TRILINEAR COORDINATES.

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TRILINEAR COORDINATES

AND OTHER METHODS OF

MODERN ANALYTICAL GEOMETRY OF TWO DIMENSIONS:

AN ELEMENTARY TREATISE,

BY THE

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ERRATA.

Page	5,	line 4	from bottom,	for OY, OX read CY, CX.
,,	15,	,, 9	,,	for k^3lm read k^3lmn.
,,	16,	,, 4	,,	for APB read APC.
,,	24,	,, 12	,,	*for* as, *read* in.
,,	48,	,, 3	,,	for $2ln \cos C$ read $2lm \cos C$.
,,	58,	,, 8	,,	for $2bcrp \cos B$ read $2carp \cos B$.
,,	73,	,, 7	,,	*for* a *read* a'.
,,	78,	,, 3	,,	*for* $\sin A \sin B \sin C$ *read* $2 \sin A \sin B \sin C$
,,	95,	,, 6	,,	*for* difference *read* reference.

,, 134, Art. 119. QOP is (three times) misprinted for AOQ.
QOQ is (twice) misprinted for AOP.

,, 202, line 7, *for* $\frac{\gamma}{\gamma_2}$ *read* $\frac{\beta}{\beta_2}$, and read equations (1), (2), (3) as follows:

$$\frac{a}{a_2\beta_1} = \frac{\beta}{\beta_1\beta_2} = \frac{\gamma}{\beta_2\gamma_1} \quad \text{(1)},$$

$$\frac{a}{\gamma_3 a_2} = \frac{\beta}{\beta_3\gamma_2} = \frac{\gamma}{\gamma_2\gamma_3} \quad \text{(2)},$$

$$\frac{a}{a_3 a_1} = \frac{\beta}{a_1\beta_3} = \frac{\gamma}{\gamma_1 a_3} \quad \text{(3)}.$$

So the first determinant ought to be

$$\begin{vmatrix} a_2\beta_1, & \beta_1\beta_2, & \beta_2\gamma_1 \\ \gamma_3 a_2, & \beta_3\gamma_2, & \gamma_2\gamma_3 \\ a_3 a_1, & a_1\beta_3, & \gamma_1 a_3 \end{vmatrix} = 0.$$

Page 281, line 10, *for* $\frac{2\Delta H}{E^3}$ *read* $-\frac{2\Delta H}{E^3}$.

PREFACE.

MODERN Analytical Geometry excels the method of Des Cartes in the precision with which it deals with the Infinite and the Imaginary. So soon, therefore, as the student has become familiar with the meaning of equations and the significance of their combinations, as exemplified in the simplest Cartesian treatment of Conic Sections, it seems advisable that he should at once take up the modern methods rather than apply a less suitable treatment to researches for which these methods are especially adapted.

By this plan he will best obtain fixed and definite notions of what is signified by the words infinite and imaginary, and much light will be thereby thrown upon his knowledge of Algebra, while at the same time, his facility in that most important subject will be greatly increased by the wonderful variety of expedient in the combination of algebraical equations which the methods of modern analytical geometry present, or suggest.

With this view I have endeavoured, in the following pages, to make my subject intelligible to those whose knowledge of the processes of analysis may be very limited; and I have devoted especial care to the preparation of the chapters on Infinite and Imaginary space, so as to render them suitable for those whose ideas of geometry have as yet been confined to the region of the Real and the Finite.

I have sought to exhibit methods rather than results,—to furnish the student with the means of establishing properties for himself rather than to present him with a repertory of isolated propositions ready proved. Thus I have not hesitated in some cases to give a variety of investigations of the same theorem, when it seemed well so to compare different methods, and on the other hand interesting propositions have sometimes been placed among the exercises rather than inserted in the text, when they have not been required in illustration of any particular process or method of proof.

In compiling the prolegomenon, I have derived considerable assistance from a valuable paper which Professor Tait contributed five years ago to the *Messenger of Mathematics*. My thanks are due to Professor Tait for his kindness in placing that paper at my disposal for the purposes of the present work, as well as to other friends for their trouble in revising proofs and collecting examples illustrative of my subject from University and College Examination Papers.

LIVERPOOL,
15 *September*, 1866.

CONTENTS.

PROLEGOMENON.

OF DETERMINANTS.

§ 1. *Introduction.*

1. If we have m equations involving a lesser number n of unknown quantities, we may determine the unknown quantities from n of the equations, and, substituting these values in the remaining $m-n$ equations, obtain $m-n$ relations amongst the coefficients of the m equations.

In other words, if we eliminate n quantities from m equations, there will remain $m-n$ equations.

2. If the equations are all simple equations, the solution can always be effected and the $m-n$ equations practically obtained.

The notation of *Determinants* supplies the means of conveniently expressing the results of such elimination, and the study of their properties facilitates the operation of reducing the results to their simplest forms.

3. It must be observed, however, that if the equations be homogeneous in the unknown quantities, or, in the case of simple equations, if every term of each equation involve one of the unknown quantities, the equations do not then involve the actual values of the unknown quantities at all, but only the ratios which they bear one to another. Thus the equations

$$3x + 4y - 5z = 0,$$
$$5x + 5y - 7z = 0,$$

are satisfied if x, y, z are proportional to 3, 4, 5, but they do not involve any statement as to the actual values of x, y, z.

In this case the number of independent magnitudes, concerning which anything is predicated in the equations, is one less than the number of unknown quantities involved in the equations. Thus each of the equations just instanced, involving the three unknown quantities x, y, z, speaks not of the actual magnitudes of those quantities, but of their ratios one to another, which are only *two* independent magnitudes, as is immediately seen by writing the equations in the form

$$3\frac{x}{z} + 4\frac{y}{z} = 5,$$

$$5\frac{x}{z} + 5\frac{y}{z} = 7,$$

where the equations are exhibited as connecting the two independent ratios $\frac{x}{z}$ and $\frac{y}{z}$.

§ 2. *Of Determinants of the second order.*

4. Def. The symbol $\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}$

is used to express the algebraical quantity $a_1b_2 - b_1a_2$, and is called a *determinant of the second order.*

The separate quantities a_1, b_1, a_2, b_2 are called the *elements* of the determinant, and may themselves be algebraically either simple or complex quantities.

Any horizontal line of elements in a determinant is called a *row*, and a vertical line is called a *column.*

Thus the determinant above written has two rows a_1, b_1 and a_2, b_2, and two columns a_1, a_2 and b_1, b_2.

5. It follows from the definition that

$$\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

Hence *a determinant of the second order is not altered by changing rows into columns and columns into rows.*

6. It follows similarly from the definition that

$$\begin{vmatrix} a_2, & b_2 \\ a_1, & b_1 \end{vmatrix} = (a_2b_1 - a_1b_2) = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

Hence in a determinant of the second order *the interchange of the two rows changes the sign of the determinant.*

So *the interchange of the two columns changes the sign of the determinant.*

7. *If* $ax + by = 0,$

and $a'x + b'y = 0,$

be two consistent equations, then will

$$\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = 0.$$

Multiplying the first equation by b' and the second by b, and subtracting, we get

$$(ab' - a'b)\,x = 0,$$

therefore $ab' - a'b = 0,$

or $\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = 0.$

Q. E. D.

8. *If* $ax + by + cz = 0,$

and $a'x + b'y + c'z = 0,$

then will

$$\frac{x}{\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}}$$

For if we multiply the first equation by c' and the second by c, and subtract, we get

$$(c'a - ca')\,x + (bc' - b'c)\,y = 0,$$

or $(ca' - c'a)\,x = (bc' - b'c)\,y,$

or $\dfrac{x}{bc' - b'c} = \dfrac{y}{ca' - c'a},$

which may be written

$$\frac{x}{\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}},$$

and therefore, by symmetry,

$$\frac{x}{\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}}.$$

Q. E. D.

9. *If* $ax + by = c,$

and $a'x + b'y = c',$

then will

$$x = -\frac{\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}}{\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}}, \textit{ and } y = -\frac{\begin{vmatrix} c, & a \\ c', & a' \end{vmatrix}}{\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}}.$$

This follows from the last proposition by writing -1 for z.

§ 3. *Of Determinants of the third order.*

10. Def. The symbol

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

is used to denote the expression

$$a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - b \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}$$

and is called a *determinant of the third order.*

11. *If* $ax + by + cz = 0,$

$a'x + b'y + c'z = 0,$

and $a''x + b''y + c''z = 0;$

then will

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0.$$

For the second and third equations give, by Art. 8,

$$\frac{x}{\begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix}} = \frac{y}{\begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a, & b \\ a'', & b'' \end{vmatrix}}.$$

Substituting these values in the first equation, we get

$$a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + b \begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix} = 0,$$

or

$$a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - b \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0.$$

Q. E. D.

12. In the foregoing proposition we eliminated the two ratios $x : y : z$ from the three given equations, and found the result in the form of a determinant.

We might have proceeded otherwise as follows:

Multiplying the three equations by λ, μ, ν (at present undetermined multipliers) and adding, we get

$$(a\lambda + a'\mu + a''\nu)\,x + (b\lambda + b'\mu + b''\nu)\,y + (c\lambda + c'\mu + c''\nu)\,z = 0,$$

which must be true for all values of λ, μ, ν.

Now by Art. 8 we know that if

$$\frac{\lambda}{\begin{vmatrix} b', & b'' \\ c', & c'' \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} b'', & b \\ c'', & c \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} b, & b' \\ c, & c' \end{vmatrix}},$$

then the coefficients of y and z in the last equation will vanish, and the equation will reduce to

$$(a\lambda + a'\mu + a''\nu)\, x = 0,$$

so that we must have

$$a\lambda + a'\mu + a''\nu = 0,$$

or substituting the values of $\lambda : \mu : \nu$

$$a \begin{vmatrix} b', & b'' \\ c', & c'' \end{vmatrix} + a' \begin{vmatrix} b'', & b \\ c'', & c \end{vmatrix} + a'' \begin{vmatrix} b, & b' \\ c, & c' \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix} = 0,$$

which is therefore the result of the elimination.

But this result must be equivalent to the result obtained by the other method. Hence the two equations

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix} = 0$$

must be identical, and therefore their first members must either be identical or differ only by a constant multiplier. But the coefficient of the term $ab'c''$ in each is seen to be + 1. Hence the two determinants are identical, or

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \equiv \begin{vmatrix} a, & a', & a'' \\ b, & b', & b'' \\ c, & c', & c'' \end{vmatrix}.$$

13. Cor. *A determinant of the third order is not affected by changing the rows into columns and the columns into rows.*

Care must, however, be taken that the first column becomes the first row, the second column the second row, and so on, and *vice versâ.*

14. Since the result of the elimination is the same in whatever order the equations be taken, it follows that the resulting equation

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0$$

is not altered in whatever order the rows of the determinant be written.

Hence the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ a'', & b'', & c'' \\ a', & b', & c' \end{vmatrix}, \quad \begin{vmatrix} a'', & b'', & c'' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}, \quad \begin{vmatrix} a'', & b'', & c'' \\ a', & b', & c' \\ a, & b, & c \end{vmatrix}, \text{ \&c.}$$

can only differ by some numerical multipliers, and since the coefficient of every term in the expansion of each of them is either $+1$ or -1, they can therefore only differ by the algebraical sign of the whole.

Since

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - b \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix},$$

and

$$\begin{vmatrix} a, & b, & c \\ a'', & b'', & c'' \\ a', & b', & c' \end{vmatrix} = a \begin{vmatrix} c', & b' \\ c'', & b'' \end{vmatrix} - b \begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix} + c \begin{vmatrix} b', & a' \\ b'', & a'' \end{vmatrix},$$

it follows from Art. 6 that

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \text{ and } \begin{vmatrix} a, & b, & c \\ a'', & b'', & c'' \\ a', & b', & c' \end{vmatrix}.$$

are of opposite algebraical sign. Hence the sign of a determinant of the third order is changed by interchanging its last two rows.

It will be seen on examination that the effect is the same if we change any other two *adjacent* rows, or two *adjacent* columns.

That is, *the sign of the determinant is changed when any two adjacent rows are interchanged, or when any two adjacent columns are interchanged.*

15. But any derangement whatever of the rows or columns may be made by a series of transpositions of adjacent rows or columns. Such a derangement will or will not affect the sign of the determinant according as it requires an odd or an even number of transpositions of adjacent rows or columns to effect it, thus.

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = -\begin{vmatrix} a', & b', & c' \\ a, & b, & c \\ a'', & b'', & c'' \end{vmatrix} = \begin{vmatrix} a', & b', & c' \\ a'', & b'', & c'' \\ a, & b, & c \end{vmatrix} = -\begin{vmatrix} a'', & b'', & c'' \\ a', & b', & c' \\ a, & b, & c \end{vmatrix}$$

$$= \begin{vmatrix} a'', & c'', & b'' \\ a', & c', & b' \\ a, & c, & b \end{vmatrix} = -\begin{vmatrix} c'', & a'', & b'' \\ c', & a', & b' \\ c, & a, & b \end{vmatrix} = \begin{vmatrix} c'', & b'', & a'' \\ c', & b', & a' \\ c, & b, & a \end{vmatrix} = \text{\&c.}$$

Similarly we may ascertain the sign of the determinant formed by any other derangement of the columns or rows.

16. *If a row or a column of a determinant be multiplied throughout by any number, the value of the determinant is multiplied by the same number.*

For the determinant may be deranged till the row or column in question becomes the first row.

Now

$$\begin{vmatrix} \mu a, & \mu b, & \mu c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = \mu a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + \mu b \begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix} + \mu c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}$$

$$= \mu \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

which proves the proposition.

17. *If two rows of the determinant be identical, or if two columns be identical, the determinant vanishes.*

For the rows and columns may be deranged until the last two rows are identical, and the determinant takes the form

$$\pm \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a', & b', & c' \end{vmatrix}$$

$$= \pm a \begin{vmatrix} b', & c' \\ b', & c' \end{vmatrix} \mp b \begin{vmatrix} a', & c' \\ a', & c' \end{vmatrix} \pm c \begin{vmatrix} a', & b' \\ a', & b' \end{vmatrix}$$

$= 0$, by Art. 4.

Therefore, &c. Q. E. D.

18. Cor. If one row be a multiple of another row or one column of another column, the determinant vanishes. For the multiplier may be divided out by Art. 16.

19. *To shew that*

$$\begin{vmatrix} a+x, & b+y, & c+z \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} + \begin{vmatrix} x, & y, & z \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

Expanding the first determinant, it takes the form

$$(a+x) \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - (b+y) \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + (c+z) \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}$$

or
$$a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - b \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}$$

$$+ x \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} - y \begin{vmatrix} a', & c' \\ a'', & c'' \end{vmatrix} + z \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}$$

or
$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} + \begin{vmatrix} x, & y, & z \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

Therefore, &c. Q. E. D.

Similarly,

$$\begin{vmatrix} a+\alpha, & b, & c \\ a'+\alpha', & b', & c' \\ a''+\alpha'', & b'', & c'' \end{vmatrix} = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} + \begin{vmatrix} \alpha, & b, & c \\ \alpha', & b', & c' \\ \alpha'', & b'', & c'' \end{vmatrix}$$

And so if each element of *any* column or row be divided into two parts, the original determinant is equal to the sum of the two determinants formed by substituting for each divided element first one of its parts and then the other.

But note that this operation cannot always be performed at once on more than one column or row.

Conversely, if a series of determinants are identical except as regards one column or one row in each, their sum is equal to the new determinant formed by retaining in their places the rows or columns that are identical, and adding together the corresponding elements of the row or column which differs.

20. *If any row of a determinant be increased by multiples of any other rows, or if any column be increased by multiples of any other columns, the value of the determinant is not altered.*

For, by Art. 19,

$$\begin{vmatrix} a+mb+nc, & b, & c \\ a'+mb'+nc', & b', & c' \\ a''+mb''+nc'', & b'', & c'' \end{vmatrix}$$

$$= \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} + \begin{vmatrix} mb, & b, & c \\ mb', & b', & c' \\ mb'', & b'', & c'' \end{vmatrix} + \begin{vmatrix} nc, & b, & c \\ nc', & b', & c' \\ nc'', & b'', & c'' \end{vmatrix},$$

$$= \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}, \text{ by Art. 18,}$$

which proves the proposition.

This theorem is of the greatest use in reducing determinants.

21. *If*

$$ax + by + cz + du = 0,$$

$$a'x + b'y + c'z + d'u = 0,$$

and

$$a''x + b''y + c''z + d''u = 0,$$

then will

$$\frac{x}{\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a, & c, & d \\ a', & c', & d' \\ a'', & c'', & d'' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a, & b, & d \\ a', & b', & d' \\ a'', & b'', & d'' \end{vmatrix}} = \frac{-u}{\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}}.$$

For if we multiply the first equation by $\begin{vmatrix} c', & c'' \\ d', & d'' \end{vmatrix}$, the second by $-\begin{vmatrix} c, & c'' \\ d, & d'' \end{vmatrix}$, the third by $\begin{vmatrix} c, & c' \\ d, & d' \end{vmatrix}$, and add, we get

$$\begin{vmatrix} a, & a', & a'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} x + \begin{vmatrix} b, & b', & b'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} y + \begin{vmatrix} c, & c', & c'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} z + \begin{vmatrix} d, & d', & d'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} u = 0,$$

or, in virtue of Art. 17,

$$\begin{vmatrix} a, & a', & a'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} x + \begin{vmatrix} b, & b', & b'' \\ c, & c', & c'' \\ d, & d', & d'' \end{vmatrix} y = 0,$$

or

$$\begin{vmatrix} a, & c, & d \\ a', & c', & d' \\ a'', & c'', & d'' \end{vmatrix} x + \begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} y = 0,$$

or

$$\frac{x}{\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a, & c, & d \\ a', & c', & d' \\ a'', & c'', & d'' \end{vmatrix}}$$

and similarly the other equations may be established.

22. *If*

$$ax + by + cz = d,$$

$$a'x + b'y + c'z = d',$$

and

$$a''x + b''y + c''z = d'',$$

then will

$$x = \frac{\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}}{\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c, & a, & d \\ c', & a', & d' \\ c'', & a'', & d'' \end{vmatrix}}{\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a, & b, & d \\ a', & b', & d' \\ a'', & b'', & d'' \end{vmatrix}}{\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}}.$$

This follows from the last proposition by writing -1 for u.

[It will be observed that these values of x, y, z obtained by solving the three simultaneous equations might have been written down by the method of cross multiplication in Algebra.]

§ 4. *Of Determinants of the fourth order.*

23. Def. The symbol

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix}$$

is used to denote the expression

$$a\begin{vmatrix} b', & c', & d' \\ b'', & c'', & d'' \\ b''', & c''', & d''' \end{vmatrix} - b\begin{vmatrix} a', & c', & d' \\ a'', & c'', & d'' \\ a''', & c''', & d''' \end{vmatrix} + c\begin{vmatrix} a', & b', & d' \\ a'', & b'', & d'' \\ a''', & b''', & d''' \end{vmatrix} - d\begin{vmatrix} a', & b', & c' \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}$$

and is called a *determinant of the fourth order.*

24. *If*

$$\begin{aligned} ax + by + cz + du &= 0, \\ a'x + b'y + c'z + d'u &= 0, \\ a''x + b''y + c''z + d''u &= 0, \\ a'''x + b'''y + c'''z + d'''u &= 0, \end{aligned}$$

then will

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix} = 0.$$

For the second, third, and fourth equations give

$$\frac{x}{\begin{vmatrix} b', & c', & d' \\ b'', & c'', & d'' \\ b''', & c''', & d''' \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a', & c', & d' \\ a'', & c'', & d'' \\ a''', & c''', & d''' \end{vmatrix}} = \frac{z}{\begin{vmatrix} a', & b', & d' \\ a'', & b'', & d'' \\ a''', & b''', & d''' \end{vmatrix}} = \frac{-u}{\begin{vmatrix} a', & b', & c' \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}}.$$

Substituting these values in the first equation, we get

$$a\begin{vmatrix} b', & c', & d' \\ b'', & c'', & d'' \\ b''', & c''', & d''' \end{vmatrix} - b\begin{vmatrix} a', & c', & d' \\ a'', & c'', & d'' \\ a''', & c''', & d''' \end{vmatrix} + c\begin{vmatrix} a', & b', & d' \\ a'', & b'', & d'' \\ a''', & b''', & d''' \end{vmatrix} - d\begin{vmatrix} a', & b', & c' \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}$$

$$\text{or} \quad \begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix} = 0. \quad \text{Q. E. D.}$$

Precisely as in the case of the determinant of the third order (Art. 13), we may shew that the value of a determinant of the fourth order is not affected by changing the rows into columns and the columns into rows.

So the results obtained in Arts. 14—20, will be seen to depend upon general principles, and to hold for determinants of the fourth order.

25. *If*
$$ax + by + cz + du + ev = 0,$$
$$a'x + b'y + c'z + d'u + e'v = 0,$$
$$a''x + b''y + c''z + d''u + e''v = 0,$$
and
$$a'''x + b'''y + c'''z + d'''u + e'''v = 0,$$
then will

$$\frac{x}{|\, b,\ c',\ d'',\ e''' \,|} = \frac{-y}{|\, a,\ c',\ d'',\ e''' \,|} = \frac{z}{|\, a,\ b',\ d'',\ e''' \,|} = \frac{-u}{|\, a,\ b',\ c'',\ e''' \,|}$$
$$= \frac{v}{|\, a,\ b',\ c'',\ d''' \,|},$$

where $|\, a,\ b',\ c'',\ d''' \,|$ denotes the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix}$$

For if we multiply the four equations respectively by

$$\begin{vmatrix} c', & c', & c' \\ d'', & d'', & d'' \\ e''', & e''', & e''' \end{vmatrix}, \ - \begin{vmatrix} c, & c, & c, \\ d'', & d'', & d'' \\ e''', & e''', & e''' \end{vmatrix}, \quad \begin{vmatrix} c, & c, & c \\ d', & d', & d' \\ e''', & e''', & e''' \end{vmatrix}, \ - \begin{vmatrix} c, & c, & c \\ d', & d', & d' \\ e'', & e'', & e'' \end{vmatrix}$$

and add, we obtain

$$\begin{vmatrix} a, & a', & a'', & a''' \\ c, & c', & c'', & c''' \\ d, & d', & d'', & d''' \\ e, & e', & e'', & e''' \end{vmatrix} x + \begin{vmatrix} b, & b', & b'', & b''' \\ c, & c', & c'', & c''' \\ d, & d', & d'', & d''' \\ e, & e', & e'', & e''' \end{vmatrix} y = 0,$$

whence

$$\frac{x}{\begin{vmatrix} b, & c, & d, & e \\ b', & c', & d', & e' \\ b'', & c'', & d'', & e'' \\ b''', & c''', & d''', & e''' \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a, & c, & d, & e \\ a', & c', & d', & e' \\ a'', & c'', & d'', & e'' \\ a''', & c''', & d''', & e''' \end{vmatrix}}$$

and similarly the other equations may be established.

§ 5. *Of Determinants of the n^{th} order.*

26. The student will have observed that the reasoning of the last two sections is perfectly general. He will have recognised the law by which a determinant of any order is defined with reference to those of the next lower order, and he will have perceived that the proofs of properties of determinants of the 3rd order given at length in § 3 will apply *mutatis mutandis* to establish corresponding properties for determinants of any order whatever, if they can be assumed to hold for the next lower order. It follows, therefore, by the principle of mathematical induction, that all those properties may be attributed to any determinant whatever.

27. Def. If we strike out one of the columns and one of the rows of a determinant of the n^{th} order, we shall obtain a new determinant of the $(n-1)^{\text{th}}$ order, which is called the *minor* of the original determinant with respect to that element which was common to the column and row.

Thus the minor of the determinant

$$\begin{vmatrix} a, & b, & c, & d & \dots \\ a', & b', & c', & d' & \dots \\ a'', & b'', & c'', & d'' & \dots \\ a''', & b''', & c''', & d''' & \dots \\ \vdots & \vdots & \vdots & \vdots & \&c. \end{vmatrix}$$

with respect to the element c'' is the determinant

$$\begin{vmatrix} a, & b, & d & \dots \\ a', & b', & d' & \dots \\ a''', & b''', & d''' & \dots \\ \vdots & \vdots & \vdots & \&c. \end{vmatrix}.$$

28. Def. The element which is common to the p^{th} column and q^{th} row of a determinant is said to occupy a positive or negative place according as $p+q$ is even or odd.

29. *The coefficient of any element of a determinant is the minor with respect to that element affected with the sign + or − according as the element occupies a positive or negative place.*

Let the determinant be

$$\begin{vmatrix} a, & b, & \dots & \dots \\ a', & b', & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & x & \dots \\ \vdots & \vdots & \vdots & \&c. \end{vmatrix}$$

and let the element x occupy the q^{th} place in the p^{th} column.

By making $q-1$ transpositions of adjacent rows and $p-1$ transpositions of adjacent columns, the determinant may be written

$$(-1)^{p+q-2} \begin{vmatrix} x & \dots & \dots & \dots \\ \vdots & a, & b, & \dots \\ \vdots & a', & b', & \dots \\ \vdots & \vdots & \vdots & \&c. \end{vmatrix},$$

whence we see that the coefficient of x is

$$\pm \begin{vmatrix} a, & b, & \dots \\ a', & b', & \dots \\ \vdots & \vdots & \&c. \end{vmatrix}$$

the sign being positive or negative according as $p+q$ is even or odd, which proves the proposition.

§ 6. *Of Determinants of unequal columns and rows.*

30. Suppose we have to eliminate n unknown quantities from a greater number m of simple equations. As we shewed in Art. 1, we shall obtain $m-n$ equations of relation amongst the coefficients.

One method of writing down these $m-n$ equations would be to take the first n of the given equations, and associate with them in order each of the remaining $m-n$ equations.

From each of the $m-n$ groups of $n+1$ equations thus formed, we might eliminate the n unknown quantities. Expressing the result by means of a determinant of the $(n+1)^{\text{th}}$ order. We should thus have $m-n$ determinants of the $(n+1)^{\text{th}}$ order each equated to zero, constituting the $m-n$ equations sought.

But it is plain we might have taken any $m-n$ combinations whatever of $n+1$ equations that could have been formed out of the m equations, provided we took care to introduce *all* the original equations. Thus there will always be a variety of forms in which the result of such elimination may be expressed.

31. Suppose, for instance, that we have to eliminate the two ratios $x : y : z$ from the five equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$
$$a_4x + b_4y + c_4z = 0,$$
$$a_5x + b_5y + c_5z = 0.$$

As the result of the elimination we shall have the three equations

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1, & a_2, & a_4 \\ b_1, & b_2, & b_4, \\ c_1, & c_2, & c_4, \end{vmatrix} = 0, \quad \begin{vmatrix} a_1, & a_2, & a_5 \\ b_1, & b_2, & b_5, \\ c_1, & c_2, & c_5, \end{vmatrix} = 0,$$

or for any one of these equations we may substitute any other equation obtained by elimination from a different set of three equations, such as

$$\begin{vmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \\ c_2, & c_3, & c_4 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} a_2, & a_4, & a_5 \\ b_2, & b_4, & b_5 \\ c_2, & c_4, & c_5 \end{vmatrix} = 0.$$

But it is convenient to express the result of this elimination briefly by the form

$$\left\| \begin{matrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \\ c_1, & c_2, & c_3, & c_4, & c_5 \end{matrix} \right\| = 0,$$

where the determinant of five columns and three rows indicates that we may select any three of the five columns to form a square determinant and equate it to zero, and the triple vertical lines indicate that three such equations may be independently formed.

32. The example given in the last paragraph will suffice to suggest to the reader the interpretation of any unequal determinant. In most general terms the definition will stand as follows:

The compound symbol

$$^{r}\left\|\begin{matrix} a_1, & a_2, & a_3 & \ldots & a_m \\ b_1, & b_2, & b_3 & \ldots & b_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1, & k_2, & k_3 & \ldots & k_m \end{matrix}\right\|^{r} = 0,$$

where the number of the quantities $a, b, c \ldots k$ is n (less than m) and the number of vertical lines bounding the determinant is r, *is to be understood as expressing a system of* r *independent equations obtained by equating to zero* r *several determinants each formed by taking* n *of the rows of the given unequal determinant.*

It will be observed, that if the system expresses the result of the elimination of n quantities out of m independent equations, we must have $r = m - n$. The notation is, however, found convenient in cases when the original equations, although m in number, are only equivalent to some lesser number m' of independent equations. In such a case we shall have $r = m' - n$.

33. As an example of the case last considered, suppose we have to eliminate the two ratios $x : y : z$ from the four equations

$$(a - a')x + (b - b')y + (c - c')z = 0,$$

$$(a' - a'')x + (b' - b'')y + (c' - c'')z = 0,$$

$$(a'' - a''')x + (b'' - b''')y + (c'' - c''')z = 0,$$

$$(a''' - a)x + (b''' - b)y + (c''' - c)z = 0.$$

These are equivalent to only three independent equations, since any one of them may be obtained from the others by simple addition.

There will therefore be only one resulting equation, which may be obtained by eliminating from *any three* of the given equations. If, however, it be desired to have a result recognising symmetrically all the four equations, we may write it

$$\begin{vmatrix} a-a', & a'-a'', & a''-a''', & a'''-a \\ b-b', & b'-b'', & b''-b''', & b'''-b \\ c-c', & c'-c'', & c''-c''', & c'''-c \end{vmatrix} = 0.$$

§ 7. *Examples.*

EXAMPLE A. *To evaluate the determinant*

$$\begin{vmatrix} \alpha^2, & \alpha, & 1 \\ \beta^2, & \beta, & 1 \\ \gamma^2, & \gamma, & 1 \end{vmatrix}.$$

Subtracting the third row from each of the others, the determinant becomes

$$\begin{vmatrix} \alpha^2-\gamma^2, & \alpha-\gamma, & 0 \\ \beta^2-\gamma^2, & \beta-\gamma, & 0 \\ \gamma^2, & \gamma, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha^2-\gamma^2, & \alpha-\gamma \\ \beta^2-\gamma^2, & \beta-\gamma \end{vmatrix} = (\alpha-\gamma)(\beta-\gamma) \begin{vmatrix} \alpha+\gamma, & 1 \\ \beta+\gamma, & 1 \end{vmatrix}$$

$$= (\alpha-\gamma)(\beta-\gamma)(\alpha-\beta).$$

EXAMPLE A′. By a similar method we may shew that

$$\begin{vmatrix} \alpha^3, & \alpha, & 1 \\ \beta^3, & \beta, & 1 \\ \gamma^3, & \gamma, & 1 \end{vmatrix} = (\alpha-\gamma)(\beta-\gamma)(\alpha-\beta)(\alpha+\beta+\gamma).$$

EXAMPLE B. *To shew that*

$$\begin{vmatrix} x+a, & x+b, & x+c \\ y+a, & y+b, & y+c \\ z+a, & z+b, & z+c \end{vmatrix} \equiv 0.$$

If $a=b$ or $b=c$ or $c=a$, two columns become identical and the determinant vanishes. Therefore if the determinant be not identically zero, $a-b$, $b-c$, $c-a$ are factors. Similarly, if $x=y$ or $y=z$ or $z=x$, two rows become identical, and so $y-z$, $z-x$, $x-y$ are factors.

But the determinant is only of the 3rd order and cannot have more than three factors. Hence it must vanish identically. Q. E. D.

EXAMPLE C. *To evaluate the determinant*

$$\begin{vmatrix} x, & x-c, & x-b \\ y-c, & y, & y-a \\ z-b, & z-a, & z \end{vmatrix}.$$

Subtracting the first column from each of the others, we get

$$\begin{vmatrix} x, & -c, & -b \\ y-c, & c, & c-a \\ z-b, & b-a, & b \end{vmatrix}$$

whence we see that the coefficient of x is

$$\begin{vmatrix} c, & c-a \\ b-a, & b \end{vmatrix}, \text{ or } ab+ac-a^2.$$

By symmetry, the coefficients of y and z must be respectively $bc+ab-b^2$ and $ca+bc-c^2$.

And the terms without x, y, z, are

$$\begin{vmatrix} 0, & -c, & -b \\ -c, & 0, & -a \\ -b, & -a, & 0 \end{vmatrix}, \text{ or } -2abc.$$

Hence the determinant may be written

$$ax(b+c-a)+by(c+a-b)+cz(a+b-c)-2abc.$$

EXAMPLE D. *To shew that*

$$\begin{vmatrix} \beta+\gamma, & \gamma+\alpha, & \alpha+\beta \\ \beta'+\gamma', & \gamma'+\alpha', & \alpha'+\beta' \\ \beta''+\gamma'', & \gamma''+\alpha'', & \alpha''+\beta'' \end{vmatrix} = 2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

The first determinant (by Art. 19) is equal to

$$\begin{vmatrix} \beta, & \gamma+\alpha, & \alpha+\beta \\ \beta', & \gamma'+\alpha', & \alpha'+\beta' \\ \beta'', & \gamma''+\alpha'', & \alpha''+\beta'' \end{vmatrix} + \begin{vmatrix} \gamma, & \gamma+\alpha, & \alpha+\beta \\ \gamma', & \gamma'+\alpha', & \alpha'+\beta' \\ \gamma'', & \gamma''+\alpha'', & \alpha''+\beta'' \end{vmatrix}$$

or, in virtue of Art. 20,

$$\begin{vmatrix} \beta, & \gamma+\alpha, & \alpha \\ \beta', & \gamma'+\alpha', & \alpha' \\ \beta'', & \gamma''+\alpha'', & \alpha'' \end{vmatrix} + \begin{vmatrix} \gamma, & \alpha, & \alpha+\beta \\ \gamma', & \alpha', & \alpha'+\beta' \\ \gamma'', & \alpha'', & \alpha''+\beta'' \end{vmatrix}$$

or again, applying the same principle,

$$\begin{vmatrix} \beta, & \gamma, & \alpha \\ \beta', & \gamma', & \alpha' \\ \beta'', & \gamma'', & \alpha'' \end{vmatrix} + \begin{vmatrix} \gamma, & \alpha, & \beta \\ \gamma', & \alpha', & \beta' \\ \gamma'', & \alpha'', & \beta'' \end{vmatrix}$$

which by Art. 15, is equal to

$$2\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}$$

Therefore, &c. Q. E. D.

EXAMPLE E. Let

$$L_1 = \begin{vmatrix} m_2, & n_2 \\ m_3, & n_3 \end{vmatrix}, \quad M_1 = \begin{vmatrix} n_2, & l_2 \\ n_3, & l_3 \end{vmatrix}, \quad N_1 = \begin{vmatrix} l_2, & m_2 \\ l_3, & m_3 \end{vmatrix}$$

$$L_2 = \begin{vmatrix} m_3, & n_3 \\ m_1, & n_1 \end{vmatrix}, \quad M_2 = \begin{vmatrix} n_3, & l_3 \\ n_1, & l_1 \end{vmatrix}, \quad N_2 = \begin{vmatrix} l_3, & m_3 \\ l_1, & m_1 \end{vmatrix}$$

$$L_3 = \begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix}, \quad M_3 = \begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix}, \quad N_3 = \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix}$$

then we have

$$l_1L_1 + m_1M_1 + n_1N_1 = l_2L_2 + m_2M_2 + n_2N_2 = l_3L_3 + m_3M_3 + n_3N$$

$$= \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} =$$

$$l_1L_1 + l_2L_2 + l_3L_3 = m_1M_1 + m_2M_2 + m_3M_3 = n_1N_1 + n_2N_2 + n_3N_3.$$

So also

$$l_1L_2 + m_1M_2 + n_1N_2 = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_3, & m_3, & n_3 \\ l_1, & m_1, & n_1 \end{vmatrix} = 0.$$

And so $$l_3L_2 + m_3M_2 + n_3N_2 = 0,$$

and four other like relations.

And similarly,

$$m_1L_1 + m_2L_2 + m_3L_3 = 0,$$

and five other like relations.

EXAMPLE F. *To shew that*

$$\begin{vmatrix} M_2, & N_2 \\ M_3, & N_3 \end{vmatrix} = l_1 \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}.$$

This may either be proved by multiplying out, or as follows.

Suppose the three equations

$$l_1x + m_1y + n_1z = 0 \ldots\ldots\ldots\ldots (1),$$
$$l_2x + m_2y + n_2z = 0 \ldots\ldots\ldots\ldots (2),$$
$$l_3x + m_3y + n_3z = 0 \ldots\ldots\ldots\ldots (3),$$

coexist,

(3) and (1) by the elimination of x gives us

$$M_2y + N_2z = 0 \ldots\ldots\ldots\ldots (4).$$

Similarly (1) and (2) gives us

$$M_3y + N_3z = 0 \ldots\ldots\ldots\ldots (5).$$

The condition that (4) and (5) coexist must be identical with the condition of coexistence of (1), (2), (3), that is, the equation

$$\begin{vmatrix} M_2, & N_2 \\ M_3, & N_3 \end{vmatrix} = 0$$

must be identical with the equation

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0.$$

And therefore these two determinants can only differ by a constant multiplier, which is determined by comparing the coefficient of n_3 in each, and found to be l_1, so that

$$\begin{vmatrix} M_2, & N_2 \\ M_3, & N_3 \end{vmatrix} = l_1 \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

Q. E. D.

EXAMPLE G. *To shew that*

$$\begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix} = \left\{ \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \right\}^2.$$

We have

$$l_1 \begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix} = \begin{vmatrix} l_1L_1, & M_1, & N_1 \\ l_1L_2, & M_2, & N_2 \\ l_1L_3, & M_3, & N_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1L_1 + m_1M_1 + n_1N_1, & M_1, & N_1 \\ l_1L_2 + m_1M_2 + n_1N_2, & M_2, & N_2 \\ l_1L_3 + m_1M_3 + n_1N_3, & M_3, & N_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} 1, & M_1, & N_1 \\ 0, & M_2, & N_2 \\ 0, & M_3, & N_3 \end{vmatrix}, \text{ by Ex. E,}$$

$$= \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} M_2, & N_2 \\ M_3, & N_3 \end{vmatrix}$$

$$= l_1 \left\{ \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \right\}^2, \text{ by the last example.}$$

therefore

$$\begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix} = \left\{ \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \right\}^2.$$

Q. E. D.

EXAMPLE H. *To eliminate x from the two equations*

$$ax^2 + bx + c = 0 \ldots\ldots(1),$$

$$a'x^2 + b'x + c' = 0 \ldots\ldots(2).$$

Multiplying each equation by x throughout, we get

$$ax^3 + bx^2 + cx = 0 \ldots\ldots(3),$$

$$a'x^3 + b'x^2 + cx = 0 \ldots\ldots(4),$$

and eliminating x^3, x^2, x from the four equations (1), (2), (3), (4), we have

$$\begin{vmatrix} 0, & a, & b, & c \\ 0, & a', & b', & c' \\ a, & b, & c, & 0 \\ a', & b', & c', & 0 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a, & c \\ a', & c' \end{vmatrix}^2 = \begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}.$$

Ex. J. *If*

$$uxx' + vyy' + wzz' + u'(yz' + y'z) + v'(zx' + z'x) + w'(xy' + x'y)$$

be zero for all values of x, y, z, then will

$$uvw - uu'^2 - vv'^2 - ww'^2 + 2u'v'w' = 0.$$

For since the given expression vanishes for all values of x, y, z, the coefficients of x, y, z must severally vanish. Therefore

$$ux' + w'y' + v'z' = 0,$$

$$w'x' + vy' + u'z' = 0,$$

$$v'x' + u'y' + wz' = 0,$$

and eliminating $x' : y' : z'$, we have

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0,$$

or
$$uvw - uu'^2 - vv'^2 - ww'^2 + 2u'v'w' = 0.$$

Q. E. D.

EXAMPLE K. *To expand the expression*

$$\begin{vmatrix} ax + ly, & c'x + n'y, & b'x + m'y \\ c''x + n''y, & bx + my, & a'x + l'y \\ b''x + m''y, & a''x + l''y, & cx + ny \end{vmatrix}$$

according to powers of x *and* y.

Putting $y = 0$, we obtain the term involving x^3, viz.

$$\begin{vmatrix} a, & c', & b' \\ c'', & b, & a' \\ b'', & a'', & c \end{vmatrix} x^3.$$

So putting $x = 0$, we obtain the term

$$\begin{vmatrix} l, & n', & m' \\ n'', & m, & l' \\ m'', & l'', & n \end{vmatrix} y^3.$$

Suppose the y, y, y in the three columns distinguished by suffixes, so that the determinant becomes

$$\begin{vmatrix} ax + ly_1, & c'x + n'y_2, & b'x + m'y_3 \\ c''x + n''y_1, & bx + my_2, & a'x + l'y_3 \\ b''x + m''y_1, & a''x + l''y_2, & cx + ny_3 \end{vmatrix} = 0.$$

Putting $y_2 = 0$ and $y_3 = 0$, we obtain for the term involving x^2y_1,

$$\begin{vmatrix} l, & c', & b' \\ n'', & b, & a' \\ m'', & a'', & c \end{vmatrix} x^2y_1.$$

Similarly, we find the terms

$$\begin{vmatrix} a, & n', & b' \\ c'', & m, & a' \\ b'', & l'', & c \end{vmatrix} x^2y_2 \text{ and } \begin{vmatrix} a, & c', & m' \\ c'', & b, & l' \\ b'', & a'', & n \end{vmatrix} x^2y_3.$$

Hence the whole coefficient of x^2y is

$$\begin{vmatrix} l, & c', & b' \\ n'', & b, & a' \\ m'', & a'', & c \end{vmatrix} + \begin{vmatrix} a, & n', & b' \\ c'', & m, & a' \\ b'', & l'', & c \end{vmatrix} + \begin{vmatrix} a, & c', & m' \\ c'', & b, & l' \\ b'', & a'', & n \end{vmatrix}.$$

Similarly the coefficient of xy^2 is

$$\begin{vmatrix} a, & n', & m' \\ c'', & m, & l' \\ b'', & l'', & n \end{vmatrix} + \begin{vmatrix} l, & c', & m' \\ n'', & b, & l' \\ m'', & a'', & n \end{vmatrix} + \begin{vmatrix} l, & n', & b' \\ n'', & m, & a' \\ m'', & l'', & c \end{vmatrix}.$$

Hence we arrive at the result, that

$$\begin{vmatrix} ax + ly, & c'x + n'y, & b'x + m'y \\ c''x + n''y, & bx + my, & a'x + l'y \\ b''x + m''y, & a''x + l''y, & cx + ny \end{vmatrix}$$

$$\equiv \begin{vmatrix} a, & c', & b' \\ c'', & b & a' \\ b'', & a'', & c \end{vmatrix} x^3$$

$$+ \left\{ \begin{vmatrix} l, & c', & b' \\ n'', & b & a' \\ m'', & a'', & c \end{vmatrix} + \begin{vmatrix} a, & n', & b' \\ c'', & m, & a' \\ b'', & l'', & c \end{vmatrix} + \begin{vmatrix} a, & c', & m' \\ c'', & b, & l' \\ b'', & a'' & n \end{vmatrix} \right\} x^2y$$

$$+ \left\{ \begin{vmatrix} a, & n', & m' \\ c'', & m, & l' \\ b'', & l'', & n \end{vmatrix} + \begin{vmatrix} l, & c', & m' \\ n'', & b, & l' \\ m'', & a'', & n \end{vmatrix} + \begin{vmatrix} l, & n', & b' \\ n'', & m, & a' \\ m'', & l'', & c \end{vmatrix} \right\} xy^2$$

$$+ \begin{vmatrix} l, & n' & m' \\ n'', & m, & l' \\ m'', & l'', & n \end{vmatrix} y^3$$

the expansion required.

EXAMPLE L. *As a particular case of the last theorem consider the determinant*

$$\begin{vmatrix} u + \kappa a^2, & w' + \kappa ab, & v' + \kappa ac \\ w' + \kappa ab, & v + \kappa b^2, & u' + \kappa bc \\ v' + \kappa ac, & u' + \kappa bc, & w + \kappa c^2 \end{vmatrix}.$$

Expanding in powers of κ, we obtain

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u, & w \end{vmatrix}$$

$$+\kappa\left\{\begin{vmatrix} a^2, & w', & v' \\ ab, & v, & u \\ ac, & u', & w \end{vmatrix} + \begin{vmatrix} u, & ab, & v' \\ w', & b^2, & u' \\ v, & bc, & w \end{vmatrix} + \begin{vmatrix} u, & w', & ac \\ w', & v, & bc \\ v, & u', & c^2 \end{vmatrix}\right\}$$

the coefficients of κ^2 and κ^3 vanishing since each determinant therein contains at least two identical columns.

But further, the coefficient of κ may be written

$$-\begin{vmatrix} 0, & a, & b, & c \\ a, & u, & w', & v' \\ b, & w', & v, & u' \\ c, & v', & u', & w \end{vmatrix}.$$

Hence the whole determinant becomes

$$\begin{vmatrix} 1, & 0, & 0, & 0 \\ a, & u, & w', & v' \\ b, & w', & v, & u' \\ c, & v', & u', & w \end{vmatrix} - \kappa \begin{vmatrix} 0, & a, & b & c \\ a, & u, & w', & v' \\ b, & w', & v, & u' \\ c, & v', & u', & w \end{vmatrix}$$

$$\text{or } -\kappa \begin{vmatrix} -\frac{1}{\kappa}, & a, & b, & c \\ a, & u, & w' & v' \\ b, & w', & v, & u' \\ c, & v', & u', & w \end{vmatrix}.$$

EXAMPLE M. *To eliminate h and k from the equations*

$$u = ha^2 + k\bar{u}^2, \quad 2u' = h(a^2 - b^2 - c^2) + 2k\bar{v}\bar{w},$$
$$v = hb^2 + k\bar{v}^2, \quad 2v' = h(b^2 - c^2 - a^2) + 2k\bar{w}\bar{u},$$
$$w = hc^2 + k\bar{w}^2, \quad 2w' = h(c^2 - a^2 - b^2) + 2k\bar{u}\bar{v}.$$

where

$$\bar{u} = u + v' + w', \quad \bar{v} = v + w' + u', \quad \bar{w} = w + u' + v'.$$

The resulting equations will be all the independent equations included in the system

$$\left| \begin{matrix} a^2, & b^2, & c^2, & a^2 - b^2 - c^2, & b^2 - c^2 - a^2, & c^2 - a^2 - b^2 \\ \bar{u}^2, & \bar{v}^2, & \bar{w}^2, & 2\bar{v}\bar{w}, & 2\bar{w}\bar{u}, & 2\bar{u}\bar{v} \\ u, & v, & w, & 2u', & 2v', & 2w' \end{matrix} \right| = 0.$$

But the first column increased by half the sum of the fifth and sixth gives us

$$\left| \begin{matrix} 0 \\ (\bar{u} + \bar{v} + \bar{w})\,\bar{u} \\ \bar{u} \end{matrix} \right|,$$

or, dividing by $\bar{u}$ throughout,

$$\left| \begin{matrix} 0 \\ \bar{u} + \bar{v} + \bar{w} \\ 1 \end{matrix} \right|,$$

which, being perfectly symmetrical, might have been equally obtained by combining the second, fourth and sixth, or the third, fourth and fifth columns. This shews that the original equations were not independent, but equivalent to only *four* equations and the result consists therefore of the *two* equations

$$\left\| \begin{matrix} a^2, & b^2, & c^2, & 0 \\ \bar{u}^2, & \bar{v}^2, & \bar{w}^2, & \bar{u} + \bar{v} + \bar{w} \\ u, & v, & w, & 1 \end{matrix} \right\| = 0.$$

[Exercises on Determinants will be found on pages 455, 456.]

TRILINEAR COORDINATES

AND OTHER METHODS OF

MODERN ANALYTICAL GEOMETRY.

CHAPTER I.

OF PERPENDICULAR COORDINATES REFERRED TO TWO AXES.

BEFORE proceeding to explain the method of trilinear coordinates, we will introduce a system which may be regarded as a connecting link between the usual Cartesian system, and the trilinear system:—it is the method of perpendicular coordinates referred to two oblique axes or lines of reference.

1. In the ordinary system of oblique coordinates, the position of any point P with reference to a pair of axes CX, CY, is determined by the lengths of two lines PM, PN measured parallel to either axis to meet the other: that is, if x, y be the oblique coordinates of P referred to CX, CY, then

$$x = PM, \text{ and } y = PN.$$

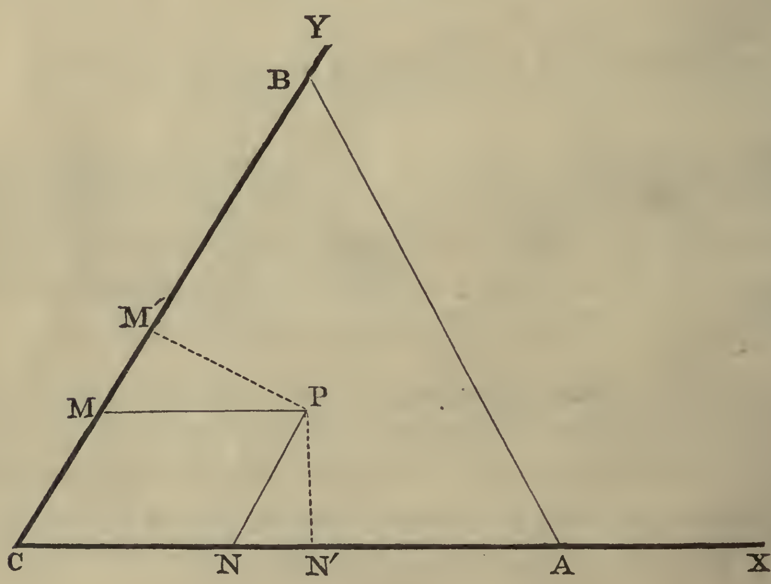
It is obvious that the position of the point would be equally well determined, if it were agreed to take as coordinates the perpendicular distances of the points from each axis, instead of measuring the distance from each axis parallel to the other. Thus, if we let fall the perpendiculars PM' and PN' on CB and CA, we observe that PM' and PN' might be used, as lawfully as PM and PN, to determine the position of P, provided it be specified beforehand which system of measurement is intended. We shall speak of PM and PN as the oblique coordinates, and PM' and PN' as the perpendicular coordinates of the point P referred to the same axes CX and CY.

We shall use α and β to denote perpendicular coordinates, reserving x and y to denote, as usual, the oblique coordinates. Thus for the point P

$$x = PM,\ y = PN:\ \alpha = PM',\ \beta = PN'.$$

When the axes are rectangular these two systems of measurement will indeed coincide, but in the case we have introduced of oblique axes they will be distinct.

Fig. 1.



The same convention as to the algebraical signs of the coordinates will hold equally in the perpendicular as in the oblique coordinates. Thus we shall consider as positive the distances from CX of all points lying on the same side with Y, and consequently the distances of all points on the opposite side will be negative. So also the positive side of CY will be that on which X lies, and the negative side the side remote from X.

2. If the angle XCY be denoted by C, then we have

$$\frac{PM'}{PM} = \sin C \text{ and } \frac{PN'}{PN} = \sin C.$$

Hence if α, β be the perpendicular coordinates of any point whose oblique coordinates are x, y, we shall have

$$\alpha = x \sin C \quad \text{ and } \beta = y \sin C;$$

or

$$x = \alpha \operatorname{cosec} C \text{ and } y = \beta \operatorname{cosec} C.$$

Consequently, if we have any relation holding good between the oblique coordinates of all points on a locus, we can, by the substitution of

$$x = \alpha \operatorname{cosec} C, \qquad y = \beta \operatorname{cosec} C,$$

obtain a relation holding good between the perpendicular coordinates of the same locus. In other words, we may transform by this substitution the oblique equation of any locus into an equation in perpendicular coordinates representing the same locus.

For example, the equation in oblique coordinates to the straight line AB cutting off intercepts $CA = b$ and $CB = a$ from the axes is known to be

$$\frac{x}{b} + \frac{y}{a} = 1.$$

Hence the equation to the same line in perpendicular coordinates will be

$$\frac{\alpha \operatorname{cosec} C}{b} + \frac{\beta \operatorname{cosec} C}{a} = 1,$$

$$\text{or } a\alpha + b\beta = ab \sin C;$$

and in a similar way any other equation might be transformed.

3. Or, instead of taking an *equation*, we might by the same substitution transform *any function whatever* of the oblique coordinates into an equivalent function of the perpendicular coordi-

nates. For example, writing the equation to the same straight line AB in the form

$$ab - ax - by = 0,$$

we can at once write down the expression for the perpendicular distance of any point (x', y') from it, viz.

$$\pm \frac{ab - ax' - by'}{\sqrt{a^2 + b^2 - 2ab \cos C}} \sin C.$$

Hence if α', β' be the perpendicular coordinates of the same point, we shall have as the expression for the distance

$$\pm \frac{ab \sin C - a\alpha' - b\beta'}{\sqrt{a^2 + b^2 - 2ab \cos C}};$$

or if c denote the distance AB, and Δ the area of the triangle ABC so that

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

and $$2\Delta = ab \sin C,$$

then the expression (for the perpendicular distance from AB of the point whose coordinates are α' and β') becomes

$$\pm \frac{2\Delta - a\alpha' - b\beta'}{c}.$$

This is given here merely as an example of transformation of coordinates from the one system to the other, but the result is one which will be seen hereafter to have an important bearing on trilinear coordinates.

4. The student will do well to examine at this stage of the subject the interpretation of some of the simpler equations connecting α and β.

(1) Consider the equation

$$\alpha = 0.$$

It is evidently satisfied by all points on the line CY and by no other: it is therefore the equation to this axis.

(2) Consider the equation

$$\alpha = d,$$

where d is a constant. It is equally obvious that this is satisfied at any point on a line parallel to CB at a distance d from it on the side towards X. Similarly the equation

$$\alpha = -d$$

is satisfied at any point on a parallel line at an equal distance on the other side of CY.

(3) Consider the equation

$$\alpha = \beta.$$

The equation speaks to us of points whose perpendicular distance from CX is equal to the perpendicular distance from CY, and of the same algebraical sign. The locus of such points is the interior bisector of the angle XCY, which is therefore the locus of the equation.

(4) Consider the equation

$$\alpha = -\beta.$$

This speaks of points whose perpendicular distances from the axes are equal, but of opposite sign. These points will be seen to lie on the *exterior* bisector of the angle XCY, which will therefore be the locus of the equation.

(5) Consider the equation

$$\alpha = m\beta.$$

Suppose P to be a point on this line, join PC and draw PM', PN' perpendiculars on CY, CX, then (by the equation)

$$PM' = m \,.\, PN',$$

and therefore

$$\frac{PM'}{PC} = m \,.\, \frac{PN'}{PC},$$

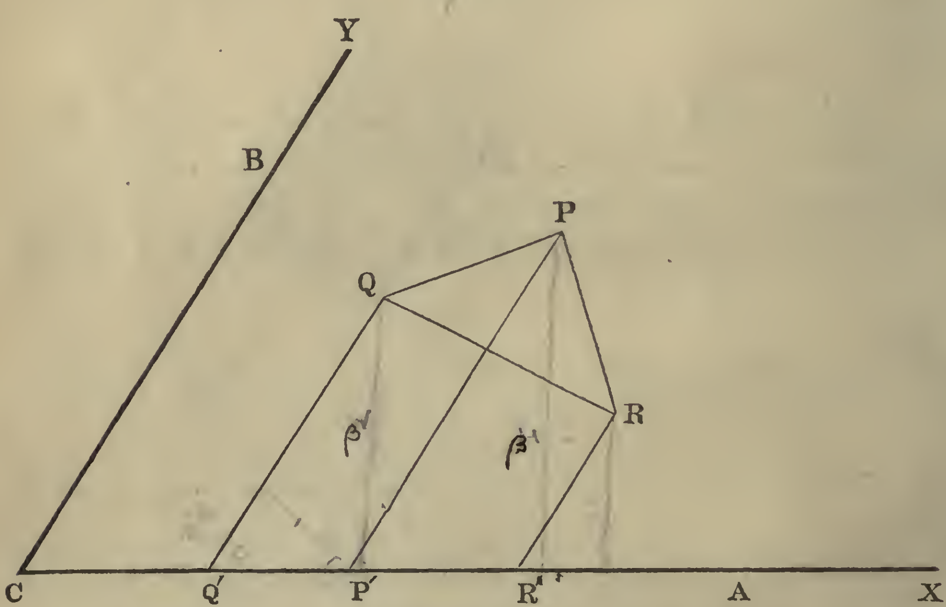
$$\sin PCY = m \,.\, \sin PCX,$$

or (in words), P lies on a straight line dividing the angle XOY into two parts PCY, PCX such that their sines are in the ratio $m : 1$.

This straight line will therefore be the locus of the equation.

5. *To find the area of a triangle the perpendicular coordinates of whose angular points are given with respect to a pair of oblique axes.*

Fig. 2.



Let PQR be the triangle and (α_1, β_1), (α_2, β_2), (α_3, β_3) the perpendicular coordinates of the angular points P, Q, R referred to the oblique axes CX, CY.

Parallel to CY draw PP', QQ', RR' to meet CX in P', Q', R'.

Then $\Delta PQR = \text{trapezium } PQQ'P' + \text{trapezium } PRR'P'$
$$- \text{trapezium } QRR'Q'.$$

But the trapezium $PQQ'P'$ stands on a base $P'Q'$ equal to $(\alpha_1 - \alpha_2) \operatorname{cosec} C$ and it has a mean altitude equal to $\frac{1}{2}(\beta_1 + \beta_2)$.

$$\text{Hence its area} = \frac{1}{2} \operatorname{cosec} C (\beta_1 + \beta_2)(\alpha_1 - \alpha_2).$$

$$\text{Similarly, area } PRR'P' = \frac{1}{2} \operatorname{cosec} C (\beta_3 + \beta_1)(\alpha_3 - \alpha_1),$$

$$\text{and area } QRR'Q' = \frac{1}{2} \operatorname{cosec} C (\beta_2 + \beta_3)(\alpha_3 - \alpha_2);$$

therefore the area of the triangle PQR

$$= \frac{1}{2} \operatorname{cosec} C \left\{(\beta_1 + \beta_2)(\alpha_1 - \alpha_2) + (\beta_3 + \beta_1)(\alpha_3 - \alpha_1) - (\beta_2 + \beta_3)(\alpha_3 - \alpha_2)\right\}$$

$$= \frac{1}{2} \operatorname{cosec} C \left\{\alpha_1 (\beta_2 - \beta_3) + \alpha_2 (\beta_3 - \beta_1) + \alpha_3 (\beta_1 - \beta_2)\right\},$$

or with determinant notation

$$= \frac{1}{2} \operatorname{cosec} C \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ 1, & 1, & 1 \end{vmatrix}.$$

Exercises on Chapter I.

(1) If (α, β) be the perpendicular coordinates of the point whose oblique coordinates (referred to the same axes) are (x, y), and (α', β') the perpendicular coordinates of the point whose oblique coordinates are (x', y'), shew that $(\alpha + \kappa\alpha', \beta + \kappa\beta')$ will be the perpendicular coordinates of the point whose oblique coordinates are $(x + \kappa x', y + \kappa y')$.

(2) Shew that the points whose perpendicular coordinates are (α, β), (α', β'), $(\alpha - \alpha', \beta - \beta')$ and $(\alpha + \alpha', \beta + \beta')$ lie all on one straight line, provided $\alpha\beta' = \alpha'\beta$.

(3) Find the equation in perpendicular coordinates to the straight line drawn from the origin at right angles to the straight line CX.

(4) Write down the equation to the other diagonal of the parallelogram two of whose sides are the axes, and one of whose diagonals has the equation

$$a\alpha + b\beta = ab \sin C.$$

What is the area of this parallelogram?

(5) The distance between two points in terms of their oblique coordinates (x, y) and (x', y') is given by the formula

$$\rho^2 = (x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos C;$$

hence write down the corresponding formula for perpendicular coordinates.

(6) Find the area of the triangle one of whose angles is at the origin, and the other two at the points whose perpendicular coordinates are (α, β) and (α', β').

(7) Find the area of a triangle whose base is of length d in the axis CX, and whose vertex is at the point whose perpendicular coordinates are (α, β).

(8) Find the perpendicular coordinates of the point bisecting the straight line joining the points whose coordinates are (α, β) and (α', β').

(9) Find the equation in perpendicular coordinates to the straight line drawn from the origin, at right angles to the straight line whose equation is

$$a\alpha + b\beta = ab \sin C.$$

CHAPTER II.

TRILINEAR COORDINATES.—THE POINT.

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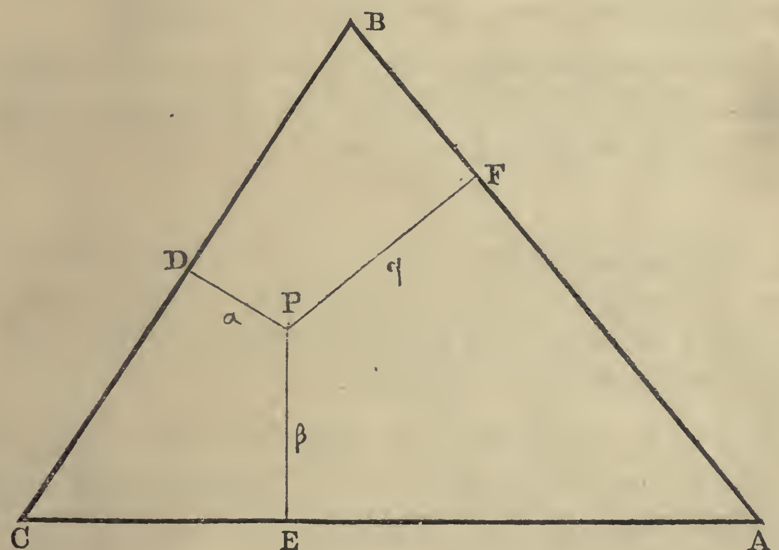
6. We will renew our construction before proceeding to the next step in the development of the system of trilinear coordinates.

Let BC, CA, AB be any three fixed straight lines forming a triangle, and let P denote some point in the plane of the triangle.

We have seen that if the perpendicular distances of P from any two fixed straight lines (CB and CA for instance) be given the point P is determinate, and that these distances may be regarded as the coordinates of P.

Fig. 3.



Now let α, β, γ denote perpendicular distances of P from the three fixed straight lines BC, CA, AB. Then, as we have seen, any two of these (e.g. α, β) may be regarded as the coordinates of P with respect to the corresponding axes (BC, CA), and the remaining perpendicular (γ) may be expressed in terms of these two (α and β), and known constants, by the method of the example, Art. 3 (or more simply, as we shall shew presently).

But there are advantages, as the sequel will shew, in regarding *all* the three perpendiculars as coordinates of P, and thus expressing P's position at once with respect to the *three* fixed axes, or *lines of reference* (as it is more usual to call them).

We shall regard as positive the distances from any line of reference to points on the same side with the intersection of the other two lines of reference, and consequently the distances of points on the other side will be negative; thus

A lies on the positive side of BC,

B .... CA,

C .... AB.

Thus all points within the triangle formed by the lines of reference (which we shall briefly call the triangle of reference) have *all* their coordinates positive, and points without the triangle of reference have either one or two coordinates negative.

It will be observed that it is impossible to find a point on the negative side of *all* the lines of reference, that is, no point has *all* its three coordinates negative.

7. We have observed that when two of the trilinear coordinates of a point are given the third may be calculated. We proceed now to investigate a simple equation, which we shall find connects these three coordinates.

Let P be any point whose coordinates are α, β, γ and join PA, PB, PC, and draw PD, PE, PF perpendiculars on BC, CA, AB respectively.

First, suppose P *within* the triangle ABC as in fig. 3; then the triangles

$$PBC + PCA + PAB = \text{whole triangle } ABC.$$

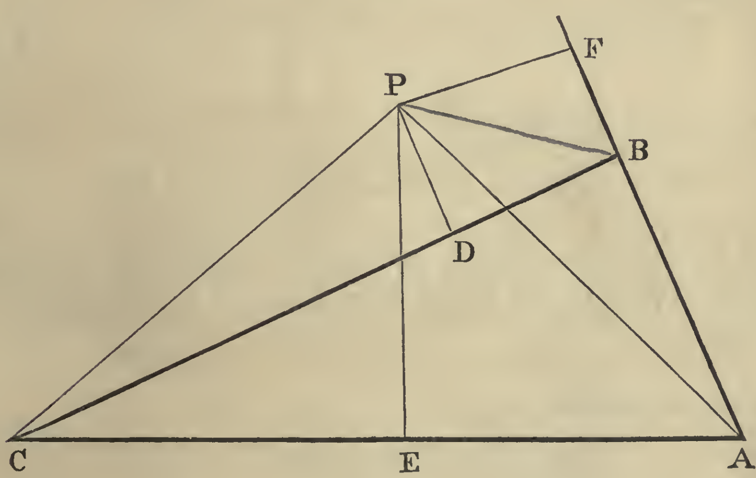
But $BC.PD$ or $a\alpha$ is the double of the area of the triangle PBC: so $b\beta$ and $c\gamma$ are the doubles of the triangles PCA and PAB;

$$\therefore\ a\alpha + b\beta + c\gamma = 2\Delta,$$

where Δ denotes the area of the triangle of reference.

Secondly, suppose P without the triangle. Let it be on the negative side of BC and on the positive sides of CA, AB, as in figure 4.

Fig. 4.



In this case the triangles

$$PCA + PAB - PBC = \text{whole triangle } ABC.$$

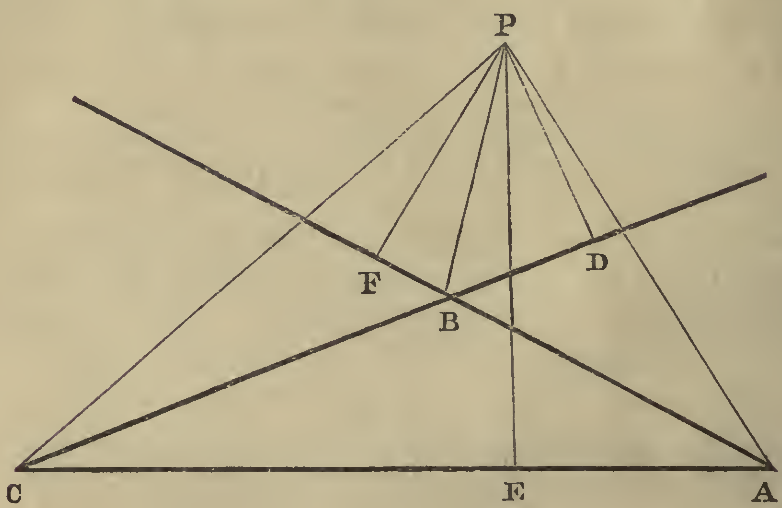
But in this case α is negative, so that the length $PD = -\alpha$; and therefore $-a\alpha$ represents the double of the area of the triangle PBC.

Hence, as before,

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

Again, suppose P on the negative side of both BC and AB, as in fig. 5.

Fig. 5.



In this case the triangles

$$PCA - PAB - PBC = \text{the triangle } ABC.$$

But in this case both α and γ are negative, and therefore $-a\alpha$ and $-c\gamma$ represent the double areas of the triangles PBC and PAB.

Hence, as before,

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

Hence we see that if α, β, γ be the coordinates of any point whatever, they are connected by the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

8. We might have deduced the result just obtained at once from the result in Art. 3, taking care to determine the ambiguous sign so that the perpendicular from the origin C should be positive in accordance with the convention of Art. 6.

Thus we should have written at once

$$\gamma = \frac{2\Delta - a\alpha - b\beta}{c},$$

or $$a\alpha + b\beta + c\gamma = 2\Delta,$$

but the proof given in Art. 7 is more simple in its character.

9. Let ρ be the radius of the circle circumscribing the triangle ABC.

Then by trigonometry,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\rho;$$

hence the equation obtained in Art. 7 may be written

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \frac{\Delta}{\rho} = S \text{ suppose.}$$

10. The equation of Art. 7, or the equivalent form just obtained plays a very important part in trilinear coordinates. It enables us to make every equation involving α, β, γ *homogeneous*, for, since

$$\frac{a\alpha + b\beta + c\gamma}{2\Delta} = 1 \quad \text{(Art. 7)},$$

we are at liberty to multiply any term we please in an equation by the fraction $\frac{a\alpha + b\beta + c\gamma}{2\Delta}$, thus raising by unity the *order* of the term. By repeating this operation we can raise every term of an equation up to the same order as the term of highest order, and thus render our equation homogeneous.

For example, if we have the equation

$$\alpha^3 + 3\alpha\gamma + 5\beta = 1,$$

we can raise every term to the third order: thus we get the homogeneous equation

$$\alpha^3 + 3\alpha\gamma\frac{a\alpha+b\beta+c\gamma}{2\Delta} + 5\beta\left(\frac{a\alpha+b\beta+c\gamma}{2\Delta}\right)^2$$
$$= \left(\frac{a\alpha+b\beta+c\gamma}{2\Delta}\right)^3,$$

which we might proceed to simplify.

·11. If the ratios of the coordinates of any point be given, the point is determinate, and the actual values of the coordinates can be found by means of the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

We may proceed thus:

Let the coordinates be proportional to $\lambda : \mu : \nu$, then

$$\frac{\alpha}{\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{\nu},$$

and therefore each of these ratios

$$= \frac{a\alpha+b\beta+c\gamma}{a\lambda+b\mu+c\nu} = \frac{2\Delta}{a\lambda+b\mu+c\nu}.$$

Hence
$$\alpha = \frac{2\lambda\Delta}{a\lambda+b\mu+c\nu},$$
$$\beta = \frac{2\mu\Delta}{a\lambda+b\mu+c\nu},$$
$$\gamma = \frac{2\nu\Delta}{a\lambda+b\mu+c\nu}.$$

·12. We may however observe that in practice we very rarely require the absolute values of the coordinates. For advantage is almost universally taken of the principle detailed in Art. 10, by means of which our equations in trilinear coordinates are homogeneous. And it is scarcely necessary to point out that a homogeneous equation in α, β, γ will not involve in any way the actual values of the quantities, but will only involve their ratios.

For example,

$$\alpha^3 - 3\alpha^2\beta + \gamma^3 = 0$$

may be written

$$\left(\frac{\alpha}{\gamma}\right)^3 - 3\left(\frac{\alpha}{\gamma}\right)^2 . \frac{\beta}{\gamma} + 1 = 0,$$

where only the ratios $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\gamma}$ are involved.

Again, if we have to substitute the coordinates of a point in a homogeneous equation it is not necessary to know more than the ratios of the coordinates.

For suppose the equation were

$$\alpha\beta\gamma - 3\alpha^2\beta + 5\beta^2\gamma + \gamma^3 = 0,$$

and suppose the coordinates of the point were known to be proportional to $l : m : n$.

The actual values of the coordinates may be supposed to be kl, km, kn, but it is not necessary generally to know the value of the multiplier k; for if we substitute kl, km, kn in the given equation, we get

$$k^3lmn - 3k^3l^2m + 5k^3m^2n + k^3n^3 = 0,$$

or, dividing by k^3 throughout,

$$lmn - 3l^2m + 5m^2n + n^3 = 0,$$

so k disappears from the final result, and therefore a knowledge of its value was unnecessary.

We shall conclude this chapter with some examples in which we shall determine the coordinates of several points related to the triangle of reference, leading to results which are continually required in the solution of problems.

·13. *To find the coordinates of the angular points of the triangle of reference.*

For the point A it is evident that $\beta = 0$ and $\gamma = 0$. Also $a\alpha = 2\Delta$, hence we can write down the coordinates

of A, $\frac{2\Delta}{a}$, 0, 0.

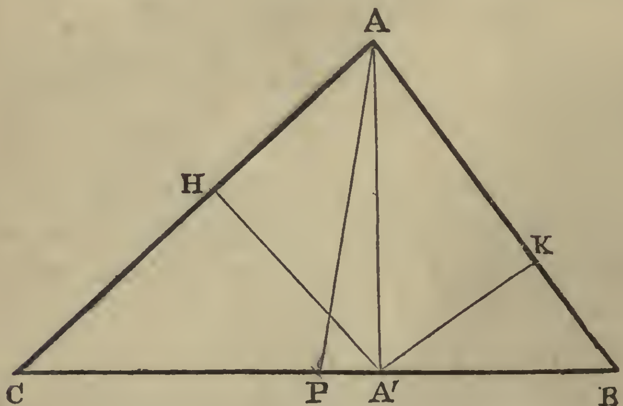
So, of B, 0, $\frac{2\Delta}{b}$, 0,

and of C, 0, 0, $\frac{2\Delta}{c}$.

N.B. The angular points of the triangle of reference are conveniently spoken of as "the points of reference."

·14. *To find the coordinates of the middle point of* BC.

Fig. 6.



Let P be the middle point, and suppose α, β, γ the coordinates of P.

Since P lies on BC we have $\alpha = 0$.

Also $b\beta$ = twice the triangle APC
= the triangle ABC,

since APB, APC on equal bases and of the same altitude are equal.

Therefore $$\beta = \frac{\Delta}{b},$$

and so $$\gamma = \frac{\Delta}{c}.$$

Hence the coordinates are

$$0, \quad \frac{\Delta}{b}, \quad \frac{\Delta}{c}.$$

15. *To find the coordinates of the foot of the perpendicular from* A *upon* BC.

Let AA' be the perpendicular and let α, β, γ be the coordinates of A'. Draw $A'H$, $A'K$ perpendicular to CA, AB respectively;

then $$\alpha = 0, \quad \beta = A'H, \quad \gamma = A'K.$$

But $$\frac{A'H}{A'A} = \cos AA'H = \cos C;$$

$$\therefore A'H = A'A \cos C;$$

$$\text{or } \beta = \frac{2\Delta}{a} \cos C;$$

$$\text{so } \gamma = \frac{2\Delta}{a} \cos B.$$

Hence the coordinates are

$$0, \quad \frac{2\Delta}{a} \cos C, \quad \frac{2\Delta}{a} \cos B.$$

16. *To find the coordinates of the centre of the inscribed circle of the triangle of reference.*

This point is equally distant from the three lines of reference;

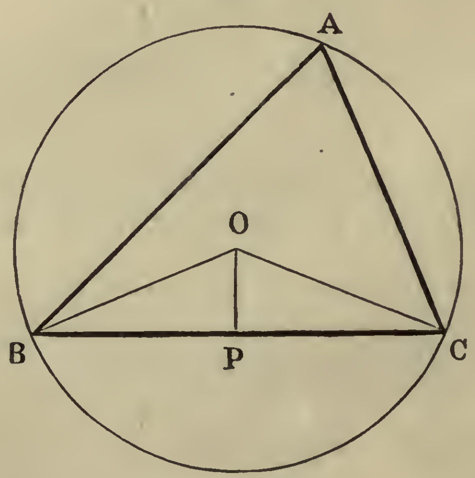
$$\therefore \alpha = \beta = \gamma;$$

$$\therefore \frac{\alpha}{1} = \frac{\beta}{1} = \frac{\gamma}{1} = \frac{a\alpha + b\beta + c\gamma}{a + b + c} = \frac{2\Delta}{a + b + c}.$$

Hence each of the coordinates is $\frac{\Delta}{s}$, where s denotes, as in Trigonometry, half the sum of the sides of the triangle.

17. *To find the coordinates of the centre of the circle circumscribing the triangle of reference.*

Fig. 7.



Let O be the centre, α, β, γ its coordinates; join OB, OC and draw OP perpendicular on BC; then (Euclid, III. 2), BC is bisected in P.

Hence the triangles OPB, OPC are equal in all respects.

Now the angle BOC at centre = twice angle BAC at circumference.

Hence $$\angle BOP = \frac{1}{2} BOC = A;$$

$$\therefore \frac{OP}{BP} = \cot A, \quad \text{or} \quad OP = BP \cot A,$$

$$\text{i.e.} \quad \alpha = \frac{a}{2} \cot A.$$

So $$\beta = \frac{b}{2} \cot B, \quad \gamma = \frac{c}{2} \cot C,$$

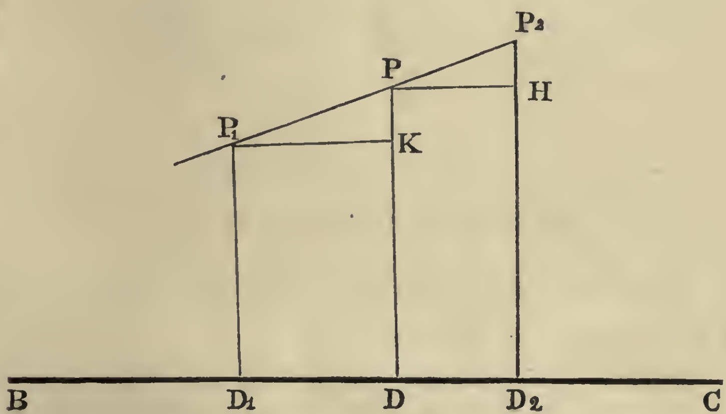
which give the required coordinates.

Cor. If ρ be the radius of the circumscribed circle, these coordinates may be expressed thus:

$$\alpha = \rho \cos A, \quad \beta = \rho \cos B, \quad \gamma = \rho \cos C.$$

18. *To find the coordinates of the point which divides in a given ratio the straight line joining two points whose coordinates are given.*

Fig. 8.



Let P_1, P_2 be the given points and $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ their coordinates, $m : n$ the given ratio. Suppose P the point required, and let (α, β, γ) be its required coordinates;

then $$P_1P : PP_2 = m : n.$$

Draw PD, P_1D_1, P_2D_2 perpendiculars on BC, and through P and P_1 draw PH and P_1K parallel to BC, and meeting P_2D_2 and PD respectively in H and K.

Then by similar triangles

$$PK : P_2H = PP_1 : P_2P = m : n,$$

i. e. $$\alpha - \alpha_1 : \alpha_2 - \alpha = m : n,$$

whence $$n\alpha - n\alpha_1 = m\alpha_2 - m\alpha;$$

or $$(m+n)\,\alpha = m\alpha_2 + n\alpha_1;$$

or $$\alpha = \frac{m\alpha_2 + n\alpha_1}{m+n};$$

Similarly we may shew that

$$\beta = \frac{m\beta_2 + n\beta_1}{m+n},$$

and
$$\gamma = \frac{m\gamma_2 + n\gamma_1}{m+n},$$

which give the required coordinates.

Cor. The coordinates of the middle point between $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are

$$\frac{\alpha_1 + \alpha_2}{2}, \quad \frac{\beta_1 + \beta_2}{2}, \quad \frac{\gamma_1 + \gamma_2}{2}.$$

Exercises on Chapter II.

(10) Find the coordinates of the points of trisection of the sides of the triangle of reference.

(11) If A' be the middle point in the side BC of the triangle of reference ABC, find the coordinates of a point P in AA', such that $AP = 2A'P$.

(12) If AA' be the perpendicular from the point of reference A upon the opposite side BC, find the coordinates of a point P, so dividing this line that

$$AP : PA' = \cos A : \cos B . \cos C.$$

(13) Find the coordinates of the centres of the circles escribed to the triangle of reference.

(14) Render the following system of equations in trilinear coordinates homogeneous:

$$(la^2 + mb^2 + nc^2)\,\alpha^2 - 4\Delta la\alpha + 4\Delta^2 l$$
$$= (la^2 + mb^2 + nc^2)\,\beta^2 - 4\Delta mb\beta + 4\Delta^2 m$$
$$= (la^2 + mb^2 + nc^2)\,\gamma^2 - 4\Delta nc\gamma + 4\Delta^2 n.$$

CHAPTER III.

TRILINEAR COORDINATES. THE STRAIGHT LINE.

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19. *To find the area of a triangle, the trilinear coordinates of whose angular points are given.*

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the coordinates of the angular points, and let A denote the area of the triangle.

Now we have already in Art. 5 found an expression for A in terms of the coordinates α and β of each angular point. We may however express that result in a form symmetrical with respect to the *three* coordinates of each point.

Thus, taking the result of Art. 5,

$$A = \frac{1}{2}\operatorname{cosec} C \left\{\alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1) + \alpha_3(\beta_1 - \beta_2)\right\}$$

$$= \frac{1}{2}\operatorname{cosec} C \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ 1, & 1, & 1 \end{vmatrix} = \frac{\operatorname{cosec} C}{2S} \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ S, & S, & S \end{vmatrix}.$$

But diminishing the last row of the determinant by the sum of the first row multiplied by $\sin A$, and the second multiplied by $\sin B$, and remembering that the relation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = S,$$

is true for the coordinates of each point, we get

$$A = \frac{\operatorname{cosec} C}{2S} \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1 \sin C, & \gamma_2 \sin C, & \gamma_3 \sin C \end{vmatrix}$$

$$= \frac{1}{2S} \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = \frac{1}{2S} \begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix}$$

a perfectly symmetrical expression for the area of the triangle.

20. *To find the condition that three points whose trilinear coordinates are given should lie on one straight line.*

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the three points.

That they should lie on a straight line is the same thing as that the area of the triangle formed by them should be zero.

Hence, by the last article, the condition is

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix} = 0.$$

21. It follows from Art. 20 that the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0$$

speaks to us of a variable point (α, β, γ) which lies on one straight line with the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$. It is manifest that the equation will be satisfied if (α, β, γ) denote any point whatever on this straight line: and that it cannot be satisfied if (α, β, γ) lie elsewhere.

Hence the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0$$

is the equation of the straight line joining the points

$$(\alpha_1, \beta_1, \gamma_1), \quad (\alpha_2, \beta_2, \gamma_2).$$

If L, M, N be equal to, *or proportional to* the minors

$$\begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}$$

of the above determinant the equation becomes

$$L\alpha + M\beta + N\gamma = 0.$$

COR. Every straight line is represented by a homogeneous equation of the first order in α, β, γ. We proceed to shew that the converse of this proposition is also true.

22. *Every homogeneous equation of the first order represents a straight line.*

Let $$l\alpha + m\beta + n\gamma = 0$$

be any homogeneous equation of the first order in α, β, γ. It shall represent a straight line.

By giving γ any value (γ_1 suppose) in the system of equations

$$l\alpha + m\beta + n\gamma = 0,$$
$$a\alpha + b\beta + c\gamma = 2\Delta,$$

we shall get corresponding values (α_1, β_1 suppose) for α and β.

Thus we can find coordinates $(\alpha_1, \beta_1, \gamma_1)$ representing a point upon the locus of the given equation.

Similarly by giving γ another value (γ_2 suppose) we can find the coordinates $(\alpha_2, \beta_2, \gamma_2)$ of another point upon the locus.

But since $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ represent points lying on the locus of the equation

$$l\alpha + m\beta + n\gamma = 0 \text{ ....}(1),$$

we have the relations

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0 \text{.......................}(2),$$

and
$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0 \text{.......................}(3).$$

By means of (2) and (3) we can eliminate the ratios $l : m : n$ from (1); thus the equation (1) will take the form

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0,$$

which we know (by Art. 21) to be the equation to a straight line.

Hence every homogeneous equation of the first order as trilinear coordinates represents a straight line.

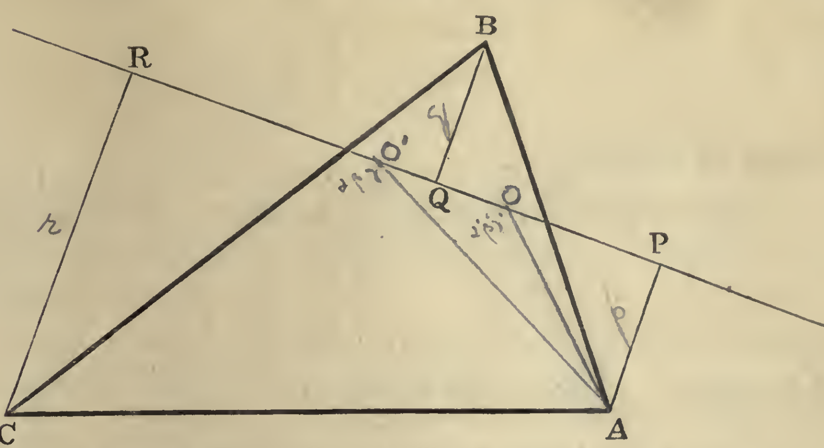
NOTE. The only apparent exception is when the two equations $l\alpha + m\beta + n\gamma = 0$ and $a\alpha + b\beta + c\gamma = 2\Delta$ are inconsistent, that is, when l, m, n are proportional to a, b, c. This case we shall discuss separately in Chapter IV.

23. By Art. 21 we are able to write down the equation to any straight line in terms of the coordinates of any two points upon it.

It is often desirable to express it in terms of any other constants which will determine the straight line. For instance, a straight line is determinate when its perpendicular distances from the three points of reference are given; we proceed to determine the equation to a straight line in terms of these three distances.

Let $AP = p$, $BQ = q$, $CR = r$ be the three perpendiculars from the angular points A, B, C upon a straight line PQR: it

Fig. 9.



is required to find the equation to the straight line in terms of these quantities p, q, r.

Let O, O' be any two points upon the straight line, and let their coordinates be α, β, γ and α', β', γ', and let ρ denote the distance between them, then

$$p\rho = \text{twice area } AOO'$$

$$= S \begin{vmatrix} \frac{2\Delta}{a}, & 0, & 0 \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} \text{ by Art. 13,}$$

therefore multiplying by $a\alpha$,

$$ap\alpha\rho = 2\Delta S \begin{vmatrix} \alpha, & 0, & 0 \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}$$

so
$$bq\beta\rho = 2\Delta S \begin{vmatrix} 0, & \beta, & 0 \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}$$

and $$cr\gamma\rho = \frac{2\Delta\aleph}{S} \begin{vmatrix} 0, & 0, & \gamma \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}$$

therefore by addition

$$(ap\alpha + bq\beta + cr\gamma)\,\rho = \frac{2\Delta\aleph}{S} \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} = 0,$$

and therefore $$ap\alpha + bq\beta + cr\gamma = 0.$$

This being a relation among the coordinates α, β, γ of any point whatever on the straight line PQR, is the equation to that straight line, and it is expressed in terms of the perpendiculars p, q, r. Therefore it is the equation required.

24. We shewed in Art. 22, that the equation

$$l\alpha + m\beta + n\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

must always represent a straight line.

In Arts. 21 and 23, we have found the equation to a straight line in terms of the coordinates of two points upon it, and in terms of its perpendicular distances from the points of reference, but both the equations thus found are particular cases of the general form (1).

Thus by comparing the various articles we are able to explain the coefficients in the general equation. We may either interpret l, m, n, as proportional to the determinants

$$\begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix} \quad \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix} \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix},$$

where $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, denote points upon the line, or we may say that they are proportional to ap, bq, cr, where p, q, r are the perpendicular distances of the line from the points of reference.

25. It should be noticed that the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0$$

will not be altered if we substitute for α_1, β_1, and γ_1, or for α_2, β_2, and γ_2, any quantities proportional to them. For this is only equivalent to multiplying the equation throughout by a fixed ratio.

Hence it is not necessary, in order to form the equation to the straight line joining two points, to know the actual coordinates of the points, but it will suffice if the *ratios of the coordinates* are given.

Thus if two points be determined by the equations

$$\frac{\alpha}{\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{\nu},$$

and

$$\frac{\alpha}{\lambda'} = \frac{\beta}{\mu'} = \frac{\gamma}{\nu'},$$

the equation to the straight line joining them will be

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} = 0$$

26. *To find the condition that three straight lines whose equations are given should pass through one point.*

Let

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$
$$l''\alpha + m''\beta + n''\gamma = 0,$$

be the three equations. If the three straight lines all pass through one point, all these equations will be satisfied by the coordinates (α', β', γ', suppose), of the point.

Hence,

$$l\alpha' + m\beta' + n\gamma' = 0,$$
$$l'\alpha' + m'\beta' + n'\gamma' = 0,$$
$$l''\alpha' + m''\beta' + n''\gamma' = 0.$$

Therefore eliminating $\alpha' : \beta' : \gamma'$ we get

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 0,$$

which will be the condition required.

27. *Every straight line passing through the point of intersection of the two straight lines whose equations are*

$$l\alpha + m\beta + n\gamma = 0 \text{ .... } (1),$$

$$l'\alpha + m'\beta + n'\gamma = 0 \text{ .... } (2),$$

will have an equation of the form

$$l\alpha + m\beta + n\gamma + \kappa (l'\alpha + m'\beta + n'\gamma) = 0 \text{ .... } (3),$$

where κ is an arbitrary constant, and by giving a suitable value to κ the equation (3) *can be made to represent any particular straight line passing through the point of intersection of* (1) *and* (2).

Suppose $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ to be the point of intersection of (1) and (2); therefore these coordinates satisfy the equations (1) and (2): therefore

$$l\bar{\alpha} + m\bar{\beta} + n\bar{\gamma} = 0,$$

$$l'\bar{\alpha} + m'\bar{\beta} + n'\bar{\gamma} = 0.$$

Multiplying the second of these by κ and adding, we get

$$l\bar{\alpha} + m\bar{\beta} + n\bar{\gamma} + \kappa (l'\bar{\alpha} + m'\bar{\beta} + n'\bar{\gamma}) = 0,$$

which shews that $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$, satisfy the equation (3).

But the equation (3) being of the first order represents a straight line. Hence it represents a straight line passing through the intersection of (1) and (2). Q. E. D. (i).

Also by giving a suitable value to κ the equation (3) will represent *any* straight line through $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$.

For suppose it be required to make it represent a straight line passing through any point $(\alpha', \beta', \gamma')$.

The condition that this point should lie on the locus is

$$l\alpha' + m\beta' + n\gamma' + \kappa (l'\alpha' + m'\beta' + n'\gamma') = 0.$$

Hence, if we give κ the value determined by this equation, i.e.,

$$\kappa = -\frac{l\alpha' + m\beta' + n\gamma'}{l'\alpha' + m'\beta' + n'\gamma'},$$

the equation (3) will represent the line joining the points $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$, and $(\alpha', \beta', \gamma')$. Hence we can determine κ so as to make the equation (3) represent *any* straight line through the point of intersection of (1) and (2). Q. E. D. (ii).

In this case the equation (3) takes the form

$$\frac{l\alpha + m\beta + n\gamma}{l\alpha' + m\beta' + n\gamma'} - \frac{l'\alpha + m'\beta + n'\gamma}{l'\alpha' + m'\beta' + n'\gamma'} = 0,$$

which is therefore the equation to the straight line joining $(\alpha', \beta', \gamma')$, to the point of intersection of the straight lines

$$l\alpha + m\beta + n\gamma = 0,$$

and

$$l'\alpha + m'\beta + n'\gamma = 0.$$

28. If we use u and v to denote the expressions

$$l\alpha + m\beta + n\gamma \text{ and } l'\alpha + m'\beta + n'\gamma;$$

the third equation of the last article will be represented by

$$u + \kappa v = 0.$$

Hence we may briefly express our result as follows.

If u and v be any functions of the first degree of the coordinates, then the equation

$$u + \kappa v = 0,$$

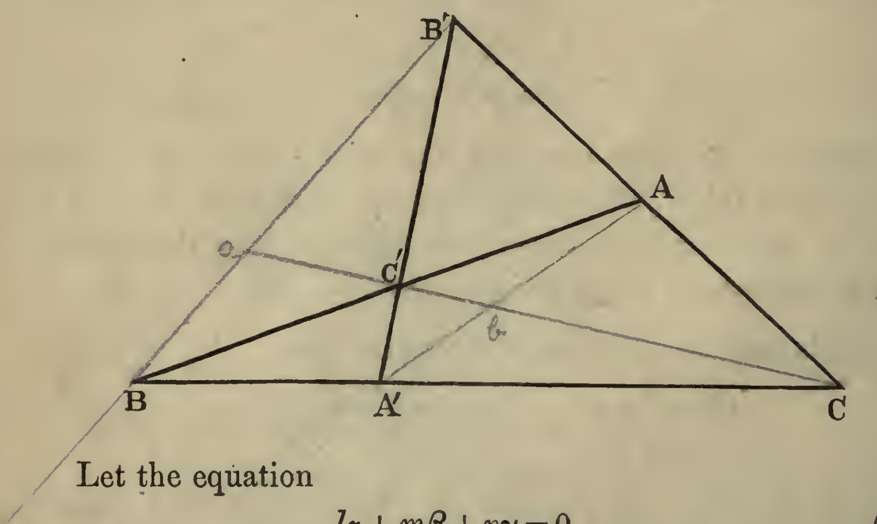
will represent a straight line passing through the intersection of the straight lines represented by

$$u = 0 \text{ and } v = 0,$$

and by giving a suitable value to κ, it will represent any *such straight line.*

29. The following affords a good illustration of the use of the foregoing article.

Fig. 10.



Let the equation

$$l\alpha + m\beta + n\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

represent a straight line meeting BC in A', CA in B', AB in C'.

Consider the equation

$$m\beta + n\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

From its present form we observe that it is a straight line passing through the intersection of $\beta = 0$ and $\gamma = 0$, that is, through A, but if we write it in the form

$$(l\alpha + m\beta + n\gamma) - l\alpha = 0,$$

we perceive that it passes through the intersection of $\alpha = 0$ and

$$l\alpha + m\beta + n\gamma = 0,$$

that is, through A'.

Hence it represents the straight line AA'.

Similarly the equations

$$n\gamma + l\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

$$l\alpha + m\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots (4),$$

will represent BB' and CC' respectively.

Further let BB', CC' meet in a; CC', AA' in b; AA', BB' in c.

Then the equation

$$m\beta - n\gamma = 0 \quad \text{.............................} \quad (5),$$

which represents a straight line through A, being equivalent to

$$l\alpha + m\beta - (n\gamma + l\alpha) = 0,$$

must pass through a.

Hence it represents the straight line Aa.

Similarly the equations

$$n\gamma - l\alpha = 0 \quad \text{.............................} \quad (6),$$

$$l\alpha - m\beta = 0 \quad \text{.............................} \quad (7),$$

will represent Bb and Cc respectively.

But further the equation (7) may be obtained from the equations (5) and (6) by addition. Hence the straight line (7) passes through the intersection of the straight lines (5) and (6).

That is, Aa, Bb, Cc meet in a point.

We conclude this chapter with some Examples of the methods we have been investigating.

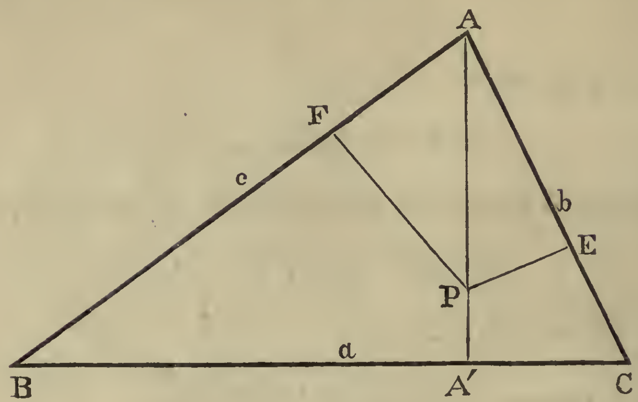
30. *To find the equation to the perpendicular from the point of reference* A *upon the line* BC.

First Method. Let AA' be the line in question. The perpendicular distances of the line from the angular points A, B, C are respectively

$$0, \quad c\cos B, \quad -b\cos C,$$

where we give opposite signs to the latter two distances, since they are measured on opposite sides of AA'.

Fig. 11.



Hence by Art. 23 the equation is

$$\beta bc \cos B - \gamma cb \cos C = 0,$$

or

$$\beta \cos B - \gamma \cos C = 0.$$

SECOND METHOD. Let $AA' = p$, then the coordinates of A are p, 0, 0, and the coordinates of A' are 0, $p \cos C$, $p \cos B$. By Art. 21 the straight line joining these points has the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ p, & 0, & 0 \\ 0, & p\cos C, & p\cos B \end{vmatrix} = 0,$$

or (dividing by p^2, and evaluating the determinant),

$$\beta \cos B - \gamma \cos C = 0,$$

which will be the equation required.

THIRD METHOD. Let P be any point in AA', and on AC let fall the perpendicular $PE = \beta$, and on BA the perpendicular $PF = \gamma$.

Then since the angle PAC is the complement of C,

$$\frac{PE}{PA} = \cos C.$$

Similarly, since the angle PAB is the complement of B,

$$\frac{PF}{PA} = \cos B;$$

therefore $$PE : PF = \cos C : \cos B,$$

or $$\beta : \gamma = \cos C : \cos B,$$

or $$\beta \cos B = \gamma \cos C,$$

a relation among the coordinates of any point P in AA', and therefore the equation to AA'.

