extremities A, E are fixed in a horizontal line by smooth joints; it is required to find the position of equilibrium.

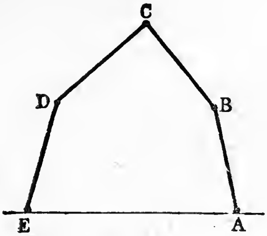

Fig. 151.

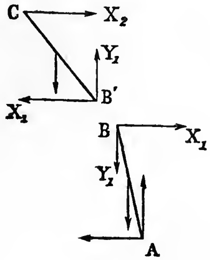

Fig. 152.

Let α be the common inclination of AB and ED to the horizon, and β that of CB and CD.

Let fig. 152 represent the bars AB and BC separated; X_2 the stress at C, which is evidently horizontal; X_1 and Y_1 the components of the stress at B. These components act on AB in directions opposite to those in which they act on BC. Finally, let W be the weight of each bar.

Resolving vertically for the equilibrium of BC,

$$Y_1 = W. \tag{1}$$

Taking moments about C for the equilibrium of BC,

$$2X_1 \sin\beta + W\cos\beta = 2Y_1\cos\beta,$$

or

$$X_1 = \tfrac{1}{2}W\cot\beta. \tag{2}$$

Taking moments about A for the equilibrium of AB,

$$(W + 2Y_1)\cos\alpha = 2X_1\sin\alpha,$$

or, substituting the values of X_1 and Y_1 from (2) and (1),

$$\tan\alpha = 3\tan\beta. \tag{3}$$

With this equation must be combined the geometrical equation which expresses that AE is equal to the sum of the horizontal projections of the bars. If the length of each bar is a, and the distance $AE = c$, we have

$$c = 2a(\cos\alpha + \cos\beta). \tag{4}$$

Equations (3) and (4) determine α and β, and therefore the position of equilibrium.

Examples.

1. A triangular system of bars, AB, BC, and CA, freely jointed at their extremities, is kept in equilibrium by three forces acting on the joints; determine the stress in each bar.

Since the forces are applied directly to the joints, the stresses will act along the bars. Let P, Q, R denote the forces applied at A, B, C, respectively; let the stresses in the sides BC, CA, AB be denoted by T_1, T_2, T_3; and let the applied forces meet in a point O.

Then for the equilibrium of the joint C, we have

$$\frac{T_1}{T_2}=\frac{\sin ACO}{\sin BCO}=\frac{a\,.\,OA\,.\sin AOC}{b\,.\,OB\,.\sin BOC},$$

a, b, c, being the sides of the triangle.

But $P:Q:R=\sin BOC:\sin COA:\sin AOB$. Therefore

$$\frac{T_1}{T_2}=\frac{a\,.\,OA\,.\,Q}{b\,.\,OB\,.\,P}, \text{ or}$$

$$T_1:T_2:T_3=\frac{a\,.\,OA}{P}:\frac{b\,.\,OB}{Q}:\frac{c\,.\,OC}{R}.$$

If O is the centroid of the triangle, we know (p. 135) that

$$P:Q:R=OA:OB:OC;$$

therefore $$T_1:T_2:T_3=a:b:c,$$

or the stresses are proportional to the sides.

If O is the orthocentre (or intersection of perpendiculars),

$$P:Q:R=a:b:c;$$

therefore $$T_1:T_2:T_3=OA:OB:OC.$$

2. A number of bars are jointed together at their extremities and form a polygon; each bar is acted upon perpendicularly by a force proportional to its length, and all these forces emanate from a fixed point. Find the magnitudes and directions of the stresses at the joints.

[This problem and the following elegant method of solution are due to Professor Wolstenholme.]

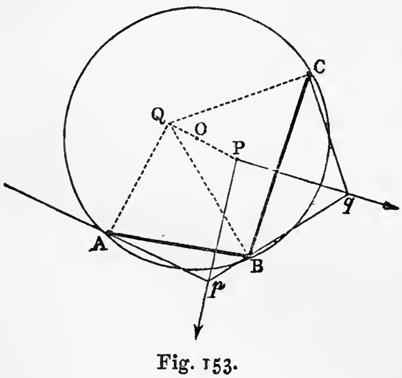

Fig. 153.

Let AB and BC (fig. 153) be any two adjacent bars of the polygon, and let P be the point from which emanate the forces, Pp, Pq, ..., acting on the bars. Then the stresses at the joints A and B, acting on AB, must meet in a point, p, on the line of action of the force Pp. Draw AQ and BQ perpendicular to the stresses in the directions Ap and Bp. Now since the sides of the triangle AQB are perpendicular to three forces which are in equilibrium, and since the side AB is proportional to the force to which it is perpendicular, the sides AQ and BQ are proportional to the forces to which they are perpendicular, that is, to the stresses at A and B, respectively.

Let q be the point in which Bp intersects Pq. Then the forces acting on the bar BC must act in the directions qB, Pq, and qC. Draw CQ.

In the triangle BQC the sides BQ and BC are perpendicular and proportional to two of three forces in equilibrium; therefore CQ is perpendicular and proportional to the third, that is, to the stress at C. In the same way it can be shown that the stress at any joint is perpendicular and proportional to the line joining the joint to Q. This point Q is, therefore, a *centre of stress* for the system. It may be shown that the polygon of bars must be inscribable in a circle. For, since the angles at A and B are right, the quadrilateral $ApBQ$ is inscribable in a circle whose diameter is pQ. If at the middle point of AB a perpendicular be drawn to AB, it will pass through the centre of the circle, and will, therefore, bisect Qp. But this perpendicular is parallel to Pp; therefore it bisects PQ in O. Also, since the stresses at A and B are proportional to QA and QB, the same point Q must be determined by considering BC and the next bar, as was determined from the bars AB and BC; consequently the point O must be the same; and since it is evident that $OB = OC \ldots$, O must be equally distant from all the vertices of the polygon, that is, the polygon must be inscribable in a circle.

The centre of stress is therefore constructed by joining P to the centre of the circumscribing circle, and producing PO to Q so that $PO = OQ$.

3. The preceding construction can be extended to the case in which the forces acting on the polygon are equally inclined, but not perpendicular, to the sides.

Let AB, BC, ... be sides of the polygon, and let forces proportional to the sides act in the lines Pb, Pc, ... so that $\angle PbB = \angle PcC = \ldots$. It is required to prove that for equilibrium the polygon must be inscribable in a circle, and to find the centre of stress. The stresses at A and B must meet in a point on the force in Pb. If, then, we draw at A and B lines, QA and QB, making with the directions of the stresses angles equal to $\angle PbB$, we shall have a triangle, QAB, the sides of which are each equally inclined to the corresponding force; and, since AB is proportional to the force in Pb, it follows that QA and QB are proportional to the stresses at A and B. It is easy

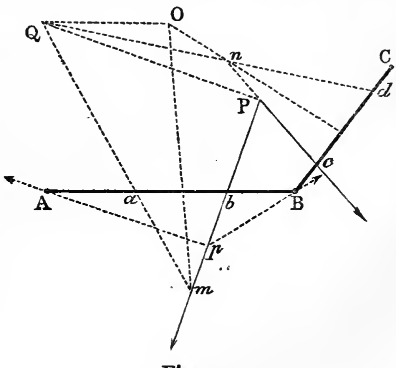

Fig. 154.

to prove that if through A and B any two lines, Ap and Bp, be drawn, meeting in a point on the right line Pb; and at A and B lines, AQ and BQ, be drawn making with Ap and Bp, respectively, angles equal to PbB, the locus of Q is a right line, ma, making $Aa = Bb$, and $\angle maB = \angle mbA$. Drawing the line Qd, in like manner, by making $Cd = Bc$ and $\angle QdB = PcC$, we obtain the point Q which is the centre of stress.

Now, since $\angle PcC = \angle PbB$, it follows that $\angle bPc$ is the supplement of $\angle B$; and since $\angle QaA = \angle QdB$, it also follows that $\angle aQn = \pi - B$. Hence the quadrilateral $mPnQ$ is inscribable in a circle, and this circle must pass through O, the point of intersection of the perpendiculars to AB and BC drawn at their middle points, since $\angle mOn$ is also the supplement of B. Hence also

$$\angle QPO = \angle QnO = \frac{\pi}{2} - ncC, \text{ and } QO = OP.$$

Again, the stresses at A and B being proportional to QA and QB, the same point Q must be determined when BC and the next bar are considered. Hence the point O is the same. But $OA = OB = OC = \dots$; therefore the polygon is inscribable in a circle.

The point P being given, if the angle which the forces through it make with the corresponding bars varies, the locus of the centre of stress, Q, is a circle concentric with that round the polygon, its radius being OP. To construct the centre of stress, then, we describe a circle round O as centre, having radius OP, and draw PQ making the $\angle OPQ =$ the complement of the angle which the forces make with the bars.

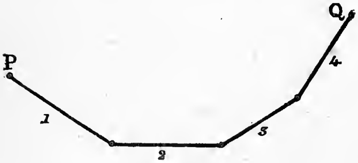

Fig. 155.

4. A system of heavy bars, freely articulated, is suspended from two fixed points, P and Q (fig. 155); determine the magnitudes and directions of the stresses at the joints.

Let the bars be denoted by the numbers 1, 2, 3, ..., and let their weights be $W_1, W_2, W_3, \dots$. Then transfer $\frac{1}{2}W_1$ and $\frac{1}{2}W_2$ to the joint connecting 1 and 2, which we shall denote by (1, 2). Transfer $\frac{1}{2}W_2$ and $\frac{1}{2}W_3$ to the joint (2, 3); $\frac{1}{2}W_3$ and $\frac{1}{2}W_4$ to (3, 4), &c. Thus all the forces act at the joints. Let T_1, $T_2, T_3, \dots$ be the tensions acting along the bars 1, 2, 3, ... on the joints, and let $S_{12}, S_{23}, S_{34}, \dots$ be the total stresses at the joints (1, 2), (2, 3), (3, 4), ... For simplicity suppose the bar 2 to be horizontal. Now, construct a force-diagram (fig. 156), by drawing a vertical line, AD, and measuring off

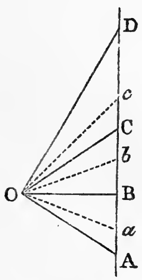

Fig. 156.

$$AB = \frac{W_1 + W_2}{2}, \quad BC = \frac{W_2 + W_3}{2}, \quad CD = \frac{W_3 + W_4}{2} \dots$$

Also take BO parallel to the bar 2 and equal to the tension T_2, which is the constant horizontal component of each of the tensions.

The lines OA, OC, OD, ... will then be parallel to the bars 1, 3, 4, ... and equal to the tensions in them. Hence if a be the inclination of 3 to the horizon,

$$\tan DOB = \frac{W_2+2W_3+W_4}{W_2+W_3} \cdot \tan a,$$

and in the same way the inclinations of the other bars may be expressed in terms of the inclination a.

Again (Art. 121), the stress S_{12} is the resultant of T_1 and $\frac{W_1}{2}$. Hence, taking $Aa = \frac{W_1}{2}$, Oa will be equal and parallel to S_{12}. Similarly, taking $Bb = \frac{W_2}{2}$, and $Cc = \frac{W_3}{2}$, the lines Ob and Oc will be equal and parallel to the stresses S_{23} and S_{34}. The tangent of the angle made by S_{23} with the horizon $= \frac{bB}{BO} = \frac{W_2}{W_2+W_3} \cdot \tan a$. Similarly for the directions of the other stresses.

If the weights of the bars are all equal, the tangents of the inclinations of the successive bars are $\tan a$, $2\tan a$, $3\tan a$, ..., and the tangents of the inclinations of the stresses are $\frac{1}{2}\tan a$, $\frac{3}{2}\tan a$, $\frac{5}{2}\tan a$, ...

5. Six equal uniform bars, freely articulated at their extremities, form a hexagon $ABCDEF$ (fig. 157). The bar ED is fixed in a horizontal position, and its middle point is connected by a string with the middle point of the lowest bar, AB, in such a manner that the bars hang in the form of a regular hexagon. Find, by a force-diagram, the tension of the string and the magnitudes and directions of the stresses at B and C.

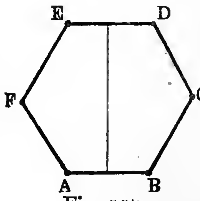

Fig. 157.

Ans. If W is the weight of each bar, the tension of the string $= 3W$; the stress at C is horizontal, and $= \frac{W}{2\sqrt{3}}$; the stress at $B = W\sqrt{\frac{13}{12}}$, and makes with the horizon an angle whose tangent $= 2\sqrt{3}$.

6. Prove that the centre of stress for the bar BC is the intersection of a perpendicular to BC at C with the line joining the middle points of AB and BC.

7. Three bars, freely articulated, form a triangle ABC, the centre of whose inscribed circle is O. Each bar is acted on by a force passing through O, proportional to the sine of half the angle subtended by the bar at O, and bisecting this angle. Prove that the stress at A makes with OA an angle whose tangent is

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}-\cos\frac{C}{2}}.$$

(This is a direct example of the Theorem of Art. 121.)

8. AB (fig. 158) is a rigid bar whose weight is neglected fixed at one extremity, A, by a smooth joint; CD is another such bar fixed at C by a smooth joint, which is vertically below A, and jointed to AB at D. From B a given weight, P, is suspended; find the magnitudes and directions of the stresses at the joints.

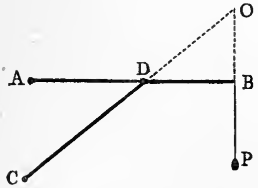

Fig. 158.

Ans. The stresses at C and D are along CD, and each $= P.\dfrac{AB.CD}{AC.AD}$; the stress at A is in AO, O being the intersection of CD produced with the vertical through B, and

$$= P.\frac{\sqrt{AB^2.AD^2+AC^2.BD^2}}{AC.AD}.$$

9. In example 5, if the bars BC and CD, AF and FE, are replaced by any bars all equally inclined to the horizon, show that the stresses at C and F will still be horizontal.

[One simple proof of this is obtained by taking moments about B for the equilibrium of BC, and about D for the equilibrium of CD. It follows then that the perpendiculars from B and D on the line of action of stress at C are equal.]

10. Two uniform heavy bars are freely jointed at a common extremity, and are fixed at their other extremities to two smooth joints in a vertical line; find the stresses at the joints.

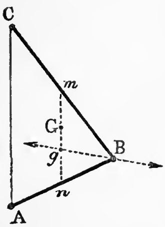

Fig. 159.

Ans. Let G (fig. 159) be the centre of gravity of the bars, m and n their middle points. It follows, by taking moments about A and C for the equilibrium of the bars separately, that the segments of AC made by the line of action of stress at B are proportional to the weights of the bars. Hence, taking $ng = mG$, the stress acts in the line gB. The stresses at A and C act, therefore, in Ag and Cg. If W is the weight of AB, the stress at $B = \frac{1}{2}W\dfrac{gB}{gn}$, and the stress at $A = \frac{1}{2}W\dfrac{gA}{gn}$. Hence the stresses at A, B, and C are proportional to gA, gB, and gC.

11. The regular hexagon of bars in example 5 rests in a vertical plane, the bar AB being fixed in a horizontal position, and the joints F and C are connected by a string; find the tension of the string, and the stresses acting on the bar FE at its extremities.

Ans. The tension $= W\sqrt{3}$ (W being the weight of each bar); the stress at $E = \frac{W}{2}\sqrt{\frac{7}{3}}$, and it makes with FE $\sin^{-1}\frac{1}{2}\sqrt{\frac{3}{7}}$; the stress at $F = W\sqrt{\frac{31}{3}}$, and it makes with FE $\sin^{-1}\frac{1}{2}\sqrt{\frac{3}{31}}$.

12. Four equal uniform heavy bars, freely jointed together at their extremities, form a square, $ABCD$; the joint A is fixed, while the diagonally opposite joints B and D are connected by a string, and the whole system rests in a vertical plane, the string being horizontal; find the tension of the string and the magnitudes and directions of the stresses on the bars at A, B, and C.

Ans. The tension $= 2W$; the stress at C is horizontal and $= \frac{1}{2}W$; the stress on the bar BC at B makes with the vertical $\tan^{-1}\frac{1}{2}$, and $= W\frac{\sqrt{5}}{2}$; the stress on AB at B makes with the vertical $\tan^{-1}\frac{3}{2}$, and $= W\frac{\sqrt{13}}{2}$; and the stress on AB at A intersects the line BD at a distance $\frac{1}{8}BD$ from B, and is equal to $\frac{5}{2}W$.

13. Six equal uniform bars, freely jointed at their extremities, form a regular hexagon, $ABCDEF$; the joint D is connected by strings with the joints F, A, and B, and the system hangs in a vertical plane, the joint D being fixed; find the tensions of the strings and the stresses at the joints.

Ans. If $W =$ weight of each bar, the tensions in the strings DB and DF are each $W\sqrt{3}$, and the tension in $DA = 2W$. Also, supposing the strings to be connected with pins distinct from the bars, the stresses at C and E are vertical and equal to $\frac{1}{2}W$, the stresses at B and F, on the bars AB and AF, are horizontal and equal to $\frac{1}{2}W\sqrt{3}$, and the stresses at A, on the bars AB and AF, are each equal to $\frac{1}{2}W\sqrt{7}$. These latter stresses act in the lines drawn from A to the middle points of the two vertical bars, BC and FE, respectively.

14. Two uniform heavy bars, AB and BC, connected by a smooth joint at B, rest each on a smooth vertical prop, the props being of the same height; find the position of equilibrium, ABC being horizontal.

Ans. If W and $2a$ are the weight and length of AB, W' and $2b$ the weight and length of BC, c the distance between the props; then x, the distance of the middle point of AB from the corresponding prop, is given by the equation

$$(W+W')x^2+[(W+W')(c-a)-W'(a+b)]x-W'a(c-a-b)=0.$$

CHAPTER IX.

EQUILIBRIUM OF ROUGH BODIES UNDER THE INFLUENCE OF FORCES IN ONE PLANE.

123.] **Criterion of the Existence of Friction.** We have already learned to regard Friction as a *passive resistance;* and every passive resistance comes into existence for the purpose of stopping some motion. Thus, the normal reaction of a surface on a body in contact with it comes into existence for the purpose of preventing the body from penetrating the surface at the point of contact; and if the circumstances of the case were so arranged that there was no tendency to this penetration, the magnitude of the force (normal resistance) required to prevent this motion would be zero.

Friction comes into existence for the purpose of preventing a certain motion—motion in the tangent plane—of a body resting against a rough surface. If the circumstances in any case of two rough bodies in contact are such that there is no tendency to slip at their point of contact, the force required to prevent this motion (friction) will not come into existence.

Generally, in the case of all passive resistances, *if there is no tendency to the displacement which a passive resistance is required to prevent, this force will not come into play.*

Hence in many cases of contact between rough bodies the conditions and circumstances are exactly the same as if the bodies were smooth; and to find whether in the contact of two bodies friction acts or not—*imagine that the bodies were smooth at their point of contact, and if no displacement would result from this supposition, friction does not come into play at that point.*

In illustration of this consider the problem in example 21, p. 155. How would the circumstances be altered if the peg Q were rough?

The peg being rough, let it be imagined to become smooth, and what motion occurs? Clearly none, supposing the board to be rigid. Hence as there is no tendency of the side AB to slip over the peg, there is no friction called into play, and the case is the same as if the peg were smooth. But if the board is not rigid, the forces acting can bend its fibres and elongate or contract them; and if we imagine the peg to become smooth, it is possible that (even a very slight) slipping might ensue at the peg, and as this slipping is prevented by the roughness, the force of friction really acts in the case, and the pressure on the hinge is modified by the assumption of smoothness at the peg.

However, even when the board is elastic, it is possible that no friction is called into play, as will be explained in Art. 130.

Rankine's hint that friction is analogous to shearing stress has been already pointed out.

124.] **The Cone of Friction.** The essential characteristic of a smooth surface is that it is capable of resisting in a normal direction only. If two rough surfaces are in contact, their mutual reaction is not constrained to assume a direction normal to the surface of contact. Each surface is capable of offering resistance to the other in any direction which does not make with the the normal to the surface of contact an angle exceeding a certain magnitude. Thus (fig. 160), let two rough bodies, A and B, be in contact at any point, P, and let PN be the normal to the surface of contact.

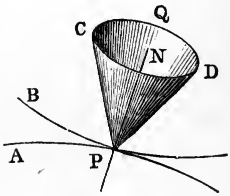

Fig. 160.

Let λ denote the greatest angle that the total resistance at P can make with PN, or, in other words, the greatest obliquity of the mutual reaction; then, describing round PN a right cone, CQD, whose semivertical angle, NPD, is equal to λ, this cone is called the *cone of friction*, and the total resistance at P can act in any direction whatever included within this cone. This angle λ is what we have called in Chap. III the *angle of friction*, and its tangent is the *coefficient of friction* for the two surfaces considered. For, if R_1 denote the normal pressure between them at P, and F the force of friction (which acts in the common tangent plane), it is clear that when the resultant of R_1 and F acts along any generator, PD, of the cone, we have

$$\frac{F}{R_1} = \tan NPD = \tan\lambda,$$

so that $\tan\lambda$ is the greatest ratio of the force of friction to the normal pressure. This quantity we have called μ.

If a rigid weightless rod, M (p. 40), be pressed against a rough surface at O, the greatest angle that the rod can make with the normal is the angle of friction. For, since the rod is acted on by only two forces, viz., the applied pressure and the total resistance at O, these must be equal and opposite, or along the rod. Hence the greatest obliquity of the rod to the normal is λ.

If the resistance to slipping is not the same in different azimuths, i.e., if it is different in different planes through the normal, the value of λ will not be the same in all these planes, and the cone of friction will not be a right circular cone.

125.] **Axiomatic Law of Friction.** We have said that the total resistance of a rigid surface is a force which can assume any magnitude. This force will in any given case be exerted by the surface to such an extent as is necessary to preserve equilibrium, but to no greater extent. It is in its nature a *passive resistance,* i.e., one which can be exerted to any extent, but which will not be exerted beyond the bare requirements of the case. Within certain limits, also, as we have seen, it can assume any direction, and in any given case it will, if possible, assume such a direction as will preserve equilibrium. In fact, in virtue of its *passive* nature, we must regard the resistance of a rough surface as an opposition called into existence by the action of external forces; and it seems clear that these forces will call into play only that amount of opposing force, exact both in magnitude and in direction, which will just counteract their own action.

The amount of assumption contained in this principle is enunciated in the following axiom:—

The total resistance which acts at any point of a rough surface will, if possible, assume such a magnitude and direction as will preserve equilibrium at that point.

This axiom is sometimes expressed thus:—*If passive resistances can give equilibrium, they will.*

126.] **Remarks on this Axiom.** Two important observations must be made on the principles contained in this axiom. Firstly, it is important to understand the circumstances which may render it impossible for the resistances of rough surfaces to preserve the equilibrium of a system in any given position. Suppose that a body, acted on by given external forces, is in contact with a rough surface at a single point, P. Then, for equilibrium, it is necessary that the resultant of the given external forces should pass through P, and that the total resistance at P should be equal and opposite to this resultant. But if the direction of the resultant makes with the normal to the surface of contact at P an angle $> \lambda$, it is impossible that the total resistance could take the required direction, and equilibrium cannot subsist.

Again, take the case in which a heavy beam, AB (fig. 161), rests against a rough horizontal and an equally rough vertical plane. Describe round the normals to the planes at A and B the cones of friction, and let the sections of these cones by the plane of the figure be rAs and pBs. Let G be the centre of gravity of the beam, and GV the vertical line through it.

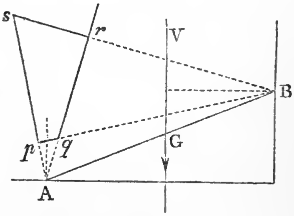

Fig. 161.

Then the beam, if in equilibrium, is so under the action of three forces, namely, the weight through G and the total resistances at A and B. These three forces must meet in a point, and if it be possible to find a point in which they can meet, the resistances will assume proper values. Now, in the figure it is impossible to find any point on GV, the line of action of the weight, the lines drawn from which to A and B could be directions of possible resistance at *both* A and B. For the portion of GV which is inside the cone of friction at B is outside the cone of friction at A, and *vice versâ*. Hence, for equilibrium, *there must be some portion of the line GV included in the space, $pqrs$, common to both cones of friction.*

Unless this condition is satisfied, it is not possible for the total resistances to give equilibrium, whatever their magnitudes may be. A possible position of equilibrium is represented in fig. 162. For, if from any point on the portion, mn, of GV which is included in the space common to both cones of friction, lines be drawn to A and B, these lines are possible directions of total resistance at A and B; and in this case the actual magnitudes and directions of the resistances at A and B cannot be determined by what is called Rational Statics.

If it be proposed to find the position of *limiting* equilibrium, that is, the position in which the beam is bordering on motion, we must make the vertical through G pass through r, as in fig. 163.

In this case there is only one point on GV which is inside both cones of friction, viz., the point r. Hence the total re-

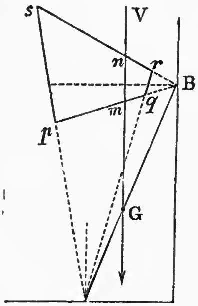

Fig. 162.

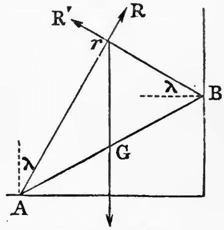

Fig. 163.

sistances act in rA and rB, and each makes the limiting angle (λ) with the corresponding normal. Moreover, both resistances are now determinate. If θ be the angle made by the beam with the horizon, we have, from the triangle ArB,

$$2 \cot rGB = \cot ArG - \cot BrG,$$

or

$$2 \tan \theta = \cot \lambda - \tan \lambda,$$

$$= \frac{1-\mu^2}{\mu},$$

which defines the position of limiting equilibrium.

It may, therefore, in certain cases be impossible for the total resistance at one or more points to preserve equilibrium; and this impossibility is always due to something in the arrangement of the figure or the external forces which requires the direction of the resistance to make with the normal to the surface of contact an angle > the angle of friction.

Again, in the axiom is contained the following important proposition :—

If a body rests against a rough surface at a point, and if the equilibrium is about to be broken by some change in the acting forces, equilibrium at that point will, if possible, be broken by a rolling instead of a sliding motion.

For, in this case, the point of the body actually in contact with the surface would be kept at rest. This part of the axiom is sometimes stated thus—*If a body can roll, it will roll, in preference to slipping.* Exactly the same considerations as before determine the possibility or impossibility of the rolling motion. Such a motion will always take place if it does not require the total resistance to make with the normal to the surface of contact an angle $> \lambda$.

For example, let us discuss the following problem:—

A heavy cubical block rests on a rough horizontal plane, and a string, attached to the middle of one of the upper edges passes over a smooth pulley, and sustains a weight which is gradually increased. Find the nature of the initial motion of the block, the string and the vertical through the centre of gravity of the block being in the same vertical plane.

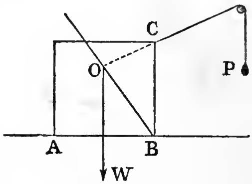

Fig. 164.

Let ABC (fig. 164) be the vertical plane in which all the forces act; CO the line of the string, intersecting the vertical through the centre of gravity of the block in O; P the suspended weight, and W the weight of the block. (Since the length of the string is immaterial, no linear magnitude can enter into the result, therefore the side of the block need not be known.)

Now in all such cases as this, it is necessary to observe the following rules:—

1. Write down the motions of the system which are *geometrically* possible.

2. Exclude those which would obviously violate any of the fundamental rules of Statics.

3. If there remain possible cases of *slipping* and *rolling* (or *turning over*), solve the problem on the supposition that equilibrium is broken in the latter way, and if this does not require too great a value of the angle of friction, equilibrium will be broken in this way.

In the present case, the following motions are *geometrically* possible:—

(α) The block may be lifted vertically off the plane.

(β) It may turn round the edge A.

(γ) It may slide in the direction AB.

(δ) It may turn round the edge B.

Now (a) is obviously excluded, because if the block is just out of contact with the horizontal plane, it is acted on by only two forces, namely, its own weight and the tension of the string. But since these cannot be equal and opposite, equilibrium cannot be broken in this way.

Suppose (β) to happen. Then the total resistance of the plane passes through A and through O. But it is impossible that three forces acting in the directions of AO, OC, and OW could be in equilibrium. Hence (β) is excluded.

The cases (γ) and (δ) remain. Now in virtue of the principle, if (δ) is possible, it will happen. Solve, then, on the supposition that the block turns round B. It is then kept in equilibrium by its weight, the tension, and the total resistance which must act in BO. If the $\angle CBO$ is less than λ, the angle of friction, the block will turn round B; but if $CBO > \lambda$, this motion is impossible, and slipping must take place in the direction AB.

To express this analytically, let θ be the angle made with the horizon by the string OC, and let fall from O a perpendicular on BC meeting BC in p. Then

$$\tan CBO = \frac{Op}{Bp} = \frac{Op}{BC - Op \cdot \tan\theta} = \frac{1}{2 - \tan\theta}.$$

Hence if μ (or $\tan\lambda$) be $> \dfrac{1}{2-\tan\theta}$, the block can turn round B, and will do so if P is gradually increased.

The magnitude of P which will just cause the tilting of the block is found by taking moments about B. We evidently obtain

$$P = \tfrac{1}{2} W \sec\theta.$$

Suppose that $CBO > \lambda$, or that $\mu < \dfrac{1}{2-\tan\theta}$. Then the increase of P will produce a sliding motion, and we can easily find the magnitude and point of application of the total resistance of the plane. Now since $CBO > \lambda$, the point, M, of application of the total resistance of the plane, is found by drawing from O a line OM making with the normal to the plane an angle $= \lambda$. The point M lies between B and the point in which the vertical through O cuts AB. P can then be determined either by taking moments about M, or by resolving vertically and horizontally. Resolving vertically, we have

$$R \cos \lambda = W - P \sin \theta;$$

resolving horizontally,

$$R \sin \lambda = P \cos \theta;$$

$$\therefore \quad \frac{P \cos \theta}{W - P \sin \theta} = \mu, \quad \text{or} \quad P = \frac{\mu W}{\cos \theta + \mu \sin \theta}.$$

The direction of the string might be so modified as to render possible either a sliding in the direction BA or a tilting over A. Thus, in fig. 165, if the line of the string intersect the line of action of the weight in a point, O, below the horizontal plane, the two motions possible are evidently one of slipping in the direction AB and one of tilting over the edge A. The latter will take place if it can. If it does, the total resistance must act in the line OA, and for this the angle DAR must be $< \lambda$. But if DAR is $< \lambda$, the block will slip in the direction AB, since the horizontal component of the tension acts in this sense. The condition for tilting over A is now evidently

$$\mu > \frac{1}{\tan \theta - 2}.$$

The values of P corresponding to both kinds of motion are calculated as before.

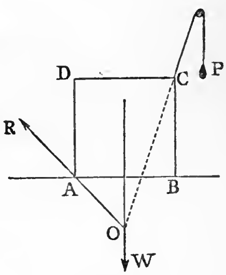

Fig. 165.

127.] **Limiting Positions of Equilibrium.** When a body rests in contact with any number of rough surfaces at several points, the equilibrium is said to be *limiting* if a slight alteration of a definite kind in the circumstances of the body would cause the equilibrium to be broken. The slight alteration referred to depends on the nature of the particular problem of equilibrium. As has been explained in Art. 46, p. 56, every statical problem relating to the equilibrium of a body is always one or other of the three following:—

(*a*) What is the least force that will sustain a body in a given position on given surfaces, or the greatest force that will allow it to rest in such a position?

(*b*) With given forces and given supporting surfaces, what is the position of equilibrium such that if this position be slightly altered, the body will not rest?

(c) With given forces, what is the least amount of roughness of the surface or surfaces which will allow the body to rest in a given position?

Thus in fig. 164 of the last Article, supposing that the angle $CBO < \lambda$, the equilibrium of the block will be limiting if $P = \frac{1}{2} W \sec \theta$; for if P is slightly increased above this value, the block will turn over B.

Again, in fig. 163 of the same Article, supposing the question to relate to the position of equilibrium, the beam AB will be in limiting equilibrium if its inclination to the horizon be $= \tan^{-1}\left(\frac{1-\mu^2}{2\mu}\right)$, because if it be slightly lowered below this position, it will slip.

Finally, if in the same figure we wish the beam to be sustained at any inclination a to the horizon between the equally rough vertical and horizontal planes, the equilibrium will be limiting if the angle of friction $= \frac{\pi}{4} - \frac{a}{2}$, because, if it be less than this, the beam will slip.

128.] **Comparative Safety of Equilibrium of a System at Different Points.** When in a system in equilibrium the directions of the total resistances at the various points of contact with rough surfaces are known, we are enabled to say at which of the points slipping is most likely to happen in case some of the circumstances should be altered.

This will be rendered clear by the following examples, taken from Jellett's "Theory of Friction," p. 61:

Two uniform beams, AC and BC, connected at C by a smooth hinge, are placed, in a vertical plane, with their lower extremities, A and B, resting on a rough horizontal plane. If equilibrium be on the point of being broken, determine how this will happen.

Fig. 143, example 4, p. 166, will represent the beams if the hinges at A and B are conceived to be removed and these points rest on the ground. Then, exactly as in that example, the direction of the mutual resistance at C is determined. Supposing AC to be the longer beam, it is clear that the angle which the total resistance, AQ, at A makes with the normal to the surface of contact (i.e., to the ground) is greater than the angle which the total resistance BQ makes with the normal at B.

For $$\frac{\tan AQn}{\tan BQn} = \frac{An}{Bn}.$$

Now $An = Am + mn$; and if $2a$, $2b$, $2c$, are the sides BC, CA, AB, we have

$$Am = b \cos \alpha,\ mn = fG = \frac{ac}{a+b} = \frac{a\,(b \cos \alpha + a \cos \beta)}{a+b},$$

$$\therefore\quad An = b \cos \alpha + \frac{a\,(b \cos \alpha + a \cos \beta)}{a+b}$$

$$= \frac{(b^2 + 2ab) \cos \alpha + a^2 \cos \beta}{a+b}.$$

Similarly $$Bn = \frac{(a^2 + 2ab) \cos \beta + b^2 \cos \alpha}{a+b};$$

therefore $$An - Bn = \frac{2ab}{a+b}(\cos \alpha - \cos \beta).$$

But since $AC > BC$, $\cos \alpha > \cos \beta$, therefore $An > Bn$.

Hence the angle $AQn > BQn$; that is, the total resistance at A makes with the normal at A an angle greater than that made by the total resistance at B with the normal at B. Consequently, if any circumstance should continually diminish the angle of friction (which is supposed to be the same for both beams) the total resistance at A would be the first to attain its limiting obliquity to the normal, and slipping would then take place at A in the direction BA, while the beam BC would turn round B.

We might inquire which of the beams will first slip if they are drawn out so as to increase the angle C, and the same result will follow, since for any given position of the beams the directions of all the resistances are determinate. In each case the angle AQn must be the first to reach the value λ, and therefore the longer beam, AC, must slip first.

The result may also be expressed thus—in any given position of rest, equilibrium is more safe at B than at A.

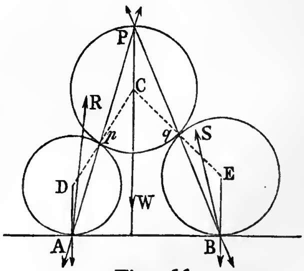

Fig. 166.

There are also cases in which the comparative safety of equilibrium can be determined, although the directions of total resistance are not completely determinate at *all* the points at contact. For example—two unequal cylinders rest on the ground at given points, A

and B (fig. 166), while a third cylinder rests on them at points p and q.

Supposing either that there is a gradual diminution of the coefficient of friction (which is the same at all the points of contact), or that the lower cylinders are gradually drawn asunder, determine the nature of the initial motion of the system.

Denote the cylinders by the letters at their centres. Then the cylinder D is kept in equilibrium by three forces—namely, 1st, its weight, which acts through A; 2nd, the total resistance of the ground, which also acts through A; and 3rd, the total resistance of the cylinder C at p. Now, since the first two forces act through A, the third must also pass through this point. Hence the total resistance at p acts in the line pA, and therefore the total resistance of the ground at A must take some intermediate (but unknown) direction, AR. In the same way, the total resistance at q is proved to act in the line qB, and the total resistance of the ground at S must act in some direction, BS, intermediate to BE and Bq. The resistances in Ap and Bq at p and q meet in a point, P, on the circumference of the upper cylinder.

Now the comparative safety of equilibrium at the different points of contact, A, B, p, q, will depend on the angles made by the total resistances at these points with the normals to the surfaces of contact; and it is manifest that since the angle $DAp > DAR$ and $DpA = DAp$, the total resistance at p makes a greater angle with the normal, DC, to the surface of contact than that which the total resistance at A makes with the normal AD. Hence equilibrium is safer at A than at p. For a similar reason, equilibrium is safer at B than at q. Consequently the final comparison is to be made between the points p and q. Now the line pq can be proved by geometry to pass through the point in which ED intersects BA; and supposing the radius $BE > AD$, this point will be at the left-hand side of the figure. Let a be the acute angle which pq makes with the ground. Then, since in the triangle pCq the base angles at p and q are equal, it is easy to see that $\angle qCW - \angle pCW = 2a$, or $qCW > pCW$. But the angle which the total resistance at q makes with the normal qC is $\frac{1}{2}qCW$, and the angle which the total resistance at p makes with the normal pC is $\frac{1}{2}pCW$; therefore if the friction were gradually and uniformly diminished everywhere, or the cylinders drawn out, the resistance at q would reach its limiting obliquity before that at p. Hence the initial motion will be a slipping of the cylinders C and E at the point q, and a motion of rotation at the other points of contact.

129.] **Virtual Work of the Total Resistance.** Suppose one rough body to roll on another through a small angle whose magnitude is regarded as an infinitesimal of the first order. Then, neglecting infinitesimals of a higher order, the point of the rolling surface in contact with the other surface is at rest during the displacement—that is, the virtual displacement of the point of application of the total resistance between the two bodies is zero. Hence for a virtual displacement which consists of a small rolling motion of one rough body on another, the total resistance will not enter into the equation of virtual work of *either* body. Of course in no case can the mutual action of two rigid bodies in contact enter into an equation of virtual work for *both* bodies.

It is a principle in Kinetics that in a motion of pure rolling of a body on a rough fixed surface no work is done between any two positions by the total resistance—a principle which the student will have no difficulty in comprehending, since for each small motion the work done by this force is imfinitesimal compared with the work done by other forces acting on the body.

130.] **Friction as Dependent on Initial Arrangements.** In dealing with natural solids, and not with strictly rigid or indeformable bodies, the existence or nonexistence of friction sometimes depends on the way in which a body or system has been placed in the position which we are considering. This will be made clear by the following example. A heavy trap door (or a beam) AB, fig. 132, p. 148, moveable about a fixed horizontal axis at A, has a rope attached at B, and this rope is also attached to any fixed point C; determine the pressure on the axis A.

The line of action of the pressure must, of course, go through O, the point of meeting of the other two forces, but beyond this we know nothing about it until we know the nature of the axis. If the axis is smooth, or if it is rough but so worn that the contact of the door with it takes place along a single line, the action between the door and the axis will consist of a force passing through the axis, as has been amply explained in Art. 114. But if the axis is rough and contact takes place all round it, the line of action of the resultant force is not generally determinate. However, even in this case this resultant force may pass through the axis. The axis being rough, let us imagine it to become smooth, and what motion results? The rope, being slightly extensible, would yield a little, and slipping would take place over a small surface at the axis; so that the supposition of smoothness alters the circumstances of the case. But suppose that (the axis being still rough) the rope has been stretched, when the door is placed in position, to such an extent that the moment of its tension about the axis is equal to the moment of the weight of the door; then clearly if we imagine the axis to become smooth, no motion will result—no slipping at the axis; and since the displacement which friction is required to prevent does not take place, friction does not act, and the case is the same as if the axis were smooth. The resultant in this case is therefore determinate.

131.] **Friction of a Pivot.** Let a cylindrical pivot, $ABCD$ (fig. 167), on the top of which a given force is applied, revolve

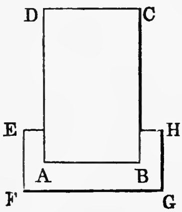

Fig. 167.

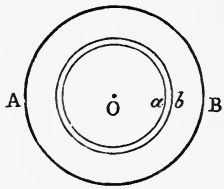

Fig. 168.

in a closely fitting bearing, $EFGH$, and let it be required to calculate the moment of the friction on the base, AB, about the axis of the pivot. Suppose fig. 168 to represent the base of the pivot, and let $P =$ the whole normal pressure on the base, which we shall suppose to be uniformly distributed over the base. Divide the area AB into a number of narrow circular strips, of which one is represented in the figure. Let $Oa = x$, $Ob = x + dx$, $OB = r$, $\mu =$ coefficient of friction. Then since the whole pressure is uniformly distributed, the pressure on the strip whose area is $= 2\pi x dx$ is $\frac{P}{\pi r^2} \cdot 2\pi x dx$, or $\frac{2Pxdx}{r^2}$. Hence the sum of the forces of friction, acting in the directions of the tangents to the strip, is $\frac{2\mu Px\,dx}{r^2}$. But since the tangents to the strip are all at the same distance from the centre, the moment of friction on the strip is equal to the sum of the forces of friction multiplied by the radius, x, of the strip. Hence the moment of friction over the whole surface is

$$\int_0^r \frac{2\mu Px^2 dx}{r^2}, \quad \text{or} \quad \frac{2}{3}\mu Pr. \tag{1}$$

If the base, instead of being a full circle, is a ring, or *collar*, whose internal and external radii are r_1 and r_2, the friction per unit of surface is $\frac{\mu P}{\pi(r_2^2 - r_1^2)}$, and the moment of friction is

$$\int_{r_1}^{r_2} \frac{2\mu Px^2\,dx}{(r_2^2 - r_1^2)}, \quad \text{or} \quad \frac{2}{3}\mu P \cdot \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}. \tag{2}$$

132.] **Wearing Away of the Step.** The piece which supports a pivot, and in which it revolves, is called a *step*. When the pivot revolves, the friction against the step wears away its own surface and that of the step. The amount of wear at any point of the step depends on the magnitude of the force of friction and the relative velocity of the rubbing surfaces at this point. Thus, suppose that ABC (fig. 169) represents a section of the step through the axis, BP, of the pivot, and that Q is any point of contact of the pivot and step. If f is the magnitude of the force of friction at Q, the wearing at Q in the direction of the normal will be proportional to f and also to the amount of rubbing surface which passes over Q in a unit of time. Supposing the pivot to revolve round its axis with an angular velocity ω, the point of the pivot in contact with Q moves in a horizontal circle with a velocity $= \omega \,.\, QM$, or $\omega \,.\, y$; QM, or y, being the perpendicular from Q on the axis of the pivot.

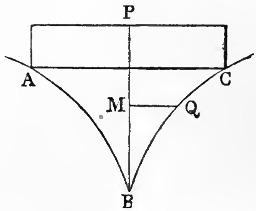

Fig. 169.

But the amount of rubbing surface which passes over Q in a unit of time is evidently proportional to the velocity at Q. Hence the normal wearing of the surface at Q is proportional to

$$\omega f y.$$

If n be the magnitude of the normal pressure per unit of surface at Q, and μ the coefficient of friction, we have $f = \mu n$.

Hence the normal wearing of the surface at Q is proportional to

$$\omega \mu n y. \tag{a}$$

133.] **Friction of a Conical Pivot.** Let ABC (fig. 170) represent a section of a conical step by a plane through the axis, BP, of the pivot, APC being the surface at which the pivot enters the step.

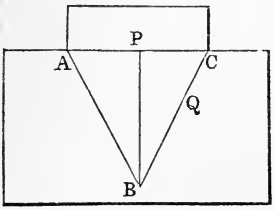

Fig. 170.

Supposing that the pressure on the top of the pivot is uniformly distributed, it will evidently be uniformly distributed over the area APC; that is, there will be a constant normal pressure, n, per unit of area on APC. Now it is impossible to determine by elementary methods the law of distribution of the pressure on the step. The following investigation proceeds on the assumption that the normal pressure per unit of area, or as it is properly called, the normal *intensity of pressure*, is constant over the surface of contact.

Let n be the constant pressure per unit of surface of the step.

If ds is a small element of the line BC at Q, the distance of which from BP is y, the corresponding elementary strip of conical surface is $2\pi y ds$, and the moment round BP of the friction on this strip is

$$2\pi y\, ds \times \mu n \times y,$$

or

$$2\mu\pi n y^2 ds.$$

Putting $ds = \dfrac{dy}{\sin\theta}$, and integrating over the surface of the step from $y = 0$ to $y = PC = r$, we have the moment of the whole friction equal to

$$\frac{2\mu\pi n r^3}{3\sin\theta}.$$

If $P =$ the whole pressure on the top of the pivot, $n = \dfrac{P}{\pi r^2}$; hence the moment of friction

$$= \frac{2\mu}{3\sin\theta}.Pr.$$

Comparing this with the result in Art. 131, we see that the moment of friction in the case of a conical is greater than in the case of a cylindrical pivot of equal radius.

134.] **The Tractory, or Anti-Friction Curve.** In the case of a conical pivot the wearing away of the step is not uniform at all points. Hence after a sufficient time the pivot will not be in perfect contact with its step. If, however, the step has such a form that the vertical wear is the same at all points, the pivot will simply sink into the piece which supports it, and remain always in contact throughout its surface with the step.

We propose to investigate the form of the step in which the vertical wear will be the same at all points. Let fig. 171 represent a section of the step through the axis of the pivot, and let CC' be the vertical wear at C, and QQ' the vertical wear at Q. Then $CC' = QQ'$, Q being any point on the curve BC. Hence the new curve, $BQ'C'$, is simply the old curve BQC moved through a vertical distance $CC' = QQ' = h$, suppose.

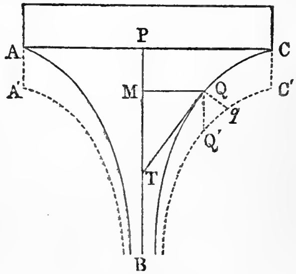

Fig. 171.

Now (Art. 132) the normal wear at Q per unit of surface is $\omega\mu ny$. Hence, if Qq is normal to the step at Q,

$$Qq = \omega\mu ny,$$

n being the normal pressure per unit of surface on APC, which we also take to be the normal pressure per unit of surface on the step.

But $QQ' = \dfrac{Qq}{\cos Q'Qq} = \dfrac{Qq}{\sin MTQ}$, QT being the tangent to the curve at Q. Hence

$$h = \frac{\omega\mu ny}{\dfrac{dy}{ds}},$$

or
$$y\frac{ds}{dy} = \frac{h}{\omega\mu n} = \text{a constant},$$

or
$$QT = \text{a constant}.$$

Therefore the curve BC is such that the length of the tangent terminated by PB, or the axis of x, is constant at all points. This curve is known as the Tractory. If $t =$ the constant length of the tangent, and PC the axis of y, we have

$$y \frac{ds}{dy} = t,$$

or

$$y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = t,$$

or

$$\frac{\sqrt{t^2 - y^2}}{y} dy = -dx,$$

the minus sign being given to the square root, because MQ diminishes as x increases. Integrating this last equation (by assuming $y = t \sin \phi$) we have for the equation of the tractory

$$t \log \frac{t - \sqrt{t^2 - y^2}}{y} + x + \sqrt{t^2 - y^2} = 0.$$

The curve approaches PB asymptotically, and the step is formed by the revolution of the curve round PB. This pivot is known as *Schiele's Anti-friction Pivot.*

EXAMPLES *.

1. A uniform rectangular board, $ABCD$ (fig. 172), rests in a vertical plane against two equally rough pegs, P and Q, in the same horizontal line, two adjacent sides of the board being each in contact with a peg. Find the position of equilibrium.

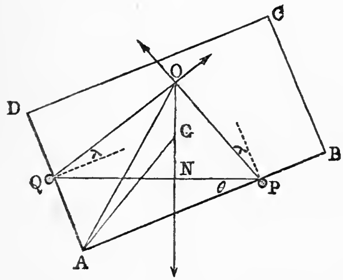

Fig. 172.

Let λ be the angle of friction, θ the inclination of the side AB to the horizon in the position of limiting equilibrium, G the centre of gravity of the board, $PQ = a$, and $AG = c$.

Then if the board is on the point of slipping down at Q and up at P, the total resistances at P and Q will act in the directions PO and QO, which are inclined at the angle λ to the normals at P and Q to the sides AB and AD, respectively. If O' (not represented in figure) be the point of meeting of the normals at P and Q, it is clear that a circle will pass through the points $APO'OQ$; and therefore $\angle OAO' = \lambda$. And since $AO' = PQ = a$, we have

$$AO = a \cos \lambda. \tag{1}$$

* Many of the following examples are due to Mr. Jellett, in whose *Theory of Friction* will be found several other instructive examples which want of space compels me to omit.

Again, since $\angle O'QP = \theta$, we have $\angle QOG = \frac{\pi}{2} - (\lambda + \theta)$, and evidently $\angle QOA = \theta$, therefore $\angle AOG = \frac{\pi}{2} - (\lambda + 2\theta)$. If $\angle GAB = a$, it is clear that $\angle AGN = \frac{\pi}{2} - (\theta + a)$. Now the position of equilibrium is found by the equation

$$AO . \sin AOG = AG . \sin AGN.$$

Substituting in this equation the value of AO from (1), we have

$$a \cos \lambda . \cos (\lambda + 2\theta) = c . \cos (a + \theta),$$

which defines the position of equilibrium.

2. A heavy uniform beam rests against a rough horizontal plane and against a rough vertical wall, the vertical plane through the beam being at right angles to the wall and the ground; determine the greatest weight that can be affixed to it at a given point, so that equilibrium may be preserved.

(a) If the beam be inclined to the vertical at an angle less than the angle of friction for the beam and the ground, equilibrium cannot be broken by attaching a weight, however great, to any point of the beam.

Let AB (fig. 173) be the beam, θ its inclination to the horizon, W its weight, $2a$ its length, P the weight suspended from the point Q in the beam, $BQ = x$, λ and λ' the angles of friction at A and B, respectively.

Draw the lines AO and BO, making the angles λ and λ' with the normals, An and Bm, at A and B.

Then when the resultant of W and P passes through O, equilibrium will be at its limit. For, if this resultant acts in a line to the left of OV, the vertical through O, it will be possible to find an infinite number of points on it such that when joined to A and B the joining lines will be possible directions of total resistance at A and B (see Art. 126).

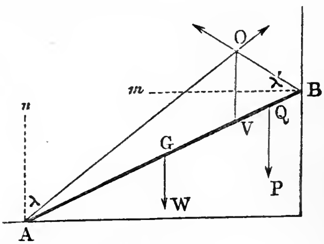

Fig. 173.

If the resultant of W and P acts in a line to the right of OV, there will be no point on it inside both cones of friction, and therefore equilibrium will be impossible. Hence for limiting equilibrium, we have by taking moments about O,

$$W . GV = P . QV,$$

G being the centre of gravity of the beam.

The lengths GV and QV are easily obtained from the data. We may observe that if the point Q lies between G and V, equilibrium can never be broken, however great P may be. For it will then be impossible by increasing P to bring the resultant of P and W to the right of OV.

These results follow also from the usual mode of solution of such a problem.

Let R and S be the normal reactions at A and B, and μ and μ' the coefficients of friction at these points. Then, resolving horizontally,

$$S = \mu R; \tag{2}$$

resolving vertically, $$R + \mu' S = P + W; \tag{3}$$

taking moments about B,

$$2aR(\cos\theta - \mu\sin\theta) = (Px + Wa)\cos\theta. \tag{4}$$

From (2) and (3) we have $R = \dfrac{P+W}{1+\mu\mu'}$,

and by substituting this value of R in (4), we get

$$P = Wa\frac{1+\mu\mu' - 2(1-\mu\tan\theta)}{2a(1-\mu\tan\theta) - x(1+\mu\mu')}. \tag{5}$$

Now it is easy to see that $BO = 2a\dfrac{\cos(\theta+\lambda)}{\cos(\lambda-\lambda')}$, and $BV = BO\dfrac{\cos\lambda'}{\cos\theta}$; therefore $BV = 2a\dfrac{1-\mu\tan\theta}{1+\mu\mu'}$, and (5) may be written

$$P = W\cdot\frac{a - BV}{BV - x},$$

from which it appears that if $x = BV$, the required force is infinite; and if $x > BV$, it is negative, or equilibrium can never be broken by any downward force.

The second part of the problem follows from (5), because if $\mu\tan\theta > 1$, or, in other words, if the angle $nAB < \lambda$, the denominator will be negative. That it is impossible to break equilibrium in this case is evident from fig. 174. For the point O is now at the right of the vertical wall, and at whatever point along AB the resultant of P and W acts, it is possible to find points on it which are within both cones of friction.

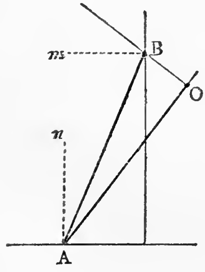

Fig. 174.

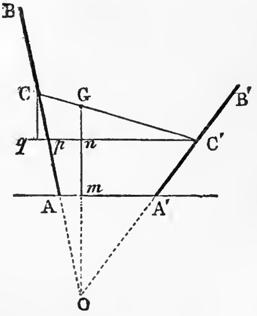

Fig. 175.

3. Two unequal uniform beams, connected by a light rope attached to their middle points, rest in a vertical plane, an extremity of each

beam resting on a rough horizontal plane. If the coefficient of friction is gradually diminished, which beam will slip first?

Let the beams be AB and $A'B'$ (fig. 175) and let C and C' be their centres, and $AB > A'B'$. Now the beam AB is in equilibrium under the influence of three forces, viz., its weight, the tension of the rope CC', and the total resistance at A; and since the first two meet in C, the third must also pass through this point, that is, the resistance at A acts along the beam. In the same way the resistance at A' acts along $A'B'$; and by considering the equilibrium of the system, we see that the vertical through G, the common centre of gravity, must pass through O, the point of intersection of the resistances. Now the angles which these resistances make with the normals at A and B are equal to mOA and mOA', respectively; and the comparative safety of the equilibrium at A and B depends on the magnitudes of these angles. Now $mOA' > mOA$. For, draw $C'q$ horizontal and Cq vertical; then, since $CG < C'G$, $qn < nC'$, and *à fortiori*, $pn < nC'$. Therefore $Am < mA'$; but $\dfrac{Am}{mA'} = \dfrac{\tan mOA}{\tan mOA'}$; therefore, $mOA' > mOA$, and if the friction were gradually diminished, the total resistance at A' would reach its limiting inclination before that at A. Hence the short beam will slip first.

4. A cylinder is supported on a rough inclined plane by a string coiled round it in a direction perpendicular to its axis, the string passing over a smooth pulley and sustaining a weight. Find the limits to the direction of the string.

Round A, the point of contact of the cylinder and plane, describe the cone of friction, the section of which by the plane of the figure is nAm, the angles nAC and CAm being each $= \lambda$.

Let OB be any direction of the string, intersecting the vertical through the centre of the cylinder in O. Then, so long as O is between the points m and n, equilibrium is possible, because AO is a possible direction of total resistance at A. There is, of course, a particular magnitude of the suspended weight, P, corresponding to the direction OB of the string, and this magnitude is found by taking moments about A. If θ is the angle made by the string, OB with the inclined plane, we have

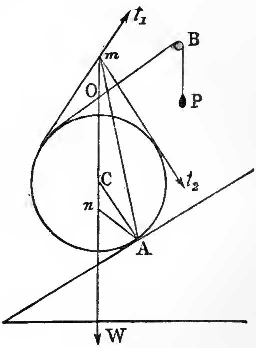

Fig. 176.

$$P = W\frac{\sin i}{2\cos^2\dfrac{\theta}{2}},$$

i being the inclination of the inclined plane.

If, the direction of the string being OB, P have a value greater or less than this, the cylinder will roll up or roll down the plane.

Drawing from m two tangents, mt_1 and mt_2, to the cylinder, we have the extreme directions of the string; that is, the point at which the string leaves the cylinder must lie between the points of contact of mt_1 and mt_2, on the upper portion of the cylinder; for it is evident that if the string leaves the cylinder at any point outside these limits, the point in which its line intersects that of W will be vertically above m, that is, outside the cone of friction.

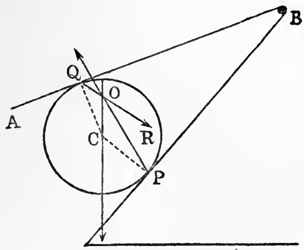

Fig. 177.

5. A heavy sphere is placed on a rough inclined plane at a point P (fig. 177), and is kept in position by a heavy rough beam, AB, which is moveable about a fixed extremity, B, the coefficient of friction for the sphere and the beam being the same as that for the sphere and plane. Supposing that the friction is gradually diminished at both points of contact, P and Q, or that the sphere is pushed further up between the plane and beam, determine the nature of the initial motion.

The total resistances at P and Q must meet in some point, O, on the vertical through C, the centre of gravity of the sphere. Beyond this, however, their directions cannot be determined. The comparative safety of equilibrium at P and Q will depend on the relative magnitudes of the angles, CPO and CQO, which the resistances at these points make with the corresponding normals. Now it is easy to show that $CQO > CPO$; for $\sin CPO = \frac{CO}{CP} \sin COP$, and $\sin CQO = \frac{CO}{CQ} \sin COR$, therefore $\frac{\sin CPO}{\sin CQO} = \frac{\sin COP}{\sin COR}$; but $COR > COP$, therefore $CQO > CPO$, and if from any cause the friction is diminished, or the sphere pushed higher up, slipping must take place at Q and rolling at P.

6. A cylinder is placed on a rough inclined plane, and a light rope is coiled round it in a plane perpendicular to its axis and containing its centre of gravity; this rope, after passing round the cylinder, is attached to the middle point, H (fig. 178), of an edge of a cubical block whose height is equal to the diameter of the cylinder. Supposing the inclination of the plane to be gradually increased, determine the manner in which equilibrium will be broken, the coefficient of friction being the same for the cylinder and plane as for the cube and plane.

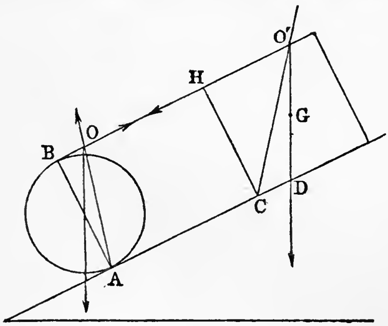

Fig. 178.

The motions which are here geometrically possible are—

(1) The cylinder may roll and the cube may turn over the edge C.
(2) The cylinder may roll and the cube may slip.
(3) The cylinder may slip and the cube may slip.
(4) The cylinder may slip and the cube may turn over.

Now if O is the point of intersection of the vertical through the centre of gravity of the cylinder with the rope, it is evident that the total resistance at A acts in the line AO. In the same way if O' is the point of intersection of the vertical through G, the centre of gravity of the cube, with the line of the rope, the total resistance of the plane on the cube must pass through O', and if D is the point in which the line of action of the weight of the cube intersects its base, the total resistance must evidently pass through some point between C and D.

Now this total resistance, wherever it acts, makes with the normal to the plane an angle greater than BAO; for $\tan BAO = \frac{1}{2}\tan i$, i being the inclination of the plane, and the angle which $O'D$ makes with the normal to the plane $= i$; hence the angle made with this normal by a line joining O' to any point between C and D is $> i$, and, *à fortiori*, $> BAO$. Consequently the cylinder can never slip before the cube, and cases 3 and 4 are to be rejected. The choice then is to be made between 1 and 2; and (see Art. 126) if the cube can turn over, it will do so. Hence we solve on the supposition that the cube turns over C, and if this does not require too great a value of the coefficient of friction, the cube will turn over.

The problem is to be solved by equating the values of the tension of the rope derived from the consideration of the equilibrium of the cylinder and that of the cube.

For the equilibrium of the cylinder take moments about A, and we have

$$T = \tfrac{1}{2} W \sin i, \tag{1}$$

T being the tension of the rope and W the weight of the cylinder.

Again, since by supposition the cube is about to turn round C, the total resistance of the plane acts through this point. Taking moments about C for the cube,

$$T \,.\, CH = W' \,.\, CG \sin\left(\frac{\pi}{4} - i\right),$$

or

$$T = \tfrac{1}{2} W' (\cos i - \sin i). \tag{2}$$

Equating the values of T in (1) and (2), we have

$$\tan i = \frac{W'}{W + W'}. \tag{3}$$

But in order that CO' may be a possible direction of total resistance, the angle HCO' must be $< \lambda$, or $\tan HCO' < \mu$. Now, it is easy to see that

$$\begin{aligned} \tan HCO' &= \frac{1 + \tan i}{2} \\ &= \tfrac{1}{2} \cdot \frac{W + 2W'}{W + W'}. \end{aligned} \tag{4}$$

Hence if $\frac{1}{2}\frac{W+2W'}{W+W'} < \mu$, equilibrium will be broken by a rolling of the cylinder and turning over of the cube. If μ is less than the quantity in (4) the cylinder will roll and the cube will slip, and there is no difficulty in determining the inclination of the plane when this happens. We may either draw from O' a line making the angle of friction, λ, with the normal to the plane, and then determine T by the triangle of forces, or resolve along and perpendicular to the plane for the equilibrium of the cube. If R is the normal reaction of the plane on the cube, we find in the latter way

$$R = W' \cos i,$$

$$\mu R = W' \sin i + T;$$

therefore $$T = W'(\mu \cos i - \sin i).$$

Equating this to the value given by (1), we have

$$\tan i = \frac{2\mu W'}{W+2W'},$$

which gives the inclination at which the cube slips.

7. Two equal carriage wheels whose centres are connected by a smooth bar are placed on a rough inclined plane; determine whether the equilibrium of the system will be best preserved by locking the hind or the fore wheel.

Let C and D (fig. 179) be the centres of the wheels, and first suppose the hind wheel to be locked. Since there is no friction between the bar CD and the axle at C, the action of the bar on the lower wheel consists of a force through C (see p. 140).

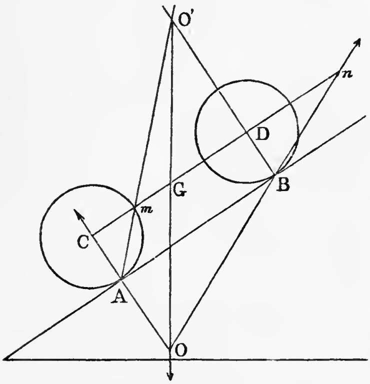

Fig. 179.

The weight of this wheel also acts through C, and therefore the total resistance at A, which is the third force keeping the wheel in equilibrium, must also act through C.

Let G be the centre of gravity of the two wheels, and consider the equilibrium of the system formed by them. There are three forces acting on the system, viz., its weight through G, the total resistance at A (which has been proved to act in a line AC), and the total resistance at B. If, then, O is the point of intersection of CA and the vertical through G, the total resistance at B must act in the line OB.

We shall now determine the inclination at which equilibrium is broken.

Since the hind wheel slips, the angle $DBn = \lambda$; also let $r =$ the radius of each wheel, $CD = 2a$, and $i =$ the inclination of the plane.

Then $$\frac{\tan COG}{\tan COn} = \frac{CG}{Cn},$$

or $$\frac{\tan i}{\mu} = \frac{a}{2a+\mu r},$$

since $Dn = r \tan DBn = \mu r$. The inclination of the plane when equilibrium is broken is therefore given by the equation

$$\tan i = \frac{\mu a}{2a+\mu r}. \tag{1}$$

Again, suppose the fore wheel alone to be locked. In this case the total resistance at B acts in the line BD, and that at A acts in AO', O' being the intersection of BD with OG. If i' is the new inclination at which equilibrium is broken, we have, since $\angle CAO' = \lambda$,

$$\frac{\tan i'}{\mu} = \frac{DG}{Dm} = \frac{a}{2a-\mu r},$$

or $$\tan i' = \frac{\mu a}{2a-\mu r}. \tag{2}$$

Now it is clear that i' is greater than i, and that, consequently, equilibrium will be safer when the fore wheel is locked than when the hind wheel is locked.

8. A cylinder is supported on a rough inclined plane by a light rope coiled round it in a plane perpendicular to its axis passing through its centre of gravity, the rope being attached to a fixed point. Find the direction of the rope in order that the inclination of the plane may be the greatest possible.

Let $O'B'$ (fig. 180) be the line of the rope, and CO' the vertical through the centre of gravity of the cylinder. Then evidently the total resistance at A, the point of contact with the plane, must act in the direction AO'. If the rope took the direction OB, which is horizontal, the direction of the total resistance would be AO, and evidently the angle $CAO < CAO'$; or, in other words, the equilibrium of the

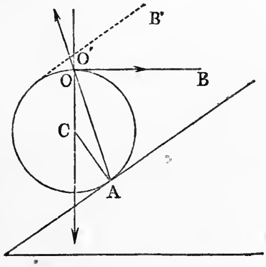

Fig. 180.

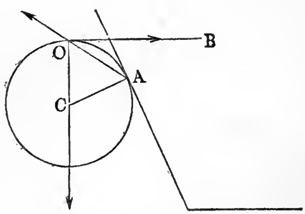

Fig. 181.

cylinder will be farther from its limit when the rope is horizontal than when it takes any other direction. For a given inclination, i, of the plane the angle $CAO = \frac{i}{2}$, and it is clear that when CAO is equal to the angle, λ, of friction, the inclination of the plane will be at its greatest. Hence the greatest inclination of the plane $= 2\lambda$.

If the coefficient of friction be > 1, the greatest inclination of the plane will be $> \frac{\pi}{2}$, and the figure of limiting equilibrium will be that represented in fig. 181, in which the angle CAO ($= \lambda$) is $> \frac{\pi}{4}$. But whether the cylinder will stay in this position or not depends on the initial arrangement. Unless the rope is pulled with such a force as to cause the resultant of this force and W to act in the line OA, equilibrium cannot be preserved by the resistance of the plane. In fact, unless this requisite tension of the rope is produced by pressing and scraping the cylinder against the plane, it would be possible for the cylinder to take a motion of and round its centre C which would keep its surface out of actual contact with the plane; and in this case the plane would not exert any resistance.

9. If in the preceding problem the rope, instead of being attached to a fixed point, is attached to a weight which hangs freely over a smooth pulley, find the conditions of equilibrium.

Let $O'B'$ (fig. 180) be the direction of the rope, P the suspended weight, W the weight of the cylinder, i the inclination of the plane, λ the angle of friction, θ the angle which the rope makes with the inclined plane.

Then for equilibrium it is necessary that AO' should be the direction of total resistance at A, and that the moments of P and W about A should be equal and opposite. Hence we must have

angle $$CAO' = \text{ or } < \lambda, \tag{1}$$

and $$P = W \frac{\sin i}{2 \cos^2 \frac{\theta}{2}}, \tag{2}$$

the second condition being equivalent to that obtained by the triangle of forces for equilibrium at O'.

If the angle $CAO' < \lambda$, and P is slightly increased above the value in (2), the initial motion will evidently be a rolling up, since moment of P about $A >$ moment of W about A; but if P is slightly diminished the rolling will be down.

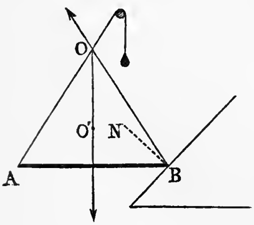

Fig. 182.

10. A heavy uniform beam, AB (fig. 182), is to be sustained in a horizontal position, one end, B, resting on a rough inclined plane, while the other end, A, is attached to a light rope which passes over a smooth pulley and sustains a weight. Find

(a) The limits to the direction of the rope, and the corresponding limiting values of the suspended weight.

(b) The least weight that will sustain the beam.

Let W be the weight of the beam, P the suspended weight, and BN the normal to the inclined plane at B. Then if AO be the line of the rope, intersecting the vertical through the centre of gravity of the beam in O, BO must be the direction of the total resistance at B; and in order that this may be a possible direction of total resistance, the angle NBO must be $< \lambda$, the angle of friction. Hence the limiting directions of the rope are obtained by drawing BO and BO' making the angle λ with BN on opposite sides. If the rope takes the direction AO' the beam must be on the point of slipping up, since the force of friction acts *down* the inclined plane; and if the direction of the rope is AO, the beam is on the point of slipping down. The corresponding magnitudes of P are easily determined by taking moments about B. Let p_1 and p_2 be the perpendiculars from B on AO and AO', respectively, a half the length of the beam, and P_1 and P_2 the corresponding values of P. Then

$$P_1 = W\frac{a}{p_1},$$

$$P_2 = W\frac{a}{p_2}.$$

The values of p_1 and p_2 can, of course, be easily expressed in terms of a, λ, and i, the inclination of the plane.

If the rope takes a direction intermediate to AO and AO', and if p is the length of the perpendicular from B on its direction, we have

$$P = W\frac{a}{p}.$$

Hence, if P is a minimum, p must be a maximum, since Wa is given. Now p will be a maximum when it is equal to AB, that is, when the rope is vertical. In this case the total resistance at B should also be vertical; but if the inclination of the plane $> \lambda$, this is impossible. Hence when $i > \lambda$, p is a maximum (*consistently with the conditions of the problem*) when the direction of the rope is AO; and therefore in this case P_1 is the least value of P.

If $i < \lambda$, the vertical at B is a possible direction of total resistance, and therefore AB is an admissible value of p. The corresponding value of P is therefore $\frac{1}{2}W$.

The student will easily see that if the angle of friction is greater than the complement of the inclination of the plane, there can be no limiting equilibrium in which the beam is about to slip up.

11. A cylinder is laid on a rough horizontal plane, and is in contact with a rough vertical wall; a string coiled round it at right angles to the axis passes over a smooth pulley and sustains a weight which is gradually increased till equilibrium is broken. Determine the nature of the initial motion. (Jellett's *Theory of Friction*, example 21, p. 214.)

Let W be the weight of the cylinder, P the suspended weight, θ the angle made by the string with the horizon, λ and λ' the angles of friction at A and B, the points of contact of the cylinder with the vertical and horizontal planes, and O the point in which the line of the string intersects the vertical through C, the centre of gravity of the cylinder.

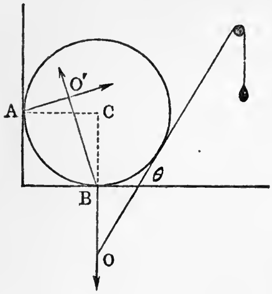

Fig. 183.

Now, in accordance with Article 126, we first consider what motions are geometrically possible. These are

(1) Rolling round A up the vertical plane.

(2) Slipping forward at B while contact ceases at A.

(3) Slipping at A and B simultaneously.

If (1) *can* happen it *will* (see Art. 126); let us suppose, therefore, that the cylinder is on the point of turning round A and coming out of contact at B. In this case there are only three forces keeping the cylinder in equilibrium, namely, W, P, and a total resistance at A. This last force should, for equilibrium, pass through O and act in the direction OA. Now whether the angle OAC is less or greater than λ, this is not a possible line of action of total resistance, because the plane cannot *pull*. Hence (1) is physically impossible.

Suppose that (2) happens. Then, as before, there are only three forces keeping the cylinder in equilibrium, namely, W, P, and the resistance at B. This last must pass through O, and must therefore act vertically. But it is obvious that such a force could not equilibrate W and P; therefore (2) is impossible.

There remains the third case, which alone is possible. To determine the value of P corresponding to limiting equilibrium, draw the lines AO' and BO' making with the normals at A and B the angles, λ and λ', of friction for the cylinder and planes. Then by taking moments about O' we easily obtain the value of P, which may also be obtained by the ordinary equations of resolution of forces. Thus, let R and R' be the normal pressures, and therefore μR and $\mu' R'$ the forces of friction, at A and B.

Taking moments about B, we have

$$R(1+\mu) = P(1-\cos\theta). \tag{1}$$

Taking moments about A,

$$R'(1-\mu') = W - P(1+\sin\theta). \tag{2}$$

Resolving horizontally,

$$\mu' R' - R = P\cos\theta. \tag{3}$$

Substituting in (3) the values of R and R' given in (1) and (2), we obtain the value of P corresponding to limiting equilibrium.

It will be a useful exercise for the student to vary the position of the pulley in such a way as to render possible a case of limiting equilibrium in which the cylinder is about to ascend the vertical plane by turning round A.

12. A heavy right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine

whether the initial motion of the cone will be one of sliding or tumbling over.

Let ABC (fig. 184) be the vertical section of the cone through its axis, CH, and let G be the centre of gravity of the cone. (GH is $\frac{1}{4}$ CH, as will appear in a subsequent Chapter.) Then, in accordance with rule 3 of Art. 126, if it is possible for the cone to turn over the point A, the cone will do so. Solve, therefore, on the supposition that equilibrium is broken by turning round A. In this case, the two forces acting on the cone are its weight and the total resistance of the plane, which, of course, passes through A; and these forces must be equal and opposite, i. e., the total resistance must act in the vertical line AG. Now this will be possible only if AG makes with the normal to the plane an angle less than the angle of friction, λ. Hence for a tumbling motion $AGH < \lambda$. But if $a = ACH$,

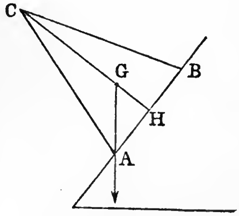

Fig. 184.

$$\tan AGH = 4 \tan a.$$

Therefore if $\mu > 4 \tan a$, the initial motion of the cone will be tumbling, and if $\mu < 4 \tan a$, the initial motion will be sliding, and this sliding will evidently occur when the inclination of the plane reaches the value λ.

13. A heavy straight rod rests on a rough horizontal plane, and at one end, perpendicularly to its length and in the horizontal plane, a force is applied with gradually increasing magnitude. Find the point about which the rod begins to turn.

(Price's *Infinitesimal Calculus*, vol. iii, p. 162.)

Let l be its length and suppose it to turn round a point at a distance z from the other extremity. Then we must equate the moment of the applied force about this point to the sum of the moments of the forces of friction acting on the different elements of the rod. Take an elementary portion of length dx at a distance x from the point round which the rod turns. The weight of this portion is $\frac{W}{l}\,dx$, and the force of friction on it is $\mu W \frac{dx}{l}$. This acts at right angles to the rod. Hence, taking the sum of the moments for all points at both sides of the turning point, we have *

$$P(l-z) = \frac{\mu W}{l}\int_0^{l-z} x\,dx + \frac{\mu W}{l}\int_0^{z} x\,dx = \frac{\mu W}{2l}\left[(l-z^2)+z^2\right].$$

But P is evidently equal to the sum of the frictions at the end adjacent to it minus the sum of those at the other end; i. e., $P = \mu W \frac{l-2z}{l}$. Hence we have

* In this simple case integration is evidently not necessary.

$$2z^2 - 4lz + l^2 = 0, \qquad \therefore \; z = \left(1 - \frac{1}{\sqrt{2}}\right) l;$$

or the turning point is at a distance $\frac{l}{\sqrt{2}}$ from the end at which the force is applied.

14. A rectangular block is placed, with one of its edges horizontal, on a rough plane, the inclination of which to the horizon is gradually increased; determine whether the equilibrium of the block will be broken by a motion of sliding or one of tumbling.

Ans. If a and b are the lengths of the edges which are not horizontal, b being the length of the edge which is perpendicular to the inclined plane, the initial motion will be one of tumbling if $\mu > \frac{a}{b}$, and of sliding if $\mu < \frac{a}{b}$.

15. A cylinder the section of which perpendicular to the axis is any given curve is to be placed, with the axis horizontal, on a rough inclined plane; how must it be placed so that it shall be least likely to slip, the cylinder being in contact with the plane along a single line?

16. An elliptic cylinder is placed, with its axis horizontal, on a rough plane inclined to the horizon at an angle less than the angle of friction; prove that the cylinder cannot rest if the eccentricity of the section perpendicular to the axis is less than $\sqrt{\frac{2 \sin i}{1 + \sin i}}$, i being the inclination of the plane.

17. A uniform beam rests with its extremities on two rough inclined planes whose line of intersection is horizontal, the vertical plane through the beam being perpendicular to this line; find the limiting position of equilibrium.

Ans. If i, i' be the inclinations of the planes, λ, λ' the angles of friction between the beam and the planes, respectively, and θ the limiting inclination of the beam to the horizon,

$$2 \tan \theta = \cot (i + \lambda) - \cot (i' - \lambda').$$

Another limiting position will be got by changing the signs of λ and λ'.

18. A heavy uniform rod rests with its extremities on the interior of a rough vertical circle; find the limiting position of equilibrium.

Ans. If $2a$ is the angle subtended at the centre by the rod, and λ the angle of friction, the limiting inclination of the rod to the horizon is given by the equation

$$\tan \theta = \frac{\sin 2\lambda}{\cos 2\lambda + \cos 2a}.$$

19. A solid triangular prism is placed, with its axis horizontal, on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the prism.

Ans. If the triangle ABC is the section perpendicular to the axis, and the side AB is in contact with the plane, A being the lower vertex, the initial motion will be one of tumbling if

$$\mu > \frac{b^2+3c^2-a^2}{4\,\Delta},$$

the sides of the triangle being a, b, c, and its area Δ. If μ is less than this value, the initial motion will be one of slipping.

20. A frustum of a solid right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the body.

Ans. If the radii of the larger and smaller sections are R and r, and h is the height of the frustum, the initial motion will be one of tumbling or slipping according as

$$\mu >< \frac{4R}{h}\cdot\frac{R^2+Rr+r^2}{R^2+2Rr+3r^2}.$$

21. An elliptic cylinder rests in limiting equilibrium between a rough vertical and an equally rough horizontal plane, the axis of the cylinder being horizontal, and the major axis of the ellipse inclined to the horizon at an angle of 45°. Find the coefficient of friction.

Ans. $$\mu = \frac{\sqrt{1+2e^2-e^4}-1}{2-e^2},$$

e being the eccentricity of the ellipse. (Employ the Theorem of Art. 116.)

22. The circumstances of the preceding problem remaining the same, except that the vertical plane is smooth, show that the coefficient of friction is $\frac{1}{2}e^2$ (Walton's *Mechanical Problems*, p. 82).

If the horizontal plane alone is smooth, is it possible for the cylinder to rest in any position?

23. A uniform beam, of which one end rests against a rough vertical wall, is supported by a light rope attached to the other end, and to a given point in the wall; find the limiting positions of equilibrium (Walton, p. 81).

Ans. If the length of the rope be n times the length of the beam, the inclination of the latter to the wall is given by the equation

$$(n^2-\mu^2-1)\tan^2\theta+4\mu\tan\theta+n^2-4=0.$$

24. In order that both limiting positions may be real, what must be the limits of n?

Ans. $2n^2$ must be $> \mu^2+5-\sqrt{(\mu^2+1)(\mu^2+9)}$, and

$< \mu^2+5+\sqrt{(\mu^2+1)(\mu^2+9)}$.

25. If n is 2, show that there is but one limiting position; and prove geometrically that if in this case the angle of friction is 60°, the limiting position is horizontal.

26. A heavy uniform beam rests with one end against a rough horizontal and the other end against an equally rough vertical plane; find the least coefficient of friction that will allow the beam to rest in all positions.

Ans. Unity.

27. In the previous question let the centre of gravity of the beam divide it into two segments, a and b, the latter segment being in contact with the vertical well; given the coefficient of friction, μ, between the beam and the ground, find the least coefficient of friction between the beam and the wall which will allow the beam to rest in all positions.

Ans. $\dfrac{a}{\mu b}$.

28. Two equal beams, AC and CB, are connected by a smooth hinge at C, and are placed in a vertical plane with their lower extremities, A and B, resting on a rough horizontal plane; from observing the greatest value of the angle ACB for which equilibrium is possible, determine the coefficient of friction for the beams and the plane (Walton's *Mechanical Problems*, p. 96, second ed.)

Ans. If the greatest value of $\angle ACB$ is β,

$$\mu = \tfrac{1}{2} \tan \frac{\beta}{2}.$$

29. Two uniform beams are placed with their lower extremities resting on a rough horizontal plane, their upper extremities resting against each other. Show how to cut a plane face from the upper extremity of one of the beams, in order that slipping may be about to ensue at their point of contact.

Ans. Determine the line of action of their mutual resistance as in p. 167; then cut a face inclined to this line at the complement of the angle of friction.

30. A cylinder is placed on a rough horizontal plane, and a uniform plank rests with one end on the ground and the other against the cylinder (the plank being at right angles to the axis of the cylinder). If the plank is gradually lowered until equilibrium is about to be broken, show that if the weight of the cylinder exceed that of the plank the latter will always slip, whatever be the dimensions of the plank and cylinder. For any position of the plank find the direction of the reaction of the ground on the cylinder.

Ans. If θ is the angle made by the plank with the ground, P = weight of plank, W = weight of cylinder, r = radius of cylinder, $2a$ = length of plank, ψ = angle made with the vertical by the reaction of the ground on the cylinder,

$$\cot \psi = \frac{2W}{P} \tan \theta + \left(1 + \frac{2W}{P}\right) \sqrt{\frac{a \sin \theta}{r - a \sin \theta}}.$$

31. A cylinder placed on a rough plane has a string coiled round it in a plane at right angles to its axis; the string after passing round

the cylinder is attached to a heavy particle which also rests on the plane. If the plane is gradually tilted up, determine the nature of the initial motion.

Ans. The cylinder will roll and the particle slip if both are equally rough; and if i is the inclination of the plane when this happens,

$$\tan i = \frac{2\mu P \cos^2 a}{W \cos 2a + 2P\cos^2 a + \mu W \sin 2a},$$

where W and P are the weights of the cylinder and the particle, μ the coefficient of friction, and $2a$ the angle between the string and the inclined plane.

32. A heavy cylinder is laid on a rough inclined plane, its axis being horizontal; a heavy uniform plank rests on the cylinder and against the inclined plane, the plank being horizontal at right angles to the axis of the cylinder, and touching the cylinder at its highest point. Supposing the inclination of the plane to be gradually increased, the horizontality of the plank being always perserved, determine the nature of the initial motion of the system and the inclination of the plane at which equilibrium is broken.

Ans. The plank will slip at its point of contact with the plane, a rolling motion taking place at the other points of contact in the system; and the inclination (i) is given by the equation

$$\left(\frac{r}{a}\cot\frac{i}{2} - 1\right)\left[P\cot\frac{i}{2}\tan(\lambda - i) - W\right] = P + W,$$

where r = radius of cylinder, $2a$ = length of plank, W = weight of cylinder, P = weight of plank, and λ = angle of friction.

33. Two particles A and B, whose weights are denoted by A and B, are connected by a string fully stretched, and placed on a rough horizontal plane, the coefficient of friction for each particle being μ. A force P, which is $< \mu(A+B)$, is applied to A in the direction BA, and its direction is gradually turned round through an angle θ in the plane. Find the nature of the initial motion of the system.

Ans. If $P < \mu\sqrt{A^2+B^2}$ and $> \mu A$, the particle A alone will slip, and this happens when $\sin\theta = \frac{\mu A}{P}$. If $P > \mu\sqrt{A^2+B^2}$, both will slip when $\cos\theta = \frac{P^2+\mu^2(B^2-A^2)}{2\mu BP}$.

34. A heavy rod is placed in any manner resting on two points A and B of a rough horizontal curve, and a string attached to a point C of the cord AB is pulled in any direction in the plane of the curve so that the rod is on the point of motion. Prove that the locus of the intersection of the lines of action of the frictions at A and B is an arc of a circle and a part of a straight line; except when C is the centre of gravity of the rod, in which case the directions of the frictions will be always parallel to the string.

35. A triangular prism, whose section by a vertical plane through its centre of gravity perpendicular to its edges is ABC, rests with its base AB on a rough horizontal plane; a rope is attached to the middle point, C, of its upper edge, and, passing over a fixed pulley in the horizontal line parallel to, and in the sense of, BA, is pulled with a gradually increasing force. Find the nature of the initial motion.

Ans. If $AB = c$, $AC = b$, and the height of the prism $= h$, the prism will tilt over the edge through A if

$$\mu > \frac{c + b\cos A}{3h};$$

otherwise it will slide.

36. A cubical block is placed on a rough inclined plane and sustained by a rope, parallel to the inclined plane, attached to the middle point of the upper edge (which is horizontal); the rope lies in the vertical plane which contains the centre of the cube and is perpendicular to the inclined plane. Show that the greatest inclination of the plane is

$$\frac{\pi}{4} + \tan^{-1}\left(\frac{\mu}{1+\mu}\right).$$

37. Two rough inclined planes slope in the same direction and intersect in a horizontal line. A cylinder placed at their intersection and touching both all along its length has a rope coiled round it in a plane through its centre of gravity perpendicular to its axis; this rope passes over a fixed pulley and is pulled with gradually increasing force. Discuss the ways in which equilibrium may be broken by varying the tension of the rope, finding (with a given position of the rope)—

(*a*) The condition that must be satisfied in order that equilibrium should be possible at all;

(*b*) The condition that the initial motion should be one of slipping on both planes;

(*c*) The value of the tension of the rope when this slipping takes place.

38. A heavy uniform circular wheel rests, in a vertical plane, against the ground at A and is in contact at B with an obstacle of given height; the wheel is to be pulled over the obstacle by means of a rope (of given direction) attached at a given point to the wheel; find—

(*a*) The condition that the initial motion of the wheel shall be a rolling over the obstacle;

(*b*) The condition that the initial motion may be slipping at A and B.

(*c*) What ultimately happens when the initial motion is slipping at A and B.

CHAPTER X.

EQUILIBRIUM OF A RIGID BODY UNDER THE ACTION OF ANY FORCES.

135.] **Resultant of any Number of Forces Applied to a Material Particle.** Let a force, P, represented in magnitude and direction by OO' (fig. 13, p. 19), act on a particle at O; let Ox, Oy, and Oz, be any three rectangular axes drawn through O; and let the angles, $O'Ox$, $O'Oy$, and $O'Oz$, which the direction of P makes with the axes of reference be denoted by α, β, and γ, respectively. From O' let fall perpendiculars, OF, OH, OD, on the planes, yz, zx, and xy, respectively, and let the parallelopiped be completed as in figure. Then the force OO' may be replaced by the forces OD and OC, by the parallelogram of forces; and OD can again be replaced by OA and OB. Hence the force P is equivalent to the three forces

$$P \cos \alpha \text{ along } Ox,$$
$$P \cos \beta \quad ,, \quad Oy,$$
and
$$P \cos \gamma \quad ,, \quad Oz.$$

The converse proposition is also evidently true—namely, that any three forces, OA, OB, OC, along Ox, Oy, Oz (whether these are mutually rectangular directions or not), give a resultant represented in magnitude and direction by the diagonal, OO', of the parallelopiped determined by the forces.

If several forces, $P_1, P_2, \ldots P_n$, act at O and make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $\ldots$ $(\alpha_n, \beta_n, \gamma_n)$, with the axes, let each of them be replaced by its three components along Ox, Oy, Oz; and if ΣX, ΣY, ΣZ denote the sums of the components along the axes, we shall have

$$\left.\begin{aligned} \Sigma X &= P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots + P_n \cos \alpha_n, \\ \Sigma Y &= P_1 \cos \beta_1 + P_2 \cos \beta_2 + \ldots + P_n \cos \beta_n, \\ \Sigma Z &= P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \ldots + P_n \cos \gamma_n, \end{aligned}\right\} \qquad (1)$$

and the whole system of forces will be replaced by the three forces, ΣX, ΣY, and ΣZ along the axes of x, y, and z. But the resultant of three forces in these directions is the diagonal of the parallelopiped determined by them. Hence, R being the magnitude of this resultant,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}, \tag{2}$$

and if θ, ϕ, ψ, be the direction-angles of R,

$$\cos\theta = \frac{\Sigma X}{R}, \quad \cos\phi = \frac{\Sigma Y}{R}, \quad \cos\psi = \frac{\Sigma Z}{R}. \tag{3}$$

136.] **Graphic Representations of the Resultant.** Since the resultant of any two forces, OA and OB, acting at O is obtained by drawing from A a line, Ab, parallel and equal to OB, and joining O to b, it follows that if a particle is acted on by n forces, OA_1, OA_2, OA_3, ... OA_n, the resultant is obtained in magnitude and direction by drawing $A_1 a_2$ parallel and equal to OA_2, $a_2 a_3$ parallel and equal to OA_3, ... $a_{n-1} a_n$ parallel and equal to OA_n, and joining O to a_n; or, in other words, the side Oa_n which closes the polygon $OA a_2 a_3 \ldots a_n$ represents the resultant in magnitude and direction. When the sides of the polygon are not all coplanar, the figure is called a *gauche polygon*. Thus the second graphic representation of the resultant of a system of coplanar forces, which has been given in p. 18, is equally applicable to non-coplanar forces. Hence, of course, it follows that a particle acted on by any set of forces which are parallel and proportional to the sides of a gauche polygon taken in order is at rest.

Again, since by the parallelogram of forces, the resultant of OA_1 and OA_2 is $2 \,.\, Og_1$, where g_1 is the middle point of $A_1 A_2$; and since the resultant of $2 Og_1$ and OA_3 is $3 Og_2$, where g_2 is determined exactly as in p. 15, it follows that Leibnitz's graphic representation of the resultant is applicable to non-coplanar forces.

This result follows also analytically; for if (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n) be the co-ordinates of the extremities A_1, A_2, ... A_n of the forces acting on the particle, it is clear that

$$\Sigma X = x_1 + x_2 + \ldots + x_n = \Sigma x = n \,.\, \bar{x},$$
$$\Sigma Y = y_1 + y_2 + \ldots + y_n = \Sigma y = n \,.\, \bar{y},$$
$$\Sigma Z = z_1 + z_2 + \ldots + z_n = \Sigma z = n \,.\, \bar{z};$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the co-ordinates of G, the centre of mass of equal masses placed at the extremities of the forces. Hence by equations (1) of Art. 135, $R = \sqrt{n^2(\bar{x}^2+\bar{y}^2+\bar{z}^2)} = n.OG$

$$R = n \, . \, OG,$$

and $$\cos\theta = \frac{\bar{x}}{OG}, \quad \cos\phi = \frac{\bar{y}}{OG}, \quad \cos\psi = \frac{\bar{z}}{OG},$$ $n\bar{x}$ $= n.OG$

which show that the resultant is represented in magnitude and direction by $n \, . \, OG$.

137.] **Transformation of Couples.** To what has been given in Chapter V on the transformation of couples it is necessary to add a few propositions relating to couples in different planes.

(α) A couple acting on a rigid body may be transferred to any plane parallel to its own.

Let AB (fig. 185) be the arm of a couple (P, P) and let $A'B'$ be *any* line parallel and equal to AB. At A' introduce two equal and opposite forces, P and P', parallel to AP, and at B introduce the same forces. The introduction of these forces will not disturb the equilibrium of the body. Draw AB' and $A'B$, which will bisect each other at O. Then the force P at A and the force P' at B' will give a resultant equal to $2P$ at O; and P at B and P' at A' will give a resultant equal and opposite to this at the same point. Hence there remain the forces P at A' and P at B'; that is, the couple (P, P) with arm AB has been moved to any plane parallel to its own.

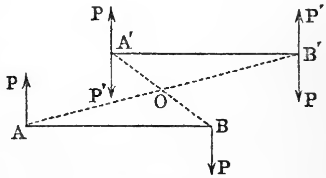

Fig. 185.

From Chapter V it is now clear that the only essential properties of a couple are (1) the constancy of its moment and (2) the parallelism of its plane; or, in other words, *the constancy of the magnitude and direction of its axis*; the actual *position* of the axis in space is of no consequence, but only its *direction*; two couples whose axes are of equal length and in the same direction are absolutely identical.

