ELEMENTS OF PLANE TRIGONOMETRY WITH COMPLETE TABLES

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Florian Cajori

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ELEMENTS OF PLANE TRIGONOMETRY

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New York
THE MACMILLAN COMPANY
1919

Set up and electrotyped. Published April, 1919.

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

THIS book carries out the chief motives which guided the authors in their larger work on Plane and Spherical Trigonom etry. On the other hand it has been entirely rewritten, and has been made still more elementary in character. The new text forms a treatment of Plane Trigonometry which is quite brief, but which nevertheless deals with the most essential topics in more than the usual detail.

This has been accomplished by omitting or curtailing certain topics that are seldom used by the student except in some special line of work. Thus all of Spherical Trigonometry and much of the detailed discussion of Trigonometric Identities and Equations is omitted. Such traditional topics as De Moivre's Theorem and infinite series were omitted from the author's larger work because they have few applications within the student's present grasp. These are of course omitted from the present book also.

Thus this treatment contains a minimum of purely theoretical matter. Its entire organization is intended to give a clear view of the immediate usefulness of trigonometry.

The solution of Triangles remains the principal motive. As such, this problem is attacked immediately and it is pushed to a definite conclusion early in the course.

More complete outlines than usual have been given for the solution of oblique triangles by means of right triangles. This method of solution was emphasized recently in the Syllabus of the War Department for instruction in the S. A. T. C. A very brief course could well close with this method of solving triangles.

Other practical problems are introduced to furnish a motive for the treatment of the general angle, the addition theorems, radian measure,.etc. Among other applications, the composition and resolution of forces, projections, and angular speed are introduced prominently.

The tables are very complete and usable. Attention is called particularly to the table of squares, square roots, cubes, etc.; by its use the Pythagorean theorem and the cosine law become practicable for actual computation. The use of the slide rule and of four-place tables is encouraged for problems that do not demand extreme accuracy. One edition of the book contains only the four-place tables. Many who use that edition find it advisable to have students purchase also the five-place tables which are published separately bound under the title The Macmillan Tables.

The authors have borne in mind constantly the needs of the beginner in trigonometry and have adapted the book to use in secondary schools as well as in colleges. Illustrative material abounds, and the explanations have been carefully worked out in great detail. The sample forms for the solution of triangles is a striking instance of this tendency.

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CONTENTS

PART I. ACUTE ANGLES AND RIGHT TRIANGLES

[See Contents, page xviii.]

ELEMENTS OF PLANE TRIGONOMETRY

PART I. ACUTE ANGLES AND RIGHT TRIANGLES

CHAPTER I

INTRODUCTION

1. Subject Matter. The word Trigonometry comes from two Greek words meaning measurement of, or by means of, triangles. The original purpose of this study was the measurement of angles and distances by indirect methods in cases in which direct measurements are inconvenient or impossible. Among such cases we may mention the determination of the heights and horizontal widths of hills, the distance across a valley or river, or the lengths of the boundaries of fields on rough or impassable ground. Trigonometry treats also the relations among the sides and angles of triangles, and the measurement of the sides, angles, and areas of triangles and of other polygons which can be separated into triangles.

2. Measurement. To measure any quantity is to determine how many times it contains some convenient unit quantity of the same kind. The expression of every measured quantity consists of these *two components:* the ***numerical measure*** and the ***name of the unit*** employed; as, 2 inches, 20 cubic centimeters, 3 pounds and 10 ounces, 7 hours and 26 minutes, 51.72 acres, 36 degrees, 7.4 feet per second, 35.8 ohms, 2.3 amperes, 110 volts, etc.

Sometimes we can make direct comparison of a quantity with the unit of measure, as when we determine the length of a segment by applying a yardstick or a steel tape to it. On the other hand we are often obliged to use indirect methods, *i.e.* to compute the numerical measure of a quantity by means of its relations to other quantities more easily measured. Thus, we find the numerical measure of the area of a triangle not by direct measurement, but rather by taking one-half the product of the numerical measures of its base and its altitude.

3. Relations to Other Subjects. Applications. It is evident that trigonometry is closely related to plane geometry on account of its use of lines, angles, triangles and other polygons. On the other hand, since the measures of the sides, angles, and areas of triangles, and the ratios of the sides, are *numbers*, trigonometry is also related to arithmetic and elementary algebra.

The applications of trigonometry are very extensive. Some of them will be given in this book. Many others are to be found in surveying, navigation, astronomy, architecture, design, geometry, mechanics, and other branches of mathematics and physics, and in military and civil engineering.

4. Graphical Solution of Triangles. For constructing triangles and measuring their parts, the student should have a

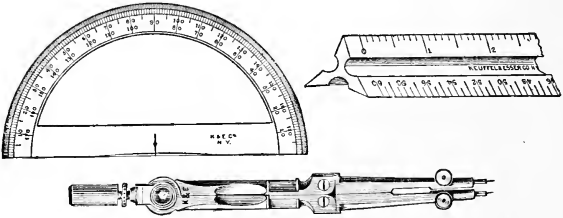

FIG. 1.

scale for measuring lengths, a *protractor* for measuring angles, and a *compass* for drawing circles, laying off arcs and equal segments.

Two triangles, or other geometric figures, are said to be **congruent** when they can be superimposed so as to coincide in all their parts.

Two figures are **similar** when their corresponding angles are equal and their corresponding sides are proportional. *Two triangles are similar if they are mutually equiangular*, but this is not necessarily true of polygons of more than three sides.

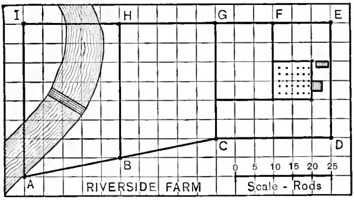

FIG. 2.

To draw a figure *to scale* is to make a drawing which shall be similar to it but smaller (or larger), as, for example, a map of a farm or a field, or the floor plan of a building.

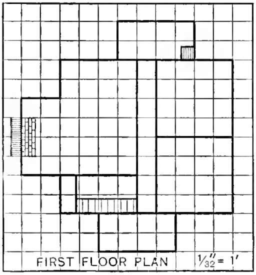

FIG. 3.

The advantage of a scale drawing is that the angles are the same as those of the figure represented, and by the scale relation marked on the drawing, any dimension of the original figure can be read off on a scale applied to the corresponding dimension of the drawing.

A builder uses the architect's plans for this purpose in constructing a building.

We know from geometry that *the other three parts of any actual* triangle are determined if any one of the following combinations is known:*

(1) *two sides and the included angle;*

(2) *two angles and any specified side;*

(3) *the three sides;*

(4) *two sides and the angle opposite one of them,*

but in the last case there may be two solutions when the given angle is acute.

When a sufficient number of parts of an actual* triangle are known, the others can be found by drawing the triangle to scale and measuring the sides with the scale and the angles with the protractor.

The process of finding the unknown parts of a triangle from any such set of given parts is called ***solving the triangle.***

EXAMPLE 1. In order to measure the width of a river, for example, it is sufficient to measure the distance AB between two points on the bank and the angles BAP and ABP made by AB with the lines joining A and B, respectively, to any point on the other bank. All of these measurements can be made from one bank of the river. Knowing AB and the angles ABP and BAP the triangle PAB can be drawn to scale; then the perpendicular PR from P to AB can be drawn and measured, whence the width PR of the stream can be determined by actual measurement in the figure. If $AB = 98$ yards, $\angle A = 38°$, and $\angle B = 65°$, PR will be found to be about 56 yards.

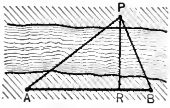

FIG. 4.

5. Preliminary Estimate. Check. In every exercise, the student should make a preliminary estimate of the unknown parts and he should keep this crude solution in mind to guide him in his work.

After the unknown parts have been found, the student should use all means at his command to *check* each answer,

*The data can be given so that it will be impossible to construct any triangle satisfying the conditions. If such data are given, the impossibility will appear when the attempt to construct the triangle is made.

since even experienced persons are liable to error in reading scales and in making computations.

In triangles drawn to scale observe the following checks:

(1) *the sum of the angles of any triangle should be 180°;*

(2) *the sum of any two sides should be greater than the third side;*

(3) *the greater of two sides should be opposite the greater of the angles opposite these sides;*

(4) *if two sides are unequal their numerical measures should be unequal in the same sense;*

(5) *the numerical measures of angles should correspond to their magnitudes;* angles of 30°, 45°, 60°, 90°, etc., are easy to judge by the eye.

These checks should reveal any *gross* error; but the student should not expect this method of solution (or any other method of computation or measurement) to give *precise* answers in the sense of having no error whatever. The purpose should be to obtain reasonably accurate results and to detect errors that are *unreasonably large.*

EXERCISES I. — GRAPHICAL SOLUTION OF TRIANGLES

Solve the following triangles by construction and measurement.

1. Two angles are 47° and 53° and the included side is 5.7
Ans. 80°, 4.2, 4.6

2. Two angles are 43° and 53° and the side opposite the latter is 6.7
Ans. 84°, 5.7, 8.3

3. Two sides are 4.3 and 5.3 and the included angle is 57°.
Ans. 51°, 72°, 4.7

4. The three sides are 4.3, 5.3, and 6.3 *Ans.* 42°, 56°, 81°.

5. Two angles are 40° and 65° and the side opposite the latter is 50.
Ans. 75°, 35.5, 53.3

6. Two angles are 30° and 105° and the included side is 7 feet 8 inches.
Ans. 45°, 5 ft., 9.7 ft.

7. Two sides are 16.9 and 40.9 and the altitude upon the third side is 12. Find the perimeter and the area. *Ans.* 108.8, 306.

8. Two angles are 30° and 100° and the shortest side is 8. Find the longest side, the altitude upon it, and the area. *Ans.* 15.8, 6.1, 48.2

9. The sides are in the ratio 3 : 4 : 5. Find the smallest and the largest angle. *Ans.* 37°, 90°.

10. The angles are in the ratio 3 : 4 : 5 and the shortest side is 30. Find the other sides. *Ans.* 37, 41.

11. The sides are 5, 7, and 8. Find the angles. *Ans.* 38°, 60°, 82°.

12. The sides are 3, 5, and 7. Find the largest angle. *Ans.* 120°.

13. Two sides are 8 and 10 and the included angle is 47°. Find the perimeter, the area, and the radius of the inscribed circle.

Ans. 25.4, 29.25, 2.3

14. From which of the following sets of given parts is it possible to construct a triangle? Do any of the sets determine more than one?

(*a*) Two angles are 41° and 59°, the side opposite the latter is 5.1

(*b*) Two sides are 1.3 and 5.6, the angle opposite the first is 66°.

(*c*) Two angles are 30° and 41°, the included side is 7.

(*d*) Two sides are 7 and 1.1, the included angle is 17°.

(*e*) The three sides are 1.1, 2.3, 3.5

(*f*) Two sides are 6 and 7, the angle opposite the first is 51°.

Ans. (*b*) and (*e*), impossible; (*f*), two.

15. Two sides are 5 and 7 and the angle opposite the latter is 60°. Find the perimeter and the area. *Ans.* 20; 17.3

6. Measurements in the Field. In surveying land, rivers, lakes, and harbors; laying out roads, ditches, the foundations of bridges, buildings, and other structures; and in many other projects of civil and military engineering, distances in the field are measured with the *chain*, or the *steel tape*. In cases where extreme accuracy is required, a long metal or wooden scale is used, and is carefully protected against, and corrected for, changes in temperature.

Angles in the horizontal plane are drawn in position on the *plane table* by means of a pair of sights on a heavy metal straightedge; or, more often both horizontal and vertical angles are sighted with the telescope of the *engineer's transit* and their measures are read off from the graduated circles of the instrument.

In determining distances and directions in an extended survey, greater accuracy can be attained by measuring the angles of certain triangles and computing the lengths of the sides, than by measuring these sides directly.

FIG 5.

A base line AB is first established and measured with great precision. Then some point C, visible from both A and B, is selected and the angles CAB and ABC are measured; another point D is next selected and the angles CBD and BCD are measured. Thus, a chain of triangles can be extended over a wide range of territory and on completing the computations the length and direction of every line in the system will be known. This process, called *triangulation*, is used by the U. S. Coast and Geodetic Survey.

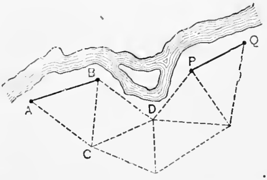

FIG. 6.

Much work has been done near the coasts and a triangulation system has been extended from the Atlantic to the Pacific.

7. Angles of Elevation and Depression. An observer at O measures the angle of ***elevation*** of an object A, higher than himself, by sighting a horizontal line OH by means of the level on the telescope of the transit and then elevating the end of the telescope until he sights A. The angle HOA through which the telescope has been turned in the vertical plane, and which is read off from the vertical graduated circle of the transit, is the angle of elevation of the object A above

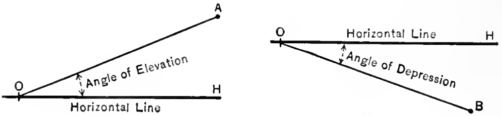

FIG. 7.

the observer at O. Similarly he measures the angle of ***depression*** of an object B, lower than himself, by first sighting the horizontal line OH and depressing the end of the telescope through the angle HOB until he sights B.

EXERCISES II. — GRAPHICAL SOLUTION OF TRIANGLES

Solve the following exercises by construction and measurement.

1. Two sides of a triangular field are 70.6 rods and 140.5 rods and the angle opposite the latter is 40°. Find the length of the fence around it.
Ans. 353.9 or 529.6

2. At a point in the street midway between two buildings their angles of elevation are 30° and 60° respectively. Find the ratio of their heights.
Ans. 1 : 3.

3. The hands of a clock are 4 and 6 inches long respectively. Find the distance between their tips at 5 : 10 o'clock. *Ans.* 6.3

4. In the triangle ABC, angle $A = 64°$, $B = 72°$, and the included side is 14. Find (a) the angle at the center of the circumscribed circle subtended by the side AB; (b) the angle at the center of the inscribed circle subtended by BC; (c) the length of the altitude from C upon AB.
Ans. 88°, 122°, 17.2

5. The diagonals of a parallelogram are 10 and 12 and they cross at an angle of 45°. Find the sides. *Ans.* 4.3, 10.1

6. The steps of a stairway have a tread of 10 in. and a rise of 7 in.; at what angle is the stairway inclined to the floor? *Ans.* 35°.

7. Two sides of a triangle are each 6 and the included angle is 120°. Find the perimeter and the area. *Ans.* 22.4, 15.6

8. Find the distance PQ across the pond (Fig. 8) from the following measurements, $AP =$ 900 ft., $AQ = 780$ ft., $PAQ = 48°$. *Ans.* 692.

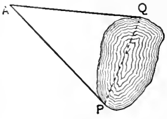

FIG. 8.

9. To determine the width AB of a hill, a point C is taken from which the points A and B on opposite sides of the hill are visible. If $AC = 200$ ft., $BC = 223$ ft., and angle $ACB = 62°$, find the width AB.

10. The angles of a triangle are in the ratio 1 : 2 : 3, and the altitude upon the longest side is 37.5. Find the perimeter and the area. *Ans.* 204.9, 1623.75.

11. Find the angles and sides of a regular five-pointed star inscribed in a circle of radius 10. *Ans.* 36°, 19.

8. Squared Paper. It is often an advantage to draw the figure on paper ruled into squares, called squared paper, or

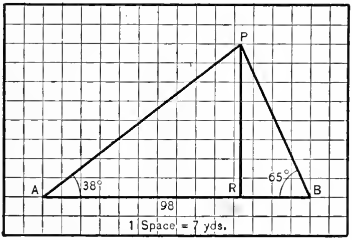

FIG. 9.

cross-section paper. The location of points is particularly easy on such paper, so that a map, for example, is readily made by using it. By suitably placing the figure, required lengths can frequently be read off at once.

Thus, if the triangle for the graphical solution of Ex. 1, § 4, be constructed on cross-section paper, the required distance, PR, Fig. 9, can be seen at once to be about 56 yards.

9. Rectangular Coördinates. If any two perpendicular rulings OY and OX of the squared paper (see Fig. 10) are selected, the position of any point P in the plane is determined by means of the distances from these two lines to the point P. The paper can be so placed that these distances are vertical and horizontal, respectively; we shall usually suppose the paper in this position.

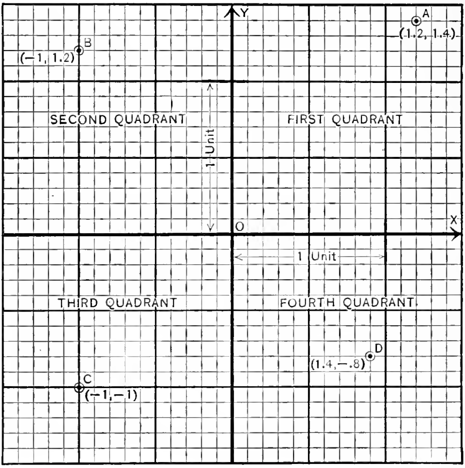

FIG. 10.

Thus, in Fig. 10, the horizontal distance from OY to the point A is 1.2 units. To avoid confusion between points at the same distance above (or below) OX but on opposite sides of OY, it is customary to call distances measured to the right of OY positive, distances to the left of OY negative; thus, B is said to be -1 unit from OY. Similarly, distances measured downwards from OX are called negative; for example, D is -0.8 from OX, and C is -1 from OX and also -1 from OY.

The two distances to any point P from OY and OX are called the ***rectangular coördinates*** of P, and are frequently denoted by the letters x and y, respectively. The horizontal distance x is called the ***abscissa*** of P; the vertical distance y is called the ***ordinate*** of P. In giving these distances it is generally understood that the first one mentioned is x, the last y.

Thus A, Fig. 10, is briefly denoted by the numbers (1.2, 1.4); B is denoted by (−1, 1.2); C by (−1, −1); D by (1.4, −0.8).

The lines OX, OY are called the ***axes of coördinates***, or simply the ***axes***. OX is called the x-axis, OY the y-axis. The point O is called the ***origin***.

The four portions into which the plane is divided by the axes are called the ***first***, ***second***, ***third***, and ***fourth quadrants***, as in Fig. 10.

To *locate* a point is to describe its position in the plane in terms of its distances from the coördinate axes; *e.g.* (− 5, 2) is a point 5 units to the left of the y-axis and 2 units above the x-axis. To *plot* a point is to mark it in proper position with respect to a pair of axes.

EXERCISES III. — SQUARED PAPER

1. Locate and plot each of the following points with respect to some pair of axes:
(*a*) (1, 2), (*b*) (2, − 3), (*c*) (4, − 7), (*d*) (− 5, 2), (*e*) (− 7, − 7), (*f*) (7, 5), (*g*) (5, 12), (*h*) (8, − 3), (*i*) (− 5, − 5), (*j*) (6, − 2).

2. Show that the line joining (5, − 4) and (− 5, 4) is bisected by the origin.

3. On what lines do all points (1, 0), (2, 0), (− 3, 0), (1.5, 0) lie? On what line do all the points (0, 0), (0, 1), (0, 2), (0, 5), (0, − 2) lie? Make a general statement about such points.

4. Find the distance from the origin to each of the points in Ex. 1, by using the folded edge of another piece of squared paper.

Compute the same distances by regarding each of them as the length of the hypotenuse of a right triangle, the lengths of whose sides can be read directly from the figure. Each of these methods can be used as a check on the other. *Ans.* (*a*) 2.2, (*b*) 3.6, (*c*) 8.1, (*d*) 5.4, (*e*) 9.9, (*f*) 8.6, (*g*) 13, (*h*) 8.5, (*i*) 7.1, (*j*) 6.3

5. Construct the triangle whose vertices are (6, 2), (8, 4), and (10, 12). Find its perimeter and its area. *Ans.* 21.8, 6.

6. Find the lengths of the segments whose end points are: (*a*) (2, 4) and (5, 8); (*b*) (4, −3) and (−1, 3); (*c*) (1, −2) and (4, 2).
Ans. 5, 7.8, 5.

7. Find the sides and diagonals of the parallelogram whose vertices are (2, 1), (5, 4), (4, 7), and (1, 4). *Ans.* $3\sqrt{2}$, $\sqrt{10}$, $2\sqrt{10}$, 4.

8. Plot the points A: (1, 0), B: (−3, 2), C: (1, 1), D: (7, 3) and determine the angle at which the line AB crosses the line CD. *Ans.* 45°.

9. Plot A: (2, 1), B: (6, −1), C: (1, 3), D: (−2, −3) and find the angle at which AB crosses CD; also find the area of the triangle whose sides are AB, CD, and BD. *Ans.* 90°, 16.8

10. Plot A: (5, −2), B: (14, 8), C: (2, 3) and find the distance from A to BC; also find the area of the triangle ABC. *Ans.* 75/13, 37.5

11. A farm is described in the deed as *N.E.* $\frac{1}{4}$ and *E.* $\frac{1}{2}$ of *N.W.* $\frac{1}{4}$, Section 5, Wayne Township, Tippecanoe County, Ind. Taking the center lines of this section as axes, make a map from the following data: A ditch crosses the farm through the points (−80, 40), (80, 80), (160, 136), distances being measured in rods. The house is at (152, 72). There are seven fields whose corners are: A, (−80, 112), (−80, 160), (−16, 112), (−16, 160); B, (−80, 40), (−16, 56), (−16, 112), (−80, 112); C, (−80, 0), (0, 0), (0, 60), (−80, 40); D, (−16, 56), (80, 80), (80, 160), (−16, 160); E, (80, 80), (160, 136), (160, 160), (80, 160); F, (80, 0), (160, 0), (160, 136), (80, 80); G, (0, 0), (80, 0), (80, 80), (0, 60). Find the area of each field and the total length of fence.
Ans. 19.2, 25.6, 25, 55.2, 26, 54, 35, (acres); 3 miles 68 rods.

12. Positions on a rectangular farm are given by their coördinates in rods, referred to two sides of the farm as axes, as follows: house (10, 4), barn (6, 4), gate of pasture (60, 20). A railroad passes between the house and barn, with a crossing at the point (3, 12). Draw a map showing these objects. Determine how much farther it is from the house to the barn by way of the crossing than along the straight line connecting them. How much farther is it from the barn to the pasture gate by way of the crossing than along a straight line? *Ans.* 15.2, 9.78

13. A certain city park is bounded by a main street, two cross streets perpendicular to it, and a stream. The distances, in feet, to the stream measured perpendicularly from the main street at 100 ft. intervals are found to be 680, 650, 525, 450, 450, 460, 540. Draw a map of the park and determine approximately its area. *Ans.* 7 acres, 9580 sq. ft.

14. To determine the height of a tree OA standing in a level field the distance $OB = 100$ ft. from the base O of the tree to a point B in the field, and the angle of elevation $OBA = 37°$, are measured. Find the height of the tree. *Ans.* 75 ft.

CHAPTER II

DEFINITIONS. SOLUTION OF RIGHT TRIANGLES

10. Tables. While the methods for solving triangles explained in Chapter I are sufficient for all cases, they are really not convenient where great accuracy is desired, since for this purpose the figure would need to be drawn on a very large scale. The method usually employed when one desires greater accuracy than can be conveniently attained by the method of construction and measurement is the method of tables. Tables are constructed which give approximately the ratios of each pair of sides *for all right triangles.* To obtain the ratio of a certain pair of sides of a right triangle with a given acute angle it is then only necessary to consult the table.

For example, it is known by geometry that if one angle of a right triangle is 30°, the side opposite this angle is one-half the hypotenuse. Hence if the hypotenuse is given, that side, and hence also the other one, can be determined. If in Fig. 11, $AB = 22.5$, and $\angle A = 30°$, then the side $BC = (1/2)(22.5) = 11.25$

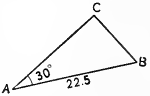

FIG. 11.

If, for an acute angle of every right triangle, the ratio of the opposite side to the hypotenuse were known to us, then we could solve every right triangle in the same manner.

It will be shown later that all oblique triangles can be cut up into right triangles in such a way that the same tables can be used in all cases for solving oblique triangles.

Since any triangle can be enlarged (or reduced) in size by drawing it on a larger (or smaller) scale, only ***the ratios of the sides*** are really important.

11. Definitions of the Ratios. As indicated in § 10, the ratio of two sides of a triangle does not depend upon the size of the triangle, but only upon the angles. Thus in the right triangles MPN, $MP'N'$, $MP''N''$ of Fig. 12, in which PN, $P'N'$, $P''N''$ are perpendicular to MN, the ratios NP/MP, $N'P'/MP'$, $N''P''/MP''$ are all equal. Moreover, if $P'''N'''$ is drawn perpendicular to MP, each of the ratios just mentioned is equal to $N'''P'''/MP'''$. (Why?) These ratios, then, depend *only on the angle α at M.* It is convenient to place the angle on a pair of axes so that the vertex falls at the origin O, one side lies along the x-axis, to the right, and the other side falls in the first quadrant. On this side take any point P at random, except O, and drop the perpendicular PM to the x-axis (see Fig. 13). Let $OP = r$; then by geometry

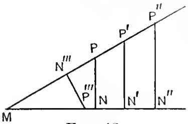

FIG. 12.

$$r = \sqrt{x^2 + y^2},^*$$

where x and y are the coördinates of the point P. The various ratios of pairs of the three quantities x, y, r are the same *for all points P* taken in the side OP of the angle α. These are:

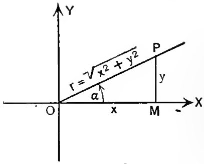

FIG. 13.

(1) $\dfrac{y}{r}$, called ***the sine of the angle*** α, written **sin** α.

(2) $\dfrac{x}{r}$, called ***the cosine of the angle*** α, written **cos** α.

(3) $\dfrac{y}{x}$, called ***the tangent of the angle*** α, written **tan** α.

The reciprocals † of these ratios are also often used:

(4) r/y is called the *cosecant of the angle* α, written csc α.
(5) r/x is called the *secant of the angle* α, written sec α.
(6) x/y is called the *cotangent of the angle* α, written ctn α.

---

* The radical sign is used to denote the *positive* square root.

† The reciprocal of a number is unity divided by the number. The reciprocal of a common fraction is the result of inverting it; thus the reciprocal of y/r is r/y. Every number has a reciprocal except 0, which has not.

These six ratios are collectively called trigonometric ***ratios*** or also trigonometric ***functions*** of the angle. Ratio – Function

Other expressions derived from these are also frequently used; for example, many engineers use the following combinations:

(7) versed sine of $\alpha = 1 - \cos\alpha$, written vers α;

(8) external secant of $\alpha = \sec\alpha - 1$, written exsec α;

(9) haversine of α = half the versed sine of α

$$= \frac{1 - \cos\alpha}{2}, \text{ written hav } \alpha;$$

and occasionally also the function coversed sine of $\alpha = 1 - \sin\alpha$, written covers α.

12. Right Triangles. In the right triangle OPM, Fig. 13, y is the side opposite the angle α, x is the side adjacent to α, and r is the hypotenuse. From the definitions (1)–(3), we see that *in any right triangle:*

(10) ***The sine of either acute angle*** $= \dfrac{\textit{side opposite}}{\textit{hypotenuse}}$;

(11) ***The cosine of either acute angle*** $= \dfrac{\textit{side adjacent}}{\textit{hypotenuse}}$;

(12) ***The tangent of either acute angle*** $= \dfrac{\textit{side opposite}}{\textit{side adjacent}}$;

and, after clearing of fractions, we find for either acute angle

(13) ***The side opposite*** = ***hypotenuse*** × ***sine***
= ***side adjacent*** × ***tangent;***

(14) ***The side adjacent*** = ***hypotenuse*** × ***cosine***
= ***side opposite*** × ***cotangent;***

(15) ***Hypotenuse*** $= \dfrac{\textit{side opposite}}{\textit{sine}} = \dfrac{\textit{side adjacent}}{\textit{cosine}}$.

The student should so thoroughly learn these statements that he can apply them instantly and confidently to any right triangle that he sees, whatever its position in the plane.

13. Elementary Relations. The trigonometric functions are connected by many simple relations. Thus:

(16) $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \text{ since } \frac{y}{x} = \frac{y}{r} \div \frac{x}{r}.$$

Similarly, the student can easily show that

$$(17)\qquad \operatorname{ctn}\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1}{\tan\alpha},$$

$$(18)\quad \sec\alpha = \frac{1}{\cos\alpha}, \qquad\qquad (19)\quad \csc\alpha = \frac{1}{\sin\alpha}.$$

Other relations will be given later.

The following examples illustrate a method of constructing an angle when one of its ratios is given.

EXAMPLE 1. Construct an acute angle whose sine is 2/7.

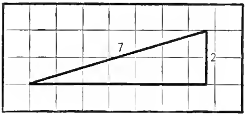

FIG. 14.

To construct such an angle draw a right triangle whose hypotenuse is 7 and one whose side is 2. This can easily be done on cross-section paper. With a radius of 7 draw a circle and mark its intersection with the horizontal ruling 2 units above the center. The angle between the horizontal diameter and the radius to this intersection is the angle required.

EXAMPLE 2. Construct an acute angle whose tangent is 3/8.

This is most easily done by drawing a triangle whose base is 8 and whose altitude is 3. The angle between the hypotenuse and base is the angle required. As in Example 1, it will be found convenient to draw the figure on cross-section paper.

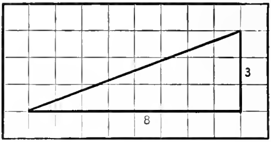

FIG. 15.

EXERCISES IV. — TRIGONOMETRIC RATIOS

1. On cross-section paper construct angles whose sines are: (*a*) 1/5; (*b*) 2/5; (*c*) 3/5; (*d*) 4/5; (*e*) 2/3; (*f*) 5/7; (*g*) 0.5

2. Is there an acute angle whose sine is any given positive number?

3. Construct angles whose tangents are: (*a*) 3/10; (*b*) 1/2; (*c*) 2/3; (*d*) 1; (*e*) 10/3; (*f*) 2; (*g*) 7.5; (*h*) 3.4; (*i*) 1.7

4. Is there an acute angle whose tangent is any given number?

5. How large, in degrees, is the acute angle whose tangent is 1?

6. How does the angle whose tangent is 2 compare with the angle whose tangent is 1? Check your answer by drawing an accurate figure.