31. *The perpendiculars from the angular points of a triangle on the opposite sides meet in a point.*

Take the triangle in question as triangle of reference, and call it ABC; then, Art. 30, the three perpendiculars will be given by the equations

$$\beta \cos B - \gamma \cos C = 0,$$

$$\gamma \cos C - \alpha \cos A = 0,$$

$$\alpha \cos A - \beta \cos B = 0,$$

of which we observe that any one can be obtained from the other two by addition; therefore by Art. 27, the three lines pass through the same point.

32. *To construct a straight line whose equation is given.*

Let $$l\alpha + m\beta + n\gamma = 0$$

be the given equation of a straight line.

It is required to construct the straight line.

The given equation will be satisfied if $\alpha = 0$, and β and γ are determined so as to satisfy the equation

$$m\beta + n\gamma = 0.$$

But if $\alpha = 0$, the corresponding values of β, γ are subject to the relation

$$b\beta + c\gamma = 2\Delta.$$

From these two equations we obtain

$$\beta = \frac{2n\Delta}{bn - cm}, \quad \gamma = \frac{-2m\Delta}{bn - cm},$$

which give the coordinates corresponding to $\alpha = 0$, of a point upon the line.

Hence we are able to construct the point where the required line meets *BC*.

Similarly we can construct the point where it meets *CA*: and by joining these two points we shall have the straight line required.

33. It will be understood that when we speak of *the straight line* $l\alpha + m\beta + n\gamma = 0$, we are using elliptical language, and mean strictly, *the straight line whose equation is*

$$l\alpha + m\beta + n\gamma = 0.$$

So we often speak of a point lying on $l\alpha + m\beta + n\gamma = 0$, when we mean that it lies on the *locus* of that equation. Or we speak of an equation passing through such and such points, when we mean that its *locus* passes through those points.

All these modes of expression are of course, speaking strictly, very loose and incorrect; but as they can hardly lead to any misconception they are not objectionable, and they shorten very much the expression of a mathematical argument.

It is convenient also to notice that just as the point whose coordinates are α, β and γ is commonly described as the point (α, β, γ), so the straight line whose equati n is $l\alpha + m\beta + n\gamma = 0$ may be spoken of as the straight line (l, m, n).

Exercises on Chapter III.

(15) Find the area of the triangle whose angular points are the middle points of the sides of the triangle of reference.

(16) Find the area of the triangle whose angular points are the feet of the perpendiculars from the points of reference on the opposite sides.

(17) Find the equations to the sides of the triangle of Ex. 16.

(18) Find the area of the triangle whose angular points are given by

$$\left.\begin{array}{r}\alpha=0\\ m\beta=n\gamma\end{array}\right\},\qquad \left.\begin{array}{r}\beta=0\\ n\gamma=l\alpha\end{array}\right\},\qquad \left.\begin{array}{r}\gamma=0\\ l\alpha=m\beta\end{array}\right\}.$$

(19) Shew that the points given by

$$\left.\begin{array}{r}\alpha=0\\ m\beta+n\gamma=0\end{array}\right\},\qquad \left.\begin{array}{r}\beta=0\\ n\gamma+l\alpha=0\end{array}\right\},\qquad \left.\begin{array}{r}\gamma=0\\ l\alpha+m\beta=0\end{array}\right\},$$

lie all on one straight line.

(20) Find the coordinates of the points of trisection of the side AB of the triangle of reference.

(21) Find the equation to a straight line cutting the lines of reference CA, AB in Q, R respectively, where $AQ=\frac{1}{3}AC$ and $AR=\frac{1}{3}AB$.

(22) A straight line cuts the sides BC, CA of a triangle ABC in P, Q and it cuts AB produced in R, shew that if $CP:CB=1:3$ and $CQ:CA=2:3$, then will $RA:AB=1:3$.

(23) Find the equation to a straight line which cuts off $\left(\frac{1}{m}\right)^{\text{th}}$ and $\left(\frac{1}{n}\right)^{\text{th}}$ respectively from the sides AB, AC of the triangle of reference, and find the coordinates of the point where it meets the side BC.

CHAPTER IV.

THE INTERSECTION OF STRAIGHT LINES. PARALLELISM. INFINITY.

34. *To find the coordinates of the point of intersection of two straight lines whose equations are given.*

Let
$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$
be the equations to the two straight lines.

Then the coordinates of the point of intersection must satisfy both equations, and the ratios of the coordinates will therefore be obtained by solving the two equations together.

Thus, eliminating γ we get

$$\frac{\alpha}{\begin{vmatrix} m, & n \\ m', & n' \end{vmatrix}} = \frac{\beta}{\begin{vmatrix} n, & l \\ n', & l' \end{vmatrix}}$$

and therefore by symmetry, each $= \dfrac{\gamma}{\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}}$,

equations which give the ratios $\alpha : \beta : \gamma$.

But to obtain the actual values of the coordinates we have to introduce the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

Thus, since

$$\frac{\alpha}{\begin{vmatrix} m, & n \\ m', & n' \end{vmatrix}}, \quad \frac{\beta}{\begin{vmatrix} n, & l \\ n', & l' \end{vmatrix}}, \quad \frac{\gamma}{\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}}$$

are equal, therefore each of them is equal to

$$\frac{a\alpha + b\beta + c\gamma}{\begin{vmatrix} a, & b, & c \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}} \quad \text{or} \quad \frac{2\Delta}{\begin{vmatrix} a, & b, & c \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}}$$

Hence
$$\alpha = \frac{2\Delta \begin{vmatrix} m, & n \\ m', & n' \end{vmatrix}}{\begin{vmatrix} a, & b, & c \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}}$$

with similar expressions for β and γ.

35. *To find the condition that two straight lines whose equations are given may be parallel.*

Let
$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$

be the two given equations.

If the two lines are parallel their point of intersection lies at an infinite distance from the triangle of reference. Hence the common denominator in the expressions for the coordinates of the point of intersection, obtained in Art. 34, must be zero.

That is
$$\begin{vmatrix} a, & b, & c \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} = 0.$$

36. *To interpret the equation*

$$a\alpha + b\beta + c\gamma = 0.$$

We shall shew first that the locus of this equation includes no point other than at infinity; and, secondly, that it includes every point at infinity.

In order to find the coordinates of points on the locus of any given equation, we have to determine values for α, β, γ, which will satisfy both the given equation and the perpetual relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

In the present case the two equations which have to be combined are inconsistent for all finite values of the variables. For, if α, β, γ are finite, we get, by subtraction,

$$0 = 2\Delta,$$

which is contrary to our original hypothesis.

But looking at the equations a little more generally, and remembering that α, β, γ may have infinite values, it appears that the result of the subtraction ought strictly to be written

$$0 \,.\, \alpha + 0 \,.\, \beta + 0 \,.\, \gamma = 2\Delta,$$

an equation which requires that one or more of the variables α, β, γ should be infinite. And from considering either of the original equations, we observe that *two* at least of these variables must be infinite, since if only one were infinite, we should have

$$a\alpha + b\beta + c\gamma = \infty.$$

But it may be asked, how can the equation $a\alpha + b\beta + c\gamma = 0$ be satisfied by points anywhere situate, since we know by the geometrical construction, Art. 7, that if α, β, γ are the coordinates of any point whatever, $a\alpha + b\beta + c\gamma$ will represent the double of the area of the original triangle?

True. But when we take any point in the plane of the triangle of reference to represent (α, β, γ), we necessarily take it at some finite distance or other from the triangle. We can make this distance as great as we please, but we can never actually make it infinite. So when we say that the equation

$$a\alpha + b\beta + c\gamma = 0$$

represents a locus lying altogether at infinity, we are not contradicting, but rather asserting this fact. For to say that every point upon the locus lies at infinity is in fact saying that no point *can be found* or drawn which shall satisfy the equation.

But the statement further implies the following: that although no finite point can be found to satisfy the equation

$$l\alpha + m\beta + n\gamma = 0$$

when l, m, n are proportional to a, b, c, yet when the ratios of l, m, n differ from those of a, b, c by the least possible difference, then such points *can* be found; and by making the difference as small as we please, the locus will recede as far as we please from the points of reference.

This is exactly the meaning which is attached to the term "infinity" in Algebra, where (for instance) the statement

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{\&c. to an infinite number of terms} = 1$$

does not mean that any number of terms which we can actually take will amount to unity, but that by taking as many terms as we please, we can make the sum as near unity as we please.

But, secondly, *any* point lying at an infinite distance from the triangle of reference may be regarded as lying upon this locus.

For, let X be *any* point at an infinite distance, and let P be any finite point, then we can conceive a straight line joining PX, and by Art. 21, Cor. it will have an equation of the form

$$l\alpha + m\beta + n\gamma = 0 \text{..................} (1).$$

Now let Q be another finite point not in the straight line PX, and let the equation to QX be

$$l'\alpha + m'\beta + n'\gamma = 0 \text{..................} (2).$$

Then since PX and QX intersect at infinity they are parallel, and therefore their equations must satisfy the condition investigated in Art. 35,

i.e.
$$\begin{vmatrix} a, & b, & c \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} = 0.$$

But this equation expresses the condition that the three equations

$$a\alpha + b\beta + c\gamma = 0,$$
$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$

should be consistent, or that their loci should have a common point. Therefore the locus of the equation

$$a\alpha + b\beta + c\gamma = 0$$

passes through the intersection of PX and QX, that is, through X; and so the same locus can be shewn to pass through *any* point whatever at infinity.

But we have already shewn that it passes through no finite point. Hence the equation

$$a\alpha + b\beta + c\gamma = 0$$

represents a locus *lying altogether at infinity, and embracing all points at infinity.*

37. It has already been seen that the equation

$$l\alpha + m\beta + n\gamma = 0$$

when the ratios $l : m : n$ have any values whatever not identical with the ratios $a : b : c$ represents a real and finite straight line.

Now since the locus is a straight line however closely the ratios $l : m : n$ approximate to the values $a : b : c$, it is a lawful form of expression to describe the limiting locus itself as a straight line.

Thus we are able briefly to express the result we have arrived at as follows:

The equation

$$a\alpha + b\beta + c\gamma = 0$$

represents the straight line passing through all points at infinity. But it must be remembered that this is but an abbreviated statement of the fact, that as $l : m : n$ approach the values $a : b : c$, the locus of the equation

$$l\alpha + m\beta + n\gamma = 0$$

will always be a straight line, which can be made to fail by as little as we please from passing through any point whatsoever and every point at an infinite distance from the lines of reference; whilst the position to which it approaches will contain no finite point whatever.

It will be observed that since $\sin A$, $\sin B$, $\sin C$ are proportional to a, b, c, the equation may be indifferently written in either of the forms

$$a\alpha + b\beta + c\gamma = 0,$$

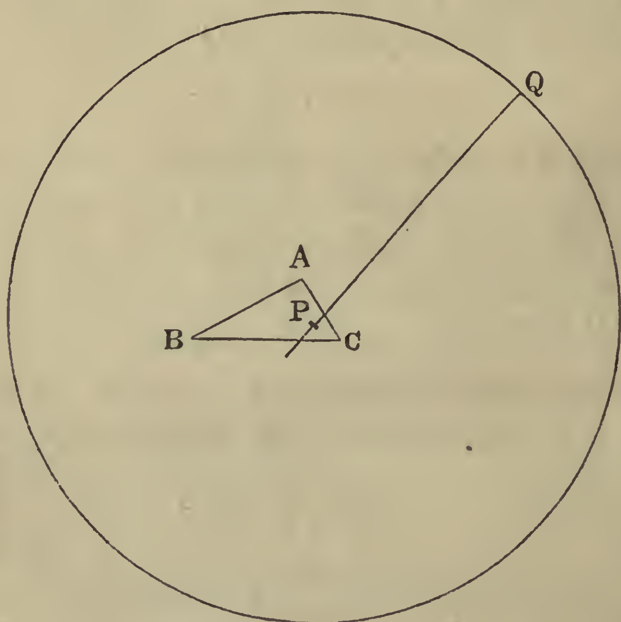
$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0.$$

38. The difficulty of conceiving such a locus as we have described, may perhaps be lessened by the following considerations.

Let ABC be the triangle of reference, and P any point at a finite distance from it.

From the centre P, at any finite radius PQ, as large as can be conveniently taken, describe a circle, and suppose that while the centre P remains fixed, the radius of this circle be gradually increased. If this enlargement be carried on indefinitely, the curvature of the circle becomes less and less, and can by sufficiently enlarging the radius be made as small as we please. Thus the arc of the circle in the neighbourhood of any point Q upon it can be made as straight as we please; and though the circle can never become actually a straight line, yet as the radius approaches an infinite length, the circle becomes in every part as nearly straight as we choose, while all its points recede to an indefinitely great distance from all finite points.

Fig. 12.



Thus we perceive that as the circle tends to become straight it tends to satisfy the same conditions as the limiting locus of the equation

$$l\alpha + m\beta + n\gamma = 0,$$

as $l : m : n$ approach the values $a : b : c$.

The consideration of this infinite circle will tend to diminish the difficulty which would naturally be felt in accepting the following proposition.

39. *Every straight line may be regarded as parallel to the straight line at infinity.*

Let $$l\alpha + m\beta + n\gamma = 0 \quad \text{.........................} \quad (1)$$

be the equation to any straight line. The straight line at infinity has the equation

$$a\alpha + b\beta + c\gamma = 0 \qquad (2).$$

And by Art. 35 the condition that (1) and (2) should represent parallel straight lines is

$$\begin{vmatrix} a, & b, & c \\ a, & b, & c \\ l, & m, & n \end{vmatrix} = 0,$$

which is identically satisfied since two rows of the determinant are the same.

Therefore *every straight line may be regarded as parallel to the straight line at infinity.* Q. E. D.

40. *To find the equation to the straight line passing through a given point and parallel to a given straight line.*

Let $(\alpha', \beta', \gamma')$ be the given point,

and $$l\alpha + m\beta + n\gamma = 0 \qquad (1)$$

the equation to the given straight line.

Let $$\lambda\alpha + \mu\beta + \nu\gamma = 0 \qquad (2)$$

be the equation required.

Then since the locus passes through $(\alpha', \beta', \gamma')$,

we have $$\lambda\alpha' + \mu\beta' + \nu\gamma' = 0 \qquad (3).$$

Also since (1) and (2) are parallel, we have

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ l, & m, & n \\ a, & b, & c \end{vmatrix} = 0,$$

or $$\lambda \begin{vmatrix} m, & n \\ b, & c \end{vmatrix} + \mu \begin{vmatrix} n, & l \\ c, & a \end{vmatrix} + \nu \begin{vmatrix} l, & m \\ a, & b \end{vmatrix} = 0 \qquad (4).$$

Eliminating λ, μ, ν from (2) by means of (3) and (4), we get

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \begin{vmatrix} m, & n \\ b, & c \end{vmatrix}, & \begin{vmatrix} n, & l \\ c, & a \end{vmatrix}, & \begin{vmatrix} l, & m \\ a, & b \end{vmatrix} \end{vmatrix} = 0,$$

the equation required.

41. *If* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ *be the coordinates of two points, and if* L, M, N *denote the determinants*

$$\begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix},$$

respectively, then will

$$\frac{\alpha_1 - \alpha_2}{\begin{vmatrix} b, & c \\ M, & N \end{vmatrix}} = \frac{\beta_1 - \beta_2}{\begin{vmatrix} c, & a \\ N, & L \end{vmatrix}} = \frac{\gamma_1 - \gamma_2}{\begin{vmatrix} a, & b \\ L, & M \end{vmatrix}} = \frac{1}{2\Delta}.$$

For
$$\begin{vmatrix} b, & c \\ M, & N \end{vmatrix} \equiv \begin{vmatrix} 0, & -c, & b \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}$$

$$= \frac{1}{c} \begin{vmatrix} 0, & -c, & 0 \\ \alpha_1, & \beta_1, & 2\Delta \\ \alpha_2, & \beta_2, & 2\Delta \end{vmatrix}$$

$$= 2\Delta\,(\alpha_1 - \alpha_2);$$

therefore
$$\frac{1}{2\Delta} = \frac{\alpha_1 - \alpha_2}{\begin{vmatrix} b, & c \\ M, & N \end{vmatrix}}$$

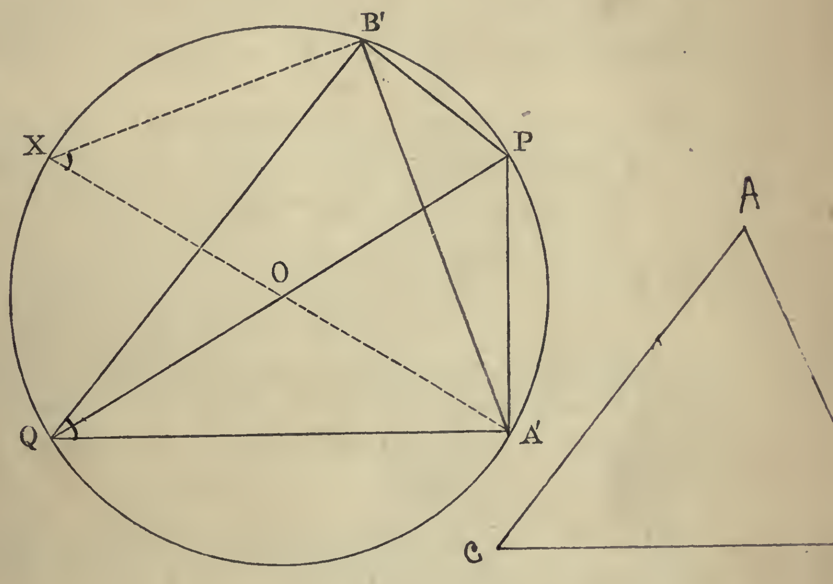
and similarly
$$= \frac{\beta_1 - \beta_2}{\begin{vmatrix} c, & a \\ N, & L \end{vmatrix}} = \frac{\gamma_1 - \gamma_2}{\begin{vmatrix} a, & b, \\ L, & M \end{vmatrix}}.$$

42. By comparing these relations with the result of Article 40, it is seen that the equation to the straight line through the point $(\alpha_1, \beta_1, \gamma_1)$ parallel to the straight line joining the points $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$

is
$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2-\alpha_3, & \beta_2-\beta_3, & \gamma_2-\gamma_3 \end{vmatrix} = 0.$$

43. *To find the distance between two points whose trilinear coordinates are given.*

Fig. 13.



Let P, Q be the two points whose given coordinates are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and let ρ be the distance between them. On PQ as diameter describe a circle, and in it draw QA', QB' parallel to CB, CA. Join PA', PB', $A'B'$ and through A' draw a diameter $A'X$. Join XB',

then $$A'B'^2 = PA'^2 + PB'^2 - 2PA'.PB' \cos A'PB'$$
$$= PA'^2 + PB'^2 + 2PA'.PB' \cos C \text{ ....(1).}$$

But the angles $A'XB'$, $A'QB'$ in the same segment are equal;

$$\therefore \angle A'XB' = \angle C,$$

and $$\therefore A'B' = A'X \sin C = PQ \sin C$$
$$= \rho \sin C;$$

also $$PA' = \alpha_1 - \alpha_2,$$

and $$PB' = \beta_1 - \beta_2.$$

Substituting these values of $A'B'$, PA', PB' in (1), we get

$$\rho^2 \sin^2 C = (\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + 2(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) \cos C.$$

Similarly we have

$$\rho^2 \sin^2 A = (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 + 2(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) \cos A,$$
$$\rho^2 \sin^2 B = (\gamma_1 - \gamma_2)^2 + (\alpha_1 - \alpha_2)^2 + 2(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2) \cos B.$$

Thus we have *three* expressions for the required distance, each of them symmetrical with respect to *two* of the coordinates of the given points. By combining these expressions in various ways, among themselves or with the identity

$$a(\alpha_1 - \alpha_2) + b(\beta_1 - \beta_2) + c(\gamma_1 - \gamma_2) = 0,$$

we might obtain a variety of expressions for the distance, symmetrical with respect to all the three coordinates of each point. Several such expressions will be found in Chapter VI.

44. *To find the distance between the two points whose coordinates are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ in a form symmetrical with respect to the determinants*

$$\begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}.$$

Let L, M, N denote these determinants, then retaining the notation of the last article, the required distance is given by

$$\rho^2 \sin^2 A = (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2 + 2(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) \cos A.$$

But by Art. 41,

$$\frac{\beta_1 - \beta_2}{\begin{vmatrix} c, & a \\ N, & L \end{vmatrix}} = \frac{\gamma_1 - \gamma_2}{\begin{vmatrix} a, & b \\ L, & M \end{vmatrix}} = \frac{1}{2\Delta};$$

therefore

$$4\Delta^2\rho^2 \sin^2 A = \begin{vmatrix} c, & a \\ N, & L \end{vmatrix}^2 + \begin{vmatrix} a, & b \\ L, & M \end{vmatrix}^2 + 2\begin{vmatrix} c, & a \\ N, & L \end{vmatrix} \cdot \begin{vmatrix} a, & b \\ L, & M \end{vmatrix} \cos A$$

$$= a^2 \{L^2 + M^2 + N^2 - 2MN \cos A - 2NL \cos B - 2LM \cos C\},$$

or remembering that $2\Delta \sin A = Sa$, (Art. 9)

$$\rho^2 = \frac{1}{S^2}\{L^2 + M^2 + N^2 - 2MN \cos A - 2NL \cos B - 2LM \cos C\},$$

$$\rho = \frac{1}{S}\sqrt{\{L^2 + M^2 + N^2 - 2MN \cos A - 2NL \cos B - 2LM \cos C\}}.$$

45. *To find the perpendicular distance of the point whose coordinates are (α, β, γ) from the straight line joining the two points whose coordinates are $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$.*

Let p be the perpendicular distance required, and ρ the distance between the last two points, then

$p\rho$ = twice area of the triangle formed by joining the three points;

$$\therefore\ p = \frac{1}{\rho}.\ \text{(this double area)},$$

and therefore in virtue of Arts. 19 and 44,

$$p = \frac{\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}}{\sqrt{\{L^2 + M^2 + N^2 - 2MN \cos A - 2NL \cos B - 2LM \cos C\}}}$$

$$= \frac{L\alpha + M\beta + N\gamma}{\sqrt{\{L^2 + M^2 + N^2 - 2MN \cos A - 2NL \cos B - 2LM \cos C\}}},$$

an expression for the perpendicular required.

46. *To find an expression for the perpendicular distance of the point $(\alpha', \beta', \gamma')$ from the straight line whose equation is*

$$l\alpha + m\beta + n\gamma = 0.$$

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be two points on the given line, and let L, M, N denote the determinants

$$\begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}, \quad \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}.$$

Then by the last article the required perpendicular is given by

$$p = \frac{L\alpha' + M\beta' + N\gamma'}{\sqrt{\{L^2 + M^2 + N^2 - 2MN\cos A - 2NL\cos B - 2LM\cos C\}}}.$$

But the equation to the straight line joining $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ may be written

$$L\alpha + M\beta + N\gamma = 0,$$

which must therefore be identical with the given equation

$$l\alpha + m\beta + n\gamma = 0.$$

Hence $$\frac{L}{l} = \frac{M}{m} = \frac{N}{n},$$

in virtue of which the expression for the perpendicular becomes

$$p = \frac{l\alpha' + m\beta' + n\gamma'}{\sqrt{\{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C\}}}.$$

Other methods of arriving at this result will be found in Chapters V. and VI.

NOTE. The expression

$$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$

is of such frequent occurrence that it will be convenient to denote it briefly by the symbol $\{l, m, n\}^2$.

47. *To find the inclination to the lines of reference of the straight line whose equation is*

$$l\alpha + m\beta + n\gamma = 0 \quad\text{.......................(1).}$$

Let θ be the inclination of the given line to the line of reference BC.

And let $$\alpha = k\beta \quad\text{.............................(2),}$$

be the equation to the parallel straight line through C. Then θ is the inclination of this line to BC, and therefore by Art. 4, (5),

$$\sin\theta = k\sin(C-\theta)$$

$$= k(\sin C\cos\theta - \cos C\sin\theta),$$

$$(1 + k\cos C)\sin\theta = k\sin C\cos\theta,$$

$$\tan\theta = \frac{k\sin C}{1 + k\cos C} \quad\text{...............(3).}$$

But since (1) and (2) are parallel, we have (Art. 35)

$$\begin{vmatrix} 1, & -k, & 0 \\ l, & m, & n \\ a, & b, & c \end{vmatrix} = 0$$

or $$(mc - bn) = (na - lc)k;$$

therefore substituting in (3),

$$\tan\theta = \frac{(mc - bn)\sin C}{(na - lc) + (mc - bn)\cos C}$$

$$= \frac{\sin C}{c} \cdot \frac{mc - bn}{m\cos C + n\cos B - l}$$

$$= \frac{m\sin C - n\sin B}{m\cos C + n\cos B - l}.$$

Similarly if ϕ and ψ are the inclinations of the same line to CA and AB, we shall have

$$\tan\phi = \frac{n\sin A - l\sin C}{n\cos A + l\cos C - m},$$

$$\tan\psi = \frac{l\sin B - m\sin A}{l\cos B + m\cos A - n}.$$

48. *To find the tangent of the angle between the two straight lines represented by the equations*

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0.$$

Let θ, θ' be their inclinations to the line of reference BC. Then if D denote the required angle between the straight lines, we have

$$D = \theta \sim \theta',$$

$$\tan D = \pm \tan(\theta - \theta') = \pm \frac{\tan\theta - \tan\theta'}{1 + \tan\theta \tan\theta'}$$

$$= \pm \frac{(m\sin C - n\sin B)(m'\cos C + n'\cos B - l') - (m'\sin C - n'\sin B)(m\cos C + n\cos B - l)}{(m\cos C + n\cos B - l)(m'\cos C + n'\cos B - l') + (m\sin C - n\sin B)(m'\sin C - n'\sin B)}$$

$$= \pm \frac{l(m'\sin C - n'\sin B) + m(n'\sin A - l'\sin C) + n(l'\sin B - m'\sin A)}{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C},$$

or (as we may write it),

$$= \pm \frac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C}.$$

49. Cor. 1. *The straight lines whose equations are*

$$l\alpha + m\beta + n\gamma = 0,$$

and
$$l'\alpha + m'\beta + n'\gamma = 0,$$

are at right angles to one another provided

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0.$$

Cor. 2. *If the equation*

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

represent two straight lines, they will be at right angles provided

$$u + v + w - 2u'\cos A - 2v'\cos B - 2w'\cos C = 0.$$

50. Obs. We shall in the course of the work give several other methods of finding the expression for the angle between two lines whose equations are given. The method in the foregoing article is generally thought to be the most convenient; but the student is recommended not to pass over, simply because they lead only to results already obtained, those other methods which we shall give, but to read them as very suggestive examples of the application of trilinear coordinates.

The methods given in Chapters V. and VI. in particular are offered as very good illustrations of the use which may be made of those forms of equations which it is the special object of those two chapters to develop.

51. *To determine the sines of the angles of a triangle the trilinear coordinates of whose angular points are given.*

Let P, Q, R be the angular points of the triangle and

$$(\alpha_1, \beta_1, \gamma_1), \quad (\alpha_2, \beta_2, \gamma_2), \quad (\alpha_3, \beta_3, \gamma_3)$$

their coordinates.

Then $$PQ \, . \, PR \sin P = 2 \text{ area } PQR;$$

therefore $$\sin P = \frac{2 \text{ area } PQR}{PQ \, . \, PR}$$

$$= S \frac{\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix}}{\{L_2, M_2, N_2\}\{L_3, M_3, N_3\}} \text{ (Arts. 19, 44)};$$

where $$L_2 \equiv \begin{vmatrix} \beta_3, & \gamma_3 \\ \beta_1, & \gamma_1 \end{vmatrix}, \quad M_2 \equiv \begin{vmatrix} \gamma_3, & \alpha_3 \\ \gamma_1, & \alpha_1 \end{vmatrix}, \quad N_2 \equiv \begin{vmatrix} \alpha_3, & \beta_3 \\ \alpha_1, & \beta_1 \end{vmatrix},$$

$$L_3 \equiv \begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}, \quad M_3 \equiv \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}, \quad N_3 \equiv \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}.$$

But
$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix} \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_1 - \alpha_2, & \beta_1 - \beta_2, & \gamma_1 - \gamma_2 \\ \alpha_1 - \alpha_3, & \beta_1 - \beta_3, & \gamma_1 - \gamma_3 \end{vmatrix}$$

$$\equiv \frac{1}{a} \begin{vmatrix} 2\Delta, & \beta_1, & \gamma_1 \\ 0, & \beta_1 - \beta_2, & \gamma_1 - \gamma_2 \\ 0, & \beta_1 - \beta_3, & \gamma_1 - \gamma_3 \end{vmatrix}$$

$$\equiv \frac{2\Delta}{a} \begin{vmatrix} \beta_1 - \beta_2, & \gamma_1 - \gamma_2 \\ \beta_1 - \beta_3, & \gamma_1 - \gamma_3 \end{vmatrix}$$

$$\equiv \frac{1}{2\Delta a} \begin{vmatrix} \begin{vmatrix} c, & a \\ N_3, & L_3 \end{vmatrix}, & \begin{vmatrix} a, & b \\ L_3, & M_3 \end{vmatrix} \\ \begin{vmatrix} c, & a \\ N_2, & L_2 \end{vmatrix}, & \begin{vmatrix} a, & b \\ L_2, & M_2 \end{vmatrix} \end{vmatrix} \quad \text{(Art. 41)}$$

$$\equiv \frac{1}{2\Delta} \begin{vmatrix} a, & b, & c \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix} \equiv \frac{1}{S} \begin{vmatrix} \sin A, & \sin B, & \sin C \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}$$

Hence
$$\sin P = \frac{\begin{vmatrix} \sin A, & \sin B, & \sin C \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}}{\{L_2, M_2, N_2\}\{L_3, M_3, N_3\}},$$

and similar expressions may be written down for $\sin Q$, and $\sin R$.

52. *To find the sine of the angle between two straight lines whose equations are given.*

Let the given equations be

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0.$$

Let $(\alpha_1, \beta_1, \gamma_1)$ denote the point of intersection of these two lines, and let $(\alpha_2, \beta_2, \gamma_2)$ be any other point on the first line, and $(\alpha_3, \beta_3, \gamma_3)$ any other point on the second.

Then if D be the angle between the lines we shall have, with the notation of the last article,

$$\sin D = \pm \frac{\begin{vmatrix} \sin A, & \sin B, & \sin C \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}}{\{L_2, M_2, N_2\}\{L_3, M_3, N_3\}}.$$

But $$\frac{L_3}{l} = \frac{M_3}{m} = \frac{N_3}{n},$$

and $$\frac{L_2}{l'} = \frac{M_2}{m'} = \frac{N_2}{n'};$$

therefore substituting

$$\sin D = \pm \frac{\begin{vmatrix} \sin A, & \sin B, & \sin C \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}}{\{l, m, n\}\{l', m', n'\}},$$

the expression required.

Other methods of arriving at these results will be given in Chapters V. and VI.

53. The expression for $\sin D$ obtained in the last article might have been deduced from the expression for $\tan D$ obtained in Art. 48; but the process of squaring and adding the numerator and denominator of that expression and resolving the result into its factors would have been tedious, so that it is perhaps more convenient to investigate the sine and tangent independently.

From a comparison of the results of Arts. 48 and 52 we can immediately write down the expression for the cosine of the angle,

viz. $\cos D$

$$= \frac{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C}{\{l, m, n\}\{l', m', n'\}}.$$

Exercises on Chapter IV.

(24) Find the coordinates of the point of intersection of the two straight lines whose equations are

$$\alpha = \gamma \cos B,$$
$$\beta = \gamma \cos A;$$

and find the equation of the straight line joining this point with the point of reference C.

(25) If the sides QR, RP, PQ of a triangle PQR be represented respectively by the equations

$$m\beta + n\gamma - 2l\alpha = 0,$$
$$n\gamma + l\alpha - 2m\beta = 0,$$
$$l\alpha + m\beta - 2n\gamma = 0;$$

find the equations to all the straight lines joining the points P, Q, R with the points of reference.

(26) Shew that the straight lines

$$(a + d)\alpha + (b + d)\beta + c\gamma = 0,$$

and $$(a + d)\alpha + (b - d)\beta + c\gamma = 0,$$

are at right angles to each other.

(27) Shew that the straight lines

$$\alpha \sin B + \beta \sin (B - C) + \gamma \sin C \cos C = 0,$$
$$\alpha \cos B + \beta \cos (B - C) + \gamma \sin^2 C = 0,$$

are parallel, and that each is parallel to the straight line

$$\alpha \sin (A - C) + \beta \sin A + \gamma \sin C \cos C = 0.$$

(28) Shew that the equations

$$\alpha \operatorname{cosec} A + \beta \operatorname{cosec} B = 0,$$
$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0,$$

represent parallel straight lines.

(29) Find the condition that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

may be parallel to the side BC of the triangle of reference.

(30) Find the condition that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

may be parallel to the bisector of the angle A of the triangle of reference.

(31) Shew that the straight lines whose equations are

$$\alpha + \gamma \cos B = 0,$$
$$\beta + \gamma \cos A = 0,$$

are parallel.

(32) Find the angle between the straight lines whose equations are

$$\alpha - \gamma \cos B = 0,$$
$$\beta - \gamma \cos A = 0.$$

(33) The perpendiculars from the middle points of the sides of the triangle of reference are given by the equations

$$\beta \sin B - \gamma \sin C + \alpha \sin (B - C) = 0,$$
$$\gamma \sin C - \alpha \sin A + \beta \sin (C - A) = 0,$$
$$\alpha \sin A - \beta \sin B + \gamma \sin (A - B) = 0.$$

(34) Straight lines are drawn from the angular points of the triangle of reference so as to pass through the point given by

$$l\alpha = m\beta = n\gamma,$$

and so as to meet the opposite sides in the points A', B', C': find the equations to the sides of the triangle $A'B'C'$.

(35) Find the equations to the sides of the triangle whose angular points are given by

$$(\alpha = 0, \text{ and } \beta + l\gamma = 0),$$
$$(\beta = 0, \text{ and } \gamma + m\alpha = 0),$$
$$(\gamma = 0, \text{ and } \alpha + n\beta = 0),$$

respectively.

(36) If O be the centre of the circle circumscribing the triangle of reference, and if AO, BO, CO be produced to meet the opposite sides in $A'B'C'$, shew that three of the four straight lines represented by the equations

$$\alpha \sec A \pm \beta \sec B \pm \gamma \sec C = 0$$

are the sides of the triangle $A'B'C'$; and construct the fourth straight line.

(37) Draw the four straight lines represented by the equations

$$\alpha \cos A \pm \beta \cos B \pm \gamma \cos C = 0.$$

(38) Draw the four straight lines represented by the equations

$$\alpha \pm \beta \pm \gamma = 0.$$

(39) Interpret the equations

$$\alpha \sin A \pm \beta \sin B \pm \gamma \sin C = 0.$$

(40) Of the four straight lines whose equations are

$$l\alpha \pm m\beta \pm n\gamma = 0$$

two intersect in P, and the other two in P'; two intersect in Q, and the other two in Q'; two intersect in R, and the other two in R'; find the coordinates of the middle points of PP', QQ', RR'; and shew that they lie on one straight line.

And find the equation to this straight line.

(41) On the three sides of a triangle ABC triangles PBC, QCA, RAB are described so that the angles QAC, RAB are equal, the angles RBA, PBC are equal, and the angles PCB, QCA are equal; prove that the straight lines, AP, BQ, CR pass through one point.

√ (42) Shew that the point determined by

$$\frac{a\alpha}{n-l}=\frac{b\beta}{l-m}=\frac{c\gamma}{m-n}$$

and the point determined by

$$\frac{a\alpha}{l-m}=\frac{b\beta}{m-n}=\frac{c\gamma}{n-l}$$

both lie at infinity, and shew that the angular distance between them, viewed from any finite point, will be a right angle if

$$a^2(m-n)^2+b^2(n-l)^2+c^2(l-m)^2=\{al,\ bm,\ cn\}^2.$$

CHAPTER V.

THE STRAIGHT LINE. THE EQUATION IN TERMS OF THE PERPENDICULARS.

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54. We have shewn that if p, q, r be the perpendicular distances of the points of reference from any straight line, the equation to this straight line will be

$$ap\alpha + bq\beta + cr\gamma = 0.$$

We proceed to consider some applications of the equation of a straight line in this form. But it will first be necessary to establish a relation which exists among the perpendiculars p, q, r.

55. *If p, q, r be the perpendicular distances of the angular points of the triangle ABC from any straight line, then will*

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr \cos A - 2bcrp \cos B$$
$$- 2abpq \cos C = 4\Delta^2.$$

Let AP, BQ, CR (fig. 14) be the perpendiculars from ABC on the straight line PQR.

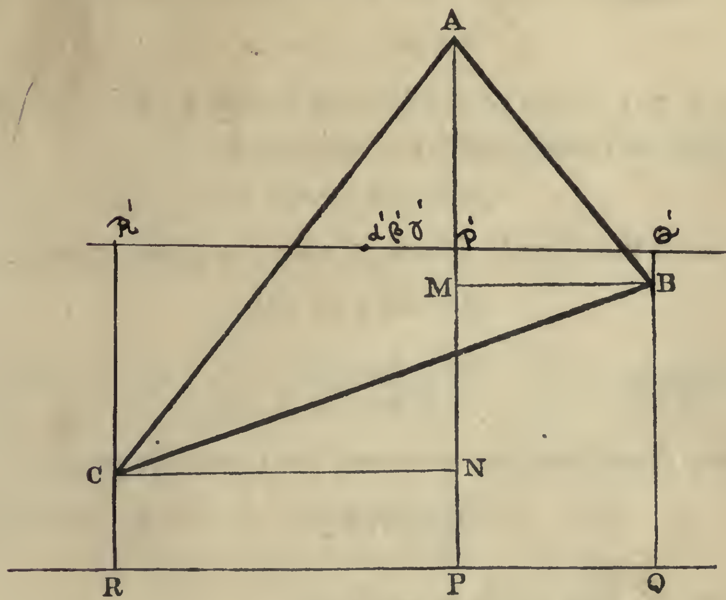
Draw BM, CN perpendiculars upon AP.

Then $$AM = p \sim q,$$

and therefore (Euclid, I. 47),

$$BM = \pm\sqrt{c^2 - (p-q)^2}.$$

Fig. 14.



Similarly $$CN = \pm\sqrt{b^2 - (r-p)^2},$$

and $$RQ = \pm\sqrt{a^2 - (q-r)^2}.$$

But (having regard to algebraical sign in relation to the direction of straight lines)

$$BM + NC + RQ = 0.$$

Hence

$$\pm\sqrt{\{a^2 - (q-r)^2\}} \pm \sqrt{\{b^2 - (r-p)^2\}} \pm \sqrt{\{c^2 - (p-q)^2\}} = 0,$$

which when cleared of radicals reduces to

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr \cos A - 2carp \cos B$$
$$- 2abpq \cos C = 4\Delta^2,$$

the relation required to be established.

N.B. With the notation introduced in Art. 46, NOTE, this result may be written

$$\{ap,\ bq,\ cr\} = 2\Delta.$$

56. *To find the perpendicular distances of the points of reference from the straight line whose given equation is*

$$l\alpha + m\beta + n\gamma = 0.$$

Let p, q, r be the perpendiculars required. Then the straight line might be represented by the equation

$$ap\alpha + bq\beta + cr\gamma = 0,$$

which must therefore be identical with the given equation

$$l\alpha + m\beta + n\gamma = 0.$$

Therefore $$\frac{ap}{l} = \frac{bq}{m} = \frac{cr}{n},$$

and since these fractions are equal, each must be equal to

$$\frac{\{ap, bq, cr\}}{\{l, m, n\}},$$

which by the last article is equal to

$$\frac{2\Delta}{\{l, m, n\}};$$

hence

$$p = \frac{2\Delta}{a} \cdot \frac{l}{\{l, m, n\}}, \quad q = \frac{2\Delta}{b} \cdot \frac{m}{\{l, m, n\}}, \quad r = \frac{2\Delta}{c} \cdot \frac{n}{\{l, m, n\}}.$$

57. *The equation to a straight line being given in the general form*

$$l\alpha + m\beta + n\gamma = 0,$$

to reduce it to the equation in terms of the perpendiculars.

We have only to multiply the equation throughout by

$$\frac{2\Delta}{\{l, m, n\}},$$

since by the last article the expression

$$2\Delta \cdot \frac{l\alpha + m\beta + n\gamma}{\{l, m, n\}}$$

is identical with

$$ap\alpha + bq\beta + cr\gamma.$$

58. *To find the perpendicular distance of the point* $(\alpha', \beta', \gamma')$ *from a straight line whose equation in terms of the perpendiculars is given.*

Let $$ap\alpha + bq\beta + cr\gamma = 0$$

be the given straight line, and let a line be drawn parallel to this through the given point $(\alpha', \beta', \gamma')$.

Then if ρ be the perpendicular distance required,

$$p \pm \rho, \quad q \pm \rho, \quad r \pm \rho$$

(the upper signs going together and the lower together) will represent the perpendicular distances of the new line from A, B, C.

Therefore the equation to the new line is

$$a\alpha(p \pm \rho) + b\beta(q \pm \rho) + c\gamma(r \pm \rho) = 0.$$

But, since this straight line passes through $(\alpha', \beta', \gamma')$,

$$a\alpha'(p \pm \rho) + b\beta'(q \pm \rho) + c\gamma'(r \pm \rho) = 0,$$

or $$(a\alpha' + b\beta' + c\gamma')\,\rho = \mp(ap\alpha' + bq\beta' + cr\gamma'),$$

and therefore

$$\rho = \pm\frac{ap\alpha' + bq\beta' + cr\gamma'}{2\Delta}$$

the expression for the distance required.

59. *To find the perpendicular distance of the point* $(\alpha', \beta', \gamma')$ *from any straight line whose equation is given in the general form*

$$l\alpha + m\beta + n\gamma = 0.$$

Let p, q, r be the perpendicular distances of the straight line from the points of reference.

Then by the last article the required distance is given by

$$\rho = \pm\frac{ap\alpha' + bq\beta' + cr\gamma'}{2\Delta}.$$

But by Art. 56,

$$\frac{l}{\{l,\ m,\ n\}} = \frac{ap}{2\Delta}, \quad \frac{m}{\{l,\ m,\ n\}} = \frac{bq}{2\Delta}, \quad \frac{n}{\{l,\ m,\ n\}} = \frac{cr}{2\Delta}.$$

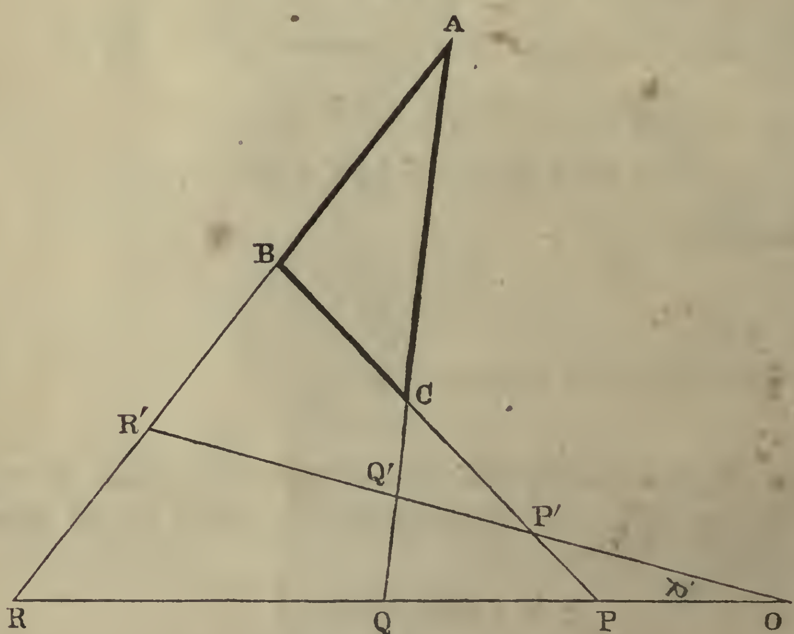
Hence the last equation becomes

$$\rho = \pm \frac{l\alpha' + m\beta' + n\gamma'}{\{l,\ m,\ n\}},$$

the same expression which we obtained by another method in Art. 46.

60. *To find the angle between two straight lines in terms of their perpendicular distances from the angular points of a triangle.*

Fig. 15.



Let D be the angle between the two straight lines $OPQR$ and $OP'Q'R'$ intersecting in O.

And let p, q, r be the perpendicular distances of the former line—and p', q', r' those of the latter—from three points ABC forming a triangle.

Then $\Delta P'OP = \Delta BOP - \Delta BOP'$,

that is, $OP.OP' \sin D = q.OP - q'OP'$.

Similarly $OP.OP' \sin D = r.OP - r'OP'$.

Hence eliminating OP'

$$(r-q)\,OP.\sin D = qr' - q'r.$$

Similarly $(p-r)\,OQ.\sin D = rp' - r'p$,

and $(q-p)\,OR.\sin D = pq' - p'q$,

therefore by addition,

$$\left\{(r-q)\,OP + (p-r)\,OQ + (q-p)\,OR\right\}\sin D = \begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix}$$

But $\Delta ABC = \Delta ARQ + CQP - BRP$,

therefore

$$2\Delta = p.(OR - OQ) + q.(OQ - OP) - r.(OR - OP)$$
$$= (r-q)\,OP + (p-r)\,OQ + (q-p)\,OR.$$

Hence $$2\Delta \sin D = \begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix}$$

which gives D in terms of the perpendiculars.

61. *To deduce the expression for the angle between the two straight lines whose equations are*

$$l\alpha + m\beta + n\gamma = 0,$$

and $$l'\alpha + m'\beta + n'\gamma = 0.$$

If p, q, r; p', q', r' be the perpendicular distances of these lines from A, B, C, we have by Art. 56,

$$\frac{ap}{l} = \frac{bq}{m} = \frac{cr}{n} = \frac{2\Delta}{\{l, m, n\}},$$

and $$\frac{ap'}{l'} = \frac{bq'}{m'} = \frac{cr'}{n'} = \frac{2\Delta}{\{l', m', n'\}}.$$

But if D be the angle between the lines, we have by the last article

$$\sin D = \frac{1}{2\Delta} \begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix}$$

therefore

$$\sin D = \frac{2\Delta}{\{l, m, n\}\{l', m', n'\}} \begin{vmatrix} \frac{l}{a}, & \frac{m}{b}, & \frac{n}{c} \\ \frac{l'}{a}, & \frac{m'}{b}, & \frac{n'}{c} \\ 1, & 1, & 1 \end{vmatrix}$$

$$= \frac{\frac{2\Delta}{abc}}{\{l, m, n\}\{l', m', n'\}} \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}$$

$$= \frac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{\{l, m, n\}\{l', m', n'\}}$$

the same expression which we otherwise obtained in Art. 52.

62. *To find the altitude of the triangle whose base is given by the equation*

$$ap\alpha + bq\beta + cr\gamma = 0,$$

and the other two sides by the equations

$$ap'\alpha + bq'\beta + cr'\gamma = 0,$$

and $$ap''\alpha + bq''\beta + cr''\gamma = 0.$$

Let h denote the altitude required, and suppose α, β, γ the coordinates of the vertex of the triangle, then, by Art. 57, 58

$$h = \frac{ap\alpha + bq\beta + cr\gamma}{2\Delta},$$

where α, β, γ are to be determined from the equations

$$\left.\begin{aligned} ap'\alpha + bq'\beta + cr'\gamma &= 0, \\ ap''\alpha + bq''\beta + cr''\gamma &= 0, \\ a\alpha + b\beta + c\gamma &= 2\Delta. \end{aligned}\right\}$$

These equations give

$$\frac{a\alpha}{\begin{vmatrix} q', & r' \\ q'', & r'' \end{vmatrix}} = \frac{b\beta}{\begin{vmatrix} r', & p' \\ r'', & p'' \end{vmatrix}} = \frac{c\gamma}{\begin{vmatrix} p', & q', \\ p'', & q'' \end{vmatrix}} = \frac{2\Delta}{\begin{vmatrix} 1, & 1, & 1 \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}.$$

But since the first three of these fractions are equal, therefore

$$\text{each} = \frac{ap\alpha + bq\beta + cr\gamma}{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}.$$

Therefore

$$h = \frac{ap\alpha + bq\beta + cr\gamma}{2\Delta} = \frac{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}{\begin{vmatrix} 1, & 1, & 1 \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}.$$

63. *To find the lengths of the sides of the same triangle.*

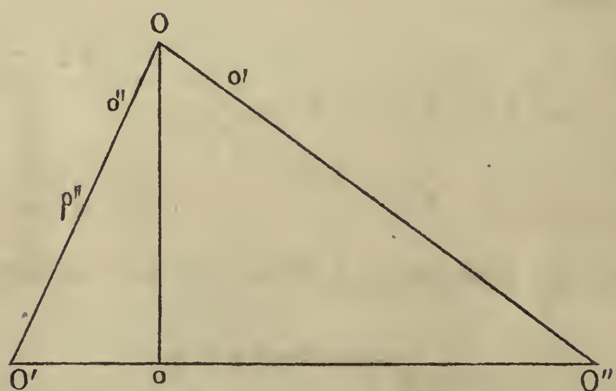
Let ρ, ρ', ρ'' denote the lengths of the sides whose equations are respectively

$$ap\alpha + bq\beta + cr\gamma = 0,$$
$$ap'\alpha + bq'\beta + cr'\gamma = 0,$$
$$ap''\alpha + bq''\beta + cr''\gamma = 0.$$

And let O, O', O'' denote the angles opposite to these sides, and Oo, $O'o'$, $O''o''$ the perpendiculars from the angles on the opposite sides. Then

$$p'' \sin O' = Oo.$$

Fig. 16.



But by Art. 60,

$$\sin O' = \pm \frac{1}{2\Delta} \begin{vmatrix} 1, & 1, & 1 \\ p, & q, & r \\ p'', & q'', & r'' \end{vmatrix};$$

and by the last article,

$$Oo = \frac{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}{\begin{vmatrix} 1, & 1, & 1 \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}};$$

therefore substituting

$$\rho'' = 2\Delta \frac{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}}{\begin{vmatrix} 1, & 1, & 1 \\ p, & q, & r \\ p'', & q'', & r'' \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}},$$

and similar expressions can be written down for the sides ρ and ρ'.

64. *To find the area of the same triangle.*

We have only to express half the rectangle contained by the base and the altitude.

Therefore by the last article

$$\text{area} = \Delta \frac{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}^2}{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} p', & q', & r' \\ p'', & q'', & r'' \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} p'', & q'', & r'' \\ p, & q, & r \\ 1, & 1, & 1 \end{vmatrix}}.$$

65. Cor. The expression just obtained is homogeneous with respect to p, q, r, and of *zero* dimensions, hence it will not be altered if we substitute for p, q, r any quantities proportional to them.

Now suppose that the equation to the base, instead of being given in the form

$$ap\alpha + bq\beta + cr\gamma = 0,$$

is given in a perfectly general form

$$l\alpha + m\beta + n\gamma = 0,$$

then $\frac{l}{a}, \frac{m}{b}, \frac{n}{c}$ are proportional to p, q, r, and may be substituted for p, q, r in the expression for the area.

And so with respect to the other two sides of the triangle.

Hence we obtain the following theorem:

If
$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$
$$l''\alpha + m''\beta + n''\gamma = 0,$$

be the equations to any three straight lines, the area of the triangle which they contain is

$$\Delta \frac{\begin{vmatrix} \frac{l}{a}, & \frac{m}{b}, & \frac{n}{c} \\ \frac{l'}{a}, & \frac{m'}{b}, & \frac{n'}{c} \\ \frac{l''}{a}, & \frac{m''}{b}, & \frac{n''}{c} \end{vmatrix}^2}{\begin{vmatrix} \frac{l}{a}, & \frac{m}{b}, & \frac{n}{c} \\ \frac{l'}{a}, & \frac{m'}{b}, & \frac{n'}{c} \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} \frac{l'}{a}, & \frac{m'}{b}, & \frac{n'}{c} \\ \frac{l''}{a}, & \frac{m''}{b}, & \frac{n''}{c} \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} \frac{l''}{a}, & \frac{m''}{b}, & \frac{n''}{c} \\ \frac{l}{a}, & \frac{m}{b}, & \frac{n}{c} \\ 1, & 1, & 1 \end{vmatrix}},$$

or (multiplying numerator and denominator by $a^3b^3c^3$),

$$abc\,\Delta \frac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^2}{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix} \begin{vmatrix} l', & m', & n' \\ l'', & m'', & n'' \\ a, & b, & c \end{vmatrix} \begin{vmatrix} l'', & m'', & n'' \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}}.$$

EXERCISES ON CHAPTER V.

(43) Find the equations to the straight lines through the angular points of the triangle of reference parallel to the straight line

$$ap\alpha + bq\beta + cr\gamma = 0.$$

(44) Find the equation to the straight line through the centre of gravity of the triangle of reference and parallel to the straight line

$$ap\alpha + bq\beta + cr\gamma = 0.$$

(45) Find the equation to the straight line bisecting AB and cutting AC at right angles.

(46) Find the equation to the straight line parallel to BC at a distance d from it on the side remote from A.

(47) Find the area of the triangle whose sides are given by the equations

$$b\beta + c\gamma = 0,$$
$$c\gamma + a\alpha = 0,$$
$$a\alpha + b\beta = 0.$$

(48) Find the area of the triangle whose sides are given by the equations

$$m\beta + n\gamma = 0,$$
$$n\gamma + l\alpha = 0,$$
$$l\alpha + m\beta = 0.$$

(49) Find the area of the triangle whose sides are given by the equations

$$-a\alpha + b\beta + c\gamma = 0,$$
$$a\alpha - b\beta + c\gamma = 0,$$
$$a\alpha + b\beta - c\gamma = 0.$$

(50) Shew that the angle between the straight lines

$$a\alpha (p + s) + b\beta (q + s) + c\gamma (r + s) = 0,$$
$$a\alpha (p' + s') + b\beta (q' + s') + c\gamma (r' + s') = 0,$$

is always the same whatever be the values of s and s'.

(51) Through the points of reference A, B, C straight lines are drawn parallel respectively to the straight lines

$$ap\alpha + bq\beta + cr\gamma = 0,$$
$$ap'\alpha + bq'\beta + cr'\gamma = 0,$$
$$ap''\alpha + bq''\beta + cr''\gamma = 0;$$

shew that they will meet in a point, provided

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ p, & q, & r, & p \\ p', & q', & r', & q' \\ p'', & q'', & r'', & r'' \end{vmatrix} = 0.$$

(52) Shew that if $p - q = c$, the straight line

$$ap\alpha + bq\beta + cr\gamma = 0$$

is at right angles to the line of reference AB.

(53) Apply the result of Art. 55 to find the equation to the straight line for which $p - q = c$ and $r = 0$.

(54) Shew that if $p - q = 0$, the straight line

$$ap\alpha + bq\beta + cr\gamma = 0$$

is parallel to the line of reference AB.

(55) Apply the result of Art. 55 to find the equation to the straight line for which $p - q = 0$ and $r = 0$.

(56) $PQRQ'$ is a parallelogram of which the diagonal QQ' coincides with the line of reference CA, and the points P, R lie on BC, AB, respectively. If the base PQ be represented by the equation

$$ap\alpha + bq\beta + cr\gamma = 0,$$

find the altitude of the parallelogram in terms of p, q, r.

CHAPTER VI.

THE EQUATIONS OF THE STRAIGHT LINE IN TERMS OF THE DIRECTION SINES.

66. When we say that the equation

$$l\alpha + m\beta + n\gamma = 0$$

represents a straight line, we do not mean that any values whatever of α, β, γ which satisfy the equation will be the coordinates of a point upon the line: for unless α, β, γ also satisfy the relation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

they will not be the coordinates of a point at all, although a point may be found having its coordinates *proportional* to them.

In other words, if the equation

$$l\alpha + m\beta + n\gamma = 0$$

is to be regarded as a relation among the coordinates of any point upon the line and not merely a relation among their ratios, we must regard the equation

$$a\alpha + b\beta + c\gamma = 2\Delta$$

as understood to be simultaneously satisfied. That is, the coordinates of any point on the straight line must satisfy the simultaneous system

$$\left.\begin{array}{l} l\alpha + m\beta + n\gamma = 0 \\ a\alpha + b\beta + c\gamma = 2\Delta \end{array}\right\}.$$

We may therefore with the greatest strictness speak of this system of two simultaneous equations, as representing the straight line, or defining the coordinates of any point on it.

Instead of these two equations we may use any equivalent pair of equations obtained by combining them. And if α', β', γ' denote known coordinates of any point upon the line, we can express the system of equations in a very convenient form.

Thus: since $(\alpha', \beta', \gamma')$ lies upon the line, we have

and
$$\left.\begin{array}{l} l\alpha' + m\beta' + n\gamma' = 0 \\ a\alpha' + b\beta' + c\gamma' = 2\Delta \end{array}\right\},$$

in virtue of which relations the original system can be put into the form

$$\left.\begin{array}{l} l(\alpha-\alpha') + m(\beta-\beta') + n(\gamma-\gamma') = 0 \\ a(\alpha-\alpha') + b(\beta-\beta') + c(\gamma-\gamma') = 0 \end{array}\right\},$$

or

$$\frac{\alpha-\alpha'}{\begin{vmatrix} m, & n \\ b, & c \end{vmatrix}} = \frac{\beta-\beta'}{\begin{vmatrix} n, & l \\ c, & a \end{vmatrix}} = \frac{\gamma-\gamma'}{\begin{vmatrix} l, & m \\ c, & a \end{vmatrix}},$$

which we may write

$$\frac{\alpha-\alpha'}{\lambda} = \frac{\beta-\beta'}{\mu} = \frac{\gamma-\gamma'}{\nu},$$

where λ, μ, ν are proportional to the determinants

$$\begin{vmatrix} m, & n \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} n, & l \\ c, & a \end{vmatrix}, \quad \begin{vmatrix} l, & m \\ a, & b \end{vmatrix},$$

or (which is the same thing), where λ, μ, ν satisfy the relations

$$a\lambda + b\mu + c\nu = 0,$$

and

$$l\lambda + m\mu + n\nu = 0.$$

67. It follows that if α', β', γ' be the coordinates of any point, the system of equations

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}$$

will represent a straight line provided

$$a\lambda+b\mu+c\nu=0,$$

and if the equation to the same straight line in the ordinary form be

$$l\alpha+m\beta+n\gamma=0,$$

the ratios $l:m:n$ will be determined by the equations

$$\lambda l+\mu m+\nu n=0,$$
$$\alpha' l+\beta' m+\gamma' n=0;$$

that is, the equation in the ordinary form will be

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \lambda, & \mu, & \nu \end{vmatrix}=0.$$

68. We proceed to obtain the equations to a straight line in the form

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu},$$

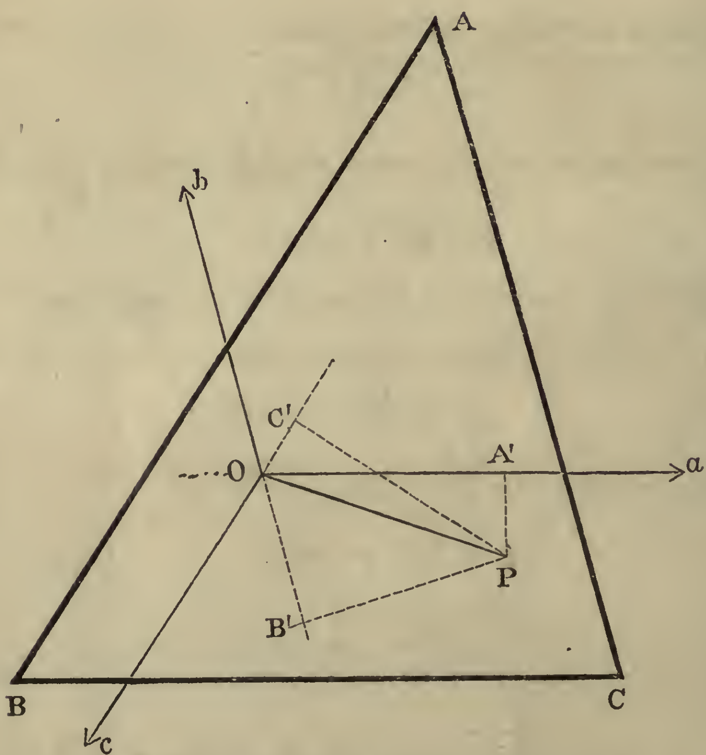
without reference to the ordinary form.

Let OP be the straight line whose equations are to be found, and let α', β', γ' be the coordinates of the fixed point O, and α, β, γ those of any point P upon the straight line: and let ρ be the distance between these two points.

It will be observed that ρ like α, β, γ is a variable quantity dependent upon the position of P.

Through O draw Oa, Ob, Oc parallel to the lines of reference BC, CA, AB respectively, and so that the angles bOc, *cOa*, *aOb* may be the supplements of the angles *A*, *B*, *C* respectively.

Fig. 17.



Let θ, ϕ, ψ denote the angles *POa*, *POb*, *POc* all measured in the same direction from the initial line *OP*.

Draw the perpendiculars *PA'*, *PB'*, *PC'*,

then $$\sin\theta = \frac{PA'}{OP} = \frac{\alpha - \alpha'}{\rho}.$$

Also, due regard being had to the algebraical signs,

$$\sin\phi = \frac{\beta - \beta'}{\rho},$$

and $$\sin\psi = \frac{\gamma - \gamma'}{\rho}.$$

Hence, if λ, μ, ν be proportional to the sines of the angles θ, ϕ, ψ, we have

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}$$

relations among the coordinates (α, β, γ) of any point on the given line, and therefore representing the given line.

And further, if λ, μ, ν be not only proportional but actually equal to the sines of the angles θ, ϕ, ψ, we may write

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho,$$

where ρ is the distance between the points (α, β, γ) and (α', β', γ').

69. DEF. The sines of the angles which any straight line makes with the three straight lines of reference may conveniently be termed the *direction sines* of the straight line.

70. *To find the relations among the direction sines of any straight line.*

Let $\lambda=\sin\theta$, $\mu=\sin\phi$, $\nu=\sin\psi$ be the direction sines of any straight line,

then (Art. 68), $\qquad \phi-\theta=\pi-C,$

and $\qquad \psi-\phi=\pi-A.$

Hence we have $\qquad \sin\theta=-\sin(C+\phi),$

and $\qquad \sin\psi=\sin(A-\phi).$

Consequently we may write

$$\left.\begin{aligned}\lambda&=-\sin(C+\phi),\\ \mu&=\sin\phi,\\ \nu&=\sin(A-\phi),\end{aligned}\right\}$$

and these equations will, on the elimination of ϕ, lead to two equations among λ, μ, ν, and the angles of the triangle of reference.

Performing the elimination between the first and second, and between the second and third equations, we get

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos C = \sin^2 C,$$

and $$\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A.$$

By symmetry we must also have

$$\nu^2 + \lambda^2 + 2\nu\lambda \cos B = \sin^2 B;$$

but this does not express any new or independent relation, being obtainable from the two former by the elimination of μ.

Also since

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu},$$

and since the simple function of the numerators,

$$a(\alpha - \alpha') + b(\beta - \beta') + c(\gamma - \gamma'),$$

is zero, the similar function of the denominators must be also zero, i.e.

$$a\lambda + b\mu + c\nu = 0,$$

a different relation among λ, μ, ν, but not an independent one, for this must also be implied in the former equations, since they were shewn to express the necessary and sufficient relations among λ, μ, ν.

Hence we arrive at the conclusion that the equations

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho$$

will represent a line passing through the point $(\alpha', \beta', \gamma')$, ρ being the distance between this point and the variable point (α, β, γ), provided, and provided only, that λ, μ, ν satisfy the conditions

$$\left.\begin{aligned} a\lambda + b\mu + c\nu &= 0, \\ \mu^2 + \nu^2 + 2\mu\nu \cos A &= \sin^2 A, \\ \nu^2 + \lambda^2 + 2\nu\lambda \cos B &= \sin^2 B, \\ \lambda^2 + \mu^2 + 2\lambda\mu \cos C &= \sin^2 C, \end{aligned}\right\}$$

which are equivalent to only *two* independent equations.

71. The required conditions of the last article are given by *any two* of the four equations just written down, or by *any two* equations that can be formed by combining them. We proceed to obtain two such which are sometimes more convenient, as involving all the coordinates symmetrically.

It will be sufficient to start with the first two equations,

$$a\lambda + b\mu + c\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) we get, transposing and squaring,

$$b^2\mu^2 + c^2\nu^2 + 2bc\mu\nu = a^2\lambda^2,$$

or

$$2\mu\nu = \frac{a^2\lambda^2 - b^2\mu^2 - c^2\nu^2}{bc}.$$

Substituting this in (2), we get

$$bc\mu^2 + bc\nu^2 + (a^2\lambda^2 - b^2\mu^2 - c^2\nu^2) \cos A = bc \sin^2 A,$$

whence

$$\lambda^2 a^2 \cos A + \mu^2 ab \cos B + \nu^2 ac \cos C = bc \sin^2 A,$$

and therefore (since $\sin A$, $\sin B$, $\sin C$ are proportional to a, b, c)

$$\lambda^2 \sin A \cos A + \mu^2 \sin B \cos B + \nu^2 \sin C \cos C$$
$$= \sin A \sin B \sin C,$$

or

$$\lambda^2 \sin 2A + \mu^2 \sin 2B + \nu^2 \sin 2C$$
$$= 2 \sin A \sin B \sin C \ldots\ldots (3),$$

a result to be remembered.

Again from (1),

$$-b\mu = a\lambda + c\nu,$$

and therefore $$\mu^2 = -\frac{a\lambda\mu + c\mu\nu}{b},$$

so $$\nu^2 = -\frac{a\lambda\nu + b\mu\nu}{c}.$$

Substituting these in (2), we get

$$ac\lambda\mu + c^2\mu\nu + ab\lambda\nu + b^2\mu\nu - 2\mu\nu\cos A = -bc\sin^2 A,$$

or $$a^2\mu\nu + ac\lambda\mu + ab\lambda\nu + bc\sin^2 A = 0,$$

or $$\mu\nu\sin A + \nu\lambda\sin B + \lambda\mu\sin C + \sin A\sin B\sin C = 0 \quad\ldots\ldots(4),$$

another notable result.

And similarly we may form *ad libitum* a variety of equations connecting λ, μ, ν, each one implicitly contained in the system of equations in Art. 70.

72. It may well be noticed that each of these equations (except the simple equation $a\lambda + b\mu + c\nu = 0$, which only involves the *ratios* of $\lambda : \mu : \nu$) furnishes us with a different expression for the distance between two points whose coordinates are given.

For let (α, β, γ), $(\alpha', \beta', \gamma')$ be the two points, and let λ, μ, ν be the direction sines of the line joining them, then

$$\frac{\alpha-\alpha'}{\lambda} = \frac{\beta-\beta'}{\mu} = \frac{\gamma-\gamma'}{\nu} = \rho,$$

and λ, μ, ν satisfy the equations of the last article;

therefore from equation (3), Art. 71, we get

$$\rho^2 = \frac{(\alpha-\alpha')^2\sin 2A + (\beta-\beta')^2\sin 2B + (\gamma-\gamma')^2\sin 2C}{\sin A\sin B\sin C},$$

and from equation (4),

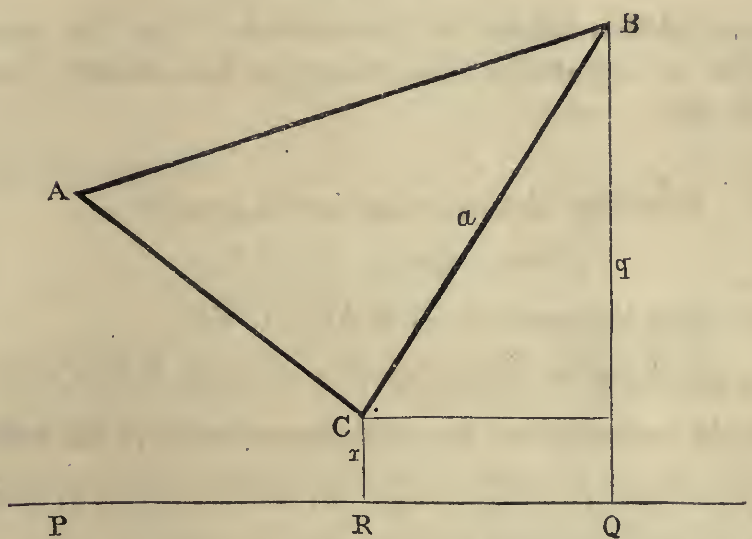
$$\rho^2 = -\frac{(\beta-\beta')(\gamma-\gamma')\sin A + (\gamma-\gamma')(\alpha-\alpha')\sin B + (\alpha-\alpha')(\beta-\beta')\sin C}{\sin A\sin B\sin C},$$

two expressions for the distance, perhaps more interesting from their symmetry than useful in practice.

73. Let p, q, r be the perpendiculars from the points of reference on the straight line whose equation is

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho.$$

Fig. 18.



Let θ be the inclination of this straight line to the line of reference BC, then

$$\lambda=\sin\theta=\pm\frac{q-r}{a};$$

therefore

$$q-r=\pm a\lambda,$$

$$r-p=\pm b\mu,$$

$$p-q=\pm c\nu,$$

the upper signs going together, and the lower together, since we must have by addition

$$0=a\lambda+b\mu+c\nu.$$

Hence, substituting in the equation

$$\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A,$$

(Art. 70), we get

$$\frac{(r-p)^2}{b^2} + \frac{(p-q)^2}{c^2} + \frac{2(r-p)(p-q)}{bc} \cos A = \sin^2 A,$$

or $c^2(r-p)^2 + b^2(p-q)^2 + 2(r-p)(p-q)\,bc \cos A = b^2c^2 \sin^2 A,$

or $$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr \cos A - 2carp \cos B$$
$$- 2abpq \cos C = b^2c^2 \sin^2 A = 4\Delta^2,$$

the same relation among the perpendiculars from the points of reference on any straight line, which we have already obtained in Art. 55.

74. If instead of substituting in the equation

$$\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A,$$

we had taken the equation (4) of Art. 71, viz.

$$\mu\nu \sin A + \nu\lambda \sin B + \lambda\mu \sin C + \sin A \sin B \sin C = 0,$$

we should have obtained our result immediately in the form

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2,$$

a form in which we shall hereafter find it useful.

Or if we had substituted in the equation (3) of Art. 71, we should have got

$$(q-r)^2 \cot A + (r-p)^2 \cot B + (p-q)^2 \cot C = 2\Delta,$$

another useful form.

75. *To find the angles between the straight lines*

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

and
$$\frac{\alpha - \alpha'}{\lambda'} = \frac{\beta - \beta'}{\mu'} = \frac{\gamma - \gamma'}{\nu'} = \rho.$$

Let ϕ, ϕ' be the angles which the straight lines make with the line of reference CA.

Then referring to Art. 70, we have

$$\sin\phi = \mu,$$

and $$\sin(A-\phi) = \nu,$$

or $$\sin A\cos\phi - \cos A\sin\phi = \nu,$$

$$\sin A\cos\phi - \mu\cos A = \nu.$$

Therefore

$$\sin A\cos\phi = \nu + \mu\cos A,$$

and $$\cos\phi = \frac{\nu+\mu\cos A}{\sin A}.$$

Similarly, we have

$$\sin\phi' = \mu',$$

$$\cos\phi' = \frac{\nu'+\mu'\cos A}{\sin A}.$$

Now if D denote the required angle between the given straight lines

$$D = \phi \sim \phi';$$

therefore

$$\sin D = \sin(\phi\sim\phi') = \pm\frac{\mu(\nu'+\mu'\cos A) - \mu'(\nu+\mu\cos A)}{\sin A}$$

$$= \pm\frac{\mu\nu'-\mu'\nu}{\sin A}\text{ ....(1).}$$

Or we may write it

$$\sin D = \frac{1}{\sin A}\begin{vmatrix}\mu, & \nu\\ \mu', & \nu'\end{vmatrix} = \frac{1}{\sin B}\begin{vmatrix}\nu, & \lambda\\ \nu', & \lambda'\end{vmatrix} = \frac{1}{\sin C}\begin{vmatrix}\lambda, & \mu\\ \lambda', & \mu'\end{vmatrix}.$$

So also

$$\cos D = \cos(\phi-\phi') = \frac{(\nu+\mu\cos A)(\nu'+\mu'\cos A)}{\sin^2 A} + \mu\mu'$$

$$= \frac{\mu\mu'+\nu\nu'+(\mu\nu'+\mu'\nu)\cos A}{\sin^2 A}\text{ ....(2),}$$

and by symmetry

$$= \frac{\nu\nu' + \lambda\lambda' + (\nu\lambda' + \nu'\lambda)\cos B}{\sin^2 B}$$

$$= \frac{\lambda\lambda' + \mu\mu' + (\lambda\mu' + \lambda'\mu)\cos C}{\sin^2 C}.$$

76. *To find the angle between the straight lines*

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu},$$

$$\frac{\alpha - \alpha'}{\lambda'} = \frac{\beta - \beta'}{\mu'} = \frac{\gamma - \gamma'}{\nu'},$$

where λ, μ, ν *and* λ', μ', ν' *are not equal but only proportional to the direction sines.*

The expressions of the last article were obtained on the supposition that λ, μ, ν and λ', μ', ν' were actual direction sines. But if we could reduce them to a form in which they would be of zero dimensions in λ, μ, ν and also in λ', μ', ν', they would still be true when these quantities are only proportional to the direction sines.

Now we have (Art. 71)

$$1 = \frac{\sqrt{\mu^2 + \nu^2 + 2\mu\nu\cos A}}{\sin A} = \frac{\sqrt{\nu^2 + \lambda^2 + 2\nu\lambda\cos B}}{\sin B}$$

$$= \frac{\sqrt{\lambda^2 + \mu^2 + 2\lambda\mu\cos C}}{\sin C},$$

and similar expressions connecting λ', μ', ν', in virtue of which the results of the last article may be written

$$\sin D = \frac{(\mu\nu' - \mu'\nu)\sin A}{\sqrt{(\mu^2 + \nu^2 + 2\mu\nu\cos A)(\mu'^2 + \nu'^2 + 2\mu'\nu'\cos A)}} = \&c.,$$

$$\cos D = \frac{\mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu)\cos A}{\sqrt{(\mu^2 + \nu^2 + 2\mu\nu\cos A)(\mu'^2 + \nu'^2 + 2\mu'\nu'\cos A)}} = \&c.,$$

whence

$$\tan D = \frac{(\mu\nu' - \mu'\nu)\sin A}{\mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu)\cos A} = \&c.$$

and these expressions for the sine, cosine and tangent of D are of zero dimensions in λ, μ, ν, and also in λ', μ', ν', and are therefore still true in the case before us when λ, μ, ν and λ', μ', ν' are only proportional to the direction sines.

77. *To deduce expressions for the angle between two lines whose equations are given in the form*

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0.$$

(Compare Arts. 48, 52, 61).

If $(\alpha', \beta', \gamma')$ be the point of intersection of these two lines, the lines may be expressed (Art. 66) by the equations

$$\frac{\alpha - \alpha'}{\begin{vmatrix} m, & n \\ b, & c \end{vmatrix}} = \frac{\beta - \beta'}{\begin{vmatrix} n, & l \\ c, & a \end{vmatrix}} = \frac{\gamma - \gamma'}{\begin{vmatrix} l, & m \\ a, & b \end{vmatrix}},$$

and

$$\frac{\alpha - \alpha'}{\begin{vmatrix} m', & n' \\ b, & c \end{vmatrix}} = \frac{\beta - \beta'}{\begin{vmatrix} n', & l' \\ c, & a \end{vmatrix}} = \frac{\gamma - \gamma'}{\begin{vmatrix} l', & m' \\ a, & b \end{vmatrix}},$$

which are of the form of the given equations of the last article.

We must consider what the functions

$$\mu\nu' - \mu'\nu,$$
$$\mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu)\cos A,$$

and

$$\mu^2 + \nu^2 + 2\mu\nu\cos A,$$

become when we substitute

$$\lambda = \begin{vmatrix} m, & n \\ b, & c \end{vmatrix} \qquad \mu = \begin{vmatrix} n, & l \\ c, & a \end{vmatrix} \qquad \nu = \begin{vmatrix} l, & m \\ a, & b \end{vmatrix}$$

and similar expressions for λ', μ', ν'.

I. We get

$$\mu\nu' - \mu'\nu = \begin{vmatrix} \mu, & \nu \\ \mu', & \nu' \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} n, & l \\ c, & a \end{vmatrix}, & \begin{vmatrix} l, & m \\ a, & b \end{vmatrix} \\ \begin{vmatrix} n', & l' \\ c, & a \end{vmatrix}, & \begin{vmatrix} l', & m' \\ a, & b \end{vmatrix} \end{vmatrix}$$

$$= a \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}.$$

See Prolegomenon, Example F.

II. $$\mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu)\cos A$$

$$= (na - lc)(n'a - l'c) + (lc - mb)(l'c - m'b)$$
$$+ \{(na - lc)(l'c - m'b) + (lc - mb)(n'a - l'c)\}\cos A$$
$$= a^2\{ll' + mm' + nn' - (mn' + m'n)\cos A$$
$$- (nl' + n'l)\cos B - (lm' + l'm)\cos C\}.$$

III. $$\mu^2 + \nu^2 + 2\mu\nu\cos A$$

is the same expression as the last with λ', μ', ν' written for λ, μ, ν respectively.

Therefore it reduces to

$$a^2\{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C\},$$

or with the notation of a former chapter,

$$a^2\{l, m, n\}^2.$$

Hence we can write down the following expressions for the trigonometrical ratios of the angle between the two straight lines whose equations are

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$

viz.
$$\sin D = \frac{\sin A}{a} \begin{vmatrix} l, & m, & n \\ l', & m', & n \\ a, & b, & c \end{vmatrix} \div \{l, m, n\}\{l', m', n'\},$$

or
$$\sin D = \frac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{\{l, m, n\}\{l', m', n'\}},$$

$\cos D$
$$= \frac{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C}{\{l, m, n\}\{l', m', n'\}},$$

and $\tan D$
$$= \frac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C}.$$

78. *To find the direction sines of the straight line whose equation is*
$$l\alpha + m\beta + n\gamma = 0.$$

Let λ, μ, ν be the direction sines required. The angle which this straight line makes with the straight line
$$l'\alpha + m'\beta + n'\gamma = 0$$

was shewn, Art. 77, to be given by
$$\sin D = \pm \frac{\begin{vmatrix} l, & m, & n, \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{\{l, m, n\}\{l', m', n'\}}.$$

Hence writing $m'=0$, $n'=0$, and dividing numerator and denominator of the fraction by l', we get

$$\lambda = \frac{\begin{vmatrix} m, & n \\ \sin B, & \sin C \end{vmatrix}}{\{l, m, n\}}.$$

So writing $l'=0$ and $n'=0$, we get

$$\mu = \frac{\begin{vmatrix} n, & l \\ \sin C, & \sin A \end{vmatrix}}{\{l, m, n\}},$$

and similarly

$$\nu = \frac{\begin{vmatrix} l, & m \\ \sin A, & \sin B \end{vmatrix}}{\{l, m, n\}},$$

which give the direction sines required.

79. Cor. 1. With the same notation the expression for $\sin D$ may be written

$$\sin D = \pm \frac{l'\lambda + m'\mu + n'\nu}{\{l', m', n'\}},$$

which therefore gives us an expression for the angle between the two lines

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

and

$$l'\alpha + m'\beta + n'\gamma = 0.$$

Cor. 2. The two straight lines expresed in Cor. 1 are at right angles provided

$$l'\lambda + m'\mu + n'\nu = \pm \{l', m', n'\}.$$

80. *To find the perpendicular distance of the point* $(\alpha', \beta', \gamma')$ *from the straight line whose equation is*

$$l\alpha + m\beta + n\gamma = 0.$$

Let $$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

be the equation to the perpendicular from $(\alpha', \beta', \gamma')$ on the given line.

Then the length of the intercept, or the value of ρ at the point of intersection of the two lines, is obtained by combining their equations; thus (1) gives us

$$\alpha=\alpha'+\lambda\rho,\quad \beta=\beta'+\mu\rho,\quad \gamma=\gamma'+\nu\rho,$$

and substituting in the given equation, we get

$$l\alpha'+m\beta'+n\gamma'+(l\lambda+m\mu+n\nu)\,\rho=0,$$

whence $$\rho=-\frac{l\alpha'+m\beta'+n\gamma'}{l\lambda+m\mu+n\nu}.$$

But since the lines are at right angles, we have by the last Cor.

$$l\lambda+m\mu+n\nu=\pm\{l,\ m,\ n\},$$

therefore $$\rho=\pm\frac{l\alpha'+m\beta'+n\gamma'}{\{l,\ m,\ n\}},$$

as before in Arts. 46, 59.

Cor. The distance of the point $(\alpha', \beta', \gamma')$ from the straight line $ap\alpha+bq\beta+cr\gamma=0$ is given by

$$\rho=\pm\frac{1}{2\Delta}(ap\alpha+bq\beta+cr\gamma),$$

since it was shewn in Art. 55 that

$$\{ap,\ bq,\ cr\}=2\Delta.$$

81. *To find the equations to the perpendicular from* $(\alpha', \beta', \gamma')$ *on the straight line whose equation is*

$$l\alpha+m\beta+n\gamma=0.$$

Let λ, μ, ν be the direction sines of the required line, and λ', μ', ν' those of the given line, then from the expression for $\cos D$ in Art. 75, since the cosine of a right angle is zero,

$$0 = \mu\mu' + \nu\nu' + (\mu\nu' + \mu'\nu) \cos A,$$

or

$$0 = \mu (\mu' + \nu' \cos A) + \nu (\nu' + \mu' \cos A).$$

But

$$\frac{\mu'}{\begin{vmatrix} n, & l \\ c, & a \end{vmatrix}} = \frac{\nu'}{\begin{vmatrix} l, & m \\ a, & b \end{vmatrix}},$$

therefore substituting

$$0 = \mu \begin{vmatrix} l, & m \cos A - n \\ a, & b \cos A - c \end{vmatrix} + \nu \begin{vmatrix} l, & m - n \cos A \\ a, & b - c \cos A \end{vmatrix}$$

But $b - c \cos A = a \cos C$, and $c - b \cos A = a \cos B$, therefore dividing by a, we get

$$0 = \mu (n - l \cos B - m \cos A) + \nu (l \cos C + n \cos A - m),$$

or

$$\frac{\mu}{m - n \cos A - l \cos C} = \frac{\nu}{n - l \cos B - m \cos A};$$

and therefore

$$= \frac{\lambda}{l - m \cos C - n \cos B}.$$

Hence the equations to the perpendicular will be

$$\frac{\alpha - \alpha'}{l - m \cos C - n \cos B} = \frac{\beta - \beta'}{m - n \cos A - l \cos C}$$

$$= \frac{\gamma - \gamma'}{n - l \cos B - m \cos A}.$$

82. The equations of the last article

$$\frac{\lambda}{l - m \cos C - n \cos B} = \frac{\mu}{m - n \cos A - l \cos C}$$

$$= \frac{\nu}{n - l \cos B - m \cos A} \quad \text{..........(1)},$$

express the conditions that λ, μ, ν may be the direction sines of a line at right angles to the line

$$l\alpha + m\beta + n\gamma = 0.$$

But since they imply the relation

$$a\lambda + b\mu + c\nu = 0,$$

they express only *one* further condition.

To find this one condition in a symmetrical form, we have from the first of the equations in (1),

$$l(\mu + \lambda \cos C) - m(\lambda + \mu \cos C) + n(\mu \cos \beta - \lambda \cos A) = 0,$$

whence dividing by C, and remembering that $a\lambda + b\mu + c\nu = 0$, we get

$$\frac{l}{a}(\mu \cos B - \nu \cos C) + \frac{m}{b}(\nu \cos C - \lambda \cos A)$$
$$+ \frac{n}{c}(\lambda \cos A - \mu \cos B) = 0,$$

or
$$\begin{vmatrix} \frac{l}{a}, & \frac{m}{b}, & \frac{n}{c} \\ \lambda \cos A, & \mu \cos B, & \nu \cos C \\ 1, & 1, & 1 \end{vmatrix} = 0,$$

or
$$\begin{vmatrix} l, & m, & n \\ \lambda \sin 2A, & \mu \sin 2B, & n \sin 2C \\ \sin A, & \sin B, & \sin C \end{vmatrix} = 0;$$

a result which the student acquainted with the differential calculus could have written down at sight from the consideration that

$$l\lambda + m\mu + n\nu$$

had to be made a minimum subject to the relations

$$\lambda^2 \sin 2A + \mu^2 \sin 2B + \nu^2 \sin 2C = 2 \sin A \sin B \sin C,$$

(equation 3 of Art. 71)

and $$\lambda \sin A + \mu \sin B + \nu \sin C = 0,$$

whence we must have

$$l\delta\lambda + m\delta\mu + n\delta\nu = 0,$$
$$\lambda \sin 2A \,.\, \delta\lambda + \mu \sin 2B \,.\, \delta\mu + \nu \sin 2C \,.\, \delta\nu = 0,$$
$$\sin A \,.\, \delta\lambda + \sin B \,.\, \delta\mu + \sin C \,.\, \delta\nu = 0,$$

and eliminating the differentials, the result is obtained.

Exercises on Chapter VI.

(57) The straight line whose direction sines are λ, μ, ν meets the line at infinity in the point given by the equations

$$\frac{\alpha}{\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{\nu}.$$

(58) Find the coordinates of the point at which the sides of the triangle of reference subtend equal angles.

If through this point three straight lines be drawn each parallel to a side and terminated by the other two sides, the rectangles contained by their segments are equal.

(59) From the point $(\alpha', \beta', \gamma')$ the straight line is drawn whose direction sines are λ, μ, ν: find the length intercepted upon this line, between the straight lines whose equations are

$$l\alpha + m\beta + n\gamma = 0,$$

and $$l'\alpha + m'\beta + n'\gamma = 0.$$

(60) Shew that if from any fixed point O there be drawn three straight lines OP, OP', OP'', whose lengths are ρ, ρ', ρ'', and whose direction sines are (λ, μ, ν), (λ', μ', ν'), $(\lambda'', \mu'', \nu'')$ respectively, then the area of the triangle $PP'P''$ will be

$$= \frac{2\Delta}{a} \begin{vmatrix} 1, & \mu\rho, & \nu\rho \\ 1, & \mu'\rho', & \nu'\rho' \\ 1, & \mu''\rho'', & \nu''\rho'' \end{vmatrix}$$

(61) From the middle points of the sides of the triangle of reference perpendiculars are drawn proportional in length to the sides; and their extremities are joined to the opposite angular points of the triangle. Shew that the three joining lines will meet in a point whose coordinates (α, β, γ) are connected by the equation

$$\frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = 0.$$

(62) From the point O, $(\alpha', \beta', \gamma')$ a straight line is drawn in any direction to meet the straight lines

$$l\alpha + m\beta + n\gamma = 0,$$

$$l'\alpha + m'\beta + \nu'\gamma = 0,$$

$$(l+l')\,\alpha + (m+m')\,\beta + (n+n')\,\gamma = 0,$$

in points P, Q, R. Shew that the ratio $OP.QR : OQ.PR$ is equal to

$$-\frac{l\alpha' + m\beta' + n\gamma'}{l'\alpha' + m'\beta' + n'\gamma'},$$

whatever be the direction of the transversal.

CHAPTER VII.

MODIFICATIONS OF THE SYSTEM OF TRILINEAR COORDINATES. AREAL AND TRIANGULAR COORDINATES.

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83. THE great principle which distinguishes the modern methods of analytical geometry from the old Cartesian methods is, as we have seen, the adoption of three coordinates instead of two to represent the position of a point, and the recognition of the power thus gained of rendering all our equations homogeneous.

This homogeneity of equations will be always attainable whatever quantities x, y, z we may use as coordinates of a point, provided the third, z, be connected with the other two by a linear equation,

$$Ax + By + Cz = D,$$

or

$$\frac{Ax + By + Cz}{D} = 1;$$

for (exactly as in the case of Art. 10, page 13) any term in an equation which is of a lower order than another may be raised by multiplying it by

$$\frac{Ax + By + Cz}{D},$$

(since this is equal to unity) and we may repeat the operation till every term is raised to the order of the highest term, and the equation is thus homogeneous.

84. We have hitherto used the perpendicular distances of the point P from the lines of reference as the coordinates of P, and we have established the relation

$$a\alpha + b\beta + c\gamma = 2\Delta \qquad (1),$$

connecting the coordinates of any point.

The position of the point would be equally determinate if we used any constant multiples of these perpendicular distances as coordinates. For instance, we might call the coordinates of P, α', β', γ', where

$$\alpha' = \lambda\alpha, \quad \beta' = \mu\beta, \quad \gamma' = \nu\gamma,$$

and the relation (1) connecting the coordinates of any point would then become

$$\frac{a\alpha'}{\lambda} + \frac{b\beta'}{\mu} + \frac{c\gamma'}{\nu} = 2\Delta \qquad (2).$$

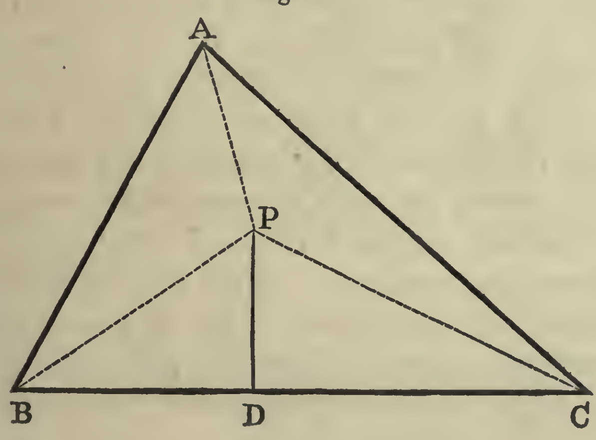
The particular case in which

$$\lambda = a, \quad \mu = b, \quad \nu = c$$

will present the advantage of a very simple relation among the coordinates, for the equation (2) reduces in this case to

$$\alpha' + \beta' + \gamma' = 2\Delta \qquad (3).$$

Fig. 19.



And the quantities α', β', γ', which in this case will be the coordinates of the point P, are capable of a simple geometrical interpretation. For if PD be the perpendicular from P on BC (fig. 19), we have

$$\alpha' = a\alpha = BC \, . \, PD$$

$$= 2\Delta \, PBC,$$

so $$\beta' = 2\Delta \, PCA,$$

and $$\gamma' = 2\Delta \, PAB.$$

The coordinates α', β', γ' of the point P are therefore the double areas of the triangles having P as vertex, and the sides of the triangle of reference as bases.

85. If α'', β'', γ'' denote the halves of α', β', γ', the equation (3) of Art. 84 gives us

$$\alpha'' + \beta'' + \gamma'' = \Delta \qquad \text{.......................(4)},$$

as the relation connecting α'', β'', γ'' if they be taken as the coordinates of P. These coordinates represent the areas of the triangles BPC, PCA, PAB, and used often to be called indifferently the *areal* or *triangular* coordinates of P with respect to the triangle ABC. These terms *areal* and *triangular* have however more recently been applied to the system of coordinates described in the next article, and authors are not uniform in their use of the expressions. It seems convenient to describe these coordinates α'', β'', γ'' which represent the actual *areas* of the triangles PBC, PCA, PAB as *areal* coordinates, observing that as they represent areas they are of two dimensions in linear magnitude. We can thus reserve the term *triangular* for the system now about to be described, although it would certainly be preferable to invent a name for them which should indicate the fact (which will immediately appear) that they are of *zero* dimensions in linear magnitude, expressing not lines nor areas but simply ratios.

86. The relation among the trilinear coordinates α, β, γ, viz.

$$a\alpha + b\beta + c\gamma = 2\Delta$$

may be written

$$\frac{a\alpha}{2\Delta} + \frac{b\beta}{2\Delta} + \frac{c\gamma}{2\Delta} = 1.$$

If therefore x, y, z denote the ratios

$$\frac{a\alpha}{2\Delta}, \quad \frac{b\beta}{2\Delta}, \quad \frac{c\gamma}{2\Delta},$$

they will be subject to the very simple relation

$$x + y + z = 1 \text{ .... } (5).$$

But since x, y, z bear constant ratios to α, β, γ they may be used as the coordinates of P (Art. 84): and on account of the simplicity of the relation (5) just obtained, very great advantages attend their use.

It will be observed that these coordinates (x, y, z) represent the ratios of the triangles PBC, PCA, PAB severally to the triangle of reference ABC. They are (not very appropriately) often spoken of as the *areal* or *triangular* coordinates of P, but as we said in the last article, we shall call them *triangular* coordinates, reserving the term *areal* for the system described in that article.

In speaking of the areas of the triangles PBC, PCA, PAB, the same convention with respect to algebraical sign will have to be adopted as in the case of the perpendicular distances of P from the lines of difference. Thus (as in Art. 6, page 10) the triangle PBC will be considered positive when it lies on the same side of the base BC as does the triangle of reference, and so for the other triangles.

87. It is important to observe that if the triangle of reference be the same, the triangular coordinates (x, y, z) and the

trilinear coordinates (α, β, γ) of any point P, are connected by the relations

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}=\frac{1}{2\Delta},$$

so that we can at once transform any equation or expression from the one system to the other.

To exemplify this, and for convenience of reference, we append a table of the principal results which we have already obtained in trilinear coordinates, together with the corresponding results for triangular coordinates.

TABLE

OF FORMULÆ AND OTHER RESULTS.

In trilinear coordinates.	*In triangular coordinates.*

(i) The coordinates of any point are connected by the relation (Art. 7),

$a\alpha+b\beta+c\gamma=2\Delta.$	$x+y+z=1.$

(ii) The coordinates of the middle point of BC are (Art. 14),

$0,\ \frac{\Delta}{b},\ \frac{\Delta}{c}.$	$0,\ \frac{1}{2},\ \frac{1}{2}.$

(iii) The coordinates of the foot of the perpendicular from A upon BC are (Art. 15),

$0,\ \frac{2\Delta}{a}\cos C,\ \frac{2\Delta}{a}\cos B.$	$0,\ \frac{b\cos C}{a},\ \frac{c\cos B}{a}.$

In trilinear coordinates.	*In triangular coordinates.*

(iv) The centre of the inscribed circle is given (Art. 16) by

$\alpha = \beta = \gamma = \dfrac{2\Delta}{a+b+c}$.	$\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{1}{a+b+c}$.

(v) The middle point between the two points

$(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, is	(x_1, y_1, z_1) and (x_2, y_2, z_2), is
$\left(\dfrac{\alpha_1+\alpha_2}{2}, \dfrac{\beta_1+\beta_2}{2}, \dfrac{\gamma_1+\gamma_2}{2}\right)$.	$\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$.