Hence the axis of a couple is what is called in modern physics a *vector*, or directed line of constant magnitude.

(β) *Convention with regard to the sense of the axis of a couple.* We have already stated that the axis is to be drawn perpendicular to the plane of the couple; but since this perpendicular might be drawn at either side of the plane, an ambiguity arises, especially in the case of several couples acting in different planes. The following convention with regard to the sense in which the axis is to be drawn, given by Thomson and Tait (*Natural Philosophy*, p. 173), is founded on a similar rule of Ampère's:—Hold a watch with its plane parallel to the plane of the couple. Then, according as the motion of the hands is contrary to, or along with, the sense in which the couple tends to turn, draw the axis of the couple through the *face* or through the *back* of the watch.

(γ) Two couples result in a single couple whose axis is found from the axes of the component couples by the parallelogram law.

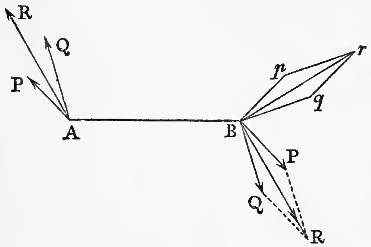

Fig. 186.

Let the planes of the couples intersect in the line AB (fig. 186) and the arm of each be made AB, by moving each couple in its own plane, and then suitably altering the forces of each couple (Art. 74, Chap. V). Let P, P be forces of one couple, and Q, Q those of the other. At B draw* Bp perpendicular to the plane $PABP$ and proportional to the moment of the couple (P, P). We may evidently take $Bp = P$, since the couples have a common arm. Draw Bq perpendicular to the plane $QABQ$ and equal to Q.

Now evidently the forces P and Q at B compound a resultant, R, equal and parallel to the resultant of P and Q at A. Hence the two couples compound a single couple.

Again, draw Br perpendicular to the plane $RABR$ and equal to R. Bp, Bq, and Br are then the axes of the couples (P, P), (Q, Q), and (R, R). But it is manifest that the figure $Bprq$ is merely the figure $BPRQ$ turned round in its own plane through a right angle. Hence Br is the diagonal of the parallelogram determined by the axes of the component couples.

Conversely, any couple may be resolved into two couples whose axes are determined from the axis of the given couple by the

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* According to the convention (β) the couples in this figure are both negative, and the axes Bp and Bq should be drawn downwards. This inaccuracy in the figure was detected too late for correction.

parallelogram law; and as in the case of forces acting at a point, any couple may be resolved into three couples whose axes are determined from the axis of the given couple by the parallelopiped law. All this follows as in Art. 135.

It is well to remark that the axis of a couple represents the moment of the forces of the couple round a line in space parallel to the axis.

(δ) To find the resultant of any number of couples acting in any planes on a rigid body.

Let the axes of the couples be all drawn, each in its proper sense according to the rule (β), at the same point, O (fig. 13), and let each axis be resolved into three components along rectangular axes Ox, Oy, Oz, drawn through O. Let $L =$ the sum of the axes in the direction Ox; then L is the axis of the component of the resultant couple in the plane yz. Let M and N be the sums of the axes in the directions Oy and Oz, respectively. Then, if G is the resultant axis,

$$G = \sqrt{L^2 + M^2 + N^2}, \tag{1}$$

and if λ, μ, ν are the direction angles of G,

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}, \tag{2}$$

equations which are exactly analogous to (2) and (3) of Art. 135.

The axes of couples are, therefore, compounded and resolved in the same manner as forces. There is this difference between forces and couples, that, while any number of couples in any planes whatever always result in a single couple, any number of forces cannot, in general, be replaced by a single force, and this difference results from the *vectorial* nature of the axis of a couple.

(ϵ) A force and a couple acting on a rigid body cannot produce equilibrium.

For, let the couple be so transferred that one of its forces, P, acts at a point on the line of action of the force, R. Then R and P at this point compound a single force which, in general, does not intersect the other force of the couple. Therefore, &c.

A force and a couple acting in the same plane are, of course, equivalent to a single force.

138.] **Theorem.** A force acting on a rigid body in a given right line can always be replaced by an equal force acting at any chosen point together with a couple.

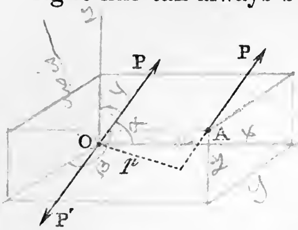


Fig. 187.

Let a force P (fig. 187) act at a point A, and let O be the chosen point. At O introduce two forces, P and P', opposite to each other and each equal and parallel to P. Then P at A and P' at O may be taken to constitute a couple whose moment is Pp, p being the perpendicular from O on the line of action of P at A. There remains, then, the force P at O; and this force together with the couple may replace P at A.

Let the axis of this couple be drawn at O; let x, y, z be the co-ordinates of A with respect to a rectangular system of axes drawn through O; and let α, β, γ, be the angles which the direction of P makes with the axes of x, y, z, respectively.

The direction cosines of OA are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, where $OA = r$, and it is easy to prove that the direction cosines of the axis of the couple (which is at once at right angles to OA and to P) are

= Y sin θ

$$\frac{y \cos\gamma - z\cos\beta}{p}, \quad \frac{z\cos\alpha - x\cos\gamma}{p}, \quad \frac{x\cos\beta - y\cos\alpha}{p}.$$

Hence, the axis of the couple being equal to Pp, the projections of the axis on the axes of x, y, and z are

$$^{*}\,P(y\cos\gamma - z\cos\beta), \quad P(z\cos\alpha - x\cos\gamma), \quad P(x\cos\beta - y\cos\alpha);$$

but it is clear from (γ), Art. 137, that these are the axes of the component couples in the planes yz, zx, and xy, into which the couple Pp can be resolved. Putting $P\cos\alpha = X$, $P\cos\beta = Y$, $P\cos\gamma = Z$, we see that the three couples are

$$Zy - Yz, \quad Xz - Zx, \quad Yx - Xy. \tag{1}$$

The force P at O may also be replaced by its three components,

$$X, Y, Z. \tag{2}$$

There is another way in which the reduction is sometimes effected.

Let P at A be resolved into its three components, X, Y, Z, and let the line of Z meet the plane (xy) in N, and let Z at A be transferred to N. Let fall Nn perpendicular to Ox; at n introduce two opposite forces Z'' and Z''', each equal and parallel to Z; and at O introduce two opposite forces, Z and Z', each equal and parallel to Z. Now the senses of positive rotation in the planes xy, yz, zx being

* Multiply the dir. cosines $y\cos\gamma - z\cos\beta$ by the length of the axis Pp ... &c. &c.

those indicated by the arrows, the forces Z at N and Z''' at n form a couple whose moment is

$$Zy \text{ parallel to plane } yz;$$

and the forces Z' at O and Z'' at n form a couple whose moment is

$$-Zx \text{ parallel to the plane } zx;$$

and in addition to these there is the force Z at O.

Similarly, the force X at A can be replaced by X at O together with two couples, Xz and $-Xy$, parallel to the planes zx and xy, respectively; and the force Y at A can be replaced by Y at O together with the couples Yx and $-Yz$ parallel to the planes xy and yz.

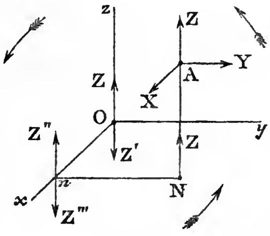


Fig. 188.

Hence P at A is replaced by the forces Y, Y, Z at O and the couples $Zy-Yz$, $Xz-Zx$, and $Yx-Xy$, parallel to the planes yz, zx, and xy, respectively.

139.] **Parallel Forces.** Suppose a rigid body to be acted on by any number of parallel forces applied at given points in the body. Take any origin, O, of co-ordinates, and through it draw three rectangular axes, that of z being parallel to the common direction of the forces. Then the force P, acting at (x_1, y_1, z_1) may be replaced, as in last Art., by

$$P_1 \text{ at } O \text{ along } Oz,$$

and the couples $\quad P_1 y_1$ and $-P_1 x_1$

parallel to the planes yz and zx.

Replacing each force in this manner, the whole system will be equivalent to a force

$$P_1 + P_2 + \ldots + P_n, \text{ or } \Sigma P \text{ at } O,$$

together with the couple

$$P_1 y_1 + P_2 y_2 + \ldots + P_n y_n, \text{ or } \Sigma Py,$$

parallel to the plane yz, and the couple

$$-P_1 x_1 - P_2 x_2 - \ldots - P_n x_n, \text{ or } -\Sigma Px,$$

parallel to the plane zx.

These two couples compound a single couple whose axis is found by drawing $OL = \Sigma Py$ and OM (in the negative sense of the axis of y) $= \Sigma Px$, and completing the parallelogram $OLGM$ (fig. 189). If OG, the diagonal is denoted by G,

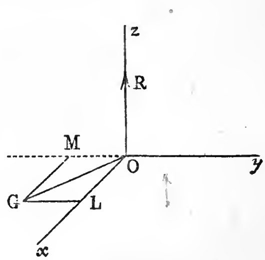


Fig. 189.

$$G = \sqrt{(\Sigma Px)^2 + (\Sigma Py)^2}$$

and
$$R = \Sigma P,$$

R being the resultant force.

140.] **Centre of Parallel Forces.** Since the resultant of two parallel forces, P_1 and P_2, acting at the points A_1 and A_2 divides the line A_1A_2 in a point g such that $\frac{A_1g}{A_2g} = \frac{P_2}{P_1}$, and since by elementary geometry (see p. 97) the distance of g from any plane $= \frac{P_1x_1 + P_2x_2}{P_1 + P_2}$, where x_1 and x_2 are the distances of A_1 and A_2 from this plane, it follows, by repeating this construction that the distances, $\bar{x}$, $\bar{y}$, $\bar{z}$, of the *centre of parallel forces* from the planes yz, zx, and xy are given by the equations

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P}.$$

141.] **Conditions of Equilibrium of a System of Parallel Forces.** A system of parallel forces has been reduced (Art. 139) to a single force, R, and a single couple, G. Now since these cannot in combination produce equilibrium (ϵ, Art. 137), we must have

$$R = 0, \text{ and } G = 0, \text{ separately.}$$

Since G cannot be $= 0$ unless $\Sigma Px = 0$ and $\Sigma Py = 0$, the conditions of equilibrium are

$$R = 0, \tag{1}$$

$$\Sigma Px = 0, \ \Sigma Py = 0. \tag{2}$$

Def. The moment of a force with respect to a plane to which it is parallel is the product of the force by its perpendicular distance from the plane.

Hence for the equilibrium of parallel forces—*The sum of the forces must vanish, and the sum of their moments with respect to every plane parallel to them must also vanish.*

Examples.

1. A heavy triangular table, ABC, is supported horizontally on three vertical props at the vertices; prove that the pressures on the props are equal.

Let P, Q, R be the pressures at A, B, C, and let a vertical plane through A and the centre of gravity of the table cut the side BC in a, its middle point. For equilibrium the sum of the moments of the forces P, Q, R, and W (the weight of the table) with respect to this plane must $= 0$. But the moments of P and W are each $= 0$, since these forces lie in the plane. Hence the moments of Q and R are equal and opposite. Now the distance of Q from the plane $= Ba \,.\, \sin AaB$, and the distance of $R = Ca \,.\, \sin AaC$; and since $Ba = Ca$, these distances are equal. Therefore $Q = R$; and similarly it can be shown that $R = P$; therefore, &c.

2. A heavy triangular plate hangs in a horizontal plane by means of three vertical strings attached to its vertices; at what point in its area must a given weight be placed so that the system of strings may be least likely to break?

At the centre of gravity of the board. For if $W =$ the weight of the board and P the sustained weight, we have

$$P + Q + R = W + P,$$

or the sum of the tensions is constant, wherever P is placed. Hence if any one is less than $\frac{1}{3}(W+P)$, some other must be greater than this value. It is evident, therefore, that the best arrangement makes each tension $= \frac{1}{3}(W+P)$; but this happens (as proved in last example) when P is placed at the centre of gravity.

3. A heavy elliptic cylinder is sustained in a vertical position by three props applied at three points on the circumference of its base; how should the props be placed in order that the cylinder may be least likely to be upset?

Let the base of the cylinder have any form, ABC (fig. 190), and let G be the projection of its centre of gravity on the plane of the base. Then, if the props are applied at A, B, and C, the perpendiculars from G on the sides of the triangle ABC must be all equal when the equilibrium is most stable. For, suppose that the cylinder is about to be upset round the line AB; then the moment of the force required to upset it is $W.p$, where W is the weight of the cylinder and p the perpendicular from G on AB. Again, suppose that the cylinder is about to be upset about AC; then the moment of the force required to upset it is $W.q$, where q is the perpendicular from G on AC. Hence if p and q are unequal, advantage will be gained by increasing the lesser of them, even though the greater must be consequently diminished; and it follows that the maximum advantage is gained when p and q are equal. In the same way it can be shown that the perpendicular from G on BC must, in the most advantageous case, be equal to that from G on AB; and therefore the perpendiculars from G on the sides ABC must be all equal.

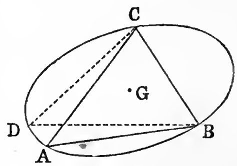


Fig. 190.

Hence the problem amounts to inscribing in a given curve a triangle on the sides of which the perpendiculars from a given point shall be equal. In the particular case in which the base is an ellipse, we have to find a circle concentric with the ellipse, such that a triangle circumscribed to the circle shall be inscribed in the ellipse.

Now (Salmon's *Conic Sections*, p. 330, 5th edition), let the ellipse have for equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, and the circle $x^2+y^2-r^2=0$; then the discriminant of $k(x^2+y^2-r^2)+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $k^3 . r^2$ $+(1+r^2\frac{a^2+b^2}{a^2b^2})k^2+\frac{r^2+a^2+b^2}{a^2b^2} . k+\frac{1}{a^2b^2}$; and the required condition being $\Theta^2=4\Delta . \Theta'$, we have two values for r, namely, $r_1=\frac{ab}{a+b}$, and $r_2=\frac{ab}{a-b}$. The first value alone is admissible, because $\frac{ab}{a-b}>b$, and the circle in this case either cuts the ellipse or entirely encloses it.

Since an infinite number of triangles can be inscribed in the ellipse and circumscribed to the circle of radius $\frac{ab}{a+b}$ (Salmon, *ibid.*), the problem is capable of an infinite number of solutions. It is easy to see that in the cases in which it is possible to have a real system of in- and circum-scribed triangles for the ellipse and the circle of radius $\frac{ab}{a-b}$, the centre of the ellipse is outside the area of the triangle. This circle is, therefore, irrelevant to our question.

4. A heavy square board, $ABCD$, of uniform thickness, is hung by three vertical strings, one of which is attached to a corner, A, of the board. The plane of the board being horizontal, find the points, E and F, in the area to which the other two strings should be attached in order that it may be most difficult to overturn the board by placing a weight anywhere on it.

It is evident that advantage is gained by taking the points E and F on the edges of the board.

Assume AE to be the direction of the line joining the points of application of two of the strings, and suppose that a weight, P, placed somewhere in the area ABE is on the point of overturning the board about the line AE. Then the tension of the string at $F=0$; and if W is the weight of the board, acting at G, the weight P required to upset it is

$$W\times\frac{\text{distance of } G \text{ from } AE}{\text{distance of } P \text{ from } AE}.$$

Hence the greater the distance of P from AE, the less the requisite value of P, or, in other words, the more easily will the board be upset. It is evident, therefore, that the applied weight should be placed at B; and in the same way, if the board is to be upset round the lines AF and FE, the applied weights should be placed at the corners D and C, respectively.

Again, in the arrangement of greatest advantage, the board is upset with equal ease round each of the lines AE, AF, and FE. For, if it be more easily upset round one of these lines than round another, advantage will be gained by making it a little more stable with regard to the first. Hence, since the weights placed at B, D, and C

are all equal, we have

$$\frac{\text{distance of } G \text{ from } AE}{\text{distance of } B \text{ from } AE} = \frac{\text{distance of } G \text{ from } AF}{\text{distance of } D \text{ from } AF} = \frac{\text{distance of } G \text{ from } EF}{\text{distance of } C \text{ from } EF}.$$

The angles EAB and FAD are, therefore, equal, and each $= \tan^{-1}(\sqrt{2}-1)$.

5. A heavy elliptic table is supported on three vertical props; how must they be placed so that it may be most difficult to upset the table by placing a weight on it?

Ans. The props must be placed at three points, A, B, C, on the circumference of the ellipse; and if γ is the eccentric angle of C, that of B is $\frac{2}{3}\pi+\gamma$, and that of A is $\frac{4}{3}\pi+\gamma$. The weight which, most advantageously applied, will then just upset the table is half its own weight.

This may be seen as follows. Assuming any line in the area as the line joining two props, the least weight that will be required to upset the table must be placed at the point of contact of a tangent parallel to the assumed line, since the weight will have maximum leverage at this point. Also, it must be equally easy to upset the table round the three lines AB, BC, CA; that is, the requisite weights placed at C', A', B', the points of contact of the tangents, must be all equal. If, then, x, y, z, be the perpendiculars from the centre on the lines BC, CA, AB, and P, Q, R the perpendiculars on the parallel tangents, we must have

$$\frac{x}{P-x} = \frac{y}{Q-y} = \frac{z}{R-z};$$

or, if α, β, γ, be the eccentric angles of A, B, C,

$$\frac{\cos\frac{\alpha-\beta}{2}}{1-\cos\frac{\alpha-\beta}{2}} = \frac{\cos\frac{\beta-\gamma}{2}}{1-\cos\frac{\beta-\gamma}{2}} = \frac{-\cos\frac{\alpha-\gamma}{2}}{1+\cos\frac{\alpha-\gamma}{2}},$$

a negative sign being used in the last, since (γ, β, α being in ascending order of magnitude) $\frac{\alpha-\gamma}{2}$ is evidently $>\frac{\pi}{2}$. Hence $\beta = \frac{2}{3}\pi+\gamma$, $\alpha = \frac{4}{3}\pi+\gamma$; and the weight required to upset the table $= W\frac{x}{P-x}$, or $\frac{1}{2}.W$ Any one position of C is, therefore, as good as any other; and if C is made the extremity of either axis, the line AB is parallel to the other at a distance equal to $\frac{1}{4}$ of the first axis from it.

6. A rectangular board is held with its plane horizontal by three vertical strings attached to three of its corners; find the point in its area at which a weight must be placed so that the tensions of the strings shall be given multiples of the weight of the board.

Ans. Let W be the weight of the board; let the strings be applied at the corners A, B, C; let $AC = 2a$, $AB = 2b$; and let the tensions of the strings at A, B, C be lW, mW, nW, respectively.

Then the weight must be placed at a point whose distances from AB and AC are

$$\frac{2n-1}{l+m+n-1}\cdot a \text{ and } \frac{2m-1}{l+m+n-1}\cdot b.$$

The magnitude of the weight is, of course, $(l+m+n-1)\,W$.

7. A uniform circular lamina is placed with its centre upon a prop; find at what points on its circumference three weights, w_1, w_2, w_3, must be placed that it may remain at rest in a horizontal position (Walton's *Mechanical Problems*, p. 73).

Ans. The angles which the weights subtend in pairs at the centre of the lamina are the supplements of the angles of a triangle whose sides are proportional to the weights.

142.] **Reduction of a System of Forces acting in any manner on a Rigid Body.** Let any origin, O (fig. 188), be assumed arbitrarily, and let any system of rectangular axes, Ox, Oy, and Oz, be drawn through it. If, then, forces P_1, P_2, P_3, ... act on the body at points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), ... each force can be replaced by three components acting at O along the axes, together with three couples whose axes coincide with the co-ordinate axes. The force P_1, for example, is equivalent to X_1, Y_1, Z_1 at O and three couples, $Z_1y_1 - Y_1z_1$, $X_1z_1 - Z_1x_1$, and $Y_1x_1 - X_1y_1$. Adding the components of the forces, and also the axes of the couples, in the directions Ox, Oy, and Oz, the whole system of forces is equivalent to

the force ΣX along Ox,
" ΣY " Oy,
and " ΣZ " Oz;

and the system of couples is equivalent to

the couple $\Sigma(Zy - Yz)$, or L, in the plane yz,
" $\Sigma(Xz - Zx)$, or M, " zx,
and " $\Sigma(Yx - Xy)$, or N, " xy,

(Of course the axes of L, M, N are drawn along the axes of x, y, and z, respectively).

Hence if R be the magnitude of the resultant of translation,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2};$$

and if G be the magnitude of the resultant couple,

$$G = \sqrt{L^2 + M^2 + N^2}.$$

The direction-cosines of R are $\frac{\Sigma X}{R}$, $\frac{\Sigma Y}{R}$, and $\frac{\Sigma Z}{R}$; and those of G are $\frac{L}{G}$, $\frac{M}{G}$, and $\frac{N}{G}$.

Thus, *any system of forces acting on a rigid body can be replaced by a single resultant force acting at an arbitrary origin, the magnitude and direction of this force being the same for all origins, and a single resultant couple the magnitude and direction of whose axis are both dependent on the origin chosen.*

It has been already remarked (Art. 137) that G is not only the axis of the resultant couple (corresponding to a resultant force acting at O), but also the sum of the moments of the forces about a line at O drawn in the direction of G; and since the axes of couples have been proved to follow the parallelopiped and parallelogram laws, it follows that the sum of the moments of the forces about this line is greater than the sum of their moments about any other line at O; and also that the sum of the moments of the forces about any other line through O is the resolved part of G in the direction of this line.

Remark. The magnitude and direction of G are constant at all points along the same right line parallel to R. For R may be supposed to act at any point in this line, and the vector G may be moved parallel to itself to the point at which R is supposed to act.

143.] **Poinsot's Central Axis.** Any system of forces acting on a rigid body has been proved to be equivalent to a single resultant force, R, acting at an arbitrary origin, O, and a single resultant couple G. Let ϕ be the angle between R and G, and resolve G into two components, OK and On (fig. 191) along and perpendicular to R, respectively. On is the axis of a couple in the plane ROx, perpendicular to On. Now let each force of this couple be made equal to R, and the arm, OP*, is consequently equal to $\frac{On}{R}$; that is,

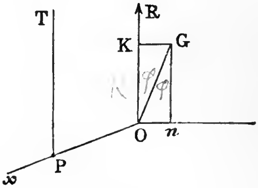


Fig. 191.

$$OP = \frac{G \sin \phi}{R}. \tag{1}$$

* The point P should be represented on the production of the line xO through O, according to the convention of Art. 137. The inaccuracy in the figure was detected too late for correction.

One of these forces may be applied at O to destroy the resultant, R, at this point, and there finally remains a resultant force, R, at P along PT (parallel to OR), together with a couple whose axis is OK, or $G \cos \phi$. Denoting OK by K, we have then

$$K = G \cos \phi. \tag{2}$$

The axis OK may, of course, be drawn at P along PT [(a), Art. 137].

Hence *the whole system of forces is equivalent to a resultant force equal to R acting along the line PT and a couple in a plane perpendicular to this line.*

The line PT thus determined is called *Poinsot's Central Axis.*

To construct Poinsot's Central Axis for any system of forces— *Reduce the forces to a resultant force, OR, acting at any origin, O, and a couple whose axis is OG; then on a line perpendicular to the plane of OR and OG measure off a length, OP*, equal to* $\frac{G \sin \phi}{R}$, *where* ϕ *is the angle between OR and OG. A line through the point P parallel to OR is the required Central Axis.*

144.] **Theorem.** A given system of forces has but one Central Axis.

This, which is sufficiently evident from the preceding construction, may be proved *ex absurdo* thus:—

Whatever origin be chosen, the resultant force acting at it is constant both in magnitude and in direction. Now, if it be possible, let the system of forces be equivalent to a resultant R acting at O and a couple whose axis is OK, and also to a resultant force R acting at O' and a couple whose axis is $O'K'$, the lines OK and $O'K'$ being, of course, in the direction of R. Now it is evident that the force R at O and the couple OK should equilibrate the *reversed* force R and *reversed* couple $O'K'$ at O'. But the couples give a single couple, $OK \backsim O'K'$, and the forces give also a couple which, being in a plane perpendicular to the first couple, cannot with it produce equilibrium. Therefore, &c.

Since this axis is unique, equation (2) of the last Article shows that for all origins the quantity $G \cos \phi$, or the *projection of the axis of the resultant couple along the direction of the resultant force is constant.*

145.] **Theorem.** The sum of the moments of the forces round Poinsot's Axis is less than the sum of their moments

* The *sense* of OP is determined by the convention of Art. 137.

round any other axis of principal moment. (Since for any origin, O, the sum of the moments round OG is greater than the sum of the moments round any other line through O (Art. 142), OG is called the *Axis of Principal Moment* at O.)

Let Oz (fig. 192) be Poinsot's Axis and OK $(= K)$ the moment of the forces round it. Let O' be any point in the body, and let it be proposed to find the principal moment at this point. $O'O$ is a line drawn through O' perpendicular to Poinsot's Axis. At O' introduce two equal and opposite forces, $O'R$ and $O'R'$, each $= R$. Then OR and $O'R'$ form a couple, whose axis, $O'n$ is perpendicular to the plane $ROO'R'$ and equal to $R.OO'$. Transfer the axis OK to $O'K'$ (Art. 137), and draw the diagonal, $O'G$, of the rectangle determined by $O'n$ and $O'K'$. Then $O'G\ (= G)$ is the axis of principal moment at O', and it is evidently $> O'K'$. Hence Poinsot's is the least principal moment.

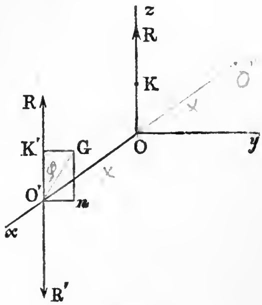


Fig. 192.

146.] **Problem.** To find the surface traced out by the axes of principal moment at points taken along a right line intersecting Poinsot's Axis perpendicularly.

Let Ox be the assumed line, and let it be taken as axis of x, Poinsot's Axis being that of z. Let $OO' = x$, and let y and z be the co-ordinates of any point on $O'G$. Then if $\phi =$ the angle $GO'K'$, we have

$$\frac{z}{y} = \cot\phi = \frac{Gn}{O'n} = \frac{K}{R\,.\,x},$$

or

$$xz = \frac{K}{R}\cdot y,$$

an equation which denotes a hyperbolic paraboloid. As the point O' moves out from O along Ox, the axes revolve towards the right; as O' moves in towards O, they revolve towards the left; and after coincidence with Poinsot's Axis at O, they revolve towards the left. At an infinite distance from O the axis of principal moment is at right angles to Poinsot's Axis.

Let it be proposed to find the surface traced out by the axes of principal moment at points taken all along an arbitrary curve in a plane perpendicular to Poinsot's Axis.

Let Q be any point on the curve whose equation in the plane xy is $f(x, y) = 0$, and let (α, β) be the co-ordinates of Q, and O the point in which Poinsot's Axis meets the plane of xy. Then the axis of principal moment at Q is constructed by drawing QN, in the plane xy, perpendicular to OQ, taking on QN a length $= R \,.\, OQ$, drawing at Q a perpendicular to the plane xy equal to K, and constructing the diagonal of the rectangle determined by these two latter lines. Suppose P to be any point on the axis of principal moment at Q, and let N be the projection of P on the plane xy. The co-ordinates of P being x, y, z, it is clear that

$$QN = z\tan\phi = \frac{R \,.\, OQ}{K} \cdot z.$$

If θ is the angle made by QN with the axis of x,

$$\alpha = x + QN\cos\theta$$

$$= x + \frac{R \,.\, OQ\cos\theta}{K} \cdot z,$$

or

$$\alpha = x + \frac{Rz}{K} \cdot \beta. \tag{1}$$

Similarly,

$$\beta = y - \frac{Rz}{K} \cdot \alpha. \tag{2}$$

Solving these equations for α and β, we have

$$\alpha = \frac{x + \frac{R}{K}yz}{1 + \frac{R^2}{K^2} \cdot z^2}, \quad \beta = \frac{y - \frac{R}{K}xz}{1 + \frac{R^2}{K^2} \cdot z^2}. \tag{3}$$

Hence, since $f(\alpha, \beta) = 0$, we have

$$f\left(\frac{x + \frac{R}{K}yz}{1 + \frac{R^2}{K^2}z^2}, \; \frac{y - \frac{R}{K}xz}{1 + \frac{R^2}{K^2}z^2}\right) = 0, \tag{4}$$

which is the equation of the surface traced out.

147.] **Theorem.** A system of forces can be reduced to two forces in an infinite number of ways. For they can be reduced to a resultant force, R, acting at any point, together with a couple. Now the forces of the couple can be made of any magnitude by varying its arm; and one of them can be combined with R. There will then remain the resultant of R and this force together with the remaining force of the couple. Therefore, &c.

148.] **Theorem.** When a system of forces is reduced to a pair of forces represented in magnitudes and lines of action by two right lines, the volume of the tetrahedron formed by these lines is constant, however the reduction is made.

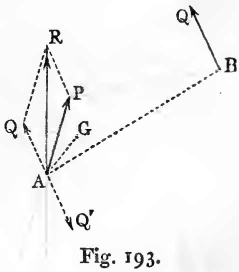


Fig. 193.

Let the system of forces be reduced to P and Q, and let these be supposed to act at the extremities, A and B, of the shortest distance between them. Now to get the force and couple corresponding to the origin A, introduce at this point two opposite forces, AQ and AQ', each equal and parallel to Q.

Compounding P and Q we get the resultant force, R; and taking the forces Q at B and Q' at A we get a couple whose axis, AG, is at right angles to the plane $QBAQ'$ and equal to $Q\,.\,AB$. Since AB is perpendicular to both P and Q, it is clear that AG is in the plane QAP and at right angles to AQ.

Now since (Art. 143) $G\cos\phi = K$, we have

$$Q\,.\,AB\,.\sin QAR = K.$$

But $\sin QAR = \dfrac{P}{R}\cdot\sin PAQ$. Hence

$$P\,.\,Q\,.\,AB\,.\sin PAQ = K\,.\,R.$$

Now the volume of the tetrahedron formed by the lines AP and BQ

$= \frac{1}{3}$ area $ABQ \times$ perpendicular from P on the plane ABQ;

$= \frac{1}{6}\, BQ\,.\,AB \times AP\,.\sin PAQ$;

$= \frac{1}{6}\, P\,.\,Q\,.\,AB\,.\sin PAQ$.

Hence if Δ denotes the volume of the tetrahedron,

$$\Delta = \tfrac{1}{6}\, K\,.\,R.$$

This theorem has been proved in various ways. For an elegant demonstration by Möbius, see *Crelle's Journal*, vol. iv, p. 179, or Jullien's *Problèmes de Mécanique Rationnelle*, vol. i, p. 71.

149.] **Symmetrical Reduction of a System of Forces.** A system of forces can be reduced to two forces equal in magnitude, equally inclined at opposite sides to Poinsot's Axis, and equally distant from this axis.

This is what Thomson and Tait call the *Symmetrical Case.*

Suppose the forces replaced by R acting along Poinsot's Axis, Oz, and a couple, K. Take any point, O' (fig. 192); draw $O'O$ perpendicular to Oz and produce it to O'' so that $O'O = OO''$. Let R acting at O be replaced by $\frac{1}{2}R$ acting at O' and $\frac{1}{2}R$ acting at O''. Also let the forces of the couple act at O' and O''; for this purpose these forces must each be made $= \frac{K}{2x}$, x being OO'. Now the resultant of $\frac{1}{2}R$ and $\frac{K}{2x}$ at O' is a force

$$= \tfrac{1}{2}\sqrt{R^2 + \frac{K^2}{x^2}},$$

acting towards the right, and the resultant of $\frac{1}{2}R$ and $\frac{K}{2x}$ at O'' is a force of the same magnitude acting towards the left of the figure.

If ω is the angle made with Poinsot's Axis by these new forces at O' and O'',

$$\tan\omega = \frac{K}{Rx}. \quad = \frac{\frac{K}{2x}}{\frac{R}{2}} = \frac{K}{Rx}$$

If we choose x so that $\frac{K}{x} = \sqrt{3}\,R$, each of the two symmetrical forces is equal to R, and they are inclined at an angle of 60° to Poinsot's Axis.

150.] **Analytical Condition for a Single Resultant.** We have just seen that a system of forces acting on a rigid body is, in general, equivalent to *two* forces. Let the forces be replaced by a single resultant force, R, acting at an arbitrary origin, O, and a couple G. Now the direction cosines of R referred to axes Ox, Oy, and Oz, are (Art. 142),

$$\frac{\Sigma X}{R}, \quad \frac{\Sigma Y}{R}, \quad \text{and} \quad \frac{\Sigma Z}{R};$$

and those of G are

$$\frac{L}{G}, \quad \frac{M}{G}, \quad \text{and} \quad \frac{N}{G}.$$

Hence, if ϕ is the angle beeween G and R, Since $\cos\phi = ll' + mm'$...

$$\cos\phi = \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{GR}. \tag{1}$$

Now if the resultant couple is in a plane containing R, one of its forces can be made to destroy R, and there will remain a single force; but if G and R are not at right angles to each other, the system of forces cannot be equivalent to a single force. The required condition is, therefore, $\cos\phi = 0$, or

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0, \qquad (2)$$

is an invar

provided that ΣX, ΣY, *and* ΣZ *do not all vanish*; for if they do, R will also vanish, and ϕ will be illusory. In fact, in this case, since L, M, and N alone exist, the system of forces is equivalent to a couple.

151.] **Theorem.** The quantity $L\Sigma X + M\Sigma Y + N\Sigma Z$ has the same value for all systems of rectangular axes assumed anywhere in space.

From (1) of the last Article it $= R\,.\,G\cos\phi$, or $R\,.\,K$, where K is Poinsot's moment (Art. 143).

Hence, if this quantity vanishes for any one set of axes, the force and the axis of the accompanying couple corresponding to any origin are at right angles.

The value of this quantity can be exhibited in another form which also shows that it is independent of any particular set of axes.

Substituting for L, M, and N the values (Art. 142), $\Sigma(Zy - Yz)$, &c., the expression becomes

$$\begin{aligned}&(Z_1 y_1 - Y_1 z_1 + Z_2 y_2 - Y_2 z_2 + \ldots)(X_1 + X_2 + \ldots)\\ &+ (X_1 z_1 - Z_1 x_1 + X_2 z_2 - Z_2 x_2 + \ldots)(Y_1 + Y_2 + \ldots)\\ &+ (Y_1 x_1 - X_1 y_1 + Y_2 x_2 - X_2 y_2 + \ldots)(Z_1 + Z_2 + \ldots);\end{aligned}$$

or, substituting for $X_1, Y_1, Z_1, \ldots$ in terms of the forces $P_1, \ldots$ and their direction-cosines,

$$\begin{aligned}&[P_1(y_1\cos\gamma_1 - z_1\cos\beta_1) + P_1(y_2\cos\gamma_2 - z_2\cos\beta_2) + \ldots]\\ &\qquad (P_1\cos a_1 + P_2\cos a_2 + \ldots) + \&c. \ldots .\end{aligned}$$

It is clear at once that the terms $P_1^2, P_2^2, \ldots$ disappear, and the products $P_1 P_2, P_1 P_3, \ldots$ alone remain.

Collecting the coefficient of $P_1 P_2$ as a typical term, we have

$$\begin{aligned}P_1 P_2\,[&(x_1 - x_2)(\cos\beta_1\cos\gamma_2 - \cos\gamma_1\cos\beta_2)\\ &+ (y_1 - y_2)(\cos\gamma_1\cos a_2 - \cos a_1\cos\gamma_2)\\ &+ (z_1 - z_2)(\cos a_1\cos\beta_2 - \cos\beta_1\cos a_2)].\end{aligned}$$

Now (see Salmon's *Geometry of Three Dimensions*, p. 31, third edition, or Frost's *Solid Geometry*, p. 39) if (P_1, P_2) denotes the angle between the directions of the forces P_1 and P_2, the quantity in brackets $= d_{12} . \sin(P_1, P_2)$, d_{12} being the shortest distance between the lines of action of the forces.

Hence

$$L\Sigma X + M\Sigma Y + N\Sigma Z = \Sigma P_1 P_2 . d_{12} . \sin(P_1, P_2). \qquad (1)$$

Again (Art. 148) the term involving $P_1 P_2$ on the right side of (1) denotes six times the tetrahedron formed by P_1 and P_2; therefore the quantity on the left side is equal to *six times the sum (with their proper signs) of the* $\frac{n(n-1)}{2}$ *tetrahedra which can be formed out of the pairs of lines representing the n forces* $P_1, P_2, \ldots P_n$.

This sum has, of course, no reference to any set of axes, and hence the necessarily *invariant* nature of $L\Sigma X + M\Sigma Y + N\Sigma Z$.

With regard to the sign to be given to any tetrahedron of the system, we define that—

The moment of a force with regard to a line is the component of the force perpendicular to the line multiplied by the shortest distance between the force and the line.

Hence $P_1 . d_{12} . \sin(P_1, P_2)$ is the moment of P_1 about the line of action of P_2. Now to determine the sign which must be given to any tetrahedron, let a watch be placed so that the direction in which either force acts passes perpendicularly from the back up through the face of the watch. If then the other force tends to produce rotation in the sense in which the hands rotate, the tetrahedron is to receive a negative sign, and if the rotation is the other way, a positive sign.

152.] **Conditions of Equilibrium of a Body Acted on by any Forces.** The forces having been reduced to a resultant of translation, R, acting at any point, together with a corresponding couple, G, since a force and a couple cannot conjointly produce equilibrium ((ϵ), Art. 137) it is necessary that

$$R = 0 \text{ and } G = 0.$$

Substituting the values of R and G given in Art. 142, we see that these two are equivalent to the following *six* conditions:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$
$$L = 0, \quad M = 0, \quad N = 0,$$

which are the analytical expressions of the fact that *the forces must have no component along any line and no moment about any axis.*

Examples.

1. When three forces keep a rigid body in equilibrium, they must be coplanar and concurrent or parallel.

Let the forces be P, Q, and R. Then the sum of their moments about every axis is zero. Take any point, p, on P and from it draw a line meeting Q in the point q, suppose. Then, since two of the forces have zero moments about this line, the moment of the third force, R, about it must $= 0$; that is, this line intersects R, in the point r, suppose.

Let another line be drawn through p meeting Q in q'. Then, as before, this line must meet R in a point, r'. Now since two points on each of the lines Q and R lie in the plane of the lines pqr and $pq'r'$, the lines Q and R must both lie in this plane.

Again, drawing any two lines across Q and R, each of these lines must intersect P; that is, P has two of its points in the plane of Q and R, and P, therefore, lies in this plane.

Finally, taking moments about the intersection of Q and R, we see that P must pass through this point; but if any two are parallel, the third must be parallel to them.

2. A rigid body is acted on by forces represented in magnitudes and lines of action by the sides of a gauche polygon taken in order; prove that the forces are equivalent to a couple, and that the sum of their moments about any line is represented by double the area of the projection of the polygon on a plane perpendicular to the line.

Let the forces be represented by the lines AB, BC, CD, ... (fig. 194), and let OQ be any axis.

On the axis take any point, O, and reduce the forces to a resultant, R, of translation at this point, together with a couple, G (Art. 142). This is done by introducing at O two forces parallel and equal to AB in opposed directions, two equal and opposite to BC, &c. Now (Art. 136) the resultant of translation vanishes, and the component couples are represented by double the areas of the triangles OAB, OBC, &c. If the axes of these couples are drawn at O, the sum of the moments of the forces about OQ will be represented by the sum of the components of the axes along OQ; but this is the same as double the sum of the projections of the areas of the triangles on a plane perpendicular to OQ; that is, the moment about OQ is represented by double the area of the projection of the polygon on a plane perpendicular to OQ.

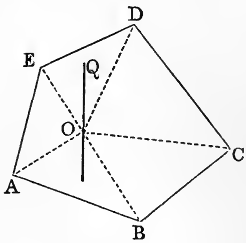


Fig. 194.

Again, since G is the greatest moment round any axis through O (Art. 142), it follows that the axis of the resultant couple is the line perpendicular to the plane on which the projected area of the polygon is a maximum.

3. When the resultant of translation vanishes, the forces will be in complete equilibrium if the sums of their moments round any three non-coplanar axes are separately equal to nothing.

For if L be the moment round the axis of x, the moment L' round a parallel axis through the point (α, β, γ) is $L+\gamma\Sigma Y-\beta\Sigma Z$. Hence $L'=L$, $M'=M$, $N'=N$; and since the moment round an axis through (α, β, γ) making angles λ, μ, ν with the axes of co-ordinates is $L'\cos\lambda+M'\cos\mu+N'\cos\nu$, it follows that the moments round all parallel axes are equal. For the three axes of moments we may take, therefore, three lines through the origin making angles $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and $(\lambda_3, \mu_3, \nu_3)$ with the axes of co-ordinates. Suppose then that

$$L\cos\lambda_1+M\cos\mu_1+N\cos\nu_1=0,$$

$$L\cos\lambda_2+M\cos\mu_2+N\cos\nu_2=0,$$

and

$$L\cos\lambda_3+M\cos\mu_3+N\cos\nu_3=0.$$

These require either that $L=M=N=0$, or

$$\begin{vmatrix} \cos\lambda_1, & \cos\mu_1, & \cos\nu_1 \\ \cos\lambda_2, & \cos\mu_2, & \cos\nu_2 \\ \cos\lambda_3, & \cos\mu_3, & \cos\nu_3 \end{vmatrix}=0.$$

The latter condition requires that the three axes of moments be in one plane. If they are not coplanar, we must have $L=M=N=0$, i.e. the forces are in equilibrium.

4. A tetrahedron is acted on by forces applied perpendicularly to the faces at their respective centroids. If the force applied to each face is proportional to the area of that face, prove that the tetrahedron is in equilibrium, the forces being supposed to act all inwards or all outwards.

Let A, B, C, D, be the vertices of the tetrahedron and denote the areas of the faces opposite these vertices by A_1, B_1, C_1, D_1, respectively. Denote also the angle between the faces A_1 and B_1 by $\widehat{A_1B_1}$. Then evidently

$$A_1=B_1\cos\widehat{A_1B_1}+C_1\cos\widehat{A_1C_1}+D_1\cos A_1D_1;$$

or if the forces perpendicular to the faces are denoted by P, Q, R, S,

$$P-Q\,.\cos\widehat{PQ}-R\,.\cos\widehat{PR}-S\,.\cos\widehat{PS}=0,$$

which shows that there is no resultant force in a direction perpendicular to the face A_1; similiarly there is no resultant force in directions perpendicular to the other faces; therefore the resultant of translation vanishes.

To show that there is no resultant couple, let each force be replaced by three equal forces acting at the angles of the corresponding face. Thus the force P is to be replaced by three forces each equal to $\frac{1}{3}P$ acting at the points B, C, D perpendicularly to the face BCD. Let us calculate the sum of the moments of the forces about the edge BC. For this purpose, let the forces $\frac{1}{3}Q$ and $\frac{1}{3}R$ at D be each resolved in the direction of the force $\frac{1}{3}P$ at this point, i. e. perpendicularly to the face BCD. Supposing the forces to act outwards, the components of $\frac{1}{3}Q$ and $\frac{1}{3}R$ are $-\frac{1}{3}Q.\cos \widehat{PQ}$ and $-\frac{1}{3}R.\cos \widehat{PR}$; therefore the sum of the moments of the forces at D about BC is proportional to

$$(A_1 - B_1 . \cos A_1\hat{}B_1 - C_1 . \cos A_1\hat{}C_1)\, p',$$

or
$$D_1 . p' . \cos A_1\hat{}D_1,$$

$$\text{or, again, } D_1 . p,$$

p' being the perpendicular from D on BC, and p the perpendicular from D on the base ABC. But this last expression is three times the volume of the tetrahedron. In the same way, the sum of the moments of the forces at A is represented by three times the volume of the tetrahedron, and as these moments are in opposite senses, the forces have no moment round the edge BC, and similarly no moment round any of the edges. Hence by the last example they are in equilibrium.

For another simple method of proof see Collignon's *Statique*, p. 354.

5. Prove that a solid body of any shape is in equilibrium if it is acted on throughout its surface by normal forces, each force being proportional to the superficial element on which it acts.

One very simple method of proof consists in imagining a surface precisely equal and similiar to that of the given body to be traced out in a weightless fluid which is subject to any pressure.

6. If a curved surface whose edge is a plane curve is acted on all over its surface by normal forces, each proportional to the element of surface on which it acts, prove that these forces have a single resultant if they all act towards the same side of the surface.

7. Forces perpendicular and proportional to the areas of the faces act at the centres of the circles circumscribing the faces of a tetrahedron; prove that they are in equilibrium, if they all act inwards or outwards.

They meet in the centre of the circumscribed sphere. The proposition is evidently true also for any polyhedron bounded by triangular faces.

Taking the results of this example and example 4 together, we see that forces proportional to the areas and perpendicular to them are in equilibrium if they act at the orthocentres of the triangular faces of any polyhedron.

8. Find the force necessary to keep a heavy door in a given position, the hinge line being inclined to the vertical and the hinges smooth.

Let i be the inclination of the hinge line to the vertical, and a the given inclination of the plane of the door to the vertical plane containing the hinge line. Then if W is the weight of the door, a the distance of its centre of gravity from the hinge line, and θ the angle between the normal to the plane of the door and the vertical, the moment of the weight about the hinge line is

$$Wa\cos\theta.$$

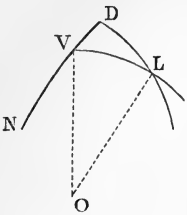


Fig. 195.

This is the moment of the required force. To find θ, let lines parallel to the hinge line and the vertical be drawn through any point, O, and through this point let a plane be drawn parallel to the plane of the door. Round O let any sphere be described; let V and L (fig. 195) be the points where these lines meet the sphere; DL the circle in which the plane of the door intersects the sphere, and N the point in which the normal, ON, to the door intersects it. Then $VL = i$, $\angle DLV = a$, and $NV = \theta$, and we have from the spherical triangle VDL

$$\sin VD = \sin i \sin a,$$

or

$$\cos\theta = \sin i \sin a,$$

since N is the pole of DL. Hence the moment of the required force is

$$Wa \sin i \sin a,$$

and when its point of application and direction are known, its magnitude is therefore known.

9. A beam can turn in every direction about one end which is fixed; the other end rests on a rough inclined plane. Find the limiting position of equilibrium. (See Walton's *Mechanical Problems*, p. 191, third edition.)

Let AB (fig. 196) be the beam, A the fixed end, DPH the rough inclined plane, PH the intersection of this plane with a horizontal plane through A, APD the vertical plane through A perpendicular to the inclined plane, BD a line parallel to PH, AO a perpendicular from A on the inclined plane, DQ a perpendicular on the horizontal plane, i the inclination of the plane, a the angle, ABO, between the beam and this plane, and μ the coefficient of friction.

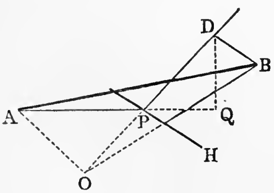


Fig. 196.

Now *suppose first that the beam is perfectly inelastic.* Then the end B describes on the inclined plane a circle whose centre is O, and if it is about to slip, the force of friction assumes a direction perpendicular to OB in the inclined plane. The extreme position of the beam will be denoted by the angle, θ or DOB, between the plane, AOB, through the beam normal to the inclined plane and the vertical plane, AOD.

The forces acting on the beam are its weight, the reaction of the smooth joint at A, and the total resistance of the inclined plane at B. This last force we shall consider as composed of a normal reaction, R, and a force of friction, μR, acting perpendicularly to BO. For the equilibrium of the beam take moments about a vertical axis through A. The moment of the normal reaction at B is $R \sin i \times BD$, or $R \sin i \,.\, BO \sin\theta$, or again $R \sin i \,.\, AB \cos a \sin\theta$. To find the moment of μR, resolve it into $\mu R \cos\theta$ along BD and $\mu R \sin\theta$ parallel to OD; and resolve this latter again into a horizontal component, $\mu R \sin\theta \cos i$, and a vertical component, $\mu R \sin\theta \sin i$. The moment of μR is then equal to the sum of the moments of $\mu R \cos\theta$ and $\mu R \sin\theta \cos i$; that is, it is equal to

$$\mu R \cos\theta \times AQ + \mu R \sin\theta \cos i \times BD.$$

Hence the equation of moments is

$$R(\sin i - \mu \cos i \sin\theta) BD = \mu R \cos\theta \,.\, AQ.$$

But $$AQ = AP + PQ = \frac{AO}{\sin i} + (OD - OP)\cos i$$

$$= \frac{AB \,.\, \sin a}{\sin i} + AB \cos i \cos a \cos\theta - AB \sin a \cot i \cos i$$

$$= AB(\sin i \sin a + \cos i \cos a \cos\theta);$$

therefore

$$(\sin i - \mu \cos i \sin\theta)\cos a \sin\theta = \mu \cos\theta\,(\sin i \sin a + \cos i \cos a \cos\theta),$$

or $$\sin i \cos a \sin\theta = \mu \cos i \cos a + \mu \sin i \sin a \cos\theta,$$

or $$\tan i \tan\theta = \mu\sqrt{1+\tan^2\theta} + \mu \tan i \tan a,$$

or finally,

$$(\tan^2 i - \mu^2)\tan^2\theta - 2\mu \tan^2 i \tan a \tan\theta + \mu^2(\tan^2 i \tan^2 a - 1) = 0. \quad (1)$$

If there is no horizontal plane through A obstructing the beam, it will be possible for the end B to describe a complete circle round O. Let us inquire the condition that the beam should rest in all possible positions. For this there must be no *limiting* position of equilibrium, or, in other words, the value of θ in (1) must be imaginary.

The required condition is, then,

$$\tan^2 i (1 + \mu^2 \tan^2 a) < \mu^2,$$

or $$\mu > \frac{\tan i}{\sqrt{1 - \tan^2 i \tan^2 a}}.$$

Let us next *suppose that the beam is elastic*, or that, in virtue of a compression of the beam, B is not constrained to move in the circle whose centre is O. Supposing, then, that the beam has been jammed against the plane, if the coefficient of friction is gradually diminished, B will begin to move in some other direction than that perpendicular to OB, and this direction will be exactly opposite to that in which the force of friction acts. Now the reaction at A, the total resistance at B, and the weight of the beam lie in one plane which must, therefore, be *the vertical plane through the beam.* The total resistance at B must, moreover, lie inside or on the cone of friction described round B. Hence if the position of the beam is such that the vertical plane through it *touches* this cone, equilibrium will be at its limit, since the line of action of the total resistance is the line of contact of the vertical plane with the cone.

Let the lines and planes of the figure be projected on a sphere described about B as centre with arbitrary radius. Then the cone of friction will appear as a small circle of angular radius, NC (fig. 197) equal to λ, the angle of friction. Let N be the point in which the normal to the inclined plane at B meets the sphere; A, the point representing the beam, and ACV the vertical plane through the beam touching the cone of friction. Now the vertical line at B lies in the vertical plane, ACV, through the beam, and it makes an angle equal to i with the normal to the inclined plane. Hence, take a point V in ACV so that $NV = i$, and we have NV, the circle answering to the vertical plane through B normal to the inclined plane (a plane which is parallel to the plane APD, fig. 196). In the spherical triangle NVC we have then

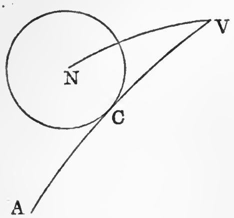


Fig. 197.

$$\sin NV \cdot \sin NVC = \sin NC,$$

or

$$\sin i \sin \theta = \sin \lambda;$$

$$\therefore \quad \sin \theta = \frac{\sin \lambda}{\sin i}.$$

This second solution suppposes that *the only condition to which the total resistance is subject is that of making with the normal an angle not greater than the angle of friction.* The supposition of perfect rigidity, on the contrary, restricts the direction of the force of friction in the inclined plane, making it perpendicular to the line OB.

10. A heavy elastic beam rests on two rough inclined planes whose intersection is a horizontal line. Show that every position of the beam may be one of equilibrium if the inclination of each plane is less than the angle of friction for that plane and the beam.

Let A (fig. 198) be one end of the beam, AN the normal to the plane on which A rests, and AV the vertical at A. Then if the beam is sufficiently elastic, it may be jammed against the planes, and the only condition to which the total resistances at its ends are subject are the conditions of making with the normals angles not greater than the corresponding angles of friction. Hence in the extreme position in which the end A is about to slip, the vertical plane through the beam must touch the cone of friction described round the normal, AN. But this is manifestly impossible, since the angle λ is $> VAN$; for the vertical line is included within the cone, and through this line no plane can be drawn to touch the cone. There can, therefore, be no *limiting* equilibrium at either end in any position of the beam.

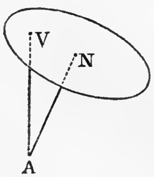


Fig. 198.

11. A particle is acted on by any number of given forces, P_1, P_2, ... ; prove that if R is their resultant,

$$R^2 = \Sigma(P^2) + 2\Sigma(P_1 . P_2 \cos P_1\hat{}P_2),$$

where $P_1\hat{}P_2$ denotes the angle between the directions of P_1 and P_2.

12. Prove that a system of forces acting on a rigid body may be replaced by two equal forces whose lines of action are perpendicular to each other, and each inclined at an angle of 45° to Poinsot's Axis: the forces act at the ends of a line bisected by this axis; the length of this line is $\frac{2K}{R}$, and each force is $\frac{R}{\sqrt{2}}$, R being the resultant of translation, and K Poinsot's moment.

13. Prove that the distance between the lines of action of the two forces which equivalently replace a given system of forces is a minimum when the forces are equal and their directions perpendicular.

14. Prove that the central axis of two forces divide the shortest distance between them into two parts which are inversely proportional to the components of the two forces along the direction of their resultant.

15. $ABCD$ is a tetrahedron; forces P, Q, R act along the edges BC, CA, AB in order, and forces P', Q', R' act along AD, BD, CD; prove that the condition for a single resultant is

$$\frac{PP'}{BC . AD} + \frac{QQ'}{CA . BD} + \frac{RR'}{AB . CD} = 0.$$

16. A rough heavy body, bounded by a curved surface, rests upon two others which themselves rest on a rough horizontal plane; show that the three centres of gravity and the four points of contact lie in one plane.

17. A heavy beam rests on two smooth inclined planes; show that their line of intersection must be perpendicular to the beam and parallel to the horizon.

18. Prove that the moment of a force represented by the right line PQ about a right line AB is six times the tetrahedron $ABPQ$ divided by AB.

19. Three equal heavy spheres hang in contact from a fixed point by strings of equal length; find the weight of a sphere of given radius which when placed upon the other three will just cause them to separate.

Ans. If W and a be the weight and radius of each of the three spheres, W' and r the weight and radius of the superincumbent sphere, and l the length of each string,

$$\frac{W'}{W' + 3W} = \sqrt{\frac{3r^2 + 6ar - a^2}{3l^2 + 6al - a^2}}.$$

20. Three spheres are placed in contact on a rough horizontal

plane, and a fourth sphere is placed upon them, there being no friction between the spheres themselves. Show that equilibrium is impossible.

21. Three equal spheres are placed in contact on a rough horizontal plane, and a fourth sphere is placed upon them, there being friction between the spheres themselves. Find the least coefficient of friction between the spheres which will allow of equilibrium.

Ans. If a is the radius of each of the equal spheres and r that of the superincumbent sphere, the least value of λ, the angle of friction, is given by the equation

$$\sin 2\lambda = \frac{2}{\sqrt{3}} \cdot \frac{a}{a+r}.$$

(The total resistance between the upper sphere and any one of the lower spheres must be capable of acting through the point of contact of the latter and the ground.)

22. Three forces whose lines of action are given, but not their magnitudes, have a single resultant. Prove that the surface traced out by the line of action of the resultant is a hyperboloid of one sheet.

(Draw any three lines across the given lines of action. Then the line of action of the resultant must always intersect these three.)

23. A heavy triangular plate of uniform thickness is suspended from a fixed point by means of three strings attached to the point and to the vertices of the plate; prove that the tension in each string is proportional to the length of the string.

(Let O be the fixed point, A, B, C the vertices of the plate, and G its centre of gravity. Then G must lie vertically under O. Take OG to represent the weight of the plate. Then, by Leibnitz's graphic representation [Art. 136], the force OG may be resolved into the forces OA, OB, OC. But a given force can have only one set of components along three given concurrent lines. Therefore, &c.)

24. At points on any right line the axes of principal moment of a given system of forces are drawn; prove that their extremities trace out another right line. (Wolstenholme's *Problems*, p. 387, 2nd edition.)

(At any point O on the given line draw R and G. Take as axes of x, y, and z the given line, the line OG, and a line at O perpendicular to R and the given line. Then at any point P on the given line at a distance x from O if the axis of principal moment be drawn the co-ordinates of its extremity will be x, G, and $Rx \sin a$, where a is the angle which R makes with the given line. Hence the extremities lie on the line $y = G$, $z = Rx \sin a$.)

25. Prove that the axes of principal moment at points along any right line whatever trace out a hyperbolic paraboloid.

(With the same axes as in last example, the surface has for equation $xy = \dfrac{G}{R \sin a} \cdot z$.)

25. Find the condition that a given right line should intersect Poinsot's axis.

Ans. If the equations of the line are $x = mz + p$, $y = nz + q$, the required condition is

$$R[mL + nM + N + q(X - mZ) - p(Y - nZ)] = K(mX + nY + Z),$$

where X is used for ΣX, &c.

(It will be found that the equations of Poinsot's axis can be put into the forms

$$x = \frac{X}{Z}z + \frac{KY - MR}{RZ}, \quad y = \frac{Y}{Z}z - \frac{KX - LR}{RZ},$$

the origin being anywhere.)

26. The first case considered in example 9 is, equally with the second, a geometrico-statical problem. Solve it without any mention of force.

[Express the condition that the vertical through the extremity A of AB is intersected by a line inclined at angle λ to the normal at B, this line lying in the plane of the normal and a perpendicular to OB.]

CHAPTER XI.

CENTROIDS [CENTRES OF GRAVITY].

153.] **Centre of Mass.** Imagine a body broken up into an indefinitely great number of infinitesimal elements of mass (without altering the relative positions of these elements) and find the mean centre of all the points at which these elements are placed, the multiple associated with each point being proportional to the element of mass at the point.

Then if the distances of the elements dm_1, dm_2, dm_3, ... from any plane are z_1, z_2, z_3, ..., the distance of the mean centre from the plane is

$$\frac{z_1\, dm_1 + z_2\, dm_2 + \dots}{dm_1 + dm_2 + \dots}, \quad \text{or} \quad \frac{\int z\, dm}{\int dm}.$$

The point thus arrived at is called the *Centre of Mass* of the body; it is also often called the *Centre of Inertia*; and the term *centroid* has lately come into use to designate it.

The distance of the centre of mass from any plane is the mean distance of the body from the plane. If each element of mass is acted on by a force proportional to the mass of the element, and these forces form a parallel system; and if w is the magnitude of the force per unit of mass, the distance of the centre of this parallel system of forces from the plane is

$$\frac{\int w z\, dm}{\int w\, dm}, \quad \text{or} \quad \frac{\int z\, dm}{\int dm},$$

since w is a constant. Thus the centre of the parallel system coincides with the centre of mass. The earth produces such a parallel system of forces on the elements of a body, and therefore the point thus arrived at has, until very recently, been universally called the *Centre of Gravity* of the body. It is only when we consider the action of such a parallel system of forces on the body as the attraction of the earth supplies that the point in question should bear the particular epithet of *Centre of Gravity*.

In numerous questions relating to the body in which the action of gravity is not considered the centre of mass plays a most important part; and it is a point possessed by the body quite independently of any force whatever acting upon it. Hence the latter term is the one most strictly appropriate to the point determined as above; and, except when the weight of the body is concerned, we shall use the terms centroid and centre of mass instead of centre of gravity.

154.] **Theorem of Moments.** If any number of masses be multiplied each by the distance of its centre of mass from any plane, the sum of the products thus obtained is equal to the total mass multiplied by the distance of its centre of mass from the plane.

The centre of mass of any number of finite masses is obtained in precisely the same manner as the centre of mass of a number of particles. Thus, if m_1 and m_2 are the masses of two bodies of any magnitudes, their centre of mass is obtained by dividing the line joining their respective centres of mass in the ratio $m_1 : m_2$, just as if two *particles* of masses m_1 and m_2 were placed at these points.

Hence the distance, $\bar{x}$, of the centre of mass of any number of finite masses from any plane (that of yz) is given by the equation

$$\bar{x} = \frac{\Sigma m x}{\Sigma m},$$

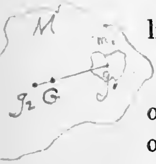
or $M.\bar{x} = \Sigma m x$, and the theorem at the head of this Article is merely the expression of this equation.

It is obvious that the formulæ which have been given for the co-ordinates of the centre of mass hold *whether the axes be rectangular or oblique*. For in Art. 79, p. 96, on which our formulæ are founded, the distances of the points $A_1, A_2, \ldots$ from the line (or plane) L may be assumed to be measured in any common direction.

It follows that if any plane be drawn through the centre of mass of a system of masses, the sum of the products obtained by multiplying each mass by the distance of its centre of mass from the plane is zero. If the plane be that of (yz), and if x' be the distance of the centre of mass of the mass m from the plane, this result is expressed by the equation

$$\Sigma m x' = 0.$$

Given the centres of mass, g_1 and g_2, of two masses, m_1 and m_2, the centre of mass of the two as one system is a point, G, on the line $g_1 g_2$ dividing it in the ratio $\frac{Gg_1}{Gg_2} = \frac{m_2}{m_1}$.



Given the centre of mass, G, of a mass M, and also the centre of mass, g_1, of a portion, m_1, of the mass, the centre of mass, g_2, of the remainder is a point on the line $g_1 G$ produced through G, such that $\frac{Gg_2}{Gg_1} = \frac{m_1}{M - m_1}$.

For assuming g_2 the centre of mass $M-m$, then the centre of mass of $M-$ and m_1 is a point G such that $\frac{g_2 G}{g_1 G}$

155.] **Density.** When a body is of the same constitution throughout, i. e., when its ultimate particles are undistinguishable from each other, and when there is the same number of them in a given volume wherever this volume is taken in the body, the body is said to be *homogeneous* or of *uniform density*; and its density is measured by the quantity of matter contained in (some selected) unit of volume. But when the particles are more or less crowded together in one region of the body than in another, instead of speaking of the density of the body, we must speak of the density *at each particular point*. To measure this, take any very small volume, dv, round the point, and let dm be the quantity of matter contained in it; then the limiting value of the ratio $\frac{dm}{dv}$, when dv (and therefore dm) is indefinitely diminished, is the *density of the body at the point considered.*

156.] **Centre of Mass of a Triangular Lamina of Uniform Thickness and Density.** Let ABC be any triangular lamina of uniform thickness and density, and let it be divided by an indefinitely great number of lines parallel to the base BC into an indefinitely great number of strips. Then the centre of mass of each strip is its middle point; and the middle points of all the strips lie on the line joining A to the middle point of BC. Hence the centre of mass of the lamina lies on this line. Similarly, the centre of mass lies on the line joining B to the middle point of CA. It is therefore *the intersection of the bisectors of the sides drawn from the opposite angles.*

Again, *the centre of mass of a uniform triangular lamina coincides with the centre of mass of three equal particles placed at its vertices.*

For, the centre of mass of the two equal particles at B and C is the middle point of BC, and the centre of mass of the three lies on the line joining this point to A. Similarly, it lies on the line joining B to the middle point of CA. Therefore, &c.

If the mass of each particle is m, the centre of mass divides the line joining A to the middle of BC in the ratio $2m:m$, *or* $2:1$. Hence *the centre of mass of a triangular lamina of uniform thickness and density lies on the bisector of any side drawn from the opposite angle at the point of trisection (nearest to the side) of the bisector.*

COR. If the distances (rectangular or oblique) of the vertices of a triangle from *any plane* are x_1, x_2, and x_3, the distance of its centre of mass from this plane is $\dfrac{x_1+x_2+x_3}{3}$.

157.] **Centre of Mass of a Triangular Pyramid of Uniform Density.** Let $ABCD$ (fig. 199) be a triangular pyramid. Now if any vertex, D, be joined to the centroid, N, of the opposite face, the joining line passes through the centroids of all triangles in which the pyramid is cut by planes parallel to this face. For, let abc be a section of the pyramid parallel to the base ABC. Draw the plane CND containing the lines CD and DN; this plane bisects the base AB in H, since (Art. 156) CN bisects AB. Let the plane CND intersect the face ABD in the right line HhD, h being the point in which this line meets ab. Then since in the triangle ABD, ab is parallel to AB, and DH bisects AB, h is the middle point of ab.

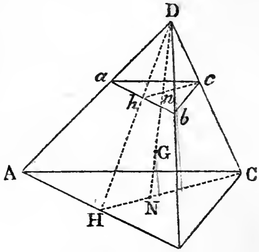


Fig. 199.

Again, if the line DN meets the plane abc in n, the points h, n, and c are in a right line. For these are evidently points common to the planes CND and abc, and since two planes intersect in a right line, the points h, n, c are in a right line—that is to say, n is a point on the bisector of the side ab drawn through c. Similarly, n is a point on the bisector of bc drawn through a; therefore n is the centroid of the triangle abc (Art. 156).

To find the centre of mass of the pyramid, let it be divided by planes parallel to ABC into an indefinitely great number of triangular laminæ. Now we have just proved that the centres of mass of all these laminæ lie on the line, DN, joining the vertex D to the centroid of the opposite base. Similarly, the centre of mass of the pyramid lies on the line joining the vertex A to the centroid of the face BCD. It is, therefore, the point, G, of intersection of lines drawn from any two vertices to the centroids of the opposite faces. But this is exactly the construction for the centre of mass of a system of four equal particles placed at the vertices of the pyramid. Hence—

The centre of mass of a triangular pyramid coincides with the centre of mass of four equal particles placed at its vertices.

Also

The centre of mass of a triangular pyramid is one-fourth of the way up the line joining the centroid of any face to the opposite vertex.

For, if at the vertices there be placed four equal particles, each of mass m, their centre of mass is found by joining D to N and taking $\frac{GN}{GD} = \frac{m}{3m} = \frac{1}{3}$, therefore $GN = \frac{1}{3} GD$, or

$$NG = \tfrac{1}{4} ND.$$

COR. 1. The *perpendicular* distance of the centre of mass of a triangular pyramid from the base is equal to $\frac{1}{4}$ height of pyramid. Since the projection of 1/4 a line is 1/4 the projection of the whole line on any line whatever.

COR. 2. If the distances (rectangular or oblique) of the vertices of a pyramid from *any plane* are x_1, x_2, x_3, x_4, the distance of the centre of mass from the plane is $\frac{x_1 + x_2 + x_3 + x_4}{4}$.

$\frac{x_1+x_2+x_3}{3}$

$\frac{x_1+x_2+x_3}{3}$

$\frac{x_1+x_2+x_3+x_4}{4}$

158.] **Centre of Mass of a Cone of Uniform Density having any Plane Base.** Consider a pyramid whose base is a polygon of any number of sides. Then, by dividing the base into triangles we can consider the whole pyramid as composed of a number of triangular pyramids. Now (Art. 157) the centre of mass of each of these pyramids lies in a plane whose distance from the base is one-fourth of the height of the pyramid; therefore the centre of mass of the whole pyramid lies in this plane—that is, its *perpendicular* distance from the base is one-fourth of the height of the pyramid.

Again, dividing the pyramid into an indefinitely great number of laminæ, as in last Art., the centres of mass of these laminæ all lie on the right line joining the vertex to the centroid of the base. Hence the centre of mass of the whole pyramid lies on this line; and by what we have just proved, it must be one-fourth of the way up this line. There is no limit to the number of sides of the polygon; hence they may form a continuous curve.

Therefore—

The centre of mass of a cone whose base is any plane curve whatever is found by joining the centroid of the base to the vertex, and taking a point one-fourth of the way up this line.

159.] **Theorem.** *If the mass of each of a system of bodies be multiplied by the square of the distance of its centre of mass from a given point, the sum of the products thus obtained is least when the given point is the centre of mass of the system of bodies.*

This theorem, which is well known in elementary geometry, admits of a very simple analytical proof.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of mass, G, of the system with reference to rectangular axes through any point, O, and let (x_1, y_1, z_1), (x_2, y_2, z_2), ..., be the co-ordinates of the centres of mass, $A_1, A_2, \ldots$, of the bodies whose masses are $m_1, m_2, \ldots$. Then

$$GA_1^2 = (\bar{x}-x_1)^2+(\bar{y}-y_1)^2+(\bar{z}-z_1)^2. \tag{1}$$

Similarly, $$GA_2^2 = (\bar{x}-x_2)^2+(\bar{y}-y_2)^2+(\bar{z}-z_2)^2, \tag{2}$$

. . . .

Multiplying these equations by $m_1, m_2, \ldots$, and adding, we have

$$\Sigma(m \,.\, GA^2) = (\bar{x}^2+\bar{y}^2+\bar{z}^2) \,.\, \Sigma m - 2\bar{x} \,.\, \Sigma mx - 2\bar{y} \,.\, \Sigma my$$
$$-2\bar{z} \,.\, \Sigma mz + \Sigma m(x^2+y^2+z^2). \tag{3}$$

Now (Art. 154),

$$\Sigma mx = \bar{x} \,.\, \Sigma m, \quad \Sigma my = \bar{y} \,.\, \Sigma m, \quad \Sigma mz = \bar{z} \,.\, \Sigma m.$$

Hence (3) becomes

$$\Sigma(m \,.\, GA^2) = \Sigma m(x^2+y^2+z^2) - (\bar{x}^2+\bar{y}^2+\bar{z}^2) \,.\, \Sigma m,$$

or $$\Sigma(m \,.\, GA^2) = \Sigma(m \,.\, OA^2) - OG^2 \,.\, \Sigma m, \tag{4}$$

from which equation it appears that $\Sigma(m \,.\, GA^2)$ is always less than $\Sigma(m \,.\, OA^2)$ by the quantity $OG^2 \,.\, \Sigma m$.

It can be shown that, if r_{12} denote the distance between the centres of mass of the masses m_1 and m_2, and M the sum of all the masses, $$M \,.\, \Sigma(m \,.\, GA^2) = \Sigma(m_1 m_2 r_{12}^2).$$

For, let the centre of mass, G, be taken as origin. Then, denoting the co-ordinates of the points $A_1, A_2, \ldots$ with reference to G by (x_1', y_1', z_1'), (x_2', y_2', z_2'), ...,

$$M\Sigma(m\,.\,GA^2) = m_1(m_1+m_2+\ldots)(x_1'^2+y_1'^2+z_1'^2)$$
$$+m_2(m_1+m_2+\ldots)(x_2'^2+y_2'^2+z_2'^2)+\ldots. \quad (5)$$

Also (Art. 154)

$$0 = m_1 x_1' + m_2 x_2' + \ldots,$$
$$0 = m_1 y_1' + m_2 y_2' + \ldots,$$
$$0 = m_1 z_1' + m_2 z_2' + \ldots.$$

Squaring each of these last three equations, adding the results together, and subtracting their sum from (5), we have

$$M\,.\,\Sigma(m\,.\,GA^2) = m_1 m_2(\overline{x_1-x_2}^2+\overline{y_1-y_2}^2+\overline{z_1-z_2}^2)+\ldots$$
$$= m_1 m_2 r_{12}^2 + \ldots$$
$$= \Sigma m_1 m_2 r_{12}^2.$$

Hence, from (4),

$$OG = \sqrt{\frac{\Sigma(m\,.\,OA^2)}{M} - \frac{\Sigma(m_1 m_2 r_{12}^2)}{M^2}},$$

under which form Lagrange expresses the distance of the centre of mass of a system of bodies from a given point (see *Mécanique Analytique*, p. 61).

Equation (4) can be employed to prove the well-known expression for the distance between the centres of the inscribed and circumscribed circles of a plane triangle, viz.

$$D^2 = R^2 - 2Rr,$$

D being the distance between the centres, and r and R being their radii, respectively.

(Suppose a system of particles at the vertices, the mass of each being proportional to the opposite side. Their centre of mass is the centre of the inscribed circle. The remainder is left to the student as an exercise.)

EXAMPLES.

1. To find the position of the centre of mass of the frustum of a pyramid.

Let the frustum be formed by the removal of the pyramid $abcD$ (fig. 199) from the whole pyramid $ABCD$; let h and H be the perpendicular heights of these pyramids, respectively; and let m and M denote their masses.

Now if the perpendicular distances of the centres of mass of the pyramid $ABCD$, the pyramid $abcD$, and the frustum, from the base ABC be denoted by z_1, z_2, and z, respectively, we have (Art. 154)

$$Mz_1 = mz_2 + (M-m)\,z. \quad (1)$$

$\frac{m}{M-m} = \frac{bc}{cd}$ ∴ $M(bc) = m(bc+cd$

or, $M(bc+ab) = m(bc+cd+ab-ab)+M(a$

i.e., $Mz_1 = mz_2 + (M-m)z$.

But $z_1 = \frac{H}{4}$, $z_2 = \frac{h}{4} + H - h = H - \frac{3}{4}h$. Also the masses of the pyramids are to each other as the cubes of their heights; therefore (1) gives

$$\frac{H^4}{4} = h^3 (H - \tfrac{3}{4}h) + (H^3 - h^3)\, z,$$

or

$$4(H^3 - h^3)\, z = H^4 - 4Hh^3 + 3h^4$$
$$= (H-h)^2 (H^2 + 2Hh + 3h^2);$$

$$\therefore \quad z = \frac{H-h}{4} \cdot \frac{H^2 + 2Hh + 3h^2}{H^2 + Hh + h^2}. \qquad (2)$$

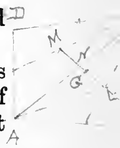
Instead of the heights we can use the square roots of the areas of the bases, to which the heights are proportional. If these areas are denoted by A and a, we have

$$z = \frac{H-h}{4} \cdot \frac{A + 2\sqrt{Aa} + 3a}{A + \sqrt{Aa} + a}. \qquad (3)$$

The centre of mass, G', of the frustum obviously lies on the line Nn (fig. 199) between N and G; and (3) evidently gives

$$NG' = \frac{Nn}{4} \cdot \frac{A + 2\sqrt{Aa} + 3a}{A + \sqrt{Aa} + a}. \qquad (4)$$

It is clear that the position of the centre of mass of the frustum of a cone standing on any plane base is also given by these equations.



2. To find the centre of mass of a board of uniform thickness and density whose figure is that of a quadrilateral.

Let $ABCD$ be the quadrilateral; draw the line AC, which divides the quadrilateral into two triangles; let L and M be the centroids of the triangles ABC and ADC, respectively; and let the line LM meet AC in N.

Then the centroid of the quadrilateral is a point, G, on LM such that
$$\frac{MG}{LG} = \frac{\text{area } ABC}{\text{area } ADC} = \frac{\text{area } ALC}{\text{area } AMC} = \frac{\text{perp. from } L \text{ on } AC}{\text{perp. from } M \text{ on } AC} = \frac{LN}{MN};$$

therefore
$$\frac{MG}{LM} = \frac{LN}{LM}, \quad \text{or} \quad MG = LN.$$

The centre of mass is therefore found by taking a point, G, on LM, such that $MG = LN$.

Another construction. The student will find little difficulty in proving the following construction. Draw the diagonals AC and BD, meeting in the point O. On AC take a point C', such that $AC' = CO$, and on BD take a point B', such that $DB' = BO$. Then the centroid of the quadrilateral is the centroid of the triangle $B'OC'$.

3. From a triangular board of uniform thickness and density the portion constituting the area of the inscribed circle is removed; prove

that the distance of the centre of mass of the remainder from any side (a) is

$$\frac{\Delta}{3as} \cdot \frac{2s^3 - 3\pi a\Delta}{s^2 - \pi\Delta},$$

Δ being the area, and s half the sum of the sides, of the board.

4. If a tetrahedron be formed by the centres of mass of any four masses, prove that each mass is proportional to the tetrahedron standing on the opposite face and having for vertex the common centre of mass of the masses.

5. If at the vertices of a triangle there be placed three masses each of which is proportional to the opposite side of the triangle, prove that their centre of mass is the centre of the circle inscribed in the triangle.

6. Prove that the centre of mass of a system of uniform bars forming a triangle is the centre of the circle inscribed in the triangle formed by the middle points of the bars.

7. A figure is formed by a right-angled triangle whose sides are a, b, and c, and the squares constructed on these sides; find the distance of the centroid of this figure from the greatest side (c).

Ans. $\dfrac{ab}{3c} \cdot \dfrac{3c^2 - 5ab}{4c^2 + ab}.$

8. Prove that the centroid of a trapezium divides the line joining the middle points of the two parallel sides in the ratio $\dfrac{a+2b}{2a+b}$, the lengths of these sides being a and b.

Prove also the following construction for the centroid:—

The vertices, in order, being A, B, C, D, and the parallel sides AB and CD, produce BA to A', and AB to B', so that $AA' = BB' = CD$; also produce DC to C', and CD to D', so that $CC' = DD' = AB$; then the point of intersection of $A'C'$ and $B'D'$ is the required centroid.

9. A right line passing through a fixed point intersects two fixed right lines; find the locus of the centroid of the triangle formed by the variable line and the two fixed lines.

Ans. If the co-ordinates of the fixed point with reference to the two fixed lines as axes are a and b, the locus is the hyperbola

$$(3x - a)(3y - b) = ab.$$

10. If the right line in the last example, instead of passing through a fixed point, cut off a triangle of constant area, find the locus of the centroid of the triangle.

Ans. If ω is the angle between the fixed lines, and k^2 the constant area, the locus is the hyperbola

$$9xy \sin\omega = 2k^2.$$

11. From a sphere of radius R is removed a sphere of radius r, the distance between their centres being c; find the centre of mass of the remainder.

Ans. It is on the line joining their centres, and at a distance $\frac{c r^3}{R^3 - r^3}$ from the centre.

12. Every body has one and only one centre of mass. Hence show that the lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

(Consider four equal particles at the vertices.)

13. From the vertices of a given triangle let perpendiculars be drawn to the opposite sides. Find the distances of the centroid of the triangle formed by the feet of these perpendiculars from the sides of the given triangle.

Ans. The distance from the side a is $\frac{1}{3} a \sin A \cos (B - C)$.

14. A thin uniform wire is bent into the form of a triangle ABC, and particles, of weights, P, Q, R, are placed at the angular points A, B, C, respectively; prove that if the centre of mass of the particles coincides with that of the wire,

$$P : Q : R = b + c : c + a : a + b.$$

(Wolstenholme's *Book of Mathematical Problems.*)

15. Find the centroid of the triangle formed by the points in which the bisectors of the angles of a given triangle meet the opposite sides.

Ans. If Δ denote the area of the given triangle, whose sides are a, b, c, the distance of the centroid from the side a is

$$\tfrac{2}{3} \Delta \frac{2a + b + c}{(a + b)(a + c)}.$$

16. A uniform wire of given length is formed into a triangle of which one angle is given; find the locus of the centre of mass of the wire referred to the sides containing the given angle as axes.

Ans. If C is the given angle, and $4l$ the length of the wire, the locus is the ellipse

$$(l - x - y)^2 + 2(l - x - y)(2l - x - y) \sin^2 \frac{C}{2} + 4xy \sin^4 \frac{C}{2} = 0.$$

17. If particles be placed at the angular points of a tetrahedron, proportional respectively to the areas of the opposite faces, their centre of mass will be the centre of the sphere inscribed in the tetrahedron.

(Wolstenholme's *Book of Mathematical Problems.*)

18. Prove that the centroid of the surface of a tetrahedron is the centre of the sphere inscribed in the tetrahedron formed by joining the centroids of the faces.

Section II.

Investigations requiring Integration.

160.] **Rule.** The general formulæ, such as that in Art. 153, for the co-ordinates of the centre of mass of a quantity of matter arranged in any manner assume particular forms according as the matter is arranged in the form of a wire of any shape, an area or thin lamina of any shape, or a solid. Then, again, they assume particular forms in each of these cases according to the manner in which the matter is supposed to be divided into elementary portions.

Many students are in the habit of remembering a special formula for each of these numerous cases; such a habit, however, is not only useless but injurious. It is much better to consider the formula of Art. 153, or the method of p. 97, as furnishing the following Rule which covers all possible cases:

Divide the given quantity of matter, in any way, into elementary portions; find the position of the centre of mass of each of these portions; then multiply the mass of each portion by the co-ordinate of its centre of mass, and take the integral of this product; and finally divide this integral by the whole quantity of matter. The result is the co-ordinate of the centre of mass required.*

161.] **Centre of Mass of the Arc of a Curve.** If the matter whose centre of mass we desire to find is arranged in the shape of the arc of any curve, the co-ordinates of its centre of mass are obtained from the formula of Art. 153, in which dm now denotes the mass of an elementary length of the curve.

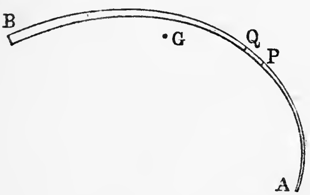


Fig. 200.

Let ds denote the length of an elementary portion of the curve contained between two points, P and Q (fig. 200); let k denote the mean area of a normal section of the curve between P and Q; and let ρ denote the density of the matter in the neighbourhood of P and Q. Then, since the quantity of matter in any space is equal to the product of the volume and the density, the quantity of matter between P and Q is $k\rho ds$.

* The co-ordinates are supposed to be such as are measured parallel to a given line. The rule would not hold if by *co-ordinate* were understood *polar co-ordinate*, for instance.

Again, the centre of mass of this element is evidently the middle point of PQ.

And since to obtain G, the centre of mass of the whole mass, the co-ordinates of this middle point must be multiplied by the infinitesimal $k\rho ds$, the co-ordinates of the centre of mass of PQ may be taken to be the same as those of P.

Replacing dm in the general formulæ by the linear element $k\rho ds$, we obtain for the position of the centre of mass of matter arranged in the form of any curve the equations

$$\bar{x} = \frac{\int k\rho x ds}{\int k\rho ds},$$

$$\bar{y} = \frac{\int k\rho y ds}{\int k\rho ds},$$

$$\bar{z} = \frac{\int k\rho z ds}{\int k\rho ds}.$$

The quantities k and ρ must be given as functions of the position of the point P before the integrations can be performed.

Examples.

1. To find the position of the centroid of a circular arc of uniform thickness and density.

Let AB be the arc, M its middle point, and O the centre of the circle. Then it is manifest from symmetry that the centroid must lie on the line OM. Take OM as axis of x. Then since k and ρ are constant, we have

$$\bar{x} = \frac{\int x ds}{\int ds},$$

x being the co-ordinate of any point, P, in the arc. Let θ be the angle POM and a the radius of the circle. Then

$$x = a\cos\theta, \text{ and } ds = a d\theta.$$

Hence
$$\bar{x} = a\frac{\int \cos\theta d\theta}{\int d\theta},$$

the integration to be extended over the whole arc. Now if the angle $BOA = 2\alpha$, the integration must be taken from $\theta = -\alpha$ to $\theta = \alpha$. Therefore

$$\bar{x} = a\frac{\sin\alpha}{\alpha}. \quad (1)$$

Hence *the distance of the centroid of the arc of a circle from the centre is the product of the radius and the chord of the arc divided by the length of the arc.*

The distance of the centroid of a semicircle from the centre is $\frac{2a}{\pi}$.

2. Find the centre of mass of a circular arc of uniform section, the density varying as the length of the arc measured from one extremity.

Let AB be the arc; let the density at any point $P = \mu . AP$, and let OA be taken as axis of x. Then if $\angle AOB = \alpha$, and $AP = s$, we have

$$\bar{x} = \frac{\int sxds}{\int sds} = a\frac{\int_0^\alpha \theta\cos\theta d\theta}{\int_0^\alpha \theta d\theta}$$ see Byerly's Tables 1

$$= 2a\frac{\alpha\sin\alpha + \cos\alpha - 1}{\alpha^2}.$$

Similarly, $$\bar{y} = \frac{\int syds}{\int sds} = a\frac{\int_0^\alpha \theta\sin\theta d\theta}{\int_0^\alpha \theta d\theta}$$ do: 123.

$$= 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}.$$

3. One extremity, A, of the arc, AB, of a curve being fixed, while the other extremity, B, varies, it is required to construct at any point the tangent to the locus of the centroid of the variable arc AB.

Let AB be a portion of the arc of any curve, and let G be the centroid of AB. Then if B' be a point on the given curve very close to B, the centroid of the whole arc AB' is obtained by joining the centroid, G, of AB to the centroid of BB', and dividing the joining line inversely as the lengths of AB and BB'. But the centroid of BB' is its middle point. Hence the centroid of AB' lies on the line joining G to the middle point of BB'. In the limit, therefore, the line joining G to its next consecutive position is the line GB, which is, then, the tangent at G to the locus of G.

4. Find the position of the centroid of the arc of a semi-cardioid.

Ans. The equation of the curve being $r = a(1+\cos\theta)$, the coordinates of its centroid referred to the axis of the curve and a perpendicular line through the cusp as axes of x and y are

$$\bar{x} = \bar{y} = \frac{4}{5}a.$$

5. Find the equation of the line joining the centroid of the arc of half a loop of a lemniscate to the double point.

Ans. The axes of x and y being the axis of the curve and a perpendicular line, the equation of the required line is

$$y = (\sqrt{2}-1).x.$$

6. Find the centroid of the arc of a semi-cycloid.

Ans. The axis of x being a tangent at the vertex, and a the radius of the generating circle,

$$\bar{x} = (\pi - \frac{4}{3})a, \quad \bar{y} = \frac{2}{3}a.$$

7. Find the distance of the centroid of the catenary

$$y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$$

from the axis of x, the curve being divided into two equal portions by the axis of y.

Ans. If $2l$ is the length of the curve and k the ordinate of its extremity, the centroid lies on the axis of y at a distance $\dfrac{kl+ac}{2l}$ from the axis of x.

8. Find a law of density of a wire of uniform section bent into the shape of a cycloid so that its centre of mass shall be half way up its axis.

Ans. If the density varies as the length of the arc measured from the vertex, the result will follow.

9. If the density of a cycloidal arc varies as the n^{th} power of the arc measured from the vertex, find the position of the centre of mass of the curve.

Ans. On the axis at a distance $2\dfrac{n+1}{n+3}a$ from the vertex, a being the radius of the generating circle.

10. One extremity of a circular arc is fixed while the other varies along the circle; trace the locus of the centroids of the varying arcs, and prove that the *algebraic* sum of the intercepts of the locus on the diameter perpendicular to that passing through the fixed extremity of the arcs is equal to half the radius.

162.] **Centroid of a Plane Area.** Let $APQB$ (fig. 201) be any curve whose equation is given, and let it be required to find the centroid of the area, $CABD$, of a lamina included between a given portion, AB, of the curve, two extreme ordinates, AC and BD, and the axis of x, the lamina being supposed of uniform thickness and density. In accordance with the rule of Art. 160, we break up the area into elementary portions. Suppose that this is done by taking rectangular strips, such as $PQNM$, included between two very close ordinates, PM and QN, and let g be the centre of mass of this strip.

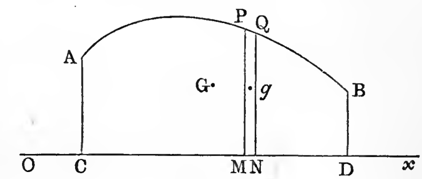


Fig. 201.

Let the co-ordinates of P be (x, y) and those of Q $(x+dx, y+dy)$; let ρ be the density and k the thickness of the lamina.

Then the mass dm, of the rectangular strip is

$$k\rho y dx.$$

Also the co-ordinates of g are $\left(x+\epsilon, \frac{y}{2}+\epsilon'\right)$, ϵ and ϵ' being extremely small quantities of the same order of magnitude as dx and dy.

Following the rule of Art. 160, to obtain the abscissa of G, the centroid of the area, we shall have to take the integral of the product

$$k\rho y (x+\epsilon) dx.$$

Now ϵdx is an infinitesimal of the second order, and is therefore to be neglected in the integral. Hence if $\bar{x}$ and $\bar{y}$ are the co-ordinates of G, we have evidently, since k and ρ are constants,

$$\bar{x} = \frac{\int xy dx}{\int y dx}, \bar{y} = \tfrac{1}{2}\frac{\int y^2 dx}{\int y dx},$$

the integrations extending over the whole area $CABD$.

EXAMPLES.

1. Find the centroid of the area of a semi-cycloid.

Taking the line joining the extremities of the arc of the whole curve as axis of x, and a perpendicular through the vertex as axis of y, the curve is given by the equations

$$x = a(\theta + \sin\theta),$$
$$y = a(1 + \cos\theta).$$

Hence $y dx = 4a^2 \cos^4 \frac{\theta}{2}\, d\theta$, and we have

$$\bar{x} = a\frac{\int_0^\pi (\theta+\sin\theta)\cos^4\frac{\theta}{2}\,d\theta}{\int_0^\pi \cos^4\frac{\theta}{2}\,d\theta}, \quad \bar{y} = a\frac{\int_0^\pi \cos^6\frac{\theta}{2}\,d\theta}{\int_0^\pi \cos^4\frac{\theta}{2}\,d\theta}.$$

To find $\int_0^\pi \theta\cos^4\frac{\theta}{2}\,d\theta$, write it

$$\frac{1}{4}\int_0^\pi \theta(1+\cos\theta)^2\,d\theta, \text{ or } \frac{1}{4}\int_0^\pi \theta\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta.$$

Now $\int \theta\cos n\theta d\theta = \frac{n\theta\sin n\theta + \cos n\theta}{n^2}$. Hence the integral in question $= \frac{3\pi^2-16}{16}$.

Again $\int_0^\pi \sin\theta\cos^4\frac{\theta}{2}\,d\theta = 2$, $\int_0^\pi \sin\frac{\theta}{2}\cos^5\frac{\theta}{2}\,d\theta = \frac{2}{3}$.

Hence $$\bar{x} = \frac{9\pi^2-16}{18\pi} \cdot a.$$

And evidently $$\bar{y} = \frac{5}{6} a.$$

2. If the ordinates of a given curve, U, be all diminished or increased in a given ratio and a new curve, U', thus formed, prove that the centroid of any portion of U' cut off by a right line is obtained by diminishing or increasing in the same ratio the ordinate of the centroid of the corresponding portion of U.

Let one right line parallel to the axis of y meet U and U' in P and P' respectively, and let another such line meet them in Q and Q'. Draw the right lines PQ and $P'Q'$; then these lines cut off corresponding portions of the two curves. From any point, M, on U draw a line parallel to the axis of y meeting the right line PQ in N, and U' and $P'Q'$ in M' and N', respectively. Denote the ordinates of M and N by y and z; then it is clear that if k is the number by which the ordinates of U are multiplied to obtain those of U', the ordinates of M' and N' are ky and kz, respectively. All these points have a common abscissa, x. An ordinate drawn with the abscissa $x+dx$ includes with the ordinate $MNM'N'$, the curve U, and the line PQ a strip of area equal to $(y-z)dx$, while the corresponding strip of the area of U' cut off by $P'Q'$ is $k(y-z)\,dx$. Again, the ordinate of the middle point of the first strip is $\frac{y+z}{2}$, and that of the middle point of the second strip is $k\frac{y+z}{2}$.