14. Construction of Small Tables. Approximate values of the trigonometric functions of a given acute angle may be

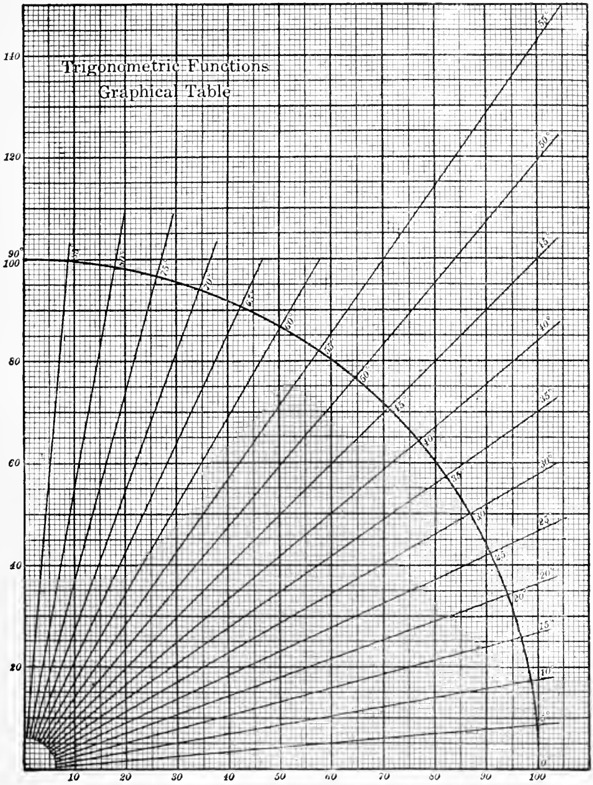

Fig. 16.

found by measurement as follows. On a sheet of squared paper, construct a quarter circle with its radius = 100, and with its center at the intersection of two heavy rulings. Draw a tangent to this circle perpendicular to the horizontal rulings. Given now any acute angle, α, lay it off above the horizontal axis with its vertex at the center of the circle. Call the points where its side crosses the circle and the tangent P and Q, respectively. Then the ordinate (y) of the point P can be read at least to units, and this divided by $r = 100$ gives the value of $\sin \alpha$ to two decimal places. Similarly, the abscissa (x) of P can be read to units, and this divided by 100 gives $\cos \alpha$. Likewise the ordinate of Q can be read to units, and this divided by 100 gives $\tan \alpha$. Finally, $\operatorname{ctn} \alpha$, $\sec \alpha$, $\csc \alpha$, can be computed as the reciprocals of $\tan \alpha$, $\cos \alpha$, $\sin \alpha$, respectively. The student will find it instructive to compute in this way, from Fig. 16, values to fill out the following table.

α	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°
$\sin \alpha$																	
$\cos \alpha$																	
$\tan \alpha$																	
$\operatorname{ctn} \alpha$																	

15. Functions of Complementary Angles. If all of this table is filled out correctly, it will be found that every number in it occurs twice; once for an angle less than 45° and once for an angle greater than 45°. This result indicates that *the sine of any angle is the cosine of its complement; and the tangent of any angle is the cotangent of its complement.*

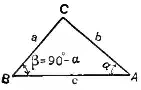

FIG. 17.

These relations will now be proved for any acute angle α. Let $\beta = 90° - \alpha$; then α and β are the acute angles of a right triangle. Denote the sides opposite α and β by a and b, respectively; and the hypotenuse by c. Then by § 12,

$$\sin\alpha = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c};$$

$$\cos\beta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{a}{c};$$

$$\tan\alpha = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b};$$

$$\text{ctn}\,\beta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{a}{b};$$

whence, remembering that $\beta = 90° - \alpha$,

(20) $$\sin\alpha = \cos\beta = \cos(90° - \alpha),$$

(21) $$\tan\alpha = \text{ctn}\,\beta = \text{ctn}\,(90° - \alpha).$$

In the same way it can be shown that

(22) $$\sec\alpha = \csc(90° - \alpha).$$

16. Applications. The values of the trigonometric ratios have been computed approximately for all acute angles, and recorded in convenient tables. These tables, together with the formulas just given, enable us to solve all cases of right triangles. On page 21 is printed a table giving the values of the ratios to three decimal places. If still greater accuracy is required, a four or a five-place table should be employed. In the following examples the three-place table is used.

Example 1. One angle of a right triangle is 38° and the hypotenuse is 12 ft. Find the lengths of each of the other sides.

Draw a figure, mark the given parts, and indicate the parts to be found by suitable letters, say x and y. The sides x and y are then respectively the side adjacent and the side opposite. To find x, note that the hypotenuse is given; hence by (14), § 12,

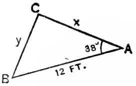

Fig. 18.

$$x = 12 \cdot \cos 38°.$$

The value of the cosine of 38° from the three place table is found to be .788 Using this value we find

$$x = 12\ (.788)$$

$$\begin{array}{r} .788 \\ 12 \\ \hline \end{array}$$

or $$x = 9.456$$

Similarly by equation (13), § 12,

$$y = 12 \cdot \sin 38^\circ$$

and from the three-place table the sine of 38° is found to be .616. Using this value we obtain

$$y = 12\ (.616)$$

$$\begin{array}{r} .616 \\ 12 \\ \hline y = \quad 7.392 \end{array}$$

As a check, the Pythagorean theorem may be used, particularly if a table of squares is available. Thus, denoting the hypotenuse by h, we *should* have

$$h = \sqrt{(9.456)^2 + (7.392)^2} = 12.002$$

This agrees reasonably well with the given value $h = 12$. Another check that is more practical is given by measurement from a good figure.

EXAMPLE 2. One side of a right triangle is 17 and the angle opposite this side is 27°; what is the length of the hypotenuse? of the other side?

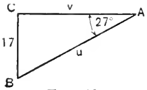

FIG. 19.

Denote the hypotenuse by u and the unknown side by v. Noting that the side *opposite* the given angle is given, find the *side adjacent*, v, by (14), § 12. To find the hypotenuse, use (15), § 12:

$$v = 17 \cdot \text{ctn } 27^\circ = 17\ (1.963)$$

$$\begin{array}{r} 1.963 \\ 17 \\ \hline 13.741 \\ 19.63 \\ \hline v = \quad 33.371 \end{array}$$

$$u = 17 \div \sin 27^\circ = 17 \div .454$$

Performing the division we find

$$u = 37.44$$

Check these answers by drawing an accurate figure.

EXAMPLE 3. The hypotenuse of a right triangle is 41 and one side is 13; find the opposite angle.

Denote the opposite angle by α, then by equation (10), § 12.

$$\sin \alpha = 13 \div 41 = .317$$

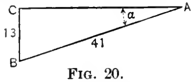

FIG. 20.

From the table (p. 21) we see that $\sin 18^\circ = .309$ and that $\sin 19^\circ = .326$, so that $\sin \alpha$ is very nearly halfway between $\sin 18^\circ$ and $\sin 19^\circ$. We judge therefore that the angle α is about halfway between 18° and 19°; hence

$$\alpha = 18^\circ.5$$

TRIGONOMETRIC FUNCTIONS TO THREE PLACES OF DECIMALS

α	$\sin\alpha$	$\sec\alpha$	$\tan\alpha$	$\operatorname{ctn}\alpha$	$\csc\alpha$	$\cos\alpha$	
0°	.000	1.000	.000	——	——	1.000	**90°**
1°	.017	1.000	.017	57.290	57.299	1.000	89°
2°	.035	1.001	.035	28.636	28.654	.999	88°
3°	.052	1.001	.052	19.081	19.107	.999	87°
4°	.070	1.002	.070	14.301	14.336	.998	86°
5°	.087	1.004	.087	11.430	11.474	.996	**85°**
6°	.105	1.006	.105	9.514	9.567	.995	84°
7°	.122	1.008	.123	8.144	8.206	.993	83°
8°	.139	1.010	.141	7.115	7.185	.990	82°
9°	.156	1.012	.158	6.314	6.392	.988	81°
10°	.174	1.015	.176	5.671	5.759	.985	**80°**
11°	.191	1.019	.194	5.145	5.241	.982	79°
12°	.208	1.022	.213	4.705	4.810	.978	78°
13°	.225	1.026	.231	4.331	4.445	.974	77°
14°	.242	1.031	.249	4.011	4.134	.970	76°
15°	.259	1.035	.268	3.732	3.864	.966	**75°**
16°	.276	1.040	.287	3.487	3.628	.961	74°
17°	.292	1.046	.306	3.271	3.420	.956	73°
18°	.309	1.051	.325	3.078	3.236	.951	72°
19°	.326	1.058	.344	2.904	3.072	.946	71°
20°	.342	1.064	.364	2.747	2.924	.940	**70°**
21°	.358	1.071	.384	2.605	2.790	.934	69°
22°	.375	1.079	.404	2.475	2.669	.927	68°
23°	.391	1.086	.424	2.356	2.559	.921	67°
24°	.407	1.095	.445	2.246	2.459	.914	66°
25°	.423	1.103	.466	2.145	2.366	.906	**65°**
26°	.438	1.113	.488	2.050	2.281	.899	64°
27°	.454	1.122	.510	1.963	2.203	.891	63°
28°	.469	1.133	.532	1.881	2.130	.883	62°
29°	.485	1.143	.554	1.804	2.063	.875	61°
30°	.500	1.155	.577	1.732	2.000	.866	**60°**
31°	.515	1.167	.601	1.664	1.942	.857	59°
32°	.530	1.179	.625	1.600	1.887	.848	58°
33°	.545	1.192	.649	1.540	1.836	.839	57°
34°	.559	1.206	.675	1.483	1.788	.829	56°
35°	.574	1.221	.700	1.428	1.743	.819	**55°**
36°	.588	1.236	.727	1.376	1.701	.809	54°
37°	.602	1.252	.754	1.327	1.662	.799	53°
38°	.616	1.269	.781	1.280	1.624	.788	52°
39°	.629	1.287	.810	1.235	1.589	.777	51°
40°	.643	1.305	.839	1.192	1.556	.766	**50°**
41°	.656	1.325	.869	1.150	1.524	.755	49°
42°	.669	1.346	.900	1.111	1.494	.743	48°
43°	.682	1.367	.933	1.072	1.466	.731	47°
44°	.695	1.390	.966	1.036	1.440	.719	46°
45°	.707	1.414	.1000	1.000	1.414	.707	**45°**
	$\cos\alpha$	$\csc\alpha$	$\operatorname{ctn}\alpha$	$\tan\alpha$	$\sec\alpha$	$\sin\alpha$	α

Example 4. The two perpendicular sides of a right triangle are 23 and 83 ; determine the acute angles and the hypotenuse.

Denote the hypotenuse by h and the angle opposite the smaller side by α ; then by equation (12) § 12,

$$\tan \alpha = 23 \div 83.$$

After performing the division it is found that

$$\tan \alpha = .277$$

As in the example above it is noticed that $\tan \alpha$ lies very nearly halfway between $\tan 15°$ and $\tan 16°$; we have, therefore, very approximately,

$$\alpha = 15°.5$$

17. Directions for Solving Triangles. In the solution of triangles, use the following procedure:

(*a*) Draw a diagram approximately to scale, indicating the given parts. Mark the unknown parts by suitable letters, and estimate their values.

(*b*) *If one of the given parts is an acute angle,* consider the relation of the known parts to the one which it is desired to find, and apply the proper one of formulas (10) ⋯ (15), § 12.

(*c*) *If two sides are given,* and one of the acute angles is desired, think of the definition of that function of the angle which employs the two given sides.

(*d*) Check each result.

EXERCISES V. — SOLUTION OF RIGHT TRIANGLES

1. One side of a right triangle is 21 ; the adjacent angle is 42° ; determine the remaining side and the hypotenuse. Check.

2. One side of a right triangle is 21 and the opposite angle is 42° ; determine the remaining side and hypotenuse. Check.

3. The hypotenuse of a right triangle is 28 ; one angle is 32°. Determine the two perpendicular sides. Check.

4. What is the angle of inclination of a roof which has half pitch ? 1/3 pitch ?

[Note. The pitch of a roof is equal to the height of the comb above the eaves divided by the total distance between the eaves.]

5. In the following triangles h denotes the hypotenuse ; the angle A is opposite the side a and the angle B is opposite the side b. Use the table to compute the unknown parts from the given parts. Check.

(*a*) $A = 61°$, $b = 41$.
(*b*) $a = 421$, $b = 401$.
(*c*) $a = 62$, $h = 125$.
(*d*) $A = 32°$, $a = 330$.
(*e*) $a = 313$, $h = 720$.
(*f*) $B = 49°$, $h = 24$.

6. Determine the height of a tower MN, if the horizontal distance EM to it is 450 ft. and the *angle of elevation* MEN is 27°. Check.

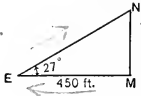

FIG. 21.

7. A vertical pole 35 ft. high casts a horizontal shadow 45 ft. long. Determine the angle of elevation of the sun above the horizon. Check.

8. An object known to be 100 ft. in height stands on the bank of a river; from the opposite bank of the river the angle of elevation of the top of the object is found to be 24°; find the width of the river. Check.

9. The radius of a circle is 7 ft. What angle will a chord of the circle ? 11 ft. long subtend at the center ? Check.

10. From the top of a cliff 92 ft. in height the angle of depression of a boat at sea is observed to be 20°. How far out is the boat ? Check.

11. To find the distance between two objects A and B, where B is in a swamp, the distance $AC = 350$ ft. is measured at right angles to the line joining them. At C an observer holds an ordinary rake with the end of the handle at his eye and with the center of the rake directed toward A. There appear then to be 6 teeth of the rake between A and B. If the teeth are one inch apart and the handle of the rake is five feet long, determine the distance between A and B.

18. The Question of Greater Accuracy. The degree of accuracy of the results obtained by using the values of the trigonometric functions to three places of decimals, while sufficient for many ordinary applications, is not satisfactory for some purposes; for example, in extended surveys, in astronomy, and in any work for which the data must be determined by using instruments of precision.

More accurate values have been calculated. The values for angles at intervals of 1′ are given to five decimal places in five-place tables.*

* Throughout this book, page references to ***Tables*** are to THE MACMILLAN TABLES. These tables may be had separately bound. They are bound with this book in the edition *with complete tables*. The edition of this book *with brief tables* contains only four-place tables, for the convenience of those who prefer the full tables separately bound.

19. Use of the Large Tables. Five-place tables are used in precisely the same manner as the small table of p. 21.

Example 1. One angle of a right triangle is 42° 20′ and the hypotenuse is 28 ft. 6 in. long. Find the remaining sides and the other angle. Draw a diagram to illustrate the problem, indicating the given parts. Denote the unknown parts by the letters a and b, as in Fig. 22.

To find b, note that it is the *side adjacent* to the given angle, and that the hypotenuse is given. Hence, by (14), § 12,

$$b = 28.5 \cos 42° 20' = 28.5 \times .73924 = 21.07$$

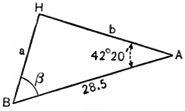

Fig. 22.

Note that a is opposite the given angle; hence by (13), § 12.

$$a = 28.5 \sin 42° 20' = 28.5 \times .67344 = 19.19$$

the sine and the cosine of 42° 20′ being found in the Tables, p. 43.

The angle β, being the complement of 42° 20′, is 47° 40′.

Example 2. The perpendicular sides of a right triangle are 22 ft. 6 in. and 54 ft., respectively. Find the hypotenuse and the angles.

Draw a diagram, indicating the given parts and lettering the parts to be found, as in Fig. 23. To find α, note that the given parts are the sides opposite and adjacent to it; hence by the *definition of tangent*, we write

$$\tan \alpha = 22.5 \div 54 = .41667$$

From the Tables, p. 33,

$$\tan 22° 37' = .41660 \text{ and } \tan 22° 38' = .41694$$

whence

$$\alpha = 22° 37'+ \text{ and } \beta, \text{ its complement, is } 67° 23'-.$$

By the Pythagorean theorem of plane geometry, using a table of squares and square roots, Tables p. 94,

$$h^2 = \overline{54}^2 + \overline{22.5}^2 = 3422.25$$

whence, $h = 58.5$ Tables, p. 103.

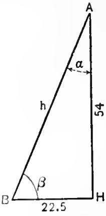

Fig. 23.

Another method of finding h is the following: Having found $\alpha = 22° 37'$, $h = 54/\cos 22° 37' = 54/.92310 = 58.498$ by (15) § 12.

However, this method is open to the objection that any error made in computing α vitiates the resulting value found for h. In general, compute each unknown part from the given parts; *i.e. do not use computed parts as data if it can be avoided.*

In solving right triangles, observe carefully the directions of § 17, p. 22, and use five-place values of the functions (Tables, pp. 22–44 and pp. 94–111) as illustrated in the preceding examples.

EXERCISES VI. — RIGHT TRIANGLES

1. Solve the following right triangles. The hypotenuse is denoted by h, other sides by other small letters, and any angle by the capital letter corresponding to the small letter that denotes the side opposite it.

(*a*) $A = 61° 17'$, $b = 1.4$ (*d*) $M = 49° 49'$, $h = 24.6$ (*g*) $p = 18.2$, $q = 50$.
(*b*) $A = 32° 31'$, $a = 33$. (*e*) $b = 4.848$, $h = 10$. (*h*) $u = 11.65$, $h = 25$.
(*c*) $A = 62.12$, $h = 254$. (*f*) $U = 63° 2'$, $u = 40$. (*i*) $m = 34.2$, $h = 100$.
Ans. (*a*) 2.56, 2.91; (*b*) 51.77, 61.39; (*c*) 14° 9'.4, 75° 50'.7, 246.29; (*d*) 18.80, 15.87; (*e*) 61°, 29°, 8.746; (*f*) 20.35, 44.88; (*g*) 20°+, 70°−, 53.21; (*h*) 27° 46'.5, 62° 13'.5, 22.12; (*i*) 70°, 20°, 93.97

2. In the following right triangles find the side not given:

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)	(*i*)	(*j*)	(*k*)	(*l*)
side	2.19	45.6	5.82	53.4	73.6	25.6	46	17.5	46.5	6.83	13.5	106
hyp.	7.75	·	9.43			54.4		45.5		9.92	35.1	535.3
side		82.5		19.2	138		110.4		42.7			
ans.	7.43	94.26	7.42	56.75	156.4	48	119.6	42	63.13	7.19	32.4	524.7

3. In each of the following right triangles find the three parts not given and the area.

(*a*) $a = 30.2$, $h = 33.3$ *Ans.* 24° 55'.1, 65° 4'.9, 14.03, 211.85
(*b*) $A = 35°$, $b = 100$. *Ans.* 70.021, 122.07, 3501.
(*c*) $h = 43$, $B = 27°$. *Ans.* 19.52, 38.31, 373.98
(*d*) $h = 176$, $A = 32°$. *Ans.* 93.26, 149.25, 6959.68
(*e*) $h = 425$, $b = 304$. *Ans.* 45° 40', 297, 45144.

4. The base of an isosceles triangle is 324 ft., the angle at the vertex is 64° 40'. Find the equal sides and the altitude. *Ans.* 302.89, 255.93

5. The shadow of a tower 200 ft. high is 252.5 ft. long. What is the angle of elevation of the sun? *Ans.* 38° 23'.

6. A chord of a circle is 21.5 ft., the angle which it subtends at the center is 41°. Find the radius of the circle. *Ans.* 30.7

7. To determine the width AB of a river, a line BC 100 rods long is laid off at right angles to a line from B to some object A on the opposite bank visible from B. The angle BCA is found to be 43° 35'. Find AB.
Ans. 95.17

8. What is the angle of elevation of a mountain slope which rises 238 ft. in one-eighth of a mile (up the slope)? *Ans.* 21° 8'+.

9. Two ships in a vertical plane with a lighthouse are observed from its top, which is 200 ft. above sea level. The angles of depression of the two ships are $15° 17'$ and $11° 22'$. Find the distance between the ships.
Ans. 262.96

10. A flagstaff stands on the top of a house. At a point 100 ft. from the house the angles of elevation of the bottom and top of the staff are respectively $21° 50'$ and $33° 3'$. Find the height of the staff. *Ans.* 25.

11. A 24-foot ladder can be so placed in a street as to reach a window 16 ft. high on one side and by turning it over on its foot it will reach a window 14 ft. high on the other side. Find the width of the street.
Ans. 37.38

12. The length of one side of a regular pentagon is 24 ft. Find the lengths of the radii of the inscribed and circumscribed circles and the area. *Ans.* 16.52, 20.42, 991.2

13. The side of a regular decagon is 10 in. long. Find the radii of the inscribed and circumscribed circles and the area.
Ans. 15.39, 16.18, 769.5

14. A round silo 21.5 feet in diameter subtends a horizontal angle of 5°. Find the distance from the observer to the silo. *Ans.* 235.7

15. In an isosceles right triangle show that lines from either base angle to the points of trisection of the opposite side cut off respectively, one-fifth and one-half the altitude from the hypotenuse to the vertex of the right angle.

CHAPTER III

TRIGONOMETRIC RELATIONS

20. Introduction. A few simple trigonometric relations have been given in §§ 12, 13, and 15. In this chapter we shall obtain others. The student should first review those already given.

21. Pythagorean Relations. The following equation between the abscissa x, the ordinate y, and the radius r is true for every point in the plane: *

(1) $$x^2 + y^2 = r^2.$$

Dividing by r^2, we obtain

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1;$$

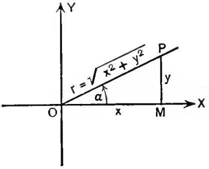

FIG. 24.

but by § 11, at least when α is acute, $x/r = \cos\alpha$, $y/r = \sin\alpha$; hence

(2) $$\sin^2\alpha + \cos^2\alpha = 1;$$

i.e. the sum of the squares of the sine and cosine of any acute angle is equal to unity.†

Dividing (1) by x^2, and then by y^2, we obtain respectively:

(3) $$1 + \tan^2\alpha = \sec^2\alpha,$$

(4) $$1 + \operatorname{ctn}^2\alpha = \csc^2\alpha.$$

Formulas (2), (3), and (4) are examples of trigonometric identities. An identity in any quantity, α, is an equation con-

* Formulas (2), (3), and (4) are called the **Pythagorean relations** because they are obtained from this equation, which is the Pythagorean theorem of plane geometry.

† This statement, as well as (3) and (4) below, will later be found to hold for all angles, for the general definitions of sine and cosine.

taining α which is satisfied by every value of α for which both members are defined. Many other examples of identities will be found in the pages that follow.

These formulas and those of § 13 are often useful in simplifying expressions or in verifying equations. Other interesting relations are given in exercises that follow.

EXAMPLE 1. To show that $\sin^4 \alpha - \cos^4 \alpha = \sin^2 \alpha - \cos^2 \alpha$.

The expression on the left is the difference of two squares and can therefore be factored; hence we have $\sin^4 \alpha - \cos^4 \alpha = (\sin^2 \alpha + \cos^2 \alpha)(\sin^2 \alpha - \cos^2 \alpha)$ which is equal to $\sin^2 \alpha - \cos^2 \alpha$, since $\sin^2 \alpha + \cos^2 \alpha = 1$.

The formulas may also be used to compute the value of one of the trigonometric functions from that of another.

EXAMPLE 2. Given $\tan \theta = 5/12$, to find $\cos \theta$.

Analytic Method. By (3), $1 + \tan^2 \theta = \sec^2 \theta$; hence, $\sec^2 \theta = 1 + 25/144 = 169/144$, or $\sec \theta = 13/12$. Hence, $\cos \theta = 12/13$, since $\cos \theta = 1/\sec \theta$.

Geometric Method. The following method is much more practical, and is easily applied to any example of this sort.

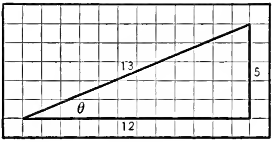

FIG. 25.

Draw a right triangle whose base is 12 and whose altitude is 5. The hypotenuse is easily found to be 13. It follows that

$$\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = 12/13.$$

EXERCISES VII. — PYTHAGOREAN RELATIONS. IDENTITIES

1. In exercises (a) – (i) determine the values of the remaining functions of the acute angle θ by each of the methods of Example 2, above.

(a) $\sin \theta = 3/5$.	(b) $\sin \theta = 1/3$.	(c) $\cos \theta = 1/3$.
(d) $\sin \theta = 5/13$.	(e) $\tan \theta = \sqrt{3}$.	(f) $\tan \theta = 3/4$.
(g) $\tan \theta = 1/m$.	(h) $\sin \theta = b/c$.	(i) $\sec \theta = 2$.

Prove the following relations for any acute angle θ:

2. $(\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$.

3. $\cos \theta \tan \theta = \sin \theta$.

4. $\tan \theta + \operatorname{ctn} \theta = \sec \theta \csc \theta$.

5. $\sin \theta \sec \theta = \tan \theta$.

6. $(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1$.

7. $(\sin^3 \theta + \cos^3 \theta) = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)$.

8. $\cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$.

9. $\sec^2 \theta \csc^2 \theta = \tan^2 \theta + \operatorname{ctn}^2 \theta + 2$.

22. Functions of 0° and 90°. If an angle of 0° be placed on coördinate axes and the construction of page 14 be made, the point P will lie on the x-axis, and we shall have

$$x = r, \qquad y = 0.$$

The functions sine, cosine, tangent, and secant of 0° are defined by the same ratios as are the corresponding functions of acute angles: hence as in (1), (2), (3), and (5), page 14,

$$\sin 0° = \frac{y}{r} = 0,\ \cos 0° = \frac{x}{r} = 1,\ \tan 0° = \frac{y}{x} = 0,\ \sec 0° = \frac{r}{x} = 1.$$

The definitions of cotangent and cosecant given for acute angles cannot be applied to 0° because $y = 0$, and therefore the divisions x/y and r/y, which occur in those definitions, are impossible.

Similarly if the angle of 90° be placed on the coördinate axes and the construction of page 14 be made, the point P will lie on the y-axis, and we shall have

$$x = 0, \qquad y = r.$$

The sine, cosine, cotangent, and cosecant of 90° are defined by the same ratios as are the corresponding functions of acute angles; hence by the definitions

$$\sin 90° = \frac{y}{r} = 1,\ \cos 90° = \frac{x}{r} = 0,\ \text{ctn } 90° = \frac{x}{y} = 0,\ \csc 90° = \frac{r}{y} = 1.$$

The definitions of tangent and secant given for acute angles cannot be applied to 90°, because $x = 0$, and the divisions y/x and r/x are impossible. We say that 0° has no cotangent or cosecant, and 90° has no tangent or secant.*

23. Functions of 30°, 45°, 60°. In plane geometry it is shown how to construct a right triangle in which one acute angle is 30°, or 45°, or 60°. From these triangles the sine, cosine, tangent, etc., of these angles can be computed.

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* It is often said that the tangent of 90°, for example, is *infinite;* this expression does not give any value to the tangent at 90°, but merely describes the fact that the tangent becomes and remains larger than any number we may name as the angle approaches 90°. Similar statements hold for the others.

To find the functions of 45°, construct an isosceles right triangle with the equal sides some convenient length m. By the Pythagorean Theorem compute the hypotenuse $= m\sqrt{2}$. Then by the definitions (10, 11) § 12,

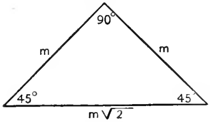


FIG. 26.

$$\sin 45° = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

and

$$\cos 45° = \frac{m}{m\sqrt{2}} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

whence by means of the relations (16, 17, 18, 19), § 12,

$$\tan 45° = \text{ctn } 45° = 1, \text{ and } \sec 45° = \csc 45° = \sqrt{2}.$$

To find the functions of 30° and 60°, *construct an equilateral triangle* of side m, and *divide it into two right triangles* by a perpendicular from one vertex to the opposite side. Apply the definitions (10), (11), § 12, to obtain the values of the functions of 30° and 60° given in the following table.

	0°	30°	45°	60°	90°	
						$\sqrt{2} = 1.414$
sin	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3} = 1.732$
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$1/\sqrt{2} = \sqrt{2}/2$
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$		$1/\sqrt{3} = \sqrt{3}/3$

These values should be memorized, since the angles 0°, 30°, 45°, 60°, and 90° occur frequently. It is easy to show that all of the relations proved in §§ 13, 15, 21, hold for the values given in this table.

24. Trigonometric Equations. An equation that is not an identity (§ 21) is sometimes called a conditional equation. Thus the equation $\sin \alpha + \cos \alpha = 1$ is ~~not~~ an identity since there are many values of α for which it is not true; there are values of α, however, which do satisfy the equation: for example, if 0° is substituted for α it will be found that the left-hand members reduce to 1 since $\sin 0° = 0$ and $\cos 0° = 1$. This equation is therefore a conditional equation but not an identity.

The simplest trigonometric equations are of the form $\sin \alpha = 1/2$, $\tan \alpha = 1/3$, etc., *i.e.* equations in which the angle α is to be determined from the value of one of the trigonometric ratios. We have already found solutions of such equations in Examples 3 and 4, § 16, and Example 1, § 19. The method there employed of looking up the value of the angle in a table can always be used for this form of equation. A trigonometric equation is therefore considered to be practically solved when it is reduced to one of these simple forms. For the present we shall consider only positive solutions not greater than 90°. Later it will be found that such equations have other solutions. (See §§ 36 and 68.)

If a trigonometric equation contains more than one of the trigonometric functions, all but one can usually be eliminated; the resulting equation may then be solved algebraically for the function which remains; the solutions may then be found by the methods explained above.

Example 1. Solve the equation $\sin^2 t - \cos^2 t = 3 \sin t - 2$. In this equation $\cos^2 t$ may be replaced by its equal $1 - \sin^2 t$; the equation then becomes a quadratic in $\sin t$, viz.:

$$2 \sin^2 t - 3 \sin t + 1 = 0.$$

This equation is *equivalent* to the given one; *i.e.* every solution of either is a solution of the other. The solutions may now be found by factoring:

$$(2 \sin t - 1)(\sin t - 1) = 0.$$

Hence we have either $\sin t - 1 = 0$, whence $\sin t = 1$, and $t = 90°$; or else, $2 \sin t - 1 = 0$, whence $\sin t = 1/2$, and $t = 30°$. There are no other solutions which do not exceed 90°.

25. Inverse Functions. A notation is sometimes needed for the angle whose sine (or any other ratio) is a given number. A notation quite frequently employed is $\sin^{-1} x$ where x is the given number. In this notation the equation $\sin \alpha = 2/7$ could also be written in the form $\alpha = \sin^{-1}(2/7)$. This equation is to be read, $\alpha =$ the angle whose sine is $2/7$.

It should be carefully noted that the (-1) of this notation is not an exponent although it is written in the position usually occupied by an exponent. Any other character written in the same position would be regarded as an ordinary exponent; thus the expression $\sin^2 \beta$ would be understood to mean, the square of the sine of the angle β.

Many prefer the notation *arcsin* x to the one given above, and this notation, though not so frequently employed as the other, is nevertheless used to a considerable extent. We shall therefore throughout this book use either notation in order to familiarize the student with both.

EXERCISES VIII. — SIMPLE TRIGONOMETRIC EQUATIONS

1. Solve the following equations by constructing a figure for each.

(*a*) $\sin x = 2/5$.	(*g*) $\cos x = .63$
(*b*) $\sin x = 1/2$.	(*h*) $\cos x = \sqrt{3}/2$.
(*c*) $\sin x = .8$	(*i*) $\sin x = 0$.
(*d*) $\sin x = .866$	(*j*) $\cos x = 0$.
(*e*) $\sin x = .48$	(*k*) $\sin x = 1$.
(*f*) $\cos x = 1/2$.	(*l*) $\cos x = 1$.

2. Prove that there is always an acute angle solution of the equation $\sin x = c$, if c is any number between 0 and 1.

3. Prove that there is always an acute angle solution of the equation $\tan x = c$, if c is any positive number whatever.

4. Find $\sin^{-1}(2/5)$ graphically.

[HINT. Compare Ex. 1(*a*).]