(vi) The area of a triangle whose angular points are given (Art. 19), is

$A = \dfrac{1}{2S}\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix}$	$A = \Delta \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$

(vii) The equation to a straight line joining two points whose coordinates are given (Art. 21), is

$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0.$	$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix} = 0.$

(viii) The equation to the straight line whose distances from the points of reference are p, q, r (Art. 23), is

$ap\alpha + bq\beta + cr\gamma = 0.$	$px + qy + rz = 0.$

(ix) The condition that the three straight lines

$l\alpha + m\beta + n\gamma = 0,$	$lx + my + nz = 0,$
$l'\alpha + m'\beta + n'\gamma = 0,$	$l'x + m'y + n'z = 0,$
$l''\alpha + m''\beta + n''\gamma = 0,$	$l''x + m''y + n''z = 0,$

In trilinear coordinates.	*In triangular coordinates.*

should meet in a point (Art. 26), is

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 0.$$

(x) The equation to the perpendicular from A on BC (Art. 30), is

In trilinear coordinates	In triangular coordinates
$\beta \cos B - \gamma \cos C = 0.$	$y \cot B - z \cot C = 0.$

(xi) The condition of parallelism (Art. 35) of the two straight lines whose equations are

In trilinear coordinates	In triangular coordinates
$l\alpha + m\beta + n\gamma = 0,$	$lx + my + nz = 0,$
$l'\alpha + m'\beta + n'\gamma = 0,$	$l'x + m'y + n'z = 0,$
is $\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix} = 0.$	is $\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 1, & 1, & 1 \end{vmatrix} = 0.$

(xii) The equation to the straight line at infinity (Art. 36),

In trilinear coordinates	In triangular coordinates
$a\alpha + b\beta + c\gamma = 0.$	$x + y + z = 0.$

(xiii) The perpendicular distance (Art. 46) of the point

In trilinear coordinates	In triangular coordinates
$(\alpha', \beta', \gamma')$	(x', y', z')

from the straight line whose equation is

In trilinear coordinates	In triangular coordinates
$l\alpha + m\beta + n\gamma = 0,$	$lx + my + nz = 0,$
is $\dfrac{l\alpha' + m\beta' + n\gamma'}{\{l, m, n\}},$	is $2\Delta \dfrac{lx' + my' + nz'}{\{al, bm, cn\}},$

where

$$\{l, m, n\}^2 \equiv l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C,$$

and therefore

$$\{al, bm, cn\}^2$$

$$\equiv a^2 (l - m)(l - n) + b^2 (m - n)(m - l) + c^2 (n - l)(n - m).$$

In trilinear coordinates.	*In triangular coordinates.*

(xiv) The perpendicular distance of the same point from the straight line whose equation is (Art. 58)

$ap\alpha + bq\beta + cr\gamma = 0,$	$px + qy + rz = 0,$
is $\dfrac{ap\alpha' + bq\beta' + cr\gamma'}{2\Delta}.$	is $px' + qy' + rz'.$

(xv) The sine of the angle D between the two straight lines whose equations are

$ap\alpha + bq\beta + cr\gamma = 0,$	$px + qy + rz = 0,$
$ap'\alpha + bq'\beta + cr'\gamma = 0,$	$p'x + q'y + r'z = 0,$

is (Art. 60)

$$\frac{1}{2\Delta}\begin{vmatrix} p, & q, & r \\ p', & q', & r \\ 1, & 1, & 1 \end{vmatrix}$$

(xvi) The sine of the angle D between the two straight lines whose equations are

$l\alpha + m\beta + n\gamma = 0,$	$lx + my + nz = 0,$
$l'\alpha + m'\beta + n'\gamma = 0,$	$l'x + m'y + n'z = 0,$

is (Art. 61)

$\dfrac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \sin A, & \sin B, & \sin C \end{vmatrix}}{\{l, m, n\}\{l', m', n'\}}.$	$2\Delta\dfrac{\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 1, & 1, & 1 \end{vmatrix}}{\{al, bm, cn\}\{al', bm', cn'\}}.$

In trilinear coordinates.	*In triangular coordinates.*

(xvii) The area of the triangle whose sides are represented by the equations

$ap\alpha + bq\beta + cr\gamma = 0,$	$px + qy + rz = 0,$
$ap'\alpha + bq'\beta + cr'\gamma = 0,$	$p'x + q'y + r'z = 0,$
$ap''\alpha + bq''\beta + cr''\gamma = 0,$	$p''x + q''y + r''z = 0,$

is (Art. 64)

$$\Delta \frac{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ p'', & q'', & r'' \end{vmatrix}^2}{\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} p', & q', & r' \\ p'', & q'', & r'' \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} p'', & q'', & r'' \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix}}.$$

(xviii) The condition that the two straight lines

$l\alpha + m\beta + n\gamma = 0,$	$lx + my + nz = 0,$
$l'\alpha + m'\beta + n'\gamma = 0,$	$l'x + m'y + n'z = 0,$

should be at right angles is (Art. 49)

$ll' + mm' + nn'$	$ll'a^2 + mm'b^2 + nn'c^2$
$- (mn' + m'n) \cos A$	$- (mn' + m'n)\, bc \cos A$
$- (nl' + n'l) \cos B$	$- (nl' + n'l)\, ca \cos B$
$- (lm' + l'm) \cos C = 0.$	$- (lm' + l'm)\, ab \cos C = 0.$

(xix) Any straight line drawn through the given point

$(\alpha', \beta', \gamma'),$	$(x', y', z'),$

may be represented by the equations (Art. 68)

$\dfrac{\alpha - \alpha'}{\lambda} = \dfrac{\beta - \beta'}{\mu} = \dfrac{\gamma - \gamma'}{\nu},$	$\dfrac{x - x'}{\lambda} = \dfrac{y - y'}{\mu} = \dfrac{z - z'}{\nu},$

In trilinear coordinates.	*In triangular coordinates.*

where λ, μ, ν are proportional to the coordinates of the point where the straight line meets infinity and are subject to the relation

$a\lambda + b\mu + c\nu = 0.$	$\lambda + \mu + \nu = 0.$

(xx) Each member of the equations in (xix) will be equal to the distance (ρ) from the given point to the point

$(\alpha, \beta, \gamma),$	$(x, y, z),$

provided λ, μ, ν satisfy a further condition which is expressed by any one of the equations (Arts. 70, 71)

$\mu^2 + \nu^2 + 2\mu\nu \cos A = \sin^2 A,$	$a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu + 1 = 0,$
$\nu^2 + \lambda^2 + 2\nu\lambda \cos B = \sin^2 B,$	$\lambda^2 bc \cos A + \mu^2 ca \cos B$
$\lambda^2 + \mu^2 + 2\lambda\mu \cos C = \sin^2 C.$	$+ \nu^2 ab \cos C = 1.$

Exercises on Chapter VII.

(63) Shew that the straight lines

$$(m + n)\, x + (n + l)\, y + (l + m)\, z = 0,$$

and

$$lx + my + nz = 0,$$

are parallel, and find their inclination to the straight line

$$(m - n)\, x + (n - l)\, y + (l - m)\, z = 0.$$

(64) If a series of different values be given to p, the equation

$$x (\cos^2\alpha + p \sin^2\alpha) + y (\cos^2\beta + p \sin^2\beta) + z (\cos^2\gamma + p \sin^2\gamma) = 0$$

will represent a series of parallel straight lines.

(65) Shew that the straight line bisecting at right angles the side BC of the triangle of reference is given by the equation

$$(x-y+z)\cot B=(x+y-z)\cot C.$$

Hence shew that the three straight lines, bisecting at right angles the three sides of the triangle of reference, meet in one point.

(66) Draw the four straight lines represented by the equations in triangular coordinates,

$$x\cot A\pm y\cot B\pm z\cot C=0.$$

(67) Draw the four straight lines represented by the equations

$$x\operatorname{cosec} A\pm y\operatorname{cosec} B\pm z\operatorname{cosec} C=0.$$

(68) Interpret the equations

$$x\pm y\pm z=0.$$

(69) Find the area of the triangle whose sides are given by the equations

$$lx+my+nz=0,$$
$$mx+ly-nz=0,$$
$$z=0.$$

(70) Find the area of the triangle whose angular points are given respectively by

$$x=0 \text{ and } y=lz,$$
$$y=0 \text{ and } z=mx,$$
$$z=0 \text{ and } x=ny.$$

If $mn=1$, the result is independent of l. Interpret this circumstance geometrically.

(71) Of the four straight lines whose equations are

$$lx \pm my \pm nz = 0,$$

two intersect in P and the other two in P', two intersect in Q and the other two in Q', two intersect in R and the other two in R'. Find the triangular coordinates of the middle points of PP', QQ', RR', and shew that they lie on one straight line.

Find the equation to this straight line, and compare the result with that of Ex. (40), page 56.

(72) Prove that the straight line represented in trilinear coordinates by the equation

$$a^3\alpha + b^3\beta + c^3\gamma = 0,$$

is parallel to the straight line represented in triangular coordinates by the equation

$$x \cot A + y \cot B + z \cot C = 0.$$

(73) ABC is a triangle: through the angular points A, B, C, straight lines bc, ca, ab are drawn forming a second triangle abc: and through a, b, c straight lines are drawn parallel to BC, CA, AB respectively, so as to form a third triangle, and so on. Shew that the areas of the triangles thus formed are in geometrical progression.

CHAPTER VIII.

ABRIDGED NOTATION OF THE STRAIGHT LINE.

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88. IN Chapter III. Art. 27, we used $u=0$ and $v=0$ to represent the equations to two straight lines, regarding u and v as abridgements or symbols of the expressions $l\alpha + m\beta + n\gamma$ and $l'\alpha + m'\beta + n'\gamma$.

But it will have been observed that the reasoning of that article would have been equally valid if $u=0$, $v=0$ had represented the equations to two straight lines expressed in triangular coordinates, or in the ordinary Cartesian system.

We may therefore write the result of that article in the following more general form.

If $u=0$, $v=0$ be the equations to two straight lines expressed in any system in which a point is determined by coordinates, then will the equation $u+\kappa v=0$ represent a straight line passing through the intersection of the former lines.

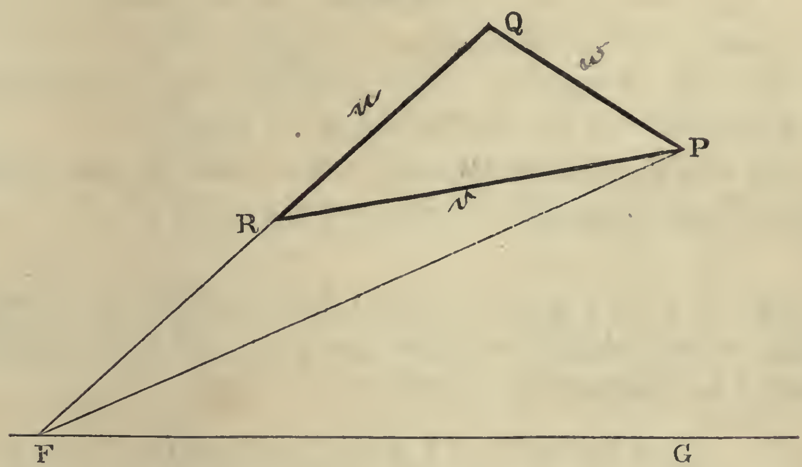
And by giving a suitable value to κ this equation can be made to represent any *straight line whatever passing through that point of intersection.*

89. The following principle will often be assumed.

If $u=0$, $v=0$, $w=0$ be the equations to three straight lines forming a triangle expressed in any system in which a point *is determined by coordinates, then any straight line whatever will be represented by an equation of the form*

$$\lambda u + \mu v + \nu w = 0.$$

Fig. 20.



For let QR, RP, PQ be the straight lines whose equations are $u=0$, $v=0$, $w=0$ respectively, and let FG be any straight line whatever whose equation it is required to find.

Since QR, RP, PQ are not parallel (*hypoth.*), one of them can be found which being produced will meet FG. Let it be QR.

Let QR produced meet FG in F. Join FP.

Then, by Art. 88, since FP passes through the intersection of $v=0$, $w=0$, it will have an equation of the form

$$v + \kappa w = 0,$$

or, as we may write it,

$$\mu v + \nu w = 0.$$

But since FG passes through the intersection of the straight line whose equations are $u=0$ and

$$\mu v + \nu w = 0,$$

it will by the same article (88) have an equation of the form

$$\lambda u + \mu v + \nu w = 0,$$

which was to be proved.

We have supposed that the three original straight lines form a triangle. All that is necessary however for the validity of the proof is *that they should not be all parallel, and that they should not meet in a point.*

If they were all parallel, one could not *necessarily* be found to intersect PG, and if they met in a point, FP and FQ would coincide, and the equation to FG could not then be determined as passing through the intersection of FP and FQ.

But the theorem is perfectly true if *two* of the original straight lines be parallel and intersect the third.

90. If $u+v+w=0$ represent the fourth side of a quadrilateral whose other three sides are $u=0$, $v=0$, $w=0$, it is required to interpret the equations

$$v \pm w = 0, \quad w \pm u = 0, \quad u \pm v = 0.$$

Let the first three sides be BC, CA, AB (fig. 21), and let the fourth side cut them (when produced) in A', B', C' respectively.

The equation $v+w=0$ represents a straight line passing through the intersection of $u=0$ and $u+v+w=0$: also through the intersection of $v=0$ and $w=0$.

It is therefore the straight line AA'.

Similarly, $\quad w+u=0$ represents BB',

and $\quad u+v=0$ represents CC'.

Again, let BB', CC', intersect in a; CC', AA' in b; AA', BB' in c.

Then the equation $v-w=0$, since it results from the subtraction of $w+u=0$ and $u+v=0$, represents a straight line passing through the intersection of BB' and CC', or a.

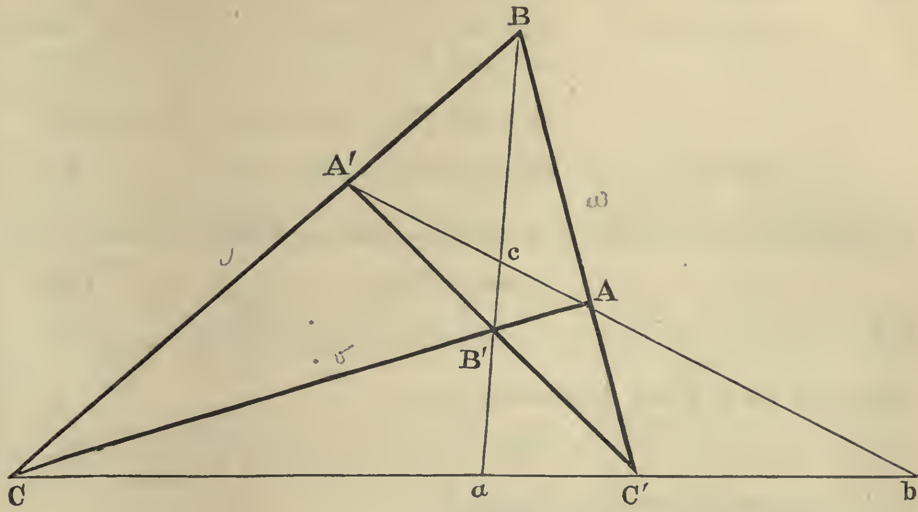
It also passes through the intersection of $v=0$ and $w=0$, or A.

Hence, $\quad v-w=0$ represents the line Aa,

so $\quad w-u=0$ .... Bb,

and $\quad u-v=0$ .... Cc.

Fig. 21.



91. The following proposition is very important.

If

$$u + v + w = 0 \quad \text{.............................}(1),$$

$$-u + v + w = 0 \quad \text{.............................}(2),$$

$$u - v + w = 0 \quad \text{.............................}(3),$$

$$u + v - w = 0 \quad \text{.............................}(4),$$

be the equations of four straight lines forming a quadrilateral, then

$$u = 0,$$

$$v = 0,$$

$$w = 0,$$

will be the three diagonals of the quadrilateral.

For $u = 0$ evidently represents a straight line passing through the intersection of (1) and (2), and also through the intersection of (3) and (4), therefore it is a diagonal of the quadrilateral.

Similarly $v = 0$ joins the intersection of (1), (3) with that of (2), (4), and so $w = 0$ joins the intersection of (1), (4) with that of (2), (3);

∴ &c. Q.E.D.

92. If the equations

$$u + v = 0 \quad\quad (1),$$

$$u + w = 0 \quad\quad (2),$$

$$u - v = 0 \quad\quad (3),$$

$$u - w = 0 \quad\quad (4),$$

represent the four sides of a quadrilateral in order, then will

$$v + w = 0 \quad\quad (5),$$

and

$$v - w = 0 \quad\quad (6),$$

represent its interior diagonals, and

$$u = 0 \quad\quad (7),$$

will represent its exterior diagonal,

and

$$v = 0 \quad\quad (8),$$

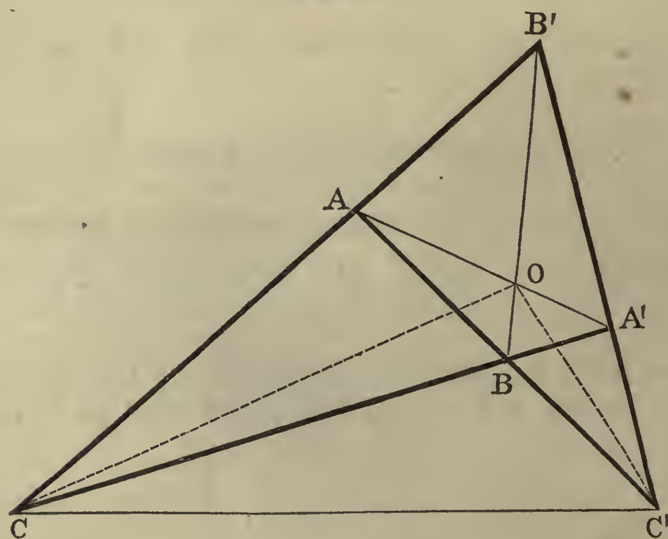
and

$$w = 0 \quad\quad (9),$$

will represent the straight lines joining the point of intersection of the two interior diagonals to the points of intersection of opposite sides.

Let AA', BB' be the interior, and CC' the exterior diagonal, so that the equations (1), (2), (3), (4) represent AB', $B'A'$, $A'B$, BA respectively, and let AA', BB' intersect in O.

Fig. 22.



Then the equation (5) may be obtained either by subtracting (1) and (4), or by subtracting (2) and (3). Therefore it represents the line joining the intersection of (1) and (4) with that of (2) and (3); i.e. the line AA'.

Similarly (6) denotes the line BB'.

But $u = 0$ passes through the intersection of (1) and (3) as well as through that of (2) and (4); therefore it represents the line CC'.

Also $v = 0$ passes through the intersection of (1) and (3) as well as through that of (5) and (6). Hence it denotes the line CO.

And similarly the equation $w = 0$ must denote the line $C'O$. Q.E.D.

The student is recommended to examine for himself the modifications which these theorems undergo when one of the straight lines is at infinity.

93. We now introduce some geometrical terms which will be found convenient.

Definitions.

I. Three or more straight lines which pass through the same point are said to be *concurrent*.

II. Three or more points which lie upon the same straight line are said to be *collinear*.

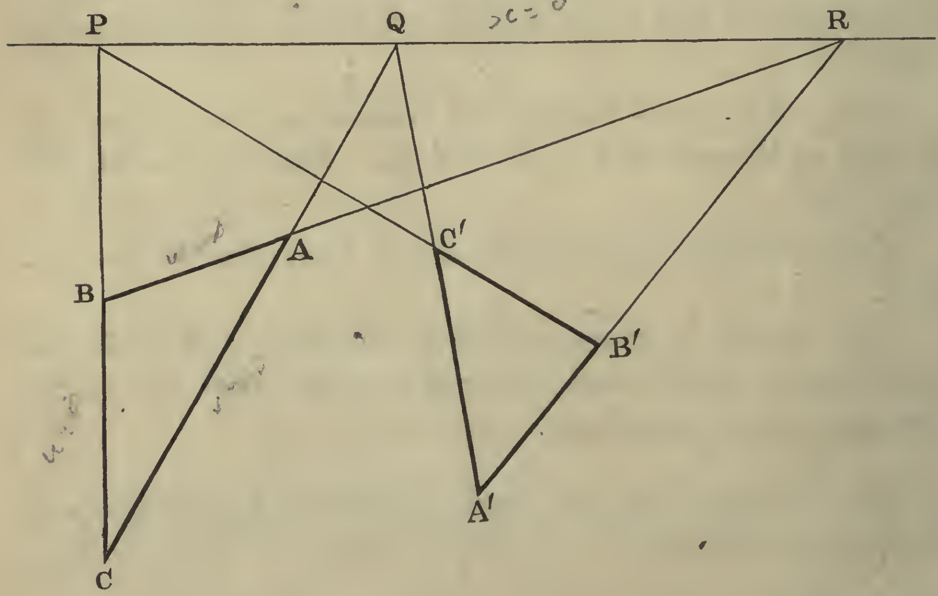
III. Two triangles ABC, $A'B'C'$ are said to be *co-polar* if AA', BB', CC' meet in a point, and this point is called the *pole* of the triangles, or the *pole* of either triangle with respect to the other.

IV. Two triangles ABC, $A'B'C'$ are said to be *co-axial* if the points of intersection of BC, $B'C'$, of CA, $C'A'$, and of AB, $A'B'$ lie in one straight line, and this straight line is called the *axis* of the two triangles, or the axis of either triangle with respect to the other.

94. *If two triangles be co-axial they will also be co-polar.*

Let ABC, $A'B'C'$ be two co-axial triangles, and let PQR be their axis; P being the point of intersection of BC, $B'C'$, Q that of CA, $C'A'$, and R that of AB, $A'B'$.

Fig. 23.



Let $u=0$, $v=0$, $w=0$ be the equations to BC, CA, AB respectively, and $x=0$ the equation to PQR.

Then, since $B'C'$ passes through the intersection of PQR and BC ($x=0$ and $u=0$), its equation may be written

$$x + lu = 0 \quad \text{(i)},$$

So the equation to $C'A'$ may be written

$$x + mv = 0 \quad \text{(ii)},$$

and the equation to $A'B'$

$$x + nw = 0 \quad \text{(iii)}.$$

From (ii) and (iii) by subtraction we obtain

$$mv - nw = 0 \quad \text{(iv)},$$

which therefore represents a straight line passing through the point of intersection of $C'A'$ and $A'B'$; i.e. through A'.

But this equation, from its form, must represent a straight line passing through the intersection of the straight lines $v=0$ and $w=0$, i.e. through A.

Hence (iv) is the equation to AA'.

Similarly, $nw - lu = 0$ .... (v),

and $lu - mv = 0$ .... (vi),

are the equations to BB' and CC' respectively. But (iv), (v), (vi) are all satisfied at the point determined by

$$lu = mv = nw.$$

Therefore AA', BB', CC' all pass through this point, and therefore the triangles are *co-polar*.

Therefore *any two co-axial triangles are also co-polar*. Q.E.D.

95. *If two triangles be co-polar, they will also be co-axial.*

Let $u=0$, $v=0$, $w=0$ be the equations to the sides of the triangle ABC, and let $A'B'C'$ be a co-polar triangle, the point O being the pole.

Let BC, $B'C'$ intersect in P, CA, $C'A'$ in Q, and AB, $A'B'$ in R. We have to shew that P, Q, R are collinear.

Let $x=0$ be the equation to PQ.

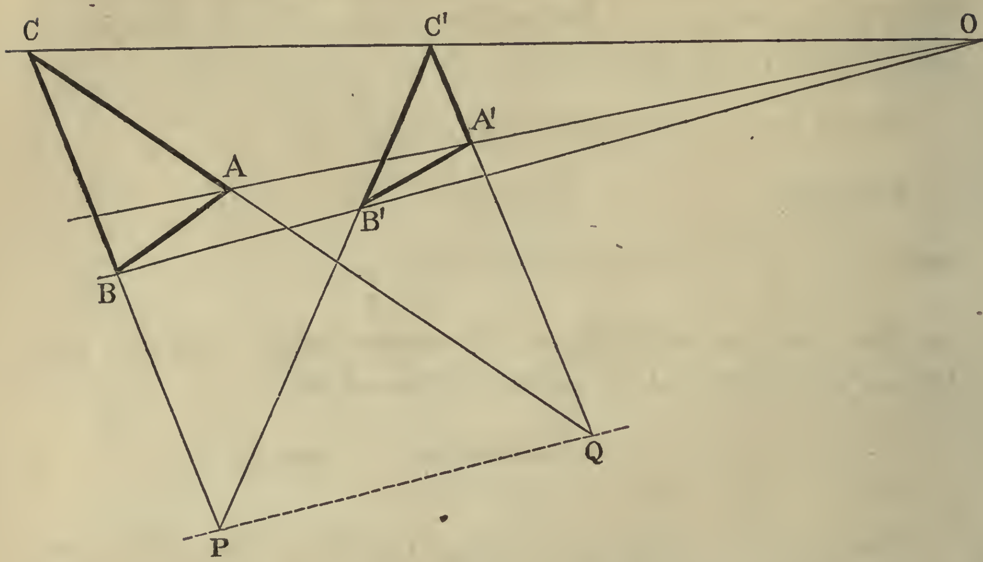
Then $B'C'$ passing through the intersection of BC and PQ has an equation which may be written

$x + lu = 0$ .... (i).

So $C'A'$ passing through the intersection of CA and PQ may be represented by the equation

$x + mv = 0$ .... (ii).

Fig. 24.



Then the equation

$$lu - mv = 0 \quad \text{(iii)}$$

must represent a straight line passing through the intersection of the straight lines (i) and (ii), as well as through the intersection of the straight lines u and v. Therefore it represents the straight line CC'.

Therefore the point O is given by the equations

$$lu = mv$$
$$= nw \text{ suppose.}$$

And therefore OA and OB will have the equations

$$mv - nw = 0 \quad \text{(iv)},$$

and

$$nw - lu = 0 \quad \text{(v)}.$$

Now consider the equation

$$x + nw = 0 \quad \text{(vi)}.$$

It must represent a straight line passing through the intersection of the straight lines (i) and (v), that is, through B'.

Similarly its locus must pass through the intersection of the straight lines (ii) and (iv), that is, through A'.

Therefore it represents the straight line $A'B'$; but by its form its locus must pass through the intersection of the straight lines w and x, i. e. AB and PQ.

Hence the three straight lines AA', BB', PQ meet in a point, or (in other words) the point of intersection R of the sides AA', BB' is collinear with P and Q. And therefore the triangles ABC, $A'B'C'$ are *co-axial.*

Hence *any two co-polar triangles are co-axial.* Q. E. D.

96. *The three straight lines which are represented in abridged notation by the equations*

$$lu + mv + nw = 0 \text{.....................} (1),$$

$$l'u + m'v + n'w = 0 \text{.....................} (2),$$

$$l''u + m''v + n''w = 0 \text{.....................} (3),$$

will be concurrent, provided

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 0.$$

For (Art. 88) if the straight line (3) pass through the intersection of (1) and (2), its equation must be obtained by adding some multiples of the first two equations.

Suppose k, k' the respective multipliers, then we must have

$$kl + k'l' = l'',$$

$$km + k'm' = m'',$$

$$kn + k'n' = n'',$$

whence, eliminating k, k', we obtain

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 0,$$

the required condition for the concurrence of the three straight lines.

97. It will be observed that the result of the last Article is precisely the same as that of Art. 26. Indeed Art. 26 is but a particular case of Art. 96.

For if we regard

$$u = 0, \quad v = 0, \quad w = 0,$$

as denoting the equations to three straight lines in trilinear coordinates, then (Art. 46) the *expressions* u, v, w, themselves denote the perpendicular distances of the point (α, β, γ) from these straight lines. And therefore if these straight lines be taken as lines of reference, u, v, w will be proportional to the new trilinear coordinates of the point (α, β, γ), and may themselves be regarded (Art. 84) as coordinates of this point, referred to the new triangle.

The equations

$$lu + mv + nw = 0,$$

$$l'u + m'v + n'w = 0,$$

$$l''u + m''v + n''w = 0,$$

need therefore be no longer regarded as abbreviated expressions, but they may be read as relations among the coordinates u, v, w, and as such may be subjected to the reasoning which in Art. 26 is applied to the relations among the coordinates α, β, γ.

98. It is interesting to observe that the method of trilinear coordinates originally grew out of the method of abridged notation applied to Cartesian coordinates exactly by the process of thought indicated in the last article. In the works of Mr Tod-

hunter and Dr Salmon, the subject will be found treated from this point of view. As far as we know, Mr Ferrers (in 1861) was the first to publish a work establishing trilinear coordinates upon an independent basis.

Exercises on Chapter VIII.

(74) If $u=0$, $v=0$, $w=0$ be the equations to three straight lines, find the equation to the straight line passing through the two points

$$\frac{u}{l}=\frac{v}{m}=\frac{w}{n} \text{ and } \frac{u}{l'}=\frac{v}{m'}=\frac{w}{n'}.$$

(75) Find the equation to the straight line passing through the intersections of the pairs of lines

$$2au+bv+cw=0, \quad bv-cw=0;$$

and

$$2bu+av+cw=0, \quad av-cw=0.$$

(76) If $s=0$ be the equation to the straight line at infinity, the equations

$$u+v+s=0, \quad -u+v+s=0,$$
$$u+v-s=0, \quad u-v+s=0,$$

represent the sides of a parallelogram whose diagonals are $u=0$ and $v=0$.

(77) Let the three diagonals of a quadrilateral be produced to meet each other in three points, and let each of these points be joined with the opposite corners of the quadrilateral: the six lines so drawn will intersect three and three in four points.

(78) If $s=0$ be the equation to the straight line at infinity, then the triangle whose sides are

$$u=0, \quad v=0, \quad w=0,$$

is co-polar with the triangle whose sides are

$$u+ls=0, \quad v+ms=0, \quad w+ns=0,$$

whatever be the values of l, m, n.

(79) If ABC, $A'B'C'$ be the two triangles in the last question, and if AA', BC intersect in D; BB', CA in E; CC', AB in F; shew that the intersections of DE and AB, EF and BC, FD and CA will be collinear.

(80) The three points determined by

$$\frac{u}{p-q}=\frac{v}{p'-q'}=\frac{w}{p''-q''},$$

$$\frac{u}{q-r}=\frac{v}{q'-r'}=\frac{w}{q''-r''},$$

$$\frac{u}{r-p}=\frac{v}{r'-p'}=\frac{w}{r''-p''},$$

are collinear.

(81) If $u=0$, $v=0$, $w=0$, $x=0$ denote the equations to four straight lines, and if the sum of the expressions u, v, w, x be identically zero, the three diagonals of the quadrilateral formed by the four straight lines will be represented by the equations

$$u+v=0, \text{ or } w+x=0,$$

$$u+w=0, \text{ or } x+v=0,$$

$$u+x=0, \text{ or } v+w=0.$$

(82) In a given triangle let *three* triangles be inscribed, by joining the points of contact of the inscribed circle, the points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the *corresponding* sides of these three triangles pass through the same point; also the triangle formed by the three points of intersection will be a *circumscribed co-polar* to the original triangle, and the *pole* will be on the *straight line* in which the sides of the given triangle meet the bisectors of its exterior angles.

CHAPTER IX.

IMAGINARY POINTS AND STRAIGHT LINES.

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99. LET $f + f'\sqrt{-1}$, $g + g'\sqrt{-1}$, $h + h'\sqrt{-1}$ be irrational values of α, β, γ which satisfy the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

If instead of being irrational the values had been rational they would have been (Art. 7) the trilinear coordinates of some real point. But being irrational *they are said to be the coordinates of an imaginary point.*

This must be taken as the definition of the term "*an imaginary point.*"

Such a point has no geometrical existence, it exists only in respect to its coordinates. In other words, when we speak of an imaginary point we are using a phrase which has no strict geometrical application, but is convenient as giving expression to an analytical result, and is very useful in enabling us often to use much more general language than we could without such a convention.

For instance, suppose we are finding the coordinates of a point of intersection of the loci of two equations (two curves). And suppose we arrive at the result

$$\alpha = a + \sqrt{b^2 - c^2}.$$

If we use only geometrical language we shall have to observe that this value of α is only real when $b > c$. Hence in stating our result we must say that there will be a point of intersection only when $b > c$. In the language of analysis which we have just introduced we shall be able to state the result more generally: we shall be able to speak of the point of intersection as always existing, but we shall observe that it is real or imaginary according as $b > c$ or $b < c$.

100. So, again, if coordinates are to be determined by the solution of a quadratic equation, we shall say that there will *always* be two solutions, but they will sometimes denote real and sometimes imaginary points.

We shall shew hereafter (Chap. XII.) that every conic section has an equation in trilinear coordinates which may be written

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \quad\text{.........(1)}.$$

Now suppose

$$l\alpha + m\beta + n\gamma = 0 \quad\text{........................(2)}$$

is the equation to a straight line, and that it is required to find the coordinates of the points of intersection of the straight line and the conic, i.e. to find coordinates which will satisfy both the equations, we shall have to solve the equations (1) and (2) simultaneously with the relation

$$a\alpha + b\beta + c\gamma = 2\Delta \quad\text{........................(3)}.$$

By means of the simple equations (2) and (3) we can express β and γ as functions of α of the first order.

Substituting these values in the quadratic (1) we shall get a quadratic equation to determine α.

This quadratic will have two roots (real or imaginary). Hence there will be two points of intersection (real or imaginary).

Hence every straight line meets every conic section in two and only two points (real or imaginary).

101. *If* $f+f'\sqrt{-1}$, $g+g'\sqrt{-1}$, $h+h'\sqrt{-1}$ *be the coordinates of an imaginary point; then* f, g, h *will be the coordinates of a real point, and* f', g', h' *will be proportional to the coordinates of a point at infinity.*

For if $f+f'\sqrt{-1}$, $g+g'\sqrt{-1}$, $h+h'\sqrt{-1}$ be the coordinates of a point they must satisfy the relation

$$a\alpha+b\beta+c\gamma=2\Delta.$$

Hence

$$(af+bg+ch)+(af'+bg'+ch')\sqrt{-1}=2\Delta,$$

and therefore equating real and imaginary parts

$$af+bg+ch=2\Delta \quad\ldots\ldots\ldots\ldots(1)$$

and $$af'+bg'+ch'=0 \quad\ldots\ldots\ldots\ldots(2).$$

But the equation (1) shews that (f, g, h) are the coordinates of a point, and (2) shews that f', g', h' satisfy the equation

$$a\alpha+b\beta+c\gamma=0$$

to the line at infinity, and therefore represent a point at infinity. Q. E. D.

Cor. The coordinates of an imaginary point at infinity will be

$$f+f'\sqrt{-1},\quad g+g'\sqrt{-1},\quad h+h'\sqrt{-1},$$

where $$af+bg+ch=0,$$

as well as $$af'+bg'+ch'=0.$$

102. Def. An equation of the form

$$(l+l'\sqrt{-1})\,\alpha+(m+m'\sqrt{-1})\,\beta+(n+n'\sqrt{-1})\,\gamma=0$$

is said to be the equation of an imaginary straight line.

We here suppose that

$$l + l'\sqrt{-1},\ m + m'\sqrt{-1},\ n + n'\sqrt{-1}$$

have no common factor, observing that such an equation as

$$(p + p'\sqrt{-1})(\lambda\alpha + \mu\beta + \nu\gamma) = 0$$

is not imaginary; it represents a real straight line.

103. *Every imaginary straight line passes through one and only one real point.*

Let the real and imaginary parts of the equation to the straight line be collected so that the equation takes the form

$$u + v\sqrt{-1} = 0 \qquad\qquad (1),$$

where u and v are rational functions of the coordinates of the first degree, and therefore

$$u = 0 \qquad\qquad (2),$$

$$v = 0 \qquad\qquad (3),$$

are the equations to two straight lines (which cannot be coincident, else the imaginary factor would divide out of the original equation and the equation would become real, which is contrary to hypothesis).

The equation (1) can only be satisfied by real values of the coordinates when u and v are each zero. (Todhunter's *Algebra*, Chap. XXV.) Hence any real point whose coordinates satisfy (1) must also satisfy (2) and (3), and must therefore lie upon both the straight lines (2) and (3).

Therefore the point of intersection of these two straight lines is the only real point which satisfies the given equation.

Hence *every imaginary straight line passes through one and only one real point.*

104. *Every imaginary point lies upon one and only one real straight line.*

For let

$$f+f'\sqrt{-1},\ g+g'\sqrt{-1},\ h+h'\sqrt{-1}$$

be the coordinates of an imaginary point.

And if possible, let

$$l\alpha+m\beta+n\gamma=0$$

be the equation to a straight line passing through it.

Then the coordinates of the point must satisfy this equation, and therefore

$$lf+mg+nh+(lf'+mg'+nh')\sqrt{-1}=0,$$

and therefore we must have

$$lf+mg+nh=0,$$
$$lf'+mg'+nh'=0,$$

whence

$$\frac{l}{\begin{vmatrix} g, & h \\ g', & h' \end{vmatrix}}=\frac{m}{\begin{vmatrix} h, & f \\ h', & f' \end{vmatrix}}=\frac{n}{\begin{vmatrix} f, & g \\ f', & g' \end{vmatrix}},$$

equations which determine $l:m:n$, and shew that only one solution is possible.

Therefore every imaginary point lies on one and only one real straight line. Q. E. D.

COR. 1. *The real straight line passing through the imaginary point whose coordinates are*

$$f+f'\sqrt{-1},\ g+g'\sqrt{-1},\ h+h'\sqrt{-1}$$

is represented by the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ f, & g, & h \\ f', & g', & h' \end{vmatrix}=0.$$

COR. 2. *The imaginary point whose coordinates are*

$$f+f'\sqrt{-1},\ g+g'\sqrt{-1},\ h+h'\sqrt{-1},$$

lies on the straight line joining the point (f, g, h) *with the point at infinity* $(f' : g' : h')$.

COR. 3. *The same real line passes through the two imaginary points*

$$f+f'\sqrt{-1},\ g+g'\sqrt{-1},\ h+h'\sqrt{-1},$$

and

$$f-f'\sqrt{-1},\ g-g'\sqrt{-1},\ h-h'\sqrt{-1}.$$

COR. 4. *If two curves intersect in a series of imaginary points, they will lie two and two upon real straight lines.*

This follows immediately from Cor. 3, when we remember that imaginary roots can only enter into an equation by pairs, the two members of every pair differing only in the sign of the imaginary part.

105. We have said (Cor. to Art. 101) that the coordinates of an imaginary point at infinity will be

$$f+f'\sqrt{-1},\ g+g'\sqrt{-1},\ h+h'\sqrt{-1},$$

where f, g, h, f', g', h' satisfy the relations

$$af+bg+ch=0,$$

$$af'+bg'+ch'=0.$$

This statement requires a little consideration.

Let us return for a moment to *real* points, and suppose λ, μ, ν are numbers satisfying the relation

$$a\lambda+b\mu+c\nu=0 \text{ ....(1)},$$

then we are accustomed to say that the equations

$$\frac{\alpha}{\lambda}=\frac{\beta}{\mu}=\frac{\gamma}{\nu} \text{ ....(2)}$$

represent a point lying at infinity, for if we suppose α', β', γ' to be the coordinates of the point determined by (2), then, since in virtue of (2), α', β', γ' are proportional to λ, μ, ν, it follows from (1) that

$$a\alpha' + b\beta' + c\gamma' = 0,$$

which shews that α', β', γ' satisfy the equation to the straight line at infinity.

But if we consider what are the actual values of these coordinates α', β', γ', we perceive (Art. 36, page 38) that two at least and generally all three of them are infinite. But no difficulty practically ensues from this, because we never want the actual coordinates of such a point, it being sufficient to know that the finite quantities λ, μ, ν are proportional to them: and it is very convenient to speak of the point at infinity whose coordinates are thus proportional to λ, μ, ν, as the point (λ, μ, ν), since the quantities λ, μ, ν, or any quantities proportional to them, satisfy all practical conditions of the coordinates of the point. For, as we have already seen, so long as we have to do with homogeneous equations we never require the actual values of the coordinates of any point, but only the ratios of those actual values, except in theorems connected with the distance of the point from other points; consequently we shall not expect ever to require the actual coordinates of a point at infinity, since its distances from all finite points are infinite and cannot therefore generally be introduced into problems.

(It should be noticed that if only two of the actual coordinates of a point be infinite and the third be finite, then two only of the quantities λ, μ, ν will have a finite magnitude, and the third will be zero).

But, to return to the imaginary points, it follows from what we have said, that whether

$$f + f'\sqrt{-1}, \;\; g + g'\sqrt{-1}, \;\; h + h'\sqrt{-1}$$

be the actual coordinates of a point at infinity or only proportional to them, we must still have

$$af + bg + ch = 0,$$
$$af' + bg' + ch' = 0.$$

And we shall find it very convenient to speak of such a point as *the point*

$$(f + f'\sqrt{-1},\ g + g'\sqrt{-1},\ h + h'\sqrt{-1}),$$

whether the quantities

$$f + f'\sqrt{-1},\ g + g'\sqrt{-1},\ h + h'\sqrt{-1}$$

be the coordinates, or be only proportional to the coordinates of the point. And indeed, since it is more convenient to deal with finite than with infinite quantities, we shall always suppose that the expressions

$$f + f'\sqrt{-1},\ g + g'\sqrt{-1},\ h + h'\sqrt{-1}$$

do represent quantities *only proportional* to the actual coordinates.

In other words, if we speak of the point (u, v, w) as a point at infinity, we mean the point at infinity determined by the equations

$$\frac{\alpha}{u} = \frac{\beta}{v} = \frac{\gamma}{w}.$$

106. Two imaginary straight lines are said to be parallel when they intersect in a point on the straight line at infinity. Hence the condition investigated in Art. 35 may be applied to imaginary straight lines.

107. Any equation of the form

$$l\alpha^2 + m\alpha\beta + n\beta^2 = 0 \text{.....................(i)},$$

represents two straight lines intersecting in the point C of the triangle of reference. For if μ_1, μ_2 be the roots of the quadratic

$$l\mu^2 + m\mu + n = 0,$$

the equation (i) can be written

$$(\alpha - \mu_1\beta)(\alpha - \mu_2\beta) = 0,$$

which shews that it represents the two straight lines whose separate equations are

$$\alpha - \mu_1\beta = 0,$$

$$\alpha - \mu_2\beta = 0.$$

The two lines will be real if μ_1 and μ_2 are real, and they will be imaginary if μ_1 and μ_2 are imaginary. Or, assuming the condition investigated in Algebra, the straight lines will be imaginary or real according as $4ln$ is or is not greater than m^2.

For example, consider the equation

$$\alpha^2 + \beta^2 + 2\alpha\beta \cos C = 0.$$

Since $1 > \cos^2 C$, it follows that this will represent two *imaginary* straight lines.

The following proposition is interesting.

108. *The two straight lines represented by the equation*

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

are parallel, each to each, to the two straight lines represented by

$$\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and also parallel, each to each, to the two straight lines represented by

$$\alpha^2 + \beta^2 + 2\alpha\beta \cos C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

For the straight lines (1) meet the line at infinity in the two points determined by

$$\left.\begin{array}{r} \beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0 \\ a\alpha + b\beta + c\gamma = 0 \end{array}\right\},$$

the latter of which gives us by the transposition,

$$a^2\alpha^2 = b^2\beta^2 + 2bc\beta\gamma + c^2\gamma^2,$$

or
$$2\beta\gamma = \frac{-b^2\beta^2 - c^2\gamma^2 + a^2\alpha^2}{bc}.$$

Substituting this value of $\beta\gamma$ in the first equation, we get

$$bc\,(\beta^2 + \gamma^2) + (b^2\beta^2 + c^2\gamma^2 - a^2\alpha^2)\cos A = 0,$$

or
$$\alpha^2 a \cos A + \beta^2 b \cos B + \gamma^2 c \cos C = 0.$$

Hence the two straight lines (1) meet the line at infinity in the two points given by the symmetrical equations

$$\left.\begin{array}{r} \alpha^2 a \cos A + \beta^2 b \cos B + \gamma^2 c \cos C = 0 \\ a\alpha + b\beta + c\gamma = 0 \end{array}\right\}.$$

By *symmetry* the straight lines (2) will meet the line at infinity in the two points given by the same two equations.

Hence the straight lines (1) and the straight lines (2) pass through the same two points at infinity, and therefore are parallel.

And similarly, each pair is parallel to the two straight lines (3). Q. E. D.

109. Def. We observe that the six straight lines represented by the equations (1), (2), (3), pass three and three through *two* imaginary points on the straight line at infinity. These two imaginary points will be found hereafter to have some very important and curious properties. We have indeed to refer to them so often that it is convenient to have a special name by which to distinguish them: and on account of properties which we shall shew hereafter to belong to them it is deemed appropriate to term them *the two "circular" points at infinity.*

By this name we shall continually refer to them.

110. *To find the ratios of the coordinates of the circular points at infinity.*

The circular points are given by the equations

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$a\alpha + b\beta + c\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equation (1) determines the ratio of β to γ, thus

$$\beta^2 + 2\beta\gamma \cos A + \gamma^2 \cos^2 A = -\gamma^2 \sin^2 A,$$

$$\beta + \gamma \cos A = \mp \gamma \sqrt{-1} \sin A,$$

$$\beta = -\gamma \{\cos A \pm \sqrt{-1} \sin A\},$$

or in virtue of the equation (2),

$$\frac{\beta}{\cos A \pm \sqrt{-1} \sin A} = \frac{\gamma}{-1} = \frac{\alpha}{\cos B \mp \sqrt{-1} \sin B},$$

the upper signs going together and the lower together.

Hence the ratios are known.

111. From considerations of symmetry we at once perceive that the result of the last article, giving the ratios of the coordinates of the two circular points as infinity may be expressed in any one of the following forms,

$$\frac{\alpha}{-1} = \frac{\beta}{\cos C \mp \sqrt{-1} \sin C} = \frac{\gamma}{\cos B \pm \sqrt{-1} \sin B},$$

$$\frac{\alpha}{\cos C \pm \sqrt{-1} \sin C} = \frac{\beta}{-1} = \frac{\gamma}{\cos A \mp \sqrt{-1} \sin A},$$

$$\frac{\alpha}{\cos B \mp \sqrt{-1} \sin B} = \frac{\beta}{\cos A \pm \sqrt{-1} \sin A} = \frac{\gamma}{-1}.$$

By simple addition we can express these ratios in a form symmetrical with respect to the three axes: but such form is more complicated, and one of the forms already written down will generally be more useful.

By multiplication we obtain

$$\frac{\alpha^3}{\cos(B-C) \pm \sqrt{-1}\sin(B-C)} = \frac{\beta^3}{\cos(C-A) \pm \sqrt{-1}\sin(C-A)}$$

$$= \frac{\gamma^3}{\cos(A-B) \pm \sqrt{-1}\sin(A-B)},$$

a result symmetrical as far as it goes, but when we come to extract the cube roots by Demoivre's Theorem we lose the symmetry, as it is found upon examination that we cannot take *similar* cube roots and write

$$\frac{\alpha}{\cos\frac{B-C}{3} \pm \sqrt{-1}\sin\frac{B-C}{3}} = \frac{\beta}{\cos\frac{C-A}{3} \pm \sqrt{-1}\sin\frac{C-A}{3}}$$

$$= \frac{\gamma}{\cos\frac{A-B}{3} \pm \sqrt{-1}\sin\frac{A-B}{3}},$$

but we must take dissimilar roots as

$$\frac{\alpha}{\cos\frac{B-C}{3} \pm \sqrt{-1}\sin\frac{B-C}{3}}$$

$$= \frac{\beta}{\cos\frac{2\pi + C - A}{3} \pm \sqrt{-1}\sin\frac{2\pi + C - A}{3}}$$

$$= \frac{\gamma}{\cos\frac{4\pi + A - B}{3} \pm \sqrt{-1}\sin\frac{4\pi + A - B}{3}},$$

a much less convenient form than those already obtained.

112. Throughout the present chapter we have spoken of trilinear coordinates, and proved the properties of imaginary points and straight lines by the aid of them.

But all that we have said applies *mutatis mutandis* to triangular coordinates: and in order to adapt our arguments to this system it is in most cases only necessary to read unity for a, b, and c severally, as well as for 2Δ.

Thus our results will take the following form.

In Triangular Coordinates.

(i) If
$$f + g + h = 1,$$
and
$$f' + g' + h' = 0,$$
then
$$f + f'\sqrt{-1}, \quad g + g'\sqrt{-1}, \quad h + h'\sqrt{-1},$$
are said to be the coordinates of an imaginary point lying within a finite distance of the lines of reference.

(ii) If
$$f + g + h = 0,$$
and
$$f' + g' + h' = 0,$$
then
$$f + f'\sqrt{-1}, \quad g + g'\sqrt{-1}, \quad h + h'\sqrt{-1},$$
are said to be the coordinates of an imaginary point lying at an infinite distance from the triangle of reference.

(iii) The results of Arts. 101—104, 106, 107 will remain unchanged.

(iv) The two circular points at infinity are represented by the equations

$$\frac{x}{-a} = \frac{y}{b \cos C \pm \sqrt{-1}\, b \sin C} = \frac{z}{c \cos B \mp \sqrt{-1}\, c \sin B},$$

or
$$\frac{x}{a \cos C \mp \sqrt{-1}\, a \sin C} = \frac{y}{-b} = \frac{z}{c \cos A \pm \sqrt{-1}\, c \sin A},$$

or
$$\frac{x}{a \cos B \pm \sqrt{-1}\, a \sin B} = \frac{y}{b \cos A \mp \sqrt{-1}\, b \sin A} = \frac{z}{-c}.$$

Exercises on Chapter IX.

(83) If the imaginary straight lines

$$u + v\sqrt{-1} = 0 \text{ and } u' + v'\sqrt{-1} = 0$$

have a real point of intersection, then the four real straight lines

$$u = 0,\ v = 0,\ u' = 0,\ v' = 0$$

are concurrent.

(84) The straight line joining the real point $(\alpha', \beta', \gamma')$ with the imaginary point

$$(f + f'\sqrt{-1}),\ g + g'\sqrt{-1},\ h + h'\sqrt{-1}$$

is represented by the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ f, & g, & h \end{vmatrix} + \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ f', & g', & h' \end{vmatrix} \sqrt{-1} = 0,$$

and will be real, provided

$$\begin{vmatrix} \alpha', & \beta', & \gamma' \\ f, & g, & h \\ f', & g', & h' \end{vmatrix} = 0.$$

(85) If the value of the multiplier k vary, the locus of the imaginary point

$$(f + kf'\sqrt{-1},\ g + kg'\sqrt{-1},\ h + kh'\sqrt{-1})$$

is a real straight line.

(86) Shew that the equation in trilinear coordinates

$$\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma\cos(B - C) + 2\gamma\alpha\cos(C - A) + 2\alpha\beta\cos(A - B) = 0,$$

represents two imaginary straight lines parallel to each other and to the real straight line

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

(87) Given $u=0$, $v=0$, $w=0$ the equations to three real straight lines in any system of coordinates, and θ, ϕ, ψ three angles together equal to three right angles, shew that the equation

$$u^2 + v^2 + w^2 + 2vw \sin\theta + 2wu \sin\phi + 2uv \sin\psi = 0$$

represents two imaginary straight lines, and find their separate equations.

(88) Shew that the equation in triangular coordinates

$$(x^2+y^2+z^2)^2 (x+y+z)^2 + 4xyz (x^2+y^2+z^2)(x+y+z) + 8x^2y^2z^2 = 0,$$

represents six imaginary straight lines parallel two and two to the three lines of reference.

CHAPTER X.

ANHARMONIC AND HARMONIC SECTION.

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113. Definitions.

(1) Let a straight line AB be divided at P into two parts in the ratio $m : 1$, and be divided at Q in the ratio $n : 1$,

then the ratio $m : n$ is called the anharmonic ratio of the section of AB in P and Q.

(2) Let an angle AOB be divided by OP into two parts whose sines are in the ratio $m : 1$, and be divided by OQ into two parts whose sines are in the ratio $n : 1$;

then the ratio $m : n$ is called the anharmonic ratio of the section of the angle AOB by OP and OQ.

114. It will be observed that if *both* the sections be external or *both* internal m and n will be of the same sign, and therefore the anharmonic ratio of the section will be *positive*. If one section be external and the other internal, m and n will be of opposite signs and the anharmonic ratio of the section will be *negative*.

115. For the sake of brevity the anharmonic ratio of the section of AB in P and Q is often spoken of as the anharmonic ratio of the range of points $APBQ$, and it is expressed by the symbol $\{APBQ\}$.

So the anharmonic ratio of the section of the angle AOB by OP and OQ is spoken of as the anharmonic ratio of the pencil of straight lines OA, OP, OB, OQ, and it is expressed by the symbol $\{O.APBQ\}$.

116. When we speak of the range of points $APBQ$ it must not be inferred that the points necessarily occur in the order in which we read them: it must be understood that AB (terminated by the points mentioned *first* and *third*) is the line which we suppose divided, and P and Q (mentioned *second* and *fourth*) are the points of section. The sections may either of them be internal or external, but we read the letters in the order in which they would come if the first section were internal and the second external. It is found most convenient to adopt this system because in a particular case of most frequent occurrence (the case of harmonic section described below) one section is always internal and the other external.

117. In expressing the ratios of lines it must be understood that AB and BA denote lengths equal in magnitude but opposite in sign.

Thus, $$\frac{AP}{BP} = -\frac{AP}{PB} = \frac{PA}{PB} = -\frac{PA}{BP}.$$

118. The anharmonic ratio of the range $APBQ$

$$= \{APBQ\} = \frac{AP}{BP} : \frac{AQ}{BQ} = \frac{AP \,.\, BQ}{AQ \,.\, BP},$$

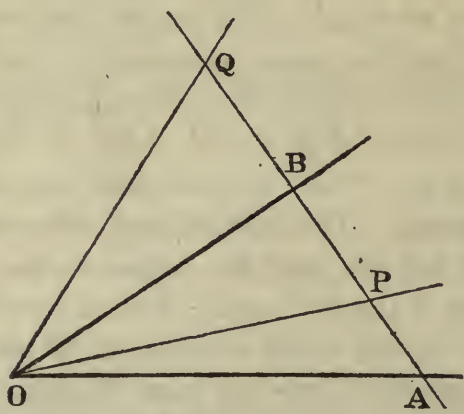
and the anharmonic ratio of the pencil OA, OP, OB, OQ,

$$= \{O.APBQ\} = \frac{\sin AOP}{\sin BOP} : \frac{\sin AOQ}{\sin BOQ}$$

$$= \frac{\sin AOP \,.\, \sin BOQ}{\sin AOQ \,.\, \sin BOP}.$$

119. *If the pencil OA, OP, OB, OQ cut any transverse straight line in the range APBQ, the anharmonic ratio of the pencil is the same as that of the range.*

Fig. 25.



For $\dfrac{\sin QOP}{\sin PAO} = \dfrac{AP}{PO}$ and $\dfrac{\sin AOQ}{\sin PAO} = \dfrac{AQ}{QO}$;

therefore $$\frac{\sin QOP}{\sin QOQ} = \frac{AP}{AQ} \cdot \frac{QO}{PO}.$$

Similarly, $$\frac{\sin BOP}{\sin BOQ} = \frac{BP}{BQ} \cdot \frac{QO}{PO};$$

therefore by division,

$$\frac{\sin QOP . \sin BOQ}{\sin QOQ . \sin BOP} = \frac{AP . BQ}{AQ . BP};$$

or $\{O . APBQ = \{APBQ\}$. Q. E. D.

120. *To shew that*

$$\{APBQ\} = \frac{1}{\{AQBP\}} = \frac{1}{\{BPAQ\}} = \{BQAP\},$$

we have only to write the values of the several anharmonic ratios as in Art. 118.

Thus $$\{APBQ\} = \frac{AP \cdot BQ}{AQ \cdot BP} = \{BQAP\};$$

and $$\{AQBP\} = \frac{AQ \cdot BP}{AP \cdot BQ} = \{BPAQ\};$$

which prove the proposition.

121. *Similarly we may prove that*

$$\{APBQ\} = \{PAQB\} = \{QBPA\} = \{BQAP\}.$$

122. *If the angle C of the triangle of reference be divided by two straight lines CP, CQ whose equations are respectively*

$$\beta = m\alpha \text{ and } \beta = n\alpha,$$

then the anharmonic ratio of the pencil CA, CP, CB, CQ is $m : n$.

For by Art. 4,

$$\frac{\sin ACP}{\sin BCP} = m \text{ and } \frac{\sin ACQ}{\sin BCQ} = n,$$

therefore by division

$$\{APBQ\} = m : n. \quad \text{Q.E.D.}$$

123. It follows that the anharmonic ratio of the section of the same angle by the two straight lines

$$l\alpha^2 + 2m\alpha\beta + n\beta^2 = 0,$$

is $$\frac{(m \pm \sqrt{m^2 - ln})^2}{ln},$$

which is real when $m^2 > ln$, i.e. when the straight lines themselves are real.

If $m^2 < ln$ the straight lines are imaginary, and unless $m = 0$, the anharmonic ratio is also imaginary.

But if $m = 0$ and l and n have the same sign the two straight lines become imaginary, but the anharmonic ratio of their section of the angle C is real and equal to negative unity.

COR. The two straight lines, whether real or imaginary, which are represented by the equation

$$\alpha^2 + \kappa\beta^2 = 0,$$

divide the angle of the triangle of reference so that the anharmonic ratio of the section is negative unity.

124. *If the angle between the straight lines* $u = 0$ *and* $v = 0$ *be divided by the straight lines* $u + \kappa v = 0$ *and* $u + \kappa' v = 0$, *the anharmonic ratio of the section is* $\kappa : \kappa'$.

Let OA, OB be the two straight lines represented by $u = 0$ and $v = 0$, and OP, OQ the two straight lines represented by $u + \kappa v = 0$ and $u + \kappa' v = 0$.

Through any point S whose coordinates are $(\alpha', \beta', \gamma')$ draw a transversal $SAPBQ$ cutting OA, OB in A and B, and OP, OQ in P and Q.

And let u', v' be what u, v become when α', β', γ' are written for α, β, γ.

Also let λ, μ, ν be the direction-cosines of the straight line SAB, and let m, n be what u, v become when λ, μ, ν are written for α, β, γ.

Then we have (as in Art. 80)

$$SA = -\frac{u'}{m}, \qquad SP = -\frac{u' + \kappa v'}{m + \kappa n},$$

$$SB = -\frac{v'}{m}, \qquad SQ = -\frac{u' + \kappa' v'}{m + \kappa' n};$$

therefore

$$\{APBQ\} = \frac{AP \, . \, BQ}{AQ \, . \, BP} = \frac{\left(\dfrac{u'}{m} - \dfrac{u' + \kappa v'}{m + \kappa n}\right) . \left(\dfrac{v'}{n} - \dfrac{u' + \kappa' v'}{m + \kappa' n}\right)}{\left(\dfrac{u'}{m} - \dfrac{u' + \kappa' v'}{m + \kappa' n}\right) . \left(\dfrac{v'}{n} - \dfrac{u' + \kappa v'}{m + \kappa n}\right)}$$

$$= \frac{\kappa\,(nu' - mv')\,(mv' - nu')}{\kappa'\,(nu' - mv')\,(mv' - nu')},$$

or $$\{APBQ\} = \frac{\kappa}{\kappa'}. \quad \text{Q.E.D.}$$

125. *To find the anharmonic ratio of the pencil formed by the four straight lines,*

$$u + \kappa v = 0, \quad u + \lambda v = 0, \quad u + \mu v = 0, \quad u + \nu v = 0.$$

Put $$\phi \equiv u + \kappa v \text{ and } \psi \equiv u + \mu v,$$

then $$u \equiv \frac{\kappa\psi - \mu\phi}{\kappa - \mu} \text{ and } v \equiv \frac{\phi - \psi}{\kappa - \mu}.$$

So the four given equations become

$$\left.\begin{aligned} &\phi = 0, \\ &\kappa\psi - \mu\phi + \lambda(\phi - \psi) = 0, \\ &\psi = 0, \\ &\kappa\psi - \mu\phi + \nu(\phi - \psi) = 0, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} &\phi = 0, \\ &\phi + \frac{\kappa - \lambda}{\lambda - \mu}\psi = 0, \\ &\psi = 0, \\ &\phi + \frac{\kappa - \nu}{\nu - \mu}\psi = 0. \end{aligned}\right.$$

Hence by the last article the anharmonic ratio is

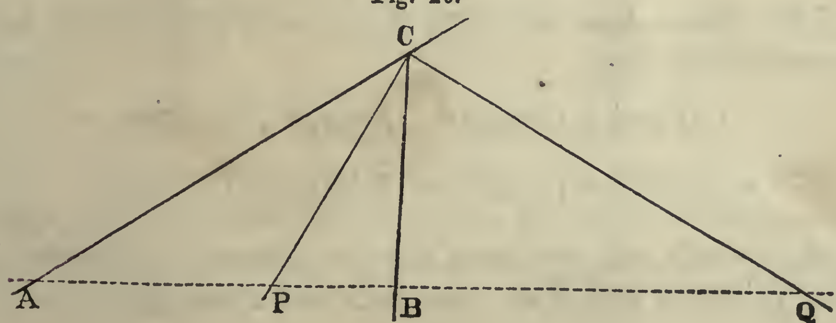
$$\frac{\kappa - \lambda}{\lambda - \mu} \div \frac{\kappa - \nu}{\nu - \mu},$$

or $$\frac{(\kappa - \lambda)(\mu - \nu)}{(\kappa - \nu)(\mu - \lambda)}.$$

126. Def. If the anharmonic ratio of any pencil or range be equal to -1 the pencil or range is called harmonic.

An example of this is obtained if the angle C of the triangle ACB be bisected internally and externally by straight lines CP, CQ meeting the base AB in P and Q.

Fig. 26.



For by Euclid VI. 3,

$$AP : PB = AC : BC,$$

$$AQ : BQ = AC : BC:$$

whence
$$\frac{AP}{PB} = \frac{AQ}{BQ},$$

or
$$\frac{AP \,.\, BQ}{PB \,.\, AQ} = 1,$$

and therefore
$$\frac{AP \,.\, BQ}{BP \,.\, AQ} = -1;$$

i.e. the range $APBQ$ is harmonic, and therefore by Art. 119 the pencil CA, CP, CB, CQ is harmonic.

127. By reference to Art. 114, it will be seen that if a line or angle be divided harmonically one of the sections must be internal and the other external.

Thus if $APBQ$ be a harmonic range one of the points P, Q will lie between A and B and the other beyond them: the four points will in fact occur either in the order in which they are read in the numerator of the fraction

$$\frac{AP \,.\, BQ}{AQ \,.\, BP},$$

(which expresses the ratio) or else in the order in which they are read in the denominator.

128. From Arts. 120, 121, it appears that if $APBQ$ be a harmonic range then

$$\{APBQ\} = \{PBQA\} = \{BQAP\} = \{QAPB\}$$
$$= \{QBPA\} = \{AQBP\} = \{PAQB\} = \{BPAQ\}:$$

in other words, we may read the four letters in any order in which neither A and B are contiguous, nor P and Q.

129. It follows immediately from Art. 122 that the straight lines whose equations are

$$\beta - m\alpha = 0, \text{ and } \beta + m\alpha = 0,$$

divide the angle C *harmonically.*

Or more generally, from Art. 124, the four straight lines

$$u = 0, \qquad u - \kappa v = 0,$$

$$v = 0, \qquad u + \kappa v = 0,$$

form a harmonic pencil.

130. *To find the equation to a straight line which shall form with the three straight lines*

$$u + \kappa v = 0, \qquad u + \lambda v = 0, \qquad u + \mu v = 0,$$

a harmonic pencil.

Let $u + \nu v = 0$, be the equation required.

Then by Art. 125,

$$\frac{(\kappa - \lambda)(\mu - \nu)}{(\kappa - \nu)(\mu - \lambda)} = -1,$$

or $$(\kappa - \lambda)(\mu - \nu) + (\kappa - \nu)(\mu - \lambda) = 0,$$

or $$(\kappa - 2\lambda + \mu)\nu + \lambda\mu - 2\mu\kappa + \kappa\lambda = 0,$$

or $$\nu = -\frac{\lambda\mu - 2\mu\kappa + \kappa\lambda}{\kappa - 2\lambda + \mu}.$$

Hence the straight line required will be represented by the equation

$$u(\kappa - 2\lambda + \mu) - v(\lambda\mu - 2\mu\kappa + \kappa\lambda) = 0.$$

131. *To establish relations among the different anharmonic ratios obtained by taking a range of four points, or a pencil of four straight lines in various orders.*

Four letters K, L, M, N can be written in 24 different orders. We have seen however, in Art. 121, that there are four different orders in which any range of points can be taken without affecting their anharmonic ratio. Hence we cannot obtain more than *six* different anharmonic ratios by taking the points in different orders. We observe also from Art. 120, that the reciprocal of any anharmonic ratio can be obtained by taking the points in a different order. Hence we cannot expect more than *three* different anharmonic ratios and their reciprocals.

We may shew this more formally as follows:

Let

$$u+\kappa v=0, \quad u+\lambda v=0, \quad u+\mu v=0, \quad u+\nu v=0,$$

be the equations to four straight lines OK, OL, OM, ON intersecting in the point O which is given by $u=v=0$.

And let

$$l \equiv \kappa\lambda+\mu\nu,$$

$$m \equiv \kappa\mu+\nu\lambda,$$

$$n \equiv \kappa\nu+\lambda\mu.$$

Then by Art. 125,

$$\{KLMN\}=\frac{(\kappa-\lambda)(\mu-\nu)}{(\kappa-\nu)(\mu-\lambda)}=\frac{m-n}{m-l},$$

or $$\{KLMN\}=-\frac{m-n}{l-m} \quad \text{.........................(1)},$$

so $$\{KMNL\}=-\frac{n-l}{m-n} \quad \text{.........................(2)},$$

and $$\{KNLM\}=-\frac{l-m}{n-l} \quad \text{.........................(3)},$$

and the ratios $\{KNML\}$, $\{KLNM\}$, $\{KMLN\}$ are (Art. 120) the reciprocals of these three, therefore

$$\{KNML\}=-\frac{l-m}{m-n} \quad \text{.........................(4)},$$

$$\{KLNM\} = -\frac{m-n}{n-l} \quad \text{.........................(5)},$$

$$\{KMLN\} = -\frac{n-l}{l-m} \quad \text{.........................(6)}.$$

It will be seen (by Art. 120 or independently) that if the letters K, L, M, N be taken in any other order besides these six, they will still give one of these same six ratios.

Hence we observe,

(i) *that by taking a range of four points in different orders we can only get six different anharmonic ratios.*

(ii) *that of these six ratios, three are the reciprocals of the other three.*

(iii) *that the ratio compounded of the first three is negative unity, and so is the ratio compounded of the other three.*

132. Cor. *If $\{KLMN\}$ be any harmonic ratio, any anharmonic ratio obtained by taking the points K, L, M, N in different order will be equal either to $+2$ or to $+\frac{1}{2}$.*

For since $\{KLMN\}$ is harmonic, therefore by equation (1) of the last article,

$$-\frac{m-n}{l-m} = -1,$$

or

$$m-n = l-m.$$

And since $m-n$ and $l-m$ are equal, each is equal to half their sum; that is,

$$\frac{m-n}{1} = \frac{l-m}{1} = -\frac{n-l}{2}.$$

Hence the equations (2), (3), (5), (6) of the last article give us

$$\{KMNL\} = \{KMLN\} = 2,$$

and

$$\{KNLM\} = \{KLNM\} = \frac{1}{2},$$

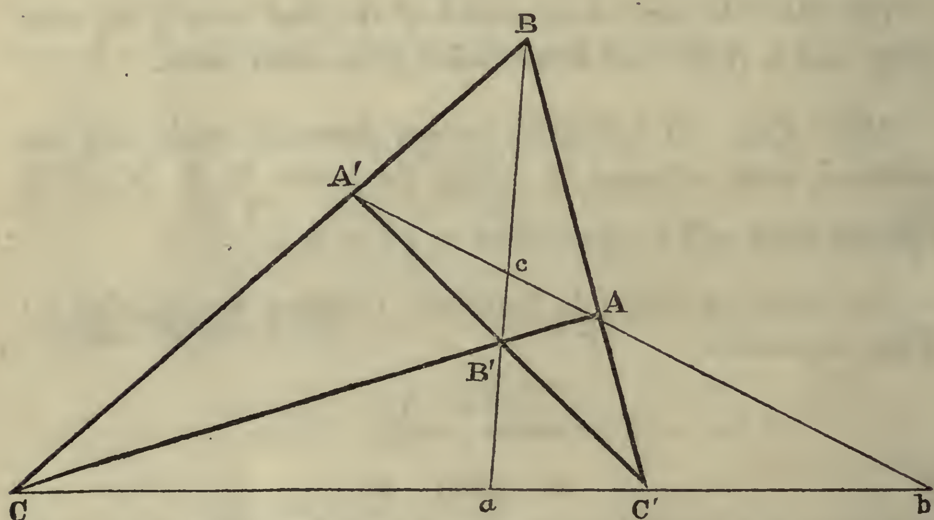
and the equation (4) shews that $\{KNML\}$ is harmonic.

Conversely. If the anharmonic ratio of a pencil or range be either 2 or $\frac{1}{2}$, we may obtain a harmonic pencil or range by taking the lines or points in a different order.

133. We proceed to establish some important harmonic properties of a quadrilateral.

Let the straight line $A'B'C'$ meet the three straight lines BC, CA, AB in the points A', B', C' respectively, so as to form a quadrilateral. Let the diagonals AA', BB', CC' be drawn and produced so as to form a triangle abc.

Fig. 27.



Let $u=0, \quad v=0, \quad w=0,$

be the equations to BC, CA, AB respectively, where u, v, w include such constant multipliers that the equation to $A'B'C'$ may be (Art. 89)

$$u+v+w=0.$$

Then, as we shewed in Art. 90, the equation

$v+w=0$ represents the line AA',

and $v-w=0$ .... Aa;

therefore by Art. 129, AA' and Aa divide harmonically the angle contained by the straight lines

$$v = 0 \ldots (AC),$$

$$w = 0 \ldots (AB).$$

Hence the pencil $\{A . BA'Ca\}$ is harmonic, and therefore (Art. 119) the range in which this pencil is cut by BB' will be harmonic, that is,

$$\{BcB'a\} = -1.$$

And similarly, $\{CaC'b\}$ and $\{AbA'c\}$ are harmonic ranges.

Again, since the range $\{BcB'a\}$ is harmonic;

therefore the pencil $\{A . BcB'a\}$ is harmonic,

and therefore the range in which this pencil is cut by the straight line $A'B'C'$ is harmonic, i.e. if Aa meet $A'B'C'$ in X, then

$$\{C'A'B'X\} \text{ is harmonic.}$$

These properties may be extended and multiplied almost without limit.

134. The following geometrical constructions are sometimes useful.

I. *Given three points in a straight line to find a fourth point completing the harmonic range.*

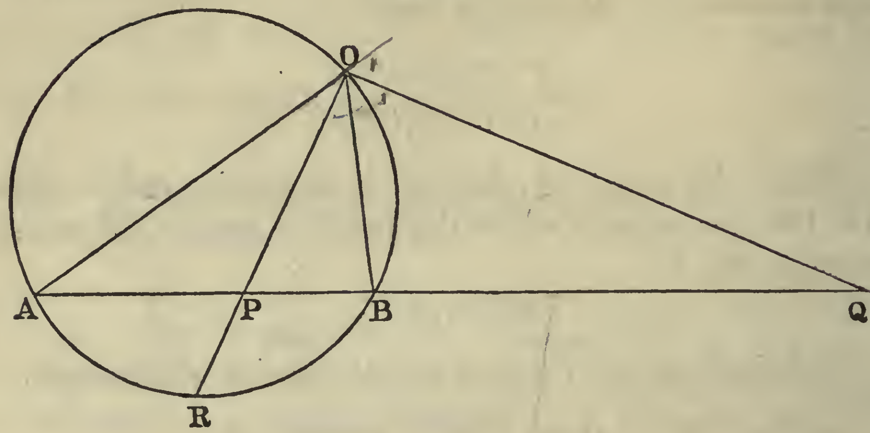
Let A, P, B be the given points, and through AB describe any circle, $ARBO$. Bisect the arc ARB in R, and join RP and produce it to meet the circumference again in O; and through O draw OQ at right angles to RO meeting AB in Q. Then Q shall be the point required.

Join AO, BO, then by Eucl. III. 27, the straight line OR bisects the interior angle AOB;

therefore OQ at right angles to it bisects the exterior angle;

therefore by Art. 126, $\{APBQ\}$ is harmonic, and Q is the point required.

Fig. 28.

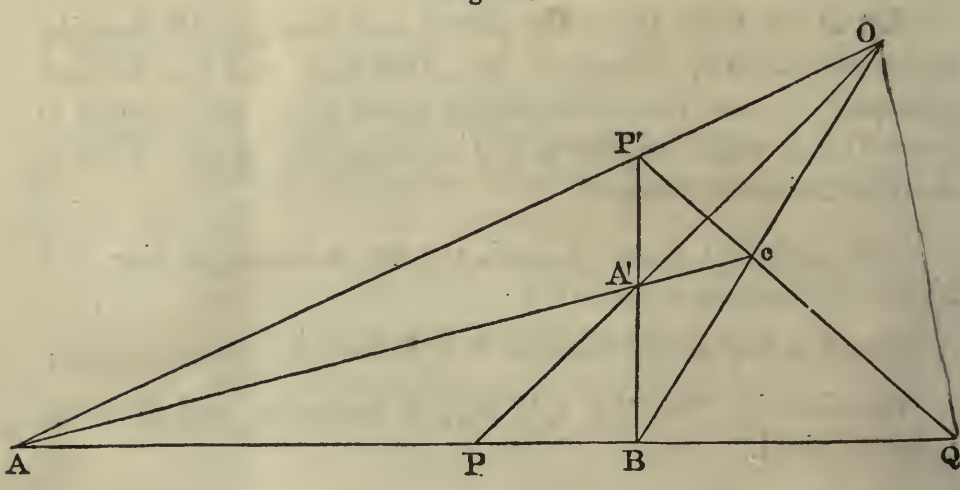


II. *Given three concurrent straight lines to find a fourth line completing the harmonic pencil.*

Let OA, OP, OB be the three given straight lines, and let them be cut by any transversal in the points A, P, B. Find a point Q completing this harmonic range and join OQ, then (Art. 119) the straight line OQ forms a harmonic pencil with OA, OP, OB, and is therefore the line required.

135. The constructions of the last article can be made by the ruler alone, without the introduction of the circle, by applying the properties proved in Art. 134. Thus:

Fig. 29.



In OA take any point P', and let OP, BP' intersect in A'; then let AA', OB intersect in c, and let cP', AB intersect in Q. Join OQ.

Then, applying to the quadrilateral $APA'P'$ the properties proved in Art. 133,

$$\{APBQ\} \text{ is a harmonic range,}$$

and therefore

$$\{O.APBQ\} \text{ is a harmonic pencil.}$$

Hence Q is the point, and OQ the line, required.

Exercises on Chapter X.

(89) If two straight lines OK and OK' intersect a system of parallel straight lines KK', LL', MM', NN' in K, L, M, N and K', L', M', N' respectively, then will

$$\{KLMN\} = \{K'L'M'N'\}.$$

(90) A point O is taken within a triangle ABC, and OA, OB, OC are drawn; and through A, B, C straight lines $B'C'$, $C'A'$, $A'B'$ are so drawn that each of the angles of the original triangle is cut harmonically. Shew that the points of intersection of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ are collinear.

(91) If through the vertex of a triangle two straight lines be drawn, one bisecting the base and the other parallel to it, they will divide the vertical angle harmonically.

(92) Any two straight lines at right angles to one another form a harmonic pencil with the straight lines joining their point of intersection with the circular points at infinity.

(93) If κ, λ, μ be in arithmetical progression the straight line $v = 0$ will form a harmonic pencil with the three straight lines

$$u + \kappa v = 0, \quad u + \lambda v = 0, \quad u + \mu v = 0.$$

(94) If κ, λ, μ be in harmonical progression the straight line $u = 0$ will form a harmonic pencil with the three straight lines

$$u + \kappa v = 0, \quad u + \lambda v = 0, \quad u + \mu v = 0.$$

(95) The four straight lines represented by the equations

$$u = 0, \quad v = 0,$$

$$lu^2 + 2muv + nv^2 = 0,$$

will form a harmonic pencil if $8m^2 = 9ln$.

(96) The angle between the two straight lines

$$lu^2 + 2muv + nv^2 = 0$$

is divided harmonically by the two straight lines

$$l'u^2 + 2m'uv + n'v^2 = 0,$$

provided ln', mm', nl' are in arithmetical progression.

(97) The anharmonic ratio of the pencil which the two straight lines

$$lu^2 + 2muv + nv^2 = 0,$$

form with the two straight lines

$$l'u^2 + 2m'uv + n'v^2 = 0,$$

is equal to

$$\frac{ln' - 2mm' + nl' \pm 2\sqrt{(m^2 - ln)(m'^2 - l'n')}}{ln' - 2mm' + nl' \mp 2\sqrt{(m^2 - ln)(m'^2 - l'n')}}.$$

CHAPTER XI.

TRANSFORMATION OF COORDINATES.

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136. SUPPOSE we have the equation to any locus referred to a triangle ABC, and suppose we wish to find the equation to the same locus referred to a new triangle $A'B'C'$.

The method of transformation will depend upon how the new triangle is given, and two cases immediately present themselves; first, the case in which the new triangle is given by the coordinates of its angular points being assigned, and secondly, the case in which the equations of its sides are given.

We proceed to discuss these two cases separately.

137. CASE I. *When the coordinates of the new points of reference are given.*

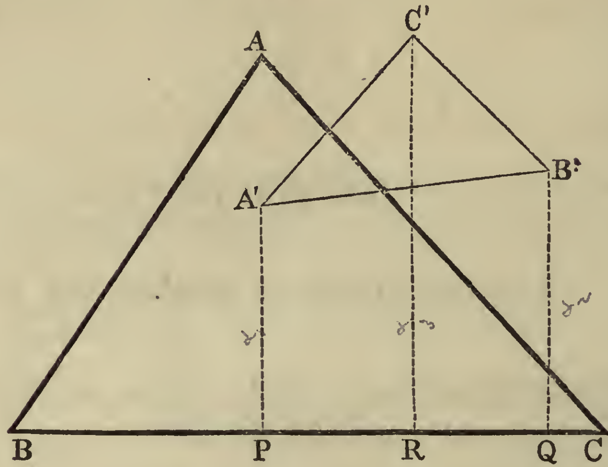
Let the coordinates of A', B', C' referred to the original triangle be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$; and let a', b', c' denote the sides of the new triangle $A'B'C'$, and Δ' its area.

Also let α', β', γ' denote the new coordinates of any point O whose old coordinates were α, β, γ.

Then α_1, α_2, α_3 are the distances of BC from the three points A', B', C' respectively; therefore the equation to the straight line BC referred to the new triangle $A'B'C'$ is (by Art. 23)

$$\alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{i}).$$

Fig. 30.



And therefore (Art. 58) the perpendicular distance of O from the line BC is

$$\frac{1}{2\Delta'}(\alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma').$$

But this perpendicular distance is the coordinate α; therefore

$$\left.\begin{aligned} \alpha &= \frac{1}{2\Delta'}(\alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma'), \\ \text{and so} \qquad \beta &= \frac{1}{2\Delta'}(\beta_1 a'\alpha' + \beta_2 b'\beta' + \beta_3 c'\gamma'), \\ \text{and} \qquad \gamma &= \frac{1}{2\Delta'}(\gamma_1 a'\alpha' + \gamma_2 b'\beta' + \gamma_3 c'\gamma'), \end{aligned}\right\} \dots\dots\dots\dots\text{(ii)}$$

equations which express the old coordinates α, β, γ of any point explicitly in terms of the new coordinates α', β', γ' of the same point.

If therefore an equation is given connecting the old coordinates α, β, γ of any point on some locus, by writing the three expressions given by (ii) instead of α, β, γ, we at once obtain a new equation connecting the new coordinates α', β', γ' of any point on the same locus: that is, we obtain the equation to the locus referred to the new triangle $A'B'C'$.

Thus if the equation to any locus referred to the triangle ABC be

$$f(\alpha, \beta, \gamma) = 0,$$

the equation to the same locus referred to the triangle $A'B'C'$ is

$$f\left(\frac{\alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma'}{2\Delta'}, \frac{\beta_1 a'\alpha' + \beta_2 b'\beta' + \beta_3 c'\gamma'}{2\Delta'}, \frac{\gamma_1 a'\alpha' + \gamma_2 b'\beta' + \gamma_3 c'\gamma'}{2\Delta'}\right) = 0.$$

But if, as is nearly always the case, the given equations be homogeneous, then $2\Delta'$ will divide out, and therefore

If any locus referred to the triangle ABC be represented by the homogeneous equation

$$f(\alpha, \beta, \gamma) = 0,$$

the same locus referred to the triangle $A'B'C'$ will have the equation

$$f(\alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma', \beta_1 a'\alpha' + \beta_2 b'\beta' + \beta_3 c'\gamma', \gamma_1 a'\alpha' + \gamma_2 b'\beta' + \gamma_3 c'\gamma') = 0.$$

138. Cor. It will be observed that the equation just obtained is necessarily of the same degree as the original equation. Hence *the degree of an equation is not altered by transformation of coordinates.*

This is a very important result.

139. This case of transformation of coordinates becomes very much simpler when *triangular* coordinates are used.

For if (x, y, z) be the triangular coordinates of any point referred to the old triangle of reference, and (x', y', z') the triangular coordinates referred to the new triangle, the angular points of the latter being

$$(x_1, y_1, z_1), \ (x_2, y_2, z_2), \ (x_3, y_3, z_3),$$

then the equations (ii) of the last article become

$$\left.\begin{aligned} x &= x_1x' + x_2y' + x_3z' \\ y &= y_1x' + y_2y' + y_3z' \\ z &= z_1x' + z_2y' + z_3z' \end{aligned}\right\}.$$

And the equation $f(x, y, z) = 0$ therefore transforms into

$$f(x_1x' + x_2y' + x_3z',\ y_1x' + y_2y' + y_3z',\ z_1x' + z_2y' + z_3z') = 0.$$

140. CASE II. *When the equations of the new lines of reference are given.*

If the sides of the new triangle be represented by the equations in terms of the perpendiculars from A, B, C; viz. by the equations

$$p_1a\alpha + q_1b\beta + r_1c\gamma = 0,$$
$$p_2a\alpha + q_2b\beta + r_2c\gamma = 0,$$
$$p_3a\alpha + q_3b\beta + r_3c\gamma = 0,$$

then the coordinates of A, B, C referred to the new triangle are respectively:

new coordinates of A, (p_1, p_2, p_3),
.... of B, (q_1, q_2, q_3),
.... of C, (r_1, r_2, r_3).

Hence, if (as before) any point O have the old coordinates (α, β, γ) and the new coordinates $(\alpha', \beta', \gamma')$, then α represents the perpendicular from $(\alpha', \beta', \gamma')$ on the line joining (q_1, q_2, q_3) and (r_1, r_2, r_3).

And $a\alpha$ is the double area of the triangle whose angular points in the new coordinates are

$$(\alpha', \beta', \gamma'),\ (q_1, q_2, q_3),\ (r_1, r_2, r_3).$$

Hence we have

$$a\alpha = \frac{1}{2S'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ q_1, & q_2, & q_3 \\ r_1, & r_2, & r_3 \end{vmatrix}$$

$$\alpha = \frac{1}{2aS'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ q_1, & q_2, & q_3 \\ r_1, & r_2, & r_3 \end{vmatrix}, \quad \beta = \frac{1}{2bS'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ r_1, & r_2, & r_3 \\ p_1, & p_2, & p_3 \end{vmatrix},$$

$$\gamma = \frac{1}{2cS'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ p_1, & p_2, & p_3 \\ q_1, & q_2, & q_3 \end{vmatrix}.$$

Therefore, *if the locus of the homogeneous equation*

$$f(\alpha, \beta, \gamma) = 0$$

be referred to the new triangle whose sides are given by

$$p_1 a\alpha + q_1 b\beta + r_1 c\gamma = 0,$$

$$p_2 a\alpha + q_2 b\beta + r_2 c\gamma = 0,$$

$$p_3 a\alpha + q_3 b\beta + r_3 c\gamma = 0,$$

its equation in terms of the new coordinates will be

$$f\left(\frac{1}{a}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ q_1, & q_2, & q_3 \\ r_1, & r_2, & r_3 \end{vmatrix}, \ \frac{1}{b}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ r_1, & r_2, & r_3 \\ p_1, & p_2, & p_3 \end{vmatrix}, \ \frac{1}{c}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ p_1, & p_2, & p_3 \\ q_1, & q_2, & q_3 \end{vmatrix}\right) = 0.$$

141. The cases which occur in practice are generally very simple. The following is an example.

Let an equation connecting the coordinates α, β, γ involve as terms or factors the three expressions

$$l\alpha + m\beta + n\gamma, \quad l'\alpha + m'\beta + n'\gamma, \quad l''\alpha + m''\beta + n''\gamma,$$

and suppose we have to transform it to the new triangle of reference whose sides are represented by the equations

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0, \quad l''\alpha + m''\beta + n''\gamma = 0.$$

Then if α', β', γ' denote the new coordinates of the point whose old coordinates are α, β, γ, we have (Art. 46)

$$\alpha' = \frac{l\alpha + m\beta + n\gamma}{\{l, m, n\}}, \quad \beta' = \frac{l'\alpha + m'\beta + n'\gamma}{\{l', m', n'\}},$$

$$\gamma' = \frac{l''\alpha + m''\beta + n''\gamma}{\{l'', m'', n''\}}.$$

Hence, in effecting our transformation, wherever the expressions

$$l\alpha + m\beta + n\gamma, \quad l'\alpha + m'\beta + n'\gamma, \quad l''\alpha + m''\beta + n''\gamma$$

occur, we have only to substitute for them

$$\kappa\alpha', \ \kappa'\beta', \ \kappa''\gamma',$$

where κ, κ', κ'' are constants and represent the expressions

$$\{l, m, n\}, \ \{l', m', n'\}, \ \{l'', m'', n''\}.$$

It follows that if the original equation be made up *entirely* of the expressions

$$l\alpha + m\beta + n\gamma, \quad l'\alpha + m'\beta + n'\gamma, \quad l''\alpha + m''\beta + n''\gamma,$$

the transformation will generally simplify it very much.

It should also be borne in mind, if

$$a\alpha + b\beta + c\gamma$$

occur as a factor or a term in the original equation, that since it is known to represent a constant quantity it cannot be transformed into an expression which would denote a variable quantity. It can therefore take no other form than

$$\kappa(a'\alpha' + b'\beta' + c'\gamma'),$$

where a', b', c' are the sides of the new triangle of reference, and κ is a constant expressing the ratio of the areas of the old and new triangle.

142. EXAMPLE.

The equation

$$(l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma)^2$$
$$= \kappa (l'\alpha + m'\beta + n'\gamma)(l''\alpha + m''\beta + n''\gamma)^2,$$

may be transformed into the much simpler form

$$\alpha'(a'\alpha' + b'\beta' + c'\gamma')^2 = \kappa'\beta'\gamma'^2,$$

by taking the straight lines

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$
$$l''\alpha + m''\beta + n''\gamma = 0,$$

as lines of reference.

We shall very often have recourse to such a transformation as this in order to simplify the equations of curves.

EXERCISES ON CHAPTER XI.

(98) Transform the equation

$$\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma \cos\phi \sin\psi + 2\gamma\alpha \cos\psi \sin\theta$$
$$+ 2\alpha\beta \cos\theta \sin\phi = 0$$

to the new triangle of reference formed by the straight lines

$$\beta \cos\phi + \gamma \sin\psi = 0,$$
$$\gamma \cos\psi + \alpha \sin\theta = 0,$$
$$\alpha \cos\theta + \beta \sin\phi = 0.$$

(99) Transform the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

to the new triangle of reference formed by the straight lines

$$u\alpha + w'\beta + v'\gamma = 0,$$
$$w'\beta + v'\gamma = 0,$$
$$(vv' - u'w')\beta - (ww' - u'v')\gamma = 0.$$

CHAPTER XII.

SECTIONS OF CONES.

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143. HAVING explained the principles of Trilinear Coordinates and exhibited the application of the method to the investigation of the properties of straight lines, we now pass on to apply the same method to curved lines, commencing with the conic sections.

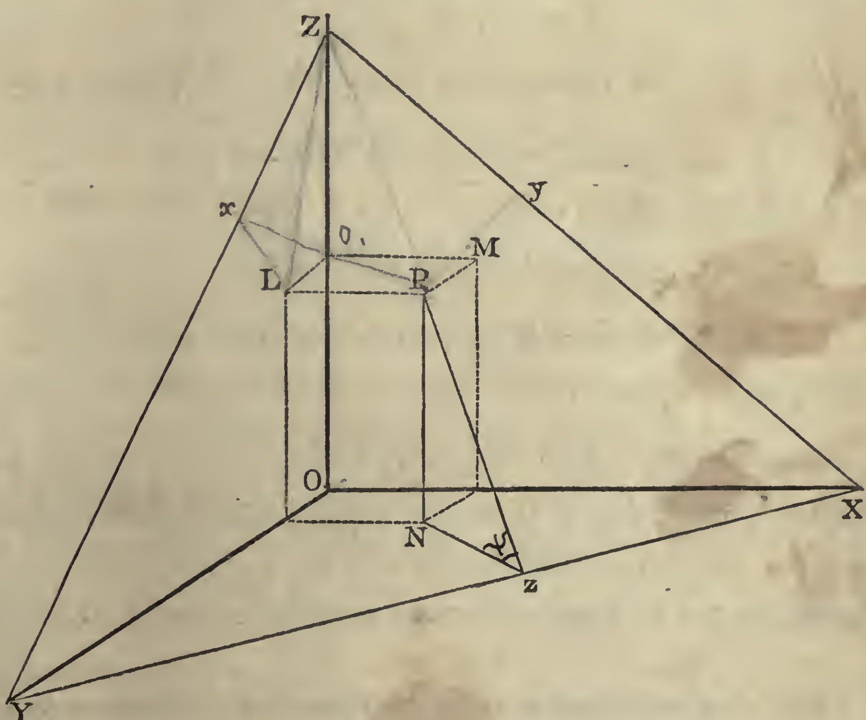
We shall endeavour to make our investigation of these curves as independent as possible of the knowledge which we have acquired of their properties by purely geometrical and other methods. At the same time the student must not expect to find in these brief chapters anything like a complete treatise on the properties of the curves, as it is our object rather to set before him such properties as can be *advantageously* treated of by trilinear coordinates, and to leave for treatment by other methods those properties to which other methods are specially applicable. Success in the solution of a problem generally depends in a great measure on the selection of the most appropriate method of approaching it; many properties of conic sections (for instance) being demonstrable by a few steps of pure geometry which would involve the most laborious operations with trilinear coordinates, while other properties are almost self-evident under the method of trilinear coordinates, which it would perhaps be actually impossible to prove by the old geometry. We shall strive to set before the student such a series of propositions as shall best illustrate the use of trilinear coordinates, and at the same time put him into possession of such properties and results as are most often called for in the solution of problems.

144. *Any plane section of a right circular cone when referred to suitable lines of reference may be expressed by an equation of the form*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

where l, m, n are not all of the same sign.

Fig. 31.



Let O be the vertex, OZ the axis of any right circular cone, and XYZ any plane cutting the cone in a curve, one of whose points is P.

Through O draw two straight lines OX, OY at right angles to OZ and to one another, meeting the plane of the section in X, Y, and let θ, ϕ, ψ be the angles which OX, OY, OZ make with the perpendicular upon XYZ.

Take XYZ as triangle of reference, and let

$$Px = \alpha, \quad Py = \beta, \quad Pz = \gamma,$$

be the trilinear coordinates of any point P on the curve.

Let PL, PM, PN be the perpendicular distances of P from the planes OYZ, OZX, OXY.

Then we have

$$\frac{PL}{PX} = \sin\theta, \quad \frac{PM}{PY} = \sin\phi, \quad \frac{PN}{PZ} = \sin\psi \text{...........(i)}.$$

But if ω be the semivertical angle of the cone

$$PN = ON . \tan\omega,$$

and since $$ON^2 = PL^2 + PM^2, \qquad \text{(Euclid, I. 47)}$$

$$PN^2 = (PL^2 + PM^2) \tan^2\omega;$$

and therefore, in virtue of (i),

$$\gamma^2 \sin^2\psi = (\alpha^2 \sin^2\theta + \beta^2 \sin^2\phi) \tan^2\omega,$$

or $$\alpha^2 \sin^2\theta \tan^2\omega + \beta^2 \sin^2\phi \tan^2\omega - \gamma^2 \sin^2\psi = 0,$$

which may be written

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

it being observed that n is of the opposite algebraical sign to l and m; but the absolute ratios of l, m, n are any whatever, since angles can be found with their sines in any assigned ratio.

145. *Any section of a right circular cone, to whatever lines it be referred, will be represented by a homogeneous equation of the second degree in trilinear coordinates.*

For we have shewn in the last article that when suitable lines of reference are chosen, the conic section will be represented by an equation of the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

a homogeneous equation of the second degree.

Now if we transform our coordinates to any other lines of reference (Art. 138, page 149), the degree of the equation will not be altered. Hence, whatever be the lines of reference, the conic will be represented by an equation of the second degree. Q. E. D.

146. Conversely.

Every equation of the second degree in trilinear coordinates represents some conic section.

For any equation of the second degree (being rendered homogeneous, Art. 10, page 13) may be written in the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Multiplying by u and re-arranging, we get

$$(u\alpha + w'\beta + v'\gamma)^2$$
$$= (w'^2 - uv)\beta^2 + 2(v'w' - uu')\beta\gamma + (v'^2 - uw)\gamma^2.$$

Now multiplying by $w'^2 - uv$ and rearranging, we get

$$(w'^2 - uv)(u\alpha + w'\beta + v'\gamma)^2 = \{(w'^2 - uv)\beta - (v'w' - uu')\gamma\}^2$$
$$+ u\{uvw + 2u'v'w' - uu'^2 - vv'^2 - ww'^2\}\gamma^2.$$

And if we take the lines

$$u\alpha + w_1\beta + v'\gamma = 0,$$

and $$(w'^2 - uv)\beta - (v'w' - uu')\gamma = 0,$$

as lines of reference instead of

$$\alpha = 0,$$

and $$\beta = 0,$$

the equation takes the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

and therefore represents a conic section by Art. 144.

Hence, every equation of the second degree in trilinear coordinates represents a section of a right circular cone.

147. *A conic section can generally be found to satisfy five simple conditions.*

For the general equation to a conic section has been seen to consist of six terms. It therefore involves six different coefficients, and by altering the ratios of the coefficients one to another we shall obtain different equations representing different conic sections.

But since an equation is not altered by altering *all* its coefficients in a constant ratio, we shall obtain all possible variations of the equation by giving any one of the coefficients an assigned value and varying the values of the remaining five.

In order to determine the equation to a conic section we have therefore *five* unknown coefficients to assign. And since each simple condition which the conic satisfies (such as the condition of passing through a particular point or touching a particular straight line) will in general give us one equation among the coefficients, it follows that we can generally so assign the coefficients as that *five* such conditions may be satisfied.

The following article illustrates this.

148. *To find the equation to the conic section which passes through five given points.*

Let

$$(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), (\alpha_3, \beta_3, \gamma_3), (\alpha_4, \beta_4, \gamma_4), (\alpha_5, \beta_5, \gamma_5)$$

be the coordinates of the five given points.

And suppose

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

to be the equation to the conic.

Then, since the five points lie upon it, we have

$$u\alpha_1^2 + v\beta_1^2 + w\gamma_1^2 + 2u'\beta_1\gamma_1 + 2v'\gamma_1\alpha_1 + 2w'\alpha_1\beta_1 = 0,$$

$$u\alpha_2^2 + v\beta_2^2 + w\gamma_2^2 + 2u'\beta_2\gamma_2 + 2v'\gamma_2\alpha_2 + 2w'\alpha_2\beta_2 = 0,$$

$$u\alpha_3^2 + v\beta_3^2 + w\gamma_3^2 + 2u'\beta_3\gamma_3 + 2v'\gamma_3\alpha_3 + 2w'\alpha_3\beta_3 = 0,$$

$$u\alpha_4^2 + v\beta_4^2 + w\gamma_4^2 + 2u'\beta_4\gamma_4 + 2v'\gamma_4\alpha_4 + 2w'\alpha_4\beta_4 = 0,$$

$$u\alpha_5^2 + v\beta_5^2 + w\gamma_5^2 + 2u'\beta_5\gamma_5 + 2v'\gamma_5\alpha_5 + 2w'\alpha_5\beta_5 = 0.$$

Hence, eliminating $u : v : w : u' : v' : w'$, we get

$$\begin{vmatrix} \alpha^2, & \beta^2, & \gamma^2, & \beta\gamma, & \gamma\alpha, & \alpha\beta \\ \alpha_1^2, & \beta_1^2, & \gamma_1^2, & \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \alpha_2^2, & \beta_2^2, & \gamma_2^2, & \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \\ \alpha_3^2, & \beta_3^2, & \gamma_3^2, & \beta_3\gamma_3, & \gamma_3\alpha_3, & \alpha_3\beta_3 \\ \alpha_4^2, & \beta_4^2, & \gamma_4^2, & \beta_4\gamma_4, & \gamma_4\alpha_4, & \alpha_4\beta_4 \\ \alpha_5^2, & \beta_5^2, & \gamma_5^2, & \beta_5\gamma_5, & \gamma_5\alpha_5, & \alpha_5\beta_5 \end{vmatrix} = 0$$

which will be the equation required.