Hence if $\bar{y}$ and $\bar{y}'$ denote the ordinates of the centroids of the portions of U and U' cut off by PQ and $P'Q'$, respectively,

$$\bar{y}' = \frac{1}{2}\frac{\int k^2(y^2-z^2)\,dx}{\int k(y-z)\,dx}$$
$$= k \,.\, \bar{y}.$$

Let PQ cut off in all positions a constant area from U; then it is evident that $P'Q'$ cuts off a constant area from U'. Suppose, moreover, that in this case the locus of the centroid of the portion of U is a curve whose equation is $f(x, y) = 0$;
then clearly the locus of the centroid of the corresponding portion of U' of constant area cut off by a right line is the curve

$$f\left(x, \frac{y}{k}\right) = 0.$$

If the lines PQ and $P'Q'$ are replaced by two curves the second of which is deduced from the first as U' was from U, the same results evidently follow.

3. Find the centroid of a quadrant of an ellipse.

Ans. $\bar{x} = \frac{4a}{3\pi}$, $\bar{y} = \frac{4b}{3\pi}$.

4. A right line cuts off a constant area from an ellipse; find the locus of the centroid of the portion cut off.

Ans. An ellipse concentric and coaxal with the given one.

5. Find the centroid of a quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$Ans.\quad \bar{x}=\frac{2.4.6.8}{3.5.7.9}\cdot\frac{2a}{\pi};\quad \bar{y}=\frac{2.4.6.8}{3.5.7.9}\cdot\frac{2b}{\pi}.$$

(Assume $x = a\cos^3\phi$, $y = b\sin^3\phi$.)

6. Find the centroid of any segment of a parabola cut off by a right line.

Ans. On the diameter conjugate to the given line at a distance from the curve equal to $\frac{3}{5}$ of the portion of the diameter intercepted by the given line.

7. Through a given point, O, is drawn a fixed right line meeting a curve in A; through O is also drawn another right line meeting the curve in P. It is required to construct at any point the tangent to the locus described by the centroid of the area AOP as the line OP varies.

Ans. Let G be the centroid of AOP, and take a point Q on OP such that $OQ=\frac{2}{3}OP$. Then GQ is the tangent to the locus at G. (See Example 3, p. 254.)

8. Find the centroid of a semi-ellipse cut off by any diameter.

Ans. It is on the diameter conjugate to the given one and at a distance $\frac{4a'}{3\pi}$ from the centre, $2a'$ being the length of this conjugate diameter.

9. Find the centroid of the area included by a parabola and two tangents.

Ans. If a and b are the lengths of the tangents (which are taken as axes of x and y), $\bar{x}=\frac{a}{5}$, $\bar{y}=\frac{b}{5}$.

(The equation of the parabola is $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$. Assume

$$x = a\cos^4\phi,\quad y = b\sin^4\phi.)$$

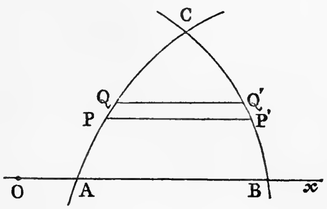


Fig. 202.

The particular manner in which it is advisable to break up the area whose centroid is required varies with the nature of the area itself. Thus, let the area be that included between the axis of x and two curves, AC and BC (fig. 202) whose equations are given. In this case the area may be broken up into thin strips, such as $PQP'Q'$, parallel to the axis of x. Let (x, y) be the co-ordinates of P and (x', y) those of P'. Then the area of the strip is $(x'-x)\,dy$, and the co-ordinates of its centroid are $\frac{1}{2}(x'+x)$ and y. Hence if no portion of the area considered is above a parallel to Ox drawn through C, the co-ordinates of its centroid are given by the equations

$$\bar{x} = \frac{1}{2}\frac{\int(x'^2-x^2)\,dy}{\int(x'-x)\,dy},\quad \bar{y} = \frac{\int y\,(x'-x)\,dy}{\int(x'-x)\,dy},$$

in which the limits of y are 0 and the ordinate of C. The values of x' and x are of course given in terms of y from the equations of the two curves.

For example, let it be required to find the centroid of the area included between a parabola and a circle described with the vertex of the parabola as centre and a radius equal to $\frac{3}{8}$ of its latus rectum. The centroid is on the axis of the parabola. Let the equation of the parabola be $y^2 = 4mx$; then the equation of the circle is $x^2+y^2 = \frac{9}{4}m^2$; and the ordinate of C, their point of intersection, is $m\sqrt{2}$.

Hence

$$\bar{x} = \frac{1}{2}\,\frac{\int_0^{m\sqrt{2}}\left(\frac{9}{4}m^2-y^2-\frac{y^4}{16m^2}\right)dy}{\int_0^{m\sqrt{2}}\left(\sqrt{\frac{9}{4}m^2-y^2}-\frac{y^2}{4m}\right)dy}$$

$$= \frac{36m}{16+27\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)},$$

as the student will find without much difficulty.

Examples.

1. Find the centroid of the area included between the arc of a semi-cycloid, the circumference of the generating circle, and the line joining the extremities of the cycloid.

Ans. The common tangent to the circle and cycloid at the vertex of the latter being taken as axis of x, the vertex being origin, and a the radius of the generating circle,

$$\bar{x} = \frac{3\pi^2-8}{4\pi}\cdot a;\quad \bar{y} = \frac{5}{4}a.$$

2. Find the locus of the centroid of the area of a parabola cut off by a variable right line drawn through the vertex.

Ans. If $4m$ is the latus rectum of the parabola, the locus is another parabola whose equation is $y^2 = \frac{5}{2}mx$.

(The student may verify the construction of Example 7, p. 258, for the tangent to this locus.)

3. Find the centroid of the portion of an ellipse cut off by a line joining the extremities of the major and minor axes.

$$Ans.\ \bar{x} = \frac{2}{3}\cdot\frac{a}{\pi-2};\quad \bar{y} = \frac{2}{3}\cdot\frac{b}{\pi-2}.$$

163.] **Graphic Construction of the Centroid of a Plane Area.** The following method of determining the centroid of any plane area is taken from Collignon's *Statique*, p. 315.

Let $APBQ$ be any plane area, and let Ox be any line in its plane. Then if the distances of the centroid from Ox and any other line in the plane are known, the position of the point is known.

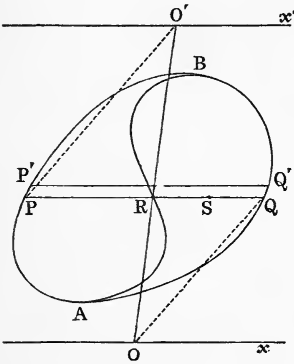


Fig. 203.

Draw any line, $O'x'$, parallel to Ox (axis of x) in the plane of the curve, and let the perpendicular distance between Ox and $O'x'$ be a. Let the area be broken up into narrow rectangular strips, such as $PP'Q'Q$, by lines parallel to the axis of x. Then if $PQ = z$, the area of the strip $= zdy$, the distance of PQ from Ox being y.

Hence the distance, $\bar{y}$, of the centroid of the area from Ox is given by the equation

$$\bar{y} = \frac{\int yzdy}{\int zdy} = \frac{\int yzdy}{A_1}, \tag{1}$$

A_1 being the area of the figure, and the values of y running from the ordinate of A to that of B, at which points the tangents are parallel to Ox. Now take any point, O, on Ox; draw OQ, and draw PO' parallel to OQ. Let the line OO' meet PQ in R. Then by similar triangles

$$\frac{QR}{RP} = \frac{OR}{RO'};\quad \therefore\ \frac{QR}{PQ} = \frac{OR}{OO'},$$

or, z' denoting the length QR,

$$az' = yz. \tag{2}$$

Let the locus of R corresponding to all strips of the given area be constructed. It will be a curve, ARB, passing through the points A and B.

Substituting the value of yz from (2) in (1), we have

$$\bar{y} = \frac{a \int z' dy}{A_1},$$

in which the limits of y are the same as before. But $\int z' dy$ is the area, A_2, between the curves ARB and AQB. Hence

$$\bar{y} = a \frac{A_2}{A_1}.$$

The distance of the centroid from Ox is therefore known. Similarly its distance from any other line can be found, and therefore the position of the point is determined.

If a point S is deduced from R in the same way as that in which R was deduced from P, and if $QS = z''$, we shall have as before

$$az'' = z'y = \frac{y^2 z}{a}.$$

If therefore the locus of S is constructed, the area included between it and AQB multiplied by a^2 will be the value of the integral $\int y^2 z dy$ extended over the original area.

By the construction of successive curves such as ARB we represent the values of $\int y^3 z dy$, $\int y^4 z dy$, &c., graphically.

An ingenious instrument founded on these principles—the Integrometer of M. Deprez—is described by Collignon in the *Annales des Ponts et Chaussées* for March, 1872.

Example.

In finding by this method the centroid of a portion of a parabola cut off by a double ordinate at a distance h from the vertex, prove that if the tangent at the vertex and the given double ordinate are taken as the lines Ox and $O'x'$, the equation of the curve ARB will be

$$h^2 y^2 = 4mx(h - 2x)^2.$$

This curve (both branches being drawn) has a loop between the values $x = 0$ and $x = \frac{1}{2}h$, and passes through the extremities of the double ordinate.

164.] **Polar Elements of a Plane Area.** Let it be required to find the centroid of a portion of a plane area bounded by a portion of any curve, AB (fig. 204), and by two extreme radii vectores, OA and OB, drawn through a given point, O. It is obvious that in this case it is advisable in applying the rule of Art. 160 to decompose the area into triangular strips, such as POQ, included between two very close radii vectores. If $OP = r$, and $\angle POx = \theta$, the element of area, POQ, is equal to

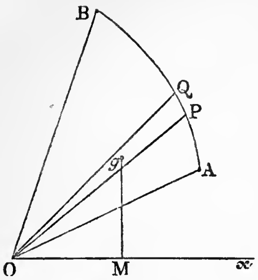


Fig. 204.

$$\tfrac{1}{2} r^2 d\theta;$$

and if the thickness and density of the lamina are uniform, the centre of mass of this element is a point g which may be considered as on OP at a distance $\frac{2}{3}r$ from O.

Hence if Ox is the axis of x, the co-ordinates of g are ultimately

$$\tfrac{2}{3} r \cos\theta, \quad \text{and} \quad \tfrac{2}{3} r \sin\theta.$$

Applying the rule of Art. 160, we then have

$$\bar{x} = \frac{2}{3}\frac{\int r^3 \cos\theta\, d\theta}{\int r^2 d\theta}; \quad \bar{y} = \frac{2}{3}\frac{\int r^3 \sin\theta\, d\theta}{\int r^2 d\theta}.$$

For example, to find the centroid of a loop of Bernouilli's Lemniscate whose equation is $r^2 = a^2 \cos 2\theta$.

The axis of the loop being taken as axis of x, the abscissa of the centroid of the whole loop is evidently the same as that of the half loop above the axis;

$$\therefore \quad \bar{x} = \frac{2a}{3}\,\frac{\int_0^{\frac{\pi}{4}} \cos^{\frac{3}{2}} 2\theta \cos\theta\, d\theta}{\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta}$$

$$= \frac{4a}{3}\int_0^{\frac{\pi}{4}} (1-2\sin^2\theta)^{\frac{3}{2}} . \, d\sin\theta.$$

Putting $\sin\theta = \dfrac{\sin\phi}{\sqrt{2}}$, this integral becomes

$$\frac{4a}{3\sqrt{2}}\int_0^{\frac{\pi}{2}} \cos^4\phi\, d\phi,$$

which $= \dfrac{4a}{3\sqrt{2}} \cdot \dfrac{1\cdot 3}{2\cdot 4} \cdot \dfrac{\pi}{2}$. Therefore

$$\bar{x} = \frac{\pi a}{4\sqrt{2}}.$$

Examples.

1. To find the centroid of a given sector of a circle.

Ans. It is on the diameter bisecting the arc, at a distance from the centre equal to $\frac{2}{3}$ of the product of the radius and the chord of the arc divided by the length of the arc.

2. Find the centroid of a portion of an equiangular spiral included by the initial line and a given radius vector.

Ans. The initial line being taken as axis of x, the equation of the spiral being $r = ae^{k\theta}$, and a being the angle of the given radius vector,

$$\bar{x} = \frac{4ka}{3(1+9k^2)} \cdot \frac{e^{3ka}\sin a + 3ke^{3ka}\cos a - 3k}{e^{2ka}-1};$$

$$\bar{y} = \frac{4ka}{3(1+9k^2)} \cdot \frac{1 - e^{3ka}\cos a + 3ke^{3ka}\sin a}{e^{2ka}-1}.$$

3. When $a = 0$ in the preceding question, find the values of $\bar{x}$ and $\bar{y}$, and explain the result.

4. Find the centroid of the portion of a parabolic area included between the axis and a radius vector drawn through the focus.

Ans. If $4m$ is the latus rectum, and t the tangent of half the angle between the given radius vector and the axis,

$$\bar{x} = \frac{2m}{3} \cdot \frac{1-\frac{1}{5}t^4}{1+\frac{1}{3}t^2}; \quad \bar{y} = \frac{2m}{3} \cdot \frac{t+\frac{1}{2}t^3}{1+\frac{1}{3}t^2}.$$

165.] **Double Integration.** When the density of the lamina varies from point to point it may be necessary to divide it into infinitesimal portions of the second order instead of strips (triangular or rectangular) whose areas are infinitesimals of the first order.

Thus, suppose that the lamina AOB (fig. 203) is not of uniform density. Then if we break it up into triangular strips, such as POQ, the element of mass will be no longer proportional to the area POQ, or $\frac{1}{2}r^2d\theta$; and, moreover, the centre of mass of the strip will not be $\frac{2}{3}r$ distant from O. 204.

Let a series of circles be described round O as centre, the distance between two successive circles of the series being dr'. These circles will divide the strip POQ into an indefinitely great number of rectangular elements; and if one of these is included between the circles of radii r' and $r'+dr'$, its area will be

$$r'dr'd\theta.$$

If ρ is the density and k the thickness of the lamina at this element, the element of mass will be

$$k\rho r' dr' d\theta.$$

Also the rectangular co-ordinates of the centre of mass of the element are ultimately $r'\cos\theta$ and $r'\sin\theta$.

Now to find the abscissa of the centre of mass we must perform the summations $\int x dm$ and $\int dm$ over the whole area considered.

The contribution to the first of these summations given by the strip POQ is evidently

$$\cos\theta\, d\theta \int_0^r k\rho r'^2 dr ;$$

and the contribution to the second is

$$d\theta \int_0^r k\rho r' dr'.$$

In each of these latter integrals the values k and ρ in terms of r' and θ must be substituted, and the integrations are to be performed on the supposition that θ is constant while r' runs from 0 to r.

The quantity $\cos\theta d\theta \int_0^r k\rho r'^2 dr'$ will then assume the shape $\phi(r, \theta) \,.\, \cos\theta d\theta$. But since the curve AB is given, r is given as a function of θ. Hence this quantity assumes the form $f(\theta) \,.\, \cos\theta d\theta$. This is the final shape of the contribution of the strip POQ. If we wish to find how much is contributed by all the strips of the area, we must integrate $f(\theta) \,.\, \cos\theta d\theta$ from $\theta = AOx$ to $\theta = BOx$.

This double process of integration—first with regard to r', and then with regard to θ—is expressed by the symbols of double integration thus:—

$$\int x\, dm = \int_\alpha^\beta \int_0^r k\rho r'^2 \cos\theta\, dr' d\theta,$$

α and β denoting the angles AOx and BOx.

Hence we obtain

$$\bar{x} = \frac{\int_\alpha^\beta \int_0^r k\rho r'^2 \cos\theta\, dr' d\theta}{\int_\alpha^\beta \int_0^r k\rho r' dr' d\theta}; \quad \bar{y} = \frac{\int_\alpha^\beta \int_0^r k\rho r'^2 \sin\theta\, dr' d\theta}{\int_\alpha^\beta \int_0^r k\rho r' dr' d\theta}.$$

Let it be required, for example, to find the centroid of the area of a cardioid in which the density at a point varies as the n^{th} power of the distance of the point from the cusp.

Here $\rho = \mu r'^n$, and k is constant; therefore, the abscissa being the same for the whole curve as for the half above the axis,

$$\bar{x} = \frac{\int_0^\pi \int_0^r r'^{n+2} \cos\theta \, dr' d\theta}{\int_0^\pi \int_0^r r'^{n+1} \, dr' d\theta}.$$

Integrating first with regard to r', we have

$$\frac{n+2}{n+3} \frac{\int_0^\pi r^{n+3} \cos\theta \, d\theta}{\int_0^\pi r^{n+2} \, d\theta}.$$

But $r = 2a\cos^2\frac{\theta}{2}$. Substituting this value and putting $\frac{\theta}{2} = \phi$, we have

$$\frac{2(n+2)}{n+3} a \,.\, \frac{\int_0^{\frac{\pi}{2}} \cos^{2n+6}\phi \,(2\cos^2\phi - 1)\, d\phi}{\int_0^{\frac{\pi}{2}} \cos^{2n+4}\phi \, d\phi}.$$

These definite integrals are well known. Dividing the numerator and denominator by $\frac{1\,.\,3\,.\,5 \ldots 2n+3}{2\,.\,4\,.\,6 \ldots 2n+4} \cdot \frac{\pi}{2}$, we have

$$\bar{x} = \frac{2(n+2)}{n+3} \left\{ 2 \,.\, \frac{(2n+5)(2n+7)}{(2n+6)(2n+8)} - \frac{2n+5}{2n+6} \right\} a$$

$$= \frac{(n+2)(2n+5)}{(n+3)(n+4)} \cdot a.$$

The centroid evidently lies on the axis of symmetry, or $\bar{y} = 0$.

Examples.

1. Find the centre of mass of a circular sector in which the density varies as the n^{th} power of the distance from the centre.

Ans. $\frac{n+2}{n+3} \cdot \frac{ac}{l}$, where a is the radius of the circle, l the length of the arc, and c the length of the chord, of the sector.

2. Find the position of the centre of mass of a circular lamina in which the density at any point varies as the n^{th} power of the distance from a given point on the circumference.

Ans. It is on the diameter passing through the given point at a distance from this point equal to $\frac{2(n+2)}{n+4}$ a, a being the radius.

Methods of double integration are also often employed when the elements of area are expressed in Cartesian co-ordinates. In this case, let the element of area at a point P, whose co-ordinates are (x', y'), be a small rectangle included between two very close lines parallel to the axis of x and two very close lines parallel to the axis of y. Then the element of area will be $dx'dy'$; and if ρ and k are the density and thickness of the lamina at the element, the element of mass,

$$dm = k\rho\, dx'dy'.$$

Also the co-ordinates of the centre of mass of this element are ultimately x' and y'. Hence

$$\bar{x} = \frac{\iint k\rho x' dx'dy'}{\iint k\rho\, dx'dy'}; \quad \bar{y} = \frac{\iint k\rho y' dx'dy'}{\iint k\rho\, dx'dy'}.$$

A single example will suffice to illustrate this method.

Let it be required to find the centre of mass of a quadrant of an ellipse included by the semi-axes, the density at any point being proportional to the product of the co-ordinates of this point.

Here $\rho = \mu . x'y'$, and since k is supposed constant,

$$\bar{x} = \frac{\iint x'^2 y' dx'dy'}{\iint x'y' dx'dy'}; \quad \bar{y} = \frac{\iint x'y'^2 dx'dy'}{\iint x'y' dx'dy'}.$$

Let the integrations be performed first over a strip parallel to the axis of y. Then we integrate with respect to y', regarding x' as constant, from $y' = 0$ to $y' = y$, the ordinate of a point on the ellipse.

Hence
$$\bar{x} = \frac{\int x'^2 y^2 dx'}{\int x' y^2 dx'}.$$

Here we must substitute the value of y in terms of x', and thus we get

$$\bar{x} = \frac{\int x'^2 (a^2 - x'^2)\, dx'}{\int x' (a^2 - x'^2)\, dx'},$$

in which summations the abscissa x' is to receive all values from 0 to a.

We easily obtain $\frac{8}{15}a$ and $\frac{8}{15}b$ for the co-ordinates of the centre of mass.

Examples may occur in which, although the density of the lamina varies from point to point, the process of double integration can be avoided by the judicious selection of an element of area.

Let it be required to find the centre of mass of a quadrant of an ellipse in which the density at any point varies as the distance of the point from the axis major.

Here by dividing the area into rectangular strips parallel to the axis major, we obtain infinitesimal elements of the *first* order throughout each of which the density is constant. Hence our equations are

$$\bar{x} = \frac{1}{2}\frac{\int x^2 y\, dy}{\int x y\, dy}; \quad \bar{y} = \frac{\int x y^2 dy}{\int x y\, dy}.$$

Making the usual eccentric angle substitutions for x and y, we find

$$\bar{x} = \frac{3}{8}a, \quad \bar{y} = \frac{3\pi}{16}b.$$

166.] **Centroid of a Surface of Revolution.** Let a plane curve AB (fig. 201) revolve round a line Ox (taken as axis of x) and generate a surface. Then the revolution of the elementary arc $PQ\,(= ds)$ generates a portion of surface whose area is $2\pi y\, ds$; and if ρ is the density of the matter in this zone and k its thickness, the element of mass is $2\pi k \rho y\, ds$. Also the centre of mass of the zone is ultimately the point M, whose abscissa is x. Hence the centroid of the surface generated (which obviously lies on the axis of revolution) is at a distance from O given by the equation

$$\bar{x} = \frac{\int k\rho x y\, ds}{\int k\rho y\, ds},$$

the integrations being extended over the whole length of the generating curve.

For example, to find the centroid of the surface of a semi-ellipsoid of revolution round the minor axis, the density of any zone being proportional to its distance from the equatoreal plane, and the thickness being constant:—

The area of a zone at a distance y from the equatoreal plane being $2\pi x\, ds$, the position of the centroid is given by the equation

$$\bar{y} = \frac{\int x y^2 ds}{\int x y\, ds},$$

the integration extending over the arc of a quadrant of the generating ellipse. Using the eccentric angle, we have

$$x = a\cos\phi, \quad y = b\sin\phi, \quad ds = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi,$$

a and b being the semi-axes of the ellipse.

Hence

$$\bar{y} = b\frac{\int_0^{\frac{\pi}{2}} \cos\phi\sin^2\phi\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi}{\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi}.$$

To find the integral in the numerator, put t for $\sin\phi$, and it becomes

$$\int_0^1 t^2\sqrt{b^2+c^2t^2}dt,$$

where $a^2-b^2=c^2$. This, again, is equal to

$$\frac{1}{c^2}\int_0^1 (b^2+c^2t^2-b^2)\sqrt{b^2+c^2t^2}.dt,$$

which $$=\frac{1}{c^2}\int_0^1 (b^2+c^2t^2)^{\frac{3}{2}}dt-\frac{b^2}{c^2}\int_0^1 (b^2+c^2t^2)^{\frac{1}{2}}dt;$$

and this, by making the first integral depend on the second, is easily proved to be

$$\left\{\frac{t(b^2+c^2t^2)^{\frac{3}{2}}}{4c^2}\right\}_0^1-\frac{b^2}{4c^2}.\int_0^1 (b^2+c^2t^2)^{\frac{1}{2}}dt.$$

The integral in this expression is one of the elementary forms in the Integral Calculus. Hence the numerator is

$$\frac{1}{8c^3}\left(2a^3c-ab^2c-b^4\log\frac{a+c}{b}\right).$$

The integral in the denominator is evidently

$$\tfrac{1}{2}\int_0^{\frac{\pi}{2}}\sqrt{b^2+c^2\sin^2\phi}\,.\,d\sin^2\phi,$$

which is equal to $\dfrac{1}{3c^2}(a^3-b^3)$.

Therefore $$\bar{y}=\frac{3b}{8}\cdot\frac{2a^3c-ab^2c-b^4\log\dfrac{a+c}{b}}{c(a^3-b^3)}.$$

For a sphere of radius a the value of $\bar{y}$ is easily proved by direct calculation to be $\frac{2}{3}a$; and the student may exercise himself in the evaluation of indeterminate forms by deducing this from the value of $\bar{y}$ given above. (For this purpose it will be advisable to put $\log\dfrac{a+c}{b}$ into the form $\frac{1}{2}\log\dfrac{a+c}{a-c}$, and expand.)

167.] **Centroid of any Portion of a Spherical Surface.** Let dS denote any portion of a spherical surface, and let $d\Sigma$ denote its projection on any plane passing through the centre of the sphere. Then, if this plane be taken as that of xy, and if z denote the distance of the centroid of the element dS from the plane, the distance of the centroid of any portion of the spherical surface from the plane is given by the equation

$$\bar{z}=\frac{\int z\,dS}{\int dS}, \tag{1}$$

the integration being extended over the whole portion of the spherical surface considered.

Now if r is the radius of the sphere, the cosine of the angle between the tangent plane to the sphere at the element dS and the plane of xy is $\frac{z}{r}$; therefore

$$d\Sigma = \frac{z}{r} dS. \qquad (2)$$

Hence $\int z\,dS = r \int d\Sigma = r\Sigma$, Σ denoting the projection of the whole spherical area considered; and making this substitution in (1), we have

$$\bar{z} = r\frac{\Sigma}{S}, \qquad (3)$$

where S is the area of that portion of the sphere whose centroid is required.

Equation (1) gives, of course, the distance of the centroid of *any* surface whose element is dS from the plane of xy; and it is clear that if the surface is generated by the motion of a sphere of constant radius whose centre moves along any curve in the plane of xy, the cosine of the angle between the tangent plane at the element dS and the plane of xy will still be $\frac{z}{r}$, since the given surface and the generating sphere have the same tangent plane. Hence equation (2) holds in this case and therefore also equation (3).

168.] **Centroid of any Surface.** Let dS denote an element of any surface, $d\Sigma$ the projection of this element on the plane of xy, and γ the angle between the plane of xy and the tangent plane to the surface at the element dS. Then if z is the distance of the centroid of dS from the plane of xy, we have

$$\bar{z} = \frac{\int z\,dS}{\int dS},$$

$$= \frac{\int z \sec\gamma \,.\, d\Sigma}{\int \sec\gamma \,.\, d\Sigma}.$$

It is not unusual to suppose the element dS cut off from the surface in the following manner.

Let m (fig. 205) be a point in the plane xy whose co-ordinates are x', y'; let mn be drawn parallel to the axis of x and equal to dx'; let mq be parallel to the axis of y and equal to dy'; and complete the rectangle $mnpq$. On the base $mnpq$ describe a prism whose edges, Mm, Nn, Pp, Qq are parallel to the axis of z. This prism will intercept on the given surface an element, *MNPQ*, which is dS. The rectangular projection, $d\Sigma$, is then *mnpq* whose area is $dx'dy'$. Substituting this value in the above equation, we have

$$z = \frac{\iint z \sec\gamma\, dx'\,dy'}{\iint \sec\gamma\, dx'\,dy'},$$

the integrations being extended over the whole projection of the given surface on the plane xy.

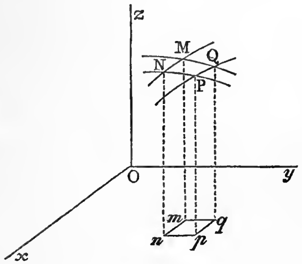


Fig. 205.

It easily follows that *the centroid of the projection (orthogonal or oblique) of any plane area on any plane is the projection of the centroid of the area.*

Take the plane on which the given area is projected as the plane of xy; let ω be the angle between this plane and the plane of the area, and let $\bar{x}$, $\bar{y}$ be co-ordinates of the centroid of the given area. Then

$$\bar{x} = \frac{\int x\,dS}{\int dS} = \frac{\int x \sec\omega \,.\, d\Sigma}{\int \sec\omega \,.\, d\Sigma}$$

$$= \frac{\int x\,d\Sigma}{\int d\Sigma},$$

since ω is the same for all elements. But the co-ordinate of the centroid of the projection is evidently given by this equation. Therefore, &c.; and a similar proof obviously holds for an oblique projection, because at all points of the given area the ratio of dS to $d\Sigma$ is constant.

Examples.

1. A section of a sphere is made by any two parallel planes; prove that the centroid of the spherical surface included is midway between them.

This is very easily proved either by direct calculation or by the application of the result of last Article. Collignon (*Statique*, p. 295) gives an elegant geometrical demonstration which depends on the fact that if a cylinder is circumscribed to a sphere along any one of its great circles, the portion of the area of the cylinder included between any two planes at right angles to its axis is equal to the portion of the area of the sphere included by these planes. By taking indefinitely close planes it follows that the spherical area may be transferred to the cylinder, and the centroid of any portion of a cylindrical area cut off by planes perpendicular to the axis is evidently midway between these planes.

Cor. The centroid of the surface of a hemisphere is at a distance equal to half the radius from the centre.

2. To find the centroid of a spherical triangle.

Let ABC be any spherical triangle, and O the centre of the sphere. Produce the sides AC and AB until they become quadrants, AE and AD, and draw the arc DE of a great circle.

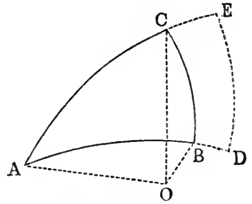


Fig. 206.

We shall find the distance of the centroid from the plane EOD, which is perpendicular to the line OA.

The projection of the area ABC on this plane is evidently the same as the projection of the sector, COB. Now if p_1 is the perpendicular arc from A on the side BC, the angle between the planes COB and EOD is $90°-p_1$; also the area of the sector COB is $\frac{1}{2}ar$, a being the length of the side BC and r the radius of the sphere. Hence if Σ denote the projection of the area of the triangle on the plane EOD,

$$\Sigma = \tfrac{1}{2}ar\sin p_1;$$

and if A, B, C denote the circular measures of the angles of the triangle, and S its area, $S = r^2(A+B+C-\pi)$.

Hence, by (3) of last Article, if x denote the distance of the centroid from the plane,

$$x = \tfrac{1}{2}\cdot\frac{a\sin p_1}{A+B+C-\pi}.$$

It is evident that x is the distance from O of the projection of the centroid on the line OA. Its projections on the lines OB and OC are obtained by writing b and p_2, c and p_3 instead of a and p_1 in this equation.

3. To find the centroid of the surface of a nearly spherical semi-ellipsoid cut off by the plane of the two greater axes.

Let the axes in order of magnitude be a, b, c, and let

$$\frac{a^2-c^2}{a^2} = k^2, \quad \frac{b^2-c^2}{b^2} = k'^2.$$

Now if $dx'dy'$ is the projection on the plane xy (which is the base of the semi-ellipsoid) of an element of surface, dS, we have

$$dS = \frac{c^2dx'dy'}{pz},$$

p being the perpendicular from the centre on the tangent plane at the element, and z the distance of the element from the plane of xy. Hence, S denoting the surface of the semi-ellipsoid, we have

$$S.\bar{z} = c^2\iint\frac{dx'dy'}{p}.$$

Again, $$\frac{1}{p^2} = \frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z^2}{c^4} = \frac{1}{c^2}\left(1-\frac{k^2x'^2}{a^2}-\frac{k'^2y'^2}{b^2}\right).$$

Therefore, rejecting all powers of k and k' beyond the second,

$$S.\bar{z} = c\iint\left(1+\frac{k^2x'^2}{2a^2}+\frac{k'^2y'^2}{2b^2}\right)dx'\,dy'.$$

Integrating from $x' = -x$ to $x' = x$, the co-ordinates of a point on the circumference of the base being x, y, we have

$$S.\bar{z} = 2c\int\left(x+\frac{k^2x^3}{6a^2}+\frac{k'^2xy^2}{2b^2}\right)dy.$$

Expressing x and y in terms of the eccentric angle, and integrating over the entire circumference, we have

$$S.\bar{z} = \pi abc\left(1-\frac{k^2+k'^2}{8}\right)$$

$$= \pi c^3\left\{1+\frac{3}{8}(k^2+k'^2)\right\}.$$

Now (Williamson's *Integral Calculus*),

$$S = \pi a^2b^2c^2\left[\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{(b^2\sin^2\theta+c^2\cos^2\theta)^{\frac{3}{2}}(a^2\sin^2\theta+c^2\cos^2\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{(a^2\sin^2\theta+c^2\cos^2\theta)^{\frac{3}{2}}(b^2\sin^2\theta+c^2\cos^2\theta)^{\frac{1}{2}}}\right],$$

which is easily proved to be $2\pi c^2\{1+\frac{4}{3}(k^2+k'^2)\}$.

Hence finally, $$\bar{z} = \frac{c}{2}\left\{1-\frac{23}{24}(k^2+k'^2)\right\}.$$

4. A parabola revolves round its axis; find the centroid of a portion of the surface between the vertex and a plane perpendicular to the axis at a distance from the vertex equal to $\frac{3}{4}$ of the latus rectum.

Ans. Its distance from the vertex $= \frac{29}{70}$ (latus rectum).

5. Find the centroid of the surface generated by the revolution of a cycloid round its axis.

Ans. It is on the axis at a distance $\frac{2(15\pi-8)}{15(3\pi-4)}\cdot a$ from the vertex, a being the radius of the generating circle.

6. Prove that the centroid of the lateral surface of the frustum of a right cone or pyramid lies in a plane whose distance from the base is $\frac{p+2p'}{3(p+p')}\cdot h$, where p and p' are the perimeters of the base and upper section, and h the height of the frustum.

169.] **Centre of Mass of a Solid of Revolution.** If the curve AB (fig. 201) revolve round Ox, the rectangular area $PQNM$ will generate a cylindrical volume equal to $\pi.PM^2.MN$, or πy^2dx. Hence if the density of the solid is uniform, we have for the position of its centre of mass (which obviously lies on Ox)

$$\bar{x} = \frac{\int xy^2\,dx}{\int y^2\,dx},$$

the integrations being extended over the whole of the area, *CABD*, of the revolving curve.

If the density varies, the element of mass may require to be taken differently. If the density is a function of x alone, i.e., if it is the same all over the rectangular strip *PQNM*, the volume may be broken up as above, and the element of mass $= \pi\rho y^2 dx$. Hence we shall have, in this case,

$$\bar{x} = \frac{\int \rho x y^2 dx}{\int \rho y^2 dx}.$$

Suppose the density to vary as y alone. Then if we take a small rectangular area, $dx' dy'$, at a point whose co-ordinates are x', y', this area will generate an element of volume equal to $2\pi y' dx' dy'$; therefore the element of mass $= 2\pi\rho y' dx' dy'$ and we have

$$\bar{x} = \frac{\iint \rho x' y' dx' dy'}{\iint \rho y' dx' dy'}.$$

The integrations are to be performed first from $y' = 0$ to $y' = y$, the ordinate of a point P on the bounding curve; and then from $x' = OC$ to $x' = OD$.

As an example, let the curve AB be a quadrant of a circle of which OA and OB are diameters, and let it be required to find the centre of mass of the solid hemisphere generated by the revolution of this quadrant round OB (taken as axis of x), firstly when the density is uniform; secondly when it is constant over a section perpendicular to OB and proportional to the distance of this section from the centre; and thirdly when it is the same at the same distance from OB, and proportional to this distance.

Firstly, we have $\bar{x} = \frac{\int xy^2 dx}{\int y^2 dx}$. Putting $x = r\cos\theta$, $y = r\sin\theta$, where r is the radius of the circle, and integrating between $\theta = 0$ and $\theta = \frac{\pi}{2}$, we have

$$\bar{x} = \frac{3}{8} r. \tag{1}$$

Secondly, since $\rho = \mu x$, we have $\bar{x} = \frac{\int x^2 y^2 dx}{\int xy^2 dx}$, which easily gives

$$\bar{x} = \frac{8}{15} r. \tag{2}$$

Thirdly, $\rho = \mu y'$, therefore

$$\bar{x} = \frac{\iint x' y'^2 dx' dy'}{\iint y'^2 dx' dy'} = \frac{\int xy^3 dx}{\int y^3 dx},$$

and the previous substitutions for x and y give

$$\bar{x} = \frac{16}{15\pi} r. \tag{3}$$

In this case the double integration might have been avoided by breaking the area up into rectangles parallel to the axis of x.

The student will do well in such examples as this to check his results as much as possible by a common-sense view of the question. Thus, having proved that the distance of the centre of mass of a homogeneous hemisphere from the centre is $\frac{3}{8}r$, it is clear that when the density of a section is directly proportional to its distance from the centre, the centre of mass of the hemisphere must be at a distance from the centre $> \frac{3}{8}r$, since the matter is most dense in the space remote from the centre; while in the third case above, since the ordinates of the portion of the curve near A are greater than those of the portion near B, and since the density increases with the ordinate, it is evident that the centre of mass must be nearer to the centre than in the homogeneous hemisphere.

The most advantageous method of breaking up a mass of varying density into elements depends entirely on the law of variation of the density, and while all these methods are embraced in the rule of Art. 160, it would be impossible to give formulæ suited to all cases.

Laplace, by assuming the change of the pressure from stratum to stratum of the earth to be proportional to the change in the square of the density, proves that if the strata of uniform density are spherical, the density of a stratum of radius x is given by the equation

$$\rho = \frac{a\rho_0}{\mu} \cdot \frac{\sin \dfrac{\mu x}{a}}{x},$$

a being the radius of the earth, ρ_0 the density of the centre, and μ a constant number.

Let it be required to find the centre of mass of a hemisphere whose density follows this law.

Here the element of mass of uniform density is the stratum included between the hemispheres of radii x and $x + dx$. Hence

$$dm = 2\pi\rho x^2 dx$$

$$= 2\pi a\rho_0 \frac{x}{\mu} \sin \frac{\mu x}{a}\, dx.$$

Also the distance of the centre of mass of this stratum from the centre is $\frac{x}{2}$ (Example 1, p. 270). Hence, the axis of x being the diameter perpendicular to the base of the hemisphere, the distance of the centre of mass from the centre is given by the equation

$$\bar{x} = \frac{1}{2}\,\frac{\int_0^a x^2 \sin\frac{\mu x}{a}\,dx}{\int_0^a x \sin\frac{\mu x}{a}\,dx}$$

$$= a\,.\,\frac{(2-\mu^2)\cos\mu + 2\mu\sin\mu - 2}{2\mu\,(\sin\mu - \mu\cos\mu)},$$

as will be easily found. When $\mu = 0$ the hemisphere is of uniform density, and the student will see that this value of $\bar{x}$ becomes $\frac{3}{8}a$, in accordance with our previous result.

Examples.

1. Find the centre of mass of a hemisphere in which the density is proportional to the n^{th} power of the distance from the centre.

Ans. It is at a distance $= \dfrac{n+3}{n+4}\cdot\dfrac{a}{2}$ from the centre, a being the radius of the hemisphere.

2. Find the centre of mass of a portion of a paraboloid of revolution cut off by a plane perpendicular to its axis.

Ans. If h is the distance of the plane of section from the vertex, $\bar{x} = \frac{2}{3}h$.

3. Find the centre of mass of a semi-ellipsoid of revolution round the minor axis, the density at any point being proportional to its distance from the base which is the plane perpendicular to the axis of revolution.

Ans. $y = \dfrac{8}{15}b$, where b is the semi-minor axis.

4. An ellipsoid of revolution round the minor axis is cut by a plane passing through this axis; find the centre of mass of the portion included between one semi-ellipsoid thus cut off and the concentric hemisphere whose diameter is the minor axis.

Ans. If a and b are the axes major and minor of the generating ellipse, the required centre of mass is on the major axis at a distance equal to $\dfrac{3}{8}\cdot\dfrac{a^2+ab+b^2}{a+b}$ from the centre.

Verify this result in two obvious cases.

170.] **Centre of Mass of any Solid.** In the solid take any point, P, whose co-ordinates are x, y, z, and also a close point, Q, whose co-ordinates are $x+dx$, $y+dy$, $z+dz$. Then evidently the volume of the parallelopiped whose diagonal is PQ and whose edges are parallel to the axes of co-ordinates is $dx\,dy\,dz$;

and if ρ is the density of the body at P the element of mass at P is $\rho dx\,dy\,dz$.

Hence the co-ordinates of the centre of mass of the solid are given by the equations

$$\bar{x} = \frac{\iiint \rho x\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint \rho y\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}, \quad \bar{z} = \frac{\iiint \rho z\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},$$

the integrations being extended over the whole solid.

It may not be necessary to take infinitesimal elements of volume of the third order. From what has preceded, the student will have learned that the best mode of breaking up the given mass into elements depends entirely on the law of density which prevails.

In many cases the symmetry of the solid enables us to simplify the problem by choosing elements of volume which are infinitesimals of the *first* order only.

The various elements of volume which it may be necessary to take are exemplified in the following problems.

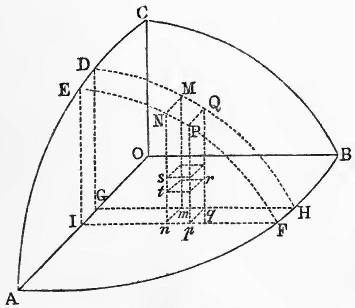


Fig. 207.

Find the centre of mass of the eighth part of an ellipsoid, ABC (fig. 207) included between its three principal planes—

(1) When the density at any point is simply a function of its distance from the principal plane BC (plane of yz).

(2) When the density at any point is a function of its distances from the two principal planes AC and BC (planes of xz and yz).

(3) When the density at any point is a function of its distances from the three principal planes.

In the first case, the density will be constant over a section DH perpendicular to OA. Hence, taking two such sections, DH and EF, at a distance dx from each other, the density of the solid between them may be considered uniform, and this portion of the solid may be taken as the element of mass.

In the second case, the density will be constant throughout a portion of the body in which x and y are constant; that is, along a perpendicular to the plane AB; and the element of mass may be taken as the prism $NQnq$, the area of whose base is $dx\,dy$, and which intersects the bounding surface in the area $NMQP$.

In the third case, the density is not the same at any two points, and the element of mass must be taken a small rectangular prism, str, whose volume is $dx\,dy\,dz$.

Examples.

1. In the problem just discussed find the centre of mass when the density at any point is proportional to its distance from the plane BC.

Here $\rho = \mu x$; also, the equation of the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the ellipse DH satisfies the equation

$$\frac{y^2}{b^2\left(1-\dfrac{x^2}{a^2}\right)} + \frac{z^2}{c^2\left(1-\dfrac{x^2}{a^2}\right)} = 1,$$

which shows that the axes GH and GD are

$$b\sqrt{1-\frac{x^2}{a^2}} \quad\text{and}\quad c\sqrt{1-\frac{x^2}{a^2}},$$

respectively. Hence, IG being $= dx$, the element of mass is

$$\pi\mu\, bc\, x\left(1-\frac{x^2}{a^2}\right)dx;$$

and since the centre of mass of this element is ultimately a point whose co-ordinates are

$$x,\quad \frac{4b}{3\pi}\sqrt{1-\frac{x^2}{a^2}},\quad\text{and}\quad \frac{4c}{3\pi}\sqrt{1-\frac{x^2}{a^2}}$$

(see Ex. 3, p. 257), we have

$$\bar{x} = \frac{\displaystyle\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)dx}{\displaystyle\int_0^a x\left(1-\frac{x^2}{a^2}\right)dx} = \frac{8}{15}a;$$

and

$$\bar{y} = \frac{4b}{3\pi}\cdot\frac{\displaystyle\int_0^a x\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx}{\displaystyle\int_0^a x\left(1-\frac{x^2}{a^2}\right)dx} = \frac{16b}{15\pi};$$

the value of $\bar{z}$ being, of course, $\dfrac{16c}{15\pi}$.

2. If the density at any point of the ellipsoid is μxy, find the centre of mass.

Taking a prismatic element of volume $NQnq$, the element of mass is

$$\mu\, xyz\,dx\,dy,$$

z being the height, Mm, of the prism.

The co-ordinates of M being x, y, z, those of the centre of mass of this prism are evidently x, y, $\dfrac{z}{2}$. Hence

$$\bar{x} = \frac{\iint x^2yz\,dx\,dy}{\iint xyz\,dx\,dy},\quad \bar{y} = \frac{\iint xy^2z\,dx\,dy}{\iint xyz\,dx\,dy},\quad \bar{z} = \tfrac{1}{2}\frac{\iint xyz^2\,dx\,dy}{\iint xyz\,dx\,dy}.$$

The integrations may be performed, first with regard to y, from $y=0$ to $y=GH$; and then with regard to x, from $x=0$ to $x=OA$.

$$\text{Now,}\qquad \iint xyz\,dx\,dy = c\iint xy\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dx\,dy;$$

and, integrating first with regard to y, we have

$$\int_0^{GH} y\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dx = \frac{b^2}{3}\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}},$$

since from the equation of the ellipse AB, the value GH of y makes $1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$ vanish. Hence

$$\iint xyz\,dx\,dy = \frac{b^2c}{3}\int_0^a x\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx = \frac{a^2b^2c}{15}.$$

In the same way,

$$\iint x^2yz\,dx\,dy = \frac{b^2c}{3}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx,$$

which, by putting $x = a\cos\phi$, is easily seen to be $\dfrac{\pi a^3b^2c}{96}$. Hence $\bar{x} = \dfrac{5\pi}{32}\cdot a$, and $\bar{y} = \dfrac{5\pi}{32}\cdot b$; and it is easily found that $\bar{z} = \dfrac{5}{8}c$.

3. If the density at any point in the solid is proportional to the product of the co-ordinates of the point, find the centre of mass.

Here, at any point we have $\rho = \mu\,.\,xyz$, and the element of mass being $\mu\, xyz\,dx\,dy\,dz$, we have

$$\bar{x} = \frac{\iiint x^2yz\,dx\,dy\,dz}{\iiint xyz\,dx\,dy\,dz},$$

with similar values of $\bar{y}$ and $\bar{z}$. If we first integrate from $z=0$ to

$z = mM$ (fig. 207), we shall have the contribution of the prism $NQnq$ to the summation. Integrating, then, with respect to z, considering x and y constant, we have

$$\iiint x^2yz\,dx\,dy\,dz = \tfrac{1}{2}\iint x^2y\,(mM)^2 .\, dx\,dy$$
$$= \frac{c^2}{2}\iint x^2y\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy,$$

since M is a point on the bounding surface of the ellipsoid. Let this latter integration be first performed with respect to y, considering x constant, from $y = 0$ to $y = GH$, and we shall then have the contribution of the mass contained between the sections DH and EF.

Now $$\int_0^{GH} y\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dy = \frac{b^2}{4}\left(1 - \frac{x^2}{a^2}\right)^2.$$

Hence $$\iiint x^2yz\,dx\,dy\,dz = \frac{b^2c^2}{8}\int_0^a x^2\left(1 - \frac{x^2}{a^2}\right)^2 dx = \frac{a^3b^2c^2}{105},$$

as easily appears by putting $x = a\cos\phi$.

It will be found without difficulty that $\iiint xyz\,dx\,dy\,dz = \dfrac{a^2b^2c^2}{48}$.

Hence $$\bar{x} = \frac{16}{35}a,\ \bar{y} = \frac{16}{35}b,\ \text{and}\ \bar{z} = \frac{16}{35}c.$$

4. Find the centre of mass of the portion of the elliptic paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2\dfrac{z}{c}$ included between the planes xz and yz and a plane perpendicular to the axis of z at a distance h from the vertex.

Ans. $$\bar{x} = \frac{16a}{15\pi}\sqrt{\frac{2h}{c}},\ \bar{y} = \frac{16b}{15\pi}\sqrt{\frac{2h}{c}},\ \bar{z} = \frac{2}{3}h.$$

5. At each point, M, in the semi-axis major of an ellipse is drawn a line perpendicular to the plane of the ellipse, its length being proportional to the distance of M from the centre; the extremity of this perpendicular is joined to the point P on one quadrant of the ellipse such that PM is perpendicular to the axis major. Find the centroid of the volume thus generated.

Ans. If at any distance, x, from the centre the perpendicular to the plane of the ellipse is kx, and if the axes of x, y, and z are the axes of the ellipse and a perpendicular to them, we have

$$\bar{x} = \frac{3\pi a}{16},\ \bar{y} = \frac{b}{4},\ \bar{z} = \frac{\pi k a}{16}.$$

6. Through a diameter of the base of a right cone are drawn two planes cutting the cone in parabolas; find the centroid of the volume of the cone included between these planes and the vertex.

Ans. It is on the axis at a distance from the vertex equal to $\frac{3}{5}$ of height of cone.

7. A plane cuts off a constant volume from an ellipsoid; find the locus of the centroid of the portion cut off.

Ans. An ellipsoid similar to the given one, and similarly placed (see Example 2, p. 257, the theorem of which is equally applicable to surfaces).

171.] **Polar Elements of Mass.** Let fig. 208 represent the portion of the volume of a solid included between its bounding surface and three rectangular co-ordinate planes. Then the solid may be broken up into elements in the following manner :—

(1) Through the axis of z draw two close planes cutting the bounding surface in curves zR and zS (called *meridians*); and let the angles ROx and SOx be denoted by ϕ and $\phi + d\phi$, respectively.

(2) Round the axis of z describe two right cones with the semi-vertical angles zOP and zOQ, equal to θ and $\theta + d\theta$, respectively.

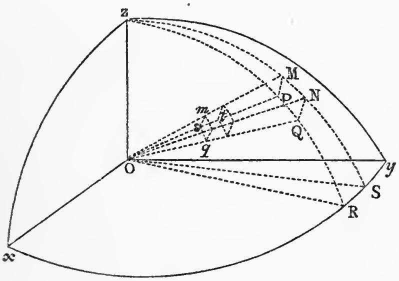


Fig. 208.

(3) With O as centre, describe two close spheres whose radii, Os and Ot, are equal to r and $r + dr$, respectively.

These planes, cones, and spheres will then determine the small rectangular parallelopiped $mstq$, whose volume $= ms \times sq \times st$.

Now, perpendiculars from m and s on Oz will each be equal to $Os \,.\, \sin zOs$, or $r \sin \theta$, and they will include an angle equal to ROS, or $d\phi$: therefore $ms = r \sin \theta \, d\phi$. Also,

$$sq = Os \,.\, \sin sOq = r d\theta \text{; and } st = dr.$$

Therefore the volume of the elementary parallelopiped $= r^2 \sin \theta dr \, d\theta \, d\phi$; and if ρ is the density of the solid at s, the element of mass is

$$\rho r^2 \sin \theta dr \, d\theta \, d\phi.$$

Again, the co-ordinates of the centre of mass of this element are ultimately the same as those of s; therefore they are

$$r \sin\theta \cos\phi, \quad r \sin\theta \sin\phi, \quad \text{and} \quad r\cos\theta;$$

and for the centre of mass of any finite portion of the solid we have

$$\bar{x} = \frac{\int\int\int \rho\, r^3 \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi}{\int\int\int \rho\, r^2 \sin\theta\, dr\, d\theta\, d\phi},$$

$$\bar{y} = \frac{\int\int\int \rho\, r^3 \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi}{\int\int\int \rho\, r^2 \sin\theta\, dr\, d\theta\, d\phi},$$

$$\bar{z} = \frac{\int\int\int \rho\, r^3 \sin\theta \cos\theta\, dr\, d\theta\, d\phi}{\int\int\int \rho\, r^2 \sin\theta\, dr\, d\theta\, d\phi},$$

the limits of integration being determined by the figure of portion of the solid considered.

The angles θ and ϕ are sometimes called the *colatitude* and *longitude*, respectively.

Examples.

1. Find the centre of mass of a portion of a solid sphere contained in a right cone whose vertex is the centre of the sphere, the density of the solid varying as the n^{th} power of the distance from the centre.

Take the axis of the cone as that of z, and any plane through it as that from which longitude is measured. Then it is clear that $\bar{x} = \bar{y} = 0$, and we have

$$\bar{z} = \frac{\int\int\int r^{n+3} \sin\theta \cos\theta\, dr\, d\theta\, d\phi}{\int\int\int r^{n+2} \sin\theta\, dr\, d\theta\, d\phi}.$$

Performing the integration first with respect to r, considering θ and ϕ constant, from $r = 0$ to $r = a$, the radius of the sphere, we have

$$\bar{z} = \frac{n+3}{n+4}\, a \frac{\int\int \sin\theta \cos\theta\, d\theta\, d\phi}{\int\int \sin\theta\, d\theta\, d\phi}.$$

Performing the integration now with respect to ϕ, the longitude, which runs from 0 to 2π, we have

$$\bar{z} = \frac{n+3}{n+4}\, a \frac{\int \sin\theta \cos\theta\, d\theta}{\int \sin\theta\, d\theta}.$$

If $\alpha =$ the semi-vertical angle of the cone, the limits of θ are 0 and α.

Therefore
$$\bar{z} = \frac{n+3}{n+4} \cdot \frac{a}{2}(1 + \cos\alpha).$$

2. Find the centre of mass of a prism whose base is a given spherical

triangle and whose vertex is the centre of the sphere on which the triangle is described.

Let O (fig. 206) be the centre of the sphere, and take OC as axis of z. From C draw the perpendicular p_3 to the base AB, and let R be the radius of the sphere.

The value of $\bar{z}$ given as a triple integral may be modified in the present case.

Let dS denote any small element of area at any point on CP; then the volume of a cone whose base is this element and vertex the centre of the sphere is $\frac{1}{3}RdS$, and the distance of its centre of mass from the plane of xy is (Art. 158) $\frac{3}{4}R\cos\theta$. Hence

$$\bar{z} = \frac{3}{4}R\frac{\int \cos\theta\, dS}{\int dS}.$$

Now $\cos\theta . dS$ is the projection of the element dS on the plane of xy; therefore the numerator is the projection of the whole area ABC on this plane, which, as in Example 2, p. 271, is $\frac{1}{2}cR\sin p_3$. Hence,

$$\bar{z} = \frac{3}{8}\frac{c\sin p_3}{A+B+C-\pi}.$$

3. A cardioid revolves round its axis; find the centre of mass of the solid generated.

Ans. It is at a distance from the cusp equal to $\frac{8}{15}$ (axis).

172.] **Theorems of Pappus.** *If a plane area revolve through any angle round a line in its plane, the volume generated is equal to the area of the revolving figure multiplied by the length of the path described by its centroid.*

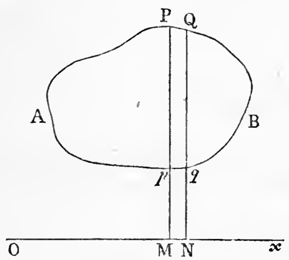


Fig. 209.

Let AB (fig. 209) be the revolving figure, and Ox the line about which it revolves. Let the area be broken up into an indefinitely great number of rectangular strips, such as $PQqp$, by lines perpendicular to Ox. Then the volume generated by this strip in revolving through an angle ω is evidently equal to

$$\frac{\omega}{2\pi}\cdot\pi(PM^2-pM^2).MN,$$

or

$$\tfrac{1}{2}\omega(y_2{}^2-y_1{}^2)\,dx,$$

denoting PM, pM, and MN by y_2, y_1, and dx. Hence if V denote the whole volume generated,

$$V = \tfrac{1}{2}\omega\int(y_2^2 - y_1^2)\,dx.$$

Now the distance of the centroid of the strip from Ox is $\frac{y_2+y_1}{2}$, and the area of the strip is $(y_2-y_1)\,dx$. Hence, denoting these quantities by y and dA respectively,

$$V = \omega\int y\,dA$$
$$= \omega A\,.\,\bar{y},$$

A denoting the whole revolving area and $\bar{y}$ the ordinate of its centroid. Now in revolving through the angle ω, the centroid of the area describes a circular arc whose length is $\omega\bar{y}$. Hence the theorem is proved.

If the axis Ox intersects the revolving figure, the theorem still applies with the convention that the volumes generated by the portions of the figure at opposite sides of Ox are affected with opposite signs.

Again *if the arc of any plane curve revolve through any angle round a line in its plane, the area of the surface generated is equal to the length of the revolving arc multiplied by the length of the path described by its centroid.*

For, the surface generated is

$$\omega\int y\,ds, \quad \text{or} \quad \omega\,L\,.\,\bar{y},$$

L being the whole length of the revolving arc and $\bar{y}$ the ordinate of its centroid. As before, $\omega\bar{y}$ is the length of the circular arc described by the centroid of the revolving arc, and the theorem is evidently proved.

If the revolving arc intersects the line Ox, the theorem is true, with the previous convention of signs.

173.] **Extension of the Theorems of Pappus.** The previous theorems can be easily extended to the case in which the plane of the revolving figure, instead of revolving round a fixed line, *rolls without sliding on any developable surface*, and the first theorem will then become—

If the plane of any plane area rolls without sliding on a developable surface, the volume generated by the area in moving from one position to another will be equal to the area of the revolving figure multiplied by the length of the path described by its centroid.

A similar enunciation gives the second theorem.

These propositions are evidently true, because in an indefinitely small motion the figure is revolving round a generating line of the developable, and for such a small motion the theorem of Pappus gives the volume generated equal to the area × small space described by its centroid. Taking the sum of all such elements of volume from one position of the figure to another, we have the theorem of this Article.

It is clear also that the theorems hold in the case of a plane area which moves in such a manner as to be always normal to the path described by its centroid. For the area may at any instant be considered as revolving round the line of intersection of two consecutive normal planes of the curve which the centroid describes, and the theorems are then directly applicable.

174.] **Volume of a Truncated Cylinder or Prism.** Let A and B denote the sections of a cylinder or prism made by any two planes. Through any line L passing through the centroid, G, of B draw any plane, B', inclined at an indefinitely small angle to B. Then G is the centroid of the section B', since this section is the projection of B made by lines parallel to the generators of the cylinder or edges of the prism, and since (Art. 168) the centroid of the projection of any plane surface is the projection of its centroid. Also the area of the section B' differs from that of B by an infinitestimal of the second order. Hence the theorems of Pappus apply, and we may consider that the area B has revolved round the line L through a small angle. But the space described by its centroid is zero; therefore the volume between the sections B and B' on one side of the line $L =$ the volume between them on the other side; in other words, infinitesimals of the second order being neglected, the volume of the prism or cylinder contained between the sections A and B is equal to that contained between the sections A and B'. Allowing B' to revolve again about L through a small angle, the same reasoning applies, and we see, finally, that for the sections A and B may be substituted *any* two passing through their respective centroids, and the included volume will be unaltered. Let two parallel sections each perpendicular to the axis of the prism or cylinder be substituted, and the included volume will be

$$\Omega \, . \, h,$$

where Ω is the area of either normal section and h the distance between them.

175.] **Equilibrium of a Heavy Body on a Horizontal Plane.**

When an indeformable body rests on a horizontal plane, the contact taking place at several points, either continuous or not, it is kept in equilibrium by two forces—namely, its own weight and the reaction of the plane. The condition necessary and sufficient for the equilibrium of such a body is that these two forces must be equal and opposite. Now this will be impossible unless the points of contact of the body with the plane can be so connected by right lines as to form a polygon within the area of which the vertical through the centre of gravity of the body intersects the plane. For, whether the plane be rough or smooth, resolve all the reactions at the points of contact vertically. Then it is evident that the resultant of the system of parallel vertical forces at the points of contact must necessarily fall within some polygon whose vertices are these points; therefore, &c.

The student must be careful to observe that this condition, though necessary in the case of a deformable system, is not sufficient (see Article 94, p. 115). Thus, in Example 14, p. 179, it is not true that the deformable system of two bars, AB and BC, will rest in any position in which their common centre of gravity falls between the props.

Examples.

1. To find the volume and surface of a tore.

(A tore is a surface generated by the revolution of a circle round a line in its plane.)

Let r be the radius of the circle, and c the distance of its centre from the axis of revolution. Then the volume of the tore is evidently $\pi r^2 \times 2\pi c$, or $2\pi^2 c r^2$; and the surface is $2\pi r \times 2\pi c$, or $4\pi^2 cr$.

2. A triangle revolves round a line in its plane; find the volume generated.

Ans. If the distances of the vertices from the lines are x_1, x_2, x_3, and A the area of the triangle, the volume $= \dfrac{2\pi A}{3}(x_1 + x_2 + x_3)$.

3. From the Theorems of Pappus deduce the volume and surface of a frustum of a right cone.

(Consider a trapezium one side of which is perpendicular to the two parallel sides.)

4. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances between the

extremities of the successive cards will form a harmonic progression. (Walton, p. 183.)

5. A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting on a horizontal plane; all the blocks above the lowest are then twisted in the same direction about an edge of the column, first the highest, then the two highest, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines. (Walton, *ibid.*)

CHAPTER XII.

THE PRINCIPLE OF VIRTUAL WORK APPLIED TO ANY SYSTEM OF BODIES.

176.] **Forces Applied to a Particle.** It has been shown in Art. 136, p. 214, that the resultant of any number of forces applied to a particle may be represented by the side required to close the polygon of the forces. And whether the polygon $OP_1 P_2 \ldots P_n$ be plane or gauche, it is clear (as in p. 67) that the sum of the projections of the sides, taken in order, along any line OA, is equal to zero.

Let the projections of the sides be denoted by $Q_1, Q_2, \ldots Q_n$. Then $Q_1 + Q_2 + \ldots + Q_n = 0$. Multiplying this by OA, an arbitrary length along the line OA, we have

$$Q_1 \cdot OA + Q_2 \cdot OA + \ldots + Q_n \cdot OA = 0.$$

But if p_1 is the projection of OA along OP_1, we have (see p. 67)

$$Q_1 \cdot OA = OP_1 . p_1.$$

If, then, the sides $OP_1, P_1 P_2, \ldots$ be denoted by $P_1, P_2, \ldots$ we have

$$P_1 . p_1 + P_2 . p_2 + \ldots + P_n . p_n = 0;$$

and if the sides represent forces, each term in this equation is the virtual work of the corresponding force for the displacement OA. Since the resultant, R, of $n-1$ of the forces is $-P_n$, we have

$$R . r = P_1 . p_1 + P_2 . p_2 \ldots;$$

and if the displacement is small, this equation is written (as in p. 71)

$$R\delta r = P_1 \delta p_1 + P_2 \delta p_2 + \ldots. \qquad (1)$$

In particular, if X, Y, Z denote the rectangular components of R, we have

$$R\delta r = X\delta x + Y\delta y + Z\delta z. \qquad (2)$$

177.] **Extension to any Number of Connected Particles.** If two particles, m_1 and m_2, are connected by a rigid inextensible rod, and are in equilibrium under the action of forces, $P_1, Q_1, \ldots$

applied to m_1 and $P_2, Q_2, \ldots$ applied to m_2, it is evident (as in p. 118) that the force arising from the connexion acts in the line joining m_1 to m_2. If, then, this force be denoted by T, and the distance between the particles by r, we have for the equilibrium of m_1

$$P_1 \delta p_1 + Q_1 \delta q_1 + \ldots + T \delta_1 r = 0,$$

$\delta_1 r$ denoting the change in r arising from an arbitrary small displacement of m_1. The equation of equilibrium of m_2 is

$$P_2 \delta p_2 + Q_2 \delta q_2 + \ldots + T \delta_2 r = 0;$$

and if in the new positions of m_1 and m_2 the distance between them remains unaltered, $\delta_1 r + \delta_2 r = 0$. Hence, by adding these equations, we obtain the equation

$$P_1 \delta p_1 + Q_1 \delta q_1 + \ldots + P_2 \delta p_2 + Q_2 \delta q_2 + \ldots = 0, \qquad (1)$$

which is free from the internal force T.

This is exactly the same as the investigation already given for coplanar forces in Chap. VI. The extension to any number of particles, that is, to any body, proceeds just as in that chapter, and the enunciation of the principle of virtual work there given applies in general without the limitation that the forces are coplanar.

If in the case of the two particles m_1 and m_2, considered above, their new positions are such that the distance between them is altered by δr, the equation of virtual work will be

$$P_1 \delta p_1 + Q_1 \delta q_1 \ldots + P_2 \delta p_2 + Q_2 \delta q_2 + \ldots + T \delta r = 0; \qquad (2)$$

and, generally, if the virtual displacement is such that the internal forces do virtual work, these forces will enter into the equation of virtual work in exactly the same manner as the applied forces. The theorem of virtual work may, therefore, be thus enunciated:—

When a material system is in equilibrium under the action of any external and internal forces, the sum of the virtual works of the external and internal forces is equal to zero for any small virtual displacement whatsoever.

Instead of saying that the total virtual work is *zero*, we should in strictness say that it is an indefinitely small quantity of the second order, the greatest of the displacements being considered as a small quantity of the first order. This has been already explained in p. 122.

The proof of the converse proposition—namely, that when the virtual work vanishes for all imagined displacements, the system is in equilibrium—has been already given in p. 122 for coplanar forces; and as the proof obviously holds for non-coplanar forces, it is unnecessary to reproduce it here.

178.] **Displacements along Smooth Surfaces.** If any body or system of connected bodies be in contact with smooth curves or surfaces, and the system be imagined to receive any small displacement along these curves or surfaces, it is clear (as in p. 71) that, since the point of application of each of the geometrical forces (reactions of the curves or surfaces) moves in a plane at right angles to the corresponding force, these forces will contribute nothing to the equation of virtual work for such a displacement.

If any of the bodies of the system are connected by strings or rods whose lengths are unaltered in the virtual displacement chosen, the tensions of these strings or rods will not enter into the equation of virtual work. But, as already explained in pp. 80 and 120, we may choose virtual displacements of the system which violate the imposed conditions at the expense of bringing into our equation the corresponding forces.

179.] **Kinematical Theorem I.** When all the points of a rigid body move parallel to a plane, the motion may be produced by a pure rotation round an axis perpendicular to this plane.

DEF. A motion of a body round an axis whereby each point in the body describes an arc of a circle having its centre on the axis and its plane perpendicular to it is called a *pure rotation.*

The position of the body will evidently be known if the positions of any two points in a plane parallel to the plane of motion are known.

Let A and B be any two points in such a plane, and suppose that after the displacement of the body they occupy the positions A' and B' (fig. 210). At the middle points of AA' and BB' erect two perpendiculars, which meet in I. Then in the triangles AIB and $A'IB'$, $AI = A'I$, $BI = B'I$, and $AB = A'B'$; therefore the triangle $A'IB'$ is nothing more than AIB turned round the point I through an angle AIA' or BIB'. Hence the line AB can be brought into its new position by a pure rotation about I, and the same is true of every point rigidly connected with A and B in the plane AIB.

If through I an axis be drawn perpendicular to the plane of motion, it is evident that the body can be brought into its new position by a pure rotation about this axis through an angle $= AIA'$, however complicated the paths along which A and B have travelled to A' and B'.

When the motion of the body is small, this axis is called the *Instantaneous Axis;* and it is obviously constructed by *drawing two planes normal to the directions of motion of any two points in the body.* The intersection of these planes is the instantaneous axis.

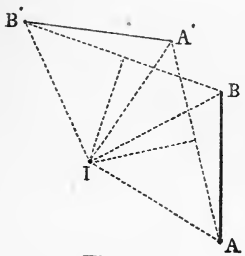


Fig. 210.

When the body is a plane figure, the point I is called the *Instantaneous Centre;* and the consideration of this point is of very extensive use in Kinematics, Statics, and Geometry.

To construct the instantaneous centre, *at any two points erect perpendiculars to the directions of motion of these points, and their intersection is the required point.*

180.] **Kinematical Theorem II.** The motion of a rigid body round a fixed point is at every instant a pure rotation round an axis.

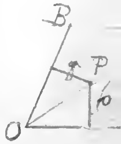
One point, O, in the body being fixed, the position of the body will be known if the positions of any two points, A and B, not in directum with O are known.

Round O let a sphere, forming part of the body or rigidly connected with it, be described with arbitrary radius, and let A and B (fig. 210) be any two points on the sphere. After the motion of the body let A' and B' be the positions of A and B. Imagine the lines AB, $A'B'$, AA', and BB' in this figure to be arcs of great circles on the sphere instead of right lines. Then, at the middle points of AA' and BB' draw two great circles perpendicular to AA' and BB', respectively, and let them meet in I. In exactly the same way as in the last theorem, we have the spherical triangles AIB and $A'IB'$ equal; that is, the latter triangle is the former turned round the axis OI through an angle AIA' or BIB'. Hence the whole body is brought by rotation through this angle round the axis OI from the old to the new position.

181.] **Kinematical Theorem III.** If a body has a motion of translation represented in magnitude and direction by a right line OA, and at the same time a motion of translation represented in magnitude and direction by a right line OB, the resulting motion of translation is represented in magnitude and direction by the diagonal, OC, of the parallelogram determined by OA and OB.

This proposition has been already illustrated in p. 6. It follows immediately that any motion of translation can be resolved by the papallelopiped law into three motions along the axes of x, y, and z, after the manner of forces.

182.] * **Kinematical Theorem IV.** If a body receives a motion of rotation round an axis OA, the rotation being represented in magnitude by OA, and at the same time a motion of rotation (of the same sign as the first) round an axis OB, the rotation being represented in magnitude by OB, the resulting motion is one of rotation round the diagonal, OC, of the parallelogram determined by OA and OB, and is represented in magnitude by this diagonal.



[The signs of rotations are determined by the rule given in Art. 137, Chapter X. We shall, for definiteness, suppose that when a watch is held with its face perpendicular to AO, so that OA passes up through the glass, the rotation about OA takes place in a sense opposite to that of the hands; and similarly for OB.]

Let P be any point on OC, p the perpendicular from P on OA, q the perpendicular from P on OB, and $k.OA$ and $k.OB$ the angular motions round OA and OB, respectively. Then in virtue of the rotation round OA, P moves upwards from the plane of the paper through a space equal to $kp.OA$; and in virtue of the rotation round OB, P moves downwards from the plane of the paper through a space equal to $kq.OB$. Therefore the whole motion of P upwards is equal to

$OA : OB :: \sin BOC : \sin A$
$:: q : p$
$\therefore p.OA = q.$

$$k(p.OA - q.OB).$$

But this is obviously zero; therefore P is at rest, and so is every point on OC. The motion is, then, a rotation round OC. Let Ω be the angular rotation of the body round OC. Then the point A moves upwards from the plane of the paper through a space equal to $\Omega . OA \sin AOC$, since $OA \sin AOC =$ the perpendicular from A on OC. But A in turning round OB moves through a space equal to $k . OB . OA \times \sin AOB$. Hence

$$\Omega . OA \sin AOC = k . OB . OA \sin AOB,$$

* Parallelogram of Angular Velocities

or
$$\Omega = k \, . \, OB \cdot \frac{\sin AOB}{\sin AOC}$$
$$= k \, . \, OC.$$

OB : OC :: sin AOC ::

Therefore the resulting angular velocity is represented by OC, if the component rotations are represented by OA and OB.

This proposition is known as the 'parallelogram of angular velocities.' It follows at once that an angular motion about any axis, OL, may be decomposed into three angular motions about three axes, Ox, Oy, and Oz. If these latter are rectangular, an angular motion ω about OL is equivalent to angular motions, $\omega \cos a$, $\omega \cos \beta$, and $\omega \cos \gamma$, of the same sign, round the axes of x, y, and z, the direction angles of OL being a, β, γ.

183.] **General Displacement of a Rigid Body.** The position of every point in a rigid body is known when the positions of any three points in it are known, provided that these points are not in one right line. The general displacement of a rigid body is, therefore, the same as that of a system of three points forming a triangle.

Let A, B, C be the positions of three points in the body before the displacement, and A', B', C' the positions occupied by these points after the displacement. Then the triangle ABC may be brought into the position $A'B'C'$ by moving A directly to A' while B and C move parallel to AA' through spaces equal to AA', and then turning the triangle about A' until B and C coincide with B' and C'. But (Art. 180) this latter motion is one of rotation round some axis through A'. Hence *the general displacement of a rigid body consists of a motion of translation which is the same for all its points, and a motion of rotation round an axis through an angle which is the same for all its points.*

To find the changes produced in the co-ordinates, x, y, z, of any point in the body by a general displacement, we may consider the motions of translation and of rotation separately.

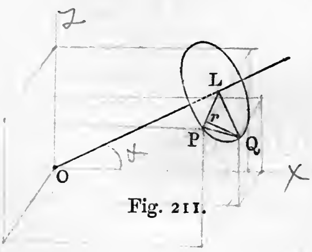


Fig. 211.

Although we shall be concerned only with small displacements, it is well to investigate the changes produced in the co-ordinates of a point by a rotation through any angle, θ, round an axis whose position is given.

Let the direction angles of the axis, OL (fig. 211), be a, β, γ; let P be the point (x, y, z) which, after the body has rotated through an angle

θ round OL, occupies the position Q; let PL $(=p)$ be the perpendicular from P on OL, and Qr a perpendicular from Q on LP. Now the x of Q is the projection of OQ on the axis of x; therefore the change in x is the projection of PQ along Ox, or the sum of the projections of Pr and rQ. But $Pr = p\,(1-\cos\theta)$, and $Qr = p\sin\theta$. (...) on $= 2\sin^2\frac{\theta}{2}\cdot p$.

Again, if the direction angles of ~~PL~~ LP are λ, μ, ν, since ~~Qr~~ rQ is at right angles to OL and ~~PL~~ LP, the direction cosines of ~~Qr~~ rQ are * $\cos\beta\cos\nu - \cos\gamma\cos\mu$, &c. Hence, if the x of Q is x', * See p. 2

$$x' - x = p\sin\theta\,(\cos\beta\cos\nu - \cos\gamma\cos\mu) - 2p\cos\lambda\sin^2\frac{\theta}{2}. \quad (1)$$

But $p\cos\lambda$ is the projection of ~~PL~~ LP along the axis of x, or the projection of OP—the projection of OL, and since

$$OL = x\cos a + y\cos\beta + z\cos\gamma,$$

$$p\cos\lambda = x - (x\cos a + y\cos\beta + z\cos\gamma)\cos a;$$

similarly

$$p\cos\mu = y - (x\cos a + y\cos\beta + z\cos\gamma)\cos\beta,$$

$$p\cos\nu = z - (x\cos a + y\cos\beta + z\cos\gamma)\cos\gamma.$$

Substituting these values in (1), we have

$$x' - x = \sin\theta\,(z\cos\beta - y\cos\gamma) + 2\sin^2\frac{\theta}{2}[(x\cos a + y\cos\beta + z\cos\gamma)\cos a - x], \quad (2)$$ †

and similar values for the changes in y and z.

If the angular rotation θ is very small, we have

$$\delta x = (z\cos\beta - y\cos\gamma)\,\delta\theta,$$

$$\delta y = (x\cos\gamma - z\cos a)\,\delta\theta,$$

$$\delta z = (y\cos a - x\cos\beta)\,\delta\theta;$$

and if the components of the rotation $\delta\theta$ along the axes be denoted by $\delta\theta_1, \delta\theta_2, \delta\theta_3$, these equations give

$$\left.\begin{aligned} \delta x &= z\,\delta\theta_2 - y\,\delta\theta_3, \\ \delta y &= x\,\delta\theta_3 - z\,\delta\theta_1, \\ \delta z &= y\,\delta\theta_1 - x\,\delta\theta_2. \end{aligned}\right\} \quad (3)$$

Of course these equations can be obtained very simply by considering the separate changes in the co-ordinates produced by successive rotations $\delta\theta_1, \delta\theta_2, \delta\theta_3$ round the axes of x, y, z, respectively. (See Routh's *Rigid Dynamics*.)

If the components of the motion of translation common to all points in the body be $\delta a, \delta b, \delta c$, the complete changes in the co-ordinates for a small displacement will be

†

$$y' - y = \sin\theta\,(x\cos\gamma - z\cos\alpha) + 2\sin^2\frac{\theta}{2}[(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\beta - y]$$

$$z' - z = \sin\theta\,(y\cos\alpha - x\cos\beta) + 2\sin^2\frac{\theta}{2}[(x\cos\alpha + y\cos\beta + z\cos\gamma)\cos\gamma - z]$$

$$\left.\begin{aligned} \delta x &= \delta a + z\delta\theta_2 - y\delta\theta_3, \\ \delta y &= \delta b + x\delta\theta_3 - z\delta\theta_1, \\ \delta z &= \delta c + y\delta\theta_1 - x\delta\theta_2. \end{aligned}\right\} \qquad (4)$$

184.] **Deduction of the Six Equations of Equilibrium.** Replacing the virtual work of each force in equation (1) of Art. 177 by the virtual work of its three components, the general equation of virtual work becomes

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0, \qquad (1)$$

and substituting in this equation the values of δx, δy, and δz given by (4), we have

$$\delta a\,.\,\Sigma X + \delta b\,.\,\Sigma Y + \delta c\,.\,\Sigma Z + \delta\theta_1\,.\,\Sigma\,(Zy - Yz)$$
$$+\delta\theta_2\,.\,\Sigma\,(Xz - Zx) + \delta\theta_3\,.\,\Sigma\,(Yx - Xy) = 0. \qquad (2)$$

Now, the displacement being quite arbitrary, its components δa, δb, δc, $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, are completely independent. Hence in (2) we may consider all of them zero except one, and the equation then gives the coefficient of this one equal to zero. Thus (2) involves the six equations

$$\Sigma X = 0,\ \Sigma Y = 0,\ \Sigma Z = 0,$$
$$\Sigma\,(Zy - Yz) = 0,\ \Sigma\,(Xz - Zx) = 0,\ \Sigma\,(Yx - Xy) = 0,$$

which are the equations of equilibrium before obtained (see p. 232).

In addition to the following Examples, the student will do well to solve some of those in p. 179 by the Principle of Work.

Examples.

1. Four rigid bars, freely jointed together at their extremities, form a quadrilateral, $ABCD$; the opposite vertices are connected by strings, AC and BD, in a state of tension; compare the tensions of these strings.

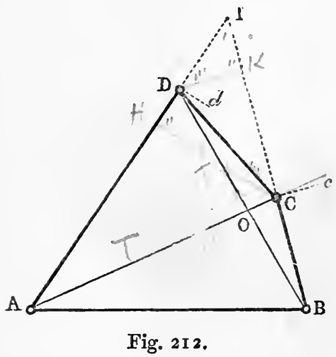


Fig. 212.

Let the bar AB be considered as fixed, and let the quadrilateral undergo any slight deformation. Then the bars AD and BC will turn round the points A and B, that is, the points D and C will describe small paths, Dd and Cc, perpendicular to AD and BC. Hence (Theorem I) the point, I, of intersection of AD and BC is the *instantaneous centre* for the bar CD, and the angles DId and CIc are equal. Denote their common value by $\delta\theta$. Then $Dd = ID \,.\, \delta\theta$, and $Cc = IC \,.\, \delta\theta$.

Now, since in the displacement of the system none of the geometrical conditions—namely, the constancy of the lengths of the bars—are violated, the stresses of the bars will not enter into the equation of virtual work. Hence if T and T' denote the tensions of the strings AC and BD, this equation will be (see p. 78),

$$T \,.\, \delta AC + T' \,.\, \delta BD = 0. \tag{1}$$

But $\delta AC =$ projection of Cc on $AC = Cc \,.\, \sin ACB = IC \,.\, \sin ACB \,.\, \delta\theta$; and similarly $\delta BD = -ID \,.\, \sin BDA \,.\, \delta\theta$. Hence (1) becomes

$= -ID\,d\theta$ $(90^\circ - d$ $= -ID \sin BDA$

$$T \,.\, IC \,.\, \sin ACB = T' \,.\, ID \,.\, \sin BDA. \tag{2}$$

Again, $$\frac{IC}{ID} = \frac{AC \sin CAD}{BD \sin CBD}.$$ Since $IC : ID :: CH : I$ $:: AC \sin CAD : BD \sin$

Substituting in (2) we obtain

$$(3) \quad T\frac{AC}{OA \,.\, OC} = T'\frac{BD}{OB \,.\, OD}.$$

$T.AC \sin CAD \sin ACB = T'.BD \sin CBD$
But $\sin CAD : \sin BDA :: OD : OA$
and " ACB : " $CBD :: OB : OC$
$\therefore T.AC.OB = T'.BD.OA.OC$
$\therefore \frac{T.AC}{OA.OC} = \frac{T'.BD}{OB.OD}$

Another solution of this problem (quoted from Euler) will be found in Walton's *Mechanical Problems*, p. 101.

✓ 2. Four rigid bars, freely jointed at their extremities, form a quadrilateral, $ABCD$; the bars AB and AD are connected by a string, $a\alpha$ in a state of tension, a being a given point in AB, and α a given point in AD; in the same way, BA and BC are connected by a string $b\beta$; CB and CD are connected by a string $c\gamma$; and DC and DA by a string $d\delta$; find the relation between the tensions of these strings.

If the lengths of the strings $a\alpha$, $b\beta$, $c\gamma$ and $d\delta$ are denoted by x, y, z, and w, and the tensions in them by X, Y, Z, W, the equation of virtual work for a slight deformation will be

$$X\delta x + Y\delta y + Z\delta z + W\delta w = 0. \tag{1}$$

Now $$x^2 = Aa^2 + A\alpha^2 - 2Aa \,.\, A\alpha \cos A = Aa^2 + A\alpha^2$$

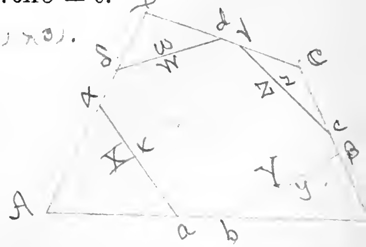
2 (3) $$-2\frac{Aa \,.\, A\alpha}{AB \,.\, AD}(AB^2 + AD^2 - BD^2);$$ See Trig. Art 92.

therefore $$x\delta x = 2\frac{Aa \,.\, A\alpha}{AB \,.\, AD} \cdot BD \,.\, \delta BD.$$

Substituting this value of δx, and similar values of δy, δx, δw, in (1), we have

$$\left(\frac{X}{x} \cdot \frac{Aa \,.\, A\alpha}{AB \,.\, AD} + \frac{Z}{z} \cdot \frac{Cc \,.\, C\gamma}{CB \,.\, CD}\right) BD \,.\, \delta BD$$

$$+\left(\frac{Y}{y} \cdot \frac{Bb \,.\, B\beta}{BA \,.\, BC} + \frac{W}{w} \cdot \frac{Dd \,.\, D\delta}{DC \,.\, DA}\right) AC \,.\, \delta AC = 0.$$



But from the last Example, we have from (1) & (3).

$$\frac{\delta BD}{\delta AC} = -\frac{BD \,.\, OA \,.\, OC}{AC \,.\, OB \,.\, OD};$$

hence, finally, $\left(\frac{X}{x}\cdot\frac{Aa\,.\,A\alpha}{AB\,.\,AD}+\frac{Z}{z}\cdot\frac{Cc\,.\,C\gamma}{CB\,.\,CD}\right)\frac{BD^2}{OB\,.\,OD}$

$$=\left(\frac{Y}{y}\cdot\frac{Bb\,.\,B\beta}{BA\,.\,BC}+\frac{W}{w}\cdot\frac{Dd\,.\,D\delta}{DC\,.\,DA}\right)\frac{AC^2}{OA\,.\,OC}.$$

For a different solution, see Walton, *ibid.*

3. Six equal heavy beams are freely jointed at their extremities; one is fixed on a horizontal plane, and the system lies in a vertical plane; the middle points of the two upper non-horizontal beams are connected by a rope in a state of tension. Show that the tension of this rope is

$$6W\cot\theta,$$

W being the weight of each beam, and θ the inclination of the non-horizontal beams to the horizon.

Let x be the length of the rope, y the height of the centre of gravity of the system, $2a$ the length of each beam, and T the tension of the rope. Then the virtual work of the tension is $-T\delta x$ (see p. 78), and the virtual work of the weight of the system is $-6W\delta y$. Hence

$$T\delta x+6W\delta y=0.$$

But $x=2a(1+\cos\theta)$, and $y=2a\sin\theta$, and the deformation imagined is one in which the upper horizontal beam moves vertically through a small space. Hence the values of y and x will be of the same forms as before, and

$$\delta x=-2a\sin\theta\delta\theta,\quad \delta y=2a\cos\theta\delta\theta.$$

Substituting these values of δx and δy, we have

$$T=6W\cot\theta.$$

4. A body receives a small general displacement parallel to one plane; find the co-ordinates of the instantaneous centre.

If the components of the motion of translation parallel to the axes of x and y are δa and δb, and the rotation is $\delta\omega$, the equations (4) of Art. 183 give for the displacement of any point whose co-ordinates are x, y,

$$\delta x=\delta a-y\delta\omega,$$

$$\delta y=\delta b+x\delta\omega.$$

Now, the displacement of the instantaneous centre is zero; hence, if (x, y) be its co-ordinates, we have

$$x=-\frac{\delta b}{\delta\omega},\quad y=\frac{\delta a}{\delta\omega}.$$

A particular case may be noticed. If any body in contact with a surface receives any small displacement parallel to one plane, the body still remaining in contact with the surface, the instantaneous centre lies on the normal to the surface of contact. In the rolling of one figure on another the point of contact is the instantaneous centre.

5. A uniform beam, AB (fig. 133, p. 149), rests as a tangent at a point P against a smooth curve in a vertical plane, one extremity, A, resting against a smooth vertical plane; find the position of equilibrium, and the nature of the curve so that the beam may rest in all positions.

Let the weight of the beam through G, and the normal reactions at A and P meet in the point O; take the vertical line AD as axis of y; and let $2a =$ the length of the beam. Then, if x is the abscissa of P, we have $AO = \dfrac{x}{\sin^2\theta}$, and also $AO = a\sin\theta$. Hence, equating these values,

$$x = a\sin^3\theta. \tag{1}$$

Now, from the equation of the given curve, θ is known in terms of x in the form

$$\theta = f(x). \tag{2}$$

From (1) and (2) the value of x, and therefore the position of equilibrium, can be found.

For example, if the curve be a circle of radius r whose centre is at a distance c from the vertical plane, we find

$$a\sin^3\theta + r\cos\theta - c = 0.$$

If $r = 0$, we get the result in Ex. 7, p. 149.

If (1) holds in all positions in which the beam is placed, every position is one of equilibrium. Now, since $\tan\theta = \dfrac{dx}{dy}$, (1) gives

$$dy = \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\,.\,x^{-\frac{1}{3}}dx,$$

and since this equation holds in all positions, we may integrate it. Hence

$$y + k = -(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}},$$

or

$$x^{\frac{2}{3}} + (y+k)^{\frac{2}{3}} = a^{\frac{2}{3}},$$

k being an arbitrary constant.

We may, without loss of generality, assume $k = 0$, and the curve will be

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The equation of virtual work shows that in this case the centre of gravity of the beam is at a constant height. For if $\bar{y}$ denote the ordinate of G, this equation is

$$W d\bar{y} = 0,$$

and since this holds in all positions, we have, by integration, $\bar{y} =$ constant.

6. Four rigid bars freely jointed at their extremities form a quadrilateral $ABCD$ (fig. 213); the middle points of the opposite pairs of bars are connected by strings, mm' and nn', in a state of tension. Compare the tensions of these strings.

Let l and l' be the lengths of the strings mm' and nn', and let the tensions in them be T and T', respectively.

Then, assuming the quadrilateral to receive any small deformation, the equation of work will be

$$T\delta l + T'\delta l' = 0. \tag{1}$$

Now, it may be left to the student as an exercise to prove that

$$l'^2 - l^2 = \tfrac{1}{2}(AB^2 + CD^2 - BC^2 - AD)^2,$$

that is, $l'^2 - l^2$ is constant however the quadrilateral may be deformed. Hence

$$l\delta l - l'\delta l' = 0; \tag{2}$$

and from (1) and (2) we have

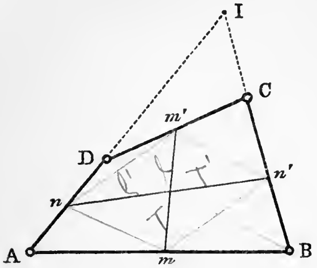


Fig. 213.

$$\frac{T}{l} + \frac{T'}{l'} = 0, \tag{3}$$

a remarkable result, since it shows that one of the tensions must be *negative*; i.e., if the bars AB and CD are pulled together, equilibrium will be impossible unless the bars AD and BC are pulled asunder.

It is well to notice an apparent exception to the result (3). The student will easily prove that if the sides AB and DC are parallel, equilibrium will be maintained by the single string mm' in any state of tension, i.e., $T' = 0$, a result which contradicts (3).

The difficulty is easily removed, however, by reverting to (1), which in the case under consideration is identically satisfied. For, since AB and CD are parallel, the line mm' passes through I, the instantaneous centre, and therefore for a slight deformation the point m' moves perpendicularly to Im', that is, to mm'. Hence $\delta l = 0$, and equation (1) is satisfied by having at once $T' = 0$ and $\delta l = 0$. The combination of (1) and (2) is therefore irrelevant.

7. A number of bars are freely jointed together at their extremities and form a polygon; each bar is acted on perpendicularly by a force proportional to its length; all the forces emanate from one point and all act inwards or all outwards; prove by virtual work that for equilibrium the polygon must be inscribable in a circle.

Let the polygon be $ADCBEF$... (fig. 213), of which the vertices E, F... are not represented in the figure. [AB is not one of the bars.]

Choose a virtual displacement in which all the bars except the three AD, DC, CB remain fixed, and let the extremities A and B be fixed in the displacement. Then I is the instantaneous centre for DC. Let O be the point from which the forces emanate; let m, n, p be the feet of perpendiculars from O on AD, DC, CB, respectively; let Q be the foot of the perpendicular from I on DC; let IQ meet mO in L and pO in M; and let the forces in Om, On, Op be $k.AD$, $k.DC$, $k.CB$.

If AD turns round A through the small angle $\delta\phi$, the displacement of D is $AD.\delta\phi$; and if DC turns round I through $\delta\omega$, the displacement of D is $ID.\delta\omega$. Hence

$$AD \, . \, \delta\phi = ID \, . \, \delta\omega.$$

Similarly $$BC \, . \, \delta\theta = IC \, . \, \delta\omega,$$

if $\delta\theta$ is the angle through which BC turns round B.

Now the equation of virtual work is

$$k \, . \, AD \, . \, Am \, . \, \delta\phi + k \, . \, DC \, . \, In \, . \, \delta\omega \, . \cos InQ - k \, . \, BC \, . \, Bp \, . \, \delta\theta = 0\,;$$

or, by the first two equations,

$$Am \, . \, ID + DC \, . \, nQ - Bp \, . \, IC = 0. \qquad (1)$$

Now $\quad Im \, . \, ID = LI \, . \, IQ, \quad$ and $\quad Ip \, . \, IC = IQ \, . \, IM\,;$

therefore $$Im \, . \, ID - Ip \, . \, IC = LM \, . \, IQ. \qquad (2)$$

Adding (2) to (1), we have

$$AI \, . \, ID - BI \, . \, IC = LM \, . \, IQ - DC \, . \, nQ.$$

But the right side of this equation is zero, since the triangles DCI and LMO are similar (nQ is the altitude of the latter). Hence the quadrilateral $ADCB$ is inscribable in a circle; and in this circle lie also the quadrilaterals $DCBE$, $CBEF$, ... and therefore the whole polygon.

8. Six equal heavy bars are freely jointed at their extremities; one bar is fixed in a horizontal position, and the system hangs in a vertical plane; the middle points of each pair of adjacent non-horizontal bars are connected by two strings in a state of tension. Show by the principle of work that, if the hexagon is regular in its position of equilibrium, the tension of each string is three times the weight of a bar.

9. Four bars whose weights may be neglected are freely articulated at their extremities and form a quadrilateral, $ABCD$, in a vertical plane. The joint A is fixed, while the lateral joints, B and D, rest each against a smooth vertical plane. A given vertical force being applied at the joint C, find the magnitudes of the reactions of the planes at B and D, and the direction and magnitude of the pressure on the joint A.