5. Express the answer to each of the exercises 1(*a*) to 1(*l*) by means of the notation $\sin^{-1}$ or $\cos^{-1}$ (or arcsin, arccos, etc.).

6. Find $\sin^{-1}(2/3)$, and also $\tan^{-1}(1/2)$ graphically.

7. Find arcsin (.66667), and also $\tan^{-1}$ (.50000) by the Tables.

Solve each of the following equations for x.

8. $2 \sin^2 x + \sin x = 1$.

[HINT. Solve this quadratic for $\sin x$. There are, of course, no solutions corresponding to values of $\sin x$ greater than 1.]

9. (*a*) $2 \sin^2 x - 5 \sin x + 2 = 0$. (*b*) $4 \cos^2 \theta + 8 \cos \theta = 5$.

10. (*a*) $\tan x = 1$. (*d*) $\tan x = -2.6$
(*b*) $\tan x = -1/2$. (*e*) $\tan x = 5.3$
(*c*) $\tan x = 2$. (*f*) $\tan x = 0$.

11. (*a*) $\tan^2 x = 3$. (*b*) $\tan^2 \theta = 6\frac{1}{4}$. (*c*) $\tan^2 \theta = 6 - 4\sqrt{2}$.

12. $2 \sin^2 x - \cos x = 1$. **13.** $\cos^2 x = \sin^2 x$.

14. $5 \sin x + 2 \cos^2 x = 5$. **15.** $\sec^2 x + \tan x = 3$.

16. If a and b are the sides of a right triangle, c the hypotenuse, and A the angle opposite a, show that the area of the triangle is equal to either of the expressions

$$\frac{ac \cos A}{2}, \quad \frac{bc \sin A}{2}.$$

17. Two straight pieces of railroad track MA and NB are to be connected by a circular track AKB with a radius of 500 ft. and center, O, tangent to MA and NB. The straight portions of the track produced intersect at a point V at an angle of 100°.

(*a*) How far back from V should the track begin to turn?

(*b*) How far from V along the bisector OV of the angle AVB is the center O?

(*c*) Find the shortest distance from V to the curved portion.

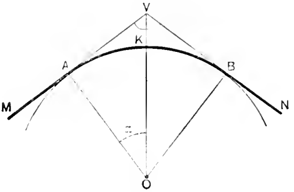


FIG. 27.

18. If, in a figure similar to that of Ex. 17, $\angle AVO$ is any angle, and $\angle VOA$ is denoted by α, and $OA = r$, show that

(*a*) $AV = r \tan \alpha$;
(*b*) $KV = r \operatorname{exsec} \alpha$;
(*c*) $AB = 2r \sin \alpha$.

19. The side b of the triangle in Ex. 16 is extended beyond A to a point D, making $AD = c$, so that ABD is isosceles. Show that

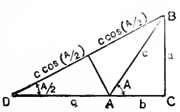


FIG. 28.

(*a*) $\angle ADB = A/2$;

(*b*) $BD = 2c \cos (A/2)$.

(*c*) From the right triangles DCB and ACB, show that

$$c \sin A = a = 2c \cos (A/2) \sin (A/2);$$

hence

$$\sin A = 2 \sin (A/2) \cos (A/2);$$

(*d*) Likewise, show that $c \cos A = b = 2c \cos^2 (A/2) - c$;
hence $\cos A = 2 \cos^2 (A/2) - 1 = \cos^2 (A/2) - \sin^2 (A/2)$.

26. Projections. The **projection** of a line segment AB upon a line l is defined to be the portion MN of the line l between perpendiculars drawn to it from A and B, respectively. The length of this projection is easily found if the length of AB and the angle α which the line AB makes with l are known. For, draw a parallel to l through A, meeting BN at C. Then ACB is a right triangle and the angle at A is α; hence by (14), § 12,

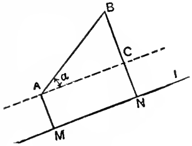


FIG. 29.

$$MN = AB \cos \alpha$$

or, *the projection of a segment upon a given line is equal to the product of the length of the segment and the cosine of the angle the segment makes with the given line.*

The projections of a segment upon the coördinate axes are frequently used. If the segment makes an angle α with the horizontal, the projections on the x and y axes are, respectively,

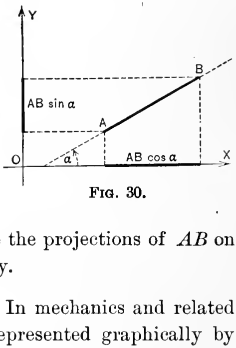


FIG. 30.

$$(5)\quad \text{Proj}_x AB = AB \cos \alpha,$$
$$\text{Proj}_y AB = AB \sin \alpha,$$

where $\text{Proj}_x AB$ and $\text{Proj}_y AB$ denote the projections of AB on the x-axis and the y-axis, respectively.

27. Applications of Projections. In mechanics and related subjects, forces and velocities are represented graphically by line segments. A force, say of 10 lb., is represented by a segment 10 units in length in the direction of the force. A velocity of 20 ft. per sec. is represented by a segment 20 units in length in the direction of motion.

The projection upon a given line l, of a segment representing a force, represents the effective force in the direction l; this is called the *component* of the given force in the direction l.

Example 1. A weight of 50 lb. is placed upon a smooth plane inclined at an angle of 27° with the horizontal. What force acting directly up the incline will be required to keep the weight at rest?

Draw to some convenient scale a segment 50 units in length directly downward to represent the force exerted by the weight. Project this segment upon a line inclined at an angle of 27° with the horizontal. The length of this projection WQ, Fig. 31, is 50 cos 63° = 22.7 nearly. This represents the component of the force down the plane. Therefore, a force of 22.7 lb. acting up the plane will be required.

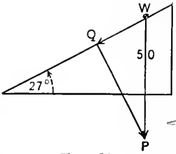


Fig. 31.

Example 2. A ladder 30 ft. long, when lying horizontal supported at its ends, will carry a safe load of 150 lb. on its middle round. Is it safe for a man weighing 190 lb. to mount it when it is so placed as to reach a window 18 ft. above the ground?

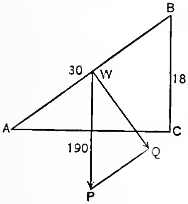


Fig. 32.

We have to find the component, perpendicular to the ladder, of the man's weight when he stands on the middle round. Let WP, drawn vertically downward from the middle point of AB, Fig. 32, represent 190 (which need not be on the same scale as AB which represents 30). Then the component perpendicular to AB is

$$WQ = 190 \cos PWQ = 190 \cos CAB.$$

Now by (11) § 12,

$$\cos CAB = AC/AB = 4/5,$$

whence

$$WQ = \frac{190 \times 4}{5} = 152,$$

which is greater than the safe load.

Example 3. A traveling crane moves with uniform speed down a shop 297 ft. long and 60 ft. wide in 1 min. 41 sec. It carries a load from one corner along the diagonal to the opposite corner. Find the speed of the crane and of the car which runs on it.

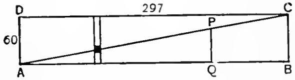


Fig. 33.

Let AP = the speed of the load along the diagonal which by the data of the problem = 3 ft. per sec. (AP need not of course be on the same scale as AB and AD). Then AQ = the speed of the crane = 3 cos PAQ = 2.94+ and QP = the speed of the car = 3 sin PAQ = .59+

EXERCISES IX. — PROJECTIONS

1. Find the horizontal and vertical projections of the segments:

(*a*) length 42, making an angle of 37° with the horizontal.

(*b*) length 5.5, making an angle of 50° with the vertical.

Ans. (*a*)33.54, 25.28; (*b*) 3.54, 4.21

2. A straight railroad crosses two north and south roadways a mile apart. The length of track between the roadways is $1\frac{1}{4}$ mi. A train travels this distance in 2 min. Find the components of the velocity of the train parallel to the roadways and perpendicular to them. Find the angle between the track and either roadway. *Ans.* $\frac{3}{8}$, $\frac{1}{2}$, 53° 7.8′

3. The eastward velocity of a certain train is 24 mi. per hour. The northward velocity is 32 mi. per hour. Find its actual velocity along the track and the angle the track makes with the east and west direction.

Ans. 40, 53° 7.8′

4. A car is drawn by means of a cable. If a force of 5000 lb. exerted along the track is required to pull the car, what force will be required when the cable makes an angle of 15° with the track? *Ans.* 5176.4

5. Find the horizontal and vertical components of a force of 30 lb. making an angle of 40° with the horizontal. *Ans.* 22.98, 19.28

6. Find the horizontal and vertical projections of the segment which joins the points $(8, -3)$ and $(-2, 7)$. *Ans.* 10, 10.

7. The stringers for a stairway are 20 ft. 7.8 in. long. The steps are to have 7 in. risers and 12 in. treads (which includes 1 in. overhang). Determine the number of steps, using the horizontal and vertical projections of the stringer to check the result. *Ans.* 19.

8. Five forces act on the point A: $(-4, 0)$ viz.: AB, AC, AD, AE, AF, and the points A, B, C, D, E, F are the vertices of a regular hexagon, center at the origin. Show that the vertical components balance, and find the sum of the horizontal components. *Ans.* 24.

9. Determine the width and height of a crate for the chair shown in Fig. 34. *Ans.* 35^+, $48\frac{5}{8}^+$.

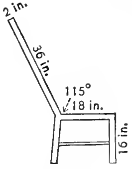


Fig. 34.

10. In surveying, the projection of a line on a north and south line is called the *latitude* of the line and the projection on an east and west line is called the *departure* of the line. Find the latitude and departure of the following lines:

(*a*) length 41 rods, bearing N 26° 15′ E. *Ans.* 36.772, 18.134

(*b*) length 487 feet, bearing E 32° 30′ S. *Ans.* 259.66, 410.73

(*c*) length 17.32 rods, bearing N 40° 45′ W. *Ans.* 13.053, 11.247

CHAPTER IV

LOGARITHMIC SOLUTIONS OF RIGHT TRIANGLES

28. The Use of Logarithms. Logarithms may be used to shorten computations involving *multiplications, divisions,* raising to *powers* or extracting *roots,* but not involving additions or subtractions. In much of the numerical work which follows, the use of logarithms is very advantageous in saving time and labor, but the student should bear in mind that logarithms are not necessary. They are merely convenient, and they belong no more to trigonometry than to arithmetic. One of the questions which a computer has to decide is whether or not it will be advantageous to use logarithms in a given problem.

At the end of this book will be found a table of the logarithms of numbers (Tables, p. 1), and a table of the logarithms of the trigonometric functions (Tables, p. 45), with explanations of their use (pp. v–xvii).* In case a review of the principles of logarithms is desired, this explanation should be studied before proceeding with the rest of this chapter.

The notation log tan 62° 51′ means the logarithm of the tangent of 62° 51′; the tangent of 62° 51′ is a number, 1.9500, and the logarithm of this number is 0.29003, as may be seen by looking up log 1.9500 in Table I. This last result is found in Table III, p. 73, which enables us to avoid the labor of looking in Tables II and I, in succession.

A formula which has been arranged so as to involve only products and quotients of powers and roots of quantities either known or easily computed from the known quantities,

* In the edition of this book *with brief tables,* only four-place tables are given. Those using that edition should refer to THE MACMILLAN TABLES, to which all page references made here apply.

is said to be *adapted to logarithmic computation.*

Thus the formula $h = \sqrt{a^2 + b^2}$, which gives the hypotenuse h of a right triangle in terms of the sides a and b, is not adapted to logarithmic computation. On the other hand, the formula

$$b = \sqrt{h^2 - a^2} = \sqrt{(h + a)\ (h - a)}$$

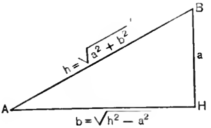


FIG. 35.

which gives one side in terms of the hypotenuse and the other side, is adapted to logarithmic computation because $(h + a)$ and $(h - a)$ are easily obtained from h and a. Thus, if the hypotenuse is 17.34 and one side is 12.27, the other side is

$$x = \sqrt{(5.07)\ (29.61)}$$

$\log 5.07$	$= 0.70501$	Tables, p. 10
$\log 29.61$	$= 1.47144$	Tables, p. 5
$\log x^2$	$= 2.17645$	
$\log x$	$= 1.08822$	
x	$= 12.252$	Tables, p. 2

The formulas (10 to 19), §§ 12, 13, are all adapted to logarithmic computation.

EXAMPLE 1. Find $a = 29.45 \sin 46° 23$

$\log 29.45$	$= 1.46909$	Tables, p. 5
$\log \sin 46° 23'$	$= 9.85972 - 10$	Tables, p. 89
$\log a$	$= 1.32881$	
a	$= 21.321$	Tables, p. 4

EXAMPLE 2. Find α from $\tan \alpha = \dfrac{675.4}{423.7}$

$\log 675.4$	$= 2.82956$	Tables, p. 13
$\log 423.7$	$= 2.62706$	Tables, p. 8
$\log \tan \alpha$	$= 0.26250$	
α	$= 57° 53'.9$	Tables, p. 78

EXAMPLE 3. Find $h = \dfrac{42.98}{\cos 15° 20'}$

$\log 42.98$	$= 11.63327 - 10$	Tables, p. 8
$\log \cos 15° 20'$	$= 9.98426 - 10$	Tables, p. 61
$\log h$	$= 1.64901$	
h	$= 44.567$	Tables, p. 8

29. Products with Negative Factors. To find by use of logarithms the product of several factors some of which are negative, the product of the same factors, *all taken positively*, is first obtained, and the sign is then determined in the usual manner by counting the number of factors with negative sign.

EXAMPLE 1. Find $x = (-115)(23.41)(-.6422)(-.1123)$

Noticing first that there are an odd number of negative factors, we may write

$$-x = (115)(23.41)(.6422)(.1123);$$

and we may compute $-x$ as follows.

$$\begin{aligned} \log 115 &= 2.06070 \\ \log 23.41 &= 1.36940 \\ \log .6422 &= 9.80767 - 10 \\ \log .1123 &= 9.05038 - 10 \\ \hline \log(-x) &= 2.28815 \\ -x &= 194.15 \qquad \text{whence } x = -194.15 \end{aligned}$$

The use of logarithms in numerical calculation is further illustrated in the following examples.

EXAMPLE 2. Find $x = \sqrt[3]{\dfrac{(87)^2\sqrt{3241}}{740050}}$

$2 \log 87 =$	3.87904	Tables, p. 17
$\frac{1}{2} \log 3241 =$	1.75534	Tables, p. 6
	5.63438	
$\log 740050 =$	5.86926	Tables, p. 14
$\log x^3 =$	$29.76512 - 30$	
$\log x =$	$9.92171 - 10$	
whence $x =$	0.83504	Tables, p. 16

EXAMPLE 3. Find $x = \sqrt{\dfrac{5.62(4.8)^{1.5}}{(.684)^{2.3}}}$

$\log 5.62 =$	0.74974	Tables, p. 10
$1.5 \log 4.8 =$	1.02186	Tables, p. 9
	$11.77160 - 10$	
$2.3 \log 0.684 =$	$9.62064 - 10$	Tables, p. 13
$\log x^2 =$	2.15096	
$\log x =$	1.07548	
whence $x =$	11.898	Tables, p. 2

EXERCISES X. — LOGARITHMS. RIGHT TRIANGLES

1. Make the following computations by logarithms

(a) $.001467 \times 96.8 \times 47.37$ *Ans.* 6.7268

(b) $.0631 \times 7.208 \times .51272$ *Ans.* 0.23317

(c) $2\sqrt[3]{5}/3^{\frac{5}{8}}$ *Ans.* 0.1364

(d) $\sqrt[7]{-0.00951}$ *Ans.* -0.5142

(e) $15.008 \times (-0.0843)/(0.06376 \times 4.248)$ *Ans.* -4.671

(f) $\sqrt{5.955} \times \sqrt[3]{61.2}/\sqrt[5]{298.54}$ *Ans.* 3.076

(*g*) $(18.9503)^{11}\ (-0.1)^{14}$ *Ans.* 1.134

(*h*) $(-0.1412)^2/\sqrt[3]{-0.00475}$ *Ans.* -0.11858

(*i*) $1/(72.32)^{\frac{2}{3}}$ *Ans.* 0.05761

(*j*) $\sqrt{(0.00812)^{\frac{2}{3}}\ (471.2)^3/\sqrt{(522.3)^8\ (0.01242)^{\frac{3}{4}}}}$ *Ans.* 0.8929

2. The following formula $d = 0.479\sqrt[5]{\dfrac{c\,l\,f^2}{h}}$ is used to determine the diameter d, of water pipe in terms of the coefficient of friction c, the length l, the flow f, and the head h. Compute d when $c = 0.02$, $l = 500$, $f = 5$, $h = 10$. *Ans.* 0.91136

3. A wire 0.1066 cm. in diameter and 27.1 cm. long is stretched 0.133 cm. by a weight of 454 grams. Find the modulus of elasticity by the formula $e = \dfrac{lw}{as}$, in which l = length, a = area of cross section, and s = the elongation produced by a weight w. *Ans.* 1.0365×10^7.

4. The flow of water over a weir is given by the formula

$$f = \frac{k}{15}\sqrt{2\,gh^5}$$

Find f when $k = 4.736$, $g = 32.2$, $h = 1.2$ *Ans.* 399.32

5. A steel bar 98.75 cm. long between supports 0.96 cm. wide and 0.74 cm. deep is deflected 1.48 cm. by a weight of 5000 grams at the middle. Find the modulus of elasticity by the formula $e = \dfrac{wl^3}{4\,bd^3h}$, in which l = length, b = breadth, d = depth, and h = the deflection due to the weight w. *Ans.* 2.0908×10^9.

6. The pressure p and the volume v of a gas at constant temperature are connected by the relation $pv^a = k$. Find p when $v = 36.36$, $a = 1.41$, $k = 12600$. *Ans.* 79.414

7. The period of a conical pendulum is given by the formula $T = 2\pi\sqrt{\dfrac{ml\cos\alpha}{w}}$. Find T when $m = 0.347$, $l = 96.8$, $\alpha = 9°\,20'$, $w = 340$. *Ans.* 1.9618

8. The volume (gal.) of a conical tank of height h (in.) and vertical angle $2\,\alpha$ is $v = \pi h^3 \tan^2\alpha/693$. Find the capacity of such a tank whose angle at the vertex is 42° 30′ and whose height is 12 ft. 5 in. *Ans.* 2267.8

9. If a ball of radius r is rolled inside a spherical surface of radius R, the time of oscillation is given by the formula $T = 2\pi\sqrt{\dfrac{7\,(R-r)}{5\,g}}$. Find the radius of a concave mirror in which a $\frac{3}{8}$ in. steel ball makes an oscillation in 1.4 sec. Take $g = 384$. *Ans.* 13.805

10. Solve by means of logarithms the following right triangles, where h denotes the hypotenuse, other small letters the sides, and the corresponding capital letters the angles opposite those sides.

(*a*) $A = 63°$; $h = 28.54$	*Ans.* 25.429, 12.957
(*b*) $P = 65° \ 25'.2$; $p = 69.25$	*Ans.* 31.676, 76.152
(*c*) $A = 28° \ 25'$; $h = 29.36$	*Ans.* 25.822, 13.972
(*d*) $U = 28° \ 40'.4$; $v = 20.71$	*Ans.* 11.326, 23.605
(*e*) $a = 735.1$; $h = 846.2$	*Ans.* 60° 18′.6, 419.14
(*f*) $r = 9.328$; $s = 6.302$	*Ans.* 55° 57′.4, 11.257
(*g*) $a = 59.68$; $h = 69.27$	*Ans.* 59° 29′.4, 35.17
(*h*) $G = 36° \ 21'$; $h = 41.376$	*Ans.* 33.325, 24.524

11. Solve the following right triangles having given

(*a*) hypotenuse = 431.8, side = 127.3	*Ans.* 17° 8′.7, 412.61
(*b*) angle = 43° 48′, side adj. = 67.92	*Ans.* 94.104, 65.133
(*c*) angle = 55° 11′, side opp. = 68.34	*Ans.* 83.242, 47.527
(*d*) hyp. = 61.14, side = 48.56	*Ans.* 37° 25′, 37.149
(*e*) angle = 49° 13′, side adj. = 72.3	*Ans.* 110.68, 83.810
(*f*) sides = 126 and 198.	*Ans.* 234.72, 32° 28¼′.
(*g*) angle = 57° 46′, side opp. = 0.688	*Ans.* 0.4338, 0.8134
(*h*) angle = 32° 15′.4, side opp. = 547.25	*Ans.* 867.12, 1025.4

12. A tree stands on the opposite side of a small lake from an observer. At the edge of the lake the angle of elevation of the top of the tree is found to be 30° 58′. The observer then measures 100 ft. directly away from the tree and finds the angle of elevation to be 18° 26′. Find the height of the tree and the width of the lake. *Ans.* 74.973, 124.94

13. From a point 250 ft. from the base of a tower on a level with the base the angle of elevation of the top is 62° 32′. Find the height.
Ans. 480.93

14. To determine the height of a tower, its shadow is measured and found to be 97.4 ft. long. A ten-foot pole is then held in vertical position and its shadow is found to be 5.5 ft. Find the height of the tower and the angle of elevation of the sun. *Ans.* 177.09, 61° 11′.4

15. Find the length of a ladder required to reach the top of a building 50 ft. high from a point 20 ft. in front of the building. What angle would the ladder in this position make with the ground? *Ans.* 53.85, 68° 12′.

16. The width of the gable of a house is 34 ft.; the height of the house above the eaves is 15 ft. Find the length of the rafters and the angle of inclination of the roof. *Ans.* 22.67, 41° 25′.4

17. Assuming the radius of the earth to be 3956 mi. find the distance to the remotest point on the surface visible from the top of a mountain 2½ mi. high. *Ans.* 140.67 mi.

CHAPTER V

SOLUTION OF OBLIQUE TRIANGLES BY MEANS OF RIGHT TRIANGLES

30. Decomposition of Oblique Triangles into Right Triangles. A general method for solving oblique triangles in all cases consists in dividing the triangle into two right triangles by a perpendicular from a vertex to the opposite side; these right triangles are then solved by the methods of the previous chapter. In all except the three side case the perpendicular can be drawn so that one of the resulting right triangles contains two of the given parts. It may sometimes happen that the perpendicular will fall outside the given triangle.

31. Case I: Given Two Angles and a Side. It is immaterial which side is given, since the third angle can be found from the fact that the sum of the three angles is 180°. Drop the perpendicular from *either extremity* of the *given side.*

EXAMPLE 1. An oblique triangle has one angle equal to 43°, another equal to 67°, and the side opposite the unknown angle equal to 51. Determine the remaining parts.

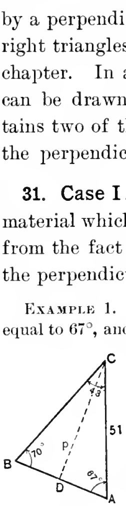


FIG. 36.

It is immediately seen that the third angle is $180° - (43° + 67°) = 70°$. To solve this triangle draw the figure approximately to scale and drop the perpendicular $CD = p$ from one extremity C of the known side to AB, the side opposite C. Denote the unknown side CB by a. In the right triangle ACD, the hypotenuse and one angle are known; hence by (13), § 12,

$$p = 51 \sin 67° = 46.95$$

An angle and the side opposite, in the right triangle BCD, are now known; hence by (15), § 12,

$$a = p/\sin 70° = 46.95/.9397 = 49.96$$

The side AB may be found in the same manner. Check as in § 5, p. 4.

If in the equation $a = p/\sin 70°$ we substitute the value $p = 51 \sin 67°$ previously found, we obtain for a the equation

$$a = \frac{51 \sin 67°}{\sin 70°}.$$

This formula is adapted to logarithmic computation. Applying the principles of logarithms we obtain

$$\log a = \log 51 + \log \sin 67° - \log \sin 70°.$$

Remembering that subtracting a logarithm is equivalent to adding the co-logarithm of the same number, we may arrange the numerical work as follows:

$$\begin{aligned} \log 51 &= 1.70757 \\ \log \sin 67° &= 9.96403 - 10 \\ \text{colog} \sin 70° &= \underline{0.02701} \\ \log a &= 1.69861 \\ a &= 49.959 \end{aligned}$$

In this solution, p was eliminated. Even if the equations are used without eliminating p, the actual value of p need not be found, since only $\log p$ is needed to complete the solution.

32. Case II: Given Two Sides and the Included Angle. The triangle can be divided into two right triangles, one of which contains two known parts, by a perpendicular from *either extremity* of the *unknown side* to the side opposite.

EXAMPLE 1. Two sides of a triangle are 26.5 and 32.8; the included angle is 52° 18′. Find the remaining parts.

In the figure let $AB = 32.8$, $AC = 26.5$, and the angle at $A = 52° 18'$. Drop a perpendicular p from B to the opposite side. Denote the unknown side by a and the segments of AC by x and y as in Fig. 37; then p, x, y, and $\tan C$ can be computed in the following order:

$p = 32.8 \sin 52° 18' = 32.8 \times .79122 = 25.952$

$x = 32.8 \cos 52° 18' = 32.8 \times .61153 = 20.058$

$y = 26.5 - x = 26.5 - 20.058 = 6.442$

$\tan C = p \div y = 25.952 \times 6.442 = 4.0286$

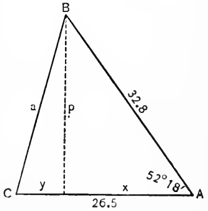


FIG. 37.

Hence from the tables,

$$C = 76° \; 3'.6$$

$$a = y \div \cos C = 6.442 \div .24101 = 26.73$$

These formulas are not well adapted to logarithmic computation. The values of p and x may be computed separately by logarithms, after which y and $\tan C$ may be found.

We use the formulas $p = c \sin A$, $x = c \cos A$, $y = b - x$, $\tan C = p \div y$. The work can be conveniently arranged in two columns, as follows:

$\log 32.8 = 1.51587$	$\log 32.8 = 1.51587$
$\log \sin A = 9.89830$	$\log \cos A = 9.78642$
$\log p = 1.41417$	$\log x = 1.30229$
$\log y = 0.80902$	$x = 20.058$
$\log \tan C = 0.60515$	$y = b - x = 6.442$
$C = 76° \; 3'.5$	$\log y = 0.80902$
$a = y \div \cos C$	$\log \cos C = 9.38190$
$a = 26.738$	$\log a = 1.42712$

33. Case III: Given the Three Sides. In this case it is not possible to divide the triangle into two right triangles in such a way that one of them contains two of the given parts; however, if a perpendicular is dropped to the *longest side* from the vertex of the *angle opposite*, the segments into which this side is divided by the perpendicular are easily computed.

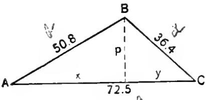


FIG. 38. c

EXAMPLE 1. The sides of a triangle are $a = 36.4$, $b = 50.8$, and $c = 72.5$ Determine the angles.

Draw a figure and drop a perpendicular from B upon AC. Denote the segments of the base by x and y as in Fig. 38; then

$$p^2 = \overline{50.8}^2 - x^2 = \overline{36.4}^2 - y^2;$$

hence $$x^2 - y^2 = \overline{50.8}^2 - \overline{36.4}^2 = 1255.68;$$

that is, $$(x - y)(x + y) = 1255.68$$

Since $$x + y = b = 72.5,$$

we have $$x - y = 1255.68 \div 72.5 = 17.32;$$

whence, adding, $$x = 44.91,$$

and, subtracting, $$y = 27.59$$

just drop

Since we now know x and y, the angles A and C are easily found. The student may complete the solution by using the formulas

$$\cos A = x \div 50.8 \qquad \cos C = y \div 36.4$$

Logarithms may be used as in the previous case to compute the separate products and quotients. The following is a convenient arrangement:

$$x^2 - y^2 = \overline{50.8}^2 - \overline{36.4}^2 = c^2 - a^2.$$

Factoring both sides gives

$$(x + y)(x - y) = b(x - y) = (c + a)(c - a)$$

or

$$x - y = (c + a)(c - a) \div b$$

$$\begin{aligned} c &= 50.8 \\ a &= \underline{36.4} \\ c + a &= 87.2 & \log(c + a) &= 1.94052 \\ c - a &= 14.4 & \log(c - a) &= 1.15836 \\ x + y = b &= 72.5 & \text{colog } b &= \underline{8.13966} \\ x - y &= 17.32 & \log(x - y) &= 1.23854 \\ x &= 44.91 \\ y &= 27.59 \\ \cos A &= x \div c & \cos C &= y \div a \\ \log x &= 1.65234 & \log y &= 1.44075 \\ \log c &= \underline{1.70586} & \log a &= \underline{1.56110} \\ \log \cos A &= 9.94648 & \log \cos C &= 9.87965 \\ A &= 27^\circ\ 51'.9 & C &= 40^\circ\ 42'.9 \end{aligned}$$

$$B = 111^\circ\ 25'.2$$

34. Case IV: Given Two Sides and the Angle Opposite One of Them. The triangle is solved by *dropping the perpendicular from the vertex of the angle included by the given sides.*

EXAMPLE 1. One angle of a triangle is 37° 20′; one side adjacent is 25.8 and the side opposite is 20.8. Solve the triangle.

First construct the given angle A and on one side of A lay off $AB = 25.8$ With B as center and radius $= 20.8$ describe an arc of a circle meeting the opposite side in two points C and C'. Either of the triangles ABC, ABC' satisfies the given conditions; the case is on this account called **the ambiguous case.**

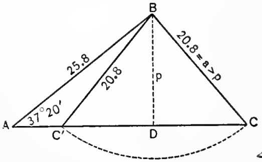


FIG. 39.

The student should note that the triangle BCC' is isosceles and that the interior angle of ABC at C is equal to the exterior angle of ABC' at C'; hence the interior angles C and C' are supplements of each other. To solve ABC draw the perpendicular $BD = p$ from B; then determine p from the right triangle ABD.

$$p = 25.8 \sin 37° \, 20' = 15.6464$$

Next determine C from the right triangle BDC;

$$\sin C = \frac{p}{a} = \frac{15.6464}{20.8} = .75223;$$

hence C is the *acute* angle whose sine is .75223; *i.e.* $C = 48° \, 47'$.

The student can complete the solution as follows:

$$AC = AD + DC;$$
$$B = 180° - (A + C).$$

Also for triangle ABC',

$$C' = 180° - C;$$
$$B' = 180° - (A + C');$$
$$AC' = AD - CD.$$

For the logarithmic solution we use the formula

$$\sin C = \frac{p}{a} = \frac{c \sin A}{a}.$$

Then the work may be arranged as follows:

$$\begin{aligned} \log c &= 1.41162 \\ \log \sin A &= 9.78280 \\ \text{colog } a &= \underline{8.68194} \\ \log \sin C &= 9.87636 \end{aligned}$$

$$\begin{aligned} C &= 48° \, 47'.1 \\ B &= 93° \, 52'.9 \\ b &= a \sin B / \sin A \\ \log a &= 1.31806 \\ \log \sin B &= 9.99900 \\ \text{colog} \sin A &= \underline{0.21720} \\ \log b &= 1.53426 \\ b &= 34.218 \end{aligned} \qquad \begin{aligned} C' &= 131° \, 12'.9 \\ B' &= 11° \, 27'.1 \\ b' &= a \sin B' / \sin A \\ \log a &= 1.31806 \\ \log \sin B' &= 9.29785 \\ \text{colog} \sin A &= \underline{0.21720} \\ \log b' &= 0.83311 \\ b' &= 6.8094 \end{aligned}$$

If, in a given problem, the side opposite the given angle is less than the perpendicular let fall upon the unknown side, there is no solution, and if it is greater than the other given side there is one solution only. The construction indicated in Ex. 1 will in all cases show the number of solutions.