149. *To find the condition that six points whose coordinates are given should lie upon one conic.*

Let the given coordinates be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$, $(\alpha_5, \beta_5, \gamma_5)$, $(\alpha_6, \beta_6, \gamma_6)$.

The points will lie upon one conic if the coordinates of one of them satisfy the equation to the conic through the other five.

Hence by the last article the condition is

$$\begin{vmatrix} \alpha_1^2, & \beta_1^2, & \gamma_1^2, & \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \alpha_2^2, & \beta_2^2, & \gamma_2^2, & \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \\ \alpha_3^2, & \beta_3^2, & \gamma_3^2, & \beta_3\gamma_3, & \gamma_3\alpha_3, & \alpha_3\beta_3 \\ \alpha_4^2, & \beta_4^2, & \gamma_4^2, & \beta_4\gamma_4, & \gamma_4\alpha_4, & \alpha_4\beta_4 \\ \alpha_5^2, & \beta_5^2, & \gamma_5^2, & \beta_5\gamma_5, & \gamma_5\alpha_5, & \alpha_5\beta_5 \\ \alpha_6^2, & \beta_6^2, & \gamma_6^2, & \beta_6\gamma_6, & \gamma_6\alpha_6, & \alpha_6\beta_6 \end{vmatrix} = 0.$$

150. *Every straight line meets every conic section in two real or imaginary points, distinct or coincident.*

Let $u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$ .... (1)

be the equation to any conic section, and

$$l\alpha + m\beta + n\gamma = 0 \text{ .... (2)}$$

the equation to any straight line.

The coordinates of their points of intersection must satisfy both the equations as well as the relation

$$a\alpha + b\beta + c\gamma = 2\Delta \text{ ....} (3).$$

Hence, to find the coordinates we may proceed theoretically thus:

From (2) and (3), which are simple equations, we may express β and γ as functions of α of the first degree. We may then substitute these values in (1), which thus becomes a quadratic equation in α. Being a quadratic it will give two values for α, real or imaginary, equal or unequal, and the simple equations (2) and (3) will then give a value for β and a value for γ corresponding to each value of α. Thus there will be determined two and only two points of intersection, which may however be real or imaginary, coincident or distinct.

151. If in the argument of the last article the straight line be at infinity the reasoning still applies. Hence every conic cuts the straight line at infinity in two real or imaginary points, either coincident or distinct.

If these two points be real and coincident the conic section is called a *Parabola*.

If they be real and distinct it is called a *Hyperbola*.

If they be imaginary it is called an *Ellipse*.

152. DEF. Tangents which do not lie altogether at infinity but have their contact at infinity are called *Asymptotes*.

It follows that an ellipse has two imaginary asymptotes and a hyperbola two real ones.

In the parabola, since the straight line at infinity meets it in two coincident points, that line is a tangent, and there can be no other tangent touching at infinity. Hence the parabola has no asymptote.

153. Students who have not commenced Analytical Geometry of Three Dimensions may omit the remainder of this chapter.

But those who have made any progress in that subject will observe that the first article in this chapter is but a particular case of the following more general theorem.

If $$f(x, y, z) = 0 \quad\text{.......................... (1)}$$

be the equation to any surface in rectangular coordinates of three dimensions, and if

$$x \cos\theta + y \cos\phi + z \cos\psi = p \quad\text{.............. (2)}$$

be the equation to any plane, then the equation to the section of the surface (1) *by the plane* (2) *will be*

$$f(\alpha \sin\theta,\ \beta \sin\phi,\ \gamma \sin\psi) = 0 \quad\text{..............(3),}$$

the coordinates being trilinear, and the lines of reference the traces of the coordinate planes upon the plane of the section.

For if P be any point upon the section, x, y, z its coordinates regarding it as a point upon the surface (1), and α, β, γ its coordinates referred to the traces of the original planes upon the plane of section, then since θ, ϕ, ψ are the angles which this plane makes with the original planes, we have

$$x = \alpha \sin\theta, \quad y = \beta \sin\phi, \quad z = \gamma \sin\psi,$$

therefore substituting in (1), we get

$$f(\alpha \sin\theta, \quad \beta \sin\phi, \quad \gamma \sin\psi) = 0 \quad\text{............ (3),}$$

a relation among the trilinear coordinates of any point in the section, and therefore the equation to the section.

154. It should be noticed that if the surface represented by the equation (1) happen to be a conical surface having its vertex at the origin, the equation (1) will be homogeneous in x, y, z, and therefore the equation (3) will be homogeneous in α, β, γ.

In all other cases these equations will be heterogeneous, and we shall generally have to make the trilinear equation (3) homogeneous by means of the identical relation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

as explained in Art. 10.

155. Cor. If $f(x, y, z)$ be of the n^{th} order, then

$$f(\alpha \sin\theta, \quad \beta \sin\phi, \quad \gamma \sin\psi) = 0$$

is of the n^{th} order: that is,

Every plane section of a surface of the n^{th} order is a curve of the n^{th} order.

The following particular case is important.

Every plane section of a surface of the second order is a conic section.

156. The identical relation among the coordinates α, β, γ may be obtained directly from the equation to the plane, thus

$$x\cos\theta + y\cos\phi + z\cos\psi = p,$$

but $$x = \alpha\sin\theta, \quad y = \beta\sin\phi, \quad z = \gamma\sin\psi;$$

therefore substituting

$$\alpha\cos\theta\sin\theta + \beta\cos\phi\sin\phi + \gamma\cos\psi\sin\psi = p,$$

or $$\alpha\sin 2\theta + \beta\sin 2\phi + \gamma\sin 2\psi = 2p.$$

Cor. *The equation to the straight line at infinity in this plane will be*

$$\alpha\sin 2\theta + \beta\sin 2\phi + \gamma\sin 2\psi = 0.$$

157. *To find the section of the ellipsoid whose equation is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by the plane whose equation is

$$x\cos\theta + y\cos\phi + z\cos\psi = p.$$

The equation to the section in trilinear coordinates is (Art. 153)

$$\frac{\alpha^2\sin^2\theta}{a^2} + \frac{\beta^2\sin^2\phi}{b^2} + \frac{\gamma^2\sin^2\psi}{c^2} = 1,$$

which has to be rendered homogeneous by means of the relation

$$x \cos\theta \sin\theta + y \cos\phi \sin\phi + z \cos\psi \sin\psi = p.$$

Hence the equation becomes

$$\frac{\alpha^2 \sin^2\theta}{a^2} + \frac{\beta^2 \sin^2\phi}{b^2} + \frac{\gamma^2 \sin^2\psi}{c^2}$$

$$- \frac{(\alpha \sin 2\theta + \beta \sin 2\phi + \gamma \sin 2\psi)^2}{4p^2} = 0.$$

COR. If the plane of section were

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

we should have

$$\cos\theta = \frac{p}{a}, \quad \cos\phi = \frac{p}{b}, \quad \cos\psi = \frac{p}{c},$$

and the equation to the section would be

$$\frac{\alpha^2}{a^2} \sin^2\theta + \frac{\beta^2}{b^2} \sin^2\phi + \frac{\gamma^2}{c^2} \sin^2\psi$$

$$- \left(\frac{\alpha}{a} \sin\theta + \frac{\beta}{b} \sin\phi + \frac{\gamma}{c} \sin\psi\right)^2 = 0,$$

or $\quad a\beta\gamma \sin\phi \sin\psi + b\gamma\alpha \sin\psi \sin\theta + c\alpha\beta \sin\theta \sin\phi = 0,$

or
$$\frac{a\beta\gamma}{\sin\theta} + \frac{b\gamma\alpha}{\sin\phi} + \frac{c\alpha\beta}{\sin\psi} = 0.$$

EXERCISES ON CHAPTER XII.

(100) Determine three straight lines which being taken as lines of reference, the equation

$$\lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta = 0$$

shall be transformed to the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

(101) Shew that the six points given by

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n},\quad \frac{\alpha}{m}=\frac{\beta}{n}=\frac{\gamma}{l},\quad \frac{\alpha}{n}=\frac{\beta}{l}=\frac{\gamma}{m},$$

$$\frac{\alpha}{n}=\frac{\beta}{m}=\frac{\gamma}{l},\quad \frac{\alpha}{l}=\frac{\beta}{n}=\frac{\gamma}{m},\quad \frac{\alpha}{m}=\frac{\beta}{l}=\frac{\gamma}{n},$$

lie all on one conic section.

(102) More generally, shew that if

$$u=0,\quad v=0,\quad w=0$$

be the equations to three straight lines, then the six points given by the equations

$$\frac{u}{l}=\frac{v}{m}=\frac{w}{n},\quad \frac{u}{m}=\frac{v}{n}=\frac{w}{l},\quad \frac{u}{n}=\frac{v}{l}=\frac{w}{m},$$

$$\frac{u}{n}=\frac{v}{m}=\frac{w}{l},\quad \frac{u}{l}=\frac{v}{n}=\frac{w}{m},\quad \frac{u}{m}=\frac{v}{l}=\frac{w}{n},$$

lie all on one conic section.

(103) Find the equation to the conic section circumscribing the pentagon, whose sides taken in order are represented by the equations in triangular coordinates,

$$x=y+z,\quad y=z+x,$$
$$x=0,\quad z=3(x+y),\quad y=0.$$

(104) Shew that the six points in which the straight lines

$$\beta\cos B=\gamma\cos C,\quad \beta\sin B=\gamma\sin C,$$
$$\gamma\cos C=\alpha\cos A,\quad \gamma\sin C=\alpha\sin A,$$
$$\alpha\cos A=\beta\cos B,\quad \alpha\sin A=\beta\sin B,$$

(drawn from the vertices of the triangle of reference) intersect the opposite sides, lie all on one conic section.

CHAPTER XIII.

ABRIDGED NOTATION OF THE SECOND DEGREE.

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158. We have already in Chapter VIII. considered the interpretation of a variety of equations obtained by combining the symbols u, v, w, &c. which we have used to denote expressions of the first degree in trilinear, triangular, or other coordinates.

But the equations of that chapter were all formed by the addition or subtraction of constant multiples of u, v, w, &c. and were therefore all of them equations of the first degree, and consequently represented straight lines.

We now go on to interpret equations of the second order obtained by combining u, v, w, &c. where these symbols themselves still represent expressions of the first degree in the coordinates, and therefore when equated severally to zero form equations to straight lines.

159. *To interpret the equation*

$$uv - \kappa wx = 0,$$

where $u = 0$, $v = 0$, $w = 0$, $x = 0$ are the equations to four straight lines and κ is any constant.

Since the equation is of the second order, it represents a conic section (Art. 146).

Moreover it is obviously satisfied if $u = 0$ and $w = 0$ are simultaneously satisfied, therefore the conic passes through the intersection of these two straight lines.

Similarly it passes through the point determined by

$$u=0 \text{ and } x=0;$$

also by

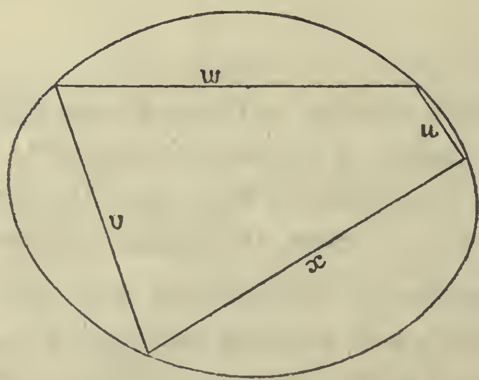
$$v=0 \text{ and } w=0,$$

and

$$v=0 \text{ and } x=0;$$

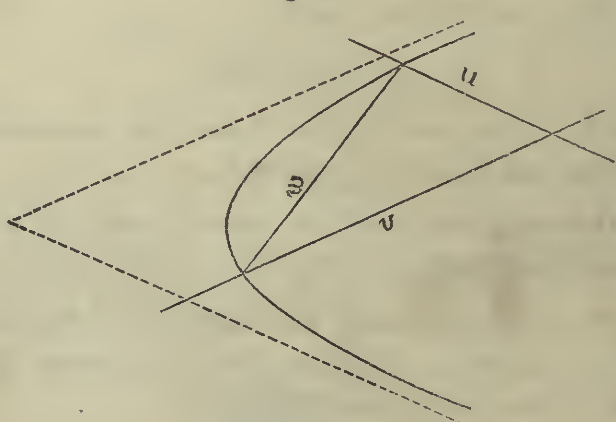
therefore the equation represents a conic circumscribing the quadrilateral in which u and v are opposite sides, and w and x opposite sides.

Fig. 32.



160. We may notice the particular case when $x=0$ represents the straight line at infinity. The equation then represents a conic passing through the points in which w is cut by u and v, and having its asymptotes parallel to these last two straight lines. This will be more clearly seen after reading Chapter XVII.

Fig. 33.

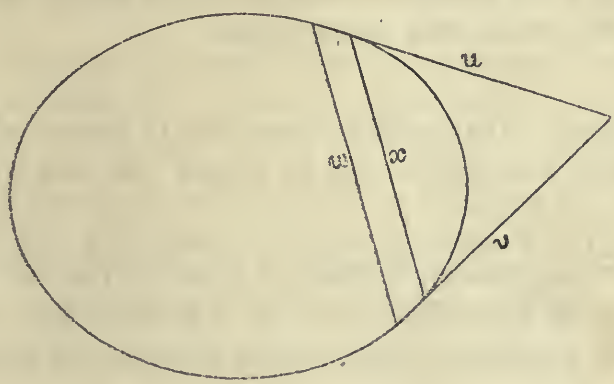


161. *To interpret the equation*

$$uv - \kappa w^2 = 0.$$

We may regard this as a particular case of the last equation, where the opposite sides $w = 0$, $x = 0$ of the quadrilateral have become coincident.

Fig. 34.



Each of the lines $u = 0$, $v = 0$ therefore meets the conic in two coincident points on the coincident straight lines $w = 0$ and $x = 0$.

Hence $u = 0$ and $v = 0$ represent tangents, and $w = 0$ represents the chord of contact.

COR. Consider the particular case

$$uv - \kappa s^2 = 0,$$

where $s = 0$ is the straight line at infinity. The lines $u = 0$, $v = 0$ are now tangents touching the conic at infinity, and are therefore asymptotes. Hence the equation $uv - \kappa s^2 = 0$ represents a hyperbola whose asymptotes are $u = 0$, $v = 0$.

162. *Any two conics may be regarded as intersecting in four real or imaginary points.*

For the two conics will be represented by two equations, each of the second degree, solving these together with the relation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

if the coordinates be trilinear, or

$$x + y + z = 1,$$

if they be triangular, we obtain four different sets of real or imaginary values for the coordinates, satisfying both the equations. These will determine four real or imaginary points lying on the two conics. But since two or more of the solutions of the equations may be identical, it follows that two or more of the points of intersection may be coincident.

163. Cor. Two conic sections have in general *six* common chords, since four points can be joined two and two in six ways.

We say *in general*, because if some of the four points of intersection be coincident, some of the common chords will become also coincident, while others will become common tangents.

Def. The chord joining *any two* of the points of intersection of two conics, together with the chord joining *the other two* points, constitute a *pair* of common chords of the two conics.

Thus if P, Q, R, S be the four points of intersection, the six common chords can be arranged in the three pairs QR, PS; RP, QS; PQ, RS.

164. Definitions. If two of the four points of intersection of two conics are coincident, the conics are said to touch one another.

If the other two are also coincident, but not coincident with the first two, the conics are said to have double contact with each other, and the common chord is then called the chord of contact.

If three of the points coincide, the points are said to have three-pointic contact, or to have the same curvature at the point of contact.

165. *To interpret the equation*

$$S - \kappa uv = 0,$$

$S = 0$ *being the equation to a conic.*

The equation

$$S - \kappa uv = 0,$$

being of the second order, represents some conic section.

To find where the straight line $u = 0$ meets it, we have to substitute $u = 0$ in the equation $S - \kappa uv = 0$: whence we must have $S = 0$;

i.e. the points of intersection of the straight line $u = 0$ with the conic to be investigated lie upon the given conic S;

i.e. $u = 0$ is a common chord of the two conics.

So $v = 0$ is a common chord of the two conics.

Hence $S - \kappa uv = 0$ represents a conic intersecting the conic $S = 0$ in the four points which lie upon the lines $u = 0$, $v = 0$.

166. *To interpret the equation*

$$S - \kappa u^2 = 0.$$

We may regard this as a particular case of the last equation, $u = 0$ and $v = 0$ coinciding, and so by reasoning similar to that in Art. 161 we conclude that it represents a conic touching the conic $S = 0$ in two points, $u = 0$ being the equation to the chord of contact.

167. It will be observed that all the equations which we have considered in the present chapter have been found to represent conics passing through some four fixed points.

Each equation has involved an undetermined constant κ, which may receive different values distinguishing the different conics which can be drawn through the same four points.

And by giving κ a suitable value, we can make the equation represent *any* conic whatever passing through the four points.

For since five points determine a conic section, any conic through the four points will be determined by one point more, and the condition that the equation may be satisfied at this point is an equation from which κ may be determined.

Hence by giving κ a suitable value, the equation $uv - \kappa wx = 0$ will represent *any* conic circumscribing the quadrilateral whose opposite sides are u, v and w, x. So $uv - \kappa w^2 = 0$ will represent *any* conic having $u = 0$, $v = 0$ as tangents, and $w = 0$ as chord of contact, and so in the other cases.

168. *If* $S = 0$ *and* $S' = 0$ *represent the equations to two conic sections, then will the equation* $S + \kappa S' = 0$ *represent a conic section passing through all the points of intersection of the first two.*

And by giving a suitable value to κ, *this equation can be made to represent* any *conic whatever passing through those points of intersection.*

For the coordinates of any point of intersection of $S = 0$ and $S' = 0$ satisfy both the equations: i.e. they make S and S' severally zero: therefore they make $S + \kappa S'$ zero: i.e. they satisfy the equation $S + \kappa S' = 0$. Hence the locus of this equation passes through every point of intersection of the given conics.

And since S and S' are of the second degree, $S + \kappa S' = 0$ is of the second degree, and therefore $S + \kappa S' = 0$ represents a conic section.

Further, by giving a suitable value to κ this equation will represent *any* conic passing through the four points of intersection of the first two. For since five points determine a conic section, any conic passing through the four points of intersection will be determined if one other point upon it be determined.

It is only necessary therefore to shew that by giving κ a suitable value, the equation $S + \kappa S' = 0$ can be made to pass through any one assigned point; which follows immediately as in the last article. For if s and s' be what S and S' become when the coordinates of the assigned point are substituted for the current coordinates, the equation $s + \kappa s' = 0$ will be the condition that the conic should pass through the assigned point.

Hence if we give κ the value $-\frac{s}{s'}$, the required condition will be fulfilled.

Exercises on Chapter XIII.

(105) Prove that the equation

$$(a\alpha + b\beta + c\gamma)^2 = (l\alpha + m\beta + n\gamma)\ (l'\alpha + m'\beta + n'\gamma)$$

represents a hyperbola, and find its centre.

(106) Interpret the equations,

(i) $k\alpha^2 = \beta\,(a\alpha + b\beta + c\gamma)$.

(ii) $(l\alpha + m\beta + n\gamma)^2$

$= (\alpha \cos A + \beta \cos B + \gamma \cos C)\ (\alpha \sin A + \beta \sin B + \gamma \sin C)$.

(iii) $b^2\alpha^2 + a^2\beta^2 = 2a\alpha\,(a\alpha + c\gamma)$.

(107) Shew that whatever be the value of κ, the locus of the equation

$$\kappa x^2 - (x+1)^2 = (\kappa - 9)\,y^2$$

will pass through four fixed points, and find their coordinates.

(108) If $s = 0$ be the equation to the straight line at infinity, and $u = 0$, $v = 0$, $w = 0$ represent three other real straight lines, the equation

$$uv + sw = 0$$

will generally represent a hyperbola; but if u and v be parallel, it will represent a parabola.

(109) With the notation of the last exercise the equation

$$u^2 + sw = 0$$

represents a parabola, and the equation

$$u^2 + v^2 + sw = 0,$$

where u and v are not parallel, represents an ellipse.

(110) If two conics have double contact with a third, the chords of contact are concurrent with a pair of common chords of the first two conics, and form with them a harmonic pencil.

(111) If three conic sections have one chord common to all, their three other common chords are concurrent.

(112) The straight line $u - 2kv + k^2w = 0$ is a tangent to the conic $uw = v^2$.

(113) The straight line $u + v + w = 0$ is a common tangent to the three conics

$$u^2 = 4vw, \quad v^2 = 4wu, \quad w^2 = 4uv.$$

(114) Three conic sections are drawn through a point P, and each touches two sides of the triangle ABC at the extremities of the third side, shew that each of the tangents at P makes a harmonic pencil with the straight lines joining P to the angular points of the triangle.

CHAPTER XIV.

CONICS REFERRED TO A SELF-CONJUGATE TRIANGLE.

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169. We shewed in Chapter XII. that by selecting suitable lines of reference any conic section might be represented by an equation of the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad \text{.....................(1).}$$

It is quite evident that if l, m, n are all positive, since the squares α^2, β^2, γ^2 are necessarily positive and cannot be all zero, the expression $l\alpha^2 + m\beta^2 + n\gamma^2$ must be positive and therefore greater than zero. Hence the coordinates of no real point can satisfy the equation, and the locus must be entirely imaginary.

If l, m, n are all negative, we may change the signs throughout, and thus arrive at the case just considered.

Hence if the equation have a real locus, two of the coefficients must be of one sign and the remaining one of the opposite sign: and therefore (by changing the signs throughout, if necessary) we can suppose two of the coefficients positive and the third negative.

We will suppose that l and m are positive and n negative, and we may write

$$l = L^2, \; m = M^2, \; n = -N^2,$$

so that the equation becomes

$$L^2\alpha^2 + M^2\beta^2 - N^2\gamma^2 = 0 \quad \text{..................(2).}$$

This may be written

$$L^2\alpha^2 + (M\beta + N\gamma)(M\beta - N\gamma) = 0,$$

whence we at once conclude, Art. 161, Chap. XIII. that the lines

$$M\beta + N\gamma = 0,$$

$$M\beta - N\gamma = 0,$$

are tangents, and the line

$$\alpha = 0$$

their chord of contact.

That is, the side BC of the triangle of reference is the chord of contact of tangents from the opposite angular point A.

Similarly, by writing the equation (2) in the form

$$M^2\beta^2 + (L\alpha + N\gamma)(L\alpha - N\gamma) = 0,$$

we conclude that the lines

$$L\alpha + N\gamma = 0,$$

$$L\alpha - N\gamma = 0,$$

are tangents, and the line

$$\beta = 0$$

their chord of contact.

That is, the side CA of the triangle of reference is the chord of contact of tangents from the opposite angular point B.

But further, the equation (2) might be written

$$(L\alpha + M\beta\sqrt{-1})(L\alpha - M\beta\sqrt{-1}) - N^2\gamma^2 = 0,$$

from which form we see that the two imaginary straight lines

$$L\alpha + M\beta\sqrt{-1} = 0,$$

$$L\alpha - M\beta\sqrt{-1} = 0,$$

(which both pass through the real point C) are tangents, and the real straight line

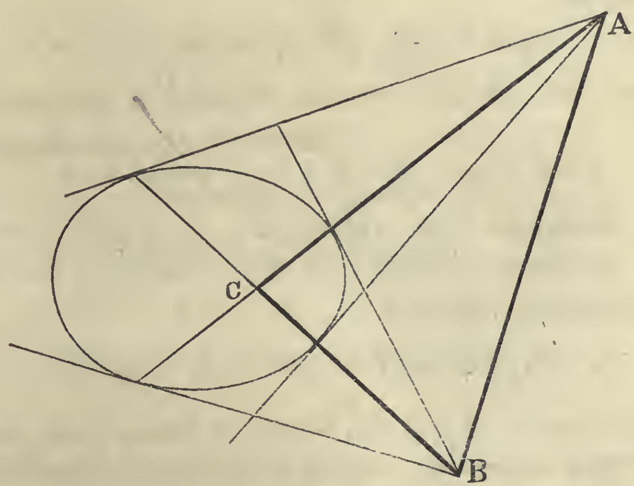
$$\gamma = 0$$

their chord of contact.

That is, the side AB of the triangle of reference is the chord of contact of imaginary tangents from the opposite angular point C.

Hence the conic is so related to the triangle of reference that each side is the chord of contact of the (real or imaginary) tangents from the opposite vertex. This is represented in figure 35.

Fig. 35.



170. DEFINITIONS. The chord of contact of real or imaginary tangents from a fixed point to a conic is called the *polar* of the point with respect to the conic.

Also a point is said to be the *pole* of that line which is its polar.

We may therefore express the result of the last article as follows:

The equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

represents a conic such that with respect to it each vertex of the triangle of reference is the pole of the opposite side.

This is often briefly expressed by saying that the triangle is *self-conjugate* with respect to the conic.

171. *From a given point a straight line is drawn in a given direction to meet the conic*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

it is required to find the lengths intercepted by the curve upon this straight line.

Let $(\alpha', \beta', \gamma')$ be the given point, and λ, μ, ν the sines of the given direction (see Chap. VI.), then the equations to the straight line are

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

whence

$$\alpha = \alpha' + \lambda\rho, \quad \beta = \beta' + \mu\rho, \quad \gamma = \gamma' + \nu\rho.$$

If we substitute these values of α, β, γ which are true for any point on the straight line in the equation to the conic, the resulting equation, viz.

$$l(\alpha' + \lambda\rho)^2 + m(\beta' + \mu\rho)^2 + n(\gamma' + \nu\rho)^2 = 0,$$

will give the values of ρ at the points of intersection, that is, the lengths of the intercepts required, measured from the given point $(\alpha', \beta', \gamma')$.

The equation is a quadratic, and may be written

$$(l\lambda^2 + m\mu^2 + n\nu^2)\rho^2 + 2(l\lambda\alpha' + m\mu\beta' + n\nu\gamma')\rho + (l\alpha'^2 + m\beta'^2 + n\gamma'^2) = 0.$$

Cor. 1. Since this quadratic has two roots, every straight line meets this—and therefore any—conic in two points (which may however be distinct, coincident or imaginary), as we proved in Chap. XII.

Cor. 2. If the point $(\alpha', \beta', \gamma')$ lie on the conic, so that

$$l\alpha'^2 + m\beta'^2 + n\gamma'^2 = 0,$$

one of the intercepts is zero, and the equation reduces to

$$(l\lambda^2 + m\mu^2 + n\nu^2)\rho + 2(l\lambda\alpha' + m\mu\beta' + n\nu\gamma') = 0,$$

which therefore gives the length of the chord drawn from $(\alpha', \beta', \gamma')$ in the given direction.

172. *To find the equation to the tangent at any point on the conic.*

Let λ, μ, ν be the direction sines of the tangent at $(\alpha', \beta', \gamma')$, and let (α, β, γ) be any point on the tangent, then (Chap. VI.),

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho \text{ .... } (1),$$

and the length of the chord in this direction is given by the equation

$$(l\lambda^2+m\mu^2+n\nu^2)\,\rho+2\,(l\lambda\alpha'+m\mu\beta'+n\nu\gamma')=0.$$

But since the direction is that of the tangent the length of the chord must be zero, therefore

$$l\lambda\alpha'+m\mu\beta'+n\nu\gamma'=0,$$

whence by the substitution of (1) we get

$$l\alpha'(\alpha-\alpha')+m\beta'(\beta-\beta')+n\gamma'(\gamma-\gamma')=0,$$

or $$l\alpha'\alpha+m\beta'\beta+n\gamma'\gamma=l\alpha'^2+m\beta'^2+n\gamma'^2$$
$$=0,$$

since $(\alpha', \beta', \gamma')$ lies on the conic.

Hence the equation

$$l\alpha'\alpha+m\beta'\beta+n\gamma'\gamma=0$$

expresses a relation among the coordinates of any point on the tangent at $(\alpha', \beta', \gamma',)$ and is therefore the equation to that tangent.

173. *To find the equation to the chord of contact of tangents drawn from a given point to the conic,* or, *to find the polar of a given point with respect to a conic.*

Let α', β', γ' be the coordinates of the given point and suppose $\alpha'', \beta'', \gamma'', \alpha''', \beta''', \gamma'''$ the coordinates of the points of contact of tangents from $(\alpha', \beta', \gamma')$ to the conic.

The equations to these tangents are by the last article

$$l\alpha''\alpha+m\beta''\beta+n\gamma''\gamma=0,$$

and $$l\alpha'''\alpha+m\beta'''\beta+n\gamma'''\gamma=0,$$

and since they pass through the given point $(\alpha', \beta', \gamma')$ we have

$$l\alpha''\alpha' + m\beta''\beta' + n\gamma''\gamma' = 0,$$
$$l\alpha'''\alpha' + m\beta'''\beta' + n\gamma'''\gamma' = 0.$$

But these two equations respectively express that $(\alpha'', \beta'', \gamma'')$ and $(\alpha''', \beta''', \gamma''')$ lie on the locus of the equation

$$l\alpha'\alpha + m\beta'\beta + n\gamma'\gamma = 0,$$

and this being of the first order is the equation to some straight line.

Therefore it is the equation to the straight line joining $(\alpha'', \beta'', \gamma'')$ and $(\alpha''', \beta''', \gamma''')$ the two points of contact, that is to the chord of contact, or the polar of the point $(\alpha', \beta', \gamma')$.

It will be observed that the tangents from $(\alpha', \beta', \gamma')$ may be imaginary, but the chord of contact or polar is always real, like the polar of C in Art. 169.

174. *To find the condition that any straight line whose equation is given should be a tangent to the conic.*

Let $$f\alpha + g\beta + h\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the equation to the straight line: and suppose $(\alpha', \beta', \gamma')$ its point of contact with the curve.

The tangent at this point is given by

$$l\alpha'\alpha + m\beta'\beta + n\gamma'\gamma = 0$$

which must be identical with (1), therefore

$$\frac{l\alpha'}{f} = \frac{m\beta'}{g} = \frac{n\gamma'}{h}.$$

But $(\alpha', \beta', \gamma')$ must also lie on the locus of (1) whence

$$f\alpha' + m\beta' + n\gamma' = 0,$$

therefore eliminating α', β', γ' we get

$$\frac{f^2}{l} + \frac{g^2}{m} + \frac{h^2}{n} = 0,$$

which will be the required relation among f, g, h in order that the given line may be a tangent.

COR. If the conic touch any one of the straight lines

$$f\alpha \pm g\beta \pm h\gamma = 0$$

it will touch them all.

175. *To find the locus of the middle points of a series of parallel chords in the conic whose equation is*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

Let λ, μ, ν be the direction sines of any one—and therefore of every one—of the series of parallel chords.

And let $(\alpha', \beta', \gamma')$ be any point on the required locus, then the chord through this point is represented by

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

and therefore the lengths of the intercepts by the conic are given by the equation

$$l(\alpha' + \lambda\rho)^2 + m(\beta' + \mu\rho)^2 + n(\gamma' + \nu\rho)^2 = 0.$$

But since $(\alpha', \beta', \gamma')$ is the middle point of the chord the two values of ρ given by this quadratic must be equal in value and opposite in sign. Therefore the coefficient of ρ must vanish in the quadratic, and therefore we have

$$l\lambda\alpha' + m\mu\beta' + n\nu\gamma' = 0$$

a relation among the coordinates of any point $(\alpha', \beta', \gamma')$ on the locus. Hence the locus is the *straight line* whose equation is

$$l\lambda\alpha + m\mu\beta + n\nu\gamma = 0.$$

Such a straight line is called a *diameter* of the conic.

176. Thus far the reasoning of the present chapter will apply equally whether we understand the coordinates to be trilinear or triangular, except that in the latter case we must modify the interpretation of λ, μ, ν in Arts. 171, 172, 175, not speaking of them as direction *sines*. [See (xix) and (xx) in the table of formulæ, pages 100, 101.] But now that we have to

introduce the straight line at infinity our equations will no longer hold for triangular coordinates until we replace a, b, c, 2Δ, severally by unity. (Art. 87.)

177. *To find the condition that the equation*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

may represent a parabola.

It is sufficient (Art. 152) to express that the line at infinity must be a tangent. Therefore by Art. 174 the condition in trilinear coordinates is

$$\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n} = 0,$$

or in triangular coordinates,

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 0.$$

178. *To find the centre of the conic.*

Let $(\alpha', \beta', \gamma')$ be the centre; then the lengths of the intercepts in the direction (λ, μ, ν) measured from the centre are given by the equation

$$\rho^2(l\lambda^2 + m\mu^2 + n\nu^2) + 2\rho(l\lambda\alpha' + m\mu\beta' + n\nu\gamma') + l\alpha'^2 + m\beta'^2 + n\gamma'^2 = 0 \quad \text{.............. (1).}$$

But since all chords through the centre are bisected in the centre the two roots of this quadratic must be equal in magnitude and opposite in sign, therefore the coefficient of ρ must vanish,

therefore $$l\lambda\alpha' + m\mu\beta' + n\nu\gamma' = 0 \quad \text{.................. (2)}$$

for all values of $\lambda : \mu : \nu$, subject to the relation

$$a\lambda + b\mu + c\nu = 0.$$

Hence we must have

$$\frac{l\alpha'}{a} = \frac{m\beta'}{b} = \frac{n\gamma'}{c} = \frac{2\Delta}{\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}},$$

which determine α', β', γ' the coordinates of the centre.

Obs. In *triangular* coordinates the centre of the conic whose equation is $lx^2+my^2+nz^2=0$ is given by the equations

$$\frac{1}{lx'}=\frac{1}{my'}=\frac{1}{nz'}=\frac{1}{l}+\frac{1}{m}+\frac{1}{n}.$$

Cor. The coordinates of the centre are infinite if

$$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0, \quad \text{or} \quad \frac{1}{l}+\frac{1}{m}+\frac{1}{n}=0,$$

(*trilinear coordinates*) (*triangular coordinates*)

But we have already seen that when the same condition holds the conic is a parabola. (Art. 177.)

Hence *the centre of a parabola is at infinity.*

179. *To find the conditions that the equation*

$$l\alpha^2+m\beta^2+n\gamma^2=0$$

should represent a circle.

Let $(\alpha', \beta', \gamma')$ be the centre, then the length of the semi-diameter in any direction will be given by the equation (1) of the last article, which in virtue of (2) becomes

$$\rho^2(l\lambda^2+m\mu^2+n\nu^2)+l\alpha'^2+m\beta'^2+n\gamma'^2=0.$$

Hence all the diameters will be equal, provided

$$l\lambda^2+\mu m^2+n\nu^2$$

be constant for all directions.

But we know (Art. 71, page 77), that

$$\lambda^2\sin 2A+\mu^2\sin 2B+\nu^2\sin 2C,$$

is constant.

Hence the diameters will be all equal, provided

$$\frac{l}{\sin 2A}=\frac{m}{\sin 2B}=\frac{n}{\sin 2C},$$

which therefore express the conditions that the conic should be a circle.

Hence

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0$$

represents the circle with respect to which the triangle of reference is self-conjugate.

Obs. The circle will be imaginary unless one of the coefficients $\sin 2A$, $\sin 2B$, $\sin 2C$ be negative (Art. 169);

i.e. unless one of the angles $2A$, $2B$, $2C$ be greater than 180°,

i.e. unless one of the angles A, B, C be greater than 90°,

i.e. unless the triangle of reference be obtuse angled.

Cor. 1. The equation in *triangular* coordinates to the circle with respect to which the triangle of reference is self-conjugate is

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = 0.$$

Cor. 2. In the case of the circle

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0,$$

the equations to give the centre become

$$\frac{\alpha' \sin 2A}{a} = \frac{\beta' \sin 2B}{b} = \frac{\gamma' \sin 2C}{c};$$

or $$\alpha' \cos A = \beta' \cos B = \gamma' \cos C,$$

which we have already seen Chap. II. are the equations to the point of intersection of the perpendiculars from the vertices of the triangle on the opposite sides.

Hence *if a triangle be self-conjugate with respect to a circle, its perpendiculars intersect in the centre of the circle.*

180. *To find the pole of any given straight line with respect to the conic, whose equation is*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

Let $$f\alpha + g\beta + h\gamma = 0 \quad\text{.......................}\quad (1)$$

be the given straight line whose pole is required, and suppose $(\alpha', \beta', \gamma')$ the pole.

The polar of this point is given by

$$l\alpha'\alpha + m\beta'\beta + n\gamma'\gamma = 0,$$

which must be identical with (1), therefore

$$\frac{l\alpha'}{f} = \frac{m\beta'}{g} = \frac{n\gamma'}{h} = \frac{2\Delta}{\frac{af}{l} + \frac{bg}{m} + \frac{ch}{n}},$$

which determine α', β', γ', the coordinates of the pole required.

COR. In trilinear coordinates the pole of the line at infinity is given by

$$\frac{l\alpha'}{a} = \frac{m\beta'}{b} = \frac{n\gamma'}{c} = \frac{2\Delta}{\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}}.$$

Hence (Art. 178) the *centre* is the pole of the line at infinity.

181. If we assume as the definition of the foci the well known geometrical property of a conic section, that the square on the semi-minor axis is equal to the rectangle contained by the focal perpendiculars on any tangent, we can readily find equations to determine the focus of a conic section with respect to which the triangle of reference is self-conjugate. Thus:

To determine the foci of the conic whose equation is

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad \text{.............................} \quad (1).$$

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be the foci required, and let κ be the semi-minor axis of the conic.

The tangent at any point $(\alpha', \beta', \gamma')$ is

$$l\alpha'\alpha + m\beta'\beta + n\gamma'\gamma = 0.$$

And expressing that κ^2 is equal to the rectangle contained by the product of the perpendicular distances of this line from $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, we get

$$\kappa^2 = \frac{(l\alpha_1\alpha' + m\beta_1\beta' + n\gamma_1\gamma')(l\alpha_2\alpha' + m\beta_2\beta' + n\gamma_2\gamma')}{l^2\alpha'^2 + m^2\beta'^2 + n^2\gamma'^2 - 2mn\beta'\gamma'\cos A - 2nl\gamma'\alpha'\cos B - 2lm\alpha'\beta'\cos C},$$

which is a relation among the coordinates $(\alpha', \beta', \gamma')$ of any point on the conic, and therefore, suppressing the accents, the equation

$$\kappa^2 = \frac{(l\alpha_1\alpha + m\beta_1\beta + n\gamma_1\gamma)(l\alpha_2\alpha + m\beta_2\beta + n\gamma_2\gamma)}{l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma\cos A - 2nl\gamma\alpha\cos B - 2lm\alpha\beta\cos C} \quad \ldots (2)$$

is the equation to the conic and therefore is identical with (1).

But this equation (2) may be written

$$\begin{aligned} l^2\alpha^2(\alpha_1\alpha_2 - \kappa^2) + m^2\beta^2(\beta_1\beta_2 - \kappa^2) + n^2\gamma^2(\gamma_1\gamma_2 - \kappa^2) \\ + mn\beta\gamma(\beta_1\gamma_2 + \beta_2\gamma_1 + 2\kappa^2\cos A) \\ + nl\gamma\alpha(\gamma_1\alpha_2 + \gamma_2\alpha_1 + 2\kappa^2\cos B) \\ + lm\alpha\beta(\alpha_1\beta_2 + \alpha_2\beta_1 + 2\kappa^2\cos C) = 0. \end{aligned}$$

Hence equating the ratios of the coefficients of this equation and the equation (1), we get

$$l(\alpha_1\alpha_2 - \kappa^2) = m(\beta_1\beta_2 - \kappa^2) = n(\gamma_1\gamma_2 - \kappa^2) = \tau \text{ (suppose)} \ldots (3),$$

and

$$\beta_1\gamma_2 + \beta_2\gamma_1 + 2\kappa^2\cos A = 0 \ldots\ldots (4),$$

$$\gamma_1\alpha_2 + \gamma_2\alpha_1 + 2\kappa^2\cos B = 0 \ldots\ldots (5),$$

$$\alpha_1\beta_2 + \alpha_2\beta_1 + 2\kappa^2\cos C = 0 \ldots\ldots (6).$$

Multiplying (5) and (6) by c and b respectively, and adding, we get

$$\alpha_1(b\beta_2 + c\gamma_2) + \alpha_2(b\beta_1 + c\gamma_1) + 2\kappa^2 a = 0,$$

or

$$\alpha_1(2\Delta - a\alpha_2) + \alpha_2(2\Delta - a\alpha_1) + 2\kappa^2 a = 0,$$

or

$$\alpha_1 + \alpha_2 = \frac{a}{\Delta}(\alpha_1\alpha_2 - \kappa^2),$$

or by equations (3),

$$\alpha_1 + \alpha_2 = \frac{a\tau}{l\Delta} \ldots\ldots (7),$$

and

$$\alpha_1\alpha_2 = \frac{\tau}{l} + \kappa^2.$$

Therefore α_1 and α_2 are given by the quadratic

$$\alpha^2 - \frac{a\tau}{l\Delta}\alpha + \frac{\tau}{l} + \kappa^2 = 0.$$

Similarly β_1 and β_2 are given by

$$\beta^2 - \frac{b\tau}{m\Delta}\beta + \frac{\tau}{m} + \kappa^2 = 0,$$

and γ_1 and γ_2 are given by

$$\gamma^2 - \frac{c\tau}{n\Delta}\gamma + \frac{\tau}{n} + \kappa^2 = 0.$$

Hence the foci of the given conic are determined by the equations

$$\alpha^2 - \frac{a\tau\alpha}{l\Delta} + \frac{\tau}{l} = \beta^2 - \frac{b\tau\beta}{m\Delta} + \frac{\tau}{m} = \gamma^2 - \frac{c\tau\gamma}{n\Delta} + \frac{\tau}{n},$$

where τ may be determined as follows.

The equation (7) gives us

$$\alpha_1 + \alpha_2 = \frac{a\tau}{l\Delta}.$$

Similarly we have $\qquad \beta_1 + \beta_2 = \dfrac{b\tau}{m\Delta},$

and $\qquad \gamma_1 + \gamma_2 = \dfrac{c\tau}{n\Delta}.$

Multiplying by a, b, c and adding, we get

$$4\Delta = \left(\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}\right)\frac{\tau}{\Delta},$$

and therefore $\qquad \tau = \dfrac{4\Delta^2}{\dfrac{a^2}{l} + \dfrac{b^2}{m} + \dfrac{c^2}{n}}.$

Hence the equations to determine the foci take the form

$$\left(\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}\right)\alpha^2 - \frac{4\Delta a\alpha}{l} + \frac{4\Delta^2}{l} = \left(\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}\right)\beta^2 - \frac{4\Delta b\beta}{m} + \frac{4\Delta^2}{m}$$
$$= \left(\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}\right)\gamma^2 - \frac{4\Delta c\gamma}{n} + \frac{4\Delta^2}{n},$$

and the corresponding equations in triangular coordinates can be immediately written down (Art. 87).

NOTE. If we had assumed the fact that the centre is the point of bisection of the straight line joining the foci, we might have written down the equation (7) and determined τ at once.

For (Art. 18, Cor. page 20) the coordinates of the centre must be

$$\frac{\alpha_1+\alpha_2}{2}, \quad \frac{\beta_1+\beta_2}{2}, \quad \frac{\gamma_1+\gamma_2}{2},$$

and therefore (Art. 178)

$$\frac{l}{a}(\alpha_1+\alpha_2)=\frac{m}{b}(\beta_1+\beta_2)=\frac{n}{c}(\gamma_1+\gamma_2)=\frac{2\Delta}{\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}}.$$

182. COR. 1. If the conic be a parabola, we have (Art. 177)

$$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0,$$

and the equations to determine the foci give one point at infinity, and reduce for the other to

$$\frac{a\alpha-\Delta}{l}=\frac{b\beta-\Delta}{m}=\frac{c\gamma-\Delta}{n},$$

each of which fractions must be equal to

$$\frac{-\Delta}{l+m+n}.$$

Hence

$$a\alpha=\Delta\frac{m+n}{l+m+n}, \quad b\beta=\Delta\frac{n+l}{l+m+n}, \quad c\gamma=\Delta\frac{l+m}{l+m+n},$$

or $$\frac{a\alpha}{m+n}=\frac{b\beta}{n+l}=\frac{c\gamma}{l+m}.$$

COR. 2. In the case of the parabola,

since $$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0,$$

it follows that the coordinates of the finite focus of any parabola with respect to which the triangle of reference is self-conjugate, will satisfy the equation

$$\frac{a^2}{a\alpha-\Delta}+\frac{b^2}{b\beta-\Delta}+\frac{c^2}{c\gamma-\Delta}=0,$$

or

$$\frac{a^2}{b\beta+c\gamma-a\alpha}+\frac{b^2}{c\gamma+a\alpha-b\beta}+\frac{c^2}{a\alpha+b\beta-c\gamma},$$

or

$$\alpha^2\sin 2A+\beta^2\sin 2B+\gamma^2\sin 2C-4(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)=0,$$

which is therefore the equation to the locus of the foci of all parabolas, with respect to which the triangle of reference is self-conjugate.

183. Def. The polar of a focus is a *Directrix.*

184. *To find the equation to the directrix corresponding to the finite focus of the parabola,*

$$l\alpha^2+m\beta^2+n\gamma^2=0.$$

The finite focus is given (Art. 182) by

$$\frac{a\alpha}{m+n}=\frac{b\beta}{n+l}=\frac{c\gamma}{l+m},$$

hence (Art. 173) its polar is represented by

$$l(m+n)\frac{\alpha}{a}+m(n+l)\frac{\beta}{b}+n(l+m)\frac{\gamma}{c}=0,$$

or

$$\left(\frac{1}{m}+\frac{1}{n}\right)\frac{\alpha}{a}+\left(\frac{1}{n}+\frac{1}{l}\right)\frac{\beta}{b}+\left(\frac{1}{l}+\frac{1}{m}\right)\frac{\gamma}{c}=0.$$

or

$$\frac{1}{l}\left(\frac{\beta}{b}+\frac{\gamma}{c}\right)+\frac{1}{m}\left(\frac{\gamma}{c}+\frac{\alpha}{a}\right)+\frac{1}{n}\left(\frac{\alpha}{a}+\frac{\beta}{b}\right)=0.$$

Cor. Since

$$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0,$$

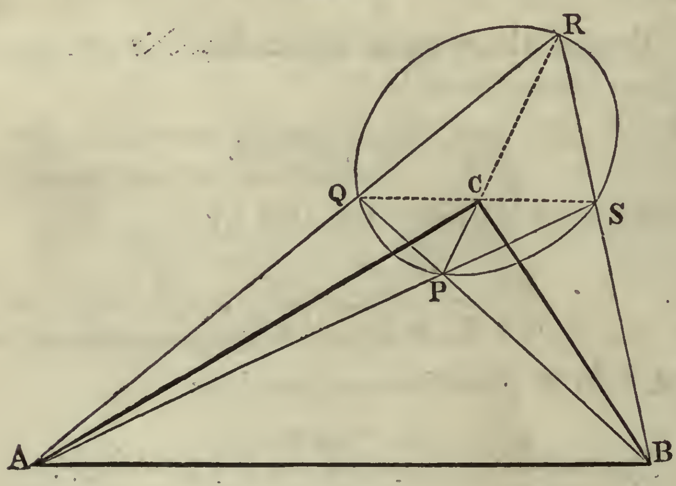
it follows that the directrix of any parabola, with respect to which the triangle of reference is self-congugate, passes through the point given by

$$\frac{1}{a^2}\left(\frac{\beta}{b}+\frac{\gamma}{c}\right)=\frac{1}{b^2}\left(\frac{\gamma}{c}+\frac{\alpha}{a}\right)=\frac{1}{c^2}\left(\frac{\alpha}{a}+\frac{\beta}{b}\right),$$

or $$\frac{\alpha}{\cos A}=\frac{\beta}{\cos B}=\frac{\gamma}{\cos C}.$$

185. *If the opposite sides of a quadrilateral be produced to meet in A and B, and its interior diagonals intersect in C, then the triangle ABC is self-conjugate with respect to any conic circumscribing the quadrilateral.*

Fig. 36.



Let $PQRS$ be the quadrilateral, take ABC as triangle of reference, and let P be determined by the equations

$$l\alpha=m\beta=n\gamma,$$

then the equation to AP is

$$m\beta-n\gamma=0,$$

and since the pencil at A is harmonic, the equation to AR is

$$m\beta+n\gamma=0.$$

So the equation to BP is

$$l\alpha - n\gamma = 0,$$

and since the pencil at B is harmonic, the equation to BR is

$$l\alpha + n\gamma = 0.$$

But these four lines are the sides of the quadrilateral. Hence any conic circumscribing the quadrilateral must have an equation of the form

$$(m\beta - n\gamma)(m\beta + n\gamma) - \kappa(l\alpha - n\gamma)(l\alpha + n\gamma) = 0,$$

or $$m^2\beta^2 - n^2\gamma^2 - \kappa(l^2\alpha^2 - n^2\gamma^2) = 0,$$

or $$\kappa l^2\alpha^2 - m^2\beta^2 + (1-\kappa)n^2\gamma^2 = 0;$$

hence the triangle of reference is self-conjugate with respect to any conic circumscribing the quadrilateral.

Cor. 1. If any number of conics intersect in four points, a triangle can be found self-conjugate with respect to all of them.

Cor. 2. If a series of conics pass through four fixed points, a suitable triangle can be found with respect to which all their equations will be of the form

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

Exercises on Chapter XIV.

(115) Shew that

$$\frac{a^2\alpha^2}{q-r} + \frac{b^2\beta^2}{r-p} + \frac{c^2\gamma^2}{p-q} = 0$$

represents a parabola.

(116) If AA', BB', CC' the diagonals of a quadrilateral, be produced to form a triangle abc, this triangle will be self-conju-

gate with respect to one conic passing through $BCB'C'$, to another passing through $CAC'A'$, and to a third passing through $ABA'B'$.

(117) Interpret the equation

$$(l\alpha + m\beta)^2 + (l\alpha - m\beta)^2 - 2n^2\gamma^2 = 0.$$

(118) Shew that all the lines

$$l\alpha \pm m\beta \pm n\gamma\sqrt{2} = 0$$

are tangents to the conic whose equation is

$$l^2\alpha^2 + m^2\beta^2 - n^2\gamma^2 = 0.$$

(119) Shew that the conic

$$u^2 - v^2 + w^2 = 0$$

circumscribes the quadrilateral whose sides in order are

$$u + v + w = 0,$$
$$-u + v + w = 0,$$
$$u - v + w = 0,$$
$$u + v - w = 0.$$

(120) Interpret the equation

$$\alpha^2 \sin 2A + \beta^2 \sin 2\beta + \gamma^2 \sin 2C = 0,$$

when the triangle of reference is right-angled.

(121) Shew that the diameter drawn through the point $(\alpha', \beta', \gamma')$ in the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

is represented by the equation

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \frac{a}{l}, & \frac{b}{m}, & \frac{c}{n} \end{vmatrix} = 0.$$

(122) The conic referred to a self-conjugate triangle and having its centre at $(\alpha', \beta', \gamma')$ is represented by the equation

$$\frac{a\alpha^2}{\alpha'} + \frac{b\beta^2}{\beta'} + \frac{c\gamma^2}{\gamma'} = 0.$$

(123) The tangents at the extremities of the chord $\alpha = \kappa\beta$ in the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

are given by the equation

$$(l\kappa\alpha + m\beta)^2 + (l\kappa^2 + m)\gamma^2.$$

(124) Find the equation to the tangents whose chord of contact is

$$f\alpha + g\beta + h\gamma = 0.$$

(125) The four straight lines of Exercise 119 are tangents to the conic

$$\frac{u^2}{m-n} + \frac{v^2}{n-l} + \frac{w^2}{l-m} = 0.$$

(126) Find the equation in triangular coordinates to the locus of the foci of all parabolas with respect to which the triangle of reference is self-conjugate.

(127) The conditions that the point of reference A should be a focus of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

are $m = n$, and $b^2 + c^2 = a^2$. In this case the triangle of reference is right-angled and the line of reference BC is the directrix corresponding to the focus A.

(128) Apply the last result to shew that the distance of any point on a conic from a focus bears a constant ratio to its distance from the corresponding directrix.

CHAPTER XV.

CONICS REFERRED TO AN INSCRIBED TRIANGLE.

186. The equation

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

represents a conic circumscribing the triangle of reference, for it is satisfied when any *two* of the coordinates are zero; and therefore each point of reference lies upon its locus.

Further, by giving suitable values to l, m, n this equation will represent *any* conic referred to an inscribed triangle (or circumscribing the triangle of reference). For by Chap. XII. every conic must have an equation of the second degree, which may be written

$$f\alpha^2 + g\beta^2 + h\gamma^2 + 2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta = 0.$$

But if the conic pass through the point of reference A, the equation must be satisfied when

$$\beta = \gamma = 0.$$

Therefore (substituting these values in the equation), we get

$$u = 0.$$

Similarly, if the conic pass through B and C, we get

$$v = 0 \text{ and } w = 0.$$

Hence the equation reduces to

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

where the values of l, m, n depend upon further conditions.

187. The equation to the conic may be written

$$l\beta\gamma + \alpha(m\gamma + n\beta) = 0.$$

Hence it is cut by the straight line $m\gamma + n\beta = 0$ in the two points where this straight line meets $\beta = 0$ and $\gamma = 0$, i.e. in two coincident points at A. Therefore $m\gamma + n\beta = 0$, or

$$\frac{\beta}{m} + \frac{\gamma}{n} = 0,$$

represents the tangent to the conic at the point of reference A.

Similarly, the tangents at the other two points of reference are given by

$$\frac{\gamma}{n} + \frac{\alpha}{l} = 0, \text{ and } \frac{\alpha}{l} + \frac{\beta}{m} = 0.$$

188. *From a given point a straight line is drawn in a given direction to meet the conic*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

it is required to find the lengths intercepted by the curve upon this straight line.

Let $(\alpha', \beta', \gamma')$ be the given point, and λ, μ, ν the sines of the given direction, then, as in Art. 171, the intercepts are given by the equation

$$l(\beta' + \mu\rho)(\gamma' + \nu\rho) + m(\gamma' + \nu\rho)(\alpha' + \lambda\rho) + n(\alpha' + \lambda\rho)(\beta' + \mu\rho) = 0,$$

which may be written

$$(l\mu\nu + m\nu\lambda + n\lambda\mu)\rho^2 + (l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta')$$
$$+ \{\lambda(m\gamma' + n\beta') + \mu(n\alpha' + l\gamma') + \nu(l\beta' + m\alpha')\}\rho = 0,$$

a quadratic giving two values for ρ, expressing the length of the two intercepts.

COR. If the point $(\alpha', \beta', \gamma')$ be on the conic, so that

$$l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta' = 0,$$

one of the intercepts is zero, and the other is given by

$$(l\mu\nu + m\nu\lambda + n\lambda\mu)\rho + \lambda(m\gamma' + n\beta') + \mu(n\alpha' + l\gamma') + \nu(l\beta' + m\alpha') = 0.$$

189. *To find the equation to the tangent at any point on the conic.*

Let λ, μ, ν be the direction sines of the tangent at $(\alpha', \beta', \gamma')$ and let (α, β, γ) be any point on the tangent, then (Chap. VI.) we have

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho\text{..................}(1),$$

and the length of the chord in this direction by the last corollary is given by

$$(l\mu\nu+m\nu\lambda+n\lambda\mu)\,\rho+\lambda\,(m\gamma'+n\beta')+\mu\,(n\alpha'+l\gamma')+\nu\,(l\beta'+m\alpha')=0.$$

But since the direction is that of the tangent, the length of the chord must be zero: therefore

$$\lambda\,(m\gamma'+n\beta')+\mu\,(n\alpha'+l\gamma')+\nu\,(l\beta'+m\alpha')=0,$$

or in virtue of (1),

$$(\alpha-\alpha')(m\gamma'+n\beta')+(\beta-\beta')(n\alpha'+l\gamma')+(\gamma-\gamma')(l\beta'+m\alpha')=0,$$

or

$$\alpha\,(m\gamma'+n\beta')+\beta\,(n\alpha'+l\gamma')+\gamma\,(l\beta'+m\alpha')=2\,(l\beta'\gamma'+m\gamma'\alpha'+n\alpha'\beta').$$

But since $(\alpha', \beta', \gamma')$ lies on the conic, we have

$$l\beta'\gamma'+m\gamma'\alpha'+n\alpha'\beta'=0\text{..................}(2),$$

and the equation becomes

$$\alpha\,(m\gamma'+n\beta')+\beta\,(n\alpha'+l\gamma')+\gamma\,(l\beta'+m\alpha')=0.$$

This equation expresses a relation among the coordinates of any point (α, β, γ) on the tangent at $(\alpha', \beta', \gamma')$ and is therefore the equation to that tangent.

COR. The equation may be written

$$\frac{\alpha}{\alpha'}\left(\frac{m}{\beta'}+\frac{n}{\gamma'}\right)+\frac{\beta}{\beta'}\left(\frac{n}{\gamma'}+\frac{l}{\alpha'}\right)+\frac{\gamma}{\gamma'}\left(\frac{l}{\alpha'}+\frac{m}{\beta'}\right)=0.$$

But (2) gives us

$$\frac{l}{\alpha'}+\frac{m}{\beta'}+\frac{n}{\gamma'}=0,$$

therefore the equation will take the form

$$\frac{\alpha}{\alpha'}\frac{l}{\alpha'}+\frac{\beta}{\beta'}\frac{m}{\beta'}+\frac{\gamma}{\gamma'}\frac{n}{\gamma'}=0,$$

or
$$\frac{l\alpha}{\alpha'^2}+\frac{m\beta}{\beta'^2}+\frac{n\gamma}{\gamma'^2}=0,$$

a form of which we shall presently give an independent investigation (Art. 198).

190. *To find the equation to the chord of contact of tangents drawn from a given point to the conic,* or, *to find the polar of a given point with respect to the conic.*

Let $(\alpha', \beta', \gamma')$ be the given point, then we may shew, precisely as in Art. 173, that the required equation is of the same form as that of the tangent at a point on the curve.

That is, the equation

$$\alpha\,(m\gamma'+n\beta')+\beta\,(n\alpha'+l\gamma')+\gamma\,(l\beta'+m\alpha')=0,$$

which, when $(\alpha', \beta', \gamma')$ is a point on the curve, represents the tangent thereat, will, when $(\alpha', \beta', \gamma')$ is a point *not* on the curve, represent the polar of that point.

191. *To find the condition that any straight line whose equation is given should be a tangent to the conic.*

Let
$$f\alpha+g\beta+h\gamma=0 \quad\text{.........................(1)}$$

be the given equation to the straight line and suppose $(\alpha', \beta', \gamma')$ its point of contact with the curve.

The tangent at $(\alpha', \beta', \gamma')$ is given by

$$\alpha\,(m\gamma'+n\beta')+\beta\,(n\alpha'+l\gamma')+\gamma\,(l\beta'+m\alpha')=0,$$

which must be identical with (1).

Therefore

$$\frac{m\gamma'+n\beta'}{f}=\frac{n\alpha'+l\gamma'}{g}=\frac{l\beta'+m\alpha'}{h},$$

or
$$\frac{\alpha'}{l(lf-mg-nh)}=\frac{\beta'}{m\,(mg-nh-lf)}=\frac{\gamma'}{n\,(nh-lf-mg)}.$$

But $(\alpha', \beta', \gamma')$ must also lie on the locus of (1), whence

$$f\alpha' + g\beta' + h\gamma' = 0.$$

Hence eliminating α', β', γ' we get

$$lf(lf - mg - nh) + mg(mg - nh - lf) + nh(nh - lf - mg) = 0,$$

or $$l^2f^2 + m^2g^2 + n^2h^2 - 2mngh - 2nlhf - 2lmfg = 0 \ldots\ldots (2),$$

which is therefore the condition required.

NOTE. The equation of condition just obtained may be written in the form

$$\sqrt{lf} + \sqrt{mg} + \sqrt{nh} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

as will be seen by clearing this latter equation of radicals, when it will be found to take the form (2).

192. COR. The equation in trilinear coordinates

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

will represent a *parabola* provided

$$\sqrt{al} + \sqrt{bm} + \sqrt{cn} = 0.$$

And the equation in triangular coordinates

$$lyz + mzx + nxy = 0,$$

will represent a *parabola* provided

$$\sqrt{l} + \sqrt{m} + \sqrt{n} = 0.$$

The remarks made in Art. 176 will apply here.

193. *To find the locus of the middle points of a series of parallel chords in the conic whose equation is*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

Let λ, μ, ν be the direction sines of the parallel chords, and let (α, β, γ) be the middle point of any one of them.

Then (Art. 188) the lengths of the intercepts measured from (α, β, γ) to the curve in the direction (λ, μ, ν) are given by the equation

$$(l\mu\nu + m\nu\lambda + n\lambda\mu)\,\rho^2 + (l\beta\gamma + m\gamma\alpha + n\alpha\beta)$$
$$+ \{\lambda\,(m\gamma + n\beta) + \mu\,(n\alpha + l\gamma) + \nu\,(l\beta + m\alpha)\}\,\rho = 0.$$

But since (α, β, γ) is the middle point of the chord, the two values of ρ, representing the intercepts, must be equal in magnitude and opposite in sign. Therefore the coefficient of ρ must vanish in the quadratic, and therefore

$$\lambda\,(m\gamma + n\beta) + \mu\,(n\alpha + l\gamma) + \nu\,(l\beta + m\alpha) = 0,$$

or $$\alpha\,(m\nu + n\mu) + \beta\,(n\lambda + l\nu) + \gamma\,(l\mu + m\lambda) = 0,$$

a relation among the coordinates α, β, γ, of the middle point of any one of the chords, and therefore the equation to the locus of the middle points.

194. *To find the centre of the conic.*

Let $(\alpha', \beta', \gamma')$ be the centre.

Then the lengths of the intercepts measured from the centre in the direction (λ, μ, ν) are given by the quadratic

$$(l\mu\nu + m\nu\lambda + n\lambda\mu)\,\rho^2 + (l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta')$$
$$+ \{\lambda\,(m\gamma' + n\beta') + \mu\,(n\alpha' + l\gamma') + \nu\,(l\beta' + m\alpha')\}\,\rho = 0 \ldots\ldots (1).$$

But since all chords are bisected in the centre, the two roots of this quadratic must be equal in magnitude and opposite in sign; therefore the coefficient of ρ must vanish, and therefore

$$\lambda\,(m\gamma' + n\beta') + \mu\,(n\alpha' + l\gamma') + \nu\,(l\beta' + m\alpha') = 0 \ldots\ldots (2),$$

for all values of $\lambda : \mu : \nu$, subject to the relation

$$a\lambda + b\mu + c\nu = 0.$$

Hence, we must have

$$\frac{m\gamma' + n\beta'}{a} = \frac{n\alpha' + l\gamma'}{b} = \frac{l\beta' + m\alpha'}{c},$$

or
$$\frac{\alpha'}{l(la-mb-nc)}=\frac{\beta'}{m(mb-nc-la)}=\frac{\gamma'}{n(nc-la-mb)}$$
$$=\frac{2\Delta}{l^2a^2+m^2b^2+n^2c^2-2mnbc-2nlca-2lmab},$$

which determine (α', β', γ') the coordinates of the centre.

N.B. If the coordinates are *triangular* instead of trilinear, we find that the centre of the conic whose equation is

$$lyz+mzx+nxy=0,$$

is given by the equations

$$\frac{x'}{l(l-m-n)}=\frac{y'}{m(m-n-l)}=\frac{z'}{n(n-l-m)}$$
$$=\frac{1}{l^2+m^2+n^2-2mn-2nl-2lm}.$$

COR. *The centre of a parabola is at infinity.*

195. *To find the conditions that the equation*

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=0,$$

should represent a circle.

Let (α', β', γ') be the centre, then the length of the semi-diameter in any direction is given by equation (1) of the last article, which in virtue of (2) reduces to

$$\rho^2(l\mu\nu+m\nu\lambda+n\lambda\mu)+l\beta'\gamma'+m\gamma'\alpha'+n\alpha'\beta'=0.$$

Hence all the diameters will be equal, provided

$$l\mu\nu+m\nu\lambda+n\lambda\mu$$

be constant for all directions.

But we know (Art. 71, page 78), that

$$\mu\nu\sin A+\nu\lambda\sin B+\lambda\mu\sin C$$

is constant for all directions. Hence the diameters will be all equal, provided

$$\frac{l}{\sin A} = \frac{m}{\sin B} = \frac{n}{\sin C},$$

which therefore express the conditions that the conic should be a circle.

Hence the equation

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0,$$

or, $$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0,$$

represents the circle circumscribing the triangle of reference.

COR. 1. In the case of this circle the equations (Art. 194) to determine the centre reduce to

$$\frac{\alpha'}{\cos A} = \frac{\beta'}{\cos B} = \frac{\gamma'}{\cos C},$$

agreeing with the result of Art. 17, Cor.

COR. 2. The equation in *triangular* coordinates to the circle which circumscribes the triangle of reference is

$$a^2yz + b^2zx + c^2xy = 0.$$

196. *To find the pole of any given straight line with respect to the conic whose equation is*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0.$$

Let $$f\alpha + g\beta + h\gamma = 0 \quad\quad (1),$$

be the equation to the given straight line, and suppose $(\alpha', \beta', \gamma')$ the pole.

The polar of this point is given by

$$\alpha(m\gamma' + n\beta') + \beta(n\alpha' + l\gamma') + \gamma(l\beta' + m\alpha') = 0,$$

which must be identical with (1).

Therefore

$$\frac{m\gamma' + n\beta'}{f} = \frac{n\alpha' + l\gamma'}{g} = \frac{l\beta' + m\alpha'}{h},$$

or $$\frac{\alpha'}{l(lf - mg - nh)} = \frac{\beta'}{m(mg - nh - lf)} = \frac{\gamma'}{n(nh - lf - mg)},$$

which determine the coordinates of the pole required.

COR. *The pole of the line at infinity is the centre of the conic.*

197. The following Theorem is attributed by Dr Salmon to M. Hermes.

If $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ *be the coordinates of any two points on the conic*

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

or $$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

the equation to the straight line joining them is

$$\frac{l\alpha}{\alpha'\alpha''} + \frac{m\beta}{\beta'\beta''} + \frac{n\gamma}{\gamma'\gamma''} = 0.$$

For this equation is satisfied at the point $(\alpha', \beta', \gamma')$, since $(\alpha'', \beta'', \gamma'')$ lies on the conic, and therefore

$$\frac{l}{\alpha''} + \frac{m}{\beta''} + \frac{n}{\gamma''} = 0.$$

So also it is satisfied at the point $(\alpha'', \beta'', \gamma'')$, since $(\alpha', \beta', \gamma')$ lies on the conic.

And it is of the first degree in α, β, γ.

Therefore it represents the straight line joining the two points $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$.

198. COR. Let the point $(\alpha'', \beta'', \gamma'')$ move up to and ultimately coincide with $(\alpha', \beta', \gamma')$: then ultimately the chord becomes the tangent at $(\alpha', \beta', \gamma')$ and the equation becomes

$$\frac{l\alpha}{\alpha'^2} + \frac{m\beta}{\beta'^2} + \frac{n\gamma}{\gamma'^2} = 0,$$

the same form of the equation to the tangent at the point $(\alpha', \beta', \gamma')$ which we deduced in the Corollary to Article 189.

199. *To find the condition that three points whose coordinates are given should lie on one conic with the three points of reference.*

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the coordinates of the given points.

Any conic passing through the points of reference may be represented by the equation

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \quad\text{.........................(1)},$$

and if it pass also through the given points we must have

$$\frac{l}{\alpha_1} + \frac{m}{\beta_1} + \frac{n}{\gamma_1} = 0 \quad\text{.........................(2)},$$

$$\frac{l}{\alpha_2} + \frac{m}{\beta_2} + \frac{n}{\gamma_2} = 0 \quad\text{.........................(3)},$$

$$\frac{l}{\alpha_3} + \frac{m}{\beta_3} + \frac{n}{\gamma_3} = 0 \quad\text{.........................(4)}.$$

Eliminating $l : m : n$ from the last three equations, we obtain

$$\begin{vmatrix} \dfrac{1}{\alpha_1}, & \dfrac{1}{\beta_1}, & \dfrac{1}{\gamma_1} \\ \dfrac{1}{\alpha_2}, & \dfrac{1}{\beta_2}, & \dfrac{1}{\gamma_2} \\ \dfrac{1}{\alpha_3}, & \dfrac{1}{\beta_3}, & \dfrac{1}{\gamma_3} \end{vmatrix} = 0,$$

which will be the condition required.

200. PASCHAL'S THEOREM. *If a hexagon be inscribed in a conic, and the pairs of opposite sides be produced to intersect, the three points of intersection are collinear.*

Let $AP_1BP_2CP_3$ be the hexagon, take ABC as triangle of reference, and let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the coordinates of P_1, P_2, P_3 respectively.

The equation to the side AP_1 is therefore

$$\frac{\beta}{\beta_1} = \frac{\gamma}{\gamma_1},$$

and the equation to the opposite side P_2C

$$\frac{\gamma}{\gamma_2} = \frac{\alpha}{\alpha_2}.$$

Hence these two sides intersect in the point given by

$$\frac{\alpha}{\alpha_2\gamma_1} = \frac{\beta}{\beta_1\gamma_2} = \frac{\gamma}{\gamma_1\gamma_2} \text{.........................} (1).$$

So the sides BP_2, P_3A intersect in the point given by

$$\frac{\alpha}{\alpha_2\alpha_3} = \frac{\beta}{\beta_3\alpha_2} = \frac{\gamma}{\gamma_2\alpha_1} \text{.........................} (2),$$

and the sides CP_3, P_1B intersect in the point given by

$$\frac{\alpha}{\alpha_3\beta_2} = \frac{\beta}{\beta_3\beta_1} = \frac{\gamma}{\gamma_1\beta_3} \text{.........................} (3).$$

But the three points represented by the equations (1), (2), (3) are collinear (Art. 20) if

$$\begin{vmatrix} \alpha_2\gamma_1, & \beta_1\gamma_2, & \gamma_1\gamma_2 \\ \alpha_2\alpha_3, & \beta_3\alpha_2, & \gamma_2\alpha_1 \\ \alpha_3\beta_2, & \beta_3\beta_1, & \gamma_1\beta_3 \end{vmatrix}$$

that is, if

$$\begin{vmatrix} \frac{1}{\alpha_3}, & \frac{1}{\beta_3}, & \frac{1}{\gamma_3} \\ \frac{1}{\alpha_1}, & \frac{1}{\beta_1}, & \frac{1}{\gamma_1} \\ \frac{1}{\alpha_2}, & \frac{1}{\beta_2}, & \frac{1}{\gamma_2} \end{vmatrix} = 0,$$

which is the condition (Art. 199) that the three points P_1, P_2, P_3 lie on the same conic with ABC.

Hence the condition that the intersections of opposite sides of a hexagon should be collinear is identical with the condition that the six angular points should lie on one conic.

This proves the proposition and its converse.

201. *Only one conic can be described passing through three given points and having its centre at another given point.*

For if we take the first three given points as points of reference for triangular coordinates, the equation to the conic may be written

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0,$$

and if (x', y', z') be the coordinates of the given centre, we have by Art. 194

$$mz' + ny' = nx' + lz' = ly' + mx',$$

whence

$$\frac{l}{x'(x' - y' - z')} = \frac{m}{y'(y' - z' - x')} = \frac{n}{z'(z' - x' - y')};$$

so that the only conic satisfying the conditions will be that represented by the equation

$$\frac{x'}{x}(x' - y' - z') + \frac{y'}{y}(y' - z' - x') + \frac{z'}{z}(z' - x' - y') = 0.$$

Obs. Since we have seen that a conic can be described so as to fulfil five simple conditions (such as passing through an assigned point) it follows that the condition of the centre being at an assigned point will count as two of these simple conditions. Such a condition may be spoken of as a double condition, or a condition of the second order.

EXERCISES ON CHAPTER XV.

(129) If
$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$
be a tangent to the conic
$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$
then the three quantities $l\lambda$, $m\mu$, $n\nu$ will be either all positive or all negative.

(130) If
$$\lambda\alpha + \mu\beta + \nu\gamma = 0$$
be a tangent to the conic
$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$
then will
$$l\alpha + m\beta + n\gamma = 0$$
be a tangent to the conic
$$\frac{\lambda}{\alpha} + \frac{\mu}{\beta} + \frac{\nu}{\gamma} = 0.$$

(131) A conic is described so as to touch in A, B, C the sides $B'C'$, $C'A'$, $A'B'$ of a triangle $A'B'C'$. Shew that AA', BB', CC' are concurrent, and that the straight line BC is divided harmonically by the straight lines AA', $B'C'$ produced if necessary.

(132) A triangle is inscribed in a conic, prove that the points are collinear in which each side intersects the tangent at the opposite angle.

(133) The six points of intersection of non-corresponding sides of a pair of co-polar triangles lie on one conic.

(134) If a triangle be self-conjugate with respect to a series of conics which all pass through a fixed point, the centres will lie on another conic which circumscribes the triangle.

(135) Determine the position of the fixed point in the last exercise in order that the locus of the centres may be the circle circumscribing the triangle.

(136) The normals at the points of reference to the conic whose equation is

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

will be concurrent, provided

$$\begin{vmatrix} l, & m, & n \\ \frac{1}{l}, & \frac{1}{m}, & \frac{1}{n} \\ \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c} \end{vmatrix} = 0.$$

(137) The equation

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

will represent a hyperbola, provided

$$a^2l^2 + b^2m^2 + c^2n^2 > 2\,(bcmn + canl + ablm).$$

(138) The tangents to the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

parallel to the line of reference BC are represented by

$$l\,(\beta + \gamma) + (\sqrt{m} \pm \sqrt{n})^2\,\alpha = 0,$$

the coordinates being triangular.

(139) The chord of contact of the tangents

$$l\,(\beta + \gamma) + (\sqrt{m} \pm \sqrt{n})^2\,\alpha = 0,$$

(whether the coordinates be trilinear or triangular) is

$$l\,(\beta - \gamma) + (m - n)\,\alpha = 0.$$

CHAPTER XVI.

CONICS REFERRED TO A CIRCUMSCRIBED TRIANGLE.

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202. THE equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \ldots\ldots (1)$$

may be written

$$(l\alpha + m\beta - n\gamma)^2 - 4lm\alpha\beta = 0,$$

and therefore (Art. 161) represents a conic section to which $\alpha = 0$ and $\beta = 0$ are tangents, and

$$l\alpha + m\beta - n\gamma = 0,$$

the chord of contact.

Similarly,

$$m\beta + n\gamma - l\alpha = 0,$$

is the chord of contact of tangents $\beta = 0$ and $\gamma = 0$, and

$$n\gamma + l\alpha - m\beta = 0,$$

the chord of contact of tangents $\gamma = 0$ and $\alpha = 0$.

Hence the equation

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \ \ldots\ldots \ (1)$$

represents a conic section, to which the lines of reference are tangents, and

$$-l\alpha + m\beta + n\gamma = 0,$$

$$l\alpha - m\beta + n\gamma = 0,$$

$$l\alpha + m\beta - n\gamma = 0,$$

the chords joining the point of contact.

203. It should be observed that if we write $-l$ for l, the equation (1) takes the form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma + 2nl\gamma\alpha + 2lm\alpha\beta = 0 \text{......} (2),$$

and the chords of contact now become

$$l\alpha + m\beta + n\gamma = 0,$$
$$l\alpha + m\beta - n\gamma = 0,$$
$$l\alpha - m\beta + n\gamma = 0.$$

So also if the equation to the conic be written

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 + 2mn\beta\gamma - 2nl\gamma\alpha + 2lm\alpha\beta = 0 \text{......} (3),$$

the chords of contact will be given by

$$l\alpha + m\beta - n\gamma = 0,$$
$$l\alpha + m\beta + n\gamma = 0,$$
$$-l\alpha + m\beta + n\gamma = 0;$$

and if the equation to the conic be written

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 + 2mn\beta\gamma + 2nl\gamma\alpha - 2lm\alpha\beta = 0 \text{......} (4),$$

the chords of contact will be given by,

$$l\alpha - m\beta + n\gamma = 0,$$
$$-l\alpha + m\beta + n\gamma = 0,$$
$$l\alpha + m\beta + n\gamma = 0.$$

Hence the four equations (1), (2), (3), (4) represent conics inscribed in the triangle of reference, and so related that all the twelve points of contact lie three and three on the four straight lines given by

$$\pm l\alpha \pm m\beta \pm n\gamma = 0.$$

This reasoning applies equally whether the coordinates be regarded as trilinear or triangular.

204. The last two articles shew that every equation of the form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 \pm 2mn\beta\gamma \pm 2nl\gamma\alpha \pm 2lm\alpha\beta = 0,$$

where we take either *one only* or *all* of the doubtful signs as negative, represents a conic inscribed in the triangle of reference. It will be observed, that if the doubtful signs be *otherwise* determined, the first member will become a perfect square and the equation will reduce to one of the forms

$$(l\alpha + m\beta + n\gamma)^2 = 0,$$

$$(-l\alpha + m\beta + n\gamma)^2 = 0,$$

$$(l\alpha - m\beta + n\gamma)^2 = 0,$$

$$(l\alpha + m\beta - n\gamma)^2 = 0.$$

In each of these cases, the locus of the equation consists of two coincident straight lines, the limiting form of a conic section when the plane of section becomes tangential to the cone along a generating line.

Such a locus will moreover meet *any* straight line in two coincident points, and will therefore, like an inscribed conic, meet each side of the triangle of reference in two coincident points. It cannot however be said to *touch* those sides in any geometrical sense.

205. Conversely, *every conic section referred to a circumscribed triangle will be represented by an equation of the form*

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 \pm 2mn\beta\gamma \pm 2nl\gamma\alpha \pm 2lm\alpha\beta = 0,$$

where the doubtful signs must be either all negative, or one negative and two positive.

For any conic section may be represented (Art. 145) by the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

But if the triangle of reference be circumscribed, $\alpha = 0$ represents a tangent, and therefore we must find two identical solutions when we combine $\alpha = 0$ with the equation to the conic.

Therefore the quadratic

$$v\beta^2 + 2u'\beta\gamma + w\gamma^2 = 0,$$

must have two equal roots.

And therefore

$$u'^2 = vw, \text{ or } u' = \pm\sqrt{vw}.$$

Similarly, since $\beta = 0$ and $\gamma = 0$ are tangents, we have

$$v' = \pm\sqrt{wu},$$

and

$$w' = \pm\sqrt{uv}.$$

Hence the equation takes the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 \pm 2\sqrt{uv}\beta\gamma \pm 2\sqrt{wu}\gamma\alpha \pm 2\sqrt{uv}\alpha\beta = 0,$$

or, writing l^2, m^2, n^2 for u, v, w,

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 \pm 2mn\beta\gamma \pm 2nl\gamma\alpha \pm 2lm\alpha\beta = 0.$$

We thus see that every conic inscribed in the triangle of reference has an equation of this form: and the doubtful signs must be either all negative or only one negative, since we found in the last Article that if they were otherwise determined, the equation would represent two coincident straight lines.

206. It will be observed that if two of the doubtful signs be positive and one negative, we can immediately make all three negative without altering the rest of the equation, *by changing the sign of one of the quantities* l, m, n. We may therefore always assume the equation to a conic referred to a circumscribed conic to be of the form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

where l, m, n may be positive or negative quantities.

It should be noticed that the equation

$$\pm\sqrt{l\alpha}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0,$$

when cleared of radicals, takes the form of the equation just written down.

So the equations (2), (3), (4) of Art. 203 are the rationalised forms of the equations

$$\pm\sqrt{-l\alpha}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0,$$

$$\pm\sqrt{l\alpha}\pm\sqrt{-m\beta}\pm\sqrt{n\gamma}=0,$$

$$\pm\sqrt{l\alpha}\pm\sqrt{m\beta}\pm\sqrt{-n\gamma}=0.$$

Thus we may always write the equation to a conic inscribed in the triangle of reference in the form

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0,$$

the coefficients l, m, n being either positive or negative, and double signs being understood before the radicals.

207. *From a given point a straight line is drawn in a given direction to meet the conic*

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0,$$

it is required to find the lengths intercepted by the curve upon this straight line.

Let $(\alpha', \beta', \gamma')$ be the given point and λ, μ, ν the sines of the given direction, then, as in Art. 171, the intercepts are given by the equation

$$l^2(\alpha'+\lambda\rho)^2+m^2(\beta'+\mu\rho)^2+n^2(\gamma'+\nu\rho)^2-2mn(\beta'+\mu\rho)(\gamma'+\nu\rho)$$
$$-2nl(\gamma'+\nu\rho)(\alpha'+\lambda\rho)-2lm(\alpha'+\lambda\rho)(\beta'+\mu\rho)=0,$$

a quadratic to determine ρ.

COR. If the point $(\alpha', \beta', \gamma')$ be on the conic, so that

$$l^2\alpha'^2 + m^2\beta'^2 + n^2\gamma'^2 - 2nm\beta'\gamma' - 2nl\gamma'\alpha' - 2lm\alpha'\beta' = 0,$$

one of the intercepts is zero, and the other is given by

$$(l^2\lambda^2 + m^2\mu^2 + n^2\nu^2 - 2mn\mu\nu - 2nl\nu\lambda - 2lm\lambda\mu)\,\rho$$
$$+ 2\,\{l\lambda\,(l\alpha' - m\beta' - n\gamma') + m\mu\,(m\beta' - n\gamma' - l\alpha') + n\nu\,(n\gamma' - l\alpha' - m\beta')\}$$
$$= 0.$$

In other words this equation gives the length of the chord drawn from $(\alpha', \beta', \gamma')$ in the direction (λ, μ, ν).

208. *To find the equation to the tangent at any point on the conic.*

Let λ, μ, ν be the direction sines of the tangent at $(\alpha', \beta', \gamma')$, and let (α, β, γ) be any point on the tangent, then (Chap. VI.) we have

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \ldots\ldots\ldots\ldots(1).$$

And the length of the chord in this direction is given by the equation of the last corollary. But since the direction is that of the tangent, the length of the chord is zero: therefore

$$l\lambda\,(l\alpha' - m\beta' - n\gamma') + m\mu\,(m\beta' - n\gamma' - l\alpha')$$
$$+ n\nu\,(n\gamma' - l\alpha' - m\beta') = 0,$$

or in virtue of (1),

$$l\,(\alpha - \alpha')\,(l\alpha' - m\beta' - n\gamma') + \&c. = 0.$$

But since $(\alpha', \beta', \gamma')$ lies on the conic, we have

$$l^2\alpha'^2 + m^2\beta'^2 + n^2\gamma'^2 - 2mn\beta'\gamma' - 2nl\gamma'\alpha' - 2lm\alpha'\beta' = 0,$$

which reduces the last equation to

$$l\alpha\,(l\alpha' - m\beta' - n\gamma') + m\beta\,(m\beta' - n\gamma' - l\alpha')$$
$$+ n\gamma\,(n\gamma' - l\alpha' - m\beta') = 0,$$

a relation among the coordinates of any point (α, β, γ) on the tangent at $(\alpha', \beta', \gamma')$, and therefore the equation to that tangent.

209. *The polar of the point* $(\alpha', \beta', \gamma')$, *or the chord of contact of tangents from that point,* may be shewn as in Art. 173 to be represented by the equation

$$l\alpha(l\alpha' - m\beta' - n\gamma') + m\beta(m\beta' - n\gamma' - l\alpha') + n\gamma(n\gamma' - l\alpha' - m\beta') = 0.$$

210. *To find the condition that any straight line whose equation is given should be a tangent to the conic.*

Let $$f\alpha + g\beta + h\gamma = 0 \quad\text{.................... (1)}$$

be the given equation to the straight line, and suppose $(\alpha', \beta', \gamma')$ its point of contact with the curve.

The tangent at $(\alpha', \beta', \gamma')$ is given by

$$l\alpha(l\alpha' - m\beta' - n\gamma') + m\beta(m\beta' - n\gamma' - l\alpha') + n\gamma(n\gamma' - l\alpha' - m\beta') = 0,$$

which must be identical with (1).

Therefore

$$\frac{l\alpha' - m\beta' - n\gamma'}{\frac{f}{l}} = \frac{m\beta' - n\gamma' - l\alpha'}{\frac{g}{m}} = \frac{n\gamma' - l\alpha' - m\beta'}{\frac{h}{n}},$$

or
$$\frac{l\alpha'}{\frac{g}{m} + \frac{h}{n}} = \frac{m\beta'}{\frac{h}{n} + \frac{f}{l}} = \frac{n\gamma'}{\frac{f}{l} + \frac{g}{m}}.$$

But $(\alpha', \beta', \gamma')$ must also lie on the locus of (1), wherefore

$$f\alpha' + g\beta' + h\gamma' = 0.$$

Hence eliminating α', β', γ', we get

$$\frac{f}{l}\left(\frac{g}{m} + \frac{h}{n}\right) + \frac{g}{m}\left(\frac{h}{n} + \frac{f}{l}\right) + \frac{h}{n}\left(\frac{f}{l} + \frac{g}{m}\right) = 0,$$

or
$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0,$$

which is therefore the condition required.

211. Cor. The equation in trilinear coordinates

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

will represent a parabola, provided

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0.$$

And the equation in triangular coordinates

$$\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$$

will represent a parabola, provided

$$l + m + n = 0.$$

212. We may shew, as in Art. 175 or 193, that the locus of the middle points of a series of parallel chords in the conic

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$$

is a straight line represented by the equation

$$l\alpha\,(l\lambda - m\mu - n\nu) + m\beta\,(m\mu - n\nu - l\lambda) + n\gamma\,(n\nu - l\lambda - m\mu) = 0,$$

where λ, μ, ν are the direction sines of the given direction, or (whether the coordinates be trilinear or triangular) where λ, μ, ν are proportional to the coordinates of the point where the parallel chords intersect at infinity.

213. *To find the centre of the conic*

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0.$$

Proceeding as in Art. 178 or 194, we find that the coordinates (α, β, γ) of the centre must satisfy the equations

$$\frac{l}{a}\,(l\alpha' - m\beta' - n\gamma') = \frac{m}{b}\,(m\beta' - n\gamma' - l\alpha') = \frac{n}{c}\,(n\gamma' - l\alpha' - m\beta'),$$

or

$$\frac{\alpha'}{bn+cm} = \frac{\beta'}{cl+an} = \frac{\gamma'}{am+bl}$$

$$= \frac{\Delta}{lbc + mca + nab},$$

which express the coordinates explicitly.

OBS. If the coordinates are *triangular* instead of *trilinear*, we find that the centre of the conic whose equation is

$$\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0,$$

is given by the equations

$$\frac{x'}{m+n}=\frac{y'}{n+l}=\frac{z'}{l+m}=\frac{1}{2(l+m+n)}.$$

214. COR. In order that the centre may be at the point $\alpha=\beta=\gamma$, which we have seen (Art. 16), is the centre of the inscribed circle, we must have

$$bn+cm=cl+an=am+bl,$$

whence
$$\frac{bcl}{b+c-a}=\frac{cam}{c+a-b}=\frac{abn}{a+b-c},$$

or
$$\frac{l}{\cos^2\frac{A}{2}}=\frac{m}{\cos^2\frac{B}{2}}=\frac{n}{\cos^2\frac{C}{2}},$$

hence the equation

$$\sqrt{\alpha}\cos\frac{A}{2}+\sqrt{\beta}\cos\frac{B}{2}+\sqrt{\gamma}\cos\frac{C}{2}=0$$

is the only equation which can represent an inscribed conic having its centre at the assigned point; therefore there is only one such conic, namely, the inscribed circle, and the equation

$$\sqrt{\alpha}\cos\frac{A}{2}+\sqrt{\beta}\cos\frac{B}{2}+\sqrt{\gamma}\cos\frac{C}{2}=0,$$

represents that circle.

Similarly, we can shew that the escribed circles having their centres at the points

$$-\alpha=\beta=\gamma,$$
$$\alpha=-\beta=\gamma,$$
$$\alpha=\beta=-\gamma,$$

are represented respectively by the equations

$$\sqrt{-\alpha}\cos\frac{A}{2} + \sqrt{\beta}\sin\frac{B}{2} + \sqrt{\gamma}\sin\frac{C}{2} = 0,$$

$$\sqrt{\alpha}\sin\frac{A}{2} + \sqrt{-\beta}\cos\frac{B}{2} + \sqrt{\gamma}\sin\frac{C}{2} = 0,$$

$$\sqrt{\alpha}\sin\frac{A}{2} + \sqrt{\beta}\sin\frac{B}{2} + \sqrt{-\gamma}\cos\frac{C}{2} = 0.$$

215. *To find the pole of any given straight line with respect to the conic*

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0.$$

Let $$f\alpha + g\beta + h\gamma = 0 \quad\text{..........................(1)}$$

be the equation to the given straight line, and suppose $(\alpha', \beta', \gamma')$ the pole.

The polar of this point is given by

$$l\alpha\,(l\alpha' - m\beta' - n\gamma') + m\beta\,(m\beta' - n\gamma' - l\alpha') + n\gamma\,(n\gamma' - l\alpha' - m\beta') = 0 \quad\text{..............................(2)},$$

which must be identical with (1).

Therefore

$$\frac{f}{l\,(l\alpha' - m\beta' - n\gamma')} = \frac{g}{m\,(m\beta' - n\gamma' - l\alpha')} = \frac{h}{n\,(n\beta' - l\alpha' - m\beta')},$$

or $$\frac{\alpha'}{ng + mh} = \frac{\beta'}{lh + nf} = \frac{\gamma'}{mf + lg},$$

which determine the coordinates of the pole required.

216. If we assume the geometrical property that every conic has a pair of foci situated at equal distances on opposite sides of the centre, and such that the rectangle contained by the perpendiculars from them upon any tangent is constant, we can

readily write down equations to give the trilinear coordinates of the foci of the conic

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0.$$

For let (α, β, γ) be the coordinates of a focus, and let $(\alpha', \beta', \gamma')$ be those of the centre.

Then, since the centre bisects the line joining the foci, the sum of the two values of α is the double of α' (Art. 18. Cor.).

But since the line of reference BC is a tangent, the rectangle represented by the product of the two values of α is equal to a constant, k^2 suppose.

Therefore the two values of α are the roots of the quadratic

$$\alpha^2 - 2\alpha'\alpha + k^2 = 0.$$

But similarly β and γ are given by the quadratics

$$\beta^2 - 2\beta'\beta + k^2 = 0,$$

and $$\gamma^2 - 2\gamma'\gamma + k^2 = 0.$$

Hence $$\alpha^2 - 2\alpha'\alpha = \beta^2 - 2\beta'\beta = \gamma^2 - 2\gamma'\gamma,$$

or substituting the values of α', β', γ' (Art. 213),

$$(bcl + cam + abn)\, \alpha^2 - 2\Delta\, (bn + cm)\, \alpha$$
$$= (bcl + cam + abn)\, \beta^2 - 2\Delta\, (cl + an)\, \beta$$
$$= (bcl + cam + abn)\, \gamma^2 - 2\Delta\, (am + bl)\, \gamma,$$

or $$\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right) \alpha^2 - \frac{2\Delta\alpha}{a} \left(\frac{m}{b} + \frac{n}{c}\right)$$
$$= \left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right) \beta^2 - \frac{2\Delta\beta}{b} \left(\frac{n}{c} + \frac{l}{a}\right)$$
$$= \left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right) \gamma^2 - \frac{2\Delta\gamma}{c} \left(\frac{l}{a} + \frac{m}{b}\right),$$

equations to determine the two values of α, β, γ for the two foci.

Obs. In *triangular* coordinates, the foci of the conic

$$\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0,$$

are given by

$$\frac{1}{a^2}\left(x^2-\frac{m+n}{l+m+n}x\right)=\frac{1}{b^2}\left(y^2-\frac{n+l}{l+m+n}y\right)$$
$$=\frac{1}{c^2}\left(z^2-\frac{l+m}{l+m+n}z\right).$$

x, y, z are absolute coords. of focus. (see p 95)

217. *To find the condition that it should be possible to find a conic touching the three straight lines of reference and three other given straight lines.*

Let
$$f_1\alpha+g_1\beta+h_1\gamma=0 \dots\dots\dots (1),$$
$$f_2\alpha+g_2\beta+h_2\gamma=0 \dots\dots\dots (2),$$
$$f_3\alpha+g_3\beta+h_3\gamma=0 \dots\dots\dots (3),$$

be the three given straight lines.

Any conic inscribed in the triangle of reference will be represented by the equation

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0.$$

If the straight line (1) be a tangent we must have (Art. 210)

$$\frac{l}{f_1}+\frac{m}{g_1}+\frac{n}{h_1}=0 \dots\dots\dots (4).$$

Similarly if (2) and (3) be tangents,

$$\frac{l}{f_2}+\frac{m}{g_2}+\frac{n}{h_2}=0 \dots\dots\dots (5),$$

$$\frac{l}{f_3}+\frac{m}{g_3}+\frac{n}{h_3}=0 \dots\dots\dots (6).$$

Hence, eliminating $l : m : n$ from the equations (4), (5), (6), we have

$$\begin{vmatrix} \frac{1}{f_1}, & \frac{1}{g_1}, & \frac{1}{h_1} \\ \frac{1}{f_2}, & \frac{1}{g_2}, & \frac{1}{h_2} \\ \frac{1}{f_3}, & \frac{1}{g_3}, & \frac{1}{h_3} \end{vmatrix} = 0,$$

which will be the condition required.

218. BRIANCHON'S THEOREM. *If a hexagon be described about a conic section, the three diagonals formed by joining opposite angular points will be concurrent.*

Let $PQ'RP'QR'$ be the hexagon.

Take three alternate sides, QR', RP', PQ' produced, for lines of reference, and let the equations of the other sides be

$$(Q'R),\ f_1\alpha + g_1\beta + h_1\gamma = 0 \text{ .... } (1),$$

$$(R'P),\ f_2\alpha + g_2\beta + h_2\gamma = 0 \text{ .... } (2),$$

$$(P'Q),\ f_3\alpha + g_3\beta + h_3\gamma = 0 \text{ .... } (3).$$

Then P is given by $\gamma = 0,\ f_2\alpha + g_2\beta = 0$,

and P' is given by $\beta = 0,\ f_3\alpha + h_3\gamma = 0$,

therefore the diagonal PP' is represented by the equation

$$f_2f_3\alpha + f_3g_2\beta + f_2h_3\gamma = 0,$$

so QQ' will have the equation

$$g_3f_1\alpha + g_3g_1\beta + g_1h_3\gamma = 0,$$

and RR' will have the equation

$$h_2f_1\alpha + h_1g_2\beta + h_1h_2\gamma = 0.$$

Hence the condition of concurrence of the three diagonals is

$$\begin{vmatrix} f_2f_3, & f_3g_2, & f_2h_3 \\ g_3f_1, & g_3g_1, & g_1h_3 \\ h_2f_1, & h_1g_2, & h_1h_2 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \frac{1}{f_1}, & \frac{1}{f_2}, & \frac{1}{f_3} \\ \frac{1}{g_1}, & \frac{1}{g_2}, & \frac{1}{g_3} \\ \frac{1}{h_1}, & \frac{1}{h_2}, & \frac{1}{h_3} \end{vmatrix} = 0,$$

which is the condition that the three straight lines (1), (2), (3) should touch the same conic with the lines of reference; which proves the proposition and its converse.

Exercises on Chapter XVI.

(140) The conics

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

and

$$l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0,$$

intersect in the points of reference and in the point given by

$$\alpha \begin{vmatrix} m, & n \\ m', & n' \end{vmatrix} = \beta \begin{vmatrix} n, & l \\ n', & l' \end{vmatrix} = \gamma \begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$$

(141) The straight line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ will be cut harmonically by the conics

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

and

$$l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0,$$

provided

$$ll'\lambda^2 + mm'\mu^2 + nn'\nu^2 - (mn' + m'n)\,\mu\nu - (nl' + n'l)\,\nu\lambda - (lm' + l'm)\,\lambda\mu = 0.$$

(142) The imaginary triangle whose sides are

$$u + \sqrt{-1}\,v = 0, \quad v + \sqrt{-1}\,w = 0, \quad w + \sqrt{-1}\,u = 0$$

is self-conjugate with respect to the conic

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 0.$$

(143) A triangle ABC is inscribed in a conic, and from each angular point straight lines are drawn parallel to the opposite sides to meet the conic again in P, Q, R; prove that QR, RP, PQ are parallel to the tangents at A, B, C.