Ans. Let F be the force applied at C, P and Q the reactions at B and D, R the pressure at A; also let a, β, γ, and δ be the inclinations of the bars AB, BC, CD, and DA to the horizon, and θ the angle made by the direction of R with the horizon. Then we shall have

$$\frac{P}{1+\cot a \tan \beta} = \frac{Q}{1+\cot \delta \tan \gamma} = \frac{F}{\tan \beta + \tan \gamma}$$

$$R = \sqrt{P^2+Q^2+F^2-2PQ}$$

$$\tan \theta = \frac{\cot \beta + \cot \gamma}{\cot a \cot \gamma - \cot \beta \cot \delta}.$$

(To get P, choose a displacement of the bars in which AD remains fixed; the intersection of AB and CD will then be the instantaneous centre.)

10. Two heavy uniform beams, AC and CB (fig. 140, p. 163), are connected by a smooth joint at C; the beam AC is moveable in a vertical plane about a smooth joint fixed at A, and the extremity B of the beam CB is capable of moving along a smooth horizontal groove whose direction passes through A. It is required to keep the system in a given position by means of a horizontal force applied at B; determine by the principle of work the requisite magnitude of this force.

Ans. If a and a' denote the angles CAB and CBA; W and W' the weights of AC and CB; and F the required force,

$$F = \frac{W + W'}{2(\tan a + \tan a')}.$$

11. Four bars, freely articulated at their extremities, form a parallelogram, $ABCD$; two forces, each equal to P, act in opposite directions in the diagonal AC, and two forces, each equal to Q, act similarly in BD. Find the figure of equilibrium.

Ans. The adjacent sides of the parallelogram being a and b, the angle between them ω, we have

$$\cos\omega = \frac{a^2+b^2}{2ab}\cdot\frac{P^2-Q^2}{P^2+Q^2}.$$

12. If the forces in Example 7 are each transferred to the middle point of the bar on which it acts, prove by virtual work that the polygon must be inscribable for equilibrium.

185.] **Lagrangian Meaning of the Virtual Moment of a Force.** We see that in the general equation (2) of virtual work, each of the displacements, δa, &c., is multiplied by a function of force *which tends to produce this displacement*. Thus $\delta\theta_1$ is multiplied by the whole moment of the forces round the axis of x, and the tendency of this moment is to produce a rotation round the axis; δa is multiplied by the whole component of the forces along the axis of x, and the tendency of this component is to produce a motion of translation in this direction. In the same way, in equation (2) of Art. 177, each force is multiplied by a variation which it tends to produce. Thus the tendency of the force P_1 is to drag its point of application in its own direction. If p_1 is the distance, OA_1, of the point, A_1, of application of the force from a fixed point, O, in the line of action of the force, the tendency of P_1 is to alter the distance p_1, and accordingly the term $P_1\delta p_1$ appears in the equation of virtual work.

Similarly, the tendency of the internal force T is to alter the distance, r, between the points m_1 and m_2, and accordingly the term $T\delta r$ enters also into the equation. Each of these terms is, in fact, the elementary work which the corresponding force tends to do, and which it *would* do if the system were displaced or deformed; and hence all such terms must appear in a complete equation of virtual work. Hence Lagrange defines the *virtual moment*, or virtual work, of a force as *the product of the force and the variation of the function which it tends to alter* (*Mécanique Analytique*, § 5, p. 29; § 9, p. 33; § 6, p. 72; § 26, p. 126, Bertrand's edition), and in every case he obtains the general equation of equilibrium of a system by adding together all such products, whether they belong to the given external forces, the geometrical forces (reactions of smooth surfaces, or forces of connexion), or to the internal forces (mutual attractions or repulsions) of the system.

This method of the solution of statical problems (which is obviously only the method of virtual work) is one of great power and generality, and its nature will be rendered more clear in the sequel.

186.] **Equations of Condition may be Replaced by Forces.** Suppose a system of n particles whose co-ordinates are connected by k equations of condition,

$$L_1 = 0,\ L_2 = 0, \dots L_k = 0, \tag{1}$$

each of these equations being of the form

$$f(x_1, y_1, z_1; x_2, y_2, z_2, \dots x_n, y_n, z_n) = 0,$$

that is, involving the co-ordinates of all the points in general. Then the equation of virtual work for the position of equilibrium of the system is

$$\Sigma (X\delta x + X\delta y + Z\delta z) = 0,$$

which, when written at full length, is

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 + \dots + X_n\delta x_n + Y_n\delta y_n + Z_n\delta z_n = 0. \tag{2}$$

Now if the virtual displacements of the particles were all independent, this equation would involve the vanishing of the coefficient of each displacement (see Art. 184); but the displacements of the particles must be such as still to satisfy the equations (1). Hence the quantities δx, &c., are connected by the k equations

$$\left.\begin{aligned}
&\frac{dL_1}{dx_1}\delta x_1 + \frac{dL_1}{dy_1}\delta y_1 + \frac{dL_1}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_1}{dx_n}\delta x_n + \frac{dL_1}{dy_n}\delta y_n + \frac{dL_1}{dz_n}\delta z_n = 0, \\
&\frac{dL_2}{dx_1}\delta x_1 + \frac{dL_2}{dy_1}\delta y_1 + \frac{dL_2}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_2}{dx_n}\delta x_n + \frac{dL_2}{dy_n}\delta y_n + \frac{dL_2}{dz_n}\delta z_n = 0, \\
&\dots\dots\dots\dots\dots\dots \\
&\frac{dL_k}{dx_1}\delta x_1 + \frac{dL_k}{dy_1}\delta y_1 + \frac{dL_k}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_k}{dx_n}\delta x_n + \frac{dL_k}{dy_n}\delta y_n + \frac{dL_k}{dz_n}\delta z_n = 0.
\end{aligned}\right\} \qquad (3)$$

Solving these k equations for any k of the displacements—suppose $\delta x_1, \delta x_2, \dots \delta x_k$—and substituting their values in (2), we obtain an equation connecting the remaining $3n-k$ displacements of the form

$$\begin{aligned}
&A_{k+1}\delta x_{k+1} + \dots + A_n \delta x_n \\
&+ B_1 \delta y_1 + \dots + B_n \delta y_n \\
&+ C_1 \delta z_1 + \dots + C_n \delta z_n = 0.
\end{aligned} \qquad (4)$$

Now, the remaining quantities, δx_{k+1}, &c., are completely independent, and therefore (see Art. 184) every coefficient in this equation must $= 0$. Thus, we obtain $3n-k$ equations involving the forces, that is, statical equations of condition. Combining these statical equations with the equations of connexion (1), we have finally $3n$ equations for the $3n$ co-ordinates of the particles. The elimination of the displacements from the equations can, however, be exhibited in a more symmetrical and useful form.

Multiply the equations (3) by $\lambda_1, \lambda_2, \dots \lambda_k$ in order, these multipliers being undetermined quantities; then add the equations together, and finally equate to zero the coefficient of every displacement in the resulting equation. Thus we shall have the following $3n$ equations :—

$$\left.\begin{aligned}
X_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \ldots + \lambda_k \frac{dL_k}{dx_1} &= 0, \\
Y_1 + \lambda_1 \frac{dL_1}{dy_1} + \lambda_2 \frac{dL_2}{dy_1} + \ldots + \lambda_k \frac{dL_k}{dy_1} &= 0, \\
Z_1 + \lambda_1 \frac{dL_1}{dz_1} + \lambda_2 \frac{dL_2}{dz_1} + \ldots + \lambda_k \frac{dL_k}{dz_1} &= 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot
\end{aligned}\right\} \qquad (5)$$

If from these equations we eliminate the k undetermined multipliers, we shall have $2n-k$ statical equations of condition, as before.

Now this method of elimination has the advantage of discovering the geometrical forces, or forces arising from the connexions, of the system. For, suppose that we suppress the condition $L_1 = 0$; then the system will begin to move; but it may be kept at rest by applying a special force to each particle.

Let the components of the force applied to m_1 be X_1', Y_1', Z_1', those of the force applied to m_2, X_2', Y_2', Z_2', and so on for all the others. The equations of equilibrium of m_1 will then be

$$X_1 + X_1' + \lambda_2 \frac{dL_2}{dx_1} + \ldots + \lambda_k \frac{dL_k}{dx_1} = 0,$$

$$Y_1 + Y_1' + \lambda_2 \frac{dL_2}{dy_1} + \ldots + \lambda_k \frac{dL_k}{dy_1} = 0,$$

$$Z_1 + Z_1' + \lambda_2 \frac{dL_2}{dz_1} + \ldots + \lambda_k \frac{dL_k}{dz_1} = 0,$$

similar equations holding for the other particles.

Subtracting each of these from the corresponding equation in (5), we have

$$X_1' = \lambda_1 \frac{dL_1}{dx_1},$$

$$Y_1' = \lambda_1 \frac{dL_1}{dy_1},$$

$$Z_1' = \lambda_1 \frac{dL_1}{dz_1}.$$

Hence,
$$X_1' : Y_1' : Z_1' = \frac{dL_1}{dx_1} : \frac{dL_1}{dy_1} : \frac{dL_1}{dz_1},$$

and
$$\sqrt{X_1'^2 + Y_1'^2 + Z_1'^2} = \lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

If, now, all the co-ordinates involved in the equation $L_1 = 0$ are considered constant except x_1, y_1, and z_1, this equation will denote a surface on which the particle m_1 is constrained to lie, and

$$\frac{dL_1}{dx_1},\quad \frac{dL_1}{dy_1},\quad \frac{dL_1}{dz_1},$$

each divided by $\sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}$,

will be the direction cosines of the normal to this surface at the point (x_1, y_1, z_1). It is evident, therefore, that the force required to keep the particle m_1 at rest, when the condition $L_1 = 0$ is suppressed, is a force acting normally to this surface, its magnitude being

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

In the same way the force required to keep m_2 at rest acts normally to the surface denoted by $L_1 = 0$ when x_2, y_2, z_2 are considered as the only variable co-ordinates in the equation, and the magnitude of this force is

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_2}\right)^2 + \left(\frac{dL_1}{dy_2}\right)^2 + \left(\frac{dL_1}{dz_2}\right)^2}.$$

If the condition $L_2 = 0$ were suppressed, it follows in like manner that forces

$$\lambda_2 \sqrt{\left(\frac{dL_2}{dx_1}\right)^2 + \left(\frac{dL_2}{dy_1}\right)^2 + \left(\frac{dL_2}{dz_1}\right)^2}, \ \&c.,$$

should be applied to the particles m_1, &c., in directions normal to the surfaces represented by the equation $L_2 = 0$ when the sole variables in it are the co-ordinates of m_1, &c., in succession. It is easy to see that

$$\lambda_1 \left(\frac{dL_1}{dx_1}\delta x_1 + \frac{dL_1}{dy_1}\delta y_1 + \frac{dL_1}{dz_1}\delta z_1\right)$$

is equal to $\quad F_1(\cos\alpha \,.\, \delta x_1 + \cos\beta \,.\, \delta y_1 + \cos\gamma \,.\, \delta z_1)$,

where F_1 is the force of connexion acting on m_1 in virtue of the condition $L_1 = 0$, and α, β, γ the direction angles of the normal to the surface denoted by $L_1 = 0$ when the co-ordinates of m_1 are regarded as the only variables in it.

Now, the multiplier of F_1 in this expression is evidently the projection of the displacement of m_1 along the normal to this surface. If this projection be denoted by δn, n being the length of the normal at the position of m_1 measured from some fixed point on the normal, we have

$$\lambda_1 \delta L_1 = F_1 \delta n,$$

in which the variation of L_1 has reference solely to the particle m_1.

Now, as the force F_1 acts along the normal, and tends directly to alter its length, or to produce the displacement δn, we may, in conformity with Lagrangian language, regard the term $\lambda_1 \delta L_1$ *as the virtual work of a force tending to vary the function* L_1.

This is true without regard to the nature of the function L_1. It may, then, be a function not only of co-ordinates, but of differential coefficients of co-ordinates. It may, for example, express the imposed condition of inextensibility in the case of a string, and then it will take the form

$$ds = \text{constant};$$

or it may express the same condition in the case of a membrane, or, finally, the incompressibility of a fluid, and then it will be

$$dx\,dy\,dz = \text{constant}.$$

Except in the case of continuous systems (such as springs, membranes, and strings), this method is not a simplification of the ordinary statical methods. Nevertheless, for the sake of showing its application in practice, we add a few examples solved by means of it, deferring its more useful application for the present.

Examples.

1. A number of heavy particles are attached at given intervals to a weightless string the extremities of which are fixed; investigate the circumstances of equilibrium (Funicular Polygon).

Let (a, b) be the co-ordinates of one of the fixed extremities, (x_1, y_1), (x_2, y_2), ... the co-ordinates of the particles taken in order from this extremity, l_{01}, l_{12}, ... the lengths of the portions of the string between these points, and W_1, W_2, ... the weights of the particles.

Then the equations of connexion of the system are

$$(a - x_1)^2 + (b - y_1)^2 = l_{01}^2,$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l_{12}^2, \text{ \&c.}$$

Hence the Lagrangian equation of virtual work is

$$W_1 \delta y_1 + W_2 \delta y_2 + \ldots - \lambda_1 \{(a - x_1)\,\delta x_1 + (b - y_1)\,\delta y_1\}$$
$$+ \lambda_2 \{(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2)\} + \ldots = 0.$$

Equating to zero the coefficients of the several displacements,

$$\lambda_1(a-x_1)-\lambda_2(x_1-x_2)=0,$$
$$\lambda_2(x_1-x_2)-\lambda_3(x_2-x_3)=0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$W_1-\lambda_1(b-y_1)+\lambda_2(y_1-y_2)=0,$$
$$W_2-\lambda_2(y_1-y_2)+\lambda_3(y_2-y_3)=0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The first set of these equations evidently give

$$\lambda_1(a-x_1)=\lambda_2(x_1-x_2)=\lambda_3(x_2-x_3)=\ldots=T, \text{ suppose,}$$

and by substituting in the remaining set,

$$\frac{b-y_1}{a-x_1}=\frac{y_1-y_2}{x_1-x_2}+\frac{W_1}{T},$$
$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_3}{x_2-x_3}+\frac{W_2}{T}.$$

But $\frac{b-y_1}{a-x_1}$ is the tangent of the inclination of the portion l_{01} of the string to the horizon. Hence we have

$$\tan\theta_{01}=\tan\theta_{12}+\frac{W_1}{T},$$
$$\tan\theta_{12}=\tan\theta_{23}+\frac{W_2}{T},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

as in p. 32. Also the tension of the string joining (a, b) to (x_1, y_1) is $\frac{\lambda_1}{l_{01}}$ acting from the first point towards the second, and so on for the other tensions.

2. Deduce by the method of Lagrange the conditions of equilibrium of a system of three particles forming a rigid triangle, each particle being acted on by given forces.

Let (x_1, y_1, z_1) be the co-ordinates of one particle, and (X_1, Y_1, Z_1) the components of the force acting on it, with similar notation for the other two particles. Then, if l_{12}, l_{23}, l_{31} denote the sides of the triangle, the equations of connexion are

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=l_{12}^2,$$
$$(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2=l_{23}^2,$$
$$(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2=l_{31}^2.$$

Hence the Lagrangian equation of equilibrium is

$$X_1\delta x_1+Y_1\delta y_1+Z_1\delta z_1+\ldots+\lambda_{12}\{(x_1-x_2)(\delta x_1-\delta x_2)$$
$$+(y_1-y_2)(\delta y_1-\delta y_2)+(z_1-z_2)(\delta z_1-\delta z_2)\}+\ldots=0,$$

the undetermined multipliers being λ_{12}, λ_{23}, and λ_{31}.

Equating to zero the coefficients of the displacements, we have

$$X_1+\lambda_{12}(x_1-x_2)-\lambda_{31}(x_3-x_1)=0, \quad (1)$$
$$Y_1+\lambda_{12}(y_1-y_2)-\lambda_{31}(y_3-y_1)=0, \quad (2)$$

$$Z_1 + \lambda_{12}(z_1 - z_2) - \lambda_{31}(z_3 - z_1) = 0, \tag{3}$$

with similar equations for the other particles.

By addition, we have at once

$$X_1 + X_2 + X_3 = 0, \text{ or } \Sigma X = 0,$$
$$Y_1 + Y_2 + Y_3 = 0, \text{ or } \Sigma Y = 0,$$
$$Z_1 + Z_2 + Z_3 = 0, \text{ or } \Sigma Z = 0,$$

which are the ordinary equations of translation.

Again, multiplying (1) by y_1 and (2) by x_1, and subtracting,

$$Y_1 x_1 - X_1 y_1 - \lambda_{12}(x_1 y_2 - y_1 x_2) - \lambda_{31}(x_1 y_3 - y_1 x_3) = 0,$$

and by taking the similar equations for the other particles, and adding, we get $\qquad \Sigma(Yx - Xy) = 0.$

Similarly, $\qquad \Sigma(Xz - Zx) = 0,$

and $\qquad \Sigma(Zy - Yz) = 0.$

These last three are the equations of moments, and they constitute with the first three six equations of equilibrium. Now these are all the conditions that can be obtained among the forces and co-ordinates. For if n particles be connected by k equations of condition, there are (Art. 186) $3n - k$ final equations. But here $n = 3$, $k = 3$, therefore $3n - k = 6$. It is to be observed that the equations of equilibrium of any rigid body must be the same in number as those for three particles forming a rigid triangle, because if three points of a rigid body are determined in position, the position of the body is determined.

3. Show that the equations of equilibrium of a system subject to given conditions may be expressed as the vanishing of the differential coefficients of a single function of the co-ordinates of the system.

Suppose that

$$(X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1) + (X_2 dx_2 + Y_2 dy_2 + Z_2 dz_2) + \dots,$$

or $\Sigma(Xdx + Ydy + Zdz), \equiv dV$ where V is a function of the co-ordinates $x_1, y_1, z_1, x_2, y_2, z_2, \dots$. Then, taking

$$U \equiv V + \lambda_1 L_1 + \lambda_2 L_2 + \dots,$$

where $L_1 = 0$, $L_2 = 0, \dots$ are the equations of condition, we shall have

$$\frac{dU}{dx_1} \equiv X_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \dots + L_1 \frac{d\lambda_1}{dx_1} + L_2 \frac{d\lambda_2}{dx_1} + \dots.$$

But since the co-ordinates make $L_1 = L_2 = \dots = 0$,

$$\frac{dU}{dx_1} \equiv X_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \dots,$$

and comparing with equations (5), we see that the equations of equilibrium are

$$\frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = 0, \dots \frac{dU}{dy_1} = 0, \quad \frac{dU}{dy_2} = 0, \text{ \&c.}$$

187.] **Distinctive Feature of the Lagrangian Method.** If the first method of eliminating the displacements described in the last article is adopted, we arrive at an equation such as (4) of that Article, from which the conditions of equilibrium are obtained by equating to zero the coefficients of the displacements. But in proceeding thus, we fail to obtain the values of the internal and geometrical forces of the system. Now these forces are, as we have seen, intimately related to the undetermined multipliers; and as these latter are found from the Lagrangian equations, it follows that—

The method of Lagrange gives not only the conditions of equilibrium, but also the internal forces of the system.

A single very elementary example will suffice to render this clear.

Two heavy particles of weights W_1 and W_2 are connected by a rigid rod, and each particle rests on a smooth inclined plane. The inclinations of the planes are i_1 and i_2 and their intersection is horizontal; find the position of equilibrium and the internal and geometrical forces.

Let the line of intersection of the planes be taken as axis of z, let the axis of y be vertical and that of x horizontal. Also let $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$ be the co-ordinates of the particles, and l the length of the rod connecting them. Then the equations of connexion are

$$y_1 - x_1 \tan i_1 = 0,$$
$$y_2 + x_2 \tan i_2 = 0,$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l^2.$$

Hence the Lagrangian equation of equilibrium is

$$-W_1 \delta y_1 - W_2 \delta y_2 + \lambda_1 (\delta y_1 - \tan i_1 . \delta x_1) + \lambda_2 (\delta y_2 + \tan i_2 . \delta x_2)$$
$$+ \tau \{(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) + (z_1 - z_2)(\delta z_1 - \delta z_2)\} = 0,$$

λ_1, λ_2, and τ being the undetermined multipliers.

Equating to zero the coefficients of the separate displacements,

$$-W_1 + \lambda_1 + \tau (y_1 - y_2) = 0,$$
$$-W_2 + \lambda_2 - \tau (y_1 - y_2) = 0,$$
$$\lambda_1 \tan i_1 - \tau (x_1 - x_2) = 0,$$
$$\lambda_2 \tan i_2 - \tau (x_1 - x_2) = 0,$$
$$\tau (z_1 - z_2) = 0.$$

From the last equation we have $z_1 - z_2 = 0$, which shows that both particles must lie in a vertical plane perpendicular to the line of intersection of the inclined planes.

If θ be the inclination of the line joining the particles to the horizon, the other equations give

$$(W_1 + W_2) \tan \theta = W_1 \cot i_2 - W_2 \cot i_1,$$

$$\tau l = \frac{W_1 \sin i_1}{\cos (i_1 - \theta)},$$

$$\lambda_1 = \frac{W_1 \cos \theta \cos i_1}{\cos (i_1 - \theta)},$$

$$\lambda_2 = \frac{W_2 \cos \theta \cos i_2}{\cos (i_2 + \theta)}.$$

The student will easily perceive from Art. 186 that τl is the tension of the rod, and $\lambda_1 \sec i_1$ and $\lambda_2 \sec i_2$ the reactions of the smooth planes. Thus we have the same values of the inclination of the rod and of the internal forces as we should have obtained by the ordinary statical methods.

Suppose now that the equation of virtual work is employed according to the first method; that is, let us write

$$W_1 \delta y_1 + W_2 \delta y_2 = 0,$$
$$\delta y_1 - \tan i_1 . \delta x_1 = 0,$$
$$\delta y_2 + \tan i_2 . \delta x_2 = 0,$$
$$(x_1 - x_2)(\delta x_1 - \delta x_2) + (y_1 - y_2)(\delta y_1 - \delta y_2) + (z_1 - z_2)(\delta z_1 - \delta z_2) = 0,$$

and eliminate the displacements without employing undetermined multipliers. Then we obtain simply the equations

$$z_1 - z_2 = 0,$$
$$(W_1 + W_2) \tan \theta = W_1 \cot i_2 - W_2 \cot i_1,$$

which define the position of equilibrium, without giving the values of the unknown forces of the system.

188.] **Potential of a System of Forces.** Let there be any number of particles, $m_1, m_2, \ldots$ acted on by forces $X_1, Y_1, Z_1, X_2, Y_2, Z_2, \ldots$, and let the co-ordinates of the particles be $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$. Then, if V be such a function of the co-ordinates that

$$\frac{dV}{dx_1} = X_1, \quad \frac{dV}{dy_1} = Y_1, \quad \frac{dV}{dz_1} = Z_1, \ldots,$$

we have

$$V = \int (X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1 + X_2 dx_2 + Y_2 dy_2 + Z_2 dz_2 + \ldots);$$

or, as it may be written for shortness,

$$V = \Sigma \int (X dx + Y dy + Z dz),$$

Σ denoting a summation of the integral for all the particles of the system. The integral may be considered either as *indefinite*, or as performed from any fixed position which the system can geometrically occupy to the position which it occupies at the moment under consideration. Of course it may happen that the forces are such that $\Sigma (X dx + Y dy + Z dz)$ is not the differential of any function of the forces and co-ordinates; when it is the differential of some function, the system belongs to what Thompson and Tait call *Conservative Systems* (*Nat. Phil.*). The function V which belongs to a conservative system is called the *Potential* of the given forces.

189.] **Stability and Instability of Equilibrium.** When a body in equilibrium under the action of given forces is slightly disturbed from its position, it will not, in general, be in equilibrium in the new position. Now the effect of the forces in the new position of the body may be either to drive it back to its original position, or to deviate it still further from this position. In the former case the equilibrium is said to be *stable*, and in the latter *unstable*. For example, take a heavy rod, AB, moveable round a smooth hinge at one end, A. If the rod is placed in a vertical position it will evidently be in equilibrium; but if the end B is vertically *above* A, a slight disturbance will cause the rod to fall from this position; while if B is vertically *below* A, after a slight disturbance the rod will revert to its original position.

190.] **Maximum and Minimum Potential.** When a body or a system of bodies assumes such a position that the potential of the forces acting on it is a maximum, the body or system is in a position of *stable* equilibrium. When, on the contrary, the position of the system is such that the potential has a minimum value, the equilibrium is unstable. Here the terms maximum and minimum are to be understood as they are defined in the Differential Calculus. The complete proof of this principle is kinetical, and it will be found at great length in the *Mécanique Analytique* (6th section of the *Dynamique*, p. 320).

In a very useful particular case, however, a statical proof may be given.

Suppose a system, subject to certain geometrical conditions, to be in equilibrium, and suppose, moreover, that, subject to these conditions, the position of the system is defined by *a single variable*.

In general (Art. 186) the equations of equilibrium are

$$L_1 = 0, \quad L_2 = 0, \ldots L_k = 0,$$

and

$$\Sigma\,(X\delta x + Y\delta y + Z\delta z) = 0.$$

Assuming the forces to have a potential, V, the last equation is

$$\delta V = 0. \qquad (\alpha)$$

Now if all the co-ordinates, $x_1, y_1, z_1, \ldots$, in conformity with the geometrical conditions, $L_1 = 0, \ldots$, are expressible in terms of a single variable, q, V is simply a function of q, and the statical equation can be written

$$\frac{dV}{dq}\cdot\delta q = 0. \qquad (\beta)$$

Hence, in the position of equilibrium $\frac{dV}{dq} = 0$, and therefore V is, *in general,* either a maximum or a minimum, since $\frac{d^2 V}{dq^2}$ will not, *in general,* vanish.

Now it has already been explained (Arts. 185 and 186), that in the equation of equilibrium the coefficient which multiplies each variation is proportional to a force which tends directly to produce this variation; therefore from (β) we see that $\frac{dV}{dq}$ is proportional to a force which tends to produce the displacement denoted by δq, or, in other words, $\frac{dV}{dq}$ is proportional to a force which tends to increase q; and (β) shows that in the position of equilibrium this force must vanish.

Suppose now that, the geometrical equations of condition being still satisfied, the system receives a small displacement for which q becomes $q + \delta q$. Then if $\frac{dV}{dq}$ is denoted by $f(q)$, the value of $\frac{dV}{dq}$ in the new position will be $f(q + \delta q)$; that is, the force called into play by the displacement is

$$f(q) + f'(q)\,\delta q,$$

or

$$\frac{dV}{dq} + \frac{d^2 V}{dq^2} \cdot \delta q.$$

But, by hypothesis, $\frac{dV}{dq} = 0$, therefore the force called into play is

$$\frac{d^2 V}{dq^2} \cdot \delta q.$$

If this force has the same sign as δq, the force called into play increases the displacement, and the equilibrium is unstable; whereas if the sign of the force is opposite to that of the displacement, the force destroys the displacement, and the equilibrium is stable. In the former case $\frac{d^2 V}{dq^2}$ is positive and V a minimum, and in the latter $\frac{d^2 V}{dq^2}$ is negative and V a maximum.

Whether the position of the system depends on a single variable or on several variables, equation (α) is satisfied in every position of equilibrium; but the vanishing of the first differential of a function of several variables is not a sufficient condition for a maximum or minimum value of the function. Hence we cannot assert that every position of equilibrium of such a system is one in which V is either a maximum or minimum. On the contrary, when the position depends on a single variable*, V is, *in general*, either a maximum or a minimum, and the equilibrium is, *in general*, either absolutely stable or absolutely unstable. A position of equilibrium is said to be *absolutely stable* when, after all possible small displacements, the system reverts to its position of equilibrium; and *absolutely unstable* when, after all possible small displacements, it deviates still further from that position.

Since maxima and minima values of a function succeed each other alternately, it is clear that the same is true of the positions of stable and unstable equilibrium of a system.

191.] **Maximum or Minimum Height of the Centre of Gravity.** When gravity is the only force acting on a system of bodies, the potential is simply

$$-W.\bar{z},$$

W denoting the weight of the system, and $\bar{z}$ the height of its centre of gravity above any fixed horizontal plane.

For if w_1 be the weight of any one body of the system and z_1 the height of its centre of gravity above a fixed horizontal plane, the virtual work of w_1 for a small increment of z_1 will be (Art. 66, p. 79) $\quad -w_1 . \delta z_1.$

Hence† $\delta V = -w_1 \delta z_1 - w_2 \delta z_2 - \dots$.

But $W.\bar{z} = w_1 . z_1 + w_2 . z_2 + \dots$; therefore $\delta V = -W.\delta \bar{z}$, and $V = -W.\bar{z}$.

Now the maximum value of V will occur when $\bar{z}$ is least; hence *when the centre of gravity of any system of bodies is in the lowest position that it can occupy consistently with the geometrical conditions of the system, that system is in a position of stable equilibrium; and when its centre of gravity is in the highest position, the system is in a position of unstable equilibrium.*

Unless the position of the system depends on a single variable, we cannot assert conversely that a position of equilibrium is one in which the height of the centre of gravity is either a maximum or a minimum.

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* The system in this case is called by French writers *un système à liaisons complètes.*

† This assumes that none of the geometrical forces required for a position of equilibrium are infinite; for the term $\lambda \delta L$ cannot be assumed to vanish, even though $\delta L = 0$, if λ is infinite.

If any bodies of the system rest on *rough* curves or surfaces, the equation of virtual work will involve the reactions of these curves or surfaces for displacements along them. Hence we have no longer the equation $W.\delta\bar{z}=0$, and the principle of maximum or minimum height of the centre of gravity does not hold.

Even when the position depends on one variable, it may happen that in a position of equilibrium the height of the centre of gravity is neither a maximum nor a minimum. Take, for example, the case of a heavy particle placed at a point of inflexion on a smooth curve in a vertical plane, the tangent at the point being horizontal. The particle is evidently in equilibrium, since for a small displacement $P\delta z$ is zero, P being the weight and z the height of the particle. But z is neither a maximum nor a minimum, and the equilibrium, accordingly, is stable for a small displacement along the upper part of the curve, and unstable for a displacement along the lower part.

When the connexions of the system are complete (see note, p. 312) the centre of gravity describes, in all positions of the system compatible with the given conditions, a curve which is sometimes very easily found. In the position of equilibrium the centre of gravity will be the point of contact of a horizontal tangent to this curve, and in this manner we can most readily perceive the nature of the equilibrium of the body.

When the connexions of the system are not complete, it may happen that its centre of gravity is constrained, in all displacements compatible with the connexions, to describe a fixed *surface*. In this case the position of equilibrium will be one in which the tangent plane to this surface at the centre of gravity is horizontal; and if the surface lies entirely below the tangent plane in the neighbourhood of the point of contact, the equilibrium will be unstable, as in the case of a curve; if the surface lies above the tangent plane, the equilibrium will be stable; and if the tangent plane intersects the surface in a real curve in the neighbourhood of contact, the equilibrium will be stable for some displacements and unstable for others.

192.] **Continuous Equilibrium.** If in all positions of the system, compatible with the geometrical conditions, the statical equation

$$\delta V = 0$$

is satisfied, every position is one of equilibrium. Writing down this equation in all positions, and adding the equations thus obtained is evidently the same thing as integrating it. Hence if all positions of the system are positions of equilibrium, the applied forces must satisfy the equation

$$V = \text{constant}.$$

In the particular case of a heavy system under the action of gravity alone, V is $-W.\bar{z}$; therefore if a system be continuously in equilibrium under the action of gravity, the centre of gravity of the system for all displacements compatible with the conditions moves in a fixed horizontal plane, or, in other words, *maintains a constant height.*

Examples.

1. A heavy beam, AB (fig. 127, p. 145) rests on two smooth inclined planes; find the nature of its equilibrium.

It is very easy to prove that if the right line AB moves between two fixed right lines, OA and OB, the given point G on AB describes an ellipse whose equation with reference to OA and OB as axes of x and y is

$$\frac{x^2}{b^2} + 2\,\frac{xy}{ab}\cos(\alpha+\beta) + \frac{y^2}{a^2} = 1.$$

The centre of this ellipse is the point O. In the position of equilibrium G is the point of contact of a horizontal tangent to this ellipse. Now two such tangents can be drawn, one above the intersection of the inclined planes and the other below it. There are, therefore, two positions of equilibrium; that with which we were concerned in the example of p. 145 is obviously the position in which G is at a maximum height, and it is, therefore, *unstable;* the other requires the planes to be prolonged below their line of intersection, and as it also requires the reactions of the planes to assume impossible directions, it is physically impossible. It would, however, be possible if the planes were replaced by smooth fixed rods to which the extremities of the beam are attached by rings. The second position of equilibrium would then be *stable.*

The impossibility in a certain case of any position of equilibrium, except one of continuous contact with either plane, which has been signalized in p. 146, is now easily explained. It occurs when the point of contact of the horizontal tangent to the ellipse locus of G falls underneath the plane (α) or the plane (β), so that it is not a possible position of G.

The problem may be solved by a purely analytical method. If z is the height of the centre of gravity of the beam, it will be easily found that in the position of equilibrium

$$\frac{d^2z}{d\theta^2}=-\frac{\sin a \sin\beta \cos\theta}{(a+b)\sin(a+\beta)}\{(a+b)^2+(a\cot a-b\cot\beta)^2\}.$$

2. Two given points of a body rest each in contact with two smooth inclined planes; show that the equilibrium of the body is unstable.

We know that if two vertices of a *given* triangle move along two fixed right lines, the locus of the third vertex is an ellipse whose centre is the intersection of the given lines.

Hence if we consider a given triangle in the body to be formed by the centre of gravity and the two points which are in contact with the planes, we see that the locus of the centre of gravity is an ellipse whose centre is at the intersection of the inclined planes. Now in the position of equilibrium the centre of gravity is the point of contact of a horizontal tangent to this ellipse. Hence the only possible position of equilibrium is one in which the height of the centre of gravity is a maximum; therefore the equilibrium is unstable; and if, as explained in the last Example, the point of contact of the tangent falls underneath either plane, the only position of equilibrium of the body is one of continuous contact with one of the planes. The student will find several particular examples of this problem in Walton's *Mechanical Problems* (pp. 164, &c.), where the solutions are analytical.

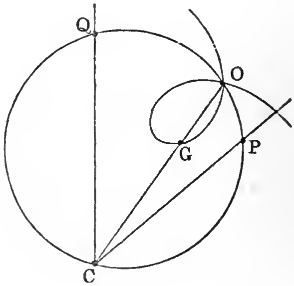

Fig. 214.

3. A heavy body has two plane surfaces, CP and CQ (fig. 214) which rest against two smooth fixed pegs, P and Q, the line PQ making any angle with the horizon; show that the positions of equilibrium are determined by drawing horizontal tangents to a Limaçon.

The centre of gravity and the pegs must lie in one vertical plane, which is that of the figure. Since P and Q are fixed points and the angle at C between the plane faces is constant, the circle described round the triangle PCQ is fixed in space. Again, let G be the centre of gravity of the body. Then since CG and CP are lines fixed in the body, the angle GCP is given; and if CG meet the circle in O, the point O is fixed in space; also the distance CG is given.

Hence in all positions of the body—i.e., in all positions of C on the circle—the centre of gravity is found by drawing the line OC from O to the circumference of the circle, and taking a constant length, CG, on this line. The curve deduced in this way from a circle is a Limaçon, which is, therefore, the locus of the centre of gravity.

A particular example has been already discussed in p. 150.

4. A heavy plane body of any shape is suspended from a smooth peg, fixed in a vertical wall, by means of a string of given length, the

extremities of which are attached to two fixed points in the body. Determine the nature of the equilibrium.

This problem, so far as the positions of equilibrium are concerned, has been already discussed (Ex. 11, p. 153). We propose here to show that there are two positions of stable and one position of unstable equilibrium. In the figure of the Example referred to, the point of contact of GP_3 with the evolute is between G and P_3; the point of contact of GP_1 is between G and P_1; and the point of contact of GP_2 is on P_2G produced. Now it is easy to see that GP_3 is a line of maximum length drawn from G to the ellipse. For, let Q be a point on the ellipse close to P_3, and let QC be the normal at Q. Then C is the centre of curvature, and therefore the point of contact of GP_3 and the evolute. Hence $CP_3 = CQ$, therefore $GP_3 = GC + CQ$, which is $> GQ$, therefore $GP_3 > GQ$, and GP_3 is, therefore, a maximum.

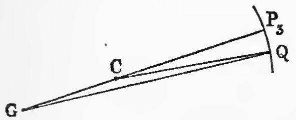

Fig. 215.

In the same way GP_1 is a maximum and GP_2 a minimum distance of G from the ellipse.

Hence, in the positions of equilibrium, GP_1 and GP_3 are maximum distances of the centre of gravity from the peg. The positions in which these lines are vertical are, therefore, positions of stable equilibrium. And since GP_2 is a minimum depth of G, the position in which GP_2 is vertical is one of unstable equilibrium.

5. To find the nature of the equilibrium of the beam in Example 5, p. 297.

Take any position of the beam (in which, of course, the lines GW, AR, and PS (p. 149) do not meet in a point). Then, if y is the ordinate of P, the point of contact of the beam and the curve, referred to a fixed horizontal axis, the ordinate of G will be

$$y + (GA - PA)\cos\theta,$$

or

$$y + a\cos\theta - x\cot\theta.$$

Denoting this by $\bar{y}$, we have

$$\frac{d\bar{y}}{d\theta} = \frac{dy}{d\theta} - a\sin\theta + \frac{x}{\sin^2\theta} - \cot\theta \cdot \frac{dx}{d\theta}.$$

Now $$\frac{dy}{dx} = \cot\theta, \quad \therefore \quad \frac{dy}{d\theta} - \cot\theta\,\frac{dx}{d\theta} = 0.$$

Hence $$\sin^2\theta\,\frac{d\bar{y}}{d\theta} = -a\sin^3\theta + x.$$

Differentiating this, and remembering that in the position of equilibrium $\dfrac{d\bar{y}}{d\theta} = 0$, we have

$$\sin^2\theta\,\frac{d^2\bar{y}}{d\theta^2} = \frac{dx}{d\theta} - 3a\sin^2\theta\cos\theta. \tag{1}$$

Again, since $\cot\theta = \frac{dy}{dx}$, we have

$$-\operatorname{cosec}^2\theta\,\frac{d\theta}{dx} = \frac{d^2y}{dx^2}.$$

But if ρ is the radius of curvature of the curve at P,

$$-\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \sin^3\theta\,\frac{d^2y}{dx^2}.$$

Therefore $\frac{d\theta}{dx} = \frac{1}{\rho\sin\theta}$, and (1) gives

$$\begin{aligned}\sin\theta\,\frac{d^2\bar{y}}{d\theta^2} &= \rho - 3a\sin\theta\cos\theta\\ &= \rho - 3PO.\end{aligned}$$

Hence, since $\sin\theta$ is necessarily positive, $\frac{d^2\bar{y}}{d\theta}$, will be positive, and $\bar{y}$ therefore a minimum if

$$\rho > 3PO.$$

The equilibrium will therefore be stable or unstable according as $\rho >$ or $< 3PO$.

To arrive at this result, it would have been sufficient to demonstrate it for a circle, which is very easily done. The curve in the neighbourhood of P may be replaced by the circle of curvature at this point.

6. Prove geometrically that the equilibrium of the beam in Example 2, p. 147, is stable.

7. Two uniform heavy rods freely jointed together at a common extremity rest on a smooth parabola whose axis is vertical and vertex upwards; find the position of equilibrium.

Ans. Let the weights of the rods be P and Q, their lengths $2a$ and $2b$, and let them make angles θ and ϕ, respectively, with the vertical in the position of equilibrium; then these angles are determined from the equations

$$\begin{aligned}Pa\sin^3\theta + (P+Q)\,m\cot\phi &= 0,\\ Qb\sin^3\phi + (P+Q)\,m\cot\theta &= 0,\end{aligned}$$

$4m$ being the latus rectum of the parabola.

[Taking the tangent at the vertex as axis of y, the abscissa of the point of intersection of two tangents, $y = tx - \frac{m}{t}$ and $y = t'x - \frac{m}{t'}$, is $-\frac{m}{tt'}$. Hence $(P+Q)\bar{x} = Pa\cos\theta + Qb\cos\phi + (P+Q)\,m\cot\theta\cos\phi$. Then $\bar{x}$ is to be a max. or min.]

8. A uniform heavy rod, AB, moveable about a smooth hinge fixed at A, has its extremity B connected with a string which, passing over a smooth pulley at a point C vertically over A, sustains a given weight

which rests on a smooth inclined plane passing through C. Find the positions of equilibrium, and the nature of each.

Ans. Let W and $2a$ be the weight and length of the rod; P the weight on the plane whose inclination to the horizon is i; $2c$ the distance AC, and θ the inclination of the rod to the vertical. Then, if $(c-a)\,W < 2\,Pc \sin i$, there will be three positions of equilibrium defined by the equations

$$\theta = 0, \ \cos\theta = \frac{W^2(a^2+c^2)-4\,P^2c^2\sin^2 i}{2ac\,W^2}, \ \text{and} \ \theta = \pi.$$

The first and last positions are stable and the intermediate one is unstable.

If $(c-a)\,W > 2\,Pc\sin i$, there is no intermediate position, and the first and last positions are unstable and stable, respectively.

9. One end of a beam rests against a smooth vertical plane, and the other on a smooth curve in a vertical plane; find the nature of the curve so that the beam may rest in all positions.

Ans. An ellipse whose axis major is the horizontal line described by the centre of gravity of the beam, the axis minor lying in the vertical plane.

10. A uniform heavy rod rests inside a smooth fixed sphere whose diameter is equal to the length of the rod. In all positions of the rod its centre of gravity is fixed; hence the rod should rest in all positions; but, except in the vertical position, it is impossible that the acting forces can give equilibrium. Explain this.

(See note, p. 313.)

11. A uniform rod rests in all positions with its extremities on two smooth curves in a vertical plane; given the equation of one, find that of the other.

Ans. Let the axis of y be vertical, $2a$ the length of the rod, h the constant height of the centre of the rod, and $x = \phi(y)$ the equation of one curve; then the equation of the other will be

$$x = \phi\,(2h-y)-2\sqrt{a^2-(h-y)^2}.$$

12. Find the general equation of a smooth curve (in a vertical plane) on which if the ends of a uniform rod are placed, the rod will rest in all positions.

Ans. If the line described by the centre of gravity is axis of x, the equation is of the form $[\phi(y^2)+x]^2+y^2 = a^2$, where $2a =$ length of rod, and $\phi(y^2)$ is a function which does not change sign with y.

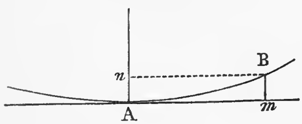

Fig. 216.

193.] **Expansion of the Abscissa and Ordinate of a Curve in Powers of the Arc.** Let A and B (fig. 216) be any points on a curve, and let Am and An be the tangent and normal at A. Also let ψ be the angle between the normals at A and B, and let

Am $(=x)$ and Bm $(=y)$ be the co-ordinates of B with reference to the tangent and normal at A as axes.

Then, by Maclaurin's Theorem we have

$$\psi = \psi_0 + s\left(\frac{d\psi}{ds}\right)_0 + \frac{s^2}{1.2}\left(\frac{d^2\psi}{ds^2}\right)_0 + \ldots,$$

s denoting the arc AB, and ψ_0, $\left(\frac{d\psi}{ds}\right)_0$, ... the values of ψ and its differential coefficients at A.

Now $\psi_0 = 0$, and $\frac{d\psi}{ds} = \frac{1}{\rho}$, where ρ is the radius of curvature. Hence

$$\psi = \frac{s}{\rho} + \frac{s^2}{1.2}\frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{s^3}{1.2.3}\frac{d^2\left(\frac{1}{\rho}\right)}{ds^2} + \ldots \tag{1}$$

the suffix being omitted, it being understood that ρ is the radius of curvature at A.

Again, we have

$$x = x_0 + s\left(\frac{dx}{ds}\right)_0 + \frac{s^2}{1.2}\left(\frac{d^2x}{ds^2}\right)_0 + \ldots;$$

also
$$\frac{d^2x}{ds^2} = -\frac{1}{\rho}\frac{dy}{ds}, \text{ and } \frac{d^2y}{ds^2} = \frac{1}{\rho}\frac{dx}{ds}.$$

But

$$\left(\frac{dx}{ds}\right)_0 = 1, \text{ and } \left(\frac{dy}{ds}\right)_0 = 0; \text{ therefore } \left(\frac{d^2x}{ds^2}\right) = 0, \left(\frac{d^2y}{ds^2}\right)_0 = \frac{1}{\rho},$$

and the successive differential coefficients are calculated with ease.

We thus obtain

$$Bn = x = s - \frac{s^3}{6\rho^2} + \frac{s^4}{8\rho^3}\frac{d\rho}{ds} + \ldots, \tag{2}$$

$$An = y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds} - \frac{s^4}{24}\left\{\frac{1}{\rho^3} - \frac{2}{\rho^3}\left(\frac{d\rho}{ds}\right)^2 + \frac{1}{\rho^2}\frac{d^2\rho}{ds^2}\right\} + \ldots. \tag{3}$$

194.] **Equilibrium of a Heavy Body resting on a Fixed Rough Surface.** Let AD (fig. 217) be a fixed rough surface on which a heavy body, AC, rests, under the action of gravity, at a single given point A; and let this body receive a slight displacement of rolling on the fixed surface.

We propose to investigate the nature of the equilibrium. The figure represents a section of the bodies made by the vertical plane through their common normal, AO, in which the rolling takes place. We suppose the normal AO to be vertical.

Then, since in the position of equilibrium the body AC is acted on by only two forces—namely, its own weight and the total resistance of the fixed surface—its centre of gravity, G, must be vertically over the point of contact.

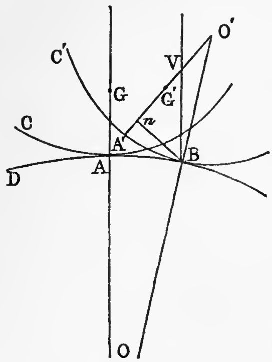

Fig. 217.

Let the point A of the rolling body come to A', and G to G', the new point of contact being B, and the new common normal OO'. Draw the vertical line BV, meeting $A'O'$ in V.

Then, if $A'V$ is $> A'G'$, the weight of the body acting through G' will produce a rotation round B which will send the body back to its original position; while, if $A'V$ is $< A'G'$, the rotation produced by the weight will be in the opposite sense, and the body will deviate still further from its original position. For stability, therefore,

$$A'V > A'G'. \tag{1}$$

Let ρ and ρ' be the radii of curvature of the curves AD and AC at A, and let ψ and ψ' be the angles AOB and $A'O'B$. Then drawing Bn perpendicular to $A'O'$, we have

$$A'V = A'n + nV = A'n + Bn \cot A'VB;$$

but $\angle A'VB = \psi + \psi'$; therefore the condition for stability is

$$A'n + Bn \cot(\psi + \psi') > A'G',$$

or, denoting $A'G'$ (or AG) by h,

$$Bn > (h - A'n) \tan(\psi + \psi'). \tag{2}$$

Now, carrying approximations as far as s^3, it will be found from equation (1) of last Article that

$$\tan(\psi + \psi') = s\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)$$
$$+ \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\},$$

s being the common length of the arcs AB and $A'B$.

Substituting this, and the values of Bn and $A'n$ from last Article, in (2), the condition for stability is

$$s - \frac{s^3}{6\rho'^2} > \left(h - \frac{s^2}{2\rho'} + \frac{s^3}{6\rho'^2}\frac{d\rho'}{ds'}\right)\left[\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)s + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)\right.$$

$$\left. + \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^3\right\}\right],$$

or
$$1 - \frac{s^2}{6\rho'^2} > h\left(\frac{1}{\rho}+\frac{1}{\rho'}\right) + h\frac{s}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right)$$

$$+ h\frac{s^2}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^3\right\} - \frac{s^2}{2\rho'}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\ldots \quad (3)$$

Neglecting all powers of s, the first condition for stability is

$$1 > h\left(\frac{1}{\rho}+\frac{1}{\rho'}\right),$$

or
$$h < \frac{\rho\rho'}{\rho+\rho'}. \quad (4)$$

If $h > \dfrac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be unstable.

A special case occurs when $h = \dfrac{\rho\rho'}{\rho+\rho'}$, and this is commonly called the 'neutral' case, or the equilibrium is said to be neutral. We shall, however, call this the *critical* case.

To find the real nature of the equilibrium in this case, we revert to the general condition (3), and neglect all powers of s beyond the first. The condition for stability now is

$$0 > \frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}.$$

Hence when $h = \dfrac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be stable or unstable according as

$$\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'} \quad (5)$$

is negative or positive.

The bodies are, however, frequently in contact at *vertices*, or points of maximum or minimum curvature, and then

$$\frac{d\frac{1}{\rho}}{ds} \text{ and } \frac{d\frac{1}{\rho'}}{ds'}$$

are both zero. Hence the condition (5) fails to determine the nature of equilibrium. Reverting to the condition (3), the terms as far as s^2 destroying each other on both sides, we see that equilibrium will be stable if

$$-\frac{1}{6\rho'^2} > \frac{h}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^3\right\} - \frac{1}{2\rho'}\left(\frac{1}{\rho}+\frac{1}{\rho'}\right),$$

or, substituting $\dfrac{\rho\rho'}{\rho+\rho'}$ for h, if

$$\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} < -\frac{(\rho+\rho')(\rho+2\rho')}{\rho^3\rho'^2}; \tag{6}$$

and in the contrary case the equilibrium will be unstable.

If the lower surface is concave, instead of convex, to the upper, the conditions are obtained by changing the sign of ρ. Thus, the equilibrium will be stable or unstable, according as

$$h < \text{or} > \frac{\rho\rho'}{\rho-\rho'},$$

and in the critical case, the equilibrium will be stable or unstable, according as

$$\frac{d\frac{1}{\rho'}}{ds'} - \frac{d\frac{1}{\rho}}{ds}$$

is negative or positive; and in case of contact at vertices the condition (6) is to be similarly modified.

If the body rest on a *plane* surface, $\rho = \infty$, and the differential coefficients of $\dfrac{1}{\rho}$ are all zero. Hence the limiting value of h for stability is ρ'; but if $h = \rho'$, the equilibrium will be stable or unstable according as $\dfrac{d\rho}{ds'}$ is positive or negative; and if the point of contact is a vertex, equilibrium will be stable or unstable, according as

$$\frac{d^2\frac{1}{\rho'}}{ds'^2}$$

is negative or positive*.

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* Different methods of arriving at the conditions for stability have been published in the *Quarterly Journal of Pure and Applied Mathematics* by Professor Curtis (vol. ix, p. 41), and Mr. Routh (vol. xi, p. 102). The kinetical method of

Examples.

1. If a cone of the same substance and of equal base with a hemisphere be fixed to the latter, so that their bases coincide, find the greatest height of the cone in order that the equilibrium may be stable, when the hemisphere rests symmetrically on a horizontal plane. (Walton's *Mechanical Problems*, p. 185.)

Ans. The height of the cone must be $< r\sqrt{3}$, r being the radius of the hemisphere.

2. Prove that any body with a plane base, resting on a fixed rough spherical surface, will, when the height of its centre of gravity has the critical value, be in unstable equilibrium.

3. A heavy body whose section in the plane of displacement is a catenary, resting on a rough horizontal plane, has its centre of gravity at the critical height; prove that the equilibrium is really stable.

(The condition (6) reduces in this case to $\dfrac{d^2\frac{1}{\rho'}}{ds'^2} < 0$ for stability.)

4. A heavy body in the shape of a paraboloid of revolution, placed on a rough horizontal plane, has its centre of gravity at the critical height; determine this height, and find the real nature of the equilibrium.

Ans. The critical height = the radius of curvature of the generating parabola at the vertex, and the equilibrium is really stable.

5. In the critical case, if both of the conditions (5) and (6) fail, prove that the equilibrium will be stable or unstable, according as

$$\frac{d^3\frac{1}{\rho}}{ds^3} + \frac{d^3\frac{1}{\rho'}}{ds'^3} - \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\left(\frac{4}{\rho} + \frac{1}{\rho'}\right)\frac{d\frac{1}{\rho'}}{ds'}$$

is negative or positive, the surfaces being convex towards each other.

6. A uniform heavy bar, AB, moveable in a vertical plane round a fixed smooth axis passing through A has a string attached to the end B; this string passes over a fixed pulley C vertically over A. Find the positions of equilibrium, and determine whether they are stable or unstable.

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treatment adopted by the latter is very exhaustive. The method in the text was employed independently by Professor Wolstenholme and the author.

It may be well to caution the student against the error of replacing the sections, AD and AC, of the surfaces in contact by their osculating circles at A. For, if we do this, the condition (5) necessarily disappears, and the application of (6) is not allowable, since, to the third power of the arc, the value of $A'n$ is not the same for the circle of curvature as for the curve AC, as at once appears from the expression for $A'n$ given by equation (3) of last Article. The nature of the equilibrium, therefore, as determined from the osculating circles is erroneous.

Ans. Let W = weight of bar, $2a$ its length, P = suspended weight, $AC = h$, $\theta = \angle CAB$. Then the positions of equilibrium are given by the equations

$$\theta = 0, \quad \cos\theta = \frac{a}{h} + \left(\frac{1}{4} - \frac{P^2}{W^2}\right)\frac{h}{a}, \quad \text{and} \quad \theta = \pi.$$

The first will be stable if $\dfrac{2h}{h-2a} > \dfrac{W}{P}$, and then the second (when it exists) will necessarily be unstable and the third stable. If the second does not exist, the third will be opposite in nature to the first.

[To find the condition for stability in this problem, take any position of the bar and calculate the moment of force tending to turn it round A. If M = this moment, and $\phi = \angle ACB$,

$$M = Ph\sin\phi - Wa\sin\theta. \qquad (1)$$

Also
$$h\sin\phi = 2a\sin(\theta+\phi). \qquad (2)$$

Now $M = 0$ in a position of equilibrium; and if $\dfrac{dM}{d\theta}$ is positive, a slight increase of θ will call into play a moment tending to restore equilibrium.

In the position $\theta = 0$, we have from (2)

$$\frac{d\phi}{d\theta} = \frac{2a}{h-2a};$$

and from (1)
$$\frac{dM}{d\theta} = Ph\frac{d\phi}{d\theta} - Wa.$$

Therefore, &c. Of course this might have been solved by Art. 191.]

7. If the equilibrium in the first position is critical, find its real nature.

Ans. It is really unstable.

[In the position $\theta = 0$, it will be found from (2) that $\dfrac{d^2\phi}{d\theta^2} = 0$,

$$\frac{d^3\phi}{d\theta^3} = -\frac{2ah(h+2a)}{(h-2a)^3}; \quad \frac{d^2M}{d\theta^2} = 0, \quad \frac{d^3M}{d\theta^3} = -.]$$

8. Determine whether the equilibrium of the beam in example 12, p. 138, is stable or unstable.

Ans. Unstable. [Either by taking the restoring moment about O, or by the maximum or minimum value of the potential; the potential $= -P(a+b)\cos\theta - Wa\sin\theta$.]

195.] **Definition of Work.** If a force *actually* displaces its point of application in such a manner that the displacement has an orthogonal projection along the direction of the force, this force is said to perform *work*; and if the force is constant during the displacement, the product of the force and the projection of the displacement along its direction is called the *work done* by the force.

Thus, suppose that during the passage of a material particle along a curve ABD (fig. 218) it is continually acted on by a force, P, constant in both magnitude and direction. Then if dp denote the projection of any elementary arc of the curve along the direction of P, the work done by P in this displacement is $P.dp$; and the work done in this passage from A to B is $\int Pdp$, or $P \times$ (the projection of AB along the direction of P), since P is constant during the motion.

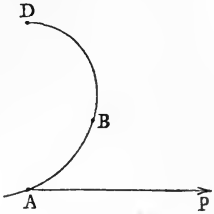

Fig. 218.

Suppose the point D to be such that AD is perpendicular to the direction of P. Then the whole work done by P on the particle during the motion from A to D is zero, *whatever be the shape of the path pursued between A and D.*

When the forces acting on a particle are variable with the position occupied by it, we have to consider the elementary work done for a small displacement of the particle; and to find the whole work done by the forces during the passage of the particle *from one position to another*, this elementary work must be integrated between the extreme positions considered.

In the most useful application of the principle of work the forces acting on a given system are functions of the co-ordinates of their points of application, and do not depend on the velocities of these points; and it is solely with forces of this description that we shall be concerned.

It must be pointed out, however, that in considering the work which such forces *are capable* of doing on a particle or system of particles while this system is displaced from one position to another, all conceptions of *time* are here left out of consideration. The work which a given system of applied forces performs on a given material system during the passage of this system by any route from the position (A) to the position (B) in no way involves the *time* or the *manner* in which the passage is effected. The different particles of the system may have in one case moved more or less swiftly than in another from the first to the second position, and yet the work done by the forces (which are functions of *co-ordinates* only) is the same in both cases.

The theorems which have already been given for virtual work apply evidently to work *actually* done.

Thus, as in p. 287, we see that for a small actual displacement of a particle occupying the position (x, y, z) the work done by the force acting on it is

$$Xdx + Ydy + Zdz,$$

the components of the force along the axes being X, Y, Z, and the components of the displacement being dx, dy, dz. The work performed on the particle in moving from one position to another is then

$$\int (Xdx + Ydy + Zdz),$$

the force acting on the particle being a function of the co-ordinates of the particle, and the integration being performed from the values of the co-ordinates in the first position to their values in the second.

If there are several particles in the system, each acted on by given forces, the work performed on the system will be

$$\Sigma \int (Xdx + Ydy + Zdz),$$

the integration being performed for each particle from its first to its second position, and Σ denoting (as in Art. 188), a summation of this integral for all the particles of the system.

The case of most usual occurrence is that in which the forces belong to a conservative system (see Art. 188), or, in other words, when

$$\Sigma (Xdx + Ydy + Zdz)$$

is the perfect differential of a function, V, of the co-ordinates of the particles acted on.

In this case if dW denote the elementary work done on the system for a small displacement, we have

$$dW = dV,$$

and the work done in the passage of the system from one position to another is given by the equation

$$W = V - V_0,$$

V_0 denoting the value of V in the first position.

Since V is a function of co-ordinates only, the value of $V - V_0$ depends merely on the original and final positions of the material system, *and not at all on the route by which the system has moved from the one to the other.*

196.] **Unit of Work.** Since, by definition, work is the product of a force and a line, the unit of work will be the product of a unit of force and a unit of length. If the unit of force is a kilogramme, and the unit of length a metre, the unit of work will be done when a force of one kilogramme drags its point of application through one metre along its line of action. Thus, if a body whose weight is a kilogramme is lifted vertically through a metre by a force which just overcomes its weight, this applied force does a unit of work, which is called a kilogramme-metre. In the same case the weight of the body does a *negative* unit of work (see Art. 54).

197.] **Energy. Potential Work.** Energy means *capacity for doing work.* This capacity is possessed by a body in motion. For the velocity of the body might be made use of for causing the body to ascend vertically against the attraction of the earth, i.e., to do work against resistance. The exact measure of the amount of work which a particle weighing w grammes moving with a velocity of v centimetres per second can do against resistance before its motion is completely destroyed is $\frac{wv^2}{2g}$ gramme-centimetres, where g is the velocity in centimetres per second acquired in one second by a body falling vertically in vacuo.

Work is always done against some resistance. The work which is done by a force in moving a particle from one position to another is done against the inertia of the particle, or its resistance to acceleration. Thus work is the equivalent of energy, and energy is reconvertible into work at the rate indicated by the expression $\frac{wv^2}{2g}$. If a system consists of any number of particles moving in any directions, its total energy is $\Sigma\frac{wv^2}{2g}$, the summation including all the particles, and the different directions of their velocities being of no account.

This is an anticipation of elementary kinetics, and is here used only for the purpose of pointing out to the student what work done on a system is converted into.

The *Potential Work* of a system of forces in any given configuration of their points of application is *the amount of work which they are capable of doing in moving their points of application from any chosen configuration to the given one.*

This chosen configuration is quite arbitrary. Thus, the quantity of work which the forces applied to a system are capable of doing during the passage of the system from one position to another is (Art. 196)

$$V - V_0,$$

V and V_0 being the potentials, or certain functions of the coordinates, belonging to the two positions considered; and the two positions may be taken as defined by the values of the function V belonging to them respectively.

The zero position of the system (that corresponding to V_0) is generally chosen in such a way that in any other position, practically considered, V shall be $> V_0$ (Thompson and Tait, *Nat. Phil.*); or, in other words, the zero position of the system is such that the work done by the acting forces in moving it to any other position considered in our investigation shall be positive.

198.] **Inclusion of Internal Forces.** When any of the bodies of a system, acted on by given forces, are connected by elastic rods or strings, or when they mutually attract or repel each other, as has been already explained (Art. 97), these forces may or may not be brought into the equation of virtual work, according to the nature of the virtual displacement chosen.

In finding the figure of equilibrium of such a system we have hitherto supposed it known, and determined the requisite conditions accordingly.

We may, however, include in the potential work of the forces not only the potential work of the external (or applied) forces, but also that of the internal forces. Thus the total potential work of the system of forces will be the sum of the works of the applied and internal forces; and equation (a) of Art. 190 shows that in the position of equilibrium *the variation of the total potential work of the forces of the system is zero.* This principle will serve to determine the figure of equilibrium of the system without presupposing it.

199.] **Criterion of Stability and Instability.** Since in a position of absolutely stable equilibrium V is a real maximum, and in a position of absolutely unstable equilibrium V is a real minimum (Art. 190), it follows that in the former case the applied and internal forces would do negative work on the system if its position were slightly altered; and in the latter they would do positive work.

A configuration of absolutely stable equilibrium is, then, such that the applied and internal forces cannot do positive work in any small displacement of the system; and a configuration of absolutely unstable equilibrium is one in which every change of position involves the doing of positive work.

And *in general* (see Art. 190) in a position of equilibrium these forces will do positive work for some displacements and negative for others.

Examples.

1. Find the work done in drawing up a Venetian blind.

Ans. Let n be the number of bars, a the interval between them, and W the weight of the blind; then the work is $\frac{n+1}{2} \cdot Wa$.

2. A and B are two fixed points which are connected by any curve, APB; at each point, P, of this curve there acts a force, F, directed towards a fixed point, O, the force being a function of the distance OP. If θ is the angle between OP and the tangent to the curve at P, and ds an element of the curve at P, prove that $\int F \cos\theta\, ds$ taken from A to B is independent of the curve.

3. Prove that the work done in dragging a heavy body up a rough inclined plane, without acceleration, by a force parallel to the plane, is equal to the work done in dragging the body along the base of the plane (supposed equally rough), together with the work done in lifting it vertically through the height of the plane.

4. A heavy body is dragged, without acceleration, up a rough inclined plane by a force whose line of action always passes through a fixed point; prove that the work done in dragging the body through a given height, h, is

$$Wh\,(1+\mu\cot i)-\mu\, Wp\cos i\,(\mu+\tan i)\log\frac{s+p\mu}{s'+p\mu},$$

where i is the inclination of plane, p the perpendicular from the fixed point on the plane, s the initial, and s' the final distance of the body from the foot of this perpendicular.

Miscellaneous Examples.

1. Two equal heavy spheres rest inside a hollow right cone, and against each other; the cone (which has no base) rests on a horizontal plane, the vertex being uppermost; only one sphere rests in contact with the ground. Find the least weight of the cone consistent with equilibrium.

Ans. Let a vertical plane through the centres of the spheres cut the cone in a triangle ABC, in which C is the vertex of the cone; let $\angle CAB = \angle CBA = \beta$; let ϕ be the angle between the line joining the centres of the spheres and the side BC; let r and c be the radii of the spheres and of the base of the cone, respectively, and W the weight of each sphere; then the least weight of the cone is

$$W\frac{\cos(\beta+\phi)}{\cos\phi}\left\{2\frac{r}{c}\left(\cos\beta\cot\frac{\beta}{2}-\cos\phi\right)-3\cos\beta\right\}.$$

2. A heavy triangular lamina, ABC, of uniform thickness and density, is suspended successively from the vertices A and B; show that if any side in the second position is at right angles to its first position

$$5c^2 = a^2+b^2.$$

(The bisectors of the sides CA and CB drawn from B and A must be at right angles to each other.)

3. A heavy rod hangs from a fixed smooth pulley by means of a string attached to its extremities; find the tension of the string. (See Example 30, p. 157.)

Ans. If W is the weight of the rod, the tension

$$= W.\frac{a}{2c}\sqrt{\frac{c^2-k^2}{a^2-c^2}},$$

with the notation of the example referred to.

4. A heavy rectangular block is laid on the less steep of two smooth inclined planes which slope in the same direction and intersect in a horizontal line, an edge of the block coinciding with the line of intersection of the planes. To the middle point of the upper edge is attached a cord which passes over a smooth pulley and sustains a weight; determine the condition of equilibrium, and supposing that in any case equilibrium is about to be broken, find how this will happen.

5. A uniform board in the shape of an isosceles triangle rests on two smooth planes equally inclined to the horizon, the base of the triangle being horizontal, and the vertex upwards; the board is cut into two equal portions by a plane passing through its vertex; find the inclination of the planes if equilibrium continues to exist.

Ans. If h is the length of the perpendicular from the vertex on the base, and c the length of the base,

$$\tan i = \frac{c}{3h}.$$

6. A solid right cone rests with its base in contact with two smooth planes equally inclined to the horizon, the base being horizontal and the vertex upwards; find the inclination of the planes such that if the cone is cut into two equal portions by a plane through the vertex, the equilibrium of the pieces will not be troubled.

Ans. If h is the height and r the radius of the base of the cone,

$$\tan i = \left(1-\frac{1}{\pi}\right)\frac{r}{h}.$$

CHAPTER XIII.

EQUILIBRIUM OF FLEXIBLE STRINGS.

200.] **Perfectly Flexible String.** A string is said to be *perfectly flexible* when at every point in its length it can be bent round all lines passing through the point perpendicularly to the tangent line without the expenditure of work.

From this definition it follows that the internal force, or mutual action between the particles at each side of any normal section of such a string, has no component in the plane of the section; this force must, therefore, be entirely normal to the section; or, in other words, *the internal force in a perfectly flexible string is at every point directed along the tangent line to the string.*

This internal force we have called the *tension* of the string, and, like all internal forces in a system, it is a mutual action between parts of the system. This has been sufficiently explained already (p. 25). In the sequel we shall use the term *flexible string* as equivalent to perfectly flexible string.

201.] **Imperfectly Flexible String.** No effort is required to bend a perfectly flexible string at any point; but if we attempt to bend an imperfectly flexible string, or a wire, we encounter a certain amount of resistance according to the degree of inflexibility or rigidity of the string or wire. If we consider the nature of the mutual forces existing between the particles on each side of a normal section of such a body, we shall find that these forces are not necessarily reducible to a single resultant at all. In the general case of a wire bent and twisted by the action of any external forces, these internal actions on the particles at one side of a section may, of course, be reduced to a single resultant force and a single couple; and the resultant force may be applied at any point in the section, the couple varying according to the point chosen. All this is evident from the general reduction of a system of forces in Chapter X.

202.] **Three Methods of Investigation.** There are three methods by which the equilibrium of a string or wire may be treated—namely,

1°. We may isolate an infinitesimal element of the body, supplying to it at each extremity the action exercised by the neighbouring portions which are imagined to be removed (see p. 161).

2°. We may apply the general condition that the variation of the whole potential work of the system of forces, internal as well as external, is zero (see p. 328).

3°. We may consider the equilibrium of any *finite* portion of the body, treating it, *when the figure of equilibrium has been assumed* (see p. 13), *as a rigid body*. (See Thomson and Tait, *Nat. Phil.*)

We begin by considering the equilibrium of a perfectly flexible string which suffers no elongation under the action of the forces which will keep it in equilibrium. Such a body is called a *flexible inextensible string*, and it is scarcely necessary to add that it exists only in the abstractions of Rational Statics.

Section I.

Flexible Inextensible Strings.

203.] **Tangential and Normal Resolutions.** Let A (fig. 219) represent a flexible inextensible string in equilibrium under the action of any system of forces applied continuously throughout the string. Then the force acting on a unit mass of matter placed at any point of the string will, in the general case, be expressed as a function of the co-ordinates of this point and their differential co-efficients with respect to the arc. Thus, if the co-ordinates of P

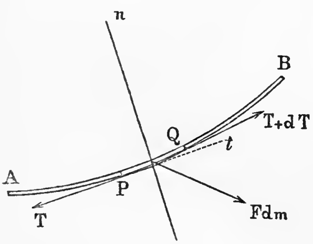

Fig. 219.

are x, y, z, the force acting on a unit mass placed at P will be

$$\phi\left(x, y, z, \frac{dx}{ds}, \ldots\right).$$

On an element containing dm units of mass the force will be

$$\phi\left(x, y, z, \frac{dx}{ds}, \ldots\right) dm.$$

We shall denote by F the coefficient of dm in this expression.

Suppose, then, that we consider an element PQ of the string, whose length is ds, apart from the rest of the string; let the mean density of the element be k, and let σ be the area of its mean section; then the mass of PQ is $k\sigma ds$, and the external force acting on it is $$k\sigma F ds.$$

Now, the element PQ is kept in equilibrium by three forces —namely, the tension (T) at P, the tension ($T+dT$) at Q, and the external force ($k\sigma F ds$), which acts at the middle point of PQ.

These three forces must be coplanar and meet in a point. Now, the two tensions act along two consecutive tangents to the string, and as the plane of two consecutive tangents to any curve in space is the *osculating plane*, we see that—

The resultant applied force at any point of a flexible string acts in the osculating plane of the string at the point.

If the string is stretched over any smooth surface by means of two forces applied at its extremities, the only applied force which is *continuously* distributed throughout the string is the reaction of the surface; and as this reaction is everywhere normal to the surface, we see that—

A string which is stretched along any smooth surface, and acted on by no external forces, except the reaction of the surface and two terminal tensions, has its osculating plane at every point normal to the surface.

The string in this case assumes the form of a shortest line, or *geodesic*, on the surface.

Let Pt be the tangent and Pn the normal at P; let $d\theta$ be the angle between the tangents at P and Q; and let ϕ be the angle between Fdm and Pt.

Then, resolving along Pt the forces acting on the element, we have $$(T+dT)\cos d\theta + k\sigma F\cos\phi\, ds - T = 0;$$

but $\cos d\theta = 1$, neglecting $(d\theta)^2$; therefore this equation gives

$$\frac{dT}{ds} + k\sigma F \cos\phi = 0, \tag{1}$$

which asserts that the rate of variation of the tension per unit of length along the string is equal to the tangential component of the applied force per unit of length.

Again, resolving the forces along Pn, the normal, we have

$$(T+dT)\sin d\theta - k\sigma F \sin\phi\, ds = 0,$$

or since ρ, the radius of curvature at P, is equal to $\frac{ds}{d\theta}$,

$$\frac{T}{\rho} - k\sigma F \sin\phi = 0, \tag{2}$$

which asserts that the curvature of the string at any point is equal to the normal force per unit of length divided by the tension.

From (1) we have $T = C - \int k\sigma F \cos\phi\, ds,$

where C is an arbitrary constant. Now, $\cos\phi\, ds$ is the projection of ds on the direction of F. Denoting this projection by df,

$$T = C - \int k\sigma F df. \tag{3}$$

But $\int k\sigma F df$ is evidently the potential of the applied forces if they are a conservative system*. Hence, if V and V_0 denote the potentials at two points in the string at which the tension are T and T_0, we have $T = T_0 - (V - V_0)$, (4)

or *the difference of the tensions at any two points is equal to the difference of the potentials*—a result which we shall find to be true also in the case in which the string rests on a smooth surface.

204.] **Cartesian Equations of Equilibrium.** Let the force F acting on the unit mass at any point P whose co-ordinates are x, y, z be resolved into three components, $X, Y, Z,$ parallel to three fixed rectangular axes. Then the components acting on the element PQ are $k\sigma X ds$, $k\sigma Y ds$, $k\sigma Z ds$. Also the components of the tension acting on the extremity P are

$$-T\frac{dx}{ds}, \quad -T\frac{dy}{ds}, \quad -T\frac{dz}{ds};$$

the components of this tension are affected with negative signs, since, when the element PQ is considered apart, the tension at

* A simple case in which the external forces are not a conservative system will be presently given. (See Art. 208.)

P will be directed towards the left-hand side of fig. 219, where the origin of co-ordinates is supposed to be.

These components of the tension will at any point be functions of the length of the arc measured from some fixed point, A, of the string up to the point considered. Thus, if $AP = s$, we shall have

$$T\frac{dx}{ds} = f(s),$$

and the component of the tension at Q is therefore $f(s+ds)$, or

$$f(s) + f'(s) \,.\, ds + f''(s)\frac{ds^2}{1\,.\,2} + \ldots,$$

or, again,

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right) . \, ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right) \cdot \frac{ds^2}{1\,.\,2} + \ldots.$$

Hence, for the equilibrium of PQ, resolving forces parallel to the axis of x, we have

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right) . \, ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right) \cdot \frac{ds^2}{1\,.\,2} + \ldots$$
$$+ k\sigma X ds - T\frac{dx}{ds} = 0,$$

or, rejecting the terms which cancel, dividing out by ds, and diminishing ds indefinitely,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + k\sigma X = 0. \qquad (1)$$

Similarly,
$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + k\sigma Y = 0, \qquad (2)$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + k\sigma Z = 0. \qquad (3)$$

Performing the differentiations, we obtain

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + k\sigma X = 0, \qquad (4)$$

$$T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + k\sigma Y = 0, \qquad (5)$$

$$T\frac{d^2z}{ds^2} + \frac{dT}{ds}\frac{dz}{ds} + k\sigma Z = 0. \qquad (6)$$

Multiplying these by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$, respectively, adding, and remembering that

$$\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}=0,$$

we obtain

$$\frac{dT}{ds}+k\sigma\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)=0, \tag{7}$$

or

$$T=C-\int k\sigma\,(Xdx+Ydy+Zdz), \tag{8}$$

which is precisely the same as (3) of last Article.

Eliminating T and $\frac{dT}{ds}$ from (4), (5), and (6) we have

$$\begin{vmatrix} \frac{d^2x}{ds^2}, & \frac{dx}{ds}, & X \\ \frac{d^2y}{ds^2}, & \frac{dy}{ds}, & Y \\ \frac{d^2z}{ds^2}, & \frac{dz}{ds}, & Z \end{vmatrix}=0,$$

the geometrical meaning of which is, that the applied force at any point is in the osculating plane of the curve at that point.

From (1), (2), and (3) can be deduced another expression for the tension. Integrating each of them, squaring, and adding the results, we have

$$T^2=(A-\int k\sigma Xds)^2+(B-\int k\sigma Yds)^2+(C-\int k\sigma Zds)^2,$$

A, B, and C being constants which must be determined after each integration by knowing the values of $T\frac{dx}{ds}$, ...at the point from which s is measured.

If (4), (5), and (6) be multiplied by $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, respectively, and added, we have

$$\frac{T}{\rho^2}+k\sigma\left(X\frac{d^2x}{ds^2}+Y\frac{d^2y}{ds^2}+Z\frac{d^2z}{ds^2}\right)=0,$$

ρ being the radius of absolute curvature at the point P.

The direction cosines of the radius of absolute curvature being

$$-\rho\frac{d^2x}{ds^2},\ -\rho\frac{d^2y}{ds^2},\ -\rho\frac{d^2z}{ds^2},$$

this equation expresses the results (2) of Art. 203.

The equations of the curve of equilibrium are found by eliminating T from equations (1), (2) and (3) in pairs. The curve is evidently given by equations

$$\frac{A-\int k\sigma X ds}{\frac{dx}{ds}} = \frac{B-\int k\sigma Y ds}{\frac{dy}{ds}} = \frac{C-\int k\sigma Z ds}{\frac{dz}{ds}}.$$

If at every point of the string

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = 0,$$

or *if the applied force is at every point at right angles to the tangent to the string, the tension will be constant throughout,* as appears from (7). This is the case, for example, when a string is stretched over any smooth surface, and acted on by no force except the reaction of the surface. Thus we prove the truth of our assumption in p. 26.

205.] **Forces in One Plane. Gravity.** When the applied forces are in one plane, the general equations of equilibrium become

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + k\sigma X = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + k\sigma Y = 0,$$

the plane of the forces being that of xy.

Let gravity be the only force acting on the string, except the terminal forces, or forces applied at the extremities. Then, taking the axis of y vertically upwards, and denoting the weight of the unit mass by g, we have $X = 0$, $Y = -g$, and the equations become

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0, \tag{1}$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) = k\sigma g. \tag{2}$$

The first equation shows that *the horizontal component of the tension is the same at all points of the string* (see p. 32).

Denoting this component by τ, we have

$$T\frac{dx}{ds} = \tau, \qquad \therefore \quad T = \tau\frac{ds}{dx}.$$

Hence, from (2)

$$\frac{d}{ds}\left(\tau\frac{dy}{dx}\right) = k\sigma g,$$

or
$$k\sigma = \frac{\tau}{g}\frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}}. \tag{3}$$

It is to be observed that $k\sigma$ is the mass per unit length of the string at the point x, y. This last equation, therefore, determines the mass per unit of length at any point when the form of the curve in which the string hangs is given; and, conversely, it determines the curve in which any string will hang when the laws of variation of its section and density are given.

If $\frac{dy}{dx}$ be denoted by p, and the independent variable changed from x to y, equation (3) becomes

$$k\sigma = \frac{\tau}{g}\frac{p\frac{dp}{dy}}{\sqrt{1+p^2}}.$$

206.] **The Common Catenary.** When the mass of a unit length of the string is everywhere constant, the form of the string is that of a curve called the *Catenary*. The name Catenary is sometimes employed to denote the form of a string in general, whatever be the law of variation of its density.

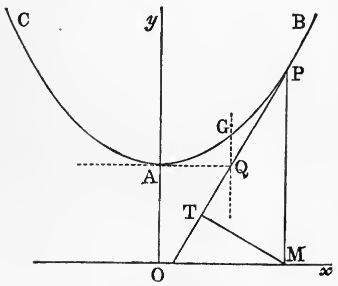

Fig. 220.

In the present case $k\sigma$ is constant—equal to m, suppose. Let $\tau = mgc$, where c is a constant length. Since at the lowest point, A (fig. 220), the tension is horizontal, τ is the tension at A, and c is the length of a portion of the string whose weight is the tension at the lowest point.

From (3) of last Art. we have

$$\frac{\frac{d^2y}{dx^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{c},$$

or
$$\frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{dx}{c}.$$

Integrating,
$$\log\left[\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{c}+c',$$

where c' is an arbitrary constant. Now, taking the axis of y passing through A, we have $x = 0$, and $\frac{dy}{dx} = 0$, simultaneously. Hence $c' = 0$, and the last equation becomes

$$\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{c}},$$

where e is the Napierian base. Solving this equation for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right);$$

and by integration
$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + c'',$$

where c'' is an arbitrary constant. Now, taking the origin, O, at a distance equal to c below A, we have $y = c$ when $x = 0$. This gives $c'' = 0$, and the equation of the catenary referred to axes chosen as above is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \qquad (1)$$

The point of intersection of these particular axes we shall in the sequel call *the origin* of the catenary.

We shall next find the length of the arc, AP, measured from A to any point, P, on the curve. If ds is the element of arc,

$$ds = \sqrt{dx^2+dy^2}$$
$$= \sqrt{1+\tfrac{1}{4}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2}\,.\,dx, \quad \text{from (1)},$$
$$= \tfrac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)dx;$$
$$\therefore\ s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right), \qquad (2)$$

no constant being added because $s = 0$ when $x = 0$.

From (1) and (2) we have

$$y^2 = s^2 + c^2, \tag{3}$$

and from (3)

$$s = y\frac{dy}{ds}.$$

Let PM and PT be the ordinate and tangent at P, and let fall a perpendicular MT on PT. Then

$$PT = y\cos MPT = y\frac{dy}{ds}; \tag{4}$$

hence

$$s = PT; \tag{5}$$

and since $y^2 = PT^2 + MT^2$, we have from (3) and (5)

$$c = MT. \tag{6}$$

Hence, given the catenary to construct its origin and horizontal axis—

On the tangent at any point, P, measure off a length, PT, equal to the arc AP; at T erect a perpendicular TM to the tangent meeting the ordinate of P in M; then the horizontal line through M is the axis of the curve.

In making a proper figure this rule will be found of great use.

The involute of the catenary which starts from the lowest point is the Tractory.

To get a point on this involute we measure on the tangent, PT, at any point, P, a length equal to the arc AP. From (5) we see, therefore, that T is a point on the involute; and since PT is a normal to the involute, its tangent at T must be TM. But from (6) TM is constant; hence the involute is a curve such that the length of the tangent between its point of contact and a fixed right line, Ox, is constant. The involute is, therefore, a tractory (see p. 195).

The tension at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point.

Consider the equilibrium of the portion AP of the string apart from the rest. This portion is kept in equilibrium by three forces—namely, the tension at P in the direction TP, the horizontal tension at A in the direction QA, and its weight acting through its centre of gravity, G. Hence the vertical through G must pass through Q. Resolving vertically, we have

$$T \cos TPM = mgs;$$

$$\therefore \quad T = mg \frac{s}{\cos TPM}$$

$$= mgy, \text{ from } (5). \tag{7}$$

Cor. It follows from this that if a uniform inextensible string hangs freely over any two smooth pegs, the vertical portions which hang over the pegs must each terminate on the horizontal axis of the catenary.

In the catenary the length of the radius of curvature at any point is equal to the length of the normal between that point and the horizontal axis.

By equation (2) of Art. 203, we have

$$\frac{T}{\rho} = mg \sin TPM,$$

which by means of (7) gives $\rho = \dfrac{y}{\sin TPM}$; but this is evidently the length of the normal between P and the axis of x.

It will be readily seen that the differential equation of the catenary can be written in the form $c^2 \dfrac{d^2y}{dx^2} = y$, and that the area $OAPM =$ twice the area of the triangle PTM.

It is well to observe that if a weight is suspended from a given point of a catenary, the continuity of the curve ceases at that point, and the portions of the string at opposite sides of the point must be treated as branches of two distinct catenaries.

207.] **The Catenary of Uniform Strength.** If the area of the normal section of the string at any point is made proportional to the tension at that point, the tendency to break will be the same at all points, and the curve is therefore called the *Catenary of Uniform Strength.*

To find its equation, we have $\sigma = \lambda T$, λ being a constant; and since $T = \tau \dfrac{ds}{dx}$, we have

$$\sigma = \lambda \tau \frac{ds}{dx}.$$

Hence (3) of Art. 205 becomes

$$g\lambda k \left(\frac{ds}{dx}\right)^2 = \frac{d^2 y}{dx^2};$$

or, denoting $g\lambda k$ by $\frac{1}{a}$, we have

$$\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{a}.$$

Integrating, $$\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{x}{a} + b,$$

where b is an arbitrary constant. Let the axis of y pass through the *lowest* point of curve, i.e. the point at which the tangent is horizontal. Then $b = 0$, and we have

$$\frac{dy}{dx} = \tan\frac{x}{a}.$$

Integrating this again,

$$\frac{y}{a} = -\log\cos\frac{x}{a} + b'.$$

Let the lowest point be taken as origin. Then $b' = 0$, and we have, finally,

$$y = a\log\sec\frac{x}{a}$$

for the equation of the catenary of uniform strength.

It is easily seen that the curve has two vertical asymptotes, each at a distance $\frac{\pi a}{2}$ from the lowest point.

The equation of this curve can be put into a remarkable form. If ρ is the radius of curvature at any point, and s the length of the arc between this and the lowest point,

$$\rho = \frac{a}{2}\left(e^{\frac{s}{a}} + e^{-\frac{s}{a}}\right),$$

an equation which can be deduced with no difficulty.

Given the whole weight (W) *of the chain**, *and the span* ($2b$), *determine the section at any point so that there shall be a constant tension* (p) *per unit of sectional area at all points.*

If A and B are the two points of support (supposed in a horizontal line), b is their common distance from the vertical axis of the curve. We have, then,

$$W = 2\int k\sigma g\,ds$$

$$= 2\lambda k g\tau\int_0^b \sec^2\frac{x}{a}\,dx$$

* A string hanging from two fixed points under the action of gravity is frequently called a *chain*.

$$= 2\tau \tan \frac{b}{a}.$$

Now evidently $\frac{1}{\lambda}$ is the tension per unit of sectional area, $= p$; and since g is the weight of a unit volume of the standard substance, kg is the weight of a unit volume of the material of the chain. Denote this last by ω. Then

$$a = \frac{1}{kg\lambda} = \frac{p}{\omega}.$$

Also, $$\sigma = \frac{T}{p} = \frac{W}{p} \cot \frac{b}{a} \cdot \frac{ds}{dx} = \frac{W}{2p} \cot \frac{b}{a} \sec \frac{x}{a}.$$

But it is easy to prove that $\sec \frac{x}{a} = \frac{1}{2}\left(e^{\frac{s}{a}} + e^{-\frac{s}{a}}\right)$.

Hence $$\sigma = \frac{W}{4p}\left(e^{\frac{\omega s}{p}} + e^{-\frac{\omega s}{p}}\right) . \cot \frac{\omega b}{p},$$

which is the expression for the area of a section at a distance s along the chain from the middle point.

The student will verify the homogeneity of this equation.

208.] **The Parabola of Suspension Bridges.** Suppose a string to be attached to two fixed points, and let each element of its length be acted on by a force in a constant direction, the magnitude of the force being proportional to the projection of the element on a line perpendicular to the direction of the force. Then it can be shown geometrically that the figure of the string is that of a parabola.

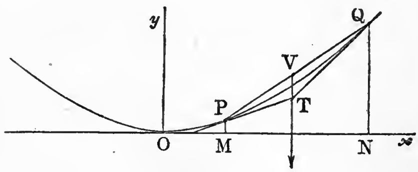

Fig. 221.

Let Oy (fig. 221) be the direction opposite to that of the force on each element; Ox a tangent to the curve, perpendicular to this direction; P and Q any two points on the string, the tangents at them being PT and QT; PM and QN perpendiculars on Ox. Consider the separate equilibrium of the portion PQ. The forces acting on it are the tensions in the directions TP and TQ, and the resultant of the parallel forces on the elements of PQ. This resultant must pass through T, and it also passes through the middle point of MN, since its constituent forces are all proportional to the elements of the line MN. Hence drawing

TV parallel to Oy, and meeting PQ in V, the point V must bisect the right line PQ.

The curve of equilibrium of the string is therefore such that *a right line drawn from the point of intersection of any two tangents parallel to a fixed direction bisects the chord joining their points of contact.*

This well known property identifies the curve with a parabola.

If we make use of the equations of equilibrium in Art. 205, we shall have $X = 0$, $Y = \mu \dfrac{dx}{ds}$, μ being the constant. There is no difficulty in arriving at the result just found.

It is to be observed that the acting forces in this case are not a conservative system. Hence the function V (see Art. 203) does not exist.

The connexion of this parabola with Suspension Bridges has been already explained in Chap. II.

209.] **String Acted on by a Central Force.** When the lines of action of the forces applied to the various elements of the string pass all through the same point, the force acting on the string is said to be *central*, and this point is called the *centre of force*. It is easy to prove that in this case the string must lie in a plane passing through the centre of force. For (Art. 203) the osculating plane at every point contains the centre of force; and since two consecutive osculating planes have a tangent line to the string common, these two planes, having in addition a point (the centre of force) common, must be identical. Hence the osculating plane is the same at all points; or the string must lie wholly in one plane.

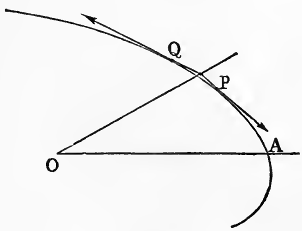

Fig. 222.

To find the form assumed by a string acted on by a given central force.

Let O (fig. 222) be the centre of force (supposed repulsive), PQ an element of the string whose equilibrium is considered apart, r the radius vector OP, θ the angle POA between OP and a fixed initial line, s the length of the arc AP, and p the perpendicular from O on the tangent at P. Then, for the equilibrium of the element PQ, taking moments about O, we have moment of tension at $P =$ moment of tension at Q;

or
$$Tp = Tp + d(Tp),$$
$$\therefore \quad Tp = h, \tag{1}$$
where h is a constant*.

Denote the tensions at P and Q by T and $T+dT$ respectively.

Resolve the forces acting on PQ along the tangent at P, denote $k\sigma$ by m, and let the central force be $mFds$. Then this force passes through the point of intersection of tangents at P and Q, and the cosine of the angle between its direction and the tangent at P is $-\frac{dr}{ds}+\epsilon$, where ϵ is indefinitely small. In the equation of resolution the component of $mFds$ is
$$mFds\left(-\frac{dr}{ds}+\epsilon\right),$$
so that ϵ may be neglected, and we have
$$dT = -mFdr. \tag{2}$$
Equations (1) and (2) determine the form of the curve.

If the central force is attractive, the sign of F must be changed in (2), and the curve of equilibrium will be convex towards O.

It is usual in problems concerning central forces to denote r by $\frac{1}{u}\cdot$ Making this substitution, and eliminating T from the above equations, we have
$$\frac{mF}{u^2}\,du = hd\left(\frac{1}{p}\right). \tag{3}$$
But (Williamson's *Differential Calculus*, Chapter XII),
$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$
Hence, denoting $\frac{mF}{u^2}$ by $\phi(u)$, and $\int\phi(u)\,du$ by $\phi_1(u)$, an arbitrary constant being implied in $\phi_1(u)$, we have from (3)
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{h^2}\{\phi_1(u)\}^2. \tag{4}$$

It is often more convenient to retain a differential equation of

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* Of course this proof holds whether the portion PQ is an element of length or a portion of any length, however great.

the second order for u*. Differentiating (4) we have, dividing out by $\frac{du}{d\theta}$, and remembering that $\phi_1'(u) = \phi(u)$,

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2}\phi_1(u) \cdot \phi(u). \qquad (5)$$

Now, since the integration of (4) gives u in terms of θ, and introduces an arbitrary constant in addition to that already involved in $\phi_1(u)$, we see that the solution of the problem involves only *two* arbitrary constants. But (5) will require two integrations to express u, and each integration will introduce an arbitrary constant. Hence it appears that in this way we get *three* arbitrary constants, instead of two. These three are, however, easily connected, since the values of u and $\left(\frac{du}{d\theta}\right)^2$ given by the complete integral of (5) must satisfy (4) for all values of u.

As an example, let it be required to discuss the form of a string of uniform section and density when the central repulsive force varies inversely as the square of the distance. In this case m is constant, and $F = \mu' u^2$, μ' being a constant which obviously denotes the magnitude of the force on a unit mass of matter placed at the unit distance from the centre of force.

Hence we have, putting $m\mu' = \mu$,

$$T = C + \mu u,$$

C being a constant. If T_0 denote the tension at a point A of the string whose distance from the centre is $\frac{1}{a}$, we have, evidently,

$$\begin{aligned} T &= T_0 - \mu a + \mu u \\ &= \mu(u + c), \text{ suppose.} \end{aligned}$$

Hence,
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2}(u + c)^2, \qquad (6)$$

which gives, by differentiation,

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu^2}{h^2}\right)u - \frac{\mu^2}{h^2}c = 0. \qquad (7)$$

First suppose that $\frac{\mu^2}{h^2} < 1$, and denote $1 - \frac{\mu^2}{h^2}$ by λ^2. Then this equation becomes
$$\frac{d^2u}{d\theta^2} + \lambda^2\left(u - \frac{1-\lambda^2}{\lambda^2}c\right) = 0,$$

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* This method of treating the equilibrium of a string acted on by a central force is taken from a paper by Professor Townsend in the *Quarterly Journal of Pure and Applied Mathematics*, 1874.

the integral of which is

$$u = \frac{1-\lambda^2}{\lambda^2} c + A \cos \lambda (\theta - a),$$

A and a being the constants of integration. Substituting this value of u in (6), we have $A = \dfrac{\sqrt{1-\lambda^2}}{\lambda^2} c$, and therefore

$$u = \frac{1-\lambda^2}{\lambda^2} c \left\{ 1 + \frac{1}{\sqrt{1-\lambda^2}} \cos \lambda (\theta - a) \right\}. \tag{8}$$

The value of a is found by putting $u = a$ and $\theta =$ the angle belonging to the point A.

