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TRIGONOMETRY

FOR

SCHOOLS AND COLLEGES

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PREFACE.

THIS book is a revision of lectures on trigonometry which we have given to our students for several years, and have furnished them during the last two years in manifolded form. The results have been so satisfactory, that we have been led to hope that the work would prove useful in a larger field. Our aim has been to adapt the work to the needs of students and teachers, with reference to other mathematical subjects both elementary and advanced.

Since in our own and a growing number of other institutions, an explicit course on a book of tables precedes the course in trigonometry, we have omitted that work, and do not publish tables. We have used some fundamental principles of algebra and geometry, with which students of trigonometry can be assumed to be familiar, without giving references.

We have not given numerical solutions of examples under the different cases of triangles, because the competent teacher does not need them, and we wish to leave him free to use such models as he prefers.

We have endeavored to lay down general conventions and definitions, to emphasize consistency in the use of the conventions, to give general demonstrations in which we have carefully aimed at logical soundness, directness, and simplicity, and to exhibit the unity of the subject as made up of its related parts. We believe that in this way the work is simplified, the student gets the ground plan of the higher analysis, and is saved from much subsequent intellectual lameness, which results from an excessive absorption of the attention on special cases. At the same time, when it seemed pedagogically helpful, we have led up to the general demonstration by the consideration of special cases.

We have given numerous examples at appropriate points, in order to admit of such selection as the teacher may find necessary for different classes, and to secure a thorough application of the theory.

The reader will find a number of original features in the book.

Our views on trigonometry and related subjects have been influenced by many American, English, French, and German treatises, and especially by our respected teachers, Professors J. M. Peirce, W. E. Byerly, and B. O. Peirce, of Harvard University.

The figures which appear in this book were drawn by Mr. D. C. Churchill of this college. We have taken some examples from other text-books.

F. A. and E. D. R., Jr.

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PLANE TRIGONOMETRY.

CHAPTER I.

A. **Linear and Angular Magnitude. Definition of Trigonometry.**

§ **1.** Magnitude is duration in time, or extent of space, or duration in time and extent of space of that which manifests itself in time and space.

§ **2.** A line is the path of a moving point. It has one dimension of space, length. The initial position of the generating point is the origin of the line. The final position is its term. The line is taken from the origin to the term, and is always determinate, unless the origin and term coincide, when it may be indeterminate. Hence every line joining two points represents at least two lines, opposites in direction, and negatives of each other. That is :

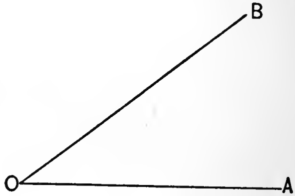
$$AB + BA = 0, \quad BA = - AB$$
$$AB = - BA.$$

It follows from this that if A_1, A_2, ... A_n, be n points arranged in any order on a line,

$$A_1A_2 + A_2A_3 + A_3A_4 + \cdots A_{n-1}A_n + A_nA_1 = kl,$$

where k is either zero, or any positive or negative integer, and l is the simplest length of the line. Linear magnitude is the extent of a line, and may be unlimited.

§ **3.** An angle is the amount of turning in a plane of a straight line about a fixed point through which it passes. The initial position of the generating line is called an axis of reference, and that part of it which forms the initial side of the angle is called the initial line of the angle. The whole of the generating line is called the rotating line, and that part of it which forms the terminal side of the angle is called the terminal line of the angle. The angle is taken *from* the initial *to* the terminal line. Hence every geometric angle must represent at least two angles, opposites in turning, and negatives of each other. That is :



$$\text{AOB} + \text{BOA} = 0, \text{ or}$$
$$\text{BOA} = -\text{AOB}$$
$$\text{AOB} = -\text{BOA}.$$

It follows from this that if $a_1, a_2, \ldots\ldots a_n$, be any n lines in a plane, and $\frac{a_l}{a_k}$ represent the angle *from* a_k *to* a_l, then

$$\frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots\ldots \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} = n\,360°,$$

where n may equal zero or any integer, positive or negative. Angular magnitude is the amount of turning of the generating line, and may be unlimited.

§ **4.** It is to be noticed that the preceding definitions of a line, and an angle, are extensions of the elementary conceptions; for they require as *essential* to the line, and the angle, two elements, magnitude and direction. The elementary conceptions require only magnitude. From these definitions corresponding extensions of all the elementary conceptions into which lines, and angles enter, necessarily follow.

§ **5.** The subject matter of Trigonometry is lines, and angles. Trigonometry is the investigation of the relations of the sides and angles of a triangle.

B. Measure-Units.

§ **6.** The *measure* of a magnitude is expressed by the number of *measure-units* the magnitude contains. Thus, the measure of a line is 3 feet, 4 rods, 2 meters, etc. Both lines and angles are measured by different measure-units.

(*a*) Linear Units and their Transformation.

§ **7.** The principal measure-units for lines are the English and the French. The equation for transformation is:

$$1 \text{ meter} \backsimeq 39.37 \text{ inches, approximately.}$$

(*b*) Angular Units and their Transformations.

§ **8.** The principal measure-units for angles are the sexagesimal, the centesimal, and the circular.

1. In the sexagesimal system:

1 right angle	$= 90$ degrees,	written	$90°$.
$1°$	$= 60$ minutes,	"	$60'$.
$1'$	$= 60$ seconds,	"	$60''$.

2. In the centesimal system:

1 right angle	$= 100$ grades,	written	100^g.
1^g	$= 100$ minutes,	"	$100^{\backprime}$.
$1^{\backprime}$	$= 100$ seconds,	"	$100^{\backprime\backprime}$.

3. The radian is that central angle in any circle whose intercepted arc is equal in length to the radius. Since in the same circle, or in equal circles, central angles are proportional to their intercepted arcs, and if δ represents the *number* of degrees in the angle, r the radius, and s the arc, then:

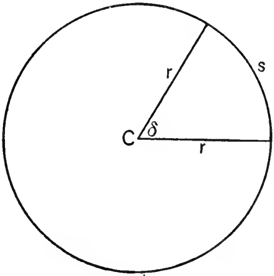


FIG. 1.

$$\frac{\delta}{360} = \frac{s}{2\pi r}; \qquad (1)$$

$$\therefore \left(\frac{s}{r}\right) = \frac{\pi}{180} \cdot \delta. \qquad (2)$$

This shows that $\left(\frac{s}{r}\right)$ varies as δ, that is, varies as the angle.

Let ϕ represent the angle in any system of measurement, then $\phi = n\left(\frac{s}{r}\right)$, where n is constant. When $s = r$, ϕ is constant; also when $s = r$, ϕ is a radian, which we will take as the angular measure-unit in the circular system, and represent by r written like an exponent; and we have, to determine n, $1 = n\left(\frac{r}{r}\right)$, or $n = 1$, which gives generally, $\phi = \frac{s}{r}$, when the angle ϕ is expressed in radians.

The preceding demonstration shows that the radian is invariable, and is equivalent to $\frac{180^\circ}{\pi}$, or $57^\circ.2957795+$. The equation, $\phi = \frac{s}{r}$, $r = \frac{s}{\phi}$, or $s = r\phi$, is of practical importance according as ϕ, r, or s, respectively, is the unknown.

If δ represents the number of degrees, γ the number of grades, and ρ the number of radians in an angle, we have, as in the preceding demonstration :

$$\frac{\delta}{360} = \frac{\gamma}{400} = \frac{s}{2\pi r}, \text{ or}$$

$$\frac{\delta}{180} = \frac{\gamma}{200} = \frac{\rho}{\pi}.$$

From which follow these six equations of transformation:

$$\delta = \frac{9}{10}\gamma, \quad \gamma = \frac{200}{\pi}\rho, \quad \rho = \frac{\pi}{180}\delta,$$

$$\gamma = \frac{10}{9}\delta, \quad \rho = \frac{\pi}{200}\gamma, \quad \delta = \frac{180}{\pi}\rho.$$

Making the symbols δ, γ, ρ in the right members of these equations successively equal to unity, we obtain the value of each unit in terms of the others. This gives the following table:

Units.	In terms of		
	o	r	g
1°	1	$\frac{\pi}{180}$	$\frac{10}{9}$
1^r	$\frac{180}{\pi}$	1	$\frac{200}{\pi}$
1^g	$\frac{9}{10}$	$\frac{\pi}{200}$	1

Degree measure is used in practical, and circular measure in theoretical work.

C. Examples. I.

1. Express $2523^\circ\ 17'\ 32''$ in grades, and radians.
2. Express $57^g\ 17^`\ 94^{``}$ in degrees, and radians.
3. Express 2.3^r in degrees, and grades.
4. Express π^r, $\frac{\pi^r}{2}$, $\frac{\pi^r}{10}$, $\frac{\pi^r}{15}$, in degrees.
5. Assuming the earth to be a sphere with a radius of 3963 miles, compute the length of a geographical mile in statute miles.
6. Find the distance, measured on the arc of a great circle, in miles, from Oberlin (lat. $41^\circ\ 17'$ N.) to the equator.
7. A man stands on the extremity of a radius of a circular platform which is supported on a vertical axis through its center, and perpendicular to its plane. The extremity of the given radius is opposite a fixed arrowhead. The platform is made to rotate about its axis counter-clockwise, and finally comes to rest. The man has moved through a distance of $\frac{1}{4}$ mi. Find in degrees the actual angle through which the platform has turned, and the apparent angle which the given radius makes with the line of the fixed arrow, the radius being 15 ft. in length.

CHAPTER II.

A. Conventions.

a. Conventions about Lines.

§ **9.** A plane is divided into four quadrants by two fixed perpendicular straight lines. The plane may always be so placed that one line is horizontal, and the other vertical. For convenience we shall speak of the lines as the horizontal, and the vertical line, or axis.

§ **10.** The point of intersection of the two lines is called the origin, and is denoted by the letter O. The horizontal axis is called the x-axis, the vertical, the y-axis.

§ **11.** The distance of any point measured *from* the y-axis parallel to the x-axis, will be represented by x, and called its abscissa; and the distance of any point measured *from* the x-axis parallel to the y-axis, will be represented by y, and called its ordinate. The distance of any point *from* the origin will be represented by r, and called its radius vector.

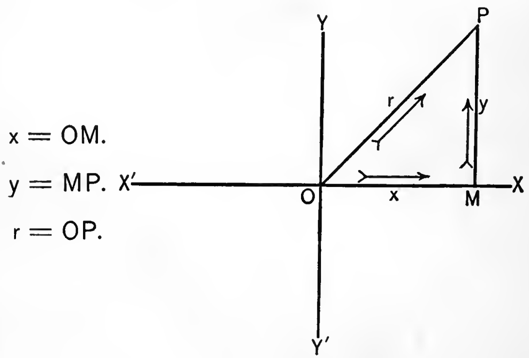


FIG. 2.

§ **12.** We will adopt the convention, that is, the AGREEMENT *that the direction* FROM *left* TO *right of* FIXED *horizontal lines is* POSITIVE, *the* OPPOSITE *direction is* NEGATIVE; *and that the direction* FROM *below* UPWARDS *of* FIXED *vertical lines is* POSITIVE, *the* OPPOSITE *direction is* NEGATIVE.

§ **13.** *By convention, the directions of the sides of an angle are said to be positive if taken from the vertex outwards, the opposite directions are negative.*

b. Conventions about Angles.

§ **14.** The turning of a line which generates an angle may be either counter-clockwise or clockwise. *Counter-clockwise rotation we will call, by convention, positive; the opposite, negative. An angle generated by positive rotation we will call positive; an angle generated by negative rotation, negative.*

§ **15.** The four quadrants are numbered the first, the second, the third, and the fourth, respectively, beginning with XOY, and taken in the order of positive rotation.

§ **16.** The initial line of any single angle will always be taken as coinciding with OX, and the angle will be said to be of that quadrant in which its terminal line lies. In any composite angle the initial line of any angle coincides with the terminal line of the angle preceding, and may, for the moment, be considered as coinciding with OX.

B. Triangle of Reference.

§ **17.** *If at any point of the* x*-axis, or of the* y*-axis, a perpendicular be erected and produced to intersect the rotating line of a given angle at the point* P, *a right triangle will be formed, which is called the* TRIANGLE OF REFERENCE *for the given angle. The abscissa of* P *is always the base, its ordinate, the perpendicular, and its radius vector, the hypotenuse of the triangle of reference.*

C. Trigonometric Functions.

§ **18.** *The six ratios which can be formed by using the three sides of the triangle of reference of a given angle, two at a time, are called the six primary trigonometric functions of the angle.*

The sine of the angle is the ratio of the perpendicular to the hypotenuse.

The cosine of the angle is the ratio of the base to the hypotenuse.

The tangent of the angle is the ratio of the perpendicular to the base.

The cotangent of the angle is the ratio of the base to the perpendicular.

The secant of the angle is the ratio of the hypotenuse to the base.

The cosecant of the angle is the ratio of the hypotenuse to the perpendicular.

§ **19.** The following figures show to the eye, how, by the definition, triangles of reference are constructed for an angle of each quadrant.

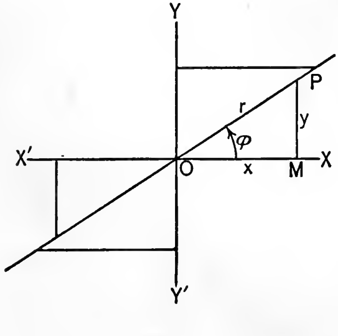


FIG. 3.

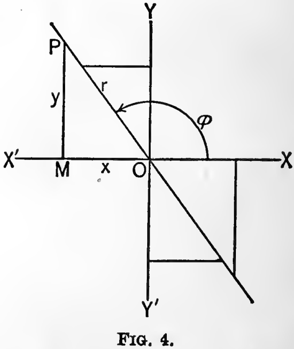


FIG. 4.

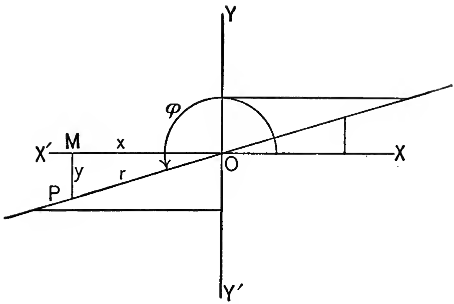


FIG. 5.

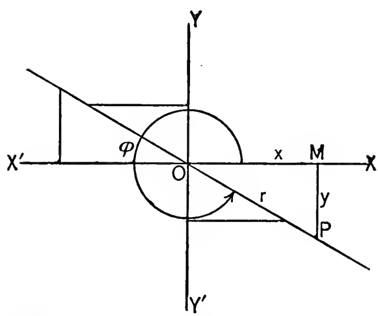


FIG. 6.

§ **20.** It is to be observed, and the student should carefully prove, that the triangle of reference for any angle includes four species, that all the triangles that can be constructed are geometrically similar, and that in passing from a triangle of one species to a triangle of any other species, either no change, or two changes of sign occur in the terms of any trigonometric function, and that, therefore, the function is unaltered. This work should be done as is shown below for the sine and cosine of an angle of the first quadrant.

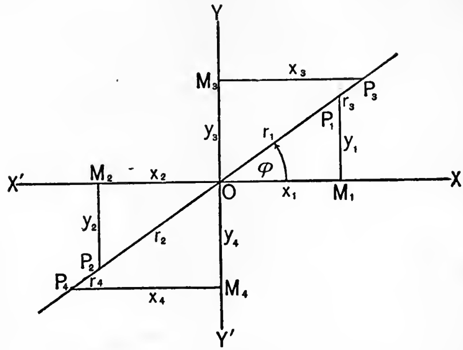


FIG. 7.

sine of ϕ, written sin ϕ, $= \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{M_1P_1}{OP_1} = \frac{y_1}{r_1} = \frac{M_2P_2}{OP_2}$

$$= \frac{y_2}{r_2} = \frac{OM_3}{OP_3} = \frac{y_3}{r_3} = \frac{OM_4}{OP_4} = \frac{y_4}{r_4}.$$

cosine of ϕ, written cos ϕ, $= \frac{\text{base}}{\text{hypotenuse}} = \frac{OM_1}{OP_1} = \frac{x_1}{r_1} = \frac{OM_2}{OP_2}$

$$= \frac{x_2}{r_2} = \frac{M_3P_3}{OP_3} = \frac{x_3}{r_3} = \frac{M_4P_4}{OP_4} = \frac{x_4}{r_4}.$$

We shall use as abbreviations for tangent of ϕ, cotangent of ϕ, secant of ϕ, and cosecant of ϕ, respectively, tan ϕ, ctn ϕ, sec ϕ, and csc ϕ.

§ **21.** Since the trigonometric functions are abstract numbers, all the arithmetical operations can be performed upon them. It is customary to indicate the nth power of sin ϕ by $\sin^n \phi$; similarly for the other trigonometric functions.

§ **22.** In addition to the primary functions previously defined, two secondary functions are usually defined as follows:

The versed-sine of ϕ, written vrs ϕ, $= 1 - \cos \phi$,
The coversed-sine of ϕ, written cvs ϕ, $= 1 - \sin \phi$.

Two others are more rarely defined as follows:

The suversed-sine of ϕ, written svs ϕ, $= (2 - \text{vrs}\ \phi)$,
$= 1 + \cos\phi$.

The sucoversed-sine of ϕ, written scs ϕ, $= (2 - \text{cvs}\ \phi)$,
$= 1 + \sin\phi$.

§ **23.** By referring to Figs. 3–6 we see that the sine, cosine, secant, and cosecant of an angle of the third quadrant are negative, because the terms of these ratios have opposite signs, and that its tangent and cotangent are positive, because the terms of these ratios have the same sign. Similarly, by a careful examination of Figs. 3–6, the student should prove the following table for the signs of the functions of angles of the four quadrants.

Functions.	Quadrants.			
	1	2	3	4
sine and cosecant	+	+	−	−
cosine and secant	+	−	−	+
tangent and cotangent . .	+	−	+	−

D. Mutual Relations of Functions.

§ **24.** From the definitions of the trigonometric functions it is evident that:

$$a.\ \sin\phi = \frac{1}{\csc\phi};\quad \cos\phi = \frac{1}{\sec\phi};\quad \tan\phi = \frac{1}{\text{ctn}\ \phi};$$

that is, the sine and the cosecant, the cosine and the secant, the tangent and the cotangent of an angle ϕ, are the reciprocals of each other, *always*.

b. $\tan\phi = \frac{\sin\phi}{\cos\phi}$; $\text{ctn}\,\phi = \frac{\cos\phi}{\sin\phi}$, *always.*

c. Since, of whatever quadrant the angle ϕ may be, that is, whatever may be the intrinsic signs of x, y, and r, x^2, y^2, and r^2 are *always* positive, and $y^2 + x^2 = r^2$, *always,*

$$\therefore \frac{y^2}{r^2} + \frac{x^2}{r^2} = 1, \quad \text{that is,} \quad \sin^2\phi + \cos^2\phi = 1, \quad \textit{always.}$$

Similarly, $\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2}$, that is, $\tan^2\phi + 1 = \sec^2\phi$, *always.*

And $\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}$, that is, $\text{ctn}^2\phi + 1 = \csc^2\phi$, *always.*

d. From the preceding relations, it is evident that some of the functions may be expressed in terms of others. It can be shown that all the functions may be expressed in terms of any one, which may be chosen at pleasure as the independent function. For example, all the functions may be expressed in terms of the tangent as follows :

$\text{ctn}\,\phi = \frac{1}{\tan\phi}$. Since $\tan^2\phi + 1 = \sec^2\phi$,

$$\therefore \sec\phi = \pm\sqrt{\tan^2\phi + 1}.$$

Since $\cos\phi = \frac{1}{\sec\phi}$, $\therefore \cos\phi = \pm\frac{1}{\sqrt{\tan^2\phi + 1}}$. Since

$\tan\phi = \frac{\sin\phi}{\cos\phi}$, $\therefore \sin\phi = \pm\frac{\tan\phi}{\sqrt{\tan^2\phi + 1}}$. And since

$\csc\phi = \frac{1}{\sin\phi}$, $\therefore \csc\phi = \pm\frac{\sqrt{\tan^2\phi + 1}}{\tan\phi}$.

It is left as an exercise for the student to complete the following table.

Functions.	In terms of					
	$\sin\phi$	$\cos\phi$	$\tan\phi$	$\text{ctn}\,\phi$	$\sec\phi$	$\csc\phi$
$\sin\phi$			$\pm\dfrac{\tan\phi}{\sqrt{\tan^2\phi+1}}$			
$\cos\phi$			$\pm\dfrac{1}{\sqrt{\tan^2\phi+1}}$			
$\tan\phi$			$\tan\phi$			
$\text{ctn}\,\phi$			$\dfrac{1}{\tan\phi}$			
$\sec\phi$			$\pm\sqrt{\tan^2\phi+1}$			
$\csc\phi$			$\pm\dfrac{\sqrt{\tan^2\phi+1}}{\tan\phi}$			

E. Examples. II.

1. Of what quadrant is each of the following angles? 38°, 317°, 2530°, -1155°, 275^g, 31^r, -13.4^r, -823°, -423^g, $625^\circ\,43'\,26''$, $761^g\,87^{\backprime}\,95^{\backprime\backprime}$, -27.364^r.

2. Determine the sign of each of the trigonometric functions of the angles in example 1.

3. If ϕ is an angle of the first quadrant, find the values of the other trigonometric functions, having given: $\sin\phi=\frac{3}{5}$, $\cos\phi=\frac{12}{13}$, $\tan\phi=\frac{7}{24}$, $\text{ctn}\,\phi=\frac{3}{4}$, $\sec\phi=3$, $\csc\phi=\frac{7}{2}$.

4. If ϕ is an angle of the second quadrant, find the other trigonometric functions, having given: $\sin\phi=\frac{1}{5}$, $\cos\phi=-\frac{2}{5}$, $\tan\phi=-\frac{1}{3}$, $\text{ctn}\,\phi=-2$, $\sec\phi=-7$, $\csc\phi=10$.

5. If ϕ is an angle of the third quadrant, find the other trigonometric functions, having given: $\sin\phi=-\frac{1}{2}$, $\cos\phi=-\sqrt{\frac{1}{2}}$, $\tan\phi=\sqrt{3}$, $\sec\phi=-5$, $\text{ctn}\,\phi=1$, $\csc\phi=-2$.

6. If ϕ is an angle of the fourth quadrant, find the other trigonometric functions, having given: $\sin\phi=-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos\phi=\frac{4}{5}$, $\tan\phi=-\sqrt{8}$, $\text{ctn}\,\phi=-2\sqrt{6}$, $\sec\phi=\dfrac{2\sqrt{2}}{\sqrt{3}+1}$, $\csc\phi=-\sqrt{5}$.

7. Determine all the possible values of the other trigonometric functions, having given: $\sin\phi = -\sqrt{\tfrac{1}{2}}$, $\cos\phi = -\sqrt{\tfrac{1}{3}}$, $\tan\phi = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, $\text{ctn}\,\phi = -4\sqrt{3}$, $\sec\phi = \sqrt{2}$, $\csc\phi = a$.

8. If ϕ is an angle of any quadrant, draw triangles of reference for the following angles: $4n\dfrac{\pi}{2}+\phi$, $(4n+1)\dfrac{\pi}{2}+\phi$, $(4n+2)\dfrac{\pi}{2}+\phi$, $(4n+3)\dfrac{\pi}{2}+\phi$.