(144) The triangle whose sides are

$$f_1\alpha + g_1\beta + h_1\gamma = 0,$$
$$f_2\alpha + g_2\beta + h_2\gamma = 0,$$
$$f_3\alpha + g_3\beta + h_3\gamma = 0,$$

will be self-conjugate with respect to the conic

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

provided

$$\left\|\begin{matrix} 0, & 0, & 0, & l, & m, & n, \\ f_1^2, & f_2^2, & f_3^2, & g_1h_1, & g_2h_2, & g_3h_3 \\ g_1^2, & g_2^2, & g_3^2, & h_1f_1, & h_2f_2, & h_3f_3 \\ h_1^2, & h_2^2, & h_3^2, & f_1g_1, & f_2g_2, & f_3g_3 \end{matrix}\right\| = 0.$$

(145) Interpret the equation in trilinear coordinates

$$\sqrt{a\alpha} + \sqrt{b\beta} + \sqrt{-c\gamma} = 0,$$

and find the coordinates of the foci of its locus.

(146) If a parabola touch the sides of a triangle its focus will lie on the circle which circumscribes the triangle.

INTRODUCTION TO CHAPTER XVII.

NOTATION, ETC.

219. STUDENTS who have not read the Differential Calculus are recommended to pay particular attention to the notation which we now introduce. Those who have read the Differential Calculus will accept it without explanation.

Let $f(x)$ denote any function whatever of x. Then the symbol $\frac{df(x)}{dx}$ (which must be regarded as a single expression not capable of resolution into numerator and denominator) is used to denote the expression derived from $f(x)$ by substituting for every power of x (suppose x^n), the next lower power multiplied by the original index (i.e. nx^{n-1}), and omitting altogether the terms which do not involve x.

Thus x will be replaced by x^0 or 1, x^2 by $2x$, x^4 by $4x^3$, and so on.

For example, if $f(x)$ denote $x^3 + 3ax^2 + 3a^2x + a^3$, then $\frac{df(x)}{dx}$ will denote $3x^2 + 6ax + 3a^2$.

So also, if $f(\alpha, \beta, \gamma)$ be any function of α, β, γ, then $\frac{df(\alpha, \beta, \gamma)}{d\alpha}$ denotes the result obtained by substituting for the powers of α according to the law enunciated above, and neglecting the terms in the original which do not involve α.

So $\frac{df(\alpha, \beta, \gamma)}{d\beta}$ denotes the result obtained by neglecting the terms which do not involve β, and substituting for the powers of β in the other terms.

It is usual when the abbreviation can be made without ambiguity to write $\frac{df}{dx}$ for $\frac{df(x)}{dx}$ and $\frac{df}{d\alpha}$ for $\frac{df(\alpha, \beta, \gamma)}{d\alpha}$.

220. The expression $\frac{df}{d\alpha}$ is called *the derived function with respect to* α of the original expression $f(\alpha, \beta, \gamma)$. So the expression $\frac{df}{d\beta}$ is called *the derived function with respect to* β, and $\frac{df}{d\gamma}$ *the derived function with respect to* γ.

Also if $f(\alpha, \beta, \gamma) = 0$ be an *equation* involving α, β, γ, then

$$\frac{df}{d\alpha} = 0$$

is called *the derived equation with respect to* α, and so on.

221. Let $f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta$, then

$$\left.\begin{aligned} \frac{df}{d\alpha} &\equiv 2u\alpha + 2w'\beta + 2v'\gamma \\ \frac{df}{d\beta} &\equiv 2v\beta + 2u'\gamma + 2w'\alpha \\ \frac{df}{d\gamma} &\equiv 2w\gamma + 2v'\alpha + 2u'\beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

Hence $$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma}$$

$$= 2\{u\alpha\alpha' + v\beta\beta' + w\gamma\gamma' + u'(\beta\gamma' + \beta'\gamma) + v'(\gamma'\alpha + \gamma\alpha') + w'(\alpha\beta' + \alpha'\beta)\}.$$

And this expression is not altered if α', β', γ' be interchanged with α, β, γ respectively.

Hence if

$$\frac{df}{d\alpha'}, \quad \frac{df}{d\beta'}, \quad \frac{df}{d\gamma'}$$

denote the derived functions of $f(\alpha', \beta', \gamma')$, then

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = \alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} \text{ ....(ii).}$$

Again from (i) we get

$$\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} = 2\{u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma$$
$$+ 2v'\gamma\alpha + 2w'\alpha\beta\} = 2f(\alpha, \beta, \gamma) \text{...(iii).}$$

Again, $f(\alpha + x, \ \beta + y, \ \gamma + z)$

$$\equiv u(\alpha + x)^2 + v(\beta + y)^2 + w(\gamma + z)^2 + 2u'(\beta + y)(\gamma + z)$$
$$+ 2v'(\gamma + z)(\alpha + x) + 2w'(\alpha + x)(\beta + y)$$
$$\equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta$$
$$+ 2x(u\alpha + v'\gamma + w'\beta) + 2y(v\beta + u'\gamma + w'\alpha) + 2z(w\gamma + u'\beta + v'\alpha)$$
$$+ ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy$$
$$\equiv f(\alpha, \beta, \gamma) + x\frac{df}{d\alpha} + y\frac{df}{d\beta} + z\frac{df}{d\gamma} + f(x, y, z) \text{ ....(iv).}$$

If we write $\lambda\rho$, $\mu\rho$, $\nu\rho$ for x, y, z, the result (iv) becomes

$$f(\alpha + \lambda\rho, \ \beta + \mu\rho, \ \gamma + \nu\rho)$$
$$\equiv f(\alpha, \beta, \gamma) + \rho\left(\lambda\frac{df}{d\alpha} + \mu\frac{df}{d\beta} + \nu\frac{df}{d\gamma}\right) + \rho^2 f(\lambda, \mu, \nu) \text{....(v).}$$

The results given in the equations (ii), (iii), (iv), (v) are of the very highest importance.

222. *To express the homogeneous function of the second degree*

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta$$

in terms of its derived functions

$$\frac{df}{d\alpha}, \quad \frac{df}{d\beta}, \quad \frac{df}{d\gamma}.$$

We have (Art. 221, equation iii)

$$2f(\alpha, \beta, \gamma) \equiv \alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} \quad \ldots\ldots\ldots\ldots(1),$$

and (equations i)

$$\frac{1}{2}\frac{df}{d\alpha} \equiv u\alpha + w'\beta + v'\gamma \quad \ldots\ldots\ldots\ldots(2),$$

$$\frac{1}{2}\frac{df}{d\beta} \equiv w'\alpha + v\beta + u'\gamma \quad \ldots\ldots\ldots\ldots(3),$$

$$\frac{1}{2}\frac{df}{d\gamma} \equiv v'\alpha + u'\beta + w\gamma \quad \ldots\ldots\ldots\ldots(4).$$

Therefore eliminating α, β, γ, from (1), (2), (3), (4), we get

$$\begin{vmatrix} 4f(\alpha, \beta, \gamma), & \frac{df}{d\alpha}, & \frac{df}{d\beta}, & \frac{df}{d\gamma} \\ \frac{df}{d\alpha}, & u, & w', & v' \\ \frac{df}{d\beta}, & w', & v, & u' \\ \frac{df}{d\gamma}, & v', & u', & w \end{vmatrix} \equiv 0,$$

or

$$4\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} f(\alpha, \beta, \gamma) \equiv - \begin{vmatrix} 0, & \frac{df}{d\alpha}, & \frac{df}{d\beta}, & \frac{df}{d\gamma} \\ \frac{df}{d\alpha}, & u, & w', & v' \\ \frac{df}{d\beta}, & w', & v, & u' \\ \frac{df}{d\gamma}, & v', & u', & w \end{vmatrix}$$

$$\equiv U\left(\frac{df}{d\alpha}\right)^2 + V\left(\frac{df}{d\beta}\right)^2 + W\left(\frac{df}{d\gamma}\right)^2$$

$$+ 2U'\frac{df}{d\beta}\frac{df}{d\gamma} + 2V'\frac{df}{d\gamma}\frac{df}{d\alpha} + 2W'\frac{df}{d\alpha}\frac{df}{d\beta},$$

where

$$U = vw - u'^2, \quad V = wu - v'^2, \quad W = uv - w'^2,$$

$$U' = v'w' - uu', \quad V' = w'u' - vv', \quad W' = u'v' - ww',$$

and so

$$f(\alpha, \beta, \gamma)$$

$$\equiv \frac{U\left(\frac{df}{d\alpha}\right)^2 + V\left(\frac{df}{d\beta}\right)^2 + W\left(\frac{df}{d\gamma}\right)^2 + 2U'\frac{df}{d\beta}\frac{df}{d\gamma} + 2V'\frac{df}{d\gamma}\frac{df}{d\alpha} + 2W'\frac{df}{d\alpha}\frac{df}{d\beta}}{4\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}}.$$

Thus every homogeneous function of the second degree of three quantities may be expressed as a homogeneous function of the second degree of its three derived functions.

CHAPTER XVII.

THE GENERAL EQUATION OF THE SECOND DEGREE.

223. Having considered in the last three chapters some special forms of the equation to a conic section, and thereby rendered the student who reads the subject for the first time familiar with the methods of treating equations of the second degree, we pass on to consider the most general case, when the conic is represented by an equation of the most general form.

But for the sake of the reader who needs not to be thus led up to the more difficult part of the subject, but prefers to investigate first the most general form of the equation and thence to deduce the particular cases, we shall make the present chapter perfectly independent of the three preceding, so that it may be read consecutively after Chapter XIII.

To this end we shall be obliged in this and the succeeding chapter to repeat some explanations and definitions which have been already given in the treatment of the special cases; but, our object being thus explained, the repetition will doubtless be excused.

224. The equation of the second degree in its most general form may be written

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

But when the coefficients of the several terms have not to be separately discussed, we shall generally denote the first member by the symbol $f(\alpha, \beta, \gamma)$ and write the equation

$$f(\alpha, \beta, \gamma) = 0.$$

225. *To find the equation to the tangent at any point on a conic section.*

Let $$f(\alpha, \beta, \gamma) = 0 \text{..........................(i)},$$

be the equation to the curve, and $(\alpha', \beta', \gamma')$ the coordinates of the given point upon it, then any straight line through the given point will be represented by the equation

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \text{......................(ii)},$$

and the lengths which the curve intercepts on this straight line will be given by

$$f(\alpha' + \lambda\rho,\ \beta' + \mu\rho,\ \gamma' + \nu\rho) = 0,$$

or, $$f(\alpha', \beta', \gamma') + \rho\left(\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}\right) + \rho^2 f(\lambda, \mu, \nu) = 0 \text{...(iii)}.$$

Since $(\alpha', \beta', \gamma')$ lies on the curve, therefore $f(\alpha', \beta', \gamma') = 0$, and one of the values of ρ is zero, as we should expect. But if the straight line be a tangent the two values of ρ must be equal, that is, each must be zero.

Hence the coefficient of ρ must also vanish in the quadratic (iii),

therefore $$\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'} = 0,$$

or since λ, μ, ν are proportional to $\alpha - \alpha'$, $\beta - \beta'$, $\gamma - \gamma'$ by (ii) we get

$$(\alpha - \alpha') \frac{df}{d\alpha'} + (\beta - \beta') \frac{df}{d\beta'} + (\gamma - \gamma') \frac{df}{d\gamma'} = 0.$$

But $$\alpha' \frac{df}{d\alpha'} + \beta' \frac{df}{d\beta'} + \gamma' \frac{df}{d\gamma'} \equiv 2f(\alpha', \beta', \gamma') \text{ (Art. 221)},$$

$$= 0,$$

since $(\alpha', \beta', \gamma')$ is on the curve; therefore

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

a relation among the coordinates of any point on the tangent at $(\alpha', \beta', \gamma')$ and therefore the equation to the tangent at that point.

Obs. If the equation to the conic be written

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

the equation to the tangent is

$$\alpha(u\alpha' + w'\beta' + v'\gamma') + \beta(v\beta' + u'\gamma' + w'\alpha') + \gamma(w\gamma' + v'\alpha' + u'\beta') = 0.$$

226. Cor. The normal at $(\alpha', \beta', \gamma')$ will be given by the equations, (Art. 81).

$$\frac{\alpha - \alpha'}{\frac{df}{d\alpha'} - \frac{df}{d\beta'}\cos C - \frac{df}{d\gamma'}\cos B} = \frac{\beta - \beta'}{\frac{df}{d\beta'} - \frac{df}{d\gamma'}\cos A - \frac{df}{d\alpha'}\cos C}$$
$$= \frac{\gamma - \gamma'}{\frac{df}{d\gamma'} - \frac{df}{d\alpha'}\cos B - \frac{df}{d\beta'}\cos A}.$$

if the coordinates are *trilinear*; or, if they be *triangular*, (Art. 87) by the equations

$$\frac{\alpha - \alpha'}{\frac{df}{d\alpha'}a^2 - \frac{df}{d\beta'}ab\cos C - \frac{df}{d\gamma'}ac\cos B} = \frac{\beta - \beta'}{\frac{df}{d\beta'}b^2 - \frac{df}{d\gamma'}bc\cos A - \frac{df}{d\alpha'}ab\cos C}$$
$$= \frac{\gamma - \gamma'}{\frac{df}{d\gamma'}c^2 - \frac{df}{d\alpha'}ca\cos B - \frac{df}{d\beta'}bc\cos A}.$$

227. *On the determination of Direction.*

Any two straight lines drawn in the same direction are parallel and have their point of intersection at infinity.

Conversely any straight lines which intersect in a point at infinity are in the same direction.

Hence every point at infinity determines a direction, and every direction may be determined by assigning the point at infinity in which straight lines drawn in that direction intersect.

It has already been remarked (page 101) that if we represent a straight line by equations of the forms

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}\text{..................}\text{(i)}$$

then λ, μ, ν will be proportional to the coordinates of the point where the straight line meets infinity, or the point where a system of straight lines parallel to the given one will intersect.

The quantities λ, μ, ν therefore determine the direction of these straight lines, and we shall henceforth speak of such a direction as the direction (λ, μ, ν), where we suppose λ, μ, ν to have such actual values as shall make each of the fractions in (i), equal to the distance ρ between the points (α, β, γ) and $(\alpha', \beta', \gamma')$.

If λ, μ, ν all have values only proportional to these values, we shall speak of the direction as the direction $(\lambda : \mu : \nu)$ instead of (λ, μ, ν). See Art. 69, *ad fin.*

When the direction $(\lambda : \mu : \nu)$ is spoken of, it must be borne in mind that λ, μ, ν satisfy the relation in *trilinear* coordinates

$$a\lambda+b\mu+c\nu=0,$$

or in *triangular* coordinates

$$\lambda+\mu+\nu=0,$$

and if the direction (λ, μ, ν) be referred to, then λ, μ, ν satisfy not only the former relation but also the non-homogeneous relations of page 101 (xx).

228. Let O be a point in which a conic section meets infinity. Any straight line drawn in the direction determined by O meets the conic in this point at infinity and therefore in one and only one finite point (Art. 150.) The number of real

directions in which straight lines can be drawn so as to cut a conic in only one real point, will thus be the same as the number of real points in which the conic meets the straight line at infinity; there is therefore one such direction for a parabola, two for a hyperbola, and none for an ellipse (Art. 151). Further, if the tangent at O lie not altogether at infinity it will be an asymptote (Art. 152). Hence in the hyperbola, any straight line parallel to an asymptote will meet the curve in only one real point, and the two asymptotes determine the two directions of such lines.

229. *To determine the direction of the tangent at any point on a conic section.*

Let $f(\alpha, \beta, \gamma) = 0$ be the equation to the given curve, and suppose λ, μ, ν the direction sines of the tangent at $(\alpha', \beta', \gamma')$.

Then the equations to the tangent will be

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho,$$

and the lengths which the curve intercepts on this line will be given by the equation

$$f(\alpha' + \lambda\rho, \quad \beta' + \mu\rho, \quad \gamma' + \nu\rho) = 0,$$

or

$$f(\alpha', \beta', \gamma') + \rho\left(\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}\right) + \rho^2 f(\lambda, \mu, \nu) = 0.$$

Since $(\alpha', \beta', \gamma')$ is on the curve, therefore $f(\alpha', \beta', \gamma') = 0$, and one of the values of ρ given by this quadratic is zero, as we should expect. But since further the straight line touches the curve at this point, we must have both values of ρ zero.

Hence $$\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'} = 0.$$

This equation, together with the relation

$$\lambda a + \mu b + \nu c = 0$$

in *trilinear* coordinates, or

$$\lambda + \mu + \nu = 0$$

in *triangular* coordinates, gives us

$$\frac{\lambda}{\begin{vmatrix} \frac{df}{d\beta'}, & \frac{df}{d\gamma'} \\ b, & c \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \frac{df}{d\gamma'}, & \frac{df}{d\alpha'} \\ c, & a \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \frac{df}{d\alpha'}, & \frac{df}{d\beta'} \\ a, & b \end{vmatrix}}$$

in *trilinear* coordinates, or

$$\frac{\lambda}{\frac{df}{d\beta'} - \frac{df}{d\gamma'}} = \frac{\mu}{\frac{df}{d\gamma'} - \frac{df}{d\alpha'}} = \frac{\nu}{\frac{df}{d\alpha'} - \frac{df}{d\beta'}}$$

in *triangular* coordinates,—which determine the ratios of λ, μ, ν, and their actual values are immediately given by one of the non-homogeneous equations of Result xx. page 101.

230. *From any point there can be drawn two real or imaginary tangents to meet any conic.*

Let $$f(\alpha, \beta, \gamma) = 0 \quad \text{.........................(1)}$$

be the equation to the conic, and $(\alpha', \beta', \gamma')$ the given point.

Suppose

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \quad \text{............... (2)}$$

the equations of a tangent to the curve, and suppose $(\alpha'', \beta'', \gamma'')$ its point of contact.

The intercepts measured from $(\alpha'', \beta'', \gamma'')$ to the curve must both be zero.

Hence the equation

$$f(\alpha'' + \lambda\rho, \quad \beta'' + \mu\rho, \quad \gamma'' + \nu\rho) = 0$$

must have both its roots $\rho = 0$;

therefore $$f(\alpha'', \beta'', \gamma'') = 0 \quad \text{........................(3)}$$

and $$\lambda \frac{df}{d\alpha''} + \mu \frac{df}{d\beta''} + \nu \frac{df}{d\gamma''} = 0 \quad \text{.................. (4).}$$

But since $(\alpha'', \beta'', \gamma'')$ lies on the straight line (2), therefore

$$\frac{\alpha''-\alpha'}{\lambda}=\frac{\beta''-\beta'}{\mu}=\frac{\gamma''-\gamma'}{\nu}.$$

Hence the equation (4) becomes

$$(\alpha''-\alpha')\frac{df}{d\alpha''}+(\beta''-\beta')\frac{df}{d\beta''}+(\gamma''-\gamma')\frac{df}{d\gamma''}=0 \quad \text{.......(5)}.$$

But by the property of homogeneous functions (Art. 221), the equation (3) gives us

$$\alpha''\frac{df}{d\alpha''}+\beta''\frac{df}{d\beta''}+\gamma''\frac{df}{d\gamma''}=0 \quad \text{.................(6)}.$$

Hence, subtracting (5) and (6),

$$\alpha'\frac{df}{d\alpha''}+\beta'\frac{df}{d\beta''}+\gamma'\frac{df}{d\gamma''}=0,$$

or, which is the same thing (Art. 221),

$$\alpha''\frac{df}{d\alpha'}+\beta''\frac{df}{d\beta'}+\gamma''\frac{df}{d\gamma'}=0 \quad \text{.................(7)}.$$

Hence the coordinates $(\alpha'', \beta'', \gamma'')$ of the point of contact of any tangent from $(\alpha', \beta', \gamma')$ to the curve, are obtained by solving simultaneously the quadratic equation (3) and the simple equation (7).

Hence there will be *two* real or imaginary solutions.

Therefore from any point there can be drawn *two* real or imaginary tangents to a conic section.

231. *To find the equation to the chord of contact of tangents drawn from a given point to a conic section.*

Let $(\alpha', \beta', \gamma')$ be the given point, and

$$f(\alpha, \beta, \gamma)=0$$

the equation to the conic.

Then if $(\alpha'', \beta'', \gamma'')$ be the point of contact of either tangent from $(\alpha', \beta', \gamma')$ to the curve, we have by equation (7) of Art. 230,

$$\alpha'' \frac{df}{d\alpha'} + \beta'' \frac{df}{d\beta'} + \gamma'' \frac{df}{d\gamma'} = 0.$$

Hence the coordinates of either point of contact satisfy the equation

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0 \quad \text{.................(8).}$$

But this is the equation to a straight line. Hence it represents the straight line through the two points of contact, that is, *the chord of contact* of tangents from $(\alpha', \beta', \gamma')$ to the conic.

It will be observed that the equation (8) represents a real straight line wherever $(\alpha', \beta', \gamma')$ be situated, i.e. the chord of contact is real whether the tangents be real or imaginary.

232. Def. The chord of contact of the real or imaginary tangents from a fixed point to a conic is called the *polar* of the fixed point with respect to the conic.

And the fixed point is called the *pole* of the straight line with respect to the conic.

We have shewn in the last article that the polar of the conic

$$f(\alpha, \beta, \gamma) = 0,$$

with respect to the point $(\alpha', \beta', \gamma')$, is represented by the equation

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0.$$

If the equation to the conic be written

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

the equation to the polar becomes

$$\alpha(u\alpha' + w'\beta' + v'\gamma') + \beta(v\beta' + u'\gamma' + w'\alpha') + \gamma(w\gamma' + v'\alpha' + u'\beta') = 0.$$

233. *To find the coordinates of the pole of any straight line with respect to a conic section.*

Let $$l\alpha + m\beta + n\gamma = 0 \qquad (1)$$

be the equation to the straight line, and

$$f(\alpha, \beta, \gamma) = 0$$

the equation to the conic.

Suppose $(\alpha', \beta', \gamma')$ the coordinates of the point required, then the polar of $(\alpha', \beta', \gamma')$ with respect to the conic is given by the equation

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

which must therefore be identical with (1).

Hence $$\frac{1}{l}\frac{df}{d\alpha'} = \frac{1}{m}\frac{df}{d\beta'} = \frac{1}{n}\frac{df}{d\gamma'},$$

or $$\frac{u\alpha' + w'\beta' + v'\gamma'}{l} = \frac{v\beta' + u'\gamma' + w'\alpha'}{m} = \frac{w\gamma' + v'\alpha' + u'\beta'}{n}.$$

Therefore

$$\frac{\alpha'}{\begin{vmatrix} l, & m, & n \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}} = \frac{\beta'}{\begin{vmatrix} l, & m, & n \\ v', & u', & w \\ u, & w', & v' \end{vmatrix}} = \frac{\gamma'}{\begin{vmatrix} l, & m, & n \\ u, & w', & v', \\ w', & v, & u', \end{vmatrix}} \qquad (2),$$

and if the coordinates be trilinear so that

$$a\alpha' + b\beta' + c\gamma' = 2\Delta,$$

each of these fractions

$$= \frac{-2\Delta}{\begin{vmatrix} 0, & l, & m, & n \\ a, & u, & w', & v' \\ b, & w', & v, & u' \\ c, & v', & u', & w \end{vmatrix}},$$

or if they be triangular so that

$$\alpha' + \beta' + \gamma' = 1,$$

then each of the equal fractions in (2)

$$= \frac{-1}{\begin{vmatrix} 0, & l, & m, & n \\ 1, & u, & w', & v' \\ 1, & w', & v, & u' \\ 1, & v', & u', & w \end{vmatrix}}.$$

Thus the actual values of the required coordinates are expressed.

Cor. The pole of the straight line

$$l\alpha + m\beta + n\gamma = 0$$

will lie upon this straight line provided

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0.$$

This is therefore the condition that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

should be a *tangent* to the conic.

234. *If a point P lie upon the polar of a point Q, the point Q will lie upon the polar of the point P.*

For let $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be any two points P and Q.

The equations to their polars are

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

$$\alpha \frac{df}{d\alpha''} + \beta \frac{df}{d\beta''} + \gamma \frac{df}{d\gamma''} = 0.$$

If P lie on the polar of Q, we have

$$\alpha' \frac{df}{d\alpha''} + \beta' \frac{df}{d\beta''} + \gamma' \frac{df}{d\gamma''} = 0,$$

which is the same thing (Art. 221) as

$$\alpha'' \frac{df}{d\alpha'} + \beta'' \frac{df}{d\beta'} + \gamma'' \frac{df}{d\gamma'} = 0,$$

which is the condition that Q should lie on the polar of P.

∴ &c. Q.E.D.

235. *If a straight line p pass through the pole of a straight line q, the straight line q will pass through the pole of the straight line p.*

For if the equations to the two straight lines be

$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (p),$$

$$l'\alpha + m'\beta + n'\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (q),$$

then, by Art. 233, the equation

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l' & m' & n', & 0 \end{vmatrix} = 0$$

expresses equally the condition that the pole of p should lie on q, and that the pole of q should lie on p, which proves the proposition.

236. The two preceding articles express the same proposition in different forms. The following corollaries follow from either article.

Cor. 1. *If a point lie on a fixed straight line, its polar will pass through a fixed point* (viz. the pole of the fixed straight line).

Or, if a series of points be collinear, their polars are concurrent.

COR. 2. *If a straight line pass through a fixed point, its pole will lie upon a fixed straight line* (viz. the polar of the fixed point).

Or, if a series of lines be concurrent, their poles are collinear.

237. *To find the equation to the two tangents drawn from a given external point to the conic whose equation is*

$$f(\alpha, \beta, \gamma) = 0.$$

Let $(\alpha', \beta', \gamma')$ be the given point P, and let $(\alpha_0, \beta_0, \gamma_0)$ be any point Q on either tangent.

Then PQ being a tangent, passes through its own pole; or P, Q and the pole of PQ are collinear.

Therefore PQ is concurrent with the polars of P and Q, (Art. 236, Cor. 1.)

But the equation to PQ is, (Art. 21)

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_0, & \beta_0, & \gamma_0 \\ \alpha', & \beta', & \gamma' \end{vmatrix} = 0,$$

and the equations to the polars of Q and P are, (Art. 231)

$$\alpha \frac{df}{d\alpha_0} + \beta \frac{df}{d\beta_0} + \gamma \frac{df}{d\gamma_0} = 0,$$

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

and therefore by the condition of concurrence, (Art. 26)

$$\begin{vmatrix} \begin{vmatrix} \beta_0, & \gamma_0 \\ \beta', & \gamma' \end{vmatrix}, & \begin{vmatrix} \gamma_0, & \alpha_0 \\ \gamma', & \alpha' \end{vmatrix}, & \begin{vmatrix} \alpha_0, & \beta_0 \\ \alpha', & \beta' \end{vmatrix} \\ \dfrac{df}{d\alpha_0}, & \dfrac{df}{d\beta_0}, & \dfrac{df}{d\gamma_0} \\ \dfrac{df}{d\alpha'}, & \dfrac{df}{d\beta'}, & \dfrac{df}{d\gamma'} \end{vmatrix} = 0,$$

a relation among the coordinates of any point $(\alpha_0, \beta_0, \gamma_0)$ on either tangent. Hence, suppressing the subscripts, we have the equation to the two tangents

$$\begin{vmatrix} \begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, & \begin{vmatrix} \gamma, & \alpha \\ \gamma', & \alpha' \end{vmatrix}, & \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} \\ \frac{df}{d\alpha}, & \frac{df}{d\beta}, & \frac{df}{d\gamma} \\ \frac{df}{d\alpha'}, & \frac{df}{d\beta}, & \frac{df}{d\gamma'} \end{vmatrix} = 0.$$

This equation may be written

$$\left(\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma}\right)\left(\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'}\right)$$

$$= \left(\alpha' \frac{df}{d\alpha'} + \beta' \frac{df}{d\beta'} + \gamma' \frac{df}{d\gamma'}\right)\left(\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma}\right),$$

or in virtue of Art. 221,

$$\left(\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'}\right)^2 = 4f(\alpha', \beta', \gamma') f(\alpha, \beta, \gamma),$$

the form in which the equation is commonly quoted.

238. The following is another method of obtaining the equation in the form just written.

To find the equation of the pair of tangents drawn from a given external point to the conic whose equation is

$$f(\alpha, \beta, \gamma) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let $(\alpha', \beta', \gamma')$ be the external point: and suppose

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \quad\ldots\ldots\ldots\ldots (2),$$

to be one of the tangents.

Then the equation of the intercepts

$$f(\alpha' + \lambda\rho, \quad \beta' + \mu\rho, \quad \gamma' + \nu\rho) = 0$$

must have two equal roots.

Therefore

$$\left(\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}\right)^2 = 4f(\alpha', \beta', \gamma') f(\lambda, \mu, \nu),$$

or substituting the equations (2), and remembering that

$$\alpha' \frac{df}{d\alpha'} + \beta' \frac{df}{d\beta'} + \gamma' \frac{df}{d\gamma'} \equiv 2f(\alpha', \beta', \gamma'),$$

we get

$$\left\{\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} - 2f(\alpha', \beta', \gamma')\right\}^2$$

$$= 4f(\alpha', \beta', \gamma') f(\alpha - \alpha', \ \beta - \beta', \ \gamma - \gamma')$$

$$= 4f(\alpha', \beta', \gamma') \left\{f(\alpha, \beta, \gamma) - \left(\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'}\right) + f(\alpha', \beta', \gamma')\right\},$$

therefore

$$\left(\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'}\right)^2 = 4f(\alpha', \beta', \gamma') f(\alpha, \beta, \gamma),$$

a homogeneous equation of the second order connecting the coordinates of any point (α, β, γ) on either of the tangents from $(\alpha', \beta', \gamma')$ to the curve, and therefore the equation to the two tangents from that point, which was required.

239. *To find the locus of the middle points of a series of parallel chords in the conic whose equation is*

$$f(\alpha, \beta, \gamma) = 0.$$

Let (λ, μ, ν) be the point of intersection at infinity of all the parallel chords: and let $(\alpha', \beta', \gamma')$ be the middle point of one of them.

Then the equation to this chord is

$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \text{ ....} (1),$$

and the lengths of the intercepts cut off by the curve are given by the quadratic

$$f(\alpha' + \lambda\rho, \ \beta' + \mu\rho, \ \gamma' + \nu\rho) = 0 \text{ ....} (2).$$

But since these intercepts are measured from the middle point of the chord, they must be equal in magnitude and opposite in sign: therefore the coefficient of ρ in the quadratic (2) must vanish: therefore we have

$$\alpha' \frac{df}{d\lambda} + \beta' \frac{df}{d\mu} + \gamma' \frac{df}{d\nu} = 0 \text{ .... (3)},$$

an equation connecting the coordinates $(\alpha', \beta', \gamma')$ of the middle point of any one of the chords, and therefore (accents suppressed,) the equation to the locus of the middle points. Since the equation (3) is of the first degree, the locus of the middle points of any system of parallel chords in a conic section is a *straight line.*

240. Def. The locus of the middle points of a series of parallel chords is called a *diameter.*

One of these chords in its limiting and evanescent position will become the tangent at the extremity of the diameter. Hence the tangent at the extremity of a diameter is parallel to the chords which the diameter bisects.

Since all chords are bisected in the centre all diameters must pass through the centre, and every straight line through the centre is a diameter.

Def. The diameter parallel to a system of parallel chords is said to be *conjugate* to the diameter which bisects those chords. Some properties of conjugate diameters will be found investigated in Chap. XVIII.

241. As a particular case of Article 239 we may observe that the diameters bisecting chords parallel to the lines of reference are represented in trilinear coordinates by the equations

$$\frac{1}{b}\frac{df}{d\beta} = \frac{1}{c}\frac{df}{d\gamma}, \quad \frac{1}{c}\frac{df}{d\gamma} = \frac{1}{a}\frac{df}{d\alpha}, \quad \frac{1}{a}\frac{df}{d\alpha} = \frac{1}{b}\frac{df}{d\beta};$$

and in triangular coordinates by the equations

$$\frac{df}{d\beta} = \frac{df}{d\gamma}, \quad \frac{df}{d\gamma} = \frac{df}{d\alpha}, \quad \frac{df}{d\alpha} = \frac{df}{d\beta}.$$

Hence the centre being the point of concurrence of diameters is represented in trilinear coordinates by the equations

$$\frac{1}{a}\frac{df}{d\alpha}=\frac{1}{b}\frac{df}{d\beta}=\frac{1}{c}\frac{df}{d\gamma},$$

and in triangular coordinates by the equations

$$\frac{df}{d\alpha}=\frac{df}{d\beta}=\frac{df}{d\gamma}.$$

Or, we may establish these equations more independently as follows.

242. *To find the centre of the conic whose equation is*

$$f(\alpha, \beta, \gamma)=0.$$

Let $(\alpha', \beta', \gamma')$ be the centre. Then since all chords through the centre are bisected in the centre, the roots of the equation

$$f(\alpha'+\lambda\rho,\quad \beta'+\mu\rho,\quad \gamma'+\nu\rho)=0$$

must be equal and opposite whatever be the direction (λ, μ, ν).

Hence $$\lambda\frac{df}{d\alpha'}+\mu\frac{df}{d\beta'}+\nu\frac{df}{d\gamma'}$$

must be zero for all values of λ, μ, ν, subject to the relation

$$a\lambda+b\mu+c\nu=0, \qquad \text{[trilinear}$$

or

$$\lambda+\mu+\nu=0. \qquad \text{[triangular}$$

Hence the centre is given by the equations

$$\frac{1}{a}\frac{df}{d\alpha'}=\frac{1}{b}\frac{df}{d\beta'}=\frac{1}{c}\frac{df}{d\gamma'}, \qquad \text{[trilinear}$$

or

$$\frac{df}{d\alpha'}=\frac{df}{d\beta'}=\frac{df}{d\gamma'}. \qquad \text{[triangular}$$

Comparing these equations with those of Art. 233, we observe that *the centre is the pole of the straight line at infinity* (which we might have inferred *à priori*, from the fact that every diameter is the chord of contact of tangents drawn from a point at infinity).

The equations can be expressed (as in the article referred to) so as to give the coordinates explicitly.

Thus we shall have in *trilinear* coordinates,

$$\frac{\alpha'}{\begin{vmatrix} w', & v, & u' \\ v', & u', & w \\ a, & b, & c \end{vmatrix}} = \frac{\beta'}{\begin{vmatrix} v', & u', & w \\ u, & w', & v' \\ a, & b, & c \end{vmatrix}} = \frac{\gamma'}{\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ a, & b, & c \end{vmatrix}} = \frac{-2\Delta}{\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}},$$

and in *triangular* coordinates,

$$\frac{\alpha'}{\begin{vmatrix} w', & v, & u' \\ v', & u', & w \\ 1, & 1, & 1 \end{vmatrix}} = \frac{\beta'}{\begin{vmatrix} v', & u', & w \\ u, & w', & v' \\ 1, & 1, & 1 \end{vmatrix}} = \frac{\gamma'}{\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ 1, & 1, & 1 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}},$$

giving the actual values of the coordinates of the centre.

243. *To find the length of the semi-diameter drawn in any given direction in the conic*

$$f(\alpha, \beta, \gamma) = 0.$$

Let $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the centre of the conic, and let λ, μ, ν define the given direction, so that

$$\frac{\alpha - \bar{\alpha}}{\lambda} = \frac{\beta - \bar{\beta}}{\mu} = \frac{\gamma - \bar{\gamma}}{\nu} = \rho$$

are the equations to the diameter.

The lengths of the intercepts are given by the equation

$$f(\bar{\alpha} + \lambda\rho, \quad \bar{\beta} + \mu\rho, \quad \bar{\gamma} + \nu\rho) = 0.$$

But since $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is the centre, the two values are equal and of opposite sign, and therefore

$$\rho^2 = -\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{f(\lambda, \mu, \nu)},$$

which gives the square on the semi-diameter required.

244. *To find the conditions that the general equation of the second degree should represent a circle.*

Let the equation be written

$$f(\alpha, \beta, \gamma) = 0,$$

then if $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the centre, the semi-diameter in direction (λ, μ, ν) is given by the equation

$$\rho^2 = -\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{f(\lambda, \mu, \nu)}.$$

If the conic be a circle all diameters are equal, and therefore $f(\lambda, \mu, \nu)$ is constant. But it will be sufficient to express that diameters in three different directions are equal, and for simplicity we will select the directions of the three lines of reference.

For the direction BC we have in *trilinear* coordinates

$$\lambda = 0, \quad \mu = \pm \sin C, \quad \nu = \mp \sin B,$$

and symmetrical values for the directions of the other lines of reference.

Hence we must have

$$f(0, \sin C, -\sin B) = f(-\sin C, 0, \sin A) = f(\sin B, -\sin A, 0),$$

or $$f(0, c, -b) = f(-c, 0, a) = f(b, -a, 0).$$

Or, if the coordinates be *triangular*, the direction BC is given by

$$\lambda = 0, \quad \mu = \frac{1}{a}, \quad \nu = -\frac{1}{a},$$

and symmetrical values may be written down for the other directions, so that the conditions become

$$\frac{f(0, 1, -1)}{a^2} = \frac{f(-1, 0, 1)}{b^2} = \frac{f(1, -1, 0)}{c^2}.$$

Obs. If the equation be written in the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

the conditions that it should represent a circle become

$$vc^2 + wb^2 - 2u'bc = wa^2 + uc^2 - 2v'ca = ub^2 + va^2 - 2w'ab$$

in *trilinear* coordinates, and in *triangular* coordinates they become

$$\frac{v + w - 2u'}{a^2} = \frac{w + u - 2v'}{b^2} = \frac{u + v - 2w'}{c^2}.$$

Or the conditions in trilinear coordinates will be given by any two of the equations

$$\frac{\frac{v}{b^2} + \frac{w}{c^2} - \frac{2u'}{bc}}{a^2} = \frac{\frac{w}{c^2} + \frac{u}{a^2} - \frac{2v'}{ca}}{b^2} = \frac{\frac{u}{a^2} + \frac{v}{b^2} - \frac{2w'}{ab}}{c^2}$$

$$= \frac{\frac{u}{a^2} + \frac{u'}{bc} - \frac{v'}{ca} - \frac{w'}{ab}}{bc \cos A} = \frac{\frac{v}{b^2} + \frac{v'}{ca} - \frac{w'}{ab} - \frac{u'}{bc}}{ca \cos B} = \frac{\frac{w}{c^2} + \frac{w'}{ab} - \frac{u'}{bc} - \frac{v'}{ca}}{ab \cos C},$$

and in triangular coordinates by any two of the equations

$$\frac{v + w - 2u'}{a^2} = \frac{w + u - 2v'}{b^2} = \frac{u + v - 2w'}{c^2}$$

$$= \frac{u + u' - v' - w'}{bc \cos A} = \frac{v + v' - w' - u'}{ca \cos B} = \frac{w + w' - u' - v'}{ab \cos C}.$$

The student will not be surprised if by other methods he obtains these conditions under other forms. In a case where only one condition is arrived at the equation expressing that condition must be identically the same however it be obtained, but a system of two simultaneous equations can always be replaced by any two of the equations formed by combining them.

245. *To find the condition that the general equation of the second degree*

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

may represent two straight lines.

Let the equation be written

$$f(\alpha, \beta, \gamma) = 0,$$

and suppose it represents two straight lines so that

$$f(\alpha, \beta, \gamma) \equiv (l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma),$$

then $$\frac{df}{d\alpha} \equiv l(l'\alpha + m'\beta + n'\gamma) + l'(l\alpha + m\beta + n\gamma).$$

So $$\frac{df}{d\beta} \equiv m(l'\alpha + m'\beta + n'\gamma) + m'(l\alpha + m\beta + n\gamma),$$

and $$\frac{df}{d\gamma} \equiv n(l'\alpha + m'\beta + n'\gamma) + n'(l\alpha + m\beta + n\gamma).$$

Hence (Art. 88) the equations

$$\frac{df}{d\alpha} = 0, \quad \frac{df}{d\beta} = 0, \quad \frac{df}{d\gamma} = 0$$

represent three straight lines passing through the point of intersection of the straight lines

$$l\alpha + m\beta + n\gamma = 0,$$

and $$l'\alpha + m'\beta + n'\gamma = 0;$$

that is, the three straight lines

$$\frac{df}{d\alpha} \equiv u\alpha + w'\beta + v'\gamma = 0,$$

$$\frac{df}{d\beta} \equiv w'\alpha + v\beta + u'\gamma = 0,$$

$$\frac{df}{d\gamma} \equiv v'\alpha + u'\beta + w\gamma = 0,$$

are concurrent; and therefore (Art. 26),

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0,$$

which will be the required condition.

246. *To find the equation to the common chords of two conics whose equations are given.*

Let $F(\alpha, \beta, \gamma) \equiv U\alpha^2 + V\beta^2 + W\gamma^2$
$$+ 2U'\beta\gamma + 2V'\gamma\alpha + 2W'\alpha\beta = 0,$$

and $f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2$
$$+ 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

be the two conics.

Any pair of common chords constitute a locus of the second order passing through points of intersection of the two conics, and must therefore be represented by an equation of the form

$$F(\alpha, \beta, \gamma) + \kappa f(\alpha, \beta, \gamma) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

where κ must be so determined that this equation may satisfy the condition of representing two straight lines.

That is, κ must be determined by the equation (Art. 245)

$$\begin{vmatrix} U + \kappa u, & W' + \kappa w', & V' + \kappa v' \\ W' + \kappa w', & V + \kappa v, & U' + \kappa u' \\ V' + \kappa v', & U' + \kappa u', & W + \kappa w \end{vmatrix} = 0 \ldots\ldots\ldots (2),$$

a cubic equation giving three values of κ for the three pairs of common chords (Art. 163).

Obs. The equation (2) may be written

$$\begin{vmatrix} U, & W' & V' \\ W', & V, & U' \\ V', & U', & W \end{vmatrix}$$
$$+ \kappa \{u(VW - U'^2) + v(WU - V'^2) + w(UV - W'^2)$$
$$+ 2u'(V'W' - UU') + 2v'(W'U' - VV') + 2w'(U'V' - WW')\}$$
$$+ \kappa^2 \{U(vw - u'^2) + V(wu - v'^2) + W(uv - w'^2)$$
$$+ 2U'(v'w' - uu') + 2V'(w'u' - vv') + 2W'(u'v' - ww')\}$$
$$+ \kappa^3 \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} = 0.$$

247. Cor. 1. In the particular case when the first conic consists of the two coincident straight lines

$$(l\alpha + m\beta + n\gamma)^2 = 0,$$

the equation for κ reduces to

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} - \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} \kappa = 0.$$

Hence *the tangents whose chord of contact is*

$$l\alpha + m\beta + n\gamma = 0,$$

have the equation

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} f(\alpha, \beta, \gamma) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (l\alpha + m\beta + n\gamma)^2 = 0.$$

Cor. 2. Since the asymptotes are the two tangents whose chord of contact is at infinity they will be represented by the equation

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u' & w, & c \\ a, & b, & c, & 0 \end{vmatrix} f(\alpha, \beta, \gamma) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (a\alpha + b\beta + c\gamma)^2 = 0,$$

if the coordinates are *trilinear*, and by the equation

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} f(\alpha, \beta, \gamma) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (\alpha + \beta + \gamma)^2 = 0,$$

if the coordinates are *triangular*.

Cor. 3. If the conic be a parabola the asymptotes lie altogether at infinity: therefore in the equations of the last corollary we must have

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

according as the coordinates are *trilinear* or *triangular*. These will therefore be the respective forms of the condition that the equation should represent a parabola.

But we shall arrive at this result more directly in the next article.

Cor. 4. The asymptotes will be at right angles to one another (Art. 49, Cor. 2) provided

$$u + v + w - 2u' \cos A - 2v' \cos B - 2w' \cos C = 0,$$

when the coordinates are *trilinear*.

This is therefore the condition that the general equation of the second degree should represent a rectangular hyperbola.

When the coordinates are *triangular* this condition becomes

$$ua^2 + vb^2 + wc^2 - 2u'bc \cos A - 2v'ca \cos B - 2w'ab \cos C = 0,$$

or $$a^2(u + u' - v' - w') + b^2(v + v' - w' - u') + c^2(w + w' - u' - v') = 0.$$

248. *To find the condition that the general equation of the second degree*

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

should represent a hyperbola, parabola, or ellipse.

I. Suppose the coordinates are *trilinear*. We shall find the coordinates of the points where the locus meets infinity by solving the given equation simultaneously with the equation

$$a\alpha + b\beta + c\gamma = 0.$$

Eliminating γ we get

$$c^2(u\alpha^2 + v\beta^2 + 2w'\alpha\beta) - 2c\,(a\alpha + b\beta)\,(v'\alpha + u'\beta) + w\,(a\alpha + b\beta)^2 = 0,$$

or
$$(wa^2 + uc^2 - 2v'ac)\,\alpha^2 + (vc^2 + wb^2 - 2u'bc)\,\beta^2$$
$$+ (w'c^2 + wab - v'bc - u'ac)\,2\alpha\beta = 0,$$

a quadratic whose roots are unequal, equal, or imaginary, according as

$$(wab + w'c^2 - v'bc - u'ac)^2 \gtreqless (wa^2 + uc^2 - 2v'ac)\,(vc^2 + wb^2 - 2u'bc),$$

that is, according as

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} \gtreqless 0.$$

Hence the given equation represents a hyperbola, parabola, or ellipse, according as the determinant just written is positive, zero, or negative.

II. Suppose the coordinates are *triangular*. The reasoning will be the same as in the other case, with unity substituted for a, b, c severally. Hence the equation will represent a hyperbola, parabola, or ellipse, according as

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

is positive, zero, or negative.

249. We may arrange the proof of the last article in a different form as follows:

Let
$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho$$

be a straight line drawn from any point (α', β', γ') so as to meet the conic at an infinite distance. Then since one of the intercepts on this line is infinite, we must have

$$f(\lambda, \mu, \nu) = 0.$$

Therefore the conic will have one, two, or no directions in which one of the radii from a finite point is infinite, according as the equation

$$f(\lambda, \mu, \nu) = 0$$

gives real and equal, real and unequal, or unreal solutions.

Eliminating ν by means of the relation

$$a\lambda + b\mu + c\nu = 0, \qquad [trilinear$$

we get

$$\lambda^2 (wa^2 + uc^2 - 2v'ca) + \mu^2 (vc^2 + wb^2 - 2u'bc)$$
$$+ 2\lambda\mu (wab + w'c^2 - u'ac - v'bc) = 0 \ldots\ldots (1).$$

Hence the conic is a hyperbola, parabola, or ellipse, (Art. 228) according as

$$(wab + w'c^2 - u'ac - v'bc)^2$$
$$\begin{matrix} > \\ = \\ < \end{matrix} (wa^2 + uc^2 - 2v'ca)(vc^2 + wb^2 - 2u'bc),$$

that is, according as

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix} \begin{matrix} > \\ = \\ < \end{matrix} 0,$$

as in the previous article.

Obs. If the coordinates are triangular the equation (1) takes the form

$$\lambda^2 (w + u - 2v') + \mu^2 (v + w - 2u')$$
$$+ 2\lambda\mu (w + w' - u' - v') = 0,$$

and the final condition becomes

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \gtreqless 0.$$

Cor. By reference to Art. 242, we see that if the conic is a parabola the centre is at infinity, and the diameters are therefore parallel. Hence the following proposition arises.

250. *To find the direction of the diameters of the* parabola

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Let (λ, μ, ν) be the required direction. The equations connecting λ, μ, ν may be expressed in a variety of forms derivable from one another in virtue of the relation among the coefficients expressing the condition that the conic is a parabola.

But one of the most useful forms may be obtained as follows:

One of the diameters is represented by the equation in trilinear coordinates

$$\frac{1}{a}\frac{df}{d\alpha} = \frac{1}{b}\frac{df}{d\beta},$$

or

$$\frac{u\alpha + w'\beta + v'\gamma}{a} = \frac{w'\alpha + v\beta + u'\gamma}{b},$$

Now λ, μ, ν are proportional to the coordinates of the point where the diameter meets the straight line at infinity. Hence we have

$$\frac{u\lambda + w'\mu + v'\nu}{a} = \frac{w'\lambda + v\mu + u'\nu}{b},$$

and $$a\lambda + b\mu + c\nu = 0,$$

whence eliminating ν, we get

$$\lambda\,(ubc + u'a^2 - v'ab - w'ac) = \mu\,(vca + v'b^2 - w'bc - u'ab),$$

and by symmetry $$= \nu\,(wab + w'c^2 - u'ca - v'bc),$$

which determine the ratios $\lambda : \mu : \nu$ required.

OBS. In triangular coordinates the result will become

$$\lambda\,(u+u'-v'-w')=\mu\,(v+v'-w'-u')=\nu\,(w+w'-u'-v').$$

251. COR. 1. The equations to the diameter of the parabola,

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0,$$

through the point $(\alpha', \beta', \gamma')$

are $$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu},$$

where λ, μ, ν are the reciprocals of

$$ubc+u'a^2-v'ab-w'ac,\ \ vca+v'b^2-w'bc-u'ab,$$
$$wab+w'c^2-u'ca-v'bc,$$

if the coordinates are trilinear, and the reciprocals of

$$u+u'-v'-w',\ \ v+v'-w'-u',\ \ w+w'-u'-v',$$

if the coordinates are triangular.

COR. 2. In the particular case of the parabola inscribed in the triangle of reference, and represented by the equation

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0,$$

where $$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=0, \qquad [\textit{trilinear}$$

or $$l+m+n=0, \qquad [\textit{triangular}$$

the equations to the diameter through $(\alpha', \beta', \gamma')$ reduce to

$$\frac{\alpha-\alpha'}{\frac{l}{a^2}}=\frac{\beta-\beta'}{\frac{m}{b^2}}=\frac{\gamma-\gamma'}{\frac{n}{c^2}}, \qquad [\textit{trilinear}$$

or $$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}. \qquad [\textit{triangular}$$

252. It will be observed that everything in these chapters applies equally whether the coordinates are trilinear or triangular, except when a restriction is specially made.

Exercises on Chapter XVII.

(147) When a conic breaks up into two right lines, the polar of any point whatever passes through the intersection of the right lines.

(148) A point moves so that the sum of the squares of its distances from n given straight lines is constant. Shew that it will describe a conic section.

(149) If all but one of the straight lines in the last exercise be parallel, this one will be a diameter of the conic, and the conjugate diameter will be parallel to the other straight lines.

(150) If the straight lines in Exercise 148 consist of two groups of parallel straight lines they will be parallel to a pair of conjugate diameters in the conic.

(151) If two conics have double contact, any tangent to the one is cut harmonically at its point of contact, the points where it meets the other, and where it meets the chord of contact.

(152) A point moves along a fixed line; find the locus of the intersection of its polars with regard to two fixed conics.

(153) Given a self-conjugate triangle with regard to a conic; if one chord of intersection with a fixed conic pass through a fixed point, the other will envelope a conic.

(154) Shew that in order to form the equations of the lines joining to $(\alpha', \beta', \gamma')$ all the points of intersection of two curves, we have only to substitute $l\alpha + m\alpha'$, $l\beta + m\beta'$, $l\gamma + m\gamma'$ in both equations, and eliminate $l : m$ from the resulting equations.

(155) The polars of the two circular points at infinity with respect to the conic $f(\alpha, \beta, \gamma) = 0$ are represented in trilinear coordinates by the equation

$$\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\} = 0.$$

(156) The locus of a point which moves so as to be always at a constant distance from its polar with respect to a conic is a curve of the fourth order, having four asymptotes parallel two and two to those of the conic, and cutting the conic in four points lying on the polars of the circular points at infinity.

(157) A conic circumscribes the triangle ABC. Any conic is described having double contact with this, and such that the bisector of the angle C is the chord of contact. Prove that the straight line in which this latter conic cuts CB and CA meets AB in a fixed point.

(158) Straight lines are drawn through a fixed point; shew that the locus of the middle points of the portions of them intercepted between two fixed straight lines is a hyperbola whose asymptotes are parallel to those fixed lines.

(159) If a conic pass through the three points of reference, and if one of its chords of intersection with a conic given by the general equation be $\lambda\alpha + \mu\beta + \nu\gamma = 0$, the other will be

$$\frac{u}{\lambda}\alpha + \frac{v}{\mu}\beta + \frac{w}{\nu}\gamma = 0.$$

(160) A conic section touches the sides of a triangle ABC in the points a, b, c; and the straight lines Aa, Bb, Cc intersect the conic in a', b', c'; shew that the lines Aa, Bb, Cc pass respectively through the intersections of Bc' and Cb', Ca' and Ac', Ab' and Ba'; and the intersections of the lines ab and $a'b'$, bc and $b'c'$, ca and $c'a'$, lie respectively in AB, BC, CA.

(161) If two triangles circumscribe a conic their angular points lie in another conic.

(162) If with a given point as centre an *ellipse* can be described so as to pass through the angular points of a triangle, then with the same point as centre another *ellipse* can be described so as to touch the sides of the triangle.

CHAPTER XVIII.

THE GENERAL EQUATION OF THE SECOND DEGREE CONTINUED.

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253. THE determinant

$$\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$$

is called the *Discriminant* of the function

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta,$$

and will be conveniently denoted by the letter H.

The minors of this determinant with respect to the terms u, v, w, u', v', w' will be represented by the letters U, V, W, U', V', W' respectively, so that

$$U \equiv vw - u'^2, \qquad V \equiv wu - v'^2, \qquad W \equiv uv - w'^2,$$

$$U' \equiv v'w' - uu', \quad V' \equiv w'u' - vv', \quad W' \equiv u'v' - ww'.$$

254. If the function

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta$$

be denoted by $f(\alpha, \beta, \gamma)$, it will be convenient and suggestive to use $F(\alpha, \beta, \gamma)$ to denote the function

$$U\alpha^2 + V\beta^2 + W\gamma^2 + 2U'\beta\gamma + 2V'\gamma\alpha + 2W'\alpha\beta.$$

Thus by Art. 222 we have

$$4H.f(\alpha, \beta, \gamma) = F\left(\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right).$$

255. The determinant

$$\begin{vmatrix} u, & w', & v', & f \\ w', & v, & u', & g \\ v', & u', & w & h \\ f, & g, & h, & 0 \end{vmatrix}$$

(where f, g, h are the coefficients of α, β, γ in the identical relation $f\alpha + g\beta + h\gamma = 1$, connecting the coordinates of any point) is called the *bordered Discriminant* of the function

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta,$$

and will be denoted by the letter K.

In *triangular* coordinates f, g, h are each unity, and we have

$$K \equiv \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

or $$-K \equiv F(1, 1, 1).$$

In *trilinear* coordinates f, g, h become

$$\frac{a}{2\Delta}, \ \frac{b}{2\Delta}, \ \frac{c}{2\Delta},$$

and we get

$$K \equiv \frac{1}{4\Delta^2} \begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}$$

or $$-K \equiv \frac{1}{4\Delta^2} F(a, b, c).$$

256. The minors of the bordered discriminant with respect to the terms f, g, h will be denoted by A, B, C, so that we have in trilinear coordinates

$$A \equiv \frac{\begin{vmatrix} w', & v, & u' \\ v', & u', & w \\ a, & b, & c \end{vmatrix}}{2\Delta}, \quad B \equiv \frac{\begin{vmatrix} v', & u', & w \\ u, & w', & v' \\ a, & b, & c \end{vmatrix}}{2\Delta}, \quad C \equiv \frac{\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ a, & b, & c \end{vmatrix}}{2\Delta},$$

and in triangular coordinates,

$$A \equiv \begin{vmatrix} w', & v, & u' \\ v', & u', & w \\ 1, & 1, & 1 \end{vmatrix}, \quad B \equiv \begin{vmatrix} v', & u', & w \\ u, & w', & v' \\ 1, & 1, & 1 \end{vmatrix}, \quad C \equiv \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ 1, & 1, & 1 \end{vmatrix}.$$

257. It should be observed that in trilinear coordinates K has -2 linear dimensions and A, B, C have each -1, while H is of zero dimensions.

But in triangular coordinates *all* these functions are of zero dimensions, giving a great advantage to the triangular system.

It will be seen that the expressions in triangular coordinates throughout this chapter will be mostly derivable from the expressions in trilinear coordinates by writing 1 for 2Δ or $\frac{1}{2}$ for Δ. So, conversely, the expressions in trilinear coordinates may often be derived from the corresponding expressions in triangular coordinates by multiplying each term by such a power of 2Δ as will produce homogeneity.

258. The student can easily verify the following results, which it is convenient to collect here for future reference.

I. For the conic whose equation is

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

we have

$$H = lmn,$$

and $\begin{cases} K = -(mn + nl + lm), & \text{[triangular} \\ K = -\dfrac{a^2mn + b^2nl + c^2lm}{4\Delta^2}. & \text{[trilinear} \end{cases}$

II. For the conic whose equation is

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

we have

$$H = \frac{1}{4} lmn;$$

and

$$K = \frac{1}{4}(l^2 + m^2 + n^2 - 2mn - 2nl - 2lm), \quad \text{[triangular}$$

$$K = \frac{a^2l^2 + b^2m^2 + c^2n^2 - 2bcmn - 2canl - 2ablm}{16\Delta^2}. \quad \text{[trilinear}$$

III. For the conic whose equation is

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

we have

$$H = -4l^2m^2n^2,$$

and

$$K = -4lmn\,(l + m + n); \quad \text{[triangular}$$

$$K = -\frac{lmn}{\Delta^2}(lbc + mca + nab). \quad \text{[trilinear}$$

IV. For the conic whose equation is

$$\alpha^2 + 2k\beta\gamma = 0;$$

we have

$$H = -k^2,$$

and

$$K = k^2 + 2k, \quad \text{[triangular}$$

$$K = \frac{a^2k^2 + 2bck}{4\Delta^2}. \quad \text{[trilinear}$$

259. *To express the equation of the straight line at infinity in terms of the derived functions*

$$\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}$$

when

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta.$$

Let

$$l\frac{df}{d\alpha} + m\frac{df}{d\beta} + n\frac{df}{d\gamma} = 0$$

be the equation required.

The first member may be written

$$(lu + mw' + nv')\,2\alpha + (lw' + mv + nu')\,2\beta + (lv' + mu' + nw)\,2\gamma,$$

which must be identical with

$$2k\,(a\alpha + b\beta + c\gamma),$$

if the coordinates be trilinear, where k is some constant.

We have, therefore,

$$\begin{aligned} lu + mw' + nv' &= ak, \\ lw' + mv + nu' &= bk, \\ lv' + mu' + nw &= ck, \end{aligned}$$

whence $$\frac{l}{A} = \frac{m}{B} = \frac{n}{C} = \frac{2\Delta k}{H},$$

and therefore the equation to the straight line at infinity becomes

$$A\frac{df}{d\alpha} + B\frac{df}{d\beta} + C\frac{df}{d\gamma} = 0,$$

which holds equally whether the coordinates be trilinear or triangular.

260. Cor. 1. The identical relation connecting the trilinear coordinates of any finite point becomes

$$A\frac{df}{d\alpha} + B\frac{df}{d\beta} + C\frac{df}{d\gamma} \equiv \frac{H}{\Delta}(a\alpha + b\beta + c\gamma)$$

$$\equiv 2H.$$

So also in triangular coordinates,

$$A\frac{df}{d\alpha} + B\frac{df}{d\beta} + C\frac{df}{d\gamma} \equiv 2H.$$

261. Cor. 2. The result of the last Cor. may be written (by Art. 221, equation ii),

$$\alpha\frac{df}{dA} + \beta\frac{df}{dB} + \gamma\frac{df}{dC} \equiv 2H,$$

which being a relation among the coordinates of any point whatever must be identical with

$$a\alpha + b\beta + c\gamma \equiv 2\Delta, \qquad \text{[}\textit{trilinear}$$

or

$$\alpha + \beta + \gamma \equiv 1. \qquad \text{[}\textit{triangular}$$

Hence in trilinear coordinates,

$$\frac{df}{dA} = \frac{aH}{\Delta}, \quad \frac{df}{dB} = \frac{bH}{\Delta}, \quad \frac{df}{dC} = \frac{cH}{\Delta},$$

and in triangular coordinates,

$$\frac{df}{dA} = \frac{df}{dB} = \frac{df}{dC} = 2H.$$

262. *Given*

$$l\alpha + m\beta + n\gamma \equiv L\frac{df}{d\alpha} + M\frac{df}{d\beta} + N\frac{df}{d\gamma}$$

to express L, M, N explicitly in terms of l, m, n.

Substituting their values for

$$\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma},$$

and equating coefficients of α, β, γ, we obtain

$$\left.\begin{aligned} l &= 2(uL + w'M + v'N) \\ m &= 2(w'L + vM + u'N) \\ n &= 2(v'L + u'M + wN) \end{aligned}\right\} \quad \text{..................... (1)},$$

and solving for L, M, N, we obtain

$$\left.\begin{aligned} L &= \frac{1}{2H}(Ul + W'm + V'n) \\ M &= \frac{1}{2H}(W'l + Vm + U'n) \\ N &= \frac{1}{2H}(V'l + U'm + Wn) \end{aligned}\right\} \quad \text{..................... (2)},$$

the required results.

263. By reference to Art. 242 it will be seen that the coordinates of the centre of the conic whose equation is $f(\alpha, \beta, \gamma) = 0$, are

$$-\frac{A}{K}, \; -\frac{B}{K}, \; -\frac{C}{K}$$

whether the coordinates be trilinear or triangular.

Hence in trilinear coordinates, (Art. 261)

$$\frac{1}{a}\frac{df}{d\alpha} = \frac{1}{b}\frac{df}{d\beta} = \frac{1}{c}\frac{df}{d\gamma} = -\frac{H}{K\Delta},$$

and in triangular coordinates

$$\frac{df}{d\alpha} = \frac{df}{d\beta} = \frac{df}{d\gamma} = -\frac{2H}{K}.$$

264. By Art. 245 the condition that the equation $f(\alpha, \beta, \gamma) = 0$ should represent two straight lines is

$$H = 0.$$

265. By Art. 247, Cor. 2, the asymptotes of the conic represented by the general equation of the second degree $f(\alpha, \beta, \gamma) = 0$ are given by

$$f(\alpha, \beta, \gamma) + \frac{H}{K} = 0.$$

266. By Art. 247, Cor. 3, the condition that the equation $f(\alpha, \beta, \gamma) = 0$ should represent a parabola is

$$K = 0.$$

By Art. 248 the equation will represent an ellipse or a hyperbola according as K is positive or negative.

267. We found in Art. 247, Cor. 4, the condition that the equation $f(\alpha, \beta, \gamma) = 0$ should represent a rectangular hyperbola.

We shall write this condition

$$E = 0,$$

so that E represents the function

$$u+v+w-2u'\cos A-2v'\cos B-2w'\cos C$$

if the coordinates are trilinear, or

$$\frac{a^2(u+u'-v'-w')+b^2(v+v'-w'-u')+c^2(w+w'-u'-v')}{2\Delta}$$

if they are triangular.

268. *If* $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ *be the centre of the conic whose equation is* $f(\alpha, \beta, \gamma)=0$, *then will*

$$f(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = -\frac{H}{K}.$$

For by Art. 261, we have identically

$$A\frac{df}{d\bar{\alpha}}+B\frac{df}{d\bar{\beta}}+C\frac{df}{d\bar{\gamma}}=2H.$$

And by Art. 263,

$$A=-K\bar{\alpha},\quad B=-K\bar{\beta},\quad C=-K\bar{\gamma}.$$

Hence

$$-K\left(\bar{\alpha}\frac{df}{d\bar{\alpha}}+\bar{\beta}\frac{df}{d\bar{\beta}}+\bar{\gamma}\frac{df}{d\bar{\gamma}}\right)=2H,$$

and therefore (Art. 221, equation 3)

$$f(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = -\frac{H}{K}.$$

Cor. It follows that

$$f(A, B, C) = -HK.$$

269. *To find the equation to the diameter through a given point.*

Let $(\alpha', \beta', \gamma')$ be the given point; then since the diameter joins this point to the centre of the conic its equation must be (Art. 21)

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ A, & B, & C \end{vmatrix}.$$

Cor. 1. The equation in terms of $\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}$, will be (Art. 262)

$$\begin{vmatrix} U, & W', & V' \\ A, & B, & C \\ \alpha', & \beta', & \gamma' \end{vmatrix} \frac{df}{d\alpha} + \begin{vmatrix} W', & V, & U' \\ A, & B, & C \\ \alpha', & \beta', & \gamma' \end{vmatrix} \frac{df}{d\beta} + \begin{vmatrix} V', & U', & W \\ A, & B, & C \\ \alpha', & \beta', & \gamma' \end{vmatrix} \frac{df}{d\gamma} = 0.$$

Cor. 2. The general equation to a diameter may be written

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} U, & W', & V' \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \frac{df}{d\alpha} + \begin{vmatrix} W', & V, & U' \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \frac{df}{d\beta} + \begin{vmatrix} V', & U', & W \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \frac{df}{d\gamma} = 0.$$

270. *The polar of any point on a diameter is parallel to the tangent at the extremity of the diameter.*

For let λ, μ, ν be proportional to the coordinates of a point on a diameter, then its polar is given by

$$\lambda \frac{df}{d\alpha} + \mu \frac{df}{d\beta} + \nu \frac{df}{d\gamma} = 0, \qquad (1)$$

and the equation to the diameter can be written

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \lambda, & \mu, & \nu \\ A, & B, & C \end{vmatrix} = 0.$$

Now let $(\alpha', \beta', \gamma')$ be the coordinates of the extremity of the diameter, then we have

$$\begin{vmatrix} \alpha', & \beta', & \gamma' \\ \lambda, & \mu, & \nu \\ A, & B, & C \end{vmatrix} = 0.$$

which expresses that the straight line (1) is parallel to the straight line

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\gamma} + \gamma' \frac{df}{d\gamma} = 0,$$

which is the tangent at $(\alpha', \beta', \gamma')$.

Thus the proposition is established.

271. *To find the condition that the diameter*

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} = 0 \text{..........................(1)}$$

should be conjugate to the diameter

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ A, & B, & C \\ \lambda', & \mu', & \nu' \end{vmatrix} = 0 \text{..........................(2).}$$

The first equation (1) may be written

$$\begin{vmatrix} U, & W', & V' \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \frac{df}{d\alpha} + \begin{vmatrix} W', & V, & U' \\ A, & B, & C \\ \lambda, & \mu, & \nu, \end{vmatrix} \frac{df}{d\beta} + \begin{vmatrix} V', & U', & W \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \frac{df}{d\gamma}$$

$$= 0 \text{ ....(3),}$$

and the second bisects chords parallel to the straight line

$$\lambda' \frac{df}{d\alpha} + \mu' \frac{df}{d\beta} + \nu' \frac{df}{d\gamma} = 0 \text{..................(4).}$$

If the diameters be conjugate these equations (3) and (4) must represent parallel straight lines; hence we must have

$$\begin{vmatrix} \begin{vmatrix} U, & W', & V' \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix}, & \begin{vmatrix} W', & V, & U' \\ A, & B, & C, \\ \lambda, & \mu, & \nu \end{vmatrix}, & \begin{vmatrix} V', & U', & W \\ A, & B, & C \\ \lambda, & \mu, & \nu \end{vmatrix} \\ A, & B, & C \\ \lambda', & \mu', & \nu' \end{vmatrix} = 0.$$

Or, if l, m, n denote the determinants

$$\begin{vmatrix} B, & C \\ \mu, & \nu \end{vmatrix}, \quad \begin{vmatrix} C, & A \\ \nu, & \lambda \end{vmatrix}, \quad \begin{vmatrix} A, & B \\ \lambda, & \mu \end{vmatrix},$$

and l', m', n' the determinants

$$\begin{vmatrix} B, & C \\ \mu', & \nu' \end{vmatrix}, \quad \begin{vmatrix} C, & A \\ \nu', & \lambda' \end{vmatrix}, \quad \begin{vmatrix} A, & B \\ \lambda', & \mu' \end{vmatrix},$$

the condition takes the form

$$Ull' + Vmm' + Wnn'$$
$$+ U'(mn' + m'n) + V'(nl' + n'l) + W'(lm' + l'm) = 0.$$

Cor. 1. From the symmetry of this result we infer that if one diameter be conjugate to a second, the second is also conjugate to the first.

Cor. 2. The equations

$$l\alpha + m\beta + n\gamma = 0,$$
$$l'\alpha + m'\beta + n'\gamma = 0,$$

will represent a pair of conjugate diameters of the conic

$$f(\alpha, \beta, \gamma) = 0,$$

provided

$$Al + Bm + Cn = 0,$$
$$Al' + Bm' + Cn' = 0,$$
$$Ull' + Vmm' + Wnn'$$
$$+ U'(mn' + m'n) + V'(nl' + n'l) + W'(lm' + l'm) = 0.$$

272. *To find the equation to the diameter parallel to the tangent at a given point.*

Let $(\alpha', \beta', \gamma')$ be the given point, $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ the centre, and (α, β, γ) any point on the diameter whose equation is required.

The tangent at $(\alpha', \beta', \gamma')$ is given by

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

and the two points (α, β, γ), $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ are equidistant from this tangent, therefore

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = \overline{\alpha} \frac{df}{d\alpha'} + \overline{\beta} \frac{df}{d\beta'} + \overline{\gamma} \frac{df}{d\gamma'}$$

$$= -\frac{2H}{K} \text{ (Arts. 261 and 263)},$$

which can be rendered homogeneous as in Art. 10.

273. *To establish equations determining the foci of the conic*

$$f(\alpha, \beta, \gamma) = 0.$$

[Def. The foci are a pair of points, equidistant on opposite sides of the centre, such that the rectangle contained by the perpendiculars from them on any tangent is constant.]

Let $(\alpha_1, \beta_1, \gamma_1)$, and $(\alpha_2, \beta_2, \gamma_2)$ be the foci, and let $(\alpha', \beta', \gamma')$ be any point on the conic.

The tangent at this point is

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

and the rectangle contained by the perpendiculars upon it from $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ is, (Art. 46)

$$\frac{\left(\alpha_1 \frac{df}{d\alpha'} + \beta_1 \frac{df}{d\beta'} + \gamma_1 \frac{df}{d\gamma'}\right)\left(\alpha_2 \frac{df}{d\alpha'} + \beta_2 \frac{df}{d\beta'} + \gamma_2 \frac{df}{d\gamma'}\right)}{\left\{\frac{df}{d\alpha'}, \quad \frac{df}{d\beta'}, \quad \frac{df}{d\gamma'}\right\}^2}.$$

But by definition this is equal to a constant area (k^2, suppose), hence we obtain

$$(\alpha_1\alpha_2 - k^2)\left(\frac{df}{d\alpha'}\right)^2 + (\beta_1\beta_2 - k^2)\left(\frac{df}{d\beta'}\right)^2 + (\gamma_1\gamma_2 - k^2)\left(\frac{df}{d\gamma'}\right)^2$$

$$+ (\beta_1\gamma_2 + \beta_2\gamma_1 + 2k^2\cos A)\frac{df}{d\beta'}\frac{df}{d\gamma'}$$

$$+ (\gamma_1\alpha_2 + \gamma_2\alpha_1 + 2k^2\cos B)\frac{df}{d\gamma'}\frac{df}{d\alpha'}$$

$$+ (\alpha_1\beta_2 + \alpha_2\beta_1 + 2k^2\cos C)\frac{df}{d\alpha'}\frac{df}{d\beta'} = 0.$$

Now this is a relation between the coordinates of any point whatever on the conic, and must therefore (accents suppressed) be the equation to the conic, and identical with the given equation which may be written (Art. 222)

$$U\left(\frac{df}{d\alpha}\right)^2 + V\left(\frac{df}{d\beta}\right)^2 + W\left(\frac{df}{d\gamma}\right)^2$$

$$+ 2U'\frac{df}{d\beta}\frac{df}{d\gamma} + 2V'\frac{df}{d\gamma}\frac{df}{d\alpha} + 2W'\frac{df}{d\alpha}\frac{df}{d\beta} = 0.$$

Hence we have

$$\frac{\alpha_1\alpha_2 - k^2}{U} = \frac{\beta_1\beta_2 - k^2}{V} = \frac{\gamma_1\gamma_2 - k^2}{W}$$

$$= \frac{\beta_1\gamma_2 + \beta_2\gamma_1 + 2k^2\cos A}{2U'} = \frac{\gamma_1\alpha_2 + \gamma_2\alpha_1 + 2k^2\cos B}{2V'}$$

$$= \frac{\alpha_1\beta_2 + \alpha_2\beta_1 + 2k^2\cos C}{2W'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

five equations, which with the two relations

$$\left.\begin{array}{l} a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta \\ a\alpha_2 + b\beta_2 + c\gamma_2 = 2\Delta \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

determine the seven quantities α_1, β_1, γ_1, α_2, β_2, γ_2 and k.

Each of the equal fractions in (1) is equal to

$$\frac{(a\alpha_1 + b\beta_1 + c\gamma_1)(a\alpha_2 + b\beta_2 + c\gamma_2)}{Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab}$$

$$= \frac{4\Delta^2}{F(a, b, c)} = -\frac{1}{K} \text{ (Art. 255.)}$$

Hence the system of equations (1) may be written

$$Kk^2 = U + K\alpha_1\alpha_2 = V + K\beta_1\beta_2 = W + K\gamma_1\gamma_2$$

$$= \frac{2U' + K(\beta_1\gamma_2 + \beta_2\gamma_1)}{-2\cos A} = \frac{2V' + K(\gamma_1\alpha_2 + \gamma_2\alpha_1)}{-2\cos B}$$

$$= \frac{2W' + K(\alpha_1\beta_2 + \alpha_2\beta_1)}{-2\cos C} \quad\text{.................. (3).}$$

But instead of using all these equations which are somewhat complicated we may combine some of them with the simpler relations

$$\alpha_1 + \alpha_2 = -\frac{2A}{K}, \quad \beta_1 + \beta_2 = -\frac{2B}{K}, \quad \gamma_1 + \gamma_2 = -\frac{2C}{K}$$

expressing the fact that the centre bisects the line joining the foci. (Art. 18. Cor.)

Thus we have

$$-K\alpha_1\alpha_2 = 2A\alpha_1 + K\alpha_1^2, \quad -K\beta_1\beta_2 = 2B\beta_1 + K\beta_1^2,$$

$$-K\gamma_1\gamma_2 = 2C\gamma_1 + K\gamma_1^2$$

so that the equations (3) give

$$K\alpha_1^2 + 2A\alpha_1 - U = K\beta_1^2 + 2B\beta_1 - V = K\gamma_1^2 + 2C\gamma_1 - W$$

which with the identical relation connecting the coordinates of any point will be sufficient to determine the coordinates $(\alpha_1, \beta_1, \gamma_1)$.

There will generally be two imaginary solutions as well as the two real ones, indicating two imaginary points having the property enunciated of the two real ones.

Cor. If the conic be a parabola, $K = 0$ and the equations reduce to

$$2A\alpha_1 - U = 2B\beta_1 - V = 2C\gamma_1 - W.$$

Obs. The equations in the form in which we have written them hold equally whether the coordinates be trilinear or triangular. If we write K, A, B, C, at length, as functions of U, V, W, &c. the equations will take the form

$$\alpha^2 (Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab)$$
$$- 4\Delta\alpha (bW' + cV' + aU) + 4\Delta^2 U$$
$$= \beta^2 (Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab)$$
$$- 4\Delta\beta (cU' + aW' + bV) + 4\Delta^2 V$$
$$= \gamma^2 (Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab)$$
$$- 4\Delta\gamma (aV' + bU' + cW) + 4\Delta^2 W$$

for trilinear coordinates, and for triangular coordinates the form

$$\frac{\alpha^2 (U + V + W + 2U' + 2V' + 2W') - 2(V' + W' + U)\alpha + U}{a^2}$$
$$= \frac{\beta^2 (U + V + W + 2U' + 2V' + 2W') - 2(W' + U' + V)\beta + V}{b^2}$$
$$= \frac{\gamma^2 (U + V + W + 2U' + 2V' + 2W') - 2(U' + V' + W)\gamma + W}{c^2}.$$

274. Definitions. A point on a conic at which the tangent is at right angles to the diameter is called a *vertex*, and the diameter through a vertex is called an *axis*.

275. *To find equations to determine the vertices of the conic whose equation is*

$$f(\alpha, \beta, \gamma) = 0.$$

Let $(\alpha', \beta', \gamma')$ be the vertex; then the tangent

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0$$

is at right angles to the diameter

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ A, & B, & C \end{vmatrix} = 0.$$

Hence if the coordinates be trilinear (Art. 49, Cor. 1).

$$\frac{df}{d\alpha'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ 1, & -\cos C, & -\cos B \end{vmatrix} + \frac{df}{d\beta'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ -\cos C, & 1, & -\cos A \end{vmatrix}$$

$$+\frac{df}{d\gamma'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ -\cos B, & -\cos A, & 1 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (1),$$

or if they be triangular,

$$a\frac{df}{d\alpha'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ a, & -b\cos C, & -c\cos A \end{vmatrix}$$

$$+b\frac{df}{d\beta'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ -a\cos C, & b, & -c\cos A \end{vmatrix}$$

$$+c\frac{df}{d\gamma'}\begin{vmatrix} \alpha', & \beta', & \gamma' \\ A, & B, & C \\ -a\cos B, & -b\cos A, & c \end{vmatrix} = 0 \ldots\ldots\ldots (1).$$

This equation, together with the relation

$$f(\alpha', \beta', \gamma') = 0,$$

will be sufficient to determine the ratios of the coordinates. And since each equation is of the second order there will be *four* solutions indicating *four* vertices.

276. *To find the equation to the axes of the conic.*

The equation (1) of the last article is a relation among the coordinates α', β', γ' of any vertex of the conic.

Therefore if we suppress the accents it will represent a locus of the second order passing through the four vertices. But it is satisfied also at the point

$$\frac{\alpha}{A}=\frac{\beta}{B}=\frac{\gamma}{C},$$

that is, at the centre. Hence it will represent a locus of the second order passing through the four vertices and the centre. But through these five points there can be only one locus of the second order (Art. 147), and the two axes constitute such a locus. Hence the equation will represent the axes.

277. The equation to the axes may be directly obtained in another form which is sometimes useful, by the following method, which depends upon the property that a conic is symmetrical with respect to an axis, and therefore the two tangents from any point on an axis are equal in length.

Suppose the coordinates trilinear.

Let λ, μ, ν, λ', μ', ν' be the direction sines of the two tangents drawn from a point $(\alpha', \beta', \gamma')$ to the conic.

The length of the first tangent is given by the equation

$$f(\alpha'+\lambda\rho, \quad \beta'+\mu\rho, \quad \gamma'+\nu\rho)=0,$$

and the roots of this equation must be equal, therefore

$$\left(\lambda\frac{df}{d\alpha'}+\mu\frac{df}{d\beta'}+\nu\frac{df}{d\gamma'}\right)^2=4f(\alpha', \beta', \gamma')\,f(\lambda, \mu, \nu).$$

Similarly,

$$\left(\lambda'\frac{df}{d\alpha'}+\mu'\frac{df}{d\beta'}+\nu'\frac{df}{d\gamma'}\right)^2=4f(\alpha', \beta', \gamma')\,f(\lambda', \mu', \nu').$$

But if $(\alpha', \beta', \gamma')$ be any point on either axis of the conic the two tangents are equal, and therefore

$$f(\lambda, \mu, \nu)=f(\lambda', \mu', \nu')\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

consequently

$$\left(\lambda\frac{df}{d\alpha'}+\mu\frac{df}{d\beta'}+\nu\frac{df}{d\gamma'}\right)^2=\left(\lambda'\frac{df}{d\alpha'}+\mu'\frac{df}{d\beta'}+\nu'\frac{df}{d\gamma'}\right)^2\ldots\ldots(2).$$

Also by the identical relations which exist among the direction sines of any straight line (Art. 70),

$$(a\lambda + b\mu + c\nu)^2 = (a\lambda' + b\mu' + c\nu')^2 \dots\dots\dots\dots(3),$$

$$\mu^2 + \nu^2 + 2\mu\nu \cos A = \mu'^2 + \nu'^2 + 2\mu'\nu' \cos A \dots\dots\dots(4),$$

$$\nu^2 + \lambda^2 + 2\nu\lambda \cos B = \nu'^2 + \lambda'^2 + 2\nu'\lambda' \cos B \dots\dots\dots(5),$$

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos C = \lambda'^2 + \mu'^2 + 2\lambda'\mu' \cos C \dots\dots\dots(6).$$

Eliminating the six quantities

$$\lambda^2 - \lambda'^2,\ \mu^2 - \mu'^2,\ \nu^2 - \nu'^2,\ \mu\nu - \mu'\nu',\ \nu\lambda - \nu'\lambda',\ \lambda\mu - \lambda'\mu'$$

from these six equations, and suppressing the accents on the coordinates, we obtain

$$\begin{vmatrix} \left(\frac{df}{d\alpha}\right)^2, & \left(\frac{df}{d\beta}\right)^2, & \left(\frac{df}{d\gamma}\right)^2, & \frac{df}{d\beta}\frac{df}{d\gamma}, & \frac{df}{d\gamma}\frac{df}{d\alpha}, & \frac{df}{d\alpha}\frac{df}{d\beta} \\ a^2, & b^2, & c^2, & bc, & ca, & ab \\ u, & v, & w, & u', & v', & w' \\ 0, & 1, & 1, & \cos A, & 0, & 0 \\ 1, & 0, & 1, & 0, & \cos B, & 0 \\ 1, & 1, & 0, & 0, & 0, & \cos C \end{vmatrix} = 0,$$

a relation of the second order between the coordinates of any point on either axis, and therefore the equation to the axes, the coordinates being trilinear.

278. Def. Two conics are said to be similar and similarly situated when the lengths of any two parallel diameters are to one another in a constant ratio.

279. *The conics represented by the equations*

$$f(\alpha, \beta, \gamma) = 0,$$

$$f(\alpha, \beta, \gamma) + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0,$$

are similar and similarly situated.

Let $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be the two centres, then (Art. 243) the squares on the semi-diameters in direction λ, μ, ν are

$$-\frac{f(\alpha', \beta', \gamma')}{f(\lambda, \mu, \nu)} \text{ and } -\frac{f(\alpha'', \beta'', \gamma'') + 2\Delta(l\alpha'' + m\beta'' + n\gamma'')}{f(\lambda, \mu, \nu)},$$

since $$a\lambda + b\mu + c\nu = 0.$$

Hence these semi-diameters are to one another in the ratio

$$\left\{\frac{f(\alpha', \beta', \gamma')}{f(\alpha'', \beta'', \gamma'') + 2\Delta(l\alpha'' + m\beta'' + n\gamma'')}\right\}^{\frac{1}{2}},$$

which is independent of λ, μ, ν, and therefore constant for all directions.

Hence if $\sigma = 0$ be the straight line at infinity, the equations

$$f(\alpha, \beta, \gamma) = 0,$$

and $$f(\alpha, \beta, \gamma) + \sigma(l\alpha + m\beta + n\gamma) = 0,$$

represent similar and similarly situated conics.

Cor. Two similar and similarly situated conics meet the line at infinity in the same points. Hence *their asymptotes are parallel.*

280. Def. Two concentric conics are said to be conjugate when the polars with respect to them of any point are parallel, and equidistant from the common centre.

281. *If $f(\alpha, \beta, \gamma) = 0$ and $f'(\alpha, \beta, \gamma) = 0$ be the equations to a pair of conjugate conics, and if H, H' be their discriminants and K, K' their bordered discriminants, then will*

$$\frac{K}{H} f(\alpha, \beta, \gamma) + \frac{K'}{H'} f'(\alpha, \beta, \gamma) \equiv -2.$$

For let $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the common centre, and $(\alpha', \beta', \gamma')$ any other point.

The polars of this point are

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} = 0,$$

and

$$\alpha \frac{df'}{d\alpha'} + \beta \frac{df'}{d\beta'} + \gamma \frac{df'}{d\gamma'} = 0.$$

Now if these be parallel and equidistant from the centre they will cut off equal and opposite intercepts from any straight line drawn from the centre. Such a straight line is represented by

$$\frac{\alpha - \bar{\alpha}}{\lambda} = \frac{\beta - \bar{\beta}}{\mu} = \frac{\gamma - \bar{\gamma}}{\nu} = \rho,$$

and the intercepts are given by

$$(\bar{\alpha} + \lambda\rho) \frac{df}{d\alpha'} + (\bar{\beta} + \mu\rho) \frac{df}{d\beta'} + (\bar{\gamma} + \nu\rho) \frac{df}{d\gamma'} = 0,$$

and

$$(\bar{\alpha} + \lambda\rho) \frac{df'}{d\alpha'} + (\bar{\beta} + \mu\rho) \frac{df'}{d\beta'} + (\bar{\gamma} + \nu\rho) \frac{df'}{d\gamma'} = 0.$$

But by Art. 260 we have

$$\bar{\alpha} \frac{df}{d\alpha'} + \bar{\beta} \frac{df}{d\beta'} + \bar{\gamma} \frac{df}{d\gamma'} = -\frac{H}{K}, \quad \bar{\alpha} \frac{df'}{d\alpha'} + \bar{\beta} \frac{df'}{d\beta'} + \bar{\gamma} \frac{df'}{d\gamma'} = -\frac{H'}{K'},$$

in virtue of which, the condition that the intercepts should be equal and opposite in sign becomes

$$\frac{K}{H}\left(\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}\right) + \frac{K'}{H'}\left(\lambda \frac{df'}{d\alpha'} + \mu \frac{df'}{d\beta'} + \nu \frac{df'}{d\gamma'}\right) = 0,$$

which must be true for all values of λ, μ, ν subject to the relation

$$a\lambda + b\mu + c\nu = 0, \qquad [\textit{trilinear}$$

or

$$\lambda + \mu + \nu = 0. \qquad [\textit{triangular}$$

But this relation will be satisfied if λ, μ, ν be proportional to

$$\alpha' - \bar{\alpha}, \quad \beta' - \bar{\beta}, \quad \gamma' - \bar{\gamma}.$$

Hence we obtain

$$\frac{K}{H}\left\{(\alpha'-\bar{\alpha})\frac{df}{d\alpha'}+(\beta'-\bar{\beta})\frac{df}{d\beta'}+(\gamma'-\bar{\gamma})\frac{df}{d\gamma'}\right\}$$

$$+\frac{K'}{H'}\left\{(\alpha'-\bar{\alpha})\frac{df'}{d\alpha'}+(\beta'-\bar{\beta})\frac{df'}{d\beta'}+(\gamma'-\bar{\gamma})\frac{df'}{d\gamma'}\right\}=0,$$

or (Art. 260) $\frac{K}{H}f(\alpha', \beta', \gamma')+\frac{K'}{H'}f'(\alpha', \beta', \gamma')=-2.$

But $(\alpha', \beta', \gamma')$ is any point whatever; hence, suppressing the accents,

$$\frac{K}{H}f(\alpha, \beta, \gamma)+\frac{K'}{H'}f'(\alpha, \beta, \gamma)\equiv-2.$$

282. Cor. 1. If $f(\alpha, \beta, \gamma)=0$ be the equation to any conic, its conjugate is represented by the equation

$$f(\alpha, \beta, \gamma)+\frac{2H}{K}(\alpha+\beta+\gamma)^2=0. \qquad [triangular$$

Cor. 2. The squares on the semi-diameters of the two curves in direction (λ, μ, ν) are

$$-\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{f(\lambda, \mu, \nu)} \text{ and } -\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})+\frac{2H}{K}}{f(\lambda, \mu, \nu)},$$

which are equal and of opposite sign, since

$$f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})=-\frac{H}{K}.$$

Hence the diameters are in the ratio $\sqrt{-1} : 1$.

Therefore *a conic and its conjugate are similar, similarly situated, and concentric conics whose linear dimensions are as* $\sqrt{-1} : 1$.

It follows that the conjugate of an ellipse is wholly imaginary. In the hyperbola, any central radius which meets the curve in real points meets the conjugate in imaginary points and *vice versa*.

283. *To find the maximum or minimum value of $f(\lambda, \mu, \nu)$ where λ, μ, ν are subject to the relations (Art. 71),*

$$\lambda^2 a\cos A + \mu^2 b\cos B + \nu^2 c\cos C = a\sin B\sin C \quad\text{.........(1)},$$

and

$$\lambda a + \mu b + \nu c = 0 \quad\text{.....................(2)},$$

the coordinates being trilinear.

Let ϕ be any maximum or minimum value of $f(\lambda, \mu, \nu)$. Equating to zero the differential of the given function, we have

$$\frac{df}{d\lambda}\delta\lambda + \frac{df}{d\mu}\delta\mu + \frac{df}{d\nu}\delta\nu = 0 \quad\text{.....................(3)},$$

and from (1) and (2)

$$\lambda\delta\lambda \,.\, a\cos A + \mu\delta\mu \,.\, b\cos B + \nu\delta\nu \,.\, c\cos C = 0 \quad\text{......(4)},$$

$$a\delta\lambda + b\delta\mu + c\delta\nu = 0 \quad\text{......(5)}.$$

Multiplying (4) and (5) by undetermined constants $2k$, $2k'$ and adding to (3), and equating the coefficients of the differentials to zero, we obtain

$$\frac{df}{d\lambda} + 2k\lambda a\cos A + 2k'a = 0 \quad\text{..................(6)},$$

$$\frac{df}{d\mu} + 2k\mu b\cos B + 2k'b = 0 \quad\text{..................(7)},$$

$$\frac{df}{d\nu} + 2k\nu c\cos C + 2k'c = 0 \quad\text{..................(8)}.$$

Multiplying these by λ, μ, ν respectively and adding, we get in virtue of (1) and (2), and by Art. 221,

$$\phi + ka\sin B\sin C = 0, \text{ or } k = -\frac{abc\phi}{4\Delta^2} \quad\text{............(9)}.$$

But the equations (6), (7), (8) may be written

$$(u + ka\cos A)\lambda + w'\mu + v'\nu + k'a = 0,$$

$$w'\lambda + (v + kb\cos B)\mu + u'\nu + k'b = 0,$$

$$v'\lambda + u'\mu + (w + kc\cos B)\nu + k'c = 0,$$

which equations, together with the relation

$$a\lambda + b\mu + c\nu = 0,$$

give upon the elimination of $\lambda : \mu : \nu : k'$,

$$\begin{vmatrix} u + ka\cos a, & w', & v', & a \\ w', & v + kb\cos B & u' & b \\ v', & u', & w + kc\cos C, & c \\ a, & b, & c, & 0 \end{vmatrix} = 0,$$

a quadratic to determine k.

This quadratic may be written

$$4\Delta^2 K - Eabck - 4\Delta^2 k^2 = 0,$$

which by the substitution of (9) gives us

$$\phi^2 - E\phi = \frac{16\Delta^4 K}{a^2b^2c^2},$$

a quadratic equation, one of whose roots will be the maximum and the other the minimum value of $f(\lambda, \mu, \nu)$.

284. *To find the maximum or minimum value of* $f(\lambda, \mu, \nu)$ *where* λ, μ, ν *are subject to the relations*

$$\lambda^2 bc\cos A + \mu^2 ca\cos B + \nu^2 ab\cos C = 1 \ldots\ldots\ldots (1),$$

and $$\lambda + \mu + \nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the coordinates being triangular.

Let ϕ be a maximum or minimum value of $f(\lambda, \mu, \nu)$.

Equating to zero the differential of the given function, we have

$$\frac{df}{d\lambda}\delta\lambda + \frac{df}{d\mu}\delta\mu + \frac{df}{d\nu}\delta\nu = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and from (1) and (2),

$$\lambda\delta\lambda \,.\, bc\cos A + \mu\delta\mu \,.\, ca\cos B + \nu\delta\nu \,.\, ab\cos C = 0 \ldots. (4),$$

$$\delta\lambda \quad + \quad \delta\mu \quad + \quad \delta\nu \quad = 0 \ldots. (5).$$

Multiplying (4) and (5) by undetermined constants $2k$, $2k'$, and adding to (3), and equating the coefficients of the differentials to zero, we obtain

$$\frac{df}{d\lambda} + 2k\lambda bc \cos A + 2k' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{df}{d\mu} + 2k\mu ca \cos B + 2k' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$\frac{df}{d\nu} + 2k\nu ab \cos C + 2k' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Multiplying these by λ, μ, ν respectively and adding, we get in virtue of (1) and (2),

$$\phi + k = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

So that the equations (6), (7), (8) may be written

$$(u - \phi bc \cos A)\,\lambda + w'\mu + v'\nu + k' = 0,$$

$$w'\lambda + (v - \phi ca \cos B)\,\mu + u'\nu + k' = 0,$$

$$v'\lambda + u'\mu + (w - \phi ab \cos C)\,\nu + k' = 0,$$

which equations, together with the relation

$$\lambda + \mu + \nu = 0,$$

give upon the elimination of $\lambda : \mu : \nu : k'$,

$$\begin{vmatrix} u - \phi bc \cos A, & w', & v', & 1 \\ w', & v - \phi ca \cos B, & u', & 1 \\ v', & u', & w - \phi ab \cos C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

a quadratic to determine ϕ.

This quadratic may be written

$$4\Delta^2\phi^2 - 2\Delta E\phi = K,$$

one of the roots of which will be the maximum, and the other the minimum value of $f(\lambda, \mu, \nu)$.

285. *To find the lengths of the axes of the conic whose equation is*

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

Let $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be the coordinates of the centre. Then the length of the semi-diameter in the direction (λ, μ, ν) is given by

$$\rho^2 = -\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{f(\lambda, \mu, \nu)},$$

and this must be a maximum or minimum.

Hence $f(\lambda, \mu, \nu)$ must be a minimum or maximum, and must therefore have one of the values of ϕ given by the quadratic of Art. 283 for trilinear coordinates, and Art. 284 for triangular coordinates.

Therefore $$\rho^2 = -\frac{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma})}{\phi}$$

$$= \frac{H}{K\phi} \text{ (Art. 268)},$$

or $$\phi = \frac{H}{K\rho^2}.$$

Now suppose the coordinates are

trilinear, or *triangular*,

so that

$$\phi^2 - E\phi = \frac{16\Delta^4}{a^2b^2c^2}K, \quad \Bigg| \quad \phi^2 - \frac{E\phi}{2\Delta} = \frac{K}{4\Delta^2}.$$

Then, substituting the value of ϕ given above,

$$K^3\rho^4 + \frac{a^2b^2c^2}{16\Delta^4}EHK\rho^2 = \frac{a^2b^2c^2}{16\Delta^4}H^2, \quad \Bigg| \quad K^3\rho^4 + 2\Delta EHK\rho^2 = 4\Delta^2H^2,$$

a quadratic equation, giving two values for ρ^2, expressing the squares on the two semi-axes.

Hence, if $\mathfrak{A}$ and $\mathfrak{B}$ denote these two semi-axes, we have

$$\mathfrak{A}\mathfrak{B} = \frac{abc}{4\Delta^2}\,\frac{H}{(-K)^{\frac{3}{2}}}, \qquad \mathfrak{A}\mathfrak{B} = 2\Delta\,\frac{H}{(-K)^{\frac{3}{2}}},$$

and

$$\mathfrak{A}^2 + \mathfrak{B}^2 = -\frac{a^2b^2c^2}{16\Delta^4}\,\frac{EH}{K^2}. \qquad \mathfrak{A}^2 + \mathfrak{B}^2 = -2\Delta\,\frac{EH}{K^2}.$$

286. Cor. Considering the quadratic in ρ^2, we observe that the two values of ρ^2 will be of opposite sign, and therefore one value of $\pm\rho$ *real* and the other *imaginary* if $K > 0$,—the condition that the conic should be a *hyperbola.*

The two values of ρ^2 will be of the same sign, and therefore *both* values of $\pm\rho$ *real* or *both imaginary* if $K < 0$,—the condition that the conic should be an *ellipse.*

And further, the ellipse will be real when E and H are of opposite sign, and imaginary when they are of the same sign.

Again, both the values of ρ^2, and therefore both values of $\pm\rho$ will be infinite if $K = 0$,—the condition that the conic should be a parabola.

Similarly, if $E = 0$, the values of ρ^2 are equal and of opposite sign, and the conic is a rectangular hyperbola, as we saw in Art. 247, Cor. 4.

So, if $H = 0$, the two values of ρ^2 are zero, and the conic degenerates into two straight lines, as we proved in Art. 245.