When $\theta = a$, $\dfrac{du}{d\theta} = 0$, and there is an apse. If the initial line be taken through the apse, and T_0 and a belong to this point, we have $c = \dfrac{T_0}{\mu} - a = (\dfrac{h}{\mu} - 1) a$, and (8) assumes the simple form

$$u = \frac{a}{1 + \frac{\mu}{h}} \left(\frac{\mu}{h} + \cos \lambda \theta \right), \tag{9}$$

which differs from the focal polar equation of a conic in having the angle multiplied by a number, λ, less than unity.

If $\dfrac{\mu^2}{h^2} > 1$, we must put $\dfrac{\mu^2}{h^2} - 1 = \lambda^2$, and putting $\mu a - T_0 = \mu c$, equation (6) becomes

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2} (u - c)^2,$$

which gives

$$u = \frac{1+\lambda^2}{\lambda^2} c + A e^{\lambda\theta} + B e^{-\lambda\theta}, \tag{10}$$

the constants A and B being connected by the equation $AB = \dfrac{1+\lambda^2}{4\lambda^4} c^2$ by (4).

Equation (10) can obviously be written

$$u = \frac{1+\lambda^2}{\lambda^2} c + \sqrt{AB} \left(\sqrt{\frac{A}{B}} e^{\lambda\theta} + \sqrt{\frac{B}{A}} e^{-\lambda\theta} \right),$$

or

$$u = \frac{1+\lambda^2}{\lambda^2} c \left\{ 1 + \frac{1}{2\sqrt{1+\lambda^2}} \left[e^{\lambda(\theta - a)} + e^{-\lambda(\theta - a)} \right] \right\}.$$

When $\theta = a$, there is an apse, and if the initial line be taken through the apse, we have, in the same manner as before,

$$u = \frac{a}{\frac{\mu}{h} + 1} \left\{ \frac{\mu}{h} + \frac{e^{\lambda\theta} + e^{-\lambda\theta}}{2} \right\}. \tag{11}$$

If $\dfrac{\mu}{h} = 1$, both (9) and (11) give $u = a$, a constant; and the figure of equilibrium is a circle.

For the remarkable analogy between the curve of equilibrium of a flexible string and the orbit of a particle under a given force, see Professor Townsend's paper, and Thomson and Tait's *Nat. Phil.*

210.] **Problem.** To find the angle between the apsides in a string which, under the action of a central force, assumes a form nearly circular.

Def. An apse is a point on a curve at which the radius vector is at right angles to the tangent.

Since the form of the string is nearly circular, u will differ from a constant value, a, by a small variable quantity, x.

Let, then, $u = a + x$. In this case $\phi_1(u) = \phi_1(a) + x\phi(a)$, neglecting higher powers of x; and $\phi(u) = \phi(a) + x\phi'(a)$. For shortness, denote $\phi_1(a)$, $\phi(a)$, and $\phi'(a)$ by ϕ_1, ϕ, and ϕ' respectively. Then (5) of last Art. becomes

$$\frac{d^2x}{d\theta^2} + a + x = \frac{1}{h^2}\{\phi\phi_1 + (\phi_1\phi' + \phi^2)x\}. \tag{1}$$

But if the string were exactly circular, x and $\frac{d^2x}{d\theta^2}$ would always $= 0$; therefore $a = \frac{\phi\phi_1}{h^2}$, or

$$\frac{1}{h^2} = \frac{a}{\phi\phi_1}. \tag{2}$$

Hence (1) becomes

$$\frac{d^2x}{d\theta^2} + \left\{1 - a\left(\frac{\phi'}{\phi} + \frac{\phi}{\phi_1}\right)\right\}x = 0. \tag{3}$$

The constant a may be chosen as the reciprocal of the radius of any circle which nearly coincides with the figure of the string; but simplicity is gained by taking it equal to the reciprocal of the radius of that circle in which the tension at each point is equal to the *mean* tension in the string.

Now in a circle of radius $\frac{1}{a}$ the tension (see (2), Art. 203) is $a\phi$; and (2) of last Art. gives T in the curve equal to $\phi_1(u)$, and therefore the mean tension $= \phi_1$. Hence

$$a\phi = \phi_1,$$

and (3) finally becomes

$$\frac{d^2x}{d\theta^2} - \frac{a\phi'}{\phi}x = 0. \tag{4}$$

If $\frac{a\phi'}{\phi}$ be positive, the value of x in terms of θ will be exponential, and the nearly circular form becomes impossible, since the value of u increases indefinitely with θ.

For the possibility of a nearly circular form $\frac{a\phi'}{\phi}$ must be negative,

and we have
$$x = A\cos\left(\sqrt{\frac{-a\phi'}{\phi}}\,\theta - a\right).$$

Hence, since at an apse $\dfrac{du}{d\theta} = 0$, or $\dfrac{dx}{d\theta} = 0$, we shall arrive at an apse whenever
$$\sin\left(\sqrt{\frac{-a\phi'}{\phi}}\,\theta - a\right) = 0,$$

and the difference between two successive values of θ which satisfy this equation is
$$\frac{\pi}{\sqrt{\dfrac{-a\phi'}{\phi}}},$$

which is, therefore, the angle between the apsides*.

211.] **String on Smooth Plane Curve.** Consider the case of an inextensible string resting on a smooth plane curve under the action of any forces in the plane of the curve, and let fig. 219 represent this case. Then into the equations of Art. 203 we have merely to introduce the normal reaction, Rds†, acting on the element PQ in the direction nP.

Resolving tangentially, we obtain
$$\frac{dT}{ds} + k\sigma F\cos\phi = 0. \tag{1}$$

Resolving normally,
$$\frac{T}{\rho} - k\sigma F\sin\phi - R = 0. \tag{2}$$

These are the most useful resolutions in the case of a string resting on a curve. Equations of resolution along arbitrary axes may, of course, be obtained by introducing the components of R into the general equations of Art. 204.

From (1) we obtain $T = C - \int k\sigma F\cos\phi\, ds$, C being a constant.

But $F\cos\phi\, ds$ is obviously the virtual work of the force F. Hence if the acting forces are conservative, and V is their potential at P, we have, as in Art. 203,
$$T = T_0 - (V - V_0).$$

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* This investigation is taken from the paper by Professor Townsend previously referred to.

† The student will observe that in considering the equilibrium of an element of length ds we represent the reaction of a curve on it by Rds, and the applied force by $k\sigma F ds$, while we represent the tension by T, and not by Tds. The reason of this is that the tension depends merely on the cross section of the element and not on its length, while the magnitude of the reaction depends evidently on the length of the element in contact with the curve.

212.] **String on Rough Plane Curve.** If the curve in the preceding Article is rough, and the string in limiting equilibrium, slipping being about to take place in the direction QP, we have merely to include among the forces acting on the element PQ a tangential force $\mu R\,ds$, the coefficient of friction being μ and the normal reaction $R\,ds$, as before.

Equations (1) and (2) of last Article now become

$$\frac{dT}{ds}+k\sigma F\cos\phi+\mu R = 0,$$

$$\frac{T}{\rho}-k\sigma F\sin\phi-R = 0.$$

213.] **String on a Smooth Surface.** When a string acted on by two terminal forces only is stretched over a smooth surface, we have seen that it assumes the form of a geodesic on the surface, and that the tension is constant throughout its length.

The general Cartesian equations of equilibrium are readily obtained by adding to the components of the given applied forces the components of the reaction of the surface.

Let R be the magnitude of the normal reaction per unit of length of the string. Then, the direction angles of the normal to the surface at the element ds being λ, μ, ν, the components of the reaction on this element parallel to the axes are $R\cos\lambda\,ds$, $R\cos\mu\,ds$, $R\cos\nu\,ds$; and (1), (2), (3) of Art. 204 become

$$\left.\begin{aligned}\frac{d}{ds}\Big(T\frac{dx}{ds}\Big)+k\sigma X+R\cos\lambda &= 0,\\ \frac{d}{ds}\Big(T\frac{dy}{ds}\Big)+k\sigma Y+R\cos\mu &= 0,\\ \frac{d}{ds}\Big(T\frac{dz}{ds}\Big)+k\sigma Z+R\cos\nu &= 0.\end{aligned}\right\}\qquad(1)$$

If we multiply these by $\cos\lambda$, $\cos\mu$, and $\cos\nu$ respectively, and add, we have

$$T\Big(\frac{d^2x}{ds^2}\cos\lambda+\frac{d^2y}{ds^2}\cos\mu+\frac{d^2z}{ds^2}\cos\nu\Big)+k\sigma N+R = 0,$$

N denoting the normal component of the applied forces measured in the same sense as R.

Now if ω is the angle between the normal to the surface and the radius of absolute curvature of the string at the point considered,

$$\frac{d^2x}{ds^2}\cos\lambda+\frac{d^2y}{ds^2}\cos\mu+\frac{d^2z}{ds^2}\cos\nu=-\frac{1}{\rho}\cos\omega,$$

where ρ is the length of the radius of curvature of the string. Hence we have

$$R=\frac{T}{\rho}\cos\omega-mN, \tag{2}$$

m being put for $k\sigma$, the mass per unit length of the string at the point considered.

If we multiply the above equations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$, respectively, and add, we obtain

$$\frac{dT}{ds}+mS=0, \tag{3}$$

where $S=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=$ the component of the applied force along the tangent to the string.

The integral of this equation, when the applied forces are conservative, gives, as in Art. 211,

$$T=T_0-(V-V_0).$$

In the particular case in which a uniform inextensible string rests on a smooth surface under the influence of gravity, this equation gives

$$T=T_0-mg(y-y_0),$$

mg being the weight of a unit length of the string, and the axis of y a vertical line. From this it follows that at all points of the string which are in the same horizontal plane the tension of the string is the same; hence the free extremities lie in the same horizontal plane.

The curve of equilibrium of the string on the surface is obtained by eliminating T and R from equations (1). If the equation of the surface is $u=0$, the result of eliminating $\frac{dT}{ds}$ and R is

$$\begin{vmatrix} T\frac{d^2x}{ds^2}+mX, & \frac{dx}{ds}, & \frac{du}{dx} \\ T\frac{d^2y}{ds^2}+mY, & \frac{dy}{ds}, & \frac{du}{dy} \\ T\frac{d^2z}{ds^2}+mZ, & \frac{dz}{ds}, & \frac{du}{dz} \end{vmatrix}=0,$$

in which the value of T must be substituted from (3).

The general results arrived at in Art. 203 can be easily verified here.

214.] **String on a Rough Surface.** If a string, acted on by no forces, is stretched over a rough surface it need not, as in the case of a smooth surface, assume the form of a geodesic or shortest line. One simple case in which it will be a geodesic is that in which it is about to slip on the surface at every point in the direction of the tangent to the string at this point.

Consider the equilibrium of an element, PQ, of the string, whose length is ds, and suppose that it is about to slip in the direction QP. The element is acted upon by three forces —namely, a tension T, at P, a tension $T+dT$, at Q, and the total resistance of the rough surface, which must pass through the intersection of the tangents at P and Q.

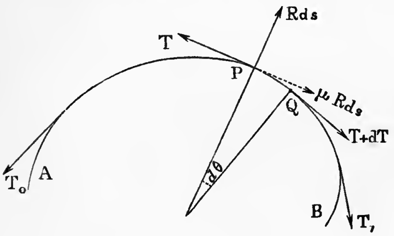

Fig. 223.

It is evident that we may consider this total resistance as acting at P, ultimately, since it is of the form $R_1\,ds$, R_1 being a finite quantity, and if it be assumed to act at any point between P and Q, its components in any directions will differ from those of the total resistance supposed to act at P by infinitesimals of the order of $(ds)^2$. Resolve the total resistance at P into a normal force, Rds, and a force in the tangent plane, μRds, μ being the coefficient of friction between the string and the surface.

Now the component μRds must act along the tangent at P, since (see p. 57) slipping is about to take place along this tangent. Hence the three forces T, $T+dT$, and μRds being all in the osculating plane of the curve at P, the remaining force, Rds, must also lie in this plane; that is, the osculating plane at every point of the curve contains the normal to the surface. Hence the string assumes the form of a geodesic.

Denoting the angle between the tangents at P and Q by $d\theta$, we have, by resolving along the tangent at P,

$$dT+\mu R\,ds = 0. \tag{1}$$

Again, resolving along the normal at P,

$$T d\theta - R\,ds = 0. \tag{2}$$

From (1) and (2) we have

$$\frac{dT}{T}+\mu d\theta = 0, \qquad \therefore \quad T = Ce^{-\mu\theta},$$

C being the constant of integration, and θ the sum of the *angles of contingence*, or angles between successive tangents to the string from any chosen point, A, to the point, P. Let T_0 be the tension at A. Then $T = T_0$ when $\theta = 0$; therefore

$$T = T_0\, e^{-\mu\theta}. \tag{3}$$

Hence, as the angle through which the string turns increases in arithmetical, the tension diminishes in geometrical, progression.

The general investigation of the equilibrium of a string on a rough surface under the action of given forces is a problem of much difficulty, and in the sequel we shall confine our attention to the case in which the string assumes the form of a plane curve on the surface.

When the string lies in one plane, θ, the sum of the angles of contingence is simply the angle between the tangents at A and P.

Suppose that (the weight of the string being neglected) two weights, P and Q, are suspended from the extremities of a string which passes over a fixed rough cylinder whose axis is horizontal, the string lying in a plane perpendicular to this axis; it is required to find the relation between P and Q when the equilibrium is limiting.

Let A (fig. 223) be the point at which the portion of the string next P leaves the cylinder, and B the point at which the portion next Q leaves it.

Then from (3) by putting $T_0 = P$ and $\theta = \pi$, we have

$$Q = Pe^{-\mu\pi}, \tag{4}$$

when P is about to overcome Q. If P is on the point of ascending, the sign of μ in this equation is to be changed.

If the string makes a complete revolution and a half round the cylinder, the value of θ corresponding to Q is 3π, and we have in this case $Q = Pe^{-3\mu\pi}$. The factor $e^{-\mu\theta}$ diminishes very rapidly as the angle increases, and thus we see how it is that a small force applied at one extremity of a rope coiled several times round a fixed rough cylinder can overcome a large force applied at the other extremity—a practical example of which occurs when the small motion of a ship in harbour is stopped by a small force applied at the extremity of a rope coiled round a fixed post. For example, if $\mu = \frac{1}{2}$, $e^{\mu\pi} = 4.8$, and $Q = \dfrac{P}{4.8}$.

215.] **Work done against Friction for a given Arc of Slipping.** If the string slips through a space δs in the direction BA, the work done against the friction, $\mu R ds$, acting on any element is $\delta s \,.\, \mu R ds$, and the work done against the friction acting all over the string is $\delta s \,.\, \int \mu R ds$.
But from (1) of last Article, $\int \mu R ds = -\int dT = T_0 - T_1$.

Hence the work done against the friction is

$$T_1 (e^{\mu\Omega} - 1) \,.\, \delta s,$$

Ω being the sum of the angles of contingence between A and B, or the angle between the tangents at these points if the curve of the string is a plane curve.

Examples.

1. A uniform chain of length l hangs over two fixed points, which are in a horizontal line; from its middle point is suspended by one end another chain of equal thickness and length l'. Supposing each of the two tangents of the former chain at its middle point to make an angle θ with the vertical, to find the distance between the two fixed points, and to show that θ can never exceed a certain value. (Walton's *Mechanical Problems*, p. 123.)

Let the fixed points be P and Q (fig. 224), $RQCPM$ the string hanging over them, CD the string of length l' suspended from C, the middle point of the first string, and $2d$ the distance PQ.

Then (Art. 206) the arcs PC and QC belong to the distinct catenaries. Suppose the semi-catenary to which PC belongs to be completed, and let A be its lowest point. Then if the portion AC were supplied to the string CPM, and the point A fixed, the string CD and the portion CQR might both be removed, and we should have the string APM hanging in equilibrium. Hence (Cor., Art. 206) PM terminates on the horizontal axis of this catenary. The same remarks apply to the portion CQR, and since the two portions CPM and CQR are exactly similar, it follows RM is the horizontal axis of the catenary AP.

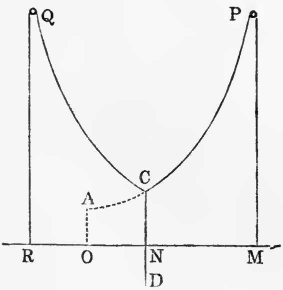

Fig. 224.

We shall next prove that

$$AC = \tfrac{1}{2}\, CD = \frac{l'}{2}.$$

Let T be the common tension of the portions CP and CQ at C. Then resolving vertically for the equilibrium of the point C,

$$2T \cos\theta = mgl'.$$

But $T = mg \,.\, CN$ (Art. 206), N being the point in which CD meets the axis. Hence $2CN \cos\theta = l'$; but it is evident from figure 220 that $CN \cos\theta = AC$; therefore $AC = \tfrac{1}{2} l'$.

Again, c being the parameter of the catenary, we have $c = AC \tan\theta$; therefore

$$c = \tfrac{1}{2} l' \tan\theta. \tag{1}$$

Also, denoting ON by x, O being the origin of the catenary, we have

$$AC = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right),$$

or

$$\frac{l'}{2} = \frac{l'}{4}\tan\theta\left(e^{\frac{2x}{l'}\cot\theta} - e^{-\frac{2x}{l'}\cot\theta}\right),$$

$$\therefore \quad 2\cot\theta = e^{\frac{2x}{l'}\cot\theta} - e^{-\frac{2x}{l'}\cot\theta}.$$

Squaring both sides of this equation, adding 4 to each side, and taking the square root, we have

$$2\operatorname{cosec}\theta = e^{\frac{2x}{l'}\cot\theta} + e^{-\frac{2x}{l'}\cot\theta};$$

which, by addition to the last equation, gives easily

$$x = \frac{l'}{2}\tan\theta\log\cot\frac{\theta}{2}. \tag{2}$$

Again,

$$AP = \frac{c}{2}\left(e^{\frac{x+d}{c}} - e^{-\frac{x+d}{c}}\right),$$

and

$$PM = \frac{c}{2}\left(e^{\frac{x+d}{c}} + e^{-\frac{x+d}{c}}\right);$$

therefore by addition we have, since $CP + PM = \tfrac{1}{2}l$,

$$\frac{l+l'}{2} = ce^{\frac{x+d}{c}}.$$

Substituting in this equation the values of c and x given by (1) and (2); and taking logarithms, we have

$$2d = l'\tan\theta\log\left(\frac{l+l'}{l'}\cdot\frac{\tan\frac{1}{2}\theta}{\tan\theta}\right), \tag{3}$$

which is the required distance between P and Q.

Since d cannot be negative, the expression whose logarithm is taken in (3) must be >1. Hence $(l+l')\tan\frac{1}{2}\theta > l'\tan\theta$; and substituting for $\tan\theta$ in terms of $\tan\frac{1}{2}\theta$, we find the limiting value of θ given by the equation

$$\tan^2\frac{\theta}{2} = \frac{l-l'}{l+l'}.$$

2. A uniform chain hangs over two smooth pegs in the same horizontal line, and at a given distance apart; find the length of the chain when the pressure on each peg is a minimum.

Let P and Q be the pegs, $2a$ the distance between them, $2l$ the length of the chain, θ the angle which the tangent to the chain at P makes with the vertical, PM the portion which hangs over the peg P, and C the lowest point of the chain.

Then $CP+PM = ce^{\frac{a}{c}}$ (by adding the values of CP and PM), or

$$\frac{l}{2} = ce^{\frac{a}{c}}, \tag{1}$$

an equation which determines l in terms of c.

Again, $CP = c \cot\theta$, and $PM = c \operatorname{cosec}\theta$, therefore by addition

$$\tan\frac{\theta}{2} = e^{-\frac{a}{c}}. \tag{2}$$

Now, the pressure on the peg P is the resultant of two equal tensions, one along PM and the other along the tangent to the chain at P. Hence, if R denote the pressure, and T the tension at P,

$$R = 2T\cos\frac{\theta}{2}.$$

Substituting for T the value $\frac{1}{2}mgc\,(e^{\frac{a}{c}}+e^{-\frac{a}{c}})$, and for $\cos\frac{\theta}{2}$ its value obtained from (2), we have

$$R = mgc\,(e^{\frac{2a}{c}}+1)^{\frac{1}{2}}. \tag{3}$$

Now, c must be determined so that R is least; hence $\frac{dR}{dc} = 0$, and we obtain easily

$$e^{\frac{2a}{c}} = \frac{c}{2a-c}, \tag{4}$$

for the determination of c in terms of a; l is then known from (1).

3. A uniform inextensible string, acted on by gravity and by two terminal tensions, rests in contact with a smooth curve in a vertical plane; find the form of this curve so that the pressure which it exerts on the string may at every point be inversely proportional to the radius of curvature.

Let vertical and horizontal lines in the plane of the curve be taken as axes of y and x, respectively, and let the concavity of the curve be upwards.

Then R being the pressure on a unit of length at any point, and T the tension at this point, we have, by resolving along the tangent,

$$dT = mg\,dy,$$

mg being the weight of a unit of length of the string. Hence

$$T = T_0 + mg\,(y-y_0), \tag{1}$$

T_0 and y_0 belonging to one end of the string.

Again, resolving normally,

$$Td\theta - mg\,dx = Rds,$$

($d\theta$ being the angle between two consecutive tangents), or

$$\frac{T}{\rho} - mg\frac{dx}{ds} = R. \tag{2}$$

Let $R = \frac{k}{\rho}$, k being a constant. Then from (1) and (2)

$$\frac{T_0 - k + mg(y - y_0)}{\rho} = mg\frac{dx}{ds},$$

or
$$\frac{y-\lambda}{\rho} = \frac{dx}{ds}, \tag{3}$$

denoting the numerator of the left-hand side of the previous equation by $mg(y-\lambda)$, for simplicity. To integrate (3), put

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+p^2}}, \text{ and } \rho = \frac{(1+p^2)^{\frac{3}{2}}}{p\frac{dp}{dy}} \text{ where } p \equiv \frac{dy}{dx}.$$

The equation then becomes
$$\frac{pdp}{1+p^2} = \frac{dy}{y-\lambda},$$

$$\therefore \; 1 + p^2 = \mu^2(y-\lambda)^2,$$

μ being the constant introduced by integration.

From this equation we have

$$\frac{dy}{\sqrt{(y-\lambda)^2 - \frac{1}{\mu^2}}} = \mu dx,$$

which gives by integration

$$y - \lambda + \sqrt{(y-\lambda)^2 - \frac{1}{\mu^2}} = be^{\mu x},$$

where b is an arbitrary constant. This equation can easily be put into the form

$$y - \lambda = \frac{b}{2}e^{\mu x} + \frac{1}{2b\mu^2}e^{-\mu x}.$$

Now, any expression of the form $Ae^{\mu x} + Be^{-\mu x}$ can be put into the form
$$C\{e^{\mu(x+a)} + e^{-\mu(x+a)}\};$$
for, identifying the two expressions, we have

$$C = \sqrt{AB}, \text{ and } e^{\mu a} = \sqrt{\frac{A}{B}}.$$

Hence we have
$$y - \lambda = \frac{1}{2\mu}\left\{b\mu e^{\mu x} + \frac{1}{b\mu}e^{-\mu x}\right\}$$

$$= \frac{1}{2\mu}\{e^{\mu(x+a)} + e^{-\mu(x+a)}\},$$

where $e^{\mu a} = b\mu$.

This is, of course, the equation of a common catenary whose parameter is $\frac{1}{\mu}$, and whose origin is the point $(\lambda, -a)$.

4. A uniform inextensible string, acted on by two terminal tensions, and any system of conservative forces in one plane, rests in contact with a smooth curve in this plane; if at every point the

pressure against the curve is inversely proportional to the radius of curvature, then, without any change in the forces, the tension at one extremity can be so varied that the constraining curve may be removed, and the string will rest in free equilibrium.

For, if V denote the potential of the applied forces at any point, we have (Art. 211)
$$T = T_0 - (V - V_0), \tag{1}$$

Again, if N denote the normal component of the applied forces at any point measured towards the convex side of the curve, and R the pressure per unit of length at this point,
$$\frac{T}{\rho} = R + N. \tag{2}$$

Suppose that $R = \dfrac{k}{\rho}\cdot$ Then, from (1) and (2) we have
$$\frac{T_0 - k - (V - V_0)}{\rho} - N = 0. \tag{3}$$

Let us now change the terminal tension T_0 into $T_0 - k$, and investigate the pressure of the curve at the point considered above. Denoting the new pressure by R', and the new tension by T', there being no change in any of the applied forces, we have
$$T' = T_0 - k - (V - V_0),$$
$$\frac{T'}{\rho} = R' + N,$$
from which
$$R' = \frac{T_0 - k - (V - V_0)}{\rho} - N;$$
but the right-hand side of this equation is zero by (3). Hence there is no pressure at any point, and the curve is one of free equilibrium. It is obvious that the last example is a particular case of this.

5. Find the law of variation of the mass per unit of length at each point of a string acted on by gravity in order that it may hang in the form of a semicircle whose diameter is horizontal.

Let $AB\,(=2a)$ be the horizontal diameter, O the centre of the semicircle, P any point on the curve, and the $\angle AOP = \theta$. Then taking horizontal and vertical lines through O as axes of x and y, respectively, we have
$$x = a\cos\theta,\ y = a\sin\theta,\ \frac{dy}{dx} = -\cot\theta,\ \frac{d\theta}{dx} = -\frac{1}{y},\ \frac{dx}{ds} = -\sin\theta = -\frac{y}{a}\cdot$$

Hence
$$\frac{d^2y}{dx^2} = \frac{1}{\sin^2\theta}\cdot\frac{d\theta}{dx} = -\frac{a^2}{y^3}\cdot$$

Also, denoting $\kappa\sigma$ in equation (3) of Art. 205 by m, we have
$$m = \frac{\tau}{g}\cdot\frac{a}{y^2},$$
which proves that the mass per unit length at any point varies inversely as the square of the depth of the point below the horizontal diameter.

6. A heavy chain of variable density, suspended from two fixed points, hangs in the form of a curve whose intrinsic equation is $s = f(\theta)$, the lowest point being origin; prove that the density at any point will vary inversely as $\cos^2\theta \,.\, f'(\theta)$. (Wolstenholme's *Book of Mathematical Problems.*)

We have here

$$\frac{dy}{dx} = \tan\theta, \quad \frac{dx}{ds} = \cos\theta, \text{ and } \frac{ds}{d\theta} = f'(\theta).$$

Hence
$$\frac{d^2y}{dx^2} = \frac{1}{\cos^2\theta}\cdot\frac{d\theta}{dx} = \frac{1}{\cos^2\theta}\frac{d\theta}{ds}\frac{ds}{dx} = \frac{1}{\cos^3\theta f'(\theta)};$$

and equation (3) of Art. 205 gives

$$m = \frac{\tau}{g\cos^2\theta f'(\theta)}.$$

7. A string is kept in equilibrium in the form of a closed curve by the action of a repulsive force tending from a fixed point, and the density at each point is proportional to the tension; prove that the repulsive force at any point is inversely proportional to the chord of curvature through the centre of force. (Wolstenholme, *ibid.*) The equations are (Art. 209),

$$Tp = h, \qquad (1)$$

$$dT = -mFdr, \qquad (2)$$

Now, $m \equiv k\sigma$, and by hypothesis $k \propto T$, and σ is constant; therefore we have $m = \mu T$, μ being a constant. Hence from (2)

$$\frac{dT}{T} = -\mu F dr. \qquad (3)$$

But from (1), $dT = -\frac{h}{p^2}dp$, therefore $\frac{dT}{T} = -\frac{dp}{p}$, and we have from (3)

$$\mu F = \frac{1}{p}\cdot\frac{dp}{dr} = \frac{2}{\gamma},$$

where γ is the chord of curvature passing through the pole (see Williamson's *Diff. Cal.*, p. 293, third ed.).

As a particular case, we may notice that the vertical chord of curvature at any point of the catenary of uniform strength (under gravity) is constant, as the student can easily prove otherwise.

8. A heavy inextensible string rests, in limiting equilibrium, on a rough curve in a vertical plane; find the tension at any point.

Let fig. 223 represent the string lying on the curve; let a horizontal line above the curve AB be the axis of x, and let the axis of y be drawn vertically downwards.

Then, if θ be the angle made by the tangent at any point, P, with the axis of x, mg the weight of a unit length of the string at P, and x, y the co-ordinates of P, we get by a tangential resolution (slipping being on the point of taking place from P to Q),

$$dT - \mu R ds + mg\,dy = 0;$$

and by a normal resolution

$$T d\theta - R ds + mg\,dx = 0.$$

Eliminating R, we obtain

$$\frac{dT}{d\theta} - \mu T = mg\left(\mu\frac{dx}{d\theta} - \frac{dy}{d\theta}\right)$$

$$= mg\left(\mu\cos\theta - \sin\theta\right)\rho, \qquad (1)$$

where ρ is the radius of curvature at P.

This is a linear differential equation of the first order, the solution of which is (Boole's *Differential Equations*, p. 39),

$$T = e^{\mu\theta}\{C + \int mg\rho\left(\mu\cos\theta - \sin\theta\right)e^{-\mu\theta}d\theta\}, \qquad (2)$$

C being a constant.

When the curve of constraint is given, ρ is known in terms of θ, and the integration may then be performed.

For example, let the string rest on a circle of radius a, one extremity being at the highest point, and free from tension.

It will be easily found that

$$\int(\mu\cos\theta - \sin\theta)\,e^{-\mu\theta}d\theta = \frac{e^{-\mu\theta}}{1+\mu^2}\{2\mu\sin\theta + (1-\mu^2)\cos\theta\},$$

therefore $\quad T = Ce^{\mu\theta} + \dfrac{mga}{1+\mu^2}\{2\mu\sin\theta + (1-\mu^2)\cos\theta\}.$

At the highest point $\theta = 0$ and $T = 0$; therefore $C = -mga\dfrac{1-\mu^2}{1+\mu^2}$.

Hence $T = \dfrac{mga}{1+\mu^2}\{2\mu\sin\theta + (1-\mu^2)\cos\theta - (1-\mu^2)e^{\mu\theta}\}.$

If the length of the string is that of a quadrant, we have $T = 0$ when $\theta = \frac{\pi}{2}$, and then μ is determined from the equation

$$e^{\frac{\mu\pi}{2}} = \frac{2\mu}{1-\mu^2}.$$

9. A, B, C are three unequally rough pegs in a vertical plane; P is the greatest weight that can be supported by a weight W when both are connected by a string (whose weight is neglected) passing over A, B, and C; Q is the greatest weight that W can support when the string passes over A and B; and R is the greatest that W can support when the string passes over B and C. Find the coefficients of friction for the pegs.

Let the inclinations of AB and BC to the vertical (measured in the same sense) be α and β, respectively; μ, μ', μ'' the coefficients of friction of A, B, C. Then, if the string passes over all the pulleys, and W hangs from A, it follows from equation (3) of Art. 214, that the tension, T, in the portion AB is $We^{\mu\alpha}$; and, by the same equation, the tension, T', in BC is $Te^{\mu'(\beta-\alpha)}$; and, finally, $P = T'e^{\mu''(\pi-\beta)}$. Hence

$$P = We^{\mu\alpha + \mu'(\beta-\alpha) + \mu''(\pi-\beta)};$$

and the equations are obviously

$$\mu a + \mu'(\beta - a) + \mu''(\pi - \beta) = \log \frac{P}{W},$$

$$\mu a + \mu'(\pi - a) = \log \frac{Q}{W},$$

$$\mu'\beta + \mu''(\pi - \beta) = \log \frac{R}{W},$$

from which μ, μ', μ'' can be found. The value of μ' is $\frac{1}{\pi} \log \frac{QR}{PW}$.

10. A heavy uniform chain rests in limiting equilibrium on a rough cycloidal arc, whose axis is vertical and vertex upwards, one extremity being at the vertex and the other at the cusp; prove that

$$e^{\frac{\mu\pi}{2}} = \frac{3}{1+\mu^2}.$$

(Wolstenholme's *Book of Math. Prob.*)

11. A uniform inextensible string whose length is l hangs in limiting equilibrium over a fixed rough cylinder of radius a whose axis is horizontal; find the lengths of the portions which hang vertically.

Ans. $\dfrac{l - \pi a}{1 + e^{-\mu\pi}} + \dfrac{2\mu a}{1+\mu^2}$, and a value obtained by changing the sign of μ in this expression.

12. Two equal weights are attached each to the extremity of a string which hangs over a rough cylinder whose axis is horizontal; find how much either weight must be increased in order that it may begin to descend, the weight of the string being neglected.

Ans. The increase of weight $= P(e^{\mu\pi} - 1)$, where P is common value of the suspended weights.

13. A string, whose weight is neglected, passes *over* any number of equally rough fixed circular pulleys in a vertical plane; show that the ratio of two weights, suspended from the extremities of the string, which just sustain each other, is the same as if only one pulley were used.

14. A heavy uniform beam is moveable in a vertical plane round a smooth hinge at one extremity, and has the other extremity attached to a cord which passes over a small rough peg placed vertically over the hinge, and sustains a given weight; find the position of limiting equilibrium, and the tension of the cord.

Ans. If $W =$ weight of beam, $P =$ suspended weight, T the tension, $2a =$ length of beam, $2c =$ distance of peg from hinge, $\theta =$ inclination of beam to vertical, and $\phi =$ inclination of cord to vertical, the position in which the beam is about to descend is given by the equations

$$c \sin\phi = a \sin(\theta - \phi),$$

$$T = Pe^{\mu(\pi-\phi)},$$

$$Wa \sin\theta = 2\, Tc \sin\phi.$$

15. Prove that the area of the normal section at any point in the catenary of uniform strength is proportional to the radius of curvature.

16. Find the law of variation of the mass per unit of length in order that a string may hang, under the action of gravity, in a parabola.

Ans. The mass at any point is proportional to the horizontal projection of the unit length at the point. (Compare Art. 208.)

17. If a string hangs, under the action of gravity, in the form of an ellipse whose axis major is horizontal, prove that the mass per unit of length at any point is $\frac{\tau}{g} \cdot \frac{b^3}{ab'y^2}$, y being the distance of the point from the axis major, and b' the length of the semi-conjugate diameter corresponding to the point.

18. One extremity of a uniform string is attached to a fixed point, and the string rests partly on a smooth inclined plane; prove that the horizontal axis of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.

19. If, in the last example, i is the inclination of the plane, a the inclination of the tangent at the fixed extremity, and l the whole length of the string, prove that the length of the portion on the plane is

$$\frac{l \cos a}{\cos i \cos (a - i)}.$$

(Walton, p. 119.)

20. Given two smooth pegs in a horizontal line, find the least length of a uniform heavy string which will rest over them.

Ans. If $2a$ is the distance between the pegs, and e the Napierian base, the least length is ae.

21. A uniform inextensible string assumes the form of a circle under the influence of a repulsive force emanating from a point on its circumference; find the law of force.

Ans. It varies inversely as the cube of the distance.

22. A uniform inextensible string is in equilibrium under the action of a central repulsive force; prove that at each point of the string this force $\propto \frac{1}{p\gamma}$, where p is the perpendicular from the centre of force on the tangent, and γ the chord of curvature passing through the centre of force.

23. If the curve of equilibrium is an ellipse whose focus is the centre of force, the force at any point $\propto \frac{1}{rb'}$, where b' is the semi-conjugate diameter corresponding to the point, and r the focal distance of the point.

24. If the string assume the form of an ellipse under the influence of a repulsive force emanating from the centre, find the law of force.

Ans. The force is directly proportional to the distance, and inversely proportional to the conjugate diameter.

25. If an inextensible string can assume the same plane figure of equilibrium under the separate action of any number of forces, it can assume this figure under their combined action.

(To prove this, suppose the string under the combined action of the forces to be constrained to a smooth curve of the given figure, and it will follow that the pressure at every point of this curve varies inversely as the radius of curvature. The theorem follows, then, from example 4.)

26. A uniform inextensible string rests against the inner side of a smooth elliptic wire, and is repelled from the foci and the centre by the following forces: $\frac{\mu}{rb'}$ and $\frac{\mu'}{r'b'}$ emanating from the foci, and $\frac{\mu'' a'}{b'}$ from the centre, the distances of a point on the string from the foci being r and r', respectively, its distance from the centre being a', and the semi-conjugate diameter corresponding to the point being b'. Find the pressure on the wire at any point.

Ans. If T_0 is the tension of the string at the extremity of the minor axis, $R =$ pressure per unit length $= \frac{a T_0 - \mu - \mu' - \mu'' a^2}{a\rho}$.

(The student will easily see from examples 4 and 25, that if the curve of constraint of a string is a possible curve of free equilibrium under the action of the given forces, the pressure will, at every point, be $\frac{C}{\rho}$, where C is a constant. The result, in this example might, therefore, be at once obtained by this principle.

By direct calculation, however, the result is obtained with little trouble. The equations of equilibrium are

$$dT + \frac{\mu}{rb'}\,dr + \frac{\mu'}{r'b'}\,dr' + \frac{\mu'' a'}{b'}\,da' = 0,$$

$$\frac{T}{\rho} + R = \left(\frac{\mu}{r} + \frac{\mu'}{r'}\right)\frac{b}{b'^2} + \frac{\mu'' ab}{b'^2};$$

and the first gives, by integration,

$$T - \frac{\mu}{a}\sqrt{\frac{r'}{r}} - \frac{\mu'}{a}\sqrt{\frac{r}{r'}} - \mu'' b' = \text{const.}).$$

The student will do well to apply the principle explained here to he kinetical examples in Walton, pp. 295, and 259 second edition.

Section II.

Flexible Extensible Strings.

216.] **Experimental Law of Extension.** The strings which we now proceed to consider are *extensible*, i. e. such as have their lengths increased when they are in a state of tension. For such strings we shall still assume the property of complete flexibility as defined in Art. 200.

The law of extension which we proceed to enunciate applies not only to flexible strings but also to straight bars of iron, steel, &c.

Let l_0 denote the length of any string or straight bar of uniform section when it is not subject to the action of any external force. This is called the *natural length* of the string or bar. Let σ be the area of the normal section, F the magnitude of the force applied at one extremity in the direction AB, of the string or bar. Then supposing the extremity A to be fixed, the force F will produce an extension, BC, of the body. Denote this extension by x. Then experience proves that *for small values* of the ratio $\frac{x}{l_0}$ in the case of solid bars there is for the same bar a constant ratio between this fraction and the quantity $\frac{F}{\sigma}$; and there is the same constancy of ratio in the case of strings, but for some of these latter bodies the value of $\frac{x}{l_0}$ may be very much greater than for bars.

Fig. 225.

We have, then,
$$\frac{F}{\sigma} = E\frac{x}{l_0}, \tag{1}$$
E being a constant quantity which is called the *modulus of elasticity* of the matter of which the string or bar is formed. Since $\frac{x}{l_0}$ is a number, it follows that E is *a force per unit of section.* This force is also known as *Young's modulus*, and it is evidently a measure of the longitudinal rigidity of the substance.

If the law expressed by equation (1) be supposed to hold for an extension x equal to l_0, and if the force applied to the body to produce this extension be called P, we have $E = \frac{P}{\sigma}$; and if σ is a section of unit area, $E = P$. The modulus of elasticity of any substance might then be defined as that force which, if applied at the extremity of a bar of the material of unit section, would double its length—this force being fictitious in the case of bars or strings for which (1) holds only within extremely narrow limits.

For bars of iron and steel this equation is true only within narrow limits—called *the limits of elasticity*—while for flexible strings of such substances as India-rubber its range is much wider. If the limiting amount of extension has not been surpassed, the body will, after a time varying with the substance, return to its original state when the stretching force F is removed. The law expressed by equation (1) is also true within narrow limits in the case of a straight bar which is *compressed* without bending.

An idea of the magnitude of the modulus of elasticity of a solid body may be formed from the fact that in the case of iron, the unit of force being a kilogramme and the unit of area a square centimetre, E is about 2,000,000. For what are commonly called elastic strings, E is of course very much smaller than for bars of iron or steel.

In the case of an elastic string it is usual to put equation (1) into another form. If l is the length which the string assumes under a tension T, we have $x = l - l_0$, and

$$\frac{T}{\sigma} = E\frac{l - l_0}{l_0},$$

or

$$l = l_0\left(1 + \frac{T}{E\sigma}\right),$$

or, as it is usually written,

$$l = l_0\left(1 + \frac{T}{\lambda}\right), \tag{2}$$

the quantity λ being called the modulus of elasticity of the string.

This quantity is obviously the force which must be applied to the string to double its length.

The law expressed by (1) or (2) is known as *Hooke's Law*, from the name of its discoverer, and is sometimes expressed in the form—*the tension of any elastic string is proportional to its extension beyond its natural length.*

217.] **Work done in slowly extending a String or Bar.** If at each instant during the extension of a string or bar the stretching force applied at the extremity is exactly equal to that which would keep the body in its state of deformation at this instant, there is continuous equilibrium between the (gradually increasing) applied force and the elastic force of the body, and therefore the total amount of work done by the applied force is equal to the work done against the internal force.

[The more advanced student will see that this would not be true if the extension were *suddenly* produced, so that oscillations would take place in the body.]

Now if x is the extension of the body at any instant, the corresponding force is $\frac{E\sigma}{l_0}x$, and the work done against this force in a further extension dx is $\frac{E\sigma}{l_0}x\,dx$. Let a be the final extension; then the total work done is

$$\int_0^a \frac{E\sigma}{l_0}\,x\,dx, \quad \text{or} \quad \frac{E\sigma a^2}{2\,l_0},$$

the extension being, of course, confined within the limits of elasticity. Now the applied force which is required to keep the body in its final state of extension is, by (1) of last Article, $\frac{E\sigma a}{l_0}$. Hence if the force applied in the final state be denoted by P, the whole amount of work done is

$$\tfrac{1}{2}Pa,$$

or half the work which would be done by the *final* force of extension in moving its point of application through a space equal to the final extension.

218.] **Equations of Equilibrium of an Extensible String.** Suppose the string to have assumed its figure of equilibrium under the action of given forces. At any point in the string let ds be the stretched length of an element whose length before the action of the forces was ds_0; and at this point let m be the mass of a unit length of density equal to that at the point, the mass per unit length at the same point in the natural state of the string being m_0.

Then since the quantity of matter in the element is unaltered by stretching,

$$m\,ds = m_0\,ds_0. \tag{1}$$

Also by Hooke's law,

$$ds = \left(1 + \frac{T}{\lambda}\right) ds_0. \tag{2}$$

But, the string having assumed its form of equilibrium, we have, as in the inextensible string,

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0, \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0, \\ \frac{d}{ds}\left(T\frac{dz}{ds}\right) + mZ = 0. \end{aligned}\right\} \tag{3}$$

Also $$ds = \sqrt{dx^2 + dy^2 + dz^2}\,; \tag{4}$$

and since the nature of the string in its original state is given, we may assume m_0 to be a given function of the position of the element ds_0 in the string; or

$$m_0 = f(s_0), \tag{5}$$

where s_0 is the length of the arc of the original string measured from some fixed point up to the element ds_0.

Now the general problem of extensible strings may be stated as follows:—*an extensible string, the law of variation of whose density in its natural state is given, is, under given circumstances, submitted to the action of given forces; find the form which it will assume.*

To solve this problem it is necessary to find two equations between x, y, z, the co-ordinates of any point in the stretched string; and as the equations just given contain, in addition to these co-ordinates, the quantities m, m_0, s, s_0, and T, these latter must be eliminated. But from the seven equations above, these five quantities may theoretically be eliminated, by differentiation or otherwise, and there will result *two* independent equations, which are the equations necessary for the determination of the curve of equilibrium.

The problem in its general form is one of great difficulty, and one which it would be practically impossible to solve. We shall, therefore, in the sequel confine our attention to the case in which the string in its natural state is such that m_0, the mass per unit length, is constant at all points, and to the case in which the acting forces are constant.

Let us first consider m_0 constant.

By multiplying the equations (3) by $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$, respectively, and adding, we have

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) = 0; \tag{6}$$

and from (1) and (2) we have $m = \dfrac{m_0}{1+\dfrac{T}{\lambda}}$. Hence (6) becomes

$$\left(1 + \frac{T}{\lambda}\right) dT + m_0 (Xdx + Ydy + Zdz) = 0. \tag{7}$$

Hence by integration,

$$\frac{\lambda}{2}\left(1 + \frac{T}{\lambda}\right)^2 + m_0 \int (Xdx + Ydy + Zdz) = \text{const.}$$

Denote the integral in this equation by V, the potential of the acting forces, and let the constant of integration be A. Then we have

$$\frac{\lambda}{2}\left(1 + \frac{T}{\lambda}\right)^2 = A - V, \tag{8}$$

or, by (2),

$$\frac{ds}{\sqrt{A-V}} = \sqrt{\frac{2}{\lambda}} \cdot ds_0, \tag{9}$$

from which the relation between s and s_0 is found, and hence the extension of the string.

Equation (8) is the analogue of (3) of Art. 230. If V' is the potential at a point of the string at which the tension is T', this equation gives

$$(T-T')\left(1 + \frac{T+T'}{2\lambda}\right) = V' - V. \tag{10}$$

The equations of the curve of equilibrium are obtained by substituting the value of T given by (8) in any two of the equations

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dx}{ds}\right) + m_0 X = 0,$$

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dy}{ds}\right) + m_0 Y = 0,$$

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dz}{ds}\right)+m_0 Z=0,$$

which are deduced from the equations (3) by substituting for m in terms of m_0.

Secondly, suppose that the applied forces, X, Y, Z, are constant. Then the first of equations (3) gives

$$T\frac{dx}{ds}=A-X\int m_0 ds_0, \qquad (11)$$

A being the constant of integration. The remaining two of these equations give

$$T\frac{dy}{ds}=B-Y\int m_0 ds_0, \quad T\frac{dz}{ds}=C-Z\int m_0 ds_0. \qquad (12)$$

Hence, by squaring and adding,

$$T^2=(A-X\int m_0 ds_0)^2+(B-Y\int m_0 ds_0)^2+(C-Z\int m_0 ds_0)^2 \quad (13)$$

This equation gives T, the tension at any point in the stretched string, in terms of the length of the arc of the unstretched string corresponding to this point; or, in other words,

$$T=\phi(s_0). \qquad (14)$$

Hence, from (2) we have

$$s=\int\left\{1+\frac{\phi(s_0)}{\lambda}\right\}ds_0,$$

which gives the relation between the stretched and unstretched lengths of any arc.

The equations of the curve are obtained from (11) and (12) by substituting for ds in terms of ds_0. Thus we have

$$dx=(A-X\int m_0 ds_0)\left\{\frac{1}{\lambda}+\frac{1}{\phi(s_0)}\right\}ds_0,$$

$$dy=(B-Y\int m_0 ds_0)\left\{\frac{1}{\lambda}+\frac{1}{\phi(s_0)}\right\}ds_0,$$

$$dz=(C-Z\int m_0 ds_0)\left\{\frac{1}{\lambda}+\frac{1}{\phi(s_0)}\right\}ds_0.$$

Integrating these equations and eliminating s_0 between them in pairs, we obtain the two equations of the curve.

As an example, let it be proposed to investigate the form of an elastic string suspended from two fixed points and acted on by gravity, the string being uniform in its natural state. Taking axes as in Art. 206, we have

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) = mg.$$

Hence $T\dfrac{dx}{ds} = \tau = m_0 gc$, suppose; and $T\dfrac{dy}{ds} = B + m_0 g s_0$. But if s_0 be measured from the lowest point, $\dfrac{dy}{ds} = 0$ and $s_0 = 0$ at the same time. Hence $B = 0$, and we have

$$T\frac{dx}{ds} = m_0 gc,$$

$$T\frac{dy}{ds} = m_0 g s_0;$$

from which $T = m_0 g\sqrt{c^2 + s_0^2}$; therefore

$$dx = \left(\frac{m_0 gc}{\lambda} + \frac{c}{\sqrt{c^2+s_0^2}}\right) ds_0,$$

$$dy = \left(\frac{m_0 g s_0}{\lambda} + \frac{s_0}{\sqrt{c^2+s_0^2}}\right) ds_0.$$

Hence, putting $\lambda = m_0 ga$, we have

$$x = \frac{cs_0}{a} + c\log\frac{s_0 + \sqrt{c^2+s_0^2}}{c}, \tag{15}$$

$$y = \frac{s_0^2}{2a} + \sqrt{c^2+s_0^2}. \tag{16}$$

The relation between x and y is obtained by eliminating s_0 from these equations.

An approximate relation between them may be obtained when the string is only slightly extensible, i. e. when λ (or a) is very great. In this case (16) gives

$$s_0^2 = (y^2 - c^2)\left(1 - \frac{y}{a} + \frac{5y^2 - c^2}{a^2}\right), \tag{17}$$

to the second order of the small quantity $\dfrac{1}{a}$.

Now, writing (15) and (16) in the forms

$$x = \frac{cs_0}{a} + \xi,\ y = \frac{s_0^2}{2a} + \eta,$$

we know that $$\eta = \frac{c}{2}\left(e^{\frac{\xi}{c}} + e^{-\frac{\xi}{c}}\right).$$

Hence $$y - \frac{s_0^2}{2a} = \frac{c}{2}\left(e^{\frac{x}{c}} \cdot e^{-\frac{s_0}{a}} + e^{-\frac{x}{c}} \cdot e^{\frac{s_0}{a}}\right)$$

$$= u - \frac{2vs_0}{a} + \frac{us_0^2}{2a^2},$$

by expanding $e^{\frac{s_0}{a}}$ and $e^{-\frac{s_0}{a}}$ as far as $\frac{1}{a^2}$ and denoting by u and v the quantities $\frac{c}{2}(e^{\frac{x}{c}}+e^{-\frac{x}{c}})$ and $\frac{c}{2}(e^{\frac{x}{c}}-e^{-\frac{x}{c}})$.

Substituting in this equation the value of s_0 given by (17)—in which it is evident that the term of the second order may be rejected if we wish to obtain y to this order only in terms of x—we obtain an equation of the form

$$y = u+\frac{P}{a}+\frac{Q}{a^2}, \tag{18}$$

in which P and Q are both functions of x and y.

Now assume $y = u+\frac{\lambda}{a}+\frac{\mu}{a^2}$, where λ and μ are functions of x alone, and substitute this value of y in every term of (18). This will give us, with a little trouble,

$$\lambda = -\frac{1}{2}v^2, \text{ and } \mu = \frac{1}{2}uv^2.$$

Hence, finally

$$y = u-\frac{v^2}{2a}+\frac{vu^2}{2a^2},$$

to the second order of the small quantity $\frac{1}{a}$.

219.] **Extensible String on Smooth Surface.** It is clear that the equations (1) of Art. 213 are applicable to an extensible string, as are also the results arrived at in that Article without integration. The result arrived at by integration, which expresses the tension in terms of the potential, is to be replaced by equation (10) of Art. 218; and from this equation it follows that if an extensible string, uniform in its natural state, rest on any smooth surface under the action of gravity, the free extremities are in the same horizontal plane.

EXAMPLES.

1. An elastic string, uniform in its natural state, is suspended from one extremity, which is fixed, and has a given weight attached to the other; find the extension of the string, taking its own weight into account.

Let W be the weight of the string, P the suspended weight, λ the modulus of elasticity, and m_0 the mass of a unit length of the unstretched string. Then the equation of equilibrium is

$$dT+m_0 g\, ds_0 = 0.$$

If l_0 is the natural length of the string, $m_0 g l_0 = W$; therefore this equation gives by integration

$$T + \frac{W}{l_0} s_0 = \text{const.}$$

When $s_0 = 0$, T is evidently $W + P$; therefore

$$T = W + P - \frac{W}{l_0} s_0.$$

Again, since $ds = (1 + \frac{T}{\lambda}) ds_0$, we have

$$ds = (1 + \frac{W+P}{\lambda} - \frac{W}{\lambda l_0} s_0) ds_0,$$

$$\therefore \quad s = (1 + \frac{W+P}{\lambda} - \frac{W}{2\lambda l_0} s_0) s_0,$$

no constant being added because $s = 0$ when $s_0 = 0$.

If $s_0 = l_0$, and l is the whole length of the stretched string, we have

$$l = l_0 (1 + \frac{W + 2P}{2\lambda}).$$

2. A heavy uniform elastic ring is placed round a smooth vertical cone; find how far it will descend.

Let W be the weight of the ring, $2\pi a$ its natural length, λ its modulus of elasticity, y the distance of the plane of the ring from the vertex of the cone in the position of equilibrium, and l the stretched length in this position. Then if the ring be shoved down through an indefinitely small vertical distance, δy, the equation of work is

$$-T\delta l + W\delta y = 0,$$

T being the tension of the ring. If α is the semi-vertical angle of the cone, $l = 2\pi y \tan \alpha$; hence $\delta l = 2\pi \tan \alpha \,.\, \delta y$, and

$$2\pi T \tan \alpha = W.$$

But, by Hooke's Law,

$$y \tan \alpha = a (1 + \frac{T}{\lambda});$$

$$\therefore \quad y = a \cot \alpha (1 + \frac{W}{2\pi\lambda} \cot \alpha).$$

3. An elastic string, uniform in its original state, is placed on any smooth curve and acted on by given forces; find its extension.

The tension at any point is determined by the equation

$$(1 + \frac{T}{\lambda}) dT + m_0 (X dx + Y dy + Z dz) = 0,$$

or

$$\lambda (1 + \frac{T}{\lambda})^2 + 2m_0 \int (X dx + Y dy + Z dz) = \text{const.} \qquad (1)$$

Let $m_0 \int (X dx + Y dy + Z dz)$ be denoted by V. Now, take any point, O, in the string as the point from which s and s_0 are measured, and let A be the value of V at a free extremity of the string. If one extremity is fixed, it will be well to measure s and s_0 from it. Putting $T=0$, $V=A$, and also $1+\frac{T}{\lambda}=\frac{ds}{ds_0}$,

(1) gives
$$\left(\frac{ds}{ds_0}\right)^2 = 1+\frac{2m_0}{\lambda}(A-V). \tag{2}$$

Suppose the curve of constraint to be given by the three equations
$$x=f_1(s), \quad y=f_2(s), \quad z=f_3(s).$$

Then (2) gives
$$\frac{ds}{\sqrt{1+\frac{2m_0}{\lambda}(A-V)}} = ds_0,$$

or, by integration,
$$\phi(s, A) = s_0+\phi(o, A), \tag{3}$$

s and s_0 being both measured from O. Let l and l_0 be the stretched and original lengths of the portion between O and the free extremity considered. Then we have
$$\phi(l, A) = l_0+\phi(o, A). \tag{4}$$

But A is evidently a function of the co-ordinates of the extremity, and these co-ordinates are, by supposition, $f_1(l)$, $f_2(l)$, $f_3(l)$; hence A is a known function of l, and by substituting its value in (4) we deduce the value of l.

4. One extremity of an elastic string, originally uniform, is fixed at the highest point of a smooth cycloid in a vertical plane, the string lying along the convex side of the curve; find the extension produced by gravity.

If the tangent at the highest point is taken as axis of x, and if $\frac{\lambda}{2m_0 g}$ is denoted by c, we find easily, for any curve of constraint,
$$\frac{ds}{\sqrt{c+h-y}} = \frac{ds_0}{\sqrt{c}},$$

h being the ordinate of the free extremity.

In the cycloid $s^2=8ay$. Substituting this value of y in the equation, and integrating, we have
$$s = 2\sqrt{2a(c+h)}\sin\left(\frac{s_0}{2\sqrt{2ac}}\right).$$

If l be the length from the fixed to the free extremity, and l_0 the natural length of the string,
$$l = 2\sqrt{2a(c+h)}\sin\left(\frac{l_0}{2\sqrt{2ac}}\right).$$

Also
$$l^2 = 8ah.$$

These equations combined give
$$l = 2\sqrt{2ac}\tan\left(\frac{l_0}{2\sqrt{2ac}}\right).$$

5. A heavy particle is attached to one end of an elastic string whose unstretched length is indefinitely small; the particle rests on a

smooth curve in a vertical plane, and the fixed end of the string is attached to a point in this curve; find the nature of the curve so that the particle may rest in all positions.

Ans. A cycloid.

6. A heavy elastic string is laid upon a smooth double inclined plane in such a manner as to remain at rest; find how much the string is stretched. (Walton, p. 140.)

Ans. If W is the weight, λ the modulus of elasticity, and c the natural length of the string, and α, α' the inclinations of the planes to the horizon, the extension is

$$\frac{W}{2\lambda}\frac{\sin\alpha\sin\alpha'}{\sin\alpha+\sin\alpha'}c.$$

[For the portion on the plane α let s and s_0 be measured from the free extremity. Then

$$T=\frac{W\sin\alpha}{c}s_0;\text{ and } ds=\left(1+\frac{T}{\lambda}\right)ds_0=\left(1+\frac{W\sin\alpha}{\lambda c}s_0\right)ds_0.$$

Hence if l is the length of the portion on the plane α, we have

$$l=l_0\left(1+\frac{W\sin\alpha}{2\lambda c}l_0\right).$$

A similar equation holds for the portion on the plane α'. Now the extension $=l+l'-l_0-l'_0$; and equating the tensions at the common summit of the planes, we have $l_0\sin\alpha=l'_0\sin\alpha'$,

$$\therefore\quad l_0=\frac{c\sin\alpha'}{\sin\alpha+\sin\alpha'},\ \&c.]$$

7. If the cone in example 2 is replaced by a smooth paraboloid of revolution, find how far the ring will descend. [By Virtual Work.]

Ans. $y=\dfrac{a}{1-\dfrac{aW}{4\pi m\lambda}}$, where $4m=$ latus rectum of generating parabola.

8. An elastic string, uniform in its original state, rests on a rough inclined plane with its upper extremity fixed; prove that its extension will lie between the limits

$$\frac{l^2}{2c}\cdot\frac{\sin(i\pm\epsilon)}{\cos\epsilon},$$

where $i=$ inclination of plane, $\epsilon=$ angle of friction, $l=$ natural length of string, and $c=$ length of a portion of the string in its natural state whose weight is the modulus of elasticity. (Wolstenholme's *Math. Prob.*)

9. A weight P just supports another weight Q by means of a fine elastic string passing over a rough circular cylinder whose axis is horizontal; λ is the modulus of elasticity, and a the radius of the cylinder; prove that the extension of the part of the string in contact with the cylinder is

$$\frac{a}{\mu}\log\frac{Q+\lambda}{P+\lambda}.\qquad\text{(Wolstenholme, } ibid.)$$

10. Two uniform ladders, freely jointed at a common extremity, rest in a vertical plane with their other extremities on a rough horizontal plane, these extremities being connected by an elastic rope; find the greatest angle between them consistent with equilibrium.

Ans. If a is the length of each ladder, $2a \sin \alpha$ the natural length of the rope, 2θ the greatest angle between the ladders, and λ the modulus of elasticity of the rope,

$$\lambda (\sin \theta - \sin \alpha) = W \sin \alpha \, (\mu + \tfrac{1}{2} \tan \theta).$$

11. A heavy uniform elastic ring is placed horizontally round a rough right cone whose axis is vertical and vertex upwards, the stretched ring being uniform; find its extreme positions of equilibrium.

$$\textit{Ans.} \quad y = a \left\{1 + \frac{W}{2\pi\lambda} \cot (\alpha \pm \epsilon)\right\}, \text{ with notation of Ex. 2.}$$

Section III.

The Method of Virtual (or Potential) Work.

220.] **Distinction between the Symbols d and δ.** In the sequel we shall use the symbol d to denote the increment which any function receives when we pass from a given point P in a body, which occupies a given position, to any indefinitely near point Q in the body, the position of the body being invariable; while by the symbol δ we shall denote the increment which the function receives as we pass from the point P when the body occupies a given position to *the same point* P in the body when it is displaced, or imagined to be displaced, from this position to any one indefinitely close to it. This use of the symbol δ has been already exemplified in the Chapters on Virtual Work.

221.] **Commutative Property of d and δ.** If V denote any function of the co-ordinates of a point P in a body, we propose to show that

$$\delta (dV) = d (\delta V).$$

This will be rendered plain by a very simple illustration.

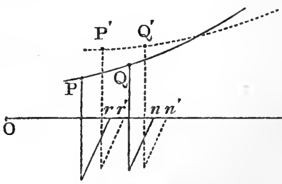

Fig. 226.

Let P and Q (fig. 226) be two very close points in a body occupying a given position, and let P' and Q' be the positions of these points when the body receives any slight displacement. Let Ox be the axis of x, and let the co-ordinates of P and Q be Or and

On, those of P' and Q' being Or' and On', measured along Ox.

Then if x is the co-ordinate of P, $dx = rn$, and $\delta x = rr'$.

Also $\delta(dx) =$ value of dx in the new position—value of dx in old position $= r'n' - rn$; and $d(\delta x) =$ value of δx for Q—value of δx for $P = nn' - rr'$. But obviously $r'n' - rn = nn' - rr'$; therefore $\delta(dx) = d(\delta x)$. From this it follows that if V is any function of x, $\delta(dV) = d(\delta V)$. For, by the elementary principles of the Differential Calculus $\delta(uv) = u\,\delta v + v\,\delta u$. Now,

$$dV = \frac{dV}{dx}\,dx,$$

$$\therefore\quad \delta(dV) = \frac{d^2V}{dx^2}\,\delta x\,dx + \frac{dV}{dx}\,\delta(dx),$$

and

$$\delta V = \frac{dV}{dx}\,\delta x;$$

$$\therefore\quad d(\delta V) = \frac{d^2V}{dx^2}\,dx\,\delta x + \frac{dV}{dx}\,d(\delta x).$$

The two expressions are, therefore, identical; and the same proof may be applied to show their identity when V is any function of the co-ordinates.

Again, since by the Differential Calculus

$$\delta(u + v + w + \dots) = \delta u + \delta v + \delta w + \dots,$$

it follows that

$$\delta\int V dx = \int \delta(V dx).$$

Suppose that any integration in which the element of arc PQ (fig. 226) is taken as the constant infinitesimal, ds, is performed over a curve, and let the integral be $\int V ds$. Then the change in the value of this integral when it is found for the same curve in a displaced position is $\delta\int V ds$. Now the infinitesimal in the new position of the curve is $P'Q'$, which is equal to PQ; therefore

$$\delta(ds) = 0,$$

and

$$\delta\int V ds = \int \delta(V ds) = \int (\delta V)\,ds,$$

that is, *the change in the value of the integral of a function = the integral of the change in the function*, both integrations being performed over the same curve, the arc of which is taken as independent variable.

The same remains true if the integration $\int V ds$ is performed over a surface or through a solid, and ds denotes the element of superficial area or of volume. Again, since by Differential Calculus, $\delta \frac{u}{v} = \frac{v\,\delta u - u\,\delta v}{v^2}$, it follows that if ds is constant,

$$\delta \frac{dx}{ds} = \frac{\delta\, dx}{ds} = \frac{d\,\delta x}{ds},$$

and
$$\delta \frac{d^2 x}{ds^2} = \delta \frac{d\frac{dx}{ds}}{ds} = \frac{d\,\delta\frac{dx}{ds}}{ds} = \frac{d^2\,\delta x}{ds^2},$$

and generally
$$\delta \frac{d^n x}{ds^n} = \frac{d^n\,\delta x}{ds^n}.$$

Example.

Every element of a solid body is multiplied by the product of its two co-ordinates x and y, and the sum of all such products is taken. If the body receives a small displacement of rotation round the axis of z, find the variation of this sum.

The element of mass at any point x, y, z being dm, the sum in question is $\int xy\,dm$. Now $\delta \int xy\,dm = \int \delta(xy)\,.\,dm = \int (x\,\delta y + y\,\delta x)\,dm$. But $\delta x = -y\,\delta\theta$, $\delta y = x\,\delta\theta$, if the angular rotation of the body is $\delta\theta$. Hence the variation $= \delta\theta \int (x^2 - y^2)\,dm$.

222.] **Method of Work Applied to a String.** *First suppose the string to be perfectly inextensible.* Now if the particles of a system are dm_1, dm_2, ..., and if they have to fulfil conditions denoted by $L_1 = 0$, $L_2 = 0$, ..., the equation of Art. 186 becomes

$$(X_1\,\delta x_1 + Y_1\,\delta y_1 + Z_1\,\delta z_1)\,dm_1 + \dots + \lambda_1\,\delta L_1 \dots = 0. \qquad (1)$$

In the present case the particles are portions ds_1, ds_2 ... of a string at points $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$, ... and each has to satisfy the condition of having its length unaltered in any displacement of the system. Hence the geometrical equations are

$$ds_1 = \text{const., \&c.};$$

and equation (1) becomes

$$(X_1\,\delta x_1 + Y_1\,\delta y_1 + Z_1\,\delta z_1)\,dm_1 + \dots + \lambda_1\,\delta\, ds_1 + \dots = 0,$$

or
$$\int (X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \int \lambda\,\delta\, ds = 0, \qquad (2)$$

the number of particles being indefinitely great.

Now, as in Art. 186, we express all the variations in terms of the variations of the co-ordinates x, y, z. For this purpose, put

$$ds^2 = dx^2 + dy^2 + dz^2,$$

$$\therefore\quad ds\,\delta\, ds = dx\,\delta\, dx + dy\,\delta\, dy + dz\,\delta\, dz,$$

or $$\delta\, ds = \frac{dx}{ds}\, d\,\delta x + \frac{dy}{ds}\, d\,\delta y + \frac{dz}{ds}\, d\,\delta z.$$

Hence (2) becomes

$$\int\left[(X\delta x + Y\delta y + Z\delta z)\,dm + \lambda\,\left(\frac{dx}{ds}\, d\,\delta x + \frac{dy}{ds}\, d\,\delta y + \frac{dz}{ds}\, d\,\delta z\right)\right] = 0. \quad (3)$$

Now $\displaystyle\int \lambda\,\frac{dx}{ds}\, d\,\delta x = \left(\lambda\,\frac{dx}{ds}\,\delta x\right)_1 - \left(\lambda\,\frac{dx}{ds}\,\delta x\right)_0 - \int \delta x\,\frac{d}{ds}\left(\lambda\frac{dx}{ds}\right).\, ds,$

by integration by parts, the term $\left(\lambda\,\frac{dx}{ds}\,\delta x\right)_1$ being the value of $\lambda\,\frac{dx}{ds}\,\delta x$ at one of the limits of integration, i.e. at one extremity of the string; and $\left(\lambda\,\frac{dx}{ds}\,\delta x\right)_0$ being its value at the other extremity.

Performing similar integrations for the other terms, (3) becomes

$$\lambda_1\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\right)_1 - \lambda_0\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\right)_0$$
$$+ \int\left[\left\{X\,dm - \frac{d}{ds}\left(\lambda\frac{dx}{ds}\right).\,ds\right\}\delta x + \left\{Y dm - \frac{d}{ds}\left(\lambda\frac{dy}{ds}\right).\,ds\right\}\delta y\right.$$
$$\left. + \left\{Z\,dm - \frac{d}{ds}\left(\lambda\frac{dz}{ds}\right).\,ds\right\}\delta z\right] = 0. \quad (4)$$

Now, as in the equation of Art. 186, we equated to zero the coefficients of δx_1, δy_1, δz_1, ..., so here we have to put the coefficients of δx, δy, and δz equal to zero for each particle of the string; that is, we put the coefficients of these quantities under the sign of integration equal to zero. Hence we have at all points

$$X\,dm - \frac{d}{ds}\left(\lambda\frac{dx}{ds}\right).\,ds = 0,$$

$$Y dm - \frac{d}{ds}\left(\lambda\frac{dy}{ds}\right).\,ds = 0,$$

$$Z\,dm - \frac{d}{ds}\left(\lambda\frac{dz}{ds}\right).\,ds = 0,$$

which equations are precisely the same as those of Art. 204, since $\frac{dm}{ds}$ is the mass per unit length at the corresponding point of the string. It appears that λ in these equations is minus the tension of the string.

The conditions of equilibrium, then, as expressed in (4), consist of two parts—namely, terms which relate to the extremities of the string (which are the terms outside the sign of integration), and terms which relate to every intermediate point in the string (which give the general equations of equilibrium above).

Equating to zero the terms outside the integral sign, we have

$$\lambda_1\left(\frac{dx}{ds}\,\delta x+\frac{dy}{ds}\,\delta y+\frac{dz}{ds}\,\delta z\right)_1-\lambda_0\left(\frac{dx}{ds}\,\delta x+\frac{dy}{ds}\,\delta y+\frac{dz}{ds}\,\delta z\right)_0=0. \quad (5)$$

Now, if the extremities of the string are fixed, they will be fixed in the displaced string, and every term of (5) vanishes since

$$\delta x_1=\delta y_1=\delta z_1=\delta x_0=\delta y_0=\delta z_0=0.$$

But if each end is perfectly free, since δx_1, δy_1, ... are quite arbitrary and independent, we must have

$$\lambda_1=0 \text{ and } \lambda_0=0,$$

i.e. each terminal tension must be zero.

If the extremity $(x_1\, y_1\, z_1)$ is constrained to lie on a fixed surface, whose equation is $u=0$, we have the displacements of this extremity connected by the equations

$$\left(\frac{dx}{ds}\right)_1\delta x_1+\left(\frac{dy}{ds}\right)_1\delta y_1+\left(\frac{dz}{ds}\right)_1\delta z_1=0,$$

$$\left(\frac{du}{dx}\right)_1\delta x_1+\left(\frac{du}{dy}\right)_1\delta y_1+\left(\frac{du}{dz}\right)_1\delta z_1=0,$$

which give by the method of undetermined multipliers,

$$\frac{\left(\frac{dx}{ds}\right)_1}{\left(\frac{du}{dx}\right)_1}=\frac{\left(\frac{dy}{ds}\right)_1}{\left(\frac{du}{dy}\right)_1}=\frac{\left(\frac{dz}{ds}\right)_1}{\left(\frac{du}{dz}\right)},$$

the geometrical meaning of which is that the direction of the string at this extremity is normal to the surface of constraint.

If the extremity is constrained to a curve whose equations are $u=0$, $v=0$, we find in the same way that at this extremity the direction of the string must be at right angles to the curve.

The method which we have just employed is the second method of Art. 202, and expresses that *the variation of the whole potential work of the external forces is zero, consistently with the geometrical condition that the distance between every two indefinitely close points in the string remains absolutely unchanged in the displaced position.* For if V is the potential, or $V-V_0$ (Art. 197), the potential work of these forces acting on a unit mass at the point x, y, z, the potential for the element dm is Vdm, and the whole potential work is $\int (V-V_0)\,dm$,

whose variation is $\delta\int V dm$, or $\int \delta V . dm$, or

$$\int (X\delta x + Y\delta y + Z\delta z)\,dm.$$

Let us, in the second place, *suppose the string to be extensible.* In this case there are no geometrical conditions to be satisfied in the displacement (or deformation) of the string. Then the equation of equilibrium will simply express the condition that in the position of equilibrium the variation of the whole potential work of applied and internal forces is zero.

Now if we consider any elementary mass, dm, whose length is ds, and whose internal force (the tension) is T, the work done by this force for a variation δds of the elementary length is (see p. 78)

$$-T\delta ds.$$

Adding together the similar terms for all the elementary masses, the variation of the potential work of the applied and internal forces is

$$\int (X\delta x + Y\delta y + Z\delta z)\,dm - \int T\delta ds,$$

which differs from (2) only in having $-T$ instead of λ. Hence the whole discussion is exactly the same as before, and the results are those arrived at in Section II.

223.] **Equipotential Surfaces.** When the applied forces are a conservative system, whose potential at any point in space is denoted by V, we have from equation (4) of Art. 203, or equation (8) of Art. 204,

$$T = K - V, \tag{1}$$

where K is a constant.

Now, since $V \equiv \phi(x, y, z)$, a function of the co-ordinates of a point, the equation

$$V = C, \tag{2}$$

where C is any constant, will denote a surface at every point of which the potential of the forces has a constant value. Moreover (1) shows that at all points on this surface T has the constant value $K-C$. Although it may happen that there is no portion of the string on the surface denoted by (2), still we shall say that the *tension* has a constant value on this surface, since T has an analytical value given by (1); and, in the same sense, we shall speak of the *tension* at any point whatever in space, although no part of the string exists at this point.

By attributing different values to C in (2), we get a series of surfaces called *Equipotential Surfaces*. These surfaces are called by French writers *Surfaces de Niveau*, or *Level Surfaces*, from the part which they play in hydrostatics. Some of the principal properties of these remarkable surfaces will be given in a subsequent Chapter.

224.] **Property of Minimum.** *If a uniform inextensible string, in equilibrium under the action of a given conservative system of forces, joins two fixed points, A and B, the variation of the integral*

$$\int T ds$$

will be zero when we pass from the curve of the string to any indefinitely close curve which passes through A and B.

Let us calculate the variation of this integral,

$$\delta\int T ds = \int \delta T . ds + T\delta\, ds)$$

$$= \int\Big\{\delta T . ds + T\Big(\frac{dx}{ds}d\delta x + \frac{dy}{ds}d\delta y + \frac{dz}{ds}d\delta z\Big)\Big\}.$$

Now, from (1) of last Art.,

$$\delta T = -\delta V = -(X\delta x + Y\delta y + Z\delta z).$$

Hence by integration by parts (as in Art. 222), we have

$$\delta\int T ds = T_1\Big(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\Big)_1 - T_0\Big(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\Big)_0$$

$$-\int\Big\{\Big[X + \frac{d}{ds}\Big(T\frac{dx}{ds}\Big)\Big]\delta x + \Big[Y + \frac{d}{ds}\Big(T\frac{dy}{ds}\Big)\Big]\delta y$$

$$+ \Big[Z + \frac{d}{ds}\Big(T\frac{dz}{ds}\Big)\Big]\delta z\Big\} ds.$$

Now the right-hand side of this equation is zero, since, the extreme points of the curve being fixed, the coefficients of T_0 and T_1 both vanish, and the coefficients of δx, δy, δz under the sign of integration vanish by the general equations of Art. 204, the mass of a unit length of the string being here taken as unity. Hence the proposition.

This theorem leads to a remarkable property of the common catenary. *Of all curves of the same length joining two given points in a vertical plane, the common catenary is that whose centre of gravity is lowest.* For if $\bar{y}$ be the depth of the centre of gravity of this curve, whose length is L, we have

$$= \frac{\int y ds}{L}.$$

But (Art. 206), $T = mgy$; therefore $\bar{y} = \dfrac{\int T ds}{mgL}$; therefore, by the theorem of this Article, we have

$$\delta \bar{y} = 0.$$

That $\bar{y}$ is in this case a minimum in the true sense of the word does not, of course, appear from this; the proof that it is so depends on the criterion for maxima and minima furnished by the Calculus of Variations, for which see Jellett's *Calculus of Variations*, p. 80. It is there proved, that when the variation of any integral of the form $\int_{x_0}^{x_1} U dx$ vanishes (the limits being fixed) the value will be, in general, an algebraic maximum or minimum according as $\dfrac{d^2 U}{d p_n^{\,2}}$ is continually $-$ or continually $+$ between the limits of integration, $\dfrac{d^n y}{dx^n}$ being denoted by p_n, and U being any function of $x, y, p_1, p_2, \dots p_n$. In the present case $U \equiv y\, ds = y\sqrt{1 + p_1^{\,2}}dx$, a change of the independent variable from s to x being necessary since it is the limits of x that are assigned. The application of the criterion is then obvious.

CHAPTER XIV.

SIMPLE MACHINES.

225.] **Functions of a Machine.** A machine may be defined either from a statical or from a kinematical point of view. Regarded statically, it is *any instrument by means of which we may change the direction, magnitude, and point of application of a given force;* and regarded kinematically, it is *any instrument by means of which we may change the direction and velocity of a given motion.*

In Statics it is usual to consider the points or machines to which forces equilibrating each other are applied as absolutely *motionless;* nevertheless, it appears from our definition of force (Art 1), that a system of forces acting at a point will be in equilibrium when the point has a uniform motion in a right line. If a particle describes any curve whatever with uniform velocity, a little reflection will show that at no point of its path can there be any force in the direction of the tangent—or, in other words, the force acting on it must everywhere be normal to the path. It follows (see Art. 195), that there is no work done by this force in the passage of its point of application from any one position to any other. Extending this a little, we shall so far anticipate the results of Kinetics as to assume that *when the parts of any machine are each in a state of uniform motion, the forces applied to the machine are in equilibrium among themselves.*

By the extension of the equilibrium of forces to this case, we comprise both the statical and kinematical definitions of a machine in the following:—a machine is *any assemblage of different pieces whose displacements, resulting from their mode of connection, depend on each other by geometrical laws, and whose object is to transform into mechanical work the result of the action of given applied forces.* (See Resal, *Mécanique Générale*, vol. iii, p. 3.)

It has been already pointed out that in applying the equation of virtual work to a system of connected bodies, advantage is gained by choosing such displacements as do not violate any of the geometrical connections of the system. This principle we shall use largely in the discussion of machines, and the displacements which we shall choose will be those which the different parts of a machine actually undergo when it is employed in doing work. Thus, instead of equations of *virtual* work, we shall have equations of *actual* work; and in future we shall speak of the principle referred to as the *Principle of Work.*

Since in the motion of a machine the work done by a force applied to any part of it depends on the magnitude and direction of the displacement of the point of application of this force, we see at once the importance of the discussion of the motions produced in the several parts of a machine by a definite motion given to some one part. This discussion, which is a problem of pure geometry, constitutes the *Kinematics of Machinery*, for which the student may consult Resal's *Mécanique Générale*, Willis's *Principles of Mechanism*, or the treatise of Reuleaux.

226.] **Moving Forces and Resistances.** Every machine is designed for the purpose of overcoming certain forces which are called *resistances;* and the forces which are applied to the machine to produce this effect are called *moving forces.* The distinction between these forces is easily drawn by the Principle of Work. For, when the machine is in motion, every moving force displaces its point of application in its own direction, while the point of application of a resistance is displaced in a direction *opposite* to that of the resistance. A moving force is, therefore, one whose elementary work is positive, and a resistance one whose elementary work is negative.

A moving force applied to a machine is often (but improperly) called a *power.* The resistances against which a machine works are divided into two classes, viz. *useful resistances* and *wasteful resistances.* The former constitute those which the machine is specially designed to overcome, while the overcoming of the latter is foreign to its purpose. For example, if a pulley is employed for the purpose of lifting a weight by means of a rope, a part of the effort employed is spent in overcoming the friction between the pulley and its spindle, and another part is spent in overcoming the rigidity of the rope.

Friction and rigidity in this case are the wasteful resistances, and the weight of the body lifted is the useful resistance.

The distinction between the resistances overcome gives also the distinction between *useful work* and (so-called) *lost work.*

Useful work is that which is performed in overcoming useful resistance, while lost work is that which is spent in overcoming wasteful resistances.

227.] **Efficiency of a Machine.** The ratio of the *useful* work yielded by a machine to the whole amount of work performed by it is called its *efficiency.*

Let W be the work done by the moving forces, W_u the useful and W_l the lost work, when the machine is moving uniformly. Then
$$W = W_u + W_l;$$
and if η denote the efficiency of the machine,
$$\eta = \frac{W_u}{W}.$$

Since some of the work expended in moving the machine must be expended in overcoming wasteful resistances, the efficiency is always less than unity, and the object of all improvements in the machine is to bring its efficiency as near unity as possible.

The *counter-efficiency* is the reciprocal of the efficiency. If the useful work to be performed is given, the amount of work to be expended on the machine is obtained by multiplying the former by the counter-efficiency.

Let P be the moving force applied at any point of a machine to perform a given amount, W_u, of useful work; let W_l be the work lost, and let s be the space through which P drives its point of application in its own direction. Then we have
$$Ps = W_u + W_l.$$

Let P_0 be the force which would perform the same amount of useful work if the wasteful resistances were removed. Then
$$P_0 s = W_u.$$
But $\eta = \dfrac{W_u}{Ps} = \dfrac{P_0}{P}$; hence the efficiency is the ratio of the force which would drive the machine against a given useful resistance, if the wasteful resistances were removed, to the force which is actually required to do so. In many cases this definition is useful in practice.

As regards the wasteful resistances in machines, the most noticeable are friction, the rigidity (or rather imperfect flexibility) of ropes, and the vibrations which are produced in the various pieces. Of these the first is that with which alone we shall be concerned. The student who desires information on the experimental laws of the rigidity of ropes may consult Coxe's translation of Weisbach's *Mechanics of Engineering and of the Construction of Machines*, vol. i, p. 363 (New York, 1872).

228.] **Simple Machines.** By simple machines are meant the Lever, the Inclined Plane, the Pulley, the Wheel and Axle, the Screw, and the Wedge. Of these, the Lever, the Inclined Plane, and the Pulley may be considered as distinct in principle, while the others are only combinations of pairs of these three.

229.] **The Lever.** A lever is a solid bar, straight or curved, which is constrained to turn round a fixed axis. This fixed axis is called the *fulcrum* of the lever. It is usual to define three kinds of levers. If the fulcrum is between the moving force and the resistance the lever is said to be of the *first kind;* if the resistance acts between the moving force and the fulcrum (as in a wheelbarrow, an oar, or a pair of nutcrackers), the lever is of the *second kind;* and if the moving force acts between the fulcrum and the resistance (as in the construction of the limbs of animals), the lever is of the *third kind.* In the last kind the moving force is always greater than the resistance to be overcome, and levers of the third kind are therefore seldom employed.

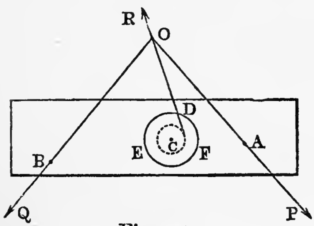

Fig. 227.

To find the efficiency of a lever, the wasteful resistance being friction—

Let the moving force, P, be applied at the point A (fig. 227) in the direction OA perpendicular to the axis, and the useful resistance at B in the direction OB, also perpendicular to the axis; let EDF be a section of the axis on which the lever turns, made by the plane of P and Q, the contact between the beam and its axis, although it may be very close, being still such that they can be considered as touching along a single line when the machine works. In this case (see Art. 114) the reaction of the axis consists of a single force touching the circle of radius $r \sin \lambda$ concentric with EDF, λ being the angle of friction for the lever and its axis; and since this reaction must also pass through O, its direction is obtained by drawing from this point a tangent to the circle.

Let p and q be the perpendiculars from C, the centre of the axis on OA and OB respectively, and let $\omega = \angle AOB$.

Then by moments about C, we have

$$Pp = Qq + Rr \sin\lambda\,;$$

also
$$R = \sqrt{P^2 + 2PQ\cos\omega + Q^2}\,;$$

$$\therefore\quad Pp = Qq + r\sin\lambda\sqrt{P^2 + 2PQ\cos\omega + Q^2}. \qquad (1)$$

If P_0 is the value of P when friction is removed,

$$P_0 p = Qq, \qquad \therefore\quad \eta = \frac{P_0}{P} = \frac{Qq}{Pp}.$$

Substituting $\frac{p}{q}\eta$ for $\frac{Q}{P}$ in (1), we have

$$pq(1-\eta) = r\sin\lambda\sqrt{p^2\eta^2 + 2pq\cos\omega\,.\,\eta + q^2},$$

which gives for the efficiency

$$\eta = \frac{q}{p}\cdot\frac{pq + r^2\cos\omega\sin^2\lambda - r\sin\lambda\sqrt{p^2 + 2pq\cos\omega + q^2 - r^2\sin^2\omega\sin^2\lambda}}{q^2 - r^2\sin^2\lambda}.$$

If the coefficient of friction is small, we shall have, approximately,

$$\eta = 1 - \frac{\mu r}{pq}\sqrt{p^2 + 2pq\cos\omega + q^2}.$$

If P and Q are parallel, $\omega = 0$, and $\eta = 1 - \mu r\left(\frac{1}{q} + \frac{1}{p}\right)$.

If the lever is of the second kind, and P and Q parallel, $\omega = \pi$, and $\eta = 1 - \mu r\left(\frac{1}{q} - \frac{1}{p}\right)$; and for a lever of the third kind, we find easily in the same circumstances

$$\eta = 1 - \mu r\left(\frac{1}{p} - \frac{1}{q}\right).$$

230.] **The Inclined Plane.** Let a moving force, P, whose direction makes an angle θ with a rough inclined plane, be employed to drag a weight Q up the plane. Then if λ is the angle of friction and i the inclination of the plane,

$$P = Q\frac{\sin(i+\lambda)}{\cos(\theta-\lambda)},$$

$$P_0 = Q\frac{\sin i}{\cos\theta};$$

$$\therefore\quad \eta = \frac{1 + \mu\tan\theta}{1 + \mu\cot i}.$$

231.] **Fixed and Moveable Pulley.** Let a flexible string pass over a smooth *fixed* pulley (that is, a pulley whose axis is fixed in space), and let a weight W be suspended from one extremity of the string, while a vertical downward force P is applied at the other extremity. Then to raise W we must have $P = W$, and in the uniform working of the machine W is raised exactly as much as the point of application of P is lowered.

Suppose, on the contrary, that one extremity of the string is fixed, that the string passes under a *moveable* pulley from which W is suspended, and that P acts vertically upward at the other extremity of the string. Then evidently $P = \frac{1}{2}W$; hence in the moveable pulley there is a gain in power. But in this case W is raised only *half* as much as the point of application of P ascends. There is, therefore, a loss in the expedition with which the work of raising the weight is performed.

232.] **Systems of Smooth Pulleys.** We shall consider three different arrangements of pulleys, as exemplifying the Principle of Virtual Work.

I. In the first system there are two blocks, A and B (fig. 228), the upper of which is fixed and the lower moveable.

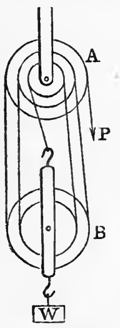

Fig. 228.

Each block contains a number of separate pulleys, of the same diameter usually, each pulley being moveable round the axis of the block in which it is. (The figure represents a section of the blocks made by a plane perpendicular to their axes, and the circumferences of the pulleys are projected on this plane.) A single rope (whose weight is neglected) is attached to the lower block and passes alternately round the pulleys in the upper and under blocks. The portion of rope proceeding from one pulley to the next is called a *ply*. In this arrangement the tension of the rope is throughout constant and equal to P, the force applied at the free extremity. The portion of the rope at which the moving force, P, is applied, is called the *tackle-fall*.

Let W be the weight to be lifted, and assume all the plies to be parallel.

Then if n is the number of plies at the lower block, we shall obviously have, neglecting the weight of the block,

$$nP = W.$$

This result follows also by the principle of work. For if p denote the length of the tackle-fall, and x the common length of the plies, we have

$$p + nx = \text{constant},$$

$$\therefore \quad dp + ndx = 0.$$

But
$$Pdp + Wdx = 0,$$
$$\therefore \quad P = \frac{1}{n} W.$$

II. Suppose each pulley to hang from a fixed block by a separate rope.

Let A (fig. 229) be the fixed pulley, n the number of moveable pulleys, and $x_1, x_2, \ldots x_n$ the distances of the centres of these latter from a horizontal plane through the centre of A.

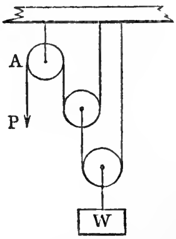

Fig. 229.

Then, p being the length (AP) of the tackle-fall,
$$2x_1 + p = \text{const.}, \qquad 2x_2 - x_1 = \text{const.}$$
$$2x_3 - x_2 = \text{const.} \ldots 2x_n - x_{n-1} = \text{const.}$$
Hence $2^n x_n + p = \text{const.}$, therefore
$$2^n dx_n + dp = 0,$$
and
$$Wdx_n + Pdp = 0,$$
$$\therefore \quad P = \frac{W}{2^n}.$$

III. Let a separate rope pass over each pulley, and let all the ropes be attached to the weight.

Neglecting the weights of the pulleys and ropes, we shall have, by resolving vertically for the equilibrium of W,
$$W = P(1 + 2 + 2^2 + \ldots + 2^{n-1}),$$
the whole number of pulleys being n; or
$$P = \frac{W}{2^n - 1}.$$

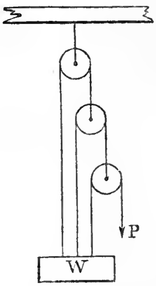

Fig. 230.

The same result follows by the principle of work. For if the distance of W from a horizontal plane through the centre of the fixed pulley is denoted by y, and if the distances of the centres of the pulleys, counting from the fixed one, are $x_1, x_2, \ldots, x_{n-1}$, we have evidently
$$y + x_1 = \text{const.}, \quad y + x_2 - 2x_1 = \text{const.} \ldots, \quad y + x_{n-1} - 2x_{n-2} = \text{const.},$$
$$y + p - x_{n-1} = \text{const.}$$

Hence, multiplying the second equation by $\frac{1}{2}$, the third by $\frac{1}{2^2}$, &c., and adding, we have $2^{n-1}y + p = \text{constant}$. Now the equation of work is
$$Wdy + Pd(p + x_{n-1}) = 0,$$
or
$$(W + P)\,dy + 2Pdp = 0;$$
and
$$2^{n-1}dy + dp = 0,$$
$$\therefore \quad P = \frac{W}{2^n - 1}.$$

233.] **The Wheel and Axle.** This consists of a horizontal cylinder, b, (fig. 231) moveable round two journals (or small cylinders projecting from the centres of its faces), one of which is represented in section at c; a wheel, a, is rigidly connected with the cylinder, and the journals rotate in fixed bearings. The machine is, in reality, a rigid combination of two pulleys, a and b, moveable about a common axis, c; and its theory is precisely the same as that of the lever. The moving force, P, is applied at the circumference of the wheel, and the useful resistance, Q, at the free extremity of a rope coiled round the axle.

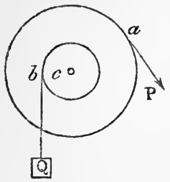

Fig. 231.

All wasteful resistances being neglected, the relation between P and Q is

$$Pa = Qb,$$

where a = radius of wheel, and b = radius of axle.

The friction of the journal (whose radius is c) against its bearing being taken into account, the relation between P and Q is

$$Pp = Qq + c\sin\lambda\sqrt{P^2 + 2PQ\cos\omega + Q^2},$$

ω being the angle between the directions of P and Q, exactly as in Art. 229; and the efficiency is the same as that investigated in the Article on the lever.