Prove the following identities:

9. $\sin\phi\csc\phi = \cos\phi\sec\phi = \tan\phi\,\text{ctn}\,\phi = 1.$

10. $\cos\phi + \text{vrs}\,\phi = \sin\phi + \text{cvs}\,\phi = \text{svs}\,\phi - \cos\phi = \text{scs}\,\phi - \sin\phi = 1.$

11. $\text{vrs}\,\phi\,\text{svs}\,\phi = 1 - \cos^2\phi = \sin^2\phi.$

12. $\text{cvs}\,\phi\,\text{scs}\,\phi = 1 - \sin^2\phi = \cos^2\phi.$

13. $\text{svs}\,\phi\,\text{scs}\,\phi = 1 + \sin\phi + \cos\phi + \sin\phi\cos\phi.$

14. $\cos\phi\tan\phi = \sin\phi.$

15. $\sin\phi\,\text{ctn}\,\phi = \cos\phi.$

16. $(\sec\phi + \tan\phi)(\sec\phi - \tan\phi) = (\csc\phi + \text{ctn}\,\phi)(\csc\phi - \text{ctn}\,\phi) = 1.$

17. $(\sin\phi + \cos\phi)^2 + (\sin\phi - \cos\phi)^2 = 2.$

18. $\tan\phi + \text{ctn}\,\phi = \sec\phi\csc\phi.$

19. $\cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1 = 1 - 2\sin^2\phi.$

20. $(\cos\phi + \sin\phi)(\csc\phi - \sec\phi) = \text{ctn}\,\phi - \tan\phi.$

21. $\sin\phi(1 + \tan\phi) + \cos\phi(1 + \text{ctn}\,\phi) = \sec\phi + \csc\phi.$

22. $\sin^4\phi + \cos^4\phi = 1 - 2\sin^2\phi\cos^2\phi.$

23. $\sin^4\phi - \cos^4\phi = \sin^2\phi - \cos^2\phi.$

24. $\tan^2\phi - \text{ctn}^2\phi = \sec^2\phi - \csc^2\phi.$

25. $(1 + \tan\phi)^2 + (1 - \tan\phi)^2 = 2\sec^2\phi.$

26. $(1 + \cos\phi)^2 + (1 + \sin\phi)^2 = 3 + 2(\sin\phi + \cos\phi).$

27. $(\sin\phi + \cos\phi)(\tan\phi + \text{ctn}\,\phi) = \sec\phi + \csc\phi.$

28. $(\sec\phi + \csc\phi)^2 = (1 + \tan\phi)^2 + (1 + \text{ctn}\,\phi)^2.$

29. $\tan\phi(1 - \text{ctn}^2\phi) + \text{ctn}\,\phi(1 - \tan^2\phi) = 0.$

30. $\sin^3\phi + \cos^3\phi = (\sin\phi + \cos\phi)(1 - \sin\phi\cos\phi).$

31. $\sin^3\phi - \cos^3\phi = (\sin\phi - \cos\phi)(1 + \sin\phi\cos\phi).$

32. $(\sec\phi - \tan\phi)^2 = \dfrac{1 - \sin\phi}{1 + \sin\phi}.$

33. $(\sec\phi + \tan\phi)^2 = \dfrac{1 + \sin\phi}{1 - \sin\phi}.$

34. $\csc\phi + \text{ctn}\,\phi = \dfrac{1}{\csc\phi - \text{ctn}\,\phi}.$

35. $(\csc\phi + \text{ctn}\,\phi)^2 = \dfrac{1 + \cos\phi}{1 - \cos\phi}.$

36. $(\csc\phi - \text{ctn}\,\phi)^2 = \dfrac{1 - \cos\phi}{1 + \cos\phi}.$

37. $\dfrac{\tan\phi + \tan\theta}{\text{ctn}\,\phi + \text{ctn}\,\theta} = \tan\phi\tan\theta.$

38. $\dfrac{\text{ctn}\,\phi + \tan\theta}{\tan\phi + \text{ctn}\,\theta} = \text{ctn}\,\phi\tan\theta.$

39. $\sin^2\phi\tan^2\phi + \cos^2\phi\,\text{ctn}^2\phi = \tan^2\phi + \text{ctn}^2\phi - 1.$

40. $\dfrac{\sin\phi + \sin\theta}{\cos\phi - \cos\theta} = \dfrac{\cos\phi + \cos\theta}{\sin\theta - \sin\phi}.$

41. $\dfrac{\tan\phi + \sec\phi - 1}{1 + \tan\phi - \sec\phi} = \tan\phi + \sec\phi.$

42. $\dfrac{1 - \sec\phi + \tan\phi}{1 + \sec\phi - \tan\phi} = \dfrac{\sec\phi + \tan\phi - 1}{\sec\phi + \tan\phi + 1}.$

43. $\dfrac{1 + \csc\phi + \text{ctn}\,\phi}{1 + \csc\phi - \text{ctn}\,\phi} = \dfrac{\csc\phi + \text{ctn}\,\phi - 1}{\text{ctn}\,\phi - \csc\phi + 1}.$

44. $(1 + \sec\phi + \tan\phi)(1 + \csc\phi + \text{ctn}\,\phi)$
$= 2(1 + \tan\phi + \text{ctn}\,\phi + \sec\phi + \csc\phi).$

CHAPTER III.

A. Variation of Trigonometric Functions.

§ 25. It is necessary to trace the changes in the values of the trigonometric functions as the angle changes. The following two examples are sufficient to show how this is done.

Beginning with an indefinitely small angle, we will trace the changes in the sine and tangent of the angle as the angle increases. One of the triangles of reference which lie in the same quadrant as the terminal line of the angle, will be used. In the discussion of the sine, the hypotenuse of the triangle of reference will be kept constant in length. The sign $\doteq$ will be used for the words "*approaches as its limit.*" 0° is the limit of an indefinitely small angle. It is evident that the sine of an indefinitely small angle must be indefinitely small, since the perpendicular of the triangle of reference must be indefinitely small; that is, $\sin a^\circ = \epsilon$, where $a^\circ \doteq 0^\circ$, and $\epsilon \doteq 0$. Since the variables $\sin a^\circ$ and ϵ are always equal however near they approach their limits, therefore their limits, $\sin 0^\circ$ and 0, are equal; that is, $\sin 0^\circ = 0$. As the angle increases between 0° and 90°, the perpendicular of the triangle of reference increases, therefore the sine of the angle increases, and *vice versa*. When an angle approaches 90° as its limit, the length of the perpendicular approaches as its limit the length of the hypotenuse; therefore, as in the preceding case, $\sin 90^\circ = 1$. As the angle increases between 90° and 180°, the perpendicular decreases, therefore the sine of the angle decreases, and *vice versa*.

When an angle approaches 180° as its limit, its sine approaches 0 as its limit, and $\sin 180^\circ = 0$. The sine of an angle in the third and fourth quadrants is negative. As the angle increases between 180° and 270°, the perpendicular increases in length, that is, decreases in algebraic value, therefore the sine decreases, and *vice versa.* By the use of limits it is evident that $\sin 270° = -1$. As the angle increases between 270° and 360°, the perpendicular decreases in length, that is, increases in algebraic value, therefore the sine increases, and *vice versa.* And by the use of limits $\sin 360° = \sin 0° = 0$. If the angle should continue to increase, the same set of values of the sine would recur periodically at intervals of 360°. This recurrence of values is called the periodicity of the sine, and will be noticed again.

§ **26.** In the discussion of the tangent, the base of the triangle of reference will be kept constant in length. By reasoning like that used for the sine it is evident that $\tan 0° = 0$, and that as the angle increases between 0° and 90° the tangent increases, and *vice versa.* As an angle approaches 90° as its limit, the perpendicular of the triangle of reference increases indefinitely, therefore the tangent increases in numerical value without limit, which is what is meant when we say $\tan 90° = \infty$. In the second quadrant the base of the triangle of reference is negative, hence the tangent is negative. As the angle increases between 90° and 180°, the perpendicular decreases, therefore the tangent increases in algebraic value, and *vice versa.* And by the doctrine of limits $\tan 180° = 0$. In the third quadrant both the base and the perpendicular of the triangle of reference are negative, hence the tangent is positive. As the angle increases between 180° and 270°, the perpendicular increases, therefore the tangent increases, and *vice versa.* As the angle approaches 270° as its limit, the perpendicular increases indefinitely, therefore the tangent increases in numerical value without limit, and we say, as before, $\tan 270° = \infty$. In the fourth quadrant, the perpendicular only of the triangle of reference is negative, hence the tangent is negative. As the angle increases between 270° and 360°, the perpendicular decreases in length, therefore the tangent increases in algebraic value, and *vice versa*. And by the doctrine of limits tan 360° = tan 0° = 0. The tangent also has a recurrence of values, at intervals of 180°.

§ **27.** The following pictures, called graphs, of the pre-

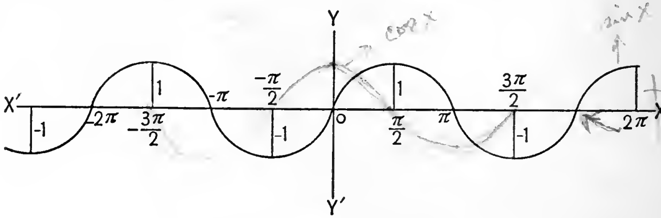


FIG. 8. — Graph of the Sine.

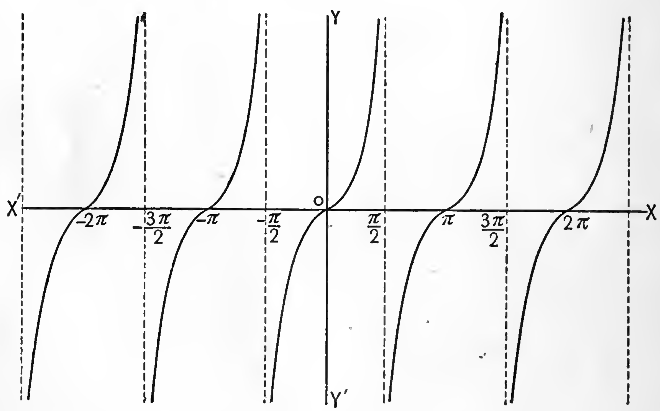


FIG. 9. — Graph of the Tangent.

ceding tracing of values are given to aid the imagination. Abscissas represent values of the angle expressed in circular measure; ordinates represent the corresponding values of the functions.

The student should now trace the changes in the values of the other trigonometric functions, and, independently, obtain a table like the following.

Angle	sin	cos	tan	ctn	sec	csc	
0°	∓ 0	1	∓ 0	$\mp \infty$	1	$\mp \infty$	−360°
Q_1	inc dec	dec inc	inc dec	dec inc	inc dec	dec inc	Q_1
90°	1	± 0	$\pm \infty$	± 0	$\pm \infty$	1	−270°
Q_2	dec inc	dec inc	inc dec	dec inc	inc dec	inc dec	Q_2
180°	± 0	-1	∓ 0	$\mp \infty$	-1	$\pm \infty$	−180°
Q_3	dec inc	inc dec	inc dec	dec inc	dec inc	inc dec	Q_3
270°	-1	∓ 0	$\pm \infty$	± 0	$\mp \infty$	-1	−90°
Q_4	inc dec	inc dec	inc dec	dec inc	dec inc	dec inc	Q_4
360°	∓ 0	1	∓ 0	$\mp \infty$	1	$\mp \infty$	0°
	sin	cos	tan	ctn	sec	csc	Angle

In this table, for an angle generated by positive rotation, *i.e.*, an increasing angle, the upper word goes with the value of the angle taken from the left-hand column; and for an angle generated by negative rotation, the lower word goes with the value of the angle taken from the right-hand column. The + and − before 0 and ∞ indicate how the function changes sign as the angle passes through the quadrant limit.

B. Line Representations of the Trigonometric Functions.

§ **28.** If in each of the primary trigonometric functions, or ratios, a positive linear unit is taken for the denominator, the number of linear units in the numerator will represent the abstract number of units in the ratio, and the line which is the numerator of the ratio will *completely represent* the function. The student should remember that the line is *not* the function, but *represents* it. This restriction gives the following constructions.

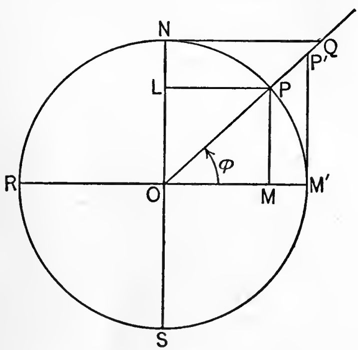


FIG. 10.

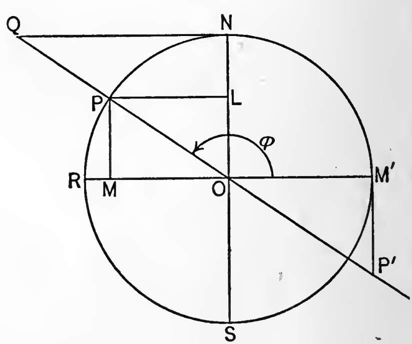


FIG. 11.

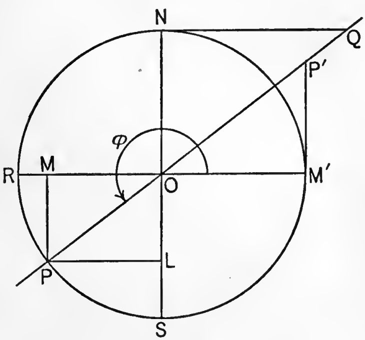


FIG. 12.

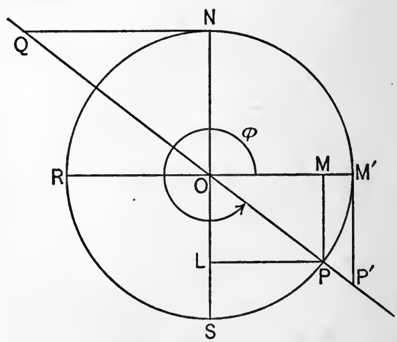


FIG. 13.

In these figures, for example, $\sin\phi = \frac{\mathsf{MP}}{\mathsf{OP}}$, but OP is taken equal to a linear unit. Hence, sin ϕ *is represented by* MP. cos ϕ *is represented by* OM. Similarly the student should prove for each quadrant, that

tan ϕ *is represented by* M′P′,
ctn ϕ *is represented by* NQ,
sec ϕ *is represented by* OP′,
csc ϕ *is represented by* OQ.

The secondary trigonometric functions are also represented by lines as follows :

vrs ϕ *is represented by* MM′,
cvs ϕ *is represented by* LN,
svs ϕ *is represented by* RM,
scs ϕ *is represented by* SL.

§ **29.** The student should ascertain the signs of the functions in the four quadrants by means of the line representatives. The accompanying figure is intended forcibly to impress on the student's mind that the sine and cosecant of an angle of a quadrant *above* the horizontal line are positive, *below* negative; that the cosine and secant of an angle *to the right* of the vertical line are positive, *to the left* negative; and that the signs of the tangent and cotangent are given by the combination of the signs of the sine and cosine, quadrant by quadrant. This is all readily seen by line representation.

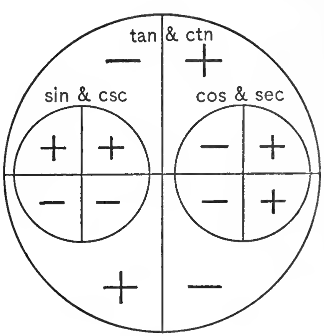


FIG. 14.

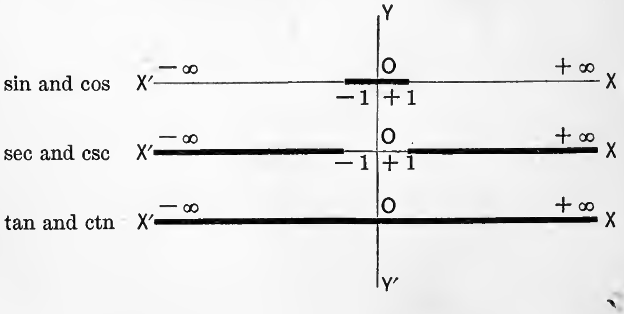
C. Inverse Trigonometric Functions.

§ **30.** If two variables are connected by any relation, either one of them may be considered as the independent variable, and the other as the dependent variable or direct function. When the dependent variable is made independent, the independent variable becomes the dependent variable and inverse function.

If x is any function of the angle ϕ, ϕ is the angle whose function is x. These two statements of the same relation, viewed from the different standpoints alluded to, are symbolized as follows: $x = f(\phi)$, and $\phi = f^{-1}(x)$, and are read, x is a function of ϕ, and ϕ is the anti-function of x. For example, if $x = \sin \phi$, ϕ *is the angle* whose sine is x, which is written $\phi = \sin^{-1} x$, and is read, ϕ is the anti-sine of x. Some writers express the inverse functional relations as follows: $\phi = \text{arc} \sin x$, $\theta = \text{arc} \cos x$, and similarly for the other functions.

D. Examples. III.

1. By representing positive numbers by distances to the right of the origin on the x-axis, and negative numbers by distances to the left of the origin, and the values which functions can have by heavy lines, prove that the following is a correct graphic representation of the possible range of values of the primary trigonometric functions.



The results of this problem are very important. The student should carefully remember both the range of *possible* and *impossible* values of the functions.

Trace the changes in value of the following expressions :

2. $\cos\phi$.	4. $\sec\phi$.	6. $\text{vrs}\,\phi$.	8. $\text{svs}\,\phi$.
3. $\text{ctn}\,\phi$.	5. $\csc\phi$.	7. $\text{cvs}\,\phi$.	9. $\text{scs}\,\phi$.

Construct an angle :

10.	(*a*) Of the	1st,	and of the	2d	quadrant,	whose sine is	$\frac{1}{2}$.
	(*b*) "	3d	"	4th	"	" "	$-\frac{1}{2}$.
11.	(*a*) "	1st	"	4th	"	" cosine is	$\frac{1}{4}$.
	(*b*) "	2d	"	3d	"	" "	$-\frac{1}{4}$.
12.	(*a*) "	1st	"	3d	"	" tangent is	2.
	(*b*) "	2d	"	4th	"	" "	-2.
13.	(*a*) "	1st	"	3d	"	" ctn is	$\frac{1}{3}$.
	(*b*) "	2d	"	4th	"	" "	$-\frac{1}{3}$.
14.	(*a*) "	1st	"	4th	"	" secant is	$\sqrt{2}$.
	(*b*) "	2d	"	3d	"	" "	$-\sqrt{2}$.
15.	(*a*) "	1st	"	2d	"	" csc is	$\sqrt{5}$.
	(*b*) "	3d	"	4th	"	" "	$-\sqrt{5}$.

Construct all possible values of :

16. $\sin^{-1}(\pm\frac{3}{5})$.	18. $\tan^{-1}(\pm 3)$.	20. $\sec^{-1}(\pm 5)$.
17. $\cos^{-1}(\pm\frac{2}{7})$.	19. $\text{ctn}^{-1}(\pm\frac{4}{3})$.	21. $\csc^{-1}(\pm 2)$.

22. Can the following be constructed? (*a*) $\sin^{-1}\frac{3}{2}$, (*b*) $\cos^{-1}(-\frac{1}{5})$, (*c*) $\tan^{-1} m$, (*d*) $\text{ctn}^{-1}(\pm 676)$, (*e*) $\sec^{-1}(\pm\frac{1}{3})$, (*f*) $\csc^{-1}\sqrt{3}$.

23. Are the following possible?

(*a*) $\tan^{-1}(\pm 36{,}947)$, (*b*) $\sec^{-1}(\pm 0.7435)$, (*c*) $\sin^{-1}(\pm 25)$.

CHAPTER IV.

A. Trigonometric Functions of Integral Multiples of $\frac{\pi}{2}, \pm \phi$.

§ **31.** The complement of an angle is the angle obtained by subtracting the given angle from 90°. The supplement of an angle is the angle obtained by subtracting the given angle from 180°. Thus, the complement of 50° is 40°; the complement of 125° is −35°. The supplement of 110° is 70°; the supplement of 200° is −20°.

Two cases of integral multiples of $\frac{\pi}{2}$ are to be distinguished; first, when the multiplier is odd; second, when it is even. Under the first case the simplest multiples of $\frac{\pi}{2}$ are, when the angle is expressed in degrees, 90° and 270°, the multipliers being 1 and 3, respectively. Under the second case the simplest examples are, similarly, 0°, 180°, and 360°, the multipliers being 0, 2, and 4, respectively. The examples enumerated lead us to seek the trigonometric functions of the special angles $\pm \phi$, $(90° \pm \phi)$, $(180° \pm \phi)$, $(270° \pm \phi)$, and $(360° \pm \phi)$ in terms of the functions of ϕ.

In comparing the functions of $(m\, 90° \pm \phi)$ with the functions of ϕ the triangles of reference will, for convenience, be made geometrically equal by keeping the hypotenuse constant in length; but it is to be observed that the demonstrations in no wise require this restriction, but depend on the underlying principle that the homologous sides of similar triangles are proportional. Also in this discussion only the primary functions will be considered.

§ **32.** The functions of ϕ have already been given. If ϕ is taken in *any* quadrant, the terminal lines of ϕ and of $(-\phi)$ will lie on the same side of the y-axis and on opposite sides of the x-axis. From the numerical equality of the angles, the triangles of reference for ϕ and for $(-\phi)$ can always be constructed geometrically equal, by drawing a single perpendicular to the x-axis and producing it in opposite directions until it cuts the terminal lines. Trigonometrically the bases and the hypotenuses of the triangles of reference are equal, the perpendiculars are negatives of each other. It follows that all the functions of $(-\phi)$ into which the perpendicular of the triangle of reference enters, will be the negatives of the same functions of ϕ, and that the others will be equal to the same functions of ϕ.

The following figure illustrates the preceding results.

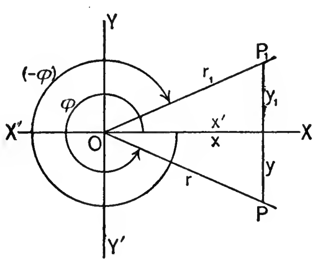


FIG. 15.

We have here $x_1 = x$, $r_1 = r$, and $y_1 = -y$. Therefore,

$$\sin(-\phi) = \frac{y_1}{r_1} = \frac{-y}{r} = -\sin\phi,\ \csc(-\phi) = \frac{r_1}{y_1} = \frac{r}{-y} = -\csc\phi.$$

$$\cos(-\phi) = \frac{x_1}{r_1} = \frac{x}{r} = \cos\phi,\ \sec(-\phi) = \frac{r_1}{x_1} = \frac{r}{x} = \sec\phi.$$

$$\tan(-\phi) = \frac{y_1}{x_1} = \frac{-y}{x} = -\tan\phi,\ \text{ctn}(-\phi) = \frac{x_1}{y_1} = \frac{x}{-y} = -\text{ctn}\,\phi.$$

§ 33. In constructing the triangle of reference for the angle $(m\,90° + \phi)$, the angle will be constructed as $(\phi + m\,90°)$. We will first consider the cases where $m = 0$, 2, and 4.

$\phi + 0.90° = \phi$, whose functions have been given by definition.

If 180° is added to ϕ in any quadrant, the triangles of reference for $(\phi + 180°)$ and ϕ are geometrically equal. The hypotenuses are equal, and the perpendiculars and bases are respectively negatives of each other. It follows that those functions of $(\phi + 180°)$ into which either the base, or the perpendicular, enters singly, are the negatives of the same functions of ϕ; and that those functions of $(\phi + 180°)$ into which both the base and the perpendicular enter are equal to the same functions of ϕ.

The following figure illustrates the preceding results.

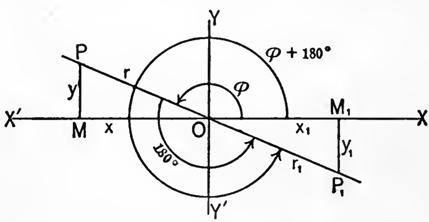


FIG. 16.

We have here $r_1 = r$, $x_1 = -x$, and $y_1 = -y$. Therefore,

$$\sin(180° + \phi) = \frac{y_1}{r_1} = \frac{-y}{r} = -\sin\phi,$$

$$\csc(180° + \phi) = \frac{r_1}{y_1} = \frac{r}{-y} = -\csc\phi,$$

$$\cos(180° + \phi) = \frac{x_1}{r_1} = \frac{-x}{r} = -\cos\phi,$$

$$\sec(180° + \phi) = \frac{r_1}{x_1} = \frac{r}{-x} = -\sec\phi.$$

$$\tan(180° + \phi) = \frac{y_1}{x_1} = \frac{-y}{-x} = \tan\phi,$$

$$\operatorname{ctn}(180° + \phi) = \frac{x_1}{y_1} = \frac{-x}{-y} = \operatorname{ctn}\phi.$$

It is to be observed that the triangle of reference for $(\phi + 180°)$ may be obtained by revolving the triangle of reference for ϕ about the origin in the plane of the figure through a positive angle of 180°. This reverses the directions, and, therefore, the signs of the base and the perpendicular.

It is also to be observed that in the preceding discussion ϕ is an angle of any quadrant, therefore is unrestricted, and may be positive or negative.