EXERCISES XI. — SOLUTION OF TRIANGLES

Find the remaining parts of the following triangles by suitably dividing each into two right triangles. Capital letters represent angles; small letters the sides opposite them.

1. (*a*) $A = 17° 17'$, $B = 37° 37'$, $c = 174$; *Ans.* 63.186, 129.81
(*b*) $A = 24° 14'$, $C = 43° 13'$, $c = 240$; *Ans.* 143.86, 323.69
(*c*) $L = 28°$, $M = 51°$, $l = 6.3$ *Ans.* 10.429, 13.173

2. (*a*) $a = 41$, $b = 51$, $C = 62°$; *Ans.* 48° 44′.7, 69° 15′.3, 48.152
(*b*) $b = 3.5$, $c = 2.6$, $A = 33°$; *Ans.* 99° 58′.9, 47° 1′.1, 1.9356
(*c*) $u = 22$, $v = 12$, $W = 42°$. *Ans.* 106° 27′.6, 31° 32′.4, 15.35

3. (*a*) $a = 7$, $b = 12$, $c = 15$; *Ans.* 27° 16′, 51° 45′.2, 100° 58′.8
(*b*) $l = 10$, $m = 14$, $n = 20$; *Ans.* 27° 39′.6, 40° 32′.2, 111° 48′.2
(*c*) $u = 3$, $v = 4$, $w = 5$. *Ans.* 36° 52′.2, 53° 7′.8, 90° 0′.0

4. (*a*) $a = 50.8$, $b = 35.9$, $A = 64°$; *Ans.* 39° 26′.0, 76° 34′.0, 54.973
(*b*) $g = 6.22$, $k = 7.48$, $G = 26°$; *Ans.* $\begin{cases} 31° 48'.9,\ 122° 11'.1,\ 12.008 \\ 148° 11'.1,\ 5° 48'.9,\ 1.438 \end{cases}$
(*c*) $b = 23.4$, $q = 19.8$, $B = 109°$; *Ans.* 53° 8′.1, 17° 51′.9, 7.5922
(*d*) $a = 213$, $b = 278$, $B = 100°$. *Ans.* 48° 59′.2, 31° 0′.8, 145.45

5. To determine the distance from a point A to an inaccessible object B, a base line $AC = 300$ ft. and the angles $BAC = 40°$, $BCA = 50°$ are measured. Find the distance AB. *Ans.* 229.8

6. To determine the distance between two trees, A, B, on opposite sides of a hill, a point C is chosen from which both trees are visible; the distances $AC = 400$ ft., $BC = 361$ ft., and the angle $ACB = 55°$ are then measured. What is the distance between the trees? *Ans.* 353.08

7. The sides of a triangular field are 43 rods, 48 rods, and 57 rods, respectively; determine the angles between the sides.
Ans. 47° 24′, 55° 15′, 77° 21′.

8. A 50-ft. chord of a circle subtends an angle of 100° at the center. A triangle is to be inscribed in the larger segment, having one of its sides 40 ft. long. How long is the other side? Is there only one solution?
Ans. 65.22

9. A triangle having one of its sides 60 ft. long is to be inscribed in the segment of Ex. 8. Determine the remaining side. How many solutions are there in this case? *Ans.* 18.88, 58.25

10. Find the length of a side of an equilateral triangle circumscribed about a circle of radius 15 inches. *Ans.* 51.96 in.

11. The angle of elevation of the top of a mountain is observed at a point in the valley to be 50° ; on going directly away from the mountain one half mile up a slope inclined 30° to the horizon, the angle of elevation of the top is found to be 20°. Find the height of the mountain.

Ans. 4529.5 ft.

12. The base of an isosceles triangle is 245.5 and each of the base angles is 68° 22′. Find the equal sides and the altitude.

Ans. 332.96, 309.51

13. The altitude of an isosceles triangle is 32.2 and each of the base angles is 32° 42′. Find the sides of the triangle. *Ans.* 100.31, 59.60

14. A chord of a circle is 100 ft. long and subtends an angle of 40° 42′ at the center. Find the radius of the circle. *Ans.* 143.78

15. From a point directly in front of a building and 150 feet away from it, the length of the building subtends an angle of 36° 44′. How long is it? *Ans.* 66.40

16. Find the perimeter and the area of a regular pentagon inscribed in a circle of radius 12. *Ans.* 70.534, 342.38

17. Find the perimeter and the area of the regular octagon formed by cutting off the corners of a square 15 inches on a side.

Ans. 49.705, 186.39

18. Find the perimeter and the area of a regular pentagon whose diagonals are 16.2 inches long. *Ans.* 50.06, 172.466

19. Find the perimeter and the area of a regular dodecagon inscribed in a circle of radius 24. *Ans.* 149.08, 1728.

20. Two chords subtend angles of 72° and 144° respectively at the center of a circle. Show that when they are parallel and on the same side of the center, the distance between the chords is one-half the radius.

21. Devise a formula for solving an isosceles triangle when the base and the base angles are given ; when the base and one of the equal sides are given ; when one of the equal sides and one of the base angles are given.

PART II. OBTUSE ANGLES AND OBLIQUE TRIANGLES

CHAPTER VI

FUNDAMENTAL DEFINITIONS AND FORMULAS

35. Obtuse Angles. The solution of oblique triangles involves obtuse * as well as acute angles. For this reason we need to be able to determine the values of the trigonometric ratios for such angles; it is not necessary, however, to enlarge our tables for this purpose, for, as will now be shown, *every ratio for an obtuse angle can be expressed in terms of some ratio of an acute angle.*

Let an obtuse angle α be placed on the coördinate axes with the vertex at the origin and one side along the x-axis to the right; then the other side will fall in the second quadrant. The ratios sin α, cos α, etc., are defined in terms of x, y, and $r = \sqrt{x^2 + y^2}$ precisely as they were for acute angles in § 11. It should be noticed, however, that since x is negative while y and r are positive, every ratio which involves x is negative for an obtuse angle; thus $x/r =$ cos α, $y/x =$ tan α, and their reciprocals, sec α and ctn α, are all negative for obtuse angles.

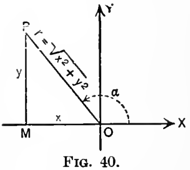


FIG. 40.

We now proceed to obtain equations similar to the equations sin $(90° - \alpha) =$ cos α, etc. (proved in § 15), which enabled us to find the values of the ratios of acute angles greater than 45° in terms of the ratios of angles less than 45°.

* An obtuse angle is an angle which is greater than 90° and less than 180°.

36. Reduction from Obtuse to Acute Angles. Let α be placed on coördinate axes as described above, and let the supplement of α be denoted by β (which is an acute angle). Lay off β in the first quadrant with one side along the x-axis. From a point P in the side of α (in second quadrant) and a point P' in the side of β (in first quadrant) at the same distance r from the origin, draw the perpendiculars PM, $P'M'$, as in Fig. 41. The value of x for the point P will be negative since P is in the second quadrant. Let its coördinates be $(-a, b)$; then, since the triangles OPM, $OP'M'$ are symmetric, the coördinates of P' are (a, b). As in § 11, we have

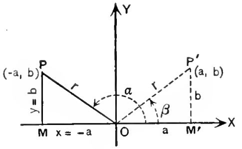


FIG. 41.

$$\sin\alpha = \frac{b}{r} = \sin\beta, \qquad \cos\alpha = -\frac{a}{r} = -\cos\beta,$$

or, since $\beta = 180° - \alpha$,

(1) $$\sin\alpha = \sin(180° - \alpha);$$

(2) $$\cos\alpha = -\cos(180° - \alpha).$$

In a similar manner it can be shown that

(3) $$\tan\alpha = -\tan(180° - \alpha).$$

It follows that if α is an obtuse angle *we find its sine by looking for the sine of its supplement, which is an acute angle*, and similarly for the other functions, always having regard for the proper sign.

EXERCISES XII.—FUNCTIONS OF OBTUSE ANGLES

1. From the accompanying figure prove the following relations:

(*a*) $\sin(90° + \alpha) = \cos\alpha$.

(*b*) $\cos(90° + \alpha) = -\sin\alpha$.

(*c*) $\tan(90° + \alpha) = -\operatorname{ctn}\alpha$.

(*d*) $\operatorname{ctn}(90° + \alpha) = -\tan\alpha$.

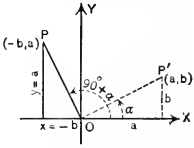


FIG. 42.

2. Construct obtuse angles whose functions have the following values:

(*a*) $\sin\theta = 1/3$. (*b*) $\tan\theta = -3/4$. (*c*) $\cos\theta = -3/5$.
(*d*) $\sin\theta = 1/2$. (*e*) $\sin\theta = \sqrt{2}/2$. (*f*) $\sin\theta = \sqrt{3}/2$.

3. Find the values of the remaining functions of the angles of Ex. 2.

4. Express the following as functions of an angle less than 45°, and look up their values in a table.

(*a*) $\sin 121°$. (*b*) $\cos 101°$. (*c*) $\tan 168°$.
(*d*) $\sin 99°$. (*e*) $\text{ctn}\, 178°$. (*f*) $\cos 154°$.
(*g*) $\cos 133° 11'$. (*h*) $\tan 144° 38'$. (*i*) $\sin 92° 3'$.

5. Solve the equation $6\cos^2 x + 7\cos x + 2 = 0$.

[To solve an equation of this type one should first regard it as an algebraic (quadratic) equation in which the unknown is $\cos x$: replacing $\cos x$ by the letter t we have the equation $6t^2 + 7t + 2 = 0$. The solutions of this equation are t (or $\cos x$) $= -\frac{1}{2}$ or $t = -\frac{2}{3}$. Then find from the tables the angles x satisfying the equations $\cos x = -\frac{1}{2}$ and $\cos x = -\frac{2}{3}$; they are $x = 120°$ or $x = 131° 48'.6$]

6. Show that the equation $\tan x = c$ has an obtuse angle solution if c is any negative number.

7. Show that the equation $\sin x = c$ has both an acute and an obtuse angle solution if c is any positive number less than 1.

8. Show that the equation $\cos x = c$ has a solution between 0° and 180° if c lies between $+1$ and -1, and that this solution is an acute angle if c is positive and an obtuse angle if c is negative.

9. Find all of the solutions between 0° and 180° for the following equations:

(*a*) $3\sin^2 x - 2\sin x - 1 = 0$. (*b*) $4\sin^2 x - 3\sin x - 1 = 0$.
(*c*) $6\sin^2 x + \sin x - 1 = 0$. (*d*) $6\sin^2 x - \sin x - 1 = 0$.

37. Geometric Relations. In the following sections certain fundamental geometric and trigonometric relations connecting the sides and angles of any triangle are given. Upon these is based a systematic method of solution of oblique triangles, which is given in the following chapter.

38. The Law of Cosines. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice their product ~~into~~ the cosine of their included angle.*

Denote the sides of a triangle by a, b, c, and the angles opposite by A, B, C; and express the square of side a in terms of b, c, and C as follows. Drop a perpendicular, p, from B to the opposite side and denote the segments of this side by x

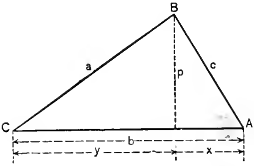


FIG. 43.

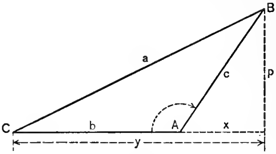


FIG. 44.

and y. By (13, 14) § 12, we have in Fig. 43,

$$p = c \sin A, \quad x = c \cos A, \quad y = b - x = b - c \cos A$$
$$a^2 = y^2 + p^2 = (b - c \cos A)^2 + c^2 \sin^2 A$$
$$= b^2 - 2\,bc \cos A + c^2 (\cos^2 A + \sin^2 A)$$

whence, since $\sin^2 A + \cos^2 A = 1$

(4) $$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

If as in Fig. 44 the side a to be found is opposite an obtuse angle A, $y = b + x$; but by (2) § 36, $x = c \cos (180° - A) = -c \cos A$; hence $y = b - c \cos A$ and $p = c \sin (180° - A) = c \sin A$, exactly as in the case considered above.

The law of cosines can be used to compute one side of a triangle when the other two sides and one angle are known, and also to find the angles when the three sides are known.

EXAMPLE 1. One angle of a triangle is 66° 25′ and the including sides are 3 and 5. Find the third side.

$$x^2 = 3^2 + 5^2 - (30\,(.4)) = 22, \;\therefore\; x = \sqrt{22} = 4.69$$

EXAMPLE 2. Two sides of a triangle are 7 and 8 and the angle opposite the former is 60°. Find the third side.

$$7^2 = x^2 + 8^2 - 16\,x\,(\tfrac{1}{2})$$

whence $x = 3$ or $x = 5$ and there are two solutions.

EXAMPLE 3. The sides of a triangle are 3, 5, and 7. Find the greatest angle.

$$7^2 = 3^2 + 5^2 - 30 \cos x$$

whence $$\cos x = -\tfrac{1}{2} \text{ and } x = 120°.$$

EXERCISES XIII. — THE COSINE LAW

1. Two sides of a triangle are 1.5 and 2.4, and their included angle is 36°. Find the third side. *Ans.* 1.48

2. Two sides of a triangle are 5 and 8 and the included angle is 135°. Find the third side. *Ans.* 5.69

3. Two sides of a triangle are 3 and 4 and the angle opposite the former is 30°. Find the third side. *Ans.* $2\sqrt{3}+\sqrt{5}$ or $2\sqrt{3}-\sqrt{5}$.

4. The sides of a triangle are 3, 5, and 6. Find the smallest angle. *Ans.* 29° 55′.6

5. The sides of a triangle are 10, 14, and 17. Find the angles. *Ans.* 36° 1′, 55° 25′, 88° 34′.

6. Two sides of a triangle are 11 and 17, and the angle opposite the former is 30°. Find the third side by the law of cosines.

7. Devise a method for finding the angle between two lines without an instrument for measuring angles. Could the law of cosines be used for this purpose?

39. The Law of Sines. *Any two sides of a triangle are to each other as the sines of the angles opposite.*

Denote the sides and angles of a triangle by a, b, c, A, B, C, as above. Prove that

$$\frac{a}{b}=\frac{\sin A}{\sin B}$$

as follows:

Drop a perpendicular from C (the angle included by the sides a and b) to the opposite side. In Fig. 45, where the

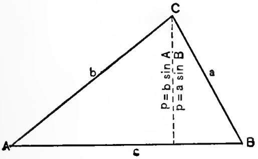


Fig. 45.

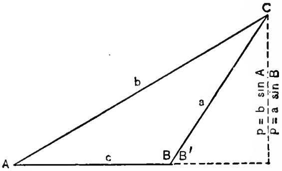


Fig. 46.

angles A and B are both acute, by (13), § 12

$$p = a \sin B \text{ and also } p = b \sin A,$$

whence $$a \sin B = b \sin A$$

and dividing through by $b \sin B$,

(5) $$\frac{a}{b}=\frac{\sin A}{\sin B}.$$

In Fig. 46, where one of the given angles is obtuse,

$$p = a \sin B' = a \sin(180° - B) = a \sin B$$

and also $p = b \sin A$, exactly as above.

If the perpendicular is drawn from one of the other vertices, say from A, the above procedure leads to

(6) $$\frac{b}{c}=\frac{\sin B}{\sin C}.$$

From equation (5), dividing each side by $\sin A$ and multiplying each side by b, we see that

$$\frac{a}{\sin A}=\frac{b}{\sin B}.$$

From (6) we see, similarly, that each of these ratios is equal to $c/\sin C$. It follows that we have

(7) $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

40. Diameter of Circumscribed Circle. It can be shown that each of the ratios in (7) (where a, b, c, stand for the numerical measures of the sides) is equal to the numerical measure of the diameter of the circumscribed circle; and this furnishes another proof of the law of sines.

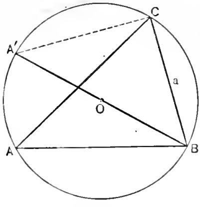


FIG. 47.

Circumscribe a circle about the triangle ABC, draw the diameter $BA' = d$, and connect $A'C$. Then angle $A'CB$ is a right angle and $A' = A$ since each is measured by one-half the arc BC. Therefore by (15), § 12,

$$d = \frac{a}{\sin A'} = \frac{a}{\sin A}$$

and similarly $d = \dfrac{b}{\sin B}$, $d = \dfrac{c}{\sin C}$.

If the angle A were obtuse we should have $A' = 180° - A$, but since $\sin(180° - A) = \sin A$, the same result holds in this case also. Therefore in general, *the diameter of the circle circumscribed about a triangle is equal to any side divided by the sine of the opposite angle.*

The law of sines can be used whenever three parts of a triangle are known, of which two are a side and the angle opposite.

Example 1. Two angles of a triangle are $10° 12'$ and $46° 36'$ and the shortest side is 10. Find the longest side.

The angle opposite the longest side is $123° 12'$ and

$$\frac{x}{\sin 123° 12'} = \frac{10}{\sin 10° 12'}$$

whence $$x = \frac{10(.83676)}{.17708} = 47.25$$

Example 2. The three sides of a triangle are 3, 5, and 7. We have seen in Ex. 3, p. 52, that the largest angle is $120°$. Find the smallest angle.

$$\frac{3}{\sin x} = \frac{7}{\sin 120°}$$

whence $$\sin x = \frac{3\sqrt{3}}{14} = .37115$$

and, since x must be acute,

$$x = 21° 47'.2$$

EXERCISES XIV. — THE SINE LAW

1. Two angles of a triangle are $19°$ and $104°$ and the side opposite the former is 20. Find the other two sides. *Ans.* 51.5, 59.6

2. The sides of a triangle are 8, 13, and 15. Find the angle opposite the second side by the law of cosines and the other two by the law of sines. *Ans.* $60°$, $32° 12'$, $87° 48$.

3. The sides of a triangle are 21, 26, 31. Find the angles as in Ex. 2. *Ans.* $56° 7'$, $42° 6'$, $81° 47'$.

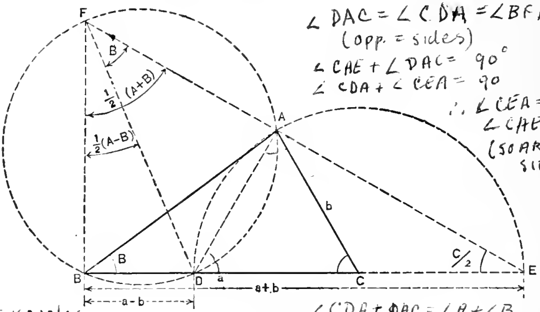
4. Compute the length of the radius of the circumscribed circle for each of the triangles in Exs. 1–3.

5. Two angles of a triangle are $38° 12'$ and $61° 10'$, and the included side is 350.6 Find the other two sides. *Ans.* 219.7, 311.3

41. The Law of Tangents. *In any triangle the difference of any two sides is to their sum as the tangent of one-half the difference of the angles opposite those sides is to the tangent of one-half their sum.*

Let ABC be any triangle having two sides a and b unequal, say $a > b$; the included angle C may be acute, right, or obtuse.

With a radius b, the *shorter* of the *given sides*, and center C, the vertex of the *included angle*, describe a circle through A



$\angle DAC = \angle CDA = \angle BFA$ (opp. = sides)
$\angle CAE + \angle DAC = 90°$
$\angle CDA + \angle CEA = 90$
$\therefore \angle CEA = \angle CAE$ (SO ARE SIDES)

$\triangle DAC$ = isoceles
$\therefore \angle C =$

FIG. 48. $\angle CDA + \angle DAC = \angle A + \angle B$

which cuts the side CB in a point D between B and C and also at a second point E beyond C. Draw EA, and at B erect a perpendicular which meets EA produced at F. On DF as a diameter construct a circle; *this circle will pass through A and B*, for FAD is a right angle since it is the supplement of DAE which is inscribed in a semicircle (SAME AS CIRCLE), and FBD is a right angle by construction. This construction is possible for any triangle in which $a > b$.

Angle $BFE = \frac{1}{2}(A + B)$ since it is the complement of angle $CEA = \frac{1}{2}C$; and $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90°$ since the sum of the angles of a triangle is 180°. Angle $DFA = B$ since each is measured by one-half the arc AD; therefore $BFD = BFE - DFA = \frac{1}{2}(A + B) - B = \frac{1}{2}(A - B)$.

In the right triangles DBF and EBF by (13), § 12,

$$a - b = BF \cdot \tan \tfrac{1}{2}(A - B),$$
$$a + b = BF \cdot \tan \tfrac{1}{2}(A + B),$$

whence

$$\textbf{(8)} \qquad \frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

$180 - C = 2 \times CEA$
$\angle \cancel{B}ACE = A + B$

This formula is still true but trivial, if $a = b$, since in that case each side reduces to zero; if $a < b$, the result would obviously be

$$(9) \qquad \frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}.$$

Since $\frac{1}{2}(A+B)$ is the complement of $\frac{1}{2}C$, (8) can be reduced to the form

$$(10) \qquad \tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \operatorname{ctn} \tfrac{1}{2} C.$$

42. Tangents of the Half-angles. The tangent of one-half any angle of a triangle can be expressed in terms of the sides as follows.

Bisect the angles of the triangle ABC and draw the inscribed circle tangent to the sides at P, Q, and R. Let r be the radius of this circle and let s stand for one-half the perimeter of the given triangle, *i.e.*

$$2s = a + b + c.$$

Then

$$AP = AR, \quad BR = BQ, \quad CQ = CP,$$

and

$$BR + BQ + CQ + CP = 2BQ + 2QC = 2a,$$

whence

$$2AP = 2s - 2a$$

and

$$AP = AR = s - a.$$

Similarly,

$$BR = BQ = s - b$$

and

$$CQ = CP = s - c.$$

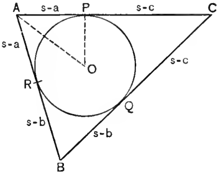


Fig. 49.

In the right triangle APO, by (12), § 12,

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}.$$

A similar result holds for the other two angles. Hence we have the three formulas:

$$(11) \quad \tan \tfrac{1}{2} A = \frac{r}{s-a}, \qquad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \qquad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

43. Radius of the Inscribed Circle. It remains to express r in terms of the sides of the triangle. In the triangle ABC produce the sides AB and AC. Bisect the angle A and the exterior angles at B and C. These bisectors meet in a point I which is the center of a circle which touches the side a, and the sides b and c produced. This circle is called an *escribed* circle of the triangle. Denote its radius by r' and mark the points of tangency P, Q, R. Then we have

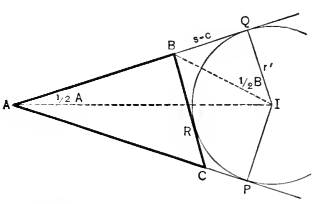


FIG. 50.

$$AQ = AP, \quad BQ = BR, \quad CP = CR,$$

therefore

$$AB + BR = AC + CR = s,$$

where s denotes half the perimeter of the given triangle. It follows that $AQ = s$ and

$$BQ = AQ - AB = s - c.$$

In the right triangle BQI,

$$\text{angle } IBQ = \tfrac{1}{2}(180° - B) = 90° - \tfrac{1}{2} B$$

and therefore angle $BIQ = \frac{1}{2} B$; then by (13, 14), § 12 and (11), § 42, in triangle BQI,

$$r' = (s - c) \operatorname{ctn} \tfrac{1}{2} B = \frac{(s-b)(s-c)}{r},$$

and in triangle AQI,

$$r' = s \tan \tfrac{1}{2} A = \frac{rs}{s-a}.$$

Equating these two values of r' and solving for r, we have,

(12) radius $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

The symmetry of this result in a, b, c shows that we shall get the same result if we produce sides c and a, or a and b.

Example 1. The sides of a triangle are 145/13, 119/13, and 156/13. Find the radius of the inscribed circle and the angles of the triangle.

We first compute the values of s, $s-a$, $s-b$, and $s-c$

$s = \frac{1}{2}(a+b+c) = 210/13$, $s-a = 65/13 = 5$, $s-b = 7$, $s-c = 54/13$.

Substituting in the formula for r we obtain

$$r = \sqrt{\frac{7 \times 5 \times (54/13)}{210/13}} = \sqrt{9} = 3,$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a} = 3/5, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b} = 3/7, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c} = 13/18\,;$$

hence from the tables we find

$$A/2 = 30^\circ\, 57'.8, \quad B/2 = 23^\circ\, 11'.9, \quad C/2 = 35^\circ\, 50'.3$$

Example 2. Two sides of a triangle are 12 and 8 and the included angle is 60°. Find the remaining angles.

Denoting the unknown angles by A and B we have

$$A + B = 180^\circ - 60^\circ = 120^\circ,$$

then by the law of tangents we have

$$\frac{12-8}{12+8} = \frac{\tan \frac{1}{2}(A-B)}{\tan 60^\circ} = \frac{\tan \frac{1}{2}(A-B)}{\sqrt{3}},$$

hence

$$\tan \tfrac{1}{2}(A-B) = \sqrt{3}/5 \text{ and } \tfrac{1}{2}\,(A-B) = 19^\circ\, 6'.4$$

Adding this result to $\frac{1}{2}(A+B) = 60^\circ$ we obtain $A = 79^\circ\, 6'.4$, and subtracting we get $B = 40^\circ\, 53'.6$

EXERCISES

1. The three sides of a triangle are 7, 12, and 15. Find the radius of the inscribed circle and the angles.

2. Determine the angles of the following triangles:

(a) $a = 5$, $b = 9$, $c = 11$. (c) $a = 10$, $b = 12$, $c = 15$.
(b) $a = 4$, $b = 8$, $c = 10$. (d) $a = 6$, $b = 8$, $c = 10$.

3. Determine the angles and third side of the following triangles:

(a) $a = 4$, $b = 8$, $C = 20^\circ$. (c) $a = 10$, $b = 12$, $C = 35^\circ$.
(b) $a = 4$, $b = 8$, $C = 40^\circ$. (d) $a = 13$, $b = 17$, $C = 44^\circ$.

4. To determine the distance between two objects A and B separated by a barrier, the distances $AC = 40$ rd., $BC = 48$ rd. are measured to a third point C. The angle $ACB = 68^\circ$ is then measured. Find the distance AB and the other angles of the triangle ABC.

CHAPTER VII

SYSTEMATIC SOLUTION OF OBLIQUE TRIANGLES

44. Analysis of Data. In the solution of oblique triangles the following cases arise:

Case I. *Given two angles and a side.*

Case II. *Given two sides and the included angle.*

Case III. *Given the three sides.*

Case IV. *Given two sides and an angle opposite one of them.*

The direction "*Solve a triangle*" tacitly assumes that a sufficient number of parts of an actual triangle are given. A proposed problem may violate this assumption and there will be no solution. Thus there is no triangle whose sides are 14, 24, and 40. An attempt to solve such an impossible problem gives rise to a contradiction such as, for example, the sine or cosine of some angle greater than 1. *Any triangle which can be constructed can be solved.*

45. Case I. Given Two Angles and a Side. In this case it is immaterial which side is given, since the third angle can be found from the fact that the sum of the three angles is 180°.

There is one and only one solution, provided the sum of the given angles is less than 180°.

The other two sides can be found, one at a time, by the law of sines (§ 39).

Example 1. Given one side of a triangle $a = 2.903$ and two of the angles $B = 79° 40'$, $C = 33° 15'$; find the remaining parts.

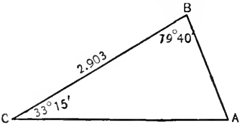


Fig. 51.

$$A = 180° - (79° 40' + 33° 15') = 67° 5'.$$

By the law of sines

$$\frac{b}{2.903} = \frac{\sin 79° 40'}{\sin 67° 5'}.$$

Many of the computations in the solution of triangles are of the following type. *To find one term of a proportion,* $\frac{a}{b} = \frac{c}{d}$, *when the other three are known*, no matter in which of the four positions the unknown stands. The student should master this problem. The following rule applies. *Imagine the means, and also the extremes, to be connected by straight lines crossing at the = sign. Multiply together the pair of knowns thus connected and divide by the known opposite the unknown.*

Applying this rule to the computation of b, the work may be written down as follows:

$$\begin{array}{rr} \sin 79^\circ 40' = & .98378 \\ & \underline{2.903} \\ & 295134 \\ & 885402 \\ & \underline{196756} \\ & 2.85591334 \end{array} \qquad \begin{array}{l} \sin 67^\circ 5' = .92107\overline{)2.85591334}\ \underline{|3.1007} \\ \qquad\qquad\qquad\qquad \underline{2\ 76321} \\ \qquad\qquad\qquad\qquad\quad 92703 \\ \qquad\qquad\qquad\qquad\quad \underline{92107} \\ \qquad\qquad\qquad\qquad\qquad 59634 \\ \qquad\qquad \therefore b = 3.1007 \end{array}$$

This work can be shortened by the use of logarithms. In all cases where the product of two or more numbers is to be divided by other numbers we can use the following principle (Tables, p. x). *Subtracting the logarithm of a number is equivalent to adding its cologarithm.*

The computation of b by logarithms may be written as follows:

$$\begin{aligned} \log 2.903 &= 0.46285 \\ \log \sin 79^\circ 40' &= 9.99290 - 10 \\ \operatorname{colog} \sin 67^\circ 5' &= 0.03571 \\ \log b &= 0.49146 \\ b &= 3.1007 \end{aligned}$$

The side c is found similarly from the proportion

$$\frac{c}{2.903} = \frac{\sin 33^\circ 15'}{\sin 67^\circ 5'}.$$

To check, apply the law of sines (§ 39), or the Law of tangents (§ 41) to the computed sides b and c.

EXERCISES XV. — CASE I

Solve the following triangles. Small letters represent sides and corresponding capital letters the angles opposite.