Economy of power is attained in the wheel and axle by diminishing b, the radius of the axle; but in this way the strength of the machine is diminished. To avoid this disadvantage a *Differential Wheel and Axle* is sometimes employed. In this instrument the axle consists of two cylinders of radii b and b' (fig. 232), and the rope, wound round the former in a sense opposite to that of watch-hand rotation (suppose), leaves it (at the point b in fig. 231), and, after passing under a moveable pulley to which the weight to be raised is attached, is wound in the opposite sense round the remaining portion (that of radius b') of the axle. The power P is applied, as before, tangentically to the wheel. For the equilibrium (or uniform motion) of the machine, the tensions of the rope in bm and $b'n$ are each equal to $\frac{1}{2}Q$; and taking moments round the centre of

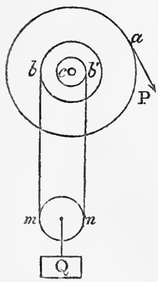

Fig. 232.

the journal, c, for the equilibrium (or uniform motion) of the rigid system consisting of the wheel and axle alone, we have

$$Pa = \tfrac{1}{2} Q (b - b').$$

Thus, by making the difference $b-b'$ small, the requisite moving force can be made as small as we please; but since the amount of *work* to be done is constant, this economy of power is accompanied by a loss in the time of performing the work. For it is easily seen that if the wheel turns through an angle $\delta\theta$, the point of application of P will describe a space $a\delta\theta$, and the weight will be raised through a space $\frac{1}{2}(b-b')\delta\theta$, which latter will be very small if $b-b'$ is very small.

234.] **The Screw.** The screw consists of a right circular cylinder on the convex circumference of which there is a uniform projecting thread, GH (fig. 234), of a *helical* form.

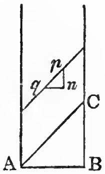

Fig. 233.

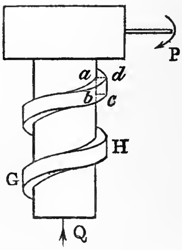

Fig. 234.

The helix is a curve traced on the circumference of a cylinder in the following manner. Take a sheet of paper on which are drawn two indefinite right lines, AB and AC, and let the paper be wound round the cylinder in such a way that the line AB coincides with the circumference of the base; then the other line, AC, will appear on the cylinder in the shape of a spiral curve which is called the *helix*. (Fig. 233 represents a projection of the helix on a plane through the axis of the cylinder.)

A screw with a *rectangular* thread (which is that represented in fig. 234) is obtained by making a small rectangular area, $abcd$, move so that one side, $a\,b$, always coincides with a generating line of the cylinder, the middle point of ab describing the helix, and the plane of the rectangle always passing through the axis of the cylinder.

If a small *triangle* is used instead of the rectangle, we should have a screw with a *triangular* thread.

Let p and q be two points on the indefinite line AC, and draw pn perpendicular to AB and qn parallel to it. Then pq becomes a portion of the arc of the helix, and qn a portion of a section of the cylinder perpendicular to its axis, pn remaining a straight line coinciding with a generator of the cylinder.

Hence the relation holding between the sides of the triangle *pqn* before the paper was wound round the cylinder will hold also after the winding. But if the angle between *AB* and *AC* is *i*, we have evidently

$$pn = qn \,.\tan i,$$

$$pq = qn \,.\sec i.$$

The thread *GH* works in a block on the inner surface of which is cut a groove which is the exact counterpart of the thread. The block in which the groove is cut is often called the *nut.* It is clear, then, that if the screw moves in the nut until the point *p* of the thread occupies the position *q*, the axis must move in its own direction through a space *pn*, and the angular rotation of the screw about its axis is $\frac{qn}{r}$, *r* being the radius of the cylinder.

Hence, if the angle $\frac{qn}{r}$ through which the screw turns is denoted by ω, we have

$$pn = \omega r \tan i, \quad pq = \omega r \sec i.$$

If $\omega = 2\pi$, or if the screw make a complete revolution, any point on the surface of the screw describes a space $2\pi r \tan i$ parallel to the axis. This is obviously the distance between two portions of the thread measured on a generator, and is called the *pitch* of the screw.

We shall consider the screw as driving a resistance *Q* applied in the direction of the axis, and the moving force, *P*, as applied in a plane perpendicular to the axis, at the extremity of an arm whose length measured from the centre of the axis is *a*.

Suppose that the screw rotates through an angle ω. Then the work done by *P* is $Pa\omega$, and the work done against *Q* is $Qr\omega \tan i$.

If no work is lost against wasteful resistance, we must have

$$Pa = Qr \tan i.$$

If there is friction between the thread and the groove, let *R* be the normal pressure at any point *p* of the thread (acting towards the under side of *pq* in the figure), and μR the friction at this point. Then, in a small angular motion, $\delta\omega$, of the screw the work done against the friction is $\mu R.pq$ (taking *pq* as an elementary portion of the thread), or $\mu R r \delta\omega \sec i$. Hence

$$Pa\delta\omega = Qr\delta\omega \tan i + \mu r \delta\omega \sec i \,\Sigma R,$$

ΣR denoting the sum of the normal reactions at all points of the thread.

But, for the equilibrium of the cylinder, resolving along its axis, we have

$$Q = \Sigma\,(R\cos i - \mu R\sin i),$$

or

$$Q = (\cos i - \mu\sin i)\,\Sigma R. \qquad (a)$$

Hence, substituting this value of ΣR in the previous equation,

$$Pa = Qr\tan(i+\lambda),$$

λ being the angle of friction.

This result could have been obtained without the principle of work by combining by (a) the equation of moments round the axis of the screw. By taking moments round the axis, we have

$$Pa = \Sigma\,(R\sin i + \mu R\cos i)\,r,$$

or

$$Pa = r\,(\sin i + \mu\cos i)\,\Sigma R. \qquad (\beta)$$

Dividing (β) by (a) we obtain the relation between P and Q.

The efficiency of the screw is evidently

$$\frac{\tan i}{\tan(i+\lambda)},$$

which will be a maximum when $i = \dfrac{\pi}{4} - \dfrac{\lambda}{2}$.

235.] **Prony's Differential Screw.** If h denote the pitch of a screw, the relation between P and Q when friction is neglected is

$$2P\pi a = Qh;$$

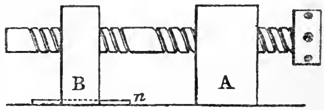

Fig. 235.

therefore economy of force in overcoming a given resistance is gained by making h very small. But it is impossible to do this in practice, and to attain the result desired a differential method is resorted to. Let the screw work in two blocks, A and B (fig. 235), the first of which is fixed and the second moveable along a fixed groove, n. Let h be the pitch of the thread which works in the block A, and h' the pitch of that which works in the block B. Then one complete revolution of the screw impresses two opposite motions on the block B—one equal to h in the direction in which the screw advances, and the other equal to h' in the opposite direction. If, then, the resistance, Q, is driven by this block, we have by the principle of work

$$2P\pi a = Q(h-h'),$$

and the requisite moving force will be diminished by diminishing $h-h'$.

236.] **The Wedge.** The wedge is a triangular prism, usually isosceles, which is used (as represented in the figure) for the purpose of separating two bodies, A and B, or parts of the same body which are kept together by some considerable force, molecular or other.

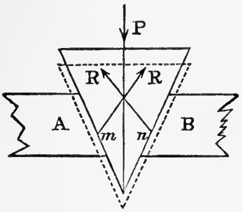

Fig. 236.

The figure represents a section of the wedge made through the line of action of the moving force, P, perpendicular to the axis of the wedge. Suppose that the line of action of P passes through the vertex of the wedge, and that slipping is about to take place; then the total resistances of the surfaces A and B against the wedge will make the angle, λ, of friction with the normals at the points, m and n, where they act; but these points are indeterminate themselves.

To find the efficiency of the wedge. Let the wedge be driven through a vertical space equal to dp, and let $2a$ be its vertical angle. Then the useful work performed is the separation of A and B in directions normal to the faces of the wedge in contact with them; in other words, the useful work is that done by the normal components of the total resistances, R. Now the point m moves vertically down through a space dp, and the projection of this displacement along the normal at m is evidently

$$\sin a . dp.$$

Hence the work done by the normal components is

$$2R\cos\lambda\sin a\, dp,$$

and the whole work expended is Pdp. Hence

$$\eta = \frac{2R\cos\lambda\sin a}{P}.$$

But by resolving vertically for the equilibrium of the wedge, we have

$$P = 2R\sin(a+\lambda);$$

$$\therefore\quad \eta = \frac{\sin a\cos\lambda}{\sin(a+\lambda)} = \frac{\tan a}{\mu+\tan a}.$$

Having given the theory of the simplest machines, we proceed to discuss a few of their most useful forms.

237.] **The Balance.** The common balance is a lever of the first kind with two equal arms, from the extremity of each of which is suspended a scale pan, the fulcrum being vertically above the centre of gravity of the beam when the latter is horizontal. Let O (fig. 237) be the fulcrum, AB the line joining the points of attachment of the scale pans to the beam, G the centre of gravity of the beam, and let AB be at right angles to OC, the line joining the fulcrum to the centre of gravity of the beam. Then, if $AC = CB = a$, $OC = h$, $OG = k$, $W =$ weight of the beam, and $\theta =$ the inclination of AB to the horizon when two weights, P and Q, are placed in the pans, we have for the position of equilibrium (by moments about O),

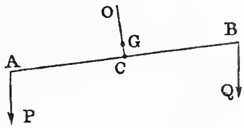

Fig. 237.

$$\tan\theta = \frac{(P-Q)a}{(P+Q)h + Wk}.$$

Now, the most important requisites for a good balance are *Sensibility* and *Stability*. The first requires that the beam should be sensibly deflected from the horizontal position by the smallest difference between the weights P and Q; hence the sensibility may be measured by the angle of deflection from the horizontal position caused by a given difference, $P-Q$. The stability of the balance is measured by the rapidity of the oscillation of the beam when it is slightly disturbed, and will be greater the smaller the time of oscillation. Hence the investigation of the stability of the balance is a kinetical problem.

For sensibility, $\tan\theta$ must be as great as possible for a given value of $P-Q$. Hence (1) a must be large, (2) h must be small, (3) W must be small, and (4) k must be small, i.e. the distance of the fulcrum from the centre of gravity of the beam must be small. The last condition is obtained in balances in which great sensibility is desired by making OC an axis along which a heavy nut moves with a screw motion; by moving the nut towards O, the centre of gravity of the machine can be made to approach the fulcrum.

The time of a small oscillation can be shown (see Thomson and Tait, p. 423) to be proportional to the square root of

$$\frac{WK^2+2Pa^2}{2Ph+Wk},$$

where K is the radius of gyration of the beam about O. For stability this must be small; it is evident that, with the exception of the third condition above, the conditions for stability are the very reverse of those for sensibility.

238.] **Roberval's Balance.** Roberval's Balance is an excellent illustration of the principle of work.

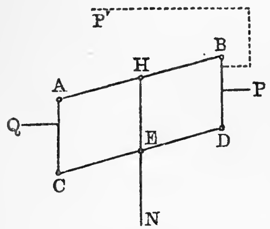

Fig. 238.

Two equal bars, AB and CD, (fig. 238) revolve round axes through their middle points, H and E, which are fixed in a vertical support, HN; these bars are connected by smooth joints to two equal bars, AC and BD, and to these latter bars are rigidly attached two plates or scale pans, P and Q, the points of attachment being any whatever, and one or both of the plates may lie *towards* the vertical support, or away from it (as in fig. 238).

Suppose P and Q to be the magnitudes of two weights placed in the pans P and Q, respectively. Then if for any displacement of the bars round the points H and E, the pans describe vertical spaces p and q, respectively, we shall have for equilibrium

$$Pp-Qq = 0.$$

Now, the bars AC and BD, being always parallel to the fixed line HE, will be always vertical, and the vertical space through which one moves up is obviously equal to that through which the other moves down. Hence $p = q$, and we have for equilibrium

$$P = Q,$$

whatever be the lengths of the pans (*provided their weights are neglected*), *whatever be their points of attachment to BD and AC, and whatever the points in the pans at which P and Q are placed.*

If the weights of the pans are taken into account, the same results follow if they are of equal weight.

If the pan P were replaced by the pan P', and the weight P placed at P', the other pan, Q, remaining unchanged, and the weights of the pans being either equal or neglected, equilibrium would still subsist—a result which seems at first sight very strange.

If the lengths AH and HB, CE and ED are not equal, it is easy to prove that $\frac{p}{q} = \frac{HB}{HA}$, and the condition of equilibrium is

$$P \,.\, HB = Q \,.\, HA.$$

239.] **Balance of Quintenz.** This is a compound balance formed of a combination of several levers, and is used for weighing very heavy loads. This machine also furnishes an admirable example of the principle of work.

AB (fig. 239) is a lever moveable about its fixed extremity, A; MN is another lever moveable about a fulcrum, F, fixed at its middle point; CD is a moveable platform, which receives the load Q, whose weight is to be found; this platform is connected with the lever MN by a rigid vertical bar, DI, articulated at D and I; and the platform further rests against the lever, AB, by an edge of contact at a fixed point, H, on the latter; finally, the two levers are connected by a rigid vertical bar, BM, articulated to both.

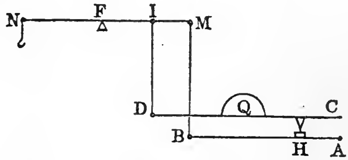

Fig. 239.

The weight, P, employed to measure Q is attached to the upper lever at N. Let the system receive any slight displacement, then the lever, AB, will turn round A through an angle $\delta\theta$, suppose, and the lever MN will turn round F through an angle $\delta\phi$.

We shall arrange the dimensions of the machine in such a manner that the platform, CD, may remain horizontal in the displacement. The vertical descent of the point H is evidently $AH \,.\, \delta\theta$, and this is also the vertical descent of the point in the platform above H.

The vertical descent of the point D is the same as that of I, and this latter is obviously $FI \,.\, \delta\phi$; hence if the platform remains horizontal,

$$FI \,.\, \delta\phi = AH \,.\, \delta\theta.$$

Again, the vertical descent of M is the same as that of B; or

$$FM \,.\, \delta\phi = AB \,.\, \delta\theta.$$

Hence from these equations we have

$$\frac{MF}{FI} = \frac{BA}{AH},$$

which is the condition for the horizontality of the platform. Denote $\frac{BA}{AH}$ by n. The equation of work is obviously

$$P \times \text{descent of } N = Q \times \text{descent of } D,$$

$$\therefore \quad P \,.\, NF\delta\phi = Q \,.\, FI\delta\phi,$$

$$\therefore \quad P = \frac{1}{n} Q,$$

or the result is the same as if Q were suspended from the point I of the upper lever.

Loads placed on the platform may all be weighed by means of a constant weight, P, by merely moving the point of suspension of this latter along the arm NF; thus, if P is suspended from the point K between N and F, we shall have

$$\frac{P}{Q} = \frac{FI}{FK}.$$

240.] **Toothed Wheels.** Motion may be transferred from one point to another and work done by means of a combination of toothed wheels, each one of which drives the next one in the series. The discussion of this kind of machinery possesses great geometrical elegance; but the space at our disposal renders it impossible to do more than give a slight sketch of the simplest case—that in which the axes of the wheels are all parallel.

For the investigation of the proper forms of teeth, the student is referred to Willis's *Principles of Mechanism*, Collignon's *Statique*, and Resal's *Mécanique Générale*.

Fig. 240 represents a toothed wheel, A_1, moveable round a horizontal axis, ab; the moving force, P, is applied by means of a handle, cd, which, when turned, causes the axis ab to rotate in its bearings at a and b and to turn the wheel A_1; this wheel causes another, B_1, in contact with it, to rotate round a horizontal axis which also moves in fixed bearings at its extremities; on this latter axis is fixed another wheel A_2, whose rotation in like manner turns B_2 on its axis, which in the figure is the axis of a cylinder to which the resistance, Q, is attached.

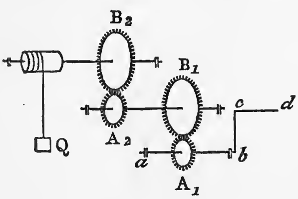

Fig. 240.

Suppose that there are n wheels, $A_1, A_2, \dots A_n$, whose radii are $a_1, a_2, \dots a_n$, and n wheels, $B_1, B_2, \dots B_n$ whose radii are $b_1, b_2, \dots b_n$; and let $bc = p$, and the radius of the cylinder (or wheel) to which Q is attached $= q$. Then if ω_1 is the angle through which the radius bc revolves, the moving force being always applied tangentially to the circle described by its point of application, the work expended is

$$Pp\omega_1;$$

and if ω_n is the angle through which, in the same time, the cylinder rotates, the weight Q will be raised through a space $q\omega_n$, and the work done against the resistance is

$$Qq\omega_n.$$

Supposing then that no work is lost either by the friction of the axes in their bearings or by the friction of the teeth against each other, we must have

$$Pp\omega_1 = Qq\omega_n, \tag{1}$$

when the machine is moving uniformly.

To determine the kinematical relation between ω_1 and ω_n, let the angle through which B_1 turns be ω_2. Then since the spaces described by the points of A_1 and B_1 which are in contact are the same, $a_1\omega_1 = b_1\omega_2$. Also if ω_3 is the angle through which B_2 turns, we have $a_2\omega_2 = b_2\omega_3$. Proceeding in this way, we have by multiplying the corresponding sides of these equations together

$$a_1 a_2 \dots a_n \,.\, \omega_1 = b_1 b_2 \dots b_n \,.\, \omega_n. \tag{2}$$

Hence from (1) and (2),

$$\frac{Q}{P} = \frac{p b_1 b_2 \dots b_n}{q a_1 a_2 \dots a_n}.$$

For the calculation of the work lost by the friction of the teeth among themselves see Collignon's *Statique*, p. 468.

CHAPTER XV.

ATTRACTIONS. THEORY OF THE POTENTIAL.

SECTION I.

Solid Distributions of Matter in General.

241.] **Universal Law of Attraction.** *Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two particles, and whose magnitude is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

This law of universal attraction is a generalization from experience, verified in its consequences as to the motions of all bodies in the universe which come within the reach of our observation.

That two particles of matter universally exercise upon each other an attractive action defined as above we observe by experiment; and this action is called *Gravitation.* Over and above this particular force, they may exert other forces, attractive or repulsive, upon each other, depending on particular states, transitory or permanent, in which they may exist in presence of each other. Among forces of the latter class are magnetic and electric attractions and repulsions, and the molecular forces of natural solids.

At present we are concerned with the bare fact that such an action as that of gravitation *is* exercised between two particles, without attempting to account either for its cause or for its precise mode of operation—that is, without any speculation as to whether it is really an *action at a distance*, or an affection of some medium intervening between them.

We shall, it is true, investigate in certain cases the consequences which would result if the particles of matter exercised on each other a force whose magnitude did not follow the natural law of the inverse square of distance; but these cases must be regarded as mere examples of an analytical method, and not as the expressions of any observed natural phenomena.

242.] **Action between Two Particles.** Let there be two particles whose masses are dm and dm', and let r be the distance between them. Then the attraction of gravitation between them is

$$\mu \frac{dm \,.\, dm'}{r^2},$$

where μ is a constant quantity depending on the unit of force adopted. Suppose that we take as a unit force that exerted by two elementary units of mass placed at a unit distance apart; then the above expression must be unity when dm, dm', and r are units. Denote the unit of mass by $[m]$, the unit of distance by $[d]$, and the unit of force by $[f]$; then the force, f, between the particles at the distance r is given by the equation

$$f = [f] \frac{dm \,.\, dm'}{[m]^2} \frac{[d]^2}{r^2}.$$

It would be tedious to introduce the unit factor $\frac{[f][d]^2}{[m]^2}$ into our equations, and we shall for the future omit it, remembering, at the same time, that it is implied in our results, and that the value of every force subsequently given must be multiplied by this unit factor.

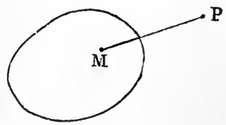

Fig. 241.

243.] **Potential due to an Attracting Solid.** Let P be any point at which a unit mass is placed; M any point in the solid at which the element of mass is dm; and r the distance PM. Then the force between the particles at P and M is $\frac{dm}{r^2}$, and the virtual work of this force is $-\frac{dm}{r^2}dr$, since the force tends to diminish r, and since dr signifies an *increment* of r.

Hence (Art. 188), if V is the potential at P,

$$V = -\Sigma dm \int \frac{dr}{r^2},$$

the sign Σ denoting a summation of the integral for all elements of mass of the solid. This evidently gives

$$V = \Sigma \frac{dm}{r};$$

but as the solid consists of an infinite number of elements, the summation here is an integration, and we have finally

$$V = \int \frac{dm}{r}.$$

If in Art. 195 V_0 is the value at infinity of the gravitation potential of a given mass, V_0 is of course zero, and from that article we have

$$W = V.$$

Hence we may define the gravitation potential of a given mass at any point to be *the quantity of work required to move a unit mass of matter from that point to an infinite distance.*

If the law of attraction is other than that of nature, let it be a function of the distance denoted by $\phi'(r)$. Then the force between P and M is $\phi'(r) \, . \, dm$, and the virtual work of this force being $-\phi'(r) \, . \, dm \, . \, dr$ (supposing the force *attractive*),

$$V = -\Sigma \, dm \int \phi'(r) \, dr = -\Sigma \, \phi(r) \, . \, dm = -\int \phi(r) \, dm,$$

$\phi(r)$ being the integral of $\phi'(r) \, dr$. For example, if the attraction is proportional to the n^{th} power of the distance,

$$V = -\frac{1}{n+1} \int r^{n+1} \, dm.$$

For a *repulsive* force given by the law $\phi'(r)$ the virtual work is $\phi'(r) \, . \, dm \, . \, dr$, and the sign of V is simply changed.

In general, then, to get the potential of any system of forces, write down the expression for their elementary virtual work, and integrate it (see *Mécanique Celeste*).

244.] **Calculation of the Potential in Special Cases.** The law of attraction considered is that of nature.

(1) Let the attracting solid consist of two particles of masses m_1 and m_2 placed at two points, N and S (fig. 35, p. 39). Then if the distances NP and SP are donoted by r_1 and r_2,

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2}, \tag{1}$$

and if the action of m_1 is repulsive,

$$V = -\frac{m_1}{r_1} + \frac{m_2}{r_2}. \tag{2}$$

(2) Let the attracting solid be a bar of uniform density and small uniform section. From P (fig. 242) let fall PO perpendicular to the bar, AB; let s denote the distance of any point M of the bar from O; let θ be the angle PMO, and let ρ and k be density and section of the bar. Then the element of mass at M is $k\rho ds$, and

$$V = k\rho \int \frac{ds}{PM}.$$

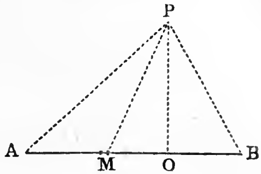

Fig. 242.

But

$$s = PO \cot\theta, \quad \therefore \quad ds = -\, PO \operatorname{cosec}^2\theta d\theta, \quad \text{and} \quad PM = PO \operatorname{cosec}\theta;$$

$$\therefore \quad V = -k\rho \int_{\pi-B}^{A} \frac{d\theta}{\sin\theta},$$

the angles PAB and PBA being denoted by A and B. Hence

$$V = k\rho \log \cot\frac{A}{2} \cot\frac{B}{2}. \tag{3}$$

This may be put into another form. If $PA = r$, $PB = r'$, and $AB = 2c$, we have from trigonometry,

$$V = k\rho \log \frac{r+r'+2c}{r+r'-2c},$$

or

$$V = k\rho \log \frac{\mu+c}{\mu-c}, \tag{4}$$

μ being the semi-axis major of the ellipse described through P, with A and B for foci.

(3) Let the attracting solid be a spherical shell of uniform density and small uniform thickness.

First suppose the point P, at which the value of the potential is required, to be outside the shell.

Let τ and ρ be the thickness and density of the shell, O its centre, and M any point on it. Then if $\angle MOP = \theta$, $OM = a$, and ϕ is the angle made by the plane MOP, with a fixed plane through OP, the element of mass at M is $\rho\tau a^2 \sin\theta d\theta d\phi$; and if $PM = r$,

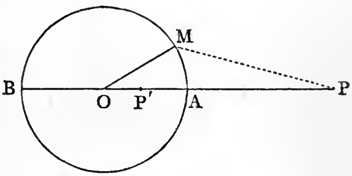

Fig. 243.

$$V = \rho\tau a^2 \iint \frac{\sin\theta d\theta d\phi}{r}$$

$$= 2\pi\rho\tau a^2 \int \frac{\sin\theta d\theta}{r},$$

by performing the integration in ϕ at once.

Now, if $OP = c$, we have

$$r^2 = a^2 - 2ac\cos\theta + c^2,$$

$$\therefore \quad r\,dr = ac\sin\theta\,d\theta,$$

and
$$V = \frac{2\pi\rho\tau a}{c}\int_{PA}^{PB} dr = \frac{4\pi\rho\tau a^2}{c} = \frac{\text{mass of shell}}{c}. \tag{5}$$

Secondly, let P be inside the sphere at P'. Then we have exactly as before

$$V = \frac{2\pi\rho\tau a}{c}\int_{P'A}^{P'B} dr = 4\pi\rho\tau a. \tag{6}$$

Since
$$P'B - P'A = a + c - (a - c) = 2c.$$

It is to be carefully noted that in this case V has the same value at all points inside the shell.

(4) Let the attracting solid be a sphere of uniform density.

First, suppose P to be outside the sphere. Let the sphere be broken up into an indefinitely great number of spherical shells, and since the potential due to each of these is given by (5), we have for the sphere (whose radius is a),

$$V = \frac{4\pi\rho a^3}{3c} = \frac{\text{mass of the sphere}}{c}. \tag{7}$$

Secondly, let P be inside the sphere. Then the potential of the sphere concentric with the given one, and passing through P, is $\dfrac{4\pi\rho c^3}{3c}$, or $\dfrac{4\pi\rho c^2}{3}$; and the potential of the portion included between these spheres must be found by dividing it into shells. Let r and dr be the radius and thickness of one of these shells; then the potential due to it at P is

$$4\pi\rho r\,dr,$$

by (6), and the integral of this from $r = c$ to $r = a$ is $2\pi\rho(a^2 - c^2)$. Adding this to the first portion of V, we have

$$V = 2\pi\rho a^2 - \frac{2}{3}\pi\rho c^2. \tag{8}$$

(5) Let the attracting solid be that inclosed between two concentric spherical surfaces of given radii, a and a', the density being uniform.

First, let P be completely outside the mass, and suppose $a > a'$. Then the potential is obviously the given mass divided by c; or

$$V = \frac{4\pi\rho(a^3 - a'^3)}{3c}. \tag{9}$$

Secondly, let P be inside the space between the bounding surfaces, i.e. inside the mass. Then evidently

$$\begin{aligned} V &= \frac{4\pi\rho(c^3 - a'^3)}{3c} + 2\pi\rho(a^2 - c^2) \\ &= 2\pi\rho(a^2 - \tfrac{1}{3}c^2) - \frac{4\pi\rho a'^3}{3c}. \end{aligned}$$

Thirdly, let P be inside the surface of radius a'. Then

$$V = 2\pi\rho\,(a^2 - a'^2).$$

245.] **Continuity of the Potential.** The gravitation potential of any attracting solid mass varies in a continuous manner from point to point in space, whether the points chosen be inside any portion of the mass or outside it.

For if r be the distance of any element of mass, dm, of the attracting body from P, the point at which the potential is required, $V = \int \frac{dm}{r}$. Let P be taken as origin, and let the position of the element dm be defined by the radius vector, r, and two angles, θ and ϕ, as in p. 280, and let ρ be the density of the element. Then $dm = \rho r^2 \sin\theta\, dr\, d\theta\, d\phi$, and

$$V = \iiint \rho r \sin\theta\, dr\, d\theta\, d\phi.$$

This form of V shows that even if r is zero, i. e. if P is inside the mass, the value of the potential is finite, no infinite term being introduced by the indefinitely close proximity of P to some of the elements of mass.

Hence the potential varies continuously throughout space, and diminishes from the vicinity of the attracting mass towards the space very remote from it in all directions.

246.] **Continuity of the First Differential Coefficients of the Potential.** At each point in space the potential of a given mass has a definite value. Let the co-ordinates of P, a particular point considered, be x, y, z. Then if x', y', z' be the co-ordinates of the attracting element dm, we have

$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2. \qquad (1)$$

And since

$$V = \int \frac{dm}{r}, \text{ we have } \frac{dV}{dx} = \int dm \frac{d\left(\frac{1}{r}\right)}{dx} = -\int \frac{x-x'}{r^3}\, dm.$$

Hence

$$\frac{dV}{dx} = -\int \frac{x-x'}{r^3}\, dm, \quad \frac{dV}{dy} = -\int \frac{y-y'}{r^3}\, dm, \quad \frac{dV}{dz} = -\int \frac{z-z'}{r^3}\, dm. \quad (2)$$

The continuity of these expressions can be shown by putting $x-x' = r\sin\theta\cos\phi$, $y-y' = r\sin\theta\sin\phi$, $z-z' = r\cos\theta$, $dm = \rho r^2 \sin\theta\, dr\, d\theta\, d\phi$, where θ and ϕ are the same as in last Article.

Then $\qquad dV = -\int \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi;$

and thus, even when the point P is inside the mass no infinite term is introduced into any of the differential co-efficients of V. Each of these differential coefficients varies, therefore, in a continuous manner throughout space, whether the points at which their values are calculated are inside the mass or outside it.

It must be carefully observed that this result has been proved true only when the attracting element of the mass is one of finite *volume*. It will be subsequently shown that if the attracting element is *superficial*, i.e. if its volume is zero, the continuity of some of the first differential coefficients of V ceases.

247.] **Discontinuity of its Second Differential Coefficients.**

Since $V = \int \frac{dm}{r}$, we have $\frac{d^2 V}{dx^2} = \int \frac{d^2\left(\frac{1}{r}\right)}{dx^2} dm$, the co-ordinates of the point, P, at which the potential is V, being x, y, z.

Now from (1) of last Art. we find

$$\frac{d^2 r}{dx^2} = \frac{1}{r} - \frac{(x-x')^2}{r^3};$$

and since
$$\frac{d^2 V}{dx^2} = \int \left\{ \frac{2}{r^3} \left(\frac{dr}{dx}\right)^2 - \frac{1}{r^2} \frac{d^2 r}{dx^2} \right\} dm,$$

$$\therefore \quad \frac{d^2 V}{dx^2} = \int \left\{ \frac{3(x-x')^2}{r^5} - \frac{1}{r^3} \right\} dm. \qquad (1)$$

Similarly
$$\frac{d^2 V}{dy^2} = \int \left\{ \frac{3(y-y')^2}{r^5} - \frac{1}{r^3} \right\} dm, \qquad (2)$$

$$\frac{d^2 V}{dz^2} = \int \left\{ \frac{3(z-z')^2}{r^5} - \frac{1}{r^3} \right\} dm. \qquad (3)$$

If in these expressions we substitute for $x-x'$, $y-y'$, $z-z'$, and dm, as in last Article, we have

$$\frac{d^2 V}{dx^2} = \int (3 \sin^2\theta \cos^2\phi - 1) \frac{\rho}{r} \sin\theta \, dr \, d\theta \, d\phi ;$$

hence, when $r = 0$, i.e. when P is inside the attracting mass, the expression under the integral sign becomes infinite, and the value of $\frac{d^2 V}{dx^2}$ ceases to be continuous from points inside to points outside the mass.

Fig. 244 represents the values of V, $\frac{dV}{dx}$, and $\frac{d^2 V}{dx^2}$, when the attracting solid is that contained between two concentric spherical surfaces whose radii are Oa' and Oa, and the point P occupies positions along a fixed diameter, Ox, varying from O to infinity. The distance of P from O is here denoted by x, which is therefore the same as c in case (5) of Art. 244.

The values of V are given by the ordinates (distances from Ox) of the continuous curve $ABCD$, of which the portion AB is a right line corresponding to the constant potential within the inner surface.

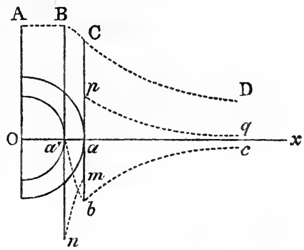

Fig. 244.

The values of $\frac{dV}{dx}$ are given by the ordinates of the continuous curve $Oa'bc$, of which Oa' corresponds to the constant zero value within the inner surface.

The values of $\frac{d^2V}{dx^2}$ are given by the ordinates of the discontinuous curve $Oa'nmpq$.

From case (5), Art. 244, when P is completely outside the mass we have $\frac{d^2V}{dc^2} = \frac{8\pi\rho(a^3 - a'^3)}{3c^3}$, and when P is inside the shell between the two surfaces

$$\frac{d^2V}{dc^2} = -\frac{4\pi\rho}{3}\left(1 + \frac{2a'^3}{c^3}\right).$$

By putting $c = a$ in the first of these values we have the value, ap, of $\frac{d^2V}{dc^2}$ when P comes to the outer surface from the outside; and putting $c = a$ in the second, we have the (negative) value, am, of $\frac{d^2V}{dc^2}$ when P comes to this surface from the inside*.

248.] **Components of Attraction.** The attraction between a unit mass at P and the element dm at M (fig. 241) is $\frac{dm}{r^2}$ in the line PM; and since PM makes with the axis of x an angle whose cosine is $\frac{x - x'}{r}$ (the co-ordinates of P and M being x, y, z and x', y', z', respectively), the component of this attraction parallel to the axis x is $-\frac{x - x'}{r^3}\,dm$. Hence if X be the attraction of the whole mass parallel to the axis of x,

$$X = -\int \frac{x - x'}{r^3}\,dm.$$

* Fig. 244 is from Thomson and Tait's *Nat. Phil.*

Similarly, if Y and Z denote the components of attraction parallel to the axes of y and z,

$$Y = -\int \frac{y-y'}{r^3}\,dm, \quad Z = -\int \frac{z-z'}{r^3}\,dm.$$

Comparing these with the differential coefficients of V (Art. 246), we see that

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}.$$

Now $\dfrac{dV}{dx}$ is the rate of variation of potential at the attracted point in a direction parallel to the axis of x; and, this direction being, of course, arbitrary, we see that—*the rate of variation of the potential at any point in any direction is the attraction in this direction on a unit mass at the point.*

If, then, generally, ds is the element of the arc of any curve at the point P,

$$\frac{dV}{ds}$$

is the attraction along the tangent to this curve at P, V being expressed as a function of s *and quantities which do not vary with* s.

If the attraction follows any other law than that of the inverse square, these results remain true. For if the attraction between P and M is $\phi'(r)\,dm$ (Art. 243), the component parallel to the axis of x is

$$-\phi'(r)\frac{x-x'}{r}\,dm, \quad \text{or} \quad -\phi'(r)\frac{dr}{dx}\,dm,$$

and we have

$$X = -\int \phi'(r)\frac{dr}{dx}\,dm = -\frac{d}{dx}\int \phi(r)\,dm = \frac{dV}{dr},$$

and similarly for all other components.

249.] **Direction of the Resultant Attraction.** If R be the magnitude of the resultant attraction, its direction cosines are $\dfrac{X}{R}, \dfrac{Y}{R}, \dfrac{Z}{R}$, or $\dfrac{dV}{dx}, \dfrac{dV}{dy}, \dfrac{dV}{dz}$, each divided by

$$\sqrt{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2}.$$

Let the value of V at the point P be denoted by C; then, V being a function of x, y, z, the equation

$$V = C$$

denotes a surface passing through P, and at each point of this surface the potential has the constant value C. Now the direction cosines of the normal to this surface are exactly the same as those obtained above for R. Hence —

At each point in space the resultant attraction on a particle is normal to the surface of constant potential passing through the point.

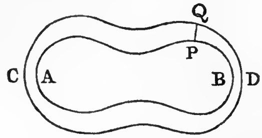

Fig. 245.

Let APB (fig. 245) be the surface of uniform potential described through P for a given attracting mass; let PQ be an element of the normal to this surface at P; and let CQD be the surface of uniform potential described through Q. Then if V is the potential at P, and V' that at Q, the resultant attraction at P in the direction PQ is the limit of the ratio

$$\frac{V'-V}{PQ};$$

or if the element of normal is denoted by dn, the resultant attraction is

$$\frac{dV}{dn}$$

in the direction in which the increments dV and dn are taken along the normal at P.

As has been already mentioned (Art. 223), surfaces of uniform potential are also called Level Surfaces, or Surfaces de Niveau, the appropriateness of this name depending on the fact that no work is done against the acting forces in displacing a particle in any manner whatever on such a surface.

250.] **Change of Attraction in passing through an Attracting Shell of Small Thickness.** It will be proved (see Example 4), that the attraction of a circular plate of uniform density (ρ) and thickness (k) on a unit mass placed on the perpendicular to the plate through its centre is $2\pi k\rho(1-\cos a)$, where a is the semivertical angle of the cone whose vertex is the attracted particle and whose base is the plate. Hence, if the particle is very close to the plate, the attraction will be

$$2\pi k\rho,$$

since a is sensibly a right angle; and this result is independent of the radius of the plate.

Let P and Q (fig. 246), be two points on the normal at opposite sides of an attracting surface of small thickness, and consider the separate attractions of a small circular plate in the vicinity of P and the remainder of the surface. The attraction of the latter portion will be sensibly the same at P as at Q; and by what precedes, the attraction of the plate at P will be a force $2\pi k\rho$ in the sense PQ, and at Q an equal force in the opposite sense. Hence it is evident that the whole attraction at P is the resultant of the whole attraction at Q and a force equal to $4\pi k\rho$ along the normal from P towards Q, where k and ρ are the thickness and density of the shell at P; so that if the attraction at Q is zero, the attraction at P is $4\pi k\rho$.

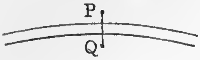

Fig. 246.

251.] **Lines and Tubes of Force.** If the element PQ (fig. 245) be indefinitely prolonged in such a manner as to be at all its points normal to the level surfaces which it meets, it becomes what Faraday called a *Line of Force*, which may therefore be defined either as a curve intersecting perpendicularly all the level surfaces, or as a curve at every point of which the resultant force is directed along the tangent to it.

If a superficial element of the level surface at P is taken, and lines of force are described along the contour of this element, these lines form a tubular surface which is called a *Tube of Force*.

As a simple example let us consider the level surfaces of a uniform bar (Art. 244).

Since $V = k\rho \log \dfrac{\mu + c}{\mu - c}$, if V is constant, μ is constant, or the axis major of the ellipse whose foci are the extremities of the bar is constant. Hence the level surface at P is an ellipsoid of revolution round AB; and since the curve drawn through P cutting at right angles a series of confocal ellipses in the plane of the figure is a hyperbola whose foci are A and B, the lines of force are hyperbolas confocal with the ellipses. Moreover, since the resultant attraction at P is normal to the ellipse through P, we see that *the attraction of the bar AB on a particle at P bisects the angle APB.*

Again (case (1), Art. 244), the level surfaces of a magnet whose magnetism is supposed to be concentrated in equal and opposite quantities at its poles are given by the equation

$$\frac{1}{r_1} - \frac{1}{r_2} = \text{const.}$$

They are obviously surfaces of revolution round the magnet bar, generated by the plane curve whose equation is the above. One of these surfaces is a plane bisecting NS perpendicularly, and the lines of force are the magnetic curves, one of which is represented in p. 39.

252.] **Surface-integral of Normal Attraction.** Let any closed surface be described so as to contain an element, dm, of attracting matter completely inside it, at a point O; and let the attraction of this element on a unit mass imagined to be placed at P, any point on the surface, be resolved, either constantly inwards or constantly outwards, along the normal to the surface at P and then multiplied by dS, an element of the surface at P. The integral of this taken all over the closed surface is called *the surface-integral of normal attraction.*

To find its value, let $OP = r$, and let θ be the angle between OP and the normal at P measured towards the interior of the closed surface. Then the normal attraction on a unit mass at P is $\frac{dm}{r^2}\cos\theta$*. Hence the surface-integral of normal attraction measured *outwards* from the surface is

$$-dm\int\frac{\cos\theta}{r^2}\,dS.$$

If a sphere of unit radius is described round O, and if lines drawn from O to the contour of dS intercept a portion of the surface of this sphere equal to $d\omega$, it is well known that

$$\cos\theta\, dS = r^2\, d\omega.$$

Hence the surface-integral becomes

$$-dm\int d\omega,$$

the integration being performed over the whole sphere since O is completely surrounded by the closed surface. But $\int d\omega =$ surface of sphere of unit radius $= 4\pi$; therefore, the surface-integral is

$$-4\pi\, dm,$$

which is independent of the position of dm.

Hence if n denote the magnitude of the normal attraction of dm on a unit mass imagined at P,

$$\int n\, dS = -4\pi\, dm;$$

and if the surface inclose any quantity of attracting matter whose mass is M_i, we have, denoting by N the normal component of its attraction on a unit mass imagined at P,

$$\int N dS = -4\pi M_i. \qquad (1)$$

* Or rather this multiplied by the unit factor, as explained in Art. 242.

If there is repulsion between the mass inside and the unit mass on the surface, we have $\int N ds = +4\pi M_i$, N being still measured *outwards*.

We shall now suppose that the element dm is completely *outside* the closed surface.

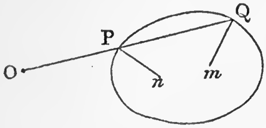

Fig. 247.

From O draw a right line OPQ meeting the surface in P and Q; let $OP = r_1$, $OQ = r_2$, $\angle QPn = \theta_1$, $\angle PQm = \theta_2$, Pn and Qm being the normals to the surface at P and Q. Then, if dS_1 is the element of surface at P, $d\omega$ the element of surface intercepted on a sphere of unit radius described round O as centre by lines from O to the contour of dS_1, and dS_2, the corresponding element of surface at Q, we have

$$\cos\theta_1\, dS_1 = r_1^2\, d\omega,$$
$$\cos\theta_2\, dS_2 = r_2^2\, d\omega;$$

hence $\frac{\cos\theta_2}{r_2^2}\, dS_2 - \frac{\cos\theta_1}{r_1^2}\, dS_1 = 0$; but the expression on the left-hand side of this equation, when multiplied by dm, is the sum of the normal attractions at P and Q, each measured inwards and multiplied by the corresponding superficial element. Hence if any closed surface is described in such a manner as to include none of the attracting matter

$$\int NdS = 0, \tag{2}$$

the integration being performed over the whole of the closed surface. In the figure we have represented a line from O as meeting the surface in two points only; but since the surface is closed, a right line must meet it in an even number of points, and it is evident that the elements of the integral considered destroy each other at the points where the line meets the surface, as in the above figure.

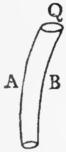

Fig. 248.

253.] **Surface-integral for a Tube of Force.** Let $PAQB$ represent any portion of a tube of force, P and Q being elements of two level surfaces intercepted by the tube. Then the atttaction on a unit mass at P is normal to the section P, and the attraction on a unit mass at Q is normal to the section Q, while at every point, A or B, on every portion of the lateral surface of the tube the attraction is wholly tangential to the surface.

Let F be the force at P, F' that at Q, and ω and ω' the areas of the sections P and Q. Then, supposing that the tube contains none of the attracting matter, equation (2) of last Article gives

$$F\omega - F'\omega' = 0, \tag{1}$$

since the only portions of the closed surface $PAQB$ which contribute elements to the surface-integral of normal attraction are the sections P and Q.

Hence, *at all points in empty space on a given line of force the resultant attraction on an imagined unit mass is inversely proportional to the normal sections of the tube of force at these points.*

This simple theorem gives the law of attraction very readily in certain cases. For example, let the attracting body be a sphere whose density is the same at the same distance from its centre. Then the lines of force are obviously right lines drawn from its centre; the tubes are therefore cones whose vertices are the centre, and since the normal sections of these cones are directly as the squares of their distances from the centre, the attraction of the sphere at any external point is inversely proportional to the square of its distance from the centre.

Again, let the attracting body be an infinite cylinder whose density is the same at the same distance from its axis. Here the lines of force are right lines emanating from the axis perpendicularly, the tubes become wedges, and the areas of their normal sections are directly proportional to their distances from the axis; hence the attraction of an infinite cylinder at an external point is inversely proportional to its distance from the axis.

Finally, for an infinite attracting plate, the tubes are cylinders and the attraction is constant at all points in empty space.

These elegant applications of equation (1) are given by Thomson and Tait (*Nat. Phil.*).

If the tube of force contain within it a quantity of the attracting matter whose mass is dq, we have by (1) of last Article

$$F\omega - F'\omega' = 4\pi\, dq. \tag{2}$$

This equation can in like manner be employed to find the resultant force inside a sphere, a cylinder, or a plate.

In the case of a sphere of uniform density, let the tube be contained between the spheres of radii r and $r+dr$. Then $dq = \rho\omega\, dr$, ρ being the density at the attracted point, and (2) becomes

$$d(F\omega) = 4\pi\rho\omega\, dr,$$

or

$$d(Fr^2) = 4\pi\rho r^2\, dr,$$

since ω is proportional to r^2. Integrating this last equation,

$$Fr^2 = \frac{3}{4}\pi\rho r^3 + C.$$

Now F is evidently zero at the centre, therefore $C = 0$, and

$$F = \frac{4}{3}\pi\rho r.$$

For a point inside an infinite cylinder at a distance r from the axis we have, since ω is ultimately a rectangle of breadth proportional to r,

$$d(Fr) = 4\pi\rho r\,dr,$$

$$\therefore \quad F = 2\pi\rho r.$$

In general, if the tube is terminated by two level surfaces whose distance measured along the lines of force forming the tube is ds, we have $dq = \rho\omega\,ds$, and (2) gives for the determination of F

$$d(F\omega) = 4\pi\rho\omega\,ds.$$

254.] **Equations of Laplace and Poisson.** We propose to prove that if V is the potential of an attracting solid mass on a unit mass at the point P (x, y, z) we shall have

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0, \text{ or } = -4\pi\rho,$$

according as there is not or is some of the mass at P. The first is Laplace's equation and the second Poisson's.

At P draw a small parallelopiped whose edges, measured from P, are dx, dy, dz, and find the surface-integral of normal attraction over the surface of this parallelopiped. The normal force, N, measured outwards from the surface on the face $dy\,dz$ passing through P is $-\frac{dV}{dx}dy\,dz$; the normal force on the opposite face is $\left(\frac{dV}{dx} + \frac{d^2V}{dx^2}dx\right)dy\,dz$; and the sum of these two is $\frac{d^2V}{dx^2}dx\,dy\,dz$. Similarly, the sums contributed, $\int N dS$, by the faces $dz\,dx$ and $dx\,dy$ are $\frac{d^2V}{dy^2}dx\,dy\,dz$ and $\frac{d^2V}{dz^2}dx\,dy\,dz$. Hence if there is no mass inside the little parallelopiped we have $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0$. If there is mass, equal to $\rho\,dx\,dy\,dz$, we have by equation (1), Art. 252,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho.$$

In this very simple way we may also find the differential equation in polar co-ordinates satisfied by V. Take as the surface over which the normal attraction is integrated the polar element of volume $msqt$ (fig. 208, p. 280). Let the polar co-ordinates of the point s be r, θ, ϕ; let the normal force on the face msq be denoted by R and the area of this face by s_1. Then this face will contribute the term $-Rs_1$ to the surface-integral; and the opposite face (through t) will contribute $Rs_1+\frac{d(Rs_1)}{dr}dr$; therefore these faces give conjointly $\frac{d(Rs_1)}{dr}dr$. Let the normal forces on the faces mst and tsq be T and S, and the areas of these faces s_2 and s_3; then the first and its opposing face will furnish the term $\frac{d(Ts_2)}{d\theta}d\theta$, and the second and its opposing face $\frac{d(Ss_3)}{d\phi}d\phi$.

Hence $$\frac{d(Rs_1)}{dr}dr+\frac{d(Ts_2)}{d\theta}d\theta+\frac{d(Ss_3)}{d\phi}d\phi=0, \text{ or } =-4\pi m_i,$$
according as there is not or is mass inside the element of volume.

Now $$R=\frac{dV}{dr},\quad T=\frac{1}{r}\frac{dV}{d\theta},\quad S=\frac{1}{r\sin\theta}\frac{dV}{d\phi};$$
$$s_1=r^2\sin\theta\,d\theta\,d\phi,\quad s_2=r\sin\theta\,dr\,d\phi,\quad s_3=r\,d\theta\,dr;$$
and the differential coefficients are, of course, all partial.

Hence, if m_i is put $=\rho r^2\sin\theta\,dr\,d\theta\,d\phi$, ρ being the density at s, we have
$$\sin\theta\frac{d}{dr}\left(r^2\frac{dV}{dr}\right)+\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right)+\frac{1}{\sin\theta}\frac{d^2V}{d\phi^2}=0,$$
or
$$=-4\pi\rho r^2\sin\theta.$$

It is usual to denote the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ by the symbol ∇.

Examples.

1. To find the attraction of a magnet on a magnetic particle whose distance from the centre of the magnet is very great compared with the length of the magnet.

Let NS (fig. 35, p. 39) be the magnet, and suppose equal and opposite quantities of magnetism, m and $-m$, to be concentrated at its poles N and S, respectively. Then, assuming a quantity μ at any point P, whose distances from N and S are r_1 and r_2, respectively,

$$V = m\mu\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Let O be the centre of the magnet,

$$OP = r,\ ON = a,\ \text{and}\ PON = \frac{\pi}{2} - \lambda.$$

Then
$$r_1^2 = r^2 - 2ar\sin\lambda + a^2,$$
$$r_2^2 = r^2 + 2ar\sin\lambda + a^2;$$

and rejecting $\left(\frac{a}{r}\right)^2$, we have

$$\frac{1}{r_1} = \frac{1}{r}\left(1 + \frac{a}{r}\sin\lambda\right),\quad \frac{1}{r_2} = \frac{1}{r}\left(1 - \frac{a}{r}\sin\lambda\right).$$

Hence
$$V = \frac{2m\mu a}{r^2}\sin\lambda.$$

If N is the attraction in the direction PO, we have $N = -\frac{dV}{dr}$, or

$$N = \frac{4m\mu a}{r^3}\sin\lambda.$$

If T is the attraction on P perpendicular to PO, we have $T = \frac{dV}{rd\lambda}$, since the element of a circular arc at P whose centre is O is $rd\lambda$;

$$\therefore\quad T = \frac{2m\mu a}{r^3}\cos\lambda.$$

If the direction of the resultant attraction at P makes an angle $\frac{\pi}{2} - i$ with OP, we have

$$\tan i = 2\tan\lambda,$$

the well-known equation which expresses the magnetic dip (i) in terms of the magnetic latitude (λ), on Biot's hypothesis of terrestrial magnetism.

2. To find the attraction of a uniform bar on a particle.

First, by the method of potentials. It has been shown (case (2), Art. 244) that

$$V = k\rho\log\frac{\mu + c}{\mu - c}, \tag{1}$$

and also that the resultant attraction acts in the bisector of the angle APB (fig. 242).

Let ds be an element of this bisector at P. Then the attraction is $\frac{dV}{ds}$; and from (1)
$$\frac{dV}{ds} = \frac{2k\rho c}{\mu^2 - c^2}\frac{d\mu}{ds}.$$

Now if $2\phi = \angle APB$, and $AP = r$, we have $\frac{dr}{ds} = \cos\phi$. Also $2\mu = r + r'$, and since along the bisector, or, in other words, along a hyperbola through P confocal with the ellipse whose axis major is

2μ, $r-r'$ is constant, we have $dr = dr'$, therefore $\dfrac{d\mu}{ds} = \dfrac{dr}{ds} = \cos\phi$

Hence
$$\frac{dV}{ds} = \frac{2k\rho c}{\mu^2-c^2}\cos\phi.$$

Again in the triangle APB, $\cos\phi = \sqrt{\dfrac{\mu^2-c^2}{rr'}}$, by an elementary formula in trigonometry. Therefore $\dfrac{dV}{ds} = \dfrac{2k\rho c}{rr'\cos\phi}$; and if y is the perpendicular from P on AB, we have $2cy = rr'\sin 2\phi$; therefore finally,
$$\frac{dV}{ds} = \frac{2k\rho\sin\phi}{y},$$
which is the resultant attraction.

Secondly, by direct calculation. The attraction of the element $k\rho ds$ at M on a unit mass at P (fig. 242) is $\dfrac{k\rho ds}{PM^2}$, or (if $\angle MPO = \psi$) $\dfrac{k\rho d\psi}{y}$, where $PO = y$. Resolve this along and perpendicular to PO, and denote the components of the resultant attraction in these directions by Y and X, respectively. Then
$$Y = \frac{k\rho}{y}\int_{-\beta}^{a}\cos\psi d\psi = \frac{k\rho}{y}(\sin a + \sin\beta),$$
$$X = \frac{k\rho}{y}\int_{-\beta}^{a}\sin\psi d\psi = \frac{k\rho}{y}(\cos\beta - \cos a),$$
where $a = \angle APO$, $\beta = \angle BPO$. From these values we have
$$R = \frac{2k\rho}{y}\sin\frac{a+\beta}{2},$$
where R is $\sqrt{X^2+Y^2}$, and
$$\frac{X}{Y} = \tan\frac{a-\beta}{2},$$
which shows that the direction of R bisects the angle APB.

3. If a circular arc of uniform thickness and density (equal to those of the bar) is described with P as centre, touching the bar at O and terminated by the lines PA and PB, the attraction of this arc at P is the same in magnitude and direction as the attraction of the bar at P.

For, draw PN to a point N on AB very near M, and let PM and PN meet the circular arc in m and n respectively. Then the attraction of the element MN on P is $k\rho\dfrac{MN}{PM^2}$. From M let fall MQ perpendicular to PN. Then
$$MN = \frac{MQ}{\sin PMO} = \frac{MQ\,.\,PM}{PO} = \frac{mn\,.\,PM^2}{Pm\,.\,PO} = \frac{mn\,.\,PM^2}{Pm^2}.$$

Therefore $k\rho\dfrac{MN}{PM^2} = k\rho\dfrac{mn}{Pm^2}$; hence the attraction of the element

MN of the bar is equal to that of the element mn of the circular arc. Therefore, &c.

4. To find the attraction of a circular plate of uniform density and small thickness on a unit mass placed anywhere on an axis through the centre of the plate perpendicular to its plane.

Let z be the distance of the attracted particle, P, from the centre, O, of the plate. Divide the plate into an infinitely great number of circular rings whose common centre is O; let r be inner radius of one of these rings, dr its breadth, a the radius of the plate, k its thickness, and ρ its density. Then the potential of the ring at P is

$$\frac{2\pi k\rho r dr}{\sqrt{z^2+r^2}},$$

since each particle of the ring is at a distance $\sqrt{z^2+r^2}$ from P.

$$\text{Hence} \quad V = 2\pi k\rho \int_0^a \frac{rdr}{\sqrt{z^2+r^2}} = 2\pi k\rho\left(\sqrt{z^2+a^2}-z\right)$$

$$\text{and the attraction} = -\frac{dV}{dz} = 2\pi k\rho\left(1-\frac{z}{\sqrt{z^2+a^2}}\right).$$

Let α be the semi-vertical angle of the cone whose base is the plate and vertex P. Then $z = a\cot\alpha$, and the attraction is

$$2\pi k\rho(1-\cos\alpha).$$

The same result follows easily by direct calculation. For if θ is the angle made with PO by lines drawn from P to the circumference of the ring of radius r, the attraction of this ring resolved along PO is $\dfrac{2\pi k\rho r dr}{z^2+r^2}\cos\theta$; but $r = z\tan\theta$, therefore this expression $= 2\pi k\rho\sin\theta\, d\theta$, the integral of which from $\theta=0$ to $\theta=\alpha$ is $2\pi k\rho(1-\cos\alpha)$.

5. To find the attraction of the frustum of a right cone on a particle placed at the vertex of the complete cone.

Let the frustum be divided into an indefinitely great number of circular plates, each of the thickness, dz. Then since α is the same for all the plates, and k in the above case is now dz,

$$\text{whole attraction} = 2\pi\rho(1-\cos\alpha)\int_{h'}^{h} dz,$$

where h and h' are the distances from P of the faces of the frustum. Hence the attraction is

$$2\pi\rho(h-h')(1-\cos\alpha),$$

and the attraction depends merely on the *thickness*, $h-h'$, of the frustum *and not its proximity to the attracted particle.*

This remarkable proposition is true also in the case of an oblique cone standing on any plane base whatever, the attracted particle being at its vertex. To prove this we have merely to show that if two plates of the same thickness, each parallel to the base, be taken anywhere in the cone, these plates exert equal attractions at the vertex.

Through the vertex, P, draw an infinite number of rays forming a very slender cone intersecting the two plates in two small similar elements of surface, dS and dS', at the points M and M', suppose. Then the attraction of dS on P is $\frac{k\rho dS}{PM^2}$, k and ρ being the thickness and density of the plate; and the attraction of dS' on P is $\frac{k\rho dS'}{PM'^2}$. These attractions are in the same line, PMM'; and since the contours of the elements dS and dS' are similar curves, $\frac{dS}{dS'} = \frac{PM^2}{PM'^2}$; therefore the attractions of these elements on P are equal. Similarly for all other corresponding elements of the plates; therefore the plates attract P equally.

The attraction of any frustum on P depends, then, only on the number of plates of given small thickness in the frustum, i.e. on the *thickness* of the frustum.

6. To find the attraction of a spherical shell of uniform density and small thickness on an external particle.

The potential has been proved (Art. 244) to be $\frac{\text{mass of shell}}{c}$, or $V = \frac{4\pi\rho\tau a^2}{c}$. Also the attraction measured in the direction in which c increases (Art. 248) is $\frac{dV}{dc}$, therefore the attraction towards the centre is $-\frac{dV}{dc}$, or

$$\frac{4\pi\rho\tau a^2}{c^2}.$$

It is the same, therefore, as if the mass of the shell were concentrated at its centre.

Hence also the attraction of the solid contained between two concentric spherical surfaces is $\frac{\text{mass of solid}}{c^2}$.

7. To find the attraction of a spherical shell on an internal particle.

Since the potential is constant inside the shell, the attraction is zero. This result is independent of the thickness of the shell. If the attracting solid is a sphere, and the attracted particle, of unit mass, is inside the sphere at a distance r from the centre, it follows that if a sphere is described concentric with the given one and passing through the particle, the portion of the solid included between this sphere and the surface of the given sphere exerts no attraction on the particle; and the attraction of the sphere of radius r is $\frac{4\pi\rho r^3}{3r^2}$, or

$$\frac{4}{3}\pi\rho r,$$

that is, the attraction of a sphere on an internal particle varies as the distance of the particle from the centre.

8. To find the attraction of a circular plate of uniform thickness and density on a particle in its plane, the law of attraction being that of the inverse cube of the distance.

From P, the attracted point, draw two very close radii vectores intercepting a narrow strip of the plate between them.

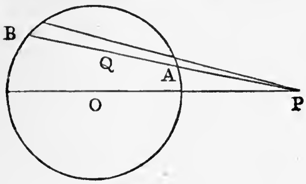

Fig. 249.

Let O be the centre of the plate, let θ be the angle OPA made by one of the radii vectores, and let $\theta + d\theta$ be the angle made by the other, with OP. Let Q be a point on PA, and $PQ = r$. Then the mass of the element at Q included between circles of radii r and $r + dr$ described with P as centre is

$$k\rho r\,dr\,d\theta,$$

k and ρ being the thickness and density of the plate.

The attraction of this element on P resolved along PO is

$$\frac{k\rho\,dr\,d\theta}{r^2}\cos\theta;$$

hence the resultant attraction is

$$k\rho\iint\frac{dr\,d\theta}{r^2}\cos\theta,$$

the integration in r being performed from $r = PA$ to $r = PB$, and that in θ from $\theta = -\sin^{-1}\frac{a}{c}$ to $\theta = \sin^{-1}\frac{a}{c}$, where a is the radius of the plate and $c = OP$, the extreme values of θ corresponding to the two tangents that can be drawn from P to the circle.

Now denoting PA by r_1 and PB by r_2, and integrating first with respect to r_1, the attraction is

$$k\rho\int\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\cos\theta\,d\theta.$$

The values of r_1 and r_2 are given by the equation

$$r^2 - 2cr\cos\theta + c^2 - a^2 = 0,$$

$$\therefore\quad \frac{1}{r_1}-\frac{1}{r_2} = \frac{2\sqrt{a^2-c^2\sin^2\theta}}{c^2-a^2}.$$

Hence the attraction is

$$\frac{2k\rho}{c(c^2-a^2)}\int_{-a}^{a}\sqrt{a^2-t^2}\,dt, \text{ or } \frac{\pi k\rho a^2}{c(c^2-a^2)},$$

where t is put for $c\sin\theta$.

In this case we might have found the attraction from the Potential. The latter is easily found by dividing the plate into rings with O as centre. If r is the radius of one of these rings we have $V = \frac{k\rho}{2}\iint\frac{r\,d\theta\,dr}{r^2-2cr\cos\theta+c^2}$. Integrating first from $\theta = 0$ to $\theta = \pi$, and doubling the result we have $V = \pi k\rho\int\frac{r\,dr}{c^2-r^2}$, in which r runs from 0 to a. Hence $V = \frac{\pi k\rho}{2}\log\frac{c^2}{c^2-a^2}$.

But V may also be easily found from the attraction, thus,

$$\frac{dV}{dc} = -\frac{\pi k\rho a^2}{c(c^2-a^2)},$$

$$\therefore \quad V = \frac{\pi k\rho}{2}\log\frac{c^2}{c^2-a^2} + \text{const.}$$

Now, since $V = \frac{1}{2}\int\frac{dm}{r^2}$, it is clear that at infinity $V=0$, or $V=0$ when $c=\infty$. This gives the const. $=0$,

$$\therefore \quad V = \frac{\pi k\rho}{2}\log\frac{c^2}{c^2-a^2}.$$

9. If V_n and V_{n-2} denote the potentials of an attracting mass when the law of attraction is the n^{th} and $(n-2)^{\text{th}}$ power of the distance, respectively, prove that

$$V_{n-2} = \frac{\nabla V_n}{(n-1)(n+2)},$$

where $\nabla \equiv \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$, the co-ordinates of the attracted particle being x, y, z.

We have $$V_n = -\frac{1}{n+1}\int r^{n+1}\,dm;$$

therefore, as in Art. 246,

$$\frac{dV_n}{dx} = -\int(x-x')\,r^{n-1}\,dm,$$

and $$\frac{d^2V_n}{dx^2} = -\int\{r^{n-1}+(n-1)(x-x')^2\,r^{n-3}\}\,dm.$$

Adding to this the similar values of $\frac{d^2V}{dy^2}$ and $\frac{d^2V}{dz^2}$, we have

$$\nabla V_n = (n-1)(n+2)V_{n-2}.$$

This equation enables us, generally, to find the potential for the $(n-2)^{\text{th}}$ power of the distance when that for the n^{th} is known; but it fails in two most important cases, namely, when $n=1$ and when $n=-2$*.

If the attracting mass is a plate, $r^2 = (x-x')^2+(y-y')^2$, and the result is easily proved to be

$$\nabla V_n = (n^2-1)\,V_{n-2}.$$

In the last example we find the potential of a circular plate for the inverse third power; hence we have at once the potentials, and therefore the attractions for the inverse fifth, seventh, &c., powers of the distance.

10. Calculate the attraction of a uniform spherical shell of small thickness on an external particle when the attraction varies as the n^{th} power of the distance.

With the notation and figure of case (3), of Art. 244, we have

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* This equation, as well as that in Ex. 19, is Mr. Jellett's.

$$V = -\frac{\rho\tau a^2}{n+1}\iint (c^2-2ca\cos\theta+a^2)^{\frac{n+1}{2}}\sin\theta d\theta d\phi$$

$$= -\frac{2\pi\rho\tau a^2}{n+1}\int_0^\pi (c^2-2ca\cos\theta+a^2)^{\frac{n+1}{2}}\sin\theta d\theta.$$

This integral is of the form $\int x^{\frac{n+1}{2}}dx$. Hence

$$V = -\frac{2\pi\rho\tau a}{(n+1)(n+3)c}\{(c+a)^{n+3}-(c-a)^{n+3}\}.$$

If we wish to find the attraction of a full sphere of radius r, we observe that τ is da, and we integrate this expression from a to 0 to $a=r$.

In each case the attraction towards the centre is $-\frac{dV}{dc}\cdot$

11. Let there be two distributions of mass denoted by M and M'; if at any point in M, where the element of mass is dM, the potential due to M' is denoted by V', and if at any point in M', where the element of mass is dM', the potential due to M is denoted by V, we shall have

$$\int V'dM = \int VdM',$$

the integration (or summation) on the left-hand side being performed throughout the mass M, and that on the right-hand side throughout M'.

For the element $V'dM$ of the first integral means that all the elements of M' are to be multiplied each by the element dM at a fixed point in M, and each of these products is to be divided by the distance between dM and the corresponding element chosen in M'; and $\int V'dM$ means, therefore, that the elements of the mass M are to be combined in pairs in all possible ways with the elements of M' and each product divided by the mutual distance of the two elements in it. But this is manifestly also the meaning of $\int VdM'$.

Of course this is also true if the elements are multiplied by *any* function of their mutual distance, and it is also true whatever elements may be denoted by dM and dM'—they, one or both, may be elements of space, for example.

Hence—*the mean potential over a spherical surface due to matter entirely outside the sphere is equal to the potential of this matter at the centre of the sphere.* (Gauss, Papers on Forces varying inversely as the square of the distance, Taylor's *Scientific Memoirs*, vol. iii. part x.)

For let mass of uniform density ρ and small uniform thickness τ, be supposed to be distributed on the sphere; let dS be an element of its surface, V the potential at this element of the attracting mass, and a the radius of the sphere. Then since the potential of a shell at an external point whose distance from the centre is r

$$= \frac{4\pi\rho\tau a^2}{r},$$

it follows that if dm is an element of the attracting matter

$$\rho\tau\int VdS = 4\pi\rho\tau a^2\int\frac{dm}{r} = 4\pi\rho\tau a^2 V_0,$$

if V_0 is the potential at the centre of the sphere. Hence

$$\frac{\int VdS}{4\pi a^2} = V_0,$$

which proves the proposition, since $\int Vds$ divided by the whole surface of the sphere is the mean value of the potential over its surface.

12. Given the whole mass of a solid, find its shape so that its attraction on a particle placed at a given point may be a maximum. It is clear that the surface of the solid must pass through the attracted point, and a little consideration shows that *the component of the attraction of any element of the solid along the direction of the resultant attraction must be constant.* Hence the surface of the solid is one of revolution round the line of action of the resultant, and the equation of the generating curve is

$$\frac{\cos\theta}{r^2} = \frac{1}{a^2} = \text{constant},$$

the attracted point being pole, and the line of action of the resultant the initial line. Hence if R be the whole attraction, and f the unit force (Art. 242),

$$R = \rho f\int\int\int\sin\theta\cos\theta\, dr\, d\theta\, d\phi = 2\pi a\rho f\int_0^{\frac{\pi}{2}}\cos^{\frac{3}{2}}\theta\sin\theta\, d\theta$$

$$= \frac{4}{5}\pi\rho a f.$$

But the mass, M, of the solid is easily found to be $\frac{4}{15}\pi\rho a^3$; therefore

$$R = \left(\frac{48\pi^2\rho^2 M}{25}\right)^{\frac{1}{3}}\cdot f.$$

The attraction of a sphere of mass M on the particle (placed on its surface) would be $\left(\frac{16\pi^2\rho^2 M}{9}\right)^{\frac{1}{3}}\cdot f$; so that the former exceeds the latter in the ratio $(\frac{27}{25})^{\frac{1}{3}}$.

13. Find an approximate value of the potential of any solid mass at a very distant point.

Let G be the centre of mass of the solid body, P the distant point, P' any point in the mass at which the element of mass is dm. Take G as origin and GP as axis of x; let $GP = r$, $GP' = r'$, and let the x of P' be x'.

Then $$V = \int\frac{dm}{\sqrt{r^2 - 2rx' + r'^2}} = \frac{1}{r}\int\left(1 - 2\frac{x'}{r} + \frac{r'^2}{r^2}\right)^{-\frac{1}{2}} dm$$

$$= \frac{1}{r}\int\left(1 + \frac{x'}{r} - \frac{r'^2}{2r^2} + \frac{3}{2}\cdot\frac{x'^2}{r^2}\right) dm,$$

neglecting all higher powers of $\frac{r'}{r}$ than the second.

Now $\int x' dm = 0$, and if we denote by λ and μ the radii of gyration of the solid about the axes of y and z, and by k its radius of gyration about GP, we have

$$\int r'^2 dm = M\frac{\lambda^2+\mu^2+k^2}{2}, \quad \int x'^2 dm = M\frac{\lambda^2+\mu^2-k^2}{2},$$

where M = mass of body.

Hence $$V = \frac{M}{r}\left(1 + \frac{\lambda^2+\mu^2-2k^2}{2r^2}\right).$$

But if k_1, k_2, k_3 are the principal radii of gyration at G, we have $\lambda^2+\mu^2+k^2 = k_1{}^2+k_2{}^2+k_3{}^2$; therefore

$$V = \frac{M}{r}\left(1 + \frac{k_1{}^2+k_2{}^2+k_3{}^2-3k^2}{2r^2}\right).$$

By differentiating this with respect to x, y, and z separately, we find the components of attraction in the directions of the principal axes at G on a unit mass at P.

14. If $V \equiv f(x, y, z)$ be a function satisfying Laplace's equation, $\nabla V = 0$, show that the function $\frac{a}{r} f\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right)$ will also satisfy it (where $r^2 = x^2+y^2+z^2$).

If O is the origin, P the point (x, y, z), Q a point on OP produced such that $OQ = \frac{a^2}{OP}$, the co-ordinates of Q are $\frac{a^2x}{r^2}$, $\frac{a^2y}{r^2}$, $\frac{a^2z}{r^2}$. Let $OQ = \rho$, let (ξ, η, ζ) be the co-ordinates of Q, and let

$$U = \frac{a}{r} f\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right) = \frac{\rho}{a} f(\xi, \eta, \zeta).$$

Then $\frac{aU}{\rho}$ satisfies the equation

$$\sin\theta \frac{d}{d\rho}\left(\rho^2 \frac{d\frac{U}{\rho}}{d\rho}\right) + \frac{d}{d\theta}\left(\sin\theta \frac{d\frac{U}{\rho}}{d\theta}\right) + \frac{1}{\sin\theta}\frac{d^2\frac{U}{\rho}}{d\phi^2} = 0.$$

But $\rho^2 \frac{d}{d\rho} = -a^2 \frac{d}{dr}$; therefore this equation becomes

$$r\sin\theta \frac{d^2(Ur)}{dr^2} + \frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \frac{1}{\sin\theta}\frac{d^2U}{d\theta^2} = 0.$$

The first term being the same as $\sin\theta \frac{d}{dr}\left(r^2 \frac{dU}{dr}\right)$, this equation is, by Art. 254, the equivalent of

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0.$$

15. A homogeneous fluid, self-attracting according to the law of nature, completely fills the space between two spherical non-concentric

surfaces one of which entirely surrounds the other; find the resultant attraction at any point of the fluid, and also the level surfaces.

Let O be the centre of the larger and O' the centre of the smaller sphere; P any point in the fluid; $OO' = c$; radius of smaller sphere $= b$; $OP = r$, $O'P = r'$; $\rho =$ density of fluid.

To calculate the resultant force at P, imagine that the place of the smaller sphere is occupied with fluid; then the larger is completely full, and there is a force $\frac{4}{3}\pi\rho r$ in the line PO towards O. Now let the effect of the fluid which we have introduced be annulled by combining with the above force the force exercised at P by a repulsive fluid of same density filling the smaller sphere. This latter force would be $\dfrac{4\pi\rho b^3}{r'^2}$ on the scale adopted; and this would act in the line $O'P$ from O'.

The resultant of these forces is the resultant force at P. If V is the potential at P,

$$dV = -\frac{4}{3}\pi\rho r dr + \frac{4\pi\rho b^3}{3r'^2}dr', \quad \text{[Art. 243]}$$

$$\therefore \quad V = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r'} + \text{const.}$$

This value is otherwise evident, since the potential at a point due to any attracting bodies is the sum of their separate potentials at the point. If a is the radius of the larger sphere (see p. 404),

$$V = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r'} + 2\pi\rho a^2.$$

The level surfaces are given by the equation

$$r^2 + \frac{2b^3}{r'} = \text{const.}$$

16. Whatever may be the law of attraction, prove that the attraction of the smaller of two concentric spheres at a point situated on the surface of the larger is to the attraction of the larger at a point situated on the surface of the smaller as the square of the radius of the smaller is to the square of the radius of the larger.

Draw any radius meeting the surface of the smaller sphere in q and the surface of the larger in Q. Let $f'(r)$ represent the law of attraction; let $a =$ radius of larger, $b =$ radius of smaller, and O be their centre. Then the attraction at Q of an element of the smaller at the point xyz is

$$f'(r)\cdot\frac{a-x}{r}\,dx\,dy\,dz$$

along QO, which is taken as axis of x.

Performing the integration with respect to x, considering y and z constant, the attraction at Q of a thin prismatic bar (parallel to OQ) of the smaller sphere is

$$[f(r_1) - f(r_2)]\,dy\,dz,$$

where r_1 and r_2 are the distances from Q of the ends, A and B, of this bar. Draw OA and OB meeting the surface of the larger sphere in A' and B', respectively; then $A'B'$ will be the axis of a prismatic bar of the larger sphere whose attraction at q is

$$[f(r_1)-f(r_2)]\,dy'dz',$$

(y' and z' being co-ordinates of A' or B') since $qA'=QA=r_1$, and $qB'=QB=r_2$. But $\frac{y}{y'}=\frac{z}{z'}=\frac{b}{a}$; therefore $\frac{dy\,dz}{dy'dz'}=\frac{b^2}{a^2}$; therefore the attractions of these corresponding bars bear to each other a constant ratio, and hence taking all such bars for the two spheres, the proposition is proved.

17. From the last proposition prove that the only law of attraction for which a homogeneous spherical shell of uniform thickness will exercise no attraction at any internal point is the law of nature.

For if such a shell exercises no internal attraction, it follows that the matter contained between the surfaces of the two spheres exercises no attraction at q, however great OQ may be. Hence however great OQ may be, the attraction of the greater at q is constant; therefore the attraction of the less sphere at $Q \propto \frac{1}{a^2}$. Q.E.D.

18. To express the amount of work done by the mutual attractive forces of the particles of a self-attracting solid when the body changes from one figure to another.

Let $m_1, m_2, \dots$ be any elements of the solid, and r_{12}, r_{13}, &c. their mutual distances. Then in the alteration of the distances between m_1 and the other elements the work done on m_1 is

$$-m_1\left(\frac{m_2}{r_{12}^{\,2}}\,dr_{12}+\frac{m_3}{r_{13}^{\,2}}\,dr_{13}+\dots\right),$$

or

$$m_1 d\left(\frac{m_2}{r_{12}}+\frac{m_3}{r_{13}}+\dots\right),\quad \text{or}\quad m_1 dV_1,$$

where V_1 is the potential of the whole mass at the position at m_1. Hence the work done on m_1 is $m_1(V_1''-V_1')$, where V_1'' denotes the potential at m_1 in the final figure of the solid, and V_1' the potential in the initial figure. Similarly the work done on m_2 is $m_2(V_2''-V_2')$; and since the term $\frac{m_1m_2}{r_{12}^{\,2}}\,dr_{12}$ is common to the expressions for the work on m_1 and the work on m_2 it is clear that whole work is expressed by

$$\tfrac{1}{2}\int V dm,$$

this integral extending over the whole solid in its first and second figures, and the first result being subtracted from the second. This expression is the potential work of the internal forces of the attracting solid.

19. If R_n and R_{n-2} denote the resultant attractions of a given solid at a given point when the law of attraction is that of the n^{th} power,

and that of the $(n-2)^{\text{th}}$ power, of the distance, respectively, prove that

$$R_{n-2} = \frac{\nabla R_n}{(n-1)(n+2)}.$$

20. Find the attraction of a circular plate of uniform thickness and density on an external particle of unit mass in its plane, the law of attraction being that of the inverse distance.

Ans. The mass of the plate divided by the distance of the particle from its centre.

21. Prove that if a material lamina attract according to the law of the inverse distance and if N is its attraction on a unit mass at any point of a closed curve, measured outwards along the normal, we shall have

$$\int N ds = 0, \text{ or } = -2\pi m_i,$$

according as there is no mass or mass m_i inside the closed curve, and hence that $\nabla V = 0$, or $= -2\pi\rho$.

22. Prove that the values of ∇V calculated for external points and for internal points do not agree for points on the surface of a solid sphere.

23. Prove that neither Laplace's nor Poisson's equation holds for points on the bounding surface of an attracting solid.

24. Find the attraction of a uniform hemispherical shell of small thickness on a unit particle placed at a distance, x, from its centre on the diameter perpendicular to the plane of the rim of the shell.

Ans. If r is the radius of the shell, ρ its density, and τ its thickness, the attraction is $\frac{2\pi\rho\tau r^2}{x^2}\left(1-\frac{r}{\sqrt{r^2+x^2}}\right)$, the unit of force being that between two units of mass at a unit distance apart.

25. If a number of uniform bars of the same section and density form any closed polygon with no re-entrant angle, prove that they produce the same potential (for the law of the inverse square) at any point inside the polygon as a polygon of bars formed by joining the feet of the perpendiculars from the given point on the sides of the given polygon.

Extend this proposition to any curve.

(See Case (2), Art. 244.)

26. If a self-attracting sphere of uniform density and radius a changes to one of uniform density and radius a', find the amount of work done by its mutual attractive forces.

Ans. The unit of work being that done when two particles, each of unit mass and placed at a unit distance apart, are drawn to an infinite distance apart, the work done will be

$$\frac{3}{5}M^2\left(\frac{1}{a}-\frac{1}{a'}\right),$$

where M is the mass of the sphere.

27. Two equal uniform bars of given sections and densities are placed parallel to each other and at right angles to the lines joining

their extremities; find the amount of work done against their mutual attraction in drawing them a given distance asunder.

Ans. If y is the distance between the bars in any position, l the length of each, m and m' are their masses, and the unit of work is the same as in last example, the work done in changing the distance from y_1 to y_2 will be the difference of the values of the expression

$$\frac{mm'}{l^2}\left(y-\sqrt{l^2+y^2}-l\log\frac{\sqrt{l^2+y^2}-l}{y}\right),$$

when y_1 and y_2 are successively put for y.

28. The gravitation potential of an attracting mass cannot have a maximum or minimum value in empty space.

[Let it have a maximum value at A. Then round A, and indefinitely near it, can be described a closed surface, at every point of which V is less than it is at A. Therefore if dn is an elementary length along the normal (measured inwards) to this surface, $\frac{dV}{dn}$ is positive all over the surface; but $N=\frac{dV}{dn}$; hence equation (2), Art. 252 is contradicted.]

29. A particle in equilibrium under the attraction of any system of masses (for the law of nature) is in unstable equilibrium.

(This follows from last example. See Art. 199. See also Clerk Maxwell's *Electricity and Magnetism*, vol. i, p. 139. The Theorem is known as Earnshaw's.)

30. If a level surface contain none of the attracting mass, the potential is constant throughout its interior, and equal (of course) to that on the surface. (Gauss, in Taylor's *Scientific Memoirs.*) For if not, it must have either a maximum or minimum value at some point within. This very simple proof is given by Thompson and Tait, *Nat. Phil.*

31. If all the attracting mass lies on or within a level surface on which the potential is zero, then in all space outside this surface the potential is constantly zero. (Gauss.)

[If possible, let the potential at any external point, P, be A, which is > 0. Then, since lines drawn from P to the given level surface meet it in points of zero potential, it is possible to find a series of points on these lines at which the potential has a constant value, $< A$ and $= B$, suppose. Also since the potential is zero at all points at infinity, it is evidently possible to describe round P a *closed* level surface on which the potential $= B$, and which includes none of the mass. This surface is subject to the result of last example, which contradicts our supposition. Therefore A cannot be > 0; and by changing the sign of every mass in the system, the supposition that A is negative may be rejected; therefore $A = 0$ in all external space.]

32. If all the attracting mass lies on or within a level surface, then in all space outside this surface the potential is less than on the surface, and has the same sign.

33. If in any portion of empty space of finite volume the potential has a constant value, it will have this value throughout all space, which can be reached without passing through any of the mass. (Gauss).

Section II.

The Attraction of Ellipsoids.

255.] **Shell bounded by Similar Surfaces.** Let $vr'p'$ and rqp be two concentric, similar, and similarly situated surfaces whose normal distance from each other is at all points very small. Suppose the space between these surfaces to be filled by attracting matter of uniform density, and let O be an attracted particle in the interior of the shell. With O as vertex let any slender cone be described, intercepting on the shell two frustums whose thicknesses measured along the generator pr of the cone are pp' and rr'. Then, since by the property of similar, similarly situated, and concentric surfaces, the intercepts pp' and and rr' are equal, we see by example 3 of last Article that the attractions of these frustums on O are equal and opposite. Hence the corresponding frustums of all such cones exert equal and opposite attractions on O; and the resultant attraction of the shell on any internal particle is therefore zero.

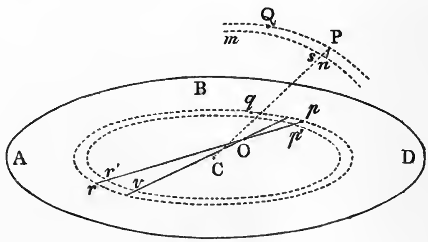

Fig. 250.

Hence, generally, if the law of attraction is that of nature, *every shell of uniform density and small thickness, bounded by similar, similarly situated, and concentric surfaces produces a constant potential at all points in its interior, and exerts, therefore, at these points no attraction.*

The same is true for a solid of uniform density and any thickness bounded by two similar, similarly situated, and concentric surfaces, since the thicknesses of the frustums intercepted between its bounding surfaces will still be equal.

256.] **Corresponding Points on Confocal Ellipsoids.** Let *rqp* and *PQ* (fig. 250) be two confocal ellipsoids, let the axes of the first be a', β', γ', and those of the second a, β, γ, let the co-ordinates of a point p on the first be x', y', z', and those of a point P on the second x, y, z. Then, if

$$\frac{x}{a} = \frac{x'}{a'}, \quad \frac{y}{\beta} = \frac{y'}{\beta'}, \quad \frac{z}{\gamma} = \frac{z'}{\gamma'},$$

the points P and p are called *corresponding* points on the ellipsoids. Also, let Q and q be two other corresponding points. Then it is very easy to prove that the distance Pq is equal to the distance Qp. (Salmon's *Geometry of Three Dimensions*, Art. 181.)

257.] **External Potential of an Ellipsoidal Shell.** Let it be required to find the potential at an external point, P, of a shell bounded by the similar, similarly situated, and concentric ellipsoids *vr′p′* and *rqp*. Through the point P describe an ellipsoid, PQ, confocal with *rqp*, and describe also an ellipsoid, *msn*, confocal with *vr′p′* and similar to PQ. This latter surface is completely determinate, since its axes must be μa, $\mu\beta$, $\mu\gamma$, and since $\mu^2(a^2-\beta^2)$ must be equal to $\mu'^2(a'^2-\beta'^2)$, where $\mu'a'$, $\mu'\beta'$, $\mu'\gamma'$ are the (given) axes of the ellipsoid *vr′p′*; or $\mu = \mu'$, since $a^2-\beta^2 = a'^2-\beta'^2$. Now, let ξ', η', ζ' be the co-ordinates of any point q on the inner shell, and ξ, η, ζ those of the corresponding point, Q, on the outer. Then if ρ is the density of each shell, the element of mass at ρ is $\rho\, d\xi' d\eta' d\zeta'$, and the potential produced by this element at P is

$$\frac{\rho\, d\xi' d\eta' d\zeta'}{Pq}.$$

But since $\dfrac{\xi'}{a'} = \dfrac{\xi}{a}$, &c., we have $d\xi' d\eta' d\zeta' = \dfrac{a'\beta'\gamma'}{a\beta\gamma}\, d\xi\, d\eta\, d\zeta$; therefore the potential of the element is

$$\frac{a'\beta'\gamma'}{a\beta\gamma}\,\frac{\rho\, d\xi\, d\eta\, d\zeta}{Pq};$$

and the potential produced at p by an element of mass at Q is

$$\frac{\rho\, d\xi\, d\eta\, d\zeta}{Qp}.$$

And since $Qp = Pq$ (Art. 256),

$$\frac{\text{potential at } P \text{ due to element of mass at } q}{\text{potential at } p \text{ due to element of mass at } Q} = \frac{a'\beta'\gamma'}{a\beta\gamma}$$

$$= \frac{\text{mass of shell } rqp}{\text{mass of shell } PQ}.$$

Taking all the elements of the inner shell, and all the corresponding elements of the outer, and thus exhausting both shells, we see that

$$\frac{\text{the potential of the inner shell at } Q}{\text{the potential of the outer shell at } p} = \frac{\text{mass of inner shell}}{\text{mass of outer shell}}.$$

Now since these shells are bounded each by similar surfaces, the potential of the outer shell is constant at all internal points, and (in virtue of the continuity of the potential) this potential is the same as the potential of the outer shell at P.

Hence the potential of an ellipsoidal shell bounded by similar surfaces is constant at all points on the surface of any ellipsoid confocal with the surface of the shell—that is, the level surfaces of an ellipsoidal shell are confocal ellipsoids, and its attraction at any point is therefore normal to the confocal ellipsoid through the point.

Let V and V' be the potentials of the shells PQ and rqp at P; then

$$V' = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} V;$$

and if x, y, z be the co-ordinates of P, we have

$$\frac{dV'}{dx} = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} \cdot \frac{dV}{dx};$$

hence the components of the attractions of the two shells in the same direction are to each other in the ratio of the masses of the shells. For this reason the calculation of the attraction of an ellipsoidal shell at an external point is reduced to that of a shell on its surface.

258.] **Attraction of an Ellipsoid at an External Point.** Let ABD (fig. 250) be a solid homogeneous ellipsoid, and let it be required to find its attraction on a unit mass placed at P. Break the ellipsoid up into an infinite number of thin shells bounded by ellipsoids similar to each other and to the surface ABD; let one of these shells be that between the surfaces $vr'p'$ and rqp. Denote this shell by (s); and describe the ellipsoids PQ and msn, similar to each other and confocal with the surfaces of (s), as in the preceding Articles. Denote this shell by (σ).

Let the axes of ABD be a, b, c; let those of rqp be ka, kb, kc, and let those of $vr'p'$ be $(k+dk)\,a$, $(k+dk)\,b$, $(k+dk)\,c$. Also, let the axes of the ellipsoid PQ be $k\sqrt{a^2+\lambda^2}$, $k\sqrt{b^2+\lambda^2}$, $k\sqrt{c^2+\lambda^2}$;

then, by Art. 257, those of msn will be $(k+dk)\sqrt{a^2+\lambda^2}$, $(k+dk)$ $\sqrt{b^2+\lambda^2}$, $(k+dk)\sqrt{c^2+\lambda^2}$. Now (Art. 250), the attraction of the shell (σ) on a unit mass at P is

$$4\pi\rho\,.\,Pn^*,$$

where Pn is the normal thickness of the shell at P. This attraction acts in the direction of the normal Pn, whose direction cosines are

$$\frac{px}{k^2(a^2+\lambda^2)},\quad \frac{py}{k^2(b^2+\lambda^2)},\quad \frac{pz}{k^2(c^2+\lambda^2)},$$

p being the length of the perpendicular from C, the centre of the ellipsoid; on the tangent plane at P, and x, y, z the coordinates of P. Hence the attraction of (σ) on P parallel to the axis of x, in the positive direction, is

$$-\frac{4\pi\rho px}{k^2(a^2+\lambda^2)}\cdot Pn. \tag{1}$$

Draw the line CP meeting the inner surface of (σ) in s. Then $\dfrac{Pn}{Ps}=\dfrac{p}{CP}$, therefore $Pn=p\,.\,\dfrac{Ps}{CP}$. But $\dfrac{Cs}{CP}=\dfrac{\text{axis of } msn}{\text{axis of } PQ}$ $=\dfrac{k+dk}{k}$; therefore $\dfrac{Ps}{CP}=-\dfrac{dk}{k}$, and $Pn=-\dfrac{pdk}{k}$.

Substituting this value in (1), we find the attraction of (σ) parallel to the axis of x to be

$$\frac{4\pi\rho p^2 x\,dk}{k^3(a^2+\lambda^2)}.$$

Multiplying this by the ratio of the mass of (s) to that of (σ), we have the component of the attraction of (s). Denoting this latter by dX, we have

$$dX=\frac{4\pi\rho\,abc\,p^2x\,dk}{k^3(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}. \tag{2}$$

Now, by the equation of the surface PQ,

$$\frac{x^2}{a^2+\lambda^2}+\frac{y^2}{b^2+\lambda^2}+\frac{z^2}{c^2+\lambda^2}=k^2.$$

Differentiating this, regarding k and λ as variables, we have

$$\frac{k^3}{p^2}\lambda d\lambda=-dk,$$

by the well-known value of the perpendicular from the centre on the tangent plane of an ellipsoid.

* The curious compensation of errors involved in the usual proof of this is well noticed by Collignon (*Dynamique*, p. 403). This simple proof is from Thomson and Tait.

Substituting this value of dk in (2), we have

$$dX = -\frac{4\pi\rho\, abc\, x\lambda\, d\lambda}{(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}.$$

To find the limits of λ, we observe that when the shell (s) is taken at the centre, $k=0$; but the axes of (σ) must be finite; and as they are $k\sqrt{a^2+\lambda^2}$, &c., the value of λ corresponding to a vanishing shell at the centre is ∞. Again, if $k=1$, or (s) is a shell at the surface ABD, we have $a^2+\lambda^2 = a_1{}^2$, where a_1 is the semi-axis of the ellipsoid confocal with ABD, and passing through P. Denote this value of λ by λ_1. Then, if M be the mass of the solid ellipsoid ABD, we have

$$X = 3Mx\int_{\infty}^{\lambda_1}\frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)^3(b^2+\lambda^2)(c^2+\lambda^2)}}; \tag{3}$$

and in the same way for the other components, Y and Z,

$$\left.\begin{aligned} Y &= 3My\int_{\infty}^{\lambda_1}\frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)^3(c^2+\lambda^2)}},\\ Z &= 3Mz\int_{\infty}^{\lambda_1}\frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)^3}}.\end{aligned}\right\} \tag{4}$$

If $L = \displaystyle\int_{\infty}^{\lambda_1}\frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}$, we have evidently

$$X = -6Mx\frac{dL}{d(a^2)},\quad Y = -6My\frac{dL}{d(b^2)},\quad Z = -6Mz\frac{dL}{d(c^2)}.$$

The expressions for X, Y, Z may be put into other forms which are useful in practice, by putting

$$\lambda = \frac{c\sqrt{1-u^2}}{u}.$$

Then

$$\left.\begin{aligned} X &= -\frac{3Mx}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\, du}{\sqrt{(1+e^2u^2)^3(1+e'^2u^2)}},\\ Y &= -\frac{3My}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\, du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)^3}},\\ Z &= -\frac{3Mz}{c^3}\int_0^{\frac{c}{c_1}}\frac{u^2\, du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)}},\end{aligned}\right\} \tag{5}$$

where $e^2 = \dfrac{a^2-c^2}{c^2}$, and $e'^2 = \dfrac{b^2-c^2}{c^2}$, the least semi-axis being c.

If the attracted particle is *on* the surface ABD of the attracting ellipsoid, the limits of u are 0 and 1, since $c_1 = c$.

If the attracted point is inside the ellipsoid, let an ellipsoid be described through it concentric with and similar to the surface ABD, and the portion between these two surfaces exerts no attraction at the point (Art. 254).

Equations (5) show that the components along the principal axes of the attraction of a homogeneous ellipsoid on a particle placed anywhere on its surface or inside its mass are of the forms

$$Ax, \quad By, \quad Cz, \tag{6}$$

where A, B, C are constant quantities.

Examples.