§ **34.** If ϕ is replaced by $(-\phi)$, the formulas of § 33 become, by the use of the formulas of § 32,

$$\sin(180° - \phi) = -\sin(-\phi) = \sin\phi,$$
$$\therefore \csc(180° - \phi) = \csc\phi.$$
$$\cos(180° - \phi) = -\cos(-\phi) = -\cos\phi,$$
$$\therefore \sec(180° - \phi) = -\sec\phi.$$
$$\tan(180° - \phi) = \tan(-\phi) = -\tan\phi,$$
$$\therefore \operatorname{ctn}(180° - \phi) = -\operatorname{ctn}\phi.$$

Observing that $(\phi - 180°) = -(180° - \phi)$, we have by the formulas of § 32, and the present section,

$$\sin(\phi - 180°) = -\sin(180° - \phi) = -\sin\phi,$$
$$\therefore \csc(\phi - 180°) = -\csc\phi.$$

Similarly for the other functions. Or, otherwise, the triangle of reference for $(\phi - 180°)$ is identical with the triangle of reference for $(\phi + 180°)$, and the functions of $(\phi - 180°)$ are the same as the functions of $(\phi + 180°)$.

§ **35.** If 360° is added to ϕ, the triangles of reference for ϕ, and $(\phi + 360°)$ coincide, and the functions of ϕ, and $(\phi + 360°)$ are identical, and we have :

$$\sin(360° + \phi) = \sin\phi, \qquad \therefore \csc(360° + \phi) = \csc\phi.$$
$$\cos(360° + \phi) = \cos\phi, \qquad \therefore \sec(360° + \phi) = \sec\phi.$$
$$\tan(360° + \phi) = \tan\phi, \qquad \therefore \text{ctn}(360° + \phi) = \text{ctn}\,\phi.$$

Replacing ϕ by $(-\phi)$ these formulas give by § 32,

$$\sin(360° - \phi) = \sin(-\phi) = -\sin\phi,$$
$$\therefore \csc(360° - \phi) = -\csc\phi.$$
$$\cos(360° - \phi) = \cos(-\phi) = \cos\phi,$$
$$\therefore \sec(360° - \phi) = \sec\phi.$$
$$\tan(360° - \phi) = \tan(-\phi) = -\tan\phi,$$
$$\therefore \text{ctn}(360° - \phi) = -\text{ctn}\,\phi.$$

Observing that $(\phi - 360°) = -(360° - \phi)$, we have by the formulas of § 32, and the present section :

$$\sin(\phi - 360°) = -\sin(360° - \phi) = \sin\phi.$$

Similarly for the other functions.

The functions of $(\phi - 360°)$ can also be directly seen to be identical with the functions of ϕ, since the triangles of reference of the two angles coincide.

§ 36. We will now consider the cases of $(m\,90° \pm \phi)$ where $m = 1$, and 3. If 90° is added to the angle ϕ of any quadrant, the triangles of reference for $(\phi + 90°)$ and ϕ, are geometrically equal. The hypotenuses are equal, the perpendicular and the base of the triangle of reference for $(\phi + 90°)$ are equal respectively to the base and the negative of the perpendicular of the triangle of reference for ϕ. It follows that any function of $(\phi + 90°)$ is the co-function of ϕ, and that it has the opposite or the same sign according as the base of the triangle of reference for $(\phi + 90°)$ does, or does not, enter into the ratio. The following figure illustrates the preceding results.

We have here, $r_1 = r$, $y_1 = x$, $x_1 = -y$.

$$\sin(\phi + 90°) = \frac{y_1}{r_1} = \frac{x}{r} = \cos\phi.$$

$$\cos(\phi+90^\circ)=\frac{x_1}{r_1}=\frac{-y}{r}=-\sin\phi.$$

$$\tan(\phi+90^\circ)=\frac{y_1}{x_1}=\frac{x}{-y}=-\text{ctn}\,\phi.$$

$$\therefore \csc(\phi+90^\circ)=\sec\phi.$$
$$\sec(\phi+90^\circ)=-\csc\phi.$$
$$\text{ctn}(\phi+90^\circ)=-\tan\phi.$$

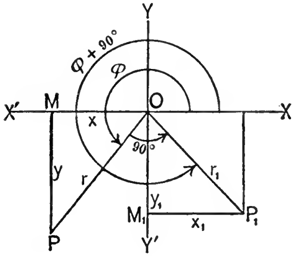


FIG. 17.

It is to be observed that the triangle of reference for $(\phi+90)$ may be obtained by revolving the triangle of reference for ϕ about the origin in the plane of the figure, through a positive angle of 90°. This converts the base into the perpendicular, and the perpendicular becomes the base changed in sign.

§ **37.** If in the formulas of § 36, ϕ is replaced by $(-\phi)$, they become by the use of the formulas of § 32:

$$\sin(90^\circ-\phi)=\cos(-\phi)=\cos\phi,\quad \therefore \csc(90^\circ-\phi)=\sec\phi.$$
$$\cos(90^\circ-\phi)=-\sin(-\phi)=\sin\phi,\quad \therefore \sec(90^\circ-\phi)=\csc\phi.$$
$$\tan(90^\circ-\phi)=-\text{ctn}(-\phi)=\text{ctn}\,\phi,\quad \therefore \text{ctn}(90^\circ-\phi)=\tan\phi.$$

These formulas make clear the meaning of the prefix "co." They show, for example, that the cosine of an angle ϕ is the *complement's* sine, [$\sin(90^\circ-\phi)$]. That is, the word "cosine" is an abbreviation for "complement's sine." Similarly for the other functions.

Since $(\phi - 90°) = -(90° - \phi)$, we have, by the formulas of § 32, $\sin(\phi - 90°) = -\sin(90° - \phi) = -\cos\phi$,

$$\therefore \csc(\phi - 90°) = -\sec\phi.$$

§ 38. If 270° is added to the angle ϕ of any quadrant, the triangles of reference for $(\phi + 270°)$ and ϕ are geometrically equal, the hypotenuses are equal, the base and the perpendicular of the triangle of reference for $(\phi + 270°)$ are equal, respectively, to the perpendicular and the negative of the base of the triangle of reference for ϕ. It follows that any function of $(\phi + 270°)$ is the co-function of ϕ, and that it has the same, or opposite sign, according as the perpendicular of the triangle of reference for $(\phi + 270°)$ does not, or does, enter into the ratio.

The following figure illustrates the preceding results.

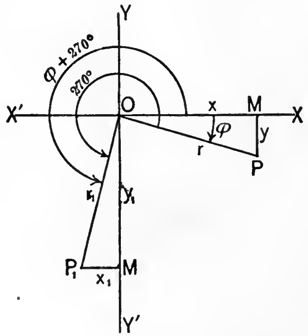


FIG. 18.

We have here, $r_1 = r,\ x_1 = y,\ y_1 = -x.$

$$\sin(\phi + 270°) = \frac{y_1}{r_1} = \frac{-x}{r} = -\cos\phi.$$

$$\cos(\phi + 270°) = \frac{x_1}{r_1} = \frac{y}{r} = \sin\phi.$$

$$\tan(\phi + 270°) = \frac{y_1}{x_1} = \frac{-x}{y} = -\operatorname{ctn}\phi.$$

$$\therefore \csc(\phi + 270°) = -\sec\phi, \qquad \sec(\phi + 270°) = \csc\phi,$$
$$\operatorname{ctn}(\phi + 270°) = -\tan\phi.$$

§ **39.** The angle $(360° + \phi)$ may be obtained by adding $180°$ to the angle $(\phi + 180°)$ of § 33, whence the triangle of reference for $(\phi + 180°)$ is turned as before, through a positive angle of $180°$, and, therefore, its base and perpendicular suffer a second reversal. And, in general, the addition of $m\ 180°$ to ϕ, produces, or does not produce, a reversal of the signs of the perpendicular and base of the triangle of reference, according as m is an odd or even integer, positive or negative.

The angle $(\phi + 270°)$ may be obtained by adding $180°$ to $(\phi + 90°)$ of § 36. This turns the triangle of reference, as before, through $180°$, and reverses the signs of both the base and perpendicular of the triangle of reference for $(\phi + 90°)$. The perpendicular and base of the triangle of reference for ϕ have been converted respectively into the base and perpendicular of the triangle of reference for $(\phi + 270°)$, and have suffered, respectively, two, and one, reversals of sign. And, in general, the addition of $(2n + 1)\ 90°$ to ϕ, converts the base and perpendicular of the triangle of reference for ϕ, into the perpendicular and base, respectively, of the triangle of reference for $\phi + (2n + 1)\ 90°$. And the base of the resulting triangle of reference has, or has not, while the perpendicular has not, or has, been reversed in sign, according as n is even or odd.

§ **40.** It should be carefully observed that in the preceding results the functions of any angle combined with a multiple of $90°$ are equal numerically to the same, or to the co-functions of the angle, according as the multiplier is one of the even numbers, 0, 2, 4; or one of the odd numbers, 1, 3.

It can be seen from the discussion of the next section that the following general relations exist:

$$\text{Function}\left(2\,n\frac{\pi}{2}\pm\phi\right)=\text{Function }\phi,\text{ numerically},$$

$$\text{Function}\left((2\,n+1)\frac{\pi}{2}\pm\phi\right)=\text{Co-function }\phi,\text{ numerically},$$

where n is zero, or any integer, positive or negative.

§ **41.** We will now give a general demonstration of the relations of the functions of $\left(m\frac{\pi}{2}\pm\phi\right)$ to the functions of ϕ, where m is any integer, positive or negative. Let x_1, y_1, r_1, and x, y, r, respectively, be the sides of the triangles of reference for $\phi+m\frac{\pi}{2}$, and ϕ. The problem divides itself into two cases:

a. When m is Even, and Equal to $2n$.

We have by § 39:

$$x_1=(-1)^n x,$$
$$y_1=(-1)^n y,$$
$$r_1=r.$$

$$\sin\left(2\,n\frac{\pi}{2}+\phi\right)=\frac{y_1}{r_1}=\frac{(-1)^n y}{r}=(-1)^n\sin\phi,$$

$$\therefore \csc\left(2\,n\frac{\pi}{2}+\phi\right)=(-1)^n\csc\phi.$$

$$\cos\left(2\,n\frac{\pi}{2}+\phi\right)=\frac{x_1}{r_1}=\frac{(-1)^n x}{r}=(-1)^n\cos\phi,$$

$$\therefore \sec\left(2\,n\frac{\pi}{2}+\phi\right)=(-1)^n\sec\phi.$$

$$\tan\left(2\,n\frac{\pi}{2}+\phi\right)=\frac{y_1}{x_1}=\frac{(-1)^n y}{(-1)^n x}=\tan\phi,$$

$$\therefore \operatorname{ctn}\left(2\,n\frac{\pi}{2}+\phi\right)=\operatorname{ctn}\phi.$$

§ **42.** Replacing ϕ by $(-\phi)$ in the formulas of § 41, we get by the formulas of § 32:

$$\sin\left(2n\frac{\pi}{2}-\phi\right)=(-1)^n\sin(-\phi)=-(-1)^n\sin\phi,$$

$$\therefore \csc\left(2n\frac{\pi}{2}-\phi\right)=-(-1)^n\csc\phi.$$

$$\cos\left(2n\frac{\pi}{2}-\phi\right)=(-1)^n\cos(-\phi)=(-1)^n\cos\phi,$$

$$\therefore \sec\left(2n\frac{\pi}{2}-\phi\right)=(-1)^n\sec\phi.$$

$$\tan\left(2n\frac{\pi}{2}-\phi\right)=\tan(-\phi)=-\tan\phi,$$

$$\therefore \text{ctn}\left(2n\frac{\pi}{2}-\phi\right)=-\text{ctn}\,\phi.$$

Formulas for functions of $\left(\phi-2n\frac{\pi}{2}\right)$ are covered by the formulas of § 41, since n may be negative as well as positive. Or, otherwise, the triangle of reference for $\left(\phi-2n\frac{\pi}{2}\right)$ is identical with the triangle of reference for $\left(\phi+2n\frac{\pi}{2}\right)$.

b. When m is Odd, and Equal to (2 n + 1).

§ **43.** We have by § 39:

$$x_1=-(-1)^n y,$$
$$y_1=(-1)^n x,$$
$$r_1=r.$$

$$\sin\left((2n+1)\frac{\pi}{2}+\phi\right)=\frac{y_1}{r_1}=\frac{(-1)^n x}{r}=(-1)^n\cos\phi,$$

$$\therefore \csc\left((2n+1)\frac{\pi}{2}+\phi\right)=(-1)^n\sec\phi.$$

$$\cos\left((2n+1)\frac{\pi}{2}+\phi\right)=\frac{x_1}{r_1}=\frac{-(-1)^n y}{r}=-(-1)^n\sin\phi,$$

$$\therefore\ \sec\left((2n+1)\frac{\pi}{2}+\phi\right)=-(-1)^n\csc\phi.$$

$$\tan\left((2n+1)\frac{\pi}{2}+\phi\right)=\frac{y_1}{x_1}=\frac{(-1)^n x}{-(-1)^n y}=-\operatorname{ctn}\phi,$$

$$\therefore\ \operatorname{ctn}\left((2n+1)\frac{\pi}{2}+\phi\right)=-\tan\phi.$$

§ **44.** Replacing ϕ by $(-\phi)$ in the formulas of § 43, we have by the formulas of § 32:

$$\sin\left((2n+1)\frac{\pi}{2}-\phi\right)=(-1)^n\cos(-\phi)\doteq(-1)^n\cos\phi,$$

$$\therefore\ \csc\left((2n+1)\frac{\pi}{2}-\phi\right)=(-1)^n\sec\phi.$$

$$\cos\left((2n+1)\frac{\pi}{2}-\phi\right)=-(-1)^n\sin(-\phi)=(-1)^n\sin\phi,$$

$$\therefore\ \sec\left((2n+1)\frac{\pi}{2}-\phi\right)=(-1)^n\csc\phi.$$

$$\tan\left((2n+1)\frac{\pi}{2}-\phi\right)=-\operatorname{ctn}(-\phi)=\operatorname{ctn}\phi,$$

$$\therefore\ \operatorname{ctn}\left((2n+1)\frac{\pi}{2}-\phi\right)=\tan\phi.$$

Formulas for functions of $\left(\phi-(2n+1)\frac{\pi}{2}\right)$ are covered by the formulas of § 43, since n may be negative as well as positive.

§ **45.** *The results of* §§ 31–44 *inclusive,* are contained in the following formulas:

$$\sin\left(2\,n\frac{\pi}{2}\pm\phi\right)=\pm(-1)^n\sin\phi,$$

$$\cos\left(2\,n\frac{\pi}{2}\pm\phi\right)=(-1)^n\cos\phi,$$

$$\tan\left(2\,n\frac{\pi}{2}\pm\phi\right)=\pm\tan\phi,$$

$$\sin\left((2\,n+1)\frac{\pi}{2}\pm\phi\right)=(-1)^n\cos\phi,$$

$$\cos\left((2\,n+1)\frac{\pi}{2}\pm\phi\right)=\mp(-1)^n\sin\phi,$$

$$\tan\left((2\,n+1)\frac{\pi}{2}\pm\phi\right)=\mp\operatorname{ctn}\phi.$$

§ 46. The formulas of § 45 may be put into the following form:

Since $(-1)^n\sin\phi=\sin[(-1)^n\phi]$ whether n is even or odd,
$(-1)^n\cos\phi=\cos[(-1)^n\phi]$ only when n is even,
$(-1)^n\tan\phi=\tan[(-1)^n\phi]$ whether n is even or odd,

we have, dividing by the sign factor of the second member of the first three formulas of § 45, and writing the second members first:

$$\sin\phi=\pm(-1)^n\sin\left(2\,n\frac{\pi}{2}\pm\phi\right)$$

$$=\sin\left(\pm(-1)^n 2\,n\frac{\pi}{2}+(-1)^n\phi\right),$$

$$\cos\phi=(-1)^n\cos\left(2\,n\frac{\pi}{2}\pm\phi\right)$$

$$=\cos\left(4\,m\frac{\pi}{2}\pm\phi\right)\text{ where } 2\,m=n,$$

$$\tan \phi = \pm \tan\left(2\,n\frac{\pi}{2} \pm \phi\right) = \tan\left(\pm 2\,n\frac{\pi}{2} + \phi\right).$$

Since n may be positive or negative at pleasure, the preceding can be written more simply :

$$\sin \phi = \sin(n\,\pi + (-1)^n\,\phi),$$
$$\cos \phi = \cos(2\,m\,\pi \pm \phi),$$
$$\tan \phi = \tan(n\,\pi + \phi).$$

§ **47.** If in the formulas of § 46, $(n\,\pi + (-1)^n\,\phi)$, $(2\,m\,\pi \pm \phi)$, $(n\,\pi + \phi)$, be successively denoted by Φ, then Φ represents all possible angles whose sines, cosines, and tangents, are respectively equal to the sine, cosine, and tangent of the particular angle ϕ. Thus, if $\sin \phi = \frac{1}{2}$, the simplest angle may be constructed whose sine is $\frac{1}{2}$, and the general solution is $\Phi = n\,\pi + (-1)^n\,\phi$, where ϕ is the angle constructed.

B. Periodicity of the Trigonometric Functions.

§ **48.** When $f(x + na) = f(x)$, where n is zero, or any positive or negative integer, $f(x)$ is said to be periodic, and "a" is called the period of $f(x)$. From the formulas of § 41, it appears that if, in them, $n = 2\,m$, where m is zero, or any positive or negative integer,

$$\sin(\phi + 2\,m\,\pi) = \sin \phi, \qquad \csc(\phi + 2\,m\,\pi) = \csc \phi,$$
$$\cos(\phi + 2\,m\,\pi) = \cos \phi, \qquad \sec(\phi + 2\,m\,\pi) = \sec \phi,$$

and that always,

$$\tan(\phi + n\,\pi) = \tan \phi, \qquad \text{ctn}(\phi + n\,\pi) = \text{ctn}\,\phi,$$

and that, therefore, the period of the sine, cosine, secant, and cosecant is $2\,\pi$; and of the tangent and cotangent is π.

C. The Functions of Some Special Angles.

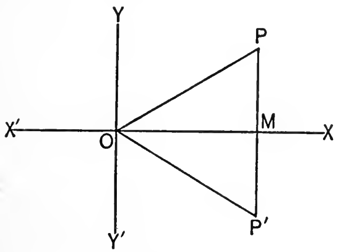


FIG. 19.

§ **49.** To find the functions of 30°. Let XOP be equal to 30°, and let OMP be its triangle of reference.

Produce PM to P′, making MP′ = PM, and draw OP′. The triangle OPP′ is geometrically equilateral, and MP $= \frac{1}{2}$ OP.

$$OM = \sqrt{\overline{OP}^2 - \overline{MP}^2} = \sqrt{\overline{OP}^2 - \frac{\overline{OP}^2}{4}} = \frac{\sqrt{3}}{2} OP.$$

From this we get,

$$\sin 30° = \frac{MP}{OP} = \frac{\frac{1}{2} OP}{OP} = \frac{1}{2}, \quad \therefore \csc 30° = 2.$$

$$\cos 30° = \frac{OM}{OP} = \frac{\frac{\sqrt{3}}{2} OP}{OP} = \frac{\sqrt{3}}{2}, \quad \therefore \sec 30° = \tfrac{2}{3}\sqrt{3}.$$

$$\tan 30° = \frac{MP}{OM} = \frac{\frac{1}{2} OP}{\frac{\sqrt{3}}{2} OP} = \frac{1}{\sqrt{3}}, \quad \therefore \text{ctn } 30° = \sqrt{3}.$$

§ **50.** To find the functions of 45°. Let XOP be equal to 45°, and let OMP be its triangle of reference. By geometry OM = MP, and $\overline{OM}^2 + \overline{MP}^2 = 2\,\overline{OM}^2 = 2\,\overline{MP}^2 = \overline{OP}^2$,

$$\therefore OM = MP = \frac{1}{\sqrt{2}} OP.$$

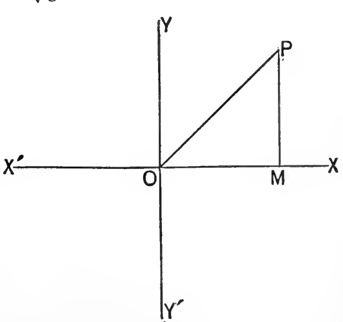


FIG. 20.

From this we get,

$$\sin 45° = \frac{MP}{OP} = \frac{\frac{1}{\sqrt{2}} OP}{OP} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}, \therefore \csc 45° = \sqrt{2}.$$

$$\cos 45° = \frac{OM}{OP} = \frac{\frac{1}{\sqrt{2}} OP}{OP} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}, \therefore \sec 45° = \sqrt{2}.$$

$$\tan 45° = \frac{MP}{OM} = 1, \therefore \text{ctn } 45° = 1.$$

Or otherwise thus : Since $45° = 90° - 45°$, we have,

$$\sin 45° = \sin (90° - 45°) = \cos 45°.$$
$$\sin^2 45° + \cos^2 45° = 2 \sin^2 45° = 2 \cos^2 45° = 1,$$
$$\therefore \sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}.$$

The other functions of 45° may be obtained from the sine and cosine of 45°. Since the angles 60°, 120°, 150°, 210°, 240°, 300°, 330°, are cases of $(m\,90° \pm 30°)$; and the angles 135°, 225°, 315°, are cases of $(m\,90° \pm 45°)$, therefore their functions can be at once obtained in terms of the known functions of 30° and 45°, respectively, by the use of the formulas of §§ 33–38, inclusive.

D. Examples. IV.

1. Find the complements and the supplements of the following angles: 57°, − 210°, − 65°, − 116°, 3^r, $-2^r.15$.

Find the functions of the following angles :

2. 60°; 3. 120°; 4. 135°; 5. 150°; 6. 210°; 7. 225°; 8. 240°; 9. 300°; 10. 315°; 11. 330°.

For convenience of reference the following table is given :

Angles.	Functions.					
	sin	cos	tan	ctn	sec	csc
0°, 360° 180°	0	± 1	0	∞	± 1	∞
30°, 150° 210°, 330°	$(\pm)\frac{1}{2}$	$({}^{+}{}_{-})\frac{1}{2}\sqrt{3}$	$(+\,-)\frac{1}{3}\sqrt{3}$	$(+\,-)\sqrt{3}$	$({}^{+}{}_{-})\frac{2}{3}\sqrt{3}$	$(\pm)2$
45°, 135° 225°, 315°	$(\pm)\frac{1}{2}\sqrt{2}$	$({}^{+}{}_{-})\frac{1}{2}\sqrt{2}$	$(+\,-)1$	$(+\,-)1$	$({}^{+}{}_{-})\sqrt{2}$	$(\pm)\sqrt{2}$
60°, 120° 240°, 300°	$(\pm)\frac{1}{2}\sqrt{3}$	$({}^{+}{}_{-})\frac{1}{2}$	$(+\,-)\sqrt{3}$	$(+\,-)\frac{1}{3}\sqrt{3}$	$({}^{+}{}_{-})2$	$(\pm)\frac{2}{3}\sqrt{3}$
90° 270°	± 1	0	∞	0	∞	± 1

In this table $(\pm)$ is an abbreviation for $(\pm\,\pm)$, $({}^{+}{}_{-})$ for $(\pm\,\mp)$, and $(+\,-)$ for $\left(\begin{smallmatrix}+ & -\\ + & -\end{smallmatrix}\right)$.

12. Find the functions of the following angles: $-30°$, 420°, 765°, 510°, $-585°$, 3360°, 1260°.

Prove that,

13. $\sin 295° = \cos 155°$.
14. $\tan 217° = \text{ctn}\,(-127°)$.
15. $\sec(-37°) = -\csc 233°$.
16. $\cos 1145° = \sin(-205°)$.
17. $\text{ctn}\,(-127°) = -\tan 503°$.
18. $\csc(-464°) = \sec 1606°$.
19. Given $\tan 200° = 0.364$, find $\sec 70°$.
20. Given $\sin 165° = \dfrac{1}{2\sqrt{2}}(\sqrt{3}-1)$, find $\tan 285°$.

