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TRIGONOMETRY

FOR

BEGINNERS

AS FAR AS THE SOLUTION OF TRIANGLES.

Printed 1886.
Second Edition 1887.
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PREFACE.

The present work is an abridgement of the more complete work on Elementary Trigonometry by the same Author. A few of the Articles have been rewritten and the order in one or two cases slightly altered.

At the request of many Teachers a Table of the Logarithms of numbers from 100 to 1000 has been inserted. It will be seen [see Exercises xxxix. and xl.] that many interesting results may be obtained by the help of this Table.

In the second Edition the Chapter on Logarithms was revised.

In the third Edition 100 Easy Miscellaneous Examples were added.

In the fourth Edition a few corrections have been made, and a short Chapter on Triangles and Circles added.

CONTENTS.

CHAPTER I.

ON MEASUREMENT.

1. IT is usual to say that we have **measured** any concrete quantity, when we have found out **how many** times it contains some familiar quantity of the same kind.

We say for example, that we have measured a line, when we have found out *how many* feet it contains. We say that we have measured a field, when we have found out *how many* acres or how many square yards it contains.

2. To know the measurement of any quantity then, we must have two things. First, we must have a *unit*, or standard of reference, of the *same kind* as the thing measured. Secondly, we must have the *measure*, or the *number of times* the thing measured contains the unit, or standard quantity.

3. Hence, the **measure** of a quantity is the **number,** and the **unit** is the **concrete quantity,** by means of which it is measured.

Example 1. *A line contains* 261 feet; that is 261 times a foot. Here the *measure* or number is 261 and the *unit* a foot.

EXAMPLES. I.

1. What is the measure of 1 mile when a chain of 66 feet is the unit? 80

2. What is the measure of an acre when a square whose side is 22 yards is the unit?

3. What is the measure of a ton when a weight of 10 stone is the unit?

4. The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as unit.

5. The measure of a certain field is 22 and the unit 1100 square yards: express the area of the field in acres.

6. Find the measure of a miles when b yards is the unit.

7. The measure of a certain distance is a when the unit is c feet. Express the distance in yards.

8. A certain sum of money has for its measures 24, 240, 960 when three different coins are units respectively. If the first coin is half a sovereign, what are the others?

4. It is explained in Arithmetic, in the application of square measure, that the measure of the area of a rectangle is found in terms of a square unit, by multiplying together the measures of the sides in terms of the corresponding linear unit.

Example. Find in square feet, *the measure of a square surface whose side is* 12 feet.

The area is 12×12 square feet $= 144 \times 1$ square foot,

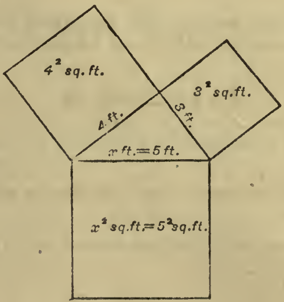
$\therefore$ the measure required is 144.

5. We shall apply this result to Euclid I. 47.

Example 1. *The sides containing the right angle of a right-angled triangle are* 3 ft. *and* 4 ft. *respectively; find the* length of the hypotenuse.

Let x be the number of feet in the hypotenuse.

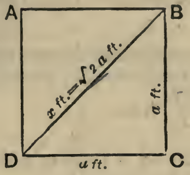
Then by Euclid I. 47, the square described on the side of x feet = the sum of the squares described on the sides of 3 feet and 4 feet respectively,



$$\therefore x^2 \text{ square feet} = 9 \text{ square feet} + 16 \text{ square feet}$$
$$= 25 \text{ square feet},$$
$$\therefore x^2 = 25,$$
$$\therefore x = 5.$$

Therefore the length of the hypotenuse is 5 feet.

Example 2. *Find the* length of the diameter *of the square one of whose sides contains a* feet.



Let $ABCD$ be the square, so that AB is a feet, and AD is a feet.

Let the diameter BD be x feet.

Then the square on DB = the sum of the squares on DA and AB.

$$\therefore x^2 \text{ sq. ft.} = a^2 \text{ sq. ft.} + a^2 \text{ sq. ft.}$$
$$\therefore x^2 = a^2 + a^2,$$
$$\therefore x^2 = 2a^2,$$
$$\therefore x = \sqrt{2} \,.\, a.$$

Thus the required length $\sqrt{2} \,.\, a$ feet $= (1{\cdot}4142 + \ldots) \times a$ ft.

EXAMPLES. II.

1. Find the length of the hypotenuse of a right-angled triangle whose sides are 6 feet and 8 feet respectively.

2. The hypotenuse of a right-angled triangle is 100 yards and one side is 60 yards : find the length of the other side.

3. One end of a rope 52 feet long is tied to the top of a pole 48 feet high and the other end is fastened to a peg in the ground. If the pole be vertical and the rope tight, find how far the peg is from the foot of the pole.

4. The houses in a certain street are 40 feet high and the street 30 feet wide: find the length of the ladder which will reach from the top of one of the houses to the opposite side of the street.

5. A wall 72 feet high is built at one edge of a moat 54 feet wide; how long must scaling ladders be to reach from the other edge of the moat to the top of the wall?

6. A field is a quarter of a mile long and three-sixteenths of a mile wide: how many cubic yards of gravel would be required to make a path 2 feet wide to join two opposite corners, the depth of the gravel being 2 inches?

7. The sides of a rectangular field are $4a$ feet and $3a$ feet respectively. Find the length of its diameter.

8. If the sides of an isosceles triangle be each $13a$ yards and the base $10a$ yards, what is the length of the perpendicular drawn from the vertex to the base?

9. Show that the perpendicular drawn from the right angle to the hypotenuse in an isosceles right-angled triangle, each of whose equal sides contains a feet, is $\frac{\sqrt{2}}{2} \cdot a$ ft.

10. If the hypotenuse of a right-angled isosceles triangle be a yards, what is the length of each side?

11. Show that the perpendicular drawn from an angular point to the opposite side of an equilateral triangle, each of whose sides contains a feet, is $\frac{\sqrt{3}}{2} \cdot a$ ft.

12. If in an equilateral triangle the length of the perpendicular drawn from an angular point to the opposite side be a feet, what is the length of the side of the triangle?

13. Find the ratio of the side of a square inscribed in a circle to the diameter of the circle.

14. Find the distance from the centre of a circle of radius 10 feet, of a chord whose length is 8 feet.

15. Find the length of a chord of a circle of radius a yards, which is distant b feet from the centre.

16. The three sides of a right-angled triangle, whose hypotenuse contains $5a$ feet, are in arithmetical progression; prove that the other two sides contain $4a$ feet and $3a$ feet respectively.

CHAPTER II.

On the Relation between the Circumference of a Circle and its Diameter.

6. The circumference of a circle is a line, and therefore it has length.

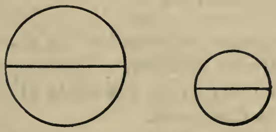
We might imagine the circumference of a circle to consist of a flexible wire; if the circular wire were cut at one point and straightened, we should have a straight line of the same length as the circumference of the circle.

7. A *polygon* is a figure enclosed by any number of straight lines.

A *regular* polygon has all its sides equal and all its angles equal.

The *perimeter* of a polygon is the sum of its sides.

8. If we have two circles in which the length of the diameter of the first is greater than the length of the diameter of the second, it is evident that the length of the circumference of the first will be greater than that of the second.



It is in fact true that when the length of one diameter = (any number of) n times that of another diameter the length of the circumference of the one = (the same number of) n times that of the other.

9. Hence when

$$\text{diameter} = n \times (\text{another diameter}),$$

then $\quad \text{circumference} = n \times (\text{the other circumference}),$

so that the ratio

$$\frac{\text{length of circumference}}{\text{length of diameter}}$$

is the same for all circles.

10. The proof of the above statement is given in more advanced works on Trigonometry. For the present the student must accept the following statements.

I. The ratio or number $\dfrac{\text{circumference}}{\text{diameter}}$ is a certain fixed number.

II. It is an *incommensurable* number.

III. It is 3·14159265 + ...

11. When we say that this number is incommensurable we mean that its exact value cannot be stated as an *arithmetical* fraction.

It also happens that we have no short *algebraical* expression such as a surd, or combination of surds, which represents it exactly.

So that we have no *numerical* expression whatever, arithmetical nor algebraical, to represent *exactly* the ratio of the circumference of a circle to its diameter.

Hence the universal custom has arisen, of denoting its *exact* value by the letter π.

12. Thus π stands *always* for **the exact value of a certain incommensurable number,** whose approximate value is **3·14159265,** which number is **the ratio of the circumference of any circle to its diameter.**

It cannot be too carefully impressed on the student's memory that π stands for this number 3·14159265...&c., and for nothing else; just as 180 stands for the number one hundred and eighty, and for nothing else.

13. We may notice that $\frac{22}{7} = 3\cdot\dot{1}4285\dot{7}$.

So that $\frac{22}{7}$ and π differ by less than a thousandth part of their value.

14. Thus in a circle of radius r

$$\frac{\text{the circumference}}{2r} = (3\cdot14159256 + \ldots) = \pi,$$

or $$\text{the circumference} = \pi \times 2r = \tfrac{22}{7} \times 2r.$$

Example 1. *The driving wheel of a locomotive engine is* 5 ft. 6 in. high. *What is its* circumference?

Here we have a circle whose diameter is $5\frac{1}{2}$ feet;

$$\therefore \text{ its circumference} = \pi \times 5\cdot5 \text{ feet},$$
$$= (3\cdot14159\ldots) \times 5\cdot5 \text{ feet},$$
$$= 17\cdot278\ldots \text{ feet}.$$

The circumference is 17 ft. 3 in. approximately.

Example 2. *A piece of wire* 1 foot *long is bent into the form of a circle; what is the* diameter *of the circle?*

Here the circumference = 1 foot,

that is $$\pi \times \text{diameter} = 1 \text{ foot},$$

$$\therefore \text{ diameter} = \frac{1 \text{ foot}}{\pi} = \tfrac{7}{22} \times 1 \text{ foot}$$
$$= \tfrac{84}{22} \text{ inches} = 3\cdot8 \text{ inches, nearly}.$$

EXAMPLES. III.

In the answers of the first 12 of the following examples $\frac{22}{7}$ is used for π.

1. Find the circumference of a circle whose diameter is one yard.
2. Find the circumference of a circle whose radius is 4 feet.
3. Find the circumference of a 48 inch bicycle wheel.
4. The circumference of a circle is 10 feet; find its diameter.
5. What must be the diameter of a locomotive driving wheel, that it may make 220 revolutions per mile?
6. How many revolutions does a 36 inch bicycle wheel make per mile?
7. How many more revolutions per mile does a 50 inch bicycle wheel make than one of 52 inches?
8. A locomotive whose driving wheel is 5 feet high has an instrument to record the number of revolutions made. What number will the instrument record in running 100 miles?

9. If the instrument in Question 8 indicates 3 revolutions per second, how many miles per hour is the engine running?

10. What is the diameter of the driving wheel of a locomotive engine which makes 4 revolutions per second when the engine is going at the rate of 60 miles per hour?

11. The large hand of the Westminster clock is 11 feet long; how many yards per day does its extremity travel? How far does the extremity move in a minute?

12. The diameter of the whispering gallery in St Paul's is 108 feet; what is its circumference?

13. Find the number of inches of wire necessary to construct a figure consisting of a circle with a regular hexagon inscribed in it, one of whose sides is 3 feet.

14. How many inches of wire would be necessary in a figure similar to that in Question 13, if the circumference of the circle were ten feet?

15. Find how many inches of wire are necessary to make a figure consisting of a circle and a square inscribed in it, when each side of the square is 2 feet.

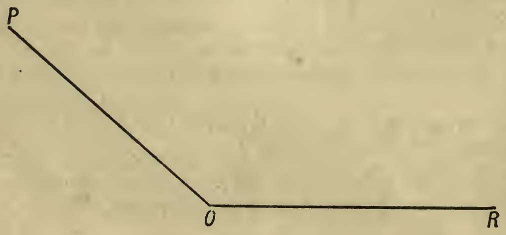
16. Find the length of string necessary to string the handle of a cricket bat; having given the diameter of the handle $=1\frac{1}{4}$ in., the length of the handle $=12$ in., the diameter of the string $=\frac{1}{40}$th of an inch.

CHAPTER III.

On the Measurement of Angles.

15. In elementary Geometry (Euclid I.—VI.) the angles considered are each always less than two right angles.

For example, in speaking of the angle ROP in Euclid we should always mean the angle less than two right angles,



not an angle measured in the opposite direction greater than two right angles.

16. In Trigonometry, by *the angle ROP* is meant, not the *present inclination* of the two lines *OR*, *OP* but the **amount of turning** which *OP* has gone through when, starting from the position *OR*, it has turned about *O* into the position *OP*.

Example. Suppose a race run round a circular course. The position of any one of the competitors would be known, if we remark that he has described a certain angle about the centre of the course. Thus, if the distance to be run is three times round, the line joining each competitor to the centre would have to describe an angle of 12 right angles.

When we remark that a competitor has described an angle of $6\frac{2}{3}$ right angles, we record not only his present position, but the total distance he has gone. He would in such a case have gone a little more than one and a half times round the course.

17. Definition. The **angle** between two lines, *OR*, *OP* is the **amount of turning** about the point *O* which one of the lines *OP* has gone through in turning from the position *OR* into the position *OP*.

18. The angle *ROP* may be the *geometrical* representative of an unlimited number of Trigonometrical angles.

(i) The angle *ROP* may represent the angle less than two right angles as in Euclid.

In this case *OP* has turned from the position *OR* into the position *OP* by turning about *O* in the direction *contrary* to that of the hands of a watch.

(ii) The angle *ROP* may represent the angle described by *OP* in turning from the position *OR* into the position *OP* in the *same* direction as the hands of a watch.

In the first case it is usual to say that the angle *ROP* is described in the *positive* direction, in the second that the angle is described in the *negative* direction.

(iii) The angle *ROP* may be the geometrical representation of any of the Trigonometrical angles formed by any number of complete revolutions in the *positive* or in the *negative* direction, added to either of the first two angles. (We shall return to this subject in Chapter VIII.)

EXAMPLES. IV.

Give a geometrical representation of each of the following angles, the starting line being drawn in each case from the turning point towards the right.

1. $+3$ right angles.
2. $+5$ right angles.
3. $+4\frac{1}{2}$ right angles.
4. $+7\frac{1}{4}$ right angles.
5. -1 right angle.
6. $10\frac{2}{3}$ right angles.
7. $-10\frac{1}{3}$ right angles.
8. $+4$ right angles.
9. -4 right angles.
10. $4n$ right angles.
11. $(4n+2)$ right angles.
12. $-(4n+\frac{1}{2})$ right angles.

19. There are two methods of measuring angles.

(i) The **rectangular** measure.

(ii) The **circular** measure.

Rectangular Measure.

20. Angles are always measured *in practice* with the **right angle** (or part of the right angle) as unit.

The reasons why the right angle is chosen for a unit are:

(i) All right angles are equal to one another.

(ii) A right angle is practically easy to draw.

(iii) It is an angle whose size is very familiar.

21. The right angle is a large angle, and it is therefore subdivided for practical purposes.

The right angle is divided into 90 equal parts, each of which is called a **degree**; each degree is subdivided into 60 equal parts, each of which is called a **minute**; and each minute is again subdivided into 60 equal parts, each of which is called a **second**.

Instruments used for measuring angles are subdivided accordingly; and the size of an angle is known when, with such an instrument, it has been observed that the angle contains a certain number of degrees, and a certain number of minutes beyond the number of complete degrees, and a certain number of seconds beyond the number of complete minutes.

Thus an angle might be recorded as containing 79 degrees + 18 minutes + 36·4 seconds.

Degrees, minutes, and seconds are indicated respectively by the symbols $^{\circ}$, $'$, $''$, and the above angle would be written

$$79^{\circ}\ \ 18'\ \ 36{\cdot}4''.$$

22. An angle given in degrees, minutes, and seconds may be expressed as the decimal of a right angle by the usual method.

Example. *Express* $39^{\circ}\ 4'\ 27''$ *as the* decimal of a right angle.

60) 27 seconds

60) 4·45 minutes

90) 39·07416666 etc. degrees

·43415740740 etc. right angles

Answer. $\cdot 43415\dot{7}4\dot{0}$ of a right angle.

NOTE. The French proposed to call the 100th part of a right angle a **grade** (written 3^g), the 100th part of a grade a minute (written $3^{\backprime}$), the 100th part of a minute a second (written $3^{\backprime\backprime}$). So that 1·437275 right angles would be read $143^g\,72^{\backprime}\,75^{\backprime\backprime}$. The decimal method of subdividing the right angles has never been used.

*EXAMPLES. V.

Express each of the following angles (i) as the decimal of a right angle, (ii) in grades, minutes, and seconds:

1. $8^{\circ}\ 15'\ 27''$.

2. $6^{\circ}\ 4'\ 30''$.

3. $97^{\circ}\ 5'\ 15''$.

4. $16^{\circ}\ 14'\ 19''$.

5. $132^{\circ}\ 6'$.

6. 49°.

Express in degrees, minutes and seconds,

7. ·01375 right angles.

8. ·0875 right angles.

9. 1·704535 right angles.

10. ·240025 right angles.

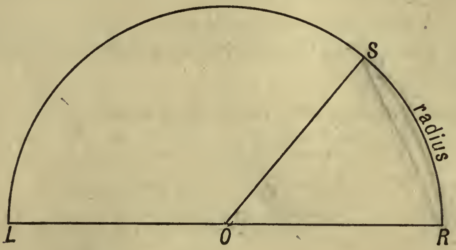
11. ·180115 right angles.

12. ·35 right angles.

ON CIRCULAR MEASURE.

23. By the following construction we get an angle of great importance in Trigonometry.

On the circumference of a circle whose centre is O



let an *arc* RS be measured so that its length is equal to the *radius* of the circle, and let R and S be joined to the centre.

24. We are about to prove (Art. 26) that this angle ROS is a *fixed* fraction of a right angle, so that all such angles are equal to one another.

We may state the same thing thus—We are about to prove that if we take any number of different circles, and measure on the circumference of each an arc equal in length to its radius, then the angles at the centres of these circles which stand on these arcs respectively, will be all of the same size.

25. DEFINITION. The angle which at the centre of a circle stands on an arc equal in length to the radius of the circle is called a **Radian.**

26. *To prove that all Radians are equal to one another.*

Since the Radian at the centre of a circle stands on an arc equal in length to the radius,

and an angle of two right angles at the centre of a circle stands on half the circumference,

and since angles at the centre of a circle are to one another as the arcs on which they stand (Euc. VI. 33),

$$\frac{\text{a radian}}{\text{2 right angles}} = \frac{\text{radius}}{\text{semi-circumference}}$$

$$= \frac{\text{diameter}}{\text{circumference}} = \frac{1}{\pi}.$$

Therefore a radian $= \frac{1}{\pi}$ of 2 right angles,

= a certain fixed fraction of 180°.

27. Thus the radian possesses the qualification most essential in a unit, viz. it is always the same.

28. The reasons why a radian is used as a unit are:

(i) All radians are equal to one another.

(ii) Its use simplifies many formulæ in Theoretical Trigonometry.

29. The system of angular measurement in which a radian is the unit is called **Circular Measure.**

Therefore the *circular measure of an angle* is the *number of radians* which the angle contains.

30. A radian $= \frac{1}{\pi} \times 2$ right angles,

$= \frac{1}{3\cdot14159\ldots}$ of 180° nearly,

$= 57\cdot2957\ldots\ldots$degrees.

31. The expression '*The angle* θ' means that θ is a *number* and some unit of an angle is implied. 'The angle 180' implies the unit of angle *a degree*. When Greek letters are used the unit of angle implied is *a radian*, thus

the angle $\theta = \theta$ radians,

the angle $\pi = \pi$ radians.

When Roman letters are used the unit implied is a degree,

the angle $A = A$ degrees.

32. Just as 30° indicates 30 degrees, so we use a little c to indicate radians, thus

$3^c = 3$ radians.

33. The student cannot too carefully notice, that unless an *angle* is obviously referred to, the letters θ, ϕ,... a, β,... stand for *mere numbers.*

Thus as we have said above (Art. 12) π *stands for a number and a number only*, viz. 3·14159......, but in the expression 'the angle π' that is 'the angle 3·14159......' there is *some unit* of angle understood. The *unit understood* here is a **radian**, and therefore 'the angle π' stands for 3·14159......^{c}, that is *two right angles.*

Hence, *when an angle is understood,* π *is a very convenient abbreviation for* **two right angles.**

34. Let D and a be the number of degrees and radians respectively in any angle, then

$$\frac{D}{180} = \frac{a}{\pi}.$$

For each fraction is the ratio of the angle to two right angles.

Example. Find the number of degrees in two radians.

Let D be the number, then

$$\frac{D}{180} = \frac{2}{\pi},$$

$$\therefore D = \frac{360}{\pi}.$$

NOTE. 2^c indicates 2 *radians.*

EXAMPLES. VI.

I. Express the following angles in rectangular measure.

1. π. 2. $\frac{3\pi}{4}$. 3. 1^c.

4. 3^c. 5. $3·14159265^c$ etc. 6. $\frac{2^c}{\pi}$.

7. θ. 8. $·00314159^c$ etc. 9. 10π.

II. Express the following angles in circular measure.

1. 180^0. 2. 360^0. 3. 60^0.

4. $22\frac{1}{2}^0$. 5. 1^0. 6. $57·295^0$ etc.

7. n^0. 8. $\frac{90^0}{\pi}$. 9. A.

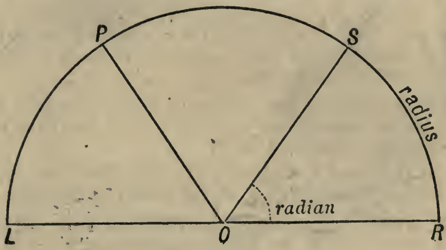
*III. Express the following angles in circular measure.

1. $33^g\ 33'\ 33\cdot\dot{3}''$. 2. 50^g. 3. $16\cdot\dot{6}^g$.
4. 1^g. 5. $1'$. 6. $10''$.
7. n^g. 8. $\frac{200^g}{\pi}$. 9. 1000^g.

*IV. Find the ratio of

1. 45^0 to $\frac{3\pi}{4}$. 2. 60^0 to 60^g.
3. 25^g to $22^0\ 30'$. 4. 24^g to 2^c.
5. $1\cdot75^c$ to $\frac{100^0}{\pi}$. 6. 1^0 to 1^c.

35. Since angles at the centre of a circle are to one another as the arcs on which they stand [Euc. VI. 33], therefore
$$\frac{\text{an angle } ROP}{\text{one radian}} = \frac{\text{arc } RP}{\text{arc } RS} = \frac{\text{arc } RP}{\text{the radius}}.$$



Hence the angle $ROP = \frac{\text{arc } RP}{\text{the radius}}$ radians.

So that the **circular measure** of an angle (at the centre of a circle) is the **ratio of its arc to the radius.**

Example. Find the number of degrees in the angle subtended by an arc 46 ft. 9 in. long, at the centre of a circle whose radius is 25 feet.

The angle stands on an arc of $46\frac{3}{4}$ ft. and the radian, at the centre of the same circle, stands on an arc of 25 feet.

$$\therefore \text{ the angle} = \frac{46\frac{3}{4}}{25} \text{ radians}, = \tfrac{187}{100} \times \frac{2 \text{ right angles}}{\pi},$$
$$= \tfrac{187}{100} \times \frac{180^0}{\pi} = 105\cdot8^0 \text{ nearly}.$$

*EXAMPLES. VII.

(In the Answers $\frac{22}{7}$ is used for π.)

1. Find the number of radians in an angle at the centre of a circle of radius 25 feet, which stands on an arc of $37\frac{1}{2}$ feet.

2. Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which stands on an arc of 5π feet.

3. Find the number of right angles in the angle at the centre of a circle of radius $3\frac{2}{11}$ inches, which stands on an arc of 2 feet.

4. Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet.

5. Find the length of an arc of eighty degrees on a circle of 4 feet radius.

6. The angle subtended by the diameter of the Sun at the eye of an observer is $32'$; find approximately the diameter of the Sun if its distance from the observer be 90,000,000 miles.

7. A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour; through what angle has it turned in 10 seconds?

8. A railway train is travelling on a curve of two-thirds of a mile radius, at the rate of 60 miles an hour; through what angle has it turned in a quarter of a minute?

9. Find approximately the number of English seconds contained in the angle which subtends an arc one mile in length at the centre of a circle whose radius is 4000 miles.

10. If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of $1''$ at the centre of the circle.

11. If in a circle whose radius is 12 ft. 6 in. an arc whose length is ·6545 of a foot subtends an angle of 3 degrees, what is the ratio of the diameter of a circle to its circumference?

12. If an arc 1·309 feet long subtend an angle of $7\frac{1}{2}$ degrees at the centre of a circle whose radius is 10 feet, find the ratio of the circumference of a circle to its diameter.

13. On a circle 80 feet in radius it was found that an angle of $22^\circ 30'$ at the centre was subtended by an arc 31 ft. 5 in. in length; hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter.

14. If the diameter of the moon subtend an angle of $30'$, at the eye of an observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of the moon, find the ratio of the diameter of the sun to that of the moon.

15. Find the number of radians in (i.e. the circular measure of) $10''$ correct to 3 significant figures. (Use $\frac{355}{113}$ for π.)

16. Find the radius of a globe such that the distance measured upon its surface between two places in the same meridian, whose latitudes differ by $1^\circ 10'$, may be one inch.

17. Two circles touch the base of an isosceles triangle at its middle point, one having its centre at, and the other passing through the vertex. If the arc of the greater circle included within the triangle be equal to the arc of the lesser circle without the triangle, find the vertical angle of the triangle.

18. By the construction in Euc. I. 1, prove that the unit of circular measure is less than 60°.

19. On the 31st December the Sun subtends an angle of $32' 36''$, and on 1st July an angle of $31' 32''$; find the ratio of the distances of the Sun from the observer on those two days.

20. Show that the measure of the angle at the centre of a circle of radius r, which stands on an arc a, is $\frac{k \cdot a}{r}$, where k depends solely on the unit of angle employed.

Find k when the unit is (i) a radian, (ii) a degree.

21. The difference of two angles is $\frac{1}{5}\pi$ and their sum 56°; find them.

22. Find the number of radians in an angle of n'.

23. Express in right angles and in radians the angles

(i) of a regular hexagon,

(ii) of a regular octagon,

(iii) of a regular quindecagon.

24. Taking for unit the angle between the side of a regular quindecagon and the next side produced, find the measures (i) of a right angle, (ii) of a radian.

25. Find the unit when the sum of the measures of a degree and of the hundredth part of a right angle is 1.

26. What is the unit when the sum of the measures of 9° and of ·05 right angles is $\frac{3}{20}$?

27. The measure of b right angles is a, find the measure of c degrees.

28. What is the unit when the sum of the measures of a right angles and of b degrees is c?

29. The three angles of a triangle have the same measure when the units are $\frac{1}{90}$ of a right angle, $\frac{1}{100}$ of a right angle and a radian respectively; find the measure.

30. The interior angles of an irregular polygon are in A. P.; the least angle is 120°; the common difference is 5°; find the number of sides.

CHAPTER IV.

THE TRIGONOMETRICAL RATIOS.

36. LET ROE be any angle (see the figure in Art. 37). In one of the lines containing the angle take any point P, and from P draw PM perpendicular to the other line OR.

Then, in the right-angled triangle OPM, formed from the angle ROE,

(i) the side MP, which is *opposite the angle under consideration*, is called the **perpendicular**;

(ii) the side OP, which is *opposite the right angle*, is called the **hypotenuse**;

(iii) the *third* side OM, which is adjacent to the right angle and to the angle under consideration, is called the **base**.

From these three,—perpendicular, hypotenuse, base,—we can form *three* different sets containing two each.

The **ratios** or *fractions* formed from these sets, viz.

(i) $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$, (ii) $\dfrac{\text{base}}{\text{hypotenuse}}$, (iii) $\dfrac{\text{perpendicular}}{\text{base}}$, and the ratios formed by inverting each of them, viz.

(iv) $\dfrac{\text{hypotenuse}}{\text{perpendicular}}$, (v) $\dfrac{\text{hypotenuse}}{\text{base}}$, (vi) $\dfrac{\text{base}}{\text{perpendicular}}$,

will be found to be of great importance in treating of any angle ROE. Accordingly to each of these six ratios has been given a separate *name* (Art. 37).

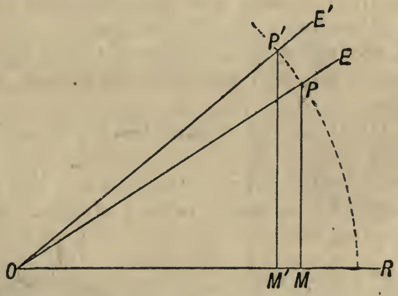
NOTE. The student should observe carefully

(i) that each *ratio*, such as $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$, is a mere *number;*

(ii) that, as we shall prove in Art. 83, these ratios remain unchanged as long as the angle remains unchanged;

(iii) that if the angle be altered ever so slightly, there is a consequent alteration in the value of these ratios.

[For, let ROE, ROE' be two angles which are nearly equal;

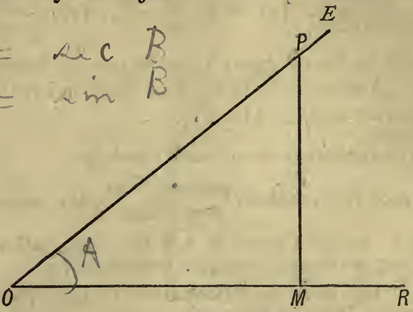


Let $OP=OP'$; then OM is *not* $=OM'$, and therefore the ratios $\dfrac{OM}{OP}$ and $\dfrac{OM'}{OP'}$ are *not* equal; also MP is *not* $=M'P'$ and therefore the ratios $\dfrac{MP}{OP}$ and $\dfrac{M'P'}{OP'}$ are *not* equal.]

(iv) that by giving names to these ratios we are enabled to apply the methods of Algebra to the Geometry of Euclid VI., just as in Chapter I. we applied the methods of Algebra to Euclid I. 47.

The student is recommended to pay careful attention to the following definitions. He should be able to write them out in the exact words in which they are printed.

37. Definition. *To define the three principal Trigonometrical Ratios of an angle.*



Let ROE be an angle.

In OE one of the lines containing the angle take any point P, and from P draw PM perpendicular to the other line OR, or, if necessary, to RO produced.

Then, in the right-angled triangle OPM, the side MP, which is *opposite the angle under consideration*, is called the *perpendicular*.

The side OP, which is *opposite the right angle*, is called the *hypotenuse*.

The *third* side OM (which is adjacent to the right angle and to the angle under consideration) is called the *base*.

Then the ratio

(i) $\frac{MP}{OP} = \frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the **sine** of the angle ROE.

(ii) $\frac{OM}{OP} = \frac{\text{base}}{\text{hypotenuse}}$ „ **cosine** „

(iii) $\frac{MP}{OM} = \frac{\text{perpendicular}}{\text{base}}$ „ **tangent** „

The *order of the letters* in MP, OM and OP indicates the direction of the lines and (as will be explained later) is an essential part of the definition.

38. If A stand for the angle ROE, these ratios are called sine A, cosine A and tangent A, and are usually abbreviated thus: **sin** A, **cos** A, **tan** A.

39. There are three other Trigonometrical Ratios, formed by *inverting* the sine, cosine and tangent respectively, which are called the cosecant, secant, and cotangent respectively.

40. *To define the three other Trigonometrical Ratios of any angle.*

The same construction and figure as in Art. 37 being made, then the ratio

(iv) $\frac{OP}{MP} = \frac{\text{hypotenuse}}{\text{perpendicular}}$ is called the **cosecant** of the angle ROE.

(v) $\frac{OP}{OM} = \frac{\text{hypotenuse}}{\text{base}}$ „ **secant** „

(vi) $\frac{OM}{MP} = \frac{\text{base}}{\text{perpendicular}}$ „ **cotangent** „

41. Thus if A stand as before for the angle ROE, these ratios are called cosecant A, secant A, and cotangent A. They are abbreviated thus,

$$\text{cosec}\, A, \qquad \sec A, \qquad \cot A.$$

42. From the definition it is clear that

$$\text{cosec}\, A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}.$$

43. The above definitions **apply to an angle of any magnitude.** (We shall return to this subject in Chapter VIII.)

For the present the student may confine his attention to angles which are each less than a right angle.

44. The powers of the Trigonometrical Ratios are expressed as follows:

$(\sin A)^2$, i.e. $\left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2$, is written $\sin^2 A$,

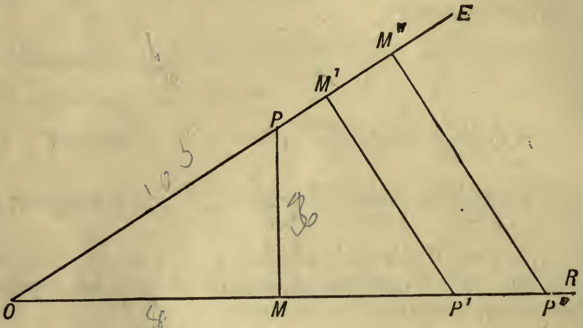
$(\cos A)^3$, i.e. $\left(\frac{\text{base}}{\text{hypotenuse}}\right)^3$, is written $\cos^3 A$,

and so on.

OP

The student must notice that '$\sin A$' is a *single symbol.* It is the *name* of a *number*, or *fraction*, belonging to the angle A; and if it be at any time convenient, we may denote *sin A* by a *single letter*, such as s or x. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, that is, for $(\sin A) \times (\sin A)$. Such abbreviations are used because they are *convenient.*

45. *The Trigonometrical Ratios are always the same for the same angle.*



Take any angle ROE; let P be any point in OE one of the lines containing the angle, and let P', P'' be any two points in OR the other line containing the angle. Draw PM perpendicular to OR, and $P'M'$, $P''M''$ perpendiculars to OE.

Then the three triangles OMP, $OM'P'$, $OM''P''$ each contain a right angle, and they have the angle at O common; therefore their third angles must be equal.

Thus the three triangles are equiangular.

Therefore the ratios $\dfrac{MP}{OP}$, $\dfrac{M'P'}{OP'}$, $\dfrac{M''P''}{OP''}$ are all equal.

(Eu. VI. 4.)

But each of these ratios is $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$ *with reference to the angle at O*; that is, they are each *sin ROE*.

Thus, sin ROE is the same *whatever* be the position of the point P on *either* of the lines containing the angle ROE.

Therefore sin ROE is always the same.

46. A similar proof holds good for each of the other ratios.

47. Also if two angles are equal, it is clear that the numerical values of their Trigonometrical Ratios will be the same.

We have already shown (Art. 36) that the values of these ratios are different for different angles.

Hence for each particular value of A, $\sin A$, $\cos A$, $\tan A$, etc. have *definite numerical values.*

Example. We shall prove (Art. 54) that

$$\sin 30^0 = \tfrac{1}{2} = \cdot 5, \quad \cos 30^0 = \frac{\sqrt{3}}{2} = \cdot 8660\ldots, \quad \tan 30^0 = \frac{1}{\sqrt{3}} = \cdot 577\ldots$$

48. In the following examples the student should notice

(i) the *angle* referred to:

(ii) that there is a *right angle* in the same triangle as the angle referred to:

(iii) the *perpendicular*, which is opposite the angle referred to, and is perpendicular to one of the lines containing the angle:

(iv) the *hypotenuse*, which is opposite the right angle:

(v) the *base*, the third side of the triangle.

Example. In the second figure on the next page, in which BDA is a right angle, find $\sin DBA$ and $\cos DBA$.

In this case

(i) DBA is the *angle.*

(ii) BDA is a *right angle* in the same triangle as the angle DBA.

(iii) DA is the *perpendicular*, for it is opposite DBA and is perpendicular to BD.

(iv) BA is the *hypotenuse.*

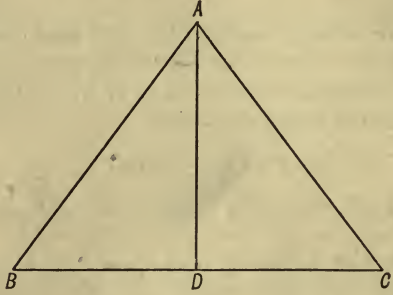
(v) BD is the *base.*

Therefore $\sin DBA$, which is $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$, $= \dfrac{DA}{BA}$,

$\cos DBA$, which is $\dfrac{\text{base}}{\text{hypotenuse}}$, $= \dfrac{BD}{BA}$.

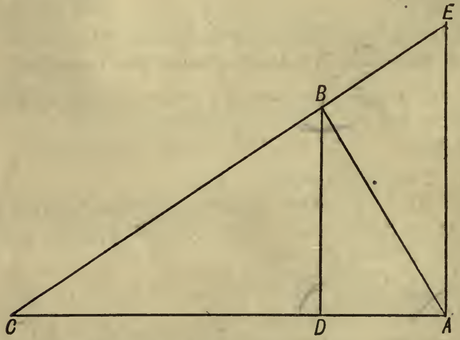
EXAMPLES. VIII.

1. Let ABC be any triangle and let AD be drawn perpendicular to BC. Write down the *perpendicular*, and the *base* when the following angles are referred to: (i) the angle ABD, (ii) the angle BAD, (iii) the angle ACD, (iv) the angle DAC.



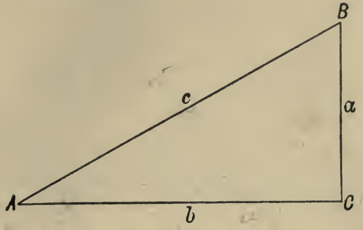
2. Write down the following ratios in the above figure; (i) $\sin BAD$, (ii) $\cos ACD$, (iii) $\tan DAC$, (iv) $\sin ABD$, (v) $\tan BAD$, (vi) $\sin DAC$, (vii) $\cos DCA$, (viii) $\tan DCA$, (ix) $\cos ABD$, (x) $\sin ACD$.

3. Let ACB be any angle and let ABC and BDC be right angles; (see next figure). Write down *two* values for each of the following ratios; (i) $\sin ACB$, (ii) $\cos ACB$, (iii) $\tan ACB$, (iv) $\sin BAC$, (v) $\cos BAC$, (vi) $\tan BAC$.



4. In the accompanying figure BDC, CBA and EAC are right angles. Write down (i) $\sin DBA$, (ii) $\sin BEA$, (iii) $\sin CBD$, (iv) $\cos BAE$, (v) $\cos BAD$, (vi) $\cos CBD$, (vii) $\tan BCD$, (viii) $\tan DBA$, (ix) $\tan BEA$, (x) $\tan CBD$, (xi) $\sin DAB$, (xii) $\sin BAE$.

5. Let ABC be a right-angled triangle such that $AB=5$ ft., $BC=3$ ft., then AC will be 4 ft.



Find the sine, cosine and tangent of the angles at A and B respectively.

In the above triangle if A stand for the angle at A and B for the angle at B, show that $\sin^2 A+\cos^2 A=1$, and that $\sin^2 B+\cos^2 B=1$.

6. If ABC be any right-angled triangle with a right angle at C, and let A, B, and C stand for the angles at A, B and C respectively, and let a, b and c be the measures of the sides opposite the angles A, B and C respectively.

Show that $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$.

Show also that $\sin^2 A+\cos^2 A=1$.

Show also that (i) $a=c \cdot \sin A$, (ii) $b=c \cdot \sin B$, (iii) $a=c \cdot \cos B$, (iv) $b=c \cdot \cos A$, (v) $\sin A=\cos B$, (vi) $\cos A=\sin B$, (vii) $\tan A=\cot B$.

7. The sides of a right-angled triangle are in the ratio 5 : 12 : 13. Find the sine, cosine and tangent of each acute angle of the triangle.