1. Find the attraction of a homogeneous ellipsoid of revolution round the minor axis (oblate spheroid) on a particle placed on its surface.

Here $a = b$, and $e = e'$ in equations (5); therefore

$$X = -\frac{3Mx}{c^3}\int_0^1 \frac{u^2\,du}{(1+e^2u^2)^2}.$$

The integral is most easily found by putting $eu = \tan\theta$. We then find

$$X = -\frac{3Mx}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Y = -\frac{3My}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Z = -\frac{3Mz}{c^3e^3}\left(e - \tan^{-1}e\right).$$

These expressions are of importance in the theory of the figure of the Earth.

2. A homogeneous fluid mass, self-attracting according to the law of nature, is acted upon at every element by a force proportional to the mass of the element and its distance from an axis passing through the centre of mass of the fluid. Prove that an ellipsoid of revolution round the axis is a possible figure of equilibrium of the fluid.

Let μr be the force emanating from the axis on a unit mass at distance r from the axis. Take the axis as axis of z, and assume the surface of the fluid to be an ellipsoid of revolution whose axes are $c\sqrt{1+e^2}$, $c\sqrt{1+e^2}$, c.

Then the x component of force on a unit mass on the surface is $(-A+\mu)x$, where A has the value in example 1. Hence if V is the potential at the surface

$$dV = (-A+\mu)\,x\,dx + (-A+\mu)\,y\,dy - Cz\,dz,$$

which is zero, since if the potential is not constant over the surface of a fluid, there will be a force in the tangent plane causing a flow from one point to another. Also by differentiating the equation of the surface, we have

$$\frac{x\,dx+y\,dy}{1+e^2}+z\,dz=0.$$

Hence we must have
$$\frac{-A+\mu}{C}=-\frac{1}{1+e^2}.$$

Substituting the values of A and C from last example, and putting $M=\frac{4}{3}\pi c^3(1+e^2)\rho$, where ρ is the density of the fluid, this equation gives

$$\frac{\mu e^3}{2\pi\rho}+3e=(3+e^2)\tan^{-1}e.$$

Put $\mu=\frac{4}{3}\pi\rho\,.\,q$; then we have

$$\frac{2qe^3+9e}{3(3+e^2)}-\tan^{-1}e=0,$$

which determines e, the eccentricity, in terms of q; and c, the least axis is known from M, the whole mass of the fluid.

There is a major limit to the value of q in order that equilibrium in the ellipsoidal form may be possible; but into the discussion of this, which is somewhat tedious, we do not enter. [See the *Mécanique Celeste*, or Besant's *Hydromechanics*.]

3. If from a solid homogeneous ellipsoid there be removed any complete ellipsoid, find the attraction at a point—(a) inside the remaining mass, (b) inside the ellipsoidal cavity.

The attraction is to be found by considering the cavity to be filled with matter of the same density as that of the rest, and then subtracting the results due to the matter which is imagined to fill the cavity.

Let the axes of the complete ellipsoid be taken as those of reference, and let the axes of the cavity make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ with them. Also let the co-ordinates of the attracted particle with reference to these axes be (x, y, z) and (x', y', z'), respectively, and let the components of attraction along these sets of axes be (X, Y, Z) and (X', Y', Z').

Then
$$X=Ax, \quad Y=By, \quad Z=Cz,$$
where A, B, C are constants; and

$$X'=A'x', \quad Y'=B'y', \quad Z'=C'z',$$

where if the attracted particle is outside the cavity, A', B', C' are variables, but if inside, constants.

The whole force parallel to the axis of x on a unit particle is obviously
$$X-(X'\cos\alpha_1+Y'\cos\alpha_2+Z'\cos\alpha_3),$$

with similar expressions for the components along the axes of y and z.

If the attracted particle is inside the cavity, the level surface passing through it is easily found. For, the virtual work of the attraction of the whole ellipsoid is

$$Xdx + Ydy + Zdz, \quad \text{or} \quad \tfrac{1}{2}d\,(Ax^2 + By^2 + Cz^2);$$

and that of the attraction of the small ellipsoid is

$$X'dx' + Y'dy' + Z'dz', \quad \text{or} \quad \tfrac{1}{2}d\,(A'x'^2 + B'y'^2 + C'z'^2).$$

Hence the level surfaces inside the cavity are given by the equation

$$Ax^2 + By^2 + Cz^2 - A'x'^2 - B'y'^2 - C'z'^2 = \text{const.}$$

They are therefore quadrics.

We could in the same way find the effect due to an ellipsoidal mass which contains in its interior another ellipsoidal mass (or nucleus) of density different from that of the remainder. If ρ and ρ' are the densities of the two portions ($\rho' > \rho$), imagine the whole to consist of a homogeneous mass of density ρ, and add the effect due to the nucleus, supposed of density $\rho' - \rho$.

4. Prove that an oblate spheroid of uniform density cannot have its own surface for one of its level surfaces.

[The condition that its own surface should be a level surface is $\tan^{-1} e = \dfrac{3e}{3 + e^2}$, which cannot be satisfied by any value of e, except zero.]

5. Prove that a prolate spheroid of uniform density cannot have its own surface for a level surface.

[By putting $e = k\sqrt{-1}$ in the last result, the required condition becomes

$$\tfrac{1}{2} \log \frac{1+k}{1-k} = \frac{3k}{3-k^2};$$

which gives by expansion

$$(3 - k^2)(1 + \tfrac{1}{3}k^2 + \tfrac{1}{5}k^4 + \ldots) = 3, \quad \text{or} \quad \frac{1}{3\,.\,5} + \frac{2k^2}{5\,.\,7} + \ldots = 0,$$

which is, of course, quite impossible.]

6. Prove that in the spheroid considered in example 2 the resultant attraction at any point on the surface is proportional to the length of the normal between that point and the axis of revolution.

7. Express gravity on the surface of such a spheroid in terms of the latitude.

[The latitude of a point on the surface is the angle made with the plane of the equator by the normal at the point.

If E denotes the value of gravity at the equator, G the value in latitude λ, and ϵ the eccentricity of the generating ellipse,

$$G = \frac{E}{\sqrt{1 - \epsilon^2 \sin^2 \lambda}};$$

so that if ϵ is small, the increase of gravity at any point above the equatoreal value is proportional to $\sin^2$ (latitude).]

8. The components of attraction of a homogeneous ellipsoid at an internal point (x, y, z) being Ax, By, Cz (as in p. 434), prove that

$$A + B + C = 4\pi\rho,$$

where ρ is the density at the point.

Section III.

Superficial Distributions.

259.] **Quantity of Electricity.** The student is supposed to be familiar with the elementary phenomena of electric attractions and repulsions—that is, with certain forces which bodies are observed to exhibit when they have been rubbed with resin, catskin, and some other substances. The mode in which these forces are brought into existence is called the process of *electrification*, and the bodies which exhibit them are said to be *electrified*. In the older theories of electricity such bodies were assumed to have been charged by electrification with a certain quantity of *fluid*, the process consisting either of directly communicating the fluid to the bodies, or of altering its arrangement within them (if they naturally possessed it themselves) in such a manner as to render possible the play of electrical forces. Whether this fluid theory is true or false, there appear to be at present some strong objections to its adoption. Following the views of Clerk Maxwell, we shall regard electrification merely as a *state* of a body, without speculating more closely as to the nature of this state.

Suppose that two small electrified bodies of equal surface, acting in exactly similar circumstances on the same third electrified body, produce exactly equal forces of attraction or of repulsion on this body; then we say that the two bodies considered have the same quantity of electricity; and if one of them attracts, while the other repels the third body, we should say that they have equal quantities of electricity *with opposite signs*. The phenomenal effect of electrification being, then, a measurable quantity, this state of electrification itself becomes also a measurable quantity—if it is, as we assume it to be, fully represented by this effect. We have thus attained the notion of equal *quantities* of electricity, as equivalent to equal states of electrification.

A unit quantity of electricity, in electrostatics, will therefore be that quantity which, when acting on an equal quantity placed at a unit distance from it, repels the latter with a unit force—the unit force called the *dyne*, or any other convenient one.

260.] **Electric Potential.** An electrified body exerts force on other electrified bodies in its neighbourhood; and, just as in distributions of matter, the potential at any point in space due to an electrified body is the amount of work which must be done against the repulsive force of the body to bring a unit of electricity from an infinite distance to the point considered. (This supposes that the proximity of the unit does not modify the state of electrification of the body—a supposition which will be allowable when the unit is small.)

An electrified body is found by experiment to be capable of impressing its state of electrification, with more or less success, on bodies which are put in contact with it, according to the nature of these bodies; and bodies which very readily allow this transference of state are called good conductors, or simply *conductors*, while those which best resist it are called non-conductors, or *dielectrics*.

We may speak of a *flow of electricity*, instead of a transference of state, if we are careful to avoid including in the expression any hypothesis of the material nature of electricity.

Where this flow can take place it will take place. It follows, therefore, that *when a conductor is in complete electrical equilibrium there must be a uniform electric potential throughout its substance.* And from this it follows by Poisson's equation that *when a conductor is in electrical equilibrium, the electricity resides entirely at the surface.* For since V is constant throughout the mass of the conductor, $\nabla V = 0$; and therefore $\rho = 0$. A distribution throughout the mass could exist only in a non-conductor. Also the external surface of the conductor itself must be a level surface of the electricity; for if not, a flow would take place from a point of high potential on it to one of low potential. In other words, no electrical force can be exerted anywhere in the conductor except at its surface of contact with a resisting (dielectric) medium. This is usually stated thus—in an electrified conductor the electricity resides wholly at the surface.

It is to be noted that in all cases the potential of the Earth is assumed as zero, and that the potential of any body in communication with the Earth by means of a conducting wire is therefore zero.

261.] **Free Charge. Induced Charge.** The quantity of electricity on a conductor is called its *charge*. A charge may be communicated to a body in two ways—either by actual contact with another charged body (such as a piece of glass which has been rubbed with catskin), or by the *influence*, at a distance, of such a charged body. In the first case the electricity on the conductor will be everywhere of one kind—viz., the kind which is on the body touched; and in the second the charge will consist of quantities of positive and negative electricity in two portions of the surface of the conductor.

262.] **Surface Density.** The electric density at any point of a surface is the limiting ratio of the quantity of electricity within a sphere whose centre is the point to the area of the surface contained within the sphere when the radius of the sphere is diminished indefinitely (Clerk Maxwell).

We have already spoken of solid distributions of matter over a surface, meaning that the thickness of the material stratum is everywhere very small. If ρ denotes the density of the matter, and k its thickness, the quantity on a small unit surface is $k\rho$. If now we imagine ρ to be increased and k to be diminished indefinitely, we shall have a truly superficial distribution, in which the product

$$k\rho$$

becomes the surface density here considered. Although this mode of conception may assist us in understanding a true superficial distribution, it is not necessary to imagine that electrical distribution is really produced in this way. We shall denote the surface density by σ.

When the density at any point of a body is zero, the body is said to be in its *natural state* at that point.

263.] **Density at Each Point of a Charged Conductor.** Through the contour of any elementary area, dS, of the surface of a conductor (or of any level surface on which electricity is distributed) let a tube of force be described; and let P and Q (fig. 248) be two very close normal sections of this tube, the first made inside, and the second outside the surface of the conductor. Then if σ is the surface density on dS, the quantity of electricity inside this tube is σdS; also if the area of the section Q is dS', and N the normal repulsion on $a +$ unit of electricity just outside the surface, the surface integral for the tube becomes

$$NdS' = 4\pi\sigma dS,$$

the value of N for the section P being, of course, zero.

Now, by taking Q sufficiently close to the surface, dS' can be made equal to dS, and this equation becomes in the limit

$$N = 4\pi\sigma, \quad \text{or} \quad \sigma = \frac{N}{4\pi}, \tag{1}$$

which determines the law of density at each point. Here N (being supposed repulsive) is of course $-\frac{dV}{dn}$, where dn is the element of normal measured outwards.

264.] **Force Exerted by an Electrified Conductor on its own Electricity.** At each point on the surface there is a certain force produced on a unit quantity of the conductor's own electricity, which, it must be very carefully observed, *is not equal to* $-\frac{dV}{dn}$.

For suppose ABD (fig. 250) to represent the surface of an electrified conductor, and take any very small element of its area at the point B. The repulsion of the remainder of the surface on a unit of electricity at B is the same as its repulsion on a unit just inside or outside B. This latter is $2\pi\sigma$, as at once appears by the method of Art. 250. Also the action of the element at B on itself is zero. Hence the resultant repulsion of the charge on a unit quantity at B is $2\pi\sigma$, and it acts in the normal at B; and if dS is the area of the small element at B, the quantity on it is σdS; on which the repulsion, dp, is $2\pi\sigma^2 dS$. Hence

$$dp = 2\pi\sigma^2 dS, \quad \text{or} \quad \frac{dp}{dS} = 2\pi\sigma^2,$$

which is the repulsion of the electricity on itself per unit of surface at a point where the density is σ.

This quantity, $2\pi\sigma^2$, is what Sir W. Thomson calls the *electric diminution of air pressure on the surface* (*Papers on Electrostatics and Magnetism*, p. 254), for the following reason:—each element of surface of an electrified soap-bubble being repelled by the force $2\pi\sigma^2$ per unit of surface, the bubble expands, just as it would do if the air pressure diminished, and, when discharged it contracts. Hence the electric diminution of air pressure at any point of a conductor is

$$2\pi\sigma^2, \quad \text{or} \quad \frac{N^2}{8\pi},$$

N being the electrical repulsion on a unit in the air just outside the point.

265.] **Theorem.** The sections of a tube of force made by the surfaces of two conductors, which mutually act on each other, contain equal quantities of opposite electricities.

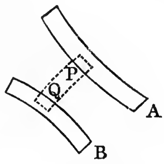

Fig. 251.

Let A and B (fig. 251) be two portions of the conductors whose adjacent surfaces, P and Q, are electrified, and let the tube of force contain the elements P and Q. Then in the substance of each conductor, no matter how thin it may be, V is a constant (Art. 260). Let the tube be prolonged to any distance in the conductors, and let it be closed at its extremities. Apply the surface integral of normal force to this tube, and let the areas cut off at P and Q be dS and dS', and the surface densities at these points σ and σ'. Now N (Art. 253) is zero all over the surface of the tube; hence

$$\sigma dS + \sigma' dS' = 0,$$

which proves the theorem.

If the surfaces are very close together we may take $dS = dS'$, and then we shall have $\sigma = -\sigma'$.

Examples.

1. To find the quantity of electricity on each of two very close parallel plates, the potential of each plate being given.

Let V and V' be their potentials, and h the distance between them (or the thickness of the dielectric). Then the surface density at any point on either is $-\frac{1}{4\pi}\frac{dV}{dn}$, where dn is the element of normal measured from the point towards the other plate. But $\frac{dV}{dn}$ is sensibly equal to $\frac{V'-V}{h}$; hence $\sigma = \frac{V-V'}{4\pi h}$; and if S be the surface of the plate and Q the quantity of electricity on it,

$$Q = \frac{(V-V')\,S}{4\pi h}.$$

If one plate communicates with the ground, its potential is zero, and Q will then be $\frac{VS}{4\pi h}$, where V is the potential of the other. This case is approximately that of the Leyden Jar.

2. To find the work done by the electric forces in the discharge of a Leyden Jar.

If one armature is connected with the ground (whose potential is zero) and the other with a source of electricity whose potential is V, the charge will be $\frac{VS}{4\pi h}$. Now on the armature whose potential is V,

the potential work of the forces, or *energy of the electrification*, is (p. 426) $\frac{1}{2}\int V dq$, or $\frac{1}{2}V\int dq$, or $\frac{1}{2}VQ$,
where Q is the whole charge on the armature and dq the elementary charge at any point. Substituting for V its value in terms of Q, we have the energy equal to

$$\frac{2\pi h Q^2}{S}.$$

This is the work done in changing the potential from V to zero; or, in other words, the work done in discharging the jar is proportional to the square of its charge.

3. To find the surface density at any point of an ellipsoidal conductor.

If we regard surface density as the limit of volume-density (Art. 262) when the thickness is indefinitely diminished, it follows at once by Art. 258 that, as the normal distance between two very close concentric, similar, and similarly placed ellipsoids is proportional to the length of the central perpendicular on the tangent plane,

$$\sigma = \lambda p,$$

where p is this perpendicular and λ a constant.

Now if Q is the whole charge on the ellipsoid, $Q = \lambda\int p\, dS$, where dS is an element of its surface on which the density of electricity is σ. But $\int p\, dS$ is obviously three times the volume of the ellipsoid, or $4\pi abc$, its axes being a, b, c. Hence

$$\sigma = \frac{Qp}{4\pi abc}.$$

4. To find the surface density at any point of an electrified circular plate.

We have at any point on the ellipsoid

$$\frac{c^2}{p^2} = c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) = c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) + 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Put $c = 0$, and we have $\frac{c}{p} = \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, so that the surface-density at any point of a charged elliptical plate is

$$\frac{Q}{4\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}},$$

and by putting $a = b$, we have for a circular plate of radius a,

$$\sigma = \frac{Q}{4\pi a\sqrt{a^2-r^2}},$$

r being the distance of the point from the centre of the plate. (For this method see Thomson's *Papers on Electrostatics*, &c., p. 179.)

5. Inside a conductor are placed any fixed electrical masses, dq_1, dq_2, … , at various points; prove that the charge induced on the

inner surface of the conductor is equal and of opposite sign to the sum of the inducing masses.

Let any closed surface be described in the body of the conductor between its inner and outer surfaces, and take the surface integral of normal force over this surface. The force at every point in the body of the conductor is necessarily zero (Art. 260); hence (Art. 252) $M_i = 0$; i.e. the sum of the charge on the internal surface and the inducing mass, $dq_1 + dq_2 + dq_3 + \ldots$, is zero. These electrical masses may obviously be themselves charges on insulated conductors.

If the conductor was in its natural state before the introduction of the inducing mass, and if there are no inducing masses outside, there will be a charge on the outer surface equal and opposite to that on the inner, and therefore equal to the inducing mass. This was discovered experimentally by Faraday.

6. An insulated conductor receives a given free charge of electricity; determine the law of distribution on the surface.

Ans. The charge spreads into an indefinitely thin layer bounded by surfaces similar to each other and to the surface of the conductor (so that the force at any internal point is zero).

7. Inside a hollow insulated conductor are placed any fixed electrical masses, and outside it any other system of electrical masses; show that however the outside masses may be arranged or varied, the total charge on the inner surface must be equal to the sum of the internal masses and of opposite sign. (Prove as in example 5.)

8. Is the result in last example modified if the conductor initially possesses a free charge?

Ans. No; because by example 6 the internal effect of this charge will be null.

266.] **Green's Equation.** Let U and V be any finite and continuous functions of the co-ordinates of a point in space, and let ∇ stand, as usual, for the operation $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$; then we shall have

$$\int U \nabla V d\omega = \int U \frac{dV}{dn} dS - \int \left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz} \right) d\omega, \ldots \quad (a)$$

where the integral on the left-hand side and the second integral on the right-hand side are taken throughout the whole of the space inside any closed surface, the element of volume of this space being denoted by $d\omega$; and the first integral on the right-hand side extends over the whole superficies of the given closed surface, the elementary length of whose normal measured *outwards* is dn. For $d\omega = dx\,dy\,dz$, and

$$\int U \nabla V d\omega = \iiint U \left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} \right) dx\,dy\,dz.$$

Integrate $U\dfrac{d^2V}{dx^2}\,dx\,dy\,dz$ with respect to x, considering y and z as constant, the extreme values of x belonging to the two (or any even number of) points, p_1 and p_2, in which the surface is intersected by the line along which y and z are constant. The contribution to the integral on the left-hand side made by a long and slender parallelopiped parallel to the axis of x will then be

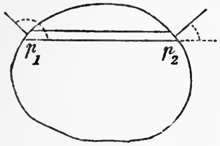

Fig. 252.

$$dy\,dz\left[\left(U\frac{dV}{dx}\right)_2-\left(U\frac{dV}{dx}\right)_1\right]-dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}\,dx, \qquad (1)$$

where the suffixes denote the values of the quantities in brackets at the points p_2 and p_1.

Now if dS_2 is the element of surface cut off by the parallelopiped at p_2, and if λ_2 is the angle (represented by dotted line) made with the axis of x by the normal measured outwards at p_2,

$$\cos\lambda_2\,.\,dS_2=dy\,dz.$$

Similarly, if dS_1 and λ_1 (measured in the same sense as λ_2) denote these quantities at p_1,

$$\cos\lambda_1\,.\,dS_1=-dy\,dz.$$

Hence (1) becomes

$$\left(U\frac{dV}{dx}\cos\lambda\,dS\right)_2+\left(U\frac{dV}{dx}\cos\lambda\,dS\right)_1-dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}\,dx\,;$$

and hence

$$\int U\frac{d^2V}{dx^2}\,d\omega=\int U\frac{dV}{dx}\cos\lambda\,dS-\iiint\frac{dU}{dx}\frac{dV}{dx}\,dx\,dy\,dz, \qquad (2)$$

λ denoting the angle made by the normal at any point with the axis of x.

In the same way, if μ and ν are the angles made by the normal with the axes of y and z, we have

$$\int U\frac{d^2V}{dy^2}\,d\omega=\int U\frac{dV}{dy}\cos\mu\,dS-\iiint\frac{dU}{dy}\frac{dV}{dy}\,dx\,dy\,dz, \qquad (3)$$

$$\int U\frac{d^2V}{dz^2}\,d\omega=\int U\frac{dV}{dz}\cos\nu\,dS-\iiint\frac{dU}{dz}\frac{dV}{dz}\,dx\,dy\,dz. \qquad (4)$$

Adding (2), (3), and (4) together, we obtain the equation (α).

Cor. Writing down the value of $\int V\nabla U\,d\omega$, and subtracting the result from (α), we obtain

$$\int(U\nabla V-V\nabla U)\,d\omega=\int\left(U\frac{dV}{dn}-V\frac{dU}{dn}\right)dS,\ldots \qquad (\beta)$$

a remarkable and very useful equation in which the volume integral on the left-hand side is changed into the surface integral on the right.

267.] **Case of Green's Equation.** In (a) of last Article let V be the potential of an attracting mass and let $U \equiv V$. Then we have, since $\nabla V = -4\pi\rho$ at every point inside the volume where there is mass (and $\nabla V = 0$ at points where there is no mass),

$$-4\pi \int V\rho\, d\omega = \int V \frac{dV}{dn}\, dS - \int R^2 d\omega,$$

where R^2 is put for $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2$, the square of the resultant force on a unit mass at the element $d\omega$ of volume.

Hence $$\int R^2 d\omega = \int V \frac{dV}{dn}\, dS + 4\pi \int V\rho\, d\omega, \tag{A}$$

an equation which will be found useful.

268.] THEOREM. *If on a conductor, removed from the influence of all electricity except its own, the total quantity of electricity is zero, the only possible distribution of the electricity is one in which the density is zero at all points*—i.e. the conductor must be in its natural state.

For if the conductor be not in its natural state, it will have a potential V different from zero; and it will be possible to describe round it and completely enclosing it a surface on which the potential has a constant value a, less than V suppose. (For on lines drawn in all possible directions from any point on the conductor we can find points at which the potential $= a$.) Denote this surface by A, and apply the equation (A) of last Article to the surface of A and the volume enclosed by it.

This equation becomes

$$\int R^2 d\omega = a \int \frac{dV}{dn}\, dS + 4\pi V \int \rho\, d\omega,$$

since wherever inside the surface A there is mass, the potential is constant—the mass existing only on the surface of the conductor. Now $\int \rho\, d\omega =$ charge on the conductor $= 0$; and

$$\int \frac{dV}{dn}\, dS = -\int N dS$$

(N being the electrical repulsion on a unit of + electricity) $= -4\pi \times$ charge on conductor $= 0$. Hence $\int R^2 d\omega = 0$, i.e. $R = 0$ at every point inside the surface A; but since at each point on the conductor the density $= \frac{R}{4\pi}$, at each point the density is zero, i.e. the conductor is in its natural state.

In the same way it can be proved that in a system of insulated conductors placed in given positions, if the total charge on each of them is zero, the only possible distribution is one in which each conductor is in its natural state.

For if possible let there be a distribution in which the potentials on the conductors are, in order of descending magnitudes, $V_1, V_2, V_3, \ldots$. Then it is evidently possible to describe round the conductor whose potential is V_1 a closed surface which will not meet any of the other conductors and on which the potential has a constant value, a, $< V_1$, and $> V_2$. Applying equation (A) to this surface and its enclosed volume, we have

$$\int R^2 d\omega = a \int \frac{dV}{dn} dS + 4\pi V_1 \int \rho\, d\omega.$$

Now $\int \rho\, d\omega = 0$, by hypothesis; therefore, as before, $R = 0$, and the first conductor is in its natural state. Proceed to the second, &c.

269.] THEOREM. *A charge of given amount can be distributed in only one way on a given conductor which is removed from the influence of all other electricity.*

For, if it be possible, let the same quantity be distributed in two different ways, and let σ and σ' be the densities at the same point in the two distributions. Reverse the density at every point in the second distribution, and superimpose this reversed on the first distribution. We have now a conductor charged with zero quantity; therefore by last Article the density at every point is zero; i.e. $\sigma = \sigma'$ at every point. Hence there is only one distribution of the given charge.

COR. The charge on a conductor removed from all influence cannot consist of positive electricity in some places and negative in others.

For one possible distribution of a charge of given amount is a distribution in which the electricity is of the same sign at all points; and by this Article this is the only distribution.

In the same way it can be shown that charges of given amounts can be distributed in only one way on any number of conductors having any fixed relative positions.

270.] **Capacity of a Conductor.** *If a given conductor has a free charge, the ratio of the amount of the charge to the potential produced by it on the surface is constant, whatever be the amount of the charge.*

For, let a given charge be divided into any number of equal parts, and let each part be spread over the surface separately. Then the law of density or distribution is the same for each part; each part produces the same potential (last Article); and since the total potential at any point is the sum of the separate potentials and the total charge the sum of the separate parts, the potential is proportional to the charge. The ratio of the charge, Q, to the potential, V, which it produces on the surface is called the *capacity* of the conductor. If we denote the capacity by C, we have

$$Q = CV.$$

The capacity may otherwise be defined as the *quantity of electricity required to charge the conductor to unit potential.*

It is evident that the capacity of a conductor depends on its relative position with respect to other conductors, whether electrified or not; for even on those which are unelectrified charges will be produced by induction.

Faraday found that the quantity of electricity imparted to a conductor in order to raise its potential from zero to a given amount depends on the medium in which the conductor is immersed. If the medium is a liquid, a greater charge will be required to produce a given potential than if the dielectric is air; and generally, *the ratio of the charges required to raise a given conductor from zero to the same potential when the conductor is placed in any medium and when it is placed in air is called the Specific Inductive Capacity of that medium.*

The specific inductive capacity is usually denoted by K; so that if C is the capacity of a conductor placed in given circumstances in air, KC is its capacity when placed in the same circumstances in the medium in question; and the charge, Q, necessary to raise the potential from 0 to V, is given by the equation

$$Q = KC.V.$$

In calculating the capacities of conductors we tacitly suppose them to be placed in air.

Examples.

1. Find the capacity of a thin circular plate freely electrified at both sides.

By example 4, p. 442, the quantity on a ring concentric with the plate and contained between a circle of radius r and one of radius $r+dr$ is $\dfrac{Q\,r\,dr}{2a\sqrt{a^2-r^2}}$. Now since the potential is constant all over the surface, it suffices to find its value at the centre. The potential produced at the centre by the above ring is $\dfrac{Qdr}{2a\sqrt{a^2-r^2}}$. Doubling this (since there is a similar ring at the other surface of the plate) we have

$$V=\frac{Q}{a}\int_0^a \frac{dr}{\sqrt{a^2-r^2}}=\frac{\pi Q}{2a}.$$

Hence $C=\dfrac{2a}{\pi}$.

2. An insulated solid sphere, A, having a given charge, Q, is surrounded by an insulated concentric spherical shell, B, having a charge Q'; find the difference of the potentials on the sphere and shell.

Let a be the radius of the sphere and b that of the shell. Then the whole charge of the sphere will be on its surface, and there will be two charges in the shell—one on its inner surface and one on its outer surface. The one on its inner surface is produced by the induction of the charged sphere, and this charge will be $-Q$ in amount, by example 5, p. 443; and since the amounts of opposite kinds of electricity separated in the body of a conductor by induction are necessarily equal, the amount of the outer charge on the shell must be $Q'-Q$.

To find the potential on the sphere, we have only to find its value at the centre. The charge on the sphere will produce potential $\dfrac{Q}{a}$ at the centre, and the charge on the shell will produce $\dfrac{Q'+Q-Q}{b}$, or $\dfrac{Q'}{b}$, at the centre. Hence if V be the potential on the sphere,

$$V=\frac{Q}{a}+\frac{Q'}{b}.$$

To find the potential produced in the shell, consider the potentials produced by the sphere and by the shell itself separately. The charge on the sphere produces (p. 404) potential $\dfrac{Q}{b}$ at all points distant b from the centre; and the charge Q' produces $\frac{Q'}{b}$. Denoting the potential on the shell by U,

$$U = \frac{Q}{b} + \frac{Q'}{b}.$$

Hence $\quad V - U = Q\left(\frac{1}{a} - \frac{1}{b}\right), \quad$ or $\quad Q = \frac{ab}{b-a}(V-U).$

If the shell is in metallic connection with the earth, $U = 0$, and the capacity of the system will be $\frac{ab}{b-a}$, which increases as $b-a$ diminishes. By this arrangement the charge accumulated may be very great. Such a compound instrument is called a *condenser*.

The capacity of an insulated sphere removed from all conductors is equal to its radius.

3. To find the capacity of a very long thin cylinder or wire.

Except near the ends, the density of the electrification will be sensibly constant, and since the potential everywhere inside is constant, we have only to find its value at the middle point of the axis. Take a section of the wire normal to its axis at a distance x from the middle point, and another section at a distance $x + dx$. Then the quantity of electricity on the surface between these sections is $2\pi\rho r dx$, where r = radius of wire and ρ = density of electrification. The potential of this at the middle point is $\frac{2\pi\rho r dx}{\sqrt{r^2+x^2}}$. Hence if l is the length of the wire,

$$V = 4\pi\rho r \int_0^{\frac{l}{2}} \frac{dx}{\sqrt{r^2+x^2}}.$$

This $= 4\pi\rho r \log\frac{l}{r}$, if r is very small compared with l. Now $Q = 2\pi\rho r l$,

$$\therefore \quad C = \frac{l}{2\log\frac{l}{r}}.$$

Hence if r is exceedingly small in comparison with l, C will be small, and if the wire is used to connect two electrified conductors, the charge on the wire may be neglected.

4. Find the capacity of a condenser consisting of a very long cylinder of radius r surrounded by another of radius R, the two being separated by a given dielectric.

Ans. $K\frac{l}{2\log\frac{r}{R}}$, where K is the specific inductive capacity of the dielectric. [This is the case of a cable.]

271.] **Case of Green's Equation.** Let V be the potential of a system of masses, M, M' (fig. 253), and let $U = \frac{1}{r}$, where r

denotes the distance of any point from a fixed point, O. Suppose, moreover, for simplicity, that S is a level surface of the system of masses and that it includes M' in its interior, while M is outside it.

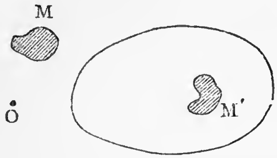

Fig. 253.

First, let O be outside the space included by S.

Then it is easy to prove that $\nabla \frac{1}{r} = 0$; and since in all parts of the space internal to S we have $\nabla V = 0$, except in those parts occupied by M', in which $\nabla V = -4\pi\rho$, where ρ is the density of M' at each point, the equation (β) of Art. 266 gives

$$-4\pi\int\frac{\rho\, d\omega}{r} = \int\frac{1}{r}\frac{dV}{dn}dS - V\int\frac{d\frac{1}{r}}{dn}dS, \qquad (\gamma)$$

dn denoting the element of normal at each point of the surface S measured outwards.

Now $\frac{d\frac{1}{r}}{dn}dS = -\frac{1}{r^2}\cos\theta\, dS$, where θ is the angle made by the normal at any point on S with the line joining this point to O; and exactly as in Art. 252, for an external point, the integral of this expression taken over the surface vanishes.

Hence (γ) gives for an external point

$$\int\frac{\rho\, d\omega}{r} = -\frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS. \qquad (\delta)$$

Secondly, let O be a point in the interior of S. In this case the distance, r, of a point in the volume from O becomes zero, and we cannot assert that $\nabla\frac{1}{r} = 0$; but this difficulty is avoided by surrounding O with an infinitely small spherical surface, and taking as the volume through which the integration is performed that contained between the given surface S and the surface of this sphere. In this way O ceases to be a point within the volume considered, and consequently $\nabla\frac{1}{r}$ is always $= 0$. We shall, however, have to perform the *surface* integration on the right-hand side of (γ) over the surface of this small sphere (on which we may consider the potential of the system as constant) as well as over S. Equation (γ) now becomes

$$-4\pi\int\frac{\rho\, d\omega}{r} = \int\frac{1}{r}\frac{dV}{dn}dS + \int\frac{1}{r'}\frac{dV'}{dn'}dS' - V\int\frac{d\frac{1}{r}}{dn}dS - V'\int\frac{d\frac{1}{r'}}{dn'}dS',$$

where V' is the potential of the whole system, M and M', at O, and dS' an element of surface of the small sphere, whose radius is r'. Now, $dS' = r'^2 ds$, where ds is the element of surface of a sphere of unit radius cut off by a cone whose base is dS' and vertex O; or, in other words, dS' is of the form $r'^2 \sin\theta\, d\theta\, d\phi$ (Art. 171). Hence, since r' is indefinitely small, $\int\frac{1}{r'}\frac{dV'}{dn'}dS' = 0$; and $V'\int\frac{d\frac{1}{r'}}{dn'}dS' = 4\pi V'$, since dn' is evidently $-dr'$. We have, then,

$$-4\pi\int\frac{\rho\, d\omega}{r} = \int\frac{1}{r}\frac{dV}{dn}dS + 4\pi V - 4\pi V',$$

or

$$-\int\frac{\rho\, d\omega}{r} = \frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS + V - V'.$$

At the point O denote the potential of the external mass, M, by V_e, and that of the inside mass M', by V_i. Then obviously $\int\frac{\rho\, d\omega}{r}$ is V_i; and this equation becomes, since $V' = V_i + V_e$,

$$-\frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS = V - V_e, \tag{ϵ}$$

while (δ) becomes

$$-\frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS = V_i. \tag{ζ}$$

EXAMPLES.

1. Any mass contained within one of its level surfaces may be distributed, according to a simple law, over this surface as a thin shell so as to produce the same effect as the given mass at all points outside the level surface.

Let the mass M' alone exist. Suppose that matter is distributed over S so that $k\rho$, the product of the density and thickness, or the surface density, if the distribution is truly superficial, at any point is equal to

$$-\frac{1}{4\pi}\frac{dV}{dn}.$$

Then the potential of this distribution at an external point, O, is

$$-\frac{1}{4\pi}\int\frac{1}{r}\frac{dV}{dn}dS.$$

But, by (ζ) of this Article, this is equal to V_i, the potential of M' at O.

Again, the whole quantity of matter on the surface is

$$-\frac{1}{4\pi}\int\frac{dV}{dn}dS;$$

but, remembering that dn is here measured outwards, this is equal to M' (Art. 252).

2. In the same case the superficial distribution which replaces M' produces constant potential at all points inside the surface.

This at once follows from (ϵ), since $V_e = 0$, there being no external mass. The constant internal potential is, therefore, the same as that on the surface.

3. Instead of the system, M, M', of which one portion, M', is internal, and the other external, to a given level surface, S, of the system, may be substituted a distribution on the surface itself, with these results:—

(1) The effect at all points outside S is the same as that of M'.

(2) The effect at all points inside S is equal and opposite to that of M. Let $k\rho$, or for an infinitely thin distribution, the surface density, σ, be $\frac{N}{4\pi}$ (Art. 263). Then these results follow at once from (ϵ) and (ζ), p. 451. In this case also the mass of the distribution $= M'$.

4. Let M and M' be two quantities, q and $-q'$, of opposite electricities concentrated at two points, I and I', and S their zero potential surface. Then, since the level surfaces are given by the equation

$$\frac{q}{r}-\frac{q'}{r'} = \text{const.},$$

the surface of zero potential is a sphere whose centre and radius are thus found:—divide the line II' internally at A so that $\frac{IA}{I'A}=\frac{q}{q'}$, and produce II' to C so that $\frac{IC}{CA}=\frac{q}{q'}$; then C is the centre and CA the radius of the sphere.

The distribution on this sphere which will produce the effect of I' at all external points and of I at all internal points is got by taking the surface density, σ, at any point P on the surface equal to $\frac{1}{4\pi}$ times the resultant of the forces $\frac{q}{IP^2}$, acting from I to P, and $\frac{q}{I'P^2}$, acting from P to I'.

Now this resultant, N, $=\frac{q}{IP^2}\cdot\frac{\sin IPI'}{\sin I'PC}$, since it acts in PC; and

$\dfrac{\sin IPI'}{\sin I'PC} = \dfrac{II'.CA}{IP.CI'}$; therefore $N = \dfrac{q}{IP^3}.\dfrac{II'.CA}{CI'}$. This is the resultant measured *inwards*; hence by Art. 263,

$$\sigma = -\frac{q}{IP^3}.\frac{II'.CA}{4\pi.CI'};$$

so that the density varies inversely as the cube of the distance from I.

It is easily proved that $CI.CI' = CA^2$, which shows that I and I' are inverse points with regard to the sphere.

5. When a conductor envelops electrical charges and also has electrical charges outside it, show that the internal charges together with the induced charge on the inner surface of the conductor form a system in equilibrium by itself, producing no action at any external point; and also that the external charges together with the induced charge on the outer surface of the conductor form a system in equilibrium by itself, producing no action at any internal point.

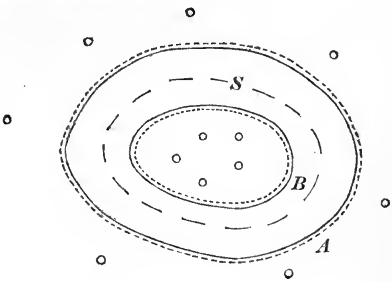

Fig. 254.

Let A and B (fig. 254) be the outer and inner surfaces of a conductor; let S be any closed surface drawn in the body of the conductor; and let the finely-dotted lines at the outside of the outer and the inside of the inner surface represent the induced charges existing on these surfaces.

The internal mass here consists of the given internal charges and the induced charge on the inner surface. Employ the equation (ζ), p. 451. Now all through the body of the conductor the resultant force $= 0$, therefore all over the surface S we have $\dfrac{dV}{dn} = 0$, therefore (ζ) gives

$$V_i = 0,$$

that is, the given internal charges and the induced charge on the inner surface give a constant zero potential at all external points, and therefore a zero force at all such points. Hence at external points the action of this internal system is null. Of course this system produces zero potential at all points in the body of the conductor also; for the surface S can be taken as close as we please to the inner surface of the conductor, and all points in the body of the conductor will be external to S. Employ now equation (ϵ). It refers to points internal to S, and it gives

$$V_e = V,$$

i. e., at all internal points the external charges together with the induced charge on the outer surface of the conductor produce a constant potential which is equal to that in the body of the conductor, and by what we have just proved this latter is entirely due to the external masses.

The conductor is therefore an *Electrical Screen* which protects charges (or other smaller charged conductors) inside it from the disturbing action of external electricity.

This explains why delicate electrical instruments are protected from external disturbance by screens of wire gauze connected with the ground. [If the conductor is connected with the ground, the constant potential inside due to the external electricity will be zero.]

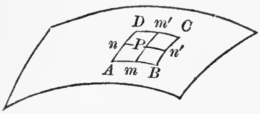

Fig. 255.

6. Calculate the surface-tension of an electrified soap-bubble.

When a membrane is acted on by forces of any kind, there will be along every line traced on the membrane a tendency of the two portions separated by this line to tear away from each other; in other words, one of these portions exercises on the other a set of internal forces along the line of separation.

In the neighbourhood of any point P of the membrane (fig. 255) consider a very small rectangular portion, $ABCD$, of the membrane isolated from the remainder. Then there will be forces exerted on its sides at their middle points, m, m', n, n', by the removed portion. These forces will, if the rectangle $ABCD$ is chosen at random, be oblique to its sides; but we shall afterwards see that it is always possible to choose the rectangle at P so that these forces are at right angles to the sides on which they act. Suppose this done. The amount of force exerted on AB is, of course, proportional to the length AB; so that if t_1 is the amount exerted on AB per unit of length, the force at m in the sense $m'\,m$ is $t_1 \times AB$. Similarly, if t_2 is the force per unit of length on AD, the force on AD is $t_2 \times AD$. The quantities t_1 and t_2 are called the *surface-tensions* at P perpendicular to AB and AD.

For the equilibrium of the rectangle resolve forces along the normal to its plane at P. Then, exactly as in Art. 203, if r_1 and r_2 are the radii of curvature of the curves mm' and nn', and N the amount of external normal force exerted at P per unit area, we have

$$N . AB \times AD = t_1 . AB \frac{mm'}{r_1} + t_2 . AD \frac{nn'}{r_2},$$

or

$$N = \frac{t_1}{r_1} + \frac{t_2}{r_2}.$$

In a soap-bubble t_1 and t_2 are evidently equal, and this equation becomes

$$N = \frac{2t}{r},$$

where t is its surface-tension and r its radius.

Now in an electrified bubble N consists of two parts—one an excess of air pressure inside over the air pressure outside, and the other the repulsion of the electricity on itself (Art. 264). Denote by p the intensity of the *excess* of air pressure and by σ the electrical density at P, and we have (see Art. 264),

$$p + 2\pi\sigma^2 = \frac{2t}{r}.$$

7. A spherical soap-bubble is electrified in such a manner that the internal pressure remains constant; find the relation between the densities of electrification when its volume has become n and m times its original value.

(Mr. Orchard, in the *Educational Times.*)

The external pressure presumably remaining constant, there will be a constant excess of pressure, p. Equate the work done by this pressure in enlarging the volume to the potential work of the electrification. Now if v is the volume of the bubble at any instant, the work done by the pressure in altering the volume by dv is pdv. Hence if Ω = original volume, the work done in the first electrification must be

$$p(n-1)\,\Omega.$$

But if V is the potential of this electrification and Q the charge, the energy of the electrification is (p. 442) $\frac{1}{2}VQ$; and evidently $V = \frac{Q}{r}$, if r is the radius of the bubble. Also if σ is the density,

$$Q = 4\pi r^2\sigma;$$

$$\therefore\quad p(n-1)\,\Omega = 8\pi^2 r^3\sigma^2.$$

But

$$\frac{4}{3}\pi r^3 = n\Omega;$$

$$\therefore\quad p(n-1) = 6n\pi\sigma^2.$$

Similarly,

$$p(m-1) = 6m\pi\sigma'^2.$$

if σ' is the density of the second electrification,

$$\therefore\quad \frac{\sigma}{\sigma'} = \sqrt{\frac{m(n-1)}{n(m-1)}}.$$

8. In the interior of a hollow conductor are placed given electrical charges, there being no external charges. Show that the induced charge on the outside of the outer surface consists of only one kind of electricity.

[$\frac{dV}{dn}$ has the same sign at all points on the surface. See ex. 32, p. 428.]

9. A spherical soap-bubble is electrified in such a manner that it is just in equilibrium when the pressures of the external and internal air are equal. Calculate the surface-tension in terms of the potential. (Mr. Orchard, *Educational Times.*)

Ans. $t = \frac{V^2}{16\pi r}.$

10. Find the law according to which a given uniform attracting bar may be distributed over any one of its level surfaces so as to produce the same attraction at all points outside the surface.

11. An electrified point I is placed in front of a given spherical conductor which is connected with the ground; find the density of the charge induced on the sphere at any point.

[In example 4 it is evident that if q' could be replaced by the superficial distribution on the sphere, this distribution could be replaced by an electrified point at I' whose charge is equal to that on the sphere; and the law of density is that given in example 4.]

272.] **Electric Images.** The theory which has been illustrated in these examples, and which is founded on equations (ϵ) and (ζ) of last Article, is Sir William Thomson's theory of *Electric Images.* If an electrified point, I, is held outside any conductor connected with the earth by a wire (so as to have zero potential) there will be induced on the conductor a certain charge of opposite electricity, the effect of which at all points outside the conductor is the same as that of an imagined electrified point, I', inside the conductor; and the effect of which at all points inside is equal and opposite to that of I.

If there is a continuous series of external points forming an electrified body, M, there will be a continuous series of imagined internal points forming an oppositely electrified body, M'; the latter is called the *electric image* of the former body in the conductor, and the distribution on the surface exerts at all external points the same effect as would be produced if this distribution were actually replaced by the image M'.

Mr. W. D. Niven has treated the subject of Electric Images on the basis of the secondary solution of Laplace's equation (ex. 14, p. 424). See the *Proceedings of the London Mathematical Society*, Dec. 1876.

CHAPTER XVI.

ANALYSIS OF STRAINS AND STRESSES.

273.] **Definitions of Strain and Stress.** When a natural solid (such as iron, wood, &c.), or any material medium, is not acted upon by any external forces, its particles assume certain determinate distances from each other, and the body is then said to be in its *natural state.* But when forces act on it either at its surface or throughout its mass, or when any disturbance is propagated through its interior, these natural distances between its particles suffer alteration, and the body is said to be in a state of *strain.* Thus a fluid exerting pressure, a medium propagating sound, and the luminiferous ether when it is propagating light are instances of a body in a state of strain.

The change of the natural distances between the particles is always attended by the production of internal forces, or, as they are called, *internal stresses,* or simply stresses; and these stresses will depend, as we shall see, both on the nature of the body and on the nature of the strain in any case.

SECTION I.

Analysis of Small Strains.

274.] **Displacements of a Rigid Body.** It has been already pointed out (Chap. X) that the general motion of a *rigid* body consists of a motion of translation which is the same for all its particles, together with a rotation round an axis through an angle which is the same for all its particles. These displacements do not alter the distance between any two particles of the body, and they are therefore unaccompanied by the development of stress in its interior. Stress results only from the *alteration* of distances between pairs of particles, and hence in treating of strains and stresses all displacements, whether of translation or of rotation, which are impressed, with common magnitude, upon all particles of the body, may be discarded; and again any such common displacement may be freely introduced if it is found convenient for the discussion.

275.] **Changes in Relative Co-ordinates.** Let a system of rectangular axes, Ox, Oy, Oz, (fig. 256) be fixed in space; through any point, P, in the natural solid under consideration let Px, Py, Pz be drawn parallel to the fixed axes. Let the particle at P be displaced to P', and suppose that the co-ordinates (x, y, z) of P referred to the axes through O are increased by small quantities, u, v, w, respectively. The co-ordinates of P' are therefore $x+u$, $y+v$, $z+w$. Now these displacements u, v, w depend on the position of the point P, i.e., they are functions of its co-ordinates depending on the law according to which the strain is produced. We have then, when the kind of strain is specified, some such equations as

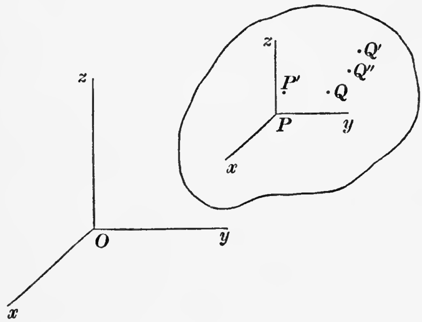

Fig. 256.

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z),$$

where f_1, f_2, f_3 are symbols of functionality.

Let Q be a particle very near P, and let its co-ordinates with reference to the axes drawn through P be (ξ, η, ζ). Then the displacements of Q parallel to the axes are obviously

$$f_1(x+\xi,\ y+\eta,\ z+\zeta),$$
$$f_2(x+\xi,\ y+\eta,\ z+\zeta),$$
$$f_3(x+\xi,\ y+\eta,\ z+\zeta),$$

that is, by Taylor's Theorem,

$$u+\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}, \quad v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz},$$
$$w+\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}.$$

Suppose Q to come to Q' by displacement. Then in considering the nature of the strain in the neighbourhood of P, we may, by last Article, impress on every particle of the body a motion of translation represented in magnitude and sense by $P'P$, so that P' will be brought back to P without in any way interfering with the strain of the solid. By drawing $Q'Q''$ equal and parallel to $P'P$, the particle which was originally at Q may now be considered to be at Q''; and a similar process is to be repeated for all other particles. The part of the strain, therefore, due to the alteration of the distance between P and Q will depend on the co-ordinates of Q'' with reference to Px, Py, Pz. These co-ordinates are, of course, the excesses of those of Q' over those of P'; and therefore the relative co-ordinates of Q'' are

$$\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}, \quad \xi\frac{dv}{dx}+\eta\left(1+\frac{dv}{dy}\right)+\zeta\frac{dv}{dz},$$
$$\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\left(1+\frac{dw}{dz}\right);$$

in other words, the changes, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ in ξ, η, ζ are

$$\Delta\xi=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}; \quad \Delta\eta=\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz};$$
$$\Delta\zeta=\xi\frac{dw}{dx}+\eta\frac{dw}{dx}+\zeta\frac{dw}{dz}. \qquad (a)$$

Cor. 1. *All particles near P which in the natural state lie in one plane will after strain also lie in one plane.* For if the co-ordinates of Q'' are denoted by ξ', η', ζ', we have

$$\xi'=\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}, \quad \eta'=\ldots, \quad \zeta'=\ldots,$$

which equations, being linear, give ξ, η, ζ linearly in terms of ξ', η', ζ'. Remembering that $\frac{du}{dx}$, $\frac{dv}{dy}$, ... are all small, these equations give $\xi=\xi'+$ small quantities of the order of $\frac{du}{dx}$, &c.; so that in any terms multiplied by $\frac{du}{dx}$, ... ξ' may be put for ξ, η' for η, and ζ' for ζ.

Hence we have, to the order of accuracy adopted,

$$\left.\begin{aligned} \xi &= \xi'\left(1-\frac{du}{dx}\right)-\eta'\frac{du}{dy}-\zeta'\frac{du}{dz}, \\ \eta &= -\xi'\frac{dv}{dx}+\eta'\left(1-\frac{dv}{dy}\right)-\zeta'\frac{dv}{dz}, \\ \zeta &= -\xi'\frac{dw}{dx}-\eta'\frac{dw}{dy}+\zeta'\left(1-\frac{dw}{dz}\right). \end{aligned}\right\} \qquad (1)$$

Therefore if all the points (ξ, η, ζ) lies in the plane

$$A\xi+B\eta+C\zeta+D=0,$$

all the points (ξ', η', ζ') will also lie in a plane. That is, every plane curve is strained into a plane curve in a different plane.

COR. 2. *All particles near P which in the natural state lie in one right line will after strain also lie in one right line.* For if we have

$$A\xi+B\eta+C\zeta+D=0 \quad \text{and} \quad A'\xi+B'\eta+C'\zeta+D'=0,$$

we shall have ξ', η', ζ', also satisfying two linear equations.

COR. 3. *Two parallel right lines in the natural state are changed into two parallel right lines (with a different direction) in the strained state.*

For one of the two lines being given by the equations

$$A\xi+B\eta+C\zeta+D=0, \;\; A'\xi+B'\eta+C'\zeta+D'=0,$$

the other will be given by two equations in which the terms D and D' alone are altered. But by substituting for ξ, η, ζ their values in terms of ξ', η', ζ', the values of D and D' do not influence the direction cosines of the line into which any one is converted by strain.

276] **Elongation in any Direction.** DEF. Supposing P and Q to be, as before, two particles in the natural state of the body, the *elongation* in the direction PQ is defined as the ratio of the change produced by strain in the distance between these same particles to the original distance between them. Hence the elongation in the direction PQ is $\dfrac{PQ''-PQ}{PQ}$, or $\dfrac{\Delta\rho}{\rho}$, if ρ denotes PQ, and $\Delta\rho$ the change in ρ.

Now
$$\rho^2 = \xi^2+\eta^2+\zeta^2,$$
$$\therefore \quad \rho\Delta\rho = \xi\Delta\xi+\eta\Delta\eta+\zeta\Delta\zeta;$$

or if we substitute for $\Delta\xi$, $\Delta\eta$, and $\Delta\zeta$ their values from last Article,

$$\rho\Delta\rho = \xi^2\frac{du}{dx} + \eta^2\frac{dv}{dy} + \zeta^2\frac{dw}{dz} + \xi\eta\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \eta\zeta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \zeta\xi\left(\frac{dw}{dx} + \frac{du}{dz}\right).$$

Let the cosines of the angles made by PQ with Px, Py, Pz be l, m, n, respectively, let

$$\frac{du}{dx} = a,\quad \frac{dv}{dy} = b,\quad \frac{dw}{dz} = c,$$

$$\frac{du}{dy} + \frac{dv}{dx} = 2s_3,\quad \frac{dv}{dz} + \frac{dw}{dy} = 2s_1,\quad \frac{dw}{dx} + \frac{du}{dz} = 2s_2,$$

and denote the elongation by ϵ; then the last equation gives

$$\epsilon = al^2 + bm^2 + cn^2 + 2lms_3 + 2mns_1 + 2nls_2. \tag{1}$$

The elongation in any direction may be graphically represented as follows:

Construct at P the quadric surface whose equation referred to the spatially fixed axes Px, Py, Pz is

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = k^2 \tag{2}$$

where k is any constant linear magnitude. If r is the length of the line PQ intercepted by this surface, we have

$$r^2(al^2 + bm^2 + cn^2 + 2lms_3 + 2mns_1 + 2nls_2) = k^2;$$

$$\therefore\quad \epsilon = \frac{k^2}{r^2}, \tag{3}$$

or *the elongation in any direction varies inversely as the square of the radius vector of the Elongation Quadric in this direction,* if we agree to call the above surface the *Elongation Quadric.*

It is possible, however, that equation (2) may fail to represent the elongation in *all* directions. For there may be contraction (negative elongation) in some directions, and then (2) will represent a hyperbolic surface, the radii of which will give as in (3) the *elongations,* while the contractions must be given by constructing the surface

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = -k^2, \tag{4}$$

which is the hyperboloid conjugate to that which gives the elongations.

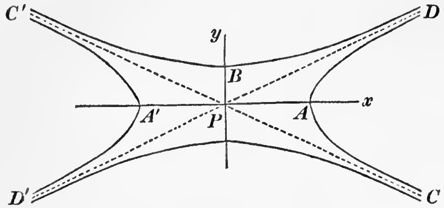

Fig. 257.

Unless, then, all lines are contracted or all lines elongated, there will really be two quadrics required, one to represent elongations and the other to represent contractions.

For example, consider the simple case in which the strain is made by drawing out all lines perpendicular to the plane yz in the same proportion, and contracting all lines perpendicular to the plane xz in the same proportion; so that

$$u = ax, \quad v = -by, \quad w = 0.$$

Then the elongation is given by the equation $\epsilon = al^2 - bm^2$. Now this expression is negative when $bm^2 > al^2$, and if we construct a surface whose equation is $a\xi^2 - b\eta^2 = 0$, i. e., two planes through the axis of z, this surface will form the boundary between lines which are elongated and lines which are contracted. The elongations are given by the radii of the surface $a\xi^2 - b\eta^2 = k^2$, a hyperbolic cylinder, the section of which by the plane xy is represented in fig. 257 by the curve $(DAC,\ D'A'C')$; and the contractions by the conjugate surface $b\eta^2 - a\xi^2 = k^2$, which is represented by $(DBC',\ D'B'C)$; the planes of no elongation or contraction being the asymptotic planes, DD', CC', of these surfaces.

All lines through P along which the elongation is the same lie on a cone whose equation is easily found from (1). For, putting $\epsilon\,(l^2+m^2+n^2)$ for ϵ, we have

$$(a-\epsilon)\,l^2 + (b-\epsilon)\,m^2 + (c-\epsilon)\,n^2 + 2\,s_3\,lm + 2\,s_1\,mn + 2\,s_2\,nl = 0;$$

and if ξ, η, ζ are the co-ordinates of any point on the line (l, m, n), we have $l : m : n = \xi : \eta : \zeta$; therefore this equation gives

$$(a-\epsilon)\,\xi^2 + (b-\epsilon)\,\eta^2 + (c-\epsilon)\,\zeta^2 + 2\,s_3\,\xi\eta + 2\,s_1\eta\zeta + 2\,s_2\zeta\xi = 0,$$

which, if ϵ is constant, denotes a cone whose vertex is P. This is called the *cone of equal elongation*. If ϵ is taken $= 0$, we get a cone of no elongation, and it is evidently (when real) the asymptotic cone both of the Elongation Quadric and of the Compression Quadric.

Cor. 1. *The elongations in the directions of the axes of* x, y, z *are, respectively,* a, b, c, or $\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}$.

Cor. 2. The elongation is the same along all parallel lines in the neighbourhood of P. For if R is any point very near P, the value of ϵ along a direction (l, m, n) at R is got by using the values of a, b, c, s_1, s_2, s_3 at R in equation (1). But these values at R differ from the values at P by infinitesimals of the second order. Therefore, &c.

Cor. 3. *Any small parallelogram or parallelopiped in the natural state in the neighbourhood of* P *is changed into another parallelogram or parallelopiped by the strain.*

For (Cor. 3, Art. 275) any two parallel lines are strained into two parallel lines, and (Cor. 2, Art. 276) they are equally elongated. Therefore, &c.

Cor. 4. *A small circle very near* P *in any plane is strained into an ellipse in a different plane.*

For, let AQB (fig. 258) be a circle in the natural state; let OA and OB be any two rectangular radii, Q any point on the circle, and QM and QN perpendiculars on OA and OB. Let the lines OA and OB become oa and ob (in a different plane) by the strain, and let Q become q. The circle will become a curve in the plane of oa and ob by Cor. 1, Art. 275. Also if qm and qn are drawn parallel to ob and oa, the lines QM and QN will become qm and qn; for M must become some point on oa (Cor. 2, Art. 275), and OB and QM must become parallel lines (Cor. 3, Art. 275).

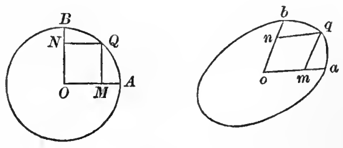

Fig. 258.

Again, if ϵ is the elongation along OA,

$$oa = (1+\epsilon)\, OA \text{ and } om = (1+\epsilon)\, OM;$$

$$\therefore \quad \frac{OM}{OA} = \frac{Om}{Oa};$$

similarly
$$\frac{ON}{OB} = \frac{on}{ob}.$$

But $$\frac{OM^2}{OA^2} + \frac{ON^2}{OB^2} = 1,$$

therefore $$\frac{om^2}{oa^2} + \frac{on^2}{ob^2} = 1,$$

which shows that the curve on which q lies is an ellipse having the lines oa and ob for conjugate semi-diameters.

Hence every pair of rectangular radii of a circle is strained into a pair of semi-conjugate diameters of an ellipse; and since among these latter there is one rectangular pair (the axes of the ellipse), it follows that *some two rectangular diameters of the circle are strained into two rectangular lines.* Hence in every plane near P can always be found two rectangular lines which are strained into two rectangular lines.

Cor. 5. *Any two small coplanar areas in the natural state are strained into two coplanar areas having the same ratio to each other as the unstrained areas.*

For let CAB and $C'A'B'$ be any two elementary rectangles in the same plane near P such that AB is parallel to $A'B'$ and AC parallel to $A'C'$. Then by Cor. 3 these will be strained into two parallelograms, cab and $c'a'b'$, such that ab is parallel to $a'b'$ and ac to $a'c'$.

Hence $$\frac{\text{area } cab}{\text{area } c'a'b'} = \frac{ac \times ab}{a'c' \times a'b'}.$$

Let ϵ be the elongation in the direction AB and ϵ' that in the direction AC; then

$$ab = (1+\epsilon)\,AB,\quad a'b' = (1+\epsilon)\,A'B';$$
$$ac = (1+\epsilon')\,AC,\quad a'c' = (1+\epsilon')\,A'C';$$

therefore $$\frac{\text{area } cab}{\text{area } c'a'b'} = \frac{AC \times AB}{A'C' \times A'B'} = \frac{\text{area } CAB}{\text{area } C'A'B'}.$$

Now, whatever be the two areas, they can each be broken up into an infinitely great number of small parallel rectangular strips, and the ratios of the strained areas of these strips being the same as those of the unstrained, the whole strained areas are to each other as the unstrained ones.

Cor. 6. *Every small sphere in the natural state is strained into a small ellipsoid.* This is evident from Cor. 4, since the sphere, being a surface every section of which is a circle, must alter into a surface every section of which is an ellipse. Nevertheless for clearness we may repeat the proof of that Cor. Let OA, OB, OC be any three rectangular semi-diameters of the sphere, Q any point on the sphere, QR a line parallel to OC terminated by the plane

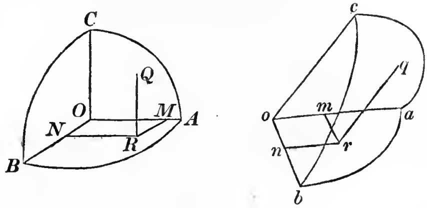

Fig. 259.

OAB, and RM, RN parallels to OB and OA. Let the lines OA, OB, OC be strained into oa, ob, oc, and Q to q. Then, by Cor. 3, Art. 276, QR, RM, and RN will be strained into qr, rm, and rn which are parallels to oc, ob, and oa terminated by the planes oab, oac, and obc. Also by Cor. 2,

$$\frac{oa}{OA} = \frac{om}{OM}, \text{ i. e., } \frac{om}{oa} = \frac{OM}{OA};$$

similarly,
$$\frac{on}{ob} = \frac{ON}{OB}, \quad \frac{qr}{oc} = \frac{QR}{OC}.$$

But
$$\frac{OM^2}{OA^2} + \frac{ON^2}{OB^2} + \frac{QR^2}{OC^2} = 1,$$

therefore
$$\frac{om^2}{oa^2} + \frac{on^2}{ob^2} + \frac{qr^2}{oc^2} = 1,$$

which shows that the surface on which q lies is an ellipsoid having oa, ob, oc for a system of conjugate semi-diameters.

Hence *every rectangular set of radii of a sphere in the natural state is strained into a system of conjugate semi-diameters of the ellipsoid into which the sphere is changed;* and it follows that there is one rectangular set which is strained into a rectangular set and altered in directions, the latter being the axes of the ellipsoid into which the sphere is strained.

COR. 7. *Any two small volumes in the natural state are strained into two small volumes which bear the same ratio to each other as the unstrained volumes.* The proof of this proceeds exactly as in Cor. 5.

277.] **Lines of no Rotation.** Let us enquire whether, with the given strain, it is possible to find a particle Q, in the natural state of the body, such that its displaced position, Q'', shall be on the line PQ. If this is so, all particles (near P) on the line PQ will retain the same direction with respect to P; i.e., the line PQ will not suffer rotation by the strain.

The direction cosines of PQ'' are $\dfrac{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}}{PQ''}, \ldots,$ and those of PQ are $\dfrac{\xi}{PQ}, \ldots.$ Hence if these are the same,

$$\frac{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}}{PQ''}=\frac{\xi}{PQ},$$

with two similar equations. Now $PQ''=(1+\epsilon)PQ$; hence

$$\left(\frac{du}{dx}-\epsilon\right)\xi+\frac{du}{dy}\cdot\eta+\frac{du}{dz}\cdot\zeta=0,$$

with two similar equations; or if l, m, n be the direction cosines of PQ, a line of no rotation,

$$\left.\begin{aligned} l\left(\frac{du}{dx}-\epsilon\right)+m\frac{du}{dy}+n\frac{du}{dz}&=0,\\ l\frac{dv}{dx}+m\left(\frac{dv}{dy}-\epsilon\right)+n\frac{dv}{dz}&=0,\\ l\frac{dw}{dx}+m\frac{dw}{dy}+n\left(\frac{dw}{dz}-\epsilon\right)&=0.\end{aligned}\right\} \quad (1)$$

By eliminating l, m, n from these equations, we obtain the cubic equation for ϵ,

$$\begin{vmatrix} \frac{du}{dx}-\epsilon, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}-\epsilon, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz}-\epsilon \end{vmatrix}=0, \quad (2)$$

which gives necessarily one real value of ϵ and may give three real values.

Hence *in the small general strain of an elastic solid there is at every point at least one line of no rotation.*

278.] **Change of Inclination of Two Lines.** In the unstrained state let there be two points, Q_1 and Q_2, very near P, and let ϕ be the angle between the lines PQ_1 and PQ_2. We propose to find the angle between the lines into which these are strained. Let $(\xi_1\ \eta_1\ \zeta_1)$ and $(\xi_2\ \eta_2\ \zeta_2)$ be the co-ordinates of Q_1 and Q_2 with reference to Px, Py, Pz (fig. 256); and supposing that the strained positions of Q_1 and Q_2 are Q_1'' and Q_2'', whose co-ordinates are $(\xi_1' \, \eta_1' \, \zeta_1')$ and $(\xi_2' \, \eta_2' \, \zeta_2')$, we have, by Art. 275,

$$\xi_1' = (1+a)\,\xi_1 + \eta_1 \frac{du}{dy} + \zeta_1 \frac{du}{dz}, \quad \eta_1' = \xi_1 \frac{dv}{dx} + (1+b)\,\eta_1 + \zeta_1 \frac{dv}{dz},$$

$$\zeta_1' = \xi_1 \frac{dw}{dx} + \eta_1 \frac{dw}{dy} + (1+c)\,\zeta_1,$$

$$\xi_2' = (1+a)\,\xi_2 + \eta_2 \frac{du}{dy} + \zeta_2 \frac{du}{dz}, \quad \eta_2' = \xi_2 \frac{dv}{dx} + (1+b)\,\eta_2 + \zeta_2 \frac{dv}{dz},$$

$$\zeta_2' = \xi_2 \frac{dw}{dx} + \eta_2 \frac{dw}{dy} + (1+c)\,\zeta_2.$$

Hence neglecting squares and products of $a, b, \ldots, \dfrac{du}{dy}, \ldots$, we have

$$\xi_1'\xi_2' + \eta_1'\eta_2' + \zeta_1'\zeta_2' = \xi_1\,\xi_2 + \eta_1\,\eta_2 + \zeta_1\,\zeta_2$$
$$+ 2(a\,\xi_1\,\xi_2 + b\,\eta_1\eta_2 + c\,\zeta_1\,\zeta_2) + (\xi_1\eta_2 + \xi_2\,\eta_1)\left(\frac{du}{dy} + \frac{dv}{dx}\right)$$
$$+ (\eta_1\,\zeta_2 + \eta_2\,\zeta_1)\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + (\zeta_1\,\xi_2 + \zeta_2\,\xi_1)\left(\frac{dw}{dx} + \frac{du}{dz}\right).$$

If ϕ' is the angle between PQ_1'' and PQ_2'',

$$\cos\phi' = \frac{\xi_1'\xi_2' + \eta_1'\eta_2' + \zeta_1'\zeta_2'}{PQ_1''\,.\,PQ_2''};$$

so that if ϵ_1 and ϵ_2 are the elongations in the directions PQ_1 and PQ_2, and $(l_1\,m_1\,n_1)$, $(l_2\,m_2\,n_2)$ the direction cosines of the lines PQ_1 and PQ_2, the above equation gives

$$(1+\epsilon_1)(1+\epsilon_2)\cos\phi' = \cos\phi + 2(a\,l_1\,l_2 + b\,m_1\,m_2 + c\,n_1\,n_2)$$
$$+ 2s_3(l_1\,m_2 + l_2\,m_1) + 2s_1(m_1\,n_2 + m_2\,n_1) + 2s_2(n_1\,l_2 + n_2\,l_1);$$

or dividing out by $(1+\epsilon_1)(1+\epsilon_2)$,

$$\cos\phi' = \cos\phi\,(1 - \epsilon_1 - \epsilon_2) + 2(a\,l_1 l_2 + b m_1 m_2 + c\,n_1 n_2)$$
$$+ 2s_3(l_1\,m_2 + l_2\,m_1) + 2s_1(m_1\,n_2 + m_2\,n_1) + 2s_2(n_1\,l_2 + n_2\,l_1), \quad (1)$$

the products of the elongations and the small quantities $a, b, \ldots$ $s_3, \ldots$ being rejected. The change in the cosine of the angle between any two rectangular lines is got by putting $\phi = \dfrac{\pi}{2}$. Denoting this change by $2s$, we have

$$s = a\,l_1\,l_2 + b\,m_1\,m_2 + c\,n_1\,n_2 + s_3(l_1\,m_2 + l_2\,m_1)$$
$$+ s_1(m_1\,n_2 + m_2\,n_1) + s_2(n_1\,l_2 + n_2\,l_1). \quad (2)$$

COR. 1. *The quantities* $2s_3$, $2s_1$, $2s_2$ *are, respectively, the cosines of the angles between the strained positions of the axes of* (x, y), (y, z), (z, x).

Cor. 2. The result at the end of Cor. 6, Art. 276, easily follows from the value of $\cos\phi'$ in (1). For if $\phi = \frac{\pi}{2}$ and also $\phi' = \frac{\pi}{2}$, the directions of the lines PQ_1 and PQ_2 are connected by the equation

$$a\,l_1 l_2 + bm_1 m_2 + cn_1 n_2 + s_3(l_1 m_2 + l_2 m_1) + s_1(m_1 n_2 + m_2 n_1) + s_2(n_1 l_2 + n_2 l_1) = 0,$$

or $$(al_1 + s_3 m_1 + s_2 n_1)\,l_2 + (s_3 l_1 + bm_1 + s_1 n_1)m_2 + (s_2 l_1 + s_1 m_1 + cn_1)\,n_2 = 0,$$

which shows that PQ_1 and PQ_2 are conjugate diameters of the quadric $$a\xi^2 + b\eta^2 + c\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = k^2,$$
k being any constant.

Cor. 3. *The quantities* $b+c$, $c+a$, $a+b$ *are, respectively, the areal dilatations, that is, the ratios of increase of small areas to their original values in the planes of* yz, zx, xy.

For since all small areas near P in the plane yz are altered in the same ratio, to determine this ratio we may take a small rectangle with lengths m and n along Py and Pz. The sides of this become (Cor. 1, Art. 276) $(1+b)\,m$ and $(1+c)\,n$, and, the cosine of the angle between them becoming s_1, the sine of this angle is 1 to the order of accuracy adopted. Hence the new area is $$(1+b)(1+c)\,mn, \quad \text{or} \quad mn + (b+c)\,mn;$$
or if A and A' are the unstrained and strained areas,

$$\frac{A'-A}{A} = b + c = \text{areal dilatation.}$$

Similarly for dilatations in the other planes.

Cor. 4. *The quantity* $a+b+c$ *is the cubical dilatation, that is, the ratio of the increase of any small volume at* P *to the unstrained magnitude of this volume.*

For since all small volumes near P are increased in the same ratio (Cor. 7, Art. 276), to determine this ratio we may take a small and rectangular parallelopiped with edges m, n, p along the axes Px, Py, Pz. These edges become $(1+a)\,m$, $(1+b)\,n$, $(1+c)\,p$, respectively, and the sines of the angles between them are each 1, to the order adopted. Hence the strained volume is

$$(1+a)(1+b)(1+c)\,mnp, \quad \text{or} \quad mnp + (a+b+c)\,mnp;$$

so that if V and V' are the unstrained and strained volumes,

$$\frac{V'-V}{V} = a + b + c.$$

COR. 5. We conclude at once that, *whatever system of rectangular lines be drawn through P, the sum, $a+b+c$, of the elongations along them is constant.*

For the ratio in which any volume is increased cannot depend on any particular set of axes of reference. This also follows from the value of ϵ given in Art. 276.

279. **Problem.** Given the components of a strain with reference to one set of rectangular axes, to find the components of the same strain with reference to any other set of rectangular axes.

The components with reference to a set of axes, Px, Py, Pz, being a, b, c, $2s_3$, $2s_1$, $2s_2$, $\left(\text{or } \frac{du}{dx}, \ldots, \frac{du}{dy}+\frac{dv}{dz}, \ldots\right)$, we wish to find them with reference to a set, Px', Py', Pz', whose direction cosines are (l, m, n), (l', m', n'), (l'', m'', n''), respectively.

The value of $\frac{du'}{dx'}$ is simply the elongation in the direction (l, m, n). Hence

$$a' = al^2+bm^2+cn^2+2s_3\,lm+2s_1\,mn+2s_2\,nl,$$

with exactly similar values of b' and c'.

Again, $\frac{du'}{dy'}+\frac{dv'}{dx'}$ is simply the cosine of the angle between the strained positions of the two lines Px', Py'; hence, by (2) of last Art.,

$$s_3' = all'+bmm'+cnn'+s_3(lm'+l'm)+s_1(mn'+m'n)$$
$$+s_2(nl'+n'l),$$

with exactly similar values of s_1' and s_2'.

Two strains having reference to two distinct sets of axes are equivalent when each produces the other; and either may be substituted for the other.

280.] **The Strain Ellipsoid.** It has been already proved (Cor. 6, Art. 276) that a small sphere in the unstrained state of the body is converted by the strain into an ellipsoid. This latter surface is called the *Strain Ellipsoid* of the given strain. We here exhibit its deduction analytically.

Let the point Q (fig. 256) be any point on a sphere of radius r and centre P. Then, Px, Py, Pz being axes of co-ordinates,

$$\xi^2+\eta^2+\zeta^2 = r^2.$$

It is required to find the surface traced out by Q'', the strained position of Q, as the latter varies on the surface of the sphere. The co-ordinates of Q'' being, as in Art. 275, ξ', η', ζ', we have by squaring and adding equations (1), p. 460,

$$\xi^2+\eta^2+\zeta^2=\left(1-2\frac{du}{dx}\right)\xi'^2+\left(1-2\frac{dv}{dy}\right)\eta'^2+\left(1-2\frac{dw}{dz}\right)\zeta'^2$$
$$-4s_3\xi'\eta'-4s_1\eta'\zeta'-4s_2\zeta'\xi',$$

or $$(a-\tfrac{1}{2})\xi'^2+(b-\tfrac{1}{2})\eta'^2+(c-\tfrac{1}{2})\zeta'^2+2s_3\xi'\eta'+2s_1\eta'\zeta'$$
$$+2s_2\zeta'\xi'+\frac{r^2}{2}=0, \quad (1)$$

which is a quadric, and necessarily an ellipsoid since a sphere must be strained into a closed surface. As we have been using $\xi\eta\zeta$ to denote running co-ordinates, we may without confusion write the equation of the strain ellipsoid

$$(\tfrac{1}{2}-a)\xi^2+(\tfrac{1}{2}-b)\eta^2+(\tfrac{1}{2}-c)\zeta^2-2s_3\xi\eta-2s_1\eta\zeta-2s_2\zeta\xi$$
$$-\tfrac{1}{2}r^2=0. \quad (2)$$

281.] **Principal Axes and Principal Elongations of a Strain.** *The principal axes of a strain at any point P are those three rectangular lines* (Cor. 6, Art. 276) *which become by the strain the axes of the strain ellipsoid;* and since in general the direction of a line is altered by the strain, the principal axes of the strain are, in general, rotated by the strain about the point P.

The principal elongations of a strain at any point P are the elongations along the principal axes. We shall denote these by e_1, e_2, e_3.

Cor. If the axes of co-ordinates at P are taken in the directions of the axes of the strain ellipsoid, the quantities s_1, s_2, and s_3 are all zero, as is evident from (2) of last Art., and the equation of this ellipsoid will be

$$(\tfrac{1}{2}-e_1)\xi^2+(\tfrac{1}{2}-e_2)\eta^2+(\tfrac{1}{2}-e_3)\zeta^2-\tfrac{1}{2}r^2=0. \quad (a)$$

282.] **Pure Strain.** A strain is said to be *pure* when the lines at P which become the axes of the strain ellipsoid are unaltered in their directions by the strain.

283.] **Conditions for a Pure Strain.** Since $a, b, c, \dots$ are infinitesimals of the first order, it follows from the value of ϵ given by equation (1), p. 461, that the elongation along the direction PQ'' (fig. 256, p. 458) may be taken as equal to the elongation along the direction PQ; so that if ϵ is the elongation in the direction of any radius vector of the strain ellipsoid, we have

$$\rho = r(1+\epsilon),$$

where ρ is the length of this radius vector and r the radius of the sphere which becomes by strain the strain ellipsoid.

Hence if the axes of this ellipsoid are α, β, γ, we have

$$\alpha = r(1+e_1),$$
$$\beta = r(1+e_2),$$
$$\gamma = r(1+e_3).$$

Now if l, m, n are the direction cosines of any axis, it is well known (see Salmon's *Geometry of Three Dimensions*, or Frost's *Solid Geometry*) that

$$\left.\begin{aligned} (\tfrac{1}{2}-a)\,l - s_3\,.\,m - s_2\,.\,n &= \lambda l, \\ -s_3\,.\,l + (\tfrac{1}{2}-b)\,m - s_1\,.\,n &= \lambda m, \\ -s_2\,.\,l - s_1\,.\,m + (\tfrac{1}{2}-c)\,n &= \lambda n\,; \end{aligned}\right\} \qquad (1)$$

the three values $(\lambda_1, \lambda_2, \lambda_3)$ of λ obtained from these equations being such that the equation of the ellipsoid referred to its own axes would be

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 - \frac{r^2}{2} = 0.$$

Hence $\quad \lambda_1 = \dfrac{r^2}{2\alpha^2} = \tfrac{1}{2} - e_1\,; \quad \lambda_2 = \tfrac{1}{2} - e_2\,; \quad \lambda_3 = \tfrac{1}{2} - e_3.$

Therefore if ϵ stands for any one of the principal elongations, e_1, e_2, e_3, the equations (1) become, for the direction of any *axis*,

$$\left.\begin{aligned} (a-\epsilon)\,l + s_3\,m + s_2\,n &= 0, \\ s_3\,l + (b-\epsilon)\,m + s_1\,n &= 0, \\ s_2\,l + s_1\,m + (c-\epsilon)\,n &= 0. \end{aligned}\right\} \qquad (2)$$

Now if there are three unrotated lines, they are given by equations (1), Art. 277; and if the same lines are determined by (2), we must have

$$\frac{du}{dy} = \frac{dv}{dx} = s_3\,; \quad \frac{du}{dz} = \frac{dw}{dx} = s_2\,; \quad \frac{dv}{dz} = \frac{dw}{dy} = s_1\,;$$

and the conditions for pure strain are that the displacements u, v, w satisfy the equations

$$\frac{du}{dy} - \frac{dv}{dx} = 0, \quad \frac{du}{dz} - \frac{dw}{dx} = 0, \quad \frac{dv}{dz} - \frac{dw}{dy} = 0. \qquad (3)$$

These are the well-known conditions that the expression

$$u\,dx + v\,dy + w\,dz,$$

in which u, v, w are functions of x, y, z, should be the perfect differential of a single function, $\phi\,(x, y, z)$. When this function exists, i.e., when the strain is pure, it is called the *Displacement Potential* of the strain.

Hence the components, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, of the strain (given in Art. 275) become when the strain is pure

$$\Delta\xi = a\xi + s_3\eta + s_2\zeta,$$
$$\Delta\eta = s_3\xi + b\eta + s_1\zeta,$$
$$\Delta\zeta = s_2\xi + s_1\eta + c\zeta,$$

i. e., the coefficient of η in $\Delta\xi$ is the same as the coefficient of ξ in $\Delta\eta$, &c.; and this is the distinguishing character of a pure strain. A pure strain is also called an *irrotational* strain.

The values of the principal elongations of a strain are the roots of the cubic equation

$$\begin{vmatrix} a-\epsilon, & s_3, & s_2 \\ s_3, & b-\epsilon, & s_1 \\ s_2, & s_1, & c-\epsilon \end{vmatrix} = 0,$$

or $$\epsilon^3 - (a+b+c)\epsilon^2 + (ab+bc+ca-s_1^2-s_2^2-s_3^2)\epsilon$$
$$+ a s_1^2 + b s_2^2 + c s_3^2 - abc - 2 s_1 s_2 s_3 = 0.$$

284.] **Theorem.** *Every strain can be resolved into a pure strain and a rotation.* By a *rotation* here is meant such a displacement as a *rigid* body undergoes in turning round an axis, and we propose to show that the general small strain at any point P of a body, may be produced by two operations, viz., first holding fixed in directions the principal axes of the strain and straining the body to a certain extent, and then rotating it as a rigid body about a certain axis.

It has been shown (p. 293) that if a rigid body receives small angular displacements $\delta\theta_1, \delta\theta_2, \delta\theta_3$ round three fixed rectangular axes, the displacements of the co-ordinates, ξ, η, ζ of any point in it are

$$\zeta\delta\theta_2 - \eta\delta\theta_3, \quad \xi\delta\theta_3 - \zeta\delta\theta_1, \quad \eta\delta\theta_1 - \xi\delta\theta_2. \tag{1}$$

(Such a displacement has, of course, no displacement potential; for if these displacements are denoted by u, v, w, we have $\frac{du}{d\eta} - \frac{dv}{d\xi}$ equal to $-2\delta\theta_3$ and not equal to zero.)

Now the component, $\Delta\xi$, of the displacement of Q along the axis Px is (Art. 275),

$$a\xi + \eta\frac{du}{dy} + \zeta\frac{du}{dz};$$

and this $$= a\xi + \tfrac{1}{2}\Big(\frac{du}{dy} + \frac{dv}{dx}\Big)\eta + \tfrac{1}{2}\Big(\frac{du}{dz} + \frac{dw}{dx}\Big)\zeta + \tfrac{1}{2}\Big(\frac{du}{dz} - \frac{dw}{dx}\Big)\zeta$$
$$- \tfrac{1}{2}\Big(\frac{dv}{dx} - \frac{du}{dy}\Big)\eta.$$

Hence, with the same values of s_1, s_2, s_3 as before, we have

$$\Delta\xi = a\xi + s_3\eta + s_2\zeta + \left\{\tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right)\zeta - \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)\eta\right\},$$

$$\Delta\eta = s_3\xi + b\eta + s_1\zeta + \left\{\tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)\xi - \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)\zeta\right\},$$

$$\Delta\zeta = s_2\xi + s_1\eta + c\zeta + \left\{\tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)\eta - \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right)\xi\right\}.$$

A comparison with (1) shows that the portions in brackets in these expressions denote rotations, as of a rigid body, about the axes through the small angles

$$\delta\theta_1 = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \quad \delta\theta_2 = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \quad \delta\theta_3 = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right), \quad (2)$$

which are equivalent to a rotation through $\sqrt{(\delta\theta_1)^2 + (\delta\theta_2)^2 + (\delta\theta_3)^2}$ about one line (p. 292); while the portions of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ outside the brackets denote a pure strain by Art. 283.

If the axes of reference, Px, Py, Pz, are chosen in the directions of the principal axes, the pure portion of the strain will be expressed by $\Delta\xi = e_1\xi$, $\Delta\eta = e_2\eta$, $\Delta\zeta = e_3\zeta$,

i.e., the pure strain is produced simply by multiplying the coordinates of every particle by the numbers $1+e_1$, $1+e_2$, $1+e_3$. A *simple elongation* of a body in a direction perpendicular to any plane means the drawing out from the plane of every particle through a distance proportional to the perpendicular from the particle on the plane, so that those particles farthest from the plane in the natural state are most drawn away, but all in the same proportion to their original distances from it.

By this Article we see that *every small strain at a point P can be produced by three successive simple elongations followed by a rotation, as of a rigid body, about an axis through P.*

285.] **Significations of s_1, s_2, s_3.** Let the axes Px and Py become by strain Px'' and Py'', fig. 260. (Of course it is supposed, as in Art. 275, that P is brought back to its original position after the strain.) All particles in the plane of Px and Py originally are in the (different) plane of Px'' and Py'' after the strain; and

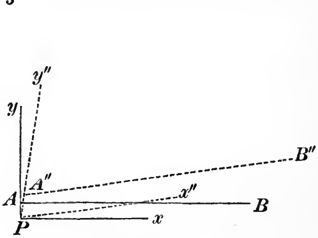

Fig. 260.

if A is a particle on the axis of y and AB a line parallel to

Px, the line of particles AB will become (Cor. 3, Art. 275) a line of particles $A''B''$ parallel to Px''. Let fall a perpendicular, $A''p$, from A'' on Px''. Then the particle (A'') which was at A has advanced in front of P parallel to the line Px'' through the distance Pp. Now $Pp = PA'' \cos x''Py'' = 2PA''.s_3$ (Cor. 1, Art. 278); and $PA'' = (1+b)PA$; therefore $Pp = 2(1+b)s_3.PA$; or, neglecting the product bs_3,

$$\frac{Pp}{PA} = 2s_3.$$

Hence the quantity $2s_3$ is the rate (per unit of distance between the two lines) at which particles on any line AB parallel to Px have slid beyond the corresponding particles on Px. Evidently it is also the rate at which sliding has taken place between particles on Py and lines parallel to Py.

Or again, imagine a little parallelopiped at P having its edges along the lines Px, Py, Pz. Then $2s_3$ is the rate at which the face parallel to that in the plane xz has slid in front of the latter; or the rate at which the face parallel to the plane yz has slid in front of the face in the plane yz.

Similarly for the values of s_1 and s_2.

Def. When a plane is held fixed in a body and all planes in the body parallel to it are slid in the same direction and sense parallel to the fixed plane, each through a distance proportional to its distance from the fixed plane, the strain so produced is called a *shearing strain*.

Those planes which are nearest to the fixed plane are least displaced, and those which are farthest from it are most displaced.

The ratio of the distance through which any plane has slid to its distance from the fixed plane is called the *amount* of the shear. Hence *the quantities* $2s_1$, $2s_2$, $2s_3$ *are the small shears of the axes of* (y,z), (z,x), (x,y) *respectively, at the point* P.

From fig. 260 it is clear that the change in the cosine of the angle between any two lines at right angles in the natural state is the shear in their plane of lines parallel to either.

286.] **Shearing Strain.** The two fundamental kinds of strain of what are called *isotropic* bodies (i. e., bodies whose constitution is the same at all points and in all directions round every point) are *Cubical Dilatation* and *Shearing Strain*. We propose, therefore, to consider this latter more particularly here.

Confining our attention to a shear, $2s_3$, of the two rectangular lines Ox and Oy, the elongation quadric would be

$$2s_3\,\xi\eta = k^2,$$

the axes of co-ordinates being the lines Ox and Oy.

But this equation denotes a hyperbola in the plane xy referred to its asymptotes; and if we alter the axes of co-ordinates to the axes of the curve, the equation referred to them will be

$$s_3(\xi^2-\eta^2) = k^2.$$

A comparison with the general equation of the elongation quadric shows that this equation denotes an elongation s_3 (half the shear) of the body along one axis of the curve accompanied by an elongation $-s_3$ (i.e., an equal *compression*) of the substance along the other axis.

Hence *the shearing strain of a body can be produced by a simple elongation (equal to half the shear) along one line and a simple compression of equal amount along a perpendicular line.*

We have been considering small displacements; but let us now consider an elongation of any amount along a line Ox, and an equal compression along a perpendicular Oy (fig. 261). Suppose that all lines in the body parallel to Ox are increased in the ratio $a:1$, and that all lines parallel to Oy are diminished in the ratio $1:a$; and consider displacements in the plane xy. There will, of course, be similar displacements in all planes parallel to xy. The displacement of the point O may be impressed in reversed direction on all points, so that O may be considered as at rest.

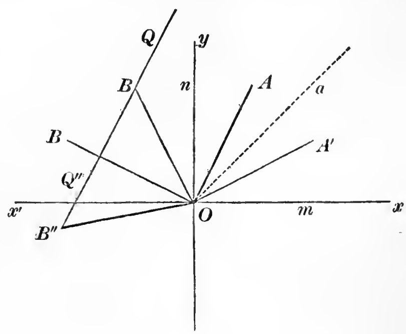

Fig. 261.

Draw OA, of any length, making the angle $AOx = \tan^{-1}a$. From A let fall An perpendicular to Oy. Then An becomes elongated by the strain parallel to Ox into $a\,.\,An$; but $a\,.\,An = nO$; therefore by this strain A is drawn out to a, Aa being parallel to Ox, and a a point on the bisector, Oa, of the angle xOy. From a draw am perpendicular to Ox. Then, by the strain parallel to Oy, am becomes shortened into $\frac{am}{a}\cdot$ Now if we draw OA' making with Ox an angle equal to AOy, this line will meet am in a point, A', such that $A'm = \frac{am}{a}\cdot$ Hence after the two strains A will come to A'; and we see that *OA' is equal in length to OA, and that they are both equally inclined to the bisector of the angle xOy.*

In the same way if OB be drawn making $\angle BOx' = \tan^{-1}a$, the length of OB will be unaltered, the point B will come to B', and the lines OB and OB' are equally inclined to the bisector of the angle $x'Oy$. Also OA is perpendicular to OB'. Hence, since parallel lines are all altered in the same ratio, all lines parallel to OA are unaltered in length, and all lines parallel to OB are unaltered in length.

Imagine a plane through OA perpendicular to the plane of the paper, and let any curve whatever be traced out in this plane. The curve will remain perfectly undistorted after the strain. For all lines perpendicular to the plane of the paper obviously remain so and are unaltered in length, and all lines parallel to the plane of the paper remain parallel to this plane, while of these latter those which are parallel to OA remain unaltered in length. Hence ordinates and abscissæ of the above-named curve parallel to OA and to a normal to the plane of the paper remain perpendicular to each other and unaltered in length. The curve, therefore, as regards magnitude and shape remains exactly as it was; its plane only is altered (to the plane through OA' perpendicular to the paper).

It follows, of course, *that all lines, whatever be their directions, in the plane through OA perpendicular to the paper remain unaltered in length.*

Similarly all lines in the plane through OB and the normal to the paper remain unaltered in magnitude; and all figures in this plane also remain undistorted.

The planes through the normal to the paper and the lines OA and OB are called *the planes of no distortion.*

Suppose that we impress on the body a common motion of rotation about the normal to the paper at O so as to bring OA' into coincidence with OA. This motion will, of course, be unaccompanied by any strain (Art. 274). Then OB' will come to OB'', and BB'' is perpendicular to OB' and parallel to OA, as is very easily seen.

Draw BQ parallel to OA. Then since the length of BQ remains unaltered, Q will come to Q'', a point such that $B''Q'' = BQ$. Hence all particles in the line BQ are slid parallel to AO through a space BB''. Now if p is the length of the perpendicular from B on OA,

$$\frac{BB''}{p} = a - \frac{1}{a},$$

as is easily found.

Consequently in this strain if *the undistorted plane OA is held fixed, every plane, BQ, parallel to it is slid parallel to it through a space proportional to the perpendicular distance between BQ and OA*; and this is the usual way of representing a shearing strain.

Of course the strain may otherwise be produced (neglecting the effect of mere rotation common to all points) by holding fast the other undistorted plane, OB, and sliding all planes parallel to it.

The plane (xy) perpendicular to the two planes of no distortion is called *the plane of the shear*; and the lines (Ox and Oy) which bisect, in the plane of the shear, the angles between the planes of no distortion are called the *axes of the shear*.

Since a sphere described about O as centre becomes an ellipsoid, and since there are two sections of an ellipsoid which are circles, the planes of these sections must be OA' and OB', the strained positions of the planes of no distortion.

The quantity, $a - \frac{1}{a}$, which is the fractional sliding per unit of distance between the parallel planes is called the *amount of the shear*.