If A, B, and C are the angles of a plane triangle, prove that:

21. $\sin A = \sin(B+C)$, $\cos A = -\cos(B+C)$, $\tan A = -\tan(B+C)$, $\sin A/2 = \cos\frac{1}{2}(B+C)$, $\cos A/2 = \sin\frac{1}{2}(B+C)$, $\tan A/2 = \text{ctn}\,\frac{1}{2}(B+C)$.
22. $\sin A/2\ \sin B/2\ \sin C/2 = \cos\frac{1}{2}(B+C)\cos\frac{1}{2}(C+A)\cos\frac{1}{2}(A+B)$.
23. $\cos A/2\ \cos B/2\ \cos C/2 = \sin\frac{1}{2}(B+C)\sin\frac{1}{2}(C+A)\sin\frac{1}{2}(A+B)$.
24. $\tan A/2\ \tan B/2\ \tan C/2 = \text{ctn}\,\frac{1}{2}(B+C)\ \text{ctn}\,\frac{1}{2}(C+A)\ \text{ctn}\,\frac{1}{2}(A+B)$.
25. $\sin(A+B+C) = 0$, $\cos(A+B+C) = -1$, $\tan(A+B+C) = 0$.
26. $\sin\frac{1}{2}(A+B+C) = 1$, $\cos\frac{1}{2}(A+B+C) = 0$, $\tan\frac{1}{2}(A+B+C) = \infty$.
27. $\sin A = -\sin(2A+B+C)$.

28. $\sin\frac{1}{2}(A - B) = \cos\frac{1}{2}(2B + C)$, $\tan\frac{1}{2}(3A + B) = \text{ctn}\frac{1}{2}(C - 2A)$.
29. $\sin n(A + B) = -(-1)^n \sin nC$.
30. $\cos n(A + B) = (-1)^n \cos nC$.
31. $\tan n(A + B) = -\tan nC$.

Find an expression for all possible angles:

32. Whose secants are the same as $\sec\phi$.
33. Whose cosecants are the same as $\csc\phi$.
34. Whose cotangents are the same as $\text{ctn}\,\phi$.
35. Obtain formulas for the functions of $(\phi - 2n\,\pi/2)$ in terms of the functions of ϕ, where n is always a positive integer.
36. The same for $[\phi - (2n + 1)\,\pi/2]$.
37. Obtain the functions of $2n\,\pi/2$, $(2n+1)\,\pi/2$, $4n\,\pi/2$, $(4n+1)\,\pi/2$, $(4n + 2)\,\pi/2$, $(4n + 3)\,\pi/2$.

Note on the General Solution of Trigonometric Equations. — A general solution of a trigonometric equation is all possible angles which satisfy the given equation. When the equation is algebraic in form, the general solution may be conveniently expressed by a finite number of general formulas, each of which gives an infinite number of values. Thus if the given equation be $\sin\phi = \frac{1}{2}$, an expression containing all possible angles, whose sines are $\frac{1}{2}$, is $n\pi + (-1)^n\,\pi/6$ (by formulas of § 47), and $\Phi = n\pi + (-1)^n\,\pi/6$ is the general solution. This is an algebraic equation of the first degree in terms of $\sin\phi$, and there is one general formula which contains an infinite number of values in itself.

If $2\sin^2\phi - 3\sin\phi + 1 = 0$ is the given equation, we have:

$$(2\sin\phi - 1)(\sin\phi - 1) = 0, \qquad \sin\phi = \tfrac{1}{2}, \qquad \sin\phi = 1,$$
$$\Phi_1 = n\pi + (-1)^n\,\pi/6, \qquad \Phi_2 = n\pi + (-1)^n\,\pi/2,$$

as the general solution. And it will be found that the number of general formulas is not greater than the degree of the equation.

Obtain general solutions of the following equations:

38. $\sin\phi = \sqrt{\tfrac{1}{2}}$.
39. $\cos\phi = \frac{1}{2}$.
40. $\tan\phi = -\sqrt{\tfrac{1}{3}}$.
41. $\sec^2\phi - \tan\phi - 1 = 0$.
42. $2\sin\phi\cos\phi - 2\sin\phi + \cos\phi - 1 = 0$.
43. $2\sin\phi = \csc\phi$.
44. $3\tan\phi - \text{ctn}\,\phi = 0$.
45. $4\cos\phi = \sec\phi$.
46. $3\sin^2\phi + \cos^2\phi - \sin\phi = 0$.
47. $\cos^3\phi - 6\cos^2\phi + 11\cos\phi - 6 = 0$.

Show that the following general formulas give the same sets of values, though not corresponding value by value, for the same value of n.

48. $\Phi = n\pi + (-1)^n 5\pi/6$, and $2n\pi + \pi/2 \pm \pi/3$.

49. $\Phi = -\pi/4 - n\pi + (-1)^n \pi/4$, $\Phi = \pi/4 + 2n\pi \pm \pi/4$.

50. Show that one value of each of the expressions, $\sin^{-1} x + \cos^{-1} x$, $\tan^{-1} x + \text{ctn}^{-1} x$, $\sec^{-1} x + \csc^{-1} x$, is $\pi/2$.

51. Find one value of each of the following expressions: $\sin^{-1} 3/5 + \sin^{-1} 4/5$, $\sec^{-1} 5 + \sec^{-1} \frac{5}{2\sqrt{6}}$, $\tan^{-1} 3 + \tan^{-1} 1/3$.

52. If ϕ is the simplest value of $\sin^{-1} x$, $\sec^{-1} x$, $\tan^{-1} x$, successively, prove that in general,

$$\begin{aligned} \sin^{-1} x + \cos^{-1} x &= [n\pi + (-1)^n \phi] + [2m\pi \pm (\pi/2 - \phi)] \\ &= [2(2m+n) \pm 1]\frac{\pi}{2} + [(-1)^n \pm 1]\phi. \\ \sec^{-1} x + \csc^{-1} x &= [2n\pi \pm \phi] + [m\pi + (-1)^m (\pi/2 - \phi)] \\ &= [2(2n+m) + (-1)^m]\pi/2 + [(-1)^{m+1} \pm 1]\phi. \\ \tan^{-1} x + \text{ctn}^{-1} x &= [n\pi + \phi] + [m\pi + \pi/2 - \phi] \\ &= [2(m+n) + 1]\pi/2, \end{aligned}$$

where m and n may be 0, or any positive or negative integers, and are independent of each other. How may m and n and the double sign be taken to give the particular solutions of problem 50 ?

CHAPTER V.

A. Projection.

§ **51.** The orthogonal projection of a finite line upon an axis of reference is the segment of the axis *taken from* the foot of the perpendicular from the origin of the line upon the axis, *to* that from the term.

It will be proved that such projection of a straight line is equal to the length of the line multiplied by the cosine of the angle between the positive directions of the axis and the line. For convenience, projection will be considered under two cases:

a. When the Line and Axis are Co-planar.

Thus, when the line OP and axes, OX and OY, lie in the same plane, the projections of OP on the axes, or fixed parallel lines, are the segments of the axes, or fixed parallel lines, taken from the feet of the perpendiculars dropped upon them from O, to those from P. These segments are obviously the base and the perpendicular, respectively, of the triangle of reference for the angle XOP.

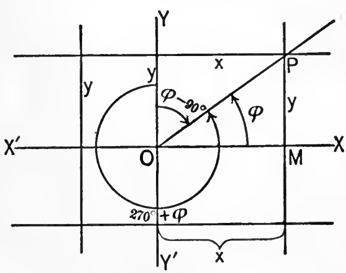


FIG. 21.

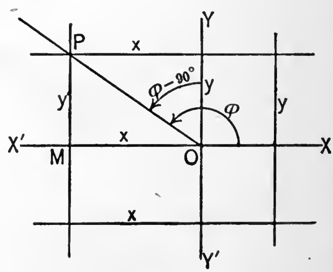


FIG. 22.

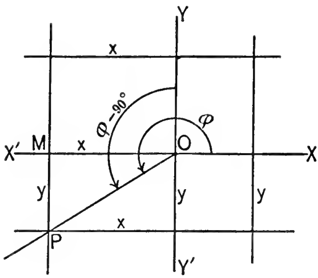


FIG. 23.

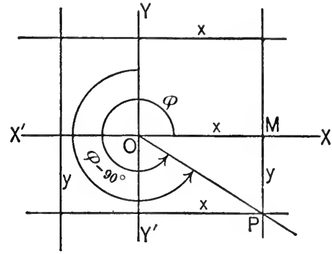


FIG. 24.

In Figs. 21–24 we have

$$\mathsf{YOP} = \mathsf{YOX} + \mathsf{XOP} = -90° + \phi + \mathsf{n}\,360°.$$

$$\sin\phi = \frac{\mathsf{MP}}{\mathsf{OP}} = \frac{\mathsf{y}}{\mathsf{r}}; \quad \sin\phi = \cos(\phi - 90° + \mathsf{n}\,360°).$$

$$\cos\phi = \frac{\mathsf{OM}}{\mathsf{OP}} = \frac{\mathsf{x}}{\mathsf{r}}. \quad \text{And these give}$$

$$\mathsf{y} = \mathsf{r}\cos(\phi - 90° + \mathsf{n}\,360°), \text{ and } \mathsf{x} = \mathsf{r}\cos\phi,$$

or $$\mathsf{MP} = \mathsf{OP}\cos\mathsf{YOP}, \text{ and } \mathsf{OM} = \mathsf{OP}\cos\mathsf{XOP}.$$

Since n may be 0 or any integer without affecting the preceding result, it will be so chosen, for uniformity, that the angle YOP shall be always the simplest positive angle.

b. When the Line and Axis are not Co-planar.

§ 52. When the line and axis are not co-planar, the angle between them is the angle taken from the positive direction of the axis to the positive direction of a line parallel to the given line, and co-planar with the axis. It is evident that if two planes o and p be passed respectively through the extremities O and P of the line OP, and perpendicular to the axis, the segment of the axis from plane o to plane p, is the projection of OP on the axis.

In the figure, O′P′ is parallel to OP, cuts the axis, and pierces o and p in O′ and P′, respectively. O′M and P′M″ lie in o and p, and cut the axis at M and M′.

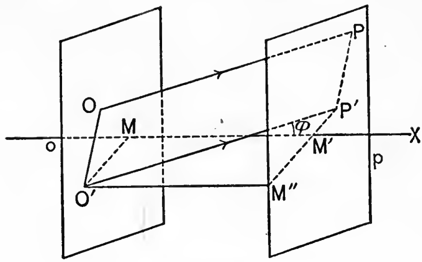


FIG. 25.

By geometry, $O'P' = OP$, $O'M'' = MM'$. $O'M$ and $P'M''$ are perpendicular to MX.

By a. $O'M'' = O'P' \cos \phi$, or $MM' = OP \cos \phi$.

§ **53.** It follows from the preceding work that the projection of the straight line OP upon any axis is equal to the sum of the projections of the parts of an uninterrupted broken line going from O to P.

The following notation will be used. The projection of a line, l, on an axis, a, will be denoted by l_a.

B. Functions of Sums and Differences of Angles.

§ **54.** In this figure, from the triangle of reference for β,

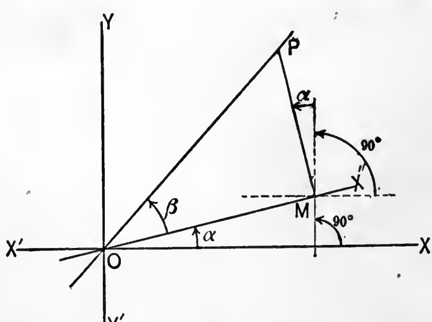


FIG. 26.

$$\sin\beta = \frac{\mathsf{MP}}{\mathsf{OP}}, \qquad \therefore \mathsf{MP} = \mathsf{OP}\sin\beta.$$

$$\cos\beta = \frac{\mathsf{OM}}{\mathsf{OP}}, \qquad \therefore \mathsf{OM} = \mathsf{OP}\cos\beta.$$

By § 53,

$$\mathsf{OP}_y = \mathsf{OM}_y + \mathsf{MP}_y.$$
$$\mathsf{OP}_x = \mathsf{OM}_x + \mathsf{MP}_x.$$

These become :

$$\mathsf{OP}\sin(\alpha+\beta) = \mathsf{OM}\sin\alpha + \mathsf{MP}\cos\alpha,^*$$

or, by substituting the values of OM and MP,

$$\mathsf{OP}\sin(\alpha+\beta) = \mathsf{OP}\sin\alpha\cos\beta + \mathsf{OP}\cos\alpha\sin\beta.$$
$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$
$$\mathsf{OP}\cos(\alpha+\beta) = \mathsf{OM}\cos\alpha + \mathsf{MP}\cos(90^\circ+\alpha),^*$$

which reduces to

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

§ 55. If in the formulas of § 54 β is replaced by $(-\beta)$ we obtain :

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

§ 56. From the formulas of §§ 54 and 55

$$\tan(\alpha\pm\beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

* The values of these projections are in all cases justified by the principle that if a line taken in the opposite direction, and for the sake of the triangle of reference for β, has its sign changed, then the projection becomes $-\mathsf{l}\cos(\phi \mp 180^\circ) = \mathsf{l}\cos\phi$, generally as before, with the l essentially negative.

$$\operatorname{ctn}(\alpha \pm \beta) = \frac{1}{\tan(\alpha \pm \beta)} = \frac{1 \mp \tan\alpha \tan\beta}{\tan\alpha \pm \tan\beta}$$
$$= \frac{\operatorname{ctn}\alpha \operatorname{ctn}\beta \mp 1}{\operatorname{ctn}\beta \pm \operatorname{ctn}\alpha}.$$

C. Functions of Multiple Angles.

§ **57.** If in the formulas of §§ 54 and 56 β is replaced by α,

$$\sin(\alpha + \alpha) = \sin 2\alpha = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha.$$
$$\cos(\alpha + \alpha) = \cos 2\alpha = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$
$$\tan(\alpha + \alpha) = \tan 2\alpha = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$
$$\operatorname{ctn}(\alpha + \alpha) = \operatorname{ctn} 2\alpha = \frac{\operatorname{ctn}\alpha\operatorname{ctn}\alpha - 1}{\operatorname{ctn}\alpha + \operatorname{ctn}\alpha} = \frac{\operatorname{ctn}^2\alpha - 1}{2\operatorname{ctn}\alpha}.$$

D. Functions of Sub-Multiple Angles.

§ **58.** By the formulas of § 57

$$\cos 2\alpha' = 1 - 2\sin^2\alpha' = 2\cos^2\alpha' - 1.$$

If in these formulas $2\alpha'$ is replaced by α,

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}, \quad \therefore 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha.$$
$$\therefore \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

And $$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1, \quad \therefore 2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha.$$
$$\therefore \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$
$$\therefore \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$
$$\therefore \operatorname{ctn}\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.$$

E. Formulas for the Sums and Differences of the Sines of two Angles, also of the Cosines of two Angles.

§ **59.** By §§ 54 and 55

$$\sin(\alpha' + \beta') + \sin(\alpha' - \beta') = 2 \sin \alpha' \cos \beta'.$$
$$\sin(\alpha' + \beta') - \sin(\alpha' - \beta') = 2 \cos \alpha' \sin \beta'.$$
$$\cos(\alpha' + \beta') + \cos(\alpha' - \beta') = 2 \cos \alpha' \cos \beta'.$$
$$\cos(\alpha' - \beta') - \cos(\alpha' + \beta') = 2 \sin \alpha' \sin \beta'.$$

In these formulas let $\alpha' + \beta' = \alpha,$

$$\alpha' - \beta' = \beta.$$

$$\alpha' = \tfrac{1}{2}(\alpha + \beta), \qquad \beta' = \tfrac{1}{2}(\alpha - \beta),$$

and the formulas become:

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta).$$
$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta).$$
$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta).$$
$$\cos \beta - \cos \alpha = 2 \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta).$$

§ **60.** $$\frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{\tan \frac{1}{2}(\alpha + \beta)}{\tan \frac{1}{2}(\alpha - \beta)}.$$

$$\frac{\sin \alpha + \sin \beta}{\cos \alpha + \cos \beta} = \tan \tfrac{1}{2}(\alpha + \beta).$$

$$\frac{\sin \alpha + \sin \beta}{\cos \beta - \cos \alpha} = \text{ctn} \tfrac{1}{2}(\alpha - \beta).$$

$$\frac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta} = \tan \tfrac{1}{2}(\alpha - \beta).$$

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \text{ctn} \tfrac{1}{2}(\alpha + \beta).$$

$$\frac{\cos \alpha + \cos \beta}{\cos \beta - \cos \alpha} = \text{ctn} \tfrac{1}{2}(\alpha + \beta) \text{ ctn} \tfrac{1}{2}(\alpha - \beta).$$

F. Examples. V.

Find the functions of :

1. $15°$. 2. $22\frac{1}{2}°$. 3. $37\frac{1}{2}°$. 4. $75°$.

Prove :

5. $\text{ctn}\, 2\alpha = \frac{1}{2}(\text{ctn}\,\alpha - \tan\alpha)$.
6. $\csc 2\alpha = \frac{1}{2}(\tan\alpha + \text{ctn}\,\alpha)$.
7. $\tan\alpha = \csc 2\alpha - \text{ctn}\, 2\alpha$.
8. $\text{ctn}\,\alpha = \csc 2\alpha + \text{ctn}\, 2\alpha$.
9. $\tan(45° + \frac{1}{2}\alpha) = \sec\alpha + \tan\alpha = \text{ctn}(45° - \frac{1}{2}\alpha)$.
10. $\tan(45° - \frac{1}{2}\alpha) = \sec\alpha - \tan\alpha = \text{ctn}(45° + \frac{1}{2}\alpha)$.
11. $\tan(45° + \frac{1}{2}\alpha)\tan(45° - \frac{1}{2}\alpha) = 1$.
12. $\tan 50° + \text{ctn}\, 50° = 2\sec 10°$.
13. $\tan 50° - \text{ctn}\, 50° = 2\tan 10°$.
14. $\cos(60° + \alpha) + \cos(60° - \alpha) = \cos\alpha$.
15. $\sin(30° + \alpha) + \sin(30° - \alpha) = \cos\alpha$.
16. $\sin(45° + \alpha) = \cos(45° - \alpha)$.

Find :

17. $\sin(\alpha + \beta + \gamma)$.
18. $\cos(\alpha + \beta + \gamma)$.
19. $\tan(\alpha + \beta + \gamma)$.
20. $\sin 3\alpha$ in terms of $\sin\alpha$.
21. $\cos 3\alpha$ in terms of $\cos\alpha$.
22. $\tan 3\alpha$ in terms of $\tan\alpha$.
23. $\sin 4\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$.
24. $\cos 4\alpha$ in terms of $\cos\alpha$.
25. $\tan 4\alpha$ in terms of $\tan\alpha$.
26. $\sin 5\alpha$ in terms of $\sin\alpha$.
27. $\cos 5\alpha$ in terms of $\cos\alpha$.

Prove :

28. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$
$$= 4\sin\tfrac{1}{2}(\beta + \gamma)\sin\tfrac{1}{2}(\gamma + \alpha)\sin\tfrac{1}{2}(\alpha + \beta).$$
29. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)$
$$= 4\cos\tfrac{1}{2}(\beta + \gamma)\cos\tfrac{1}{2}(\gamma + \alpha)\cos\tfrac{1}{2}(\alpha + \beta).$$
30. $\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma = \dfrac{\sin(\alpha + \beta + \gamma)}{\cos\alpha\cos\beta\cos\gamma}$.
31. $\text{ctn}\,\alpha + \text{ctn}\,\beta + \text{ctn}\,\gamma - \text{ctn}\,\alpha\,\text{ctn}\,\beta\,\text{ctn}\,\gamma = \dfrac{-\cos(\alpha + \beta + \gamma)}{\sin\alpha\sin\beta\sin\gamma}$.
32. $\sin^2 x + \sin^2 y + \sin^2 z + \sin^2(x + y + z)$
$$= 2[1 - \cos(y + z)\cos(z + x)\cos(x + y)].$$

33. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 (x + y + z)$
$$= 2\,[1 + \cos (y + z) \cos (z + x) \cos (x + y)].$$

G. Examples. VI.

1. If γ^g, δ°, and ρ^r be the three measures of an angle, prove that
$$\gamma - \delta = \frac{20\,\rho}{\pi}.$$

2. A bicycle is propelled on a circular arc of $\frac{1}{4}$ mile radius at the rate of 30 miles per hour. Through what angle has it turned in two minutes?

Prove the following identities:

3. $(\tan\phi + \text{ctn}\,\phi - 1)(\sin\phi + \cos\phi) = \dfrac{\sec\phi}{\csc^2\phi} + \dfrac{\csc\phi}{\sec^2\phi}.$

4. $\sin^2\phi\,(1 + n\,\text{ctn}^2\phi) + \cos^2\phi\,(1 + n\tan^2\phi)$
$$= \sin^2\phi\,(n + \text{ctn}^2\phi) + \cos^2\phi\,(n + \tan^2\phi).$$

5. $\sin^6\phi - \cos^6\phi = (2\sin^2\phi - 1)(1 - \sin\phi\cos\phi)(1 + \sin\phi\cos\phi).$

6. $\dfrac{(\tan\phi - 1)(\csc\phi - 1)}{\text{ctn}\,\phi - 1} + \dfrac{\text{ctn}\,\phi + 1}{(\tan\phi + 1)(\csc\phi + 1)} = 0.$

7. Find $\sin 18^\circ$ and $\cos 18^\circ$. (Use the geometric properties of the decagon.)

8. Find $\sin 54^\circ$ and $\cos 54^\circ$.

Prove the following identities:

9. $\sin 65^\circ + \cos 65^\circ = \sqrt{2}\cos 20^\circ.$

10. $\sin 78^\circ - \sin 18^\circ + \cos 132^\circ = 0.$

11. $\dfrac{\sin\phi\sin 2\phi + \sin 2\phi\sin 5\phi + \sin 3\phi\sin 10\phi}{\cos\phi\sin 2\phi + \sin 2\phi\cos 5\phi - \cos 3\phi\sin 10\phi} = -\tan 7\phi.$

12. $\dfrac{\sin\phi + \sin 3\phi + \sin 5\phi + \sin 7\phi}{\cos\phi + \cos 3\phi + \cos 5\phi + \cos 7\phi} = \tan 4\phi.$

13. $\dfrac{\sin\phi\sin 2\phi + \sin\phi\sin 4\phi + \sin 2\phi\sin 7\phi}{\sin\phi\cos 2\phi + \sin 2\phi\cos 5\phi + \sin\phi\cos 8\phi} = \tan 5\phi.$

14. $\dfrac{\cos(\alpha - 3\beta) - \cos(3\alpha - \beta)}{\sin 2\alpha + \sin 2\beta} = 2\sin(\alpha - \beta).$

15. $\dfrac{\sin(\alpha + 3\beta) + \sin(3\alpha + \beta)}{\sin 2\alpha + \sin 2\beta} = 2\cos(\alpha - \beta).$

16. $\dfrac{a\sin(\alpha - \beta) + b\sin\alpha + a\sin(\alpha + \beta)}{a\cos(\alpha - \beta) + b\cos\alpha + a\cos(\alpha + \beta)} = \tan\alpha.$

17. $-\sin(\alpha + \beta + \gamma) + \sin(-\alpha + \beta + \gamma) + \sin(\alpha - \beta + \gamma)$
$$+ \sin(\alpha + \beta - \gamma) = 4\sin\alpha\sin\beta\sin\gamma.$$

18. $\cos(\alpha + \beta + \gamma) + \cos(-\alpha + \beta + \gamma) + \cos(\alpha - \beta + \gamma)$
$$+ \cos(\alpha + \beta - \gamma) = 4\cos\alpha\cos\beta\cos\gamma.$$

19. $8\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{-\alpha+\beta+\gamma}{2}\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}$
$= \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 4\cos\alpha\cos\beta\cos\gamma + 1.$

20. $\frac{\text{ctn}\,(n-2)\,\phi\;\text{ctn}\,n\,\phi + 1}{\text{ctn}\,(n-2)\,\phi - \text{ctn}\,n\,\phi} = \frac{1}{2}(\text{ctn}\,\phi - \tan\phi).$

21. $\frac{\sin n\,\phi - \sin(n-2)\,\phi}{\cos(n-2)\,\phi - \cos n\,\phi} = \text{ctn}\,(n-1)\,\phi.$

Justify the following statements:

22. $\sin^{-1}a \pm \sin^{-1}b = \sin^{-1}[a\sqrt{1-b^2} \pm b\sqrt{1-a^2}].$

23. $\cos^{-1}a \pm \cos^{-1}b = \cos^{-1}[ab \mp \sqrt{(1-a^2)(1-b^2)}].$

24. $\tan^{-1}a \pm \tan^{-1}b = \tan^{-1}\frac{a \pm b}{1 \mp ab}.$

25. $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \tan^{-1}\frac{a+b+c-abc}{1-ab-bc-ca}.$

26. $\tan^{-1}a + \tan^{-1}b + \tan^{-1}\frac{1-a-b-ab}{1+a+b-ab} = \frac{\pi}{4}.$

27. $a\sqrt{1-b^2} + b\sqrt{1-a^2} = 1$, given $\sin^{-1}a + \sin^{-1}b = \frac{\pi}{2}.$

28. $\sin^{-1}a - \sin^{-1}b = \cos^{-1}[ab + \sqrt{1-a^2-b^2+a^2b^2}].$

29. $a^2 = \frac{1}{2}$, given $\tan^{-1}\frac{a+1}{a+2} + \tan^{-1}\frac{a-1}{a-2} = \frac{\pi}{4}$

30. $a + b + c = abc$, given $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c = \pi.$

31. $2\sin\frac{\phi}{2} = +\sqrt{1+\sin\phi} + \sqrt{1-\sin\phi}$, given $\phi = 200°.$

32. $2\sin\frac{\phi}{2} = -\sqrt{1+\sin\phi} - \sqrt{1-\sin\phi}$, given $450° < \phi < 630°.$

33. $2\cos\frac{\phi}{2} = -\sqrt{1+\sin\phi} + \sqrt{1-\sin\phi}$ between fixed limits. Determine the limits.