1. $B = 50^\circ 30'$, $C = 122^\circ 9'$, $a = 72$. *Ans.* 334.28, 476.51

2. $F = 82^\circ 20'$, $G = 43^\circ 20'$, $f = 48$. *Ans.* 33.097, 39.165

3. $M = 79^\circ 59'$, $N = 44^\circ 41'$, $p = 477$. *Ans.* 340.73, 398.39

4. $P = 37° 58'$, $Q = 65° 2'$, $r = 133.2$ *Ans.* 84.103, 110.579

5. $A = 70° 55'$, $K = 52° 9'$, $a = 48.09$ *Ans.* 42.645, 40.031

6. $A = 51° 47'$, $B = 66° 20'$, $c = 337.6$ *Ans.* 300.73, 350.58

7. $A = 48° 10'$, $B = 54° 10'$, $c = 38.7$ *Ans.* 29.516, 32.116

8. $B = 38° 12'$, $C = 61° 10'$, $a = 70.12$ *Ans.* 43.949, 62.257

9. $U = 46° 36'$, $V = 124° 18'$, $w = 1001.$ *Ans.* 4598.6, 5228.4

10. $B = 21° 16'$, $C = 113° 34'$, $d = 20.93$ *Ans.* 10.705, 27.053

11. $B = 62° 42'$, $M = 52° 22'$, $a = 39.75$ *Ans.* 38.995, 34.753

12. $B = 58° 20'$, $G = 61° 2'.3$, $g = 8.75$ *Ans.* 8.512, 8.715

13. $C = 43° 50'.4$, $Q = 69° 30'.2$, $c = 73.05$ *Ans.* 96.685, 97.123

14. $G = 75° 2'.7$, $K = 43° 44'.3$, $k = 81.5$ *Ans.* 103.32, 113.89

15. Two observers, facing each other 3 kilometers apart and at the same altitude, find the angles of elevation of a Zeppelin to be 57° 20′ and 64° 30′, respectively. Find the height. *Ans.* 2.683

16. A diagonal of a parallelogram is 18.56 and it makes angles 26° 30′ and 38° 40′ with the sides. Find the sides and the area of the parallelogram. *Ans.* 9.125, 12.777, 105.81

17. A lighthouse was observed from a ship to be N. 16° W.; after sailing due east 4.5 miles, the lighthouse was N. 48° W. Find the distance from the lighthouse to the ship in both positions. *Ans.* 5.682, 8.163

18. The side of a hill is inclined at an angle of 22° 37′ to the horizon. A flagstaff at the top of the hill subtends an angle of 13° 17′ from a point at the foot of the hill, and an angle of 18° 2′ from a point 100 ft. directly up the hill. Find the height of the flagstaff. *Ans.* 95.053

19. To find the distance from a station A to an inaccessible point B, a base line $AC = 500$ ft., and the angles $ACB = 68° 18'$, $CAB = 58° 28'$ are measured. Find the distance AB.

20. To find the height of an inaccessible object AB, a base line $CD = 250$ ft. is measured directly toward the object: also the angles of elevation $ADB = 48° 20'$ and $ACB = 38° 40'$. Find the height AB.

46. Case II. Given Two Sides and the Included Angle.

There is always one and only one solution.

The obvious method of solution is to find the third side by the law of cosines (§ 38), and then the other two angles by the law of sines (§ 39).

Example 1. Two sides of a triangle are 10 and 11, and the included angle is 35° 24′. Find the other parts.

Draw a figure, denote the unknown side by a, and the unknown angles by B, C. Then we may write

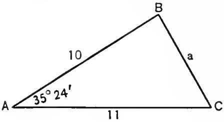


FIG. 52.

$$a^2 = \overline{10}^2 + \overline{11}^2 - 2(10)(11)\cos 35° 24',$$
$$a^2 = 221 - (220)(.81513).$$

Then $a^2 = 41.6714$, whence $a = 6.4553$ (Tables, p. 104).

To find B and C by the law of sines, we have

$$\frac{\sin B}{\sin 35° 24'} = \frac{11}{6.4553}, \quad \text{and} \quad \frac{\sin C}{\sin 35° 24'} = \frac{10}{6.4553},$$

whence on computing (see Example 1, § 45)

$$B = 80° 47'.0, \qquad C = 63° 48'.8$$

CHECK: $A + B + C = 179° 59'.8$

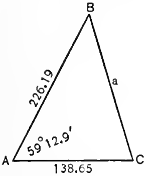


FIG. 53.

EXAMPLE 2. Two sides of a triangle are 138.65 and 226.19, and the included angle is 59° 12′.9. Find the third side.

Construct the triangle as in Fig. 53.

$$a^2 = \overline{138.65}^2 + \overline{226.19}^2 - 2(138.65)(226.19)\cos 59° 12'.9$$

While this is not adapted to logarithms, nevertheless logarithms can be used to compute separately the three terms on the right; for the moment call the third term, x.

$\log 138.65 = 2.14192$	$\log 226.19 = 2.35447$
2	2
4.28384	4.70894
$(138.65)^2 = 19224$	$(226.19)^2 = 51161$
51161	$\log 2 = 0.30103$
70385	$\log \cos 59° 12'.9 = 9.70911$
$x = 32102$	2.14192
$a^2 = 38283$	2.35447
$a = 195.66$	$\log x = 4.50653$
(Tables, p. 95)	$x = 32102$

47. Logarithmic Solution of Case II. When two sides and the included angle are given, a triangle can be completely solved by logarithms by finding first the other two angles by the law of tangents (§ 41).

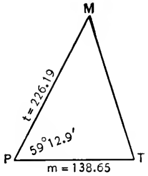


FIG. 54.

EXAMPLE 1. In a triangle MPT, side $m = 138.65$, side $t = 226.19$, and the included angle $P = 59° 12'.9$. Find the other parts.

Applying the law of tangents to the given sides, noting that $t > m$,

$$\frac{t-m}{t+m} = \frac{\tan\frac{1}{2}(T-M)}{\tan\frac{1}{2}(T+M)}.$$

In this proportion three terms are known since $T + M = 180° - P$. The work may be set down as follows.

$$\begin{array}{rll}
t = & 226.19 & \\
m = & 138.65 & \\
t - m = & 87.54 & \log(t-m) = 1.94221 \\
t + m = & 364.84 & \operatorname{colog}(t+m) = 7.43790 - 10 \\
\frac{1}{2}(T+M) = \frac{1}{2}(180° - P) = & 60° 23'.55 & \log\tan\frac{1}{2}(T+M) = \underline{0.24546} \\
\frac{1}{2}(T-M) \phantom{= \frac{1}{2}(180° - P)} = & \underline{22° 53'.5} & \log\tan\frac{1}{2}(T-M) = 9.62557 - 10 \\
\therefore\ T = & 83° 17' & \\
M = & 37° 30' &
\end{array}$$

The side p can now be found by solving the proportion

$$\frac{p}{138.65} = \frac{\sin 59° 12'.9}{\sin 37° 30'}$$

$$\begin{aligned}
\log 138.65 &= 2.14192 \\
\log\sin 59° 12'.9 &= 9.93404 - 10 \\
\operatorname{colog}\sin 37° 30' &= \underline{0.21555} \\
\log p &= 2.29151
\end{aligned}$$

from which $p = 195.66$ Compare Example 2, § 46.

EXERCISES XVI. — CASE II

1. Solve the following triangles by using the law of cosines:

(a) $a = 22$, $b = 12$, $C = 42°$. *Ans.* $106° 27'.7$, $31° 32'.4$, 15.35
(b) $a = 14$, $c = 16$, $B = 52°$. *Ans.* $56° 13'$, $71° 47'$, 13.27
(c) $l = 28$, $m = 36$, $N = 125°$. *Ans.* $23° 46'.6$, $31° 13'.4$, 56.89
(d) $a = 21$, $b = 24$, $c = 28$. *Ans.* $46° 51'.6$, $56° 30'.3$, $76° 38'.1$
(e) $a = 2.2$, $b = 4.2$, $c = 5.5$ *Ans.* $21° 16'.9$, $43° 51'.4$, $114° 51'.7$
(f) $l = 13$, $m = 16$, $n = 20$. *Ans.* $40° 27'.1$, $52° 59'.6$, $86° 33'.3$
(g) $u = 41$, $v = 51$, $W = 61°$. *Ans.* $69° 57'.3$, $49° 2'.7$, 47.48
(h) $b = 3.5$, $c = 2.6$, $A = 33°$. *Ans.* $47° 1'.3$, $99° 58'.7$, 1.935

2. Two sides of a triangle are 2.1 and 3.5 and the included angle is $53° 8'$. Determine the remaining parts. *Ans.* $36° 52'$, $90°$, 2.4

3. How long is a rod which subtends an angle of $60°$ at a point which is 5 ft. from one end of the rod and 8 ft. from the other? *Ans.* 7 ft.

4. How long is a rod which subtends an angle of $120°$ at a point 3 ft. from one end and 5 ft. from the other? *Ans.* 7 ft.

5. Solve each of the following triangles, using logarithms:

(*a*)	$a = 52.8$,	$b = 25.2$,	$C = 124° 34'$.	*Ans.* $17° 11'.1$, 70.233
(*b*)	$b = 55.1$,	$c = 45.2$,	$A = 16° 16'$.	*Ans.* $47° 14'.1$, 17.246
(*c*)	$l = 131$,	$m = 72$,	$N = 39° 46'$.	*Ans.* $31° 19'.9$, 88.568
(*d*)	$a = 35$,	$b = 21$,	$C = 48° 48'$.	*Ans.* $36° 44'.4$, 26.415
(*e*)	$u = 604$,	$v = 291$,	$W = 106° 19'$.	*Ans.* $22° 9'.5$, 740.45
(*f*)	$a = 23.45$,	$b = 18.44$,	$D = 81° 50'$.	*Ans.* $56° 56'.4$, $41° 13'.6$, 27.696
(*g*)	$u = .6238$,	$v = .2347$,	$C = 108° 30'$.	*Ans.* $53° 49'.2$, $17° 40'.8$, 0.7329

6. Two sides of a triangle are 22.531 and 34.645; the included angle is $43° 31'$. Determine the remaining parts.

Ans. $40° 16'.7$, $96° 12'.3$, 23.716

7. To determine the distance between two objects A and B separated by a hill, the distances $AC = 300$ ft., $BC = 277$ ft., and the angle $ACB = 65° 47'$, are measured. From these measurements find the distance AB. *Ans.* 313.94

8. Two objects, A, B, are separated by an impassable swamp. A station C is selected from which distances in a straight line can be measured to each of the objects. These distances are found to be $CA = 341$ ft. 7 in., $CB = 237$ ft. 5 in., and the angle ACB is found to be $53° 11'$. Find the distance AB. *Ans.* 275.4

9. Two objects, A, B, are separated by a building. To determine the direction of the line joining them, a point C is taken from which both A and B are visible and the distances $AC = 200$ ft., $BC = 137$ ft. 9 in., and the angle $ACB = 52° 25'$ are measured. Determine the angle which AB makes with AC. Also the distance AB. *Ans.* $43° 15'.9$, 159.27

10. To determine the distance between two objects A and B, a base line $CD = 350$ ft. in the same plane as A and B is measured, and the angles $BCD = 40° \ 42'$, $ACB = 30° \ 30'$, $ADB = 51° \ 12'$, $ADC = 32° 41'$, are observed. Find the distance AB.

Ans. 273.4

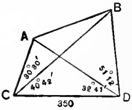


Fig. 55.

48. Case III. Given the Three Sides.

There is one and only one solution, provided the sum of any two of the given sides is greater than the third side.

The law of cosines, applied to the side opposite the required angle, will always give a solution; and if the sides are small, or if only one angle is required, it is often the best method.

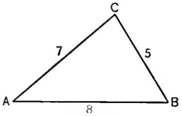


FIG. 56.

EXAMPLE 1. Find the angles of the triangle whose sides are 5, 7, 8.

By the law of cosines:

$$5^2 = 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cos A,$$
$$7^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos B,$$
$$8^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos C,$$

whence

$$\cos A = \tfrac{11}{13} = .84615^{+}, \quad \cos B = \tfrac{1}{2}, \quad \cos C = \tfrac{1}{7} = .14286^{-}$$

Hence

$$A = 32^\circ\, 12'.3, \quad B = 60^\circ, \quad C = 81^\circ\, 47'.2$$

CHECK: $A + B + C = 179^\circ\, 59'.5$

EXAMPLE 2. The sides of a triangle are 2431, 3124, and 2314. Find the largest angle.

$$\overline{3124}^2 = \overline{2314}^2 + \overline{2431}^2 - 2(2314)(2431) \cos \alpha,$$

$$\cos \alpha = \frac{\overline{2314}^2 + \overline{2431}^2 - \overline{3124}^2}{2(2314)(2431)}.$$

Call the numerator x and the denominator y. Then the solution may be carried out by logarithms as follows:

$$\begin{array}{rr}
\log 2314 = & 3.36436 \\
 & 2 \\ \hline
 & 6.72872
\end{array}
\qquad
\begin{array}{rr}
\log 2431 = & 3.38578 \\
 & 2 \\ \hline
 & 6.77156
\end{array}
\qquad
\begin{array}{rr}
\log 3124 = & 3.49471 \\
 & 2 \\ \hline
 & 6.98942
\end{array}$$

$$\begin{array}{rr}
\overline{2314}^2 = & 5354500 \\
\overline{2431}^2 = & 5909600 \\ \hline
 & 11264100 \\
\overline{3124}^2 = & 9759250 \\ \hline
x = & 1504850
\end{array}
\qquad\qquad
\begin{array}{rr}
\log 2 = & 0.30103 \\
\log 2314 = & 3.36436 \\
\log 2431 = & 3.38578 \\ \hline
\log y = & 7.05117
\end{array}$$

$$\begin{array}{rr}
\log x = & 6.17750 \\
\log y = & 7.05117 \\ \hline
\log \cos \alpha = & 9.12633 - 10
\end{array}
\qquad \therefore\ \alpha = 82^\circ\, 18'.8$$

49. Logarithmic Solution of Case III. To compute by the aid of logarithms the three angles of a triangle whose sides are known, we first find the radius of the inscribed circle by the formula of § 43:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

and then compute the angles by the formulas of § 42:

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

Example. Find the angles of the triangle whose sides are 2314, 2431, and 3124.

The work may be arranged as follows

$$\begin{array}{rlrl} a &= 2314 & s &= 3934.5 \\ b &= 2431 & s-a &= 1620.5 \\ c &= \underline{3124} & s-b &= 1503.5 \\ 2s &= 7869 & s-c &= \underline{810.5} \\ & & 2s &= 7869.0 \ (\textit{Check}) \end{array}$$

Computation of log r

$$\begin{array}{rl} \text{colog } s &= 6.40512 - 10 \\ \log(s-a) &= 3.20965 \\ \log(s-b) &= 3.17710 \\ \log(s-c) &= \underline{2.90875} \\ \log r^2 &= 5.70062 \end{array}$$

$$\begin{array}{rl} \log r &= 2.85031 \\ \log(s-a) &= \underline{3.20965} \\ \log\tan\tfrac{1}{2}A &= 9.64066 - 10 \\ \tfrac{1}{2}A &= 23^\circ\, 36'.8 \end{array} \qquad \begin{array}{rl} \log r &= 2.85031 \\ \log(s-b) &= \underline{3.17710} \\ \log\tan\tfrac{1}{2}B &= 9.67321 - 10 \\ \tfrac{1}{2}B &= 25^\circ\, 13'.8 \end{array}$$

$$\begin{array}{rl} \log r &= 2.85031 \\ \log(s-c) &= \underline{2.90875} \\ \log\tan\tfrac{1}{2}C &= 9.94156 - 10 \\ \tfrac{1}{2}C &= 41^\circ\, 9'.4 \end{array}$$

Then $A = 47^\circ\, 13'.6$, $B = 50^\circ\, 27'.6$, $C = 82^\circ\, 18'.8$

Check: $\tfrac{1}{2}(A + B + C) = 90^\circ\, 00'.0$

EXERCISES XVII. — CASE III

1. In each of the following triangles, the three sides are given. Find the smallest angle.

(a) 1, 2, 3. *Ans.* 0°
(b) 3, 5, 7. *Ans.* 68° 12′.8
(c) 3, 4, 5. *Ans.* 36° 52′.2
(d) 13, 14, 15. *Ans.* 53° 7′.8
(e) 35, 41, 47. *Ans.* 46° 15′.1
(f) 4.7, 5.1, 5.8 *Ans.* 50° 35 .3
(g) 48.3, 53.2, 62.7 *Ans.* 48° 24′.4
(h) 1.9, 3.4, 4.9 *Ans.* 16° 25′.6
(i) 32.1, 36.1, 40.2 *Ans.* 49° 24′.0
(j) 5.29, 6.41, 7.02 *Ans.* 46° 7′.0

2. Solve each of the following triangles, using logarithms:

(a) $a = 22.2$, $b = 31.82$, $c = 40.64$
Ans. 32° 54′.6, 51° 8′.8, 95° 56′.6

(b) $a = 27.53$, $b = 18.93$, $c = 30.14$
Ans. 63° 31′, 37° 59′.1, 78° 29′.9

(c) $a = 523.8,\quad b = 566.2,\quad c = 938.4$
Ans. $29°\,17'.3,\ 31°\,55'.5,\ 118°\,47'.3$

(d) $l = 3.171,\quad m = 5.331,\quad n = 5.101$
Ans. $35°\,18'.3,\ 76°\,18'.6,\ 68°\,23'.1$

(e) $u = 40.04,\quad v = 50.56,\quad w = 70.12$
Ans. $34°\,7'.2,\ 45°\,5'.9,\ 100°\,46'.8$

(f) $p = 38.2,\quad b = 45.36,\quad d = 26.54$
Ans. $57°\,14'.7,\ 87°,\ 35°\,45'.2$

(g) $m = .126,\quad n = .3226,\quad c = .253$
Ans. $21°\,11,\ 112°\,17'.8,\ 46°\,31'.2$

(h) $a = .0506,\quad b = .1234,\quad c = .0936$
Ans. $21°\,56',\ 114°\,21'.4,\ 43°\,42'.6$

(i) $u = 167,\quad v = 321,\quad w = 231.$
Ans. $29°\,56'.4,\ 106°\,24'.3,\ 43°\,39'.3$

(j) $u = 196.1,\quad v = 264.1,\quad w = 135.4$ *Ans.* $46°\,3'.6,\ 29°\,48'.8$

3. Find the angle subtended by a rod 16.2 ft. long at the observer's eye, which is 11.9 ft. from one end and 17.6 ft. from the other.
Ans. $73°\,44'$.

4. To determine without an instrument for measuring angles the angle between two lines meeting at C, the distances $CA = 500$ ft. and $CB = 700$ ft. are measured; AB is then found to be 633 ft. Find $\angle ACB$.
Ans. About 61°.

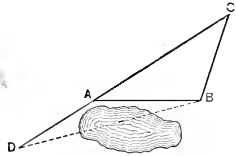


Fig. 57.

5. A piece of land is bounded by three intersecting streets, on which the property has a frontage of 312 ft., 472 ft., and 511 ft. respectively. Find the angles at which these streets cross.
Ans. $64°\,28'.4,\ 77°\,40'.4,\ 37°\,51'.4$

6. In Fig. 57 $AB = 316.8$ ft., $BC = 226.4$ ft., $AC = 431.6$ ft., and $AD = 280.4$ ft. Find BD.
Ans. 576.1

50. Case IV. The Ambiguous Case. Here we have given two sides and the angle opposite one of them; *i.e.* an angle, a side adjacent, and the side opposite.

The number of solutions (two, one, or none) is best determined by the geometrical construction of the triangle from the data.

Construct an angle AGQ, equal to the given angle which we shall at first suppose to be acute; on one of its sides lay off GA equal to the given adjacent side and drop a perpendicular AP, to the other side GQ. Then with A as center and with a radius equal to the given opposite side draw an arc.

If, as in Fig. 58 (a), this arc does not reach GQ, there is *no solution;* if it is tangent to GQ, as in Fig. 58 (b), there is *one solution;* if it cuts GQ twice, as in Fig. 58 (c), there are *two solutions;* if it cuts GQ once, as in Fig. 58 (d), there is *one solution;* and finally if the given angle is obtuse, there is *no solution* when the radius of the arc is less than GA and *one solution* when it is greater.

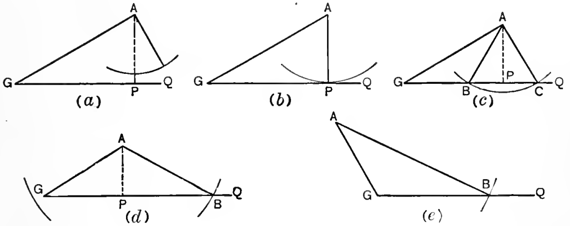


FIG. 58.

The results may be collected for reference as follows: Let G = the given angle, (adj.) = the given adjacent side, (opp.) = the given opposite side; then

I. *When G is acute, compute $p = (adj.) \sin G$; then if $(opp.) < p$ there is no solution; if $(opp.) = p$, one solution; if $p < (opp.) < (adj.)$, two solutions; and if $(opp.) > (adj.)$ one solution.*

II. *When G is right or obtuse, if $(opp.) \lesseqgtr (adj.)$, there is no solution but if $(opp.) > (adj.)$, one solution.*

The practical method, however, in the case of any given problem is *to construct the triangle approximately to scale.*

Having determined the number of solutions, the unknown parts can be computed by the law of sines.

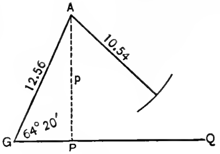


FIG. 59.

EXAMPLE 1. Two sides are 12.56 and 10.54 and the angle opposite the latter is $64° 20'$. Solve the triangle.

Construct the angle $G = 64° 20'$ and lay off $GA = 12.56$ and draw AP. A glance at the tables (p. 34) shows that $\sin G > .9$, whence $p > .9 \times 12.56 > 11$. Therefore, no solution.

EXAMPLE 2. In the triangle ABC, $a = 301.35$, $c = 352.11$, and $A = 33° 17'$. Determine the remaining parts.

Construct angle $A = 33° 17'$, lay off $AB = 352.11$, and draw BP.

Without any tables whatever, we know that $\sin 33° 17' < .7$ and therefore $p < .7 \times 360 < 260$, and therefore there are two solutions.

$$\frac{\sin C}{\sin 33° 17'} = \frac{352.11}{301.35}.$$

$$\begin{aligned} \log \sin 33° 17' &= 9.73940 - 10 \\ \log 352.11 &= 2.54668 \\ \text{colog } 301.35 &= 7.52093 - 10 \\ \hline \log \sin C &= 9.80701 - 10 \\ C &= 39° 53'.0 \end{aligned}$$

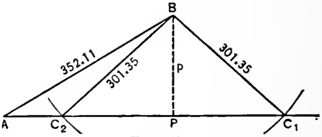


FIG 60.

There are *two* angles less than 180° having a given *sine;* therefore $C_1 = 39° 53'$ and $C_2 = 140° 7'$.

From this point on we have to solve *two* distinct triangles, viz.: ABC_1 and ABC_2. Call AC_1, b_1, and AC_2, b_2, angle ABC_1, B_1 and angle ABC_2, B_2. Then $B_1 = 106° 50'$ and $B_2 = 6° 36'$.

$$\frac{b_1}{301.35} = \frac{\sin 106° 50'}{\sin 33° 17'}.$$

$$\begin{aligned} \log 301.35 &= 2.47907 \\ \log \cos 16° 50' &= 9.98098 - 10 \\ \text{colog } \sin 33° 17' &= 0.26060 \\ \hline \log b_1 &= 2.72065 \\ b_1 &= 525.59 \end{aligned}$$

$$\frac{b_2}{301.35} = \frac{\sin 6° 36'}{\sin 33° 17'}.$$

$$\begin{aligned} \log 301.35 &= 2.47907 \\ \log \sin 6° 36' &= 9.06046 - 10 \\ \text{colog } \sin 33° 17' &= 0.26060 \\ \hline \log b_2 &= 1.80013 \\ b_2 &= 63.114 \end{aligned}$$

EXAMPLE 3. Two sides of a triangle are 5 and 7 and the angle opposite the latter is 120°. Solve the triangle.

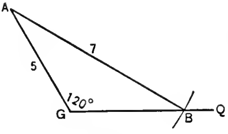


FIG. 61.

Construct the angle $G = 120°$, lay off $GA = 5$. It is at once obvious that a circle center at A, of radius 7, will cut GQ once and only once, at B.

Let the student complete the solution, finding by the law of sines, angle B and the side GB. *Ans.* $38° 12'.8$, 3.

EXERCISES XVIII. — CASE IV

Solve each of the following triangles, using logarithms; if two solutions exist, obtain both of them.

1. $a = 17.16$, $b = 14.15$, $B = 42°$. *Ans.* $83° 45'.7$, 21.022

2. $a = 54$, $b = 48.6$, $A = 31° 14'$. *Ans.* $120° 56'.9$, 89.314

3. $u = 971$, $v = 1191$, $U = 51° 15'$. *Ans.* $55° 41'.8$, 1028.5

4. $l = 281$, $m = 152$, $L = 103°$. *Ans.* $45° 11'.6$, 204.61

5. $b = 13.12$, $c = 7.22$, $B = 39° 54'$. *Ans.* $20° 40'.2$, 17.814

6. $p = 48$, $q = 36.1$, $Q = 45° 50'$. *Ans.* $61° 39'.5$, 44.293

7. $m = 10.08$, $n = 5.82$, $M = 21° 31'$. *Ans.* $146° 15'.4$, 15.264

8. $t = 93.99$, $s = 91.97$, $T = 120° 35'$. *Ans.* $2° 1'.3$, 3.85

9. $a = 309$, $b = 360$, $A = 21° 14'.4$ *Ans.* $133° 47'.7$, 615.67

10. $k = 91.06$, $m = 77.04$, $K = 51° 9'.1$ *Ans.* $87° 37'.9$, 116.82

11. One diagonal of a parallelogram is 68 ft. long and makes an angle of $30° 20'$ with the other diagonal; one side is 22 ft. long. Find the length of the other side. *Ans.* 48.107 or 74.450

12. In a certain town the streets intersect at an angle of $82° 14'$. It is desired to know the distance between two objects, A and B, which lie on a line parallel to one set of streets and which are separated by a large building. A line $AC = 200$ ft. is measured along a side line parallel to the other set of streets, and $CB = 222$ ft. is then measured. Determine AB. *Ans.* 127.09

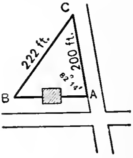


Fig. 62.

13. The pilot of a ship S sees a lighthouse H on the shore; by measuring the angle of elevation of the top of the lighthouse, and knowing its height, he determines that it is 8950 ft. from his ship. At the ship an angle of $2° 40'$ is subtended by a line connecting the lighthouse with a light L on the shore known to be 575 ft. from the lighthouse. Find the angle SLH and thus determine exactly the position of the ship with reference to the shore. Practically, how may he tell which of the two possible solutions is actually correct? *Ans.* $46° 24'$, $133° 36'$.

14. Suppose a, b, and A are given; let x represent the third side. Apply the law of cosines to side a and determine under what conditions the resulting equation in x will have (1) no real root, (2) one positive real root, (3) two positive real roots. Consider separately the two cases when A is acute and when A is obtuse and compare results with the statements of § 50, p. 68.

CHAPTER VIII

AREAS—APPLICATIONS—PROBLEMS

51. Areas of Triangles. It is shown in plane geometry that the area A,* of a triangle is equal to one-half the product of any side and the altitude from the opposite vertex.

(1) *The area of a triangle is equal to one-half the product of the base and altitude.*

52. Area from Two Sides and the Included Angle. If we have two sides and the included angle, a, b, and C, and drop

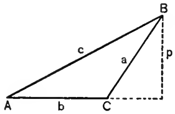


FIG. 63.

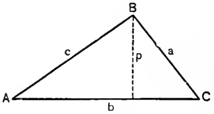


FIG. 64.

a perpendicular upon one of the given sides, as p upon b, then $p = a \sin C$ and by (1) $\mathbf{A} = \frac{1}{2} b (a \sin C)$; whence

(2) *The area of a triangle is equal to one-half the product of any two sides into the sine of their included angle.*

53. Area from Three Sides. If the three sides are given, draw lines from the vertices to the center of the inscribed circle dividing the triangle into three triangles having a common altitude, r. By (12), § 43,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

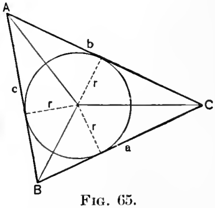


FIG. 65.

* The area is denoted by the boldface type A in distinction from the angle A.

The sum of the bases of the three triangles is $a + b + c = 2\,s$. Therefore their combined area is, by (1),

$$(3) \qquad \mathsf{A} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

Hence we have the rule:

(3) *Add the three sides and take half the sum; from the half sum subtract the three sides severally; take the product of the half sum and the three remainders and extract the square root.*

54. Illustrative Examples. The area is most conveniently found in other cases by solving the triangle sufficiently to secure the data required by one of the three rules given above, all of which are adapted to logarithmic computation.

EXAMPLE 1. One side of a triangle is 50, the angle opposite is 10° 12′, and another angle is 46° 36′. Find the area.

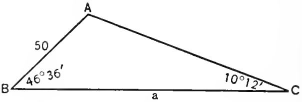


FIG. 66.

The third angle, A, Fig. 66, is then 123° 12′. If we knew a or b, we should know two sides and the included angle. By the law of sines,

$$\frac{a}{50} = \frac{\sin 123^\circ\, 12'}{\sin 10^\circ\, 12'}.$$

By rule (2) the area,

$$\mathsf{A} = \tfrac{1}{2} \cdot 50 \cdot a \cdot \sin 46^\circ\, 36',$$

$$\begin{aligned} \log 50 &= 1.69897 \\ \log \cos 33^\circ\, 12' &= 9.92260 - 10 \\ \text{colog} \sin 10^\circ\, 12' &= 0.75182 \\ \hline \log a &= 2.37339 \\ \log 25 &= 1.39794 \\ \log \sin 46^\circ\, 36' &= 9.86128 - 10 \\ \hline \log \mathsf{A} &= 3.63261 \\ \mathsf{A} &= 4291.5 \end{aligned}$$

EXAMPLE 2. Two sides of a triangle are 35 and 50 and the angle opposite the latter is 28° 30′. Find the area.

On constructing the triangle, Fig. 67, it is evident that there is only one solution and B is acute. By the law of sines,

$$\frac{\sin B}{\sin 28^\circ\, 30'} = \frac{35}{50}.$$

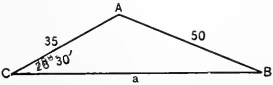


FIG. 67.

$$\begin{aligned} \log \sin 28^\circ\, 30' &= 9.67866 - 10 \\ \log 35 &= 1.54407 \\ \text{colog}\, 50 &= 8.30103 - 10 \\ \hline \log \sin B &= 9.52376 - 10 \\ B &= 19^\circ\, 30'.7 \\ \text{whence } A &= 131^\circ\, 59'.3 \end{aligned}$$

We now know two sides and the included angle and

$A = \frac{1}{2} \cdot 50 \cdot 35 \cdot \sin 131° 59'.3$	$\log 25 = 1.39794$
	$\log 35 = 1.54407$
	$\log \cos 41° 59'.3 = 9.87115 - 10$
$A = 650.37$	$\log A = 2.81316$

EXAMPLE 3. The sides of a triangle are 13, 37, and 40. Find the area. Using (3), we have,

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

and the computation may be made as follows:

	$s = 45$	$\log = 1.65321$
$a = 13$	$s - a = 32$	$\log = 1.50515$
$b = 37$	$s - b = 8$	$\log = 0.90309$
$c = 40$	$s - c = 5$	$\log = 0.69897$
$2s = 90$	*Check* $= 90$	$\log A = 2.38021$
		$A = 2400$

EXERCISES XIX. — AREAS

Find the area of the following triangles:

1. $a = 829,\quad b = 592,\quad C = 62°.$ *Ans.* 216,661.
2. $a = 713,\quad b = 987,\quad c = 1255.$ *Ans.* 351,105.
3. $B = 25°,\quad C = 68°,\quad b = 392.$ *Ans.* 168,331.
4. $p = 231,\quad q = 195,\quad P = 47°.$ *Ans.* 22,440.
5. $u = 8,\quad v = 5,\quad W = 60°.$ *Ans.* 17.32
6. $k = 72.3,\quad K = 52° 35,\quad M = 63° 17'.$ *Ans.* 2648.7
7. $l = .582,\quad m = .601,\quad n = .427$ *Ans.* 0.11765
8. $b = 21.5,\quad c = 30.456,\quad D = 41° 22'.$ *Ans.* 216.37
9. $u = 41,\quad v = 401,\quad w = 408.$ *Ans.* 8160.
10. $p = 62.4,\quad q = 20.5,\quad r = 44.5$ *Ans.* 262.08
11. $A = 60°,\quad b = 30,\quad a = 70.$ *Ans.* 1039.23
12. $a = 78.35,\quad B = 34° 22',\quad C = 66° 11'.$ *Ans.* 1613.3
13. $p = 26.6,\quad q = 35.2,\quad R = 73°.$ *Ans.* 447.7

14. Find the area of a triangular field having one of its sides 15 rods in length, and the two adjacent angles, respectively, 70° and 69° 40′. *Ans.* 153.16

15. The area of a triangular plat of ground is one acre. Two of its sides are 127 yd. and 150 yd., respectively. Find the angle between them. *Ans.* 30° 32′.4

16. The length of the bisector of one of the acute angles of an isosceles right triangle is 4. Find the area. *Ans.* 4.

55. Composition and Resolution of Forces and Velocities. We saw in § 27 that forces and velocities may be represented graphically by straight line segments. The length of such a segment represents the magnitude of the force or velocity, and its direction the direction of the force or velocity.