If the strain is small, $a = 1 + s$, where s is a small quantity; and $\frac{1}{a} = 1 - s$, nearly, so that the amount of the shear $= 1 + s - (1 - s) = 2s$, which agrees with the analytical result at the beginning of this Article.

The expression for the displacement in a shearing strain can be simplified by taking the fixed plane as that of xy and the axis of x in the direction of the sliding. Then

$$u = 2sy, \quad v = 0, \quad w = 0;$$

so that a shear is a homogeneous strain, but not a pure one (Art. 283).

287.] **Traction and Torsion.** Suppose a cylindrical bar of an isotropic body to have its base held fixed while the bar is pulled in the direction of its length. Then each particle of the bar will be displaced in a direction parallel to the axis through a distance proportional to the natural distance of the particle from the fixed base; and in addition, the particle will be displaced towards the axis through a distance proportional to its natural distance from the axis. That is, at each point there will be uniform elongation and uniform contraction. Hence if the axis of the bar is taken as that of z, and the axes of x and y are in the plane of the fixed base,

$$u = -ax, \quad v = -ay, \quad w = cz$$

will express the displacements of any point, the quantities a and c being constant throughout the bar. This is the case of *Traction*. Suppose that, the base being still held fixed, the free extremity is twisted round through any angle (measured by the angle through which any diameter of the section revolves); then every other normal section of the bar will turn through an angle proportional to the distance, z, of this section from the fixed base.

If l = length of bar, a = angle through which its free end is twisted, every point in the section considered will be twisted through an angle $a\frac{z}{l}$. Hence the displacements of a point x, y in this section are (the twisting taking place from axis of x towards axis of y)

$$u = -\frac{azy}{l}, \quad v = \frac{azx}{l}, \quad w = 0.$$

This strain is called *Torsion*.

288.] **Lines of Flow and Vortex Lines.** Just as a Line of Force has been defined (p. 410) as a curve at every point of which the resultant force of attraction of a system is directed along the tangent, so a *Line of Flow* is defined to be a curve at every point of which the resultant displacement of the particle existing there is directed along the tangent.

Again, we have seen that the whole strain at any point can be produced by a pure strain together with a rotation round an axis through the point. A curve such that at every point of it the rotation corresponding to that point takes place round the tangent is called a *Vortex Line.*

In analogy with a Tube of Force, we have a *Tube of Flow.* If through points constituting the contour of any area we draw Lines of Flow, these lines form a surface called a Tube of Flow. Similarly if through the points constituting the contour of any area we draw Vortex Lines, these lines will make a surface which may be called a *Vortex Tube.*

When the normal section of the Vortex Tube is everywhere very small, it is called a *Vortex Filament.* Such a filament, AB, is represented in figure 262.

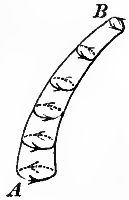

Fig. 262.

289.] **Equipotential Surfaces.** When the strain at every point is irrotational, the quantity $u\,dx + v\,dy + w\,dz$ is a perfect differential of a function $\phi(x, y, z)$. Describe in the body a series of surfaces the equation of any one of which is

$$\phi(x, y, z) = C. \tag{1}$$

Then by giving C a series of different values we shall have a series of surfaces, exactly analogous to the equipotential surfaces of an attracting mass (Chap. XV); and these equipotential surfaces of strain will be related to the lines of flow exactly as the equipotential surfaces of attraction are to the lines of force; that is, at every point the line of flow is perpendicular to the equipotential surface. For the direction cosines of the normal to the surface (1) at any point (x, y, z) are proportional to $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$; i.e., to u, v, w. But u, v, w being the components of the displacement, are of course proportional to the direction cosines of the line of flow. Therefore, &c.

The potential function of any small strain being ϕ, we see that $\frac{d\phi}{dx}$ is the displacement parallel to the axis of x; and since the axis of x may be in any direction, *the displacement in any direction is the rate of variation, per unit of length, of potential in this direction.*

It follows that the resultant displacement (which is perpendicular to the surface $\phi = C$) is $\frac{d\phi}{dn}$, where n denotes length measured along the normal to the surface, and the displacement is measured in the same sense as n.

Let two very close equipotential surfaces, $\phi = C_1$, $\phi = C_2$, be described. Denote these by ϕ_1 and ϕ_2. Then at all points on ϕ_1 the resultant displacement is inversely proportional to the normal distance at this point between the surfaces ϕ_1 and ϕ_2.

For if at any point on the surface ϕ_1 the normal distance between it and ϕ_2 is Δn, the displacement is $\dfrac{\Delta\phi}{\Delta n}$ or $\dfrac{\phi_2-\phi_1}{\Delta n}$. But for all points considered $\phi_2-\phi_1 = C_2 - C_1 =$ a constant; therefore the displacement varies inversely as Δn.

290.] **Circulation.** Suppose any curve, AB, to be traced out in the body, and let the displacement of each particle, P, on the curve between A and B be resolved along the tangent to the curve at P (the resolution taking place between A and B in a sense opposite to that of watch-hand rotation); then the sum obtained between A and B by multiplying this resolved part of displacement by the element, ds, of the curve at P and adding all such products, is called the *circulation* between A and B. Hence, by definition, the circulation from B to A is equal and opposite to the circulation from A to B.

The components of the displacement parallel to the axes being, as before, u, v, w, and the direction cosines of the tangent to the curve at P being $\dfrac{dx}{ds}$, $\dfrac{dy}{ds}$, $\dfrac{dz}{ds}$, the circulation is

$$\int\left(u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds}\right)ds, \quad \text{or} \quad \int(u\,dx+v\,dy+w\,dz),$$

the integral being taken from A to B.

Supposing that there is no rotation, or, in other words, that there is a displacement potential which has a value ϕ_1 at A and ϕ_2 at B, the circulation from A to B is $\phi_2-\phi_1$; it therefore depends merely on the co-ordinates of A and B, and not at all on the curve between them, along which it is taken.

If the curve is closed, B coincides with A, and the circulation is zero, it being still supposed that the strain is irrotational. If A and B are any two points on an equipotential surface, the circulation along any path from one to the other is zero.

We now proceed to consider the case in which rotation exists, and to prove the following fundamental theorem:—

The circulation round any small plane curve described round any *point, P, in the body is equal to twice the product of the area of the curve and the component of rotation at P perpendicular to the plane of the curve.*

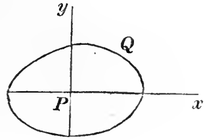

Fig. 263.

Let Q (fig. 263) be any point on the small curve whose plane is taken as that of xy; denote the components of the displacement of P by u, v, w; and the co-ordinates of Q with reference to P by ξ, η. Then the displacements of Q parallel to the axes are

$$u+\xi\frac{du}{dx}+\eta\frac{du}{dy},\quad v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy},\quad w+\xi\frac{dw}{dx}+\eta\frac{dw}{dy};$$

and the component of these along the tangent at Q is

$$\left(u+\xi\frac{du}{dx}+\eta\frac{du}{dy}\right)\frac{d\xi}{ds}+\left(v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy}\right)\frac{d\eta}{ds}.$$

When this is multiplied by ds and integrated, we shall have (since u, v, $\frac{du}{dx}$, ... are constant for all points on the curve)

$$u\int d\xi+v\int d\eta+\frac{du}{dx}\int\xi d\xi+\frac{dv}{dy}\int\eta d\eta+\frac{dv}{dx}\int\xi d\eta+\frac{du}{dy}\int\eta d\xi,$$

of which all the integrals except the last two vanish, since the curve is closed. Now $\int\xi d\eta=$ area of curve $=A$; and $\int\eta d\xi=-A$, since the two integrations are carried round at the same time from x to y. Hence the circulation $=A\left(\frac{dv}{dx}-\frac{du}{dy}\right)$

$$=2A\,.\,d\theta_3,$$

(p. 473) $d\theta_3$ being the rotation round axis of z at P, i.e., perpendicular to the plane of the curve.

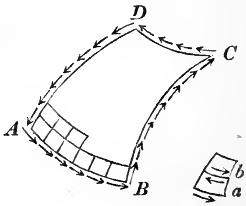

Fig. 264.

Suppose that any surface, plane or curved, bounded by any curve, $ABCD$, (fig. 264) is traced out in the body and that at each point on this surface we take the component of rotation round the normal to the surface, multiply this component by the element of superficial area at the point, and take the sum of all such products. This sum is called the *surface-integral of normal rotation.* The normal must be supposed to be drawn away from the same side of the surface at every point, and the rotation is supposed to take place opposite to that of the hands of a watch held so that the normal passes up through its face.

It is very easy to prove that this *surface-integral of rotation is equal to one half the circulation round the edge, ABCD, of the surface.* For, let the surface be broken up into an indefinitely great number of little plane areas. Then the sum of the circulations round these areas is twice the surface integral of rotation (by what has just been proved). But the circulations in the common portions of every two contiguous areas are directly opposed, and therefore mutually destructive, as is seen by drawing any two such little areas, a and b, apart; hence the circulation exists only along lines which do not form common parts of contiguous areas, i.e., along the edge which bounds the surface.

If the surface has no bounding edge, i.e., if it is a closed surface, the surface-integral of rotation over it is zero.

If the surface, without being closed, is such that at every point of it the rotation takes place about a tangent line to the surface, the circulation round its bounding edge is zero. Such a surface is that of a vortex filament (fig. 262, p. 479); or that represented in fig. 265, which consists of a vortex tube whose ends are any two irregular curves whatever. The sum of the circulations round the terminal sections D and E of this tube, *estimated in the cyclical order indicated round the contour in* fig. 264, is zero, i.e., *the circulations,* estimated as represented by the arrows in fig. 262, *round any two sections whatever of a vortex tube are equal*; or, in other words, the circulation round any section, normal or oblique, plane or tortuous, of a vortex tube is constant.

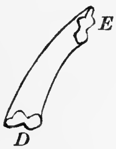

Fig. 265.

Examples.

1. Prove analytically that the shear of any two rectangular lines intersecting at any point is equal to the difference between the elongations along the internal and external bisectors of the angle between them.

Let the axes of co-ordinates be the principal axes of the strain at the point. Then the value of s given in equation (2), Art. 278, becomes

$$s = e_1 ll' + e_2 mm' + e_3 nn',$$

the direction-cosines of the lines being (l, m, n) and (l', m', n'), and the shear $2s$. Now the direction-cosines of one bisector are $l-l'$, $m-m'$, $n-n'$, each divided by the square root of the sum of the squares of these quantities, i. e., by $\sqrt{2}$, since the lines are rectangular; and the direction cosines of the other bisector are $l+l'$, $m+m'$, $n+n'$, each divided by $\sqrt{2}$. Let ϵ and ϵ' be the elongations along these bisectors. Then, by Art. 276,

$$2\epsilon = e_1(l-l')^2 + e_2(m-m')^2 + e_3(n-n')^2,$$
$$2\epsilon' = e_1(l+l')^2 + e_2(m+m')^2 + e_3(n+n')^2;$$

therefore $\qquad \epsilon'-\epsilon = 2(e_1 ll' + e_2 mm' + e_3 nn'),$

or $\qquad \epsilon'-\epsilon = 2s,$

which proves the proposition.

2. Find the pair of rectangular lines in a given plane for which the shear is greatest.

In any plane the elongation is greatest along one axis of the conic in which this plane cuts the Elongation Quadric, and least along the other. Therefore the *difference* of elongation along two rectangular lines is greatest for this pair; and therefore, by last example, the shear of the two rectangular lines of whose angle these axes are the external and internal bisectors is greatest.

Hence the shear in a given plane is greatest for two lines making angles of 45° *with the axes of the conic in which the given plane cuts the Elongation Quadric.*

The magnitude of the shear for any two rectangular lines in the plane is easily found and represented by a curve.

Let the axes of x and y be taken in the given plane and coincident with the axes of the section of the Elongation Quadric in the plane. Then s_3 must $= 0$ for these axes. Also let one of two lines along which we wish to find the shear make an angle θ with the axis of x. Then in the expression for s (Art. 278) we have $l_1 = \cos\theta$, $m_1 = \sin\theta$, $l_2 = -\sin\theta$, $m_2 = \cos\theta$, $n_1 = n_2 = 0$; therefore

$$s = \tfrac{1}{2}(b-a)\sin 2\theta,$$

or $$2s = (b-a)\sin 2\theta = \text{shear},$$

which of course shows that the shear is a maximum along lines bisecting the angles between the axes of the section. The curve whose polar equation is $r = (b-a)\sin 2\theta$ consists of four loops, one in each quadrant, and its radius-vector gives the shear for any directions, denoted by θ and $\dfrac{\pi}{2}+\theta$.

It follows that the two rectangular lines whose shear is absolutely the greatest at a point in the body are those in the plane of the greatest and least axes of the Elongation Quadric (or of the Strain Ellipsoid) and making angles of 45° with them, and that their shear is e_3-e_1, if we assume e_1, e_2, e_3 to be in ascending order of magnitude.

3. Prove that a simple elongation in any direction is equivalent to a uniform cubical dilatation together with two shears, each having the given direction for one axis, the other axes being at right angles to it and to each other.

Consider a cube whose three edges at the point O are Ox, Oy, Oz, and suppose the given simple elongation, ϵ, to take place along Ox. We may consider this as $\frac{1}{3}\epsilon+\frac{1}{3}\epsilon+\frac{1}{3}\epsilon$ along Ox, and we may suppose an elongation $\frac{1}{3}\epsilon$ along Oy together with an elongation $-\frac{1}{3}\epsilon$ (or a contraction) in the sense of yO; and similarly $\frac{1}{3}\epsilon$ and $-\frac{1}{3}\epsilon$ in Oz. Now $\frac{1}{3}\epsilon$ along Ox, Oy, and Oz (and of course along all lines parallel to these) constitutes (p. 468) a cubical dilatation ϵ; while $\frac{1}{3}\epsilon$ along Ox and $-\frac{1}{3}\epsilon$ along Oy constitute (Art. 286) a shear, whose amount is $\frac{2}{3}\epsilon$ (Art. 286). Therefore, &c.

4. Resolve a simple elongation ϵ in a given direction into its components with reference to three rectangular axes.

Ans. If the direction-cosines of the direction of elongation with reference to the three axes are l, m, n, the elongations and shears to which ϵ is equivalent are

$$\epsilon l^2,\ \epsilon m^2,\ \epsilon n^2,\ 2\epsilon lm,\ 2\epsilon mn,\ 2\epsilon nl.$$

For, if ξ, η, ζ be the co-ordinates of any point before strain, the length of the perpendicular from this point on the plane through the origin perpendicular to the direction (l, m, n) is $l\xi+m\eta+n\zeta$; and the point (ξ, η, ζ) is drawn out along this perpendicular through a distance $\epsilon(l\xi+m\eta+n\zeta)$. The projection of this distance along the axis of x is $\epsilon l(l\xi+m\eta+n\zeta)$; hence the strained co-ordinates (ξ', η', ζ'), are

$$\xi'=\xi+\epsilon l(l\xi+m\eta+n\zeta),\quad \eta'=\eta+\epsilon m(l\xi+m\eta+n\zeta),$$
$$\zeta'=\zeta+\epsilon n(l\xi+m\eta+n\zeta).$$

Comparing these values of ξ', η', ζ' with those given at p. 459, we see that

$$\epsilon l^2=a,\quad \epsilon m^2=b,\quad \epsilon n^2=c,\quad \epsilon lm=s_3,\quad \epsilon mn=s_1,\quad \epsilon nl=s_2,$$

which are the required components of the elongation with reference to the axes.

5. Find the condition that, in the general small strain, there should be two planes of no elongation.

Ans. $\begin{vmatrix} a, & s_3, & s_2 \\ s_3, & b, & s_1 \\ s_2, & s_1, & c \end{vmatrix}=0.$ Hence one of the principal elongations must be zero (see p. 472).

6. Given two small strains, $(a, b, c, 2s_1, 2s_2, 2s_3)$, $(a', b', c', 2s_1', 2s_2', 2s_3')$, find the resulting elongation quadric and strain ellipsoid.

Ans. In the previous equations of these surfaces put $a+a'$ for a, &c., s_3+s_3' for s_3, &c.

7. Resolve a shear, $2s$, of two given rectangular lines into its components along three given rectangular axes.

Ans. If the direction-cosines of the two given lines with reference to the given axes are (l, m, n), (l', m', n'), the components are

$$2sll',\ 2smm',\ 2snn',\ 2s(lm'+l'm),\ 2s(mn'+m'n),\ 2s(nl'+n'l).$$

8. Find the conditions that a strain whose components with

reference to three given rectangular axes are given should be equivalent to a shear.

Ans. $\begin{vmatrix} a, & s_3, & s_2 \\ s_3, & b, & s_1 \\ s_2, & s_1, & c \end{vmatrix} = 0$ and $a+b+c=0$.

The first of these expresses that the product of the three principal elongations is zero, and the second that their sum (the cubical dilatation) is zero. Hence the principal elongations are of the forms $e, -e, 0$.

9. Given the components of strain with reference to the principal axes of the strain, find the components of the same strain with reference to any set of rectangular axes.

Ans.

$$a = e_1 l^2 + e_2 m^2 + e_3 n^2,\ b = e_1 l'^2 + e_2 m'^2 + e_3 n'^2,\ c = e_1 l''^2 + e_2 m''^2 + e_3 n''^2,$$
$$s_1 = e_1 l'l'' + e_2 m'm'' + e_3 n'n'',\ s_2 = e_1 ll'' + e_2 mm'' + e_3 nn'',$$
$$s_3 = e_1 ll' + e_2 mm' + e_3 nn'.$$

10. Find the Vortex Lines in the case of Torsion.

Ans. The rotations at any point are

$$\delta\theta_1 = -\frac{ax}{2l},\ \delta\theta_2 = -\frac{ay}{2l},\ \delta\theta_3 = \frac{az}{l}.$$

Hence the differential equations of the Vortex Lines are

$$\frac{dx}{x} = \frac{dy}{y} = -\frac{dz}{2z}.$$

The Vortex Lines are therefore the intersections of $\frac{y}{x} = c_1$ and $x^2 z = c_2$. The vortex line at any point lies in the plane through this point and the axis about which the torsion takes place.

11. When the small strain $(a, b, c, 2s_1, 2s_2, 2s_3)$ is equivalent to a shear, find the magnitude of the shear.

Ans. If $2s$ is the shear, $s = \sqrt{s_1^2 + s_2^2 + s_3^2 + \frac{1}{2}(a^2 + b^2 + c^2)}$. To get this equate the components in example 7 to $a, b, c, 2s_3, \ldots$ Squaring and adding the last three we have

$$s^2(1 - l^2 l'^2 - \ldots + 2ll'mm' + \ldots) = s_1^2 + s_2^2 + s_3^2;$$

or $$s^2[1 - 2(l^2 l^2 + m^2 m'^2 + n^2 n'^2)] = s_1^2 + s_2^2 + s_3^2;$$

therefore the rest follows from the first three.

12. Prove that torsion is equivalent to shear at each point, and find its amount.

Ans. Let P be the point considered, PO the perpendicular (of length r) from P on the axis of torsion, and let the strain be expressed as in Art. 287; then the amount of the shear is $\frac{ar}{l}$, and the strain is a shear of the line drawn through P parallel to the axis of torsion and a line perpendicular to this one and to PO.

13. Find the areal dilatation on a plane the direction-cosines of whose normal are l, m, n.

Ans. $a + b + c - (al^2 + bm^2 + cn^2 + 2lms_3 + 2mns_1 + 2nls_2)$.

Section II.

Analysis of Stresses.

291.] **Intensity of a Stress.** If a force whose magnitude is P acts over an area S in such a way that there is all over the area the same force on the same amount of area, the force is said to be *uniformly distributed* over the area; and the *intensity* of force on the area is $\frac{P}{S}$, i.e., the rate at which the force is distributed per unit of area. Thus the atmospheric pressure on any area at the surface of the earth is roughly 15 lbs. on every square inch, and if the unit of force is a pound weight and the unit of length an inch, the intensity of atmospheric pressure is represented by the number 15.

If force acts over an area in such a way that there is not the same amount exerted on the same area everywhere, the distribution is not uniform; and in this case we can speak only of the intensity of force *at each particular point.* If about any point we describe a very small area, ds, on which we may assume the distribution of force to be constant, and if dF is the amount of force on it, the intensity of force at the point selected is $\frac{dF}{ds}$. An instance of this occurs when the area pressed is any non-horizontal area in a heavy liquid. The intensity of pressure at points in the upper part of the area is less than the intensity at points in the lower part.

292.] **Stress at a Point.** At any point, P, of the body consider a small plane surface of area ds and any position. This may be regarded as separating the part (A) of the body at one side of it from the part (B) at the other side. Then the particles in this element plane, when the body is strained in any manner, are subject to certain forces proceeding from the particles at the side (A) and resulting from the elongation or contraction of the natural distances. The resultant of these forces is called the stress on the side (A) of the element plane.

The particles in the element plane are also subject to forces proceeding from particles at the side (B) of the plane; and the resultant of these latter is, of course, a stress equal and opposite to the first-mentioned stress.

The resultant stress (on either side of the element plane) divided by the area, ds, is the intensity of stress on the plane; and the resultant stress may be either normal to the plane, oblique to it, or in it.

If at the same point P we consider a small plane surface of the same area as before, but of different position, the resultant stress on it will, generally speaking, be different both in magnitude and in direction from the previous stress. In the case of a perfect fluid body the magnitude of the stress is constant and its direction is normal to the element plane, whatever be the position of the latter at the point P.

Hence in the case of a strained body the term 'stress at a point' has no definite meaning until we specify the element plane on which the stress acts.

293.] **Equilibrium of an Element.** At any point, O (fig. 13, p. 19) whose co-ordinates with reference to three fixed rectangular axes are (x, y, z) let a very small rectangular parallelopiped of the substance be separated in imagination from the rest of the body by means of element planes perpendicular to the fixed axes; and through O draw the lines Ox, Oy, Oz parallel to the fixed axes. We may then, if we actually produce on the faces of this element the stresses which are produced on them by the neighbouring portions of the body, consider the equilibrium of the element apart from the remainder*. Let the stress per unit of area on the face $BOCF$ have for components along Ox, Oy, Oz the values P_x, P_y, P_z, respectively; let the corresponding components for the plane $AOCH$ be Q_x, Q_y, Q_z; and let those for the face $AOBD$ be R_x, R_y, R_z. The stress on each face may be supposed to be applied at the middle point of the face, and each component is supposed to be measured in the positive sense of the corresponding axis.

Let $OA = dx$, $OB = dy$, $OC = dz$. Now these component

* In considering the equilibrium of an element of a fluid body it is customary to say that we consider it as *solidified* and acted on by the stresses (pressures) which the fluid exerts on its surface. This solidification is however wholly unnecessary and misleading—if, indeed, it is not actually wrong. The element while regarded as forming part of the body is not solidified, but is kept in its condition by the very forces which, by supposition, are produced on it by other means. If these forces were by themselves sufficient in the one case, they must be so in the other without the aid of solidification.

stresses are all functions of the position of O, i.e., each of them is some function of (x, y, z). And the cō-ordinates of A are $(x+dx, y, z)$; so that if $P_x = f(x, y, z)$, the P_x for the face $DAHO'$ is $f(x+dx, y, z)$, i.e., it is $P_x + \frac{dP}{dx}dx$, neglecting $(dx)^2$, &c. This component is, of course, directed in the sense AO, since the stresses produced on the faces $BOCF$ and $DAHO'$ by the portions of the body removed are opposed. Hence the components of intensity of stress on $DAHO'$ are

$$-(P_x + \frac{dP_x}{dx}dx), \quad -(P_y + \frac{dP_y}{dx}dx), \quad -(P_z + \frac{dP_z}{dx}dx).$$

Similarly for the components of intensity of stress on the faces $DBFO'$ and $HCFO'$. To get the whole amount of stress in any direction on any face, the intensity in this direction must, of course, be multiplied by the area of the face. Let us calculate the whole amount of stress parallel to Ox exerted on the parallelopiped. The face $BOCF$ will contribute $P_x \times dy\,dz$, while the opposite face, $HADO'$, will contribute $-(P_x + \frac{dP_x}{dx}dx)\,dy\,dz$; and the sum of these is $-\frac{dP_x}{dx} \times dx\,dy\,dz$. The face $AOCH$ will give a stress $Q_x \times dz\,dx$ parallel to Ox, and the opposite face will give $-(Q_x + \frac{dQ_x}{dy}dy)\,dz\,dx$; and the sum of these is $-\frac{dQ_x}{dy}dx\,dy\,dz$; similarly, the faces $AOBD$ and $HCFO'$ will give $-\frac{dR_x}{dz}dx\,dy\,dz$. Hence the whole stress force acting on the element in the direction Ox is

$$-\left(\frac{dP_x}{dx} + \frac{dQ_x}{dy} + \frac{dR_x}{dz}\right)dx\,dy\,dz.$$

Some external force (gravity, or other) may also act on each element of the body. Such a force will always be proportional to the quantity of matter in the element. Suppose ρ to be density of the body at O; then, approximately, the quantity of matter in the parallelopiped is $\rho\,dx\,dy\,dz$. Let the components of the external force which is felt at O along the axes of x, y, z be X, Y, Z, per unit of mass. Then the component of the external force along Ox exerted on the element is $\rho X dx\,dy\,dz$. Equating to zero the sum of the components along Ox of all forces exerted on the element, we have

Similarly,

$$\left.\begin{aligned}\frac{dP_x}{dx}+\frac{dQ_x}{dy}+\frac{dR_x}{dz}&=\rho X.\\ \frac{dP_y}{dx}+\frac{dQ_y}{dy}+\frac{dR_y}{dz}&=\rho Y,\\ \frac{dP_z}{dx}+\frac{dQ_z}{dy}+\frac{dR_z}{dz}&=\rho Z;\end{aligned}\right\}\qquad(1)$$

the last two equations being obtained by resolving forces along the axes of y and z.

In a strained medium in which the stress on every plane is normal the equations of equilibrium are

$$\frac{dP}{dx}=\rho X,\ \frac{dQ}{dy}=\rho Y,\ \frac{dR}{dz}=\rho Z,$$

since the tangential components P_y, P_z, Q_x, ... are zero; and if, in addition, as in a perfect fluid, the intensity of stress is the same on all element planes at a given point, $P=Q=R$, and these equations become the well-known hydrostatical equations

$$\frac{dP}{dx}=\rho X,\ \frac{dP}{dy}=\rho Y,\ \frac{dP}{dz}=\rho Z.$$

For any kind of body we obtain another valuable set of equations by expressing the equilibrium of the *moments* of the forces acting on the parallelopiped. For example, take moments about the line joining the middle points of the opposite faces $BOCF$ and $DAHO'$. The external force* acting on the parallelopiped may be considered to act at its middle point; it will therefore contribute nothing to the moments about the axis chosen. Neither will the stresses on the faces $BOCF$ and $DAHO'$, since these stresses act at the middle points of the faces. Of the stresses on the faces $AOCH$ and $DBFO'$ the

* It is important for the student to distinguish two species of external force acting on any body. There may be external forces which act only at *particular points on its surface*—as, for example, when a beam rests against the ground and against a wall, the reactions of the ground and wall—and there may be external forces which affect *every element* inside the body—as, in the same case, the attraction of the earth which produces a force (the weight) on each element of the beam. The latter are called *continuous forces*. Thus a strained body may be affected by both—the above beam, if slightly flexible, will be bent. The forces (per unit of mass), X, Y, Z, in equations (1) belong exclusively to the second kind. Forces of the first kind do not enter into these equations; they are like the terminal tensions of a string, and are required for determining the values of constants which occur in the integrals of the differential equations (1) of equilibrium.

components $Q_z \times dx\,dz$ and $-\left(Q_z + \frac{dQ_z}{dy}dy\right)dx\,dz$, which are parallel to Oz will alone contribute moments. The moment of the first is $Q_z \times dx dz \times \frac{dy}{2}$, or $\frac{1}{2}Q_z dx dy dz$, and the moment (in the same sense) of the second is $\left(Q_z + \frac{dQ_z}{dy}dy\right)dx dz \times \frac{dy}{2}$, or $\frac{1}{2}Q_z dx\,dy\,dz$, neglecting the term $dx(dy)^2dz$. The sum of these moments is $Q_z dx\,dy\,dz$.

Again, of the stresses on the faces $AOBD$ and $HCFO'$ the components, $R_y \times dx dy$ and $-\left(R_y + \frac{dR_y}{dz}dz\right)dx dy$, will alone contribute; and the sum of their moments is $R_y dx dy dz$, which is obviously in the sense opposite to that of the previous moment.

Hence equating the sum of these moments to zero,

$$\left.\begin{aligned} Q_z &= R_y. \\ \text{Similarly,}\qquad P_y &= Q_x, \\ R_x &= P_z, \end{aligned}\right\} \tag{2}$$

which are obtained by taking moments about the lines joining the middle points of the faces ($AOBD$, $HCFO'$) and ($AOCH$, $DBFO'$), respectively.

The stress (per unit of area) on the face $BOCF$ can be resolved into two, viz., one normal to the face and the other in the face. The first is P_x, and the second (which is the shearing force on the face) is $\sqrt{P_y^2 + P_z^2}$. Equations (2) obviously assert that if we take any two element planes at right angles to each other at any point of the body, *the component along the normal to the second of the stress per unit area on the first is equal to the component along the normal to the first of the stress per unit area on the second.* We shall now see that this very important result is true for two element planes inclined at any angle to each other.

To save a multiplicity of symbols, use N_1 for P_x, N_2 for Q_y, N_3 for R_z, T_3 for P_y and Q_x, T_2 for P_z and R_x, T_1 for Q_z and R_y; N standing for normal and T for tangential intensity of stress.

Consider now the equilibrium of a tetrahedral element of the body included between the plane ABC (fig. 13, p. 19), and the planes BOC, AOC, AOB. Let the components along Ox, Oy, Oz of the stress per unit area on the triangular face ABC be P, Q, R; and let the direction-cosines of the perpendicular on this plane be l, m, n. Resolve along Ox the forces acting on the tetrahedral element. The face BOC will contribute $N_1 \times BOC$ (where BOC means the area of the face); the face AOC will contribute $T_3 \times AOC$; the face AOB will contribute $T_2 \times AOB$; the face ABC will contribute $-P \times ABC$; and the external force $\rho X \times \frac{1}{6} OA . OB . OC$. Hence

$$P \times ABC = N_1 \times BOC + T_3 \times AOC + T_2 \times AOB + \tfrac{1}{6} \rho X \times OA . OB . OC.$$

Divide out by ABC.

Now $$\frac{BOC}{ABC} = l, \quad \frac{AOC}{ABC} = m, \quad \frac{AOB}{ABC} = n, \quad \frac{OA . OB . OC}{ABC} = 2l . OA.$$

Therefore $P = lN_1 + mT_3 + nT_2 + \frac{1}{3} \rho X \times l . OA.$

But by taking all dimensions of the element very small, *the term $\frac{1}{3} \rho X . OA$ proceeding from the external force, ultimately vanishes*, and we have accurately

Similarly,
$$\left.\begin{aligned} P &= lN_1 + mT_3 + nT_2 . \\ Q &= lT_3 + mN_2 + nT_1, \\ R &= lT_2 + mT_1 + nN_3, \end{aligned}\right\} \qquad (3)$$

by resolving along Oy and Oz. These very important equations give us the intensity of stress in magnitude and direction on any assigned element plane when the stresses on *three* rectangular element planes are known; they are, in fact, *the composition and resolution of stress.*

Any one of these equations (3) suffices for the proof of the important general theorem of projection already referred to. For P is the projection, along the normal to the element plane BOC, of the intensity of stress on the element plane ABC, and

$$lN_1 + mT_3 + nT_2$$

is the projection, along the normal to the latter plane, of the intensity of stress on the former. This theorem is true therefore for any two element planes at a point.

Remark. The components of stress on an element plane at the bounding surface of the body are to be equated to the components of the external force applied to the surface at the element.

Cor. It follows immediately from this theorem of the projections of two stresses that if there is at a point in the body any plane on which the stress is zero, the lines of action of the stresses on all other planes at this point lie in this plane of zero stress.

When the stress on an element plane, ϖ, exercised by the part, A, of the body on one side of it consists of a force whose component normal to ϖ is directed from this plane towards the part A, the stress on ϖ is called *tension*; and when the normal component is directed from A to ϖ, it is called *pressure*. All fluid stress is pressure. In general at every point inside a strained body there will be some planes on which the stress is pressure, and others on which the stress is tension.

It may assist the student to understand the nature of the action of stress on an element plane if we draw a figure representing the equilibrium of these stresses on an element of the body. Thus if we take the elementary parallelopiped OO' (fig. 13, p. 19) to be a cube, and also take (as we may) the stress on any face as acting at its middle point, the forces in the plane of xy may be represented as in fig. 266, which is that of a section of the cube through its centre and parallel to the plane of xy. If there were no stresses on planes parallel to xy, this figure would completely represent the equilibrium of the cubical element. (Since the faces have been all taken as equal in area, the *intensities* of stresses are proportional to the stresses acting on them.)

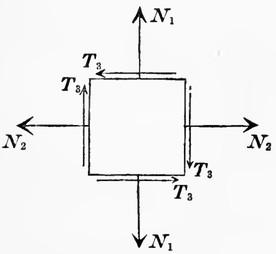

Fig. 266.

It is evident, of course, that when the stresses on *any three* planes at a point (rectangular or not) are known, the stress on every plane at this point can be found both in magnitude and in line of action. For we may consider the equilibrium of the tetrahedral element contained by the assumed plane and the three given ones, and the required force will be equal and opposite to the resultant of three given forces.

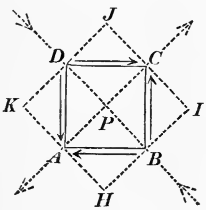

Fig. 267.

Let it, for example, be given that the stress at any point P is a shearing stress in each of two rectangular planes, there being no stress on planes perpendicular to both of them. Suppose that all planes in the neighbourhood of P which are perpendicular to the plane of the paper and parallel to CD (fig. 267) are subject to a shearing stress, and that all planes parallel to AD and perpendicular to the paper are also subject to shearing stress, and that planes parallel to the paper are not subject to stress. The intensities of these shearing stresses are obviously equal (either by what precedes, or by considering the equilibrium of a small prism whose base is the square $ABCD$ and whose edges are perpendicular to the paper. The equality of moments round an axis through P perpendicular to the figure gives the equality of the intensities of these shears). Let their common intensity be S, and suppose them represented by the arrows.

Draw the plane AC, and consider the equilibrium of the portion ABC of the body (or rather of a little right prism whose base is ACB). It is kept in equilibrium by the forces S acting in the lines DC and DA and by the stress on the face AC. This last must (since it may be supposed to act at the middle point of AC) act in the line PD from P to D. If h is the height of the prism, the areas of its faces are $h \times AC$, $h \times CD$, $h \times DA$; so that the forces acting in DC and DA are $S \times h \times DC$ and $S \times h \times DA$; and their resultant, F, which is equal and opposite to the stress on AC, is given by the equation

$$F = \sqrt{S^2 \times h^2 \times DC^2 + S^2 \times h^2 \times DA^2} = S \times h \times AC;$$

$$\therefore \quad \frac{F}{h \times AC} = S,$$

i. e., the intensity of stress on the face AC is equal to the intensity of the shearing stress on each of the other two faces; moreover, the stress on AC is normal to AC. This stress is the action of the portion of the body at the right-hand side of AC on the particles in the plane AC, and since it acts in the sense PD, it is a *pressure*. Hence if the portion of the body at the right-hand side of AC, or of any plane parallel to it and near it be removed, a pressure of intensity S must be applied to the plane in the sense PD. The action of the part of the body at the left-hand side of AC, or of any parallel to and near it, consists, of course, of a *pressure* in the opposite sense; so that if we draw two element planes HI and JK parallel to AC and consider the portions of the body at the right of the first and at the left of the second as removed, two pressures (indicated by the arrows pointing to B and D) must be applied to the portion of the body contained between these planes.

Similarly, by drawing BD and considering the equilibrium of the prism standing on the base BCD, we see that the action of the portion of the body at the lower side of BD on the particles in this face consists of a normal stress of intensity S directed in the sense CP, i. e., *towards* the parts considered as removed; in other words, this stress is a *tension*. Consequently if we isolate in imagination a small prism of the body standing on the square $HIJK$, we regard it as acted on by two *pressures* on its faces HI and JK, and by two *tensions* on its faces IJ and KH.

The state of stress of the body at P may just as well be produced by applying normal stress (pressure), of the same intensity as the shearing stress, to all planes parallel to AC and near it, and normal stress (tension), of same intensity, to all planes parallel to BD and near it; in other words, *we may substitute this state of stress for the shearing stress.*

Hence *a shearing stress on two rectangular planes at any point produces equal normal stresses of opposite signs (pressure and tension) and of intensities equal to* that of the shearing stress on the two planes which bisect the angles between them.*

This result follows, of course, from equations (3) by taking the lines from P perpendicular to CD and BC as axes of x and y, and putting $N_1 = 0,\ N_2 = 0,\ N_3 = 0,\ T_1 = 0,\ T_2 = 0,\ T_3 = S,$ $l = m = \frac{1}{\sqrt{2}},\ n = 0.$ From these equations also we deduce the magnitude and line of action of the stress on any plane near P.

The student will do well, however, to deduce from the figure the stress on any plane through (or near) P perpendicular to the figure.

294.] **Problem.** *Given the condition of stress of a body at any point in it with reference to one set of rectangular planes, to find the condition of stress at the same point with reference to any other set of rectangular planes.*

Let the given stresses at a point O, on three rectangular planes of xy, yz, zx, be $N_1, N_2, N_3, T_1, T_2, T_3$, as in last Article. Then the components along the axes of x, y, z of the stress per unit area on an element plane at the point the direction-cosines of whose normal are l, m, n are given by equations (3) of last

* Compare with the corresponding result in the case of shearing *strain*. The shearing strain may be replaced by two simple elongations, the magnitude of each being *half* that of the shear. (See p. 447.)

Article. The resolved part, T, of this stress along any line whose direction-cosines are λ, μ, ν is $\lambda P+\mu Q+\nu R$; i.e.,

$$T=l\lambda N_1+m\mu N_2+n\nu N_3+(l\mu+m\lambda)T_3+(m\nu+n\mu)T_1 \\ +(n\lambda+l\nu)T_2. \quad (1)$$

If the line along which the stress is resolved is the normal to the element plane itself, the component, N, is $lP+mQ+nR$; i.e.,

$$N=l^2N_1+m^2N_2+n^2N_3+2lmT_3+2mn\,T_1+2nl\,T_2. \quad (2)$$

Let it be required to find the intensities of stress on three other rectangular element planes at O whose normals are Ox', Oy', Oz'; and let the direction-cosines of these normals with respect to Ox, Oy, Oz be (l, m, n), (l', m', n'), (l'', m'', n''), respectively. Denote the components of the intensity of stress on the plane $y'z'$ by N_1' along Ox', T_3' along Oy', and T_2' along Oz'; the components of the intensity of stress on the plane $z'x'$ by T_3' along Ox', N_2' along Oy', and T_1' along Oz'; and those of the intensity of stress on the plane $x'y'$ by T_2' along Ox', T_1' along Oy', and N_3' along Oz'.

Then N_1' is given by (2); N_2' is obtained by using (l', m', n') for (l, m, n) in (2); N_3' by using (l'', m'', n'') for (l, m, n) in (2); T_3' by using (l', m', n') for (λ, μ, ν) in (1); T_2' by using (l'', m'', n'') for (λ, μ, ν) in (1); and T_1' by using (l', m', n') for (l, m, n), and (l'', m'', n'') for (λ, μ, ν) in (1).

It will be seen from this that in transforming from one set of rectangular axes Ox, Oy, Oz to another, the quantities N_1, N_2, N_3, T_3, T_1, T_2 transform like x^2, y^2, z^2, xy, yz, zx.

The system of stress, thus calculated, on the new planes may be substituted for the original system of stress—the two systems are, in other words, perfectly equivalent, and either will produce the other.

295.] **Cone of Shearing Stress.** The expression (2) for the normal component of intensity of stress on a plane may for all values of l, m, n (i.e., for all element planes at the point considered) retain a positive value. In this case the normal component of stress is a *tension* on all planes. Or the expression may be negative for all planes, and then the normal stress will be *pressure* all round. Or, finally, it may be positive for some directions and negative for others. It will then be zero for some directions; i.e., there will be planes on which the stress is entirely tangential. The directions of the normals to these planes are given by the equation

$$N_1 l^2 + N_2 m^2 + N_3 n^2 + 2 T_3 lm + 2 T_1 mn + 2 T_2 nl = 0,$$

and therefore the normals trace out the cone

$$N_1 x^2 + N_2 y^2 + N_3 z^2 + 2 T_3 xy + 2 T_1 yz + 2 T_2 zx = 0, \qquad (1)$$

the planes themselves tracing out the cone whose generators are perpendicular to the generators of this cone. This latter cone, when it exists, is called the *Cone of Shearing Stress.*

296.] **Principal Planes of a Stress.** The angle between the direction of stress and the plane on which it acts depends on the plane chosen. Let us try whether, with any given stress, it is possible to find a plane on which the stress is normal.

If F is the resultant stress on a plane the direction-cosines of whose normal are (l, m, n), and if F acts in the normal, $P = lF$, $Q = mF$, $R = nF$, and equations (3) of Art. 293 become

$$\left.\begin{aligned} lN_1 + mT_3 + nT_2 &= lF, \\ lT_3 + mN_2 + nT_1 &= mF, \\ lT_2 + mT_1 + nN_3 &= nF; \end{aligned}\right\} \qquad (1)$$

and these give, by elimination of the direction-cosines, the cubic for F

$$\begin{vmatrix} N_1 - F, & T_3, & T_2 \\ T_3, & N_2 - F, & T_1 \\ T_2, & T_1, & N_3 - F \end{vmatrix} = 0,$$

or $$F^3 - (N_1 + N_2 + N_3) F^2 + (N_1 N_2 + N_2 N_3 + N_3 N_1 - T_1^2 - T_2^2 - T_3^2) F - (N_1 N_2 N_3 - N_1 T_1^2 - N_2 T_2^2 - N_3 T_3^2 + 2 T_1 T_2 T_3) = 0.$$

This equation, as is well known, gives three real values of F, and equations (1) will give the direction-cosines of the planes subject to these normal stresses. The coefficients of this equation have, as is also well known, the same values, no matter what three rectangular planes are taken as those of reference.

All theorems, therefore, concerning stress may be simplified by supposing that we have selected as planes of reference the three on which the stresses are normal. These are called the *principal planes* of the stress at the point considered. Let the stresses on them (per unit area, of course) be denoted by A, B, C.

The equations (1) which determine the planes and magnitudes of the principal stresses show that these planes are the principal planes of the quadric

$$N_1 x^2 + N_2 y^2 + N_3 z^2 + 2 T_3 xy + 2 T_1 yz + 2 T_2 zx = f, \qquad (2)$$

f being any constant force magnitude.

The equation of the tangent plane to this quadric at the point x', y', z' is

$$(N_1x' + T_3y' + T_2z')\,x + (T_3x' + N_2y' + T_1z')\,y$$
$$+ (T_2x' + T_1y' + N_3z')\,z = f.$$

Let a normal be drawn to any element plane at the point, O, considered, and let r be the length of this normal from O to the surface of this quadric. Then by putting lr, mr, nr for x', y', z', the tangent plane at the extremity of this normal is (by the values of P, Q, R in p. 491)

$$Px + Qy + Rz = \frac{f}{r}. \tag{3}$$

The direction cosines of the perpendicular from O on this plane are $\frac{P}{F}$, $\frac{Q}{F}$, $\frac{R}{F}$, where F is the resultant stress (per unit area) on the element plane; and these show that the resultant stress acts in this perpendicular. Again, if p is the length of the perpendicular from O on the plane (3), we have

$$F = \frac{f}{pr}, \tag{4}$$

the value of the resultant stress.

If the axes of the quadric (2) are taken as those of co-ordinates, we have

$$N_1 = A,\ N_2 = B,\ N_3 = C,\ T_1 = T_2 = T_3 = 0;$$

and the quadric has for equation

$$Ax^2 + By^2 + Cz^2 = f.$$

The cone traced out by the normals to the planes of shearing stress is obviously the asymptotic cone of the quadric (2); and if this cone is real its reciprocal cone (the cone of shearing stress) will separate the planes on which the stress is pressure from those on which it is tension. When the cone is imaginary, all planes at the point O will be subject to stress of one kind—either pressure or tension.

When the cone is real, the quadric (2) must be accompanied by another whose equation is obtained by merely changing f to $-f$, as has been explained in the analogous case of strain (p. 461).

Another graphic mode of connecting the stress on a plane with the position of the plane is this. Let the principal planes be taken as the co-ordinate planes; then the components of the intensity of stress on any plane (l, m, n) are by equations (3), p. 491,

$$\left.\begin{aligned} P &= lA, \\ Q &= mB, \\ R &= nC. \end{aligned}\right\} \qquad (5)$$

Hence $\dfrac{P^2}{A^2}+\dfrac{Q^2}{B^2}+\dfrac{R^2}{C^2}=1$. Of course P, Q, R are the co-ordinates of the extremity of the line representing the intensity of stress on the plane (l, m, n). Hence the extremities of lines representing in magnitude and direction the intensities of stresses on all planes at O lie on the ellipsoid

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1, \qquad (6)$$

whose semi-axes are in magnitudes and directions the principal intensities of stress at O.

If a tangent plane be drawn to this ellipsoid parallel to the plane whose stress is considered, the length of the perpendicular from the centre on the tangent plane represents the magnitude of the intensity of stress, as is obvious by squaring and adding the sides of equations (5).

The ellipsoid (6) may for shortness be called the *Stress Ellipsoid*.

In proving general properties of stress simplicity is, of course, gained by taking the principal axes of the stresses as those of reference. Thus, with these axes, the cone of shearing stress is

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0,$$

and that traced out by the normals to planes of shearing stress is $Ax^2+By^2+Cz^2=0$; so that for the reality of these cones (i.e. *for the existence of planes subject wholly to shearing stress*) the principal stresses must consist either of one tension and two pressures, or two tensions and one pressure. With any system of axes the equation of the cone of shearing stress is

$$\begin{vmatrix} N_1 & T_3 & T_2 & x \\ T_3 & N_2 & T_1 & y \\ T_2 & T_1 & N_3 & z \\ x & y & z & 0 \end{vmatrix} = 0.$$

297.] **Work done in Strain.** We propose to investigate the work done in the strain of any small volume of the body.

About the point P (fig. 256, p. 458) let any small closed surface be drawn in the natural state of the body. Let ds be any element of this surface, and let the direction cosines of the normal to this element, measured outwards, be l, m, n. Then the components of *intensity* of stress (resulting from strain) on the element plane ds being P, Q, R, and the final displacements of the mean point of the element being (see Art. 275) $\Delta\xi, \Delta\eta, \Delta\zeta$, the work done in the displacement of the element will be (see Art. 217, p. 366)

$$\tfrac{1}{2}(P\Delta\xi + Q\Delta\eta + R\Delta\zeta)\,ds.$$

Hence the work done in the strain of the volume contained in the whole surface is

$$\tfrac{1}{2}\int(P\Delta\xi + Q\Delta\eta + R\Delta\zeta)\,ds.$$

Substituting for P its value (p. 491), the term Pds becomes $(lN_1 + mT_3 + nT_2)\,ds$.

But if $d\sigma_1$, $d\sigma_2$, $d\sigma_3$ are the projections of ds on the planes of yz, zx, and xy, respectively, $lds = d\sigma_1$, $mds = d\sigma_2$, $nds = d\sigma_3$; so that the work done becomes

$$\tfrac{1}{2}\int(N_1\Delta\xi + T_3\Delta\eta + T_2\Delta\zeta)\,d\sigma_1 + \tfrac{1}{2}\int(T_3\Delta\xi + N_2\Delta\eta + T_1\Delta\zeta)\,d\sigma_2 + \tfrac{1}{2}\int(T_2\Delta\xi + T_1\Delta\eta + N_3\Delta\zeta)\,d\sigma_3.$$

The intensities of stress $N_1, N_2, \ldots$ may be considered as constant over the surface and taken outside the integral signs. Also substituting for $\Delta\xi, \Delta\eta, \Delta\zeta$ their values (Art. 275), we have

$$\int\Delta\xi\,d\sigma_1 = \int\left(\xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}\right)d\sigma_1 = \frac{du}{dx}\int\xi\,d\sigma_1 + \frac{du}{dy}\int\eta\,d\sigma_1 + \frac{du}{dz}\int\zeta\,d\sigma_1.$$

Now, the surface being closed, $\int\xi\,d\sigma_1 = d\omega =$ volume enclosed by surface; and $\int\eta\,d\sigma_1 = \int\zeta\,d\sigma_1 = 0$, since, the normal being always drawn outwards, the elementary projections $d\sigma_1$ on one side of the plane yz must be given a sign opposite to the sign of those on the other side.

In this way we have also

$$\int\eta\,d\sigma_2 = \int\zeta\,d\sigma_3 = d\omega;\ \int\xi\,d\sigma_2 = \int\xi\,d\sigma_3 = \ldots = 0.$$

Hence the work of straining the element of volume considered is

$$\tfrac{1}{2}(N_1 a + N_2 b + N_3 c + 2T_1 s_1 + 2T_2 s_2 + 2T_3 s_3)\,d\omega,$$

where $a, b, c, 2s_1, 2s_2, 2s_3$ are, as usual, the simple elongations and shears of the strain. If we use the principal elongations and stresses, the work is

$$\tfrac{1}{2}(Ae_1 + Be_2 + Ce_3)\,d\omega.$$

Examples.

1. To resolve a shearing stress of intensity S, which is exerted on two given rectangular planes at any point into its components with reference to any three rectangular planes at the point.

Let O (fig. 13, p. 19) be the point, and suppose that the stress on all planes parallel to $BOCF$ is a shearing stress of intensity S, and that the stress on all planes parallel to $AOCH$ is also a shearing stress of (necessarily) the same intensity (see p. 493), while there is no stress on planes parallel to $AOBD$.

Let the direction-cosines of the normals, OA, OB, OC, to these planes with reference to three rectangular axes of x, y, z, be (l, m, n), (l', m', n'), (l'', m'', n''). Then for the system of planes on which the stresses are given we have $N_1' = N_2' = N_3' = 0$, and also $T_1' = T_2' = 0$, since there is no stress on $AOBD$. Therefore if P', Q', R' are the components along OA, OB, OC of the intensity of stress on a plane whose direction-cosines with respect to these lines are λ, μ, ν, we have

$$P' = \mu S,\ Q' = \lambda S,\ R' = 0.$$

Hence the components along OA, OB, OC of the intensity stress on the plane yz are

$$P' = l'S,\ Q' = lS,\ R' = 0\,;$$

and N_1 is the sum of the components of these along the axis of x;

therefore $\quad N_1 = lP' + l'Q' + l''R' = 2ll'S.$

Also $\quad T_3 = mP' + m'Q' + m''R' = (lm' + l'm)\,S,$

$$T_2 = nP' + n'Q' + n''R' = (ln' + l'n)\,S\,;$$

and hence the components of the given shearing stress are

$$2ll'S,\ 2mm'S,\ 2nn'S,\ (lm' + l'm)\,S,\ (ln' + l'n)\,S,\ (mn' + m'n)\,S.$$

(Compare with the resolution of a shearing strain, p. 484.)

2. Two normal stresses on two rectangular planes are combined with two shearing stresses on the same planes; find the principal planes and intensities of the resultant stress.

Let fig. 266, p. 492, represent the normal stresses N_1 and N_2 acting on planes at right angles to each other and to the plane of the paper, and combined with shearing stresses, T_3, or S, in these planes. (Of course the figure represents the *equilibrium* of an element of the body.) Since there is no stress on any plane parallel to the plane of the paper, the stress on *every* plane lies in the plane of the paper (p. 491). Then $T_3 = S$, $N_3 = T_1 = T_2 = 0$; and the principal planes are obviously perpendicular to the plane of the paper. Let the normal to any plane perpendicular to the paper make an angle θ with the direction of N_1. Then the components of stress on this plane are

$$P = N_1 \cos\theta + S \sin\theta,$$
$$Q = S \cos\theta + N_2 \sin\theta.$$

For a principal plane $P = F.\cos\theta$, $Q = F.\sin\theta$, where F is a principal stress. Hence

$$(N_1 - F).\cos\theta + S.\sin\theta = 0,$$
$$S.\cos\theta + (N_2 - F).\sin\theta = 0.$$

From these equations we find the two principal intensities of stress to be

$$\tfrac{1}{2}\left[N_1 + N_2 \pm \sqrt{(N_1 - N_2)^2 + 4S^2}\right],$$

and the directions of the principal planes are given by the equation

$$\tan 2\theta = \frac{2S}{N_1 - N_2}.$$

3. If the stress on a plane is wholly a shearing stress, prove that its line of action is the line of contact of the plane with the cone of shearing stress, and find its magnitude.

Since $P = lA$, $Q = mB$, $R = nC$, a point whose co-ordinates are P, Q, R will lie on the cone $\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0$, if $Al^2 + Bm^2 + Cn^2 = 0$; that is, the extremity of the line representing the intensity of stress will lie on the cone of shearing stress if the stress is wholly shearing. Therefore, &c. Since the magnitudes of all stress intensities are represented by the radii vectores of the ellipsoid $\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1$, the intensities of shearing stress will be represented by the radii vectores of this ellipsoid measured along the edges of the cone of shearing stress.

4. If at any point in a body the principal stresses consist of two tensions of intensities A and B $(A > B)$ and a pressure of intensity C, prove that the maximum intensity of shearing stress is $\sqrt{AC}$, and find the plane on which it is exerted.

$$\textit{Ans.}\quad l = \sqrt{\frac{C}{A+C}},\ m = 0,\ n = \sqrt{\frac{A}{A+C}}.$$

5. If at any point in a body the principal stresses consist of a tension of intensity A and two pressures of intensities B and C $(B > C)$, prove that the maximum intensity of shearing stress is $\sqrt{AB}$, and find the plane on which it is exerted.

$$\textit{Ans.}\quad l = \sqrt{\frac{B}{A+B}},\ m = \sqrt{\frac{A}{A+B}},\ n = 0.$$

6. Find the conditions that a stress whose components with respect

to any three rectangular axes are given should produce shearing stress on two planes only, and these rectangular.

Ans. $\begin{vmatrix} N_1, & T_3, & T_2 \\ T_3, & N_2, & T_1 \\ T_2, & T_1, & N_3 \end{vmatrix} = 0$, and $N_1+N_2+N_3 = 0$; the first expresses that the product of the three principal stresses $= 0$, and the second that their sum $= 0$; so that one principal stress must be zero and the other two a tension and a pressure of equal intensities.

7. Given the components of the stress with reference to the principal axes of the stress, find the components of the same stress with reference to any set of rectangular axes.

Ans. $N_1 = Al^2+Bm^2+Cn^2,\ N_2 = Al'^2+Bm'^2+Cn'^2,$
$N_3 = Al''^2+Bm''^2+Cn''^2.$
$T_1 = Al'l''+Bm'm''+Cn'n'',\ T_2 = All''+Bmm''+Cnn'',$
$T_3 = All'+Bmm'+Cnn'.$

Section III.

Expression of Stress in terms of Strain.

298.] **Coefficients of Elasticity.** The strain at any point depends, in the first instance, on the nine quantities

$$\frac{du}{dx},\ \frac{du}{dy},\ \frac{du}{dz},\ \frac{dv}{dx},\ \frac{dv}{dy},\ \frac{dv}{dz},\ \frac{dw}{dx},\ \frac{dw}{dy},\ \frac{dw}{dz}.$$

Now the strain being small, we may evidently assume that if these components of strain are all increased in the same ratio, the stress components which correspond to them will all be increased in the same ratio. Hence each of the six stress components, N_1, N_2, N_3, T_1, T_2, T_3, is a *linear* function of the nine strain components; so that we have, for example,

$$N_1 = c_1\frac{du}{dx} + c_2\frac{du}{dy} + c_3\frac{du}{dz} + c_4\frac{dv}{dx} + c_5\frac{dv}{dy} + c_6\frac{dv}{dz} + c_7\frac{dw}{dx} + c_8\frac{dw}{dy} + c_9\frac{dw}{dz},$$

with similar values of N_2, &c. In this way we should have fifty-four distinct coefficients, c_1, c_2, ..., expressing the stress in terms of the strain.

A first reduction in this number is obvious; for in the strain the terms $\frac{du}{dy}$ and $\frac{dv}{dx}$ always go together in the form $\frac{du}{dy}+\frac{dv}{dx}$, which is a shear; and similarly we have two other pairs, which are also shears. Hence, as the strain really involves only *six* components, a, b, c, $2s_1$, $2s_2$, $2s_3$, each stress component is a linear function of only six quantities; and there are therefore only thirty-six distinct coefficients.

There is a further reduction of this number to twenty-one in all cases of strain, irrespective of the nature of the strained body—a reduction which is thus made by Green (see the *Mathematical Papers of the late George Green*, pp. 249, &c.).

The work done in bringing a body from any one state of strain to any other must be simply a function of the quantities which define the magnitudes of the two strains; i.e., it cannot depend on the order or nature of the series of states of strain, through which the body may pass from the first state to the second; in other words, the stresses must be a conservative system (see p 309). For, if this were not the case, we might bring the body from a state (A) to a state (B) through a certain series of states by the expenditure of a certain amount, W, of work, and then (by constraint, implying no expenditure of work) make it return from (B) to (A) through another series of states, and in this series we might receive from the stresses an amount, $W+W'$, of work done against external resistances. Each cycle of changes would therefore *create* an amount of work, and perpetual motion would be possible. The reasoning would be conclusive were it not for the fact (well pointed out and explained by Thomson and Tait, *Nat. Phil.*) that compression (as a rule) generates heat and extension (as a rule) causes a loss of heat; and this alteration of temperature at every moment affects the elasticity of the body, and therefore the stresses. Hence even when the body is at two different times in the same state of strain, the stresses may not be the same in these states; and the above reasoning for the existence of a potential of stress falls to the ground.

If, however, the states of strain are produced slowly, so that the temperature may be sensibly constant, the stresses will always be the same in the same state of strain; and the work done in strain will be simply a function of the strain.

By p. 499, the work done in the very small strain

$$(da, \dots ds_1, \dots)$$

of an element of volume $d\omega$ is

$$\tfrac{1}{2}(N_1 da + N_2 db + N_3 dc + 2T_1 ds_1 + 2T_2 ds_2 + 2T_3 ds_3)\, d\omega;$$

and if $\phi(a, b, c, s_1, s_2, s_3)$ is the potential of the strain per unit of volume, this work must be

$$\left(\frac{d\phi}{da}da+\frac{d\phi}{db}db+\ldots+\frac{d\phi}{ds_1}ds_1+\ldots\right)d\omega.$$

Hence $$N_1=2\frac{d\phi}{da}, \quad N_2=2\frac{d\phi}{db}, \ldots T_1=\frac{d\phi}{ds_1}, \ldots$$

Since $N_1, \ldots$ are linear functions of $a, \ldots$, ϕ is obviously a homogeneous quadratic function of the six components of strain, and it has therefore twenty-one distinct coefficients, which are those entering into the values of the components of stress.

For the particular case of Isotropic Bodies (p. 474) these coefficients reduce to *two*, as has been differently shown by Green, Lamé, and Rankine. Green's method consists essentially in so determining the constants in ϕ that it shall be symmetrical all round each of three axes—as it must be for isotropic, as distinguished from crystalline, bodies.

299.] **Method of Cauchy.** This simple method consists in assuming that at every point in a strained isotropic body the principal axes of the strain coincide with the principal axes of the stress. Here then we have

$$s_1=s_2=s_3=0, \quad T_1=T_2=T_3=0.$$

Also we can assume

$$A=(\lambda+2\mu)e_1+\lambda e_2+\lambda e_3,$$

where λ and μ are constants; for e_2 and e_3 must evidently have the same coefficient in the value of A, since the body is elastically symmetric with regard to the axes of y and z (and, of course, with regard to *all* axes) and the plane on which N_1 acts is also symmetrically placed with respect to them. Thus

$$\left.\begin{aligned} A&=\lambda\theta+2\mu e_1,\\ B&=\lambda\theta+2\mu e_2,\\ C&=\lambda\theta+2\mu e_3,\end{aligned}\right\} \qquad (1)$$

where $\theta=e_1+e_2+e_3=$ the cubical dilatation, and e_1, e_2, e_3 are the principal elongations.

It is required to express the components, $N_1, N_2, N_3, T_1, T_2, T_3$, of the stress at the point considered in the body with reference to three rectangular axes at the point and the corresponding components of the strain. Let (l, m, n), &c., be the direction-cosines of the new axes with reference to the principal axes of strain and stress. Then by multiplying both sides of equations

(1) by l^2, m^2, n^2, respectively, and adding, we have by example 7, p. 502, and example 9, p. 485,

Similarly
$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu a. \\ N_2 &= \lambda\theta + 2\mu b, \\ N_3 &= \lambda\theta + 2\mu c. \end{aligned}\right\} \qquad (2)$$

And by multiplying the sides of equations (1) by $l'l''$, $m'm''$, $n'n''$, and adding, we have by the same examples

Similarly,
$$\left.\begin{aligned} T_1 &= 2\mu s_1. \\ T_2 &= 2\mu s_2, \\ T_3 &= 2\mu s_3. \end{aligned}\right\} \qquad (3)$$

300.] Method of Thomson and Tait. If a spherical portion of an isotropic body be subject to pressure of uniform intensity all over its surface, it must in yielding retain its spherical form, i.e. it experiences no distortion. And if a cube of it be subject to shearing stresses in the planes of its faces, it must, for a small strain, undergo distortion (into the shape of a slightly oblique parallelopiped) without alteration of volume, and the amount of this distortion (defined as in Art. 286) must be the same no matter to what side of any face the shearing stress is parallel.

Consequently the elastic quality of a completely isotropic body depends on two, and only two, constants which are the same throughout its mass—viz., its resistance to dilatation (or compression) and its resistance to distortion.

Resistance to Dilatation. To find this constant, let a uniform tension (or pressure) of intensity N be applied all over the surface of any portion of the body and let it produce a small dilatation (or compression) of this portion, the amount of this dilatation being θ (defined as in Cor. 4, Art. 278); then the resistance to change of volume is

$$\frac{N}{\theta}.$$

This resistance (since θ is a number) is *a force per unit of area.*

Resistance to Distortion. To find this constant, let a shearing stress of intensity S be applied to any pair of parallel planes, and let the amount of the shear (defined as in Art. 286), be denoted by $2s$; then the resistance to distortion is

$$\frac{S}{2s}.$$

This resistance (since s is a number) is *a force per unit of area.*

Denote these two coefficients respectively by k and μ.

The values of the shearing stresses, T_1, T_2, T_3, in terms of the shears (given in equations (3) of last Art.) follow at once.

To find the stresses called into play by a simple elongation, a, along the axis of x, resolve this elongation exactly as in example 3, p. 483, into a cubical dilatation a together with two shears. Now, by our above definition, the dilatation will cause a normal intensity of stress equal to ka on each face of a cubical element whose edges coincide with Ox, Oy, and Oz at the point O.

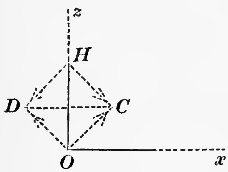

Fig. 268.

Consider the elongation $\frac{1}{3}a$ along Ox and the accompanying contraction $\frac{1}{3}a$ along Oz. These give shears each equal to $\frac{2}{3}a$ on the planes $OCHD$ inclined at angles of 45° to Ox and Oz; and these shears will, by the above definition, give rise to shearing stresses each of intensity $\frac{2}{3}\mu a$ on these planes. Again, by p. 493, these shearing stresses will give rise to normal stresses each of intensity $\frac{2}{3}\mu a$ on planes parallel to OH and CD; and it is obvious that the normal stress on the plane OH (or rather the plane through OH perpendicular to the paper) produced by the portion of the body to the right of OH will be *tension*, i. e., it will be in the sense Ox; while on the plane CD (or Ox) the normal stress produced by the portion of the body at the upper side of the figure will be *pressure*, i. e., it will be in the sense zO.

Similarly by considering the other shear (that which consists of elongation $\frac{1}{3}a$ along Ox and contraction $\frac{1}{3}a$ along Oy) we have a further normal tension equal to $\frac{2}{3}\mu a$ on the plane perpendicular to Ox; and normal pressure $\frac{2}{3}\mu a$ on the plane perpendicular to Oy. Hence the elongation a gives normal stresses

$$(k+\tfrac{4}{3}\mu)\,a, \quad (k-\tfrac{2}{3}\mu)\,a, \quad (k-\tfrac{2}{3}\mu)\,a,$$

on the planes perpendicular to Ox, Oy, Oz, respectively.

Similarly the elongation b (which is along Oy) gives normal stresses

$$(k-\tfrac{2}{3}\mu)\,b, \quad (k+\tfrac{4}{3}\mu)\,b, \quad (k-\tfrac{2}{3}\mu)\,b$$

in the same directions; and the remaining elongation, c, gives

$$(k-\tfrac{2}{3}\mu)\,c, \quad (k-\tfrac{2}{3}\mu)\,c, \quad (k+\tfrac{4}{3}\mu)\,c.$$

Hence we have

$$N_1 = (k+\tfrac{4}{3}\mu)\,a+(k-\tfrac{2}{3}\mu)\,b+(k-\tfrac{2}{3}\mu)\,c;$$

$$\left.\begin{aligned} \text{or} \qquad N_1 &= (k-\tfrac{2}{3}\mu)\,\theta + 2\mu a; \\ \text{and} \qquad N_2 &= (k-\tfrac{2}{3}\mu)\,\theta + 2\mu b, \\ N_3 &= (k-\tfrac{2}{3}\mu)\,\theta + 2\mu c, \end{aligned}\right\} \tag{A}$$

where $\theta \equiv a+b+c =$ the cubical dilatation.

EXAMPLES.

1. To express Young's modulus in terms of the resistances to dilatation and distortion.

Let a bar of the body be subject to traction, as in Art. 287.

Then we have $N_3 = (k-\frac{2}{3}\mu)(c-2a) + 2\mu c$; $N_1 = N_2 = (k-\frac{2}{3}\mu)(c-2a) - 2\mu a$. But the intensity of the elongating stress is N_3, and the elongation (per unit of length) is c; therefore if $E =$ Young's modulus,

$$E = \frac{N_3}{c}.$$

Also since there is no force on the sides of the bar

$$N_1 = N_2 = 0, \quad \therefore \quad 2a = \frac{k-\frac{2}{3}\mu}{k+\frac{1}{3}\mu}\,c, \text{ and } \frac{N_3}{c} = (k-\tfrac{2}{3}\mu)\left(1-2\frac{a}{c}\right) + 2\mu,$$

$$\therefore \quad E = \frac{9k\mu}{3k+\mu}.$$

When a bar is elongated, it thus appears that there is lateral contraction (a) in all directions perpendicular to the axis of the bar, and the ratio of this to the elongation (c) is

$$\frac{3k-2\mu}{2(3k+\mu)}.$$

2. One end of a bar of isotropic material is held fixed, and the bar hangs vertically; find its elongation caused by its weight.

Let AB be the bar in its natural state, P a point in AB at a distance z from A; let $A'B'$ represent the elongated bar, and let P' be the displaced position of P.

Then the intensity of stress on a normal section at $P' = E\dfrac{dw}{dz}$, where E is Young's modulus. But if ω is the area of the section at P', the intensity of stress $= \dfrac{\text{weight of length } PB}{\omega} = \dfrac{W}{\omega}\dfrac{l-z}{l}$, where W and l are the weight and length of the bar.

Hence
$$E\frac{dw}{dz} = \frac{W}{\omega}\,\frac{l-z}{l}.$$

$$\therefore \quad w = \frac{W}{E\omega l}(lz - \tfrac{1}{2}z^2) + C,$$

where C is a constant. Now the value of w for the fixed end is zero, therefore $C = 0$; and the value of w for the free end, B, is the amount of elongation. Hence, putting $z = l$,

$$\text{amount of elongation} = \frac{Wl}{2E\omega}.$$

It is immaterial whether ω means the section of the bar $A'B'$ or the section of AB, since these areas differ by a small quantity of the first order.

3. To find stresses produced at any point in a circular cylinder which undergoes torsion round its axis.

With the notation of p. 478, we have by Art. 299.

$$N_1 = 0, \quad N_2 = 0, \quad N_3 = 0,$$

$$T_1 = \frac{\mu a}{l}x, \quad T_2 = -\frac{\mu a}{l}y, \quad T_3 = 0.$$

The torsion may be produced either by fixing one end of the cylinder and applying a couple to the other end, or by applying two equal and opposite couples to the ends, each of which is free. By considering the equilibrium of a portion of the cylinder between one end and a section made at any point, O, (fig. 269) on the axis perpendicularly to the axis, we see that the stress system exerted over this section by the remaining portion of the cylinder must be a couple equal in amount to the applied couple, (F, F).

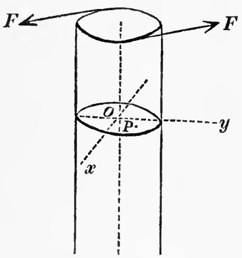

Fig. 269.

Let the fixed axes of x and y at O be Ox and Oy, and let P be a point in the section whose co-ordinates are x and y. Then the above values of the intensities of stress show that on the element area dS at P the two components of stress on the *lower* side of dS are $\frac{\mu a}{l} ydS$ in the direction Ox, and $\frac{\mu a}{l} xdS$ in the direction yO. The sum of their moments about Oz is $\frac{\mu a}{l}(x^2 + y^2)\,dS$ in a sense opposite to that of the applied couple. Hence if the moment of this couple is denoted by T,

$$\frac{\mu a}{l}\int r^2 dS = T,$$

where $r = OP$, and the integration is extended over the whole area of the section at O. Now $\int r^2 dS$ is the moment of inertia, I, of the section about Oz. Therefore

$$\frac{\mu a}{l} I = T.$$

Let $\frac{a}{l}$, which is the rate of twist per unit length of the cylinder be denoted by τ, and we have

$$\mu\tau I = T. \qquad (a)$$

If the cylinder is solid (having no hollow part), $I = \frac{\pi r^4}{2}$. The result in equation (a) is known as Coulomb's Law.

4. To show that Coulomb's Law cannot apply to a non-circular cylinder when it is acted on only by twisting couples at its extremities.

In order that the law of torsion strain expressed by the equations

$$u = -\tau yz,\ v = \tau xz,\ w = 0$$

may hold we shall show that force must be applied over the bounding surface of the cylinder parallel to its axis.

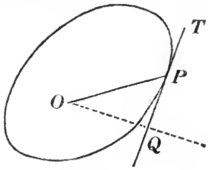

Fig. 270.

Let fig. 270 represent a section of the cylinder perpendicular to its axis, the axis passing through O; let P be a point on the bounding surface, PT the tangent to the section, and OQ a perpendicular to PT. Let OQ be taken as axis of x, the axis of z being the axis of the cylinder; and let us calculate the stress on an element plane which touches the bounding surface at P. We have for this plane $l = 1$, $m = 0$, $n = 0$; and equations (3) p. 491 give (by last example)

$$P = 0,\ Q = 0,\ R = -\mu\tau y = -\mu\tau \,.\, PQ;$$

i.e., the stress on this plane is proportional to PQ, and there must be an applied force to balance this stress, since there is none of the material of the cylinder at the right-hand side of the plane. (See *Remark*, p. 491.)

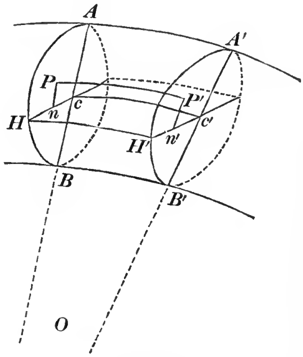

Fig. 271.

5. Let there be a straight solid bar or beam subject to a slight bending strain such that the fibres (*mean fibres*) which lie in a certain plane, although bent, are not elongated, and that the elongation (positive or negative) along every other fibre is proportional to its (positive or negative) distance from this plane, the bending of all fibres taking place parallel to a single plane which cuts the normal section of the bar perpendicularly. It is required to find for any normal section the sum of the moments, round the line in which it intersects the plane of the mean fibres, of the stresses which are exerted at the section by the strained fibres.

Suppose that after the bending any one section, AHB (fig. 271), is brought by a motion as of a rigid body (Art. 274) back to its old position, and let a neighbouring section then occupy the position $A'H'B'$. Let HH', cc' be two of the mean fibres which reach across from one of the sections to the other. Then the original distance between the sections is HH' or cc'. Let this be denoted by ds. If PP' is any other fibre reaching across, Pn and $P'n'$ the perpendiculars from P and P' on the right lines cH and $c'H'$, the elongation along PP' (i.e. $\frac{PP'-ds}{ds}$) is proportional to Pn. Let the planes of the sections AHB and $A'H'B'$ intersect in a line OL, let ρ denote the length of the radius of curvature (cO) of the bent mean fibres (cc' or nn'), and $Pn = y$. Then evidently

$$\frac{PP'}{nn'} = \frac{\rho+y}{\rho},$$

$$\therefore \quad \frac{PP'-nn'}{nn'} = \frac{y}{\rho},$$

which is the elongation along PP'. For fibres at the lower side of cH, there is contraction, or negative elongation, and for these y is reckoned as negative.

Now, by Hooke's Law, if we consider a small prism whose sides are the fibres emanating from points on a very small area, $d\sigma$, at the point P, the longitudinal stress of this prism is (p. 364)

$$\frac{Ey}{\rho}\,d\sigma.$$

The moment of this force about cH is $\frac{Ey^2}{\rho}d\sigma$; therefore the sum of these moments all over the section AHB is $\frac{E}{\rho}\int y^2 d\sigma$, or

$$\frac{EI}{\rho}, \tag{a}$$

where I is the Moment of Inertia of the section AHB about the line cH.

Remark. If the end of a beam merely rests against a fixed surface, there will be no Bending Moment at this end, and $\rho = \infty$ at it. But if the end is tangentially fixed there will be a Bending Moment at it, and its curvature will not be zero.

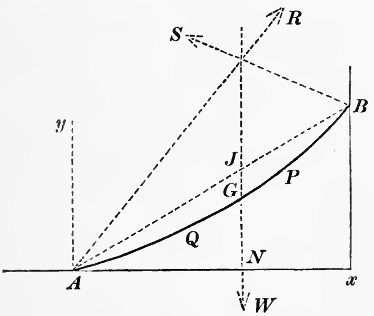

Fig. 272.

6. A uniform slightly elastic beam rests, in non-limiting equilibrium, with one end on the ground and the other against a vertical wall, the vertical plane through the beam being at right angles to the wall; find the form of the mean fibre of the beam. Let AB (fig. 272) be the beam; GN the vertical through its centre of gravity, G; R and S the reactions of the ground and wall; ϕ the angle made by R with the vertical; α the angle which the tangent to the beam at A makes with the horizon; h and

k the distances, Ax and Bx, of the extremities from the line of intersection of the ground and wall.

Let P be any point in the beam, at which we shall calculate the Bending Moment, i.e., the sum of the moments of all the forces acting on the beam between P and A; let the horizontal and vertical lines through A be taken as axes of x and y; let Q be any point between P and A; let the co-ordinates of P and Q be (x, y) and (x', y'), respectively; let the original length of the beam be l, and its weight W.

Then the weight of an element of length, ds', at Q is $\frac{W}{l}\,ds'$, and the moment of this force tending to produce curvature at P round a line (such as cH in fig. 271) perpendicular to the plane of the figure is

$$-\frac{W}{l}(x-x')\,ds'.$$

Also the moment of R about this axis is

$$R\,(x\cos\phi - y\sin\phi).$$

Hence if ρ is the radius of curvature of the mean fibre at P, we have

$$\frac{EI}{\rho} = R\,(x\cos\phi - y\sin\phi) - \frac{W}{l}\int(x-x')\,ds', \tag{1}$$

the integration being performed from A to P.

If P is taken very close to A, the Bending Moment on the right side of (1) is zero, therefore ρ at $A = \infty$, i.e., A is a point of inflexion; and B is also a point of inflexion for a similar reason.

Assume $$y = x\tan a + a_3x^3 + a_4x^4 + a_5x^5 \ldots, \tag{2}$$

where a_3, a_4, a_5, ... are all very small quantities; there being no term in x^2 since $\frac{d^2y}{dx^2} = 0$ ($\rho = \infty$) at A.

From (2), we find

$$\frac{ds}{dx} = \sec a + \sin a\,(3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots)$$

$$\frac{d^2y}{dx^2} = 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots.$$

Now $\frac{1}{\rho} = \dfrac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}$; and if we neglect products of a_3, a_4, ...,

we shall have $\frac{1}{\rho} = \cos^3 a\,(6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots)$.

Also

$$\int_0^x (x-x')\frac{ds'}{dx'}\cdot dx' = \tfrac{1}{2}x^2\sec a + \tfrac{1}{4}a_3\sin a\,.\,x^4 + \tfrac{1}{5}a_4\sin a\,.\,x^5 + \ldots.$$

Making these substitutions in (1), and equating to zero the coefficient of every power of x, we have

$$a_3 = \frac{R \sin \phi \, (\cot \phi - \tan a)}{6EI \cos^3 a},$$

$$a_4 = - \frac{W}{24lEI \cos^4 a},$$

while a_5, a_6, ... are of the order $\frac{1}{(EI)^2}$ and may be neglected.

Also at the extremity B, $\frac{d^2y}{dx^2}$ must be zero; therefore

$$a_3 + 2ha_4 = 0;$$

and the equation of the mean fibre is

$$y = x \tan a + \frac{W}{24lEI}(2hx^3 - x^4) \sec^4 a.$$

By putting k and h for y and x, this equation gives

$$\tan a = \frac{k}{h} - \frac{Wh^3}{24hlEI} \sec^4 a.$$

Putting $\sec a = \frac{\sqrt{k^2+h^2}}{h}$ in the same term, we get

$$\tan a = \frac{k}{h} - \frac{Wl'^4}{24hlEI},$$

where l' is used for $\sqrt{k^2+h^2}$.

Substituting this value of a in the equation of the mean fibre, we have

$$y = \frac{k}{h}x - \frac{Wl'^4}{24lh^4EI}(h^3x - 2hx^3 + x^4),$$

which is the equation of the mean fibre, to the first power of $\frac{1}{EI}$.

It will be easily found that AN, the abscissa of the centre of gravity of the beam, is

$$\frac{h}{2}\left(1 + \frac{Wkl'^2}{60lEI}\right).$$

7. A rigid bar is supported nearly horizontally on three given vertical props which are slightly elastic; to determine the pressures on these props.

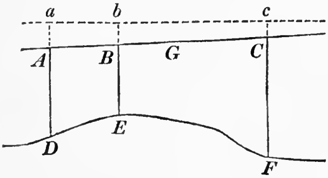

Fig. 273.

Suppose that the props are fixed in the ground at D, E, and F (fig. 273), and that their extremities were originallly a, b, c, which are in a horizontal line; but that when the shrinking has taken place, their extremities, A, B, C, lie in a line slightly inclined to the horizon. Let their original lengths be p, q, r, so that $Aa = \delta p$, $Bb = \delta q$, $Cc = \delta r$; let the pressures on them at A, B and C be P, Q, and R; let G be the centre of gravity of the bar and W its weight.

Then we have

$$P+Q+R = W, \text{ and } P \,.\, GA + Q \,.\, GB - R \,.\, GC = 0, \tag{1}$$

the second being obtained by moments about G,

Now if the areas of the normal sections of the props are α, β, γ, we have (Art. 216)

$$\frac{P}{\alpha} = E\frac{\delta p}{p}, \quad \frac{Q}{\beta} = E\frac{\delta q}{q}, \quad \frac{R}{\gamma} = E\frac{\delta r}{r}, \tag{2}$$

supposing that Young's modulus is the same for all.

Again, we must express the fact that ABC is a right line. Drawing through C a parallel to abc, we have

$$\frac{\delta p - \delta r}{\delta q - \delta r} = \frac{AC}{BC},$$

$$\therefore \quad BC \,.\, \delta p - AC \,.\, \delta q + AB \,.\, \delta r = 0, \tag{3}$$

or, by (2),

$$\frac{p \,.\, BC}{\alpha} P - \frac{q \,.\, AC}{\beta} Q + \frac{r \,.\, AB}{\gamma} R = 0. \tag{4}$$

The three equations (1) and (4) determine P, Q, R.

8. A heavy rigid slab is supported nearly horizontally on four given vertical props; to determine the pressures on these props.

Let the extremities, A, B, C, D, of the props when the shrinking has taken place be represented in fig. 212, p. 294; let the original lengths of the props be p, q, r, s; let the perpendiculars from A and C on the diagonal BD be p' and r'; let those from B and D on AC be q' and s'; let the perpendiculars from G, the centre of gravity of the slab, on AC and BD be x and y; let P, Q, R, S be the pressures on the props, whose sections are α, β, γ, δ, respectively; and let $W=$ weight of slab. Then we have obviously the statical equations

$$P+Q+R+S = W, \quad Pp' - Rr' - Wx = 0, \quad Qq' - Ss' + Wy = 0, \tag{1}$$

[G is supposed to lie within the area AOD] the two latter being equations of moments round BD and AC.

We must now express the fact that A, B, C, D lie in one plane.

To do this we shall calculate the vertical descent, $\delta\xi$, of the point O from the descents of A and C and also from those of B and D. Just as in last example, we have

$$\frac{\delta p - \delta r}{\delta\xi - \delta r} = \frac{AC}{OC} = \frac{p'+r'}{r'}, \quad \therefore \quad \delta\xi = \frac{r'\delta p + p'\delta r}{p'+r'}.$$

Similarly

$$\delta\xi = \frac{s'\delta q + q'\delta s}{q'+s'};$$

therefore

$$\frac{r'\delta p + p'\delta r}{p'+r'} = \frac{s'\delta q + q'\delta s}{q'+s'}. \tag{2}$$

Also, as before, $\dfrac{P}{\alpha} = E\dfrac{\delta p}{p}$, &c., therefore (2) becomes

$$\frac{pr'}{\alpha(p'+r')} P - \frac{qs'}{\beta(q'+s')} Q + \frac{rp'}{\gamma(p'+r')} R - \frac{sq'}{\delta(q'+s')} S = 0. \tag{3}$$

The four equations (1) and (3) determine the pressures.

9. When the external forces have a potential (for the law of inverse square), prove that the cubical dilatation satisfies the equation

$$\nabla\theta = 0,$$

and that each component (u) of displacement satisfies the equation

$$\nabla^2 u = 0,$$

where $\nabla \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$.

These results follow easily. For if X, Y, Z in equations (1), p. 489, are $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$, and if $\nabla V = 0$, we obtain $\nabla\theta = 0$ by differentiating the first of these with respect to x, the second with respect to y, the third with respect to z, and adding, using the values of N_1, T_3, &c., given in equations (2) and (3) of p. 505.

10. If $U = 0$ is the equation of the surface of any solid subject to strain, but having no superficial stress prove that at all points on the surface,

$$N_1 U_1 + T_3 U_2 + T_2 U_3 = 0,$$

$$T_3 U_1 + N_2 U_2 + T_1 U_3 = 0,$$

$$T_2 U_1 + T_1 U_2 + N_3 U_3 = 0,$$

where $U_1 \equiv \frac{dU}{dx}$, $U_2 \equiv \frac{dU}{dy}$, $U_3 \equiv \frac{dU}{dz}$; and that the stresses on all planes passing any point on the surface lie in the tangent plane at this point.

11. Investigate an expression, in terms of stress alone, for the work done in the small strain of a body.

It has been shown (p. 500) that the work done in the strain of an element, $d\omega$, of volume is $\frac{1}{2}(Ae_1 + Be_2 + Ce_3)\,d\omega$. Now in equations (A) p. 507, using the principal stresses A, B, C for N_1, N_2, N_3, and the principal strains e_1, e_2, e_3 for $\frac{du}{dx}$, $\frac{dv}{dy}$, $\frac{dw}{dz}$ we have $A + B + C = 3k\theta$; and multiplying them by A, B, C and adding, we have

$$2\mu(Ae_1 + Be_2 + Ce_3) = A^2 + B^2 + C^2 - \frac{k - \frac{2}{3}\mu}{3k}(A + B + C)^2.$$

Therefore if $A + B + C \equiv S$, and $AB + BC + CA \equiv \Sigma$,

$$Ae_1 + Be_2 + Ce_3 = \frac{S^2}{E} - \frac{\Sigma}{\mu},$$

where E is Young's modulus. Hence the whole work of deformation is

$$\tfrac{1}{2}\int \left(\frac{S^2}{E} - \frac{\Sigma}{\mu}\right) d\omega,$$

the integration being extended throughout the whole body.

If we do not employ the principal stresses and strains, but those having reference to a given set of axes, the same expression gives the work, and S will stand for

$$N_1+N_2+N_3 \text{ and } \Sigma \text{ for } N_1N_2+N_2N_3+N_3N_1-T_1^2-T_2^2-T_3^2.$$

This expression for the work of deformation is *Clapeyron's Theorem.* (See Lamé's *Leçons sur L'Élasticité*, p. 83.)

12. Find the work done in the uniform compression of a body.

Ans. If P is the intensity of external pressure exerted all over the surface, V the original volume and V' the final volume, the work is $\frac{1}{2}P(V-V')$.

[In a uniform compression $u=-ax$, $v=-ay$, $w=-az$; and of course $\theta = \dfrac{V-V'}{V}$. The principal stresses are equal at all points, and each $=P$.]

13. Prove that, although the volume of a solid body may not have changed during a small strain, there is work done in its deformation, and find an expression for this work.

Ans. The work $= \dfrac{1}{2\mu}\int(N_1^2+N_2^2+N_3^2+2T_1^2+2T_2^2+2T_3^2)\,d\omega$, or $\dfrac{1}{2\mu}\int(A^2+B^2+C^2)\,d\omega$ if we express it in terms of the principal stresses at each point; and this cannot possibly vanish unless all the components of internal stress vanish. (Lamé, p. 85.)

[Assuming no change of volume at any element,

$$\theta = 0, \text{ therefore } N_1+N_2+N_3 = 0.$$

In a fluid the stresses are all of the same kind (pressures) therefore the work $=0$ if $\theta = 0$.]

14. If throughout a body there is only one principal stress (A), which is constant, prove that the work of deformation is

$$\frac{VA^2}{2E},$$

where V is its volume. (Lamé, p. 83.)

15. A weight is placed on an ordinary rectangular table which rests on the ground; calculate the pressures on the four legs, supposing that the legs may be treated as rigid in comparison with the ground.

Ans. If the adjacent sides at any corner A are b and a, and if x and y are the distances from these sides, respectively, of the point of application of the resultant of the sustained weight and the weight of the table, the pressure on the leg through A is

$$\frac{W}{2}\left(\frac{3}{2}-\frac{x}{a}-\frac{y}{b}\right),$$

where $W=$ sum of sustained weight and weight of table.

16. Prove that a circular cylinder can be subject to the strain

$$u = -\tau yz, \qquad v = \tau zx, \qquad w = cxy,$$

(its axis being axis of z) provided that surface stress parallel to the axis is supplied.

17. Determine the components of strain as quadratic functions of the co-ordinates so that at all points we shall have

$$N_1 = N_2 = T_1 = T_3 = 0\,;$$

and show that such strain will require the application of external surface stress.

[Assume $u = px + qy + rz + \frac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)$, with similar values of v and w; then let the equations be satisfied at all points, i. e., equate to zero the coefficient of each variable.]

18. Construct a diagram of the work done in slowly extending a cylindrical bar.

[On the axis of x measure off from the origin a length, OA, equal to l_0, the natural length of the bar; at A draw a line making with the axis of x an angle whose tangent is the numerical value of $\frac{E\sigma}{l_0}$ (see p. 364). The ordinate, PM, of this line at any point, P, will represent the force which produces a length in OM in the bar; and the area of the triangle APM represents the work of extension. The result in p. 366 is graphically evident.]

19. A slightly elastic beam rests horizontally at any number of points against fixed vertical props, and is loaded uniformly between each successive pair of props. Prove that if M_1, M_2, M_3 denote the bending moments at three successive points, A_1, A_2, A_3, of support, we shall have

$$8(a+b)M_2 + 4aM_1 + 4bM_3 = wa^3 + w'b^3,$$

where $a = A_1A_2$, $b = A_2A_3$, $w =$ load per unit length throughout A_1A_2, $w' =$ load per unit length in A_2A_3.

[The Bending Moment at any point $= EI\,\frac{d^2y}{dx^2}$, since, $\frac{dy}{dx}$ being everywhere small, we may neglect its square.

This is known as *The Equation of Three Moments.*]

[The four following examples were communicated to the Author by the Rev. Professor Townsend.]

20. A horizontal beam, supported at both ends, being loaded with any number of isolated weights, if the bending moments be equal at any pair of contiguous weights, P and Q, they are equal throughout the entire interval PQ.

21. A uniform load, PQ, is moved along a horizontal beam supported at both ends, A and B; prove that at a given point, O, in the beam the bending moment will be greatest when PQ occupies such a position that $\frac{OP}{OQ} = \frac{OA}{OB}$.

22. A uniform beam is *tangentially fixed* at both extremities A and B, D is its point of greatest deflection, C is the foot of the perpendicular from D on AB; X is any point in the line AB; a perpendicular to AB at X meets the bent beam in Y and the circular arc through A, D, B in Z.

Prove that
$$XY = \frac{XZ^2}{CD}.$$

23. A uniform beam is supported by four equidistant props, two of which are terminal; prove that the two points of inflexion of its middle segment lie on the horizontal line of the props.

Miscellaneous Examples.

1. Let the magnetic curves of a magnet be described, and suppose electric currents to run in wires coinciding with the curves.

Prove that if C is the strength of the current in any wire and k the constant sum of cosines (see p. 39) corresponding to it, the force which it will exert on either pole of the magnet is proportional to
$$C\sqrt{2k-k^2}.$$

[Hence the curve which cuts the magnet perpendicularly exerts the maximum force.]

2. If the walls of a room and an insulated electrified body inside it are at the same potential, prove that no electrical effects (attractions or repulsions) will be observed in the room.

3. A uniform beam, AB, is supported horizontally at two points, C and D, in its length, C being adjacent to A and D to B. Prove that if two circles be described with C and D for centres and CA and DB for radii, respectively, the two points of inflexion of the beam are the two *limiting points* of the coaxal system determined by the circles. (Rev. Professor Townsend).

4. A force, R, given in magnitude, line of action, and sense, is resolved into two components, P, Q, which are subject solely to the condition of passing each through a given point; find a relation (involving only given quantities) between P, Q, and R.

5. Two equal bars, OA and OC, are freely jointed at the fixed point O; four equal bars forming a lozenge, $ABCD$, are freely jointed at A, B, C, and D, and the system (called a *Peaucellier's Cell*) is held in equilibrium by two forces applied at B and D. If the force at D is of constant magnitude in all positions of the cell, as it suffers deformation about O, prove that the force at B will be one varying inversely as the square of the distance OB. (Mr. G. H. Darwin, *Proceedings of the London Math. Soc.*, April 8, 1875. See the same paper for Mr. Darwin's most ingenious mechanical description of the

Equipotential Lines of any number of magnetic poles by means of Peaucellier's Cells).

6. A given system of forces is to be reduced to two inclined at the angle a; prove that the shortest distance between their lines of action cannot be less than $\frac{2G}{R}\cot\frac{a}{2}$. (Wolstenholme's *Book of Math. Prob.*, p. 387, second ed.)

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