Solve for ϕ in the following equations:

34. $a\cos\phi + b\sin\phi = 1.$

Suggestion: $a\cos\phi + b\sin\phi = a\left(\cos\phi + \frac{b}{a}\sin\phi\right).$ Put $\frac{b}{a} = \tan\alpha$ and get $\cos(\phi - \alpha) = \frac{\cos\alpha}{a}$, from which the general value of ϕ can be found.

35. $\sin\phi + \cos\phi = 1.$

36. $\sin\phi - \cos\phi = 1.$

37. $\sin\phi + \cos\phi = \sqrt{\tfrac{1}{2}}.$

38. $\sin\phi + \sqrt{3}\cos\phi = 1.$

39. $\sqrt{3}\sin\phi - \cos\phi = \sqrt{2}.$

40. $\sin\phi + 2\cos\phi = \frac{1}{3}$.

41. $\cos(\alpha + \phi) - \sin(\alpha + \phi) = \sqrt{2}\cos\beta$.

Construct the graphs of the following expressions:

42. $\sin\phi + \cos\phi$.

43. $\sin\phi - \cos\phi$.

44. $\tan\phi + \operatorname{ctn}\phi$.

45. $\tan\phi - \operatorname{ctn}\phi$.

46. $\sec\phi + \csc\phi$.

47. $\sec\phi - \csc\phi$.

48. $\sec\phi + \tan\phi$.

49. $\sec\phi - \tan\phi$.

CHAPTER VI.

A. Right Triangles.

§ **61.** To solve a triangle is to find its unknown from its known parts. It is shown in plane geometry that to determine a triangle three parts, one of which is a side, must be given. In right triangles the right angle counts as one of the three given parts.

a. Cases.

§ **62.** The following table gives all the cases which can arise in the solution of right triangles.

Case.	Given.	Required.
1	x, y	ϕ, $90° - \phi$, r
2	r, x or y	ϕ, y or x, $90° - \phi$
3	r, ϕ or $90° - \phi$	x, y, $90° - \phi$ or ϕ
4	x or y, ϕ or $90° - \phi$	r, $90° - \phi$ or ϕ, y or x

b. Solutions.

Since the angles ϕ and $90° - \phi$ are acute, and, therefore, of the first quadrant, all their functions are positive. We will

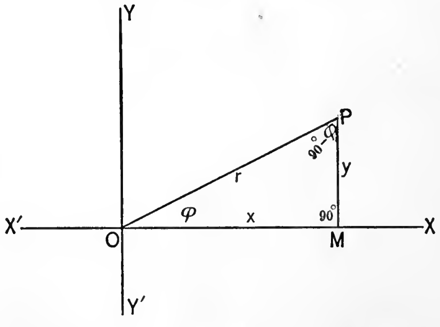


FIG. 27.

express all solutions in terms of the given parts, but sometimes, in order to use logarithmic computation, it will be necessary to express an unknown part in terms of a given part, and a part already found.

Case 1. By § 18, $\tan\phi = \text{ctn}\,(90^\circ - \phi) = \frac{y}{x}\cdot$

Also, $$r = +\sqrt{x^2 + y^2}.$$

But in order to use logarithms we take $r = x\sec\phi = y\csc\phi$.

Case 2. By § 18, $\cos\phi = \sin\,(90^\circ - \phi) = \frac{x}{r}\cdot$

Also, $$y = +\sqrt{r^2 - x^2} = +\sqrt{(r + x)\,(r - x)}.$$
$$\{\log y = \tfrac{1}{2}\,[\log\,(r + x) + \log\,(r - x)]\}.$$

If y is given, the acute angles and x can be found in a similar way.

Case 3. By § 18, $$x = r\cos\phi.$$ $$y = r\sin\phi.$$

Solve similarly if the other acute angle is given.

Case 4. By § 18, $$r = x\sec\phi.$$ $$y = x\tan\phi.$$

Solve similarly if y is given.

B. Examples. VII.

Find the unknown parts of the following right triangles, having given:

1. $x = 273,\ y = 115.$
2. $x = 23,\ y = 7.$
3. $r = 97,\ x = 16.$
4. $r = 42,\ y = 37.$
5. $r = 67,\ \phi = 43^\circ\, 25'.$
6. $r = 19,\ 90^\circ - \phi = 27^\circ\, 17'.$
7. $x = 17,\ \phi = 47^\circ\, 32'.$
8. $y = 211,\ 90^\circ - \phi = 47^\circ\, 16'.$

C. Practical Applications to Problems on Heights and Distances.

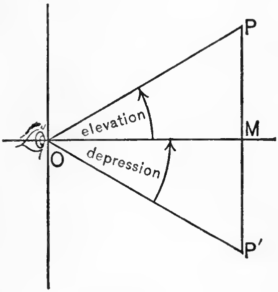


FIG. 28.

§ 63. The angle of elevation, or depression, of a point P is the angle between a horizontal line and the line of vision to the point P, both lines being taken in the same vertical plane. In the accompanying figure the eye of the observer is supposed to be at O. In the case of a heavenly body its angle of elevation is called its altitude. In other cases altitude means the height of a point. Many problems on heights and distances can be made to depend upon the solution of right triangles.

D. Examples. VIII.

1. When the sun's altitude is 30°, the length of the shadow of Bunker Hill monument on the horizontal plane is 381.1 ft. How high is the monument?

2. At the distance of 75 ft. from its base, the angle of elevation of the top of a tree standing on a horizontal plain is 63° 26′. How high is the tree?

3. A steeple is 120 ft. high. What is the angle of elevation of its top at a distance of 75 ft. from its base?

4. A man on the top of a tower 250 ft. high observes an object whose angle of depression is 53° 27′. How far is the object from the base of the tower?

5. At two points, 75 yards from each other, in a horizontal line, and in the same vertical plane with the top of a hill, the angles of elevation are observed to be 45° and 60°. Find the height of the hill and the distance from the nearer point to the point directly under the top of the hill.

6. A besieging party comes to a channel at a point directly opposite a fortified castle of the enemy. They measure away from the shore, and in a line with the castle, 500 yards. The angles of elevation of the top of the castle are observed to be 16′ 15″, and 15′ 50″ at the extremities of the base line respectively. How high is the castle? If the maximum

range of their guns is ten miles, will there be any use in their trying to storm the castle? (log csc $25'' = 3.9165$.)

7. From the top of a tower 175 ft. high two objects on the horizontal plane, in a line with the base of the tower, are observed to have angles of depression of $60°$ and $70°$ respectively. What are their distances from the foot of the tower, and from each other?

8. At two points, a feet from each other, in a horizontal line, and in the same vertical plane with the top of a tower, the angles of elevation are observed to be α and β, respectively, where $\alpha > \beta$. Find the height, h, of the tower, and the distance, b, from the nearer point of observation to the base of the tower.

Ans. $h = a \sin\alpha \sin\beta \csc(\alpha \mp \beta)$, $b = a \cos\alpha \sin\beta \csc(\alpha \mp \beta)$, according as the points are on the same, or on opposite sides of the tower.

9. Use the formulas obtained in Problem 8 to solve Problems 5 and 6.

10. From the top of a tower h ft. high two objects in the horizontal plane, in a line with the base of the tower, are observed to have angles of depression of α and β, respectively, where $\alpha > \beta$. What are their distances from the foot of the tower, and from each other?

11. A farmer wishes to make a ditch 100 rds. long on level land. If the inclination of the bottom of the ditch to the horizon is $12'$, and the depth of the upper end of the ditch is 12 in., what is the depth of the lower end?

12. A winding railroad is built around a cylindrical tower whose radius is 100 ft. and height 500 ft. If the maximum inclination to the horizon which can safely be used is $10°$, how many times will the road wind around the tower? If the road is 3 ft. wide, what is the length of a line midway between the rails?

13. The length of a road (ascent 1 ft. in 5), from the foot to the top of a hill, is $1\frac{3}{8}$ mi. What will be the length of a zigzag road (ascent 1 ft. in 12)?

CHAPTER VII.

A. Some General Formulas for Oblique Triangles.

§ **64.** In treating the oblique triangle we limit the discussion to geometrical triangles. We shall denote the sides of a triangle by a, b, c, and the angles opposite these sides by A, B, C, respectively.

a. Theorem of Sines.

§ **65.** In any plane triangle the sides are proportional to the sines of the opposite angles.

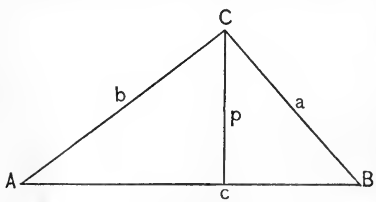


FIG. 29.

Let ABC represent any plane triangle. Let p represent a perpendicular let fall from any vertex upon the opposite side.

By § 18,
$$p = b \sin A,$$
$$p = a \sin B,$$
$$\therefore b \sin A = a \sin B.$$

Similarly,
$$c \sin B = b \sin C.$$
$$a \sin C = c \sin A.$$

These equations may be written:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

or
$$a : b : c :: \sin A : \sin B : \sin C.$$

b. Theorem of Tangents.

§ 66. From $$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ of § 65,}$$

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

$$\therefore \frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \text{ by § 60.}$$

Similarly, $$\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}.$$

$$\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}.$$

c. Theorem of Cosines.

§ 67. Formulas for one side of a triangle, in terms of the other two sides, and their included angle.

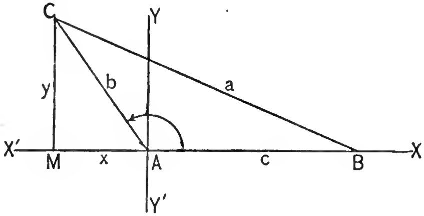


FIG. 30.

Whatever allowable value of A is used,

$$\frac{x}{b} = \cos A, \text{ or } x = b \cos A.$$

$$a^2 = \overline{MC}^2 + (MA + AB)^2$$
$$= y^2 + (c - x)^2$$
$$= y^2 + x^2 + c^2 - 2cx$$
$$= b^2 + c^2 - 2bc \cos A.$$

Similarly, $$b^2 = c^2 + a^2 - 2ca \cos B.$$
$$c^2 = a^2 + b^2 - 2ab \cos C.$$

d. Formulas for one Side of a Triangle, in Terms of the Adjacent Angles, and the other two Sides.

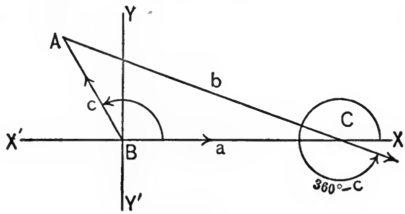


FIG. 31.

§ 68. By § 53,
$$a_a = c_a + b_a,$$
or
$$a = c \cos B + b \cos (360° - C).$$
$$a = b \cos C + c \cos B.$$
Similarly,
$$b = c \cos A + a \cos C.$$
$$c = a \cos B + b \cos A.$$

e. Formulas for the Functions of $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$.

§ 69. By § 67,
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$
$$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \frac{b^2 + c^2 - a^2}{2\,bc}}{2}} \text{ by § 58,}$$
$$= \sqrt{\frac{a^2 - (b - c)^2}{4\,bc}}$$
$$= \sqrt{\frac{(a - b + c)(a + b - c)}{4\,bc}}.$$
Put
$$s = \frac{a + b + c}{2}.$$
Then,
$$s - a = \frac{-a + b + c}{2}.$$
$$s - b = \frac{a - b + c}{2}.$$

$$s - c = \frac{a + b - c}{2}.$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{2(s - b)\, 2(s - c)}{4bc}}$$

$$= +\sqrt{\frac{(s - b)(s - c)}{bc}}.$$

Similarly, $\sin \tfrac{1}{2} B = +\sqrt{\dfrac{(s - c)(s - a)}{ca}}.$

$$\sin \tfrac{1}{2} C = +\sqrt{\frac{(s - a)(s - b)}{ab}}.$$

By § 58, $\cos \tfrac{1}{2} A = \sqrt{\dfrac{1 + \dfrac{b^2 + c^2 - a^2}{2\,bc}}{2}}$

$$= \sqrt{\frac{(a + b + c)(-a + b + c)}{4\,bc}}$$

$$= +\sqrt{\frac{s(s - a)}{bc}}.$$

Similarly, $\cos \tfrac{1}{2} B = +\sqrt{\dfrac{s(s - b)}{ca}}.$

$$\cos \tfrac{1}{2} C = +\sqrt{\frac{s(s - c)}{ab}}.$$

Since $\tan \tfrac{1}{2} A = \dfrac{\sin \tfrac{1}{2} A}{\cos \tfrac{1}{2} A},$

$$\tan \tfrac{1}{2} A = +\sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

Similarly, $\tan \tfrac{1}{2} B = +\sqrt{\dfrac{(s - c)(s - a)}{s(s - b)}}.$

$$\tan \tfrac{1}{2} C = +\sqrt{\frac{(s - a)(s - b)}{s(s - c)}}.$$

f. Formulas for the Area of a Triangle, and the Radii of the Inscribed and Circumscribed Circles.

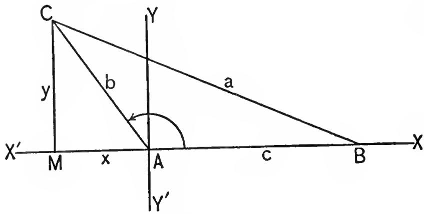


Fig. 32.

§ **70.** By geometry, $\mathcal{A} = \frac{1}{2}yc$ where $\mathcal{A}$ represents the area of the triangle.

But,
$$y = b \sin A.$$
$$1. \quad \mathcal{A} = \tfrac{1}{2} bc \sin A.$$

By § 65,
$$b = \frac{a \sin B}{\sin A}.$$
$$c = \frac{a \sin C}{\sin A}.$$
$$\therefore 2. \quad \mathcal{A} = \tfrac{1}{2}\frac{a^2 \sin B \sin C}{\sin A} = \tfrac{1}{2}\frac{a^2 \sin B \sin C}{\sin (B + C)}.$$

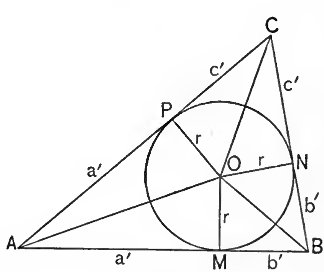


Fig. 33.

Let O be the center of the inscribed circle whose radius is r.

By geometry, $AM = AP\ [= a']$.

$$BM = BN\ [= b'].$$

$$CN = CP\ [= c'].$$

$$b' + c' = a.$$

$$c' + a' = b.$$

$$a' + b' = c.$$

Adding and dividing by 2,

$$a' + b' + c' = s.$$

$$\therefore a' = s - a.$$

$$b' = s - b.$$

$$c' = s - c.$$

$$\therefore \tan\frac{A}{2} = \frac{r}{s-a}.$$

$$\tan\frac{B}{2} = \frac{r}{s-b}.$$

$$\tan\frac{C}{2} = \frac{r}{s-c}.$$

$$\therefore 3.\quad r = (s-a)\tan\frac{A}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\therefore 4.\quad \mathcal{A} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

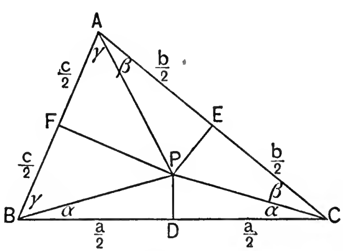


FIG. 34.

Let P be the center of the circumscribed circle whose radius is R. By geometry we have the parts as designated in the figure.

$$\alpha + \beta = C.$$
$$\beta + \gamma = A.$$
$$\gamma + \alpha = B.$$
$$\therefore \alpha + \beta + \gamma = \tfrac{1}{2}(A + B + C) = 90°.$$
$$\beta + \gamma = A.$$
$$\therefore \alpha = 90° - A.$$

$$\cos \alpha = \sin A = \frac{\frac{a}{2}}{R}.$$

$$\therefore R = \frac{a}{2} \csc A.$$

By 1, $$\csc A = \frac{bc}{2\mathcal{A}}.$$

$$\therefore 5.\quad R = \frac{abc}{4\mathcal{A}}.$$

This demonstration, in connection with § 65, gives incidentally the interesting relation

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

The formulas of this chapter enable us to solve any plane triangle when the necessary parts are given, and also to find its area.

B. Oblique Triangles.

a. Cases.

§ **71.** There are four cases.

1. Given two angles and a side.
2. Given two sides and an angle opposite one of them.
3. Given two sides and the included angle.
4. Given three sides.

b. Solutions.

Case 1. The third angle is the supplement of the sum of the two given angles. The required sides are given by the formulas of § 65. For example, if a is the given side,

$$b = a \sin B \csc A. \qquad c = a \sin C \csc A.$$

Case 2. If a, b, and A are given,

$$\sin B = \frac{b \sin A}{a} \text{ by § 65.}$$

Since B is found from its sine, and since there are always two angles, each less than 180°, which have the same sine, B may have two values, and the triangle may have two solutions. But if A is obtuse, there cannot be more than one solution. The cases may be discriminated as follows :

(*1*) A *acute.*

(1) Two solutions when $b > a > b \sin A$.

(2) One solution when $a > b$, $a = b$ (isosceles), $a = b \sin A$ (right angled).

(3) No solution when $a < b \sin A$.

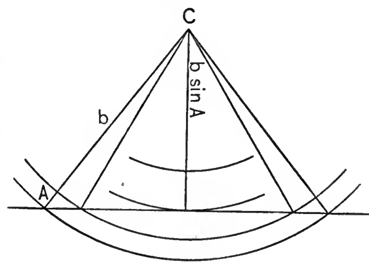


FIG. 35.

(*2*) A *obtuse.*

(1) One solution when $a > b$.

(2) No solution when $b \geqq a$.

Case 3. Given a, b, and C.

By § 66, $\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \operatorname{ctn} \frac{1}{2} C.$

Since, $\tan \frac{1}{2}(A + B) = \operatorname{ctn} \frac{1}{2} C.$

This gives $\tan \frac{1}{2}(A - B)$, and therefore $\frac{1}{2}(A - B)$.

Then,

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A.$$

$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B.$$

The required side can now be found by § 65.

Case 4. Given a, b, and c.

By § 69 or § 70 the angles can be found.

The tangent formulas should be used in practice.

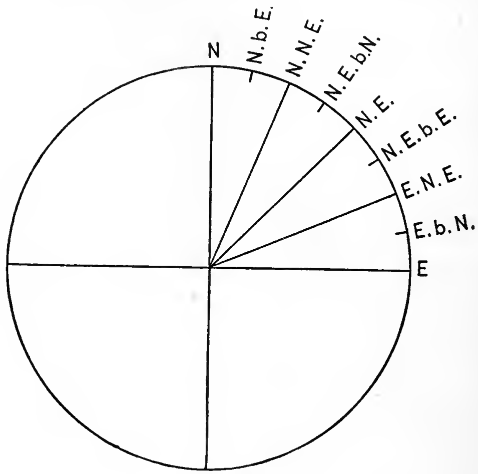


FIG. 36.

C. Examples. IX.

1. A = 67° 24′, B = 39° 18′, a = 15.
2. C = 25° 19′, A = 74° 30′, c = 23.
3. a = 90, b = 100, A = 60°.
4. b = 65, a = 124, A = 117°.
5. c = 96, b = 45, C = 30°.
6. a = 25, b = 19, C = 59°.
7. c = 43, b = 29, A = 40°.
8. a = 25, b = 33, c = 44.
9. a = 37, b = 14, c = 29.
10. a = 29, b = 57, c = 27.
11. Are the following triangles possible? If impossible, what conditions are violated? If possible, have they one or two solutions?

(1) $A = 95° 27'$, $B = 84°, 35'$, $a = 73$.
(2) $a = 47$, $b = 53$, $A = 91° 16'$.
(3) $b = 27$, $c = 54$, $B = 30°$.
(4) $a = 65$, $b = 70$, $A = 58°$.
(5) $a = 34$, $b = 79$, $c = 44.5$.

12. The angle of elevation of a balloon which is ascending uniformly and vertically, when it is one mile high is observed to be 35° 20′; 20 minutes later the elevation is observed to be 55° 40′. How fast is the balloon moving?

13. A base line, AB, 1000 ft. long is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC and CBA are observed to be 65° 37′ and 53° 4′ respectively. Find the breadth of the river at C.

14. A column on a pedestal 20 ft. high subtends an angle 45° to a person on the ground; on approaching 20 ft. it again subtends an angle 45°. Find the height of the column.

15. A and B are two points on opposite sides of a swamp, and C is a point visible from both A and B. AC is 15 rods, and BC 12 rods; the angle ACB is 60°. Find the distance AB.

16. Two mountains, each one mile high, are just visible from each other over the sea. If the radius of the earth is taken equal to 4000 miles, how far are the mountains apart?

17. Two men standing at the same point, C, observe the horizontal angle subtended by the line joining two inaccessible objects, A and B; they then move away, one in the direction AC to D, the other in the direction BC to E, until each observes the horizontal angle to be half what it was before. $ACB = 30°$, $CD = 100$, $CE = 200$. Determine AB.

18. A base line 400 yards long, whose upper end is 8 yards higher than the lower, is measured in the same vertical plane with the top of a hill. The angles of elevation of the top of the hill, measured at the lower and upper ends of the base line, are found to be 5° 17′ and 3° 17′ respectively. Find the height of the hill.

19. A pole 100 ft. high stands in the center of a horizontal equilateral triangle. At the top of the pole each side subtends an angle 60°. Find the length of a side of the triangle.

20. A regular pyramid on a square base has an edge of 150 ft. in length. The side of the base is 200 ft. long. Find the inclination of the face to the base.

21. The distance between two lighthouses, one of which bears E from the other, is 4 miles. The bearings of the lighthouses from a ship are

E.b.N. and N.W.b.W. respectively. What is the distance of the ship from each lighthouse?

22. Two cruisers start from the same point, and steam at the rate of 19 and 23 knots respectively, one S.E., and the other S.$\frac{1}{2}$W. How far apart are they in five hours?

23. A man standing on a horizontal plain at a distance a feet from a tower observed that a flagstaff on the tower subtends an angle α, and that on walking 2 b feet towards the tower the flagstaff again subtends the same angle. Prove that the height of the flagstaff is $2(a - b)\tan\alpha$ feet.

24. A tower and a spire on its top subtend equal angles at a point whose distance from the foot of the tower is a. If h is the height of the tower, prove that the height of the spire is $h\,\frac{a^2 + h^2}{a^2 - h^2}$.

25. A lighthouse is N.b.W. of a ship. After the ship sails 21 miles E.S.E., the bearing of the lighthouse is N.W. What is the distance of the lighthouse from each position of the ship?