287. *To find the length of the latus rectum in a parabola represented by the equation*

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

In the general case, we have by the preceding articles,

$\mathfrak{A}^2+\mathfrak{B}^2=-\frac{a^2b^2c^2}{16\Delta^4}\,\frac{EH}{K^2}$,	$\mathfrak{A}^2+\mathfrak{B}^2=-2\Delta\,\frac{EH}{K^2}$,
and $\mathfrak{A}^{\frac{4}{3}}\mathfrak{B}^{\frac{4}{3}}=\frac{a^{\frac{4}{3}}b^{\frac{4}{3}}c^{\frac{4}{3}}}{(2\Delta)^{\frac{8}{3}}}\,\frac{H^{\frac{4}{3}}}{K^2}$.	$\mathfrak{A}^{\frac{4}{3}}\mathfrak{B}^{\frac{4}{3}}=(2\Delta)^{\frac{4}{3}}\,\frac{H^{\frac{4}{3}}}{K^2}$.
Therefore by division, $\left(\frac{\mathfrak{A}}{\mathfrak{B}^2}\right)^{\frac{2}{3}}+\left(\frac{\mathfrak{B}}{\mathfrak{A}^2}\right)^{\frac{2}{3}}=-\frac{a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}E}{(2\Delta)^{\frac{4}{3}}H^{\frac{1}{3}}}$.	$\left(\frac{\mathfrak{A}}{\mathfrak{B}^2}\right)^{\frac{2}{3}}+\left(\frac{\mathfrak{B}}{\mathfrak{A}^2}\right)^{\frac{2}{3}}=-\frac{E}{(2\Delta)^{\frac{1}{3}}H^{\frac{1}{3}}}$.

In the case of the parabola (when $K=0$), one of the fractions

$$\frac{\mathfrak{A}}{\mathfrak{B}^2} \text{ and } \frac{\mathfrak{B}}{\mathfrak{A}^2},$$

is the reciprocal of the semi-latus rectum, and the other is zero. Hence if $\mathfrak{L}$ be the semi-latus rectum of the parabola, we obtain

$\mathfrak{L}^2=\frac{-(2\Delta)^4H}{a^2b^2c^2E^3}$.	$\mathfrak{L}^2=\frac{-2\Delta H}{E^3}$.

288. *To find the area of an ellipse whose equation is given.*

Let

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

represent an ellipse whose semi-axes are $\mathfrak{A}$ and $\mathfrak{B}$.

Then the area $=\pi\mathfrak{A}\mathfrak{B}$

$$=\frac{\pi abcH}{4\Delta^2(-K)^{\frac{3}{2}}}, \qquad \text{[trilinear}$$

or

$$=\frac{2\pi\Delta H}{(-K)^{\frac{3}{2}}}. \qquad \text{[triangular}$$

289. *To find the equation to the greatest ellipse which can be inscribed in the triangle of reference.*

Using triangular coordinates, let the equation to the ellipse be

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0.$$

The area $$= \frac{2\pi\Delta H}{(-K)^{\frac{3}{2}}};$$

therefore $$(\text{area})^2 \propto \frac{H^2}{K^3}.$$

But $$H = -4l^2m^2n^2,$$

$$K = -4lmn\,(l+m+n).$$

Therefore $$(\text{area})^2 \propto \frac{lmn}{(l+m+n)^3}.$$

But $\frac{lmn}{(l+m+n)^3}$ will have either a maximum or minimum value (by symmetry) when $l = m = n$, and *minima* values occur when

$$l = 0 \text{ or } m = 0 \text{ or } n = 0,$$

leading to the inference that $l = m = n$ produces a maximum.

Hence the equation of the maximum ellipse will be

$$\sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} = 0,$$

or $$\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta = 0.$$

Obs. In trilinear coordinates the equation will be

$$\sqrt{a\alpha} + \sqrt{b\beta} + \sqrt{c\gamma} = 0,$$

or $$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\beta\gamma - 2ca\gamma\alpha - 2ab\alpha\beta = 0.$$

Cor. Similar reasoning shews that the least ellipse circumscribing the triangle of reference is represented in triangular coordinates by the equation

$$\beta\gamma + \gamma\alpha + \alpha\beta = 0,$$

and in trilinear coordinates by the equation

$$\frac{\beta\gamma}{a} + \frac{\gamma\alpha}{b} + \frac{\alpha\beta}{c} = 0.$$

290. If H, H' be the discriminants, K, K' the bordered discriminants of the equations to two conics, and $E = 0$, $E' = 0$ the conditions that the equations should represent rectangular hyperbolas: then

(I) *If the conics be similar, K, K' will be in the duplicate ratio of E, E', and*

(II) *If the conics be also similarly situated, we shall have $K=K'$ and $E=E'$, and the linear dimensions of the two conics will be in the subduplicate ratio of $H : H'$.*

Suppose the conics are similar, and that the linear dimensions of the first are to those of the second as $1 : x$.

Then since the sum of the squares on the axes must be in the duplicate ratio of the linear dimensions, therefore (Art. 285)

$$\frac{HE}{K^2} : \frac{H'E'}{K'^2} = 1 : x^2 \qquad (1).$$

But the rectangle contained by the axes must also be in the duplicate ratio of the linear dimensions, and therefore (Art. 285)

$$\frac{H}{K^{\frac{3}{2}}} : \frac{H'}{K'^{\frac{3}{2}}} = 1 : x^2 \qquad (2).$$

Comparing (1) and (2), we obtain

$$\frac{E}{K^{\frac{1}{2}}} = \frac{E'}{K'^{\frac{1}{2}}},$$

or

$$K : K' = E^2 : E'^2 \qquad (3).$$

Next, suppose that the conics are not only similar but also similarly situated, then the squares on parallel diameters must be in the duplicate ratio of the linear dimensions; therefore we have, (in virtue of Arts. 279, and 268),

$$\frac{H}{K} : \frac{H'}{K'} = 1 : x^2 \qquad (4).$$

Hence comparing (2) and (4), we have

$$K = K' \qquad (5),$$

and therefore in virtue of (3),

$$E = E' \qquad (6),$$

and from (2),

$$H : H' = 1 : x^2,$$

or $$H^{\frac{1}{2}} : H'^{\frac{1}{2}} = 1 : x \quad \text{.......................... (7).}$$

Thus the propositions are proved.

291. Cor. If H and H' be of opposite signs the ratio of the linear dimensions is an imaginary quantity, and therefore in whatever direction diameters can be drawn, terminated in real points in one of the conics, parallel diameters in the other conic will meet the conic in imaginary points, and vice versa. An instance is afforded in a pair of conjugate hyperbolas.

292. We will conclude this chapter by observing that the equation to any two straight lines satisfies the condition of representing a conic section.

If the two straight lines are imaginary but intersect in a real point, the equation may be regarded as representing an indefinitely small ellipse, and the two imaginary straight lines will be the ultimate form of the imaginary branches of the ellipse which have now coincided with the asymptotes.

If the straight lines be real they may be regarded as the limiting case of the hyperbola, when the imaginary part connecting the two real branches has become evanescent.

We may state the result as follows:

If the real part of a conic degenerates into a point, the imaginary part will become two straight lines coincident with the asymptotes, and if the imaginary part becomes evanescent, the real part will become two straight lines coincident with the asymptotes.

These cases of conic sections will be evidently obtained by cutting a cone by a plane passing through the vertex, the first case occurring when the plane passes between the two sheets of the cone, the second when it intersects the sheets.

EXERCISES ON CHAPTER XVIII.

(163) With the notation of Art. 254, shew that

$$f\left(\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right) = 4F(\alpha, \beta, \gamma).$$

(164) If $f(\alpha, \beta, \gamma) = 0$ be the equation to a conic in trilinear coördinates, the polars with respect to it of the points of reference will form a triangle whose area is

$$\frac{1}{8}\frac{abcH^2}{ABC\Delta^2}.$$

Verify the result in the particular case of the conic represented by

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

(165) If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be any three points, their polars with respect to the conic $f(\alpha, \beta, \gamma) = 0$ form a triangle whose area is

$$\frac{\dfrac{abcH^2}{8\Delta^2}\begin{vmatrix}\alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\\ \alpha_3, & \beta_3, & \gamma_3\end{vmatrix}}{\begin{vmatrix}A, & B, & C\\ \alpha_2, & \beta_2, & \gamma_2\\ \alpha_3, & \beta_3, & \gamma_3\end{vmatrix}\begin{vmatrix}A, & B, & C\\ \alpha_3, & \beta_3, & \gamma_3\\ \alpha_1, & \beta_1, & \gamma_1\end{vmatrix}\begin{vmatrix}A, & B, & C\\ \alpha_1, & \beta_1, & \gamma_1\\ \alpha_2, & \beta_2, & \gamma_2\end{vmatrix}}.$$

(166) Each of a series of parallel chords in a conic is divided so that the rectangle under the segments is constant. Shew that the points of section lie upon a similar, similarly situated and concentric conic.

(167) Each of a series of parallel chords in a conic is divided so that the algebraical sum of the reciprocals of the segments is constant. Shew that the points of section lie upon a similar and similarly situated conic which cuts the original conic at the extremities of the diameter bisecting the parallel chords.

(168) A straight line is drawn from the focus of a conic to meet the tangent at a constant angle; find the locus of the

point of intersection, and shew that, in the case of the parabola, the locus will *always* touch it, but in the case of the other two curves it will only touch them (in one or two points) under certain conditions.

(169) If A', B', C', K' be the values of the functions A, B, C, K (Arts. 255, 256) for the equation

$$f(\alpha, \beta, \gamma) + 2(a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0,$$

shew that

$$A - A' = l(c^2v + b^2w - 2bcu') - m(abw - acu' - bcv' + c^2w') \\ - n(acv - abu' + b^2v' - bcw'),$$

with similar expressions for $B - B'$ and $C - C'$, the coordinates being trilinear. Hence deduce $K = K'$.

(170) With the notation of the last exercise, shew that

$$l(A - A') + m(B - B') + n(C - C') = F,$$

where $$F \equiv f\left(\begin{vmatrix} b, & c \\ m, & n \end{vmatrix}, \begin{vmatrix} c, & a \\ n, & l \end{vmatrix}, \begin{vmatrix} a, & b \\ l, & m \end{vmatrix}\right).$$

Hence the result of the last exercise may be written

$$A - A' = \frac{1}{2}\frac{dF}{dl}, \quad B - B' = \frac{1}{2}\frac{dF}{dm}, \quad C - C' = \frac{1}{2}\frac{dF}{dn}.$$

(171) If H' be the discriminant of the equation

$$f(\alpha, \beta, \gamma) + 2(l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0,$$

then

$$H' - H = l(A + A') + m(B + B') + n(C + C').$$

(172) If θ be the angle between the asymptotes of the conic $f(\alpha, \beta, \gamma) = 0$, then

$$\sin^2\theta = \frac{8\Delta K}{E^2}\sin A \sin B \sin C, \qquad [\textit{trilinear}$$

or $$\sin^2\theta = \frac{4K}{E^2}. \qquad [\textit{triangular}$$

CHAPTER XIX.

CIRCLES.

293. *To find the equation in trilinear coordinates to a circle whose radius and centre are given.*

Let r be the given radius and α', β', γ' the coordinates of the given centre.

And let σ denote the expression

$$\frac{a\alpha + b\beta + c\gamma}{2\Delta},$$

so that $\sigma = 0$ is the equation to the straight line at infinity.

Let (α, β, γ) be any point on the circle: then since r is the distance between the points (α, β, γ) and $(\alpha', \beta', \gamma')$ we have (Art. 72),

$$(\alpha - \alpha')^2 \sin 2A + (\beta - \beta')^2 \sin 2B + (\gamma - \gamma')^2 \sin 2C$$
$$= 2r^2 \sin A \sin B \sin C,$$

an equation which may be written in the homogeneous form

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$- 2\sigma (\alpha\alpha' \sin 2A + \beta\beta' \sin 2B + \gamma\gamma' \sin 2C)$$
$$+ \sigma^2 (\alpha'^2 \sin 2A + \beta'^2 \sin 2B + \gamma'^2 \sin 2C - 2r^2 \sin A \sin B \sin C) = 0,$$

a relation among the coordinates of any point $(\alpha', \beta', \gamma')$ on the circle and therefore the equation to the circle.

Obs. In triangular coordinates, if σ denote $\alpha+\beta+\gamma$ the equation to the circle at a distance r from the centre $(\alpha', \beta', \gamma')$ will become

$$\alpha^2 \cot A + \beta^2 \cot B + \gamma^2 \cot C$$
$$-2\sigma(\alpha\alpha' \cot A + \beta\beta' \cot B + \gamma\gamma' \cot C)$$
$$+\sigma^2\left(\alpha'^2 \cot A + \beta'^2 \cot B + \gamma'^2 \cot C - \frac{r^2}{2\Delta}\right) = 0.$$

294. If we render the equation in trilinear coordinates, of the last article, homogeneous with respect to the coordinates $(\alpha', \beta', \gamma')$ of the centre, it may be written

$$\alpha^2(\beta'^2 + \gamma'^2 + 2\beta'\gamma' \cos A - r^2 \sin^2 A)$$
$$+\beta^2(\gamma'^2 + \alpha'^2 + 2\gamma'\alpha' \cos B - r^2 \sin^2 B)$$
$$+\gamma^2(\alpha'^2 + \beta'^2 + 2\alpha'\beta' \cos C - r^2 \sin^2 C)$$
$$+2\beta\gamma\{(\alpha'^2 - r^2) \sin B \sin C - (\beta' + \alpha' \cos C)(\gamma' + \alpha' \cos B)\}$$
$$+2\gamma\alpha\{(\beta'^2 - r^2) \sin C \sin A - (\gamma' + \beta' \cos A)(\alpha' + \beta' \cos C)\}$$
$$+2\alpha\beta\{(\gamma'^2 - r^2) \sin A \sin B - (\alpha' + \gamma' \cos B)(\beta' + \gamma' \cos A)\} = 0.$$

295. Referring to the equations of Art. 293 it will be observed that the radius r is only introduced in *the coefficient of* σ^2. Hence it follows that

If $O=0$ be the equation to any circle, any concentric circle will have the equation $O+k\sigma^2=0$, where k is some constant.

296. Again, referring to the same equation, the terms which are independent of σ are free from α', β', γ' and r, they will therefore be the same for all circles. Hence

If $O=0$ be the equation to any circle, any other circle will be represented by the equation $O+u\sigma=0$, where $u=0$ is the equation to some straight line.

This is equivalent to saying that any two circles have a common chord at infinity, or meet the line at infinity in the same two points, a property which is exhibited more explicitly in the next article.

297. *Every circle passes through the two imaginary points called the circular points at infinity.* (See the definition Art. 109.)

For any circle is represented (Art. 293) by the equation

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$- 2\sigma(\alpha\alpha' \sin 2A + \beta\beta' \sin 2B + \gamma\gamma' \sin 2C)$$
$$+ \sigma^2 (\alpha'^2 \sin 2A + \beta'^2 \sin 2B + \gamma'^2 \sin 2C - 2r^2 \sin A \sin B \sin C) = 0.$$

And substituting $\sigma = 0$, we find the points of intersection with the line at infinity to be determined by

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0;$$

and we shewed in Art. 108 that this equation determined on the straight line at infinity the same two points as any equation of the system

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0,$$
$$\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B = 0,$$
$$\alpha^2 + \beta^2 + 2\alpha\beta \cos C = 0,$$

the loci of which intersect in the two circular points; which proves the proposition.

298. Conversely, *every conic which passes through the two circular points is a circle.*

For if not, take any three points P, Q, R on the conic, and describe a circle about the triangle PQR: then the conic and the circle intersect in the points P, Q, R as well as in the two circular points, i.e. they intersect in five points, which is impossible since two loci of the second order can only intersect in four real or imaginary points (Art. 162). Hence no conic but a circle can pass through the two circular points.

299. *If $O = 0$, $O' = 0$ be the equations to two circles, then will $kO + k'O' = 0$ also represent a circle.*

For by Art. 168 the last equation represents a locus passing through all the points of intersection of the first two. But, by Art. 297, the loci of the first two equations intersect in the circular points, therefore the locus of the third passes through these two points, and therefore the locus is a circle (Art. 298).

300. Since the straight line at infinity meets a circle in the two circular points, it cannot meet it in any other point. Hence the other two points of intersection of any two circles are not at infinity. They may be real or imaginary, but must be finite.

Def. The straight line joining the two finite points of intersection of two circles is called the *Radical axis* of the two circles.

301. *Any two real circles, whether they intersect in real or imaginary points, have a real radical axis.*

For let $O=0$ be the equation to one of the circles. Then the other may be represented by the equation

$$O+u\sigma=0,$$

where $u=0$ is the equation to a real straight line (Art. 296).

But (Art. 165) $u=0$, $\sigma=0$ are a pair of common chords of the two circles, and we know that the two points of intersection at infinity lie on the locus of $\sigma=0$, therefore $u=0$ joins the two finite points, or $u=0$ is the radical axis.

Hence any two circles have a real radical axis.

Cor. We see at once (from symmetry or otherwise) that the radical axis of two circles is at right angles to the straight line joining the centres.

302. *The three radical axes of three circles are concurrent.*

For if the three circles be represented by the equations

$$O+u\sigma=0,$$
$$O+v\sigma=0,$$
$$O+w\sigma=0,$$

their radical axes will have the equations

$$v-w=0, \quad w-u=0, \quad u-v=0,$$

and therefore all pass through the point given by

$$u=v=w.$$

303. *If a system of circles have a common radical axis, the polars with respect to them of any fixed point are concurrent.*

Let $O = 0$ be the equation to any circle of the system, and let $u = 0$ be the equation to the common radical axis.

Then, by giving different values to k, every circle in the system will be represented by the equation

$$O + ku\sigma = 0 \text{ ....(1)}.$$

Now let $(\alpha', \beta', \gamma')$ be the fixed point, and let O', u', and σ' denote what O, u, and σ become when α', β', γ' are substituted for α, β, γ.

Then the polar of the point $(\alpha', \beta', \gamma')$ with respect to the circle (1) will be represented by

$$\alpha \frac{dO'}{d\alpha'} + \beta \frac{dO'}{d\beta'} + \gamma \frac{dO'}{d\gamma'} + k(u\sigma' + u'\sigma) = 0 \text{ ......(2)},$$

and it will therefore pass through the point given by

$$\alpha \frac{dO'}{d\alpha'} + \beta \frac{dO'}{d\beta'} + \gamma \frac{dO'}{d\gamma'} = 0 \text{ ....(3)}$$

and $$u\sigma' + u'\sigma = 0 \text{ ....(4)},$$

whatever be the value of k.

Hence the polars of the point $(\alpha', \beta', \gamma')$ with respect to all the conics are concurrent.

304. For convenience of reference we will now recapitulate the results arrived at in the foregoing chapters respecting some particular cases of circles (see Chapters XIV. XV. XVI.).

I. *The circle with respect to which the triangle of reference is self-conjugate* is represented (Art. 179) by the equation in trilinear coordinates

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0,$$

which is obviously a particular case of the equation of Art. 293.

The centre of this circle is given by

$$\alpha \cos A = \beta \cos B = \gamma \cos C,$$

and is the point of intersection of perpendiculars from the angular points of the triangle of reference on the opposite sides.

In triangular coordinates the same circle is represented by the equation

$$\alpha^2 \cot A + \beta^2 \cot B + \gamma^2 \cot C = 0,$$

and its centre is given by

$$\alpha \cot A = \beta \cot B = \gamma \cot C.$$

II. *The circle passing through the three points of reference* is given (Art. 195) by the equation in trilinear coordinates

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0,$$

and its centre is given by the equations

$$\frac{\alpha}{\cos A} = \frac{\beta}{\cos B} = \frac{\gamma}{\cos C}.$$

The same circle is represented in triangular coordinates by the equation

$$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta = 0,$$

and its centre is given by

$$\frac{\alpha}{\sin 2A} = \frac{\beta}{\sin 2B} = \frac{\gamma}{\sin 2C}.$$

III. *The circle inscribed in the triangle of reference* is represented (Art. 214) by the equation in trilinear coordinates

$$\sqrt{\alpha} \cos \frac{A}{2} + \sqrt{\beta} \cos \frac{B}{2} + \sqrt{\gamma} \cos \frac{C}{2} = 0,$$

and its centre is given by

$$\alpha = \beta = \gamma.$$

In triangular coordinates the same circle is represented by the equation

$$\sqrt{\alpha \cot \frac{A}{2}} + \sqrt{\beta \cot \frac{B}{2}} + \sqrt{\gamma \cot \frac{C}{2}} = 0,$$

and the centre is given by

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c}.$$

IV. *The circles escribed to the triangle of reference* (i. e. touching one side of the triangle and the other sides produced) are represented (Art. 214) by the equations

$$\sqrt{-\alpha}\cos\frac{A}{2}+\sqrt{\beta}\sin\frac{B}{2}+\sqrt{\gamma}\sin\frac{C}{2}=0,$$

$$\sqrt{\alpha}\sin\frac{A}{2}+\sqrt{-\beta}\cos\frac{B}{2}+\sqrt{\gamma}\sin\frac{C}{2}=0,$$

$$\sqrt{\alpha}\sin\frac{A}{2}+\sqrt{\beta}\sin\frac{B}{2}+\sqrt{-\gamma}\sin\frac{C}{2}=0,$$

and their centres are given respectively by

$$-\alpha=\beta=\gamma,$$

$$\alpha=-\beta=\gamma,$$

$$\alpha=\beta=-\gamma.$$

The same circles are represented in triangular coordinates by the equations

$$\sqrt{-\alpha\cot\frac{A}{2}}+\sqrt{\beta\tan\frac{B}{2}}+\sqrt{\gamma\tan\frac{C}{2}}=0,$$

$$\sqrt{\alpha\tan\frac{A}{2}}+\sqrt{-\beta\cot\frac{B}{2}}+\sqrt{\gamma\tan\frac{C}{2}}=0,$$

$$\sqrt{\alpha\tan\frac{A}{2}}+\sqrt{\beta\tan\frac{B}{2}}+\sqrt{-\gamma\cot\frac{C}{2}}=0,$$

and their centres are given respectively by

$$-\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c},$$

$$\frac{\alpha}{a}=-\frac{\beta}{b}=\frac{\gamma}{c},$$

$$\frac{\alpha}{a}=\frac{\beta}{b}=-\frac{\gamma}{c}.$$

305. It should be observed that

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$+ 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)$$
$$\equiv 2(\alpha \sin A + \beta \sin B + \gamma \sin C)(\alpha \cos A + \beta \cos B + \gamma \cos C).$$

Hence the straight line represented in trilinear coordinates by the equation

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0,$$

is the radical axis of the circle circumscribing the triangle of reference, and the circle with respect to which that triangle is self-conjugate.

306. *To find the equation to the circle which passes through the middle points of the sides of the triangle of reference.*

Let the equation be (Art. 293)

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$= (l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C).$$

Since the circle passes through the middle point of BC it must be satisfied by

$$\alpha = 0, \quad \beta \sin B = \gamma \sin C,$$

hence we get

$$\cot B + \cot C = \frac{m}{\sin B} + \frac{n}{\sin C},$$

or

$$\sin A = m \sin C + n \sin B.$$

Similarly, since the circle bisects CA and AB,

$$\sin B = n \sin A + l \sin C,$$
$$\sin C = l \sin B + m \sin A,$$

these equations give

$$l = \cos A, \quad m = \cos B, \quad n = \cos C.$$

Hence the required equation is

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$= (\alpha \cos A + \beta \cos B + \gamma \cos C)(\alpha \sin A + \beta \sin B + \gamma \sin C) \ldots\ldots (1),$$

or, in virtue of the identity of Art. 305,

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$= 2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) \quad \ldots\ldots\ldots (2),$$

or again, by comparison of (1) and (2),

$$2(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C)$$
$$= (\alpha \cos A + \beta \cos B + \gamma \cos C)(\alpha \sin A + \beta \sin B + \gamma \sin C) \ldots (3).$$

All these three forms of the equation are useful. The second shews that the circle passes through the points of intersection of the circumscribed circle, and the circle with respect to which the triangle of reference is self-conjugate: the first and third shew that its radical axis with respect to either of these circles is the straight line

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

307. *To find where the circle which bisects the sides of the triangle of reference meets those sides again.*

The equation to the circle is

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$- 2\beta\gamma \sin A - 2\gamma\alpha \sin B - 2\alpha\beta \sin C = 0.$$

Putting $\alpha = 0$, we get

$$\beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma \sin A = 0,$$

or

$$\beta^2 \sin B \cos B + \gamma^2 \sin C \cos C - \beta\gamma (\sin B \cos C + \sin C \cos B) = 0,$$

or $$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos B) = 0,$$

shewing that the circle meets BC in the two points determined by

$$\alpha = 0, \quad \beta \sin B = \gamma \sin C,$$

and $$\alpha = 0, \quad \beta \cos B = \gamma \cos C,$$

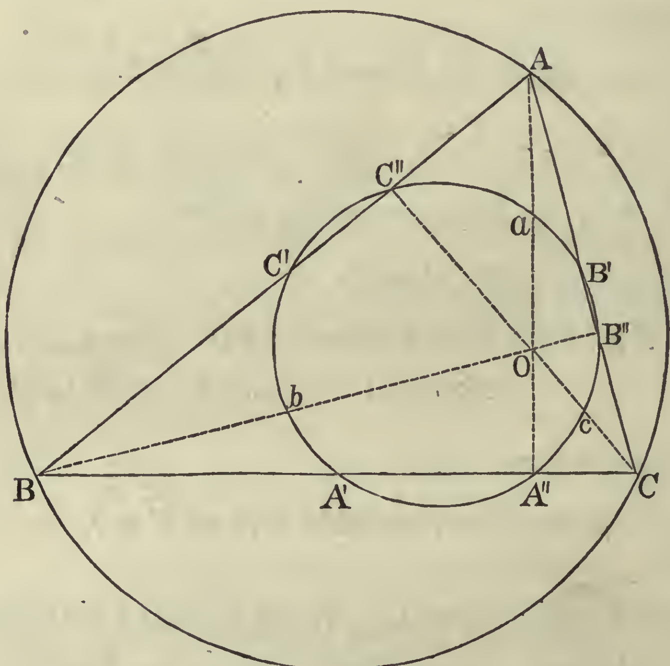
that is (Arts. 14, 15), in the middle point of BC, and in the foot of the perpendicular from A. And similarly for the other sides.

Hence *the circle which bisects the sides of the triangle of reference passes also through the feet of the perpendiculars from the opposite angles.*

308. But this circle has other significant properties with respect to the triangle.

Thus, let A', B', C' (fig. 37) be the middle points of the sides of the triangle ABC, and let AA'', BB'', CC'' be the perpendiculars from the opposite angles, and O their point of intersection. Bisect OA, OB, OC in a, b, c respectively.

Fig. 37.



Then, since A'', B'', C'' are the feet of the perpendiculars from the angular points on the sides of the triangle OBC, therefore by the last article the circle through A'', B'', C'' will bisect the sides of the triangle OBC, and therefore will pass through b and c. Similarly it will pass through a. Hence the same circle which passes through A', B', C', A'', B'', C'' passes also through a, b, c. In consequence of all these nine points lying on its circumference, this circle is called the *nine-points' circle* of the triangle ABC.

309. It is easily seen that the property last proved is only a particular case of the following more general theorem.

Any straight line drawn from O, the point of intersection of the perpendiculars of the triangle ABC to meet the circumference of the circumscribed circle is bisected by the nine-points' circle.

310. Any circle may be represented (Art. 296) by the equation

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0 \ldots (1).$$

Hence if

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \ldots\ldots (2),$$

be the equation to a circle, this equation must be identical with

$$k(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + \left(\frac{u\alpha}{a} + \frac{v\beta}{b} + \frac{w\gamma}{c}\right)(a\alpha + b\beta + c\gamma) = 0 \ldots (3).$$

311. From the result stated in the last article we may readily deduce the conditions (already established by a different method in Art. 244), that the general equation of the second degree should represent a circle.

For, comparing the equations (2) and (3), and equating coefficients, we get

$$2u' = ka + \frac{vc}{b} + \frac{wb}{c},$$

$$2v' = kb + \frac{wa}{c} + \frac{uc}{a},$$

$$2w' = kc + \frac{ub}{a} + \frac{va}{b},$$

or

$$-kabc = vc^2 + wb^2 - 2u'bc = wa^2 + uc^2 - 2v'ca = ub^2 + va^2 - 2u'ab,$$

the same conditions as we found in the article referred to.

312. The equation in trilinear coordinates to the circle inscribed in the triangle of reference may be written

$$\alpha^2 \cos^4 \frac{A}{2} + \beta^2 \cos^4 \frac{B}{2} + \gamma^2 \cos^4 \frac{C}{2} - 2\beta\gamma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \&c. = 0.$$

Hence (Art. 310) it must take the form

$$k (a\beta\gamma + b\gamma\alpha + c\alpha\beta)$$
$$+ \left(\frac{\alpha}{a} \cos^4 \frac{A}{2} + \frac{\beta}{b} \cos^4 \frac{B}{2} + \frac{\gamma}{c} \cos^4 \frac{C}{2}\right) (a\alpha + b\beta + c\gamma) = 0,$$

where (by Art. 311),

$$-abck = c^2 \cos^4 \frac{B}{2} + b^2 \cos^4 \frac{C}{2} + 2bc \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}$$
$$= \left(c \cos^2 \frac{B}{2} + b \cos^2 \frac{C}{2}\right)^2$$
$$= s^2,$$

s denoting as in trigonometry $\frac{1}{2}(a+b+c)$; therefore the equation to the circle becomes

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{abc}{s^2} \left(\frac{\alpha}{a} \cos^4 \frac{A}{2} + \frac{\beta}{b} \cos^4 \frac{B}{2} + \frac{\gamma}{c} \cos^4 \frac{C}{2}\right) (a\alpha + b\beta + c\gamma)$$
$$= \frac{1}{abc} \left\{a\alpha (s-a)^2 + b\beta (s-b)^2 + c\gamma (s-c)^2\right\} (a\alpha + b\beta + c\gamma).$$

By a similar method we may shew that the equations to the escribed circle opposite to A may be written

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{abc}{(s-a)^2} \left(\frac{\alpha}{a} \cos^4 \frac{A}{2} + \frac{\beta}{b} \sin^4 \frac{B}{2} + \frac{\gamma}{c} \sin^4 \frac{C}{2}\right) (a\alpha + b\beta + c\gamma),$$

or $$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{1}{abc} \left\{a\alpha s^2 + b\beta (s-c)^2 + c\gamma (s-b)^2\right\} (a\alpha + b\beta + c\gamma),$$

and by symmetry the other two escribed circles may be represented by the equations

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{1}{abc}\left\{a\alpha(s-c)^2 + b\beta s^2 + c\gamma(s-a)^2\right\}(a\alpha + b\beta + c\gamma),$$

and $$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{1}{abc}\left\{a\alpha(s-b)^2 + b\beta(s-a)^2 + c\gamma s^2\right\}(a\alpha + b\beta + c\gamma).$$

313. Cor. The radical axes of the circumscribed circle and the several inscribed (or escribed) circles are represented in trilinear coordinates by the equations

$$a\alpha(s-a)^2 + b\beta(s-b)^2 + c\gamma(s-c)^2 = 0,$$
$$a\alpha s^2 + b\beta(s-c)^2 + c\gamma(s-b)^2 = 0,$$
$$a\alpha(s-c)^2 + b\beta s^2 + c\gamma(s-a)^2 = 0,$$
$$a\alpha(s-b)^2 + b\beta(s-a)^2 + c\gamma s^2 = 0,$$

and therefore in triangular coordinates by the equations

$$\alpha(s-a)^2 + \beta(s-b)^2 + \gamma(s-c)^2 = 0,$$
$$\alpha s^2 + \beta(s-c)^2 + \gamma(s-b)^2 = 0,$$
$$\alpha(s-c)^2 + \beta s^2 + \gamma(s-a)^2 = 0,$$
$$\alpha(s-b)^2 + \beta(s-a)^2 + \gamma s^2 = 0.$$

314. The equation to the nine-points' circle may be written (Art. 306)

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{1}{2}(\alpha\cos A + \beta\cos B + \gamma\cos C)(a\alpha + b\beta + c\gamma),$$

and the equation to the inscribed circle is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \frac{1}{abc}\left\{a\alpha(s-a)^2 + b\beta(s-b)^2 + c\gamma(s-c)^2\right\}(a\alpha + b\beta + c\gamma),$$

therefore their radical axis is represented by

$$\alpha\left\{\cos A-\frac{2\,(s-a)^2}{bc}\right\}+\beta\left\{\cos B-\frac{2\,(s-b)^2}{ca}\right\}+\gamma\left\{\cos C-\frac{2\,(s-c)^2}{ab}\right\}=0,$$

or

$$\frac{\alpha}{bc}(c-a)(a-b)+\frac{\beta}{ca}(a-b)(b-c)+\frac{\gamma}{ab}(b-c)(c-a)=0,$$

or
$$\frac{a\alpha}{b-c}+\frac{b\beta}{c-a}+\frac{c\gamma}{a-b}=0\ldots\ldots\ldots\ldots(1).$$

Similarly the radical axes of the nine-points' circle with each of the several escribed circles will be seen to have the equations

$$\frac{a\alpha}{b-c}+\frac{b\beta}{c+a}-\frac{c\gamma}{a+b}=0\ldots\ldots\ldots\ldots(2)$$

$$-\frac{a\alpha}{b+c}+\frac{b\beta}{c-a}+\frac{c\gamma}{a+b}=0\ldots\ldots\ldots\ldots(3),$$

$$\frac{a\alpha}{b+c}-\frac{b\beta}{c+a}+\frac{c\gamma}{a-b}=0\ldots\ldots\ldots\ldots(4).$$

315. It will be observed that the equation (1) of the last article satisfies the condition (Art. 210) of tangency to the inscribed circle.

Hence the radical axis of the nine-points' circle and the inscribed circle is a tangent to the latter, and therefore the circles touch one another, and the radical axis is the tangent at the point of contact.

So the equations (2), (3), (4) of the last article will be seen to represent tangents to the escribed circles; hence the nine-points' circle touches these circles also.

That is, *the nine-points' circle of any triangle touches all the four circles which touch the sides of the triangle, or the sides produced.*

316. It should be observed that the trilinear equation

$$(a\alpha + b\beta + c\gamma)^2 = 0$$

satisfies all the conditions of representing a circle.

Its radius is infinite, and the equations to give the centre reduce to

$$\frac{\alpha}{0} = \frac{\beta}{0} = \frac{\gamma}{0},$$

shewing that the centre is indeterminate.

It represents therefore the circle spoken of in Art. 38, being the limiting form of a circle drawn from any centre at a distance which is indefinitely increased.

Every point at infinity lies upon the locus, which indeed is geometrically identical with the straight line at infinity as explained in Art. 38, but analytically it must rather be regarded as equivalent to two straight lines, both lying altogether at infinity.

We shall have much more to say about this circle in the chapters on tangential coordinates and polar reciprocals.

317. The equation

$$(a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) = 0 \quad \text{.................. (1)},$$

also satisfies the conditions of representing a circle.

To explain this, consider any straight line

$$l\alpha + m\beta + n\gamma = 0 \quad \text{........................ (2)},$$

and suppose a circle is described from any centre O so as to touch this straight line.

If the distance of the centre from the straight line be increased, the circumference in contact with the straight line becomes less curved and tends to coincide with the straight line; and by increasing the radius indefinitely the circumference will coincide as nearly as we please with the straight line. Hence *ultimately* the part of the circle remaining in finite space coincides with the straight line, while there is still another part, at an infinite distance, which since its curvature is indefinitely small will ultimately coincide with the straight line at infinity.

Hence the equation (1) which we know represents the straight line (2) and the straight line at infinity, represents the ultimate form of an infinite circle:—which accounts for its satisfying the circular criteria.

318. Before we leave the subject of circles, we ought to observe as a particular case of Art. 292 that the equation to the straight lines joining any real point to the circular points at infinity satisfies all the conditions of representing a circle.

To take the very simplest case in ordinary Cartesian coordinates, the equation $x^2+y^2=0$ may be said to represent two imaginary straight lines through the origin, or to represent an indefinitely small circle at the origin.

So in trilinear coordinates the equation

$$\beta^2+\gamma^2+2\beta\gamma\cos A=0,$$

represents the two imaginary straight lines joining the point of reference A to the circular points at infinity, but it is also the equation to the indefinitely small circle round A.

Perhaps this circumstance is best represented by the following statement:

When the radius of a circle is indefinitely decreased the real part of the circle degenerates into a point, and the imaginary branches into two straight lines joining that point to the two circular points at infinity.

319. *To find the equation to an indefinitely small circle at the point* $(\alpha', \beta', \gamma')$.

The equations to the straight lines joining this point to the circular points are by Arts. 21 and 110,

$$\begin{vmatrix} -1, & \cos C\pm\sqrt{-1}\sin C, & \cos B\mp\sqrt{-1}\sin B \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}=0,$$

or if A, B, C denote respectively the determinants

$$\begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \quad \begin{vmatrix} \gamma, & \alpha \\ \gamma', & \alpha' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}$$

$$-A + B\cos C + C\cos B \pm \sqrt{-1}\,(B\sin C - C\sin B) = 0,$$

and therefore the two straight lines are given by

$$(A - B\cos C - C\cos B)^2 + (B\sin C - C\sin B)^2 = 0,$$

$$\text{or } \{A,\ B,\ C\}^2 = 0.$$

But by the last article these two imaginary straight lines coincide with the indefinitely small circle at $(\alpha', \beta', \gamma')$, therefore that circle is given by the equation

$$\left\{\begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \quad \begin{vmatrix} \gamma, & \alpha \\ \gamma', & \alpha' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}\right\}^2 = 0.$$

320. Cor. 1. A circle such that $(\alpha', \beta', \gamma')$ is the pole of the straight line

$$l(\alpha + \alpha') + m(\beta + \beta') + n(\gamma + \gamma') = 0,$$

is represented by the equation

$$\left\{\begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \quad \begin{vmatrix} \gamma, & \alpha \\ \gamma', & \alpha' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}\right\}^2$$

$$+ k(l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

321. Cor. 2. A particular case of considerable importance is that of the indefinitely small circle at the intersection of the perpendiculars of the triangle of reference.

This point is given by the equations

$$\alpha\cos A = \beta\cos B = \gamma\cos C,$$

and the equation to the circle reduces to

$$\{a\alpha\cos A,\ b\beta\cos B,\ c\gamma\cos C\}^2 = 0,$$

or in triangular coordinates

$$\{\alpha\cos A,\ \beta\cos B,\ \gamma\cos C\}^2 = 0.$$

322. Cor. 3. The equation to any circle may be written

$$\{a\alpha \cos A,\ b\beta \cos B,\ c\gamma \cos C\}^2 = (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma)$$

in trilinear coordinates, or

$$\{\alpha \cos A,\ \beta \cos B,\ \gamma \cos C\}^2 = (l\alpha + m\beta + n\gamma)(\alpha + \beta + \gamma)$$

in triangular coordinates, where

$$l(\alpha + \alpha') + m(\beta + \beta') + n(\gamma + \gamma') = 0$$

represents the polar of the point of intersection of the perpendiculars from the angular points of the triangle of reference on the opposite sides.

Exercises on Chapter XIX.

(173) The equation in triangular coordinates to the circle whose centre is at the point of reference A and whose radius is r may be written

$$(\beta + \gamma)(c^2\beta + b^2\gamma) - a^2\beta\gamma = r^2(\alpha + \beta + \gamma)^2,$$

or $(a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta) - (\alpha + \beta + \gamma)(c^2\beta + b^2\gamma) + r^2(\alpha + \beta + \gamma)^2 = 0.$

(174) The circles described on the sides of the triangle of reference as diameters are represented in trilinear coordinates by the equations

$$\beta\gamma = \alpha(\alpha \cos A - \beta \cos B - \gamma \cos C),$$

$$\gamma\alpha = \beta(\beta \cos B - \gamma \cos C - \alpha \cos A),$$

$$\alpha\beta = \gamma(\gamma \cos C - \alpha \cos A - \beta \cos B).$$

Deduce *Euclid*, III. 31.

(175) The polar of the centre of one circle with respect to another circle is parallel to the radical axis of the two circles.

(176) The distance of a point from its polar with respect to a circle is double of its distance from the radical axis of that circle and an indefinitely small circle described about the point as centre.

(177) A circle is described through the centres of three other circles: shew that the radical axes of the new circle with each of the others will be concurrent provided

$$r_1^2 \sin 2\theta_1 + r_2^2 \sin 2\theta_2 + r_3^2 \sin 2\theta_3 = 4\Delta,$$

where r_1, r_2, r_3 are the radii of the original circles, and θ_1, θ_2, θ_3 the angles which each pair of centres subtend at the remaining centre, and Δ the area of the triangle formed by the centres.

(178) The equation of the circle which passes through the three centres of the escribed circles of the triangle of reference is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + (a + b + c)(\beta\gamma + \gamma\alpha + \alpha\beta) = 0.$$

(179) The equations to the three circles which pass through the centre of the circle inscribed in the triangle of reference and through the centres of two of the escribed circles are

$$b\beta^2 + c\gamma^2 - a\alpha^2 + (b + c - a)(\beta\gamma - \gamma\alpha - \alpha\beta) = 0,$$
$$c\gamma^2 + a\alpha^2 - b\beta^2 + (c + a - b)(\gamma\alpha - \alpha\beta - \beta\gamma) = 0,$$
$$a\alpha^2 + b\beta^2 - c\gamma^2 + (a + b - c)(\alpha\beta - \beta\gamma - \gamma\alpha) = 0,$$

the coordinates being trilinear.

(180) Circles are drawn through the angular points of a triangle, through the centres of the three escribed circles, and through the centres of the inscribed and each two of the escribed circles; shew that the radical axes of these circles will meet the sides of the triangle at the points where they are cut by the straight lines bisecting the angles.

(181) Find the equations of the circles whose diameters are (i) the perpendiculars drawn from the angles of a triangle to the opposite sides, (ii) the lines bisecting the angles and terminated by the opposite sides, and (iii) the lines joining the angles with the middle points of the opposite sides.

(182) Find the locus of a point such that, if parallels be drawn through it to the three sides of a triangle, the sum of the rectangles under the three pairs of intercepts on each line respec-

tively, between the point and the two sides which it meets, shall be equal to a given rectangle.

(183) From a point in the plane of a given triangle draw perpendiculars to its sides, and find the locus of this point, when the area of the triangle formed by joining the feet of the perpendiculars is constant.

(184) The four tangents to the nine-points' circle of any triangle at the points where it touches the inscribed and escribed circles, all touch the maximum ellipse which can be inscribed in the triangle.

(185) If $f(\alpha, \beta, \gamma) = 0$ be the trilinear equation to a circle, the equation

$$\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\}^2 = 0$$

will represent an indefinitely small concentric circle.

(186) If $f(\alpha, \beta, \gamma) = 0$ be the trilinear equation to a circle at distance r from the centre $(\alpha', \beta', \gamma')$, then will

$$-\frac{1}{K} = \frac{\alpha'^2 - r^2}{U} = \frac{\beta'^2 - r^2}{V} = \frac{\gamma'^2 - r^2}{W}$$

$$= \frac{\beta'\gamma' + r^2 \cos A}{U'} = \frac{\gamma'\alpha' + r^2 \cos B}{V'} = \frac{\alpha'\beta' + r^2 \cos C}{W'}.$$

CHAPTER XX.

QUADRILINEAR COORDINATES.

323. In investigating theorems which involve four straight lines symmetrically, it is often advantageous to take them all as lines of reference using a system of four coordinates.

In the present chapter we shall exhibit and exemplify the use of such four coordinates.

324. Let α', β', γ' be trilinear coordinates of a point O, and let δ' be its distance from the straight line whose equation is

$$l\alpha + m\beta + n\gamma = 0;$$

then (Art. 46),

$$l\alpha' + m\beta' + n\gamma' = \{l, m, n\}\,\delta',$$

or

$$l\alpha' + m\beta' + n\gamma' + r\delta' = 0 \quad\ldots\ldots\ldots\ldots(1).$$

Also, if a, b, c, d be the sides of the quadrilateral formed by the three lines of reference and the other line, we have

$$a\alpha' + b\beta' + c\gamma' + d\delta' = \text{twice the area of the quadrilateral,}$$
$$= S, \text{ (suppose)} \quad\ldots\ldots\ldots\ldots(2).$$

Thus if α', β', γ', δ' be the perpendiculars from any point on the four sides of a quadrilateral, they are connected by the relations

$$l\alpha' + m\beta' + n\gamma' + r\delta' = 0,$$

$$a\alpha' + b\beta' + c\gamma' + d\delta' = S,$$

where l, m, n, r, a, b, c, d, S are constant functions of the sides and angles of the quadrilateral.

20—2

325. The perpendiculars α', β', γ', δ' might be used as the quadrilinear coordinates of the point O: but it will generally be found preferable to use certain multiples of these perpendiculars instead of the perpendiculars themselves to express the position of the point. In the few cases when it is advantageous to use the perpendiculars we shall expressly call them *perpendicular* coordinates. In all other cases when we speak of quadrilinear coordinates, the system set forth in the next article must be understood to be referred to.

326. Let $l\alpha'=\alpha$, $m\beta'=\beta$, $n\gamma'=\gamma$, $r\delta'=\delta$; then α, β, γ, δ being constant multiples of the perpendiculars, α', β', γ', δ' may be used as coordinates to express the position of the point O with respect to the lines of reference. (See Art. 105.) We shall call α, β, γ, δ the quadrilinear coordinates of the point O.

Also, referring to the notation of Art. 324, let

$$\frac{a}{lS}=A,\quad \frac{b}{mS}=B,\quad \frac{c}{nS}=C,\quad \frac{d}{rS}=D,$$

then the equations (1) and (2) of that article become

$$\alpha+\beta+\gamma+\delta=0\ldots\ldots\ldots\ldots\ldots(1),$$

$$A\alpha+B\beta+C\gamma+D\delta=1\ldots\ldots\ldots\ldots\ldots(2),$$

two equations which will always connect the quadrilinear coordinates of any point.

327. The equation (1) of the last article plays a very important part in the manipulation of equations in quadrilinear coordinates. It will be well to consider at once one or two cases of its application.

I. It enables us, at any time, to reduce our quadrilinear equations to trilinear, by eliminating one of the variables (δ suppose) by the substitution

$$\delta=-(\alpha+\beta+\gamma).$$

And since this transformation cannot affect the degree of an equation, it follows that, as in trilinear coordinates, an equation of the first degree represents a straight line, of the second a conic section, &c.

II. It gives us the power of eliminating, from any equation, terms of any particular argument that we may wish to be free from. For instance, we may without loss of generality eliminate from the general equation of the second degree, the terms whose arguments are α^2, β^2, γ^2, δ^2, by substituting

$$\alpha^2 = -(\alpha\beta + \alpha\gamma + \alpha\delta),$$

$$\beta^2 = -(\beta\gamma + \beta\delta + \beta\alpha),$$

$$\gamma^2 = -(\gamma\delta + \gamma\alpha + \gamma\beta),$$

$$\delta^2 = -(\delta\alpha + \delta\beta + \delta\gamma).$$

Hence there is no objection to our assuming the general equation of the second degree of the form

$$\lambda\beta\gamma + \lambda'\alpha\delta + \mu\gamma\alpha + \mu'\beta\delta + \nu\alpha\beta + \nu'\gamma\delta = 0,$$

or we may omit some of these terms and retain some of the others.

III. It must be observed that the same locus can be represented by an indefinite number of different equations, formed from one another by substitutions of the equation (1). Hence we are not at liberty to infer that because two different equations

$$l\alpha + m\beta + n\gamma + r\delta = 0,$$

and $$l'\alpha + m'\beta + n'\gamma + r'\delta = 0,$$

(suppose) represent the same straight line, the coefficients of like terms in the two are proportional: we may infer no more than that the two equations, together with

$$\alpha + \beta + \gamma + \delta = 0,$$

form a system of only two independent equations.

Thus too, the equation to the straight line at infinity is not restricted to the form

$$A\alpha + B\beta + C\gamma + D\delta = 0,$$

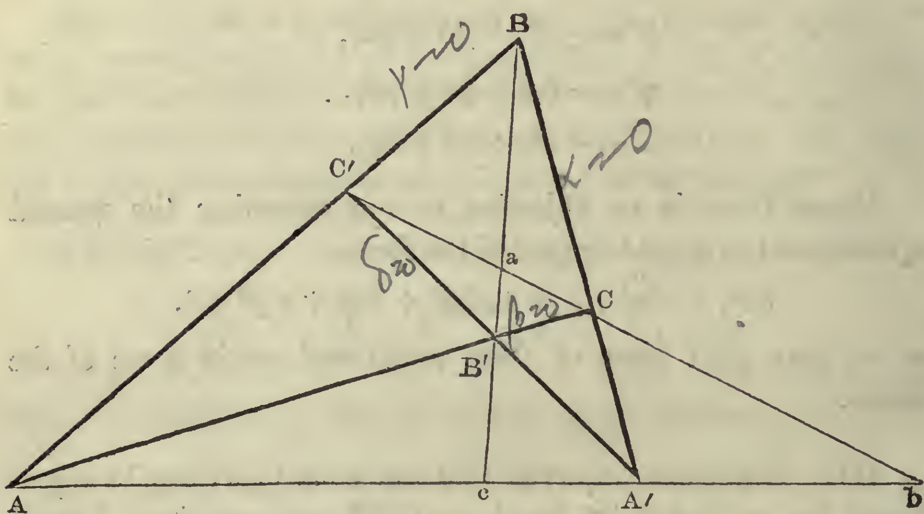
but, more generally, that straight line is represented by any equation of the form

$$(A+\kappa)\,\alpha+(B+\kappa)\,\beta+(C+\kappa)\,\gamma+(D+\kappa)\,\delta=0,$$

κ being any constant.

328. We shall assume the following construction throughout the present chapter.

Fig. 38.



Let AA', BB', CC' be the three diagonals of the complete quadrilateral formed by the lines of reference, and let $A'BC$, $AB'C$, ABC', $A'B'C'$ be the lines respectively denoted by

$$\alpha=0,\ \ \beta=0,\ \ \gamma=0,\ \ \delta=0.$$

Or, in other words, let ABC be the triangle formed by the lines

$$\alpha=0,\ \ \beta=0,\ \ \gamma=0,$$

and let A', B', C' be the points where these lines are respectively cut by the fourth line ($\delta=0$).

Also let BB', CC' intersect in a, CC', AA' in b, and AA', BB' in c, so that abc is the triangle formed by the diagonals of the parallelogram.

329. To interpret the constants A, B, C, D in equation (2) of Article 326, it is not necessary to trace the course by which they were introduced. The following interpretation is deduced immediately from the equations and is sufficient.

In virtue of the relation

$$\alpha + \beta + \gamma + \delta = 0 \qquad \text{(1)}$$

the equation

$$A\alpha + B\beta + C\gamma + D\delta = 1 \qquad \text{(2)}$$

may be written

$$(A - D)\,\alpha + (B - D)\,\beta + (C - D)\,\gamma = 1 \qquad \text{(3)}.$$

But α, β, γ are multiples of the trilinear coordinates of the point O; hence the equation (3) expresses implicitly an identical relation amongst the trilinear coordinates of the point, and therefore must be equivalent to the other forms in which the identical relation can be written (Art. 84), therefore the three terms

$$(A - D)\,\alpha,\ \ (B - D)\,\beta,\ \ (C - D)\,\gamma,$$

must represent respectively the ratios of the triangles OBC, OCA, OAB to the whole triangle ABC.

So by eliminating α we should prove that

$$(B - A)\,\beta,\ \ (C - A)\,\gamma,\ \ (D - A)\,\delta,$$

represent the ratios of the triangles OAB', OAC', $OB'C'$ to the whole triangle $AB'C'$.

Similarly,

$$(A - B)\,\alpha,\ \ (C - B)\,\gamma,\ \ (D - B)\,\delta,$$

represent the ratios of the triangles $OA'B$, OBC', $OC'A'$ to the whole triangle $O'BC'$.

And $$(A - C)\,\alpha,\ \ (B - C)\,\beta,\ \ (D - C)\,\delta,$$

represent the ratios of the triangles $OA'C$, $OB'C$, $OA'B'$ to the whole triangle $A'B'C$.

The student will observe that these results hold while α, β, γ, δ are the quadrilinear coordinates of any point whatever.

330. It will be easily seen that the coordinates of the six angular points of reference are as follows:

$$\text{of } A \ldots\ldots \alpha = -\delta = \frac{1}{A-D}, \quad \beta = \gamma = 0;$$

$$\text{of } B \ldots\ldots \beta = -\delta = \frac{1}{B-D}, \quad \gamma = \alpha = 0;$$

$$\text{of } C \ldots\ldots \gamma = -\delta = \frac{1}{C-D}, \quad \alpha = \beta = 0;$$

$$\text{of } A' \ldots\ldots \beta = -\gamma = \frac{1}{B-C}, \quad \alpha = \delta = 0;$$

$$\text{of } B' \ldots\ldots \gamma = -\alpha = \frac{1}{C-A}, \quad \beta = \delta = 0;$$

$$\text{of } C' \ldots\ldots \alpha = -\beta = \frac{1}{A-B}, \quad \gamma = \delta = 0.$$

Cor. The coordinates of the middle points of the diagonal AA' will be (by Art. 18),

$$\alpha = \frac{1}{2(A-D)}, \quad \beta = \frac{1}{2(B-C)}, \quad \gamma = \frac{-1}{2(B-C)}, \quad \delta = \frac{-1}{2(A-D)},$$

so the middle point of BB' is given by

$$\alpha = \frac{-1}{2(C-A)}, \quad \beta = \frac{1}{2(B-D)}, \quad \gamma = \frac{1}{2(C-A)}, \quad \delta = \frac{-1}{2(B-D)},$$

and the middle point of CC' is given by

$$\alpha = \frac{1}{2(A-B)}, \quad \beta = \frac{-1}{2(A-B)}, \quad \gamma = \frac{1}{2(C-D)}, \quad \delta = \frac{-1}{2(C-D)}.$$

331. We proceed to interpret some of the simplest equations connecting quadrilinear coordinates.

In virtue of equation (1) of Art. 326, the equations

$$\beta + \gamma = 0 \quad \text{and} \quad \alpha + \delta = 0 \ldots\ldots\ldots\ldots\ldots(1)$$

must be identically equivalent, and therefore represent the same straight line. But from their form, the first represents a straight line through A, and the second represents a straight line through A'. Hence either equation must represent the diagonal AA'.

Similarly either of the equations

$$\gamma + \alpha = 0 \quad \text{and} \quad \beta + \delta = 0 \text{..............} (2)$$

represents the diagonal BB', and either of the equations

$$\alpha + \beta = 0 \quad \text{and} \quad \gamma + \delta = 0 \text{..............} (3)$$

represents the diagonal CC'.

Again, the locus of the equation $\beta - \gamma = 0$ must pass through the intersection of BB', CC' as well as through the point A.

Hence	$\beta - \gamma = 0$	represents	Aa.
So	$\gamma - \alpha = 0$	....	Bb,
	$\alpha - \beta = 0$	....	Cc,
	$\alpha - \delta = 0$	....	$A'a$,
	$\beta - \delta = 0$	....	$B'b$,
and	$\gamma - \delta = 0$	....	$C'c$.

332. The equations in the last article immediately lead us to some of the most important harmonic properties of a quadrilateral.

Thus from the form of the equations we observe (Art. 129) that the lines AA' and Aa divide the angle at A harmonically. So also the pencils

$$\{B\,.\,AB'Cb\}, \quad \{C\,.\,BC'Ac\}$$

as well as the pencils

$$\{A'\,.\,CAB'a\}, \quad \{B'\,.\,ABC'b\}, \quad \{C'\,.\,BCA'c\}$$

are harmonic.

333. From the form of the equations in Art. 331, we observe that the straight lines

$$Aa, \; B'b, \; C'c$$

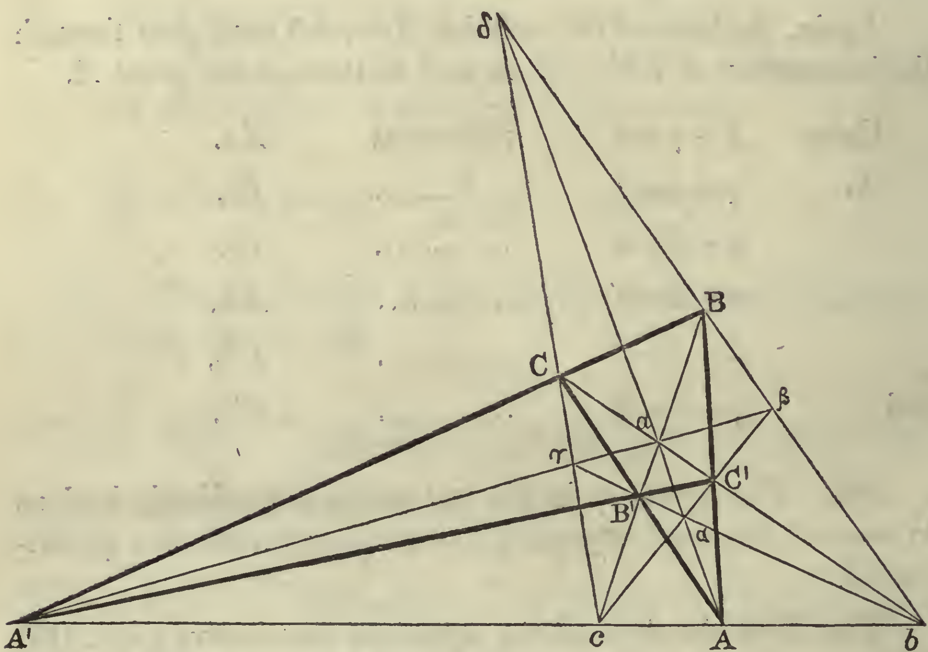
are concurrent. So

$$A'a,\quad Bb,\quad C'c,$$
$$A'a,\quad B'b,\quad Cc,$$
$$Aa,\quad Bb,\quad Cc,$$

are concurrent systems.

Let α (Fig. 39) be the point of intersection of the first set, β, γ, δ those of the other sets respectively,

Fig. 39.



then the points $\alpha, \beta, \gamma, \delta$ are given respectively by the systems of equations

$$\beta = \gamma = \delta,$$
$$\gamma = \delta = \alpha,$$
$$\delta = \alpha = \beta,$$
$$\alpha = \beta = \gamma.$$

334. *To find the condition that three points whose quadrilinear coordinates are given should be collinear.*

Let $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$, $(\alpha''', \beta''', \gamma''', \delta''')$, be the three points.

Suppose they lie upon a straight line whose equation free from δ is

$$l\alpha + m\beta + n\gamma = 0.$$

Then

$$l\alpha' + m\beta' + n\gamma' = 0,$$

$$l\alpha'' + m\beta'' + n\gamma'' = 0,$$

$$l\alpha''' + m\beta''' + n\gamma''' = 0,$$

therefore

$$\begin{vmatrix} \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha''', & \beta''', & \gamma''' \end{vmatrix} = 0 \quad\quad (1).$$

So if we had written the equation to the straight line free from γ we should have found the condition in the form

$$\begin{vmatrix} \alpha', & \beta', & \delta' \\ \alpha'', & \beta'', & \delta'' \\ \alpha''', & \beta''', & \delta''' \end{vmatrix} = 0,$$

which is obviously identical with (1), since

$$\alpha' + \beta' + \gamma' + \delta' \equiv 0,$$

$$\alpha'' + \beta'' + \gamma'' + \delta'' \equiv 0,$$

$$\alpha''' + \beta''' + \gamma''' + \delta''' \equiv 0.$$

Thus it will be convenient to write the condition in the form

$$\begin{vmatrix} \alpha', & \beta', & \gamma', & \delta' \\ \alpha'', & \beta'', & \gamma'', & \delta'' \\ \alpha''', & \beta''', & \gamma''', & \delta''' \end{vmatrix} = 0,$$

the unequal determinant denoting that we may take any three of the four columns to form a determinant equal to zero.

335. *The middle points of the three diagonals of a quadrilateral are collinear.*

Taking the sides of the quadrilateral as lines of reference and using the coordinates of the middle points obtained in

Art. 330, the condition that the points should be collinear will be (by the last article)

$$\begin{vmatrix} \frac{1}{A-D}, & \frac{1}{B-C}, & \frac{-1}{B-C} \\ \frac{1}{A-C}, & \frac{1}{B-D}, & \frac{-1}{A-C} \\ \frac{1}{A-B}, & \frac{-1}{A-B}, & \frac{1}{C-D} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} B-C, & A-D, & D-A \\ D-B, & C-A, & B-D \\ C-D, & D-C, & A-B \end{vmatrix} = 0,$$

which is seen to be an identity by the addition of its rows.

∴ &c. Q.E.D.

336. It may be shewn, as in trilinear coordinates, that the polar of the point $(\alpha', \beta', \gamma', \delta')$ with respect to a conic whose equation is

$$f(\alpha, \beta, \gamma, \delta) = 0,$$

is represented by the equation

$$\alpha \frac{df}{d\alpha'} + \beta \frac{df}{d\beta'} + \gamma \frac{df}{d\gamma'} + \delta \frac{df}{d\delta'} = 0.$$

Obs. If $(\alpha', \beta', \gamma', \delta')$ lie upon the conic this equation will represent the tangent thereat.

337. Referring to Art. 159 we observe that the general equation in quadrilinear coordinates to a conic passing through the four points B, B', C, C', is

$$\beta\gamma + k\alpha\delta = 0.$$

So the general equation to a conic through C, C', A, A', is

$$\gamma\alpha + k\beta\delta = 0,$$

and to a conic through A, A', B, B'

$$\alpha\beta + k\gamma\delta = 0.$$

338. As an example we may prove the well known theorem:

If a system of conics pass through four fixed points, the polars with respect to them of any fixed point are concurrent.

Take the four fixed points as the angular points B, C, B', C' of the quadrilateral of reference.

Then the equation to any conic in the system may be written

$$\beta\gamma + k\alpha\delta = 0.$$

Let $(\alpha', \beta', \gamma', \delta')$ be the fixed point, then the polar is given by the equation

$$\beta\gamma' + \beta'\gamma + k(\alpha\delta' + \alpha'\delta) = 0,$$

and therefore it passes always through the point determined by

$$\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0, \quad \text{and} \quad \frac{\alpha}{\alpha'} + \frac{\delta}{\delta'} = 0,$$

a fixed point since α', β', γ', δ' are constant.

Hence if a system of conics pass through four fixed points the polars with respect to them of any fixed point are concurrent.

COR. Let two of the four points be the circular points at infinity, then the theorem reduces to the following, which we proved otherwise in Art. 303.

If a system of circles have a common radical axis the polars with respect to them of any fixed point are concurrent.

339. *To find the equations to the tangents at B, C, B', C' to the conic whose equation is*

$$\beta\gamma + k\alpha\delta = 0.$$

The coordinates of B are (Art. 329)

$$\alpha = \gamma = 0, \quad \beta = -\delta = \frac{1}{B-D}.$$

Hence, applying Art. 336 the equation to the tangent at B will be

$$\gamma - k\alpha = 0.$$

So the tangent at B' will be given by

$$\beta - k\delta = 0.$$

Similarly, the tangents at C, C' will be given by

$$\beta - k\alpha = 0,$$
$$\gamma - k\delta = 0.$$

340. Cor. From the form of the equations to the tangents we observe, that in *any* conic passing through B, C, B', C',

$$\left.\begin{array}{l}\text{the tangents at } B \text{ and } C\\ \ldots\ldots\ldots\ldots\ldots\ B' \ \ldots\ C'\end{array}\right\}\text{ meet on } Aa;$$

$$\left.\begin{array}{l}\ldots\ldots\ldots\ldots\ldots\ B \ \ldots\ C'\\ \ldots\ldots\ldots\ldots\ldots\ B' \ \ldots\ C\end{array}\right\}\ldots\ldots\ A'a;$$

$$\left.\begin{array}{l}\ldots\ldots\ldots\ldots\ldots\ B \ \ldots\ B'\\ \ldots\ldots\ldots\ldots\ldots\ C \ \ldots\ C'\end{array}\right\}\ldots\ldots\ AA'.$$

A great number of well known properties follow from these results. We will enunciate two of them.

I. *If a quadrilateral* ($BCB'C'$) *be inscribed in a conic, and another quadrilateral* ($\beta\alpha\gamma\delta$) *be described touching the conic in the angular points of the former one, the four interior diagonals of the two quadrilaterals meet in one point* (a) *and the two exterior diagonals coincide* (AA').

II. *If a quadrilateral be inscribed in a conic, the points of intersection of opposite sides and the points of intersection of the tangents at opposite angles are collinear.* (Camb. Math. Tripos, 1847.)

341. The following proposition exemplifies the use of *perpendicular* quadrilinear coordinates (Art. 325).

To shew that the circles circumscribing the triangles $AB'C'$, $A'BC'$, $A'B'C$, ABC pass all through one point.

In perpendicular coordinates, all the distances being positive towards the interior of the quadrilateral $BCB'C'$ (fig. 39), the equations to the four circles will be (Art. 195)

$$(AB'C'), \quad \frac{\sin C'}{\beta} + \frac{\sin B'}{\gamma} - \frac{\sin A}{\delta} = 0,$$

$$(A'BC'), \quad \frac{\sin C'}{\alpha} - \frac{\sin A'}{\gamma} + \frac{\sin B}{\delta} = 0,$$

$$(A'B'C), \quad \frac{\sin B'}{\alpha} + \frac{\sin A'}{\beta} + \frac{\sin C}{\delta} = 0,$$

$$(ABC), \quad \frac{\sin A}{\alpha} + \frac{\sin B}{\beta} + \frac{\sin C}{\gamma} = 0.$$

And these equations will be satisfied by the same values of $\alpha, \beta, \gamma, \delta$, provided

$$\begin{vmatrix} 0, & \sin C', & \sin B', & -\sin A \\ \sin C', & 0, & -\sin A', & \sin B \\ \sin B', & \sin A', & 0, & \sin C \\ \sin A, & \sin B, & \sin C, & 0 \end{vmatrix} = 0 \text{.........(1)}.$$

But if a, b, c, d denote the sides $BC, CB', C'B, B'C'$ respectively of the closed figure $BCB'C'$, we obtain by projecting the sides upon lines at right angles to each of them in order

$$-b\sin C + c\sin B + d\sin A' = 0,$$

$$-a\sin C + c\sin A + d\sin B' = 0,$$

$$-a\sin B + b\sin A + d\sin C' = 0,$$

$$a\sin A' - b\sin B' + c\sin C' = 0,$$

whence eliminating $a : b : c : d$,

$$\begin{vmatrix} 0, & \sin C, & \sin B, & \sin A' \\ \sin C, & 0, & \sin A, & \sin B' \\ \sin B, & -\sin A, & 0, & \sin C' \\ -\sin A', & \sin B', & \sin C', & 0 \end{vmatrix} = 0 \text{.........(2)}.$$

The equations (1) and (2) are the same, each of the determinants being equal to

$$(\sin A \sin A' - \sin B \sin B' + \sin C \sin C')^2.$$

Hence the condition (1) is satisfied, and therefore the four circles meet in a point.

342. To find the anharmonic ratio of the range in which the lines of reference are cut by a given line.

Let $$l\alpha + m\beta + n\gamma + r\delta = 0$$

be the given line, and let it meet $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ in L, M, N, R.

Then the equations to BL, BM, BN, BR will be

$$\alpha = 0,\quad (l-r)\,\alpha + (n-r)\,\gamma = 0,$$

$$\gamma = 0,\quad (l-m)\,\alpha + (n-m)\,\gamma = 0.$$

Hence $$\{LMNR\} = \frac{(n-r)\,(n-m)}{(l-r)\,(l-m)}\,.$$

343. *To find the anharmonic ratio of the pencil formed by joining any given point O to the four points of reference B, C, B′, C′.*

Let $(\alpha', \beta', \gamma', \delta')$ be the given point O. Then the four straight lines are represented by

$$(OB),\quad \frac{\gamma}{\gamma'} - \frac{\alpha}{\alpha'} = 0,$$

$$(OB'),\quad \frac{\beta}{\beta'} - \frac{\delta}{\delta'} = 0,$$

$$(OC),\quad \frac{\alpha}{\alpha'} - \frac{\beta}{\beta'} = 0,$$

$$(OC'),\quad \frac{\delta}{\delta'} - \frac{\gamma}{\gamma'} = 0.$$

Now let $$u \equiv \frac{\gamma}{\gamma'} - \frac{\alpha}{\alpha'} \text{ and } v \equiv \frac{\beta}{\beta'} - \frac{\delta}{\delta'}.$$

Then the equations to the four straight lines become

$$u = 0, \quad v = 0, \quad \gamma' u - \delta' v = 0, \quad \beta' u - \alpha' v = 0.$$

Hence $$\{O \,.\, BCB'C'\} = \frac{\alpha'\delta'}{\beta'\gamma'}.$$

344. Cor. If O lie upon the conic

$$\beta\gamma + \kappa\alpha\delta = 0,$$

so that

$$\beta'\gamma' + \kappa\alpha'\delta' = 0,$$

then we have

$$\{O \,.\, BCB'C'\} = -\frac{1}{\kappa},$$

or, *the anharmonic ratio of the pencil formed by joining any point on a conic to four fixed points on the same is constant.*

And conversely, *if the anharmonic ratio of the pencil formed by joining a variable point to four fixed points is constant, the locus of the variable point is a conic passing through the four fixed points.*

Exercises on Chapter XX.

(187) The four coordinates of a point cannot be all positive in the ordinary system of quadrilinear coordinates. But in perpendicular coordinates the four coordinates of a point may be of the same sign.

(188) The general equation to a conic inscribed in the quadrilateral of reference may be written

$$(\mu - \nu)^2 (\beta\gamma + \alpha\delta) + (\nu - \lambda)^2 (\gamma\alpha + \beta\delta) + (\lambda - \mu)^2 (\alpha\beta + \gamma\delta) = 0.$$

(189) The equation

$$\lambda\beta\gamma + \lambda'\alpha\delta + \mu\gamma\alpha + \mu'\beta\delta + \nu\alpha\beta + \nu'\gamma\delta = 0$$

will represent a parabola, provided

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 0 \\ 1, & 0, & \lambda, & \mu, & \nu, & A \\ 1, & \lambda, & 0, & \nu', & \mu', & B \\ 1, & \mu, & \nu', & 0, & \lambda', & C \\ 1, & \nu, & \mu', & \lambda', & 0, & D \\ 0, & A, & B, & C, & D, & 0 \end{vmatrix} = 0.$$

(190) The equation

$$\beta\gamma + \kappa\alpha\delta = 0$$

will represent a parabola, provided

$$\kappa^2 (B-C)^2 + \kappa\{(A-B)^2 + (C-D)^2 + (A-C)^2 + (B-D)^2\} + (A-D)^2 = 0.$$

(191) Through any four points on a parabola another parabola can be drawn unless the four points lie on two parallel straight lines.

(192) The general equation of a conic circumscribing the triangle formed by the three diagonals of the quadrilateral of reference may be written

$$\lambda(\beta\gamma - \alpha\delta) + \mu(\gamma\alpha - \beta\delta) + \nu(\alpha\beta - \gamma\delta) = 0.$$

(193) The equation to the parabola inscribed in the quadrilateral of reference is

$$(B-C)^2(A-D)^2(\beta\gamma + \alpha\delta) + (C-A)^2(B-D)^2(\gamma\alpha + \beta\delta) + (A-B)^2(C-D)^2(\alpha\beta + \gamma\delta) = 0.$$

(194) The conic passing through the four points of reference B, C, B', C' and through the fifth point $(\alpha', \beta', \gamma', \delta')$, is represented by the equation

$$\frac{\beta\gamma}{\beta'\gamma'} - \frac{\alpha\delta}{\alpha'\delta'} = 0,$$

and its tangent at the point $(\alpha', \beta', \gamma', \delta')$ is represented by

$$\frac{\alpha}{\alpha'} - \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} + \frac{\delta}{\delta'} = 0.$$

(195) If two conics circumscribing $CAC'A'$, and $ABA'B'$, intersect in a point O, the tangents at O divide the angle between OA, OA' harmonically.

(196) If the point of intersection of a pair of common chords of two conics be joined to the points of contact of a common tangent, the pencil thus formed is harmonic.

(197) If four common tangents be drawn to a pair of conics which intersect in real points, and if the four points of contact with one of the conics be joined in all possible ways by straight lines, the three points of intersection of these straight lines coincide with the points of intersection of the six common chords of the two conics.

CHAPTER XXI.

CERTAIN CONICS RELATED TO A QUADRILATERAL.

345. We shall use the term *tetragram* to describe the figure contained by four straight lines indefinitely produced, and not regarded in any particular order.

We shall use the word *quadrilateral* when we speak of the four-sided figure contained by four straight lines taken in a particular order.

Thus a tetragram has three diagonals, but a quadrilateral has two proper diagonals and an exterior diagonal.

Thus any four straight lines α, β, γ, δ forming a tetragram, form three quadrilaterals according to the order in which we take the sides, viz.:

(1) $\gamma\alpha\beta\delta$, with α opposite to δ,

(2) $\alpha\beta\gamma\delta$, with β opposite to δ,

(3) $\beta\gamma\alpha\delta$, with γ opposite to δ.

One of these quadrilaterals will generally be *proper*, another *sectant*, and the third *re-entrant*.

Thus, retaining the construction of the last chapter (Fig. 39, page 314) the four straight lines $A'BC$, $B'CA$, $C'AB$, $A'B'C'$ form one tetragram, but they form three quadrilaterals, viz.:

(1) $BCB'C'B$, proper,

(2) $ACA'C'A$, sectant,

(3) $BCB'C'B$, re-entrant.

346. DEFINITIONS. Amongst the conics passing through the four points B, C, B', C' there is one which touches at B, C, B', C' the four straight lines Bb, Cc, $B'b$, $C'c$: this is called the *critical circumscribed conic* of the quadrilateral B, C, B', C'.

So the critical circumscribed conic of the quadrilateral $CAC'A'$ touches the four straight lines Cc, Aa, $C'c$, $A'a$, and the critical circumscribed conic of the quadrilateral $ABA'B'$ touches the four straight lines Aa, Bb, $A'a$, $B'b$.

Amongst the conics inscribed in the quadrilateral $BCB'C'$ there is one whose points of contact lie on the two chords Aa, $A'a$: this is called the *critical inscribed conic* of the quadrilateral $BCB'C'$.

So the critical inscribed conic of the quadrilateral $CAC'A'$ has Bb, Bb' as chords of contact, and the critical inscribed conic of the quadrilateral $ABA'B'$ has Cc, Cc' as chords of contact.

OBS. The critical circumscribed conic of a *square* is the circumscribed circle, and the critical inscribed conic is the inscribed circle.

347. It follows from the definitions that the critical circumscribed conic of any quadrilateral is the critical inscribed conic of the quadrilateral formed by the tangents at the angular points.

And (similarly) the critical inscribed conic of any quadrilateral is the critical circumscribed conic of the quadrilateral formed by joining its points of contact.

348. *To shew the existence of a critical circumscribed conic with respect to any quadrilateral.*

Let $BCB'C'$ be the quadrilateral, and let

$$\beta\gamma + \kappa\alpha\delta = 0$$

be any circumscribed conic. Then the tangents at the angular points are given by

$$\gamma - \kappa\alpha = 0, \quad \beta - \kappa\alpha = 0, \quad \gamma - \kappa\delta = 0, \quad \beta - \kappa\delta = 0,$$

(Art. 339). Therefore in the particular case when $\kappa = 1$, the equations to these tangents are

$$\gamma - \alpha = 0, \quad \beta - \alpha = 0, \quad \gamma - \delta = 0, \quad \beta - \delta = 0,$$

which (Art. 331) represent the lines

$$Bb, \; Cc, \; B'b, \; C'c.$$

Therefore the conic

$$\beta\gamma + \alpha\delta = 0$$

represents the critical circumscribed conic of the quadrilateral: and therefore there is such a conic with respect to any quadrilateral.

Cor. 1. The three equations

$$\beta\gamma + \alpha\delta = 0,$$
$$\gamma\alpha + \beta\delta = 0,$$
$$\alpha\beta + \gamma\delta = 0,$$

represent the critical circumscribed conics of the three quadrilaterals $BCB'C'$, $CAC'A'$, $ABA'B'$.

349. Cor. 2. Since (Art. 346) the critical inscribed conic of any quadrilateral is the critical circumscribed conic of the quadrilateral formed by joining the points of contact assigned in the definition, it follows that there always exists a critical inscribed conic with respect to any quadrilateral.

350. *To find the equation to the critical inscribed conic of the quadrilateral $BCB'C'$.*

Since $\beta = 0$, $\gamma = 0$ represent tangents, and $\alpha - \delta = 0$ their chord of contact (Def.), the equation to the conic must be (Art. 161) of the form

$$(\alpha - \delta)^2 = \kappa\beta\gamma,$$

or $$(\alpha + \delta)^2 = \kappa\beta\gamma + 4\alpha\delta \quad \text{..........................(1).}$$

Similarly, since $\alpha=0$, $\delta=0$ represent tangents, and $\beta-\gamma=0$ their chords of contact (Def.), the equation to the conic must be of the form

$$(\beta+\gamma)^2=4\beta\gamma+\kappa'\alpha\delta \quad \text{.........................(2)}.$$

But $\quad \alpha+\beta+\gamma+\delta=0,$

and therefore $\quad (\alpha+\delta)^2\equiv(\beta+\gamma)^2.$

Hence the equations (1) and (2) will be identical if $\kappa=\kappa'=4$, or the equation

$$(\alpha+\delta)^2=4\,(\beta\gamma+\alpha\delta) \quad \text{.........................(3)},$$

or

$$(\beta+\gamma)^2=4\,(\beta\gamma+\alpha\delta) \quad \text{.........................(4)},$$

represents a conic fulfilling all the conditions required by the definition.

Hence, either of these equations, or any other equation obtained by combining them with the equation

$$\alpha+\beta+\gamma+\delta=0,$$

will represent the critical inscribed conic of the quadrilateral $BCB'C'$.

351. Cor. Comparing the equation (3) or (4) with the equation to the critical circumscribed conic (Art. 348)

$$\beta\gamma+\alpha\delta=0,$$

we observe that *the critical inscribed and circumscribed conics with respect to the same quadrilateral have double contact with one another, the chord of contact being the diagonal joining the intersections of opposite sides of the quadrilateral.*

We shall however see immediately (Art. 353) that the points of contact are imaginary whenever the quadrilateral is real.

352. One form of the equation to the critical inscribed conic of the quadrilateral $BCB'C'$, obtained from equations (3) and (4) of the last article by addition and transposition, is

$$\alpha^2+\beta^2+\gamma^2+\delta^2=6\,(\beta\gamma+\alpha\delta) \quad \text{................(1)}.$$

So the critical inscribed conics of the quadrilaterals $CAC'A$ and $ABA'B'$ are

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 6\,(\gamma\alpha + \beta\delta) \quad \text{.........(2)},$$

and

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 6\,(\alpha\beta + \gamma\delta) \quad \text{.........(3)}.$$

But the critical inscribed conics of the same quadrilaterals are given by the equations

$$\beta\gamma + \alpha\delta = 0 \quad \text{.........(4)},$$

$$\gamma\alpha + \beta\delta = 0 \quad \text{.........(5)},$$

$$\alpha\beta + \gamma\delta = 0 \quad \text{.........(6)},$$

hence the equation

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0 \quad \text{.........(7)},$$

represents a conic passing through the points of intersection of the critical inscribed and circumscribed conics of each quadrilateral.

But the critical inscribed and circumscribed conics of each quadrilateral have double contact. Hence the conic represented by equation (7) has double contact with all those six conics in the six points where the three inscribed touch the corresponding circumscribed conics.

353. The conic represented by the equation

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$$

is necessarily imaginary when the quadrilateral is real, since each term is essentially positive.

Hence the six points of contact of the inscribed and circumscribed critical conics (which may be conveniently termed the critical points of the tetragram) are imaginary, since they lie upon the imaginary conic

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0.$$

354. *To find where the conic whose equation is*

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$$

cuts the lines of reference.

To find where the conic cuts the line $A'B'C'$ we have to put $\delta = 0$,

therefore $\qquad \alpha^2 + \beta^2 + \gamma^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$

but when $\delta = 0$, $\qquad \alpha + \beta + \gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

Hence, eliminating α,

$$\beta^2 + 2\beta\gamma + \gamma^2 = 0.$$

Hence the two points divide $B'C'$, so that the anharmonic ratio of the section is unity (Art. 123).

But if we eliminate β between (1) and (2), we find that the two points are given by

$$\gamma^2 + 2\gamma\alpha + \alpha^2 = 0,$$

and therefore they divide $C'A'$ so that the anharmonic ratio of the section is unity.

Similarly, by eliminating γ we may shew that the same two points divide $A'B'$ so that the anharmonic ratio of the section is unity.

Under these circumstances the points are said to form with A', B', C' an equi-anharmonic system.

And, similarly, each of the other sides of the tetragram can be shewn to be cut by the conic in two points forming with the three vertices in the same side an equi-anharmonic system.

355. The conic

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$$

is thus seen to cut each of the diagonals of the tetragram in the two critical points on that diagonal, and to cut each of the three sides in two points which form with the three vertices in that side an equi-anharmonic system. On account of these properties the conic has been named by Professor Cremona (who seems to have been the first to discover and investigate it) the *fourteen-points' conic* of the tetragram.

356. It is easily seen that the straight lines

$$A'a,\ B'b,\ C'c,\ Aa,\ Bb,\ Cc,$$

are the polars of the points A, B, C, A', B', C', with respect to the fourteen-points' conic.

It follows that

$$AA',\ BB',\ CC'$$

are the polars of the points

$$a,\ b,\ c,$$

and the lines of reference

$$A'BC,\ AB'C,\ ABC',\ A'B'C',$$

are the polars of the points

$$\alpha,\ \beta,\ \gamma,\ \delta.$$

It follows that the tangents from the seven points a, b, c, α, β, γ, δ touch the conic in the fourteen points from which it derives its name.

357. *If the equations to four straight lines in any system of coordinates be*

$$\pm u \pm v \pm w = 0 \text{.......................} (1),$$

the fourteen-points' conic of the tetragram which they form will be represented by

$$u^2 + v^2 + w^2 = 0 \text{.......................} (2).$$

For if we write

$$\left.\begin{array}{l} \alpha \equiv -u+v+w \\ \beta \equiv \quad u-v+w \\ \gamma \equiv \quad u+v-w \\ \delta \equiv -u-v-w \end{array}\right\} \text{.......................} (3),$$

we have

$$\alpha + \beta + \gamma + \delta = 0,$$

and therefore the equations (3) will be the relations by which to transform to quadrilinear coordinates, having the four given lines as lines of reference.

But in quadrilinear coordinates the fourteen-points' conic is given by

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0,$$

therefore by (3) it will be represented in the original coordinates by

$$(-u + v + w)^2 + (u - v + w)^2 + (u + v - w)^2 + (u + v + w)^2 = 0,$$

or $$u^2 + v^2 + w^2 = 0. \quad \text{Q. E. D.}$$

EXERCISES ON CHAPTER XXI.

(198) The critical circumscribed conics with respect to the quadrilaterals $CAC'A'$, $ABA'B'$ have double contact at A and A'.

(199) The chords of contact Aa, $A'a$ of the critical inscribed conic of the quadrilateral $BCB'C'$ are common chords of the other two critical inscribed conics of the same tetragram.

(200) Shew that the conics

$$\alpha^2 = \beta\gamma, \quad \beta^2 = \gamma\alpha, \quad \gamma^2 = \alpha\beta,$$

have three points common to them all, two of which also lie on the fourteen-points' conic of the tetragram of reference.

CHAPTER XXII.

TANGENTIAL COORDINATES. THE STRAIGHT LINE AND POINT.

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358. Let us agree to determine a straight line by its perpendicular distances p, q, r from the three points of reference, just as hitherto we have determined a point by its perpendicular distances from the three lines of reference.

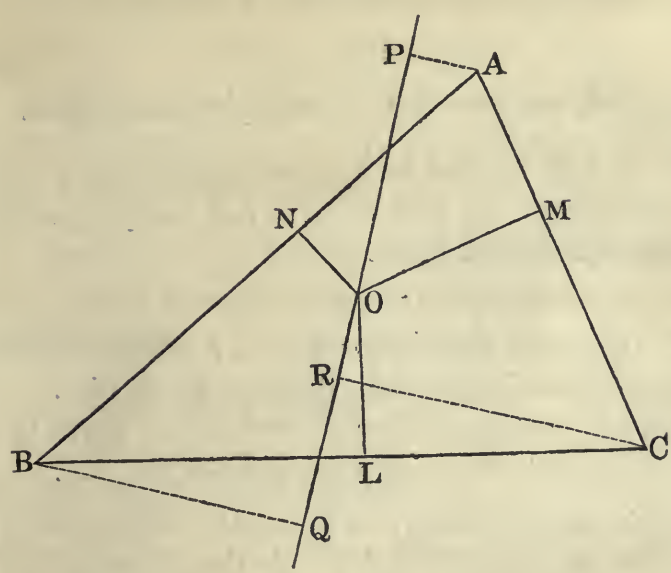
We may with propriety speak of these quantities p, q, r as the coordinates of the line. Thus we shall use the symbol (p, q, r) to denote the line whose coordinates are p, q, r, or which lies at perpendicular distances p, q, r from the points of reference.

Such coordinates are called tangential coordinates.

359. When we commenced with the coordinates of a point, a straight line was determined by passing through two points; so now, when we commence with the coordinates of a straight line, a point will be determined as lying on two straight lines.

Again, as we formerly defined the equation of a line as an equation satisfied by the coordinates of all points on the line, so now we shall define the equation to a point as an equation satisfied by the coordinates of all straight lines passing through the point.

Fig. 40.



360. We have seen (Chap. v.) that if p, q, r be the perpendicular distances of a straight line from the points of reference, and α, β, γ the perpendicular distances of a point O in the line from the lines of reference, then will

$$ap\alpha + bq\beta + cr\gamma = 0.$$

Further, if the straight line be determined by the quantities p, q, r being given, this equation constitutes a relation among the coordinates of *any* point upon the line, and is therefore the equation to the line.

But if instead of p, q, r being known quantities entering into the coefficients of the equation which connects the variables α, β, γ, these latter be known (as being the coordinates of a fixed point O), then the same equation

$$a\alpha p + b\beta q + c\gamma r = 0$$

will constitute a relation among p, q, r which will hold for *any* straight line passing through the point O, and will therefore be the equation to the point O according to the definition of the last article.

That is, *any point whose trilinear coordinates are* α, β, γ *is represented in tangential coordinates by the equation*

$$a\alpha p + b\beta q + c\gamma r = 0,$$

the triangle of reference being the same for both systems.

Cor. It follows that in tangential coordinates every point has an equation of the first degree, and every equation of the first degree represents a point.

361. Of course the distances p, q, r will be regarded as of the same algebraical sign when they are all on the same side of the line on which they are let fall, and any two will be of opposite sign when they are on opposite sides of the line.

But it is never necessary to determine which side of the line shall be the positive side and which the negative, nor to give any one of the coordinates by itself any absolute sign, since all our equations in tangential coordinates are either homogeneous, or if their terms be of different orders they are at least all of even orders or all of odd, so that a change in the absolute signs of p, q, r would have no effect. This is a direct consequence of the circumstance that the coordinates p, q, r of any line are identically connected by a relation of the second order,

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2,$$

(Art. 74) and not like the coordinates of a point by a *simple* equation.

362. *To find the equation in trilinear coordinates to the straight line whose tangential coordinates are* p, q, r.

Since p, q, r are the perpendicular distances of the straight line from the points of reference, therefore by Chap. v., the equation to the straight line is

$$ap\alpha + bq\beta + cr\gamma = 0.$$

Cor. The equation to the same straight line in triangular coordinates is

$$p\alpha + q\beta + r\gamma = 0.$$

363. *To find the trilinear coordinates of the point of intersection of two straight lines whose tangential coordinates are given.*

Let the given coordinates of the two lines be (p_1, q_1, r_1), (p_2, q_2, r_2), and suppose (α, β, γ) the trilinear coordinates of their point of intersection.

Then since (α, β, γ) is a point on a straight line whose perpendicular distances from the points of reference are p_1, q_1, r_1, therefore (Chap. v.)

$$ap_1\alpha + bq_1\beta + cr_1\gamma = 0.$$

Similarly, $ap_2\alpha + bq_2\beta + cr_2\gamma = 0.$

Therefore we have

$$\frac{a\alpha}{\begin{vmatrix} q_1, & r_1 \\ q_2, & r_2 \end{vmatrix}} = \frac{b\beta}{\begin{vmatrix} r_1, & p_1 \\ r_2, & p_2 \end{vmatrix}} = \frac{c\gamma}{\begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix}},$$

equations which determine the ratios of the coordinates required.

Cor. In virtue of Art. 360 it follows that the tangential equation to the same point is

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{vmatrix} = 0,$$

a result which we presently establish (Art. 366) without reference to the trilinear system.

364. *To find the coordinates of the straight line joining two points whose equations are given.*

Let $$lp + mq + nr = 0,$$

and $$l'p + m'q + n'r = 0,$$

be the equations to the two points.

Then the coordinates of the straight line joining them must satisfy both equations (Art. 359), and therefore their ratios are given by

$$\frac{p}{\begin{vmatrix} m, & n \\ m', & n' \end{vmatrix}} = \frac{q}{\begin{vmatrix} n, & l \\ n', & l' \end{vmatrix}} = \frac{r}{\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}}.$$

To find the absolute values of these coordinates we shall have to substitute their ratios in the relation which we found in Chap. VI. (Arts. 73, 74) connecting the perpendiculars upon any straight line. That relation, as we there shewed, can be written in any of the various forms,

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr \cos A - 2carp \cos B - 2abpq \cos C = 4\Delta^2,$$

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2,$$

$$(q-r)^2 \cot A + (r-p)^2 \cot B + (p-q)^2 \cot C = 2\Delta,$$

or with the notation of Art. 46, it may be written

$$\{ap, \ bq, \ cr\}^2 = 4\Delta^2,$$

the form in which we shall generally quote it.

365. It appears from the foregoing article that by solving together the equations of any two points we may determine the coordinates of the straight line joining them.

Hence any two equations of the first degree taken simultaneously will determine a straight line, viz. the straight line joining the two points which the equations represent separately. Therefore two equations may be spoken of as the equations of a straight line. For example, the straight line (p', q', r') may be said to be given by the equations

$$\frac{p}{p'} = \frac{q}{q'} \text{ and } \frac{q}{q'} = \frac{r}{r'},$$

or

$$\frac{p}{p'} = \frac{q}{q'} = \frac{r}{r'}.$$

366. *To find the equation of the point of intersection of two straight lines whose coordinates are given.*

In other words, to find the relation among the perpendiculars p, q, r from the points of reference upon any straight line passing through the point of intersection of the given straight lines.

Let (p_1, q_1, r_1), (p_2, q_2, r_2) be the given straight lines, and suppose

$$lp + mq + nr = 0 \text{.........................} (1),$$

the equation of their point of intersection O.

Then this equation expresses a relation satisfied by the coordinates of any straight line passing through O.

But (p_1, q_1, r_1) passes through O, therefore

$$lp_1 + mq_1 + nr_1 = 0 \text{.....................} (2).$$

Similarly, (p_2, q_2, r_2) passes through O, and therefore

$$lp_2 + mq_2 + nr_2 = 0 \text{.....................} (3).$$

Hence eliminating $l : m : n$ from the equations (1), (2), (3), we get

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{vmatrix} = 0,$$

a relation among the coordinates p, q, r, and therefore the equation of the point O.

COR. 1. The equation just obtained will not be affected if p_1, q_1, r_1 or p_2, q_2, r_2 be multiplied by any constant ratio. Hence if the coordinates of one straight line be only proportional to three given quantities p_1, q_1, r_1, and those of another straight line proportional to p_2, q_2, r_2, the equation of their point of intersection is still

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{vmatrix} = 0.$$

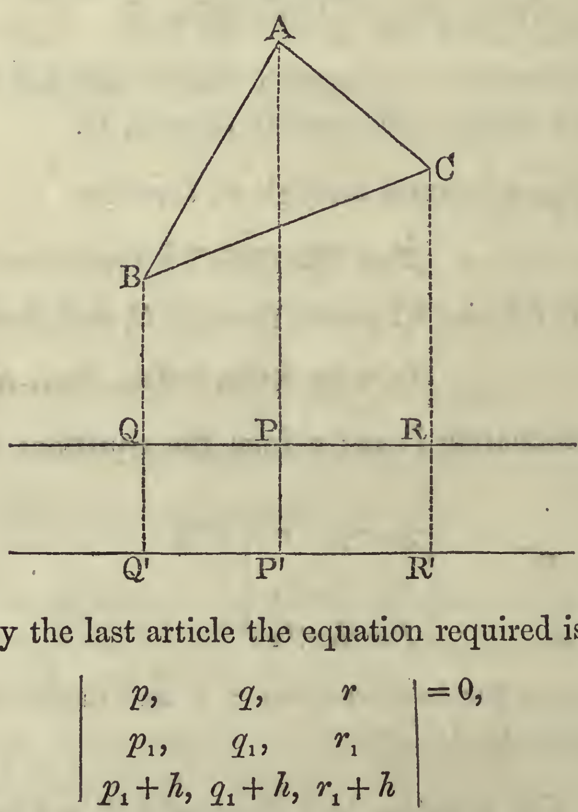
367. *To find the equation of the point where a given straight line meets the line at infinity.*

Let (p_1, q_1, r_1) be the given straight line, then

$$(p_1 + h,\ q_1 + h,\ r_1 + h)$$

will be a parallel straight line (fig. 41), and these will therefore intersect in the point required.

Fig. 41.



Hence by the last article the equation required is

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ p_1 + h, & q_1 + h, & r_1 + h \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ h, & h, & h \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} p, & q, & r \\ p_1, & q_1, & r_1 \\ 1, & 1, & 1 \end{vmatrix} = 0.$$

368. Cor. The last equation is satisfied if $p=q=r$, for then the first and third rows of the determinant become identical.

Hence all points at infinity lie upon the straight line given by $p=q=r$.

Hence $$p=q=r$$

are the equations of the straight line at infinity.

369. *To find the coordinates of a straight line passing through a given point and parallel to a given straight line.*

Let $$lp+mq+nr=0 \quad \text{.........................(1)}$$

be the equation to the given point, and let (p_1, q_1, r_1) be the given straight line.

The coordinates of any straight line parallel to (p_1, q_1, r_1) may be written (fig. 41) p_1+h, q_1+h, r_1+h.

If this straight line pass through the point (1) we must have

$$lp_1+mq_1+nr_1+(l+m+n)h=0,$$

therefore $$h=-\frac{lp_1+mq_1+nr_1}{l+m+n};$$

therefore $$p_1+h=\frac{m(p_1-q_1)+n(p_1-r_1)}{l+m+n},$$

$$q_1+h=\frac{n(q_1-r_1)+l(q_1-p_1)}{l+m+n},$$

$$r_1+h=\frac{l(r_1-p_1)+m(r_1-q_1)}{l+m+n},$$

which are therefore the coordinates required.

370. *To find the distance of the point whose equation is*

$$lp+mq+nr=0 \quad \text{.........................(1)}$$

from the straight line whose coordinates are (p', q', r').

Let h be the distance, then the line parallel to (p', q', r') through the given point will have the coordinates

$$(p' \pm h,\ q' \pm h,\ r' \pm h).$$

These must satisfy the equation (1), therefore

$$lp' + mq' + nr' \pm (l + m + n)\, h = 0,$$

$$\text{or} \quad h = \pm \frac{lp' + mq' + nr'}{l + m + n}.$$

371. *To find the equation to a point which divides in a given ratio the straight line joining two given points.*

Let $1 : k$ be the given ratio, and

$$lp + mq + nr = 0, \text{ and } l'p + m'q + n'r = 0,$$

the equations to the given points.

Suppose p, q, r the coordinates of any straight line through the required point. Then the perpendicular distances of the given points from this line are in the given ratio.

Therefore by the last article,

$$\pm \frac{lp + mq + nr}{l + m + n} : \mp \frac{l'p + m'q + n'r}{l' + m' + n'} = 1 : k,$$

the two expressions for the distance having opposite signs, since the two points are on opposite sides of the straight line.

Therefore

$$k \frac{lp + mq + nr}{l + m + n} + \frac{l'p + m'q + n'r}{l' + m' + n'} = 0,$$

a relation among the coordinates of any straight line through the required point, and therefore the equation to the required point.