8. The sides of a right-angled triangle are in the ratio $1 : 2 : \sqrt{3}$. Find the sine, cosine and tangent of each acute angle of the triangle.

9. Prove that if A be either of the angles of the above two triangles $\sin^2 A+\cos^2 A=1$.

10. ABC is a right-angled triangle, C being the right angle. AB is 2 ft. and AC is 1 foot; find the length of BC, and thence find the value of $\sin A$, $\cos A$, and $\tan A$.

11. ABC is a right-angled triangle, C being the right angle; $AB=\sqrt{2}$ ft. and $AC=1$ ft.; prove that $\sin A=\cos A=\sin B=\cos B$.

12. ABC is a right-angled triangle, C being the right angle; $AC=1$ ft. and $AB=\sqrt{3}$ feet; find AC and $\sin A$ and $\sin B$.

CHAPTER V.

On the Trigonometrical Ratios of certain Angles.

49. The Trigonometrical Ratios of an angle are *numerical quantities simply*, as their name ratio implies. They are in nearly all cases incommensurable numbers.

Their practical value has been found for all angles between 0 and 90°, which differ by 1′; and a list of these values will be found in any volume of Mathematical Tables.

It will be an advantage for the student to see a volume of Mathematical Tables that he may understand what is meant.

It will not be necessary for each student to procure a copy, as in nearly all examples the necessary quotations from the Tables are given.

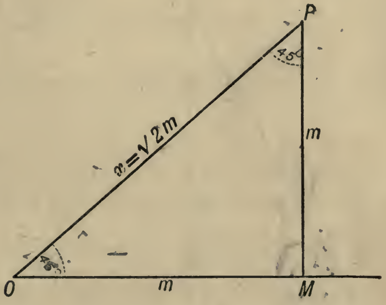
A well arranged and useful set of Tables is that published by Messrs Chambers, of Edinburgh.

50. The finding the values of these Ratios has involved a large amount of labour; but, as the results have been published in Tables, the finding the Trigonometrical Ratios does not form any part of a student's work, except to exemplify the method employed.

51. The general method of finding Trigonometrical Ratios belongs to a more advanced part of the subject than the present, but there are certain angles whose Ratios can be found in a simple manner.

52. *To find the* sine, cosine *and* tangent *of an angle of* **45°**.

When one angle of a right-angled triangle is 45°, that is, the half of a right angle, the third angle must also be 45°. Hence 45° is one angle of an *isosceles* right-angled triangle.



Let *POM* be an isosceles triangle such that *PMO* is a right angle, and $OM = MP$. Then $POM = OPM = 45°$.

Let the measures of *OM* and of *MP* each be *m*. Let the measure of *OP* be *x*.

Then $$x^2 = m^2 + m^2 = 2m^2;$$

$$\therefore\ x = \sqrt{2}\,.\,m.$$

Hence, $$\textbf{sin 45}^0 = \sin POM = \frac{MP}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

$$\textbf{cos 45}^0 = \cos POM = \frac{OM}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

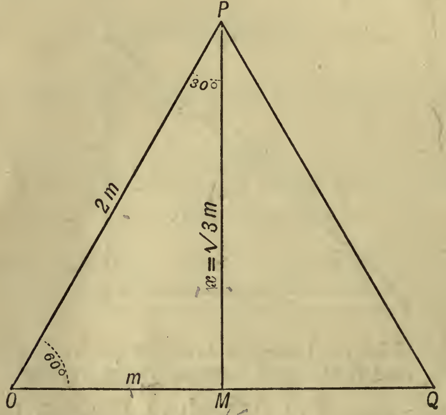
$$\textbf{tan 45}^0 = \tan POM = \frac{MP}{OM} = \frac{m}{m} = \frac{1}{1} = 1.$$

53. *To find the* sine, cosine *and* tangent *of* 60°.

In an *equilateral* triangle each of the equal angles is 60°, because they are each one-third of 180°. And if we draw a perpendicular from one of the angular points of the triangle to the opposite side we get a right-angled triangle in which one angle is 60°.

Let OPQ be an equilateral triangle. Draw PM perpendicular to OQ. Then OQ is bisected in M.

Let the measure of OM be m; then that of OQ is $2m$, and therefore that of OP is $2m$.



Let the measure of MP be x.

Then $$x^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2,$$

$$\therefore\ x = \sqrt{3}\,.\,m.$$

Hence, $$\sin 60^0 = \sin POM = \frac{MP}{OP} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$

$$\cos 60^0 = \cos POM = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2},$$

$$\tan 60^0 = \tan POM = \frac{MP}{OM} = \frac{\sqrt{3}\,.\,m}{m} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

54. *To find the* sine, cosine *and* tangent *of* 30°.

With the same figure and construction as above we have

Hence, $\sin 30^0 = \sin OPM = \frac{MO}{PO} = \frac{m}{2m} = \frac{1}{2}$,

$$\cos 30^0 = \cos OPM = \frac{PM}{PO} = \frac{\sqrt{3} \cdot m}{2m} = \frac{\sqrt{3}}{2},$$

$$\tan 30^0 = \tan OPM = \frac{MO}{PM} = \frac{m}{\sqrt{3} \cdot m} = \frac{1}{\sqrt{3}}.$$

55. *To find the* sine, cosine *and* tangent *of* 0^0.

Let ROP be a small angle. Draw PM perpendicular to OR, and let OP be always of the same length, so that P lies on a circle whose centre is O.

Then if the angle ROP be diminished, we can see that MP is diminished also, and that consequently $\frac{MP}{OP}$, which is $\sin ROP$, is diminished. And, by diminishing the angle ROP sufficiently, *we can make MP as small as we please*, and therefore *we can make* $\sin ROP$ *smaller than any assignable number however small that number may be.*

This is what is meant when it is said that the value to which $\sin ROP$ approaches as the angle is diminished, is 0. This is expressed by saying, $\sin 0^0 = 0$ ....i.

Again, as the angle ROP diminishes, OM approaches OP in length; and $\cos ROP$, which is $\frac{OM}{OP}$, approaches in value to $\frac{OP}{OP}$, i.e. to 1.

This is expressed by saying, $\cos 0^0 = 1$ ....ii.

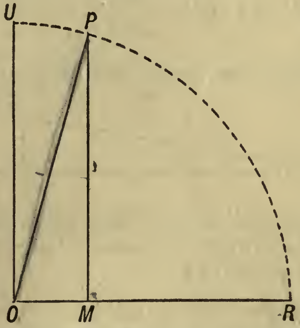
Also, $\tan ROP$ is $\frac{MP}{OM}$; and we have seen that MP approaches 0, while OM does not; $\therefore$ $\tan ROP$ approaches 0.

This is expressed by saying, $\tan 0^0 = 0$..............iii.

56. *To find the* sine, cosine *and* tangent *of* 90°.

Let ROU be a right angle $= 90°$.

Draw ROP nearly a right angle; draw PM perpendicular to OR, and let OP be always of the same length, so that P lies on a circle whose centre is O.



Then, as the angle ROP approaches to ROU, we can see that MP approaches OP, while OM continually diminishes.

Hence when ROP approaches 90°, sin ROP, which is $\frac{MP}{OP}$, approaches in value to $\frac{OP}{OP}$, that is to $\frac{1}{1}$, i.e. to 1.

Hence we say that $\sin 90° = 1$ ....i.

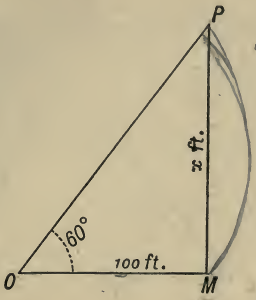
Again, when ROP approaches 90°, cos ROP, which is $\frac{OM}{OP}$, approaches in value to $\frac{0}{OP}$, that is to 0.

Hence we say that $\cos 90° = 0$ ....ii.

Again, when ROP approaches 90°, tan ROP which is $\frac{MP}{OM}$ approaches in value to $\frac{OP}{\text{a quantity which approaches } 0}$.

But in any fraction, whose numerator does not diminish, the smaller the denominator, the greater the value of that fraction; and if the denominator continually diminishes, the value of the fraction continually increases.

Example 1. *At a point* 100 feet *from the foot of a tower, the angle of elevation of the top of the tower is observed to be* 60°. *Find the* height *of the top of the tower above the point of observation.*



Let O be the point of observation; let P be the top of the tower; let a horizontal line through O meet the foot of the tower at the point M. Then $OM = 100$ feet, and the angle $MQP = 60°$. Let MP contain x feet.

Then $$\frac{MP}{OM} = \tan MOP = \tan 60° = \sqrt{3}.$$

$$\therefore \frac{x}{100} = \sqrt{3}.$$

$$\therefore x = 100 . \sqrt{3} = 100 \times 1{\cdot}7320 \text{ etc.}$$

$$= 173{\cdot}2.$$

Therefore the required height is 173·2.

Example 2. *At a point* 100 yds. *from the foot of a building, I measure the angle of elevation of the top, and find that it is* 23° 15′; *what is the height of the building?*

As in Example 1 let the height be x yards.

Then $\dfrac{x}{100} = \tan 23° 15'$.

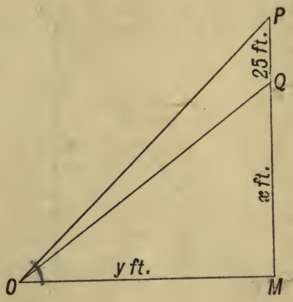
From the Table of tangents we find that

$$\tan 23° 15' = {\cdot}4296339.$$

Hence $x = 100 \times {\cdot}4296339 = 42{\cdot}96339$.

The height of the building = 43 yds. *nearly*. *Ans.*

Example 3. *A flagstaff*, 25 feet *high, stands on the top of a cliff; from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be* 47° 12′ *and* 45° 13′ *respectively. Find the* height *of the cliff.*



Let O be the point of observation, PQ the flagstaff.

Let a horizontal line through O meet the vertical line PQ produced in M.

Then $QP = 25$ feet, $MOP = 47^\circ\ 12'$, $MOQ = 45^\circ\ 13'$.

Let $MQ = x$ feet; let $OM = y$ feet.

Then $\dfrac{MP}{OM} = \tan 47^\circ\ 12'$, $\therefore \dfrac{x+25}{y} = \tan 47^\circ\ 12'$,

and $\dfrac{MQ}{OM} = \tan 45^\circ\ 13'$, $\therefore \dfrac{x}{y} = \tan 45^\circ\ 13'$.

Hence, by division, $\therefore \dfrac{x+25}{x} = \dfrac{\tan 47^\circ\ 12'}{\tan 45^\circ\ 13'}$.

In the Tables we find that

$$\tan 47^\circ\ 12' = 1{\cdot}0799018, \text{ and } \tan 45^\circ\ 13' = 1{\cdot}0075918,$$

$$\therefore 1 + \frac{25}{x} = \frac{1{\cdot}0799018}{1{\cdot}0075918} = 1 + \frac{{\cdot}0723100}{1{\cdot}0075918},$$

$$\therefore \quad \frac{x}{25} = \frac{1{\cdot}0075918}{{\cdot}0723100} = \frac{100759}{7231}.$$

$$\therefore x = \frac{2518975}{7231} = 348 \text{ nearly.}$$

Therefore the cliff is 348 feet high.

EXAMPLES X.

NOTE. The answers are given correct to three significant figures.

1. At a point 179 feet in a horizontal line from the foot of a column, the angle of elevation of the top of the column is observed to be 45^0. What is the height of the column?

2. At a point 200 feet from, and on a level with the base of a tower, the angle of elevation of the top of the tower is observed to be 60^0: what is the height of the tower?

3. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30^0: find the height of the cliff.

4. From the top of a tower 117 feet high the angle of depression of the top of a house 37 feet high is observed to be 30^0: how far is the top of the house from the tower?

5. A man 6 ft. high stands at a distance of 4 ft. 9 in. from a lamp-post, and it is observed that his shadow is 19 ft. long. Find the height of the lamp.

6. The shadow of a tower in the sunlight is observed to be 100 ft. long, and at the same time the shadow of a lamp-post 9 ft. high is observed to be $3\sqrt{3}$ ft. long. Find the angle of elevation of the sun, and the height of the tower.

7. From a point P on the bank of a river, just opposite a post Q on the other bank, a man walks at right angles to PQ to a point R so that PR is 100 yards; he then observes the angle PRQ to be $32^0\ 17'$: find the breadth of the river. ($\tan 32^0\ 17' = \cdot 6317667$.)

8. I walk 1000 ft. away from a tower and observe the elevation of the top to be $15^0\ 30'$; what is the height of the tower?

($\tan 15^0\ 30' = \cdot 2773245$.)

9. A fine wire 300 ft. long is attached to the top of a spire and the inclination of the wire to the horizon when held tight is observed to be 40^0; find the height of the spire. $\sin 40^0 = \cdot 6428$.

10. A vertical pole 30 ft. high stands on the bank of a river; at the point on the other bank just opposite the pole the angle of elevation of the top of the pole is 21^0; find the breadth of the river. ($\cot 21^0 = 2\cdot 6051$.)

11. A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands the angles of elevation of the top and bottom of the flagstaff are observed to be 60^0 and 45^0 respectively: find the height of the house above the point of observation.

12. From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be 45^0 and 30^0 respectively; if the

line joining the ships points directly to the foot of the cliff, find the distance between the ships.

13. A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles of elevation of the top and bottom of the tower are observed to be 75^0 and 60^0 respectively; find the height of the cliff. (Tan $75^0 = 2 + \sqrt{3}$).

14. A man walking along a straight road observes at one milestone a house in a direction making an angle 30^0 with the road, and that at the next milestone the angle is 60^0: how far is the house from the road?

15. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30^0. He then goes 400 yards along the bank to B and finds that BC makes with BA an angle of 60^0; find the breadth of the river.

16. A building on a square base $ABCD$ has two of its sides, AB and CD, parallel to the bank of a river. An observer, standing at E on the other side of the river so that DAE is a straight line, finds that AB subtends at his eye an angle of 45^0. Having walked a yards parallel to the bank, he finds that DE subtends an angle whose tangent is $\sqrt{2}$. Show that $DB = a$ yards.

17. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be $45^0\,13'$ and $47^0\,12'$ respectively; find the height of the hill. ($\tan 45^0\,13' = 1\cdot0075918$. $\tan 47^0\,12' = 1\cdot0799018$.)

18. From each of two stations, East and West of each other, the altitude of a balloon is observed to be 45^0, and its bearings to be respectively N.W. and N.E.: if the stations be 1 mile apart, determine the height of the balloon.

19. The angle of elevation of a balloon from a station due south of it is 60^0; and from another station due west of the former and distant a mile from it it is 45^0. Find the height of the balloon.

20. An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. If $2a$ be the base of the triangle, b its height, and 30^0 the altitude of the sun, find the tangent of half the angle at the apex of the shadow.

21. The length of the shadow of a vertical stick is to the length of the stick as $\sqrt{3} : 1$. If the stick be turned about its lower extremity in a vertical plane, so that the shadow is always in the same direction, find what will be the angle of its inclination to the horizon when the length of the shadow is the same as before.

22. What distance in space is travelled in an hour in consequence of the earth's rotation, by a person situated in latitude 60^0? (Earth's radius = 4000 miles.)

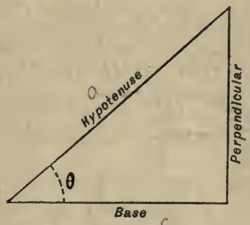
CHAPTER VI.

On the Relations between the Trigonometrical Ratios of One Angle.

63. The following relations are evident from the definitions:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

To prove $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.



We have $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}}$,

and $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$;

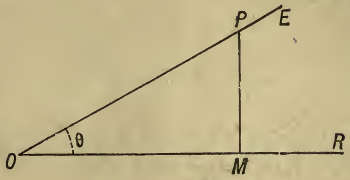
$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{\text{perpendicular}}{\text{base}} = \tan\theta.$$

64. We may prove similarly $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Or thus, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$.

65. *To prove that* $\cos^2\theta + \sin^2\theta = 1$.

Let ROE be any angle θ.



In OE take any point P, and draw PM perpendicular to OR. Then with respect to θ, MP is the perpendicular, OP is the hypotenuse, and OM is the base;

$$\therefore \sin^2\theta = \frac{MP^2}{OP^2}, \quad \cos^2\theta = \frac{OM^2}{OP^2}.$$

We have to prove that $\sin^2\theta + \cos^2\theta = 1$.

Now $$\sin^2\theta + \cos^2\theta = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2}$$

$$= \frac{MP^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1,$$

since $$MP^2 + OM^2 = OP^2. \qquad \text{[Euc. I. 47.]}$$

Similarly we may prove that

$$1 + \tan^2\theta = \sec^2\theta,$$

and that $$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

66. The following is a List of Formulæ with which the student must make himself familiar:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\tan^2\theta + 1 = \sec^2\theta,$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

67. In proving Trigonometrical identities it is often convenient to express the other Trigonometrical Ratios *in terms of the sine and cosine.*

Example. Prove that $\tan A + \cot A = \sec A \cdot \operatorname{cosec} A$.

Since $\tan A = \dfrac{\sin A}{\cos A}$, $\cot A = \dfrac{\cos A}{\sin A}$,

$\sec A = \dfrac{1}{\cos A}$, $\operatorname{cosec} A = \dfrac{1}{\sin A}$,

we have $\tan A + \cot A = \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}$

$= \dfrac{\sin^2 A + \cos^2 A}{\cos A \cdot \sin A} = \dfrac{1}{\cos A \cdot \sin A}$ [Art. 65.]

$= \sec A \cdot \operatorname{cosec} A$.

68. It is sometimes convenient to express all the Ratios in terms of the *sine* only; or in terms of the *cosine* only.

Example i. *Prove that* $\sin^4\theta + 2\sin^2\theta\cos^2\theta = 1 - \cos^4\theta$.

By Art. 65, we have $\sin^2\theta = 1 - \cos^2\theta$, hence

$$\sin^4\theta + 2\sin^2\theta\cos^2\theta = (1-\cos^2\theta)^2 + 2(1-\cos^2\theta)\times\cos^2\theta$$
$$= (1 - 2\cos^2\theta + \cos^4\theta) + (2\cos^2\theta - 2\cos^4\theta)$$
$$= 1 - \cos^4\theta. \quad \text{Q. E. D.}$$

Example ii. Express $\sin^4\theta + \cos^4\theta$ in terms of $\cos\theta$.

$$\sin^4\theta + \cos^4\theta = (1-\cos^2\theta)^2 + \cos^4\theta$$
$$= (1 - 2\cos^2\theta + \cos^4\theta) + \cos^4\theta$$
$$= 1 - 2\cos^2\theta + 2\cos^4\theta.$$

NOTE. $(1 - \cos\theta)$ is called the **versed sine** of θ, and is written versin θ.

EXAMPLES. XI.

Prove the following statements.

1. $\cos A \cdot \tan A = \sin A$.
2. $\cot A \cdot \tan A = 1$.
3. $\cos A = \sin A \cdot \cot A$.
4. $\sec A \cdot \cot A = \operatorname{cosec} A$.
5. $\operatorname{cosec} A \cdot \tan A = \sec A$.

6. $(\tan A+\cot A)\sin A\,.\,\cos A=1.$

7. $(\tan A-\cot A)\sin A\,.\,\cos A=\sin^2 A-\cos^2 A.$

8. $\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A.$

9. $(\sin A+\cos A)^2=1+2\sin A\,.\,\cos A.$

10. $(\sin A-\cos A)^2=1-2\sin A\,.\,\cos A.$

11. $\cos^4 B-\sin^4 B=2\cos^2 B-1.$

12. $(\sin^2 B+\cos^2 B)^2=1.$

13. $(\sin^2 B-\cos^2 B)^2=1-4\cos^2 B+4\cos^4 B.$

14. $1-\tan^4 B=2\sec^2 B-\sec^4 B.$

15. $(\sec B-\tan B)(\sec B+\tan B)=1.$

16. $(\operatorname{cosec}\theta-\cot\theta)(\operatorname{cosec}\theta+\cot\theta)=1.$

17. $\sin^3\theta+\cos^3\theta=(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta).$

18. $\cos^3\theta-\sin^3\theta=(\cos\theta-\sin\theta)(1+\sin\theta\cos\theta).$

19. $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\,.\,\cos^2\theta.$

20. $(\sin^6\theta-\cos^6\theta)=(2\sin^2\theta-1)(1-\sin^2\theta+\sin^4\theta).$

21. $\dfrac{\tan A+\tan B}{\cot A+\cot B}=\tan A\,.\,\tan B.$

22. $\dfrac{\cot\alpha+\tan\beta}{\tan\alpha+\cot\beta}=\cot\alpha\,.\,\tan\beta.$

23. $\dfrac{1-\sin A}{1+\sin A}=(\sec A-\tan A)^2.$

24. $\dfrac{1+\cos A}{1-\cos A}=(\operatorname{cosec}A+\cot A)^2.$

25. $2\operatorname{versin}\theta-\operatorname{versin}^2\theta=\sin^2\theta.$

26. $\operatorname{versin}\theta\,(1+\cos\theta)=\sin^2\theta.$

Express in terms of (i) $\cos\theta$, (ii) of $\sin\theta$,

27. $\cos^4\theta-\sin^4\theta.$

28. $(\sin^2\theta-\cos^2\theta)^2.$

29. $1-\tan^4\theta.$

30. $\sin^6\theta+\cos^6\theta.$

31. $\tan^2\theta+\cot^2\theta.$

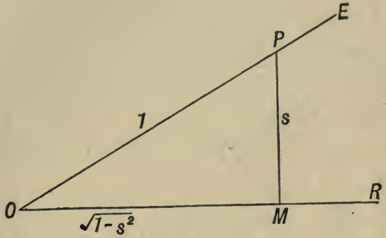
32. $1+\cot^4\theta.$

33. $1+\cot^2\theta-\operatorname{cosec}^2\theta.$

34. $2\tan^4\theta-4\sin^2\theta.$

69. All the Trigonometrical Ratios of an angle can be expressed in terms of any one of them.

Example 1. *To express all the trigonometrical ratios of an angle in terms of the* **sine.**



Let ROE be any angle θ.

We can take P anywhere in the line OE; so that we can make *one* of the lines, OP, OM, or MP any length we please.

Let us take OP so that its measure is 1, and let s be the measure of MP; so that $\sin\theta$, which is $\frac{MP}{OP}$, $=\frac{s}{1}$; or, $s = \sin\theta$.

Let x be the measure of OM.

Then since $$OM^2 = OP^2 - MP^2,$$

$$\therefore\ x^2 = 1 - s^2,$$

$$\therefore\ x = \sqrt{1-s^2}.$$

Hence $$\cos\theta = \frac{OM}{OP} = \frac{\sqrt{1-s^2}}{1} = \sqrt{1-\sin^2\theta},$$

$$\tan\theta = \frac{MP}{OM} = \frac{s}{\sqrt{1-s^2}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

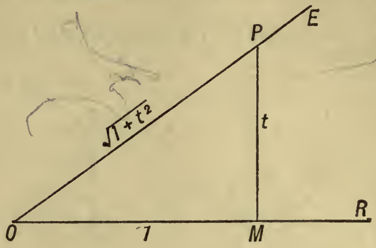
and so on.

NOTE. The solution of the equation $x^2 = 1 - s^2$, gives

$$x = \pm\sqrt{1-s^2},$$

and therefore the ambiguity ($\pm$) must stand before each of the root symbols in the above. This ambiguity, as will be explained later on, is of great use when the magnitude of the angle is not limited. When we limit A to be less than a right angle we have no use for the negative sign.

Example 2. *To express all the other trigonometrical ratios of an angle in terms of the* **tangent**.



In this case $\tan\theta = \frac{MP}{OM}$.

Take P so that the measure of OM is 1, and let t be the measure of MP; so that $\tan\theta$, which is $\frac{MP}{OM}$, $=\frac{t}{1}$; or, $t = \tan\theta$.

Then we can show that the measure of OP is $\sqrt{1+t^2}$.

Hence,
$$\sin\theta = \frac{MP}{OP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}},$$
$$\cos\theta = \frac{OM}{OP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}},$$
and so on.

70. The same results may be obtained by the use of the formulæ on p. 69.

Example. $\cos^2\theta + \sin^2\theta = 1$, $\therefore \cos^2\theta = 1 - \sin^2\theta$,
$$\therefore \cos\theta = \sqrt{1-\sin^2\theta}.$$

Again $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$, and so on.

EXAMPLES. XII.

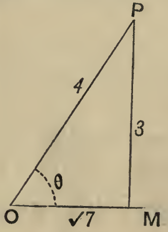
1. Express all the other Ratios of A in terms of $\cos A$.
2. Express all the other Ratios of A in terms of $\cot A$.
3. Express all the other Ratios of A in terms of $\sec A$.
4. Express all the other Ratios of A in terms of $\operatorname{cosec} A$.
5. Use the formulæ of Art. 66 to express all the other Trigonometrical Ratios of A in terms of $\sin A$.
6. Use the formulæ of Art. 66 to express all the other Trigonometrical Ratios of A in terms of the $\tan A$.

71. *Given one of the Trigonometrical Ratios of an angle less than a right angle, we can find all the others.*

Since all the Trigonometrical Ratios of an angle can be expressed in terms of any one of them, it is clear that if the numerical value of any one of them be given, the numerical value of all the rest can be found.

Example. Given $\sin\theta = \frac{3}{4}$, *find the other Trigonometrical Ratios of* θ.

Let ROE be the angle θ. Take P on OE so that the measure of OP is 4. Draw PM perpendicular to OR.



Then since $\sin\theta = \frac{3}{4}$ $\left(\text{so that } \frac{MP}{OP} = \frac{3}{4}\right)$, and since the measure of OP is 4, therefore the measure of MP must be 3.

Let x be the measure of OM;

then $$OM^2 = OP^2 - MP^2,$$
$$\therefore\ x^2 = 4^2 - 3^2 = 16 - 9 = 7.$$
$$\therefore\ x = \sqrt{7}.$$

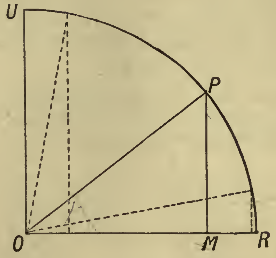
Therefore the measure of OM is $\sqrt{7}$. Hence,

$$\cos\theta = \frac{OM}{OP} = \frac{\sqrt{7}}{4}, \quad \tan\theta = \frac{MP}{OM} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}, \quad \cot\theta = \frac{\sqrt{7}}{3}.$$

EXAMPLES. XIII.

1. If $\sin A = \frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.
2. If $\cos B = \frac{1}{3}$, find $\sin B$ and $\cot B$.
3. If $\tan A = \frac{4}{3}$, find $\sin A$ and $\sec A$.
4. If $\sec\theta = 4$, find $\cot\theta$ and $\sin\theta$.
5. If $\tan\theta = \sqrt{3}$, find $\sin\theta$ and $\cos\theta$.
6. If $\cot\theta = \frac{2}{\sqrt{5}}$, find $\sin\theta$ and $\sec\theta$.
7. If $\sin\theta = \frac{b}{c}$, find $\tan\theta$.
8. If $\tan\theta = a$, find $\sin\theta$ and $\cos\theta$.
9. If $\sec\theta = a$, find $\sin\theta$ and $\cot\theta$.
10. If $\sin\theta = a$, and $\tan\theta = b$, prove that $(1 - a^2)(1 + b^2) = 1$.
11. If $\cos\theta = h$, and $\tan\theta = k$, find the equation connecting h and k.

72. *To trace the changes in the magnitude of* sin A *as* A *increases from* 0° *to* 90°.



Take a line OR, of any length; and describe the quadrant RPU of the circle whose centre is O and radius OR.

Draw the right angle ROU, cutting the circle in U.

Let OP make any angle ROP ($=A$) with OR; draw PM perpendicular to OR.

Then $$\sin A = \frac{MP}{OP}.$$

When the angle A is 0°, MP is zero, and when A is 90°, MP is equal to OP; and as A continuously increases from 0° to 90°, MP increases continuously from zero to OP; also OP is always equal to OR.

Therefore, when $A = 0°$, the fraction $\frac{MP}{OP}$ is equal to $\frac{0}{OP}$, that is 0; when $A = 90°$ the fraction $\frac{MP}{OP}$ is equal to $\frac{OP}{OP}$, that is 1; and as A continuously increases from 0° to 90°, the *numerator* of the fraction $\frac{MP}{OP}$ *continuously increases* from zero to OP, while the denominator is unchanged, and therefore the *fraction* $\frac{MP}{OP}$, which is $\sin A$, *increases continuously* from 0 to 1.

73. *To trace the changes in the magnitude of* tan A *as* A *increases from* 0° *to* 90°.

With the same construction and figure as in the last article, we have

$$\tan A = \frac{MP}{OM}.$$

When the angle A is 0°, MP is zero; when A is 90°, MP is equal to OP; and as the angle continuously increases from 0° to 90°, MP increases continuously from zero to OP.

When the angle A is 0°, OM is equal to OP; when A is 90°, OM is zero; and as A continuously increases from 0° to 90°, OM continuously decreases from OP to zero.

Hence, when A is 0°, the fraction $\frac{MP}{OM}$ is equal to $\frac{0}{OM}$, that is 0; when A is 90°, the fraction $\frac{MP}{OM}$ is equal to $\frac{OP}{0}$, that is 'infinity' (see Art. 56); and as A continuously increases from 0° to 90°, the numerator continuously increases from zero to OP, while the denominator continuously diminishes from OP to zero; so that the fraction $\frac{MP}{OM}$, which is tan A, *continuously* increases from 0 until it is *greater than any assignable numerical quantity.*

EXAMPLES. XIV.

1. Show that as A continuously increases from 0° to 90°, cos A continuously diminishes from 1 to 0.

2. Trace the changes in the magnitude of sec θ as θ increases from 0 to $\frac{\pi}{2}$.

3. Trace the changes in the magnitude of sin A as A diminishes from 90° to 0°.

4. Trace the changes in the magnitude of cot θ as θ increases from 0 to $\frac{\pi}{2}$.

On the Solution of Trigonometrical Equations.

74. A Trigonometrical equation is an equation in which there is a letter, such as θ, which stands for an *angle* of unknown magnitude.

The *solution* of the equation is the process of finding an angle which, if it be substituted for θ, satisfies the equation.

Example 1. *Solve* $\cos\theta = \frac{1}{2}$.

This is a Trigonometrical equation. To solve it we must find some angle such that its cosine is $\frac{1}{2}$.

We know that $\cos 60^\circ = \frac{1}{2}$.

Therefore if 60° be put for θ the equation is satisfied.

$$\therefore\ \theta = 60^\circ \text{ is a solution of the equation.}$$

Example 2. *Solve* $\sin\theta - \operatorname{cosec}\theta + \frac{3}{2} = 0$.

The usual method of solution is to express all the Trigonometrical Ratios in terms of one of them.

Thus we put $\dfrac{1}{\sin\theta}$ for $\operatorname{cosec}\theta$, and we get

$$\sin\theta - \frac{1}{\sin\theta} + \frac{3}{2} = 0.$$

This is an equation in which θ, and therefore $\sin\theta$ is unknown. It will be convenient if we put x for $\sin\theta$, and then solve the equation for x as an ordinary algebraical equation. Thus we get

$$x - \frac{1}{x} + \frac{3}{2} = 0,$$

or,

$$x^2 + \frac{3x}{2} = 1.$$

Whence $\quad x = -2, \text{ or } x = \frac{1}{2}$.

But x stands for $\sin\theta$.

Thus we get $\quad \sin\theta = -2, \text{ or } \sin\theta = \frac{1}{2}$.

The value -2 is inadmissible for $\sin\theta$, for there is no angle whose sine is numerically greater than 1.

$$\therefore\ \sin\theta = \tfrac{1}{2}.$$

But $\quad \sin 30^\circ = \frac{1}{2}$.

$$\therefore\ \sin\theta = \sin 30^\circ.$$

Therefore *one* angle which satisfies this equation for θ is 30°.

EXAMPLES. XV.

Find one angle which satisfies each of the following equations.

1. $\sin\theta=\frac{1}{\sqrt{2}}$.
2. $4\sin\theta=\operatorname{cosec}\theta$.
3. $2\cos\theta=\sec\theta$.
4. $4\sin\theta-3\operatorname{cosec}\theta=0$.
5. $4\cos\theta-3\sec\theta=0$.
6. $3\tan\theta=\cot\theta$.
7. $3\sin\theta-2\cos^2\theta=0$.
8. $\sqrt{2}\sin\theta=\tan\theta$.
9. $2\cos\theta=\sqrt{3}\cot\theta$.
10. $\tan\theta=3\cot\theta$.
11. $\tan\theta+3\cot\theta=4$.
12. $\tan\theta+\cot\theta=2$.
13. $2\sin^2\theta+\sqrt{2}\cos\theta=2$.
14. $2\cos^2\theta+\sqrt{2}\sin\theta=2$.
15. $3\tan^2\theta-4\sin^2\theta=1$.
16. $2\sin^2\theta+\sqrt{2}\sin\theta=2$.
17. $\cos^2\theta-\sqrt{3}\cos\theta+\frac{3}{4}=0$.
18. $\cos^2\theta+2\sin^2\theta-\frac{5}{2}\sin\theta=0$.

MISCELLANEOUS EXAMPLES. XVI.

1. Prove that $3\sin 60^\circ-4\sin^3 60^\circ=4\cos^3 30^\circ-3\cos 30^\circ$.

2. Prove that $\tan 30^\circ(1+\cos 30^\circ+\cos 60^\circ)=\sin 30^\circ+\sin 60^\circ$.

3. If $2\cos^2\theta-7\cos\theta+3=0$, show there is only one value of $\cos\theta$.

4. Find $\cos\theta$ from the equation $8\cos^2\theta-8\cos\theta+1=0$.

5. Find $\sin\theta$ from the equation $8\sin^2\theta-10\sin\theta+3=0$, and prove that one value of θ is $\frac{\pi}{6}$.

6. Find $\tan\theta$ from the equation $12\tan^2\theta-13\tan\theta+3=0$.

7. If $3\cos^2\theta+2.\sqrt{3}.\cos\theta=5\frac{1}{4}$, show that there is only one value of $\cos\theta$, and that one value of θ is $\frac{\pi}{6}$.

8. Prove that the value of $\sin^4\theta+\cos^4\theta+2.\sin^2\theta.\cos^2\theta$ is always the same.

9. Simplify $\cos^4 A+2.\sin^2 A.\cos^2 A$.

10. Express $\sin^6 A+\cos^6 A$ in terms of $\sin^2 A$ and powers of $\sin^2 A$.

11. Express $1+\tan^4\theta$ in terms of $\cos\theta$ and its powers.

12. Prove that $\frac{\cos A+\cos B}{\sin A-\sin B}+\frac{\sin A+\sin B}{\cos A-\cos B}=0$.

13. Express $(\sec A-\tan A)^2$ in terms of $\sin A$.

14. Trace the changes in $\operatorname{cosec}\theta$ as θ increases from 0 to $\frac{1}{2}\pi$.

15. Trace the changes in $\cot\theta$ as θ decreases from $\frac{1}{2}\pi$ to 0.

16. Solve $\quad 2\sin(\theta+\phi)=\sqrt{3}, \quad 2\cos(\theta-\phi)=\sqrt{3}$.

CHAPTER VII.

On the Use of the Signs + and −.

75. The student is probably aware that, in the application of Algebra to Problems concerning *distance*, we sometimes find that the solution of an equation gives the measure of a *distance* with the sign − before it.

Example. Let M, N, O be places in a straight line; let the distance from M to N be 3 miles, and the distance from N to O, 3 miles.

One man A starting from M, rides towards O at the rate of 10 miles an hour, while another man B starting simultaneously from N, walks towards O at the rate of 4 miles an hour;

If Q be the point at which they meet, *how far is Q beyond O?*

Let P be any point beyond O, and let x be the number of miles in OP. We wish to find x, i.e. the measure of OP, so that P may coincide with Q, the point at which A overtakes B.

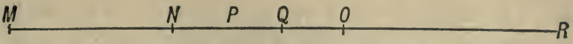
When A arrives at P, he has ridden $6+x$ miles. The time occupied at the rate of 10 miles an hour is $\frac{6+x}{10}$ hours.

When B arrives at P, he has walked $3+x$ miles. The time occupied at the rate of 4 miles an hour is $\frac{3+x}{4}$ hours.

When P is the point at which they meet, these times are equal, so that
$$\frac{6+x}{10}=\frac{3+x}{4}; \text{ whence } x=-1.$$

Thus the required *number of miles* has the sign − before it; and we have failed to find a point *beyond O* at which A overtakes B.

76. Such a result can generally be *interpreted* by altering the statement of the problem, thus:



Example. Taking the former example, let us alter the question as follows:

If Q be the point at which A overtakes B, how far is Q **to the left** of O?

Let P be any point *to the left* of O, and let x be the number of miles in OP.

We wish to find x (i.e. the measure of OP), so that P may coincide with Q, the point at which A overtakes B.

When A arrives at P, he has ridden $6 - x$ miles.

When B arrives at P, he has walked $3 - x$ miles.

Proceeding as before, we get

$$\frac{6-x}{10} = \frac{3-x}{4}; \text{ whence } x = +1.$$

Therefore if P is to coincide with Q (the point at which A overtakes B), OP must be one mile *to the left* of O.

77. The consideration of such examples as the above has suggested, that the sign − may be made use of, in the application of Algebra to Geometry, to represent a **direction** exactly opposite to that represented by the sign +.

Accordingly the following Rule, or Convention, has been made.

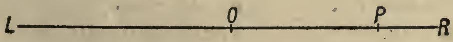
RULE. Any straight line AB being given, then

lines drawn parallel to AB in **one direction** shall be **positive**; that is, shall be *represented algebraically* by their *measures with the sign + before them:*

lines drawn parallel to BA in the **opposite direction** shall be **negative**; that is, shall be *represented algebraically* by their *measures with the sign − before them.*

78. We may choose for the positive direction in each case that direction which is most convenient.

Example. Let LR be a straight line parallel to the printed lines in the page,



and let the lines drawn in the direction from L to R in the figure, that is, from the left-hand towards the right, be considered *positive.* Then by the above rule, lines drawn in the direction from R to L, that is, from right to left, must be *negative.*

79. In naming a line by the letters at its extremities, we can indicate by the **order of the letters** the direction in which the line is supposed to be drawn.

Example. Let O and P be two points in the line LR as in the figure, and let the measure of the distance between them be a.

Then OP, i.e. the line drawn from O to P, which is in the *positive* direction, is *represented algebraically* by $+a$.

While PO, i.e. the line drawn from P to O, which is in the *negative* direction, is *represented algebraically* by $-a$.

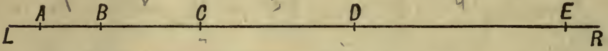
80. Hence in using the two letters at its extremities to represent a line, the student will find it advantageous always to pay careful attention to the **order of the letters.**

Example. Let LR be a straight line parallel to the printed lines in the page.

Let A, B, C, D, E be points in LR, such that the measures of AB, BC, CD, DE, are 1, 2, 3, 4 respectively.

Find the algebraical representation of

(i) $AC+CB$, (ii) $AD+DC-BC$.



(i) The algebraical representation of AC is $+3$,
the algebraical representation of CB is -2.

Hence that of $AC+CB$ is $+3-2$; that is, $+1$†.

(ii) The algebraical representation of AD is $+6$, that of DC is -3, and that of BC is $+2$.

Therefore that of $AD+DC-BC=6-3-2=+1$.

This is equivalent to that of AB.

EXAMPLES. XVII.

In the above figure, find the algebraical representation of

1. $AB+BC+CD$. **2.** $AB+BC+CA$.

3. $BC+CD+DE+EC$. **4.** $AD-CD$.