26. Find the area of a quadilateral field whose sides, AB, BC, CD, are 20, 15, and 18 respectively, and whose angles, B and C, are 85° and 116° respectively.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

A. Introduction.

a. Geometrical Principles.

§ 72. 1. Every tangent to a sphere is perpendicular to the radius drawn to the point of contact, and is also tangent to that great circle arc in whose plane it lies.

2. Two angles in space which have their sides parallel respectively, and lying in the same or in opposite directions from their vertices, are equal.

In any spherical triangle $\frac{\mathsf{ABC}}{\mathsf{abc}}$:

3. $$\mathsf{A} = \pi - \mathsf{a}', \quad \mathsf{B} = \pi - \mathsf{b}', \quad \mathsf{C} = \pi - \mathsf{c}',$$
$$\mathsf{a} = \pi - \mathsf{A}', \quad \mathsf{b} = \pi - \mathsf{B}', \quad \mathsf{c} = \pi - \mathsf{C}',$$

where $\frac{\mathsf{A'B'C'}}{\mathsf{a'b'c'}}$ is the polar triangle of $\frac{\mathsf{ABC}}{\mathsf{abc}}$.

4. The greater side is opposite the greater angle, and conversely.

5. The sum of two sides is greater than the third.

That is, $\mathsf{a} + \mathsf{b} > \mathsf{c}, \ \mathsf{b} + \mathsf{c} > \mathsf{a}, \ \mathsf{c} + \mathsf{a} > \mathsf{b}.$

6. $$2\pi > \pi + \mathsf{A} > \mathsf{B} + \mathsf{C},$$
$$2\pi > \pi + \mathsf{B} > \mathsf{C} + \mathsf{A},$$
$$2\pi > \pi + \mathsf{C} > \mathsf{A} + \mathsf{B}.$$

7. $$2\pi > a + b + c > 0.$$

8. $$3\pi > A + B + C > \pi.$$

b. Sufficiency of the Convex Spherical Triangle.

The solution of every spherical triangle may be made to depend upon the solution of a triangle in which the parts are each separately less than π. This is made evident by the following summary:

$$\text{If } \begin{Bmatrix} s_k \\ a_k \end{Bmatrix} \text{ denote that } k \begin{Bmatrix} \text{sides} \\ \text{angles} \end{Bmatrix}$$

of a spherical triangle are $> \pi$, the possible combinations are 16, of which 8 are excluded, and 8 give spherical triangles:

1. $s_0 a_0$	~~$s_1 a_0$~~	5. $s_2 a_0$	7. $s_3 a_0$
~~$s_0 a_1$~~	3. $s_1 a_1$	~~$s_2 a_1$~~	~~$s_3 a_1$~~
~~$s_0 a_2$~~	4. $s_1 a_2$	~~$s_2 a_2$~~	~~$s_3 a_2$~~
2. $s_0 a_3$	~~$s_1 a_3$~~	6. $s_2 a_3$	8. $s_3 a_3$

These eight cases are represented graphically by the following three figures. The numbers accompanying the figures correspond respectively to the cases.

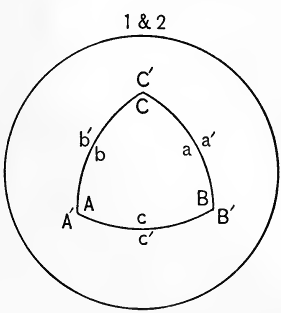


FIG. 37.

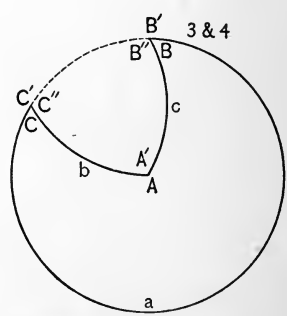


FIG. 38.

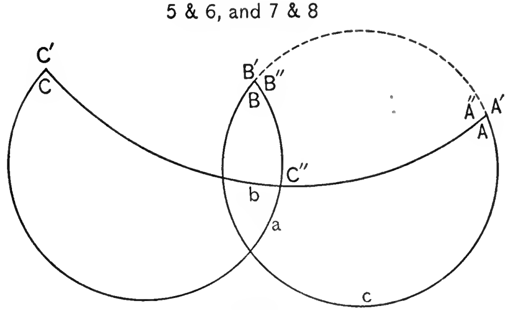


FIG. 39.

B. General Theory of Spherical Trigonometrical Relations.

§ **73.** It is shown in geometry that a spherical triangle is determined if any three parts, not excluding the three angles, are properly given. In the general case, the three parts are independent of each other, and relations between the parts of the general triangle cannot, therefore, exist unless the parts enter into such relations at least four at a time. For example, if $c = f(b)$, or if $c = f(a, b)$, c is determined, and, therefore, three parts are determined as soon as a and b are given in either case; that is, the triangle is determined by two parts. But if $A = f(a, b, c)$, A is not determined until three independent parts, a, b, and c, have been given.

§ **74.** Let us inquire how many species of relations we may expect to meet. In the relations of parts taken four at a time we shall pick out four and always exclude two parts. The number of species of parts retained will be the same as the number of species of parts excluded. For convenience, it will be easier to consider the latter. The number of combinations of six things taken two at a time is $\frac{6 \times 5}{1 \times 2} = 15$. The six parts of the triangle are a, b, c, A, B, C. These fifteen combinations are ab, ac, aA, aB, aC, bc, bA, bB, bC, cA, cB, cC, AB, AC, BC. Excluding those which are repetitions, we find only four distinct species represented by 1. BC, 2. cC, 3. cB, 4. bc, and these give us, therefore, to enter into relations four at a time the four distinct species where the parts retained are the complementary combinations of the parts excluded, viz.:

Retained	1. abcA,	2. abAB,	3. abAC,	4. ABCa.
Excluded	1. BC,	2. cC,	3. cB,	4. bc.

§ **75.** To find the number of species of parts taken five at a time, we may proceed as above.

The number of combinations of six things taken one at a time is six. The six combinations are a, b, c, A, B, C. Here are only two distinct cases, 1. C, 2. c, and we have:

Retained	1. abcAB,	2. abABC.
Excluded	1. C,	2. c.

When the parts are taken six at a time there can be only one species of relations.

C. Systems of Equations demanded by the Preceding Theory.

a. Equations in which the Parts enter Four at a Time.

These are by § 74 of four species.

§ **76.** 1. *Species one. In which we have three sides and an angle.*

Let ABC be any spherical triangle. Let O be the center of the sphere. OA, OB, OC, are radii of the sphere. In the plane OAB draw DB and OM perpendicular to AO. AT, the tangent to the arc, AB, at A is also perpendicular to OA. Hence AT, DB, and OM are parallel lines. In the plane

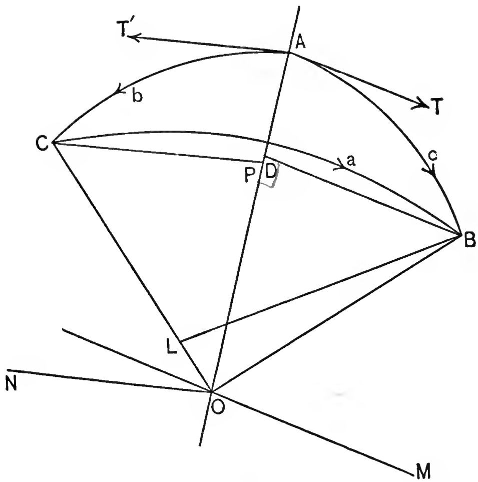


FIG. 40.

OAC draw PC and ON perpendicular to OA; then AT, the tangent to the arc AC at A, PC, and ON are parallel. And the angle NOM = the angle T′AT = the angle A of the spherical triangle ABC.

§ 77. By § 53,

$$OB_{oc} = OD_{oc} + DB_{oc},$$

$$\therefore OB \cos a = OD \cos b + DB \cos COM.$$

And

$$OC_{OM} = OP_{OM} + PC_{OM},$$

$$\therefore OC \cos MOC = OC \cos COM = 0 + PC \cos A.$$

But

$$PC = OC \sin b,$$

hence

$$OC \cos COM = OC \sin b \cos A,$$

or

$$\cos COM = \sin b \cos A.$$

Also

$$OD = OB \cos c,$$

$$DB = OB \sin c.$$

Substituting these values in the equation,

$$OB \cos a = OD \cos b + DB \cos COM,$$

we get $$\text{OB} \cos a = \text{OB} \cos b \cos c + \text{OB} \sin b \sin c \cos A,$$
or $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Advancing the letters, $$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

§ **78.** From these three equations all other equations whatever, which are true of any spherical triangle, may be obtained by algebraic transformations, and are therefore dependent or derived equations. For as three parts of a spherical triangle may be taken arbitrarily, but when they are taken the triangle is fixed, and three parts must be taken to fix the triangle, there remain three parts unknown. Now the number of simultaneous and independent equations required to determine three unknown quantities is three. As many as three must be given, but not more than three equations can exist independently. If, then, we have a system or systems of simultaneous equations involving three unknown quantities, not more than three of the equations can be taken to be independent, and as many as three of them must be taken to be independent, if the system is determinately solvable. From this it is evident that we might take any one of the systems of three equations, where the three equations are independent of each other, as the three fundamental equations of spherical trigonometry. But we will choose the preceding three equations of § 77, as the fundamental equations. We shall now show how the other systems of equations are derived from them.

2. *Species two. Equations involving two sides and the angles opposite to them.*

§ **79.** Here, *e.g.*, we must retain abAB, and exclude cC. By adding and subtracting the first two of the equations of § 77, we have:

$$(\cos a + \cos b)(1 - \cos c) = \sin c (\sin b \cos A + \sin a \cos B).$$
$$(\cos a - \cos b)(1 + \cos c) = \sin c (\sin b \cos A - \sin a \cos B).$$

Multiplying these two equations together, and dividing by $\sin^2 c = 1 - \cos^2 c$, which we have a right to do, since $c \neq 0$, $\sin c \neq 0$, and therefore $\sin^2 c \neq 0$, and we get:

$$\cos^2 a - \cos^2 b = \sin^2 b \cos^2 A - \sin^2 a \cos^2 B,$$

or expressing in terms of sines,

$$\sin^2 b \sin^2 A = \sin^2 a \sin^2 B, \quad \therefore \sin b \sin A = \pm \sin a \sin B,$$

but as all the factors involved in the equation are positive, we must exclude the minus sign, and write:

$$\sin b \sin A = \sin a \sin B.$$

Advancing the letters,

$$\sin c \sin B = \sin b \sin C,$$
$$\sin a \sin C = \sin c \sin A.$$

These equations may also be put in the forms:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = r.$$

$$\sin a = r \sin A, \qquad \sin b = r \sin B, \qquad \sin c = r \sin C.$$

3. *Species three. Equations involving two sides and an angle opposite to one of them.*

§ **80.** We must, *e.g.*, retain abAC, and exclude cB.

In the first of the equations of § 77, substitute for cos c its value given by the third equation, and for sin c, its value, $\frac{\sin a \sin C}{\sin A}$, given by the third one of the equations of § 79.

We get:

$$\cos a = \cos b (\cos a \cos b + \sin a \sin b \cos C) + \frac{\sin b \sin a \sin C \cos A}{\sin A},$$

$$\cos a (1 - \cos^2 b) = \cos a \sin^2 b = \cos b \sin a \sin b \cos C + \sin b \sin a \sin C \operatorname{ctn} A.$$

Dividing both sides of the equation by sin a sin b, since sin b sin c $\neq$ 0,

$$\text{ctn } a \sin b = \cos b \cos C + \sin C \text{ ctn } A.$$

Advancing the letters,

$$\text{ctn } b \sin c = \cos c \cos A + \sin A \text{ ctn } B,$$
$$\text{ctn } c \sin a = \cos a \cos B + \sin B \text{ ctn } C.$$

If instead of retaining abAC, and excluding cB, we had retained acAB, and excluded bC, we should simply have put c in the place of b, and B in the place of C, and instead of the first of the preceding equations, we should have had:

$$\text{ctn } a \sin c = \cos c \cos B + \sin B \text{ ctn } A.$$

Advancing,
$$\text{ctn } b \sin a = \cos a \cos C + \sin C \text{ ctn } B,$$
$$\text{ctn } c \sin b = \cos b \cos A + \sin A \text{ ctn } C.$$

4. *Species four. Equations involving three angles and a side.*

§ **81.** Here we must, *e.g.*, retain ABCa, and exclude bc.

If we divide the equation at the beginning of the preceding section by sin b instead of sin a sin b, there results:

$$\cos a \sin b = \cos b \sin a \cos C + \sin a \sin C \text{ ctn } A.$$

Replacing sin b by r sin B, sin a by r sin A, sin c $= \dfrac{\sin a \sin C}{\sin A}$ by r sin C, and dividing by r, we have:

$$\cos a \sin B = \cos b \sin A \cos C + \sin C \cos A.$$

If cos c is replaced in the second of the equations of § 77 by its value taken from the third, the result will differ from the preceding in having a changed into b, and A into B, and it will be:

$$\cos b \sin A = \cos a \sin B \cos C + \sin C \cos B.$$

Multiplying the latter equation by cos C, and equating the terms cos b sin A cos C, there results:

$$\cos a \sin B - \sin C \cos A = \cos a \sin B \cos^2 C + \sin C \cos C \cos B,$$

or $$\cos a \sin B (1 - \cos^2 C) = \cos a \sin B \sin^2 C$$
$$= \sin C \cos A + \sin C \cos C \cos B.$$

Dividing by sin C, since $\sin C \neq 0$,

$$\cos a \sin B \sin C = \cos A + \cos B \cos C,$$

or $$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Advancing, $$\cos B = -\cos C \cos A + \sin C \sin A \cos b,$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

b. Equations in which the Parts enter Five at a Time.

1. *Species one. Equations involving three sides and two angles.*

§ **82.** We retain here, *e.g.*, abcAB, and exclude C.

If in the first of the equations of § 77, we substitute for cos b, its value as given by the second of those equations,

$$\cos a = (\cos c \cos a + \sin c \sin a \cos B) \cos c + \sin b \sin c \cos A,$$

or $$\cos a (1 - \cos^2 c) = \cos a \sin^2 C = \sin c \sin a \cos c \cos B$$
$$+ \sin b \sin c \cos A.$$

Dividing by sin c, since $\sin c \neq 0$,

$$\cos a \sin c = \cos c \sin a \cos B + \sin b \cos A,$$

or as usually written,

$$\sin b \cos A = \cos a \sin c - \cos c \sin a \cos B.$$

Advancing, $$\sin c \cos B = \cos b \sin a - \cos a \sin b \cos C,$$
$$\sin a \cos C = \cos c \sin b - \cos b \sin c \cos A.$$

If instead of putting cos b in the first of the equations of § 77, we had put it in the third, the results would have appeared with a and c, and A and C, interchanged, or

$$\sin b \cos C = \cos c \sin a - \cos a \sin c \cos B,$$

and also, $$\sin c \cos A = \cos a \sin b - \cos b \sin a \cos C,$$
$$\sin a \cos B = \cos b \sin c - \cos c \sin b \cos A.$$

2. *Species two. Equations involving three angles and two sides.*

§ 83. Here we retain, *e.g.*, ABCab, and exclude c.

The first of these equations has already been obtained in § 81. It is:

$$\cos a \sin B = \cos b \sin A \cos C + \sin C \cos A.$$

Advancing,
$$\cos b \sin C = \cos c \sin B \cos A + \sin A \cos B,$$
$$\cos c \sin A = \cos a \sin C \cos B + \sin B \cos C.$$

The leading equation of the second division of this set has also been obtained in § 81. It is:

$$\cos b \sin A = \cos a \sin B \cos C + \sin C \cos B.$$

Advancing,
$$\cos c \sin B = \cos b \sin C \cos A + \sin A \cos C,$$
$$\cos a \sin C = \cos c \sin A \cos B + \sin B \cos A.$$

c. Equations in which the Parts enter Six at a Time.

§ 84. Here there is but one species. An example may suffice. From § 77 we can obtain,

$$\cos a \cos b \cos c - (\cos b \cos c + \sin b \sin c \cos A)$$
$$(\cos c \cos a + \sin c \sin a \cos B)(\cos a \cos b + \sin a \sin b \cos C) = 0,$$
$$\cos A \cos B \cos C \sin^2 a \sin^2 b \sin^2 c = (\cos a - \cos b \cos c)$$
$$(\cos b - \cos c \cos a)(\cos c - \cos a \cos b).$$

D. On the Use of the Polar Triangle in Establishing Relations.

§ 85. The relations
$$A = \pi - a', \quad a = \pi - A',$$
$$B = \pi - b', \quad b = \pi - B', \text{ where } \begin{matrix} A'B'C' \\ a'b'c' \end{matrix}$$
$$C = \pi - c', \quad c = \pi - C',$$

is the polar triangle of $\begin{matrix} ABC \\ abc \end{matrix}$, often afford an easy means of establishing a relation. For example, we may say that the formulas of § 77 are transformed into those of § 81. This is shown as follows: Express that the formulas of § 77 hold for the polar triangle. Then the first one gives:

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Substituting for a′, b′, c′, A′, the values given above, and we have:

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a),$$

or $$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a),$$

or $$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

The equations of § 79 are transformed into themselves; that is, we get nothing different by using the polar triangle with them. The first half of the formulas of § 80 are transformed into the second half, and the second half into the first half. Thus the first one of the first half,

$$\text{ctn } a \sin b = \cos b \cos C + \sin C \text{ ctn } A,$$

is transformed into

$$-\text{ctn } A \sin B = \cos B \cos c - \sin c \text{ ctn } a,$$

or $$\text{ctn } a \sin c = \cos c \cos B + \sin B \text{ ctn } A,$$

which is the first one of the second half. The formulas of § 82 are transformed into those of § 83, and *vice versa,* as the student may satisfy himself, by writing out the leading equations and making the substitutions. As this work can be done at a glance when once understood, it is evident that the method affords an easy and powerful means of writing down new formulas without any work at all. It is a method that ought, at least, to be applied to every new formula obtained, for it may enable one to double the number of his formulas without great labor.

General Note on Systems of Trigonometric Equations. — The preceding investigations have shown that the number of species of trigonometric equations or formulas is exactly determinable. And while complete sets of the various species have been given, it is to be observed that other sets might be derived, and that the number of derived sets of the various species is unlimited.

CHAPTER II.

A. Formulas for Spherical Right, and Quadrantal Triangles.

§ **86.** In the formulas of §§ 77–81, making an angle, and a side, as C, and c, successively equal to 90°, we have the following formulas for spherical right, and quadrantal triangles:

a. **Spherical Right Triangles.**		*b.* **Quadrantal Triangles.**	
1. $\cos c = \cos a \cos b.$	§ 77	1′. $\cos C = -\cos A \cos B.$	§ 81
2. $\sin A = \dfrac{\sin a}{\sin c}.$	§ 79	2′. $\sin a = \dfrac{\sin A}{\sin C}.$	§ 79
3. $\sin B = \dfrac{\sin b}{\sin c}.$	§ 79	3′. $\sin b = \dfrac{\sin B}{\sin C}.$	§ 79
4. $\cos A = \dfrac{\tan b}{\tan c}.$	§ 80	4′. $\cos a = -\dfrac{\tan B}{\tan C}.$	§ 80
5. $\cos B = \dfrac{\tan a}{\tan c}.$	§ 80	5′. $\cos b = -\dfrac{\tan A}{\tan C}.$	§ 80
6. $\tan A = \dfrac{\tan a}{\sin b}.$	§ 80	6′. $\tan a = \dfrac{\tan A}{\sin B}.$	§ 80
7. $\tan B = \dfrac{\tan b}{\sin a}.$	§ 80	7′. $\tan b = \dfrac{\tan B}{\sin A}.$	§ 80
8. $\cos a = \dfrac{\cos A}{\sin B}.$	§ 81	8′. $\cos A = \dfrac{\cos a}{\sin b}.$	§ 77
9. $\cos b = \dfrac{\cos B}{\sin A}.$	§ 81	9′. $\cos B = \dfrac{\cos b}{\sin a}.$	§ 77
10. $\cos c = \operatorname{ctn} A \operatorname{ctn} B.$	§ 81	10′. $\cos C = -\operatorname{ctn} a \operatorname{ctn} b.$	§ 77

c. Attention is called to the Following Deductions from the Formulas of § 86.

§ 87. 1. From 1 and 1′ it appears that c will be of the first or second quadrant, and C of the second or first quadrant, according as a and b, or A and B, respectively, are of the same or of different quadrants.

2. From 6 and 7 it appears that an oblique angle and its opposite side, and from 6′ and 7′ that a non-quadrantal side, and its opposite angle, are of the same quadrant.

B. Solution of Spherical Right and Quadrantal Triangles.

§ 88. In the right spherical triangle, the right angle, and in the quadrantal triangle, the quadrant, counts as one of the three given parts. There are six cases to be treated.

a. Right Triangle.					*b.* Quadrantal Triangle.				
Case.	Given.	Required.	Formulas § 86. *a.*	Check.	Case.	Given.	Required.	Formulas § 86. *b.*	Check.
1	ab	cAB	1, 6, 7	10	1	AB	Cab	1′, 6′, 7′	10′
2	ac	bAB	1, 2, 5	9	2	AC	Bab	1′, 2′, 5′	9′
3	aA	bcB	6, 2, 8	3	3	Aa	BCb	6′, 2′, 8′	3′
4	aB	bcA	7, 5, 8	4	4	Ab	BCa	7′, 5′, 8′	4′
5	cA	abB	2, 4, 10	7	5	Ca	ABb	2′, 4′, 10′	7′
6	AB	abc	8, 9, 10	1	6	ab	ABC	8′, 9′, 10′	1′

§ 89. The remarks of § 87, taken in connection with the formulas given in § 88 for the solution of the various cases, show that of the six cases, only the third is ambiguous. In this case the required parts must be obtained from their sines. Since each required part has two values, supplements of each other, there might apparently be eight triangles; but as b and B (for the right spherical triangle) must be of the same quadrant, only two combinations of a, b, and B are allowable, and when a and b are determined, c is determined, and, therefore there cannot be more than two solutions. If a differs less than A from 90°, the triangle is impossible, since the sine of each required part would be greater than unity. An analogous consideration holds for the corresponding case of the quadrantal triangle.

§ **90.** A complete set of formulas could be deduced for spherical isosceles triangles from the general formulas of §§ 77–81, by assuming that two sides are equal, but as they can be easily solved by the formulas for right triangles, we will not give a special set of formulas.

C. Some Astronomical Conceptions.

a. Celestial Sphere.

1. *Definition.* The Celestial Sphere is a sphere of indefinitely great radius, concentric with the earth, and in whose surface, the heavenly bodies are, for astronomical uses, supposed to be.

Its axis, called the Celestial Axis, coincides with the axis of the earth, and its ends are the North and South Poles of the celestial sphere.

2. *Great Circles.*

(1) The Sensible Horizon is that small circle of the celestial sphere, which is tangent to the earth at the observer's standpoint.

The Rational Horizon is that great circle of the celestial sphere, which is parallel to the sensible horizon.

Because the radius of the celestial sphere is indefinitely great, we are justified, in practice, in identifying the sensible and the rational horizon, and the observer's standpoint and the center of the celestial sphere. Accordingly we shall, without distinction, speak only of the horizon.

The upper and lower poles of the horizon are called the Zenith and Nadir, respectively.

(2) Vertical Circles are the great circles which pass through the zenith and nadir.

(3) The Equinoctial, or Celestial Equator, is that great circle of the celestial sphere which is perpendicular to the celestial axis. Obviously its poles are the celestial poles.

(4) Hour Circles, or Declination Circles, are great circles passing through the celestial poles.

(5) The Celestial Meridian is that hour circle which passes through the zenith and nadir. It coincides with the terrestrial meridian of the observer's standpoint.

The intersections of the celestial meridian and horizon are the North and South Points.

(6) The Prime Vertical is that vertical circle which is perpendicular to the celestial meridian.

The points of intersection of the prime vertical and the horizon are the East and West Points.

(7) The Ecliptic is that great circle of the celestial sphere in which the sun appears to move.

The intersections of the equinoctial and ecliptic are called the Vernal and Autumnal Equinoxes. The sun appears to be at these points on about March 21, and September 21, respectively.