Cor. The middle point between the points

$$lp + mq + nr = 0, \text{ and } l'p + m'q + n'r = 0,$$

is given by the equation

$$\frac{lp + mq + nr}{l + m + n} + \frac{l'p + m'q + n'r}{l' + m' + n'} = 0.$$

372. The principles of abridged notation explained in Chapter VIII. for trilinear coordinates are equally applicable to tangential coordinates.

As we there used $u=0$, $v=0$, $w=0$ to represent equations to straight lines expressed in their most general form, so now we shall use the same expressions to denote the most general forms of the equation to the point in tangential coordinates.

373. *If* $u=0$, $v=0$ *be equations to two points in tangential coordinates, then*

$$u+\kappa v=0,$$

(where κ *is an arbitrary constant) will represent a point lying on the straight line joining the two points.*

For if p, q, r be the coordinates of this point they satisfy the equations $u=0$ and $v=0$; that is, their substitution makes u and v severally vanish, therefore it must make $u+\kappa v$ vanish; that is, p, q, r satisfy the equation

$$u+\kappa v=0,$$

and therefore this equation represents a point on the line (p, q, r). Q.E.D.

374. *If the line joining the points* $u=0$ *and* $v=0$ *be divided by the points* $u+\kappa v=0$, *and* $u+\kappa' v=0$, *the anharmonic ratio of the section is* $\kappa : \kappa'$.

Let A, B be the two points represented by $u=0$ and $v=0$, and P, Q the two points represented by $u+\kappa v=0$ and $u+\kappa' v=0$.

Let (p', q', r') be the coordinates of any straight line whatever, and let u', v' be what u, v become when p', q', r' are written for p, q, r, and let m, n be what u, v become when unity is written for each of these letters p, q, r.

Then the perpendicular distances of the points A, B, P, Q from the straight line (p', q', r') are respectively

$$\frac{u'}{m}, \quad \frac{v'}{n}, \quad \frac{u'+\kappa v'}{m+\kappa n}, \quad \frac{u'+\kappa' v'}{m+\kappa' n}.$$

and therefore the distances AP, AQ, BP, BQ are proportional (by similar triangles) to the differences

$$\frac{u'}{m}-\frac{u'+\kappa v'}{m+\kappa n},\quad \frac{u'}{m}-\frac{u'+\kappa' v'}{m+\kappa' n},\quad \frac{v'}{n}-\frac{u'+\kappa v'}{m+\kappa n},\quad \frac{v'}{n}-\frac{u'+\kappa' v'}{m+\kappa' n},$$

therefore

$$\{APBQ\}=\frac{AP.BQ}{AQ.BP}=\frac{\left(\frac{u'}{m}-\frac{u'+\kappa v'}{m+\kappa n}\right)\cdot\left(\frac{v'}{n}-\frac{u'+\kappa' v'}{m+\kappa' n}\right)}{\left(\frac{u'}{m}-\frac{u'+\kappa' v'}{m+\kappa' n}\right)\cdot\left(\frac{v'}{n}-\frac{u'+\kappa v'}{m+\kappa n}\right)}$$

$$=\frac{\kappa\,(nu'-mv')\,(mv'-nu')}{\kappa'\,(nu'-mv')\,(mv'-nu')},$$

or $$\{APBQ\}=\frac{\kappa}{\kappa'}.\quad \text{Q.E.D.}$$

375. *To find the anharmonic ratio of the range of the four points whose equations are*

$$u+\kappa v=0,\quad u+\lambda v=0,\quad u+\mu v=0,\quad u+\nu v=0.$$

The proof of Article 125 (p. 137) applies *verbatim*. Thus we find that the anharmonic ratio required is

$$\frac{(\kappa-\lambda)\,(\mu-\nu)}{(\kappa-\nu)\,(\mu-\lambda)}.$$

376. It follows, as in Art. 123, that the line joining the points $u=0$, $v=0$, is divided by the two points

$$lu^2+2muv+nv^2=0;$$

so that the anharmonic ratio is

$$\frac{(m\pm\sqrt{m^2-ln})^2}{ln}.$$

Exercises on Chapter XXII.

(201) The coordinates of the line of reference BC are

$$0, \quad 0, \quad \frac{2\Delta}{a}.$$

(202) The coordinates of the perpendicular from A on BC are

$$0, \quad \pm b \cos C, \quad \mp c \cos B.$$

(203) The coordinates of the straight line through A parallel to BC are

$$0, \quad \frac{2\Delta}{a}, \quad \frac{2\Delta}{a}.$$

(204) The straight line joining A to the middle point of BC is given by

$$p = 0, \quad q + r = 0.$$

(205) The equation $q + r = 0$ represents the middle point of the side BC of the triangle of reference.

(206) The equation $q \tan B + r \tan C = 0$ represents the foot of the perpendicular from the point of reference A upon BC.

(207) The equation $mq + nr = 0$ represents a point P in the line BC such that $BP : PC = n : m$.

(208) The equation $mq - nr = 0$ represents a point P in the line BC produced, such that $PB : PC = n : m$.

(209) The equation $q - r = 0$ represents the point of intersection (at infinity) of straight lines parallel to BC.

(210) The equation $p + q + r = 0$ represents the point of intersection of the straight lines which join the angular points of the triangle of reference to the middle points of the opposite sides.

(211) The equation

$$p \tan A + q \tan B + r \tan C = 0$$

represents the point of intersection of the perpendiculars from the angular points on the opposite sides of the triangle of reference.

(212) The equations

$$\pm p \sin A \pm q \sin B \pm r \sin C = 0$$

represent the centres of the inscribed and escribed circles of the triangle of reference.

(213) The equation

$$p \sin 2A + q \sin 2B + r \sin 2C = 0$$

represents the centre of the circle passing through the points of reference.

(214) The equation

$$(q + r) \sin 2A + (r + p) \sin 2B + (p + q) \sin 2C = 0$$

represents the centre of the nine-points' circle of the triangle of reference.

(215) Apply tangential coordinates to shew that the middle points of the three diagonals of a complete quadrilateral are collinear.

(216) The straight line joining the points

$$lp + mq + nr = 0, \quad l'p + m'q + n'r = 0$$

is divided harmonically in the points

$$\frac{lp + mq + nr}{l + m + n} \pm \kappa \frac{l'p + m'q + n'r}{l' + m' + n'} = 0.$$

CHAPTER XXIII.

TANGENTIAL COORDINATES. CONIC SECTIONS.

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377. Definition. The equation to a curve in tangential coordinates is a relation among the coordinates of any straight line which touches the curve.

The equation to a curve is therefore satisfied by the coordinates of any tangent to the curve; and any straight line whose coordinates satisfy the equation is a tangent to the curve.

378. We have already seen that the identical relation connecting the coordinates of any straight line may be written in any of the forms

$$(q-r)^2 \cot A + (r-p)^2 \cot B + (p-q)^2 \cot C = 2\Delta,$$

$$a^2 (p-q)(p-r) + b^2 (q-r)(q-p) + c^2 (r-p)(r-q) = 4\Delta^2,$$

or
$$\{ap,\ bq,\ cr\}^2 = 4\Delta^2.$$

It should be noticed, that if

$$O \equiv \{ap,\ bq,\ cr\}^2,$$

then
$$\frac{dO}{dp} \equiv 2a\,(ap - bq \cos C - cr \cos B),$$

$$\frac{dO}{dq} \equiv 2b\,(bq - cr \cos A - ap \cos C),$$

$$\frac{dO}{dr} \equiv 2c\,(cr - ap \cos B - bq \cos A),$$

and
$$\frac{dO}{dp} + \frac{dO}{dq} + \frac{dO}{dr} \equiv 0.$$

379. *To find the equation to the circle whose centre is at the point*

$$lp + mq + nr = 0,$$

and whose radius is ρ.

Let p, q, r be the coordinates of any tangent to the circle. Then since ρ is the distance of the tangent from the centre, we have (Art. 370)

$$\rho = \pm \frac{lp + mq + nr}{l + m + n};$$

and rendering this homogeneous by the relation

$$\{ap,\ bq,\ cr\}^2 = 4\Delta^2,$$

we get
$$\{ap,\ bq,\ cr\}^2 = \frac{4\Delta^2}{\rho^2}\left(\frac{lp + mq + nr}{l + m + n}\right)^2,$$

a relation among the coordinates of any tangent and therefore the equation to the circle.

380. *The general equation to a circle is therefore*

$$\{ap,\ bq,\ cr\}^2 = (\lambda p + \mu q + \nu r)^2,$$

and its radius is

$$\frac{2\Delta}{\lambda + \mu + \nu},$$

and the equation to its centre is

$$\lambda p + \mu q + \nu r = 0.$$

For comparing the equation just written down with the form which we investigated, we have

$$\lambda p + \mu q + \nu r \equiv \frac{2\Delta}{\rho}\,\frac{lp + mq + nr}{l + m + n};$$

therefore

$$\frac{\lambda\rho}{2\Delta} = \frac{l}{l + m + n}, \quad \frac{\mu\rho}{2\Delta} = \frac{m}{l + m + n}, \quad \frac{\nu\rho}{2\Delta} = \frac{n}{l + m + n},$$

and by addition,

$$\frac{(\lambda + \mu + \nu)\,\rho}{2\Delta} = 1, \text{ or } \rho = \frac{2\Delta}{\lambda + \mu + \nu}.$$

381. A particular case of the equation to a circle occurs when $\lambda = \mu = \nu = 0$, or when $\rho = \infty$.

In this case the equation takes the form

$$\{ap, \ bq, \ cr\}^2 = 0,$$

or $a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 0 \ldots(1)$,

which is evidently satisfied when $p = q = r$, shewing that any straight line lying altogether at infinity is a tangent.

But since the coordinates p, q, r of any *finite* straight line satisfy the relation

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 4\Delta^2,$$

which is inconsistent with (1), we see that no finite straight line is a tangent to the circle.

The circle is in fact that described in Article 38, and would be represented in trilinear coordinates by the equation

$$(a\alpha + b\beta + c\gamma)^2 = 0.$$

The centre is given by $Op + Oq + Or = 0$, and is indeterminate: the radius ρ is infinite.

We shall speak of this circle briefly as *the great circle*.

382. Some writers speak of the equation

$$a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q) = 0,$$

as representing only the two circular points at infinity: and some correct results are deduced from giving it this interpretation.

The discrepancy is precisely analogous to that which attaches to the interpretation of the trilinear equation

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A = 0,$$

or to the Cartesian equation

$$x^2 + y^2 = 0.$$

It has already been pointed out (Art. 318) that either of these equations represents two imaginary straight lines intersecting in a real point, but is also the limiting form of the equation to an evanescent circle at that real point. We explained that a complete description of the locus of such an equation of the second order must recognise the fact that when the real part of a conic section degenerates into a point, the imaginary branches become two straight lines through the point; and the equation to any two imaginary straight lines intersecting in a real point—so soon as it is regarded as representing a locus of the second order at all—must be regarded as representing the ultimate conic evanescent at the real point and having the two straight lines as imaginary branches.

In the present case we have to deal with the ultimate conic at the opposite limit. Instead of the diameters becoming indefinitely small they have become indefinitely great: but as before the asymptotes are imaginary, and in the limit the imaginary branches of the curve coincide with them. And just as in the former case, the equation to the conic could in a partial view be regarded as only representing the imaginary asymptotes, so in this case the tangential equation to the conic may be regarded as representing only the two circular points at infinity, which are at the same time the points of contact of the asymptotes and their polars with respect to the curve.

We must again refer to the chapter on reciprocal polars, where this point is more fully discussed.

383. *To find the equation to the conic section whose foci are at the points*

$$lp + mq + nr = 0,$$

$$l'p + m'q + n'r = 0,$$

and whose conjugate or minor axis is 2ρ.

Let (p, q, r) be any tangent to the conic; then since ρ^2 is equal to the rectangle under the focal perpendiculars on any tangent, we have

$$\rho^2 = \frac{lp + mq + nr}{l + m + n} \cdot \frac{l'p + m'q + n'r}{l' + m' + n'},$$

and, rendering this homogeneous by the relation

$$\{ap,\ bq,\ cr\}^2 = 4\Delta^2,$$

we get

$$\{ap,\ bq,\ cr\}^2 = \frac{4\Delta^2}{\rho^2}\,\frac{(lp+mq+nr)\,(l'p+m'q+n'r)}{(l+m+n)\,(l'+m'+n')},$$

a relation among the coordinates of any tangent, and therefore the equation to the conic.

384. The general equation to a conic may therefore be written

$$up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq = 0,$$

and the foci are given by the equation

$$up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq + k\,\{ap,\ bq,\ cr\}^2 = 0,$$

where k is to be so determined that the left-hand member of this equation may be resolvable into two factors.

385. OBS. The equation to give k is

$$\begin{vmatrix} u+ka^2, & w'-kab\cos C, & v'-kca\cos B \\ w'-kab\cos C, & v+kb^2, & u'-kbc\cos A \\ v'-kca\cos B, & u'-kbc\cos A, & w+kc^2 \end{vmatrix} = 0.$$

The coefficient of k^3 in this cubic vanishes, and the equation reduces to a quadratic giving two values for k, indicating two pairs of foci. One will be a real pair, the other an imaginary pair.

Or, viewing the equation for k in a more general aspect, it has *three* roots, one of which is infinite. There will therefore be three pairs of foci, the two pairs just spoken of and another pair represented by the equation

$$\{ap,\ bq,\ cr\}^2 = 0,$$

to which we must in this case give its partial interpretation, as representing the two circular points.

Hence every conic may be said to have *six* foci, two coinciding with the circular points, two real ones whose geometrical properties are known, and two other imaginary ones.

When we speak of the *four* foci of a conic, it will be understood that we neglect the two circular points which arise from the interpretation of the evanescent term in the cubic for k.

386. *To find the coordinates of the tangents drawn from a given point to a given conic*, we have only to solve simultaneously the equations to the point and the conic and we shall get two solutions for the ratios of the coordinates of the tangents required.

387. *To find the condition that a given point may lie upon a conic*, we must construct the equation for the coordinates of the two tangents from the point, and express the condition that the quadratic thus constructed may have equal roots.

388. *The imaginary tangents drawn to a circle from its centre touch all concentric circles and the great circle at infinity.*

For let

$$\lambda p + \mu q + \nu r = 0 \qquad (1)$$

be the centre: then by giving different values to k, the equation

$$\{ap,\ bq,\ cr\}^2 + k(\lambda p + \mu q + \nu r)^2 = 0 \qquad (2)$$

will represent any circle in the concentric series, and the coordinates of the tangents from the centre are obtained by solving simultaneously equations (1) and (2). Hence they are given by

$$\left.\begin{aligned}(\lambda p + \mu q + \nu r)^2 &= 0\\ \text{and}\qquad \{ap,\ bq,\ cr\}^2 &= 0\end{aligned}\right\} \qquad (3),$$

which are independent of k, shewing that the same imaginary tangents touch all the concentric circles.

But the equation

$$\{ap,\ bq,\ cr\}^2 = 0$$

represents the great circle, and therefore the equations (3) determine the coordinates of the tangents from the given centre to the great circle. Hence this circle has imaginary tangents in common with any concentric series.

389. Cor. The four common tangents to the great circle and any other coincide two and two, for they coincide with the two tangents to the latter circle from its centre.

390. *The common tangents to the great circle and any conic intersect, two and two, in the foci of the conic.*

Let $\{ap,\ bq,\ cr\}^2 = (\lambda p + \mu q + \nu r)(\lambda' p + \mu' q + \nu' r)$ ....(1)
be the equation to any conic.

The common tangents to this conic and the great circle will be obtained by solving together the equation (1) and the equation

$$\{ap,\ bq,\ cr\}^2 = 0 \text{ ....(2).}$$

These tangents are therefore four in number.

From the equations (1) and (2) we obtain

$$(\lambda p + \mu q + \nu r)(\lambda' p + \mu' q + \nu' r) = 0.$$

Hence the four tangents pass through one or other of the points represented by this equation, i. e. through one or other of the foci of the conic.

Therefore, &c. Q. E. D.

391. Cor. 1. We may adopt the following definition of the foci of a conic.

The four common tangents to any conic and the great circle at infinity intersect in six points which are called the foci of the conic.

Two of these six foci are the circular points, as we saw in Art. 385. Hence every real or imaginary tangent to the great circle passes through one or other of the circular points.

392. Cor. 2. The common tangents to two confocal conics pass two and two through the foci and touch the great circle at infinity.

393. *To find the equation to the centre of the conic whose tangential equation is*

$$f(p, q, r) = 0.$$

Let (p', q', r') be the coordinates of any diameter, and suppose $(p' + h, q' + h, r' + h)$ a parallel tangent. Then, since $(p' + h, q' + h, r' + h)$ is a tangent these coordinates must satisfy the equation to the curve, therefore

$$f(p' + h, q' + h, r' + h) = 0,$$

an equation to determine h.

We may write it

$$f(p', q', r') + h\left(\frac{df}{dp'} + \frac{df}{dq'} + \frac{df}{dr'}\right) + h^2 f(1, 1, 1) = 0;$$

and since the two values of h must be equal and of opposite sign we have

$$\frac{df}{dp'} + \frac{df}{dq'} + \frac{df}{dr'} = 0.$$

But (p', q', r') is *any* diameter: therefore every diameter passes through the point whose equation is

$$\frac{df}{dp} + \frac{df}{dq} + \frac{df}{dr} = 0,$$

therefore this is the equation to the centre which was required.

394. Cor. 1. If the equation to the conic be written

$$\{ap, bq, cr\}^2 + k(lp + mq + nr)(l'p + m'q + n'r) = 0,$$

the equation to the centre becomes

$$\frac{lp + mq + nr}{l + m + n} + \frac{l'p + m'q + n'r}{l' + m' + n'} = 0,$$

a result which we might have inferred *à priori* from the property that the centre bisects the line joining the foci.

395. Cor. 2. If we write the equation to the conic in the general form

$$up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq = 0,$$

the equation to the centre takes the form

$$\bar{u}p + \bar{v}q + \bar{w}r = 0,$$

where $\quad \bar{u} \equiv u + v' + w', \quad \bar{v} \equiv v + w' + u', \quad \bar{w} \equiv w + u' + v'.$

396. Cor. 3. If $f(p, q, r) = 0$ represent a circle this equation must (Art. 380) be identical with

$$\{ap, bq, cr\}^2 - k(\bar{u}p + \bar{v}q + \bar{w}r)^2 = 0.$$

Hence we must have (see Prolegomenon,)

$$\left\| \begin{matrix} u, & v, & w, & 1 \\ \bar{u}^2, & \bar{v}^2, & \bar{w}^2, & \bar{u} + \bar{v} + \bar{w} \\ a^2, & b^2, & c^2, & 0 \end{matrix} \right\| = 0,$$

which, therefore, express the conditions that the general equation of the second degree should represent a circle.

397. *To find the coordinates of the diameter parallel to a given straight line.*

Let (p', q', r') be the given straight line, and suppose

$$(p' + h, \quad q' + h, \quad r' + h),$$

the parallel diameter. Then these coordinates must satisfy the equation to the centre, therefore

$$\bar{u}(p' + h) + \bar{v}(q' + h) + \bar{w}(r' + h) = 0,$$

$$\therefore \; h = -\frac{\bar{u}p' + \bar{v}q' + \bar{w}r'}{\bar{u} + \bar{v} + \bar{w}};$$

hence the required coordinates are

$$\frac{\bar{v}(p' - q') + \bar{w}(p' - r')}{\bar{u} + \bar{v} + \bar{w}}, \quad \frac{\bar{w}(q' - r') + \bar{u}(q' - p')}{\bar{u} + \bar{v} + \bar{w}}, \quad \frac{\bar{u}(r' - p') + \bar{v}(r' - q')}{\bar{u} + \bar{v} + \bar{w}}.$$

398. *To find the condition that the equation*

$$f(p, q, r) \equiv up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq = 0,$$

should represent a parabola.

The necessary and sufficient condition is that the line at infinity should be a tangent.

Therefore $p = q = r$ must satisfy the equation.

Therefore $$f(1, 1, 1) = 0,$$

or $$u + v + w + 2u' + 2v' + 2w' = 0,$$

or $$\bar{u} + \bar{v} + \bar{w} = 0,$$

the required condition.

399. Cor. If the equation to the conic be written

$$\{ap, bq, cr\}^2 + (\lambda p + \mu q + \nu r)(\lambda' p + \mu' q + \nu' r) = 0,$$

the condition becomes

$$(\lambda + \mu + \nu)(\lambda' + \mu' + \nu') = 0,$$

shewing that a conic is a parabola if either focus lie at infinity.

400. *To interpret the equation*

$$p \frac{df}{dp'} + q \frac{df}{dq'} + r \frac{df}{dr'} = 0 \text{ .... (1)}$$

with respect to the conic $f(p, q, r) = 0$.

I. Suppose that the straight line (p', q', r') is a tangent to the conic.

Then $$f(p', q', r') = 0,$$

or, as we may write it,

$$p' \frac{df}{dp'} + q' \frac{df}{dq'} + r' \frac{df}{dr'} = 0,$$

which shews that the equation (1) represents some point on the tangent (p', q', r').

Now let (p'', q'', r'') be the other tangent from this point. Then since it passes through the point (1), we have

$$p''\frac{df}{dp'}+q''\frac{df}{dq'}+r''\frac{df}{dr'}=0,$$

or
$$p'\frac{df}{dp''}+q'\frac{df}{dq''}+r'\frac{df}{dr''}=0,$$

which shews that (p', q', r') passes through the point given by

$$p\frac{df}{dp''}+q\frac{df}{dq''}+r\frac{df}{dr''}=0\text{.....................} (2).$$

But since (p'', q'', r'') is a tangent, we have

$$f(p'', q'', r'')=0,$$

or
$$p''\frac{df}{dp''}+q''\frac{df}{dq''}+r''\frac{df}{dr''}=0,$$

which shews that (p'', q'', r'') also passes through the point (2).

Hence the point (2) is the point of intersection of the tangents (p', q', r') and (p'', q'', r''); that is, it coincides with the point (1), therefore the equations (1) and (2) are identical.

Therefore,

$$\frac{\frac{df}{dp'}}{\frac{df}{dp''}}=\frac{\frac{df}{dq'}}{\frac{df}{dq''}}=\frac{\frac{df}{dr'}}{\frac{df}{dr''}},$$

of which a solution (and since they are simple equations, the only solution) is evidently

$$\frac{p'}{p''}=\frac{q'}{q''}=\frac{r'}{r''},$$

or the tangents (p', q', r'), (p'', q'', r'') coincide. Hence the given equation represents the point of contact of the tangent (p', q', r').

But II. suppose (p', q', r') be not a tangent, then let (p_1, q_1, r_1), (p_2, q_2, r_2) be the tangents at the points where (p', q', r') meets the conic.

Then by the Case I. their points of contact are given by the equations

$$p\frac{df}{dp_1}+q\frac{df}{dq_1}+r\frac{df}{dr_1}=0,$$

$$p\frac{df}{dp_2}+q\frac{df}{dq_2}+r\frac{df}{dr_2}=0.$$

And since these points both lie upon (p', q', r'), we have

$$p'\frac{df}{dp_1}+q'\frac{df}{dq_1}+r'\frac{df}{dr_1}=0$$

and

$$\left.p'\frac{df}{dp_2}+q'\frac{df}{dq_2}+r'\frac{df}{dr_2}=0,\right\}$$

or

$$p_1\frac{df}{dp'}+q_1\frac{df}{dq'}+r_1\frac{df}{dr'}=0$$

and

$$\left.p_2\frac{df}{dp'}+q_2\frac{df}{dq'}+r_2\frac{df}{dr'}=0,\right\}$$

which shew that (p_1, q_1, r_1), (p_2, q_2, r_2) pass through the point given by

$$p\frac{df}{dp'}+q\frac{df}{dq'}+r\frac{df}{dr'}=0;$$

that is, the equation

$$p\frac{df}{dp'}+q\frac{df}{dq'}+r\frac{df}{dr'}=0$$

represents the point of intersection of tangents at the extremities of the chord (p', q', r').

Therefore always—

The pole of the straight line (p', q', r') *is represented by the equation*

$$p\frac{df}{dp'}+q\frac{df}{dq'}+r\frac{df}{dr'}=0.$$

401. Cor. 1. If the equation to the conic be written

$$up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq = 0,$$

the pole of the line (p', q', r') is given by

$$p(up' + w'q' + v'r') + q(vq' + u'r' + w'p') + r(wr' + v'p' + u'q') = 0.$$

402. Cor. 2. With respect to the great circle the pole of the straight line (p', q', r') is

$$ap(ap' - bq'\cos C - cr'\cos B) + bq(bq' - cr'\cos A - ap'\cos C)$$
$$+ cr(cr' - ap'\cos B - bq'\cos A) = 0,$$

which is satisfied if $p = q = r$.

Hence *the pole of any straight line with respect to the great circle is at infinity.*

403. Cor. 3. The equation of the last corollary becomes indeterminate if $p' = q' = r'$.

Hence *the pole of the straight line at infinity with respect to the great circle is indeterminate,* as we shewed otherwise in Art. 381.

It also follows from Cor. 2, in virtue of Art. 234, that *the polar of any finite point with respect to the great circle is the straight line at infinity.*

404. *To find the coordinates of a diameter of a conic conjugate to a given diameter.*

Let $f(p, q, r) = 0$ be the given conic, and (p', q', r') the given diameter.

Let $(p' + h, q' + h, r' + h)$ and $(p' - h, q' - h, r' - h)$ be the parallel tangents.

Then their points of contact (or poles) are given by the equations

$$p'\frac{df}{dp}+q'\frac{df}{dq}+r'\frac{df}{dr}+h\left(\frac{df}{dp}+\frac{df}{dq}+\frac{df}{dr}\right)=0,$$

and $$p'\frac{df}{dp}+q'\frac{df}{dq}+r'\frac{df}{dr}-h\left(\frac{df}{dp}+\frac{df}{dq}+\frac{df}{dr}\right)=0.$$

The conjugate diameter joins these two points: hence its coordinates will be obtained by solving together these two equations. Hence the coordinates are given by

$$\left.\begin{aligned} p'\frac{df}{dp}+q'\frac{df}{dq}+r'\frac{df}{dr}&=0 \\ \text{and}\qquad \frac{df}{dp}+\frac{df}{dq}+\frac{df}{dr}&=0\end{aligned}\right\}\text{.................(1)},$$

or by $$\frac{\frac{df}{dp}}{q'-r'}=\frac{\frac{df}{dq}}{r'-p'}=\frac{\frac{df}{dr}}{p'-q'}\text{...............(2)}.$$

The equations (2), with the identical relation (Art. 364), determine the coordinates p, q, r required.

Cor. The first of the equations (1) shews that the conjugate diameter passes through the pole (at infinity) of the original diameter. Hence we might express the definition of conjugate diameters thus:

Two diameters of a conic are said to be conjugate when each passes through the pole of the other.

405. If the equation to the conic be written in the form

$$up^2+vq^2+wr^2+2u'qr+2v'rp+2w'pq=0,$$

the equations to determine the diameter conjugate to a diameter (p', q', r') become

$$\frac{up+v'r+w'q}{q'-r'}=\frac{vq+w'p+u'r}{r'-p'}=\frac{wr+u'q+v'p}{p'-q'},$$

or

$$\frac{p}{\begin{vmatrix} q'-r', & w', & v' \\ r'-p', & v, & u' \\ p'-q', & u', & w \end{vmatrix}} = \frac{q}{\begin{vmatrix} q'-r', & v', & u \\ r'-p', & u', & w' \\ p'-q', & w, & v' \end{vmatrix}} = \frac{r}{\begin{vmatrix} q'-r', & u, & w' \\ r'-p', & w', & v \\ p'-q', & v', & u' \end{vmatrix}}.$$

COR. If the equation $f(p, q, r) = 0$ represent a parabola, (p', q', r') will be a diameter, provided

$$\frac{q'-r'}{\bar{u}} = \frac{r'-p'}{\bar{v}} = \frac{p'-q'}{\bar{w}}.$$

406. *To find the asymptotes of the conic*

$$f(p, q, r) = 0.$$

Let (p', q', r') be an asymptote.

Then, since (p', q', r') is a tangent whose point of contact is at infinity, these coordinates must satisfy the equation to the conic, and the coordinates of infinity must satisfy the equation to the pole of this tangent.

Hence $$f(p', q', r') = 0,$$

and $$\frac{df}{dp'} + \frac{df}{dq'} + \frac{df}{dr'} = 0.$$

The first of these is a quadratic, and the second is a simple equation; the coordinates of the two asymptotes will therefore be obtained by solving them together.

COR. It appears therefore that the coordinates of the asymptotes of a conic are obtained by solving together the equation to the conic and the equation to its centre. Hence (Art. 386) the asymptotes are the tangents to the curve from its centre.

407. *To shew that the equation*

$$lqr + mrp + npq = 0$$

represents a conic inscribed in the triangle of reference.

The equation is satisfied if $q=0$, $r=0$ are satisfied.

But these equations represent the side BC. Hence BC is a tangent to the conic. So the other sides are tangents.

Therefore &c. Q. E. D.

408. *To shew that the triangle of reference is self-conjugate with respect to the conic*

$$lp^2+mq^2+nr^2=0.$$

By Art. 400, the equation to the pole of the line (p', q', r') is

$$lpp'+mqq'+nrr'=0.$$

Hence, putting $q'=0$, $r'=0$, the pole of the side BC of the triangle of reference is given by

$$p=0,$$

that is, it is the point A.

Hence each side of the triangle of reference is the polar of the opposite angular point.

Therefore &c. Q. E. D.

409. *To find the general equation to a conic circumscribing the triangle of reference.*

Let $$up^2+vq^2+wr^2+2u'qr+2v'rp+2w'pq=0$$

be the equation of a conic passing through the points of reference.

The tangents from $p=0$ are given by

$$vq^2+wr^2+2u'qr=0,$$

and these must be coincident;

therefore $$u'=\pm\sqrt{vw},$$

so $$v'=\pm\sqrt{wu},$$

and $$w'=\pm\sqrt{uv}.$$

Hence writing l^2, m^2, n^2 for u, v, w, the equation becomes

$$l^2p^2 + m^2q^2 + n^2r^2 \pm 2mnqr \pm 2nlrp \pm 2lmpq = 0,$$

and, as in Art. 205, the doubtful signs must be taken either all negative or only one negative, or else the equation would degenerate into two simple equations.

Hence the general equation of a conic circumscribing the triangle of reference may be written

$$\sqrt{lp} + \sqrt{mq} + \sqrt{nr} = 0.$$

410. *If $S = 0$ be the equation to a conic, and $u = 0$, $v = 0$ the equations to two points, it is required to interpret the equation*

$$S + \kappa uv = 0,$$

where κ is an arbitrary constant.

Let (p, q, r) be one of the tangents from the point $u = 0$ to the conic $S = 0$.

Then (p, q, r) satisfy both the equations

$$S = 0 \text{ and } u = 0,$$

and therefore (p, q, r) satisfy the equation

$$S + \kappa uv = 0.$$

Hence either tangent from $u = 0$ to $S = 0$ is a tangent to the conic

$$S + \kappa uv = 0.$$

Similarly, either tangent from $v = 0$ to $S = 0$ is a tangent to the same conic.

Hence the equation represents a conic section, so related to the given conic that two of the common tangents intersect in $(u = 0)$, and the other two in $(v = 0)$.

411. *To interpret the equation*

$$uv + \kappa wx = 0,$$

where $u = 0,\ v = 0,\ w = 0,\ x = 0$

are the equations to points.

Suppose (p, q, r) the straight line joining the points

$$u = 0,\ w = 0,$$

then these coordinates satisfy both the equations

$$u = 0,\ w = 0,$$

and therefore satisfy the equation

$$uv + \kappa wx = 0.$$

But this equation being of the second order represents a conic section. Hence it represents a conic section touching the straight line joining the points

$$u = 0,\ w = 0.$$

Similarly, the conic touches the line joining

$$u = 0,\ x = 0,$$

and the line joining

$$v = 0,\ w = 0,$$

and the line joining

$$v = 0,\ x = 0.$$

Hence it represents a conic inscribed in the quadrilateral whose angular points are

$$u = 0,\ w = 0,\ v = 0,\ x = 0$$

in order.

412. *To interpret the equation*

$$uv + \kappa w^2 = 0.$$

As in the last case, this is a conic touching the lines joining the points $(u = 0,\ w = 0)$ and $(v = 0,\ w = 0)$.

Moreover, the tangents from $u=0$ to the curve, are given by

$$\left.\begin{array}{r} u=0 \\ w^2=0 \end{array}\right\},$$

and therefore are coincident. Hence $(u=0)$ lies on the curve.

Similarly, $(v=0)$ lies on the curve.

Hence the equation represents a conic section passing through the points $u=0$, $v=0$, and whose tangents at those points intersect in $w=0$.

413. *To interpret the equation*

$$lvw + mwu + nuv = 0,$$

where $u=0$, $v=0$, $w=0$ *are the equations of three points.*

Being of the second order the equation represents some conic.

The equation is satisfied when $v=0$ and $w=0$.

Hence the straight line joining $v=0$ and $w=0$ is a tangent to the conic.

Similarly, the straight line joining $w=0$ and $u=0$, and the straight line joining $u=0$, $v=0$, are tangents.

Hence the equation represents a conic inscribed in the triangle whose angular points are $u=0$, $v=0$, $w=0$.

414. By comparison with Art. 408, it will be seen that the equation

$$lu^2 + mv^2 + nw^2 = 0,$$

represents a conic, with respect to which the triangle formed by joining the points

$$u=0, \quad v=0, \quad w=0,$$

is self-conjugate.

So it may be shewn as in Art. 409, that the equation

$$\sqrt{lu} + \sqrt{mv} + \sqrt{nw} = 0,$$

represents a conic circumscribing the same triangle.

415. It will be necessary for the student to distinguish between a curve of the n^{th} order and a curve of the n^{th} class. The following definitions are usually given.

DEF. 1. A curve is said to be of the n^{th} order when any straight line meets it in n real or imaginary points.

DEF. 2. A curve is said to be of the n^{th} class when from any point there can be drawn to it n real or imaginary tangents.

A curve of the n^{th} order will therefore be represented by an equation of the n^{th} degree in trilinear or triangular coordinates, and a curve of the n^{th} class will be represented by an equation of the n^{th} degree in tangential coordinates.

We have shewn that every conic section is both of the second order (Art. 145) and of the second class (Art. 230).

EXERCISES ON CHAPTER XXIII.

(217) The equation

$$p^2 \tan A + q^2 \tan B + r^2 \tan C = 0,$$

represents the circle with respect to which the triangle of reference is self-conjugate.

(218) The circle circumscribing the triangle of reference has the equation

$$\sqrt{p} \sin A + \sqrt{q} \sin B + \sqrt{r} \sin C = 0.$$

(219) The circles escribed to the triangle of reference are given by the equations

$$\begin{aligned} - sqr + (s-c)\,rp + (s-b)\,pq &= 0, \\ (s-c)\,qr - srp + (s-a)\,pq &= 0, \\ (s-b)\,qr + (s-a)\,rp - spq &= 0. \end{aligned}$$

(220) The circle inscribed in the triangle of reference is given by

$$(s-a)\,qr+(s-b)\,rp+(s-c)\,pq=0.$$

(221) The equation to the nine-points' circle of the triangle of reference is

$$\{ap,\ bq,\ cr\}^2=\{ap\cos(B-C)+bq\cos(C-A)+cr\cos(A-B)\}^2,$$

or $$a\sqrt{q+r}+b\sqrt{r+p}+c\sqrt{p+q}=0.$$

(222) The general equation to a conic bisecting the sides of the triangle of reference is

$$(m-n)^2p^2+(n-l)^2q^2+(l-m)^2r^2+2\,(l^2qr+m^2rp+n^2pq)$$
$$=2\,(mn+nl+lm)\,(qr+rp+pq).$$

(223) The conic which touches the sides of the triangle of reference at their middle points has the equation

$$qr+rp+pq=0.$$

(224) The point $fp+gq+hr=0$, lies on the conic

$$lp^2+mq^2+nr^2=0,$$

provided $$\frac{f^2}{l}+\frac{g^2}{m}+\frac{h^2}{n}=0.$$

(225) The point $fp+gq+hr=0$, lies on the conic

$$lqr+mrp+npq=0,$$

provided $$\sqrt{lf}+\sqrt{mg}+\sqrt{nh}=0.$$

(226) The point $fp+gq+hr=0$, lies on the conic

$$\sqrt{lp}+\sqrt{mq}+\sqrt{nr}=0,$$

provided $$\frac{l}{f}+\frac{m}{g}+\frac{n}{h}=0.$$

(227) The six straight lines joining the non-corresponding vertices of two co-polar triangles touch one conic.

(228) The points given by the equation

$$l(q+r)+(\sqrt{m}\pm\sqrt{n})^2 p=0,$$

lie upon the conic $lqr+mrp+npq=0$, and the pole of the chord joining them is given by the equation

$$l(q-r)+(m-n)p=0.$$

(229) Shew that the conic

$$\sqrt{lp}+\sqrt{mq}+\sqrt{nr}=0,$$

is inscribed in the triangle whose angular points are

$$mq+nr-lp=0,\ \ nr+lp-mq=0,\ \ lp+mq-nr=0.$$

(230) If conics are inscribed in a quadrilateral the poles of any fixed straight line lie on another straight line.

(231) Shew that the conic

$$u^2-v^2+w^2=0,$$

is inscribed in the quadrilateral whose angular points in order are

$$u+v+w=0,$$
$$-u+v+w=0,$$
$$u-v+w=0,$$
$$u+v-w=0.$$

(232) The conic

$$\frac{u^2}{m-n}+\frac{v^2}{n-l}+\frac{w^2}{l-m}=0,$$

circumscribes the same quadrilateral.

(233) The equation

$$(l^2 - mn)(p^2 - qr) + (m^2 - nl)(q^2 - rp) + (n^2 - lm)(r^2 - pq) = 0,$$

represents a parabola, passing through the points

$$mp + nq + lr = 0 \quad \text{and} \quad np + lq + mr = 0,$$

and touching the straight lines joining these points to the point

$$lp + mq + nr = 0.$$

(234) The points of contact of tangents from the point $lp + mq + nr = 0$ to the conic $f(p, q, r) = 0$, are given by the equation

$$\begin{vmatrix} u, & w', & v', & l \\ w', & v, & u', & m \\ v', & u', & w, & n \\ l, & m, & n, & 0 \end{vmatrix} f(p, q, r) + \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} (lp + mq + nr)^2 = 0.$$

(235) Two conics have double contact and the common tangents intersect in O. If P be any point on the exterior conic the tangent at P and the tangents from P to the interior conic form with the straight line OP, a harmonic pencil.

CHAPTER XXIV.

POLAR RECIPROCALS.

416. If we refer to the proof of Pascal's Theorem in Art. 200, and to that of Brianchon's Theorem in Art. 218, we shall observe that by interpreting the coordinates as tangential instead of trilinear in the proof of either theorem, we should obtain a proof of the other.

And so in many other cases, the same equations being written down and the same eliminations or other processes being performed, we shall arrive by the selfsame work at two different theorems, differing by the interpretation which we give to the coordinates and to the equations into which they enter.

This is the strict analytical method of applying the principle of duality, the principle by which every theorem concerning the configuration of points has another theorem corresponding to it concerning the configuration of straight lines. And in working with equations either in trilinear or tangential coordinates, we ought always to be on the watch for properties which may be suggested by supposing our coordinates to belong to the opposite system.

But when we use geometrical methods, and arrive at properties of points or lines without the aid of equations, we have not generally any symbols capable of a double interpretation by which we may take advantage of the principle of duality. In this case, therefore, since we cannot obtain a double result by a double interpretation of symbolical expressions, it is useful to consider by what means we can transform a single result so as to arrive at the corresponding theorem.

The method by which we can most directly effect this transformation is called the method of *Polar Reciprocals.* As a geometrical method it does not strictly enter into the scope of the present work, and therefore we shall not greatly enlarge upon its application. We shall, however, explain the fundamental principles upon which these transformations are made, both because we shall thereby obtain an opportunity of exhibiting the significance of many of the equations in tangential coordinates, and because the nomenclature which the method introduces is often employed in the statement of propositions of importance in the analytical methods.

417. As an example of the double interpretation of results to which we have just referred, we will arrange in parallel columns two important propositions connected together by the principle of duality, and give their common method of proof, using trilinear coordinates for the one proposition and tangential for the other. To shew the identity of the work, we will use the same letters x, y, z to represent triangular coordinates in the first column and tangential coordinates in the second.

If two triangles be inscribed in one conic, their sides will touch one conic.	*If two triangles circumscribe one conic, their angular points will lie on one conic.*

Take one of the triangles as triangle of reference, and let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the

angular points	sides

of the other. And let the conic have the equation

trilinear] $$\frac{l}{x}+\frac{m}{y}+\frac{n}{z}=0,$$ [*tangential*

then the equations to the

sides	angular points

of the second triangle will be

$$\frac{lx}{x_2x_3}+\frac{my}{y_2y_3}+\frac{nz}{z_2z_3}=0,$$

$$\frac{lx}{x_3x_1}+\frac{my}{y_3y_1}+\frac{nz}{z_3z_1}=0,$$

$$\frac{lx}{x_1x_2}+\frac{my}{y_1y_2}+\frac{nz}{z_1z_2}=0.$$

Now the equation

$$\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0$$

represents any conic

inscribed in } circumscribing

the first triangle (the triangle of reference) and it will also

be inscribed in } circumscribe

the second triangle provided λ, μ, ν be determined so as to satisfy the equations

$$\frac{\lambda x_2x_3}{l}+\frac{\mu y_2y_3}{m}+\frac{\nu z_2z_3}{n}=0,$$

$$\frac{\lambda x_3x_1}{l}+\frac{\mu y_3y_1}{m}+\frac{\nu z_3z_1}{n}=0,$$

$$\frac{\lambda x_1x_2}{l}+\frac{\mu y_1y_2}{m}+\frac{\nu z_1z_2}{n}=0,$$

which are consistent equations provided

$$\begin{vmatrix} \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2} \\ \frac{1}{x_3}, & \frac{1}{y_3}, & \frac{1}{z_3} \end{vmatrix}=0.$$

And this is seen to be identically satisfied, since by hypothesis

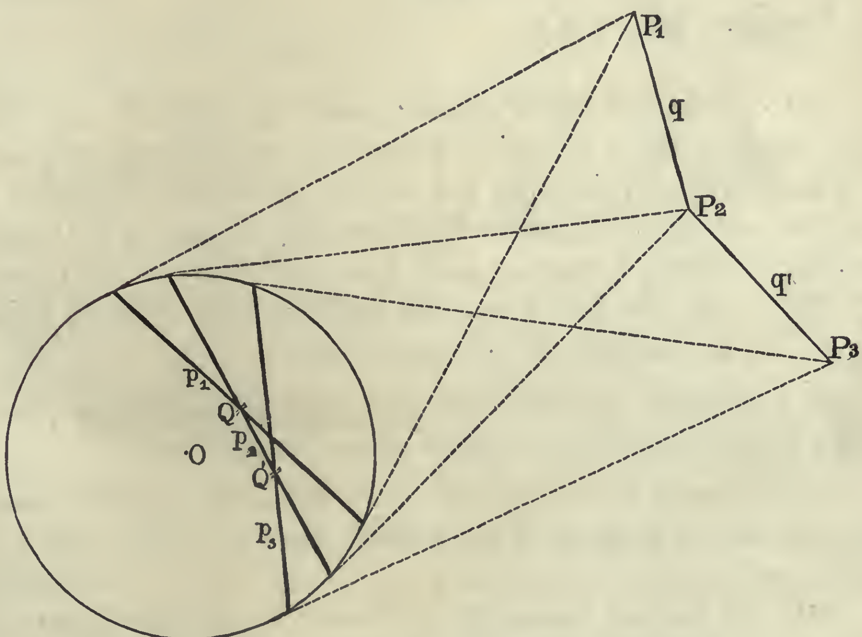
$$\frac{l}{x_1}+\frac{m}{y_1}+\frac{n}{z_1}=0, \quad \frac{l}{x_2}+\frac{m}{y_2}+\frac{n}{z_2}=0, \quad \frac{l}{x_3}+\frac{m}{y_3}+\frac{n}{z_3}=0.$$

Therefore, &c. Q.E.D.

418. Let the points P_1, P_2, P_3 ...... be the poles of the straight lines p_1, p_2, p_3 ...... respectively, with respect to a conic O.

If the points P_1, P_2, P_3 ...... all lie upon one straight line, we know that the straight lines p_1, p_2, p_3 ...... will all pass through one point. But if otherwise, the points P_1, P_2, P_3 ......

Fig. 42.



in order may be regarded as the angular points of a polygon, and the straight lines p_1, p_2, p_3 ...... in order may be regarded as the sides of another polygon.

This second polygon is called the reciprocal of the first polygon with respect to the conic O.

419. *If with respect to a conic, the reciprocal of the polygon* G *be the polygon* g; *the reciprocal of the polygon* g *will be the polygon* G.

For let P_1, P_2 be any two adjacent angular points of the polygon G, and p_1, p_2 their polars: then since g is the reciprocal of G, p_1, p_2 are sides of the polygon g.

Let the sides p_1, p_2 intersect in Q, and let the polar of Q be q, then since p_1, p_2 intersect in Q, their poles P_1, P_2 lie on q the polar of Q.

Hence Q is an angular point of the second polygon, and its polar q is a side of the first polygon.

Therefore the polars of all the angular points of the second polygon are sides of the first. Therefore the reciprocal of the second polygon is the first polygon.

Therefore, &c. Q. E. D.

420. If the number of angular points P_1, P_2, P_3 ...... of the first polygon be indefinitely increased, so that the polygon becomes ultimately a curve, the number of sides of the second polygon will likewise increase indefinitely, so that it will also become ultimately a curve. And if we regard any of the points P_1, P_2 ...... on the first curve, the corresponding straight lines p_1, p_2 ...... are tangents to the second curve.

So if n points on either curve lie upon a straight line, then will n tangents to the other curve pass through a point.

Consequently *if one curve be of the* m^{th} *order and* n^{th} *class, the other will be of the* n^{th} *order and* m^{th} *class.*

421. It follows from Art. 419 that if any curvilinear or other locus F be the reciprocal of another locus f with respect to a conic O, the locus f will also be the reciprocal of the locus F. The two loci are said to correspond to each other with respect to the conic O.

It is convenient to speak of the centre of the conic O as the *centre of reciprocation.*

422. The following theorems follow immediately from the principles we have laid down.

(i). A point	*corresponds to*	(i). a straight line.
(ii). The point of intersection of two straight lines.		(ii). The straight line joining the corresponding points.
(iii). Collinear points.		(iii). Concurrent straight lines.
(iv). A polygon of n sides.		(iv). A polygon of n sides.
(v). The angular points of a polygon.		(v). The sides of the corresponding polygon.
(vi). A curve of the m^{th} order and n^{th} class.		(vi). A curve of the m^{th} class and n^{th} order.
(vii). A point on a curve.		(vii). A tangent to the corresponding curve.
(viii). The point of contact of a tangent.		(viii). The tangent at the corresponding point.
(ix). A chord joining two points.		(ix). The point of intersection of the corresponding tangents.
(x). The chord of contact of two tangents.		(x). The point of intersection of tangents at the corresponding points.
(xi). A curve inscribed in a polygon.		(xi). A curve circumscribing the corresponding polygon.
(xii). A point of intersection of two curves.		(xii). A common tangent to two curves.

(xiii). *corresponds to* (xiii).

Two curves which touch one another. i.e. Which have a common point and the same tangent thereat.	Two curves which touch one another. i.e. Which have a common tangent and the same point of contact.
(xiv). Two curves having double contact.	(xiv). Two curves having double contact.
(xv). The chord of contact.	(xv). The point of intersection of the common tangents.
(xvi). A double point on a curve*. i.e. A point at which there are two tangents.	(xvi). A double tangent* to the corresponding curve. i.e. A tangent having two points of contact.
(xvii). A point of osculation*.	(xvii). A point of osculation.
(xviii). A point Q in which the tangent at P cuts the curve.	(xviii). A tangent q drawn from the point of contact of a tangent p.
(xix). A point of inflexion*. Obtained from the last case by making Q coincide with P.	(xix). The tangent at a point of inflexion. Obtained from the last case by making q coincide with p.
(xx). A curve having r points of inflexion.	(xx). A curve having r points of inflexion.
(xxi). The straight line at infinity.	(xxi). The centre of reciprocation.
(xxii). A point at infinity.	(xxii). A straight line through the centre of reciprocation.

* See the Definitions *infrà* Chap. XXVI.

(xxiii).	*corresponds to*	(xxiii).
An asymptote. i.e. A tangent at infinity.		The point of contact of a tangent from the centre of reciprocation.
(xxiv). Parallel straight lines.		(xxiv). Points collinear with the centre of reciprocation.

423. The foregoing properties apply to all curves whatsoever: we proceed now to state some which apply to conic sections in particular.

Since a conic section is of the second order and of the second class (Art. 415), it follows immediately from (vi) that

(xxvi).		(xxvi).
A conic section	*corresponds to*	a conic section.
(xxvii). The pole of a straight line with respect to any conic. This follows from (x).		(xxvii). The polar of the corresponding point with respect to the corresponding conic.
(xxviii). The centre of a conic. This follows from the preceding by supposing the line to be at infinity. See (xxi).		(xxviii). The chord of contact of tangents from the centre of reciprocation to the corresponding conic.
(xxix). Parallel tangents. See (xxiv).		(xxix). The extremities of a chord through the centre of reciprocation.
(xxx). Concentric conics.		(xxx). Conics with respect to which the polars of the centre of reciprocation coincide.
(xxxi). A pair of conjugate diameters in a conic. i. e. Two lines each of which is the polar of the point where the other meets infinity.		(xxxi). Two points which with the centre of reciprocation form a self-conjugate triangle with respect to the corresponding conic.

(xxxii). *corresponds to*	(xxxii).
The points where a conic meets the straight line at infinity.	The tangents to the corresponding conic from the centre of reciprocation.
(xxxiii).	(xxxiii).
A hyperbola. i.e. A conic meeting the straight line at infinity in real points.	A conic having its convexity towards the centre of reciprocation. i.e. Having real tangents from that point.
(xxxiv).	(xxxiv).
An ellipse. i.e. A conic meeting the straight line at infinity in imaginary points.	A conic having its concavity towards the centre of reciprocation. i.e. Having imaginary tangents from that point.
(xxxv).	(xxxv).
A parabola. i.e. A conic meeting the straight line at infinity in coincident points.	A conic passing through the centre of reciprocation. i.e. Having coincident tangents from that point.

424. *Given the equation of a curve in triangular coordinates, it is required to find the equation to its polar reciprocal with reference to the conic*

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \text{ ....(1).}$$

Let $$f(\alpha, \beta, \gamma) = 0$$

be the equation to the given curve: let $(\alpha', \beta', \gamma')$ be any point upon it so that

$$f(\alpha', \beta', \gamma') = 0 \text{.......................(2),}$$

and let p, q, r be the tangential coordinates of the straight line corresponding to $(\alpha' \beta', \gamma')$.

Then the equation to this straight line in triangular coordinates is

$$p\alpha + q\beta + r\gamma = 0.$$

But since it is the polar of $(\alpha', \beta', \gamma')$ with respect to (1), its equation may be written

$$l\alpha'\alpha + m\beta'\beta + n\gamma'\gamma = 0,$$

therefore
$$\frac{l\alpha'}{p} = \frac{m\beta'}{q} = \frac{n\gamma'}{r},$$

and substituting in (2), we get

$$f\left(\frac{p}{l}, \frac{q}{m}, \frac{r}{n}\right) = 0,$$

a relation amongst p, q, r, and therefore the tangential equation of the curve reciprocal to the given one.

Cor. 1. The conic of reciprocation being its own reciprocal is represented in tangential coordinates by the equation

$$\frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} = 0.$$

The centre of reciprocation is given in triangular coordinates by the equations

$$l\alpha = m\beta = n\gamma,$$

and in tangential coordinates by the equation

$$\frac{p}{l} + \frac{q}{m} + \frac{r}{n} = 0.$$

Cor. 2. The reciprocal of the curve whose tangential equation is

$$f(p, q, r) = 0$$

is represented in triangular coordinates by the equation

$$f(l\alpha, m\beta, n\gamma) = 0.$$

Cor. 3. With respect to the conic

$$\alpha^2 + \beta^2 + \gamma^2 = 0,$$

the equations

$$f(\alpha, \beta, \gamma) = 0, \text{ and } f(p, q, r) = 0$$

represent corresponding curves.

But it must be borne in mind that if the lines of reference are real, the conic of reciprocation is here imaginary. The centre of reciprocation is however the real point

$$\alpha = \beta = \gamma, \text{ or } p + q + r = 0,$$

viz. the centre of gravity of the triangle.

425. We are now in a position to illustrate the apparent discrepancy alluded to in Art. 382 as to the interpretation of the equation in tangential coordinates

$$\{ap, \ bq, \ cr\}^2 = 0.$$

We have seen (Art. 111) that the circular points at infinity are given in trilinear coordinates by the equations

$$\frac{\alpha}{-1} = \frac{\beta}{\cos C \mp \sqrt{-1} \sin C} = \frac{\gamma}{\cos B \pm \sqrt{-1} \sin B}.$$

Their polars with respect to the conic

$$f(\alpha, \beta, \gamma) = 0$$

are therefore represented by the equations

$$\left(-\frac{df}{d\alpha} + \frac{df}{d\beta}\cos C + \frac{df}{d\gamma}\cos B\right) \mp \sqrt{-1}\left(\frac{df}{d\beta}\sin C - \frac{df}{d\gamma}\sin B\right) = 0,$$

or $$\left\{\frac{df}{d\alpha}, \ \frac{df}{d\beta}, \ \frac{df}{d\gamma}\right\}^2 = 0.$$

So in triangular coordinates the polars of the circular points at infinity with respect to the conic

$$f(\alpha, \beta, \gamma) = 0,$$

are given by the equation

$$\left\{a\frac{df}{d\alpha}, \ b\frac{df}{d\beta}, \ c\frac{df}{d\gamma}\right\}^2 = 0.$$

Consider the particular case of the conic whose equation is

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0, \qquad [triangular$$

and whose centre is given (Art. 178) by the equation

$$l\alpha = m\beta = n\gamma.$$

The polars with respect to it of the circular points are represented by the equation

$$\{la\alpha,\ \ mb\beta,\ \ nc\gamma\}^2 = 0.$$

But since the circular points are imaginary points at infinity, their polars must be imaginary and pass through the centre of the conic.

Hence the equation

$$\{la\alpha,\ \ mb\beta,\ \ nc\gamma\}^2 = 0$$

represents two imaginary straight lines intersecting in the real point

$$l\alpha = m\beta = n\gamma.$$

But it follows from Art. 292 that the same equation may be more completely viewed as representing an evanescent conic section, whose real branch has degenerated into the point

$$l\alpha = m\beta = n\gamma,$$

and whose imaginary branches have become two imaginary straight lines.

Now suppose we reciprocate the locus of this equation

$$\{la\alpha,\ \ mb\beta,\ \ nc\gamma\}^2 = 0$$

with respect to the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

If we interpret the locus as two imaginary straight lines (the polars of the circular points) it will reciprocate into two imaginary points (the circular points themselves). On the other hand, if we regard the locus as a conic (evanescent at the centre of reciprocation) it will reciprocate into a conic (the great circle at infinity).

Now by Art. 424, the equation to the reciprocal is

$$\{ap,\ bq,\ cr\}^2 = 0.$$

Hence the ambiguity in the interpretation of this equation is accounted for, and is seen to be the direct result of the ambiguity in the interpretation of any equation to an evanescent conic, which may always be regarded as equally representing two imaginary straight lines.

426. We can now continue our table as follows:

(xxxvi). Infinity	*corresponds to*	(xxxvi). the centre of reciprocation.
(xxxvii). The great circle at infinity.		(xxxvii). An evanescent conic at the centre of reciprocation.
(xxxviii). The circular points at infinity.		(xxxviii). The polars of the circular points at infinity with respect to the evanescent conic.
(xxxix). The foci of a conic.		(xxxix). The chords joining the four points in which the corresponding conic is cut by the lines corresponding to the circular points.

427. *If a pencil of straight lines be reciprocated into a range of points, the anharmonic ratio of the range is the same as that of the pencil.*

Take the lines of reference so that the conic of reciprocation may have the equation

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

And let $u=0$, $u+\kappa v=0$, $v=0$, $u+\kappa' v=0$, be the equations to the straight lines forming the pencil, then the anharmonic ratio is $\frac{\kappa}{\kappa'}$. (Art. 124).

But the same equations with p, q, r written for α, β, γ will represent in tangential coordinates the range of four points. (Art. 424, Cor. 3).

Therefore (Art. 374) the anharmonic ratio of the range is $\frac{\kappa}{\kappa'}$, the same as that of the pencil. Q. E. D.

428. Cor. 1. If four straight lines a, b, c, d (not necessarily concurrent) be cut by another straight line p in four points forming a range whose anharmonic ratio is k, the points corresponding to a, b, c, d being joined to the point corresponding to p will form a pencil whose range is also k: and conversely.

For by Art. 422 (ii), the points in the range correspond to the lines in the pencil.

Note. If a, b, c, d, p represent straight lines, it is often convenient to use the symbol $\{p\,.\,abcd\}$ to denote the anharmonic ratio of the points in which the straight lines a, b, c, d intersect the straight line p.

429. Cor. 2. If p, a, b, c, d be the tangents to a conic at the points P, A, B, C, D respectively, then will

$$\{p\,.\,abcd\}=\{P\,.\,ABCD\}.$$

430. *With respect to a circle, any circle reciprocates into a conic, having a focus at the centre of reciprocation.*

Take a triangle self-conjugate with respect to the circle of reciprocation as triangle of reference.

Then in triangular coordinates the circle of reciprocation has the equation (Art. 179),

$$\alpha^2\cot A+\beta^2\cot B+\gamma^2\cot C=0\ldots\ldots\ldots\ldots\ldots(1),$$

and any other circle may be represented by the equation (Art. 322),

$$\{\alpha \cos A,\ \beta \cos B,\ \gamma \cos C\}^2 = (l\alpha + m\beta + n\gamma)(\alpha + \beta + \gamma) \ldots (2).$$

Now by Art. 424 the reciprocal of the circle (2) with respect to the circle (1) is represented in tangential coordinates by the equation

$$\{p \sin A,\ q \sin B,\ r \sin C\}^2$$

$$= (lp \tan A + mq \tan A + nr \tan A)(p \tan A + q \tan B + r \tan C),$$

or
$$\{ap,\ bq,\ cr\}^2$$

$$= \frac{a^2}{\sin^2 A}(lp \tan A + mq \tan B + nr \tan C)(p \tan A + q \tan B + r \tan C).$$

But (Art. 383) this represents a conic whose foci are given by

$$lp \tan A + mq \tan B + nr \tan C = 0,$$

and
$$p \tan A + q \tan B + r \tan C = 0;$$

the latter of which is the equation to the centre of reciprocation.

Hence the reciprocal of a circle with respect to a circle is a conic, having a focus at the centre of reciprocation.

Cor. Conversely, any conic reciprocated with respect to a circle having a focus as centre, corresponds to a circle.

431. We can tabulate our results as follows:

Reciprocal Loci with respect to a Circle.

(xl).	(xl).
A hyperbola having the centre of reciprocation as focus.	A circle having the centre of reciprocation without it. See (xxxiii).

(xli). An ellipse having the centre of reciprocation as focus.	(xli). A circle having the centre of reciprocation within it. (See xxxiv).
(xlii). A parabola having the centre of reciprocation as focus.	(xlii). A circle passing through the centre of reciprocation. See (xxxv).
(xliii). The directrix of the conic.	(xliii). The centre of the circle.
(xliv). The great circle at infinity.	(xliv). The evanescent circle at the centre of reciprocation.
(xlv). The circular points at infinity.	(xlv). The straight lines joining the centre of reciprocation to the circular points.
(xlvi). The foci of a conic. See Art. 391.	(xlvi). The chords joining the four points in which the corresponding conic is cut by radii from the centre of reciprocation to the circular points.
(xlvii). The focus of reciprocation.	(xlvii). The straight line at infinity.

432. Observing that the polar of a point P with respect to a circle whose centre is O, is the common chord of that circle, and the circle on OP as diameter, and is therefore at right angles to OP, it follows that the angle which two points subtend at the centre is equal to the angle between their polars.

Hence, when we reciprocate with respect to a circle, *the angle between two straight lines is equal to the angle which the corresponding points subtend at the centre of reciprocation.*

433. Moreover the distances of a point and its polar from the centre of the circle contain a rectangle equal to the square on the radius.

Hence, when we reciprocate with respect to a circle, *the distances of different points from the centre of reciprocation are inversely proportional to the distances of the corresponding lines.*

434. By the aid of this property it is easy to calculate the magnitude of the conic corresponding to any circle with respect to another circle.

For let k be the radius of the circle of reciprocation, r the radius of the circle to be reciprocated, and h the distance between their centres. And suppose a and b the semi-axes of the conic, and e its excentricity.

By symmetry, the line joining the centres of the circles must be the axis of the conic, and the perpendiculars on the tangents at the vertices lie along this line.

We have, therefore,

$$a(1+e)=\frac{k^2}{r-h}, \quad \text{and } a(1-e)=\frac{k^2}{r+h},$$

whence, $$a=\frac{k^2r}{r^2-h^2}, \quad b=\frac{k^2}{\sqrt{(r^2-h^2)}}, \quad e=\frac{h}{r}.$$

435. It thus appears that the excentricity of the reciprocal conic is independent of the radius of the circle of reciprocation. The magnitude of this circle therefore only affects the magnitude, not the form of the resulting figure. Thus it happens in many cases that the magnitude of the circle of reciprocation does not affect a proposition, and it is therefore often convenient to speak briefly of *reciprocation with respect to a point O*, when we mean reciprocation with respect to a circle drawn at an undefined distance from the centre O.

We will now give some examples of the manner in which the method of polar reciprocals is applied in the solution of problems.

436. *Four fixed tangents are drawn to a conic: to prove that the anharmonic ratio of the points in which they are cut by any variable tangent is constant.*

Let a, b, c, d denote four fixed tangents to a conic, and let p and q be any other tangents. Reciprocate the figure with respect to a *focus:* then the tangents a, b, c, d correspond to four fixed points A, B, C, D on a circle, and p, q to any other points P, Q on the same circle.

Now, by *Eucl.* III. 21, the chords joining A, B, C, D subtend the same angles at P as at Q.

Hence, $$\{P.ABCD\} = \{Q.ABCD\};$$

therefore by Art. 428,

$$\{p.abcd\} = \{q.abcd\}.$$ Q. E. D.

437. *Four fixed points are taken on a conic: to prove that the anharmonic ratio of the pencil joining them to any variable point on the same conic is constant.*

Let A, B, C, D denote four fixed points on a conic, and let P, Q be any other points. Reciprocate the figure with respect to *any point;* then the points A, B, C, D, P, Q correspond to tangents a, b, c, d, p, q to another conic, and therefore by the last proposition

$$\{p.abcd\} = \{q.abcd\}.$$

Hence by Art. 428,

$$\{P.ABCD\} = \{Q.ABCD\}.$$ Q. E. D.

438. *An ellipse is inscribed in a quadrilateral: to prove that any two opposite sides subtend supplementary angles at either focus.*

Reciprocate the whole figure with respect to a circle having the focus as centre. Then, by Art. 430, the conic corresponds to a circle, and the circumscribed quadrilateral to an inscribed quadrilateral. By *Eucl.* III. 22, any two opposite angles of this quadrilateral are equal to two right angles. Hence, by Art. 432, any two opposite sides of the corresponding quadrilateral subtend at the centre of reciprocation angles which are together equal to two right angles. Hence the proposition is proved.

439. The following "corresponding theorems" will suffice to shew how the principal properties of conic sections may be deduced from the simplest properties of the circle by the method of polar reciprocals:

Two tangents to a circle are equally inclined to their chord of contact.	Two tangents to a conic measured from their point of intersection subtend equal angles at a focus.
Two tangents to a circle are equally inclined to the diameter through their point of intersection.	The segments of any chord of a conic, measured from the directrix subtend equal angles at a focus.
Parallel tangents to a circle touch it at the extremities of a diameter.	Tangents at the extremities of a focal chord intersect in the directrix.
A chord which subtends a right angle at a fixed point on a circle passes through the centre.	Tangents to a parabola at right angles to one another intersect on the directrix.
In any circle the sum of the perpendiculars from a fixed point on a pair of parallel tangents is constant.	In any conic the sum of the reciprocals of the segments of any focal chord is constant.
If chords of a circle be drawn through a fixed point, the rectangle contained by the segments is constant.	The rectangle contained by the perpendiculars from the focus of a conic on a pair of parallel tangents is constant.

440. The following corresponding theorems illustrate the nature of the great circle at infinity:

All real points on an evanescent conic coincide.	All real tangents to the great circle coincide with the straight line at infinity.
All real tangents to an evanescent conic meet in a point.	All real points on the great circle lie on the straight line at infinity.

All imaginary points on an evanescent conic lie on one of two imaginary straight lines.

All imaginary tangents to the great circle pass through one of two imaginary points (viz. the circular points).

441. The following will be also seen to be reciprocal theorems:

Pascal's Theorem. If a hexagon be inscribed in a conic the points of intersection of opposite sides are collinear.

Brianchon's Theorem. If a hexagon circumscribe a conic the straight lines joining opposite vertices are concurrent.

If a quadrilateral circumscribe a conic, the intersections of its opposite sides and of its diagonals will be the vertices of a self-conjugate triangle.

If a quadrilateral be inscribed in a conic, the intersections of its opposite sides and of its diagonals will be the vertices of a self-conjugate triangle.

If two triangles be polar reciprocals with respect to any conic, the intersections of the corresponding sides lie on a straight line.

If two triangles be polar reciprocals with respect to any conic, the straight lines which join their corresponding vertices meet in a point.

Exercises on Chapter XXIV.

(236) If a conic touch the sides BC, CA, AB of a triangle in the points A', B', C', then at either focus ~~BC, CA, AB~~ subtend equal angles: so also do CA, AB', BC', and so do AB, $B'C$, CA'.

(237) If two tangents to a parabola meet the directrix in Z, Z', and if S be the focus, the angle ZSZ' or its supplement is double of the angle between the tangents.

(238) In the plane of the figure 38 (page 310) any point O is taken, and through A, A', a straight lines AP, $A'P'$, ap are drawn so as to make the pencils

$$\{A \,.\, BPCO\},\quad \{A' \,.\, BP'C'O\},\quad \{a,\, bpcO\}$$

harmonic. Shew that these three straight lines are concurrent.

(239) If two conics have a common focus and directrix, the tangent and focal radius at any point on the exterior conic divide harmonically the tangents from that point to the interior conic.

(240) With a given point as focus four conics can be drawn so as to pass through three given points, and another conic can be described having the same focus and touching the first four conics.

(241) S is the common focus of two conics, and S_1, S_2 are the poles with respect to either of the directrices of the other. Shew that S, S_1, S_2 are collinear.

(242) Four conics are described each touching the three sides of one of the four triangles ABC, BCD, CAD, ABD, and all having a common focus S: shew that they all have a common tangent.

(243) The reciprocal of a parabola with regard to a point on the directrix is an equilateral hyperbola.

(244) The intersection of perpendiculars of a triangle circumscribing a parabola is a point on the directrix.

(245) The intersection of perpendiculars of a triangle inscribed in an equilateral hyperbola lies on the curve.

(246) The tangents from any point to two confocal conics are equally inclined to each other.

(247) The locus of the pole of a fixed line with regard to a series of confocal conics is a straight line.

(248) On a fixed tangent to a conic are taken a fixed point A and two moveable points P, Q, such that AP, AQ subtend equal angles at a fixed point O. From P, Q are drawn two other tangents to the conic, prove that the locus of their point of intersection is a straight line.

(249) Chords are drawn to a conic, subtending a right angle at a fixed point; prove that they all touch a conic, of which that point is a focus.

(250) Prove that two ellipses which have a common focus cannot intersect in more than two points.

(251) OA, OB are common tangents to two conics which have a common focus S, and ASB is a focal chord. Shew that if the second tangents from A and B to one conic meet in C, and those to the other conic meet in D, then C, D, S are collinear.

(252) If two conics circumscribe a quadrilateral and have double contact with another conic, the tangents at the extremities of the chords of contact intersect in two points which divide harmonically one of the diagonals of the quadrilateral.

(253) Three conic sections have a common tangent, and each touches two sides of the triangle ABC at the extremities of the third side; shew that if the sides of this triangle meet the common tangent in A', B', C', each of the points of contact of that tangent will form with A', B', C' a harmonic range.

(254) A triangle ABC is inscribed in a conic, and the tangents at the angular points A, B, C are produced to meet the opposite sides in P, Q, R. From these points other tangents are drawn to touch the conic in A', B', C'. Shew that if the tangents at A, B, C form a triangle abc, and the tangents at A', B', C' form a triangle $a'b'c'$, then A, a, a' are collinear, so are B, b, b', and so are C, c, c'.

CHAPTER XXV.

CONICS DETERMINED BY ASSIGNED CONDITIONS.

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442. We shewed in Art. 147 that a conic can generally be found to satisfy five simple conditions, each condition giving rise to an equation connecting the coefficients in the general equation to a conic. It will, however, be observed that if any of these equations are of the second or a higher order, we shall have two or more solutions indicating two or more conics satisfying the given conditions.

Again, in Art. 201 we gave an example of a double condition, when we shewed that if the centre of a conic be assigned this is equivalent to two simple conditions being given: and it will presently be seen that conditions may occur equivalent to three or four or five simple conditions.

In order therefore that we may in all cases be able to judge of the sufficiency of any assigned conditions to determine a conic, it will be desirable

(1°) To determine what conditions shall be regarded as simple conditions, classifying them according to the nature of the relations to which they give rise, among the coefficients of the general equation.

(2°) To consider how many conics can be drawn to fulfil five simple conditions when the classes of those conditions are assigned, and

(3°) To analyse more complicated conditions, and to determine to how many simple conditions they are equivalent, assigning the class of those simple conditions.

443. We shall only find it necessary to make two classes of simple conditions, which we shall distinguish as *point-conditions* and *line-conditions.* We shall find that all other conditions of common occurrence may be regarded either as particular cases of these two, or as made up of repetitions of them.

444. DEF. We shall call two points *conjugate* with respect to a conic when each lies on the polar of the other, and we shall call two straight lines *conjugate* when each passes through the pole of the other.

445. Let

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$$

be the equation in trilinear coordinates to a conic section, and let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be any two points in the same plane.

The equation

$$u\alpha_1\alpha_2 + v\beta_1\beta_2 + w\gamma_1\gamma_2 + u'(\beta_1\gamma_2 + \beta_2\gamma_1) + v'(\gamma_1\alpha_2 + \gamma_2\alpha_1) + w'(\alpha_1\beta_2 + \alpha_2\beta_1) = 0,$$

may be written in either of the forms

$$\alpha_1 \frac{df}{d\alpha_2} + \beta_1 \frac{df}{d\beta_2} + \gamma_1 \frac{df}{d\gamma_2} = 0,$$

or

$$\alpha_2 \frac{df}{d\alpha_1} + \beta_2 \frac{df}{d\beta_1} + \gamma_2 \frac{df}{d\gamma_1} = 0,$$

and expresses the condition (Art. 232) that each of the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ lies on the polar of the other with respect to the conic, or that the two points are conjugate with respect to the conic.

It will be observed that when the two points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ are given, the condition that they should be conjugate furnishes us with a simple equation, connecting the six coefficients in the general equation to a conic. Five such conditions will therefore suffice to determine the five ratios of the coefficients in the equation, and therefore to determine the conic.

446. Let $$l_1\alpha + m_1\beta + n_1\gamma = 0,$$
and $$l_2\alpha + m_2\beta + n_2\gamma = 0,$$
be two straight lines. Then the equation

$$\begin{vmatrix} u, & w', & v', & l_1 \\ w', & v, & u', & m_1 \\ v', & u', & w, & n_1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0$$

will express the condition (Art. 233) that each passes through the pole of the other, or that the two straight lines are conjugate with respect to the conic.

It will be seen that this equation is a quadratic in u, v, w, u', v', w'. Hence when two straight lines are given, the condition that they should be conjugate furnishes us with a quadratic equation connecting the six coefficients in the general equation to the conic. Therefore if a condition such as this be substituted for one of the conditions in the case last considered, there will be an ambiguity in the determination of the coefficients of the trilinear equation unless it happen that the quadratic have equal roots.

447. Let
$$f(p, q, r) \equiv up^2 + vq^2 + wr^2 + 2u'qr + 2v'rp + 2w'pq = 0,$$
be the equation to a conic section in tangential coordinates, and let (p_1, q_1, r_1), (p_2, q_2, r_2) be any two straight lines in the same plane.

The equation
$$up_1p_2 + vq_1q_2 + wr_1r_2 + u'(q_1r_2 + q_2r_1) + v'(r_1p_2 + r_2p_1) + w'(p_1q_2 + p_2q_1) = 0,$$
may be written in either of the forms

$$p_1 \frac{df}{dp_2} + q_1 \frac{df}{dq_2} + r_1 \frac{df}{dr_2} = 0,$$

or
$$p_2 \frac{df}{dp_1} + q_2 \frac{df}{dq_1} + r_2 \frac{df}{dr_1} = 0,$$

and expresses the condition (Art. 232) that each of the straight lines (p_1, q_1, r_1), (p_2, q_2, r_2) passes through the pole of the other with respect to the conic, or that the two straight lines are conjugate with respect to the conic.

It will be observed that when the two straight lines (p_1, q_1, r_1), (p_2, q_2, r_2) are given, the condition that they should be conjugate furnishes us with a simple equation connecting the six coefficients in the general equation to a conic. Five such conditions will therefore suffice to determine the five ratios of the six coefficients in the equation, and therefore to determine the conic.

448. Let
$$l_1 p + m_1 q + n_1 r = 0,$$
and
$$l_2 p + m_2 q + n_2 r = 0,$$
be two points. Then the equation

$$\begin{vmatrix} u, & w', & v', & l_1 \\ w', & v, & u', & m_1 \\ v', & u', & w, & n_1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0$$

will express the condition (Art. 233) that each lies on the polar of the other, or that the two points are conjugate with respect to the conic.

It will be seen that this equation is a quadratic in u, v, w, u', v', w'. Hence when two points are given, the condition that they should be conjugate furnishes us with a quadratic equation connecting the six coefficients in the general equation to a conic. Therefore if a condition such as this be substituted for one of the conditions in the case last considered, there will be an ambiguity in the determination of the coefficients of the tangential equation unless it happen that the quadratic have equal roots.

449. Def. The condition that a conic be such that with respect to it two given points are conjugate, is called a *point-condition.*

The condition that a conic be such that with respect to it two given straight lines are conjugate, is called a *line-condition.*

450. *To fulfil five point-conditions there can be drawn one and only one conic.*

For, using trilinear coordinates, each of the five conditions will furnish us with a simple equation (Art. 445) connecting the coefficients of the general equation to a conic. These five equations will determine the five ratios of the coefficients without ambiguity, and therefore will determine one and only one conic fulfilling the given conditions.

451. *To fulfil four point-conditions and one line-condition there cannot be drawn more than two conics.*

For, using trilinear coordinates, each of the four point-conditions will furnish us with a simple equation (Art. 445) connecting the coefficients of the general equation. And the line-condition will furnish us with a fifth equation, a quadratic (Art. 446), connecting the same coefficients. These five equations will determine the five ratios of the coefficients, but since one is a quadratic there will in general be two solutions, indicating two conics fulfilling the given conditions.

452. *To fulfil three point-conditions and two line-conditions there cannot be drawn more than four conics.*

For, using trilinear coordinates, each of the three point-conditions will furnish us with a simple equation (Art. 445) connecting the coefficients of the general equation and the two line-conditions will furnish us with two more equations, quadratics (Art. 446), connecting the same coefficients. These five equations will determine the five ratios of the coefficients, but since two are quadratics there will in general be four solutions, indicating four conics fulfilling the given conditions.

453. *To fulfil three line-conditions and two point-conditions there cannot be drawn more than four conics.*

For, using tangential coordinates, each of the three line-conditions will furnish us with a simple equation (Art. 447) connecting the coefficients of the general equation, and the two point-conditions will furnish us with two more equations, quadratics (Art. 448), connecting the same coefficients. These five equations will determine the five ratios of the coefficients, but since two are quadratics there will in general be four solutions, indicating four conics fulfilling the given conditions.

454. *To fulfil four line-conditions and one point-condition there cannot be drawn more than two conics.*

For, using tangential coordinates, each of the four line-conditions will furnish us with a simple equation (Art. 447) connecting the coefficients of the general equation. And the point-condition will furnish us with a fifth equation, a quadratic (Art. 448), connecting the same coefficients. These five equations will determine the five ratios of the coefficients, but since one is a quadratic there will in general be two solutions, indicating two conics fulfilling the given conditions.

455. *To fulfil five line-conditions there can be drawn one and only one conic.*

For, using tangential coordinates, each of the five conditions will furnish us with a simple equation (Art. 447) connecting the coefficients of the general equation to a conic. These five equations will determine the five ratios of the coefficients without ambiguity, and therefore will determine one and only one conic fulfilling the given conditions.

456. It remains that we should analyse the conditions most usually assigned, and determine to how many point- or line-conditions they may severally be equivalent. We shall then be able to apply the five preceding articles to determine how many conics (at most) can be drawn in cases where such conditions are given.

I. *Given a point on a conic.*

Since a point on a conic lies on its own polar it is conjugate to itself. This therefore is equivalent to *one point-condition.*

II. *Given a tangent to a conic.*

Since a tangent to a conic passes through its own pole it is conjugate to itself. This therefore is equivalent to *one line-condition.*

III. *Given a diameter.*

Any diameter passes through the centre, which is the pole of the straight line at infinity. Hence a diameter and the straight line at infinity are conjugate lines. This therefore is equivalent to *one line-condition.*

IV. *Let a given point be the pole of a given straight line with respect to a conic.*

Let P be the given point and QR the given straight line. Then the polar of P passes through Q, which is one point-condition; and the polar of P passes through R, which is another. Hence the data are equivalent to two point-conditions.

Or we may reason thus: the pole of QR lies on PQ, which is one line-condition, and the pole of QR lies on PR, which is another. Hence the data are equivalent to two line-conditions.

Therefore the pole of a given straight line being given may be regarded as equivalent to *two point-conditions or two line-conditions.*

V. *Given a point on a conic and the tangent thereat.*

This is a particular instance of the last case, the given pole lying on the given polar. It is therefore equivalent to *two point-conditions or two line-conditions.*

VI. *Given an asymptote.*

This is an instance of the last case, the given point being at infinity. It is therefore equivalent to *two point-conditions or two line-conditions.*

VII. *Given the direction of an asymptote.*

In this case one of the points in which the conic meets the straight line at infinity is given. It is therefore an instance of (I) and is equivalent to *one point-condition.*

VIII. *Given that the conic is a parabola,*

Or that the line at infinity is a tangent. This is an instance of (II) and is therefore equivalent to *one line-condition.*

IX. *Given that the conic is a circle,*

Or that it passes through the two circular points. By (I) this is equivalent to *two point-conditions.*

X. *Given the centre.*

The centre is the pole of the straight line at infinity; hence this case is an instance of (IV) and is therefore equivalent to *two point-conditions or two line-conditions.*

XI. *Given a self-conjugate triangle.*

A triangle is self-conjugate if each pair of angular points are conjugate. Hence this case is equivalent to three point-conditions.

Or again, a triangle is self-conjugate if each pair of sides are conjugate lines. Hence it is equivalent to three line-conditions.

Therefore a self-conjugate triangle being given, constitutes *three point-conditions or three line-conditions.*

XII. *Given in position (not in magnitude) a pair of conjugate diameters.*

A pair of conjugate diameters form with the straight line at infinity a self-conjugate triangle. Hence this is an instance of (XI) and is equivalent to *three point-conditions or three line-conditions.*

XIII. *Given the directions of a pair of conjugate diameters.*

The points where any lines in these directions meet the line at infinity are conjugate points. Hence this is equivalent to *one point-condition.*

XIV. *Given in position an axis.*

The axis is a diameter, and this being given is equivalent to one line-condition. But the direction of the conjugate diameter is known to be at right angles to this, which gives by (XIII) a point-condition. Therefore that an axis be given in position is equivalent to *one point-condition and one line-condition.*

XV. *Given in position the two axes.*

This is no more than a case of (XIV) and is equivalent to *three point-conditions or three line-conditions.*

XVI. *Given a focus.*

The two tangents from the given point to the great circle at infinity are tangents to the conic. Hence two tangents are given, and therefore by (II) the data are equivalent to *two line-conditions.*

XVII. *Given a similar and similarly situated conic.*

Since similar and similarly situated conics are those which meet the straight line at infinity in the same points, this is equivalent to two points being given. Hence by (I) it may be treated as *two point-conditions.*

457. When a conic has to be drawn subject to conditions having reference to another conic, we may often estimate the value of the conditions by considering the particular case in which the latter conic reduces to two straight lines. Thus:

XVIII. *Given a conic having double contact with the required one.*

Consider the case when the given conic reduces to two straight lines. Then we have two tangents given, furnishing two line-conditions. Hence we may infer that generally a conic having double contact with the required one being given is equivalent to *two line-conditions.*

The following are examples of the application of our results:

458. *Only one parabola can be inscribed in a given quadrilateral.*

That the required conic is a parabola is one line-condition (VIII); that it touch the sides of the quadrilateral gives four more. Hence we have five line-conditions, and therefore (Art. 455) the conic is absolutely determined.

459. *Not more than two parabolas can be described about a given quadrilateral.*

That the required conic is a parabola is a line-condition (VIII); that it circumscribe the quadrilateral gives four point-conditions (I). Hence (Art. 451) not more than two solutions are possible.

460. *Two conics can generally be described with given foci and passing through a given point.*

For the foci give four line-conditions (XVI); and the point gives a point-condition (I). Hence (Art. 454) there will generally be two solutions.

461. *Only one conic can be described with given foci so as to touch a given straight line.*

For the foci give four line-conditions (XVI), and the tangent gives a fifth (II). Hence (Art. 455) there is only one solution.

462. *Only one conic can be described with a given centre, with respect to which a given triangle shall be self-conjugate.*

For the self-conjugate triangle may be regarded as giving three point-conditions (XI), and the given centre as giving two more (X). Hence (Art. 450) there will be only one solution.

Exercises on Chapter XXV.

(255) Given two tangents and their chord of contact, shew that only one conic can be described so as to touch a given straight line.

(256) Given two tangents and their chord of contact, shew that only one conic can be described so as to pass through a given point.

(257) Two confocal conics cannot have a common tangent.

(258) Three confocal conics cannot have a common point.

(259) Two concentric conics cannot circumscribe the same triangle.

(260) Two concentric conics cannot be inscribed in the same triangle.

(261) Only two conics can be described about a triangle having an axis in a given straight line.

(262) Only two conics can be inscribed in a triangle and have an axis in a given straight line.

(263) Four circles can generally be described through a given point so as to have double contact with a given conic.

(264) Four circles can generally be described so as to touch a given straight line and have double contact with a given conic.

(265) Only one conic can be described having double contact with a given conic, and such that a given triangle is self-conjugate with respect to it.

(266) One conic can generally be inscribed in a given quadrilateral so as to have its centre on a given straight line.

CHAPTER XXVI.

EQUATIONS OF THE THIRD DEGREE.

463. Definitions. The curve determined by an equation of the third degree in trilinear coordinates, or in any other system in which a point is represented by coordinates, is called a *cubic curve*, or a *cubic locus*.

The curve determined by an equation of the third degree in tangential coordinates where a straight line is represented by coordinates, is called a *cubic envelope*.

464. *Every straight line meets a cubic locus in three points, real or imaginary, coincident or distinct: and from every point there can be drawn to a cubic envelope three tangents real or imaginary, coincident or distinct.*

For to find the points of intersection (or the tangents) we have to solve together the equation to the given straight line (or the given point), which is of the first degree, and the equation to the curve, which is of the third degree. Hence we shall have three solutions real or imaginary, equal or unequal.

All the solutions however cannot be imaginary, since imaginary roots enter into an equation by pairs. One at least must be real, and the other two either both real or both imaginary. Hence *every straight line meets any cubic locus in one or three real points, and from every point there can be drawn either one or three real tangents to any cubic envelope.*

Cor. By Art. 415, a cubic locus is a curve of the third *order*, and a cubic envelope is a curve of the third *class*.

465. DEFINITIONS.

I. A point in which two branches of a curve intersect, or at which there are two distinct tangents, is called a *double point.*

II. A point in which more than two branches intersect, or at which there are more than two distinct tangents, is called a *multiple point.*

A multiple point is said to be of the n^{th} order when n branches intersect in it, or when n tangents can be drawn at it.

III. When a closed branch of a curve becomes indefinitely small so as to constitute an isolated point satisfying the conditions of a point on the curve, it is called a *conjugate point.* From the consideration that a conjugate point is an indefinitely small oval, it follows that any straight line through it must be regarded as the ultimate position of a chord of the oval. Any such straight line will therefore satisfy the condition of meeting the curve in two coincident points.

IV. A *cusp* is a point on a curve at which two branches meet a common tangent and stop at that point. Any straight line through a cusp must be regarded as cutting both branches at the cusp, and therefore satisfies the condition of meeting the curve in two coincident points.

V. If the two branches having the common tangent be continued through the point, then the point is called a *point of osculation.*

VI. A point at which a curve crosses its tangent, is called a *point of inflexion.*

If P be a point of inflexion and Q be another point on the curve very near to P, the straight line QP being produced through P, will meet the curve again in another point Q', very near to P. If this straight line turn about the fixed point P until it ultimately coincide with the tangent, since it must ultimately be a tangent to the branch on each side of the point P, it follows that as Q approaches P so also will Q', and that they will both simultaneously arrive at coincidence with P. Hence the tangent at a point of inflexion may be regarded as meeting the curve in three coincident points.

All the foregoing are often classed together as *singular points*.

VII. A *double tangent* to a curve is a tangent which touches the curve in two distinct points.

466. It will be observed from the definitions in the last article, that a double point, cusp, and conjugate point are marked by the same property, that any straight line through such a point meets the curve in two coincident points, and that a tangent thereat meets the curve in three coincident points. But they are distinguished by the property that the two tangents at a double point are distinct, at a cusp—coincident, and at a conjugate point—imaginary.

Again, a cusp and a point of inflexion are both characterised by the property that the tangent at such a point meets the curve in three coincident points, but they are distinguished by the fact that a straight line other than the tangent meets the curve in only one point at a point of inflexion, but in two points at a cusp.

467. *A cubic curve cannot have more than one double point, cusp or conjugate point.*

For, if possible, let it have two such points P and Q, and join them by a straight line. Then this straight line cuts the curve in two coincident points at P, and in two coincident points at Q (Art. 465), i.e. in four points altogether,

Which is impossible (Art. 464).

468. *A cubic curve cannot have a double tangent.*

For such a tangent, touching at P and at Q, would meet the curve in two coincident points at P and in two coincident points at Q, i.e. in four points altogether,

Which is impossible (Art. 464).

469. *A cubic curve cannot have a point of osculation.*

For the tangent at a point of osculation, touching both branches of the curve, would meet it altogether in four points, which is impossible (Art. 464).

470. The general homogeneous equation of the third degree in three coordinates consists of ten terms, viz. the three terms whose arguments are

$$x^3,\ y^3,\ z^3,$$

the six, $x^2y,\ x^2z;\ y^2z,\ y^2x;\ z^2x,\ z^2y;$

and the one, $xyz.$

If the coefficient of any one of these terms be arbitrarily assigned, those of the remaining nine will be undetermined constants.

Hence the general equation of the third degree involves nine undetermined constants, and can therefore generally be made to satisfy nine independent conditions.

Hence a curve represented by an equation of the third degree can generally be drawn through nine given points, or otherwise made to satisfy nine given conditions.

471. If the nine conditions be given, the equation to the curve can generally be determined. If any less number (r suppose) of conditions be given, a series of curves can generally be drawn to satisfy them, and their general equation will involve the complementary number $(9-r)$ of undetermined constants.

For example, we shall shew in the next article that the general equation in trilinear coordinates to a curve of the third order, circumscribing the triangle of reference ABC, and whose tangents at A, B, C are represented by the equations

$$u=0,\ \ v=0,\ \ w=0,$$

respectively, has for its equation

$$\alpha\beta\gamma + lu\alpha^2 + mv\beta^2 + nw\gamma^2 = 0,$$

which involves *three* undetermined constants l, m, n, the number of given conditions having been *six*.

472. The general equation of the third degree in trilinear coordinates may be written

$$\alpha\beta\gamma + \alpha^2(l_1\alpha + m_1\beta + n_1\gamma) + \beta^2(l_2\alpha + m_2\beta + n_2\gamma) + \gamma^2(l_3\alpha + m_3\beta + n_3\gamma) = 0.$$

If we take three points on the curve as the angular points of the triangle of reference, then since the equation must be satisfied by any of the systems

$$(\beta = 0,\ \gamma = 0),\ (\gamma = 0,\ \alpha = 0),\ (\alpha = 0,\ \beta = 0),$$

we obtain

$$l_1 = 0,\ m_2 = 0,\ n_3 = 0,$$

and the equation reduces to

$$\alpha\beta\gamma + \alpha^2(m_1\beta + n_1\gamma) + \beta^2(n_2\gamma + l_2\alpha) + \gamma^2(l_3\alpha + m_3\beta) = 0.$$

The tangent to the locus of this equation at the point A $(\beta = 0,\ \gamma = 0)$ is readily seen to be given by the equation

$$m_1\beta + n_1\gamma = 0.$$

Similarly, $\quad n_2\gamma + l_2\alpha = 0$

and $\quad l_3\alpha + m_3\beta = 0$

represent the tangents at the points B and C.

Hence the equation

$$\alpha\beta\gamma + lu\alpha^2 + mv\beta^2 + nw\gamma^2 = 0$$

represents a cubic touching at the points of reference the straight lines

$$u = 0,\ v = 0,\ w = 0.$$

Or, more generally, if

$$x = 0,\ y = 0,\ z = 0$$

represent any equations to straight lines, then

$$xyz + lux^2 + mvy^2 + nwz^2 = 0$$

is the general equation of a cubic to which

$$u=0, \quad v=0, \quad w=0$$

are tangents, and

$$x=0, \quad y=0, \quad z=0$$

the chords of contact.

473. Similarly, in tangential coordinates,

$$xyz + lux^2 + mvy^2 + nwz^2 = 0$$

is the general equation to a curve of the third class, on which

$$u=0, \quad v=0, \quad w=0$$

are the points of contact of tangents intersecting in the points

$$x=0, \quad y=0, \quad z=0.$$

474. *To find the general equation in trilinear coordinates to a cubic curve having a double point, cusp, or conjugate point at one of the points of reference.*

The general equation to a cubic curve may be written

$$\alpha\beta\gamma + \alpha^2(l_1\alpha + m_1\beta + n_1\gamma) + \beta^2(l_2\alpha + m_2\beta + n_2\gamma) + \gamma^2(l_3\alpha + m_3\beta + n_3\gamma) = 0.$$

If the point A be a double point, cusp, or conjugate point, any straight line through A must meet the cubic in two coincident points at A. Any such straight line may be represented by the equation

$$\beta = \kappa\gamma.$$

Hence, substituting for β in the general equation, the resulting equation must have two roots $\gamma = 0$. Hence the terms involving α^3 and α^2 must vanish, and therefore we must have

$$l_1 = 0, \quad m_1 = 0, \quad n_1 = 0;$$

these are therefore the conditions that the locus of the equation (1) should have a double point, cusp, or conjugate point at A.

When these conditions are satisfied we may express the constants differently, and write the equations

$$\alpha (f\beta^2 + g\beta\gamma + h\gamma^2) + l\beta^3 + m\beta^2\gamma + n\beta\gamma^2 + r\gamma^3 = 0,$$

which is therefore the general equation to a cubic having a double point, cusp, or conjugate point given by

$$\beta = 0, \quad \gamma = 0.$$

475. *To find the equation to the tangents at the double point or cusp to the cubic curve whose equation is*

$$\alpha (f\beta^2 + g\beta\gamma + h\gamma^2) + l\beta^3 + m\beta^2\gamma + n\beta\gamma^2 + r\gamma^3 = 0 \text{......(1)}.$$

Let $\beta = \kappa\gamma$ be a tangent at A, then substituting in the equation, the resulting equation

$$\alpha\gamma^2 (f\kappa^2 + g\kappa + h) + \gamma^3 (l\kappa^3 + m\kappa^2 + n\kappa + r) = 0$$

must have all three roots equal, $(\gamma = 0)$.

Hence $$f\kappa^2 + g\kappa + h = 0,$$

giving the two values for κ corresponding to the two tangents.

The equation to the tangents is therefore

$$f\beta^2 + g\beta\gamma + h\gamma^2 = 0.$$

If the two roots of the quadratic be equal, the point A will be a cusp: if they be real and unequal it will be a double point: if they be imaginary it will be a conjugate point.

Cor. The equation

$$\alpha (f\beta^2 + g\beta\gamma + h\gamma^2) + l\beta^3 + m\beta^2\gamma + n\beta\gamma^2 + r\gamma^3 = 0$$

represents a cubic having at A a double point, conjugate point, or cusp, according as $g^2 - 4fh$ is positive, negative, or zero.

476. *Every curve of the third order which has a cusp is also of the third class.*

Let there be a curve of the third order having a cusp A, and let B be *any point whatever* in its plane. We have to shew that only three tangents can be drawn from B to the curve.

Let BC be one of the tangents from the point B and let it meet the tangent at the cusp in C. Then if we refer the cubic to the triangle ABC its equation may be written (Art. 475)

$$\alpha\beta^2 + l\beta^3 + m\beta^2\gamma + n\beta\gamma^2 + r\gamma^3 = 0.$$

But since $\alpha = 0$ is a tangent the equation

$$l\beta^3 + m\beta^2\gamma + n\beta\gamma^2 + r\gamma^3 = 0,$$

must have two of its roots equal, and may therefore be written

$$l(\beta + \mu\gamma)(\beta + \nu\gamma)^2 = 0.$$

Hence the equation to the cubic may be written

$$\alpha\beta^2 + l(\beta + \mu\gamma)(\beta + \nu\gamma)^2 = 0.$$

Now let $\alpha = \kappa\gamma$ be any tangent from B to the curve. Substituting for α in the equation to the curve the resulting equation

$$\kappa\beta^2\gamma + l(\beta + \mu\gamma)(\beta + \nu\gamma)^2 = 0,$$

must have two of its roots equal.

The condition that this should be the case will be found to be

$$4\mu\kappa^3 + l\kappa^2(8\mu^2 + 20\mu\nu - \nu^2) + 4l^2\kappa(\mu - \nu)^3 = 0,$$

a cubic equation giving three values of κ of which one is zero. Then there are three tangents from B to the curve, one of which is the known tangent $\alpha = 0$.

Hence from any point, only three tangents can be drawn to a cubic which has a cusp. Q.E.D.

477. If a cubic curve have three real points of inflexion, the tangents at which do not meet in a point, we may take those tangents as lines of reference for trilinear coordinates.

Each line of reference will now meet the cubic in three coincident points; therefore if we substitute $\alpha = 0$ in the equation to the cubic the resulting equation must have three equal roots; that is, the terms free from α must form a perfect cube, $(m\beta + n\gamma)^3$ suppose. So the terms free from β must form a perfect cube, which (since the coefficient of γ^3 is already known to be n^3) may be written $(n\gamma + l\alpha)^3$. Similarly the terms free from γ must be $(l\alpha + m\beta)^3$. Hence the equation may be written

$$(m\beta + n\gamma)^3 + (n\gamma + l\alpha)^3 + (l\alpha + m\beta)^3 - l\alpha^3 - m\beta^3 - n\gamma^3 + h\alpha\beta\gamma = 0,$$

or, expressing the constant h differently,

$$(l\alpha + m\beta + n\gamma)^3 + k\alpha\beta\gamma = 0.$$

(The argument would not hold if one or more of the points of inflexion were imaginary, as in such case *different* cube roots of n^3, &c. might be involved.)