To find the effect of two simultaneous velocities, let us suppose that a body moves along a straight track with a velocity of 4 units per second and that each point of the track moves with a velocity of 3 units per second along a line making an angle of 60° with the track. What is the position of the body at the end of 1 second? To answer this question draw a segment 4 units long to represent the magnitude and direction of the velocity of the body along the track, and from the ends of this segment draw segments AC, BD, each 3 units in length and making an angle of 60° with AB to represent the magnitude and direction of the velocity of the ends of the track. The track will then take the position CD at the end of 1 second. But since the body moves along the track at the rate of 4 units per second, it will reach the point D at the end of 1 second. That is, it will reach the same point as if it had moved along the diagonal AD with a speed represented by the length of the diagonal. The velocity represented by AD is called the **resultant** of the velocities represented by AB and AC. AB and AC are called **components.** The length of AD can be computed by solving the triangle ABD.

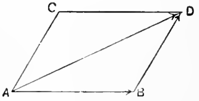


FIG. 68.

The resultant of any two velocities may be found by drawing from a common point A, segments AB, AC to represent the given velocities in magnitude and direction and then completing the parallelogram $ABCD$. The diagonal AD represents the resultant. This fact is often called the **parallelogram law.**

The resultant of two forces is found by a similar construction. This diagram is known as the **parallelogram of forces.**

56. Illustrative Examples. EXAMPLE 1. The angle between the directions of two forces of 19 lb. and 26 lb. is 54°. Find the magnitude and direction of their resultant.

The forces may be represented by segments 19 units long and 26 units long, respectively, and making the angle of 54° with each other. If the parallelogram is completed which has these segments for two of its intersecting sides, the diagonal extending from their intersection to the opposite corner will represent the resultant both in magnitude and in direction. This diagonal is a side of a triangle having two sides equal to 19 and 26, respectively, with an included angle of 126° (the supplement of 54°). Hence we can find the magnitude and direction of the resultant.

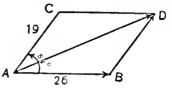


FIG. 69.

EXAMPLE 2. Two forces of 51 lb. and 73 lb. have a resultant of 80 lb. Find the angle between them.

In this case, in the parallelogram of forces, the diagonal and two intersecting sides are known; the angle opposite the diagonal is determined by Case III. The required angle is the supplement of this one.

EXAMPLE 3. A weight of 100 lb. is supported by two cords AW, 3 ft. long, and BW, 5 ft. long, attached to a horizontal beam at A and B, 7 ft. apart. Find the tensions, s in AW and t in BW.

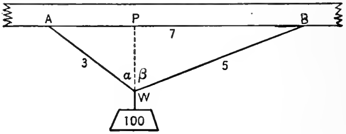


FIG. 70.

Since the weight acts vertically, we need the angles α and β which AW and BW make with the vertical WP. Solving the triangle ABW, we find $A = 38° 12'.8$, $B = 21° 47'.2$, whence $\alpha = 51° 47'.2$ and $\beta = 68° 12'.8$

To construct Fig. 71, draw AC making the angle $\alpha = 51° 47'.2$ with the vertical and similarly draw AD making $\beta = 68° 12'.8$. Take $AV = 100$ on some convenient scale and draw VE parallel to AD and VF parallel to AC. Then AE represents s and AF, t; because a force of 100 lb. acting upward at A must be the resultant of s and t since the point A is at rest. In the triangles AVE and AVF we have enough data to find $s = 104.8$ and $t = 90.7$

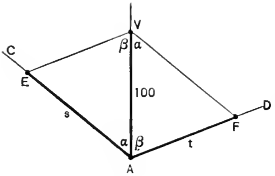


FIG. 71.

EXERCISES XX. — VECTORS

1. Solve Example 1, above. *Ans.* 40.22, $22° 28'.1$ with AB.

2. Complete the solution of Example 2. *Ans.* $101° 51'.4$

3. Compute α, β, s, and t of Example 3.

4. Check the answers to Example 3 by finding, (*a*) the sum of the vertical components of s and t; (*b*) their horizontal components.

5. Three forces of 13 lb., 22 lb., and 28 lb., respectively, are in equilibrium. Determine the angles which they make with one another. [HINT. Study Example 2.] *Ans.* $76° 45'.6$, $130° 6'.4$, $153° 7'.8$

6. Find the resultant of two forces of 30 lb. and 40 lb. acting at an angle of 60° with each other. *Ans.* 60.83

7. A ball rolls along the diagonal of the floor of a car from the back to the front with a speed of 30 ft. per second. The car is moving forward with a speed of 40 ft. per second. Find the actual speed of the ball if the car is 7 ft. wide and 30 ft. long. *Ans.* 69.55

8. Two forces are acting on a block resting on the ground as shown in the figure. What horizontal force could replace them? *Ans.* 139.85

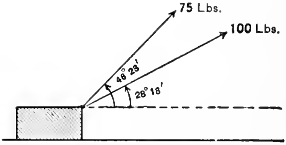


FIG. 72.

9. A point is kept at rest by forces of 6, 8, 11 lb. Find the angle between each pair. *Ans.* $77° 21'.9$, $147° 50'.6$, $134° 47'.6$

10. A boat is rowed across a river at the rate of 3.5 mi. per hour; the river flows at the rate of 4.8 mi. per hour. Find the speed of the boat and the direction of its motion. *Ans.* 5.94, $36° 6'$ with shore.

11. A ship is sailing 10 mi. per hour and a sailor climbs the mast 200 ft. high in 30 sec. Find his speed relative to the earth, and the direction of his motion. *Ans.* 966.7 ft. per min., $24° 26'.6$ with the vertical.

12. A train is going 15 mi. per hour northward; a man crosses the car eastward 12 ft. per second. Find his speed relative to the ground, and his direction. *Ans.* 25.06 N., $28° 36'.6$ E.

13. A ball rolling along the floor 10 ft. per second is struck so that its speed is increased 2 ft. per second, and the direction of motion is changed 45°. What speed and direction of motion is due to the stroke alone? *Ans.* 8.6 ft. per second, $80° 30'$.

14. A river flows 4 mi. per hour, and a motor boat goes 6 mi. per hour. In what direction must the boat be pointed to go straight across the river, and what will be its speed? *Ans.* $63° 36'.7$, 8.06 mi. per hour.

15. An oarsman rows his boat due north 5 miles an hour. There is a breeze of 12 miles an hour from the southeast. Determine the resulting speed and direction of the boat if the resistance of the water damps the effect of the wind one-third. *Ans.* 6.03 mi. per hour, N. 27° 58′ W.

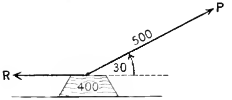


FIG. 73.

16. A weight of 400 lb. is drawn along the ground by a force of 500 lb. attached as shown in Fig. 73. What pressure does it exert on the ground? If the resisting force R (due to friction) is 1 % of the pressure on the ground, what resultant force is effective in moving the weight forward?

Ans. 150 lb., 408 lb.

EXERCISES XXI. — MISCELLANEOUS PROBLEMS

1. Solve the following triangles:

(a)	$a = 10.34$,	$B = 5^\circ\ 7'.6$,	$C = 19^\circ\ 49'$.
(b)	$a = 36.423$,	$b = 14.578$,	$C = 68^\circ\ 14'$.
(c)	$l = 14.236$,	$m = 13.761$,	$N = 45^\circ\ 11'$.
(d)	$a = 734.34$,	$B = 108^\circ\ 6'$,	$C = 61^\circ\ 7'$.
(e)	$u = 32.19$,	$v = 69.182$,	$U = 69^\circ\ 17'$.
(f)	$a = .75632$,	$b = .62751$,	$C = 84^\circ\ 48'$.
(g)	$c = 454.72$,	$J = 11^\circ\ 11'$,	$C = 57^\circ\ 37'$.
(h)	$a = 474.17$,	$b = 1008.8$,	$c = 940.25$
(i)	$a = 100.37$,	$c = 95.376$,	$B = 100^\circ\ 58'$.
(j)	$d = 391.68$,	$D = 25^\circ\ 36'$,	$B = 68^\circ\ 13'$.
(k)	$a = 622.02$,	$b = 293.22$,	$A = 100^\circ$.
(l)	$u = 375.64$,	$v = 438.79$,	$w = 133.94$
(m)	$a = .010231$,	$c = .0047233$,	$A = 44^\circ\ 58'$.
(n)	$a = 476.53$,	$P = 40^\circ\ 17'$,	$A = 39^\circ\ 14'$.
(o)	$b = 94.961$,	$a = 88.234$,	$c = 12^\circ$.
(p)	$b = .43124$,	$a = .53467$,	$A = 99^\circ\ 59'$.

Answers to the preceding exercises.

(a) 2.1909, 8.3119
(b) 88° 11′.2, 23° 34′.8, 33.844
(c) 69° 44′.6, 65° 4′.4, 10.764
(d) 3730.7, 3436.7
(e) No solution.
(f) 53° 25′.2, 41° 46′.8, 0.93795
(g) 104.43, 502.02
(h) 27° 52′.6, 84° 7′.7, 67° 59′.7
(i) 40° 43′.3, 38° 18′.7, 151.04
(j) 904.48, 841.76
(k) 27° 39′.6, 52° 20′.4, 500.01
(l) 53° 49′.8, 109° 26′.4, 16° 43′.8
(m) 115° 59′.5, 19° 2′.5, 0.013013
(n) 487.13, 740.85
(o) 64° 44′.5, 103° 15′.5, 20.284
(p) 52° 35′.5, 27° 25′.5, 0.25005

2. A pole 17 ft. high has a mark 8 ft. 4 in. from the ground. Find the angle subtended by each part at a point 20 in. from the ground and 53 ft. 4 in. from the pole. *Ans.* $8° 54'.9$

3. The diagonals of a parallelogram are 22 ft. and 31 ft., and the angle between them is $51° 12'$. Determine the sides of the parallelogram. *Ans.* 23.977, 12.148

4. A biplane is observed from the ground and from an upper window of a building 60 ft. directly above. The angles of elevation are found to be $10° 42'$ and $9° 58'$. Find the distance from each point to the airship. *Ans.* 4606.4, 4617.2

5. Two sides of a triangle are 63 and 81, and the included angle is $54°$. Find the length of the bisector of the largest angle. *Ans.* 51.015

6. The sides of a triangle are 22, 35, 44. Find the length of the median to the longest side. *Ans.* 20.5

7. Two sides of a triangle are 7.2 and 8.1 and the angle opposite the latter is $32° 41'$. Find the radius of the circumscribed circle. *Ans.* 15.

8. The three sides of a triangle are 26, 28, and 30. Find the radius of the inscribed circle. *Ans.* 8.

9. The angles of a triangle are to each other as $1:2:3$; the altitude upon the longest side is 45. Find the sides. *Ans.* 90, 51.96, 103.92

10. The sides of a triangle are to each other as $2:3:4$. Find the angles. *Ans.* $28° 57'.3$, $46° 34'$, $104° 28'.7$

11. To determine the distance between two objects A and B that have a barrier between them, a distance $AC = 200$ ft. is measured to a point C, from which both objects are visible. The distance $BC = 321$ ft. and the angle $ACB = 68° 41'$. Find the distance AB. *Ans.* 310.43

12. To find the distance between two objects A and B situated on opposite sides of a lake, the distance $AC = 250$ ft. and the angles $CAB = 44° 13'$, $ACB = 51° 9'$, are measured. Find AB. *Ans.* 195.55

13. An object B is wholly inaccessible and is invisible from a certain point A. To find the distance AB, two points C and D, from which B can be seen, are selected on a line through A. If $CD = 243$ ft., $CA = 102$ ft., $\angle DCB = 68° 56'$, $\angle CDB = 48° 22'$, find AB. *Ans.* 192.9

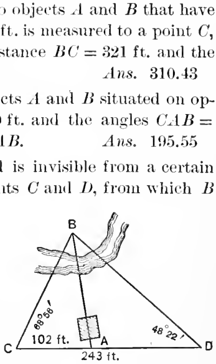


FIG. 74.

14. It is desired to know the height of an object AB. A line $CD = 250$ ft., in a horizontal plane with the base A of the object, is measured, also the angle of elevation $ACB = 13° 22'$, and the angles $DCA = 35° 37'$ and $CDA = 64° 28'$. Determine the height AB. *Ans.* 54.44

15. A tall building stands at the foot of a hill. From a point on the side of the hill the angle of depression of the base of the building is observed to be $14° 36'$, and the angle of elevation of the top is $21° 43'$. A level line from the instrument meets the building 19 ft. 7 in. above the base. Find the height of the building. *Ans.* 49.52

16. A balloon is observed, at the moment it passes over a level road, from two points in the road an eighth of a mile apart. The angles of elevation from the two points are $33° 17'$ and $42° 6'$. Find the distances of the balloon from the two observers. *Ans.* 374.31, 427.26

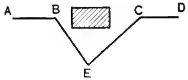


FIG. 75.

17. In surveying, it is sometimes desired to extend such a line as AB in the figure beyond an obstacle. If at B a right turn of $58°$, $BE = 126$ ft., and at E a left turn of $110°$ are laid off, compute EC, and the angle (right turn) at C. *Ans.* 135.6, 52°.

18. To find the distance PQ in Fig. 76, a base line AB is measured $= 518$ ft. At A the angles $PAQ = 43° 18'$ and $QAB = 48° 32'$ are measured and checked by measuring $PAB = 91° 50'$, and at B, $ABP = 38° 43'$, $PBQ = 41° 28'$, $ABQ = 80° 11'$. Find PQ by two methods. *Ans.* 451.39

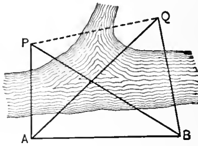


FIG. 76.

19. Find the distance AC, Fig. 77, through a thicket, having measured $AB = 20.71$ rods, $BC = 18.87$ rods, angle $ABC = 55° 12'$. *Ans.* 18.40

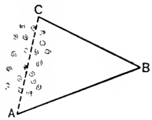


FIG. 77.

20. From two points A and B, 300 ft. apart on the deck of a ship, a second ship, S, is observed. The angles $ABS = 85° 18'$, and $BAS = 83° 47'$ are measured. What is the distance between the ships? *Ans.* 2496, 2502, av. 2499.

21. How far to the side of a target 1300 ft. away should a gunner aim from a ship going 15 mi. per hour, if the speed of the bullet is 2000 ft. per second and he fires when he is directly opposite? *Ans.* 14.3

22. From a railway train going 50 mi. per hour a bullet is fired 1000 ft. per second at an angle of $75° 28'.3$ with the track ahead. Find its speed and direction. *Ans.* $71° 29'.2$, 1020.9 ft. per second.

23. A man in a railway car going 45 mi. per hour observes the raindrops falling at an angle of $30°$ with the vertical. Assuming that the raindrops are actually falling vertically, find their speed. *Ans.* 77.9

24. The resultant of two forces is 10 lb.; one of the forces is 8 lb. and makes an angle of 36° with the resultant. Find the magnitude of the other force. *Ans.* 5.88

25. A horse pulls a canal boat by a rope which makes an angle of 25° 35′ with the tow path. What size of engine would propel the boat at the same speed? (Assume that the horse is doing one "horse power.") *Ans.* 0.9+

26. A man climbs a hill inclined (on the average) 32° with the horizontal. His pocket barometer shows that at the end of $2\frac{1}{2}$ hr. he has increased his elevation 2750 ft. Find his average speed up the slope. *Ans.* 2075.8

27. The sides of a triangular field are 82.7 rods, 91.4 rods, and 104.3 rods. Determine the area of the field and the angles between the sides. *Ans.* 226.39 A., 49° 27′.4, 57° 7′.6, 73° 25′.

28. Find the area of a triangular piece of ground having two angles, respectively, 73° 10′ and 90° 50′, and the side opposite the latter 150.6 rods. *Ans.* 18.7 A.

29. Find the areas of triangles which have the following given parts:

(*a*) $a = 116.082$,	$b = 100$,	$C = 118° 15'.7$
(*b*) $b = 100$,	$A = 76° 38'.2$,	$C = 40° 5'$.
(*c*) $u = 31.325$,	$v = 13° 57'$,	$U = 53° 11'.3$
(*d*) $a = 408$,	$b = 41$,	$c = 401$.
(*e*) $a = .9$,	$b = 1.2$,	$c = 1.5$

Ans. (*a*) 5112.1 (*b*) 3506.8 (*c*) 136.13 (*d*) 8160 (*e*) .54

30. Three circles whose radii are 2, 3, 10, respectively, are tangent externally. Find the area of the triangle formed by joining their centers. *Ans.* 30.

31. Prove that the area of the triangle formed by joining the centers of any three circles which are tangent externally is a mean proportional between the sum and the product of their radii. See § 53.

32. Prove that one-half the product of the three sides of any triangle is equal to the product of its area into the diameter of its circumscribed circle. See §§ 40 and 52.

33. Prove that the area of any triangle is equal to the product of the radii of its inscribed and circumscribed circles into the sum of the sines of its angles. See §§ 40 and 53.

PART III. THE GENERAL ANGLE

CHAPTER IX

DIRECTED ANGLES — RADIAN MEASURE

57. Directed Lines and Segments. As explained in elementary algebra, it is often convenient to select one direction on a straight line as the *positive direction;* the other is then called the *negative direction.* Thus, if two forces act along the same line, but in opposite directions, it is convenient to call one positive and the other negative.

Two segments are said to have the *same sense* if they lie on the same line or on parallel lines, and if both are positive or both are negative. Two segments are said to be of *opposite sense* if they lie on the same line or on parallel lines, and if one is positive and the other is negative.

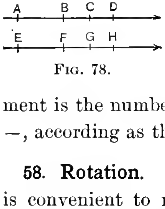


Fig. 78.

Thus, in Fig. 78, $AB = EF$, while $AC = -GE$ and $CB = -FG$.

The numerical measure of a directed segment is the number of units in its length with the sign + or −, according as the segment is positive or negative.

58. Rotation. Directed Angles. In describing rotation, it is convenient to regard angles as positive or negative in a manner analogous to that explained in § 57 for line-segments.

An angle is thought of as generated by the rotation of one of its sides about the vertex as center; its first position is called the *initial side*, the final position is called the *terminal side.* An angle generated by a rotation opposite to the motion of the hands of a clock (*counterclockwise*), is said to be *positive;*

an angle generated by a *clockwise* rotation, is said to be *negative.**

Angles may be of any magnitude, positive or negative. Thus, in Fig. 79, α, β, δ are positive angles; γ is negative; β is greater than a straight angle; and δ is greater than 360°, or a complete revolution. In rotating parts of machinery, such angles have a very vivid meaning. Thus, a wheel which rotates 370° per second has a very different speed from that of a wheel which rotates 10° per second.

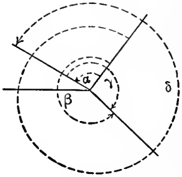


FIG. 79.

59. Placing Angles on Rectangular Axes. To place any given angle on a pair of rectangular axes in the plane of the angle, put the vertex at the origin and the initial side on the x-axis extending to the right; the terminal side will then fall in one of the four quadrants (or, if the angle is a multiple of a right angle, on one of the axes). If the terminal side falls in the first quadrant, the angle is said to be *an angle in the first quadrant*, etc. In Fig. 80, α is a positive angle in the first quadrant, β is a negative angle in the fourth quadrant, δ is a positive angle in the fourth quadrant.

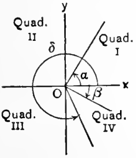


FIG. 80.

EXERCISES XXII.—DIRECTED LINES AND ANGLES

1. What angle will the minute hand of a clock generate in 2 hr. 24 min. 10 sec.?

2. A flywheel is running steadily at the rate of 450 revolutions per minute. What angle does one of its spokes generate in 2 sec.? In 1.2 sec.?

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* Either of these directions may of course be chosen as the positive direction of rotation, the other is then the negative direction. The choice here made is the customary one for angles; but in many kinds of machinery, the other sense of rotation is considered positive, as in the case of a clock.

3. Find the sum, or *resultant*, of two forces that act in the same line whose intensities (measured in pounds) are -5 and $+10$, respectively. Draw a figure to represent the solution.

4. If three forces of intensities $+7$, -15, $+2$ (lb.), respectively, act on a body in the same line, find the resultant force. Draw a figure.

5. If a man walks with a speed of 4 mi. per hour toward the rear of a train going 35 mi. per hour, find his actual speed. Draw a figure.

6. A man's gains and losses (indicated by $-$) in business in successive months are \$250, $-$\$118, \$35, \$712, $-$\$15. Find the total gain and the average gain per month. Draw a figure.

7. By means of a ruler and a protractor, construct the following angles and their sums; check by adding their numerical measures.

(*a*) $-75°$ and $125°$. (*b*) $66°$ and $-30°$. (*c*) $45°$ and $30°$, and $70°$.
(*d*) $-60°$ and $-36°$. (*e*) $485°$ and $55°$. (*f*) $-750°$ and $30°$.

8. With some two of the angles just given verify $\alpha + \beta = \beta + \alpha$.

9. (*a*) Construct $27° + 85° + (-45°) + 135°$.

(*b*) Construct $-150° + 96° + 24° + (-80°)$.

10. If a wheel is rotating 120° per second, how many revolutions does it make per minute? how many per hour? How many degrees does it turn through per minute?

11. Express an angular speed of 2.5 revolutions per second in degrees per second; in revolutions per minute; in degrees per minute.

12. A flywheel rotates at the rate of 40 revolutions per minute. Through what angle does one of its spokes turn in a second?

13. Reduce an angular speed of 3.4 revolutions per second to degrees per second; to degrees per minute; to revolutions per minute.

14. Find the angular speed of the rotation of the earth on its axis (*a*) in revolutions per minute; (*b*) in degrees per second.

15. Construct a right triangle whose sides are 3 and 4; construct an angle which is 3 times the smaller angle of this triangle.

16. Construct the following angles and place them on the axes, (*a*) $-150°$; (*b*) $285°$; (*c*) $480°$; (*d*) $570°$; (*e*) $-225°$; (*f*) $-450°$.

17. In what quadrant is each of the following angles: $459°$, $682°$, $725°$, $-100°$, $-1090°$, $\pm 85°$, $\pm 95°$, $\pm 175°$, $\pm 185°$, $\pm 265°$, $\pm 275°$, $\pm 355°$?

18. Taking $\alpha = 60°$, $\beta = -300°$, $\gamma = -50°$, $\delta = 310°$ draw a figure showing that α differs from β, and also that γ differs from δ by $360°$.

19. Find the angle between $0°$ and $360°$ which differs from each of the following angles by a multiple of $360°$:

(*a*) $-42° 13'$; (*b*) $-842°$; (*c*) $364° 23'$; (*d*) $2700°$.

60. Measurement of Angles. An angle may be named and used before it is expressed in any system of measurement. Thus, we may refer to an angle A of a right triangle whose perpendicular sides are 16 in. and 24 in., respectively; and we can compute $\tan A = 24/16 = 1.5$, etc., without measuring A in terms of any unit angle. General theorems like the law of sines remain true in any system of measurement.

The unit angle (see § 2) chiefly used in Geometry and Trigonometry is the *degree* with its subdivisions minute, tenth of minute, second, with which the student is familiar. It is often convenient to use another unit angle called the *radian*.

61. Radian Measure of Angles.* A **radian** is a positive angle such that when its vertex is placed at the center of a circle, the intercepted arc is equal in length to the radius.

This unit is thus a little less than one of the angles of an equilateral triangle; in fact it follows from the geometry of the circle, since the length of a semicircumference is πr, that

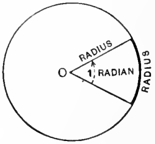

FIG. 81.

(1) π ***radians*** $= \mathbf{180°}$, where $\pi = \mathbf{3.14159}$,

whence 1 radian $= 57° \, 17' \, 44''.806$, or $57°.3$ approximately.

It is easy to change from degrees to radians and *vice versa* by means of relation (1), which should be remembered. Conversion tables for this purpose are printed in Tables, pp. 91–93.

62. Use of Radian Measure. It is shown in geometry that two angles at the center of a circle are to each as their intercepted arcs; therefore if an angle at the center is measured in radians and if the radius and the intercepted arc are measured in terms of the same linear unit, their numerical measures satisfy the simple relation:

(2) $$\textit{arc} = \textit{angle} \times \textit{radius}.$$

* Sometimes also called circular measure.

In other words, the number of linear units in the arc is equal to the product of the number of radians in the angle by the number of linear units in the radius.

EXAMPLE 1. Find the difference in latitude of two places on the same meridian 200 mi. apart, taking the radius of the earth as 4000 mi.

Angle = arc/radius = 1/20 in radians = $2° 51' 53''$, approximately.

63. Angular Speed. In a rotating body a point P, which is at a distance r from the axis of rotation, moves through a distance $2\pi r$ during each revolution or through a distance r while the body turns through an angle of one radian. Therefore if v is the linear (actual) speed of P (in linear units per time unit, *e.g.* feet per second), and if ω is the angular speed of the rotating body (in radians per time unit, *e.g.* radians per second), then their numerical measures satisfy the relation

$$v = r \cdot \omega; \tag{3}$$

hence *the angular speed of a rotating body is numerically equal to the actual speed of a point one unit from the axis of rotation.*

Engineers usually express the angular speed of the rotating parts of machinery in revolutions per minute (R. P. M.) or revolutions per second (R. P. S.). These are easily reduced to radians per minute (or per second) by remembering that one revolution equals 2π radians.

EXAMPLE 1. A flywheel of radius 2 ft. rotates at an angular speed of 2.5 R. P. S. Find the linear speed of a point on the rim.

In radians per second, $\omega = 2.5 \times 2\pi = 5\pi$, and for a point 2 ft. from the axis of rotation $v = 2 \times 5\pi = 31.416$ ft. per second.

EXAMPLE 2. Find the angular speed of a 34-inch wheel on an automobile going 20 mi. per hour.

Every time the wheel turns through a radian the car goes forward 17 in. (the length of the radius), and 20 mi. per hour = 352 in. per second; therefore the wheel turns through 352/17 = 20.7, radians per second.

64. Notation. In measuring angles in radian measure we shall adopt the practice universal in advanced work and write only the *numerical measure* of the angle in terms of the unit *one radian.* Thus in the expression $\tan x$, the letter x will denote a *number* (the numerical measure of an angle) rather than the angle itself. See § 2.

When necessary, to call attention to the fact that radian measure is intended, the symbol $(^{(r)})$ is appended to the numerical measure, thus :

$$\begin{aligned} 1^{(r)} &= 1 \text{ radian} = 57° \, 17' \, 44''.8, \\ 2^{(r)} &= 2 \text{ radians} = 114° \, 35' \, 29''.6, \\ \pi^{(r)} &= \pi \text{ radians} = 180° = 2 \text{ rt. } \measuredangle\text{s}, \\ (\pi/2)^{(r)} &= \pi/2 \text{ radians} = 90° = 1 \text{ rt. } \angle, \end{aligned}$$

and so forth.

As it happens that the acute angles whose trigonometric functions are most easily recalled without consulting tables are simple fractional parts of 180°, the number π often appears as a factor of the numerical measure of angles. In this system, for example, $\sin(\pi/2) = 1$, $\cos(\pi/3) = 1/2$, $\tan(\pi/4) = 1$, etc.

The use of pure numbers, such as 2 or π in place of an angle is precisely similar to the use of 10 for 10 feet or 10 inches in expressing lengths. The student should supply the *unit of measurement* (radians or feet or inches), and should not confuse the number π $(= 3.14159 \cdots)$ with the angle whose measure is π *radians*, as he should not confuse the number 10 with the distance 10 *feet*.

EXERCISES XXIII. — ANGULAR SPEED — RADIAN MEASURE

1. Express the following angles in degrees, minutes, and seconds:

(*a*) $\pi^{(r)}/4$; (*b*) $\pi^{(r)}/6$; (*c*) $2\pi^{(r)}/3$; (*d*) $3^{(r)}$.

2. Express the following angles in radians:

(*a*) 25°; (*b*) 30°; (*c*) 35°; (*d*) 28° 39′; (*e*) 114° 35′.

3. How far short of one revolution is $6^{(r)}$?

4. To gain ability to judge the size of angles in circular measure, express approximately (to within 1°) angles whose sizes are $1^{(r)}$, $4^{(r)}$, $5^{(r)}$, $8^{(r)}$. Draw an angle which is about your impression of an angle of $2^{(r)}$, and measure it with a protractor. Do not revise your figures.

5. If a vehicle moves at the rate of 15 ft. per second, through what angle does one of its wheels, 3 ft. in diameter, revolve in 1 sec.?

Ans. $10^{(r)}$.

6. If the linear speed of a vehicle is 30 mi. per hour, what is the angular speed of one of its wheels which is 4 ft. in diameter?

Ans. 22 radians per second.

7. A wheel 5 ft. in diameter is connected by a belt 40 ft. in length with a wheel 4 ft. in diameter. If the large wheel makes 30 revolutions per minute, how often does the seam of the belt pass this wheel? What is the angular speed of the smaller wheel?

Ans. $5\frac{1}{11}$ sec., $3\frac{13}{14}$ radians per second.

8. Find the angular distance on the earth between two points whose distance from each other, on the arc of a great circle, is 800 miles. [Take the radius of the earth to be 4000 miles.] *Ans.* $11° 27' 33''$.

9. Find the distance in miles between two points on the earth's surface whose angular distance is $1°$; between two points whose angular distance is 0.25 radians. *Ans.* 69.81, 1000.