The angle between the equinoctial and ecliptic is called the Obliquity of the Ecliptic. It is denoted by the letter e. $e = 23° 27'$, nearly.

(8) The hour circle passing through the equinoxes is called the Colure.

(9) Latitude Circles are the great circles which pass through the poles of the ecliptic.

(10) That latitude circle which passes through the equinoxes we shall call the Prime Latitude Circle.

b. Spherical Coördinates.

1. *Definition.* The Spherical Coördinates of a heavenly body are its angular distances from two fixed, mutually perpendicular, great circles of reference.

For convenience the great circle arcs which measure the angular distances, are used.

2. *Systems of Spherical Coördinates.*

(1) Using the horizon and meridian as reference circles, we get Altitude and Azimuth, respectively; altitude taken from the horizon from 0° to 90°, and azimuth taken from the south point measured by way of the west from 0° to 360°. The complement of the altitude is called the Zenith Distance.

(2) Using the equinoctial and meridian we get Declination and Hour Angle, respectively; declination taken from the equinoctial north or south from 0° to 90° or — 90°, and hour angle taken from the meridian measured by way of the west from 0° at noon to 360°, or 24 hrs. The complement of the declination is called the Polar Distance.

(3) Using the equinoctial and colure we get Declination and Right Ascension, respectively; right ascension taken from the vernal equinox by way of the east from 0° to 360°, or 24 hrs.

(4) Using the ecliptic and prime latitude circle we get Celestial Latitude and Celestial Longitude, respectively; celestial latitude taken from the ecliptic from 0° to 90° or — 90°, and celestial longitude taken from the vernal equinox by way of the east from 0° to 360°.

It is important not to confound celestial latitude and longitude with terrestrial.

Figs. 41 and 42 represent the celestial sphere, with its center at O.

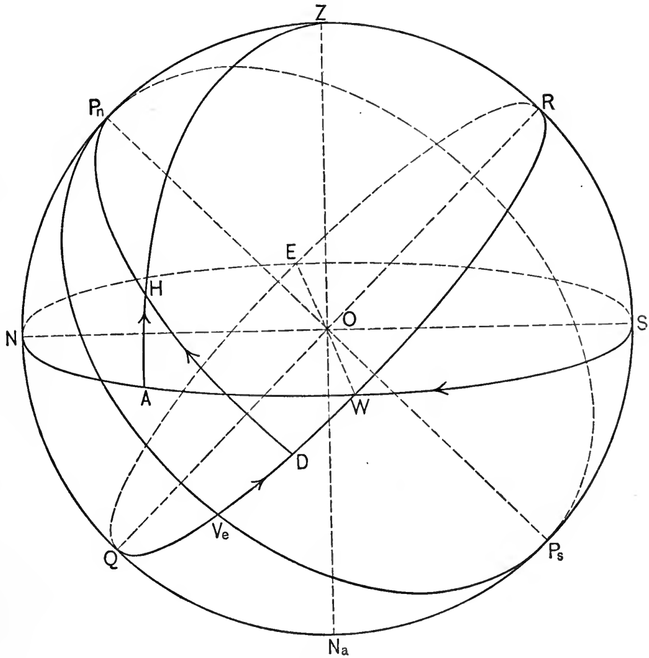


FIG. 41.

In Fig. 41, NWS represents the horizon; EQR the celestial equator; Z, N_a, and P_n, P_s, their respective poles; V_e, the vernal equinox; and H, a heavenly body.

AH = altitude . . . . denoted by h.
SWA = azimuth . . . . " a.
DH = declination . . . . " δ.
Angle RP_nH = Arc RWD = hour angle . . . . " t.
V_eD = right ascension . . . . " α.
RZ = NP_n ≐ terrestrial latitude of observer's standpoint " ϕ.
ZH = 90° − h = zenith distance.
P_nH = 90° − δ = polar distance.

The triangle P_nHZ is called the Astronomical Triangle; the angle at H, the Position Angle. The student should carefully note the parts of this triangle.

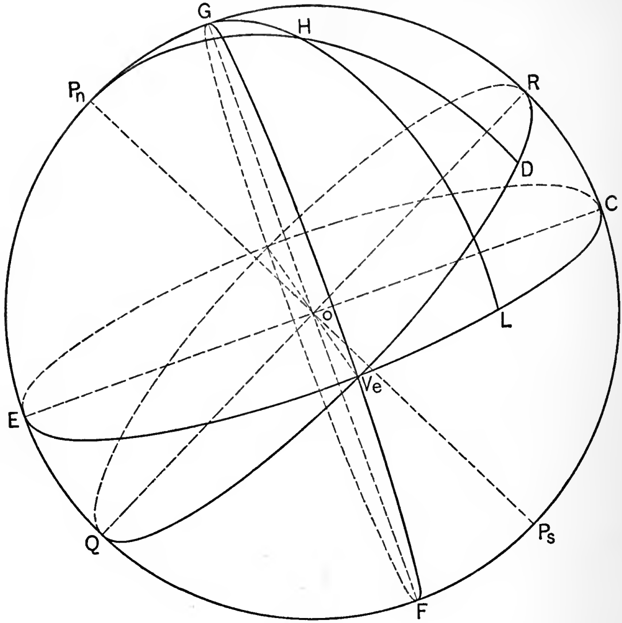


FIG. 42.

In Fig. 42, QV_eR, represents the equinoctial; EV_eC, the ecliptic; P_n, P_s, and F, G, their respective poles; V_e, the vernal equinox; and H, a heavenly body.

DH = declination.
V_eD = right ascension.
LH = celestial latitude, denoted by β.
V_eL = celestial longitude, " λ.
$CR = P_nG$ = obliquity of the ecliptic.

The student should carefully note the parts of the triangle P_nHG.

D. Examples. X.

1. $a = 116°$, $b = 16°$, $C = 90°$.
2. $A = 60° 47'$, $B = 57° 16'$, $c = 90°$.
3. $a = 28°$, $b = 28°$, $B = 79°$.

4. $a = 33° 40'$, $A = 43° 21'$, $C = 90°$.
5. $A = 105° 53'$, $a = 104° 54'$, $c = 90°$.
6. $a = 61°$, $B = 123° 40'$, $C = 90°$.
7. $A = 139°$, $b = 143°$, $c = 90°$.
8. $a = 69° 18'$, $c = 84° 27'$, $C = 90°$.
9. $A = 70° 12'$, $C = 106° 25'$, $c = 90°$.
10. $c = 69° 25'$, $A = 54° 54'$, $C = 90°$.
11. $C = 134°$, $a = 143°$, $c = 90°$.
12. $A = 46° 59'$, $B = 57° 59'$, $C = 90°$.
13. $a = 174° 12'$, $b = 94° 08'$, $c = 90°$.

For a right spherical triangle prove, when $C = 90°$:

14. $\tan^2 \frac{b}{2} = \tan \frac{1}{2}(a + c) \tan \frac{1}{2}(c - a)$.
15. $\tan^2 (45° - \frac{1}{2} A) = \tan \frac{1}{2}(c - a) \operatorname{ctn} \frac{1}{2}(c + a)$.
16. $\tan^2 \frac{1}{2} B = \sin (c - a) \csc (c + a)$.
17. $\tan^2 (45° - \frac{1}{2} B) = \tan \frac{1}{2}(A - a) \tan \frac{1}{2}(A + a)$.
18. $\tan^2 (45° - \frac{1}{2} b) = \sin (A - a) \csc (A + a)$.
19. $\tan^2 (45° - \frac{1}{2} c) = \tan \frac{1}{2}(A - a) \operatorname{ctn} \frac{1}{2}(A + a)$.
20. $\operatorname{ctn}^2 \frac{1}{2} a = \dfrac{\tan \frac{1}{2}(90° + B - A)}{\tan \frac{1}{2}(B + A - 90°)}$.
21. $\operatorname{ctn}^2 \frac{1}{2} b = \tan \frac{1}{2}(90° + A - B) \operatorname{ctn} \frac{1}{2}(B + A - 90°)$.
22. $\operatorname{ctn}^2 \frac{1}{2} c = -\cos (B - A) \sec (B + A)$:

23–31. Obtain formulas for the quadrantal triangle corresponding to formulas of problems 14–22.

32. A spherical square is a spherical quadrilateral which has equal sides and equal angles. The diagonals divide it into four equal spherical right triangles. Given a side a of the spherical square, find an angle A.

33. A line makes with a plane an angle α; through the foot of the line and in the plane a line is drawn making an angle β with the projection of the first line on the plane; find the angle γ made by the second line with the first.

34. Given the number of sides of a regular spherical polygon equal to n and each angle equal to A; find a side of the polygon and the polar radii of the inscribed and circumscribed circles.

35. Find the angles between the adjacent faces of each of the five regular polyhedrons.

36. The base of a regular pyramid is an octagon; each angle at the vertex of the pyramid is 30°; find the angle α between two adjacent lateral faces, and the angle β between any lateral face and the base.

37. From the longitude of the sun (λ) and the obliquity of the ecliptic (e) find the sun's right ascension (α) and declination (δ).

38. From the latitude (ϕ) of a place on the earth's surface and the declination (δ) of the sun on a given day, find the times and places of the rising and setting of the sun and its distance from the zenith at noon.

When are days and nights equal?

(Neglect the effect of refraction.)

39. When does the solution of Ex. 38 become impossible? When indeterminate? And what follows for places so situated on the earth's surface as to give these results?

40. When and where does the sun rise in Oberlin (latitude 41° 17′) on the longest day, June 21 ($\delta = +23° 27'$)? When and where on the shortest day, Dec. 21 ($\delta = -23° 27'$)?

41. At a place on the earth's surface whose latitude is ϕ, a rod OA, pointed toward the north pole, makes with a horizontal plane an angle, $BOA = \phi$; let OC be the position of the shadow of the rod at a given time of the day; find the angle BOC.

What does the result become for $\phi = 41° 17'$, $t = 2$ hrs. $= 30°$?

42. Through O (Ex. 41) pass a plane perpendicular to OB; extend the rod through this vertical plane so that its shadow may fall upon it; find the angle S which the shadow makes with the projection of the rod on the plane at a given time t. Find S for $\phi = 41° 17'$, $t = 3$ hrs. $= 45°$.

43. At what time of a given day (δ given) at a given place (lat. $= \phi$) is the sun exactly east or west?

44. Find the altitude (h) and the azimuth (a) of the sun at a given place and on a given day at 6 A.M.

45. Find the shortest distance (*i.e.*, length of great circle arc) from Oberlin, long. 82° 14′, to New Haven, long. 72° 55′, the latitude of each place being 41° 17′.

CHAPTER III.

A. Formulas for Spherical Oblique Triangles.

a. Formulas for Functions of ½ A, ½ B, ½ C.

§ 91. From the formulas of § 77,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\therefore \cos \tfrac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{\cos a - (\cos b \cos c - \sin b \sin c)}{2 \sin b \sin c}}$$ § 58

$$= \sqrt{\frac{\cos a - \cos (b + c)}{2 \sin b \sin c}} = \sqrt{\frac{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}{\sin b \sin c}}.$$ § 59

Let $\frac{a + b + c}{2} = s,$

and $$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

Similarly, $$\cos \tfrac{1}{2} B = \sqrt{\frac{\sin s \sin (s - b)}{\sin c \sin a}}.$$

$$\cos \tfrac{1}{2} C = \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}.$$

§ 92. $$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$ §§ 58 and 59

Similarly, $$\sin \tfrac{1}{2} B = \sqrt{\frac{\sin (s - c) \sin (s - a)}{\sin c \sin a}}.$$

$$\sin \tfrac{1}{2} C = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}}.$$

§ **93.** From the formulas of §§ 91 and 92,

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}}.$$

$$\tan \tfrac{1}{2} C = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}.$$

The student may prove that in each formula of the last section the positive square root is to be taken, and that the number under the radical sign in each case is positive.

§ **94.** From the formulas of § 81, or from the formulas of the last three sections, by using the properties of polar triangles, the student may prove the following formulas :

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}.$$

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}.$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

Where $$S = \frac{A+B+C}{2}.$$

The student may write the corresponding formulas for b and c, and prove that in each formula the positive square root is to be taken, and that the number under the radical sign in each case is positive.

b. Gauss's Equations.

§ **95.** From §§ 54 and 55

$$\cos \tfrac{1}{2}(A \pm B) = \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \mp \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

Substituting for

$$\cos \tfrac{1}{2} A,\ \cos \tfrac{1}{2} B,\ \sin \tfrac{1}{2} A, \text{ and } \sin \tfrac{1}{2} B,$$

their values given in §§ 91 and 92 :

$$\cos \tfrac{1}{2}(A \pm B) = \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$\mp \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$= \frac{\sin s \mp \sin(s-c)}{\sin c}\sin\frac{C}{2}.$$

Taking the upper sign,

$$\cos \tfrac{1}{2}(A + B) = \frac{\sin\left(\frac{a+b}{2}+\frac{c}{2}\right) - \sin\left(\frac{a+b}{2}-\frac{c}{2}\right)}{2\sin\frac{c}{2}\cos\frac{c}{2}}\sin\frac{C}{2}$$

$$= \frac{2\cos\tfrac{1}{2}(a+b)\sin\frac{c}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}\sin\frac{C}{2} \qquad \S\ 59$$

$$= \frac{\cos\tfrac{1}{2}(a+b)}{\cos\frac{c}{2}}\sin\frac{C}{2}.$$

Taking the lower sign,

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin s + \sin(s-c)}{\sin c}\sin\frac{C}{2}$$

$$= \frac{\sin\tfrac{1}{2}(a+b)}{\sin\frac{c}{2}}\sin\frac{C}{2}.$$

Similarly from

$$\sin \tfrac{1}{2}(A \pm B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \pm \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B.$$

Substituting as above,

$$\sin \tfrac{1}{2}(A \pm B) = \frac{\sin(s-b) \pm \sin(s-a)}{\sin c}\cos\frac{C}{2}$$

$$= \frac{\sin\left(\frac{c}{2}+\frac{a-b}{2}\right) \pm \sin\left(\frac{c}{2}-\frac{a-b}{2}\right)}{2\sin\frac{c}{2}\cos\frac{c}{2}}\cos\frac{C}{2}.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{c}{2}} \cos \frac{C}{2},$$

$$\text{and } \sin \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{c}{2}} \cos \frac{C}{2}.$$

These formulas may be put in a form in which they may be more easily remembered. They can be written as follows:

$$1.\quad \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c}.$$

$$2.\quad \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c}.$$

$$3.\quad \frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c}.$$

$$4.\quad \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c}.$$

Here observe: (1) All the angles are half angles. (2) In the left members of the equations we have A, B, C; in the right members a, b, c. (3) In each left member are co-functions, in the right, the same functions. (4) In each equation to a sign + in the numerator of one member corresponds a cosine in the numerator of the other member. And in the same way to a sign − corresponds a sine.

c. Napier's Analogies.

§ 96. If we divide member by member, 1 of the Gauss equations by 3, 2 by 4, 4 by 3, and 2 by 1, we obtain:

$$1.\quad \frac{\tan \frac{1}{2}(A+B)}{\operatorname{ctn} \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

$$2.\quad \frac{\tan \frac{1}{2}(A-B)}{\operatorname{ctn} \frac{1}{2}C} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

$$3.\quad \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}.$$

$$4.\quad \frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}.$$

(1) Here the parts enter five at a time, and in the respective members we have A, B, C and a, b; or a, b, c, and A, B. (2) When A, B, C, enter the left members we have there the co-functions, tangent and

cotangent, but the same functions, tangents, when a, b, c, enter the left members. In the right members are the same functions, sines or cosines. (3) The angles are half angles. (4) In the right members are sines or cosines according as the signs — or + are in the numerators of the left members. (5) In the right members the angles are differences in the numerators, sums in the denominators.

d. l'Huilier's Formula.

§ 97. If in 1 and 3, § 95, $\frac{A+B+C}{2}-\frac{\pi}{2}$ be represented by E,

we get
$$\frac{\cos\left(\frac{C}{2}-E\right)}{\cos\frac{C}{2}}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}},$$

and
$$\frac{\sin\left(\frac{C}{2}-E\right)}{\sin\frac{C}{2}}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}.$$

By division and composition the first of these gives

$$\frac{\cos\left(\frac{C}{2}-E\right)-\cos\frac{C}{2}}{\cos\left(\frac{C}{2}-E\right)+\cos\frac{C}{2}}=\frac{\cos\frac{1}{2}(a-b)-\cos\frac{c}{2}}{\cos\frac{1}{2}(a-b)+\cos\frac{c}{2}}.\ \therefore,\ \S\ 60,$$

$$-\tan\tfrac{1}{2}(C-E)\tan\tfrac{1}{2}(-E)=-\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}[-(s-a)],$$
or,
$$\tan\tfrac{1}{2}(C-E)\tan\tfrac{1}{2}E=\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-a).$$

In a similar manner the second equation gives,

$$\operatorname{ctn}\tfrac{1}{2}(C-E)\tan\tfrac{1}{2}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c);$$

multiplying these equations together, term by term, we have,

$$\tan^2\tfrac{1}{2}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c).$$

e. Formula for the Polar Radius of a Circumscribed Circle of a Spherical Triangle.

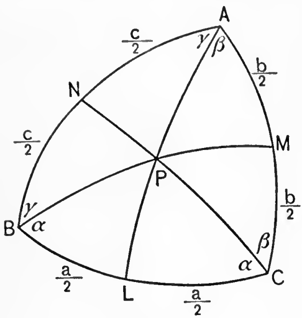


FIG. 43.

§ **98.** The construction is analogous to that for the similar problem in the case of a plane triangle. P, the pole of the required circle is the intersection of perpendicular great circle arcs erected to the sides at their middle points.

$$PA = PB = PC = R,$$

the required radius.

$$\beta + \gamma = A, \quad \gamma + \alpha = B, \quad \alpha + \beta = C,$$

$$\therefore \alpha + \beta + \gamma = S,$$

$$\alpha = S - A, \qquad \beta = S - B, \qquad \gamma = S - C.$$

In the right spherical triangle LPC, by §§ 86 and 94,

$$\tan R = \frac{\tan \frac{a}{2}}{\cos (S - A)} = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}$$

$$= \frac{1}{-\cos S} \tan \tfrac{1}{2} a \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c, \text{ by § 94.}$$

f. Formula for the Polar Radius of the Inscribed Circle of a Spherical Triangle.

§ **99.** Bisect the angles of the triangle by great circle arcs. From the point of intersection of the angle bisectors draw perpendicular arcs of great circles to the sides of the triangle. Since L, M, and N are points of tangency, $BN = BL$, $AN = AM$, $CM = CL$, $a' + b' = c$, $b' + c' = a$, $c' + a' = b$, adding and dividing by 2,

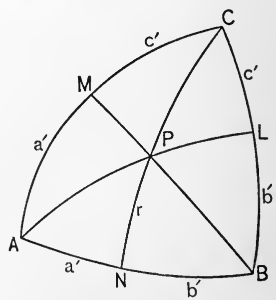


FIG. 44.

$$a' + b' + c' = \frac{a+b+c}{2} = s,$$

$$\therefore a' = s - (b' + c') = (s - a).$$

In the right triangle ANP, $\tan \frac{A}{2} = \frac{\tan r}{\sin a'} = \frac{\tan r}{\sin (s-a)}$, § 86

$$\therefore \tan r = \tan \frac{A}{2} \sin (s-a)$$

$$= \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}} \qquad \text{§ 93}$$

$$= \sin s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C.$$

B. Plane Trigonometry a Special Case of Spherical Trigonometry.

§ 100. Let ABC be any spherical triangle of which the sides a, b, and c are measured in some linear unit. Let r be the radius of the sphere, and α, β, and γ the angles at the center corresponding to the arcs a, b, and c, respectively.

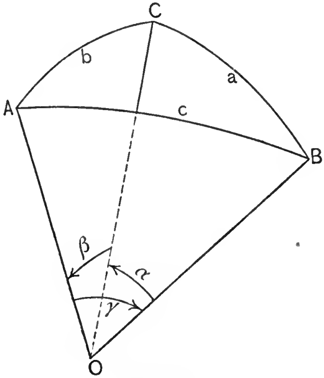


FIG. 45.

Then $\alpha = \frac{a}{r}$, $\beta = \frac{b}{r}$, $\gamma = \frac{c}{r}$, by § 8, 3.

If now r increases while a, b, and c remain fixed in length, α, β, and γ will diminish; and if r increases indefinitely, α, β, and γ will approach zero as a limit. And the limit as r increases indefinitely of

$$r \sin \alpha = \lim_{\alpha \doteq 0} r\alpha \frac{\sin \alpha}{\alpha} = \lim_{\alpha \doteq 0} a \frac{\sin \alpha}{\alpha} = a \lim_{\alpha \doteq 0} \frac{\sin \alpha}{\alpha} = a.$$

Similarly as r increases indefinitely $\lim r \sin \frac{\alpha}{2} = \frac{a}{2}$.

We have by § 77 $\cos a = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos A$. $\cos a = 1 - 2\sin^2\frac{a}{2}$, by § 58. Substituting this value for $\cos a$ and similar values for $\cos \beta$ and $\cos \gamma$ in the above equation and reducing we have:

$$\sin^2\frac{a}{2} = \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} - 2\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2} - \tfrac{1}{2}\sin\beta\sin\gamma\cos A.$$

Multiplying this equation through by r^2 and taking the limits of both members as r increases indefinitely we have:

$$\left(r\sin\frac{a}{2}\right)^2 = \left(r\sin\frac{\beta}{2}\right)^2 + \left(r\sin\frac{\gamma}{2}\right)^2 - \frac{2}{r^2}\left(r\sin\frac{\beta}{2}\right)^2\left(r\sin\frac{\gamma}{2}\right)^2 - \tfrac{1}{2}(r\sin\beta)(r\sin\gamma)\cos A.$$

$$\therefore \left(\frac{a}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - \tfrac{1}{2}bc\cos A, \text{ or } a^2 = b^2 + c^2 - 2bc\cos A,$$

an equation which with two others similar to it, and obtained in the same way, may be considered the fundamental equations of plane trigonometry. This has led us to the broadest generalization of which we are capable at this stage, but the student will learn subsequently that all trigonometry of the sphere and plane is wrapped up in one quaternion equation, an equation which in the quaternion calculus is very elementary. This equation is:

$$S(V\gamma a V a\beta) = a^2 S\beta\gamma - S\gamma a S a\beta,$$

and being interpreted with reference to a sphere it gives at once the three equations of § 77, which are the fundamental equations of spherical trigonometry, which includes plane trigonometry as a special case.

C. Spherical Triangles.

a. Cases.

§ 101. With any three parts given, a spherical triangle may be solved, and the equations of § 77 theoretically determine the solution. In practice, however, derived equations adapted to logarithmic computation are used. The three given parts may be selected in $\frac{6.5.4}{1.2.3} = 20$ ways, as follows:

ABC	ACa	Aac	BCc	Cab
ABa	ACb	Abc	Bab	Cac
ABb	ACc	BCa	Bac	Cbc
ABc	Aab	BCb	Bbc	abc

Of these, there are only six distinct cases.

Excluding repetitions, we have:

1. abc, three sides.
2. ABC, three angles.
3. abC, two sides and included angle.
4. ABc, two angles and included side.
5. abA, two sides and angle opposite one.
6. ABa, two angles and side opposite one.

The number of these cases may again be reduced by one half, by the use of the polar triangle, since, for example, cases 2, 4, and 6 could be made to depend upon cases 1, 3, and 5, respectively. But it is practically better to know how to treat six cases.

b. Solutions.

Case 1. Given a, b, and c. In order that the triangle should be possible, the following inequalities must be satisfied:

$$2\pi > a + b + c > 0,$$

$$\begin{array}{ll} a + b > c > 0 & \pi > a > 0 \\ b + c > a > 0 & \pi > b > 0 \\ c + a > b > 0 & \pi > c > 0 \end{array}$$

If these inequalities are satisfied, the solution is determinate in a single way, and the angles are given by the formulas:

$$\tan\frac{A}{2}=\frac{\tan r}{\sin(s-a)}, \quad \tan\frac{B}{2}=\frac{\tan r}{\sin(s-b)}, \quad \tan\frac{C}{2}=\frac{\tan r}{\sin(s-c)},$$

where $\tan r=+\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$. § 99

If only a single angle is desired, it may be computed from one of the formulas,

$$\tan\frac{A}{2}=+\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}, \text{ etc.} \qquad \S\ 93$$

As a check, the formulas

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C} \qquad \S\ 79$$

may be used.