5. $AD+DB+BE$. **6.** $BC-AC+AD-BD$.

7. $CD+DB+BE$. **8.** $CD-BD+BA+AC+CE$.

† By $AC+CB$ (attention being paid to *direction*), we mean 'Go from A to C and from C to B.' The result is equivalent to starting from A and stopping at B, *i.e.* equivalent to AB.

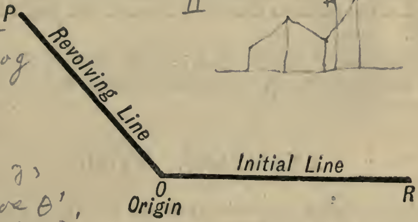
CHAPTER VIII.

On the Use of the Signs + and − in Trigonometry.

81. In **Trigonometry** in order conveniently to treat of angles of *any* magnitude, we proceed as follows.

We take a fixed point O, called the **origin**; and a fixed straight line OR, called the **initial line.**

The angle of which we wish to treat is described by a line OP, called the **revolving line.** This line OP *starts from* the initial line OR, and turns about O through an angle ROP of any proposed magnitude into the position OP.



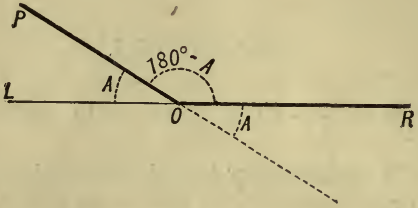
82. We have already said in Art. 18

(i) that, when an angle ROP is described by OP turning about O in the direction *contrary to that of the hands of a watch*, the angle ROP is said to be **positive**; that is, *is represented algebraically* by its *measure with the sign + before it.*

(ii) that, when an angle ROP is described by OP turning about O in the *same* direction as the hands of a watch, the angle is said to be **negative**; that is, is *represented algebraically* by its *measure with the sign − before it.*

Example. $(180^0 - A)$ indicates

(i) the angle described by OP turning about O from the position OR in the positive direction until it has described an angle of $(180 - A)$ degrees.



Or, (ii) the angle described by OP turning about O, from the position OR, in the positive direction until it has described an angle of 180^0 (when it has turned into the position OL), and then turning back from OL in the negative direction through the angle $-A$ into the position OP.

Or, (iii) the angle described by OP turning about O from the position OR, in the negative direction through the angle $-A$, and then turning back in the positive direction through the angle 180^0, into the position OP.

The student should observe that in each of these three ways of regarding the angle $(180^0 - A)$, the *resulting* angle ROP is the same.

EXAMPLES. XVIII.

Draw a figure giving the position of the revolving line after it has turned through each of the following angles.

1. 270^0. **2.** 370^0. **3.** 425^0. **4.** 590^0.

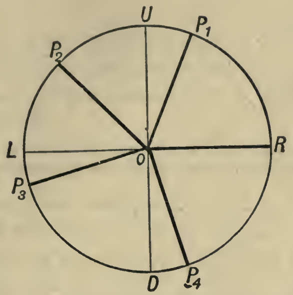
5. -30^0. **6.** -330^0. **7.** -480^0. **8.** -750^0.

9. $\frac{27\pi}{4}$. **10.** $2n\pi + \frac{\pi}{6}$. **11.** $(2n+1)\pi + \frac{\pi}{3}$.

12. $(2n+1)\pi - \frac{\pi}{4}$. **13.** $2n\pi - \frac{\pi}{2}$. **14.** $(2n+1)\pi - \frac{\pi}{2}$.

NOTE. $n\pi$ always stands for a whole number of two right angles.

83. It is often convenient to keep the revolving line of the same length.



In this case the point P lies always on the circumference of a circle whose centre is O.

Let this circle cut the lines LOR, UOD in the points L, R, U, D respectively.

The circle $RULD$ is thus divided at the points R, U, L, D into four **Quadrants**, of which

RU is called the **first Quadrant.**
UL is called the **second Quadrant.**
LD is called the **third Quadrant.**
DR is called the **fourth Quadrant.**

Hence we say that, in the figure,

the angle ROP_1 is an angle of the first Quadrant.
ROP_2 ,, ,, second Quadrant.
ROP_3 ,, ,, third Quadrant.
ROP_4 ,, ,, fourth Quadrant.

84. When we are told that an angle is of some particular Quadrant, say the third, we know that the position in which the revolving line *stops* is in the third Quadrant. But there is an unlimited number of *angles* having this same final position of OP.

Example. 25^0; 385^0 i.e. 360^0+25^0; 745^0 i.e. $2\times360^0+25^0$; -335 i.e. -360^0+25^0 are each an angle of the first Quadrant, and are all represented *geometrically* by the *same* final position of OP.

85. Let A be an angle between 0° and 90°, and let n be any *whole* number, positive or negative.

Then

(i) $2n \times 180^\circ + A$ represents algebraically an angle whose revolving line is in the *first* Quadrant.

(ii) $2n \times 180^\circ - A$ represents algebraically an angle of the *fourth* Quadrant.

[For $2n \times 180^\circ$ represents some number n of *complete* revolutions of OP; so that after describing $n \times 360^\circ$, OP is again in the position OR.]

(iii) $(2n+1) \times 180^\circ - A$ represents algebraically an angle of the *second* Quadrant.

(iv) $(2n+1) \times 180^\circ + A$ represents algebraically an angle of the *third* Quadrant.

[For after describing $(2n+1) \times 180^\circ$, OP is in the position OL.]

The corresponding expressions in circular measure are

(i) $2n\pi + \theta$; (ii) $2n\pi - \theta$; (iii) $(2n+1)\pi - \theta$; (iv) $(2n+1)\pi + \theta$.

EXAMPLES. XIX.

State in which Quadrant the revolving line will be after describing the following angles:

1. 120°. **2.** 340°. **3.** 490°.

4. -100°. **5.** -380°. **6.** -1000°.

7. $\frac{2\pi}{3}$. **8.** $10\pi + \frac{\pi}{4}$. **9.** $9\pi - \frac{3\pi}{4}$.

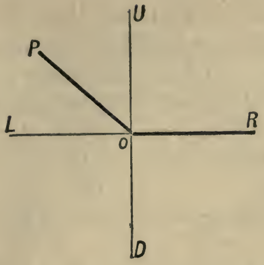
10. $2n\pi - \frac{\pi}{4}$. **11.** $(2n+1)\pi + \frac{2\pi}{3}$. **12.** $n\pi + \frac{\pi}{6}$.

86. The principal **directions of lines** with which we are concerned in **Trigonometry** are as follows;

i. that parallel to the initial line OR (OR is usually drawn from O towards the right hand, parallel to the printed lines in the page; and RO is produced to L.)

ii. that parallel to the line DOU, which is drawn through O at right angles to LOR;

iii. that parallel to the revolving line OP.



Accordingly we make the following **rules:**

I. Any line drawn parallel to LR in the direction **from left to right** is to be **positive**; and *consequently* (Art. 112) any line drawn parallel to RL in the *opposite* direction, i.e. from **right to left,** is to be **negative.**

II. Any line drawn parallel to DU in the direction from D to U, **upwards,** is to be **positive**; and consequently any line drawn parallel to UD in the *opposite* direction, i.e. **downwards,** is to be **negative.**

III. Any line drawn parallel to the revolving line in the direction from O to P is to be **positive,** and consequently any line drawn in the direction from P to O is to be **negative.**

NOTE. The student must notice that the revolving line OP *carries its positive direction round with it,* so that the line 'OP' is *always positive.*

87. We said, in Art. 43, that **the definitions of the Trigonometrical Ratios** (on pp. 20, 21), **apply to angles of any magnitude.** We have only to remark that it is generally convenient to take P on the revolving line; that PM is drawn perpendicular to the other line *produced if necessary;* and that the **order of the letters** in MP, OP, OM is an essential part of the definition.

The **order of the letters** P, M, O in the expressions $\frac{MP}{OP}$, etc., is therefore of great importance.

88. We proceed to show that the Trigonometrical Ratios of an angle vary in **Sign** according to the **Quadrant** in which the revolving line of the angle happens to be.

From the definition we have, with the usual letters,

$$\sin ROP = \frac{MP}{OP},\ \cos ROP = \frac{OM}{OP},\ \tan ROP = \frac{MP}{OM}.$$

Fig. I.

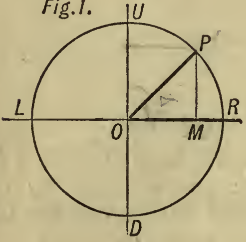


Fig. II.

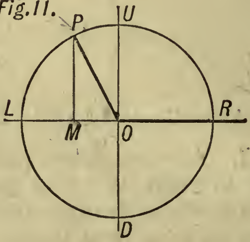


Fig. III.

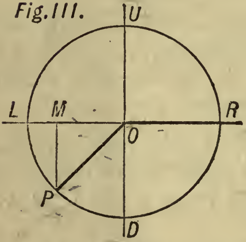
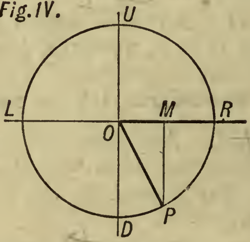


Fig. IV.



I. When OP is in the **first** Quadrant (Fig. I.).

MP is *positive* because from M to P is *upwards* (Rule II. p. 55.)

OM is *positive* because from O to M is *towards the right*, (Rule I.).

OP is *positive.* (Rule III.).

Hence, if A be any angle of the *first* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{positive};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{positive}.$$

II. When OP is in the **second** Quadrant (Fig. II.).

MP is *positive*, because from M to P is *upwards*,

OM is *negative*, because from O to M is *towards the left*.

OP is *positive*.

Hence, if A be any angle of the *second* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{negative};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{negative}.$$

III. When OP is in the **third** Quadrant (Fig. III.) MP is *negative*, OM is *negative*, OP is *positive*.

So that, if A be any angle of the *third* Quadrant,

sin A is *negative*, cos A is *negative*, tan A is *positive*.

IV. When OP lies in the **fourth** Quadrant (Fig. IV.) MP is *negative*, OM is *positive*, OP is *positive*.

So that, if A be any angle of the *fourth* Quadrant,

sin A is *negative*, cos A is *positive*, tan A is *negative*.

89. The table given below exhibits the results of the last Article.

Quadrant ...	I.	II.	III.	IV.
Sine	+	+	–	–
Cosine	+	–	–	+
Tangent ...	+	–	+	–

The student should notice that for any particular Quadrant the three *signs* of sine, cosine, and tangent are unlike their signs for any other Quadrant.

EXAMPLES. XX.

State the *sign* of the sine, cosine, and tangent of each of the following angles:

1. 60°.	2. 135°.	3. 265°.
4. 275°.	5. -10°.	6. -91°.
7. -193°.	8. -350°.	9. -1000°.
10. $2n\pi+\frac{1}{4}\pi$.	11. $2n\pi+\frac{3}{4}\pi$.	12. $2n\pi-\frac{1}{6}\pi$.

90. The NUMERICAL VALUES through which the Trigonometrical Ratios of the angle ROP pass, as the line OP turns through the *first* Quadrant, are **repeated** as OP turns through each of the other Quadrants.

Thus as OP turns through the second Quadrant from U to L, Fig. II. p. 56 (OP being always of the same length) MP and OM pass through the same succession of numerical values through which they pass, as OP turns through the first Quadrant in the *opposite* direction from U to R.

Example 1. *Find the sine, cosine and tangent of* 120°.

120° is an angle of the second Quadrant.

Let the angle ROP be 120° (Fig. II. p. 56).

Then the angle $POL=180^\circ-120^\circ=60^\circ$.

Hence, $\sin 120^\circ=\frac{MP}{OP}=\sin 60^\circ$ *numerically*, and in the *second* Quadrant the sine is *positive*.

Therefore $\sin 120^\circ=\frac{\sqrt{3}}{2}$ ....(i).

Again, $\cos 120^\circ=\frac{OM}{OP}=\cos 60^\circ$ *numerically*, and in the *second* Quadrant the cosine is *negative*.

Therefore $\cos 120^\circ=-\frac{1}{2}$..................(ii).

Similarly, $\tan 120^\circ=-\sqrt{3}$ ....(iii).

Example 2. *Find the sine, cosine and tangent of* 225°.

225° is an angle of the third Quadrant.

Let the angle ROP be 225° (Fig. III. p. 56).

Here the angle $POL=225^\circ-180^\circ=45^\circ$.

Therefore the Trigonometrical Ratios of $225^\circ=$ those of 45° *numerically;* and in the *third* Quadrant the sine and cosine are each *negative* and the tangent is *positive*.

Hence, $\sin 225^\circ=-\frac{1}{\sqrt{2}}$; $\cos 225^\circ=-\frac{1}{\sqrt{2}}$; $\tan 225^\circ=1$.

91. The cosecant, secant and cotangent of an angle A have the same *sign* as the sine, cosine, and tangent of A respectively.

EXAMPLES. XXI.

Find the algebraical value of the sine, cosine and tangent of the following angles:

1. 150^0. 2. 135^0. 3. -240^0. 4. 330^0.
5. -45^0. 6. -300^0. 7. 225^0. 8. -135.
9. 390^0. 10. 750^0. 11. -840^0. 12. 1020^0.
13. $2n\pi+\frac{\pi}{4}$. 14. $(2n+1)\pi-\frac{\pi}{3}$. 15. $(2n-1)\pi+\frac{\pi}{6}$.

Find the four smallest angles which satisfy the equations

16. $\sin A=\frac{1}{2}$. 17. $\sin A=\frac{1}{\sqrt{2}}$. 18. $\sin A=\frac{\sqrt{3}}{2}$. 19. $\sin A=-\frac{1}{2}$.

Find four angles between zero and +8 right angles which satisfy the equations

20. $\sin A=\sin 20^0$. 21. $\sin\theta=-\frac{1}{\sqrt{2}}$. 22. $\sin\theta=-\sin\frac{\pi}{7}$.

23. Prove that 30^0, 150^0, -330^0, 390^0, -210^0 have the same sine.

24. Show that each of the following angles has the same cosine:

$$-120^0,\quad 240^0,\quad 480^0,\quad -480^0.$$

25. The angles 60^0 and -120^0 have one of the Trigonometrical Ratios the same for both; which of the ratios is it?

26. Can the following angles have any one of their Trigonometrical Ratios the same for all? -23^0, 157^0 and -157^0.

92. *Proposition. To trace the changes in the magnitude and sign of sin* A, *as* A *increases from* 0^0 *to* 360^0.

Take the figure and construction of page 56.

As A increases from 0^0 to 90^0, MP increases from zero to OP, and is positive.

Therefore $\sin A$ increases from 0 to 1 and is positive.

As A increases from 90^0 to 180^0, MP decreases from OP to zero, and is positive.

Therefore $\sin A$ decreases from 1 to 0 and is positive.

As A increases from 180° to 270°, MP increases from zero to OP, and is negative.

Therefore $\sin A$ increases numerically from 0 to 1, and is negative.

As A increases from 270° to 360°, MP decreases from OP to zero, and is negative.

Therefore $\sin A$ decreases numerically from 1 to 0 and is negative.

*EXAMPLES. XXII.

Trace the changes in sign and magnitude as A increases from 0° to 360° of

1. $\cos A$.
2. $\tan A$.
3. $\cot A$.
4. $\sec A$.
5. $\operatorname{cosec} A$.
6. $1 - \sin A$.
7. $\sin^2 A$.
8. $\sin A \cdot \cos A$.
9. $\sin A + \cos A$.
10. $\tan A + \cot A$.
11. $\sin A - \cos A$.

93. Def. One angle is said to be the **complement** of another, when the two angles added together make up a *right angle*.

Example 1. The complement of A is $(90^\circ - A)$.

Example 2. The complement of 190° is $(90^\circ - 190^\circ) = -100^\circ$.
For $190^\circ + (90^\circ - 190^\circ) = 90^\circ$.

Example 3. The complement of $\frac{5\pi}{4}$ is $\left(\frac{\pi}{2} - \frac{5\pi}{4}\right) = -\frac{3\pi}{4}$.

94. *To prove that the sine of an angle* A *is equal to the cosine of its complement* $(90^\circ - \text{A})$.

Let A be less than 90°, and let ROP be A.

Draw PM perpendicular to OR. [See figure, p. 20.]

Then since $PMO = 90^\circ$, therefore $POM + OPM = 90^\circ$, and therefore $OPM = (90^\circ - A)$.

Now, $\sin A = \frac{MP}{OP} = \cos OPM = \cos(90^\circ - A)$. Q.E.D.

EXAMPLES. XXIII.

Find the complements of

1. 30°.
2. 190°.
3. 90°.
4. 350°.
5. -25°.
6. -320°.
7. $\frac{3}{4}\pi$.
8. $-\frac{1}{6}\pi$.

Prove by drawing a figure in each case

9. $\sin 70^0 = \cos 20^0$. **10.** $\cos 47^0\, 16' = \sin 42^0\, 44'$.

11. $\tan 79^0 = \cot 11^0$. **12.** $\sec 36^0 = \text{cosec}\, 54^0$.

If A be less than 90^0, prove

13. $\cos A = \sin (90^0 - A)$. **14.** $\tan A = \cot (90^0 - A)$.

15. $\sec A = \text{cosec}\, (90^0 - A)$. **16.** $\cot A = \tan (90^0 - A)$.

If A, B, C be the angles of a triangle, so that $A + B + C = 180^0$, prove

17. $\cos \frac{1}{2}A = \sin \frac{1}{2}(B + C)$. **18.** $\cos \frac{1}{2}B = \sin \frac{1}{2}(A + C)$.

19. $\sin \frac{1}{2}C = \cos \frac{1}{2}(A + B)$. **20.** $\sin \frac{1}{2}A = \cos \frac{1}{2}(B + C)$.

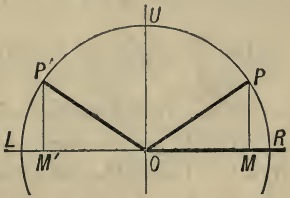
95. **Def.** One angle is said to be the **supplement** of another when their sum is two right angles.

Thus $(180^\circ - A)$ is the **supplement** of A.

If A, B, C be the angles of a triangle, $(A + B + C) = 180^\circ$, so that $(B + C)$ is the **supplement** of A.

96. *To prove that the sine of an angle = the sine of its supplement*, when the angle is less than 180°.

Let ROP be the angle A, take LOP' also $= A$, then $ROP' = (180^\circ - A)$.



Take $OP = OP'$ and draw PM, $P'M'$ perpendicular to ROL, then the triangle POM, $P'OM'$ are equal in all respects, since they are equiangular and $OP = OP'$.

Hence $$\frac{MP}{OP} = \frac{M'P'}{OP'};$$

that is, $\sin ROP = \sin ROP'$; or, $\sin A = \sin (180^\circ - A)$.

Also $$\frac{OM}{OP} = -\frac{OM'}{OP'};$$

that is, $\cos ROP = -\cos ROP'$; or, $\cos A = -\cos (180^\circ - A)$.

EXAMPLES. XXIV.

Prove, drawing a separate figure in each case, that

1. $\sin 60^\circ = \sin 120^\circ$.
2. $\sin 340^\circ = \sin(-160^\circ)$.
3. $\sin(-40^\circ) = \sin 220^\circ$.
4. $\cos 320^\circ = -\cos(-140^\circ)$.
5. $\cos(-380^\circ) = -\cos 560^\circ$.
6. $\cos 195^\circ = -\cos(-15^\circ)$.

If A, B, C be the angles of a triangle, prove

7. $\sin A = \sin(B + C)$.
8. $\sin C = \sin(A + B)$.
9. $\cos B = -\cos(A + C)$.
10. $\cos A = -\cos(C + B)$.

Prove by means of a figure that

11. $\sin(-A) = -\sin A$.
12. $\cos(-A) = \cos A$.
13. $\sin(90^\circ + A) = \cos A$.
14. $\cos(90^\circ + A) = -\sin A$.
15. $\tan(180^\circ + A) = \tan A$.

CHAPTER IX.

On the Trigonometrical Ratios of Two Angles.

97. We proceed to establish the following fundamental formulæ:

$$\left.\begin{aligned} \sin(A + B) &= \sin A \,.\, \cos B + \cos A \,.\, \sin B \\ \cos(A + B) &= \cos A \,.\, \cos B - \sin A \,.\, \sin B \\ \sin(A - B) &= \sin A \,.\, \cos B - \cos A \,.\, \sin B \\ \cos(A - B) &= \cos A \,.\, \cos B + \sin A \,.\, \sin B \end{aligned}\right\} \ldots \text{(i)}.$$

Here, A and B are angles; so that $(A + B)$ and $(A - B)$ are also angles.

Hence, $\sin(A + B)$ is the sine of an angle, and must not be confounded with $\sin A + \sin B$.

Sin $(A + B)$ is a single fraction.

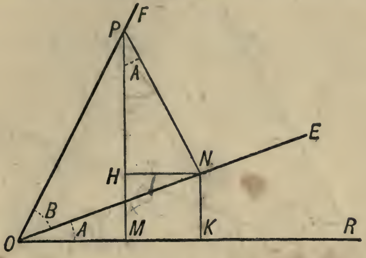
Sin $A + \sin B$ is the sum of two fractions.

The student should notice that the words of the two proofs of Arts. 98, 99 are very nearly the same.

98. *To prove that*

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

and that $$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$



Let ROE be the angle A, and EOF the angle B. Then in the figure, ROF is the angle $(A+B)$.

In OF, *the line which bounds the compound angle* $(A+B)$, take any point P, and from P draw PM, PN at right angles to OR and OE respectively. Draw NH, NK at right angles to MP and OR respectively. Then the angle

$$NPH = 90^0 - HNP = HNO = ROE = A\dagger.$$

Now

$$\sin(A+B) = \sin ROF = \frac{MP}{OP} = \frac{MH+HP}{OP} = \frac{KN}{OP} + \frac{HP}{OP}$$

$$= \frac{KN \,.\, \mathrm{ON}}{\mathrm{ON} \,.\, OP} + \frac{HP \,.\, \mathrm{NP}}{\mathrm{NP} \,.\, OP} = \frac{KN}{ON} \cdot \frac{ON}{OP} + \frac{HP}{NP} \cdot \frac{NP}{OP}$$

$$= \sin ROE \,.\, \cos EOF + \cos HPN \,.\, \sin EOF$$

$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B.$$

Also

$$\cos(A+B) = \cos ROF = \frac{OM}{OP} = \frac{OK - MK}{OP} = \frac{OK}{OP} - \frac{HN}{OP}$$

$$= \frac{OK \,.\, \mathrm{ON}}{\mathrm{ON} \,.\, OP} - \frac{HN \,.\, \mathrm{NP}}{\mathrm{NP} \,.\, OP} = \frac{OK}{ON} \cdot \frac{ON}{OP} - \frac{HN}{NP} \cdot \frac{NP}{OP}$$

$$= \cos ROE \,.\, \cos EOF - \sin HPN \,.\, \sin EOF$$

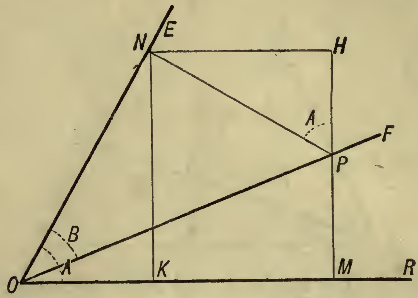
$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$

† Or thus. On OP as diameter describe a circle; this will pass through M and N, because the angles OMP and ONP are right angles; therefore MPN and MON are angles in the same segment; so that the angle $MPN = MON = A$.

99. *To prove that*

$$\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B,$$

and that $\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B.$



Let ROE be the angle A, and FOE the angle B. Then in the figure, ROF is the angle $(A-B)$.

In OF, *the line which bounds the compound angle* $(A-B)$, take any point P, and from P draw PM, PN at right angles to OR and OE respectively. Draw NH, NK at right angles to MP and OR respectively. Then the angle

$$NPH = 90^\circ - HNP = HNE = ROE = A\dagger.$$

Now

$$\sin(A-B) = \sin ROF = \frac{MP}{OP} = \frac{MH-PH}{OP} = \frac{KN}{OP} - \frac{PH}{OP}$$

$$= \frac{KN \,.\, ON}{ON \,.\, OP} - \frac{PH \,.\, NP}{NP \,.\, OP} = \frac{KN}{ON} \cdot \frac{ON}{OP} - \frac{PH}{NP} \cdot \frac{NP}{OP}$$

$$= \sin ROE \,.\, \cos FOE - \cos HPN \,.\, \sin FOE$$

$$= \sin A \,.\, \cos B - \cos A \,.\, \sin B.$$

Also

$$\cos(A-B) = \cos ROF = \frac{OM}{OP} = \frac{OK+KM}{OP} = \frac{OK}{OP} + \frac{NH}{OP}$$

$$= \frac{OK \,.\, ON}{ON \,.\, OP} + \frac{NH \,.\, NP}{NP \,.\, OP} = \frac{OK}{ON} \cdot \frac{ON}{OP} + \frac{NH}{NP} \cdot \frac{NP}{OP}$$

$$= \cos ROE \,.\, \cos FOE + \sin HPN \,.\, \sin FOE$$

$$= \cos A \,.\, \cos B + \sin A \,.\, \sin B.$$

† Or thus. On OP as diameter describe a circle, this will pass through M and N, because the angles OMP and ONP are right angles; therefore the angles MPN and MON together make up two right angles; so that the angle $HPN = MON = A$.

Example. Find the value of sin 75^0.

$$\sin 75^0 = \sin(45^0 + 30^0)$$
$$= \sin 45^0 . \cos 30^0 + \cos 45^0 . \sin 30^0$$
$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}.$$

EXAMPLES. XXV.

X

1. Show that $\cos 75^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

2. Show that $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

3. Show that $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

4. Show that $\tan 75^0 = 2 + \sqrt{3}$.

5. If $\sin A = \frac{4}{5}$ and $\sin B = \frac{3}{5}$, find a value for $\sin(A+B)$ and for $\cos(A-B)$.

6. If $\sin A = \cdot 6$ and $\sin B = \frac{5}{13}$, find a value for $\sin(A+B)$ and for $\cos(A+B)$.

7. When $\sin A = \frac{1}{\sqrt{5}}$ and $\sin B = \frac{1}{\sqrt{10}}$, then one value of $(A+B)$ is 45^0.

8. Prove that $\sin 75^0 = \cdot 9659...$

9. Prove that $\sin 15^0 = \cdot 2588...$

10. Prove that $\tan 15^0 = \cdot 2679...$

100. It is important that the student should become thoroughly familiar with the formulæ proved on the last two pages, and that he should be able to work examples involving their use.

EXAMPLES. XXVI.

Prove the following statements.

1. $\sin(A+B) + \sin(A-B) = 2 \sin A . \cos B$.
2. $\sin(A+B) - \sin(A-B) = 2 \cos A . \sin B$.
3. $\cos(A+B) + \cos(A-B) = 2 \cos A . \cos B$.
4. $\cos(A-B) - \cos(A+B) = 2 \sin A . \sin B$.
5. $\frac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$.

6. $\tan\alpha+\tan\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha\,.\,\cos\beta}$.

7. $\tan\alpha-\tan\beta=\dfrac{\sin(\alpha-\beta)}{\cos\alpha\,.\,\cos\beta}$.

8. $\cot\alpha+\tan\beta=\dfrac{\cos(\alpha-\beta)}{\sin\alpha\,.\,\cos\beta}$.

9. $\cot\alpha-\tan\beta=\dfrac{\cos(\alpha+\beta)}{\sin\alpha\,.\,\cos\beta}$.

10. $\tan\alpha+\cot\beta=\dfrac{\cos(\alpha-\beta)}{\cos\alpha\,.\,\sin\beta}$.

11. $\dfrac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}$.

12. $\dfrac{\tan\theta\,.\,\tan\phi+1}{1-\tan\theta\,.\,\tan\phi}=\dfrac{\cos(\theta-\phi)}{\cos(\theta+\phi)}$.

13. $\dfrac{\tan\theta+\cot\phi}{\cot\phi-\tan\theta}=\cos(\theta-\phi)\,.\,\sec(\theta+\phi)$.

14. $\dfrac{\cot\theta+\cot\phi}{\cot\theta-\cot\phi}=-\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}$.

15. $\dfrac{\tan\theta\,.\,\cot\phi+1}{\tan\theta\,.\,\cot\phi-1}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}$.

16. $\dfrac{1+\cot\gamma\,.\,\tan\delta}{\cot\gamma-\tan\delta}=\tan(\gamma+\delta)$.

17. $\dfrac{1-\cot\gamma\,.\,\tan\delta}{\cot\gamma+\tan\delta}=\tan(\gamma-\delta)$.

18. $\dfrac{\tan\gamma\,.\,\cot\delta-1}{\tan\gamma+\cot\delta}=\tan(\gamma-\delta)$.

19. $\dfrac{\tan\gamma\,.\,\cot\delta+1}{\cot\delta-\tan\gamma}=\tan(\gamma+\delta)$.

20. $\dfrac{\cot\delta-\cot\gamma}{\cot\gamma\,.\,\cot\delta+1}=\tan(\gamma-\delta)$.

21. $\tan^2\alpha-\tan^2\beta=\dfrac{\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)}{\cos^2\alpha\,.\,\cos^2\beta}$.

22. $\cot^2\alpha-\tan^2\beta=\dfrac{\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)}{\sin^2\alpha\,.\,\cos^2\beta}$.

23. $\dfrac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\,.\,\tan^2\beta}=\tan(\alpha+\beta)\,.\,\tan(\alpha-\beta)$.

24. $\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha$.

25. $\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$.

26. $\sin(A-45^0)=\dfrac{\sin A-\cos A}{\sqrt{2}}$.

27. $\sqrt{2}\,.\,\sin(A+45^0)=\sin A+\cos A$.

28. $\cos A-\sin A=\sqrt{2}\,.\,\cos(A+45^0)$.

29. $\cos(A+45^0)+\sin(A-45^0)=0$.

30. $\cos(A-45^0)=\sin(A+45^0)$.

31. $\sin(\theta+\phi)\,.\,\cos\theta-\cos(\theta+\phi)\,.\,\sin\theta=\sin\phi$.

32. $\sin(\theta-\phi)\,.\,\cos\phi+\cos(\theta-\phi)\,.\,\sin\phi=\sin\theta$.

33. $\cos(\theta+\phi)\,.\,\cos\theta+\sin(\theta+\phi)\,.\,\sin\theta=\cos\phi$.

34. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\,.\,\tan\phi}=\tan\theta.$

35. $\dfrac{\tan(\theta+\phi)-\tan\theta}{1+\tan(\theta+\phi)\,.\,\tan\theta}=\tan\phi.$

36. $2\sin\left(\alpha+\dfrac{\pi}{4}\right).\cos\left(\beta-\dfrac{\pi}{4}\right)=\cos(\alpha-\beta)+\sin(\alpha+\beta).$

37. $2\sin\left(\dfrac{\pi}{4}-\alpha\right).\cos\left(\dfrac{\pi}{4}+\beta\right)=\cos(\alpha-\beta)-\sin(\alpha+\beta).$

38. $\cos(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\left(\dfrac{\pi}{4}+\alpha\right).\cos\left(\dfrac{\pi}{4}+\beta\right).$

39. $\cos(\alpha+\beta)-\sin(\alpha-\beta)=2\sin\left(\dfrac{\pi}{4}-\alpha\right).\cos\left(\dfrac{\pi}{4}-\beta\right).$

40. $\sin nA\,.\cos A+\cos nA\,.\sin A=\sin(n+1)A.$

41. $\cos(n-1)A\,.\cos A-\sin(n-1)A\,.\sin A=\cos nA.$

42. $\sin nA\,.\cos(n-1)A-\cos nA\,.\sin(n-1)A=\sin A.$

43. $\cos(n-1)A\,.\cos(n+1)A-\sin(n-1)A\,.\sin(n+1)A=\cos 2nA.$

101. The following formulæ are important:

$$\left.\begin{aligned}\tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}\\ \tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\,.\tan B}\end{aligned}\right\}\ \ldots\ldots\ldots\text{(ii)}.$$

The proof of the first is given below. The student should prove the second in a similar manner.

Example. To prove $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\,.\tan B}$.

(i) By using the results of Arts. 98, 99, we have

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\,.\cos B+\cos A\,.\sin B}{\cos A\,.\cos B-\sin A\,.\sin B}.$$

Divide the numerator and the denominator of this fraction each by $\cos A\,.\cos B$, and we get

$$\tan(A+B)=\frac{\dfrac{\sin A\,.\cos B}{\cos A\,.\cos B}+\dfrac{\cos A\,.\sin B}{\cos A\,.\cos B}}{\dfrac{\cos A\,.\cos B}{\cos A\,.\cos B}-\dfrac{\sin A\,.\sin B}{\cos A\,.\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\,.\tan B}.\qquad\text{Q.E.D.}$$

EXAMPLES. XXVII.

1. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{4}$, prove that $\tan(A+B) = \frac{6}{7}$, and $\tan(A-B) = \frac{2}{9}$.

2. If $\tan A = 1$ and $\tan B = \dfrac{1}{\sqrt{3}}$, prove that $\tan(A+B) = 2+\sqrt{3}$.

3. Prove that $\tan 15^0 = 2-\sqrt{3}$.

4. If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$, prove that $\tan(A+B) = 1$. What is $(A+B)$ in this case?

5. If $\tan A = m$ and $\tan B = \dfrac{1}{m}$, prove that $\tan(A+B) = \infty$. What is $(A+B)$ in this case?

Prove the following statements:

6. $\cot(A+B) = \dfrac{\cot A \,.\, \cot B - 1}{\cot A + \cot B}$.

7. $\cot(A-B) = \dfrac{\cot A \,.\, \cot B + 1}{\cot B - \cot A}$.

8. $\cot\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{\cot\theta + 1}{1 - \cot\theta}$.

9. $\dfrac{\cot\theta - 1}{\cot\theta + 1} = \cot\left(\theta + \dfrac{\pi}{4}\right)$.

10. $\tan\left(\theta - \dfrac{\pi}{4}\right) + \cot\left(\theta + \dfrac{\pi}{4}\right) = 0$.

11. $\cot\left(\theta - \dfrac{\pi}{4}\right) + \tan\left(\theta + \dfrac{\pi}{4}\right) = 0$.

12. If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, prove that $\tan(\alpha+\beta) = 1$.

13. $\dfrac{\tan(n+1)\phi - \tan n\phi}{1 + \tan(n+1)\phi \,.\, \tan n\phi} = \tan\phi$.

14. $\dfrac{\tan(n+1)\phi + \tan(1-n)\phi}{1 - \tan(n+1)\phi \,.\, \tan(1-n)\phi} = \tan 2\phi$.

15. If $\tan\alpha = m$ and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

16. If $\tan\alpha = (a+1)$ and $\tan\beta = (a-1)$, then $2\cot(\alpha-\beta) = a^2$.

17. If $\alpha+\beta+\gamma = 90^0$, then $\tan\gamma = \dfrac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$.

102. From pages 63 and 64 we have

$$\left.\begin{aligned}\sin(A+B) &= \sin A \,.\, \cos B + \cos A \,.\, \sin B\\ \sin(A-B) &= \sin A \,.\, \cos B - \cos A \,.\, \sin B\\ \cos(A+B) &= \cos A \,.\, \cos B - \sin A \,.\, \sin B\\ \cos(A-B) &= \cos A \,.\, \cos B + \sin A \,.\, \sin B\end{aligned}\right\} \ldots\ldots \text{(i)}.$$

From these by addition and subtraction we get

$$\left.\begin{aligned}\sin(A+B) + \sin(A-B) &= 2\sin A \,.\, \cos B\\ \sin(A+B) - \sin(A-B) &= 2\cos A \,.\, \sin B\\ \cos(A+B) + \cos(A-B) &= 2\cos A \,.\, \cos B\\ \cos(A-B) - \cos(A+B) &= 2\sin A \,.\, \sin B\end{aligned}\right\}$$

Now put S for $(A+B)$,

and put T for $(A-B)$:

Then $S+T=2A$, and $S-T=2B$,

so that $A=\dfrac{S+T}{2}$, and $B=\dfrac{S-T}{2}$.

Hence the above results may be written

$$\left.\begin{aligned}\sin S + \sin T &= 2\sin\frac{S+T}{2} \,.\, \cos\frac{S-T}{2}\\ \sin S - \sin T &= 2\cos\frac{S+T}{2} \,.\, \sin\frac{S-T}{2}\\ \cos S + \cos T &= 2\cos\frac{S+T}{2} \,.\, \cos\frac{S-T}{2}\\ {}^*\cos T - \cos S &= 2\sin\frac{S+T}{2} \,.\, \sin\frac{S-T}{2}\end{aligned}\right\} \ldots \text{(iii)}.$$

103. The formulæ (iii) are most important, and the student is recommended to get thoroughly familiar with them *in words*, as on the next page;

* If A and B are each less than 90^0, then S, which is their ***sum***, is greater than T, their *difference*. Therefore if S be less than 90^0, $\cos S$ is less than $\cos T$; so that $\cos T - \cos S$ is *positive*.

(1) *The sum of the sines of two angles equals twice the sine of half their sum into the cosine of half their difference.*

(2) *The difference of the sines of two angles equals twice the cosine of half their sum into the sine of half their difference.*

(3) *The sum of the cosines of two angles equals twice the cosine of half their sum into the cosine of half their difference.*

(4) *The difference of the* †*cosines of two angles equals twice the sine of half their sum into the sine of half their difference.*

† Note. The difference of the cosines of two angles is the cosine of the smaller angle – the cosine of the greater angle.

104. It will be convenient to refer to the formulæ (i) as the 'A, B' formulæ, and to the formulæ (iii) as the 'S, T' formulæ.

EXAMPLES. XXVIII.

Prove the following statements:

1. $\sin 60^0 + \sin 30^0 = 2 \sin 45^0 . \cos 15^0$.
2. $\sin 60^0 + \sin 20^0 = 2 \sin 40^0 . \cos 20^0$.
3. $\sin 40^0 - \sin 10^0 = 2 \cos 25^0 . \sin 15^0$.
4. $\cos \frac{\pi}{3} + \cos \frac{\pi}{2} = 2 \cos \frac{5\pi}{12} . \cos \frac{\pi}{12}$.
5. $\cos \frac{\pi}{3} - \cos \frac{\pi}{2} = 2 \sin \frac{5\pi}{12} . \sin \frac{\pi}{12}$.
6. $\sin 3A + \sin 5A = 2 \sin 4A . \cos A$.
7. $\sin 7A - \sin 5A = 2 \cos 6A . \sin A$.
8. $\cos 5A + \cos 9A = 2 \cos 7A . \cos 2A$.
9. $\cos 5A - \cos 4A = -2 \sin \frac{9A}{2} . \sin \frac{A}{2}$.
10. $\cos A - \cos 2A = 2 \sin \frac{3A}{2} . \sin \frac{A}{2}$.
11. $\frac{\sin 2\theta + \sin \theta}{\cos \theta + \cos 2\theta} = \tan \frac{3\theta}{2}$.
12. $\frac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} = \cot \frac{3\theta}{2}$.
13. $\frac{\sin 3\theta + \sin 2\theta}{\cos 2\theta - \cos 3\theta} = \cot \frac{\theta}{2}$.

14. $\dfrac{\sin\theta+\sin\phi}{\cos\theta-\cos\phi}=\dfrac{\cos\theta+\cos\phi}{\sin\phi-\sin\theta}$.

15. $\cos(60^\circ+A)+\cos(60^\circ-A)=\cos A$.

16. $\cos(45^\circ+A)+\cos(45^\circ-A)=\sqrt{2}\,.\,\cos A$.

17. $\sin(45^\circ+A)-\sin(45^\circ-A)=\sqrt{2}\,.\,\sin A$.

18. $\cos(30^\circ-A)-\cos(30^\circ+A)=\sin A$.