10. Find the length of the subtended arc of an angle of 3.46 radians at the center of a circle of radius 5. *Ans.* 17.3

11. Find the length of the subtended arc of an angle of $55°$ at the center of a circle of radius 3. *Ans.* 2.8798

12. Find the angle at the center which subtends an arc of 3 ft. on a circle of radius 4 ft. Express the angle in radians and in degrees, and compare the work done in the two cases. *Ans.* $\frac{3}{4}$ radian $= 42°.97+$

13. Reduce to radian measure by means of Tables IV, p. 91:

(*a*) $23° 40'$; (*b*) $68° 45' 20''$; (*c*) $138° 35' 15''$.

Ans. 0.4130612, 1.2000109, 2.4188082

14. Reduce to degree measure by means of the Tables pp. 92–93:

(*a*) $3.46^{(r)}$; (*b*) $.256^{(r)}$; (*c*) $.0127^{(r)}$; (*d*) $8.24^{(r)}$.

Ans. $198° 14' 36''.2$, $14° 40' 3''.8$, $43' 39''.5$, $472° 7' 2''$

15. Reduce the following angular speeds to degrees per second; to revolutions per second; to revolutions per minute:

(*a*) $4.5^{(r)}$ per sec.; (*b*) $2.48^{(r)}$ per sec.; (*c*) $10.54^{(r)}$ per sec.

Ans. (*a*) 257.83, 0.7162, 42.972; (*b*) 142.09, 0.3947, 23.682; (*c*) 603.90, 1.6775, 100.65

CHAPTER X

FUNCTIONS OF ANY ANGLE

65. Resolution of Forces. Projections. In § 26, p. 34, we saw how to find the components of a force, or a velocity, on any line, as the projection of the force on that line; and we saw that the components of a force F on each of two perpendicular axes, even when the angle α is obtuse, are

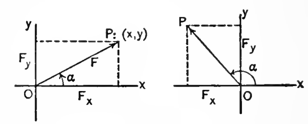

FIG. 82.

(1) $F_x = \text{Proj}_x\, F = F \cos \alpha,$
$F_y = \text{Proj}_y\, F = F \sin \alpha.$

If several forces occur in the same problem, some of them may make an angle α greater than 180° with the positive direction OX. It is convenient to *define* $\cos \alpha$ and $\sin \alpha$ for angles greater than 180° so that the equations (1) remain true. If we do so, the projection on the two axes of any directed segment of length r joining the origin O to a point P are

$$(2) \qquad x = \text{Proj}_x\, r = r \cos \alpha, \qquad y = \text{Proj}_y\, r = r \sin \alpha,$$

where α is the angle between the positive direction OX and the positive direction OP, and may be an angle of any size, positive or negative. Hence the desired definitions are:

$$(3) \qquad \cos \alpha = \frac{x}{r}, \qquad \sin \alpha = \frac{y}{r}.$$

These definitions are consistent with those already given, §§ 11, 35, for the sine and the cosine; *i.e.* in case $0° \leqq \alpha \leqq 180°$, they determine the same values as the earlier definitions.

66. General Definitions. Trigonometric Functions of Any Angle. The definitions of $\sin \alpha$ and $\cos \alpha$ given in § 65 have, of course, no necessary dependence upon forces. Each is a number which depends only on the magnitude and sign of the angle. A purely geometric definition of these and of the other trigonometric functions of any angle α, consistent with the definitions of §§ 11, 35, and with the fundamental relations between them, such as $\tan\alpha = \sin\alpha/\cos\alpha$, $\sin^2\alpha + \cos^2\alpha = 1$, the reciprocal relations, etc., may be made as follows:

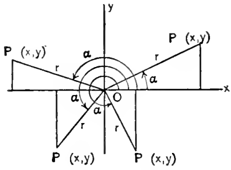

Fig. 83.

Place the given angle on a pair of rectangular axes, and select any point P whose coördinates are (x, y) on the terminal side at a distance $r > 0$ from the origin. Then

(4) $\sin\alpha = \dfrac{y}{r} = \dfrac{\text{ordinate}}{\text{radius}},$

(5) $\cos\alpha = \dfrac{x}{r} = \dfrac{\text{abscissa}}{\text{radius}},$

(6) $\tan\alpha = \dfrac{y}{x} = \dfrac{\text{ordinate}}{\text{abscissa}}\cdot$ provided $x \neq 0$; *

(7) $\operatorname{ctn}\alpha = \dfrac{x}{y} = \dfrac{\text{abscissa}}{\text{ordinate}},$ provided $y \neq 0$;

(8) $\sec\alpha = \dfrac{r}{x} = \dfrac{\text{radius}}{\text{abscissa}},$ provided $x \neq 0$;

(9) $\csc\alpha = \dfrac{r}{y} = \dfrac{\text{radius}}{\text{ordinate}},$ provided $y \neq 0$.

Three additional functions sometimes used are:

(10) The versed sine of α: $\operatorname{vers}\alpha = 1 - \cos\alpha$.

(11) The haversine of α: $\operatorname{hav}\alpha = \frac{1}{2}(1 - \cos\alpha)$.

(12) The external secant of α: $\operatorname{exsec}\alpha = \sec\alpha - 1$.

and also the coversed sine of $\alpha = 1 - \sin\alpha$.

* The exceptions noted are based on the general principle that a fractional expression does not represent a number if its denominator is zero.

By these definitions every angle has a sine and a cosine, because in the ratios y/r and x/r the denominator r is never zero. There is no secant or tangent* for 90°, or for 270°, or for any angle whose terminal side coincides with either the positive or negative end of the y-axis, because the denominator x in the ratios r/x, y/x, is zero. Similarly, there is no cosecant or cotangent for 0° or for 180°, or for any angle whose terminal side coincides with the positive or negative end of the x-axis. There exists a tangent, cotangent, secant, and cosecant for every angle except those just mentioned.

If two angles differ by any multiple of 360° it is evident that any one of the trigonometric functions will have the same value for both of them because the initial sides of the two angles (when placed on the axes) will coincide, and also their terminal sides. It follows that for a point P on the common terminal side the values of x, y, and r are the same for both angles; hence the ratio which defines any given function will be the same for both angles.

For example: $\sin(-295°) = \sin 65°$, $\cos(-315°) = \cos 45°$, $\tan 1476° = \tan 36°$, $\sin(\theta - 180°) = \sin(180° + \theta)$, $\cos(x - 90°) = \cos(270° + x)$, $\tan(360° - y) = \tan(-y)$.

67. Algebraic Signs of Trigonometric Functions. The sine of any angle in the first or second quadrant is positive, because the ordinate of any point above the x-axis is positive; the sine of any angle in the third or fourth quadrant is negative, because the ordinate of any point below the x-axis is negative.

The cosine of any angle in the first or fourth quadrant is positive, because the abscissa of any point to the right of the

* To say that 90° has no tangent does not mean that the tangent of 90° is zero. When we say that an article has no value we mean that it has a value and that value is zero. Not so here. Since the general definition of tangent does not apply to 90°, we could, if we found it convenient, define tan 90°, but we do not; we leave it undefined. Often it is said $\tan 90° = \infty$, but this does not mean that 90° has a tangent; it means that as an angle α increases from 0° to 90°, $\tan \alpha$ increases without limit, *and that before α reaches 90°.*

y-axis is positive; similarly, the cosine of any angle in the second or third quadrant is negative.

Similarly, the signs of $\tan\alpha$, $\operatorname{ctn}\alpha$, $\sec\alpha$, $\csc\alpha$, etc., may be determined directly from a figure; they are as follows:

QUADRANT	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\operatorname{ctn}\alpha$	$\sec\alpha$	$\csc\alpha$
1st	+	+	+	+	+	+
2d	+	−	−	−	−	+
3d	−	−	+	+	−	−
4th	−	+	−	−	+	−

NOTE. (1) $\tan\alpha$ is positive (negative) when $\sin\alpha$ and $\cos\alpha$ have like (unlike) signs; (2) reciprocals have the same sign.

EXERCISES XXIV.—FUNCTIONS OF THE GENERAL ANGLE

1. By placing the angles on the axes, show from the definitions that

(a) $\sin 225° = -\sqrt{2}/2$, $\cos 225° = -\sqrt{2}/2$.

(b) $\sin 150° = 1/2$, $\cos 150° = -\sqrt{3}/2$.

(c) $\sin 330° = -1/2$, $\cos 330° = \sqrt{3}/2$.

(d) $\sin(-315°) = \sqrt{2}/2$, $\cos(-315°) = \sqrt{2}/2$.

(e) $\sin(-1020°) = \sqrt{3}/2$, $\cos(-1020°) = 1/2$.

(f) $\sin 180° = 0$, $\sin(n \cdot 180°) = 0$; for $n = \pm 1, \pm 2, \pm 3, \cdots$.

(g) $\cos 90° = 0$, $\cos[(2n-1)90°] = 0$; for $n = \pm 1, \pm 2, \pm 3, \cdots$.

2. Which of the following are positive and which negative? $\sin 72°$, $\sin 352°$, $\sin 850°$, $\tan 128°$, $\sec 260°$, $\sin(-20°)$, $\cos(-380°)$, $\sin(-260°)$, $\cos 160°$, $\operatorname{ctn} 280°$, $\cos 33°$, $\csc 91°$, $\cos(-40°)$, $\tan(-140°)$, $\cos(-400°)$.

3. Prove for *any angle* α that $\sin^2\alpha + \cos^2\alpha = 1$. [Use $x^2 + y^2 = r^2$.]

Prove each of the other **Pythagorean** relations for *any angle* α:

$1 + \tan^2\alpha = \sec^2\alpha$, if $\cos\alpha \neq 0$; $1 + \operatorname{ctn}^2\alpha = \csc^2\alpha$, if $\sin\alpha \neq 0$.

4. Prove that $\operatorname{ctn}\alpha$, $\sec\alpha$, $\csc\alpha$ are the reciprocals of $\tan\alpha$, $\cos\alpha$, $\sin\alpha$, respectively, for all values of α for which both are defined.

5. (a) Prove that the sine of any angle in the first or second quadrant is between 0 and 1. (b) Prove that the cosine of any angle in the 1st or 4th quadrant is between 0 and 1.

6. Prove that if an angle is not an odd multiple of a right angle its sine is between -1 and $+1$; and conversely. For what angles is $\sin\alpha = +1$; $\sin\alpha = -1$; $\cos\alpha = +1$?

7. Show that $\tan \alpha = \sin \alpha / \cos \alpha$ for all values of α, if $\cos \alpha \neq 0$.

8. Show that $\tan \alpha$ and $\operatorname{ctn} \alpha$ may have any values whatever.

9. Show that $\operatorname{vers} \alpha$ and $\operatorname{hav} \alpha$ are always positive or zero.

10. If an angle α starts at 0° and gradually increases to 360°, show that the behavior of $\sin \alpha$ and $\cos \alpha$ will be as indicated in this table:

α	0°	$0^\circ <\ < 90^\circ$	90°	$90^\circ < \alpha < 180^\circ$	180°	$180^\circ < \alpha < 270^\circ$	270°	$270^\circ < \alpha < 360^\circ$	360°
$\sin \alpha$	0	increases to	1	decreases to	0	decreases to	-1	increases to	0
$\cos \alpha$	1	decreases to	0	decreases to	-1	increases to	0	increases to	1

11. By placing the angles indicated on rectangular axes determine the numbers to fill the blanks in the following table:

α	30°	45°	60°	120°	135°	150°	210°	225°	240°	300°	315°	330°
$\sin \alpha$												
$\cos \alpha$												

12. Assuming that the sun passes directly overhead, trace the change in the length of the shadow of an object from dawn to sunset. Which trigonometric function do you think of in this problem?

13. Assuming the results of Exs. 10 and 11, derive from them the variation of the tangent from 0° to 360° and its values at each of the angles mentioned in Ex. 11. Do the same for $\operatorname{ctn} \alpha$, $\sec \alpha$, $\csc \alpha$.

68. Reading of Tables. Sine and Cosine of $-\theta$ and $90^\circ + \theta$. In order to find the value of any one of the trigonometric functions of a given angle we consult the tables. In the tables the values of the different functions are printed only up to 45°. To find the sine of an acute angle greater than 45° we make use of the relation $\sin \alpha = \cos (90^\circ - \alpha)$. The tables are arranged to facilitate this by having the angles above 45° printed at the bottom of the page, and the column headings changed from sine to cosine, etc. (See *Tables*, p. 22.)

If we wish to find the sine of an angle greater than 90°, we must find a way to express the sine in terms of some function of an acute angle. We proceed to find expressions for the values of the sine and the cosine of the angles $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, and $360° - \theta$.

To construct these angles we draw a circle of radius r with its center at the origin and draw the diameters HN, KS making the angle θ with the y-axis to the right and left, and also the diameters PM, TL making the angle θ with the x-axis above and below. The angles XOH, XOK; XOL, XOM; XON, XOS; and XOT, are the angles mentioned above, and XOP is the angle θ. Denote the coördinates of the point P by (a, b); then because the triangle OAH is congruent to the triangle OCP the coördinates of the point H are (b, a), and in the same way the coördinates of the points K, L, M, N, S, T are easily seen to be as indicated in the figure. We are now able to read off the values of the trigonometric functions of the various angles from the figure, in terms of a, b, and r;

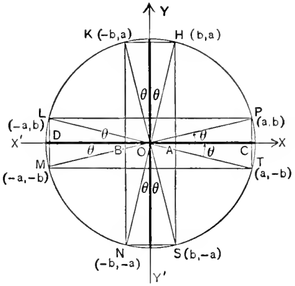

FIG. 84.

thus $\sin\theta = b/r,$

$$\cos(90° - \theta) = b/r, \qquad \cos(90° + \theta) = -b/r,$$
$$\sin(180° - \theta) = b/r, \qquad \sin(180° + \theta) = -b/r,$$
$$\cos(270° - \theta) = -b/r, \qquad \cos(270° + \theta) = b/r,$$
$$\sin(360° - \theta) = \sin(-\theta) = -b/r.$$

Hence we have

$$\cos(90° - \theta) = \sin\theta \qquad \cos(90° + \theta) = -\sin\theta$$
$$\sin(180° - \theta) = \sin\theta \qquad \sin(180° + \theta) = -\sin\theta$$
$$\cos(270° + \theta) = \sin\theta \qquad \cos(270° - \theta) = -\sin\theta$$
$$\sin(360° - \theta) = \sin(-\theta) = -\sin\theta$$

Similarly we obtain from the figure $\cos\theta = a/r$,

$$\sin(90° - \theta) = a/r, \qquad \sin(90° + \theta) = a/r,$$
$$\cos(180° - \theta) = -a/r, \qquad \cos(180° + \theta) = -a/r,$$
$$\sin(270° - \theta) = -a/r, \qquad \sin(270° + \theta) = -a/r,$$
$$\cos(360° - \theta) = \cos(-\theta) = a/r.$$

Whence,

$$\sin(90° - \theta) = \cos\theta \qquad \cos(180° - \theta) = -\cos\theta$$
$$\sin(90° + \theta) = \cos\theta \qquad \cos(180° + \theta) = -\cos\theta$$
$$\cos(360° - \theta) = \cos(-\theta) \qquad \sin(270° - \theta) = -\cos\theta$$
$$= \cos\theta \qquad \sin(270° + \theta) = -\cos\theta$$

These formulas together with the fact mentioned in § 66, that a function of an angle α has the same value as the same function of any angle that differs from α by a multiple of 360°, are sufficient to enable one to find the value of any one of the functions of any angle from the tables.*

EXAMPLE 1. Find the sine of 793° 22′.

The angle 73° 22′ differs from the given angle by 720°, which is a multiple of 360°; hence the required value is the same as sin 73° 22′. From the tables this value is found to be .95816

69. Solution of Trigonometric Equations. We are now able to give the general solutions of the equations $\sin\theta = c$ and $\cos\theta = c$ where c is any number lying between $+1$ and -1. In the first place, it is clear that there are two and only two angles between 0° and 360° which will satisfy either of these equations. For in the above figure there are only two points of the circle for which x has a given value between $+r$ and $-r$, and likewise, only two points for which y has a given value between $+r$ and $-r$; a radius drawn to either of these two points will be the terminal side of an angle between 0° and 360°, satisfying the first equation if y is chosen so that $y/r = c$ and satisfying the second if x is chosen so that $x/r = c$. To obtain the general solution we add or subtract any whole

* The proofs given above are for the case in which θ is an acute positive angle. The formulas, however, are true for *any value* of θ whatever.

multiple of 360° to either of the solutions just found. The solutions which lie between 0° and 360° can be found from the tables by means of the formulas given above.

70. Illustrative Examples on Composition and Resolution of Forces.

EXAMPLE 1. Find the components, R_x, R_y, of the resultant of two forces, the first of 12 lb. acting at an angle of 30° with the horizontal, the second of 20 lb. acting at an angle 60° with the horizontal.

SOLUTION. To solve this we make use of the principle that the projection *on any line* of the resultant of any number of forces is the algebraic sum of the projections of the component forces.

By equations (1), § 65, the horizontal component of the first is $12 \cos 30°$, and of the second, $20 \cos 60°$: hence

$$R_x = 12 \cos 30° + 20 \cos 60° = 10.392 + 10.000 = 20.392$$

In a similar manner we find

$$R_y = 12 \sin 30° + 20 \sin 60° = 6.000 + 17.320 = 23.320$$

We can easily find the magnitude of the resultant from the equation

$$R^2 = R^2_x + R^2_y = (20.392)^2 + (23.320)^2 = 959.665$$

Hence

$$R = \sqrt{(959.665)} = 30.979$$

The direction of the resultant is given by the equation

$$\tan \theta = R_y \div R_x = 23.320 \div 20.392 = 1.1436$$

Hence

$$\theta = 48° 50'.$$

EXAMPLE 2. Find the magnitude and the direction of the resultant of the two forces $F = (17, 128°)$, $G = (24, 213°)$.

[NOTE. The notation (24, 213°) means a force of magnitude 24 acting at an angle of 213° with the positive x-axis.]

The method of solution is the same as in Example 1; we find

$$F_x = 17 \cos 128° = -17 \sin 38° \qquad \text{(by § 68)}.$$
$$G_x = 24 \cos 213° = -24 \cos 33° \qquad \text{(by § 68)}.$$

Hence

$$R_x = -17 \sin 38° - 24 \cos 33° = -10.466 - 20.128 = -30.594$$

Similarly we obtain

$$R_y = 17 \cos 38° - 24 \sin 33° = 13.396 - 13.071 = .325$$
$$R = \sqrt{(R^2_x + R^2_y)} = 30.611$$

$$\theta = \arctan (.325/-30.594) = \arctan (-.01062) = 180° - 36'.4 = 179° 23'.6$$

EXERCISES XXV. — READING OF TABLES — REDUCTION TO FUNCTIONS OF ACUTE ANGLES

1. Express the following as functions of acute angles not greater than 45°. Make use of congruent angles whenever advantageous:

(*a*) $\sin 150° 21'$. (*b*) $\cos 125° 15'$. (*c*) $\tan 283° 45'$.
(*d*) $\text{ctn}\,(-36° 16')$. (*e*) $\sec 460°$. (*f*) $\csc\,(-210° 20')$.
(*g*) $\sin\,(-943° 24')$. (*h*) $\cos 551° 23'$. (*i*) $\tan\,(-546° 28')$.

2. From the tables find the values of the following logarithms:

(*a*) $\log\,(-\cos 161° 11')$. (*b*) $\log \sin 161° 11'$.
(*c*) $\log\,(-\sin 217° 17')$. (*d*) $\log\,(-\cos 252° 48')$.

[Note that the numbers in parentheses in (*a*), (*c*), and (*d*) are positive; if the minus sign were absent, each of them would be negative. Negative numbers have no real logarithms.]

3. Compute the values of the following expressions by logarithms:

(*a*) $2.35 \sin 148° 23'$. (*b*) $24.8 \cos 160° 40'$. (*c*) $16.2 \cos 320° 45'$.

4. Solve the following trigonometric equations:

$$(a)\quad \cos^2 t - \sin^2 t = \sin t.$$

Solution. In this equation $\cos^2 t$ may be replaced by its equal $1 - \sin^2 t$; the equation then becomes a quadratic in $\sin t$, viz.: $2\sin^2 t + \sin t - 1 = 0$. This equation is *equivalent* to the given one; *i.e.* every solution of either is a solution of the other. The solutions may now be found by factoring:

$$(2\sin t - 1)(\sin t + 1) = 0.$$

Hence we have either $\sin t + 1 = 0$, whence $\sin t = -1$, and $t = 270°$ or $t = 270° + k\,360°$; or else $2\sin t = 1$, whence $\sin t = 1/2$ and $t = 30° + k\,360°$ or $t = 150° + k\,360°$. There are no other solutions.

(*b*) $2\sin^2 x - \cos x = 1$. (*g*) $\sec^2 x + \tan x = 3$.
(*c*) $\cos^2 x = \sin^2 x$. (*h*) $4\sec^2 x + \tan x = 7$.
(*d*) $\cos 2x + 5\sin x = 3$. (*i*) $\tan x + \text{ctn}\, x = 2$.
(*e*) $\cos 2x - \sin x = 1/2$. (*j*) $\sin x + 3 = \csc x$.
(*f*) $5\sin x + 2\cos^2 x = 5$. (*k*) $\sin 2x \cos x = \sin x$.

5. Find the resultant (R, θ) of three forces (100, 350°), (150, 490°), (200, 720°), where (F, α) indicates a force of magnitude F and direction α.

6. Find the components on the axes of a force of magnitude 5.74 lb. which makes an angle of 215° 20′ with the positive end of the x-axis.

7. Find the magnitude and the direction of a force whose components on two perpendicular axes are $F_x = 25.46$, $F_y = 38.72$

CHAPTER XI

THE ADDITION FORMULAS

71. The Addition Formulas. In the reduction of certain trigonometric expressions to simpler or more convenient forms it is sometimes desirable to express a trigonometric function of the sum or difference of two angles in terms of functions of the separate angles forming the sum or difference. Without reflection the student might think that $\sin(\alpha+\beta)$ would be equal to $\sin\alpha+\sin\beta$ by analogy with the formula $\frac{1}{2}(a+b) = \frac{1}{2}a+\frac{1}{2}b$, but a trial of one or two special cases will show this is not always true; thus, $\sin(60°+30°)$ is equal to one, but $\sin 60° + \sin 30°$ is equal to $\frac{1}{2}\sqrt{3}+\frac{1}{2}$, which is greater than one. In order to find the correct formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ we make use of the theory of directed quantities as explained in §§ 26, 55, 57, 58, and 65.

Suppose a force of magnitude A makes an angle α with the positive x-axis, while another force of magnitude B makes an angle $\alpha+90°$ with this axis; then the resultant R of A and B is represented by the diagonal OP of the rectangle of which A and B are two sides. The y-component, R_y of this resultant is

$$(1)\quad R_y = A\sin\alpha + B\sin(\alpha+90°) = A\sin\alpha + B\cos\alpha.$$

Similarly, the x-component of R is

$$(2)\quad R_x = A\cos\alpha + B\cos(\alpha+90°) = A\cos\alpha - B\sin\alpha.$$

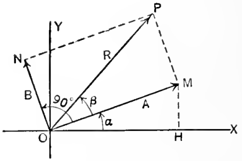

FIG. 85.

Now by § 65,

$$(3)\qquad R_x = R\cos(\alpha+\beta),\qquad R_y = R\sin(\alpha+\beta),$$

where β is the angle between A and the resultant R.

Inserting these values in formulas (1) and (2) we find

(4) $$R \sin(\alpha+\beta) = A \sin\alpha + B\cos\alpha.$$

(5) $$R\cos(\alpha+\beta) = A\cos\alpha - B\sin\alpha.$$

Moreover, from the figure, $A = R\cos\beta$, $B = R\sin\beta$.

Substituting these values in (4) and (5) and dividing through by R we finally obtain the formulas

(6) $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

(7) $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

It should be carefully noticed that, although in the figure the angles α and β are acute angles, the proof does not at all depend on this fact. Formulas (6) and (7) are therefore true for all values of the angles α and β.

72. The Subtraction Formulas. It can be shown in a manner exactly similar to the preceding that we have also

(8) $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

(9) $$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

It is easy to derive (8) and (9) directly from (6) and (7), however. Thus, if, in (6), we replace β by $-\beta$ we obtain

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta),$$

or, since by § 68, $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

which is (8). We prove (9) in a similar manner from (7).

These formulas are also true for all values of the angles α and β. They are examples of trigonometric identities involving two angles.

73. Reduction of $A\cos\alpha \pm B\sin\alpha$. Such expressions as $A\cos\alpha \pm B\sin\alpha$ which appeared in formulas (1) and (2) of the previous article arise in various connections; for example, a combination of two vibrations gives rise to such a form.

It is possible, and often convenient, to reduce such expressions to the product of a single number, and the sine (or the cosine) of the sum of two angles. The method depends on formulas (6) and (7) and upon the fact that any two numbers are proportional to the sine and the cosine of some angle.

EXAMPLE 1. Express $3\cos\alpha + 4\sin\alpha$ in the form $k\sin(\alpha+\beta)$.

To solve this we first find an angle whose sine and cosine are proportional to 3 and 4. We may clearly choose an angle β so that $\sin\beta = \frac{3}{5}$, and $\cos\beta = \frac{4}{5}$; hence we may write

$$\begin{aligned} 3\cos\alpha + 4\sin\alpha &= 5(\tfrac{3}{5}\cos\alpha + \tfrac{4}{5}\sin\alpha) \\ &= 5(\sin\beta\cos\alpha + \cos\beta\sin\alpha). \end{aligned}$$

Hence by formula (6) we have

$$3\cos\alpha + 4\sin\alpha = 5\sin(\beta+\alpha).$$

From the tables $\beta = 36° 52'$.

EXERCISES XXVI. — ADDITION FORMULAS

1. Given $\sin\alpha = 3/5$, $\sin\beta = 5/13$; find $\sin(\alpha+\beta)$.

(a) When α and β are both acute; (b) when α and β are both obtuse.

2. Find $\sin(45° + x)$, $\cos(45° + x)$, $\sin(30° + x)$, $\cos(30° + x)$ in terms of $\sin x$ and $\cos x$.

3. Given that x and y are both obtuse angles and that $\sin x = 1/2$, $\sin y = 1/3$; find $\sin(x+y)$ and $\cos(x+y)$.

4. Use the addition formulas to express $\sin(90° + \alpha)$ and $\cos(90° + \alpha)$ in terms of $\sin\alpha$ and $\cos\alpha$.

5. Prove that $\sin(60° + x) - \cos(30° + x) = \sin x$.

6. Express $\sin(\alpha+\beta+\theta)$ in terms of sines and cosines of α, β, and θ.

[HINT. Let $\phi = \alpha + \beta$ and obtain $\sin(\phi+\theta)$; then replace ϕ by its value, $\alpha+\beta$.]

7. Express $\cos(\alpha+\beta+\theta)$ in terms of sines and cosines of α, β, and θ.

8. Reduce the combination of two simple harmonic motions $5\cos t - 12\sin t$ to the form $r\cos(t+\theta)$.

9. Reduce $3\sin t + 4\cos t$ to the form $r\sin(t+\theta)$.

10. Reduce each of the following to the product of a number and the sine or the cosine of a single angle:

(a) $\sin x - 2\cos x$.

(b) $3\cos y - 4\sin y$.

(c) $5\cos\theta + 12\sin\theta$.

(d) $3\sin t - 3\cos t$.

(e) $\sqrt{3}\cos x - \sin x$.

(f) $\sin y + .5\cos y$.

(g) $.7\cos\theta - \sin\theta$.

(h) $.55667\sin c + .5\cos c$.

11. Given two forces of intensities 2 and 3 that make angles of 30° and 120°, respectively, with the positive x-axis; find the horizontal and the vertical components of their resultant without finding the resultant itself; find the same quantities by using the resultant.

12. Given $.56 \sin c + .5 \cos c = -.34$, find an angle θ, and a number r, such that $.56 \sin c + .5 \cos c = r \sin (c + \theta)$, by means of § 70. Then, from $r \sin (c + \theta) = -.34$, find $\sin (c + \theta)$, and therefore (from the Tables) find $c + \theta$. Hence find c.

74. Double Angles. Since formulas (6) and (7), § 71, are true for all angles, they hold when $\alpha = a$, any angle whatever, and $\beta = a$, the same angle; hence,

$$\sin (a + a) = \sin a \cos a + \cos a \sin a,$$

and $$\cos (a + a) = \cos a \cos a - \sin a \sin a.$$

Therefore the following formulas hold for any angle whatever:

(10) $$\sin 2\alpha = 2 \sin \alpha \cos \alpha;$$

(11) $$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha;$$

or, since $\sin^2 a + \cos^2 a = 1$,

(12) $$\cos 2\alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1.$$

75. Tangent of a Sum or of a Difference. Since formulas (6) and (7) hold for all values of α and β, the formula

$$\frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

holds good for all values of α and β except those which make $\cos (\alpha + \beta) = 0$, *i.e.* except when $\alpha + \beta = 90°$, or $270°$, or an angle that differs from one of these by an integral number of times 360°. For example, it does not hold for $\alpha = 47°$, $\beta = 43°$.

Dividing both numerator and denominator by $\cos \alpha \cos \beta$, we obtain the formula

(13) $$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta},$$

which holds for all angles α and β such that α, β, and $\alpha + \beta$ have tangents.

Similarly from formulas (8) and (9), we obtain

(14) $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta},$$

which holds for all angles α and β such that α, β, and $\alpha - \beta$ have tangents.

From formulas (10) and (11) we find

(15) $$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha},$$

which holds for every angle α such that α and 2α have tangents. The same formula may be obtained directly from (13) by putting α in place of β.

76. Applications. The formulas of this chapter are frequently used for reducing expressions whose values are to be calculated, to a form in which logarithms may be used.

Example. Suppose the height of an object CD is to be determined and that it is not convenient to measure a base line bearing directly toward the base C. The following method is then sometimes employed. The angle of elevation α is measured from some convenient point A; a line $AB = d$ is then measured at right angles to the line AC; finally the angle of elevation, β, is observed from B. The height h can then be determined by solving a succession of triangles. With the aid of the formulas of this chapter it is frequently possible in such cases to reduce the calculation to a single logarithmic computation.

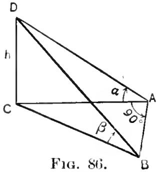

Fig. 86.

In the case just mentioned we have

$$BC = h \operatorname{ctn}\beta \qquad AC = h \operatorname{ctn}\alpha,$$

$$\begin{aligned} d^2 &= \overline{BC}^2 - \overline{AC}^2 = h^2(\operatorname{ctn}^2\beta - \operatorname{ctn}^2\alpha) \\ &= h^2(\operatorname{ctn}\beta - \operatorname{ctn}\alpha)(\operatorname{ctn}\beta + \operatorname{ctn}\alpha) \\ &= h^2\frac{(\sin\alpha\cos\beta - \cos\alpha\sin\beta)(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}{\sin^2\alpha\sin^2\beta}; \end{aligned}$$

hence, using formulas (6) and (8), we have

$$h = \frac{d\sin\alpha\sin\beta}{\sqrt{\sin(\alpha - \beta)\sin(\alpha + \beta)}}.$$

Let the student show, by opening a book and studying the dihedral angle formed by two leaves, that $\alpha > \beta$.