Case 2. Given A, B, and C. If the following inequalities,

$$3\pi > A+B+C > \pi,$$

$$2\pi > \pi + A > B + C, \qquad \pi > A > 0,$$
$$2\pi > \pi + B > C + A, \qquad \pi > B > 0,$$
$$2\pi > \pi + C > A + B, \qquad \pi > C > 0,$$

are satisfied, the three sides may be obtained determinately in a single way, from the formulas :

$$\tan\frac{a}{2}=\tan R\cos(S-A),$$

$$\tan\frac{b}{2}=\tan R\cos(S-B),$$

$$\tan\frac{c}{2}=\tan R\cos(S-C),$$

where $\tan R=+\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$. § 98

The formulas of § 79 may be used as a check. If only a single side is desired, one of the formulas of § 94 may be used.

Case 3. Given a, b, and C. If the inequalities,

$$\pi > a > 0, \qquad \pi > b > 0, \qquad \pi > C > 0,$$

are satisfied, we may solve the triangle, which will be determinate in a single way, as follows: A and B may be found by using the formulas:

$$\tan \frac{A+B}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \operatorname{ctn} \frac{C}{2},$$

$$\tan \frac{A-B}{2} = \frac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} \operatorname{ctn} \frac{C}{2}. \qquad \S\ 96$$

These give us $\frac{A+B}{2} = m$, $\frac{A-B}{2} = n$, whence $A = m + n$, $B = m - n$.

c may be found from the formula of § 96:

$$\tan \frac{c}{2} = \frac{\sin \frac{A+B}{2}}{\sin \frac{A-B}{2}} \tan \frac{a-b}{2}.$$

As a check use § 79.

If the side c alone is desired, it may be found from the formula, § 77,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

which may be adapted to logarithmic computation by the aid of an auxiliary angle, as follows:

$$\cos c = \cos b\,(\cos a + \sin a \tan b \cos C).$$

If $\tan b \cos C$ is put equal to $\tan x = \frac{\sin x}{\cos x}$, we have the two formulas,

$$\cos c = \frac{\cos b \cos (a - x)}{\cos x}, \text{ and } x = \tan^{-1} (\tan b \cos C),$$

or a single formula,

$$\cos c = \frac{\cos b \cos [a - \tan^{-1} (\tan b \cos C)]}{\cos \tan^{-1} (\tan b \cos C)}.$$

In the same manner, A and B can be computed singly. By § 80, we have

$$\text{ctn } a \sin b = \cos b \cos C + \sin C \text{ ctn } A,$$
$$\text{ctn } b \sin a = \cos a \cos C + \sin C \text{ ctn } B,$$

or

$$\sin C \text{ ctn } A = \text{ctn } a (\sin b - \cos b \tan a \cos C),$$
$$\sin C \text{ ctn } B = \text{ctn } b (\sin a - \cos a \tan b \cos C).$$

If we put
$$\tan b \cos C = \tan x,$$
$$\tan a \cos C = \tan y,$$

these formulas become

$$\text{ctn } A = \frac{\text{ctn } a}{\sin C} \cdot \frac{\sin (b - y)}{\cos y} = \frac{\text{ctn } C \sin (b - y)}{\sin y},$$

$$\text{ctn } B = \frac{\text{ctn } b}{\sin C} \cdot \frac{\sin (a - x)}{\cos x} = \frac{\text{ctn } C \sin (a - x)}{\sin x},$$

or

$$\text{ctn } A = \frac{\text{ctn } C \sin [b - \tan^{-1} (\tan a \cos C)]}{\sin \tan^{-1} (\tan a \cos C)},$$

$$\text{ctn } B = \frac{\text{ctn } C \sin [a - \tan^{-1} (\tan b \cos C)]}{\sin \tan^{-1} (\tan b \cos C)}.$$

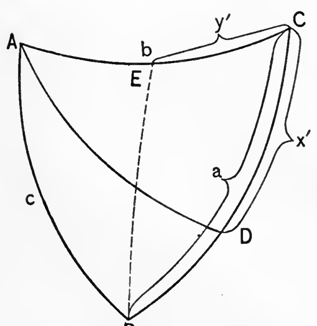


FIG. 46.

The employment of the auxiliary angles x and y is equivalent to decomposing the triangle $\begin{matrix}ABC\\abc\end{matrix}$ into two right triangles. For if AD is an arc perpendicular to BC from A,

$$\tan x' = \tan b \cos C, \qquad \text{§ 86, 4}$$

whence
$$x = x'.$$

Also,
$$\cos AD = \frac{\cos b}{\cos x}, \qquad \text{§ 86, 1}$$

$$\tan AD = \frac{\sin x}{\text{ctn } C}. \qquad \text{§ 86, 6}$$

And in the triangle ADB we have

$$\cos c = \cos AD \cos (a - x) = \frac{\cos b \cos (a - x)}{\cos x},$$

and $$\operatorname{ctn} B = \frac{\sin (a - x)}{\tan AD} = \frac{\operatorname{ctn} C \sin (a - x)}{\sin x}.$$

In the same way, if BE is a perpendicular arc from B to AC, $y = y'$, and the angle A may be found similarly as the angle B.

Case 4. Given A, B, and c. If the inequalities $\pi > A > 0$, $\pi > B > 0$, $\pi > c > 0$, are satisfied, the triangle is possible, and determinate in a single way. We may find a, and b by the following formulas:

$$\tan \frac{a + b}{2} = \frac{\cos \frac{A - B}{2}}{\cos \frac{A + B}{2}} \tan \frac{c}{2},$$

§ 96

$$\tan \frac{a - b}{2} = \frac{\sin \frac{A - B}{2}}{\sin \frac{A + B}{2}} \tan \frac{c}{2}.$$ These give

$$\frac{a + b}{2} = p, \quad \frac{a - b}{2} = q, \qquad \text{whence} \qquad a = p + q, \quad b = p - q.$$

C may be found from the formula of § 96,

$$\operatorname{ctn} \frac{C}{2} = \frac{\sin \frac{a + b}{2}}{\sin \frac{a - b}{2}} \tan \frac{A - B}{2}.$$ Use § 79 as a check.

If only the angle C is required, it may be obtained by the formula, § 81,

$$\begin{aligned}\cos C &= -\cos A \cos B + \sin A \sin B \cos c\\ &= \cos B\,(-\cos A + \sin A \tan B \cos c)\\ &= \frac{\cos B \sin (A - u)}{\sin u}\\ &= \frac{\cos B \sin [A - \text{ctn}^{-1}(\tan B \cos c)]}{\sin \text{ctn}^{-1}(\tan B \cos c)},\end{aligned}$$

if $\text{ctn}\, u = \tan B \cos c$.

Similarly the sides a and b may be computed, if in the formulas,

$$\begin{aligned}\text{ctn}\, a \sin c &= \cos c \cos B + \sin B\, \text{ctn}\, A,\\ \text{ctn}\, b \sin c &= \cos c \cos A + \sin A\, \text{ctn}\, B,\end{aligned} \qquad \S\ 80$$

we put
$$\begin{aligned}\text{ctn}\, u &= \tan B \cos c,\\ \text{ctn}\, v &= \tan A \cos c,\end{aligned}$$

and obtain,
$$\text{ctn}\, a = \frac{\text{ctn}\, c \cos [B - \text{ctn}^{-1}(\tan A \cos c)]}{\cos \text{ctn}^{-1}(\tan A \cos c)},$$

$$\text{ctn}\, b = \frac{\text{ctn}\, c \cos [A - \text{ctn}^{-1}(\tan B \cos c)]}{\cos \text{ctn}^{-1}(\tan B \cos c)}.$$

Similarly as in Case 3, it may be shown that the use of the auxiliary angles u and v, is equivalent to decomposing the triangle $\begin{matrix}ABC\\ abc\end{matrix}$ into two right triangles. The student may prove $\begin{matrix}\text{that angle } u = \text{angle } BAD,\\ \text{and angle } v = \text{angle } ABE,\end{matrix}$ in Fig. 46.

Case 5. Given a, b, and A.

(1) If the inequalities, $\pi > a > 0$, $\pi > b > 0$, $\pi > A > 0$, are satisfied, there will always be one solution, when b differs more than a from $\frac{\pi}{2}$.

(2) Unless the inequalities $\frac{\sin A \sin b}{\sin a}$ $[= \sin B] < 1$, $\pi > \begin{matrix}a\\ A\end{matrix} > \frac{\pi}{2}$, or else $\frac{\pi}{2} > \begin{matrix}a\\ A\end{matrix} > 0$, and $\pi > b > 0$, are satisfied when a differs more than b from $\frac{\pi}{2}$, there will be no solu-

tion ; and when they are satisfied, and a and b thus related, there will always, necessarily, be two solutions.

These statements may be proved as follows : It is evident that there can be no solution under (2), unless the inequalities, $\pi > \frac{a}{A} > \frac{\pi}{2}$, or else $\frac{\pi}{2} > \frac{a}{A} > 0$, are satisfied under the conditions. For, in any spherical triangle with real parts, if one side differs more than another side from $\frac{\pi}{2}$, its opposite angle is of the same quadrant with it.

This appears from the fact that the parts of the triangle must satisfy the relation $\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$, § 77, and if a differs more than b from $\frac{\pi}{2}$, $\cos a > \cos b$, numerically, and if c is real, $\cos c < 1$, numerically, and, therefore, $\cos a > \cos b \cos c$, numerically, and hence, cos A and cos a have the same sign, and A and a are of the same quadrant.

To prove the remaining statements, we consider the equation

$$\sin c = \frac{\cos a - \cos b \cos c}{\sin b \cos A},$$

where we let $\frac{\cos a}{\sin b \cos A} = m$, $\frac{\cos b}{\sin b \cos A} = n$, and get the following equation in sin c :

$$(1 + n^2) \sin^2 c - 2\, m \sin c + m^2 - n^2 = 0.$$

The product of the roots is $\frac{m^2 - n^2}{n^2 + 1}$.

The roots are :

$$\sin c_1 = \frac{1}{n^2 + 1} (m + n\, (n^2 + 1 - m^2)^{\frac{1}{2}}),$$

$$\sin c_2 = \frac{1}{n^2 + 1} (m - n\, (n^2 + 1 - m^2)^{\frac{1}{2}}).$$

According as $\left\{\begin{matrix} b \\ a \end{matrix}\right\}$ differs more than $\left\{\begin{matrix} a \\ b \end{matrix}\right\}$ from $\frac{\pi}{2}$, that is according as we have $\left\{\begin{matrix} (1) \\ (2) \end{matrix}\right\}$, $n^2 \gtrless m^2$, and in (2) $m > 0$.

According as we have $\left\{\begin{matrix} (1) \\ (2) \end{matrix}\right\}$, the product of the roots, $\frac{m^2 - n^2}{n^2 + 1}$, is therefore $\left\{\begin{matrix} \text{negative} \\ \text{positive} \end{matrix}\right\}$; and its form shows that the factors, *i.e.*, the roots, must be real, and that, therefore $n^2 + 1 - m^2 > 0$; and in $\left\{\begin{matrix} (1) \\ (2) \end{matrix}\right\}$ that the roots must be $\left\{\begin{matrix} \text{positive and negative} \\ \text{both positive or both negative} \end{matrix}\right\}$. And in (2) both roots must be positive, since, as $m > 0$, a single root at least must be positive.

Again, $2\,m^2n^2\,(n^2 + 1 - m^2) \leqslant m^4n^4 + (n^2 + 1 - m^2)^2$,

and

$$4\,m^2n^2\,(n^2 + 1 - m^2) \leqslant m^4n^4 + 2\,m^2n^2\,(n^2 + 1 - m^2) + (n^2 + 1 - m^2)^2.$$

Taking positive values of m, n, and $n^2 + 1 - m^2$,

$$2\,mn\,(n^2 + 1 - m^2)^{\frac{1}{2}} \leqslant n^2 + 1 - m^2 + m^2n^2,$$

$$\therefore 2\,mn\,(n^2 + 1 - m^2)^{\frac{1}{2}} \leqslant n^4 + 2\,n^2 + 1 - m^2 - n^2(n^2 + 1 - m^2),$$

$$\therefore m^2 + n^2\,(n^2 + 1 - m^2) + 2\,mn\,(n^2 + 1 - m^2)^{\frac{1}{2}} \leqslant (n^2 + 1)^2,$$

$$\therefore \frac{m + n\,(n^2 + 1 - m^2)^{\frac{1}{2}}}{n^2 + 1} \leqslant 1.$$

From this it is seen that *no one* of all the possible values of $\sin c$, in either (1) or (2), can be numerically greater than unity.

In (1), there is one solution only, for one value of $\sin c$ is negative, and B must be of the same quadrant with b.

In (2), there are two solutions, real and distinct, in $\sin c$, unless $n(n^2+1-m^2)^{\frac{1}{2}}=0$, *i.e.*, unless $b=\frac{\pi}{2}$, or else $\cos^2 a - \cos^2 b = \sin b \cos A$, when the two solutions in $\sin c$ analytically coincide. For, with this possible exception, we have proved that $1 > \frac{\sin c_1}{\sin c_2} > 0$, and $\sin c_1 \neq \sin c_2$. If $n=0$, *i.e.*, if $b=\frac{\pi}{2}$, the two analytically coincident solutions in $\sin c$ give two real triangles, in which the values of c are supplementary, as shown in § 89.

If $n^2+1-m^2=0$, *i.e.*, if $\cos^2 a - \cos^2 b = \sin b \cos A$, the condition $\sin B < 1$, is violated, and the solutions are not real.*

It might seem as if $\sin c_1$, and $\sin c_2$, would give two values of c_1 and of c_2, which taken with the two values of B, obtained from $\sin B$, would give eight triangles, but the formulas of § 96, which we use in finding C and c, show that only one value of c can consist with one value of B. (See examples XI. 24, 25, 26.)

We may now consider that we have

	bc_1A	B_1C_1	
a	bc_2A	B_2C_2	where $\pi > b, c_1, c_2, A > 0$.
	given	required	

We may find B by the formula,

$$\sin B = \frac{\sin A \sin b}{\sin a}. \qquad \S\ 79$$

If two values of B are admissible, they are supplements of each other and may be denoted by B_1 and B_2. When B is found, c and C may be obtained from the formulas :

* It is easily shown in the Theory of Functions that $\sin^{-1} x$, if $x > 1$, is a complex angle.

$$\tan\frac{c}{2} = \frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}\tan\frac{a-b}{2},$$

§ 96

$$\operatorname{ctn}\frac{C}{2} = \frac{\sin\frac{a+b}{2}}{\sin\frac{a-b}{2}}\tan\frac{A-B}{2}.$$

If there are two solutions we have:

$$\tan\frac{c_1}{2} = \frac{\sin\frac{A+B_1}{2}}{\sin\frac{A-B_1}{2}}\tan\frac{a-b}{2},$$

$$\tan\frac{c_2}{2} = \frac{\sin\frac{A+B_2}{2}}{\sin\frac{A-B_2}{2}}\tan\frac{a-b}{2},$$

with two similar formulas for C_1 and C_2.

Case 6. Given A, B, and a.

(1) If the inequalities, $\pi > A > 0$, $\pi > B > 0$, $\pi > a > 0$, are satisfied, there will always be one solution, when B differs more than A from $\frac{\pi}{2}$.

(2) Unless the inequalities,

$$\frac{\sin a \sin B}{\sin A}[=\sin b] < 1, \quad \pi > {}^{A}_{a} > \frac{\pi}{2},$$

or else $\quad \frac{\pi}{2} > {}^{A}_{a} > 0,$

and $\quad \pi > B > 0,$

are satisfied when A differs more than B from $\frac{\pi}{2}$, there will be no solution; and when they are satisfied, and A and B thus related, there will always, necessarily, be two solutions.

These statements might be proved similarly as in the preceding case, taking the equation,

$$\sin C = \frac{\cos A + \cos B \cos C}{\sin B \cos a}. \qquad \S\ 81$$

But it is not necessary to do this. If A, B, and a are given, a′, b′, and A′ of the polar triangle are given. If the polar triangle has two solutions, the given triangle will have two solutions, and conversely. The polar triangle will have two solutions, when a′ differs more than b′ from $\frac{\pi}{2}$. The given triangle will therefore have two solutions, when $\pi - A$ differs more than $\pi - B$ from $\frac{\pi}{2}$, *i.e.*, when A differs more than B from $\frac{\pi}{2}$. We may find b by the formula

$$\sin b = \frac{\sin a \sin B}{\sin A} \qquad \S\ 79$$

and if two values of b are admissible they are supplements of each other, and may be denoted by b_1 and b_2. c and C may be found from the formulas :

$$\tan \frac{c}{2} = \frac{\sin \frac{A + B}{2}}{\sin \frac{A - B}{2}} \tan \frac{a - b}{2},$$

$$\operatorname{ctn} \frac{C}{2} = \frac{\sin \frac{a + b}{2}}{\sin \frac{a - b}{2}} \tan \frac{A - B}{2}. \qquad \S\ 96$$

Similarly if there are two solutions.

D. Examples. XI.

Apply the conditions of possibility, and solve the following triangles, having given:

1. $a = 124° 12'.5$, $b = 54° 18'$, $c = 97° 12'.5$.
2. $a = 74° 26'$, $b = 85° 12'$, $c = 29° 47'$.
3. $A = 20° 10'$, $B = 55° 52'$, $C = 114° 20'$.
4. $A = 130°$, $B = 110°$, $C = 80°$.
5. $a = 35° 37'$, $b = 59° 12'$, $C = 124° 18'$.
6. $A = 54° 55'$, $b = 69° 25'$, $c = 109° 46'$.
7. $A = 26° 59'$, $B = 39° 45'$, $c = 154° 47'$.
8. $A = 57° 35'$, $B = 120° 48'$, $c = 124° 18'$.
9. $a = 50°$, $b = 40°$, $A = 80°$.
10. $a = 150° 57'$, $b = 134° 16'$, $A = 144° 23'$.
11. $A = 113° 39'$, $B = 123° 40'$, $a = 65° 40'$.
12. $A = 52° 50'$, $B = 66° 07'$, $a = 59° 28'$.
13. Are the following triangles possible? If impossible, what conditions are violated?
    (1) $a = 160° 25'$, $b = 92° 23'$, $c = 64° 49'$.
    (2) $A = 137° 18'$, $B = 123° 15'$, $C = 74° 36'$.
    (3) $a = 2° 29'$, $b = 37'.5$, $C = 179° 48'$.
    (4) $A = 5° 41'$, $B = 15° 50'$, $c = 163° 34'$.
    (5) $A = 88° 49'$, $a = 16° 34'$, $b = 16° 30'$.
    (6) $a = 30° 08'$, $b = 54° 03'$, $A = 60° 01'$.
    (7) $a = 117° 29'$, $A = 174° 17'$, $B = 173° 25'$.
14. Prove that the polar radii of the inscribed circle of a triangle, and of the circumscribed circle of its polar triangle are complements of each other.
15. If R, R_a, R_b, R_c, are the polar radii of the circles circumscribed about the triangles ABC, BCA', CAB', ABC', respectively, where AA', BB', CC', are diameters of the sphere, prove that

$$\begin{aligned} -\operatorname{ctn} R \cos S &= \operatorname{ctn} R_a \cos (S - A) \\ &= \operatorname{ctn} R_b \cos (S - B) \\ &= \operatorname{ctn} R_c \cos (S - C) \\ &= \sqrt{-\cos S \cos (S - A) \cos (S - B) \cos (S - C)}. \\ &[= L, \text{ for brevity.}] \end{aligned}$$

16. If r, r_a, r_b, r_c, are, respectively, the polar radii of the circles inscribed in the triangles of the preceding problem, prove that

$$\tan r \sin s = \tan r_a \sin (s - a) = \tan r_b \sin (s - b) = \tan r_c \sin (s - c)$$
$$= \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$
$[= l$, for brevity.$]$

Using the notation of examples 15 and 16, prove the following:

17. $\tan r \tan r_a \tan r_b \tan r_c = l^2$.
18. $\text{ctn}\, R\, \text{ctn}\, R_a\, \text{ctn}\, R_b\, \text{ctn}\, R_c = L^2$.
19. $\tan r_a \tan r_b \tan r_c = l \sin s$.
20. $\text{ctn}\, R_a\, \text{ctn}\, R_b\, \text{ctn}\, R_c = - L \cos S$.
21. $\tan r_a + \tan r_b + \tan r_c - \tan r = \dfrac{4 L \sin S}{\sin A \sin B \sin C}$.
22. $\tan R_a + \tan R_b + \tan R_c - \tan R = 2\, \text{ctn}\, r$.
23. $\text{ctn}\, R - \text{ctn}\, R_a - \text{ctn}\, R_b - \text{ctn}\, R_c = \dfrac{2 L \cos s}{l}$.
24. Show that if the inequalities of § 101. *b*. Case 5. (1), are satisfied,

$$\frac{\sin A \sin b}{\sin a} [= \sin B] < 1.$$

Show that under § 101. *b*. Case 5. (2), the two analytically coincident solutions in $\sin c$:

25. Give two triangles whose sum is equal to a lune, if $n = 0$.
26. Violate the condition

$$\frac{\sin A \sin b}{\sin a} [= \sin B] < 1,$$

for real triangles, if $n^2 + 1 - m^2 = 0$.

27. Find the volume V of an oblique parallelopiped, having given the three conterminous edges l, m, n, and the three angles α, β, γ, which the edges make with each other.

In the following problems the earth is assumed to be a sphere whose radius is 3963 miles.

28. Given the latitudes and longitudes of three places on the earth's surface, show how to find the area of the triangle of which the given places are the vertices.

29. Colorado extends from 37° to 40° N., and from 102° to 107° W. What is its area in statute square miles?

30. If Pennsylvania is assumed to extend from 39° 43′ to 42° N., and from 75° to 80° 35′ W., what is its area in statute square miles?

31. If Ohio is assumed to extend from 39° to 41° 30′ N., and from 80° 35′ to 84° 40′ W., what is its area in statute square miles?

32. Given the latitudes and longitudes of two places on the earth's surface, show how to find the shortest distance between them.

33. What is the shortest distance in statute miles, from Cambridge, Mass. (lat. 42° 23′ N., long. 77° 08′ W.), to Oberlin (lat. 41° 17′ N., long. 82° 11′ W.)?

34. What is the shortest distance in statute miles from Philadelphia (lat. 39° 57′ N., long. 75° 10′ W.) to Oberlin?

35. Given the shortest distance between two places, and their latitudes, show how to find the difference in their local time.

36. The shortest distance between Sandy Hook (lat. 40° 28′ N.) and Cork Harbor (lat. 51° 47′ N.) is 2726 geographical miles. When it is 6 A.M. at Sandy Hook, what time is it at Cork Harbor?

37. Given ϕ, δ, and t. What is the formula for the position angle of the astronomical triangle?

38. At Oberlin Observatory, the declination of a star is 54° 25′, its hour angle 10 h. 20 m. What is its position angle?

39. Find the distance between the sun and moon at Greenwich mean noon on June 20, 1896, when their respective right ascensions are 5 h. 58 m. 11.09 s. and 13 h. 58 m. 21.65 s., and their declinations 23° 27′ 14″ and 17° 06′ 15″.3.

40. Find the distance between Uranus and Neptune at Greenwich mean noon on March 31, 1896, when their respective right ascensions are 15 h. 26 m. 59.06 s. and 4 h. 58 m. 8 s., and their declinations 18° 30′ 34″ and 21° 16′ 40″.4.

41. Find the distance between Venus and Jupiter at Greenwich mean noon on Sept. 10, 1896, when their respective right ascensions are 12 h. 21 m. 38.16 s. and 9 h. 55 m. 29.76 s., and their declinations 1° 09′ 19″.9 and 13° 27′ 06″.3.

42. Find the moon's right ascension and declination at Greenwich mean midnight Oct. 29, 1896, when her true longitude is 124° 53′ 30″.7, and the obliquity of the ecliptic is 23° 27′ 16″.78.

43. Find the declination of a star whose longitude is 47° 25′ and latitude 51° 36′. (Take $e = 23° 27′$.)

44. Find the latitude of a star whose longitude is 27° 19′, and declination 32° 48′.

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