19. $\dfrac{\sin\theta-\sin\phi}{\cos\phi-\cos\theta}=\cot\dfrac{\theta+\phi}{2}$.

20. $\dfrac{\sin\theta-\sin\phi}{\sin\theta+\sin\phi}=\cot\left(\dfrac{\theta+\phi}{2}\right).\tan\left(\dfrac{\theta-\phi}{2}\right)$.

105. It is important that the student should be thoroughly familiar with the second set of formulæ on p. 69.

Written as follows, they may be regarded as the inverse of the 'S, T' formulæ.

$$\left.\begin{aligned}2\sin A\,.\,\cos B&=\sin(A+B)+\sin(A-B)\\2\cos A\,.\,\sin B&=\sin(A+B)-\sin(A-B)\\2\cos A\,.\,\cos B&=\cos(A+B)+\cos(A-B)\\2\sin A\,.\,\sin B&=\cos(A-B)-\cos(A+B)\end{aligned}\right\}\ldots\text{(iv)}.$$

EXAMPLES. XXIX.

Express as the sum or as the difference of two trigonometrical ratios the ten following expressions:

1. $2\sin\theta\,.\,\cos\phi$.

2. $2\cos\alpha\,.\,\cos\beta$

3. $2\sin 2\alpha\,.\,\cos 3\beta$.

4. $2\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)$.

5. $2\sin 3\theta\,.\,\cos 5\theta$.

6. $2\cos\dfrac{3\theta}{2}\,.\,\cos\dfrac{\theta}{2}$.

7. $\sin 4\theta\,.\,\sin\theta$.

8. $\cos\dfrac{5\theta}{2}\,.\,\sin\dfrac{3\theta}{2}$.

9. $2\cos 10^\circ\,.\,\sin 50^\circ$.

10. $\cos 45^\circ\,.\,\sin 15^\circ$.

11. Simplify $2\cos 2\theta\,.\,\cos\theta-2\sin 4\theta\,.\,\sin\theta$.

12. Simplify $\sin\dfrac{5\theta}{2}\,.\,\cos\dfrac{\theta}{2}-\sin\dfrac{9\theta}{2}\,.\,\cos\dfrac{3\theta}{2}$.

13. Simplify $\sin 3\theta+\sin 2\theta+2\sin\dfrac{3\theta}{2}\,.\,\cos\dfrac{\theta}{2}$.

14. Prove that $\sin\dfrac{11\theta}{4}\,.\,\sin\dfrac{\theta}{4}+\sin\dfrac{7\theta}{4}\,.\,\sin\dfrac{3\theta}{4}=\sin 2\theta\,.\,\sin\theta$.

CHAPTER X.

ON THE TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES.

106. To express the Trigonometrical Ratios of the angle $2A$ in terms of those of the angle A.

Since $\sin(A+B) = \sin A . \cos B + \cos A . \sin B$;

$$\therefore \sin(A+A) = \sin A . \cos A + \cos A . \sin A;$$

$$\therefore \sin 2A = 2 \sin A . \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also, since $\cos(A+B) = \cos A . \cos B - \sin A . \sin B$;

$$\therefore \cos(A+A) = \cos A . \cos A - \sin A . \sin A;$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A \quad \ldots\ldots\ldots\ldots \quad (2).$$

But $\qquad 1 = \cos^2 A + \sin^2 A$;

$$\therefore 1 + \cos 2A = 2\cos^2 A,$$

$$\text{and } 1 - \cos 2A = 2\sin^2 A.$$

The last two results are usually written

$$\cos 2A = 2\cos^2 A - 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\text{and } \cos 2A = 1 - 2\sin^2 A \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

Again, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$;

$$\therefore \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A . \tan A};$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

107. These five formulæ are very important,

$$\left.\begin{aligned}
\sin 2A &= 2\sin A \,.\, \cos A \;\ldots\ldots\ldots\ldots\; (1),\\
\left.\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ \cos 2A &= 2\cos^2 A - 1\\ \cos 2A &= 1 - 2\sin^2 A\end{aligned}\right\} &\;\ldots\ldots\ldots\ldots\; (2),\ (3),\ (4),\\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A} \;\ldots\ldots\ldots\ldots\; (5)
\end{aligned}\right\}\ (\text{v}).$$

108. The following result is important,

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A \,.\, \cos A}{2\cos^2 A} = \tan A.$$

109. The student must notice that A is *any* angle, and therefore these formulæ will be true whatever we put for A.

Example. Write $\frac{A}{2}$ instead of A, and we get

$$\sin A = 2\sin\frac{A}{2} \,.\, \cos\frac{A}{2} \;\ldots\ldots\ldots\ldots\; (1),$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \;\ldots\ldots\ldots\ldots\; (2),$$

and so on.

EXAMPLES. XXX.

Prove the following statements:

1. $2\,\text{cosec}\, 2A = \sec A \,.\, \text{cosec}\, A.$

2. $\dfrac{\text{cosec}^2 A}{\text{cosec}^2 A - 2} = \sec 2A.$

3. $\dfrac{2-\sec^2 A}{\sec^2 A} = \cos 2A.$

4. $\cos^2 A\,(1-\tan^2 A) = \cos 2A.$

5. $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}.$

6. $\dfrac{2\tan B}{1+\tan^2 B} = \sin 2B.$

7. $\tan B + \cot B = 2\,\text{cosec}\, 2B.$

8. $\dfrac{1-\tan^2 B}{1+\tan^2 B} = \cos 2B.$

9. $\cot B - \tan B = 2\cot 2B.$

10. $\dfrac{\cot^2 B + 1}{\cot^2 B - 1} = \sec 2B.$

11. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

12. $\left(\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right)^2 = 1 - \sin\theta.$

13. $\cos^2\dfrac{\theta}{2}\left(1+\tan\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta.$

14. $\sin^2\dfrac{\theta}{2}\left(\cot\dfrac{\theta}{2} - 1\right)^2 = 1 - \sin\theta.$

15. $\left(\dfrac{\tan\frac{\theta}{2}+1}{\tan\frac{\theta}{2}-1}\right)^2=\dfrac{1+\sin\theta}{1-\sin\theta}$.

16. $\dfrac{\sin\beta}{1+\cos\beta}=\tan\dfrac{\beta}{2}$.

17. $\dfrac{\sin\beta}{1-\cos\beta}=\cot\dfrac{\beta}{2}$.

18. $\dfrac{1-\cos\beta}{1+\cos\beta}=\tan^2\dfrac{\beta}{2}$.

19. $\dfrac{1+\sec\beta}{\sec\beta}=2\cos^2\dfrac{\beta}{2}$.

20. $\operatorname{cosec}\beta-\cot\beta=\tan\dfrac{\beta}{2}$.

21. $\dfrac{\cos 2x}{1+\sin 2x}=\dfrac{1-\tan x}{1+\tan x}$.

22. $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$.

23. $\dfrac{\cos x}{1+\sin x}=\dfrac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1}$.

24. $\dfrac{\cos x}{1-\sin x}=\dfrac{\cot\frac{x}{2}+1}{\cot\frac{x}{2}-1}$.

25. $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x}=\cot\dfrac{x}{2}$.

26. $\dfrac{\cos^3\alpha+\sin^3\alpha}{\cos\alpha+\sin\alpha}=\dfrac{2-\sin 2\alpha}{2}$.

27. $\dfrac{\cos^3\alpha-\sin^3\alpha}{\cos\alpha-\sin\alpha}=\dfrac{2+\sin 2\alpha}{2}$.

28. $\cos^4\alpha-\sin^4\alpha=\cos 2\alpha$.

29. $\cos^6\alpha+\sin^6\alpha=\dfrac{1+3\cos^2 2\alpha}{4}$.

30. $\cos^6\alpha-\sin^6\alpha=\dfrac{(3+\cos^2 2\alpha)\cos 2\alpha}{4}$.

31. $\dfrac{\sin 3\beta}{\sin\beta}-\dfrac{\cos 3\beta}{\cos\beta}=2$.

32. $\dfrac{\cos 3\beta}{\sin\beta}+\dfrac{\sin 3\beta}{\cos\beta}=2\cot 2\beta$.

33. $\dfrac{\sin 4\beta}{\sin 2\beta}=2\cos 2\beta$.

34. $\dfrac{\sin 5\beta}{\sin\beta}-\dfrac{\cos 5\beta}{\cos\beta}=4\cos 2\beta$.

35. $\dfrac{\sin\frac{5\pi}{12}}{\sin\frac{\pi}{12}}-\dfrac{\cos\frac{5\pi}{12}}{\cos\frac{\pi}{12}}=2\sqrt{3}$.

36. $\tan(45^0+A)-\tan(45^0-A)=2\tan 2A$.

37. $\tan(45^0-A)+\cot(45^0-A)=2\sec 2A$.

38. $\dfrac{\tan^2(45^0+A)-1}{\tan^2(45^0+A)+1}=\sin 2A$.

39. $\dfrac{\sec A+\tan A}{\sec A-\tan A}=\tan\left(45^0+\dfrac{A}{2}\right).\cot\left(45^0-\dfrac{A}{2}\right)$.

40. $\dfrac{\cos(A+45^0)}{\cos(A-45^0)}=\sec 2A-\tan 2A$.

41. $\tan B=\dfrac{\sin B+\sin 2B}{1+\cos B+\cos 2B}$.

42. $\tan B=\dfrac{\sin 2B-\sin B}{1-\cos B+\cos 2B}$.

110. The following two formulæ should be remembered:

$$\left.\begin{aligned}\sin 3A &= 3\sin A - 4\sin^3 A\\ \cos 3A &= 4\cos^3 A - 3\cos A\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

NOTE. The similarity of these two results is apt to cause confusion. This may be avoided by observing that the second formula must be true when $A=0^0$; and then $\cos 3A=\cos 0^0=1$. In which case the formula gives $\cos 0^0=4\cos 0^0-3\cos 0^0$, or $1=4-3$, which is true.

The first formula may be proved thus:

$$\begin{aligned}\sin 3A=\sin(2A+A)&=\sin 2A\,.\,\cos A+\cos 2A\,.\,\sin A\\ &=(2\sin A\,.\,\cos A)\cos A+(1-2\sin^2 A)\sin A\\ &=2\sin A\,.\,\cos^2 A+\sin A-2\sin^3 A\\ &=2\sin A(1-\sin^2 A)+\sin A-2\sin^3 A\\ &=2\sin A-2\sin^3 A+\sin A-2\sin^3 A\\ &=3\sin A-4\sin^3 A.\end{aligned}$$

The second formula may be proved in a similar manner.

Example. Prove that $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$.

$$\tan 3A=\tan(2A+A)=\frac{\tan 2A+\tan A}{1-\tan 2A\,.\,\tan A}$$

$$=\frac{\dfrac{2\tan A}{1-\tan^2 A}+\tan A}{1-\dfrac{2\tan A}{1-\tan^2 A}\,.\,\tan A}=\frac{2\tan A+\tan A-\tan^3 A}{1-\tan^2 A-2\tan^2 A}$$

$$=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$$

EXAMPLES. XXXI.

Prove the following statements:

1. $\dfrac{\sin 3A}{\sin A}=2\cos 2A+1.$

2. $\dfrac{\cos 3A}{\cos A}=2\cos 2A-1.$

3. $\dfrac{3\sin A-\sin 3A}{\cos 3A+3\cos A}=\tan^3 A.$

4. $\cot 3A=\dfrac{\cot^3 A-3\cot A}{3\cot^2 A-1}.$

5. $\dfrac{\sin 3A-\sin A}{\cos 3A+\cos A}=\tan A.$

6. $\dfrac{\sin 3A-\cos 3A}{\sin A+\cos A}=2\sin 2A-1.$

7. $\dfrac{\sin 3A+\cos 3A}{\cos A-\sin A}=2\sin 2A+1.$

8. $\dfrac{1}{\tan 3A-\tan A}+\dfrac{1}{\cot A-\cot 3A}=\cot 2A.$

9. $\left(\dfrac{3\sin A-\sin 3A}{3\cos A+\cos 3A}\right)^2=\left(\dfrac{\sec 2A-1}{\sec 2A+1}\right)^3.$

10. $\dfrac{1-\cos 3A}{1-\cos A}=(1+2\cos A)^2.$

CHAPTER XI.

On Logarithms.

111. In Algebra it is explained

(i) that the *multiplication* of different powers of the same quantity is effected by *adding* the *indices* of those powers;

(ii) that *division* is effected by *subtracting* the *indices;*

(iii) that *involution* and *evolution* are respectively effected by the *multiplication* and *division* of the *indices.*

Example 1. Let $m=a^h$, $n=a^k$,

then
$$m\times n=a^h\times a^k=a^{h+k} \quad\text{.........(i)},$$
$$m\div n=a^h\div a^k=a^{h-k} \quad\text{.........(ii)},$$
$$\left.\begin{aligned} m^3&=(a^h)^3=a^{3h},\\ \sqrt[4]{m}&=m^{\frac{1}{4}}=(a^h)^{\frac{1}{4}}=a^{\frac{h}{4}} \end{aligned}\right\} \quad\text{.........(iii)}.$$

Example 2. *Given that* $347=10^{2\cdot5403295}$ * and $461=10^{2\cdot6637009}$, *prove that* $347\times461=10^{5\cdot2040304}$.

We have
$$347\times461=10^{2\cdot5403295}\times10^{2\cdot6637009}$$
$$=10^{2\cdot5403295+2\cdot6637009}$$
$$=10^{5\cdot2040304}.$$
Q.E.D.

EXAMPLES. XXXII.

1. If $m=a^h$, $n=a^k$, express in terms of a, h and k,

(i) $m^2\times n^3$. (ii) $m^4\div n^5$. (iii) $\sqrt[3]{m^4\times n^5}$. (iv) $\{\sqrt[4]{m^5\times n^3}\}^2$.

2. If $453=10^{2\cdot6560982}$ and $650=10^{2\cdot8129134}$, find the indices of the powers of 10 which are equal to

(i) 453×650. (ii) $(453)^4$. (iii) $650^3\times453^2$. (iv) $\sqrt[3]{453}$.
(v) $\sqrt{453}\times\sqrt[6]{650}$. (vi) $\sqrt[5]{453}\times(650)^3$. (vii) $\sqrt{453\times650}$.

3. Express in powers of 2 the numbers, 8, 32, $\frac{1}{2}$, $\frac{1}{16}$, ·125, 128.

4. Express in powers of 3 the numbers, 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, $\cdot\dot{1}$, $\frac{1}{81}$.

* The nnmber 347 lies between 100 and 1000, i. e. between 10^2 and 10^3. Hence, if there is a power of 10 which is equal to 347, its index must be greater than 2 and less than 3, i.e. equal to 2 + a fraction.

112. Suppose that some convenient number (such as 10) having been chosen, we are given a **list** of the **indices** of the powers of that number, which are equivalent to every whole number from 1 up to 100000. Such a list could be used to shorten Arithmetical calculations.

Example 1. Multiply 3759 by 4781 and divide the result by 2690.

Looking in our *list* we should find $3759 = 10^{3 \cdot 5750723}$, $4781 = 10^{3 \cdot 6795187}$, $2690 = 10^{3 \cdot 4297523}$.

Therefore $3759 \times 4781 \div 2690 = 10^{3 \cdot 5750723} \times 10^{3 \cdot 6795187} \div 10^{3 \cdot 4297523}$ $= 10^{3 \cdot 5750723 + 3 \cdot 6795187 - 3 \cdot 4297523} = 10^{3 \cdot 8248387}$.

The list will give us that $10^{3 \cdot 8248387} = 6680 \cdot 9$.

Therefore the answer correct to five significant figures is 6680·9.

Example 2. Simplify $3^6 \times 2^{10} \div \sqrt[3]{17601}$.

The *list* gives $2 = 10^{\cdot 3010300}$, $3 = 10^{\cdot 4771213}$ and $17601 = 10^{4 \cdot 2455373}$.

Thus $3^6 \times 2^{10} \div \sqrt[3]{17601} = (10^{\cdot 4771213})^6 \times (10^{\cdot 3010300})^{10} \div (10^{4 \cdot 2455373})^{\frac{1}{3}}$ $= 10^{2 \cdot 8627278} \times 10^{3 \cdot 0103000} \div 10^{1 \cdot 4151791} = 10^{2 \cdot 8627278 + 3 \cdot 0103000 - 1 \cdot 4151791} = 10^{4 \cdot 4578487}$.

And from our list we find $10^{4 \cdot 4578487} = 28697$, nearly.

EXAMPLES. XXXIII.

Given that $2 = 10^{\cdot 3010300}$, $3 = 10^{\cdot 4771213}$ and $7 = 10^{\cdot 8450980}$, find the indices of the powers of 10 equivalent to the following numbers.

1. $2^2, 3^2, 2^3, 2 \times 3, 2^4, 7^2$.

2. 14, 16, 18, 24, 27, 42.

3. 10, 5, 15, 25, 30, 35.

4. 36, 40, 48, 50, 200, 1000.

5. $3^{10} \times 7^{10} \div 2^{20}$, $2^{12} \times 3^{20} \div 7^{11}$.

6. $\sqrt[3]{21} \times \sqrt[4]{18}$, $\sqrt[2]{\cdot 49 \times 4^5} \times \sqrt[3]{3^4 \times 2^{10}}$.

7. Find approximately the numerical value of $\sqrt[10]{42}$, having given that $10^{\cdot 1623249} = 1 \cdot 4532$ nearly.

8. Find approximately the numerical value of $\sqrt[3]{(42)^4} \times \sqrt[4]{(42)^3}$, having given that $10^{3 \cdot 38177} = 2408 \cdot 6$.

9. Find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$, (ii) of $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$, having given that $10^{\cdot 6615067} = 4 \cdot 5868$ and $10^{-\cdot 0285094} = \cdot 93646$.

10. Find the value of $(67 \cdot 21)^{\frac{3}{5}} \times (49 \cdot 62)^{\frac{1}{5}} \times (3 \cdot 971)^{-\frac{7}{5}}$, having given that $67 \cdot 21 = 10^{1 \cdot 8274339}$, $49 \cdot 62 = 10^{1 \cdot 6956568}$, $3 \cdot 971 = 10^{\cdot 5988999}$ and $10^{\cdot 5971310}$ $= 3 \cdot 9549$.

11. Find the area of a square field whose side is 640·12 feet, having given that $640 \cdot 12 = 10^{2 \cdot 8062614}$ and that $10^{5 \cdot 6125228} = 40975 \cdot 3$.

12. Find the edge of a solid cube which contains 42601 cubic inches, having given $42601 = 10^{4 \cdot 6294198}$ and $10^{1 \cdot 5431399} = 34 \cdot 925$.

13. Find the edge of a solid cube which contains 34·701 cubic inches, having given that $34 \cdot 701 = 10^{1 \cdot 5403420}$, and $10^{\cdot 5134473} = 3 \cdot 2617$.

14. Find the volume of the cube the length of one of whose edges is 47·931 yds.; given $47 \cdot 931 = 10^{1 \cdot 6806165}$, $10^{5 \cdot 0418495} = 110115$.

113. The powers of any other number than 10 might be used in the manner explained above, but 10 is the most *convenient* number, as will presently appear.

114. This method, in which the *indices* of the powers of a certain fixed number (such as 10) are made use of, is called the *Method of Logarithms*.

Indices thus used are called **logarithms.**

The *fixed number* whose powers are used is called the **base.** Hence we have the following definition:

DEF. The **logarithm** of a number to a given *base* is the **index** of that power of the base, which is equal to the given number.

If l be the logarithm of the number n to the base a, then $a^l = n$.

115. The **notation** used is $\log_a n = l$.

Here, $\log_a n$ is an abbreviation for the words 'the logarithm of the number n to the base a.' And this means, as we have explained above, 'the **index** of that power of a which is equal to the number n.'

Example 1. What is the logarithm of $a^{\frac{3}{2}}$ to the base a?

That is, what is the **index** of the power of a which is $a^{\frac{3}{2}}$?
The index is $\frac{3}{2}$; therefore $\frac{3}{2}$ is the required logarithm, or

$$\log_a a^{\frac{3}{2}} = \frac{3}{2}.$$

Example 2. What is the logarithm of 32 to the base 2?

That is, what is the **index** of the power of 2 which is equal to 32?
Now $32 = 2^5$, $\therefore$ the required index is 5; or $\log_2 32 = 5$.

The use of Logarithms is based upon the following propositions:—

I. The logarithm of the product of two numbers is equal to the logarithm of one of the numbers + the logarithm of the other.

For, let $\log_a m = x$ and $\log_a n = y$, then, $m = a^x$, $n = a^y$,
$$\log_a (m \times n) = \log_a (a^x \times a^y) = \log_a (a^{x+y}) = x + y = \log_a m + \log_a n.$$

II. The logarithm of the quotient of two numbers is the logarithm of the dividend − the logarithm of the divisor.

For, $\log_a \left(\frac{m}{n}\right) = \log_a \left(\frac{a^x}{a^y}\right) = \log_a (a^{x-y}) = x - y$ [as above]

$$= \log_a m - \log_a n.$$

III. The logarithm of a number raised to a power k is k times the logarithm of the number.

For, $\log_a (m^k) = \log_a \{(a^x)^k\} = \log_a (a^{kx}) = kx = k$ times $\log_a m$.

Examples. Given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 3 = \cdot 4771213$, $\log_{10} 7 = \cdot 8450980$, *find the values of the following:*

(i) $\log_{10} 6 = \log_{10} (2 \times 3) = \log_{10} 2 + \log_{10} 3$
$= \cdot 3010300 + \cdot 4771213 = \cdot 7781513$. [by I.]

(ii) $\log_{10} \frac{7}{3} = \log_{10} 7 - \log_{10} 3 = \cdot 8450980 - \cdot 4771213$
$= \cdot 3679767$. [by II.]

(iii) $\log_{10} 3^5 = 5$ times $\log_{10} 3 = 5 \times \cdot 3010300 = 1 \cdot 5051500$. [by III.]

(iv) $\log_{10} \sqrt[3]{\frac{3 \times 4}{7}} = \log_{10} \left(\frac{3 \times 5}{7}\right)^{\frac{1}{3}} = \frac{1}{3}$ of $\log_{10} \frac{3 \times 5}{7}$ [by III.]

$= \frac{1}{3}$ of $(\log 3 + \log 4 - \log 7) = \frac{1}{3}$ of $\{\cdot 4771213 + \text{twice } \cdot 30103 - \cdot 8450980\}$
$= \frac{1}{3}$ of $\cdot 2340833 = \cdot 0780278$. [by I. and II.]

(v) $\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - \cdot 3010300 = \cdot 6989700$.

EXAMPLES. XXXIV.

1. Find the logarithms to the base a of a^3, $a^{\frac{10}{3}}$, $\sqrt[4]{a}$, $\sqrt[3]{a^2}$, $\frac{1}{a^{\frac{5}{2}}}$.

2. Find the logarithms to the base 2 of 8, 64, $\frac{1}{2}$, $\cdot 125$, $\cdot 015625$, $\sqrt[3]{64}$.

3. Find the logarithms to the base 3 of 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, $\cdot\dot{1}$, $\frac{1}{81}$.

4. Find the logarithms to base 4 of 8, $\sqrt[3]{16}$, $\sqrt{\cdot 5}$, $\sqrt[3]{\cdot 015625}$.

5. Find the value of
$\log_2 8$, $\log_2 \cdot 5$, $\log_3 243$, $\log_5 (\cdot 04)$, $\log_{10} 1000$, $\log_{10} \cdot 001$.

6. Find the value of $\log_a a^{\frac{4}{3}}$, $\log_b \sqrt[3]{b^2}$, $\log_8 2$, $\log_{27} 3$, $\log_{100} 10$.

If $\log_{10} 2 = \cdot 30103$, $\log_{10} 3 = \cdot 4771213$, $\log_{10} 7 = \cdot 845098$, find the values of

7. $\log_{10} 6$, $\log_{10} 42$, $\log_{10} 16$. **8.** $\log_{10} 49$, $\log_{10} 36$, $\log_{10} 63$.

9. $\log_{10} 200$, $\log_{10} 600$, $\log_{10} 70$. **10.** $\log_{10} 5$, $\log_{10} 3 \cdot \dot{3}$, $\log_{10} 50$.

11. $\log_{10} 35$, $\log_{10} 150$, $\log_{10} \cdot 2$. **12.** $\log_{10} 3 \cdot 5$, $\log_{10} 7 \cdot 29$, $\log_{10} \cdot 081$.

13. Given $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 7$, find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$.

(ii) of $\sqrt[10]{2} \times 3^{-\frac{4}{5}} \times 7^{\frac{7}{11}}$

[$\cdot 6615067 = \log_{10} 4 \cdot 5868$; $-\cdot 0285094 = \log_{10} \cdot 93646$].

14. Prove that (i) $\log \{\sqrt[3]{2} \times \sqrt[4]{7} \div \sqrt[5]{9}\} = \frac{1}{3} \log 2 + \frac{1}{4} \log 7 - \frac{2}{5} \log 3$,

(ii) $\log \{\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}\} = \frac{1}{10} \log 2 - \frac{5}{4} \log 3 + \frac{7}{11} \log 7$.

Common Logarithms.

116. That System of Logarithms whose base is **10**, is called the **Common** System of Logarithms.

In speaking of logarithms hereafter, *common* logarithms are referred to unless the contrary is expressly stated.

We shall assume that a power of 10 can be found which is practically equivalent to any number.

117. The indices of these powers of 10, *i.e.* the Common Logarithms, are in general *incommensurable* numbers.

Their value for every whole number, from 1 to 100000, has been calculated to 7 significant figures. Thus any calculation made with the aid of logarithms is as exact as the most carefully observed measurement.

118. Now, the greater the index of any power of 10, the greater will be the numerical value of that power; and the less the index, the less will be the numerical value of the power.

Hence, if one number be less than another, the logarithm of the first will be less than the logarithm of the second.

But the student should notice that logarithms (or indices) are *not proportional* to the corresponding numbers.

Example. 1000 is less than 10000; and the logarithm to base 10 of the first is 3 and of the second is 4.

But 1000, 10000, 3, 4 are not in proportion.

119. We know from Algebra that $1 = 10^0$,

$$\begin{array}{rcl} 10 = 10^1 & \text{and that} & \cdot 1 = \frac{1}{10} = 10^{-1} \\ 100 = 10^2 & \ldots\ldots\ldots & \cdot 01 = \frac{1}{100} = 10^{-2} \\ 1000 = 10^3 & \ldots\ldots\ldots & \cdot 001 = \frac{1}{1000} = 10^{-3} \\ 10000 = 10^4 & \ldots\ldots\ldots & \cdot 0001 = \frac{1}{10000} = 10^{-4} \end{array}$$

and so on.

Hence, the logarithm of 1 is 0.

The (common) logarithm of any number greater than 1 is *positive*.

The logarithm of any positive number less than 1 is *negative*.

120. We observe also

that the logarithm of any number between 1 and 10 is a positive decimal fraction;

that the logarithm of any number between 10 and 100, *i. e.* between 10^1 and 10^2, is of the form 1 + a decimal fraction;

that the logarithm of any number between 1000 and 10000, *i. e.* between 10^3 and 10^4, is of the form 3 + a decimal fraction;

and so on.

121. We observe also

that the logarithm of any number between 1 and ·1, *i.e.* between 10^0 and 10^{-1}, can be written in the form -1 + a decimal fraction;

that the logarithm of any number between ·1 and ·01, *i.e.* between 10^{-1} and 10^{-2}, can be written in the form -2 + a decimal fraction;

and so on.

Example 1. How many digits are contained in the integral part of the number whose logarithm is 3·67192?

The number is $10^{3\cdot67192}$ and this is greater than 10^3, *i.e.* greater than 1000, and it is less than 10^4, *i. e.* less than 10000. Therefore the number lies between 1000 and 10000, and therefore the integral part of it contains 4 figures.

Example 2. Given that $3=10^{\cdot4771213}$, find the number of the digits in the integral part of 3^{20}.

We have $$3=10^{\cdot4771213},$$

$$\therefore\ 3^{20}=(10^{\cdot4771213})^{20}=10^{9\cdot542426 0}$$

Therefore there are 10 digits in the integral part of 3^{20}; for it is greater than 10^9 and less than 10^{10}.

Example 3. Supposing that the decimal part of the logarithm is to be kept positive, find the integral part of the logarithm of ·0001234.

This number is greater than ·0001 *i.e.* than 10^{-4} and less than ·001, *i. e.* than 10^{-3}.

Therefore its logarithm lies between -3 and -4, and therefore it is -4 + a fraction; the integral part is therefore -4.

EXAMPLES. XXXV.

NOTE. The *decimal* part of a logarithm is to be kept *positive.*

1. Write down the integral part of the common logarithms of 17601, 361·1, 4·01, 723000, 29.

2. Write down the integral part of the common logarithms of ·04, ·0000612, ·7963, ·001201. (See Note above.)

3. Write down the integral part of the common logarithms of 7963, ·1, 2·61, 79·6341, 1·0006, ·00000079.

4. How many digits are there in the integral part of the numbers whose common logarithms are respectively

3·461, ·3020300, 5·4712301, 2·6710100?

5. Give the position of the first *significant* figure in the numbers whose logarithms are $-2+{\cdot}4612310$, $-1+{\cdot}2793400$, $-6+{\cdot}1763241$.

6. Give the position of the first *significant* figure in the numbers whose common logarithms are 4·2990713, ·3040595, 2·5860244, $-3+{\cdot}1760913$, $-1+{\cdot}3180633$, ·4980347.

7. Given that $2=10^{{\cdot}3010300}$, find the number of digits in the integral part of 8^{10}, 2^{12}, 16^{20}, 2^{100}.

8. Given that $\log 7={\cdot}8450980$, find the number of digits in the integral part of 7^{10}, 49^{6}, $343^{\frac{100}{3}}$, $(\frac{10}{7})^{20}$, $(4{\cdot}9)^{12}$, $(3{\cdot}43)^{10}$.

9. Find the position of the first significant figure in

$$\sqrt[10]{2},\ (\tfrac{1}{2})^{10},\ (\tfrac{10}{7})^{20},\ ({\cdot}02)^{4},\ ({\cdot}49)^{6}.$$

10. Find the position of the first significant figure in the numerical value of 20^{7}, $({\cdot}02)^{7}$, $({\cdot}007)^{2}$, $(3{\cdot}43)^{\frac{1}{10}}$, $({\cdot}0343)^{8}$, $({\cdot}0343)^{\frac{1}{10}}$.

122. PROP. *To prove that when two numbers expressed in the decimal notation have the same digits (so that they differ only in the position of the decimal point), their logarithms to the base* 10 *differ only by an integer.*

The decimal point in a number is moved by multiplying or dividing the number by some *integral* power of 10.

Let the numbers be m and n; then $m=n\times 10^{k}$ when k is a whole number (positive or negative); then

$$\log m=\log (n\times 10^{k})=\log n+\log 10^{k}$$
$$=\log n+k.$$

That is $\log m$ and $\log n$ differ by an integer. Q. E. D.

Example i. $\log 1779{\cdot}2=\log \{(1{\cdot}6792)\times 10^{3}\}=\log 1{\cdot}6792+\log 10^{3}$ $=\log 1{\cdot}6792+3$.

Example ii. Given that $\log 1{\cdot}7692 = {\cdot}247776$, find (i) $\log 16792$, (ii) $\log {\cdot}0016792$, (iii) $\log 167{\cdot}92$.

Here

$$\log 16792 = \log(1{\cdot}6792 \times 10^4) = 4{\cdot}247776,$$
$$\log {\cdot}0016792 = \log(1{\cdot}6792 \times 10^{-3}) = -3 + {\cdot}247776,$$
$$\log 167{\cdot}92 = \log(1{\cdot}6792 \times 10^2) = 2{\cdot}247776.$$

123. It is *convenient* to keep the *decimal part* of common logarithms always *positive*, because then the *decimal part* of the logarithms of any numbers expressed by the same digits will be *always* the same.

124. The decimal part of a logarithm is called the **mantissa.**

125. The integral part is called the **characteristic.**

126. The characteristic of a logarithm can be always obtained by the following rule, which is evident from page 81.

RULE. The characteristic of the logarithm of a number greater than unity is **one less** than the number of *integral* figures in that number.

The characteristic of a number less than unity is negative, and (when the number is expressed as a decimal,) is **one more** than the number of cyphers between the decimal point and the first significant figure to the right of the decimal point.

127. When the characteristic is negative, as for example in the logarithm $-3 + {\cdot}1760913$, the logarithm is abbreviated thus, $\bar{3}{\cdot}1760913$.

Example 1. The characteristics of 36741, 36·741, ·0036741, 3·6741 and ·36741 are respectively 4, 1, -3, 0, and -1.

Example 2. Given that the mantissa of the logarithm of 36741 is 5651510, we can at once write down the logarithm of any number whose digits are 36741.

Thus

$$\log 3674100 = 6{\cdot}5651510,$$
$$\log 36741 = 4{\cdot}5651510,$$
$$\log 367{\cdot}41 = 2{\cdot}5651510,$$
$$\log {\cdot}36741 = \bar{1}{\cdot}5651510,$$
$$\log {\cdot}00036741 = \bar{4}{\cdot}5651510,$$

and so on.

128. In any set of tables of common logarithms the student will find the *mantissa only* corresponding to any set of digits.

It would obviously be superfluous to give the *characteristic.*

129. It is most important to remember to keep the **mantissa** always **positive.**

Example. Find the fifth root of ·00065061.

Here $\log_{10}\cdot 00065061 = \bar{4}\cdot 8133207$,

$\therefore \log_{10}(\cdot 00065061)^{\frac{1}{5}} = \frac{1}{5}(\bar{4}\cdot 8133207) = \frac{1}{5}(-4 + \cdot 8133207)$

$= \frac{1}{5}(-5 + 1\cdot 8133207) = -1 + \cdot 3626641 = \bar{1}\cdot 3626641$,

and $\bar{1}\cdot 3626641 = \log\cdot 23050$,

$\therefore$ the fifth root of ·00065062 = ·23050 nearly.

EXAMPLES. XXXVI.

1. Write down the logarithms of 776·43, 7·7643, ·00077643 and 776430. (The table gives opposite the numbers 77643, the figures 8901023.)

2. Given that $\log_{10} 59082 = 4\cdot 7714552$, write down the logarithms of 5908200, 5·9082, ·00059082, 590·82 and 5908·2.

3. Find the fourth root of ·0059082, having given that

$\log 5\cdot 9082 = \cdot 7714552;\ 4\cdot 4428638 = \log_{10} 27724.$

4. Find the product of ·00059082 and ·027724, having given that ·21431 = log 16380 (cf. Question 3).

5. Find the 10th root of ·077643 (cf. Question 1), having given that ·8890102 = log 7·7448.

6. Find the product of $(\cdot 27724)^2$ and ·077643. (See Questions 1 and 3; 7758288 = log 59680.)

MISCELLANEOUS EXAMPLES. XXXVII.

1. Find $\log_2 8$, $\log_5 1$, $\log_8 2$, $\log_7 1$, $\log_{32} 128$.

2. Show that the logarithms of all except eight of the numbers from 1 to 30 inclusive, can be calculated in terms of log 2, log 3 and log 7.

3. Show that the logarithms of the numbers 1 to 10 inclusive may be found in terms of the logarithms of 8, 14, 21.

4. The mantissa of the log of 85762 is 9332949. Find the log of

$\sqrt[11]{\cdot 0085762}.$

Find how many figures there are in the integral part of $(85762)^{11}$.

5. Find the product of 47·609, 476·09, ·47609, ·000047609, having given that $\log 4{\cdot}7609 = {\cdot}6776891$ and ${\cdot}7107564 = \log 5{\cdot}1375$.

6. What are the characteristics of the logarithm of 3742 to the bases 3, 6, 10 and 12 respectively?

7. Having given that $\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$ and $\log 7 = {\cdot}8450980$, solve the following equations:

(i) $2^x \times 3^{4x} = 7^2$, (ii) $3^{2x} = 128 \times 7^{4-x}$,

(iii) $12^x = 49$, (iv) $2^{8x} = 21^{4-3x}$.

8. Given $\log_{10} 7$, find $\log_7 490$.

9. Given $\log_{10} 3$, find $\log_9 270$.

10. Given $\log_{10} 2$, find $\log_5 10$.

11. Given $\log_8 9 = a$, $\log_2 5 = b$, $\log_5 7 = c$; find the logs to base 10 of numbers 1 to 7 inclusive.

12. How many positive integers are there whose logarithms to base 2 have 5 for a characteristic?

13. If a be an integer, how many positive integers are there whose logs to base a have 10 for their characteristic?

14. Given log 2 and log 7, find the eleventh root of $(39{\cdot}2)^2$.

$$\log 1{\cdot}9485 = {\cdot}289688.$$

15. Prove that $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

16. Prove that $2 \log a + 2 \log a^2 + 2 \log a^3 \ldots + 2 \log a^n = n(n+1) \log a$.

17. Prove that $\log_a b \,.\, \log_b a = 1$; and that $\log_a b \,.\, \log_b c \,.\, \log_c a = 1$.

18. Prove that $\log_a r = \log_a b \,.\, \log_b c \,.\, \log_c d \ldots \log_q r$.

19. Given that the integral part of $(3{\cdot}456)^{100000}$ contains 53856 digits, find log 345·6 correct to five places of decimals.

20. Given that the integral part of $(3{\cdot}981)^{100000}$ contains sixty thousand digits, find log 39810 correct to five places of decimals.

21. If the number of births in a year be $\frac{1}{45}$ of the population at the beginning of the year, and the number of deaths $\frac{1}{60}$, find in what time the population will be doubled.

Given log 2, log 3, and that $\log 241 = 2{\cdot}3820170$.

22. Prove that $\log s + \log (s-a) - \log b - \log c = 2 \log \sqrt{\dfrac{s(s-a)}{bc}}$.

23. Prove that $\log (a^2 + x^2) + \log (a+x) + \log (a-x) = \log (a^4 - x^4)$.

24. Prove that $\log \sin 4A = \log 4 + \log \sin A + \log \cos A + \log \cos 2A$.

CHAPTER XII.

On the Use of Mathematical Tables.

130. The Logarithms referred to in this chapter, and in future throughout the book, are *Common* Logarithms.

131. Books of Mathematical Tables usually give an explanation of their own contents, but there are some points common to all such Tables which we proceed to explain.

132. The student will be supposed to have access to a book containing the following:

(i) A list of the logarithms of all whole numbers from 1 to 99999, calculated to seven significant figures;

(ii) A list of the numerical values, calculated to seven significant figures, of the Trigonometrical Ratios of all angles, between 0° and 90°, which differ by $1'$;

(iii) A list of the logarithms of these Ratios calculated to seven significant figures.

These will be found in Chambers' *Mathematical Tables.*

133. We have said that logarithms are in general incommensurable numbers. Their values can therefore only be given approximately.

If the value of any number is given to seven significant figures, then the **error** (i.e. the difference between the *given* value and the *exact* value of the number) is less than a millionth part of the number.

Example. 3·141592 is the value of π correct to seven significant figures. The *error* is less than ·000001; for π is less than 3·141593, and greater than 3·141592.

The ratio of ·000001 to 3·141592 is equal to 1 : 3141592. The ratio of ·000001 to π is less than this; i. e. much less than the ratio of one to one million.

134. An actual measurement of any kind must be made with the greatest care, with the most accurate instruments, by the most skilful observers, if it is to attain to anything like the accuracy represented by 'seven significant figures.'

Therefore the value of any quantity given correct to 'seven significant figures' is exact for all practical purposes.

135. We are given in the Tables the logarithms of all numbers from 1 to 99999; that is, of any number having *five* significant figures.

A Table consisting of the logarithms of all numbers from 1 to 9999999 (i.e. of any number having *seven* significant figures) would be *a hundred times as large.*

136. There is however a rule by which, if we are given a *complete* list of the logarithms of numbers having *five* significant figures, we can find the logarithms of numbers having *six* or *seven* significant figures.