EXERCISES XXVII. — SECONDARY FORMULAS — APPLICATIONS

1. Find $\sin 15°$, $\cos 15°$, $\tan 15°$ from the known values of $\sin 30°$, $\cos 30°$, $\tan 30°$, and $\sin 45°$, $\cos 45°$, $\tan 45°$. [HINT. $15° = 45° - 30°$.]

2. Find $\tan 75°$, $\tan 105°$, $\sin 165°$, $\cos 255°$. [HINT. $75° = 45° + 30°$.]

3. Given $\sin 36° 52' = .6$; find the sine, cosine, and tangent of $66° 52'$; find $\sin 73° 44'$.

4. Given $\tan 26° 34' = .5$; find sine, cosine, tangent of $71° 34'$; find $\tan 53° 8'$.

5. Given $\sin \alpha = 5/13$ and $90° < \alpha < 180°$; $\cos \beta = 8/17$ and $0° < \beta < 90°$; find $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$, $\tan(\alpha + \beta)$, $\sin 2\alpha$, $\cos 2\beta$.

6. Given $\tan \alpha = 15/8$ and $0° < \alpha < 90°$; $\cos \beta = 4/5$ and $270° < \beta < 360°$; find $\sin(\alpha - \beta)$, $\cos(\beta - \alpha)$, $\tan 2\alpha$, $\cos 2\beta$.

7. Given $\sin \alpha = 1/3$ and $90° < \alpha < 180°$; find $\sin(135° - \alpha)$ and $\tan 2\alpha$.

8. The angular elevation of an object from an upper window is observed to be α. The angular elevation from a point on the ground h feet directly beneath the window is β. Show that the height of the object is $h \sin \beta \cos \alpha \div \sin(\beta - \alpha)$.

9. To determine the difference in elevation of two stations, a flagstaff of known height h is held at the upper of two stations and the angles of elevation of its top and bottom are observed to be α and β, respectively. Show that the difference in elevation of the two stations is $h \tan \beta \div (\tan \alpha - \tan \beta)$; reduce this expression to a form convenient for logarithmic computation.

10. A tree leans directly toward two points of observation distant a and b, respectively, from its foot. The angles of elevation of the top of the tree from these two points are α and β. Show that the perpendicular height of the tree is $(b - a) \div (\cot \beta - \cot \alpha)$; reduce this expression to a form suitable for logarithmic computation.

11. Prove that $\sin 3\alpha = \sin \alpha (3 - 4 \sin^2 \alpha) = \sin \alpha (4 \cos^2 \alpha - 1)$, and state for what values of α it holds. Use formulas (6) and (7).

12. Prove that $\cos 3\alpha = \cos \alpha (4 \cos^2 \alpha - 3) = \cos \alpha (1 - 4 \sin^2 \alpha)$, and state for what values of α it holds. Use formulas (6) and (7).

13. Prove that $\tan 3\alpha = \dfrac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$ and show that it holds for all values of α such that α and 3α have tangents.

14. Prove that $\sin(45° + \alpha) \sin(45° - \alpha) = 1/2 \cos 2\alpha$ for all values of α.

15. Prove that $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$ for all values of α and β.

16. Prove that $\cos(\alpha + \beta) \cos \beta + \sin(\alpha + \beta) \sin \beta = \cos \alpha$.

77. Functions of Half Angles. The formulas

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

and

$$\cos^2 \alpha - \sin^2 \alpha = \cos 2\alpha$$

are true for all values of α. If we subtract one of these from the other, and if we also add them, we obtain the formulas:

$$2 \sin^2 \alpha = 1 - \cos 2\alpha, \tag{16}$$

$$2 \cos^2 \alpha = 1 + \cos 2\alpha. \tag{17}$$

These formulas are true for all values of α; for $\alpha = \alpha'/2$ they become

$$2 \sin^2 (\alpha'/2) = 1 - \cos \alpha'$$

and

$$2 \cos^2 (\alpha'/2) = 1 + \cos \alpha',$$

or since these are true for all values of α', we may write

$$\sin (\alpha/2) = \pm\sqrt{\frac{1 - \cos \alpha}{2}}, \tag{18}$$

$$\cos (\alpha/2) = \pm\sqrt{\frac{1 + \cos \alpha}{2}}, \tag{19}$$

which hold good for all values of α. The same formulas may be obtained from (12) by solving for $\sin (\alpha'/2)$, or for $\cos (\alpha'/2)$, after putting $\alpha'/2$ for α.

From (18) and (19) we get by division

$$\tan \alpha/2 = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}, \tag{20}$$

which hold for all values of α except when a denominator vanishes. The ambiguity of sign of the radical is determined in a given case by the fact that $\tan (\alpha/2)$ is positive or negative according as $\alpha/2$ is or is not in the first or second quadrant.

The relations between an angle and its half are frequently useful in problems that relate to a chord of a circle and the angle which it subtends at the center; this occurs, for example, in laying out railroad curves where it is convenient to make measurements along chords of the curve. This is illustrated in some of the exercises below. The relations are also useful in simplifying trigonometric expressions and in adapting formulas to logarithmic computation.

EXERCISES XXVIII. — HALF-ANGLE FORMULAS

1. Find the sine, the cosine, and the tangent of $22° 30'$ from the known values of $\sin 45°$, $\cos 45°$, $\tan 45°$.

2. Find the sine, cosine, and tangent of $15°$.

3. Given that $\sin \alpha = 4/5$, and that α is an acute angle; find $\sin (\alpha/2)$ and $\tan (\alpha/2)$.

4. Given $\tan 26° 34' = 1/2$; find $\tan 13° 17'$.

5. Given $\tan 36° 52' = 3/4$; find sine, cosine, and tangent of $18° 26'$.

6. If r denotes the radius of the circle in the accompanying figure, c a chord, and θ the angle which c subtends at the center; show that $\sin (\theta/2) = c/(2\, r)$.

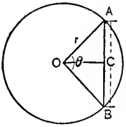

Fig. 87.

7. In the figure, draw the line BD tangent to the circle, and AD perpendicular to BD from the opposite end of the chord BA. Show that (a) $\angle ABD = \theta/2$; (b) $BD = AB \cos (\theta/2) = 2\, r \sin (\theta/2) \cos (\theta/2) = r \sin \theta$.

8. Prove that $\tan (45° + \alpha/2) = \sec \alpha + \tan \alpha$, if $\tan \alpha$ exists.

9. Prove that $\tan (45° + \alpha/2) \tan (45° - \alpha/2) = \tan 45°$ if $\tan \alpha$ exists.

10. Prove that $\tan (\alpha/2) + 2 \sin^2 (\alpha/2) \operatorname{ctn} \alpha = \sin \alpha$, if $\sin \alpha \neq 0$.

11. Prove that $\tan (\alpha/2) + \operatorname{ctn} (\alpha/2) = 2 \csc \alpha$, if $\sin \alpha \neq 0$.

12. Prove that $[\sin (\alpha/2) + \cos (\alpha/2)]^2 = 1 + \sin \alpha$ for all values of α.

13. Prove that $[\sin (\alpha/2) - \cos (\alpha/2)]^2 = 1 - \sin \alpha$ for all values of α.

14. In the figure, COA is a diameter of a circle of radius r; $AOP = \alpha$ is any acute angle; $OCP = \alpha/2$, by geometry; and PB is perpendicular to OA. Show that

$$OB = r \cos \alpha, \quad BP = r \sin \alpha, \quad BA = r \operatorname{vers} \alpha, \quad CB = r(1 + \cos \alpha),$$

$$CP = \sqrt{\overline{PB}^2 + \overline{CB}^2} = r\sqrt{2(1 + \cos \alpha)}.$$

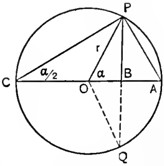

Fig. 88.

15. From Ex. 14, show that the functions of $\alpha/2$ can be read directly from the figure in the form:

$$\sin(\alpha/2)=\frac{r\sin\alpha}{r\sqrt{2(1+\cos\alpha)}}=\sqrt{\frac{1-\cos\alpha}{2}};$$

$$\cos(\alpha/2)=\frac{1+\cos\alpha}{\sqrt{2(1+\cos\alpha)}}=\sqrt{\frac{1+\cos\alpha}{2}};$$

$$\tan(\alpha/2)=\frac{\sin\alpha}{1+\cos\alpha}=\frac{\sqrt{1-\cos^2\alpha}}{1+\cos\alpha}=\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{1-\cos\alpha}{\sin\alpha}.$$

16. If a numerical value of any function of α is given, all the other functions of α and of $\alpha/2$ can be found geometrically from Ex. 14. Thus, if $\sin\alpha=4/5$ is given, lay off $OP=5$, $BP=4$; then $OB=\sqrt{5^2-4^2}=3$. Hence, $CB=8$, $BA=2$; and $CP=\sqrt{\overline{CB}^2+\overline{BP}^2}=\sqrt{8^2+4^2}=\sqrt{80}$. It follows that

$$\sin\alpha=4/5,\quad \cos\alpha=3/5,\quad \tan\alpha=4/3,$$
$$\sin(\alpha/2)=4/\sqrt{80}=1/\sqrt{5}=\sqrt{5}/5,$$
$$\cos(\alpha/2)=8/\sqrt{80}=2/\sqrt{5}=2\sqrt{5}/5,$$
$$\tan(\alpha/2)=4/8=1/2.$$

17. Find the remaining functions of α and those of $\alpha/2$ by means of Ex. 16, if $\cos\alpha=5/13$; if $\tan\alpha=1/3$.

18. The remaining functions of $(\alpha/2)$ and those of α can be found when any function of $\alpha/2$ is given from the figure of Ex. 14, by dropping a perpendicular from O to CP. Do this if $\tan(\alpha/2)=3/4$.

19. Since, in the figure of Ex. 14, by geometry $\overline{BP}^2=CB\cdot BA$, show that

$$(1+\cos\alpha)\operatorname{vers}\alpha=\sin^2\alpha.$$

20. Derive trigonometric formulas from the geometric identities (Ex. 14):

$$BP\cdot PA=\overline{AB}^2,\qquad BP\cdot CP=\overline{CB}^2.$$

78. Factor Formulas. In adapting trigonometric formulas to logarithmic computation it is often desirable to express the sum (or difference) of two sines (or cosines) as the product of other functions.

Example 1. Reduce $\sin 35^\circ+\sin 15^\circ$ to the form $2\sin 25^\circ\cos 10^\circ$.

To do this, set $\quad x+y=35^\circ,\quad x-y=15^\circ,$
and solve for x and y: $\quad x=25^\circ,\ y=10^\circ.$
Then
$$\sin(x+y)=\sin x\cos y+\cos x\sin y,$$
$$\sin(x-y)=\sin x\cos y-\cos x\sin y;$$
whence, adding, $\sin(x+y)+\sin(x-y)=2\sin x\cos y$;
substituting $x=25^\circ$, $y=10^\circ$, we get $\sin 35^\circ+\sin 15^\circ=2\sin 25^\circ\cos 10^\circ$.

EXAMPLE 2. Reduce $\sin s - \sin(s-c)$ to a product, where
$$s=(a+b+c)/2.$$

Let $x+y=s$, $x-y=s-c$; then $x=(a+b)/2$, $y=c/2$, and
$$\sin(x+y)=\sin x\cos y+\cos x\sin y,$$
$$\sin(x-y)=\sin x\cos y-\cos x\sin y;$$
subtracting $\quad \sin(x+y)-\sin(x-y)=2\cos x\sin y,$
whence $\quad \sin s-\sin(s-c)=2\cos[(a+b)/2]\sin(c/2).$

EXERCISES XXIX. — FACTORING

1. Reduce each of the following forms to products:

(*a*) $\sin 70^\circ - \sin 10^\circ$. (*b*) $\sin 70^\circ + \sin 50^\circ$.
(*c*) $\sin 13^\circ + \sin 41^\circ$. (*d*) $\sin 34^\circ - \sin 19^\circ$.
(*e*) $\cos 26^\circ - \cos 35^\circ$. (*f*) $\sin 43^\circ + \sin 28^\circ$.
(*g*) $\cos 20^\circ + \cos 10^\circ$. (*h*) $\cos 51^\circ - \sin 11^\circ$.
(*i*) $\dfrac{\sin 15^\circ + \cos 45^\circ}{\cos 45^\circ - \sin 15^\circ}$. (*j*) $\dfrac{\sin 28^\circ + \sin 12^\circ}{\cos 28^\circ + \cos 12^\circ}$.
(*k*) $\dfrac{\sin 64^\circ + \sin 16^\circ}{\sin 64^\circ - \sin 16^\circ}$. (*l*) $\dfrac{\sin 80^\circ - \sin 40^\circ}{\cos 40^\circ - \cos 80^\circ}$.

2. Prove that $\cos(x+y)+\cos(x-y)=2\cos x\cos y$.

3. Prove that $\cos(x+y)-\cos(x-y)=-2\sin x\sin y$.

4. Prove that
$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$
by substituting $A=x+y$, $B=x-y$ in Ex. 2.

5. Prove by means of Ex. 3 that
$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

6. By the method of Example 1, § 78, show that
$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

7. By the method of Example 2, § 78, show that
$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

8. Prove $\dfrac{\sin x+\sin y}{\sin x-\sin y}=\tan\dfrac{x+y}{2}\operatorname{ctn}\dfrac{x-y}{2}.$

9. Prove $\dfrac{\cos x+\cos y}{\cos x-\cos y}=-\operatorname{ctn}\dfrac{x+y}{2}\operatorname{ctn}\dfrac{x-y}{2}.$

10. Prove $\dfrac{\sin\theta+\sin 2\theta}{\cos\theta-\cos 2\theta}=\operatorname{ctn}(\theta/2).$

11. Prove $\dfrac{\sin(2x-3y)+\sin 3y}{\cos(2x-3y)+\cos 3y}=\tan x.$

12. $\sin(45° + x) + \sin(45° - x) = \sqrt{2}\cos x.$

13. $\sin 3x + \sin 5x = 2\sin 4x\cos x.$

14. If $a + b + c = 2s$, show that

(*a*) $\cos(b - c) - \cos a = 2\sin(s - b)\sin(s - c)$;

(*b*) $\cos a - \cos(b + c) = 2\sin s\sin(s - a)$;

(*c*) $\dfrac{\sin s - \sin(s-c)}{\sin s + \sin(s-c)} = \dfrac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}.$

15. $\dfrac{\tan x\tan y}{\tan x - \tan y} = \dfrac{\sin x\sin y}{\sin(x-y)}.$

16. The so-called "method of offsets" for laying out a circular track is illustrated in the adjoining figure. The track OAB is tangent at O to OB', and the distances OA', $A'B'$, $A'A$, CB, are easily shown to be as marked in the figure, where $\alpha/2 = \angle AOA'$ is half the angle at the center subtended by a 100-foot chord.

In practice, the line $OA'B'$ is run, and A' and B' marked. Show that $B'B$, the distance actually to be laid off from B', is

$$B'B = A'A + CB = 200\sin\alpha\cos(\alpha/2).$$

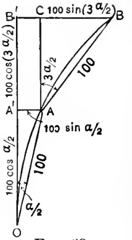

FIG. 89.

CHAPTER XII

GRAPHS OF TRIGONOMETRIC FUNCTIONS

79. Scales and Units. The graph of the function $\sin x$ is a curve passing through all points whose coördinates (x, y), satisfy the equation $y = \sin x$. The graph of any other trigonometric function as $\cos x$, $\tan x$, etc., is similarly determined.

The radian is the unit angle commonly used in plotting the graphs and in the further study of the trigonometric functions in the Calculus and in other advanced mathematical subjects. Unless otherwise specified, the equation $y = \sin x$ is understood to mean that y is the sine of x radians * as explained in § 64.

In plotting curves it is of advantage in many ways to make the horizontal and vertical scale units the same, and this should be done if not too inconvenient.†

80. Plotting Points. In Table V are given the values of the sine, cosine, and tangent of acute angles measured in radians which are very convenient for plotting the graphs of these functions on cross-section paper.

81. Graph of $\sin x$. Draw a pair of coördinate axes and choose the scale unit = 10 small divisions of the cross-section paper. Take from Table V the sines of the angles in the first quadrant for each tenth radian and tabulate:

x	0	.1	.2	.3	.4	.5	etc. . . .	1.5	1.57
$y = \sin x$	0	.099	.198	.295	.389	.479	etc.		1.000

* In any case, $y = \sin x$ means that y is the sine of x units of angle. The right angle, the 60° angle, the 45° angle, the degree, or any other angle might be chosen as the unit, if it were convenient.

† If we were to take the two scale units the same in plotting the curve $y = \sin x$ where the unit angle is the degree, one arch of the curve would be 180 units long and only 1 unit high.

Plot these points and draw a smooth curve through them as OA in the figure.

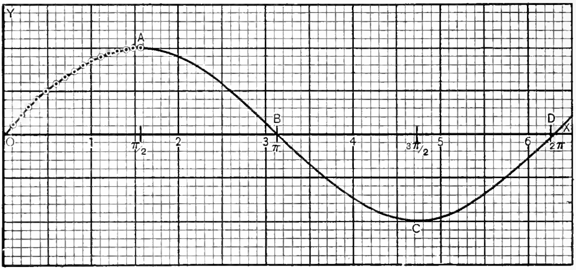

FIG. 90.

It is readily seen by the principles of § 68 that the extension of the curve through the second, third, and fourth quadrants is as shown by AB, BC, and CD; and that the curve extends to the left and to the right of the origin in a succession of arches such as OAB, BCD, etc.

The graph of $\sin x$ can be drawn without the aid of Table V as follows: Choose a convenient scale unit and lay off on the x-axis $OP = \frac{\pi}{2} = 1.57$ approximately, and divide this segment into a convenient number of equal parts, 15 say; the points of division correspond to $x = 0, \frac{\pi}{30}, \frac{2\pi}{30}, \frac{3\pi}{30}, \cdots, \frac{\pi}{2}$. Take from a table of sines, such as the one printed on p. 21 for example, the sines of the angles in the first quadrant for each 6° and tabulate:

x	0	$\frac{\pi}{30}$	$\frac{2\pi}{30}$	$\frac{3\pi}{30}$	$\frac{4\pi}{30}$	etc. . . .	$\frac{\pi}{2}$
$y = \sin x$	0	.105	.208	.309	.407	etc. . . .	1.000

Plot these points and draw a smooth curve through them.

The same methods may be used, with obvious modifications, to plot the graphs of $\cos x$, $\tan x$, and in fact any one of the trigonometric functions.

82. Mechanical Construction of the Graph. If an angle of x radians be laid off at the center of a unit circle (*i.e.* a circle whose radius is the scale unit), as AOB in Fig. 91, the numerical measure of the arc AB is the number of radians in the angle, *i.e.* x; the measure of CB is $\sin x$, the measure of AD is $\tan x$, the measure of OC is $\cos x$, and the measure of OD is $\sec x$.

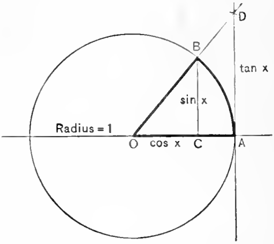

FIG. 91.

These facts can be used to construct the graphs of these functions without the use of any tables whatever. If we lay off on the x-axis a segment equal in length to the arc AB and at its end point erect a perpendicular equal to CB, its end point will lie on the graph of $\sin x$. It remains to show how to lay off a line segment approximately equal in length to a circular arc. If the arc AB is a known part of the quadrant AQ whose measure is 1.5708^-, the measure of AB can be computed and laid off with a scale. This will be the case if B is one of the points of division which divide the quadrant into a number of equal arcs. But even if the ratio of AB to AQ is unknown, provided $AB < AQ$ it can be approximately rectified as follows.

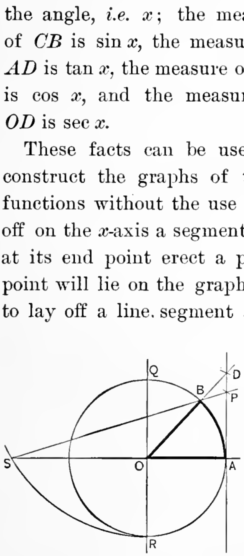

FIG. 92.

With Q as center, and the diameter QR as radius, strike an arc cutting AO produced in S; draw SB cutting AD in P.

Then the length of AP is approximately equal to the length of the arc AB.*

To use this method for constructing the graphs of $\sin x$, $\cos x$, etc., draw the unit circle, as in Fig. 93, tangent to the y-axis at the origin and divide the radian arc into a convenient number of equal parts, say 5, by lines from S to the points .2, .4, .6, etc., on the y-axis (the last division of the quadrant will of course be only .17$^+$ long). Mark the points 0, .2, .4, .6, .8,

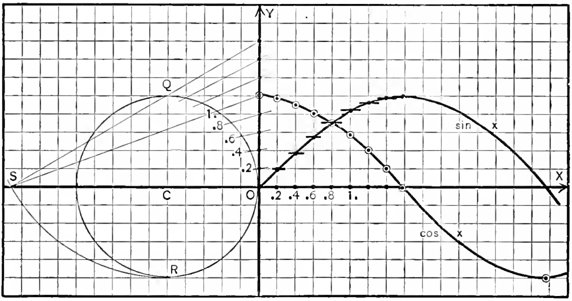

FIG. 93.

1, 1.2, 1.4, 1.57, on the x-axis and erect perpendiculars equal to the ordinates of the corresponding points on the arc. These give points on the graph of $\sin x$.

By erecting perpendiculars to the x-axis equal to the horizontal distances from CQ of the corresponding points on the arc we shall get points on the graph of $\cos x$.

By drawing radiating lines from the center C of the unit circle through the points of division of the arc we can lay off the tangents of these arcs on the y-axis and construct the graph

* The proof of this cannot be given until the student has studied Calculus. The distance AP is greater than x, but the error is less than $.017\,x^3$. The greatest error, about .017, occurs when AB is an arc of about 74° 29′, or when $x = 1.3^{(r)}$ approximately. The error for a 45° arc is .007 and for a quadrant, .006.

of $\tan x$; and in an obvious manner (see Fig. 91) the graph of $\sec x$ can be drawn. These graphs can be extended through the other three quadrants, and to the left of the y-axis, as in § 81. If the angle increases beyond 2π (radians) the values of all the trigonometric functions repeat themselves and the graph from $x=2\pi$ to $x=4\pi$ will be a repetition of those from $x=0$ to $x=2\pi$.

Functions which repeat themselves as x increases are called **periodic functions.** The **period** is the smallest amount of increase in x which produces the repetition of the value of the function. Thus, $\sin x$ is a periodic function with a period of 2π, while the period of $\tan x$ is π.

EXERCISES XXX. — GRAPHS OF TRIGONOMETRIC FUNCTIONS

1. Plot the graphs of the following functions using Table V, and Table VI when necessary.

(*a*) $\cos x$	(*b*) $\tan x$	(*c*) $\text{vers}\, x$
(*d*) $\text{ctn}\, x$	(*e*) $\sec x$	(*f*) $\csc x$
(*g*) $\sin^2 x$	(*h*) $\cos^2 x$	(*i*) $\sqrt{\sin x}$

2. Plot the graphs of the following functions without the use of tables: (*a*) $\cos x$ (*b*) $\tan x$ (*c*) $\sec x$

3. Plot the graph of $\cos x$ by dividing the second quadrant of the unit circle into fifths of a radian (see Fig. 91) and making use of the fact that $\cos x = \sin(\pi/2 + x)$.

4. Plot on the same axes the graphs of $\sin x$, $\sin \frac{1}{2} x$, $\sin 2x$, and $2 \sin x$.

5. Plot on the same axes the graphs of $\cos x$, $\cos \frac{1}{3} x$, $\cos 3x$, and $3 \cos x$.

6. Discuss the graphs of $\sin x/n$, $\sin nx$, and $n \sin x$ (where n is a natural number) in view of the results of Ex. 4 and 5.

7. Plot the graph of $\sin x + \cos x$ by adding the corresponding ordinates of the curves $y = \sin x$ and $y = \cos x$ plotted on the same axes.

8. Plot the graphs of the following functions by adding ordinates:

(*a*) $\sin x - \cos x$	(*b*) $2 \sin x + \cos x$
(*c*) $\tan x - 2 \sin x$	(*d*) $-\cos x$ (*i.e.* $0 - \cos x$)
(*e*) $x + \sin x$	(*f*) $x - \cos x$

9. Plot on the same axes the graphs of $\sin x$, and $\sin(x - \pi/6)$.

10. Plot on the same axes the graphs of $\sin x$, $\cos x$, and $\cos(x - \pi/2)$.

83. Inverse Functions. We have seen in § 69 that the equation

(1) $$y = \sin x$$

can be solved for x if y is any number whatever between -1 and $+1$, and that there are an infinite number of solutions. Any one of these solutions is denoted by *

(2) $$x = \arcsin y.$$

If we suppose that the angle is measured in radians, (2) means that *x is the number of radians in an angle (or arc) whose sine is y;* it is read "arc sine y" or "an angle whose sine is y."

Likewise **arccos y** denotes *an angle whose cosine is y;* **arctan y** denotes *an angle whose tangent is y.*

The expressions $y = \sin x$, $x = \arcsin y$, are two aspects of one relation, just as are the two statements "A is the uncle of B" and "B is the nephew of A"; either one implies the other; both mean the same thing.

As we wish to study the *arcsine function,* and in particular to compare it with the sine function, it is convenient and customary to think of it as depending on the same variable x, and write

(3) $$y = \arcsin x, \quad [i.e.\ x = \sin y].$$

We note that (3) is obtained from (1) by two steps, (*a*) *solving* (1) *for x;* and (*b*) *interchanging x and y in* (2). Two functions so related that each can be obtained from the other in this manner are called **inverse functions**; each is the inverse of the other.

In the same sense, $y = \cos x$ and $y = \text{arccos } x$; $y = \tan x$ and $y = \arctan x$; $y = \sec x$ and $y = \text{arcsec } x$; $y = \text{vers } x$ and $y = \text{arcvers } x$; etc., are inverse functions.

84. Graphical Representation of Inverse Functions. Since the equations

(1) $$y = \sin x \quad \text{and} \quad (2) \quad x = \arcsin y$$

* The notation $\sin^{-1} y$ also is used very frequently to denote $\arcsin y$; it is necessary to notice carefully that $\sin^{-1} y$ does *not* mean $(\sin y)^{-1}$.

are equivalent, the same pairs of values of x and y which satisfy one of them satisfy the other. Hence either of these

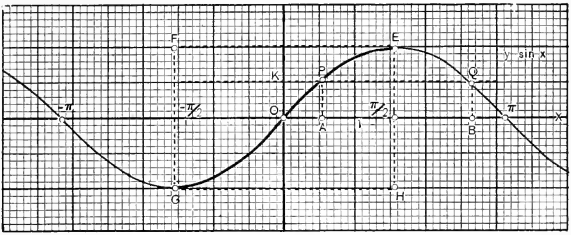

Fig. 94.

two equivalent equations is represented graphically by the curve drawn in Fig. 94.

From the manner in which equation

$$(3) \qquad y = \arcsin x$$

is derived from (2) it follows that the graph of arcsin x is obtained from the graph of sin x by interchanging the x- and y-axes; or, what gives the same result, by leaving the axes fixed and rotating the curve through an angle of 180° about the line through the origin which makes an angle of 45° with the x-axis. The result is shown in Fig. 96, p. 116.

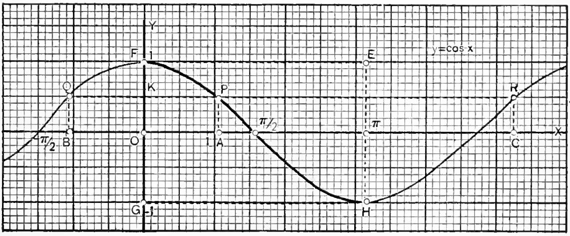

Fig. 95.

Similarly from the graph of cos x, Fig. 95, we derive the graph of arccos x in Fig. 97; and in the same way the graphs of arctan x, arcsec x, arcctn x, arccsc x, arcvers x, can be drawn from those of tan x, sec x, ctn x, csc x, vers x.

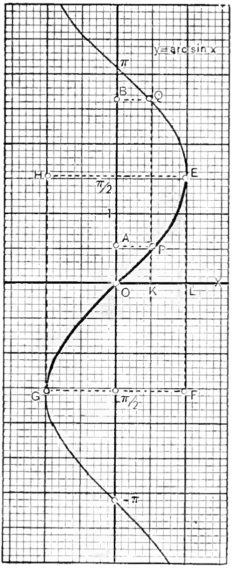

Fig. 96.

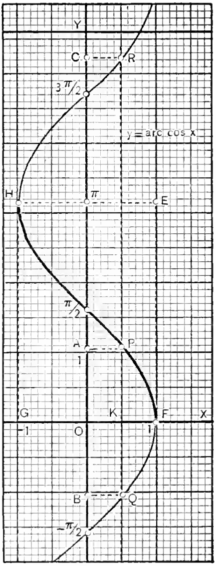

Fig. 97.

EXERCISES XXXI. — INVERSE FUNCTIONS

1. Draw the graph of $y = \arcsin x$ as in § 82.

2. Draw the graph of $y = \arccos x$ as in § 82.

3. Draw the graph of $y = \arctan x$.

4. Draw the graph of $y = \operatorname{arcsec} x$.

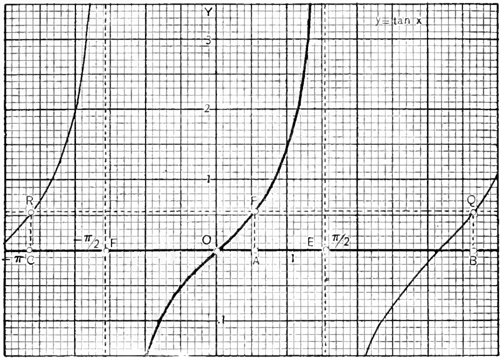
Y
y = tan x
3
2
1
R
P
Q
$-\pi/2$
F
O
E
$\pi/2$
$-\pi$
C
A
1
B
-1

Fig. 98.

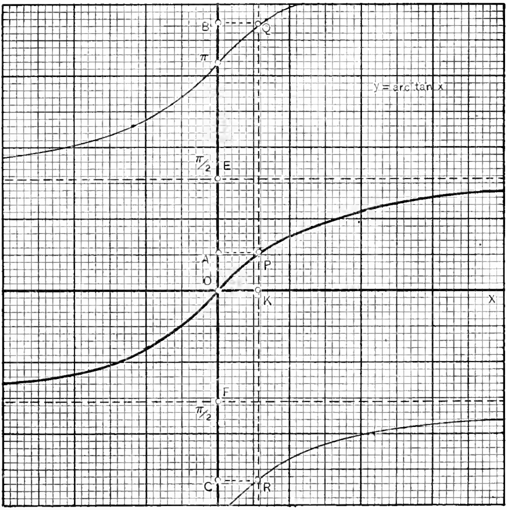
B
Q
π
y = arc tan x
$\pi/2$
E
A
P
O
K
X
F
$\pi/2$
C
R

Fig. 99.