Example. Suppose we require the logarithm of 4·804213.

From the Tables we find

$$\log 4{\cdot}8042 = {\cdot}6816211, \qquad \text{i.e. } 4{\cdot}8042 = 10^{{\cdot}6816211\cdots},$$
$$\log 4{\cdot}8043 = {\cdot}6816301, \qquad 4{\cdot}8043 = 10^{{\cdot}6816301\cdots},$$

The number 4·804213 lies *between* the two numbers 4·8042, 4·8043 whose logarithms are found in the Tables, so that the required logarithm must lie *between* the two given logarithms.

Therefore we suppose that

$$\log 4{\cdot}804213 = {\cdot}6816211 + d, \qquad \text{i.e. } 4{\cdot}804213 = 10^{{\cdot}6816211\cdots+d}.$$

137. The **RULE** is as follows. **The differences between three numbers are proportional to the corresponding differences between the logarithms of those numbers,** provided that the *differences* between the numbers are *small* compared with the numbers.

Example. Thus in the above example 4·8042, 4·8043 and 4·804213 are three numbers; ·6816211, ·6816301 and ·6816211 + d are their three logs.

The **difference** between the first and second numbers is ·0001.

The **difference** between the first and third numbers is ·000013.

The **difference** between the logarithms of the first and second numbers is ·000009.

The **difference** between the logarithms of the first and third numbers is d.

By the Rule these differences are in proportion

$$\therefore \cdot 0001 : \cdot 000013 = \cdot 000009 : d,$$

or $$100 : 13 = \cdot 000009 : d;$$

whence $$d = \cdot 00000117...,$$

$$\therefore \log 4\cdot 804213 = \cdot 6816211 + \cdot 00000117...$$

$$= \cdot 68162227 = \cdot 6816223 \text{ (to seven figures).}$$

138. We shall refer to the above rule as the **Rule of Proportional Differences.**

It is often called also 'The Principle of Proportional Parts.'

139. In Art. 197 we said that numbers are *not* proportional to their Logarithms. Hence the differences of numbers and the corresponding differences of their logarithms cannot be *exactly* in proportion. The rule is however true for all practical purposes. The proof of the rule belongs to a higher part of the subject than the present.

140. In the above example we said that

$$6\cdot 68162227 = 6\cdot 6816223;$$

and for this reason. We are *retaining* only *seven* significant figures in the decimal part of the logarithm.

If we put 6·6816222 for 6·68162227 the 'error' is greater than ·00000007.

If we put 6·6816223 for 6·68162227 the 'error' is less than ·00000003.

Thus the second error is less than the first.

In such a case, 1 must be added to the last digit which is retained, when the first digit which is neglected is 5 or greater than 5.

141. We give two more specimen examples.

Example 1. *Find the logarithm of* ·004804213.

We first find as before, by the rule of proportional differences, that

$$\log 4\cdot 804213 = \cdot 6816223$$

$$\therefore \log \cdot 004804213 = \bar{3}\cdot 6816223.$$

Example 2. *Find the number whose logarithm is* 2·5354291.

In the Table we find that

$$\cdot 5354207 = \log 3{\cdot}4310 \ldots\ldots (i).$$

and
$$\cdot 5354334 = \log 3{\cdot}4311 \ldots\ldots (ii).$$

Let
$$\cdot 5354291 = \log (3{\cdot}4310 + d) \ldots\ldots (iii).$$

Here we have three logarithms and three numbers.

The **difference** between the first and second logs is ·0000127.

The **difference** between the first and third logs is ·0000084.

The **difference** between the first and second numbers is ·0001.

The **difference** between the first and third numbers is d.

By the Rule these four **differences** are in proportion,

$$\therefore \cdot 0000127 : \cdot 0000084 = \cdot 0001 : d,$$

or,
$$127 : 84 = \cdot 0001 : d;$$

$$\therefore d = \cdot 0001 \times \tfrac{84}{127} = \cdot 0000661, \text{ etc.}$$

Therefore from (iii) $\cdot 5354291 = \log (3{\cdot}4310 + \cdot 000066)$

$$= \log 3{\cdot}431066.$$

Hence, $2{\cdot}5354291 = \log 343{\cdot}1066,$

or, the required number is 343·1066.

EXAMPLES. XXXVIII.

1. Find log 7·65432, having given that log 7·6543 = ·8839055, log 7·6544 = ·8839112.

2. Find log 564·123, having given that log 5·6412 = ·7513715, log 5·6413 = ·7513792.

3. Find log ·0008736416, having given that log 8·7364 = ·9413325, log 8·7365 = ·9413375.

4. Find log 6437125, having given that log 6·4371 = ·8086903, log 6·4372 = ·8086970.

5. Find log 3·72456, having given that log 37245 = 4·5710680, log 37246 = 4·5710796.

6. Find the number whose logarithm is ·5686760, having given that ·5686710 = log 3·7040, ·5686827 = log 3·7041.

7. Find the number whose logarithm is 4·6602987, having given that ·6602962 = log 4·5740, 6603057 = log 4·5741.

8. Find the number whose logarithm is 6·3966938, having given that ·3966874 = log 2·4928, ·3967049 = log 2·4929.

9. Find the number whose logarithm is $\overline{4}{\cdot}6431150$, having given that ·6431071 = log 4·3965, ·6431170 = log 4·3966.

10. Find the number whose logarithm is ·7550480, having given that 3·7550436 = log 5689·1, 2·7550512 = log 568·92.

142. On pages 91 to 94 will be found a Table of the Logarithms of all numbers from 100 to 1000.

We proceed to give Examples involving the use of this table.

Example. Find to three significant figures the diagonal of a cube whose side is 14·7 inches.

Let x be the number of inches in the diagonal,

then $$x^2 = 3 \times (14{\cdot}7)^2$$
$$\therefore\ x = \sqrt{3} \times 14{\cdot}7$$
$$\therefore\ \log x = \tfrac{1}{2} \log 3 + \log 14{\cdot}7$$
$$= \tfrac{1}{2}({\cdot}47712) + 1{\cdot}16732 \text{ [from the table.]}$$
$$= {\cdot}23856 + 1{\cdot}16732 = 1{\cdot}40588 = \log 25{\cdot}46 \text{ nearly.}$$

Thus the diagonal is 25·46 inches (nearly).

143. By the aid of the rule of proportional parts we can work correctly to four figures by the aid of the table given.

Example. Find log 347·6.

From the table $$\log 3{\cdot}47 = {\cdot}54033$$
$$\log 3{\cdot}48 = {\cdot}54158$$
$$\text{difference for } {\cdot}01 = {\cdot}00125$$
$$\therefore \text{ difference for } {\cdot}006 = {\cdot}00075$$
$$\therefore\ \log 347{\cdot}6 = 2{\cdot}54108.$$

EXAMPLES. XXXIX.

Find the values of the following correct to four significant figures:

1. $\sqrt[3]{451}$. 2. $\sqrt[5]{802}$. 3. $(273)^{\frac{4}{9}} \times (234)^{\frac{1}{4}}$.

4. $(451)^{\frac{3}{5}} \times (231)^{\frac{4}{3}}$. 5. $\left(\dfrac{192{\cdot}5}{84}\right)^3$. 6. $\dfrac{(34{\cdot}79)^{\frac{2}{3}}}{(41{\cdot}25)^{\frac{3}{2}}}$.

7. $\dfrac{(24{\cdot}76)^{\frac{2}{7}}}{({\cdot}0045)^{\frac{3}{2}}}$. 8. $\dfrac{7{\cdot}89}{{\cdot}0345} \times (89130)^{\frac{1}{7}}$. 9. $\dfrac{\frac{3}{2}\sqrt{(5{\cdot}2)}}{5\sqrt{(11{\cdot}31)}} \times (\tfrac{3}{7})^{-\frac{1}{2}}$.

10. $\sqrt[5]{\left\{\dfrac{2\sqrt{(34)}}{3\sqrt{(791)}}\right\}}$. 11. $\dfrac{\sqrt[4]{3}}{\sqrt[6]{3}}$. 12. $\left(\dfrac{21^3 \times 45^5}{2^7 \times 3^9}\right)^{\frac{1}{2}}$.

Solve the equations correct to 4 figures.

13. $10^x = 421$. 14. $(\tfrac{21}{20})^x = 3$. 15. $(\tfrac{203}{200})^{2x} = 2$.

16. $(\tfrac{26}{25})^x = 3$. 17. $\log 37^{x+3} = 3{\cdot}412$. 18. $x = 10\sqrt[3]{(31{\cdot}2)}$.

TABLE OF THE LOGARITHMS OF ALL NUMBERS FROM 100 TO 1000.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
100	00000	143	15534	186	26951	229	35983	272	43457
101	00432	144	15836	187	27184	230	36172	273	43616
102	00860	145	16137	188	27416	231	36361	274	43775
103	01284	146	16435	189	27646	232	36549	275	43933
104	01703	147	16732	190	27875	233	36736	276	44091
105	02119	148	17026	191	28103	234	36922	277	44248
106	02531	149	17319	192	28330	235	37107	278	44404
107	02938	150	17609	193	28556	236	37291	279	44560
108	03342	151	17898	194	28780	237	37475	280	44716
109	03743	152	18184	195	29003	238	37658	281	44870
110	04139	153	18469	196	29226	239	37840	282	45025
111	04532	154	18752	197	29447	240	38021	283	45179
112	04922	155	19033	198	29667	241	38202	284	45332
113	05308	156	19312	199	29885	242	38382	285	45484
114	05690	157	19590	200	30103	243	38561	286	45637
115	06070	158	19866	201	30320	244	38739	287	45788
116	06446	159	20140	202	30535	245	38917	288	45939
117	06819	160	20412	203	30750	246	39094	289	46090
118	07188	161	20683	204	30963	247	39270	290	46240
119	07555	162	20951	205	31175	248	39445	291	46389
120	07918	163	21219	206	31387	249	39620	292	46538
121	08279	164	21484	207	31597	250	39795	293	46687
122	08636	165	21748	208	31806	251	39967	294	46835
123	08991	166	22011	209	32015	252	40140	295	46982
124	09342	167	22272	210	32222	253	40312	296	47129
125	09691	168	22531	211	32428	254	40483	297	47276
126	10037	169	22789	212	32634	255	40654	298	47422
127	10380	170	23045	213	32838	256	40824	299	47567
128	10721	171	23300	214	33041	257	40993	300	47712
129	11059	172	23553	215	33244	258	41162	301	47857
130	11394	173	23805	216	33446	259	41330	302	48001
131	11727	174	24055	217	33646	260	41497	303	48144
132	12057	175	24304	218	33846	261	41664	304	48287
133	12385	176	24551	219	34044	262	41830	305	48430
134	12710	177	24797	220	34242	263	41996	306	48572
135	13033	178	25042	221	34439	264	42160	307	48714
136	13354	179	25285	222	34635	265	42325	308	48855
137	13672	180	25527	223	34830	266	42488	309	48996
138	13988	181	25768	224	35025	267	42651	310	49136
139	14301	182	26007	225	35218	268	42813	311	49276
140	14613	183	26245	226	35411	269	42975	312	49415
141	14921	184	26482	227	35602	270	43136	313	49554
142	15229	185	26717	228	35793	271	43297	314	49693

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
315	49831	361	55751	407	60959	453	65610	499	69810
316	49969	362	55871	408	61066	454	65705	500	69897
317	50106	363	55991	409	61172	455	65801	501	69984
318	50243	364	56110	410	61278	456	65896	502	70070
319	50379	365	56229	411	61384	457	65992	503	70157
320	50515	366	56348	412	61470	458	66087	504	70243
321	50651	367	56467	413	61595	459	66181	505	70329
322	50786	368	56585	414	61700	460	66276	506	70415
323	50920	369	56703	415	61805	461	66370	507	70501
324	51055	370	56820	416	61909	462	66464	508	70586
325	51188	371	56937	417	62014	463	66558	509	70672
326	51322	372	57054	418	62118	464	66652	510	70757
327	51455	373	57171	419	62221	465	66745	511	70842
328	51587	374	57287	420	62325	466	66839	512	70927
329	51720	375	57403	421	62428	467	66932	513	71011
330	51851	376	57519	422	62531	468	67025	514	71096
331	51983	377	57634	423	62634	469	67117	515	71181
332	52114	378	57749	424	62737	470	67210	516	71265
333	52244	379	57864	425	62839	471	67302	517	71349
334	52375	380	57978	426	62941	472	67394	518	71433
335	52504	381	58093	427	63043	473	67486	519	71517
336	52634	382	58206	428	63144	474	67578	520	71600
337	52763	383	58320	429	63246	475	67669	521	71684
338	52892	384	58433	430	63347	476	67761	522	71767
339	53020	385	58546	431	63448	477	67852	523	71850
340	53148	386	58659	432	63548	478	67943	524	71933
341	53275	387	58771	433	63649	479	68034	525	72016
342	53403	388	58883	434	63749	480	68124	526	72099
343	53529	389	58995	435	63849	481	68215	527	72181
344	53656	390	59106	436	63949	482	68305	528	72263
345	53782	391	59218	437	64048	483	68395	529	72346
346	53908	392	59329	438	64147	484	68485	530	72428
347	54033	393	59439	439	64246	485	68574	531	72509
348	54158	394	59550	440	64345	486	68664	532	72591
349	54283	395	59660	441	64444	487	68753	533	72673
350	54407	396	59770	442	64542	488	68842	534	72754
351	54531	397	59879	443	64640	489	68931	535	72835
352	54654	398	59988	444	64738	490	69020	536	72916
353	54777	399	60097	445	64836	491	69108	537	72997
354	54900	400	60206	446	64933	492	69197	538	73078
355	55023	401	60314	447	65031	493	69285	539	73159
356	55145	402	60423	448	65128	494	69373	540	73239
357	55267	403	60530	449	65225	495	69461	541	73320
358	55388	404	60638	450	65321	496	69548	542	73400
359	55509	405	60746	451	65418	497	69636	543	73480
360	55630	406	60853	452	65514	498	69723	544	73560

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
545	73640	591	77159	637	80414	683	83442	729	86273
546	73719	592	77232	638	80482	684	83506	730	86332
547	73799	593	77305	639	80550	685	83569	731	86392
548	73878	594	77379	640	80618	686	83632	732	86451
549	73957	595	77452	641	80686	687	83696	733	86510
550	74036	596	77525	642	80754	688	83759	734	86570
551	74115	597	77597	643	80821	689	83822	735	86629
552	74194	598	77670	644	80889	690	83885	736	86688
553	74273	599	77743	645	80956	691	83948	737	86747
554	74351	600	77815	646	81023	692	84011	738	86806
555	74429	601	77887	647	81090	693	84073	739	86864
556	74507	602	77960	648	81158	694	84136	740	86923
557	74586	603	78032	649	81224	695	84198	741	86982
558	74663	604	78104	650	81291	696	84261	742	87040
559	74741	605	78176	651	81358	697	84323	743	87099
560	74819	606	78247	652	81425	698	84385	744	87157
561	74896	607	78319	653	81491	699	84448	745	87216
562	74974	608	78390	654	81558	700	84510	746	87274
563	75051	609	78462	655	81624	701	84572	747	87332
564	75128	610	78533	656	81690	702	84634	748	87390
565	75205	611	78604	657	81757	703	84696	749	87448
566	75281	612	78675	658	81823	704	84757	750	87506
567	75358	613	78746	659	81889	705	84819	751	87564
568	75435	614	78817	660	81954	706	84880	752	87622
569	75511	615	78888	661	82020	707	84942	753	87680
570	75587	616	78958	662	82086	708	85003	754	87737
571	75664	617	79029	663	82151	709	85065	755	87795
572	75740	618	79099	664	82217	710	85126	756	87852
573	75815	619	79169	665	82282	711	85187	757	87910
574	75891	620	79239	666	82347	712	85248	758	87967
575	75967	621	79309	667	82413	713	85309	759	88024
576	76042	622	79379	668	82478	714	85370	760	88081
577	76118	623	79449	669	82543	715	85431	761	88138
578	76192	624	79518	670	82607	716	85491	762	88196
579	76268	625	79588	671	82672	717	85552	763	88252
580	76343	626	79657	672	82737	718	85612	764	88309
581	76418	627	79727	673	82802	719	85673	765	88366
582	76492	628	79796	674	82866	720	85733	766	88423
583	76567	629	79865	675	82930	721	85794	767	88480
584	76641	630	79934	676	82995	722	85854	768	88536
585	76716	631	80003	677	83059	723	85914	769	88593
586	76790	632	80072	678	83123	724	85974	770	88649
587	76864	633	80140	679	83189	725	86034	771	88705
588	76938	634	80209	680	83251	726	86094	772	88762
589	77012	635	80277	681	83315	727	86153	773	88818
590	77085	636	80346	682	83378	728	86213	774	88874

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
775	88930	820	91381	865	93702	910	95904	955	98000
776	88986	821	91434	866	93752	911	95952	956	98046
777	88042	822	91487	867	93802	912	95999	957	98091
778	89098	823	91540	868	93852	913	96047	958	98137
779	89154	824	91593	869	93902	914	96095	959	98182
780	89209	825	91645	870	93952	915	96142	960	98227
781	89265	826	91698	871	94002	916	96190	961	98272
782	89321	827	91751	872	94052	917	96237	962	98318
783	89376	828	91803	873	94101	918	96284	963	98363
784	89432	829	91855	874	94151	919	96332	964	98408
785	89487	830	91908	875	94201	920	96379	965	98453
786	89542	831	91960	876	94250	921	96426	966	98498
787	89597	832	92012	877	94300	922	96473	967	98543
788	89653	833	92065	878	94349	923	96520	968	98588
789	89708	834	92117	879	94399	924	96567	969	98632
790	89763	835	92169	880	94448	925	96614	970	98677
791	89818	836	92221	881	94498	926	96661	971	98722
792	89873	837	92273	882	94547	927	96708	972	98767
793	89927	838	92324	883	94596	928	96754	973	98811
794	89982	839	92376	884	94645	929	96801	974	98856
795	90037	840	92428	885	94694	930	96848	975	98900
796	90091	841	92480	886	94743	931	96895	976	98945
797	90146	842	92531	887	94792	932	96941	977	98990
798	90200	843	92583	888	94841	933	96988	978	99034
799	90255	844	92634	889	94890	934	97035	979	99078
800	90309	845	92686	890	94949	935	97081	980	99123
801	90363	846	92737	891	94988	936	97128	981	99167
802	90417	847	92788	892	95036	937	97174	982	99211
803	90472	848	92840	893	95085	938	97220	983	99255
804	90526	849	92891	894	95134	939	97267	984	99300
805	90580	850	92942	895	95182	940	97313	985	99344
806	90634	851	92993	896	95231	941	97359	986	99388
807	90687	852	93044	897	95279	942	97405	987	99432
808	90741	853	93095	898	95328	943	97451	988	99476
809	90795	854	93146	899	95376	944	97497	989	99520
810	90849	855	93197	900	95424	945	97543	990	99564
811	90902	856	93247	901	95472	946	97589	991	99607
812	90956	857	93298	902	95521	947	97635	992	99651
813	91009	858	93349	903	95569	948	97681	993	99695
814	91062	859	93399	904	95617	949	97727	994	99739
815	91116	860	93450	905	95665	950	97772	995	99782
816	91169	861	93500	906	95713	951	97818	996	99826
817	91222	862	93551	907	95761	952	97864	997	99870
818	91275	863	93601	908	95809	953	97909	998	99913
819	91328	864	93651	909	95856	954	97955	999	99957

Example (i). *Find the* amount at Compound Interest *on* £1 *for* 8 years *at* 5 per cent.

To find the amount for 1 year we multiply by $\frac{105}{100}$, i.e. by $\frac{21}{20}$.

The amount for 2 years will be £$\frac{21}{20}\times\frac{21}{20}$ and the amount for 8 years$=(\frac{21}{20})^8$.

Let x be the required amount in pounds, then

$$x=(\tfrac{21}{20})^8$$

$$\therefore \log x=8(\log 21-\log 20)$$

$$=8(1\cdot32222-1\cdot30103)=8(\cdot02119)$$

$$=\cdot16852=\log 1\cdot474\ldots$$

Hence, to find the amount at Compound Interest for 8 years at 5 per cent. we multiply the Principal expressed in pounds by 1·474 + ...

Example (ii). *In how many years will the Principal be doubled at* 5 per cent. *Compound Interest?*

Let x be the number of years, then

$(\frac{21}{20})^x$ is the amount at the end of x years,

hence $(\frac{21}{20})^x=2$,

or $x(\log 21-\log 20)=\log 2 \quad \therefore x=\frac{\cdot30103}{\cdot02119}=\frac{30103}{2119}=14\cdot2.$

EXAMPLES. XL.

1. Find the Compound Interest on £100 for 10 years at 4 per cent.

2. Find the Compound Interest on £1 for 8 years at 5 per cent.

3. In how many years will a sum of money be doubled at 3 per cent. Compound Interest?

4. In how many years will a sum of money be doubled at 4 per cent. Compound Interest?

5. Find the present value of £100 to be paid 8 years hence reckoning Compound Interest at 4 per cent.

6. If the number of births in a town are 25 per 1000 and the deaths 20 per 1000 annually, in how many years will the population be doubled?

7. On the birth of an infant £1000 is invested at Compound Interest in the Funds (3 per cent. payable half-yearly); calculate what it will be worth when the child is 21 years old.

8. In what time will a sum of money treble itself at 3 per cent. Compound Interest payable half-yearly?

9. A sum of 1 shilling lent on condition of 1 penny interest being paid monthly, accumulates at Compound Interest at the same rate for 12 years; what will be then the amount?

10. A man puts by 2*d.* at the end of the second week of the year, 4*d.* at the end of the fourth week, 8*d.* at the end of the sixth week; what sum would be put by for the last fortnight in the year?

11. A train starting from rest has at the end of 1 second velocity ·001 ft. per sec. and at the end of each second its velocity is greater by one-third than at the end of the preceding second; find the velocity in miles per hour at the end of 25 seconds.

12. The volume of a sphere is $\frac{4}{3}\pi \times$ (cube of the radius); find the diameter of the sphere which contains a cubic yard.

144. The same Rule of Proportional Differences is used in the case of **angles** and their **Trigonometrical Ratios**; and therefore also in the case of **angles** and the **logarithms of their Ratios.**

Thus the (small) differences between three angles are assumed to be proportional to the corresponding differences between the sines of those three angles; also, proportional to the corresponding differences between the logarithms of the sines of those angles.

145. Sines and cosines are always less than unity, as also are the tangents of all angles between 0° and 45°.

The logarithms of these Ratios must therefore have *negative* characteristics.

To avoid the inconvenience of having to print these negative characteristics, the whole number 10 is added to each logarithm of the Trigonometrical Ratios, before it is set down in the Table.

The numbers thus recorded are called the **tabular logarithms** of the sine, cosine, etc., of an angle.

They are indicated by the letter 'L.'

Thus L *sin* $31^\circ\ 15'$, stands for the tabular logarithm of sin $31^\circ\ 15'$, and is equal to $\{\log(\sin 31^\circ\ 15') + 10\}$.

The words **logarithmic sine** are used as abbreviation for **tabular logarithm of the sine.**

Thus in the tables we find $L \sin 31^\circ\ 15' = 9{\cdot}7149776$.

Therefore $\log(\sin 31^\circ\ 15') = 9{\cdot}7149776 - 10 = \bar{1}{\cdot}7149776$.

Example 1. *Find* $\sin 31^0\,6'\,25''$.

The Tables give $\qquad \sin 30^0\,6' = \cdot5165333$ .... (i),

$\qquad\qquad\qquad \sin 31^0\,7' = \cdot5167824$ .... (ii).

Let $\qquad \sin 31^0\,7'\,25'' = \cdot5165333 + d$ .... (iii).

The difference between the first two angles is 60″.

The difference between the first and third angle is 25″.

The differences between the corresponding sines are ·0002491 and *d*.

By the Rule these four differences are in proportion.

Therefore $\qquad 60'' : 35'' = \cdot0002491 : d,$

$$\therefore\ d = \cdot0002491 \times \tfrac{25}{60} = \cdot0001038.$$

Hence from (iii) $\sin 31^0\,7'\,25'' = \cdot5165333 + \cdot0001038 = \cdot5166371.$

Example 2. Find the angle whose logarithmic cosine is 9·7858083.

The table gives $\qquad 9\cdot7857611 = L\cos 52^0\,22'$ ....(i),

$\qquad\qquad\qquad 9\cdot7859249 = L\cos 52^0\,21'$ ....(ii).

The cosine diminishes as the angle increases. Hence corresponding to an increase in the angle there is a diminution of the cosine.

Hence, let $\qquad 9\cdot7858083 = L\cos(50^0\,22' - D)$ ....(iii).

Subtracting the first tabular logarithm from the second the difference is ·0001638.

Subtracting the first tabular logarithm from the third, the difference is ·0000472.

Subtracting the first angle from the second, the difference is − 60″.

Subtracting the first angle from the third, the difference is − *D*.

By the Rule these four differences are in proportion.

Therefore $\qquad \cdot0001638 : \cdot0000472 = -60'' : -D.$

$$\therefore\ D = 60'' \times \tfrac{472}{1638} = 17\cdot3''.$$

Hence $\qquad 9\cdot7858083 = L\cos(52^0\,22' - 17'')$

$$= L\cos 52^0\,21'\,43''.$$

EXAMPLES. XLI.

1. Find $\sin 42^0\,21'\,30''$

having given that $\qquad \sin 42^0\,21' = \cdot6736577$

$\qquad\qquad\qquad \sin 42^0\,22' = \cdot6738727.$

2. Find $\cos 47^0\,38'\,30''$

having given that $\qquad \cos 47^0\,38' = \cdot6738727$

$\qquad\qquad\qquad \cos 47^0\,39' = \cdot6736577.$

3. Find $\cos 21^0\,27'\,45''$

having given that $\qquad \cos 21^0\,27' = \cdot9307370$

$\qquad\qquad\qquad \cos 21^0\,28' = \cdot9306306.$

4. Find the angle whose sine is ·6666666
having given that ·6665325 = sin 41° 48′
·6667493 = sin 41° 49′.

5. Find the angle whose cosine is ·3333333
having given that ·3332584 = cos 70° 32′
·3335326 = cos 70° 31′.

6. Find the angle whose cosine is ·25
having given that ·2498167 = cos 75° 32′
·2500984 = cos 75° 31′.

7. Find L sin 45° 16′ 30″
having given that L sin 45° 16′ = 9·8514969
L sin 45° 17′ = 9·8516220.

8. Find L tan 27° 13′ 45″
having given that L tan 27° 13′ = 9·7112148
L tan 27° 14′ = 9·7115254.

9. Find L cot 36° 18′ 20″
having given that L cot 36° 18′ = 10·1339650
L cot 36° 19′ = 10·1337003.

10. Find the angle whose Logarithmic tangent is 9·8464028
having given that 9·8463018 = L tan 35° 4′
9·8465705 = L tan 35° 5′.

11. Find the angle whose Logarithmic cosine is 9·9448230
having given that 9·9447862 = L cos 28° 17′
9·9448541 = L cos 28° 16′.

12. Find the angle whose Logarithmic cosecant is 10·4274623
having given that 10·4273638 = L cosec 21° 57′
10·4276774 = L cosec 21° 56′.

146. Problems in which each of the lines involve contains an *exact* number of feet, and each angle an *exac* number of degrees, **do not occur** in practical work.

As from time to time the skill of observers and of in strument-makers has increased, so also has the number c significant figures by which observations have been recordec

Thus the want was felt of some method by which th labour involved in the multiplication and division of lon numerical quantities could be avoided. In the year 1614 Scotch mathematician, John Napier, Baron of Merchiston proposed his method of 'Logarithms'; i.e. the method c representing numbers by **indices**; 'which, by reducing t a few days the labour of many months, doubles, as i were, the life of an astronomer, besides freeing him fron the errors and disgust inseparable from long calculations *Laplace*.

147. We shall now give a few examples of the **practical** use of logarithms.

Example 1. The sides containing the right angle C in a right-angled triangle ABC contain 3456·4 ft. and 4543·5 ft. respectively; find the angles of the triangle, and the length of the hypotenuse.

Let a, b, c be the lengths of the sides of the triangle opposite the angles A, B, C respectively. See figure, p. 25.

Then $a = 3456{\cdot}4$ feet, $b = 4543{\cdot}5$ feet.

$$\tan A = \frac{a}{b} = \frac{3456{\cdot}4}{4543{\cdot}5}.$$

In the Tables we find

$$\log 3456{\cdot}4 = 3{\cdot}5386240.$$
$$\log 4543{\cdot}5 = 3{\cdot}6573905.$$
$$\therefore \log \frac{a}{b} = \log a - \log b.$$
$$= 3{\cdot}5386240 - 3{\cdot}6573905.$$
$$\therefore \log \tan A = \bar{1}{\cdot}8812335.$$
$$\therefore L \tan A = 9{\cdot}8812335.$$

In the Tables we find

$$9{\cdot}8810522 = L \tan 37^0 \, 15'.$$
$$9{\cdot}8813144 = L \tan 37^0 \, 16'.$$

Whence we find by the Rule of Proportional Differences
$9{\cdot}8812335 = L \tan 37^0 \, 15' \, 42''$.

$$\therefore A = 37^0 \, 15' \, 42''.$$

Also $B = (90^0 - A)$, $\quad \therefore B = 52^0 \, 44' \, 18''$,

and
$$\frac{c}{a} = \operatorname{cosec} A = \operatorname{cosec} 37^0 \, 15' \, 42'',$$

$$\therefore \log c = \log a + \log \operatorname{cosec} 37^0 \, 15' \, 42''$$
$$= \log a + L \operatorname{cosec} 37^0 \, 15' \, 42'' - 10$$
$$= 3{\cdot}5386240 + 10{\cdot}2179174 - 10$$
$$= 3{\cdot}7565414$$
$$= \log 5708{\cdot}8,$$

$\therefore$ the hypotenuse contains 5708·8 feet.

Thus we have found the angles and the third side of the triangle.

148. There are some formulæ which are seldom used i
practical work, because they are not adapted to logarithmi
calculation. They are those in which powers of quantitie
are connected by the signs + or −.

Example. In the above example we might have found the lengt
of the hypotenuse by means of the formula $c^2 = a^2 + b^2$.

But we should have had to go through the process of calculatin
by multiplication the values of a^2 and b^2.

For this reason, a formula which consists entirely o
factors is always preferred to one which consists of **term**
when any of those terms contain any power of the quantitie
involved.

If in the above example the lengths of the hypotenuse c and of on
side a were given, then the formula $b^2 = c^2 - a^2 = (c-a)(c+a)$ will giv
the length of b. For $\log b^2 = \log\{(c-a)(c+a)\}$,

or, $2\log b = \log(c-a) + \log(c+a)$.

And the values of $(c+a)$ and $(c-a)$ are easily written down fro
the given values of c and a.

EXAMPLES. XLII.

In the following questions A, B, C are the angles of a right-angle
triangle of which C is a right angle, and a, b, c are the lengths of th
sides opposite those angles respectively.

1. Given that $a = 1046{\cdot}7$ yards, $c = 1856{\cdot}2$ yards, $C = 90^0$, find A.
   $\log 1046{\cdot}7 = 3{\cdot}0198222$, $\log 1856{\cdot}2 = 3{\cdot}2686248$,
   $L \sin 34^0\,19' = 9{\cdot}7510991$,
   $L \sin 34^0\,20' = 9{\cdot}7512842$.

2. Given that $a = 843{\cdot}2$ feet, $C = 90^0$, and $A = 34^0\,15'$, find c.
   $\log 843{\cdot}2 = 2{\cdot}9259306$, $L \operatorname{cosec} 34^0\,15' = 10{\cdot}2496421$,
   $\log 1{\cdot}4982 = {\cdot}17557$.

3. Given that $a = 4845$ yards, $b = 4742$ yards, and $C = 90$, find A.
   $\log 4845 = 3{\cdot}6852938$, $\log 4742 = 3{\cdot}6759615$,
   $L \tan 45^0\,36' = 10{\cdot}0090965$, $L \tan 46^0\,37' = 10{\cdot}0093492$.

4. Given that $c = 8762$ feet, $C = 90$, and $A = 37^0\,10'$, find a and b.
   $\log 8762 = 3{\cdot}9426032$, $L \sin 37^0\,10' = 9{\cdot}7811344$,
   $L \cos 37^0\,10' = 9{\cdot}9013938$, $\log 5{\cdot}2934 = {\cdot}72373$,
   $\log 6{\cdot}9823 = {\cdot}843997$.

5. Given that $b = 1694{\cdot}2$ chains, $C = 90^0$, and $A = 18^0\,47'$, find a.
   $\log 1694{\cdot}2 = 3{\cdot}2289647$, $L \cot 18^0\,47' = 10{\cdot}4683893$,
   $\log 5{\cdot}7620 = {\cdot}76057$.

6. Given that $a = 1072$ chains, $c = 4849$ chains, and $C = 90^0$, find b
   $\log 5921 = 3{\cdot}7723951$, $\log 3777 = 3{\cdot}5771470$,
   $\log 4{\cdot}729 = {\cdot}67477$.

7. Given that $b=841$ feet, $c=3762$ feet, and $C=90^0$, find a.
$\log 4603=3\cdot6630410$, $\log 2921=3\cdot4655316$,
$\log 3\cdot6668=\cdot56428$.

8. Given that $a=7694\cdot5$ chains, $b=8471$ chains, $C=90^0$, find A and c.
$\log 7694\cdot5=3\cdot8861804$, $\log 8471=3\cdot9279347$,
$L \tan 42^0\,15'=9\cdot95824$, $L \operatorname{cosec} 42^0\,15'=10\cdot1723937$,
$\log 1\cdot1444=\cdot05857$.

MISCELLANEOUS EXAMPLES. XLIII.

1. A balloon is at a height of 2500 feet above a plain and its angle of elevation at a point in the plain is $40^0\,35'$. How far is the balloon from the point of observation? $L \operatorname{cosec} 40^0\,35'=10\cdot18672$.

2. A tower standing on a horizontal plain subtends an angle of $37^0\,19'$ at a point in the plain distant $369\cdot5$ feet from the foot of the tower. Find the height of the tower. $L \tan 37^0\,19'=9\cdot88210$.

3. The shadow of a tower on a horizontal plain in the sunlight is observed to be $176\cdot2$ feet and the elevation of the sun at that moment is $33^0\,12'$. Find the height of the tower. $L \tan=9\cdot81583$.

4. From the top of a tower $163\cdot5$ feet high by the side of a river the angle of depression of a post on the opposite bank of the river is $29^0\,47'$. Find the distance of the post from the foot of the tower.
$L \cot 39^0\,47'=10\cdot67952$.

5. Given $a=673$, $b=416$ chains, $C=90^0$, find A and B.
$L \tan 58^0\,17'=10\cdot20900$.

6. Given $a=576$, $c=873$ chains, $C=90^0$, find b and A.
$L \sin 41^0\,17'=9\cdot81940$, $L \cos 41^0\,17'=9\cdot87590$.

7. From the top of a light-house $112\cdot5$ feet high, the angles of depression of two ships, when the line joining the ships points to the foot of the light-house, are $27^0\,18'$ and $20^0\,36'$ respectively. Find the distance between the ships.
$L \cot 27^0 18'=10\cdot28723$, $L \cot 20^0\,36'=10\cdot42496$.

8. From the top of a cliff the angles of depression of the top and bottom of a light-house $97\cdot25$ feet high are observed to be $23^0\,17'$ and $24^0\,19'$ respectively. How much higher is the cliff than the light-house?
$L \tan 23^0\,17'=9\cdot63379$, $L \tan 24^0\,19'=6\cdot65501$.

9. Find the distance in space travelled in an hour, in consequence of the earth's rotation, by St Paul's Cathedral. (Latitude of London $=51^0\,25'$, earth's diameter $=7914$ miles.)
$L \cos 51^0\,25'=9\cdot79494$.

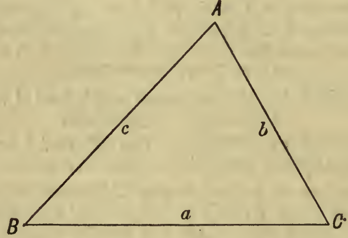
10. The angle of elevation of a balloon from a station due south of it is $47^0\,18'$, and from another station due west of the former and distant 671 feet from it the elevation is $41^0\,14'$. Find the height of the balloon.
$\cot 47^0\,18'=\cdot92277$, $\cot 41^0\,14'=1\cdot14095$.

CHAPTER XIII.

On the Relations between the Sides and Angles of a Triangle.

149. The **three sides** and the **three angles** of any **triangle,** are called its six **parts.**

By the letters A, B, C we shall indicate *geometrically*, the three angular points of the triangle ABC; *algebraically*, the three angles at those angular points respectively.



By the letters a, b, c we shall indicate the measures of the sides BC, CA, AB opposite the angles A, B, C respectively.

150. **I.** We know that, $A+B+C=180^\circ$. [Euc. I. 32.]

151. Also if A be an angle of a triangle, then A may have any value between 0° and 180°. Hence,

(i) **sin A** must be positive (and less than 1),

(ii) **cos A** may be positive or negative (but must be numerically less than 1),

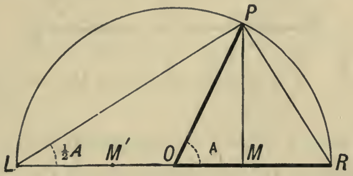
(iii) **tan A** may have any value whatever, positive or negative.

APPENDIX.

In some Examinations, as for instance that of the 2nd stage, Mathematics, of the South Kensington Science and Art Department, Chapters IX. and X. of this book (the A, B; S, T; and $2A$ formulæ) are not required. As, however, the student is required to solve Triangles by the aid of Logarithms he must use [see Arts. 158, 159, 161, 162] the two following propositions. The proofs here given are deduced from Euclid III.

Prop. I. *To prove that*

$$\cos A = 2\cos^2 \tfrac{1}{2}A - 1 = 1 - 2\sin^2 \tfrac{1}{2}A.$$



Let ROP be the angle A; with O as centre and any radius OR describe the semicircle RPL; join PL, PR, and draw PM perpendicular to LOR.

Then $$POM = OLP + OPL = 2OLP,$$
$$\therefore\ OLP = \tfrac{1}{2}POM = \tfrac{1}{2}A.$$

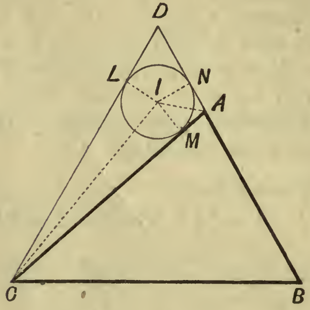
Now, $$\cos A = \frac{OM}{OP} = \frac{LM - LO}{OP} = \frac{2LM}{2OP} - \frac{OP}{OP}$$
$$= 2 \cdot \frac{LM}{LP} \cdot \frac{LP}{LR} - 1 = 2\cos OLP \cdot \cos OLP - 1$$
$$= 2\cos^2 \tfrac{1}{2}A - 1 \quad \text{..............................(i)}$$
$$= 2(1 - \sin^2 \tfrac{1}{2}A) - 1$$
$$= 1 - 2\sin^2 \tfrac{1}{2}A \quad \text{..............................(ii).}$$

Note. $$\sin A = \frac{MP}{OP} = 2 \cdot \frac{MP}{LP} \cdot \frac{LP}{2OP} = 2 \cdot \frac{MP}{LP} \cdot \frac{LP}{LR}$$
$$= 2\sin OLP \cdot \cos OLP = 2\sin \tfrac{1}{2}A \cdot \cos \tfrac{1}{2}A.$$

[See Art. 161.]

PROP. II. *To prove that in any triangle*

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$



Let ABC be a triangle of which the angle B is greater than C.

Make the angle $BCD = B$ and produce BA to D.

In the triangle ACD inscribe the circle LMN, centre I, touching the sides in L, M, N; join IL, IM, IN, IA, IC.

Then $ICM=\frac{1}{2}LCM=\frac{1}{2}(DCB-ACB)=\frac{1}{2}(B-C)$,

$$IAM=\tfrac{1}{2}DAC=\tfrac{1}{2}(180^\circ-CAB)=(90^\circ-\tfrac{1}{2}A),$$

$$CM=CL=CD-LD=BD-ND=BN=BA+AM;$$

$$\therefore\ CM=\tfrac{1}{2}(CM+BA+AM)=\tfrac{1}{2}(AC+AB)=\tfrac{1}{2}(b+c),$$

and $$AM=AC-CM=b-\tfrac{1}{2}(b+c)=\tfrac{1}{2}(b-c).$$

Hence $$\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}=\frac{\tan ICM}{\tan(90^\circ-\frac{1}{2}A)}=\frac{\tan ICM}{\tan IAM}$$

$$=\frac{\frac{IM}{CM}}{\frac{IM}{AM}}=\frac{AM}{CM}=\frac{\frac{1}{2}(b-c)}{\frac{1}{2}(b+c)}=\frac{b-c}{b+c}.\qquad \text{Q.E.D.}$$

152. Also, if we are given the value of

(i) **sin A**, there are **two** angles, each less than 180°, which have the given positive value for their sine.

(ii) **cos A**, or (iii) **tan A**, then there is only **one** value of A, which value can be found from the Tables.

153. $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$. Therefore $\frac{A}{2}$ is less than 90°, and its Trigonometrical Ratios are all positive. Also, $\frac{A}{2}$ is known, when the value of any one of its ratios is given. Similar remarks of course apply to the angles B and C.

Example 1. *To prove* $\sin(A+B)=\sin C$. [Art. 96.]

$$A+B+C=180°;\ \therefore\ A+B=180°-C,$$

and $\therefore\ \sin(A+B)=\sin(180°-C)=\sin C.$ [p. 61.]

Example 2. *To prove* $\sin\frac{A+B}{2}=\cos\frac{C}{2}$.

Now $\frac{A+B+C}{2}=90°.\quad \therefore\ \frac{A+B}{2}=90°-\frac{C}{2},$

and $\therefore\ \sin\frac{A+B}{2}=\sin\left(90°-\frac{C}{2}\right)=\cos\frac{C}{2}.$ [Art. 94.]

EXAMPLES. XLIV.

Find A from each of the six following equations, A being an angle of a triangle.

1. $\cos A=\frac{1}{2}$. **2.** $\cos A=-\frac{1}{2}$. **3.** $\sin A=\frac{1}{2}$.

4. $\tan A=-1$. **5.** $\sqrt{2}\sin A=1$. **6.** $\tan A=-\sqrt{3}$.

Prove the following statements, A, B, C being the angles of a triangle.

7. $\sin(A+B+C)=0$. **8.** $\cos(A+B+C)=-1$.

9. $\sin\frac{1}{2}(A+B+C)=1$. **10.** $\cos\frac{1}{2}(A+B+C)=0$.

11. $\tan(A+B)=-\tan C$. **12.** $\cot\frac{1}{2}(B+C)=\tan\frac{1}{2}A$.

13. $\cos(A+B)=-\cos C$. **14.** $\cos(A+B-C)=-\cos 2C$.

15. $\tan A-\cot B=\cos C\,.\sec A\,.\operatorname{cosec} B$.

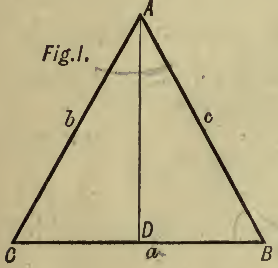
16. $\frac{\sin A-\sin B}{\sin A+\sin B}=\tan\frac{C}{2}\,.\tan\frac{A-B}{2}$.

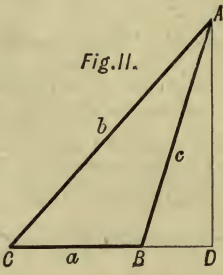
17. $\frac{\sin 3B-\sin 3C}{\cos 3C-\cos 3B}=\tan\frac{3A}{2}$.

154. **II.** *To prove* $a = b \cos C + c \cos B.$

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary.

There will be three cases. Fig. i. when both B and C are acute angles; Fig. ii. when one of them (B) is obtuse; Fig. iii. when one of them (B) is a right angle. Then,





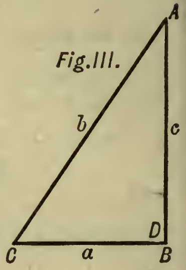


Fig. i. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b \cos C$,

and $\dfrac{DB}{AB} = \cos ABD$; or, $DB = c \cos B$,

$\therefore\ a = CD + DB = b \cos C + c \cos B.$

Fig. ii. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b \cos C$,

$\dfrac{BD}{AB} = \cos ABD$; or, $BD = c \cos (180° - B)$,

$\therefore\ a = CD - BD = b \cos C - c \cos (180° - B)$

$= b \cos C + c \cos B.$

Fig. iii. $a = CB = b \cos C$

$= b \cos C + c \cos B.$ [For, $\cos B = \cos 90° = 0.$]

Similarly it may be proved that,

$b = c \cos A + a \cos C$; $c = a \cos B + b \cos A.$

155. **III.** *To prove that in any triangle, the sides are proportional to the sines of the opposite angles;* or, *To prove that* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary. Then,

Fig. i. $AD = b \sin C$; for, $\frac{AD}{AC} = \sin C$ [Def.];

also $AD = c \sin B$; for, $\frac{AD}{AB} = \sin B$.

$\therefore b \sin C = c \sin B$;

or, $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Fig. ii. $AD = b \sin C$,

and $AD = c \sin ABD = c \sin (180° - B)$.

$\therefore AD = c \sin B$;

$\therefore b \sin C = c \sin B$;

or, $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Fig. iii. $AB = AC \,.\, \sin C$; or, $c = b \sin C$;

$\therefore \frac{c}{\sin C} = \frac{b}{\sin B}$. [For $\sin B = \sin 90° = 1$.]

Similarly it may be proved that

$$\frac{a}{\sin A} = \frac{b}{\sin B};$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \text{Q.E.D.}$$

156. **IV.** *To prove that* $a^2 = b^2 + c^2 - 2bc\cos A$.

Take one of the angles A. Then of the other two, one must be acute. Let B be an acute angle. From C draw CF perpendicular to BA, or to BA produced if necessary.

There will be three figures according as A is less, greater than, or equal to a right angle. Then,

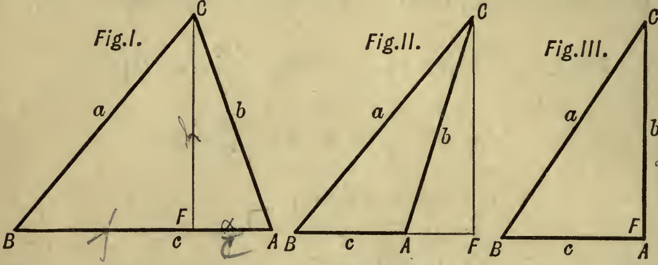


Fig. i. $BC^2 = CA^2 + AB^2 - 2 \cdot BA \cdot FA$; [Euc. II. 13]

or, $a^2 = b^2 + c^2 - 2c \cdot FA$

$= b^2 + c^2 - 2cb\cos A$. [For $FA = b \cdot \cos A$.]

Fig. ii. $BC^2 = CA^2 + AB^2 + 2 \cdot BA \cdot AF$; [Euc. II. 12]

or, $a^2 = b^2 + c^2 + 2cb\cos FAC$

$= b^2 + c^2 - 2bc\cos A$. [For $FAC = 180° - A$.]

Fig. iii. $BC^2 = CA^2 + AB^2$; [Euc. I. 47]

or, $a^2 = b^2 + c^2 - 2bc\cos A$. [For $\cos A = \cos 90° = 0$.]

Similarly it may be proved that

$$b^2 = c^2 + a^2 - 2ca\cos B,$$

and that

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

157. **V.** Hence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

158. **VI.** *To prove that* $\sin^2 \dfrac{A}{2} = \dfrac{(a + c - b)(a + b - c)}{4bc}$.

Since $\cos A = 1 - 2\sin^2 \dfrac{A}{2}$, [Art. 109]

and $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$; [Art. 157]

$$\therefore\ 2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc-(b^2+c^2-a^2)}{2bc} = \frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{\{a-(b-c)\}\{a+(b-c)\}}{2bc},$$

$$\therefore\ \sin^2\frac{A}{2} = \frac{(a+c-b)(a+b-c)}{4bc}. \quad \text{Q.E.D.}$$

159. To prove that $\cos^2\frac{A}{2} = \frac{(a+b+c)(b+c-a)}{4bc}$.

Since $\cos A = 2\cos^2\frac{A}{2} - 1$; [Art. 109]

$$\therefore\ 2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}; \quad \text{[Art. 157]}$$

$$\therefore\ \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{(b+c+a)(b+c-a)}{4bc}. \quad \text{Q.E.D.}$$

160. **VII.** Now let s stand for $\frac{a+b+c}{2}$, so that $(a+b+c) = 2s$.

Then, $(b+c-a) = (b+c+a-2a) = (2s-2a) = 2(s-a)$,

and $(c+a-b) = (c+a+b-2b) = (2s-2b) = 2(s-b)$,

and $(a+b-c) = (a+b+c-2c) = (2s-2c) = 2(s-c)$.

Then the result of Arts. 158, 159 may be written

$$\sin^2\frac{A}{2} = \frac{2(s-b)\,2(s-c)}{4bc}; \text{ or, } \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

and $\cos^2\frac{A}{2} = \frac{2s\,2(s-a)}{4bc}$; or, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$,

and so on.

Hence,
$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}.$$

Example. Write down the corresponding formulæ for $\sin\frac{B}{2}$, for $\cos\frac{B}{2}$, and for $\tan\frac{B}{2}$.

161. **VIII.** Again,

$$\sin A = 2\sin\frac{A}{2}\cdot\cos\frac{A}{2}; \qquad \text{[Art. 109]}$$

$$\therefore \sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The letter S usually stands for $\sqrt{s(s-a)(s-b)(s-c)}$, so that the above may be written $\frac{\sin A}{a} = \frac{2S}{abc}$.

Similarly, $$\frac{\sin B}{b} = \frac{2S}{abc} = \frac{\sin C}{c}.$$

162. **IX.** *To prove that* $\frac{b-c}{b+c}\cdot\cot\frac{A}{2} = \tan\frac{B-C}{2}$.

Since $\frac{b}{\sin B} = \frac{c}{\sin C}$, let each of these fractions $= d$.

Then $b = d\sin B$, and $c = d\sin C$.

$$\therefore \frac{b-c}{b+c} = \frac{d\sin B - d\sin C}{d\sin B + d\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\sin\frac{B-C}{2}\cdot\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}} = \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}$$

$$= \frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}. \qquad \left[\text{Since } \tan\frac{B+C}{2} = \tan\left(90^\circ - \frac{A}{2}\right).\right]$$

$$\therefore \frac{b-c}{b+c}\cdot\cot\frac{A}{2} = \frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}\cdot\cot\frac{A}{2} = \tan\frac{B-C}{2}. \qquad \text{Q.E.D.}$$

Similarly,

$$\frac{c-a}{c+a}\cdot\cot\frac{B}{2}=\tan\frac{C-A}{2},\quad \frac{a-b}{a+b}\cdot\cot\frac{C}{2}=\tan\frac{A-B}{2}.$$

163. The student is advised to make himself thoroughly familiar with the following **formulæ**:

$$a=b\cos C+c\cos B \quad\text{.......... (ii)},$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\left[=d\right]=\frac{abc}{2S} \quad\text{.......... (iii)},$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \quad\text{.......... (v)},$$

$$\left.\begin{aligned}\sin\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{bc}}\\ \cos\frac{A}{2}&=\sqrt{\frac{s(s-a)}{bc}}\end{aligned}\right\} \quad\text{.......... (vii)},$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2S}{bc} \quad\text{.......... (viii)},$$

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2} \quad\text{.......... (ix)}.$$

EXAMPLES. XLV.

In any triangle ABC prove the following statements:

1. $\dfrac{\sin A+2\sin B}{a+2b}=\dfrac{\sin C}{c}$.

2. $\dfrac{\sin^2 A-m\,.\sin^2 B}{a^2-m\,.\,b^2}=\dfrac{\sin^2 C}{c^2}$.

3. $a\cos A+b\cos B-c\cos C=2c\cos A\,.\cos B$.

4. $(a+b)\sin\dfrac{C}{2}=c\cos\dfrac{A-B}{2}$.

5. $(b-c)\cos\dfrac{A}{2}=a\sin\dfrac{B-C}{2}$.

6. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.

7. $\dfrac{a-b}{c}=\dfrac{\cos B-\cos A}{1+\cos C}$.

8. $\dfrac{b+c}{a}=\dfrac{\cos B+\cos C}{1-\cos A}$.

9. $\sqrt{bc\sin B\,.\sin C}=\dfrac{b^2\sin C+c^2\sin B}{b+c}$.

10. $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$.

11. $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C$.

12. $\tan A=\dfrac{a\sin C}{b-a\cos C}$.

13. $\dfrac{\tan B}{\tan C}=\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}$.

14. $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C = 3abc$.

15. $a\cos(A+B+C) - b\cos(B+A) - c\cos(A+C) = 0$.

16. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$.

17. $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = s$.

18. $\tan\frac{B}{2} \cdot \tan\frac{C}{2} = \frac{b+c-a}{b+c+a}$.

19. $\tan\frac{A}{2}(b+c-a) = \tan\frac{B}{2}(c+a-b)$.

20. $c^2 = (a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}$.

MISCELLANEOUS EXAMPLES. XLVI.

1. Simplify the formulæ

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos\tfrac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

in the case of an equilateral triangle.

2. The sides of a triangle are as $2 : \sqrt{6} : 1+\sqrt{3}$, find the angles.

3. The sides of a triangle are as 4, $2\sqrt{2}$, $2(\sqrt{3}-1)$, find the angles.

4. Given $C=120^0$, $c=\sqrt{19}$, $a=2$, find b.

5. Given $A=60^0$, $b=4\sqrt{7}$, $c=6\sqrt{7}$, find a.

6. Given $A=45^0$, $B=60^0$ and $a=2$, find c.

7. The sides of a triangle are as $7 : 8 : 13$, find the greatest angle.

8. The sides of a triangle are 1, 2, $\sqrt{7}$, find the greatest angle.

9. The sides of a triangle are as $a : b : \sqrt{(a^2+ab+b^2)}$, find the greatest angle.

10. When $a : b : c$ as $3 : 4 : 5$, find the greatest and least angles; given $\cos 36^0\, 52' = \cdot 8$.

11. If $a=5$ miles, $b=6$ miles, $c=10$ miles, find the greatest angle. [$\cos 49^0\, 33' = \cdot 65$.]

12. If $a=4$, $b=5$, $c=8$, find C; given that $\cos 54^0\, 54' = \cdot 575$.

13. $a : b = \sqrt{3} : 1$, and $C=30^0$; find the other angles.

14. Given $C=18^0$, $a=\sqrt{5}+1$, $c=\sqrt{5}-1$, find the other angles.

15. If $b=3$, $C=120^0$, $c=\sqrt{13}$, find a and the sines of the other angles.

16. Given $A=105^0$, $B=45^0$, $c=\sqrt{2}$, solve the triangle.

17. Given $B=75^0$, $C=30^0$, $c=\sqrt{8}$, solve the triangle.

18. Given $B=45^0$, $c=\sqrt{75}$, $b=\sqrt{50}$, solve the triangle.

19. Given $B=30^0$, $c=150$, $b=50\sqrt{3}$, show that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the third side in the greatest of these triangles.

20. Are there two triangles in which $B=30^0$, $c=150$, $b=75$?

21. If the angles adjacent to the base of a triangle are $22\frac{1}{2}^0$ and $112\frac{1}{2}^0$, show that the perpendicular altitude will be half the base.

22. If $a=2$, $b=4-2\sqrt{3}$, $c=\sqrt{6}(\sqrt{3}-1)$, solve the triangle.

23. If $A=9^0$, $B=45^0$, $b=\sqrt{6}$, find c.

24. Given $B=15^0$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$, solve the triangle.

25. Given $\sin B=\cdot25$, $a=5$, $b=2\cdot5$, find A. Draw a figure to explain the result.

26. Given $C=15^0$, $c=4$, $a=4+\sqrt{48}$, solve the triangle.

27. Two sides of a triangle are $3\sqrt{6}$ yards and $3(\sqrt{3}+1)$ yards, and the included angle 45^0, solve the triangle.

28. If $C=30^0$, $b=100$, $c=45$, is the triangle ambiguous?

29. Prove that if $A=45^0$ and $B=60^0$ then $2c=a(1+\sqrt{3})$.

30. The cosines of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{3}{5}$, find the ratio of the sides.

CHAPTER XIV.

ON THE SOLUTION OF TRIANGLES.

164. The problem known as the **Solution of Triangles** may be stated thus: *When a sufficient number of the parts of a triangle are given, to find the magnitude of each of the other parts.*

165. When **three** parts of a Triangle (one of which must be a side) are given, the other parts can in general be determined.

There are four cases.

I. Given **three sides.** [Compare Euc. I. 8.]

II. Given **one side** and **two angles.** [Euc. I. 26.]

III. Given **two sides** and the **angle between** them. [Euc. I. 4.]

IV. Given **two sides** and the **angle opposite** one of them. [Compare Euc. VI. 7.]

Case I.

166. Given **three sides,** a, b, c. [Euc. I. 8; VI. 5.]

We find two of the angles from the formulæ

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

The third angle $C = 180 - A - B$.

167. In practical work we proceed as follows:

$$\log\tan\frac{A}{2} = \log\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

or,

$$L\tan\frac{A}{2} - 10 = \tfrac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\}.$$

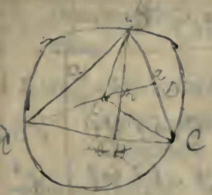
Similarly,

$$L\tan\frac{B}{2} - 10 = \tfrac{1}{2}\{\log(s-c) + \log(s-a) - \log s - \log(s-b)\}.$$

168. Either of the formulæ $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ may also be used as above.

The $\sin\frac{A}{2}$ and the $\cos\frac{A}{2}$ formulæ are either of them as convenient as the $\tan\frac{A}{2}$ formulæ, when *one* of the angles only is to be found. If *all* the angles are to be found the tangent formula is convenient, because we can find the L tangents of two half angles from the same *four* logs, viz. $\log s$, $\log(s-a)$, $\log(s-b)$, $\log(s-c)$. To find the L sines of *two* half angles we require the *six* logarithms, viz. $\log(s-a)$, $\log(s-b)$, $\log(s-c)$, $\log a$, $\log b$, $\log c$.

Example. Given $a=275{\cdot}35$, $b=189{\cdot}28$, $c=301{\cdot}47$ chains, find A and B.



$OD \perp BC$

$\angle DOC = \angle A$

radius $= R$.

$\sin DOC = \sin A = \frac{a/2}{R}$

$\therefore R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A} = \frac{abc}{4S}$

$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}}$

$= \frac{S}{s(s-a)}$. $\tan\frac{B}{2} = \frac{S}{s(s-c)}$.

$\tan\frac{C}{2} = \frac{S}{s(s-c)}$.

$\tan\frac{A}{2} = \frac{\frac{abc}{4R}}{s(s-a)}$

$= \frac{abc}{4R\,s(s-a)}$

Lėm

Case **I.**

Given.	Number,	Log	log tan	
½		1.4171		
s − a	7	0.8451	0.5740	75–
s − b	175	2.2430	9.1761	8°–
s − c	234	2.3692	9.0499	6°–
S	416	*7.3089		
r		2.5372		
2,s	832	$r = \sqrt{\frac{P}{S}}$	A	150°–
→ a	409		B	17°–
→ b	241	$\tan\frac{1}{2}A = \frac{r}{s-a}$	C	12°–
→ c	182		a + B + C = 1	

* co-logarithm

$P = (s-a)(s-b)(s-c)$

$\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-a)(s-b)(s-c)}{S(s-a)^2}}$

$= \frac{r}{s-a}$

$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

Parts given represented by →

Example. Given $a=275{\cdot}35$, $b=189{\cdot}28$, $c=301{\cdot}47$ chains, find A and B.

Here, $s=383{\cdot}05$, $s-a=107{\cdot}70$, $s-b=193{\cdot}77$, $s-c=81{\cdot}58$.

Then

$$L \tan\frac{A}{2}=10+\tfrac{1}{2}\{\log 193{\cdot}77+\log 81{\cdot}58-\log 383{\cdot}05-\log 107{\cdot}70\}$$
$$=10+\tfrac{1}{2}\{2{\cdot}2872865+1{\cdot}9115837-2{\cdot}5832555-2{\cdot}0322157\}$$
$$=9{\cdot}7916995 \qquad \text{[from the Tables]},$$

whence $\frac{A}{2}=31^0\,45'\,28{\cdot}5''$; $\therefore A=63^0\,30'\,57''$. Again,

$$L \tan\frac{B}{2}=10+\tfrac{1}{2}\{\log 81{\cdot}58+\log 107{\cdot}70-\log 383{\cdot}05-\log 193{\cdot}77\}$$
$$=9{\cdot}5366287=L\tan 18^0\,59'\,9{\cdot}8'';$$

$\therefore B=37^0\,58'\,20''$; $C=180^0-A-B=78^0\,30'\,43''$.

169. This Case may also be solved by the formula

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

But this formula is not adapted for logarithmic calculation, and therefore is seldom used in practice.

It may sometimes be used with advantage, when the given lengths of a, b, c each contain less than three digits.

Example. Find the greatest angle of the triangle whose sides are 13, 14, 15.

Let $a=15$, $b=14$, $c=13$. Then the greatest angle is A.

$$\text{Now, } \cos A=\frac{14^2+13^2-15^2}{2\times 14\times 13}=\frac{140}{2\times 14\times 13}=\tfrac{5}{13}={\cdot}384615$$
$$=\cos 67^0\,23', \text{ nearly.}$$

[By the Table of natural cosines.]

$\therefore$ the greatest angle $=67^0\,23'$.

EXAMPLES. XLVII.

1. If $a=352{\cdot}25$, $b=513{\cdot}27$, $c=482{\cdot}68$ yards, find the angle A, having given

$\log 674{\cdot}10=2{\cdot}8287243$, $\log 321{\cdot}85=2{\cdot}5076535$,
$\log 160{\cdot}83=2{\cdot}2063401$, $\log 191{\cdot}42=2{\cdot}2819873$,
$L\tan 20^0\,38'=9{\cdot}5758104$, $L\tan 20^0\,39'=9{\cdot}5761934$.

2. Find the two largest angles of the triangle whose sides are 484, 376, 522 chains, having given that

$$\log 6{\cdot}91 = {\cdot}8394780,\quad \log 3{\cdot}15 = {\cdot}4983106,$$
$$\log 2{\cdot}07 = {\cdot}3159703,\quad \log 1{\cdot}69 = {\cdot}2278867,$$
$$L\tan 36^\circ 46' 6'' = 9{\cdot}8734581,\quad L\tan 31^\circ 23' 9'' = 9{\cdot}7853745.$$

3. If $a=5238$, $b=5662$, $c=9384$ yards, find the angles A and B, having given

$$\log 1{\cdot}0142 = {\cdot}0061236,\quad \log 4{\cdot}904 = {\cdot}6905505,$$
$$\log 4{\cdot}48 = {\cdot}6512780,\quad \log 7{\cdot}58 = {\cdot}8796692,$$
$$L\tan 14^\circ 38' = 9{\cdot}4168099,\quad L\tan 15^\circ 57' = 9{\cdot}4560641,$$
$$L\tan 14^\circ 39' = 9{\cdot}4173265,\quad L\tan 15^\circ 58' = 9{\cdot}4565420.$$

4. If $a=4090$, $b=3850$, $c=3811$ yards, find A, having given

$$\log 5{\cdot}8755 = {\cdot}7690448,\quad \log 3{\cdot}85 = {\cdot}5854607,$$
$$\log 1{\cdot}7855 = {\cdot}2517599,\quad \log 3{\cdot}811 = {\cdot}5810389,$$
$$L\cos 32^\circ 15' = 9{\cdot}9272306,\quad L\cos 32^\circ 16' = 9{\cdot}9271509.$$

5. Find the greatest angle in a triangle whose sides are 7 feet, 8 feet, and 9 feet, having given

$$\log 3 = {\cdot}4771213,\quad L\cos 36^\circ 42' = 9{\cdot}9040529,$$
$$\log 1{\cdot}4 = {\cdot}146128,\quad \text{diff. for } 60'' = {\cdot}0000942.$$

6. Find the smallest angle of the triangle whose sides are 8 feet, 10 feet, and 12 feet, having given that

$$\log 2 = {\cdot}30103,\quad L\sin 20^\circ 42' = 9{\cdot}5483585,\quad \text{diff. for } 60'' = {\cdot}0003342.$$

7. If $a : b : c = 4 : 5 : 6$, find C, having given

$$\log 2 = {\cdot}3010300,\quad \log 3 = {\cdot}4771213,$$
$$L\cos 41^\circ 25' = 9{\cdot}8750142,\quad \text{diff. for } 60'' = {\cdot}0001115.$$

8. The sides of a triangle are 2, $\sqrt{6}$, and $1+\sqrt{3}$, find the angles.

9. The sides of a triangle are 2, $\sqrt{2}$, and $\sqrt{3}-1$, find the angles.

Sem

Case II.

170. Given **one side** and **two angles**, as a, B, C. [Euc. I. 26; VI. 4.]

First, $A = 180^\circ - B - C$; which determines A.

Next, $$\frac{b}{\sin B} = \frac{a}{\sin A},\quad \text{or, } b = \frac{a \cdot \sin B}{\sin A};$$

and, $$\frac{c}{\sin C} = \frac{a}{\sin A},\quad \text{or, } c = \frac{a \cdot \sin C}{\sin A}.$$

These determine b and c.

171. In practical work we proceed as follows:

$$\text{Since } b = \frac{a \cdot \sin B}{\sin A},$$

$$\therefore \log b = \log \frac{a \cdot \sin B}{\sin A}$$

$$\therefore \log b = \log a + \log(\sin B) + 10 - (10 + \log \sin A),$$

or, $$\log b = \log a + L \sin B - L \sin A.$$

Similarly, $\log c = \log a + L \sin C - L \sin A$.

Example. Given that $c = 1764{\cdot}3$ feet, $C = 18^0\,27'$, and $B = 66^0\,39'$, find b.

From the Tables we find $\log 1764{\cdot}3 = 3{\cdot}2465724$.

$$L \sin 18^0\,27' = 9{\cdot}5003421, \quad L \sin 66^0\,39' = 9{\cdot}9628904;$$

$$\therefore \log b = 3{\cdot}2465724 + 9{\cdot}9628904 - 9{\cdot}5003421$$

$$= 3{\cdot}7091207 = \log 5118{\cdot}2;$$

$$\therefore b = 5118{\cdot}2 \text{ feet.}$$

EXAMPLES. XLVIII.

1. If $A = 53^0\,24'$, $B = 66^0\,27'$, $c = 338{\cdot}65$ yards, find C and a, having given that

$$L \sin 53^0\,24' = 9{\cdot}9046168, \quad \log 3{\cdot}3865 = {\cdot}5297511,$$
$$L \sin 60^0\,\ 9' = 9{\cdot}9381851, \quad \log 3{\cdot}1346 = {\cdot}4961821,$$
$$\log 3{\cdot}1347 = {\cdot}4961960.$$

2. If $A = 48^0$, $B = 54^0$, and $c = 38$ inches, find a and b, having given that

$$\log 38 = 1{\cdot}5797836, \quad \log 2{\cdot}88704 = {\cdot}4604527,$$
$$\log 3{\cdot}14295 = {\cdot}4973368, \quad L \sin 54^0 = 9{\cdot}9079576,$$
$$L \sin 78^0 = 9{\cdot}9904044, \quad L \sin 48^0 = 9{\cdot}8710735.$$

3. Find c, having given that $a = 1000$ yards, $A = 50^0$, $C = 66^0$, and that

$$L \sin 50^0 = 9{\cdot}8842540, \quad L \sin 66^0 = 9{\cdot}9607302,$$
$$\log 1{\cdot}19255 = {\cdot}0764762.$$

4. Find b, having given that $B = 32^0\,15'$, $C = 21^0\,47'\,20''$, $a = 34$ feet.

$$\log 3{\cdot}4 = {\cdot}531479, \quad L \sin 32^0\,15' = 9{\cdot}727228,$$
$$\log 2{\cdot}241 = {\cdot}350442, \quad L \sin 54^0\,\ 2' = 9{\cdot}908141,$$
$$\log 2{\cdot}242 = {\cdot}350636, \quad L \sin 54^0\,\ 3' = 9{\cdot}908233.$$

5. Find a, b, C, having given $A = 72^0\,4'$, $B = 41^0\,56'\,18''$, $c = 24$ feet.

$$\log 2{\cdot}4 = {\cdot}3802112, \quad L \sin 72^0\,4' = 9{\cdot}9783702,$$
$$\log 1{\cdot}755 = {\cdot}2442771, \quad L \sin 41^0\,56'\,10'' = 9{\cdot}8249725,$$
$$\log 1{\cdot}756 = {\cdot}2445245, \quad L \sin 41^0\,56'\,20'' = 9{\cdot}8249959,$$
$$\log 2{\cdot}4995 = {\cdot}3978531, \quad L \sin 65^0\,59' = 9{\cdot}9606739,$$
$$\log 2{\cdot}4996 = {\cdot}3978701, \quad L \sin 66^0 = 9{\cdot}9607302.$$

Case III.

172. Given **two sides** and the **included angle**, as b, c, A. [Euc. I. 4; VI. 6.]

First, $B+C=180^\circ-A$. Thus $(B+C)$ is determined.

Next,
$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

Thus $(B-C)$ is determined.

And B and C can be found when the values of $(B+C)$ and $(B-C)$ are known.

Lastly,
$$\frac{a}{\sin A}=\frac{b}{\sin B}, \text{ or } a=\frac{b\,.\sin A}{\sin B}.$$

Whence a is determined.

173. In practical work we proceed as follows:

Since $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$,

$$\therefore \log\left(\tan\frac{B-C}{2}\right)+10$$
$$=\log(b-c)-\log(b+c)+\log\left(\cot\frac{A}{2}\right)+10,$$

or, $L\tan\dfrac{B-C}{2}=\log(b-c)-\log(b+c)+L\cot\dfrac{A}{2}$.

Also, since
$$a=\frac{b\,.\sin A}{\sin B},$$

$\therefore\ \log a=\log b+L\sin A-L\sin B$, as in Case II.

Example. Given $b=456\cdot12$ chains, $c=296\cdot86$ chains, and $A=74^\circ 20'$, find the other angles.

Here, $b-c=159\cdot26$, $b+c=752\cdot98$.

From the Table we find

$$\log 159\cdot26=2\cdot2021067, \text{ and } \log 752\cdot98=2\cdot8767834,$$
$$L\cot 37^\circ 10'=10\cdot1202593;$$

$$\therefore L\tan\frac{B-C}{2}=2\cdot2021067-2\cdot8767834+10\cdot1202593$$
$$=9\cdot4455826=L\tan 15^\circ 35' 18''.$$

$$\therefore B-C=31^\circ 10' 36'', \text{ and } B+C=180^\circ-74^\circ 20'.$$

Thus $\quad B+C=105^\circ 40'$;

$$\therefore 2B=136^\circ 50' 36''; \quad 2C=74^\circ 29' 24'',$$

or, $\quad B=68^\circ 25' 18''$; or, $C=37^\circ 14' 42''$.

174. The formula $a^2 = b^2 + c^2 - 2bc \cos A$ may be used in simple cases.

Example. If $b = 35$ feet, $c = 21$ feet, and $A = 50^0$, find a, given that

Case III.

Given	Number Angle	Log
→ b	3.984	
→ a	48.72	1.6877
→ C	68° – 34′	9.9689 log
A	136° – 57′	0.0193 * " "
+ B	111° – 26′	1.6759 (C = 47.4
~~+ b~~	~~52.704~~	~~* a~~ [illegible]
− b	44.736	1.6508
(A + B)	55° – 43′	0.1664
(A – B)	51° – 14′ ←	0.0952
[A	4° – 27′	[C = 47.41
B	136° – 57′	

Given represented by →

* colog.

Case III.

172. Given **two sides** and the **included angle**, as b, c, A.

[Euc. I. 4; VI. 6.]

174. The formula $a^2=b^2+c^2-2bc\cos A$ may be used in simple cases.

Example. If $b=35$ feet, $c=21$ feet, and $A=50^0$, find a, given that $\cos 50^0=\cdot 643$.

Here $$a^2=35^2+21^2-2\times 35\times 21\times \cos 50^0;$$

$$\therefore \frac{a^2}{7^2}=5^2+3^2-2\times 5\times 3\times \cos 50^0,$$

$$=25+9-30\times \cdot 643, \quad =14\cdot 71.$$

$$\frac{a}{7}=3\cdot 82 \text{ nearly; or, } a=26\cdot 74=\text{about } 26\tfrac{3}{4} \text{ feet.}$$

EXAMPLES. XLIX.

1. Find B and C, having given that $A=40^0$, $b=131$, $c=72$.
$\log 5\cdot 9 = \cdot 7708520$, $L\cot 20^0 = 10\cdot 4389341$,
$\log 2\cdot 03 = \cdot 3074960$, $L\tan 38^0 36' = 9\cdot 9021604$,
$L\tan 38^0 37' = 9\cdot 9024195$.

2. Find A and B, having given that $a=35$ feet, $b=21$ feet, $C=50^0$.
$\log 2 = \cdot 301030$, $L\tan 28^0 11' = 9\cdot 729020$,
$L\tan 65^0 = 10\cdot 331327$, $L\tan 28^0 12' = 9\cdot 72923$.

3. If $b=19$ chains, $c=20$ chains, $A=60^0$, find B and C, having given that $\log 3\cdot 9 = \cdot 591065$, $L\tan 2^0 32' = 8\cdot 645853$,
$L\cot 30^0 = 10\cdot 238561$, $L\tan 2^0 33' = 8\cdot 648704$.

4. Given that $a=376\cdot 375$ chains, $b=251\cdot 765$ chains, and $C=78^0 26'$, find A and B. $L\cot 39^0 13' = 10\cdot 0882755$,
$\log 1\cdot 2461 = \cdot 0955529$, $L\tan 13^0 39' = 9\cdot 3853370$,
$\log 6\cdot 2814 = \cdot 7980565$, $L\tan 13^0 40' = 9\cdot 3858876$.

5. If $a=135$, $b=105$, $C=60^0$, find A, having given that
$\log 2 = \cdot 3010300$, $L\tan 12^0 12' = 9\cdot 3348711$,
$\log 3 = \cdot 4771213$, $L\tan 12^0 13' = 9\cdot 3354823$.

6. If $a=21$ chains, $b=20$ chains, $C=60^0$, find c.

7. Find c in the triangle of Example 5.

8. In a triangle the ratio of two sides is 5 : 3 and the included angle is $76^0 30'$. Find the other angles.
$\log 2 = \cdot 3010300$, $L\cot 35^0 15' = 10\cdot 1507464$,
$L\tan 19^0 28' 50'' = 9\cdot 5486864$.

Case IV.

175. Given **two sides** and the **angle opposite** one of them, as b, c, B. [Omitted in Euc. I.; Euc. VI. 7.]

First, since $\frac{c}{\sin C} = \frac{b}{\sin B}$; $\therefore \sin C = \frac{c \sin B}{b}$.

C must be found from this equation.

When C is known, $A = 180^\circ - B - C$,

and, $a = \frac{b \sin A}{\sin B}$.

Which solves the triangle.

176. We remark, however, that the angle C, found from the **trigonometrical** equation *sin* $C =$ *a given quantity*, where C is an angle of a triangle, has **two** values, one less than 90°, and one greater than 90°. [Art. 152.]

The question arises, Are both these values admissible?

This may be decided as follows:

If B is not less than 90°, C must be less than 90°; and the smaller value for C **only** is admissible.

If B is less than 90° we proceed thus.

1. If b is less than $c \sin B$, then $\sin C$, which $= \frac{c \sin B}{b}$, is greater than 1. This is impossible. Therefore if b is less than $c \sin B$, there is **no** solution whatever.

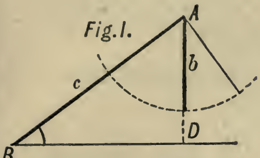
2. If b is equal to $c \sin B$, then $\sin C = 1$, and therefore $C = 90^\circ$; and there is only **one** value of C, viz. 90°.

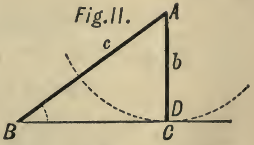
3. If b is greater than $c \sin B$, and less than c, then B is less than C, and C may be obtuse or acute. In this case C may have either of the values found from the equation $\sin C = \frac{c \sin B}{b}$. Hence there are **two** solutions, and the triangle is said to be **ambiguous.**

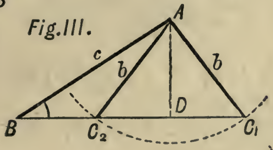
4. If b is equal to or greater than c, then B is equal to or greater than C, so that C must be an acute angle; and the smaller value for C **only** is admissible.

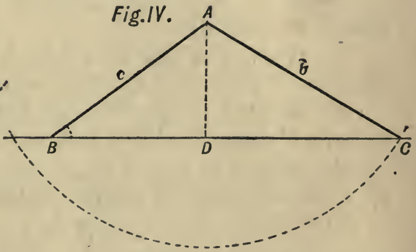
177. The same results may be obtained **geometrically.**

Construction. Draw $AB = c$; make the angle $ABD =$ the given angle B; with centre A and radius $= b$ describe a circle; draw AD perpendicular to BD.









Then $AD = c \sin B$.

1. If b is less than $c \sin B$, i.e. less than AD, the circle will not cut BD at all, and the construction **fails.** (Fig. I.)

2. If b is equal to AD, the circle will *touch* the line BD in the point D, and the required triangle is the **right-angled** triangle ABD. (Fig. II.)

3. If b is greater than AD and less than AB, i.e. than c, the circle will cut the line BD in *two* points C_1, C_2 each on the same side of B. And we get **two** triangles ABC_1, ABC_2 each satisfying the given condition. (Fig. III.)

4. If b is equal to c, the circle cuts BD in B and in one other point C; if b is greater than c the circle cuts BD in *two* points, but on *opposite* sides of B. In either case there is only **one** triangle satisfying the given condition. (Fig. IV.)

178. We may also obtain the same results **algebraically**, from the formula $b^2 = c^2 + a^2 - 2ca \cos B$.

In this b, c, B are given, a is unknown. Write x for a and we get the **quadratic equation**

$$x^2 - 2c\cos B \,.\, x = b^2 - c^2.$$

Whence, $x^2 - 2c\cos B \,.\, x + c^2\cos^2 B = b^2 - c^2 + c^2\cos^2 B$

$$= b^2 - c^2\sin^2 B;$$

$$\therefore\; x = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

Let a_1, a_2 be the two values of x thus obtained, then

$$\left.\begin{aligned} a_1 &= c\cos B + \sqrt{b^2 - c^2\sin^2 B} \\ a_2 &= c\cos B - \sqrt{b^2 - c^2\sin^2 B} \end{aligned}\right\}.$$

Which of these two solutions is admissible may be decided as follows:

1. When b is less than $c\sin B$, then $(b^2 - c^2\sin^2 B)$ is negative, so that a_1, a_2 are **impossible** quantities.

2. When b is equal to $c\sin B$, then $(b^2 - c^2\sin^2 B) = 0$, and $a_1 = a_2$; thus the *two* solutions become **one**.

3. When b is greater than $c\sin B$, then the **two** values a_1, a_2 are different and positive unless

$$\sqrt{b^2 - c^2\sin^2 B} \text{ is } > c\cos B,$$

i.e. unless $b^2 - c^2\sin^2 B > c^2\cos^2 B$,

i.e. unless $b^2 > c^2$.

4. When b is equal to c, then $a_2 = 0$; if b is greater than c, then a_2 is negative and is therefore inadmissible. In either of these cases a_1 is the **only** available solution.

179. We give two examples. In the first there are two solutions, in the second there is only one.

Example 1. Find A and C, having given that $b = 379{\cdot}41$ chains, $c = 483{\cdot}74$ chains, and $B = 34^\circ 11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}6846120 + 9{\cdot}7496148 - 2{\cdot}5791088 \\ &= 9{\cdot}8551180 = L\sin 45^\circ 45'; \end{aligned}$$

$$\therefore\; C = 45^\circ 45', \text{ or, } 180^\circ - 45^\circ 45' = 134^\circ 15'.$$

Since b is less than c, each of these values is admissible.

When $C = 45^\circ 45'$, then $A = 100^\circ 4'$.

When $C = 134^\circ 15'$, then $A = 11^\circ 34'$.

Example 2. Find A and C, when $b=483{\cdot}74$ chains, $c=379{\cdot}14$ chains, and $B=34^{0}\,11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}5791088 + 9{\cdot}7496148 - 2{\cdot}6846120 \\ &= 9{\cdot}6441116 = L\sin 26^{0}\,9'; \end{aligned}$$

$$\therefore\ C=26^{0}\,9',\ \text{or},\ 180^{0}-26^{0}\,9'=153^{0}\,51'.$$

Since b is greater than c, C must be less than 90^{0}, and the larger value for C is inadmissible.

[It is also clear that $(153^{0}\,51'+34^{0}\,11')$ is $> 180^{0}$].

$$\therefore\ C=26^{0}\,9',\ A=119^{0}\,40'.$$

EXAMPLES. L.

1. If $B=40^{0}$, $b=140{\cdot}5$ feet, $a=170{\cdot}6$ feet, find A and C.
$\log 1{\cdot}405={\cdot}1476763$, $L\sin 40^{0}=9{\cdot}8080675$,
$\log 1{\cdot}706={\cdot}2319790$, $L\sin 51^{0}\,18'=9{\cdot}8923342$,
$L\sin 51^{0}\,19'=9{\cdot}8924354$.

2. Find B and C, having given that $A=50^{0}$, $b=119$ chains, $a=97$ chains, and that $\log 1{\cdot}19={\cdot}075547$, $L\sin 50^{0}=9{\cdot}884254$,
$\log 9{\cdot}7={\cdot}986772$, $L\sin 70^{0}=9{\cdot}972986$,
$L\sin 70^{0}\,1'=9{\cdot}973032$.

3. Find B, C, and c, having given that $A=50^{0}$, $b=97$, $a=119$ (see Example 2). $\log 1{\cdot}553={\cdot}191169$, $L\sin 38^{0}\,38'\,24''=9{\cdot}795479$,
$L\sin 88^{0}\,37'\,24''=9{\cdot}999876$.

4. Find A, having given that $a=24$, $c=25$, $C=65^{0}\,59'$, and that
$\log 2{\cdot}5={\cdot}3979400$, $L\sin 65^{0}\,59'=9{\cdot}9606739$,
$\log 2{\cdot}4={\cdot}3802112$, $L\sin 61^{0}\,16'=9{\cdot}9429335$,
$L\sin 61^{0}\,17'=9{\cdot}9430028$.

5. If $a=25$, $c=24$, and $C=65^{0}\,59'$, find A, B and the greater value of b. $\log 1{\cdot}755={\cdot}2442771$, $L\sin 72^{0}\,4'=9{\cdot}9783702$,
$\log 1{\cdot}756={\cdot}2445245$, $L\sin 72^{0}\,5'=9{\cdot}9784111$,
$L\sin 41^{0}\,56'\,10''=9{\cdot}8249725$,
$L\sin 41^{0}\,56'\,26''=9{\cdot}8249959$ (see Example 4.)

6. Supposing the data for the solution of a triangle to be as in the three following cases (α), (β), (γ), point out whether the solution will be ambiguous or not, and find the third side in the obtuse-angled triangle in the ambiguous case:

(α) $A=30^{0}$, $a=125$ feet, $c=250$ feet,
(β) $A=30^{0}$, $a=200$ feet, $c=250$ feet,
(γ) $A=30^{0}$, $a=200$ feet, $c=125$ feet.

$\log 2={\cdot}3010300$, $L\sin 38^{0}\,41'=9{\cdot}7958800$,
$\log 6{\cdot}0389={\cdot}7809578$, $L\sin 8^{0}\,41'=9{\cdot}1789001$,
$\log 6{\cdot}0390={\cdot}7809650$.

180. In the following Examples the student must find the necessary logarithms etc. from the Tables.

MISCELLANEOUS EXAMPLES. LI.

1. Find A when $a=374\cdot5$, $b=576\cdot2$, $c=759\cdot3$ feet.
2. Find B when $a=4001$, $b=9760$, $c=7942$ yards.
3. Find C when $a=8761\cdot2$, $b=7643$, $c=4693\cdot8$ chains.
4. Find B when $A=86^\circ\,19'$, $b=4930$, $c=5471$ chains.
5. Find C when $B=32^\circ\,58'$, $c=1873\cdot5$, $a=764\cdot2$ chains.
6. Find c when $C=108^\circ\,27'$, $a=36541$, $b=89170$ feet.
7. Find c when $B=74^\circ\,10'$, $C=62^\circ\,45'$, $b=3720$ yards.
8. Find b when $B=100^\circ\,19'$, $C=44^\circ\,59'$, $a=1000$ chains.
9. Find a when $B=123^\circ\,7'\,20''$, $C=15^\circ\,9'$, $c=9964$ yards.

Find the other two angles in the six following triangles.

10. $C=100^\circ\,37'$, $b=1450$, $c=6374$ chains.
11. $C=52^\circ\,10'$, $b=643$, $c=872$ chains.
12. $A=76^\circ\,2'\,30''$, $b=1000$, $a=2000$ chains.
13. $C=54^\circ\,23'$, $b=873\cdot4$, $c=752\cdot8$ feet.
14. $C=18^\circ\,21'$, $b=674\cdot5$, $c=269\cdot7$ chains.
15. $A=29^\circ\,11'\,43''$, $b=7934$, $a=4379$ feet.
16. The difference between the angles at the base of a triangle is $17^\circ\,48'$, and the sides subtending those angles are $105\cdot25$ feet and $76\cdot75$ feet; find the third angle.
17. If $b:c=4:5$, $a=1000$ yards and $A=37^\circ\,19'$, find b.

The student will find some Examples of Solution of Triangles without the aid of logarithms, in an Appendix.

CHAPTER XV.

On the Measurement of Heights and Distances.

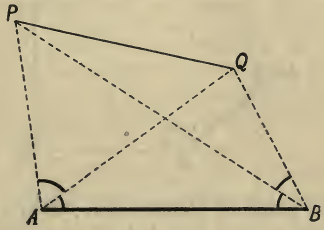
181. We have said (Art. 58) that the measurement, with scientific accuracy, of a line of any considerable length involves a long and difficult process.

On the other hand, sometimes it is required to find the *direction* of a line that it may point to an object which is not visible from the point from which the line is drawn. As, for example, when a tunnel has to be constructed.

By the aid of the Solution of Triangles
we can find the length of the distance between points which are inaccessible;
we can calculate the magnitude of angles which cannot be practically observed;
we can find the relative heights of distant and inaccessible points.

The method on which the Trigonometrical Survey of a country is conducted affords the following illustration.

182. *To find the distance between two distant objects.*



Two convenient positions A and B, on a level plain as far apart as possible, having been selected, the distance between A and B is measured with the greatest possible care. This line AB is called the **base** line. (In the survey of England, the base line is on Salisbury Plain, and is about 36,578 feet long.)

Next, the two distant objects, P and Q (church spires, for instance), visible from A and B, are chosen.

The angles PAB, PBA are observed. Then by Case II. Chapter XIV, the lengths of the lines PA, PB are calculated.

Again, the angles QAB, QBA are observed; and by Case II. the lengths of QA and QB are calculated.

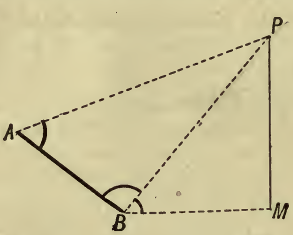
Thus the lengths of PA and QA are found.

The angle PAQ is observed; and then by Case III. the length of PQ is calculated.

183. Thus the distance between two points P and Q has been found. The points P and Q are not necessarily accessible; the only condition being that P and Q must be *visible* from both A and B.

184. In practice, the points P and Q will generally be accessible, and then the line PQ, whose length has been calculated, may be used as a new base to find other distances.

185. *To find the height of a distant object above the point of observation.*



Let B be the point of observation; P the distant object. From B measure a base line BA of any convenient length, in any convenient direction; observe the angles PAB, PBA, and by Case II. calculate the length of BP. Next observe at B the 'angle of elevation' of P; that is, the angle which the line BP makes with the horizontal line BM, M being the point in which the vertical line through P cuts the horizontal plane through B.

Then PM, which is the vertical height of P above B can be calculated, for $PM = BP . \sin MBP$.

Example 1. *The distance between a church spire* A *and a milestone* B *is known to be* 1764·3 *feet; C is a distant spire. The angle* CAB *is* 94° 54′, *and the angle* CBA *is* 66° 39′. *Find the distance of* C *from* A.

ABC is a triangle, and we know one side c and two angles (A and B), and therefore it can be solved by Case II.

The angle $ACB = 180^0 - 94^0\,54' - 66^0\,39'$

$= 18^0\,27'$.

Therefore the triangle is the same as that solved on page 115. Therefore $AC = 5118{\cdot}2$ feet.

Example 2. *If the spire* C *in the last Example stands on a hill, and the angle of elevation of its highest point is observed at* A *to be* $4^0\,19'$; *find how much higher* C *is than* A.

The required height $x = AC \,.\, \sin 4^0\,19'$ and AC is 5118·2 feet,

$$\therefore \log x = \log (AC \,.\, \sin 4^0\,19')$$
$$= \log 5118{\cdot}2 + L \sin 4^0\,19' - 10$$
$$= 3{\cdot}7091173 + 8{\cdot}8766150 - 10$$
$$= 2{\cdot}5857323 = \log 385{\cdot}24.$$

Therefore $x = 385$ ft. 3 in. nearly.

EXAMPLES. LII.

(Exercises x. and xliii. consist of easy Examples on this subject.)

1. Two straight roads inclined to one another at an angle of 60^0, lead from a town A to two villages B and C; B on one road distant 30 miles from A, and C on the other road distant 15 miles from A. Find the distance from B to C. *Ans.* 25·98 m.

2. Two ships leave harbour together, one sailing N.E. at the rate of $7\frac{1}{2}$ miles an hour and the other sailing North at the rate of 10 miles an hour. Prove that the distance between the ships after an hour and a half is 10·6 miles.

3. A and B are two consecutive milestones on a straight road and C is a distant spire. The angles ABC and BAC are observed to be 120^0 and 45^0 respectively. Show that the distance of the spire from A is 3·346 miles.

4. If the spire C in the last question stands on a hill, and its angle of elevation at A is 15^0, show that it is ·8[illegible]6 of a mile higher than A.

,866

5. If in Question (3) there is another spire D such that the angles DBA and DAB are 45^0 and 90^0 respectively and the angle DAC is 45^0; prove that the distance from C to D is $2\frac{3}{4}$ miles very nearly.

6. A and B are two consecutive milestones on a straight road, and C is the chimney of a house visible from both A and B. The angles CAB and CBA are observed to be $36^0\,18'$ and $120^0\,27'$ respectively. Show that C is 2639·5 yards from B,

$\log 1760 = 3{\cdot}2455127$	$L \sin 36^0\,18' = 9{\cdot}7723314$
$\log 2639{\cdot}5 = 3{\cdot}42152$	$L \operatorname{cosec} 23^0\,15' = 10{\cdot}4036846.$

7. A and B are two points on opposite sides of a mountain, and C is a place visible from both A and B. It is ascertained that C is distant 1794 feet and 3140 feet from A and B respectively and the angle ACB is $58^0\,17'$. Show that the angle which the line pointing from A to B makes with AC is $86^0\,55'\,49''$,

$\log 1346 = 3{\cdot}1290451$	$L \cot 29^0\,8'\,30'' = 10{\cdot}2537194$
$\log 4934 = 3{\cdot}6931991$	$L \tan 26^0\,4'\,19'' = 9{\cdot}6895654.$

8. A and B are two hill-tops 34920 feet apart, and C is the top of a distant hill. The angles CAB and CBA are observed to be $61^0\ 53'$ and $76^0\ 49'$ respectively. Prove that the distance from A to C is 51515 feet,

$$\log 34920 = 4{\cdot}5430742 \qquad L \sin 76^0\ 49' = 9{\cdot}9884008$$
$$\log 51515 = 4{\cdot}71193 \qquad L \operatorname{cosec} 41^0\ 18' = 10{\cdot}1804552.$$

9. From two stations A and B on shore, 3742 yards apart, a ship C is observed at sea. The angles BAC, ABC are simultaneously observed to be $72^0\ 34'$ and $81^0\ 41'$ respectively. Prove that the distance from A to the ship is 8522·7 yards,

$$\log 3742 = 3{\cdot}5731038 \qquad L \sin 81^0\ 41' = 9{\cdot}9954087$$
$$\log 8522{\cdot}7 = 3{\cdot}9005774 \qquad L \operatorname{cosec} 25^0\ 45' = 10{\cdot}3620649.$$

10. The distance between two mountain peaks is known to be 4970 yards, and the angle of elevation of one of them when seen from the other is $9^0\ 14'$. How much higher is the first than the second? $\sin 9^0 14' = {\cdot}1604555$. *Ans.* 797·5 yards.

11. Two straight railways intersect at an angle of 60^0. From their point of intersection two trains start, one on each line, one at the rate of 40 miles an hour. Find the rate of the second train that at the end of an hour they may be 35 miles apart. *Ans.* Either 25 or 15 miles an hour. (Art. 264.)

12. A and B are two positions on opposite sides of a mountain; C is a point visible from A and B; AC and BC are 10 miles and 8 miles respectively, and the angle BCA is 60^0. Prove that the distance between A and B is 9·165 miles.

13. In the last question, if the angle of elevation of C at A is 8^0, and at B is $2^0\ 48'\ 24''$: show that the height of A above B is one mile very nearly.

$$\sin 8^0 = {\cdot}1391731 \quad \sin 2^0\ 48'\ 24'' = {\cdot}0489664.$$

14. Show that the angles which a tunnel going through the mountain from A to B, in Questions 12 and 13, would make (i) with the horizon, (ii) with the line joining A and C, are respectively $6^0\ 16'$ and $49^0\ 6'\ 24''$.

$$\sin 6^0\ 16' = {\cdot}1091; \ \tan 10^0\ 53'\ 36'' = {\cdot}192450.$$

15. A and B are consecutive milestones on a straight road; C is the top of a distant mountain. At A the angle CAB is observed to be $38^0\ 19'$; at B the angle CBA is observed to be $132^0\ 42'$, and the angle of elevation of C at B is $10^0 15'$. Show that the top of the mountain is 1243·5 yards higher than B.

$$L \sin 38^0\ 19' = 9{\cdot}7923968 \qquad \log 1760 = 3{\cdot}2455127$$
$$L \operatorname{cosec} 8^0\ 59' = 10{\cdot}8064659 \qquad \log 1243{\cdot}5 = 3{\cdot}09465$$
$$L \sin 10^0 15' = 9{\cdot}2502822.$$

16. A base line AB, 1000 feet long, is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC and CBA are observed to be $65^0\ 37'$ and $53^0\ 4'$ respectively.

Prove that the perpendicular breadth of the river at C is 829·87 feet; having given

$L \sin 65^\circ 37' = 9{\cdot}9594248, \quad L \sin 53^\circ 4' = 9{\cdot}9027289$

$L \operatorname{cosec} 61^\circ 19' = 10{\cdot}0568589, \quad \log 8{\cdot}2987 = {\cdot}91901.$

MISCELLANEOUS EXAMPLES. LIII.

1. A man walking along a straight road at the rate of three miles an hour sees, in front of him at an elevation of 60° a balloon which is travelling horizontally in the same direction at the rate of six miles an hour; ten minutes after he observes that the elevation is 30°. Prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

2. A person standing at a point A, due south of a tower built on a horizontal plain, observes the altitude of the tower to be 60°. He then walks to a point B due west from A and observes the altitude to be 45°, and then at the point C in AB produced he observes the altitude to be 30°. Prove that $AB = BC$.

3. The angle of elevation of a balloon, which is ascending uniformly and vertically, when it is one mile high is observed to be $35^\circ 20$; 20 minutes later the elevation is observed to be $55^\circ 40'$. How fast is the balloon moving?

Ans. $3 (\sin 20^\circ 20') (\sec 55^\circ 40') (\operatorname{cosec} 35^\circ 20')$ miles per hour.

4. The angular elevation of a tower at a place A due south of it is 30°; and at a place B due west of A, and at a distance a from it, the elevation is 18°; show that the height of the tower is

$$a\{2 + 2\sqrt{5}\}^{-\frac{1}{2}}.$$

5. The angular elevation of the top of a steeple at a place due south of it is 45°, and at another place due west of the former station and distant a feet from it the elevation is 15°; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$ feet.

6. A tower stands at the foot of an inclined plane whose inclination to the horizon is 9°; a line is measured up the incline from the foot of the tower of 100 feet in length. At the upper extremity of this line the tower subtends an angle of 54°. Find the height of the tower. *Ans.* 114·4 ft.

7. The altitude of a certain rock is observed to be 47°, and after walking 1000 feet towards the rock, up a slope inclined at an angle of 32° to the horizon, the observer finds that the altitude is 77°. Prove that the vertical height of the rock above the first point of observation is 1034 ft. $\sin 47^\circ = {\cdot}73135$.

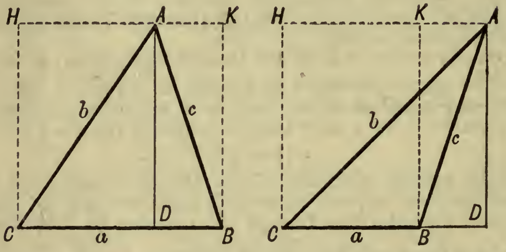
8. At the top of a chimney 150 feet high standing at one corner of a triangular yard, the angle subtended by the adjacent sides of the yard are 30° and 45° respectively; while that subtended by the opposite side is 30°. Show that the lengths of the sides are 150 ft. 86·6 ft. and 106 ft. respectively.

CHAPTER XVI.

ON TRIANGLES AND CIRCLES.

186. *To find the* **Area** *of a Triangle.*

The area of the triangle ABC is denoted by Δ.



Through A draw HK parallel to BC, and through ABC draw lines AD, BK, CH perpendicular to BC.

The area of the triangle ABC is half that of the rectangular parallelogram $BCHK$ [Euc. I. 41].

$$\text{Therefore } \Delta = \frac{BC \cdot CH}{2} = \frac{BC \cdot DA}{2}$$

$$= \frac{a \cdot b \sin C}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But $\quad \sin C = \dfrac{2}{ab} \cdot \sqrt{s(s-a)(s-b)(s-c)}$;

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)} = S \ \ldots\ldots\text{(ii)}.$$

187. *To find the Radius of the* **Circumscribing Circle.**

Let a circle $AA'CB$ be described about the triangle ABC. Let R stand for its radius. Let O be its centre. Join BO, and produce it to cut the circumference in A'. Join $A'C$.

Then, Fig. I. the angles BAC, $BA'C$ in the same segment are equal; Fig. II. the angles BAC, $BA'C$ are supplementary; also the angle BCA' in a semicircle is a right angle.

Fig. I.

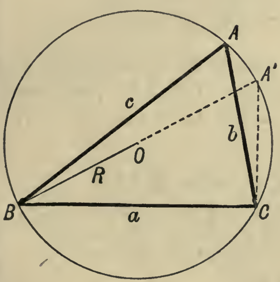
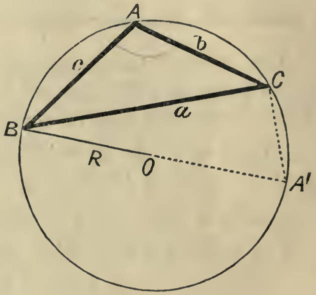


Fig. II.



Therefore $\dfrac{CB}{A'B} = \sin CA'B = \sin CAB = \sin A$,

or, $\dfrac{a}{2R} = \sin A$; $\quad \therefore\ 2R = \dfrac{a}{\sin A}$.

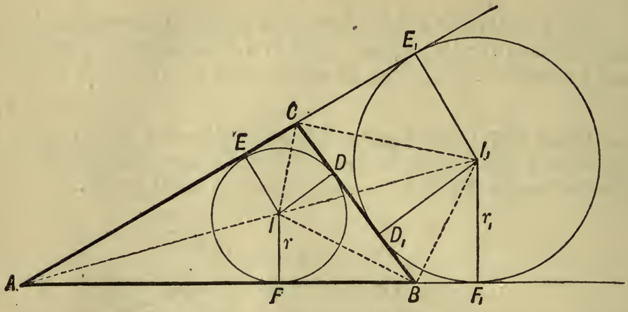
188. Similarly, it may be proved that

$$2R = \frac{b}{\sin B}; \text{ and that } 2R = \frac{c}{\sin C}.$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$

Thus d, the value of each of these fractions, is the diameter of the circumscribing circle.

189. *To find the radius of the* **Inscribed Circle.**



Let D, E, F be the points in which the circle inscribed in the triangle ABC touches the sides. Let I be the centre of the circle; let r be its radius. Then $ID = IE = IF = r$.

The area of the triangle ABC

$$= \text{area of } IBC + \text{area of } ICA + \text{area of } IAB.$$

And the area of the triangle $IBC = \frac{1}{2}ID \,.\, BC = \frac{1}{2}r \,.\, a$,

$$\therefore \text{ area of } ABC = \tfrac{1}{2}ID \,.\, BC + \tfrac{1}{2}IE \,.\, CA + \tfrac{1}{2}IF \,.\, AB$$

$$= \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc;$$

or,

$$\Delta = \tfrac{1}{2}r\,(a + b + c) = \tfrac{1}{2}r \,.\, 2s = rs.$$

$$\therefore \quad r = \frac{\Delta}{s} = \frac{S}{s}.$$

190. A circle which touches one of the sides of a triangle and the other two sides produced is called an **Escribed Circle** of the triangle.

191. *To find the radius of an* **Escribed Circle.**

Let an escribed circle touch the side BC and the sides AC, AB produced in the points D_1, E_1, F_1 respectively. Let I_1 be its centre, r_1 its radius. Then

$$I_1D_1 = I_1E_1 = I_1F_1 = r_1.$$

The area of the triangle ABC

$$= \text{area of } ABI_1C - \text{area of } I_1BC,$$

$$= \text{area of } I_1CA + \text{area of } I_1AB - \text{area of } I_1BC,$$

or $$\Delta = \tfrac{1}{2}I_1E_1 . CA + \tfrac{1}{2}I_1F_1 . AB - \tfrac{1}{2}I_1D_1 . BC$$
$$= \tfrac{1}{2}r_1 b + \tfrac{1}{2}r_1 c - \tfrac{1}{2}r_1 a$$
$$= \tfrac{1}{2}r_1 (b + c - a) = \tfrac{1}{2}r_1 (2s - 2a) = r_1 (s - a).$$
$$\therefore\ r_1 = \frac{\Delta}{s-a} = \frac{S}{s-a}.$$

192. Similarly if r_2 and r_3 be the radii of the other two escribed circles of the triangle ABC, then

$$r_2 = \frac{S}{s-b};\quad r_3 = \frac{S}{s-c}.$$

EXAMPLES. LIV.

(1) Find the area of the triangle ABC when

(i) $a=4$, $b=10$ feet, $C=30^0$.
(ii) $b=5$, $c=20$ inches, $A=60^0$.
(iii) $c=66\frac{2}{3}$, $a=15$ yards, $b=17^0\,14'$ [$\sin 17^0\,14' = \cdot 29626$].
(iv) $a=13$, $b=14$, $c=15$ chains.
(v) $a=10$, the perpendicular from A on $BC=20$ feet.
(vi) $a=625$, $b=505$, $c=904$ yards.

(2) Find the Radii of the Inscribed and each of the Escribed Circles of the triangle ABC when $a=13$, $b=14$, $c=15$ feet.

(3) Show that the triangles in which (i) $a=2$, $A=60^0$; (ii) $b=\frac{2}{3}.\sqrt{3}$, $B=30^0$ can be inscribed in the same circle.

(4) Prove that $R=\dfrac{abc}{4S}$; find R in the triangle of (2).

(5) Prove that if a series of triangles of equal perimeter are described about the same circle, they are equal in area.

(6) If $A=60^0$, $a=\sqrt{3}$, $b=\sqrt{2}$, prove that the area $=\frac{1}{4}(3+\sqrt{3})$.

(7) Prove that each of the following expressions represents the area of the triangle ABC:

(i) $\dfrac{abc}{4R}$. (ii) $2R^2 \sin A . \sin B . \sin C$.
(iii) rs. (iv) $Rr(\sin A + \sin B + \sin C)$.
(v) $\frac{1}{2}a^2 \sin B . \sin C . \operatorname{cosec} A$. (vi) $ra \operatorname{cosec} \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.
(vii) $(rr_1r_2r_3)^{\frac{1}{2}}$. (viii) $\frac{1}{2}(a^2 - b^2)\sin A . \sin B . \operatorname{cosec}(A - B)$.

Prove the following statements:

(8) If a, b, c are in A.P., then $ac=6rR$.

(9) The area of the greatest triangle, two of whose sides are 50 and 60 feet, is 1500 sq. feet.

(10) If the altitude of an isosceles triangle is equal to the base, R is five-eighths of the base.

EXAMPLES FOR EXERCISE. LV.

1. Define the terms sine, cotangent; and prove that if A be *any* angle, $$\sin^2 A + \cos^2 A = 1.$$

If $\tan A = \frac{3}{4}$, find $\sin A$ and $\cos A$.

2. Find the sine, cosine and tangent of 30^0.

In a triangle ABC the angle C is a right angle, the angle A is 60^0, and the length of the perpendicular let fall from C on AB is 20 feet; find the length of AB.

3. Prove geometrically that $\cos(180^0 - A) = -\cos A$.

Find A if $2 \sin A = \tan A$.

4. Prove

(1) $$\sin(A+B) \,.\, \sin(A-B) = \sin^2 A - \sin^2 B;$$

(2) $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

5. Prove that
$$\cos^2 A - \cos A \cos(60^0 + A) + \sin^2(30^0 - A) = \frac{3}{4}.$$

6. Find the greatest side of the triangle of which one side is 2183 feet and the adjacent angles are $78^0\,14'$ and $71^0\,24'$.

$\log 2183 = 3{\cdot}3390537$, $\log 42274 = 4{\cdot}6260733$,
$L \sin 78^0\,14' = 9{\cdot}9907766$, $\log 42275 = 4{\cdot}6260836$.
$L \sin 30^0\,22' = 9{\cdot}7037486$,

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7. Express the other trigonometrical ratios in terms of the cosine.

8. Prove
$$\sin(180 + A) = -\sin A;$$
$$\tan(90 + A) = -\cot A.$$

9. Write down the sines of all the angles which are multiples of 30^0 and less than 360^0.

10. Prove
$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

11. If $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$, when A is less than 90^0.

If $\sin A = \frac{4}{5}$, prove that $\tan A + \sec A = 3$, when A is less than 90^0.

12. The length of the greatest side of a triangle is 1035·43 feet, and the three angles are 44^0, 66^0, and 70^0. Solve the triangle, having given

$L \sin 44^0 = 9{\cdot}8417713$, $L \sin 66^0 = 9{\cdot}9607302$,
$L \sin 70^0 = 9{\cdot}9729858$, $\log 1035{\cdot}43 = 3{\cdot}0151212$,
$\log 765432 = 5{\cdot}8839067$, $\log 10066 = 4{\cdot}0028656$.

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13. Express the other trigonometrical ratios in terms of the cotangent.

14. Prove that
$$\cos(180^\circ - A) = -\cos A;$$
$$\operatorname{cosec}(180^\circ + A) = -\operatorname{cosec} A.$$

15. Write down the tangents of all the angles which are multiples of 30° and less than 360°.

16. If $\tan A + \sec A = 3$, prove that $\sin A = \frac{4}{5}$, when A is less than 90°.

If $\sin A = \frac{3}{5}$, prove that $\tan A + \sec A = 2$, when A is less than 90°.

17. Find the sines of the three angles of the triangle whose sides are 193, 194, and 195 feet.

18. Investigate the following formulæ:

(1) $$\cos\frac{3A}{2} = (2\cos A - 1)\cos\tfrac{1}{2}A;$$

(2) $$\cos\theta - \cos(\theta + \delta) = \sin\theta\sin\delta\,(1 + \cot\theta\tan\tfrac{1}{2}\delta).$$

---

19. Define the secant of an angle.

Prove the formula $\dfrac{1}{\sec^2 A} + \dfrac{1}{\operatorname{cosec}^2 A} = 1.$

If $\sin A = \frac{1}{3}$, find $\sec A$.

20. Find the logarithms of $\sqrt{(32)}$ and of $\cdot 03125$ to the base $\sqrt[3]{2}$.

21. Express the sine, cosine, and tangent of each of the angles $1962^\circ, 2376^\circ, 2844^\circ$, in terms of the trigonometrical functions of angles lying between 0 and 45°.

22. Prove the formula to express the cosine of the sum of two angles in terms of the sines and cosines of those angles.

Express $\cos 5\alpha$ in terms of $\cos\alpha$.

23. Find solutions of the equations

(i) $\sec\theta\operatorname{cosec}\theta - \cot\theta = \sqrt{3}$;

(ii) $\sin 2\theta - \sin\theta = \cos 2\theta + \cos\theta$.

24. A ring 10 inches in diameter is suspended from a point 1 foot above its centre by six equal strings attached to its circumference at equal intervals; find the cosine of the angle between two consecutive strings.

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25. Define 1°. Assuming that $\frac{22}{7}$ is the circular measure of two right angles, express the angle A° in circular measure.

Find the number of degrees in the angle whose circular measure is $\cdot 1$.

26. Find the trigonometrical ratios of the angle whose cosine is $\frac{3}{5}$.

27. Prove that

$$(1)\quad \cos(180^0+A)=\cos(180^0-A);$$
$$(2)\quad \tan(90^0+A)=\cot(180^0-A).$$

28. Prove $\sin x(2\cos x-1)=2\sin\frac{x}{2}\cos\frac{3x}{2}$.

29. Express $\log_{10}5{\cdot}832$, $\log_{10}\sqrt[3]{(35)}$ and $\log_{10}{\cdot}304\dot{8}$ in terms of $\log_{10}2$, $\log_{10}3$, $\log_{10}7$.

30. If the angle opposite the side a be 60^0, and if b, c be the remaining sides of the triangle, prove that

$$(a+b+c)(b+c-a)=3bc.$$

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31. Assuming $\frac{22}{7}$ to be the circular measure of two right angles, express in degrees the angle whose circular measure is θ. Find the number of degrees in an angle whose circular measure is $\frac{1}{3}$.

32. Shew from the definitions of the trigonometrical function that $\sin^2A+\cot^2A+\cos^2A=\operatorname{cosec}^2A$.

Prove that $$\frac{\tan A+\sec A+1}{\tan A+\sec A-1}=\frac{\sec A+1}{\tan A}.$$

33. Prove $\sin x(2\cos x+1)=2\cos\frac{x}{2}\sin\frac{3x}{2}$.

34. Find the logarithms of $\sqrt{(27)}$ and ${\cdot}\dot{0}3\dot{7}$ to the base $\sqrt[3]{3}$.

35. If $(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)=3\sin A\sin B$, and $A+B+C=180^0$, prove that $C=60^0$.

36. Given $A=18^0$, $B=144^0$, and $b=1$, solve the triangle.

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37. Give the trigonometrical definition of an angle.

What angle does the minute-hand of a clock describe between twelve o'clock and 20 minutes to four?

38. Express the cosine and the tangent of an angle in terms of the sine.

The angle A is greater than 90^0 but less than 180^0, and $\sin A=\frac{1}{3}$. Find $\cos A$.

39. Find all the values of θ between 0 and 2π for which

$$\cos\theta+\cos 2\theta=0.$$

40. If in a triangle $a\cos A=b\cos B$, the triangle will be either isosceles or right-angled.

41. The sides are 1 foot and $\sqrt{3}$ feet respectively, and the angle opposite to the shorter side is 30^0; solve the triangle.

42. The sides of a triangle are 2, 3, 4. Find the greatest angle, having given

$$\log 2={\cdot}3010300,$$
$$\log 3={\cdot}4771213,$$
$$L\tan 52^0.15'=10{\cdot}1111004,$$
$$L\tan 52^0.14'=10{\cdot}1108395.$$

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43. Distinguish between Euclid's definition of an angle and the trigonometrical definition.

What angle does the minute-hand of a clock describe between half-past four and a quarter-past six?

44. Express the sine and the cosine of an angle in terms of the tangent.

The angle A is greater than 180^0 but less than 270^0, and $\tan A = \frac{1}{2}$. Find $\sin A$.

45. Prove (i) $\sin 2A = \dfrac{2 \cot A}{1 + \cot^2 A}$.

(ii) Show that if $A + B + C = 90^0$,

$$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

46. Find all the values of θ between 0 and 2π for which

$$\sin \theta + \sin 2\theta = 0.$$

47. If in a triangle $b \cos A = a \cos B$, show that the triangle is isosceles.

48. The sides are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is 30^0; solve the triangle.

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49. Express in degrees, minutes, etc. (1) the angle whose circular measure is $\frac{1}{20}\pi$; (2) the angle whose circular measure is 5.

If the angle subtended at the centre of a circle by the side of a regular heptagon be the unit of angular measurement, by what number is an angle of 45^0 represented?

50. Prove that

$$(\sin 30^0 + \cos 30^0)(\sin 120^0 + \cos 120^0) = \sin 30^0.$$

51. Prove the formulæ:

(1) $\cos^2(\alpha + \beta) - \sin^2\alpha = \cos\beta \cos(2\alpha + \beta)$;

(2) $1 + \cot\alpha \cot\frac{1}{2}\alpha = \operatorname{cosec}\alpha \cot\frac{1}{2}\alpha$.

52. Find solutions of the equations:

(1) $5 \tan^2 x - \sec^2 x = 11$; (2) $\sin 5\theta - \sin 3\theta = \sqrt{2} . \cos 4\theta$.

53. Two sides of a triangle are 10 feet and 15 feet in length, and the angle between them is 30^0. What is its area?

54. Given that

$$\sin 40^0 29' = 0{\cdot}6492268, \quad \sin 40^0 30' = 0{\cdot}6494480,$$

find the angle whose sine is 0·6493000.

---

55. Express in circular measure (1) 10′, (2) $\frac{1}{8}$ of a right angle.

If the angle subtended at the centre of a circle by the side of a regular pentagon be the unit of angular measurement, by what number is a right angle represented?

56. If $\sec \alpha = 7$, find $\tan \alpha$ and $\operatorname{cosec} \alpha$.

57. Prove the formulæ:

(1) $\cos^2(\alpha - \beta) - \sin^2(\alpha + \beta) = \cos 2\alpha \cos 2\beta$;

(2) $1 + \tan \alpha \tan \frac{1}{2}\alpha = \sec \alpha$.

58. Find solutions of the equations:

(1) $5 \tan^2 x + \sec^2 x = 7$; (2) $\cos 5\theta + \cos 3\theta = \sqrt{2} \cdot \cos 4\theta$.

59. The lengths of the sides of a triangle are 3 feet, 5 feet, and 6 feet. What is its area?

60. Given that

$$\sin 38^0\, 25' = 0{\cdot}6213757, \quad \sin 38^0\, 26' = 0{\cdot}6216036,$$

find the angle whose sine is (0·6215000).

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61. Which is greater, 76^g or $1{\cdot}2^c$? [Art. 32.]

62. Determine geometrically $\cos 30^0$ and $\cos 45^0$.

If $\sin A$ be the arithmetic mean between $\sin B$ and $\cos B$, then

$$\cos 2A = \cos^2(B + 45^0).$$

63. Establish the following relations:

(1) $\tan^2 A - \sin^2 A = \tan^2 A \sin^2 A$;

(2) $\cot A - \cot 2A = \operatorname{cosec} 2A$;

(3) $\dfrac{\sin(x + 3y) + \sin(3x + y)}{\sin 2x + \sin 2y} = 2 \cos(x + y)$.

64. Express $\log_{10} \sqrt{(28)}$, $\log_{10} 3{\cdot}888$, $\log_{10} {\cdot}174\dot{2}$ in terms of $\log_{10} 3$, $\log_{10} 5$, $\log_{10} 7$.

65. Prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$, and deduce the expression for $\cos(A + B)$.

Show that

$$\sin A \cos(B + C) - \sin B \cos(A + C) = \sin(A - B) \cos C.$$

66. One side of a triangular lawn is 102 feet long, its inclinations to the other sides being $70^0\, 30'$, $78^0\, 10'$ respectively. Determine the other sides and the area. $L \sin 70^0\, 30' = 9{\cdot}974$, $\log 102 = 2{\cdot}009$, $L \sin 78^0\, 10' = 9{\cdot}990$, $\log 185 = 2{\cdot}267$, $L \sin 31^0\, 20' = 9{\cdot}716$, $\log 192 = 2{\cdot}283$, $\log 2 = {\cdot}301$, $\log 9234 = 3{\cdot}965$.

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67. Which is greater, 126^0 or the angle whose circular measure is 2·3?

68. Establish the following relations:

(1) $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$;

(2) $\tan A + \cot 2A = \operatorname{cosec} 2A$;

(3) $\dfrac{\cos(x - 3y) - \cos(3x - y)}{\sin 2x + \sin 2y} = 2 \sin(x - y)$.

69. Given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 9 = \cdot 9542425$; find without using tables, $\log_{10} 5$, $\log_{10} 6$, $\log_{10} \cdot 0216$ and $\log_{10} \sqrt[5]{(\cdot 375)}$.

70. Prove that $\sin 30^0 + \sin 120^0 = \sqrt{2} \cos 15^0$.

71. Establish the identities:

(1) $1 + \cos A + \sin A = \sqrt{2(1 + \cos A)(1 + \sin A)}$;

(2) $\operatorname{cosec} 2A = \dfrac{\operatorname{cosec}^2 A}{2\sqrt{\operatorname{cosec}^2 A - 1}}$;

(3) $\sin \dfrac{2\pi}{7} + \sin \dfrac{4\pi}{7} - \sin \dfrac{6\pi}{7} = 4 \sin \dfrac{\pi}{7} \sin \dfrac{3\pi}{7} \sin \dfrac{5\pi}{7}$.

72. The sides of a triangular lawn are 102, 185, and 192 feet in length, the smallest angle being approximately $31^0 20'$. Find its other angles and its area.

$\log 102 = 2 \cdot 009$, $\quad L \sin 31^0 20' = 9 \cdot 716$,
$\log 185 = 2 \cdot 267$, $\quad L \sin 70^0 30' = 9 \cdot 974$,
$\log 192 = 2 \cdot 283$, $\quad L \sin 78^0 10' = 9 \cdot 990$,
$\log 2 = \cdot 301$, $\log 9234 = 3 \cdot 965$.

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73. If the circumference of a circle be divided into five parts in arithmetical progression, the greatest part being six times the least, express in radians the angle each subtends at the centre.

74. Define the sine of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\quad \sin(90^0 + A) = \cos A$,
and $\quad \cos(90^0 + A) = -\sin A$,

and by means of these deduce the formulæ

$$\sin(180^0 + A) = -\sin A, \quad \cos(180^0 + A) = -\cos A.$$

75. Prove the formulæ:

(1) $\cot^2 A = \operatorname{cosec}^2 A - 1$;

(2) $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A$.

Verify (2) when $A = 30^0$.

76. Evaluate to 4 significant figures by the aid of the table of logarithms $\dfrac{7 \cdot 891}{\cdot 0345} \times \sqrt[7]{(\cdot 008931)}$.

77. If $\sin B$ be the geometric mean between $\sin A$ and $\cos A$, then

$$\cos 2B = 2 \cos^2 (A + 45^0).$$

78. The lengths of two of the sides of a triangle are 1 foot and $\sqrt{2}$ feet respectively, the angle opposite the shorter side is 30^0. Prove that there are two triangles which satisfy these conditions; find their angles, and show that their areas are in the ratio $\sqrt{3} + 1 : \sqrt{3} - 1$.

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79. If the circumference of a circle be divided into six parts in arithmetical progression, the greatest being six times the least, express in radians the angle each subtends at the centre.

80. Define the tangent of an angle, wording your definition so as to include angles of any magnitude.

Prove that $\tan(90^\circ + A) = -\cot A$, and by means of this formula deduce the formula $\tan(180^\circ + A) = \tan A$.

81. Compute by means of tables the value of

$$\frac{6 \cdot 12}{\cdot 4131} \times \sqrt[5]{54 \cdot 17}.$$

82. Prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$, and deduce the expression for $\sin(A+B)$.

Show that

$$\cos A \cos(B+C) - \cos B \cos(A+C) = \sin(A-B)\sin C.$$

83. Establish the identities:

(1) $1 + \cos A - \sin A = \sqrt{2(1+\cos A)(1-\sin A)}$;

(2) $\sec 2A = \dfrac{\sec^2 A}{2 - \sec^2 A}$;

(3) $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7} + 4\cos\dfrac{\pi}{7}\cos\dfrac{3\pi}{7}\cos\dfrac{5\pi}{7} + 1 = 0.$

84. Two adjacent sides of a parallelogram 5 in. and 3 in. long respectively, include an angle of 60°. Find the lengths of the two diagonals and the area of the figure.

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85. Investigate the following formulæ:

(1) $\sin\dfrac{3A}{2} = (1 + 2\cos A)\sin\frac{1}{2}A$;

(2) $\sin(\theta + \delta) - \sin\theta = \cos\theta \sin\delta(1 - \tan\theta \tan\frac{1}{2}\delta)$.

86. Prove that

(1) $\sin 10^\circ + \sin 50^\circ = \sin 70^\circ$;

(2) $\sqrt{3} + \tan 40^\circ + \tan 80^\circ = \sqrt{3}\tan 40^\circ \tan 80^\circ$;

(3) if $A + B + C = 180^\circ$,

$$\frac{\sin A - \sin B \cos C}{\cos B} = \frac{\sin B - \sin A \cos C}{\cos A}.$$

87. Prove by means of the logarithmic table that

$$\frac{1}{73^{-\frac{1}{7}}} = 1 \cdot 846 \text{ nearly.}$$

88. The length of one side of a triangle is 1006·62 feet and the adjacent angles are 44° and 70°. Solve the triangle, having given

$L \sin 44^\circ = 9 \cdot 8417713$, $L \sin 70^\circ = 9 \cdot 9729858$,
$L \sin 66^\circ = 9 \cdot 9607302$, $\log 1006 \cdot 62 = 3 \cdot 0028656$,
$\log 7654321 = 6 \cdot 8839067$, $\log 103543 = 5 \cdot 0151212$.

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89. Find the length of the arc of a circle whose radius is 8 feet which subtends at the centre an angle of 50^0, having given

$$\pi = 3\cdot1416.$$

90. Prove that $\sin A = -\sin(A - 180^0)$.

Find the sines of 30^0 and 2010^0.

91. Given that the integral part of $(3\cdot1622)^{100000}$ contains fifty thousand digits, find $\log_{10} 31622$ to five places of decimals.

92. Prove that

(1) $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$;

(2) $\cos^2 A + \sin^2 A \cos 2B = \cos^2 B + \sin^2 B \cos 2A$.

93. Prove that in any triangle

$$a^2 \cos 2B + b^2 \cos 2A = a^2 + b^2 - 4ab \sin A \sin B.$$

94. If $a = 123$, $B = 29^0\ 17'$, $C = 135^0$, find c, having given

$\log 123 = 2\cdot0899051$, $\log 2 = \cdot3010300$,
$\log 3211 = 4\cdot5066403$, diff. for $1 = 1352$.

$$L \sin 15^0\ 43' = 9\cdot4327777.$$

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95. Define the unit of circular measure, and prove that it is an invariable angle.

If an arc of 12 feet subtend at the centre of a circle an angle of 50^0, what is the radius of the circle, π being equal to $3\cdot1416$?

96. Express the cosine and cotangent in terms of the cosecant.

If $\cot A + \operatorname{cosec} A = 5$, find $\cos A$.

97. Given that the integral part of $(3\cdot981)^{100000}$ contains sixty thousand digits, calculate $\log_{10} 39810$ correct to 5 places of decimals.

98. Prove that

(1) $\sin^2 A + \sin^2 B + 2\sin A \sin B \cos(A+B) = \sin^2(A+B)$;

(2) $\sin^2 A - \cos^2 A \cos 2B = \sin^2 B - \cos^2 B \cos 2A$.

99. On the birth of an infant £1500 is invested so that it may accumulate at Compound Interest (3 per cent. per annum payable half-yearly) during the child's minority; calculate by logarithms the amount at the end of 21 years.

100. Prove that in any triangle

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

ANSWERS TO THE EXAMPLES.

I. **1.** 80. **2.** 10. **3.** 16. **4.** $109\frac{11}{21}$. **5.** 5 acres.
6. $\frac{1760a}{b}$. **7.** $\frac{a \cdot c}{3}$ yds. **8.** A shilling and a three-penny piece.

II. **1.** 10 ft. **2.** 80 yds. **3.** 20 ft. **4.** 50 ft.
5. 90 ft. **6.** $20\frac{10}{27}$ nearly. **7.** $5a$ feet. **8.** $12a$ yards.
10. $\frac{\sqrt{2}}{2}a$ yards. **12.** $\frac{2\sqrt{3}}{3}a$ feet. **13.** $1 : \sqrt{2}$. **14.** $\sqrt{84}$ ft.
15. $2\sqrt{9a^2 - b^2}$ ft.

III. **1.** $3\frac{1}{7}$ yds. **2.** $25\frac{1}{7}$ ft. **3.** $150\frac{6}{7}$ in. **4.** $3\frac{2}{11}$ ft.
5. $7\frac{7}{11}$ ft. **6.** 560. **7.** $15\frac{1}{2}$ nearly. **8.** 33600. **9.** $32\frac{1}{7}$.
10. 7 ft. **11.** $553\frac{1}{7}$, 13·8 in. **12.** $339\frac{3}{7}$ ft. **13.** 443 in.
14. 235 in. **15.** 203 in. **16.** 1886 in.

V. **1.** ·09175 of a right angle $= 9^g 17' 50''$.
2. ·0675 „ $= 6^g 75'$.
3. 1·07875 „ $= 107^g 87' 50''$.
4. $\cdot 18042\dot{9}01234567\dot{9}$ $= 18^g 4' 29''$, etc.
5. $1\cdot 46\dot{7}$ „ $= 146^g 77' 77\cdot\dot{7}''$.
6. $\cdot 5\dot{4}$ „ $= 54^g 44' 44\cdot\dot{4}''$.
7. $1^0 14' 15''$. **8.** $7^0 52' 30''$. **9.** $153^0 24' 29\cdot 34''$.
10. $21^0 36' 8\cdot 1''$. **11.** $16^0 12' 37\cdot 26''$. **12.** $31^0 30'$.

VI. I. (1) 2 right angles or 180^0. (2) $\frac{3}{2}$ of a right angle.
(3) $\frac{2}{\pi}$ right angles. (4) $\frac{6}{\pi}$ right angles. (5) 2 right angles.
(6) $\frac{4}{\pi^2}$ right angles. (7) $\frac{2\theta}{\pi}$ right angles. (8) ·002 of a right angle.
(9) 20 right angles.

II. (1) π. (2) 2π. (3) $\frac{\pi}{3}$. (4) $\frac{\pi}{8}$. (5) $\frac{\pi}{180}$. (6) 1^c. (7) $\frac{n}{180}\pi$.
(8) $\frac{1}{2}^c$. (9) $\frac{A\pi}{180}$.

III. (1) $\frac{\pi}{6}$. (2) $\frac{\pi}{4}$. (3) $\frac{\pi}{12}$. (4) $\frac{\pi}{200}$. (5) $\frac{\pi}{20000}$.
(6) $\frac{\pi}{200000}$. (7) $\frac{n\pi}{200}$. (8) 1^c. (9) 5π.

IV. (1) $\frac{1}{3}$. (2) $\frac{10}{9}$. (3) 1. (4) $\frac{50}{3\pi}$. (5) $\frac{63}{20}$. (6) $\frac{\pi}{180}$.

VII. **1.** $\frac{3}{2}$. **2.** 90. **3.** $4\frac{4}{5}$. **4.** $112\frac{1}{2}$ ft. **5.** $5\frac{37}{63}$ ft.
6. 838000 miles. **7.** $\frac{1}{9}$ radian $= 6\frac{4}{11}$ degrees. **8.** $21\frac{21}{44}$ degrees.
9. $51\frac{6}{11}''$. **10.** about 34 yds. **11.** 1 : 3·1416. **12.** 3·1416.
13. $3·141\dot{6}$. **14.** 400 : 1. **15.** ·0000484.... **16.** $49\frac{1}{11}$ in.
17. $\frac{\pi}{2}$ i. e. a right angle. **19.** 473 : 489.

20. (i) $k=1$, (ii) $k=\frac{180}{\pi}$. **21.** 38^0, 18^0. **22.** $\frac{n\pi}{10800}$.

23. (i) 120^0, $133·\dot{3}^g$, $\frac{2\pi}{3}$, (ii) 135^0, 150^g, $\frac{3\pi}{4}$, (iii) 156^0, $173·\dot{3}^g$, $\frac{13\pi}{15}$.

24. (i) $3\frac{3}{4}$, (ii) $\frac{15}{2\pi}$. **25.** $\frac{180}{16}^0$. **26.** a right angle. **27.** $\frac{ac}{90b}$.

28. $\frac{9a+10b}{10c}$ degrees. **29.** $\frac{1800\pi}{19\pi+1800}$. **30.** 9 or 16.

VIII. **1.** (i) DA, BD. (ii) DB, AD. (iii) DA, CD. (iv) DC, AD.

2. (i) $\frac{DB}{AB}$. (ii) $\frac{DC}{CA}$. (iii) $\frac{CD}{AD}$. (iv) $\frac{DA}{BA}$. (v) $\frac{DB}{AD}$.
(vi) $\frac{DC}{AC}$. (vii) $\frac{CD}{CA}$. (viii) $\frac{DA}{CD}$. (ix) $\frac{BD}{BA}$. (x) $\frac{DA}{CA}$.

3. (i) $\frac{DB}{CB}$, $\frac{BA}{CA}$. (ii) $\frac{CD}{CB}$, $\frac{CB}{CA}$. (iii) $\frac{DB}{CD}$, $\frac{BA}{CB}$.
(iv) $\frac{DB}{AB}$, $\frac{BC}{AC}$. (v) $\frac{AD}{AB}$, $\frac{AB}{AC}$. (vi) $\frac{DB}{AD}$, $\frac{BC}{AB}$.

4. (i) $\frac{DA}{BA}$. (ii) $\frac{BA}{EA}$ or $\frac{AC}{EC}$. (iii) $\frac{DC}{BC}$. (iv) $\frac{AB}{AE}$.
(v) $\frac{AD}{AB}$ or $\frac{AB}{AC}$. (vi) $\frac{BD}{BC}$. (vii) $\frac{DB}{CD}$, or $\frac{BA}{CB}$, or $\frac{AE}{CA}$.
(viii) $\frac{DA}{BD}$. (ix) $\frac{BA}{EB}$ or $\frac{AC}{EA}$. (x) $\frac{DC}{BD}$. (xi) $\frac{DB}{AB}$ or $\frac{BC}{AC}$.
(xii) $\frac{BE}{AE}$.

5. $\sin A=\frac{3}{5}$, $\cos A=\frac{4}{5}$, $\tan A=\frac{3}{4}$; $\sin B=\frac{4}{5}$, $\cos B=\frac{3}{5}$, $\tan B=\frac{4}{3}$.
7. Of the smaller angle, the sine $=\frac{5}{13}$, cosine $=\frac{12}{13}$, tangent $=\frac{5}{12}$.
Of the larger angle, the sine $=\frac{12}{13}$, cosine $=\frac{5}{13}$, tangent $=\frac{12}{5}$.

8. Of the smaller angle, the sine $=\frac{1}{2}$, cosine $=\frac{\sqrt{3}}{2}$, tangent $=\frac{1}{\sqrt{3}}$.

Of the larger angle, the sine $=\frac{1}{2}\sqrt{3}$, cosine $=\frac{1}{2}$, tangent $=\sqrt{3}$.

10. $Bc=\sqrt{3}$; $\sin A=\frac{1}{2}\sqrt{3}$, $\cos A=\frac{1}{2}$, $\tan A=\sqrt{3}$.

12. $AC=\sqrt{2}$; $\sin A=\sqrt{\frac{2}{3}}$, $\sin B=\dfrac{1}{\sqrt{3}}$.

X. 1. 179 ft. 2. 346 ft. 3. 86·6 ft. 4. 138·5 ft.
5. $7\frac{1}{2}$ ft. 6. 60^0, 173 ft. 7. 63·17 yds. 8. 277·3 ft.
9. 192·8 ft. 10. 78 ft. 11. 34·15 ft.
12. 73·2 ft. 13. 86·6 ft. 14. ·866 miles = 1524 yds.
15. 173·2 yds. 17. 373 ft. 18. 3733 ft.
19. $\frac{1}{2}\sqrt{6}$ miles = 6465 ft. 20. $\dfrac{\sqrt{3}\,.\,a}{3b}$. 21. 30^0.
22. About 523·6 miles.

XI. 27. $2\cos^2\theta-1$, $1-2\sin^2\theta$. 28. $(1-2\cos^2\theta)^2$, $(2\sin^2\theta-1)^2$.

29. $\dfrac{2\cos^2\theta-1}{\cos^4\theta}$, $\dfrac{1-2\sin^2\theta}{(1-\sin^2\theta)^2}$.

30. $1-3\cos^2\theta\,(1-\cos^2\theta)$, $1-3\sin^2\theta\,(1-\sin^2\theta)$.

31. $\dfrac{1-2\cos^2\theta+2\cos^4\theta}{\cos^2\theta\,(1-\cos^2\theta)}$, $\dfrac{1-2\sin^2\theta+2\sin^4\theta}{\sin^2\theta\,(1-\sin^2\theta)}$.

32. $\dfrac{1-2\cos^2\theta+2\cos^4\theta}{(1-\cos^2\theta)^2}$, $\dfrac{1-2\sin^2\theta+2\sin^4\theta}{\sin^4\theta}$. 33. 0.

34. $\dfrac{2\,(1-\cos^2\theta)\,(1-\cos^2\theta-2\cos^4\theta)}{\cos^4\theta}$, $\dfrac{2\sin^2\theta\,(5\sin^2\theta-2-2\sin^4\theta)}{(1-\sin^2\theta)^2}$.

XII. 1. $\sin A=\sqrt{1-\cos^2 A}$, $\tan A=\dfrac{\sqrt{1-\cos^2 A}}{\cos A}$,

$\cot A=\dfrac{\cos A}{\sqrt{1-\cos^2 A}}$, $\sec A=\dfrac{1}{\cos A}$, $\operatorname{cosec} A=\dfrac{1}{\sqrt{1-\cos^2 A}}$.

2. $\sin A=\dfrac{1}{\sqrt{1+\cot^2 A}}$, $\cos A=\dfrac{\cot A}{\sqrt{1+\cot^2 A}}$, $\tan A=\dfrac{1}{\cot A}$,

$\sec A=\dfrac{\sqrt{1+\cot^2 A}}{\cot A}$, $\operatorname{cosec} A=\sqrt{1+\cot^2 A}$.

3. $\sin A=\dfrac{\sqrt{\sec^2 A-1}}{\sec A}$, $\cos A=\dfrac{1}{\sec A}$, $\tan A=\sqrt{\sec^2 A-1}$,

$\cot A=\dfrac{1}{\sqrt{\sec^2 A-1}}$, $\operatorname{cosec} A=\dfrac{\sec A}{\sqrt{\sec^2 A-1}}$.

4. $\sin A = \dfrac{1}{\operatorname{cosec} A}$, $\cos A = \dfrac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$, $\tan A = \dfrac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$,

$\cot A = \sqrt{\operatorname{cosec}^2 A - 1}$, $\sec A = \dfrac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}$.

5. $\cos A = \sqrt{1 - \sin^2 A}$, $\tan A = \dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$, $\cot A = \dfrac{\sqrt{1 - \sin^2 A}}{\sin A}$,

$\sec A = \dfrac{1}{\sqrt{1 - \sin^2 A}}$, $\operatorname{cosec} A = \dfrac{1}{\sin A}$.

6. $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1 + \tan^2 A}}$, $\cot A = \dfrac{1}{\tan A}$,

$\sec A = \sqrt{1 + \tan^2 A}$, $\operatorname{cosec} A = \dfrac{\sqrt{1 + \tan^2 A}}{\tan A}$.

XIII. 1. $\frac{3}{4}$, $\frac{5}{3}$. 2. $\dfrac{2\sqrt{2}}{3}$, $\dfrac{1}{2\sqrt{2}}$. 3. $\frac{4}{5}$, $\frac{5}{3}$.

4. $\dfrac{1}{\sqrt{15}}$, $\dfrac{\sqrt{15}}{4}$. 5. $\dfrac{\sqrt{3}}{2}$, $\frac{1}{2}$. 6. $\dfrac{\sqrt{5}}{3}$, $\frac{3}{2}$. 7. $\dfrac{b}{\sqrt{c^2 - b^2}}$.

8. $\dfrac{a}{\sqrt{a^2+1}}$, $\dfrac{1}{\sqrt{a^2+1}}$. 9. $\dfrac{\sqrt{a^2-1}}{a}$, $\dfrac{1}{\sqrt{a^2-1}}$.

11. $h^2(1+k^2) = 1$.

XIV. 2. $\sec\theta$ increases continuously from 1 to ∞.
3. $\sin A$ diminishes continuously from 1 to 0.
4. $\cot\theta$ diminishes continuously from ∞ to 0.

XV. 1. 45^0. 2. 30^0. 3. 45^0. 4. 60^0. 5. 30^0.
6. 30^0. 7. 30^0. 8. 0^0, or 45^0. 9. 90^0, or 60^0. 10. 60^0.
11. 45^0. 12. 45^0. 13. 90^0, or 45^0. 14. 45^0. 15. 45^0.
16. 45^0. 17. 30^0. 18. 30^0.

XVI. 3. The value 3 is inadmissible. 4. $\frac{1}{4}(2 \pm \sqrt{2})$.
5. $\frac{3}{4}$, or $\frac{1}{2}$. 6. $\frac{3}{4}$, or $\frac{1}{3}$. 7. The value $-\frac{1}{6}(7\sqrt{3})$ is inadmissible.
9. $1 - \sin^4 A$. 10. $1 - 3\sin^2\theta + 3\sin^4\theta$.
11. $\dfrac{1 - 2\cos^2\theta + 2\cos^4\theta}{\cos^4\theta}$. 13. $\dfrac{1 - \sin A}{1 + \sin A}$.
14. $\operatorname{cosec}\theta$ decreases continuously from ∞ to 1.
15. $\cot\theta$ increases continuously from 0 to ∞.
16. $\theta = \frac{1}{4}\pi$, $\phi = \frac{1}{12}\pi$.

XVII. 1. +6. 2. 0. 3. +2. 4. +3.
5. +10. 6. 0. 7. +7. 8. +7.

XIX. 1. The second. 2. The fourth. 3. The second.
4. The third. 5. The fourth. 6. The first.
7. The second. 8. The first. 9. The first.
10. The fourth. 11. The fourth.
12. The first, if n be even, the third, if n be odd.

XX. 1. +, +, +. 2. +, −, −. 3. −, −, +.
4. −, +, −. 5. −, +, −. 6. −, −, +.
7. +, −, −. 8. +, +, +. 9. +, +, +.
10. +, +, +. 11. +, −, −. 12. −, +, −.

XXI. 1. $+\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$. 2. $+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$.

3. $+\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$. 4. $-\frac{1}{2}, +\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$.

5. $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, -1$. 6. $-\frac{\sqrt{3}}{2}, +\frac{1}{2}, +\sqrt{3}$.

7. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1$. 8. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1$.

9. $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$. 10. $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$.

11. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, +\sqrt{3}$. 12. $-\frac{\sqrt{3}}{2}, +\frac{1}{2}, -\sqrt{3}$.

13. $+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, +1$. 14. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$.

15. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$. 16. $30^0, 150^0, -210^0, -330^0$.

17. $45^0, 135^0, -225^0, -315^0$. 18. $60^0, 120^0, -240, -300^0$.

19. $-30^0, -150^0, 210^0, 330^0$. 20. $20^0, 160^0, 380^0, 520^0$.

21. $\frac{5}{4}\pi, \frac{7}{4}\pi, \frac{13}{4}\pi, \frac{15}{4}\pi$. 22. $\frac{8}{7}\pi, \frac{13}{7}\pi, \frac{22}{7}\pi, \frac{27}{7}\pi$. 25. The tan.

26. No.

XXIII. 1. 60^0. 2. -100^0. 3. 0^0. 4. -260^0.
5. 115. 6. 410^0. 7. $-\frac{1}{4}\pi$. 8. $\frac{2}{3}\pi$.

XXIX. 1. $\sin(\theta+\phi)+\sin(\theta-\phi)$. 2. $\cos(\alpha-\beta)+\cos(\alpha+\beta)$.

3. $\sin(2\alpha+3\beta)+\sin(2\alpha-3\beta)$. 4. $\cos 2\alpha+\cos 2\beta$.

5. $\sin 8\theta-\sin 2\theta$. 6. $\cos\theta+\cos 2\theta$. 7. $\frac{1}{2}(\cos 3\theta-\cos 5\theta)$.

8. $\frac{1}{2}(\sin 4\theta-\sin\theta)$. 9. $\sin 60^0+\sin 40^0$.

10. $\frac{1}{2}(\sin 60^\circ - \sin 30^\circ)$. 11. $2\cos 3\theta \cos 2\theta$.

12. $-\cos 4\theta \sin 2\theta$. 13. $4\cos^2\frac{\theta}{2}\sin 2\theta$.

XXXII. 1. (i) a^{2h+3k}. (ii) a^{4h-5k}. (iii) $a^{\frac{4h}{3}+\frac{5k}{3}}$. (iv) $a^{\frac{5h}{2}+\frac{3k}{2}}$.
2. (i) 5·4690116. (ii) 10·6243928. (iii) 13·7509366. (iv) ·8853661. (v) 1·7968680. (vi) 8·9699598. (vii) 2·7345058.
3. $2^3, 2^5, 2^{-1}, 2^{-4}, 2^{-3}, 2^7$. 4. $3^2, 3^4, 3^{-1}, 3^{-3}, 3^{-2}, 3^{-4}$.

XXXIII. 1. ·60206, ·9542426, ·90309, ·7781513, 1·20412, 1·690196. 2. 1·146128, 1·20412, 1·2552726, 1·3802113, 1·4313639, 1·6232493. 3. 1, ·69897, 1·1760913, 1·39794, 1·4771213, 1·5440680.
4. 1·5563026, 1·60206, 1·6812413, 1·69897, 2·30103, 3.
5. 7·201593, 3·858708. 6. ·7545579, 2·989843. 7. 1·4532.
8. 2408·6. 9. (i) 4·5868. (ii) ·93646. 10. 3·9549.
11. 40975·3 sq. ft. 12. 34·925 in. 13. 3·2617 in.
14. 110115 cub. yds.

XXXIV. 1. 3, $\frac{10}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $-\frac{5}{2}$. 2. 3, 6, −1, −3, −6, 2.
3. 2, 4, −1, −3, −2, −4. 4. $\frac{3}{2}$, $\frac{2}{3}$, $-\frac{1}{4}$, −1.
5. 3, −1, 5, −2, 3, −3. 6. $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$.
7. ·7781513, 1·6232493, 1·20412.
8. 1·6901960, 1·5563026, 1·7993406.
9. 2·30103, 2·7781513, 1·845098. 10. ·69897, 5228787, 1·69897.
11. 1·544068, 2·1760913, −1+·30103.
12. ·5440680, ·8627278, −2+·9084852.

XXXV. 1. 4, 2, 0, 5, 1. 2. −2, −5, −1, −3.
3. 3, −1, 0, 1, 0, −7. 4. 4, 1, 6, 3.
5. the second decimal place, the first dec. pl., the sixth dec. pl.
6. ten thousands, units, hundreds, third dec. pl., first dec. pl., units.
7. 10, 4, 25, 31. 8. 9, 11, 85, 4, 9, 6.
9. units, fourth dec. pl., thousands, seventh dec. pl., second dec. pl.
10. tenth integral pl., twelfth dec. pl., fifth dec. pl., units, twelfth dec. pl., first dec. pl.

XXXVI. 1. 2·8901023, ·8901023, $\bar{4}$·8901023, 5·8901023.
2. 6·7714552, ·7714552, $\bar{4}$·7714552, 2·7714552, 3·7714552.
3. ·27724... 4. ·00001638... 5. ·77448... 6. ·005968...

XXXVII. **1.** 3, 0, $\frac{1}{3}$, 0, $\frac{7}{5}$. **4.** $\bar{1}$·8121177, 55.
5. ·51375. **6.** 7, 4, 3, 3.

7. (i) $x=\dfrac{2\log 7}{\log 2+4\log 3}$. (ii) $x=\dfrac{7\log 2+4\log 7}{2\log 3+\log 7}$.

(iii) $x=\dfrac{2\log 7}{2\log 2+\log 3}$. (iv) $x=\dfrac{4(\log 3+\log 7)}{8\log 2+3(\log 3+\log 7)}$.

8. $2+\dfrac{1}{\log_{10}7}$. **9.** $\frac{3}{2}+\dfrac{1}{\log_{10}3}$. **10.** $\dfrac{1}{1-\log_{10}2}$.

11. $0,\ \dfrac{1}{b+1},\ \dfrac{3a}{2b+2},\ \dfrac{2}{b+1},\ \dfrac{b}{b+1},\ \dfrac{3a+2}{2b+2},\ \dfrac{bc}{b+1}$.

12. 63 − 31 = 32. **13.** $(a^{11}-a^{10})$ integers. **14.** 1·9485 nearly.
19. 2·53855. **20.** 4·59999. **21.** 167 years.

XXXVIII. **1.** ·8839066. **2.** 2·7513738. **3.** $\bar{4}$·9413333.
4. 6·8086920. **5.** ·5710750. **6.** 3·70404. **7.** 45740·26.
8. 2492837. **9.** ·000439658. **10.** 5·689158.

XXXIX. **1.** 7·669. **2.** 3·809. **3.** 47·32. **4.** 55460.
5. 12·03. **6.** ·04023. **7.** 8287. **8.** 1165. **9.** ·3107.
10. ·6731. **11.** 1·096. **12.** 823·6. **13.** 2·624.
14. 22·51. **15.** 23·28. **16.** 28·01. **17.** − ·8243. **18.** 1407.

XL. **1.** £48. **2.** £·477 = 9*s.* 6½*d.* **3.** 23·4. **4.** 17·7.
5. £73·07. **6.** 140 years. **7.** £1869. **8.** 36·9 years.
9. £5066 *about.* **10.** *About* 67,100,000 pence.
11. ·0679 miles per hour. **12.** 1·24 yds.

XLI. **1.** ·6737652. **2.** ·6737652. **3.** ·9306572.
4. 41° 48′ 37″. **5.** 70° 31′ 43·6″. **6.** 75° 31′ 21″.
7. 9·8515594. **8.** 9·7114477. **9.** 10·1338768.
10. 35° 4′ 23″. **11.** 28° 16′ 27·5″. **12.** 21° 56′ 41″.

XLII. **1.** 34° 19′ 31·8″. **2.** 1498·2 ft. **3.** 45° 36′ 56″.
4. 5293·4 ft., 6982·3 ft. **5.** 576·2 chains. **6.** 4729 chains.
7. 3666·8 feet. **8.** 42° 15′, 11444 chains.

XLIII. **1.** 3843 ft. **2.** 281·7 ft. **3.** 115 ft.
4. 286 ft. **5.** 58° 17′, 31° 42′. **6.** 656 chains, 41° 17′.
7. 81 ft. **8.** 1942 ft. **9.** 646·7 miles. **10.** 1000 ft.

XLIV. **1.** 60°. **2.** 120°. **3.** 30°. **4.** 135°.
5. 45°. **6.** 120°.

XLVI. **1.** $\cos A = \frac{1}{2}$, $\cos\frac{1}{2}A = \frac{1}{2}\sqrt{3}$. **2.** 45^0, 60^0, 75^0.
3. 135^0, 30^0, 15^0. **4.** 3. **5.** 14. **6.** $1+\sqrt{3}$. **7.** 120^0.
8. 120^0. **9.** 120^0. **10.** 90^0, $36^0\,52'$. **11.** $130^0\,27'$.
12. $125^0\,6'$. **13.** 120^0. **14.** $A=54^0$ or 126^0, $B=108^0$ or 36^0.
15. $a=1$. **16.** $C=30^0$, $a=\sqrt{3}+1$, $b=2$. **17.** $A=75^0$, $a=b=\sqrt{3}+1$.
18. $C=60^0$ or 120^0. **19.** $100\sqrt{3}$. **20.** No.
22. $A=105^0$, $C=60^0$, $B=15^0$. **23.** $\frac{1}{2}\sqrt{3}(\sqrt{5}+1)$. **24.** $A=90^0$ or 60^0, $C=75^0$ or 105^0, $a=2\sqrt{2}$ or $\sqrt{6}$. **25.** 30^0 or 150^0.
26. $A=45^0$ or 135^0, $B=30^0$ or 120^0, $b=\sqrt{2}(1+\sqrt{3})$ or $\sqrt{6}(1+\sqrt{3})$.
27. 60^0, 75^0, 6 yds. **28.** It is impossible. **30.** $15 : 8\sqrt{3} : 4\sqrt{5}+6$.

XLVII. **1.** $41^0\,16'\,51{\cdot}5''$. **2.** $73^0\,32'\,12''$, $62^0\,46'\,18''$.
3. $29^0\,17'\,16''$, $31^0\,55'\,31''$. **4.** $64^0\,31'\,58''$. **5.** 73^0, $23'\,54{\cdot}4''$.
6. $41^0\,24'\,34{\cdot}6''$. **7.** $82^0\,49'\,9''$. **8.** 75^0, 60^0, 45^0. **9.** 135^0, 30^0, 15^0.

XLVIII. **1.** 313·46 yds. **2.** 28·87 in., 31·43 in. **3.** 1192·55 yds.
4. 22·415 ft. **5.** 24·995 = 25 ft. nearly, 17·559 ft., $65^0\,59'\,42''$.

XLIX. **1.** $108^0\,36'\,30''$, $31^0\,23'\,30''$. **2.** $93^0\,11'\,49''$, $36^0\,48'\,11''$.
3. $57''\,27'\,25{\cdot}4''$, $62^0\,32'\,34{\cdot}6''$. **4.** $64^0\,26'\,47''$, 37^0, $7'$, $13''$.
5. $72^0\,12'\,59''$. **6.** 20·5 chains. **7.** 122·7. **8.** $71^0\,13'\,50''$, $32^0\,16'\,10''$.

L. **1.** $A=51^0\,18'\,21''$, $C=88^0\,41'\,39''$; or $A=128^0\,41'\,39''$, $C=11^0\,18'\,21''$.
2. $B=70^0\,0'\,56''$, $C=59^0\,59'\,4''$; or $B=109^0\,59'\,4''$, $C=20^0\,0'\,56''$.
3. $B=38^0\,38'\,24''$, $C=91^0\,21'\,36''$, $c=155{\cdot}3$. **4.** $61^0\,16'\,10''$.
5. $A=72^0\,4'\,48''$, $B=41^0\,56'\,12''$; or $A=107^0\,55'\,12''$, $B=6^0\,5'\,48''$, $b=17{\cdot}56$. **6.** β is ambiguous; 60·3893 ft.

LI. The angles are given correct to the nearest second.
1. $28^0\,35'\,39''$. **2.** $104^0\,44'\,39''$. **3.** $32^0\,20'\,48''$.
4. $43^0\,40'$. **5.** $128^0\,23'\,13''$. **6.** 106531 ft.
7. 3437·6 yds. **8.** 1728·2 chains. **9.** 25376 yds.
10. $A=66^0\,27'\,48''$, $B=12^0\,55'\,12''$. **11.** $A=92^0\,12'\,53''$, $B=35^0\,37'\,7''$.
12. $B=29^0\,1'\,40''$, $C=74^0\,55'\,50''$. **13.** $B=70^0\,35'\,24''$; or $109^0\,24'\,36''$.
14. $B=51^0\,56'\,17''$; or $128^0\,3'\,43''$. **15.** $B=62^0\,6'\,10''$; or $117^0\,53'\,50''$.
16. Very nearly 90^0. **17.** 1319·6 yds.

LV. **1.** $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$. **2.** $\frac{80}{3}\sqrt{3}$ ft. = 46·19... ft.
3. $A = n\times 180^0$; or, $n360^0 \pm 60^0$. **6.** 4227·47 feet.
9. 30^0, 60^0, 90^0, 120^0, etc. have for sine $\frac{1}{2}$, $\sqrt{\frac{3}{2}}$, 1, $\sqrt{\frac{3}{2}}$, $\frac{1}{2}$, 0, $-\frac{1}{2}$, $-\sqrt{\frac{3}{2}}-1$, $-\sqrt{\frac{3}{2}}$, $-\frac{1}{2}$ respectively.

12. The other sides are 765·4321 ft.; 1006·6 ft.

15. 30^0, 60^0, 90^0, etc. have for tan $\frac{1}{3}\sqrt{3}$, $\sqrt{3}$, ∞, $-\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$, 0, $\frac{1}{3}\sqrt{3}$, ∞, $-\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$ respectively.

17. $\frac{168}{193}$, $\frac{168}{195}$, $\frac{32592}{193\times195}$. 19. $\sec A=\frac{3}{4}\sqrt{2}$. 20. (i) $\frac{15}{2}$; (ii) -15.

21. $+\sin 18^0$, $-\cos 18^0$, $-\tan 18^0$; $-\sin 30^0$, $-\cos 36^0$, $+\tan 36^0$; $-\sin 36^0$, $+\cos 36^0$, $-\tan 36^0$. 22. $\cos 5a=16\cos^5 a-20\cos^3 a+5\cos a$.

23. (i) 0, $n\pi$, $\frac{1}{3}\pi$; (ii) $\cos\theta=\frac{1}{2}$, or, $\sin(\theta-45^0)=\frac{1}{\sqrt{2}}$.

24. $\frac{1}{26}\sqrt{651}=\cdot981\ldots$ 25. $\frac{11}{630}A$ radians; $5\cdot72965^0$.

26. sine, $\frac{4}{5}$; tan, $\frac{4}{3}$; cot, $\frac{3}{4}$; cosec, $\frac{5}{4}$; sec, $\frac{5}{3}$.

29. (i) $6\log_{10}3+3\log_{10}2-3$; (ii) $\frac{1}{3}\{\log_{10}7+1-\log_{10}2\}$; (iii) $3\log_{10}7+3\log_{10}2-2\log_{10}3-2$. 31. $\frac{630}{11}\theta$ deg.; $19\cdot09854^0$.

34. $\frac{9}{2}$; -9. 36. $C=18^0$, $a=c=2\div\sqrt{(10-2\sqrt{5})}$.

37. -1320^0. 38. $-\frac{2}{3}\sqrt{2}$. 39. π; $\frac{1}{3}\pi$; $\frac{5}{3}\pi$.

41. 1 foot, 120^0, 30^0; or 2 feet, 60^0, 90^0. 42. $104^0\,28\,39''$.

43. -630^0. 44. $-\frac{1}{5}\sqrt{5}$. 46. 0; π; $\frac{2}{3}\pi$; $\frac{4}{3}\pi$.

48. $\frac{1}{2}\{\sqrt{6}\pm\sqrt{2}\}$ and 15^0, 135^0; or, 105^0, 45^0.

49. 9^0; $286^0 . 28' . 41\cdot16''$; $\frac{7}{8}$.

52. (i) $n\pi\pm\frac{1}{3}\pi$. (ii) $\frac{1}{2}n\pi\pm\frac{1}{8}\pi$, or $n\pi+(-1)^n\frac{1}{4}\pi$.

53. $37\frac{1}{2}$ sq. ft. 54. $40^0 . 29' . 19\cdot85''$.

55. $\frac{1}{1080}\pi$; $\frac{1}{10}\pi$; $\frac{5}{4}$. 56. $\tan\alpha=4\sqrt{3}$, $\operatorname{cosec}\alpha=\frac{7}{12}\sqrt{3}$.

58. (i) $n\pi\pm\frac{1}{4}\pi$. (ii) $\frac{1}{2}n\pi\pm\frac{1}{8}\pi$; or, $2n\pi\pm\frac{1}{4}\pi$.

59. $2\sqrt{14}$ sq. ft. 60. $38^0 . 25' . 32\cdot725''$.

61. 1·2 radians $=76\cdot39416^g$.

64. (i) $1-\log_{10}5+\frac{1}{2}\log_{10}7$; $1-4\log_{10}5+5\log_{10}3$;
$2-5\log_{10}5-2\log_{10}3+2\log_{10}7$.

66. 192 ft., 185 ft. and 9234 sq. ft.

67. 2·3 radians $=131\cdot779926^0$.

69. ·6989700; ·7781513; $\bar{2}\cdot3344538$; $\bar{1}\cdot9148063$.

72. $78^0\,10'$, $70^0\,30'$, 9234 sq. ft.

73. $\frac{4}{35}\pi$; $\frac{9}{35}\pi$; $\frac{14}{35}\pi$; $\frac{19}{35}\pi$; $\frac{24}{35}\pi$. 76. 116·6.

78. 135^0, 15^0; or 45^0, 105^0. 79. $\frac{2}{21}\pi$; $\frac{4}{21}\pi$; $\frac{6}{21}\pi$; $\frac{8}{21}\pi$; $\frac{10}{21}\pi$; $\frac{12}{21}\pi$.

81. 32·92... 84. 7 ft.; $\sqrt{19}$ ft.; $\frac{15}{4}\sqrt{3}$ sq. ft.

88. 1035·43 ft.; 765·4321 ft.; 66^0. 89. 6·981 feet.

90. $\frac{1}{2}$; $-\frac{1}{2}$. 91. 4·49999. 94. 3210·793. 95. 13·751 ft.

96. $\frac{12}{13}$. 97. 4·59999. 99. £2803 nearly.

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