478. Cor. The points of contact of the tangents, or the points of inflexion themselves are given by

$$(\alpha = 0,\ m\beta + n\gamma = 0),\ (\beta = 0,\ n\gamma + l\alpha = 0),\ (\gamma = 0,\ l\alpha + m\beta = 0).$$

Hence they all lie on the straight line

$$l\alpha + m\beta + n\gamma = 0.$$

Therefore, *if p, q, r be the tangents at three real points of inflexion P, Q, R on a cubic, then either p, q, r are concurrent, or P, Q, R are collinear.*

479. *If a cubic curve have three real points of inflexion they will be collinear.*

For if not we may take them as points of reference for trilinear coordinates. Then since the tangents at the three points are concurrent (Art. 478) we may represent them by the equations

$$m\beta - n\gamma = 0, \quad n\gamma - l\alpha = 0, \quad l\alpha - m\beta = 0.$$

Hence the equation to the cubic may be written (Art. 472)

$$lmn\alpha\beta\gamma + \lambda l^2\alpha^2 (m\beta - n\gamma) + \mu m^2\beta^2 (n\gamma - l\alpha) + \nu n^2\gamma^2 (l\alpha - m\beta) = 0.$$

And since the line $m\beta - n\gamma = 0$ is a tangent at a point of inflexion, it meets the cubic in three coincident points; therefore we must have

$$1 - \mu + \nu = 0.$$

Similarly, since $n\gamma - l\alpha = 0$ and $l\alpha - m\beta = 0$ are tangents at points of inflexion,

$$1 - \nu + \lambda = 0,$$

$$1 - \lambda + \mu = 0.$$

But these equations are inconsistent, as we find by adding them together.

Hence the points of reference cannot be points of inflexion. Therefore &c. Q. E. D.

480. The theorems of the following articles, being expressed in a most general form in abridged notation, will be found very useful in interpreting equations of the third degree, and will often enable us to recognise by simple inspection the existence of singular points.

481. *If* $u = 0,\ v = 0,\ w = 0,\ x = 0,\ y = 0,\ z = 0$ *are the equations of six straight lines, then the equation*

$$uvw = kxyz$$

will represent a cubic locus passing through the nine points given by the intersection of the straight lines

$$(u = 0,\ x = 0),\ (u = 0,\ y = 0),\ (u = 0,\ z = 0),$$

$$(v = 0,\ x = 0),\ (v = 0,\ y = 0),\ (v = 0,\ z = 0),$$

$$(w = 0,\ x = 0),\ (w = 0,\ y = 0),\ (w = 0,\ z = 0);$$

and k can be determined so as to make the equation represent a cubic passing through any tenth point.

The proof follows immediately as in the corresponding proposition respecting conics, Art. 159.

But we may observe with respect to our result that it is only because the first nine points lie three and three on six straight lines that we are able to describe a cubic passing through a tenth point. If the first nine points had been unconnected and perfectly general they would have sufficed to determine the cubic absolutely, as we shewed in Art. 470.

482. *So if*

$$u=0,\ v=0,\ w=0,\ x=0,\ y=0,\ z=0$$

represent points in tangential coordinates, the equation

$$uvw = kxyz$$

represents a cubic envelope touching the nine straight lines

$$(u=0,\ x=0),\ (u=0,\ y=0),\ (u=0,\ z=0),$$
$$(v=0,\ x=0),\ (v=0,\ y=0),\ (v=0,\ z=0),$$
$$(w=0,\ x=0),\ (w=0,\ y=0),\ (w=0,\ z=0);$$

and k can be determined so as to make the equation represent a cubic envelope touching any tenth straight line.

483. Consider the equation

$$u^2w = kxyz. \qquad [trilinear$$

This is a particular case of the equation of Art. 481, the straight lines $u=0$ and $v=0$ being coincident. The equation represents a cubic locus to which the straight lines

$$x=0,\ y=0,\ z=0$$

are tangents, their points of contact lying all on the straight line $u=0$, and the other points where they meet the curve lying on the straight line $w=0$.

484. So the equation

$$u^2w = kxyz \qquad [tangential$$

represents a cubic envelope passing through the points

$$x=0,\ y=0,\ z=0,$$

and touching at those points the straight lines

$$(u=0,\ x=0),\ (u=0,\ y=0),\ (u=0,\ z=0),$$

and also touching the straight lines

$$(w=0,\ x=0),\ (w=0,\ y=0),\ (w=0,\ z=0),$$

485. Consider the equation

$$u^3 = kxyz. \qquad [trilinear$$

This is a particular case of the last equation, the straight lines $u=0$ and $w=0$ being coincident. The equation represents a cubic locus having three points of inflexion in the straight line $u=0$, the tangents at those points of inflexion being given by

$$x=0, \quad y=0, \quad z=0.$$

For each of the straight lines

$$x=0, \quad y=0, \quad z=0$$

meets the curve in three coincident points determined by $u^3=0$.

486. So the equation

$$u^3 = kxyz \qquad [tangential$$

represents a cubic envelope having points of inflexion at

$$x=0, \quad y=0, \quad z=0,$$

the tangents at these points intersecting in the point $u=0$.

487. Consider the equation

$$u^3 = kx^2y. \qquad [trilinear$$

This equation represents a cubic locus in which $x=0$ is the tangent at a cusp, $y=0$ the tangent at a point of inflexion, and $u=0$ the chord of contact.

For $x=0$, $y=0$ both meet the cubic in three coincident points on the line $u=0$, but $u=0$ cuts it in two points on $x=0$, and in only one on $y=0$. Hence $(u=0,\ x=0)$ must be a cusp, and $(u=0,\ y=0)$ a point of inflexion.

488. So the equation

$$u^3 = kx^2y \qquad [tangential$$

represents a cubic envelope having a cusp at $x=0$, and a point of inflexion at $y=0$, the tangents at these points intersecting in $u=0$.

489. Consider the equation

$$uv^2 = kxy^2. \qquad [trilinear$$

The straight lines $u = 0$ and $x = 0$ are tangents: their points of contact lying respectively on $y = 0$ and $v = 0$, and their point of intersection also lying on the cubic, and there is a singular point at the intersection of $v = 0$ and $y = 0$.

For $v = 0$ meets the curve in two incident points lying on $y = 0$, and $y = 0$ meets it in two coincident points lying on $v = 0$. Hence at the point of intersection ($y = 0$, $v = 0$) both lines satisfy the condition of meeting the cubic in two coincident points; hence this point must be a double point, cusp, or conjugate point.

490. So the equation

$$uv^2 = kxy^2 \qquad [tangential$$

represents a cubic envelope to which the straight line ($v=0$, $y=0$) is a double tangent, and the points $u = 0$, $x = 0$ are points of contact of tangents from $y = 0$ and $v = 0$.

491. The cubic represented by the equation

$$x^3 + y^3 + z^3 + 3kxyz = 0,$$

deserves special attention, as an example of a curve free from double points, cusps and conjugate points.

The straight lines $x = 0$, $y = 0$, $z = 0$ meet the cubic in nine points, lying by threes on twelve straight lines.

The straight line $x = 0$ meets the cubic in the points given by

$$y^3 + z^3 = 0,$$

that is (if i denote one of the imaginary cube roots of unity), in the three points

$$(x = 0,\ y + z = 0),\ (x = 0,\ y + iz = 0),\ (x = 0,\ y + i^2z = 0).$$

We will call these points respectively X, X', X''.

So the straight lines $y=0$, $z=0$ meet the cubic in the points

$$(y=0,\ z+x=0),\ (y=0,\ z+ix=0),\ (y=0,\ z+i^2x=0),$$
$$(z=0,\ x+y=0),\ (z=0,\ x+iy=0),\ (z=0,\ x+i^2y=0),$$

which we will denote by the letters Y, Y', Y'', Z, Z', Z'' respectively.

Now it is easily seen that

$$\begin{array}{llll}
X, & Y, & Z & \text{lie on}\quad x+y+z=0,\\
X', & Y', & Z' & \ldots\ldots\quad x+iy+i^2z=0,\\
X'', & Y'', & Z'' & \ldots\ldots\quad x+i^2y+iz=0,\\
X, & Y', & Z'' & \ldots\ldots\quad ix+y+z=0,\\
X, & Y'', & Z' & \ldots\ldots\quad i^2x+y+z=0,\\
X'', & Y, & Z' & \ldots\ldots\quad x+iy+z=0,\\
X', & Y, & Z'' & \ldots\ldots\quad x+i^2y+z=0,\\
X', & Y'', & Z & \ldots\ldots\quad x+y+iz=0,\\
X'', & Y', & Z & \ldots\ldots\quad x+y+i^2z=0.
\end{array}$$

And we know that X, X', X'' lie on the straight line $x=0$; Y, Y', Y'' on $y=0$; Z, Z', Z'' on $z=0$.

Hence the nine points lie by threes on twelve straight lines. Q. E. D.

492. *These nine points are points of inflexion.*

Let the tangent at the point $x=0$, $y+iz=0$, be $y+iz=\mu x$.

Then the equation

$$(y+iz)^3+\mu^3(y^3+z^3)+3k\mu^2yz(y+iz)=0\ldots\ldots(1)$$

must have two equal roots $(y+iz=0)$.

Hence $y+iz=0$ must satisfy the equation

$$(y+iz)^2+\mu^3(y^2-iyz+i^2z^2)+3k\mu^2yz=0,$$

whence $\mu=i^2k$ and this equation reduces to

$$(y+iz)^2=0,$$

shewing that all the three roots of (1) are equal.

Hence $$y + iz = i^2kx$$

represents a tangent meeting the curve in three coincident points at the point $(x = 0,\ y + iz = 0)$. Hence there is a point of inflexion. Thus all the nine points

$$\begin{matrix} X, & X', & X'', \\ Y, & Y', & Y'', \\ Z, & Z', & Z'', \end{matrix}$$

are points of inflexion, and the tangents are given respectively by the equations

$$\begin{matrix} y + z = kx, & y + iz = i^2kx, & y + i^2z = ikx, \\ z + x = ky, & z + ix = i^2ky, & z + i^2x = iky, \\ x + y = kz, & x + iy = i^2kz, & x + i^2y = ikz. \end{matrix}$$

ON THE INFINITE BRANCHES OF CUBIC CURVES.

493. Since every straight line meets a curve of the third order in either one or three real points, the straight line at infinity meets it in one or three real points.

And since any system of parallel straight lines meet the line at infinity in one point, there is always at least one system of parallel straight lines which meet the curve on the line at infinity, and therefore only meet it in two other points (real or imaginary), and there *may be three* such directions or systems of parallel straight lines.

If P be the point in which such a system of parallel straight lines intersect at infinity, one of these straight lines through P will generally be the tangent at P and therefore an asymptote. Hence a straight line which meets a cubic in only two finite points is generally parallel to an asymptote.

We say generally, because it may happen that the tangent at P at infinity lies altogether at infinity. In this case lines in the direction P will meet the curve in two finite points, but will not be parallel to an asymptote, except in the sense in which all straight lines are parallel to the straight line at infinity.

It follows that there can generally be one asymptote drawn to a cubic curve, and that there may be as many as three asymptotes.

The only cases in which there can be no asymptote will occur when the straight line at infinity meets the curve in three coincident points, at a cusp or a point of inflexion. (See Arts. 502—504.)

494. All possible cases may be analysed according to the nature of the three points in which the straight line at infinity cuts the cubic.

I. *If two of these points be imaginary and one real*, there will be two imaginary and one real asymptote.

II. *If all the points be real and distinct*, they will determine the direction of three asymptotes.

III. *If all the points be coincident*, either the straight line at infinity is a tangent at a point of inflexion or a cusp, and there is no asymptote, or else it is one of the tangents at a double point, in which case the other tangent at the double point is an asymptote.

IV. *If two of the points be coincident at P*, the third point will always determine the direction of the only asymptote, and unless P be a singular point, the straight line at infinity will be the tangent at P, and all straight lines in direction P cut the curve in two finite points.

It may happen however that P is a double point, or a conjugate point, in which case all straight lines in direction P, cutting the curve in two coincident points at infinity, will cut it in only one finite point.

We proceed to consider some typical examples of all these cases. We shall use abridged notation throughout, each of the symbols u, v, w, x, y, z denoting expressions of the most general form which, when equated to zero, represent straight lines; and we shall use $\sigma = 0$ to denote the equation to the straight line at infinity.

495. Consider the equation

$$uv\sigma = kxyz.$$

Each of the straight lines $x = 0$, $y = 0$, $z = 0$ meets the locus in two finite points lying on the straight lines $u = 0$, $v = 0$ and in one point at infinity. Hence the asymptotes are parallel to the straight lines $x = 0$, $y = 0$, $z = 0$.

496. Consider the equation

$$u^2\sigma = kxyz.$$

The straight lines $x = 0$, $y = 0$, $z = 0$ are now tangents parallel to the asymptotes, their points of contact lying in the straight line $u = 0$.

497. Consider the equation

$$u\sigma^2 = kxyz.$$

Each of the straight lines $x = 0$, $y = 0$, $z = 0$ meets the locus in two points at infinity and in one point on the straight line $u = 0$. Hence the equation represents a cubic having $x = 0$, $y = 0$, $z = 0$ as asymptotes, the points in which they cut the curve again lying on the straight line $u = 0$.

498. Consider the equation

$$\sigma^3 = kxyz.$$

This is a particular case of the last equation, $u = 0$ now coinciding with the line at infinity. It therefore represents a cubic having three asymptotes which do not cut the curve in any finite points, the asymptotes being tangents at points of inflexion.

499. Consider the equation

$$uv\sigma = kx^2y.$$

The straight line at infinity meets the locus of this equation in two coincident points on $x = 0$, and in a third point on $y = 0$. The locus further cuts $x = 0$ in two finite points lying on $u = 0$ and $v = 0$. Hence the straight line at infinity is a tangent at the point given by $x = 0$, and there is only one asymptote, its direction being given by $y = 0$.

If $y = mu$ and $y = nv$ be the straight lines through ($u = 0$, $y = 0$) and ($v = 0$, $y = 0$) parallel to the straight line $x = 0$, so that

$$y - mu \equiv m'x + \mu\sigma,$$

and

$$y - nu \equiv n'x + \nu\sigma$$

identically, then the parabola whose equation is

$$k^2m^2n^2x^2 = kmn\sigma\,(mu + nv - y) + m'n'\sigma^2$$

will be found to meet the cubic in five coincident points at infinity.

This parabola having five-pointic contact with the cubic at infinity will serve the purpose of an asymptote in approximating to the form of that infinite branch of the cubic to which the linear asymptote is not an approximation.

Such a parabola is called a *Parabolic asymptote.*

500. Consider the equation

$$u\sigma^2 = kx^2y.$$

In this case also, the straight line at infinity meets the curve in three real points two of which are coincident. As before the distinct point at infinity determines an asymptote (its equation now being $y = 0$), but the two coincident points now indicate a singular point. Thus the straight line at infinity will not be itself a tangent, but the two coincident points upon it will be the points of contact of two tangents real or imaginary according as the singular point is a double point or conjugate point.

If $u = \mu y$ be the equation to the straight line through ($u = 0$, $y = 0$) parallel to $x = 0$, it is easily seen that the straight lines $x = \pm\sqrt{\mu}\sigma$ are the two tangents at the singular point, forming two real or imaginary asymptotes according as μ is positive or negative.

We have spoken of the singular point either as a double point or a conjugate point. If it happen to be a cusp the two straight lines $x = \pm\sqrt{\mu}\sigma$ must coincide and μ must be either zero or infinite, i.e. either $u = 0$ or $y = 0$, must represent a straight line parallel to $x = 0$. These two cases we proceed to consider separately.

501. Consider the equation

$$(x + \sigma)\,\sigma^2 = kx^2y.$$

This is a particular instance of the last case, the straight line $u = 0$ being now parallel to $x = 0$. As we saw in considering the former case, the straight line at infinity meets the curve in a cusp at the point determined by $x = 0$, and in a distinct point at $y = 0$. We have now $\mu = 0$, shewing that there are two coincident asymptotes represented by the equation $x = 0$, tangents at the cusp, as well as a distinct asymptote $y = 0$.

502. Consider the equation

$$u\sigma^2 = kx^2(x + \sigma).$$

This is a particular instance of the case of Art. 500, the straight line $y = 0$ being now parallel to $x = 0$. The three points in which the curve meets the straight line at infinity are now coincident, that line being itself the tangent at the cusp. There is therefore no asymptote, but there will be two branches, each touching the straight line at infinity at the cusp.

503. Consider the equation

$$uv\sigma = kx^3.$$

We observe that $u=0$, $v=0$, $\sigma=0$ are the tangents at three points of inflexion lying on the straight line $x=0$. There is no asymptote, since the straight line at infinity is a tangent at the point of inflexion at infinity.

504. The equation

$$u^2\sigma = kx^3$$

represents a curve, only differing from the last in that $u=0$ and $v=0$ coincide, and the two finite points of inflexion have coalesced into a cusp.

505. The examples which we have given will suffice to shew the student how to analyse any cubic whose equation is given, as to its infinite branches and its singular points. We have exhibited types of curves having every variety of asymptote and every variety of singular point, and other examples in which different combinations of asymptotes and singular points occur may be analysed by analogous methods.

506. We will conclude with the following proposition which is very important.

All cubics which pass through eight fixed points pass also through a ninth.

Let $\phi(\alpha, \beta, \gamma)=0$ and $\psi(\alpha, \beta, \gamma)=0$ be the equations to two cubics which pass through eight given points. Since nine points determine a cubic, any other cubic through the eight points will generally be determined if another point upon it be assigned.

Consider the cubic which passes through the eight given points, and the ninth point $(\alpha', \beta', \gamma')$. The equation

$$\frac{\phi(\alpha, \beta, \gamma)}{\phi(\alpha', \beta', \gamma')} = \frac{\psi(\alpha, \beta, \gamma)}{\psi(\alpha', \beta', \gamma')} \quad \text{.................} \quad (1)$$

must represent it. For since the functions $\phi(\alpha, \beta, \gamma)$ and $\psi(\alpha, \beta, \gamma)$ are of the third degree, this equation is of the third degree and therefore represents some cubic. But it is satisfied at the point (α, β, γ), and also at all points of intersection of the two cubics $\phi(\alpha, \beta, \gamma) = 0$ and $\psi(\alpha, \beta, \gamma) = 0$, and therefore at the eight given points. But the two cubics $\phi(\alpha, \beta, \gamma) = 0$ and $\psi(\alpha, \beta, \gamma) = 0$ intersect in nine points altogether. Hence the locus of the equation (1) passes not only through the assigned point $(\alpha', \beta', \gamma')$ and the eight given points, but it passes also through a ninth fixed point on each of the original cubics. Therefore all cubics through eight fixed points pass also through a ninth.

Exercises on Chapter XXVI.

(267) The only cubic having *three* double points consists of three straight lines forming a triangle.

(268) The only cubic having *two* double points consists of a straight line intersecting a conic.

(269) Shew that the straight lines $u = 0$, $v = 0$ are tangents to the cubic

$$uvw = ku^3 + k'v^3 \qquad [\textit{trilinear}$$

at a double point.

(270) Shew that the equation

$$uvw = ku^3 + k'v^3 \qquad [\textit{tangential}$$

represents a curve having a double tangent.

(271) The cubic

$$u^2v = ku^2w + k'v^2w$$

has a conjugate point or a double point according as k, k' are of the same or of opposite signs.

(272) Find the six points of intersection of the conic

$$\frac{\alpha^2+\beta^2+\gamma^2}{l^2+m^2+n^2}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{mn+nl+lm},$$

and the cubic

$$\frac{\alpha^3+\beta^3+\gamma^3}{l^3+m^3+n^3}=\frac{\alpha\beta\gamma}{lmn}.$$

(273) If a triangle be inscribed in a cubic so as to have its sides parallel to the asymptotes of the cubic; and if the tangents at B and C intersect in P, those at C, A in Q; those at A, B in R; the three straight lines PA, QB, RC will be concurrent.

(274) If a cubic curve consist of three equal and symmetrical branches having contact with three asymptotes which form an equilateral triangle, the algebraical sum of the reciprocals of the perpendiculars from any point on the three sides of the equilateral triangle formed by joining the vertices of the three branches, is constant.

(275) If two cubics touch one another in three collinear points, their other points of intersection will be collinear.

(276) If a series of conics have the same three asymptotes, the points of intersection of any two lie on a straight line, and all these straight lines are concurrent.

(277) If on any cubic the points P, P', P'' be collinear, and Q, Q', Q'' be collinear, and R, R', R'' be collinear, and if also P, Q, R be collinear and P', Q', R' be collinear, then will P'', Q'', R'' be also collinear.

(278) If a cubic have three asymptotes, one of which cuts it in a finite point, another must do so also.

(279) If two and only two asymptotes cut the curve in finite points, the other asymptote is parallel to the straight line joining those points of intersection.

(280) If a curve of the third order have a double point A, and be cut by any straight line in B, C, D; and if when ABC is taken as the triangle of reference, the tangents at A are represented by the equation

$$P\beta^2 + Q\beta\gamma + R\gamma^2 = 0,$$

and the tangents at B and C by the equations

$$P\alpha + N\gamma = 0, \text{ and } M\beta + R\alpha = 0,$$

shew that the equation to the straight line AD is

$$N\beta + M\gamma = 0,$$

and find the equation to the curve.

(281) Shew that the cubic

$$x^2y = \sigma(lx^2 + my^2 + n\sigma^2)$$

has a parabolic asymptote whose equation is

$$x^2 = m\sigma y.$$

(282) Shew that the cubic which circumscribes the triangle of reference and passes through the six points in which the two straight lines

$$\lambda\alpha + \mu\beta + \nu\gamma = 0, \quad \frac{\alpha}{\lambda} + \frac{\beta}{\mu} + \frac{\gamma}{\nu} = 0$$

intersect the three straight lines

$$l\alpha + m\beta + n\gamma = 0, \quad m\alpha + n\beta + l\gamma = 0, \quad n\alpha + l\beta + m\gamma = 0,$$

(the coordinates being triangular) has its asymptotes parallel to the last three straight lines.

(283) The cubic which circumscribes the triangle of reference and touches the straight lines

$$l^2\alpha = \beta + \gamma, \quad m^2\beta = \gamma + \alpha, \quad n^2\gamma = \alpha + \beta$$

at points lying on the straight line

$$l\alpha + m\beta + n\gamma = 0$$

(the coordinates being triangular) has its asymptotes parallel to the given tangents.

(284) If (λ, μ, ν) be a point on the curve

$$lx(y^2+z^2)+my(z^2+x^2)+nz(x^2+y^2)+kxyz=0,$$

the point $\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}\right)$ will also lie upon it.

Shew that the tangents at these two points intersect on the curve.

(285) If (λ, μ, ν) be a point on the cubic

$$(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+3kxyz=0,$$

then all the twelve points

$$(\lambda, \mu, \nu),\ (\mu, \nu, \lambda),\ (\nu, \lambda, \mu),\ (\lambda, \nu, \mu),\ (\mu, \lambda, \nu),\ (\nu, \mu, \lambda),$$

$$\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}\right),\ \left(\frac{1}{\mu}, \frac{1}{\nu}, \frac{1}{\lambda}\right),\ \left(\frac{1}{\nu}, \frac{1}{\lambda}, \frac{1}{\mu}\right),$$

$$\left(\frac{1}{\lambda}, \frac{1}{\nu}, \frac{1}{\mu}\right),\ \left(\frac{1}{\mu}, \frac{1}{\lambda}, \frac{1}{\nu}\right),\ \left(\frac{1}{\nu}, \frac{1}{\mu}, \frac{1}{\lambda}\right)$$

lie upon it.

(286) If $x=0$, $y=0$, $z=0$ be the equation of any three straight lines, and if

$$P \equiv x^3+y^3+z^3,$$

$$Q \equiv yz^2+y^2z+zx^2+z^2x+xy^2+x^2y,$$

$$R \equiv xyz,$$

and if P', Q', R' denote what P, Q, R become when any constant quantities a, b, c are substituted for x, y, z, shew that the equation

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ \lambda, & \mu, & \nu \end{vmatrix} = 0,$$

(where λ, μ, ν are arbitrary), represents a series of cubic curves passing through nine fixed points of which six lie upon a conic and the other three upon a straight line.

(287) Find the values of λ, μ, ν in the last exercise, in order that the cubic may break up into the conic and the straight line.

(288) In curves of the third order the locus of the middle points of chords parallel to an asymptote which does not cut the curve is a straight line.

(289) In curves of the third order the locus of the middle points of chords parallel to an asymptote which cuts the curve is a hyperbola.

(290) The tangents to a cubic at three collinear points will meet the cubic again in collinear points.

(291) If a cubic have three asymptotes which do not meet in a point, its equation referred to the asymptotes will be in triangular coordinates

$$\alpha\beta\gamma = (l\alpha + m\beta + n\gamma)(\alpha + \beta + \gamma)^2.$$

(292) A cubic circumscribes a triangle ABC, and cuts the sides BC, CA, AB again in A', B', C' respectively. Shew that if the chords AA', BB', CC' are concurrent, so also are the tangents at A, B, C.

(293) The general equation to a cubic touching the conic

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

in the three points of reference is

$$k\alpha\beta\gamma + (\lambda\alpha + \mu\beta + \nu\gamma)(l\beta\gamma + m\gamma\alpha + n\alpha\beta) = 0.$$

(294) If a conic touch a cubic in three points the three chords of contact will cut the cubic again in collinear points.

(295) The general equation in triangular coordinates to a cubic which cuts each of the sides of the triangle of reference in only two finite points, viz. the points which lie on the nine-points' circle, is

$$k\alpha\beta\gamma + \alpha \cot A(\alpha^2 - \beta^2 - \gamma^2) + \beta \cot B(\beta^2 - \gamma^2 - \alpha^2)$$
$$+ \gamma \cot C(\gamma^2 - \alpha^2 - \beta^2) = 0.$$

INTRODUCTION TO CHAPTER XXVII.

GENERAL PROPERTIES OF HOMOGENEOUS FUNCTIONS.

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507. In the following chapter a knowledge of the principles of the Differential Calculus will be required on the part of the student.

We call attention at once to some of the principal results which we shall assume, referring to treatises on the Differential Calculus for the discussion and proof.

508. *If $f(\alpha, \beta, \gamma)$ be a homogeneous function of α, β, γ of the n^{th} degree, then will*

$$\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} \equiv n \, . \, f(\alpha, \beta, \gamma).$$

Cor. 1. Since $\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}$ are themselves homogeneous functions of the n^{th} degree, we have

$$\alpha \frac{d^2 f}{d\alpha^2} + \beta \frac{d^2 f}{d\alpha \, d\beta} + \gamma \frac{d^2 f}{d\alpha \, d\gamma} = (n-1) \frac{df}{d\alpha},$$

$$\alpha \frac{d^2 f}{d\alpha \, d\beta} + \beta \frac{d^2 f}{d\beta^2} + \gamma \frac{d^2 f}{d\beta \, d\gamma} = (n-1) \frac{df}{d\beta},$$

$$\alpha \frac{d^2 f}{d\alpha \, d\gamma} + \beta \frac{d^2 f}{d\beta \, d\gamma} + \gamma \frac{d^2 f}{d\gamma^2} = (n-1) \frac{df}{d\gamma}.$$

Cor. 2. By the elimination of α, β, γ from the last four equations, we obtain

$$\left|\begin{array}{cccc} \frac{n}{n-1} f(\alpha, \beta, \gamma), & \frac{df}{d\alpha}, & \frac{df}{d\beta}, & \frac{df}{d\gamma} \\ \frac{df}{d\alpha}, & \frac{d^2f}{d\alpha^2}, & \frac{d^2f}{d\alpha\, d\beta}, & \frac{d^2f}{d\gamma\, d\alpha} \\ \frac{df}{d\beta}, & \frac{d^2f}{d\alpha\, d\beta}, & \frac{d^2f}{d\beta^2}, & \frac{d^2f}{d\beta\, d\gamma} \\ \frac{df}{d\gamma}, & \frac{d^2f}{d\gamma\, d\alpha}, & \frac{d^2f}{d\beta\, d\gamma}, & \frac{d\gamma}{d\gamma^2} \end{array}\right| = 0.$$

Cor. 3. If $f(\alpha, \beta, \gamma) = 0$ be a homogeneous equation of any degree, then will

$$\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} = 0,$$

and
$$\left|\begin{array}{cccc} 0, & \frac{df}{d\alpha}, & \frac{df}{d\beta}, & \frac{df}{d\gamma} \\ \frac{df}{d\alpha}, & \frac{d^2f}{d\alpha^2}, & \frac{d^2f}{d\alpha\, d\beta}, & \frac{d^2f}{d\gamma\, d\alpha} \\ \frac{df}{d\beta}, & \frac{d^2f}{d\alpha\, d\beta}, & \frac{d^2f}{d\beta^2}, & \frac{d^2f}{d\beta\, d\gamma} \\ \frac{df}{d\gamma}, & \frac{d^2f}{d\gamma\, d\alpha}, & \frac{d^2f}{d\beta\, d\gamma}, & \frac{d^2f}{d\gamma^2} \end{array}\right| = 0.$$

509. The following notation is very convenient.

The symbol

$$\left(\lambda \frac{d}{d\alpha} + \mu \frac{d}{d\beta} + \nu \frac{d}{d\gamma}\right) f(\alpha, \beta, \gamma)$$

denotes the same thing as

$$\lambda \frac{df}{d\alpha} + \mu \frac{df}{d\beta} + \nu \frac{df}{d\gamma},$$

where $\frac{df}{d\alpha}$, $\frac{df}{d\beta}$, $\frac{df}{d\gamma}$ are the derived functions of $f(\alpha, \beta, \gamma)$ with respect to α, β, γ.

The symbol

$$\left(\lambda \frac{d}{d\alpha} + \mu \frac{d}{d\beta} + \nu \frac{d}{d\gamma}\right)^2 f(\alpha, \beta, \gamma),$$

or

$$\left(\lambda^2 \frac{d^2}{d\alpha^2} + \mu^2 \frac{d^2}{d\beta^2} + \nu^2 \frac{d^2}{d\gamma^2}\right.$$

$$\left. + 2\mu\nu \frac{d^2}{d\beta\, d\gamma} + 2\nu\lambda \frac{d^2}{d\gamma\, d\alpha} + 2\lambda\mu \frac{d^2}{d\alpha\, d\beta}\right) f(\alpha, \beta, \gamma),$$

denotes the same thing as

$$\lambda^2 \frac{d^2 f}{d\alpha^2} + \mu^2 \frac{d^2 f}{d\beta^2} + \nu^2 \frac{d^2 f}{d\gamma^2} + 2\mu\nu \frac{d^2 f}{d\beta\, d\gamma} + 2\nu\lambda \frac{d^2 f}{d\gamma\, d\alpha} + 2\lambda\mu \frac{d^2 f}{d\alpha\, d\beta},$$

where $\frac{d^2 f}{d\alpha^2}$, $\frac{d^2 f}{d\beta\, d\gamma}$ &c. are the second derived functions of $f(\alpha, \beta, \gamma)$.

Similarly,

$$\left(\lambda \frac{d}{d\alpha} + \mu \frac{d}{d\beta} + \nu \frac{d}{d\gamma}\right)^n f(\alpha, \beta, \gamma)$$

denotes the expression obtained by expanding

$$\left(\lambda \frac{d}{d\alpha} + \mu \frac{d}{d\beta} + \nu \frac{d}{d\gamma}\right)^n,$$

as if each of the expressions $\lambda \frac{d}{d\alpha}$, $\mu \frac{d}{d\beta}$, $\nu \frac{d}{d\gamma}$ were an algebraical term, and then replacing every such term as

$$\left(\lambda \frac{d}{d\alpha}\right)^{n-p-q} \left(\mu \frac{d}{d\beta}\right)^p \left(\nu \frac{d}{d\gamma}\right)^q f(\alpha, \beta, \gamma)$$

which occurs, by

$$\lambda^{n-p-q} \mu^p \nu^q \frac{d^n f}{d\alpha^{n-p-q}\, d\beta^p\, d\gamma^q}.$$

So also

$$\left(\lambda\frac{d}{d\alpha}+\mu\frac{d}{d\beta}+\nu\frac{d}{d\gamma}\right)\left(\lambda'\frac{d}{d\alpha}+\mu'\frac{d}{d\beta}+\nu'\frac{d}{d\gamma}\right)f(\alpha,\beta,\gamma)$$

will denote

$$\lambda\lambda'\frac{d^2f}{d\alpha^2}+\mu\mu'\frac{d^2f}{d\beta^2}+\nu\nu'\frac{d^2f}{d\gamma^2}+(\mu\nu'+\mu'\nu)\frac{d^2f}{d\beta\, d\gamma}+\&c.$$

510. *If $f(\alpha, \beta, \gamma)$ be a homogeneous function of α, β, γ of the $(p+q)^{\text{th}}$ degree, then*

$$\frac{1}{\lfloor p}\left(x\frac{d}{d\alpha}+y\frac{d}{d\beta}+z\frac{d}{d\gamma}\right)^p f(\alpha,\beta,\gamma)$$

$$\equiv\frac{1}{\lfloor q}\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)^q f(x,y,z).$$

COR. As a particular case, if $f(\alpha, \beta, \gamma)$ be a homogeneous function of α, β, γ of the n^{th} degree, then

$$\left(x\frac{d}{d\alpha}+y\frac{d}{d\beta}+z\frac{d}{d\gamma}\right)^n f(\alpha,\beta,\gamma)=\lfloor n\, f(x,y,z),$$

and

$$\left(\alpha\frac{d}{d\alpha}+\beta\frac{d}{d\beta}+\gamma\frac{d}{d\gamma}\right)^n f(\alpha,\beta,\gamma)=\lfloor n\, f(\alpha,\beta,\gamma).$$

511. *If $f(\alpha, \beta, \gamma)$ be a homogeneous function of α, β, γ of the n^{th} degree, then will*

$$\begin{aligned}f(\alpha+x,\ \beta+y,\ \gamma+z) &= f(\alpha,\beta,\gamma)+\left(x\frac{d}{d\alpha}+y\frac{d}{d\beta}+z\frac{d}{d\gamma}\right)f(\alpha,\beta,\gamma)\\ &\quad+\frac{1}{\lfloor 2}\left(x\frac{d}{d\alpha}+y\frac{d}{d\beta}+z\frac{d}{d\gamma}\right)^2 f(\alpha,\beta,\gamma)\\ &\quad+\frac{1}{\lfloor 3}\left(x\frac{d}{d\alpha}+y\frac{d}{d\beta}+z\frac{d}{d\gamma}\right)^3 f(\alpha,\beta,\gamma)\\ &\quad+\&c.+f(x,y,z).\end{aligned}$$

COR. So we have

$$f(\alpha+\lambda\rho,\ \beta+\mu\rho,\ \gamma+\nu\rho)$$

$$=f(\alpha,\beta,\gamma)+\rho\left(\lambda\frac{d}{d\alpha}+\mu\frac{d}{d\beta}+\nu\frac{d}{d\gamma}\right)f(\alpha,\beta,\gamma)$$

$$+\frac{\rho^2}{\lfloor 2}\left(\lambda\frac{d}{d\alpha}+\mu\frac{d}{d\beta}+\nu\frac{d}{d\gamma}\right)^2 f(\alpha,\beta,\gamma)$$

$$+\&c.$$

$$+\rho^n f(\lambda,\mu,\nu).$$

This last expansion is of the greatest utility, and will be frequently applied in the following chapter.

CHAPTER XXVII.

THE GENERAL EQUATION OF THE n^{th} DEGREE.

512. It is our purpose, in this concluding chapter, to exhibit in its most general form the method by which the investigation of the fundamental properties of any curve must be carried on, when the curve is presented under an equation in trilinear coordinates of any degree whatever. We shall obtain equations to give the direction of the curve at any point, to determine the points of inflexion and the singular points, the tangents at the singular points and the asymptotes, and the curvature at any point whatever; but we shall not attempt any detailed discussion of the properties of the several classes of curves in general, as such a discussion properly demands by its magnitude to be treated by itself, and from its intricacy cannot with propriety find a place in an elementary treatise such as the present.

513. Let
$$f(\alpha, \beta, \gamma) = 0 \quad\text{.........................}(1)$$
represent the general homogeneous equation of the n^{th} degree in trilinear coordinates.

And let $(\alpha', \beta', \gamma')$ be any point whatever, and let
$$\frac{\alpha - \alpha'}{\lambda} = \frac{\beta - \beta'}{\mu} = \frac{\gamma - \gamma'}{\nu} = \rho \quad\text{..................}(2)$$
represent any straight line drawn through the point $(\alpha', \beta', \gamma')$ to meet the locus of the equation (1).

The lengths of the intercepts measured from $(\alpha', \beta', \gamma')$ are given by

$$f(\alpha' + \lambda\rho,\ \beta' + \mu\rho,\ \gamma' + \nu\rho) = 0.$$

And if $(\alpha'', \beta'', \gamma'')$ be any other point on the same line, the intercepts measured from $(\alpha'', \beta'', \gamma'')$ are given by

$$f(\alpha'' + \lambda\rho,\ \beta'' + \mu\rho,\ \gamma'' + \nu\rho) = 0.$$

This equation may be written

$$f(\alpha'', \beta'', \gamma'') + \rho\left(\lambda \frac{df}{d\alpha''} + \mu \frac{df}{d\beta''} + \nu \frac{df}{d\gamma''}\right) + \text{higher powers of } \rho = 0.$$

Now suppose the straight line is a tangent to the curve, and that $(\alpha'', \beta'', \gamma'')$ is the point of contact, then two of the roots of the last equation must be zero, and we have

$$f(\alpha'', \beta'', \gamma'') = 0 \quad\text{.........................(3)}$$

and

$$\lambda \frac{df}{d\alpha''} + \mu \frac{df}{d\beta''} + \nu \frac{df}{d\gamma''} = 0 \quad\text{..................(4).}$$

But since $(\alpha'', \beta'', \gamma'')$ lies on the locus of the equation (2), we have

$$\frac{\alpha'' - \alpha'}{\lambda} = \frac{\beta'' - \beta'}{\mu} = \frac{\gamma'' - \gamma'}{\nu},$$

in virtue of which, the equation (4) becomes

$$(\alpha'' - \alpha') \frac{df}{d\alpha''} + (\beta'' - \beta') \frac{df}{d\beta''} + (\gamma'' - \gamma') \frac{df}{d\gamma''} = 0 \quad\text{........(5).}$$

But from (3), by the property of homogeneous functions (Art. 508), we have

$$\alpha'' \frac{df}{d\alpha''} + \beta'' \frac{df}{d\beta''} + \gamma'' \frac{df}{d\gamma''} = 0,$$

hence the equation (5) becomes

$$\alpha' \frac{df}{d\alpha''} + \beta' \frac{df}{d\beta''} + \gamma' \frac{d\gamma}{d\gamma''} = 0 \quad\text{..................(6).}$$

But $(\alpha'', \beta'', \gamma'')$ is the point of contact of *any* tangent from $(\alpha', \beta', \gamma')$ to the curve; hence *all the points of contact of tangents from* $(\alpha', \beta', \gamma')$ *to the locus of the equation*

$$f(\alpha, \beta, \gamma) = 0$$

lie upon the locus of the equation

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = 0 \quad \text{.................... (7).}$$

But again, since $(\alpha', \beta', \gamma')$ is *any* point upon the tangent at $(\alpha'', \beta'', \gamma'')$, therefore the equation (6) may be also read as stating that *the equation to the tangent at any point* $(\alpha'', \beta'', \gamma'')$ *on the locus of the equation*

$$f(\alpha, \beta, \gamma) = 0$$

is represented by the equation

$$\alpha \frac{df}{d\alpha''} + \beta \frac{df}{d\beta''} + \gamma \frac{df}{d\gamma''} = 0 \quad \text{................ (8).}$$

514. *If* λ, μ, ν *be the direction sines of the tangent at any point* $(\alpha', \beta', \gamma')$ *on the curve* $f(\alpha, \beta, \gamma)$, *then will*

$$\frac{\lambda}{\begin{vmatrix} \frac{df}{d\beta'}, & \frac{df}{d\gamma'} \\ \sin B, & \sin C \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \frac{df}{d\gamma'}, & \frac{df}{d\alpha'} \\ \sin C, & \sin A \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \frac{df}{d\alpha'}, & \frac{df}{d\beta'} \\ \sin A, & \sin B \end{vmatrix}}$$

$$= \frac{1}{\left\{\frac{df}{d\alpha'}, \frac{df}{d\beta'}, \frac{df}{d\gamma'}\right\}}.$$

For by equation (4) of the last article, we have

$$\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'} = 0,$$

and by the identical relation (Chap. VI.),

$$\lambda \sin A + \mu \sin B + \nu \sin C = 0.$$

Therefore

$$\frac{\lambda}{\begin{vmatrix} \frac{df}{d\beta'}, & \frac{df}{d\gamma'} \\ \sin B, & \sin C \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \frac{df}{d\gamma'}, & \frac{df}{d\alpha'} \\ \sin C, & \sin A \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \frac{df}{d\alpha'}, & \frac{df}{d\beta'} \\ \sin A, & \sin B \end{vmatrix}} \ldots\ldots(1).$$

But $$\mu^2 + \nu^2 + 2\mu\nu\cos A = \sin^2 A,$$

whence each of the equal fractions in (1)

$$= \frac{1}{\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\}},$$

the coordinates being trilinear.

Therefore, &c. Q. E. D.

515. The locus of the equation (7) of Art. 513,

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = 0,$$

containing all the points of contact of tangents from $(\alpha', \beta', \gamma')$ to the curve represented by the equation

$$f(\alpha, \beta, \gamma) = 0$$

may be conveniently called the *first polar* curve of the point $(\alpha', \beta', \gamma')$ with respect to the original curve.

It will be observed that this polar curve is of an order lower by unity than the original curve, and when the original curve is a conic section, the first polar curve of any point, becomes the straight line which we have been accustomed in previous chapters to speak of as the *polar* of that point with respect to the conic.

516. If we take the equation to the first polar of the point $(\alpha', \beta', \gamma')$ with respect to the given curve, and form the equation to the first polar of the same point with respect to this new curve, the locus of this equation is called the *second polar* of the point $(\alpha', \beta', \gamma')$ with respect to the given curve. Similarly

the polar with respect to the second polar is called the *third polar*, and so on.

The equation to the first polar of the point $(\alpha', \beta', \gamma')$ being

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the equation to the second polar will be

$$\alpha'^2 \frac{d^2f}{d\alpha^2} + \beta'^2 \frac{d^2f}{d\beta^2} + \gamma'^2 \frac{d^2f}{d\gamma^2} + 2\beta'\gamma' \frac{d^2f}{d\beta d\gamma} + 2\gamma'\alpha' \frac{d^2f}{d\gamma d\alpha} + 2\alpha'\beta' \frac{d^2f}{d\alpha d\beta} = 0,$$

or, as we may write it,

$$\left(\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}\right)^2 f(\alpha, \beta, \gamma) = 0 \ldots\ldots\ldots (2).$$

So the equation to the third polar may be written

$$\left(\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}\right)^3 f(\alpha, \beta, \gamma) = 0 \ldots\ldots\ldots (3),$$

and the equation to the r^{th} polar,

$$\left(\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}\right)^r f(\alpha, \beta, \gamma) = 0.$$

517. It has been observed that the first polar is of an order one less than that of the original curve. So each polar is of an order one less than that of the preceding polar, and therefore, the original curve being of the n^{th} order, its $(n-1)^{\text{th}}$ polar will be a *straight line* represented by the equation

$$\left(\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}\right)^{n-1} f(\alpha, \beta, \gamma) = 0.$$

518. As our condition of tangency we have simply expressed that a line should meet a curve in two coincident points. It is obvious that this condition will be satisfied by any line through a cusp, multiple point, or conjugate point.

Hence every cusp, multiple point, or conjugate point in any curve will lie upon the polar curve of any point whatever.

Therefore the cusps, multiple points, and conjugate points of the curve whose equation is

$$f(\alpha, \beta, \gamma) = 0$$

lie upon the curve

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = 0,$$

where the ratios $\alpha' : \beta' : \gamma'$ may have any values whatever.

Hence the coordinates of all these singular points must satisfy simultaneously the equations

$$\frac{df}{d\alpha} = 0, \quad \frac{df}{d\beta} = 0, \quad \frac{df}{d\gamma} = 0.$$

519. *If a point lie upon a fixed straight line, its first polar with respect to a curve of the n^{th} order will pass through $(n-1)^2$ fixed points.*

Let the point $(\alpha', \beta', \gamma')$ lie upon the straight line

$$l\alpha + m\beta + n\gamma = 0.$$

Its first polar curve with respect to the curve $f(\alpha, \beta, \gamma) = 0$, is represented by

$$\alpha' \frac{df}{d\alpha} + \beta' \frac{df}{d\beta} + \gamma' \frac{df}{d\gamma} = 0.$$

Therefore, since

$$l\alpha' + m\beta' + n\gamma' = 0,$$

the polar curve passes through the points given by

$$\frac{1}{l} \frac{df}{d\alpha} = \frac{1}{m} \frac{df}{d\beta} = \frac{1}{n} \frac{df}{d\gamma}.$$

And these equations represent the intersection of two curves each of the $(n-1)^{\text{th}}$ order, and therefore give $(n-1)^2$ points.

Therefore, &c. Q.E.D.

COR. The polar curve of any point at infinity passes through the $(n-1)^2$ points given by

$$\frac{1}{a} \frac{df}{d\alpha} = \frac{1}{b} \frac{df}{d\beta} = \frac{1}{c} \frac{df}{d\gamma}.$$

520. *Of the tangents at singular points.*

If a point $(\alpha', \beta', \gamma')$ can be found whose coordinates satisfy simultaneously the three equations

$$\frac{df}{d\alpha'}=0,\quad \frac{df}{d\beta'}=0,\quad \frac{df}{d\gamma'}=0\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

that point is, as we have seen, a double point, cusp or conjugate point, and any straight line through it cuts the curve in two coincident points.

The equation to a straight line meeting the curve in two coincident points at $(\alpha', \beta', \gamma')$, viz.

$$\alpha\frac{df}{d\alpha'}+\beta\frac{df}{d\beta'}+\gamma\frac{df}{d\gamma'}=0,$$

becomes indeterminate, and in order to find a true tangent at this point, we must express that such a line meets the curve in *three* coincident points.

Thus if

$$\frac{\alpha-\alpha'}{\lambda}=\frac{\beta-\beta'}{\mu}=\frac{\gamma-\gamma'}{\nu}=\rho\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

be any straight line through $(\alpha', \beta', \gamma')$ the equation to determine the intercepts on this line may be written

$$f(\alpha', \beta', \gamma')+\rho\left(\lambda\frac{d}{d\alpha'}+\mu\frac{d}{d\beta'}+\nu\frac{d}{d\gamma'}\right)f(\alpha', \beta', \gamma')$$

$$+\frac{1}{\lfloor 2}\rho^2\left(\lambda\frac{d}{d\alpha'}+\mu\frac{d}{d\beta'}+\nu\frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')$$

$$+\frac{1}{\lfloor 3}\rho^3\left(\lambda\frac{d}{d\alpha'}+\mu\frac{d}{d\beta'}+\nu\frac{d}{d\gamma'}\right)^3 f(\alpha', \beta', \gamma')$$

$$+\&c.=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

of which the first two terms vanish since $f(\alpha', \beta', \gamma')$ and its first derived functions are zero.

Hence in order that the straight line may be a tangent, λ, μ, ν must be such as to make the third coefficient vanish and thus make *three* of the roots zero.

So we must have

$$\left(\lambda \frac{d}{d\alpha'} + \mu \frac{d}{d\beta'} + \nu \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma') = 0,$$

or in virtue of (2)

$$\left\{(\alpha - \alpha') \frac{d}{d\alpha'} + (\beta - \beta') \frac{d}{d\beta'} + (\gamma - \gamma') \frac{d}{d\gamma'}\right\}^2 f(\alpha', \beta' \gamma') = 0 \ldots (4).$$

Now if we expand the first member of this equation, the terms of the second order in α, β, γ are

$$\left(\alpha \frac{d}{d\alpha'} + \beta \frac{d}{d\beta'} + \gamma \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma').$$

The coefficient of α is

$$-2\left(\alpha' \frac{d}{d\alpha'} + \beta' \frac{d}{d\beta'} + \gamma' \frac{d}{d\gamma'}\right) \frac{d}{d\alpha'} f(\alpha', \beta', \gamma'),$$

which (since $\frac{df}{d\alpha'}$ is a homogeneous function of the $(n-1)^{\text{th}}$ degree) becomes

$$-2(n-1)\frac{df}{d\alpha'},$$

and vanishes in virtue of (1).

So the coefficients of β and γ vanish, and the remaining terms are

$$\left(\alpha' \frac{d}{d\alpha'} + \beta' \frac{d}{d\beta'} + \gamma' \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma'),$$

which similarly vanish.

So the whole equation (4) reduces to

$$\left(\alpha \frac{d}{d\alpha'} + \beta \frac{d}{d\beta'} + \gamma \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma') = 0 \text{..........(5)},$$

a relation among the coordinates of a point on any tangent at $(\alpha', \beta', \gamma')$, and therefore the equation to the two tangents (real or imaginary).

521. That the equation just obtained does really represent two straight lines is immediately seen by applying the criterion of Art. 245.

That criterion requires that

$$\begin{vmatrix} \frac{d^2f}{d\alpha'^2}, & \frac{d^2f}{d\alpha' d\beta'}, & \frac{d^2f}{d\gamma' d\alpha'} \\ \frac{d^2f}{d\alpha' d\beta'}, & \frac{d^2f}{d\beta'^2}, & \frac{d^2f}{d\beta' d\gamma'} \\ \frac{d^2f}{d\gamma' d\alpha'}, & \frac{d^2f}{d\beta' d\gamma'}, & \frac{d^2f}{d\gamma'^2} \end{vmatrix} = 0 \text{............(6)}.$$

Now, by multiplying the three rows of the determinant by α', β', γ', and adding, we obtain

$$\left(\alpha' \frac{d}{d\alpha'} + \beta' \frac{d}{d\beta'} + \gamma' \frac{d}{d\gamma'}\right) \frac{df}{d\alpha'}, \quad \left(\alpha' \frac{d}{d\alpha'} + \beta' \frac{d}{d\beta'} + \gamma' \frac{d}{d\gamma'}\right) \frac{df}{d\beta'},$$

$$\left(\alpha' \frac{d}{d\alpha'} + \beta' \frac{d}{d\beta'} + \gamma' \frac{d}{d\gamma'}\right) \frac{df}{d\gamma'},$$

or by the property of homogeneous functions (Art. 506)

$$(n-1)\frac{df}{d\alpha'}, \quad (n-1)\frac{df}{d\beta'}, \quad (n-1)\frac{df}{d\gamma'},$$

each of which terms vanishes in virtue of (1).

Hence the condition (6) is satisfied, and therefore the equation (5) represents two straight lines.

522. *To distinguish between a double point, cusp or conjugate point.*

If $(\alpha', \beta', \gamma')$ be a double point, cusp or conjugate point, the real or imaginary tangents thereat are given by

$$\left(\alpha \frac{d}{d\alpha'} + \beta \frac{d}{d\beta'} + \gamma \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma') = 0.$$

According as the point is a double point, cusp or conjugate point, the tangents will be real and distinct, real and coincident, or imaginary; and therefore they will meet any straight line in real and distinct, real and coincident, or imaginary points.

Consider the straight line $\alpha = 0$: it meets the tangents in the points given by

$$\beta^2 \frac{d^2f}{d\beta'^2} + 2\beta\gamma \frac{d^2f}{d\beta' d\gamma'} + \gamma^2 \frac{d^2f}{d\gamma'^2} = 0,$$

which are real, coincident, or imaginary, according as

$$\left(\frac{d^2f}{d\beta' d\gamma'}\right)^2 > = < \frac{d^2f}{d\beta'^2} \frac{d^2f}{d\gamma'^2}.$$

Hence if $(\alpha', \beta', \gamma')$ be a double point,

$$\begin{vmatrix} \frac{d^2f}{d\beta^2}, & \frac{d^2f}{d\beta\, d\gamma} \\ \frac{d^2f}{d\beta\, d\gamma}, & \frac{d^2f}{d\gamma^2} \end{vmatrix}, \quad \begin{vmatrix} \frac{d^2f}{d\gamma^2}, & \frac{d^2f}{d\gamma\, d\alpha} \\ \frac{d^2f}{d\gamma\, d\alpha}, & \frac{d^2f}{d\alpha^2} \end{vmatrix}, \quad \begin{vmatrix} \frac{d^2f}{d\alpha^2}, & \frac{d^2f}{d\alpha\, d\beta} \\ \frac{d^2f}{d\alpha\, d\beta}, & \frac{d^2f}{d\beta^2} \end{vmatrix}$$

will be negative: if it be a cusp, they will vanish: if it be a conjugate point, they will be positive.

523. Cor. If $(\alpha', \beta', \gamma')$ be a cusp, the tangent thereat is represented by any one of the equations

$$\alpha \frac{d^2f}{d\alpha'^2} + \beta \frac{d^2f}{d\alpha' d\beta'} + \gamma \frac{d^2f}{d\alpha' d\gamma'} = 0,$$

$$\alpha \frac{d^2f}{d\beta' d\alpha'} + \beta \frac{d^2f}{d\beta'^2} + \gamma \frac{d^2f}{d\beta' d\gamma'} = 0,$$

$$\alpha \frac{d^2f}{d\gamma' d\alpha'} + \beta \frac{d^2f}{d\gamma' d\beta'} + \gamma \frac{d^2f}{d\gamma'^2} = 0,$$

which are identical since

$$\begin{vmatrix} \frac{d^2f}{d\beta'^2}, & \frac{d^2f}{d\beta'\,d\gamma'} \\ \frac{d^2f}{d\beta'\,d\gamma'}, & \frac{d^2f}{d\gamma'^2} \end{vmatrix} = \begin{vmatrix} \frac{d^2f}{d\gamma'^2}, & \frac{d^2f}{d\gamma'\,d\alpha'} \\ \frac{d^2f}{d\gamma'\,d\alpha'}, & \frac{d^2f}{d\alpha'^2} \end{vmatrix} = \begin{vmatrix} \frac{d^2f}{d\alpha'^2}, & \frac{d^2f}{d\alpha'\,d\beta'} \\ \frac{d^2f}{d\alpha'\,d\beta'}, & \frac{d^2f}{d\beta'^2} \end{vmatrix} = 0.$$

524. *At a cusp, the first polar of any point whatever touches the curve.*

Let (l, m, n) be any point whatever, and let $(\alpha', \beta', \gamma')$ be a cusp.

The first polar of the point (l, m, n) is given by

$$l\frac{df}{d\alpha} + m\frac{df}{d\beta} + n\frac{df}{d\gamma} = 0.$$

The tangent to this curve at the point $(\alpha', \beta', \gamma')$ is

$$\alpha\left(l\frac{d^2f}{d\alpha'^2} + m\frac{d^2f}{d\alpha'\,d\beta'} + n\frac{d^2f}{d\alpha'\,d\gamma'}\right)$$
$$+\beta\left(l\frac{d^2f}{d\alpha'\,d\beta'} + m\frac{d^2f}{d\beta'^2} + n\frac{d^2f}{d\beta'\,d\gamma'}\right)$$
$$+\gamma\left(l\frac{d^2f}{d\alpha'\,d\gamma'} + m\frac{d^2f}{d\beta'\,d\gamma'} + n\frac{d^2f}{d\gamma'^2}\right) = 0\,\ldots\ldots\ldots(1).$$

But the equations

$$\alpha\frac{d^2f}{d\alpha'^2} + \beta\frac{d^2f}{d\alpha'\,d\beta'} + \gamma\frac{d^2f}{d\alpha'\,d\gamma'} = 0,$$
$$\alpha\frac{d^2f}{d\beta'\,d\gamma'} + \beta\frac{d^2f}{d\beta'^2} + \gamma\frac{d^2f}{d\beta'\,d\gamma'} = 0,$$
$$\alpha\frac{d^2f}{d\gamma'\,d\alpha'} + \beta\frac{d^2f}{d\gamma'\,d\beta'} + \gamma\frac{d^2f}{d\gamma'^2} = 0,$$

are any one of them the equation to the tangent at the cusp. Therefore the equation (1) which is derived from them by addition represents the same tangent.

Hence at the cusp the first polars of all points touch one another and touch the curve.

525. *To determine the points of inflexion on a curve of the n^{th} order.*

Let $(\alpha', \beta', \gamma')$ be a point of inflexion, and λ, μ, ν the direction sines of the tangent thereat.

The equation giving the lengths of the intercepts on this tangent is

$$f(\alpha', \beta', \gamma') + \rho\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)f(\alpha', \beta', \gamma')$$
$$+\frac{1}{2}\rho^2\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)f(\alpha', \beta', \gamma')$$
$$+ \&c. = 0.$$

Three of the roots of this equation must be evanescent. We have therefore

$$\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma') = 0 \text{............}(1)$$

as well as

$$\lambda\frac{df}{d\alpha'} + \mu\frac{df}{d\beta'} + \nu\frac{df}{d\gamma'} = 0 \text{.................}(2).$$

The equations (1) and (2) determine the ratios of λ, μ, ν, the direction sines of the point of inflexion, the two roots of the resulting quadratic being equal, since we have (Art. 506, Cor. 3)

$$\begin{vmatrix} \frac{d^2f}{d\alpha'^2}, & \frac{d^2f}{d\alpha' d\beta'}, & \frac{d^2f}{d\gamma' d\alpha'^2}, & \frac{df}{d\alpha'} \\ \frac{d^2f}{d\alpha' d\beta'}, & \frac{d^2f}{d\beta'^2}, & \frac{d^2f}{d\beta' d\gamma'}, & \frac{df}{d\beta'} \\ \frac{d^2f}{d\gamma' d\alpha'}, & \frac{d^2f}{d\beta' d\gamma'}, & \frac{d^2f}{d\gamma'^2}, & \frac{df}{d\gamma'} \\ \frac{df}{d\alpha'}, & \frac{df}{d\beta'}, & \frac{df}{d\gamma'}, & 0 \end{vmatrix} \equiv 0.$$

But λ, μ, ν must satisfy the relation

$$a\lambda + b\mu + c\nu = 0 \qquad\qquad (3),$$

(the coordinates being trilinear). Hence we have three equations (1), (2), (3) from which to eliminate the ratios of λ, μ, ν; and the resulting equation will constitute the condition amongst the coordinates $(\alpha', \beta', \gamma')$ in order that they may represent a point of inflexion.

But instead of performing the elimination directly, we may write down the result at once from indirect considerations. For we have seen that the equations (1) and (2) lead to a quadratic having equal roots. Now λ, μ, ν might as well be determined from the equations (1) and (3), hence these also must lead to a quadratic having equal roots.

Hence
$$\begin{vmatrix} \frac{d^2f}{d\alpha'^2}, & \frac{d^2f}{d\alpha'\,d\beta'}, & \frac{d^2f}{d\gamma'\,d\alpha'}, & a \\ \frac{d^2f}{d\alpha'\,d\beta'}, & \frac{d^2f}{d\beta'^2}, & \frac{d^2f}{d\beta'\,d\gamma'}, & b \\ \frac{d^2f}{d\alpha'\,d\gamma'}, & \frac{d^2f}{d\beta'\,d\gamma'}, & \frac{d^2f}{d\gamma'^2}, & c \\ a, & b, & c, & 0 \end{vmatrix} = 0,$$

which must be one form of the equation resulting from the elimination.

The points of inflexion will be obtained by solving this equation simultaneously with the equation

$$f(\alpha', \beta', \gamma') = 0.$$

And of these two equations, the one is of the $3(n-2)^{\text{th}}$ degree, and the other of the n^{th}.

Hence there will be in general $3n(n-2)$ points of inflexion on a curve of the n^{th} order.

526. *Of multiple points and the tangents thereat.*

We have seen that the intercepts which the curve makes on the straight line drawn from $(\alpha', \beta', \gamma')$ in the direction λ, μ, ν are given by the equation

$$f(\alpha', \beta', \gamma') + \rho\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)f(\alpha', \beta', \gamma')$$
$$+ \frac{1}{\lfloor 2}\rho^2\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')$$
$$+ \frac{1}{\lfloor 3}\rho^3\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)^3 f(\alpha', \beta', \gamma')$$
$$+ \&c. = 0 \quad\ldots\ldots\ldots\ldots(1).$$

Now suppose $(\alpha', \beta', \gamma')$ is a point such that $f(\alpha', \beta', \gamma')$ and its differential coefficients with respect to α', β', γ' up to those of the $(r-1)^{\text{th}}$ order inclusive, all vanish identically.

The first r terms in the equation (1) will disappear, and the equation will take the form

$$\rho^r\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)^r f(\alpha', \beta', \gamma')$$
$$+ \text{higher powers of } \rho = 0,$$

shewing that *any* straight line through the point $(\alpha', \beta', \gamma')$ meets the curve in r coincident points thereat. Such a point is called a multiple point of the r^{th} order, the double points considered in the preceding articles constituting the particular case when $r = 2$.

The tangents at a multiple point of the r^{th} order will meet the curve in $r+1$ coincident points; and, as in Art. 520, it will be seen that their directions are given by the equation

$$\left(\lambda\frac{d}{d\alpha'} + \mu\frac{d}{d\beta'} + \nu\frac{d}{d\gamma'}\right)^r f(\alpha', \beta', \gamma') = 0;$$

and the equation to the r tangents themselves is

$$\left(\alpha\frac{d}{d\alpha'} + \beta\frac{d}{d\beta'} + \gamma\frac{d}{d\gamma'}\right)^r f(\alpha', \beta', \gamma') = 0.$$

527. *To find the directions of the asymptotes of the curve whose equation is*

$$f(\alpha, \beta, \gamma) = 0 \quad\ldots\ldots\ldots\ldots(1).$$

Let λ, μ, ν be the direction sines of an asymptote. Then if $(\alpha', \beta', \gamma')$ be any point whatever, one of the roots of the equation

$$f(\alpha' + \lambda\rho,\ \beta' + \mu\rho,\ \gamma' + \nu\rho) = 0 \quad\text{.....................}\quad (2)$$

must be infinite, and therefore

$$f(\lambda, \mu, \nu) = 0 \quad\text{...}\quad (3).$$

Solving this equation simultaneously with the identical relation

$$a\lambda + b\mu + c\nu = 0, \qquad [\textit{trilinear}$$

$$\lambda + \mu + \nu = 0, \qquad [\textit{triangular}$$

we obtain the ratios $\lambda : \mu : \nu$, determining the direction of an asymptote.

Cor. Since the equation (3) is of the n^{th} order there will in general be n solutions determining the directions of n asymptotes.

528. *To find the equation to an asymptote of the same curve.*

Let λ, μ, ν be the direction sines of an asymptote determined as in the last article, and suppose $(\alpha', \beta', \gamma')$ any point on the asymptote. Then two of the radii from this point in the direction λ, μ, ν must be infinite, and therefore the equation

$$f(\alpha' + \lambda\rho,\ \beta' + \mu\rho,\ \gamma' + \nu\rho) = 0 \quad\text{.....................}\quad (2)$$

must have two infinite roots.

Hence we must not only have

$$f(\lambda, \mu, \nu) = 0,$$

but also

$$\alpha' \frac{df}{d\lambda} + \beta' \frac{df}{d\mu} + \gamma' \frac{df}{d\nu} = 0 \quad\text{..............................}\quad (3).$$

Now this is a relation among the coordinates $(\alpha', \beta', \gamma')$ of any point on the asymptote. Hence, suppressing the accents, the equation to the asymptote is

$$\alpha \frac{df}{d\lambda} + \beta \frac{df}{d\mu} + \gamma \frac{df}{d\nu} = 0.$$

529. *To find the radius of curvature at any point on a plane curve.*

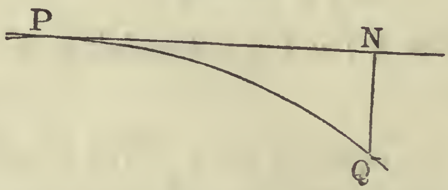
Let P be the given point, (α, β, γ) its coordinates, and

$$f(\alpha, \beta, \gamma) = 0 \quad \text{.................................(1)}$$

the equation to the given curve.

Let Q be a point on the curve near to P, and draw QN perpendicular on the tangent at P.

Fig. 43.



Let λ, μ, ν be the direction sines of the straight line PQ, and let $PQ = \delta s$, so that the coordinates of Q are

$$\alpha' + \lambda\delta s, \;\; \beta' + \mu\delta s, \;\; \gamma' + \nu\delta s.$$

Then we have (Art. 46)

$$QN = \frac{(\alpha' + \lambda\delta s)\dfrac{df}{d\alpha'} + (\beta' + \mu\delta s)\dfrac{df}{d\beta'} + (\gamma' + \nu\delta s)\dfrac{df}{d\gamma'}}{\left\{\dfrac{df}{d\alpha'}, \dfrac{df}{d\beta'}, \dfrac{df}{d\gamma'}\right\}};$$

but $(\alpha', \beta', \gamma')$ lies on the locus of the given equation (1), and therefore

$$\alpha'\frac{df}{d\alpha'} + \beta'\frac{df}{d\beta'} + \gamma'\frac{df}{d\gamma'} = 0,$$

hence

$$QN = \frac{\left(\lambda\dfrac{df}{d\alpha'} + \mu\dfrac{df}{d\beta'} + \nu\dfrac{df}{d\gamma'}\right)\delta s}{\left\{\dfrac{df}{d\alpha'}, \dfrac{df}{d\beta'}, \dfrac{df}{d\gamma'}\right\}}.$$

Now if ρ be the radius of curvature at P, and if Q move up to and ultimately coincide with P, we have

$$2\rho = lt.\frac{PQ^2}{QN} = lt.\frac{(\delta s)^2}{QN},$$

therefore

$$2\rho = lt. \frac{\left\{\frac{df}{d\alpha'}, \frac{df}{d\beta'}, \frac{df}{d\gamma'}\right\} \delta s}{\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}} \quad\text{..........................(3).}$$

But since Q lies on the given curve, we have

$$f(\alpha' + \lambda\delta s,\ \beta' + \mu\delta s,\ \gamma' + \nu\delta s) = 0,$$

or $f(\alpha', \beta', \gamma') + \left(\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'}\right) \delta s$

$$+ \frac{1}{2}\left(\lambda \frac{d}{d\alpha'} + \mu \frac{d}{d\beta'} + \nu \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')\, \delta s^2$$

+ higher powers of $\delta s = 0$.

But we have $\quad f(\alpha', \beta', \gamma') \equiv 0,$

therefore

$$\lambda \frac{df}{d\alpha'} + \mu \frac{df}{d\beta'} + \nu \frac{df}{d\gamma'} + \frac{1}{2}\left(\lambda \frac{d}{d\alpha'} + \mu \frac{d}{d\beta'} + \nu \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')\, \delta s$$

+ higher powers of $\delta s = 0$.

Hence, substituting in (3) and diminishing δs indefinitely

$$\rho = -\frac{\left\{\frac{df}{d\alpha'}, \frac{df}{d\beta'}, \frac{df}{d\gamma'}\right\}}{\left(\lambda \frac{d}{d\alpha'} + \mu \frac{d}{d\beta'} + \nu \frac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')}.$$

But in the limit λ, μ, ν are the direction sines of the tangent at $(\alpha', \beta', \gamma')$, and therefore (Art. 514),

$$\frac{\lambda}{\begin{vmatrix} \frac{df}{d\beta'}, & \frac{df}{d\gamma'} \\ \sin B, & \sin C \end{vmatrix}} = \frac{\mu}{\begin{vmatrix} \frac{df}{d\gamma'}, & \frac{df}{d\alpha'} \\ \sin C, & \sin A \end{vmatrix}} = \frac{\nu}{\begin{vmatrix} \frac{df}{d\alpha'}, & \frac{df}{d\beta'} \\ \sin A, & \sin B \end{vmatrix}}$$

$$= \frac{1}{\left\{\frac{df}{d\alpha'}, \frac{df}{d\beta'}, \frac{df}{d\gamma'}\right\}}.$$

Hence, if L, M, N denote the determinants

$$\begin{vmatrix} \dfrac{df}{d\beta'}, & \dfrac{df}{d\gamma'} \\ \sin B, & \sin C \end{vmatrix}, \quad \begin{vmatrix} \dfrac{df}{d\gamma'}, & \dfrac{df}{d\alpha'} \\ \sin C, & \sin A \end{vmatrix}, \quad \begin{vmatrix} \dfrac{df}{d\alpha'}, & \dfrac{df}{d\beta'} \\ \sin A, & \sin B \end{vmatrix},$$

we obtain

$$\rho = -\frac{\left\{\dfrac{df}{d\alpha'}, \dfrac{df}{d\beta'}, \dfrac{df}{d\gamma'}\right\}^3}{\left(L\dfrac{d}{d\alpha'} + M\dfrac{d}{d\beta'} + N\dfrac{d}{d\gamma'}\right)^2 f(\alpha', \beta', \gamma')}.$$

This result in a slightly different form has been recently published by Mr Walton in the *Quarterly Journal of Mathematics*, Vol. VIII. p. 41 (June 1866), in a paper on Curvature, to which the reader is referred.

530. *To determine the coordinates of the centre of curvature at any point on a plane curve.*

Let $(\alpha', \beta', \gamma')$ be the given point, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ the centre of curvature.

The equations to the normal are (Art. 226)

$$\frac{\alpha - \alpha'}{\dfrac{df}{d\alpha'} - \dfrac{df}{d\beta'}\cos C - \dfrac{df}{d\gamma'}\cos B} = \frac{\beta - \beta'}{\dfrac{df}{d\beta'} - \dfrac{df}{d\gamma'}\cos A - \dfrac{df}{d\alpha'}\cos C}$$

$$= \frac{\gamma - \gamma'}{\dfrac{df}{d\gamma'} - \dfrac{df}{d\alpha'}\cos B - \dfrac{df}{d\beta'}\cos A} = \rho.$$

And if ρ represent the length of the radius of curvature, the point (α, β, γ) lying on the normal at the distance ρ from the point $(\alpha', \beta', \gamma')$ must be the centre of curvature.

Hence, we have

$$\frac{\bar{\alpha} - \alpha'}{\dfrac{df}{d\alpha'} - \dfrac{df}{d\beta'}\cos C - \dfrac{df}{d\gamma'}\cos B} = \frac{\bar{\beta} - \beta'}{\dfrac{df}{d\beta'} - \dfrac{df}{d\gamma'}\cos A - \dfrac{df}{d\alpha'}\cos C}$$

$$= \frac{\bar{\gamma} - \gamma'}{\frac{df}{d\gamma'} - \frac{df}{d\alpha'}\cos B - \frac{df}{d\beta'}\cos A}$$

$$= -\frac{\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\}^3}{\left\{L\frac{d}{d\alpha'} + M\frac{d}{d\beta'} + N\frac{d}{d\gamma'}\right\}^2 f(\alpha', \beta', \gamma')},$$

where L, M, N denote the determinants

$$\begin{vmatrix} \frac{df}{d\beta'}, & \frac{df}{d\gamma'} \\ \sin B, & \sin C \end{vmatrix}, \quad \begin{vmatrix} \frac{df}{d\gamma'}, & \frac{df}{d\alpha'} \\ \sin C, & \sin A \end{vmatrix}, \quad \begin{vmatrix} \frac{df}{d\alpha'}, & \frac{df}{d\beta'} \\ \sin A, & \sin B \end{vmatrix}$$

as in the last article.

These equations determine the coordinates $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ required.

531. *A curve of the n^{th} order can generally be found to satisfy $\frac{n(n+3)}{2}$ simple conditions.*

The homogeneous equation of the n^{th} degree in its most general form may be written

$$\alpha^n + \alpha^{n-1}(a_1\beta + a_2\gamma) + \alpha^{n-2}(b_1\beta^2 + b_2\beta\gamma + b_3\gamma^2)$$
$$+ \alpha^{n-3}(c_1\beta^3 + c_2\beta^2\gamma + c_3\beta\gamma^2 + c_4\gamma^3) + \&c. = 0.$$

Whence we observe that there will be one term involving α^n, two involving α^{n-1}, three involving α^{n-2}, four involving α^{n-3}, and so on, and $(n+1)$ not involving any power of α.

Hence the whole number of terms is

$$\frac{(n+1)(n+2)}{2}.$$

There are, therefore, $\frac{(n+1)(n+2)}{2}$ coefficients involving $\frac{(n+1)(n+2)}{2} - 1$, or $\frac{n(n+3)}{2}$ ratios, and therefore by reasoning analogous to that of Art. 147, a curve of the n^{th} order can generally be found to pass through $\frac{n(n+3)}{2}$ given points, or to fulfil this number of simple conditions.

532. *All curves of the n^{th} order which pass through* $\frac{n(n+3)-2}{2}$ *fixed points pass also through* $\frac{(n-1)(n-2)}{2}$ *other fixed points.*

Let $\phi(\alpha, \beta, \gamma) = 0$ and $\psi(\alpha, \beta, \gamma) = 0$ be the equations to two curves of the n^{th} order passing through $\frac{n(n+3)-2}{2}$ given points. And let any other curve of the n^{th} order be determined by passing through these points and the point $(\alpha', \beta', \gamma')$, thus making the requisite number $\frac{n(n+3)}{2}$ of points altogether.

The equation to this curve will be

$$\frac{\phi(\alpha, \beta, \gamma)}{\phi(\alpha', \beta', \gamma')} = \frac{\psi(\alpha, \beta, \gamma)}{\psi(\alpha', \beta', \gamma')} \qquad \text{(1)},$$

for this equation is satisfied at the point $(\alpha', \beta', \gamma')$ and at any point of intersection of the given curves.

But the two given curves being each of the n^{th} order intersect in n^2 points, and the locus of the equation (1) passes through all these points, i.e. through the $\frac{n(n+3)-2}{2}$ given points, and the remaining $\frac{(n-1)(n-2)}{2}$ points of intersection. Hence all curves of the n^{th} order which pass through $\frac{n(n+3)-2}{2}$ fixed points pass also through $\frac{(n-1)(n-2)}{2}$ other fixed points. Q. E. D.

533. For a more extensive discussion of the properties of the locus of the general equation of the n^{th} degree, the reader is referred to Dr Salmon's *Higher Plane Curves*.

Exercises on Chapter XXVII.

(296) Shew that the curve

$$u^2v^2 = (u - v)^2 yz$$

has a point of osculation at the intersection of the straight lines $u = 0$ and $v = 0$, the tangent being $u = v$.

(297) Shew that the curve

$$(\mu - \nu)\,\alpha^n + (\nu - \lambda)\,\beta^n + (\lambda - \mu)\,\gamma^n = 0$$

touches the straight line

$$(\mu - \nu)\,\alpha + (\nu - \lambda)\,\beta + (\lambda - \mu)\,\gamma = 0$$

at the point $\alpha = \beta = \gamma$.

(298) Shew that the curve

$$(\mu - \nu)\,\alpha^{2n} + (\nu - \lambda)\,\beta^{2n} + (\lambda - \mu)\,\gamma^{2n} = 0$$

touches the conic

$$(\mu - \nu)\,\alpha^2 + (\nu - \lambda)\,\beta^2 + (\lambda - \mu)\,\gamma^2 = 0$$

at the four points

$$\pm\,\alpha = \pm\,\beta = \pm\,\gamma,$$

and that the common tangents are represented by the equations

$$\pm\,(\mu - \nu)\,\alpha \pm (\nu - \lambda)\,\beta \pm (\lambda - \mu)\,\gamma = 0.$$

(299) Find the asymptotes of the curve whose trilinear equation is

$$\alpha^2\,(a\alpha + b\beta + c\gamma)^2 = k\,(b\beta + c\gamma)^2\,(a\beta\gamma + b\gamma\alpha + c\alpha\beta),$$

and shew that the curve passes through the circular points at infinity.

(300) If a conic be inscribed in the triangle of reference so that one focus lies on the conic

$$\sqrt{\frac{\alpha}{l}}+\sqrt{\frac{\beta}{m}}+\sqrt{\frac{\gamma}{n}}=0,$$

then the other focus will lie on the curve

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0.$$

(301) The general equation to a curve of the fourth order having double points at the points of reference is

$$\frac{\lambda}{\alpha^2}+\frac{\mu}{\beta^2}+\frac{\nu}{\gamma^2}+\frac{l\alpha+m\beta+n\gamma}{\alpha\beta\gamma}=0.$$

(302) The tangents at the double points in the last exercise are given by the equations

$$\frac{\beta^2}{\mu}+\frac{\gamma^2}{\nu}+\frac{l\beta\gamma}{\mu\nu}=0,\quad \frac{\gamma^2}{\nu}+\frac{\alpha^2}{\lambda}+\frac{m\gamma\alpha}{\nu\lambda}=0,\quad \frac{\alpha^2}{\lambda}+\frac{\beta^2}{\mu}+\frac{n\alpha\beta}{\lambda\mu}=0.$$

(303) The general equation to a curve of the fourth order having cusps at the points of reference is

$$\frac{l^2}{\alpha^2}+\frac{m^2}{\beta^2}+\frac{n^2}{\gamma^2}-\frac{2mn}{\beta\gamma}-\frac{2nl}{\gamma\alpha}-\frac{2lm}{\alpha\beta}=0,$$

and the equations to the tangents at the cusps are

$$\frac{\beta}{m}=\frac{\gamma}{n},\quad \frac{\gamma}{n}=\frac{\alpha}{l},\quad \frac{\alpha}{l}=\frac{\beta}{m}.$$

(304) If a curve of the fourth order have three cusps their tangents are concurrent.

(305) The tangents drawn from the double point ($\beta=0$, $\gamma=0$) to meet the curve

$$\frac{\lambda}{\alpha^2}+\frac{\mu}{\beta^2}+\frac{\nu}{\gamma^2}+\frac{l}{\beta\gamma}+\frac{m}{\gamma\alpha}+\frac{n}{\alpha\beta}=0$$

are represented by the equation

$$\beta^2(m^2-4\lambda\nu)+2\beta\gamma(mn-2\lambda l)+\gamma^2(n^2-4\lambda\mu)=0.$$

(306) If the equation

$$\frac{\lambda}{\alpha^2} + \frac{\mu}{\beta^2} + \frac{\nu}{\gamma^2} = 0$$

represent a real curve, it has a conjugate point and two double points at the points of reference.

(307) The curve whose equation is

$$\frac{1}{\sqrt{l\alpha}} + \frac{1}{\sqrt{m\beta}} + \frac{1}{\sqrt{n\gamma}} = 0$$

passes through all the points of intersection of the three conics

$$\frac{m}{\beta} + \frac{n}{\gamma} = \frac{l}{\alpha}, \quad \frac{n}{\gamma} + \frac{l}{\alpha} = \frac{m}{\beta}, \quad \frac{l}{\alpha} + \frac{m}{\beta} = \frac{n}{\gamma}.$$

(308) The tangent to the same curve at the point $(\alpha', \beta', \gamma')$ is represented by the equation

$$\frac{\frac{\alpha}{\alpha'}}{\frac{l}{\alpha'} - \frac{m}{\beta'} - \frac{n}{\gamma'}} + \frac{\frac{\beta}{\beta'}}{\frac{m}{\beta'} - \frac{n}{\gamma'} - \frac{l}{\alpha'}} + \frac{\frac{\gamma}{\gamma'}}{\frac{n}{\gamma'} - \frac{l}{\alpha'} - \frac{m}{\beta'}} = 0.$$

(309) The equation

$$\alpha^3 (m'\beta^2 + n''\gamma^2 + 2l\beta\gamma) + \beta^3 (n'\gamma^2 + l''\alpha^2 + 2m\gamma\alpha)$$
$$+ \gamma^3 (l'\alpha^2 + m''\beta^2 + 2n\alpha\beta) + \alpha\beta\gamma (\lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta) = 0$$

is the general equation to a curve of the fifth order referred to a triangle formed by joining three double points.

(310) The tangents at the points of reference in the last exercise are given by the equations

$$m'\beta^2 + n''\gamma^2 + 2l\beta\gamma = 0, \qquad n'\gamma^2 + l''\alpha^2 + 2m\gamma\alpha = 0,$$
$$l'\alpha^2 + m''\beta^2 + 2n\alpha\beta = 0.$$

(311) The general equation to a curve of the fifth order referred to the triangle formed by joining three cusps is

$$\alpha^3 (m\beta + n'\gamma)^2 + \beta^3 (n\gamma + l'\alpha)^2 + \gamma^3 (l\alpha + m'\beta)^2$$
$$+ \alpha\beta\gamma (\lambda\beta\gamma + \mu\gamma\alpha + \nu\alpha\beta) = 0.$$

(312) The general equation to a curve of the n^{th} order having double points at the points of reference is

$$\alpha\beta\gamma\,.\,f(\alpha, \beta, \gamma) + \beta^2\gamma^2\,.\,\phi(\beta, \gamma) + \gamma^2\alpha^2\,.\,\psi(\gamma, \alpha) + \alpha^2\beta^2\,.\,\chi(\alpha, \beta) = 0,$$

where $f(\alpha, \beta, \gamma)$ is any function whatever of the $(n-3)^{\text{th}}$ degree of the three coordinates, and $\phi(\beta, \gamma)$, $\psi(\gamma, \alpha)$, $\chi(\alpha, \beta)$ any functions whatever of the $(n-4)^{\text{th}}$ degree of two coordinates each.

(313) The two tangents at the double point $(\beta = 0, \gamma = 0)$ in the last exercise are represented by the equation

$$\beta^2\,.\,\chi(1, 0) + \beta\gamma\,.\,f(1, 0, 0) + \gamma^2\,.\,\psi(0, 1) = 0.$$

MISCELLANEOUS EXERCISES.

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(314) Shew that

$$\begin{vmatrix} u, & w', & v' \\ -w', & v, & u' \\ -v', & -u', & w \end{vmatrix} = uvw + uu'^2 + vv'^2 + ww'^2.$$

(315) Shew that

$$\begin{vmatrix} a^2, & -ab, & -ca \\ -ab, & b^2, & -bc \\ -ca, & -bc, & c^2 \end{vmatrix} + 2 \begin{vmatrix} 0, & c^2, & b^2 \\ c^2, & 0, & a^2 \\ b^2, & a^2, & 0 \end{vmatrix} \equiv 0.$$

(316) Shew that

$$\begin{vmatrix} \beta + \gamma - \alpha, & \gamma + \alpha - \beta, & \alpha + \beta - \gamma \\ \beta' + \gamma' - \alpha', & \gamma' + \alpha' - \beta', & \alpha' + \beta' - \gamma' \\ \beta'' + \gamma'' - \alpha'', & \gamma'' + \alpha'' - \beta'', & \alpha'' + \beta'' - \gamma'' \end{vmatrix} \equiv 4 \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}$$

(317) Prove that if $n > 2$, the determinant

$$\begin{vmatrix} x_1 - a_1, & x_1 - a_2, & \dots & x_1 - a_n \\ x_2 - a_1, & x_2 - a_2, & \dots & x_2 - a_n \\ x_3 - a_1, & x_3 - a_2, & \dots & x_3 - a_n \\ \vdots & \vdots & & \vdots \\ x_n - a_1, & x_n - a_2, & \dots & x_n - a_n \end{vmatrix} \equiv 0.$$

(318) Prove that

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & z^2, & y^2 \\ 1, & z^2, & 0, & x^2 \\ 1, & y^2, & x^2, & 0 \end{vmatrix}$$

$$\equiv (x+y+z)(x-y-z)(y-z-x)(z-x-y).$$

(319) Shew that if

$$H \equiv \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix} \text{ and } K \equiv \begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}$$

then will

$$\begin{vmatrix} Ha^2+Ku, & Hab+Kw', & Hca+Kv' \\ Hab+Kw', & Hb^2+Kv, & Hbc+Ku' \\ Hca+Kv', & Hbc+Ku', & Hc^2+Kw \end{vmatrix} = 0.$$

(320) Prove that the determinant of the $(n+1)^{\text{th}}$ order

$$\begin{vmatrix} 0, & 1, & 1, & 1, & \ldots \\ 1, & 0, & a+b, & a+c, & \ldots \\ 1, & b+a, & 0, & b+c, & \ldots \\ 1, & c+a, & c+b, & 0, & \ldots \\ \vdots & \vdots & \vdots & \vdots & \&c. \end{vmatrix}$$

$$= -(-2)^{n-1}\, abc \ldots \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right).$$

(321) Prove that

$$\begin{vmatrix} \dfrac{b+c}{a}, & \dfrac{a}{b+c}, & \dfrac{a}{b+c} \\ \dfrac{b}{c+a}, & \dfrac{c+a}{b}, & \dfrac{b}{c+a} \\ \dfrac{c}{a+b}, & \dfrac{c}{a+b}, & \dfrac{a+b}{c} \end{vmatrix} = \frac{2(a+b+c)^3}{(b+c)(c+a)(a+b)}.$$

(322) If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the trilinear coordinates of three points, the ratios of the sides of the triangle of reference are given by the equations

$$\frac{a}{\begin{vmatrix} 1, & 1, & 1 \\ \beta_1, & \beta_2, & \beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix}} = \frac{b}{\begin{vmatrix} 1, & 1, & 1 \\ \gamma_1, & \gamma_2, & \gamma_3 \\ \alpha_1, & \alpha_2, & \alpha_3 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1, & 1, & 1 \\ \alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2, & \beta_3 \end{vmatrix}}.$$

(323) If (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be the coordinates of three points, and

$$(\kappa\alpha, \kappa\alpha', \kappa\alpha''), \quad (\kappa'\beta, \kappa'\beta', \kappa'\beta''), \quad (\kappa''\gamma, \kappa''\gamma', \kappa''\gamma'')$$

the coordinates of three other points, shew that

$$\frac{a}{\kappa} + \frac{b}{\kappa'} + \frac{c}{\kappa''} = a + b + c.$$

Also prove that if the first three points are collinear, so also are the other three.

(324) If in a homogeneous equation in trilinear coordinates the sum of the coefficients on each side of the equation be the same, then the equation will be satisfied by the coordinates of the centre of the circle inscribed in the triangle of reference.

(325) ABC is a triangle, right-angled at C: draw AE, BF perpendicular and equal to AC, BC respectively; join AF, BE, and draw CD perpendicular to AB. Then the three lines AF, BE, CD will be concurrent.

(326) The straight line whose equation is

$$l\alpha + m\beta + n\gamma = 0$$

meets the lines of reference BC, CA, AB in the points A', B', C' respectively; and AO, BO, CO meet the same lines in the points P, Q, R respectively, O being given by the equations

$$\lambda\alpha = \mu\beta = \nu\gamma.$$

Find the equations to the straight lines $A'Q$, $B'P$.

(327) If, with the construction of Ex. (326), the straight lines $B'R$, $C'Q$ intersect in X, the straight lines $C'P$, $A'R$ in Y, and the straight lines $A'Q$, $B'P$ in Z, find the equations to the sides of the triangle XYZ.

(328) If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to the sides of a triangle, find the equations to the straight lines joining the centre of the circumscribed circle with the centres of the inscribed and escribed circles.

(329) If two straight lines be given by equations of the form

$$l\alpha + m\beta + n\gamma = 0,$$

what are the equations to the lines which pass through their intersection and bisect the angles between them?

(330) If upon the sides of a triangle as diagonals, parallelograms be described having their sides parallel to two given straight lines, the other diagonals of the parallelograms will meet in a point.

(331) If straight lines be drawn bisecting the interior angles of a quadrilateral, shew that they will form another quadrilateral whose diagonals pass through the intersections of the opposite sides of the first.

Shew further, that if three of the straight lines pass through a point, the fourth will also pass through that point.

(332) If p, q, r be the distances of a variable straight line from the vertices of the triangle ABC, shew that the value of the determinant

$$\begin{vmatrix} 0, & 0, & 1, & 1, & 1 \\ 0, & 0, & p, & q, & r \\ 1, & p, & 0, & c^2, & b^2 \\ 1, & q, & c^2, & 0, & a^2 \\ 1, & r, & b^2, & a^2, & 0 \end{vmatrix}$$

is constant.

(333) If the anharmonic ratio of four collinear points A, P, B, Q be μ, shew that

$$\frac{\mu}{AP} - \frac{1}{AQ} = \frac{\mu - 1}{AB}.$$

(334) Through each angle of a triangle let two straight lines be drawn, equally inclined to the bisectors of those angles, but the inclination not necessarily the same for each of the three; then the straight lines joining the intersections of these lines will meet the corresponding sides of the triangle in three collinear points.

(335) Three straight lines AD, AE, AF are drawn through a fixed point A, and fixed points B, C, D are taken in AD. Any straight line through C intersects AE and AF in E and F; and BE, DF intersect in P; DE, BF in Q. Shew that the loci of P and Q are straight lines passing through A, and if AD be harmonically divided in the points B, C, D, the loci of P and Q coincide and form with the lines AD, AE, AF a harmonic pencil.

(336) Find the locus of a point, the sum of the perpendiculars from which on a series of given straight lines shall be equal to a given line.

(337) If the two sides BC, $B'C'$ of any hexagon $ABCA'B'C'$ intersect on the diagonal AA' produced, and the two sides CA', $C'A$ on the diagonal BB' produced; then will the remaining sides AB, $A'B'$ intersect on the diagonal CC' produced.

(338) The two conics which have B, C for foci, and pass through A are represented in tangential coordinates by the equations

$$\{ap,\ bq,\ cr\}^2 + 4bcqr\cos^2\frac{A}{2} = 0,$$

and

$$\{ap,\ bq,\ cr\}^2 - 4bcqr\sin^2\frac{A}{2} = 0.$$

(339) Find the equation to the straight line joining the middle points of the diagonals of the quadrilateral formed by the triangle of reference and the polar of the point $(\alpha', \beta', \gamma')$ with respect to the circle circumscribing that triangle.

(340) If $u=0$ be a tangent to a conic $S=0$, the two conics $S+ku^2=0$, $S+k'u^2=0$ have four-pointic contact.

(341) If a series of conics have four-pointic contact at a fixed point, and if from any point on the common tangent other tangents be drawn to the conics, their points of contact are collinear.

(342) Two conics have four-pointic contact at a fixed point P, and through P a variable straight line is drawn cutting the conics in Q and R: find the locus of the intersection of the tangents at Q and R.

(343) The triangle whose sides are

$$m\beta+n\gamma=0,\quad l\alpha-2n\gamma=0,\quad l\alpha-2m\beta=0$$

is self-conjugate with respect to the conic

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0.$$

(344) A conic section is described round a triangle ABC; lines bisecting the angles of this triangle meet the conic in the points A', B', C' respectively; express the equations to

$$A'B,\ A'C,\ A'B'.$$

(345) The triangle whose sides are

$$\frac{m\beta}{q}+\frac{n\gamma}{r}=0,\quad \frac{n\gamma}{r}+\frac{l\alpha}{p}=0,\quad \frac{l\alpha}{p}+\frac{m\beta}{q}=0,$$

will be self-conjugate with respect to the conic

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0,$$

provided $$p+q+r=0.$$

(346) The straight lines which bisect the angle of a triangle meet the opposite sides in the points P, Q, R respectively; find the equation to an ellipse described so as to touch the sides of the triangle in these points.

(347) If a conic section be described about any triangle, and the points where the lines bisecting the angles of the tri-

angle meet the conic be joined, the intersection of the sides of the triangle so formed with the corresponding sides of the original triangle lie in a straight line.

(348) If from any point perpendiculars be drawn on the three sides of any triangle, the area of the triangle formed by joining the feet of the perpendiculars bears a constant ratio to the rectangle under the segments of a chord of the circle circumscribing the triangle, drawn through the point.

(349) If conics pass through two fixed points and touch at another fixed point, the common tangents to any pair of them intersect on a straight line passing through the point of contact.

(350) If two conics have four-pointic contact at A, and if any straight line touch one conic in A' and cut the other in B, C; and if AB, AC cut the former conic in C', B', then AA', BB', CC' are concurrent.

(351) If a conic circumscribe a triangle and if three conics be described having four-pointic contact with the first at the angular points, and touching the opposite sides, the straight lines joining the points of contact to the opposite angular points are concurrent.

(352) If $$\sqrt{lx} + \sqrt{my} + \sqrt{nz} = 0$$

be the equation in triangular coordinates to a parabola, the equations

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n}$$

will represent a straight line meeting it in only one finite point.

(353) Find the equation to the hyperbola conjugate to the hyperbola represented by the equation

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0.$$

(354) If through the extremities of one side of a triangle any circle be described, cutting the other sides in two points, and these points be joined; shew that the locus of the intersection of

the diagonals of the quadrilateral thus formed will be a conic passing through the angular points of the triangle.

(355) From the middle points of the sides of a triangle draw perpendiculars, proportional in length to those sides, and join the ends of the perpendiculars with the opposite angles of the triangle; then the locus of the point of intersection of the joining lines will be a conic described about the triangle.

(356) The locus of a point from which the two tangents to a given conic are at right angles is a circle.

(357) Two conics have double contact, shew that the locus of the poles with respect to the first, of tangents to the second, is another conic having double contact with both at the same points.

(358) If a conic be inscribed in the triangle of reference and also touch the straight line whose trilinear equation is

$$\frac{\alpha}{\lambda}+\frac{\beta}{\mu}+\frac{\gamma}{\nu}=0,$$

its centre will lie on the chord of contact of tangents from the point $(\lambda : \mu : \nu)$ to the circle circumscribing the triangle of reference.

(359) If a series of conics be inscribed in a quadrilateral, their centres will lie upon the straight line joining the middle points of the diagonals.

(360) $AA'B'B$ is a quadrilateral inscribed in a conic. Two tangents PP', QQ' meet the diagonals AB', $A'B$ in the points P, P', Q, Q' respectively. Shew that a conic can be described so as to touch AA', BB', and also to pass through the four points P, P', Q, Q'.

(361) If three conics circumscribe the same quadrilateral, the common tangent to any two is cut harmonically by the third.

(362) If a series of parabolas are inscribed in a triangle the poles of any fixed straight line lie on another straight line.

(363) If $ABCD$ be a quadrilateral circumscribing a conic, and P, Q be any two points on the conic, then a conic can be described touching AP, AQ, CP, CQ, and passing through the points B and D. Also the pole of PQ with respect to the first conic will be the pole of BD with respect to the second.

(364) Given four tangents to a conic, find the locus of the foci.

(365) If P and Q be any points on a conic, S and H the foci, a circle can be inscribed in the quadrilateral formed by SP, SQ, HP, HQ having its centre at the pole of PQ with respect to the conic.

(366) SS', HH', OO' are the diagonals of a quadrilateral circumscribing a conic which cuts SS' in A, A'; HH' in B, B'; OO' in I, I'. Shew that if SS', HH' intersect in C, CI, CI' are the tangents at I, I'. Also if D, D' be the points of contact of the sides SH, and E, E' the points of contact of the sides SH', $S'H'$, the tangent at any other point P will meet DE in a point Z such that $\{S . POZO'\}$ is harmonic. Prove also that if the tangents to the conic from the point Q divide the angle OQO', harmonically, the locus of Q will be a conic passing through O, O' and having its centre at C.

(367) Suppose O, O' in the last exercise are the circular points at infinity, and consider what focal properties will be derived.

(368) Apply the property proved in Art. 417 to shew that if a parabola be inscribed in a triangle its focus lies on the circle circumscribing the triangle.

(369) Two conics intersect in a point O and touch the sides of a quadrilateral whose diagonals are AA', BB', CC'. If OP, OQ be the tangents at O, then pencils $\{O.APA'Q\}$, $\{O . BPB'Q\}$, $\{O . CPC'Q\}$ are harmonic. *Hence prove* that two confocal conics intersect at right angles.

(370) Given a focus and two tangents to a conic section, shew that the chord of contact passes through a fixed point.

(371) Shew that if $fl+gm+hn\equiv 0$, the locus of the foci of the conics represented by the equation

$$\frac{1}{l\alpha}+\frac{1}{m\beta}+\frac{1}{n\gamma}=0$$

is a cubic curve: and find its equation.

(372) Two imaginary parabolas can be drawn having their foci at the circular points at infinity, and intersecting in the centres of the inscribed and escribed circles of a given triangle.

(373) If a variable conic to which a fixed triangle is always self-conjugate always passes through the centre of the circle inscribed in the triangle, the locus of its centre will be the circle circumscribing the triangle.

(374) B, C are the foci of a conic P; C, A those of a conic Q; A, B those of a conic R; and common tangents to Q, R intersect in A', common tangents to R, P in B', and common tangents to P, Q in C'. Shew that the systems of points A', B', C'; A', B, C; A, B', C; A, B, C' are collinear.

(375) If $f(\alpha, \beta, \gamma)=0$ be the trilinear equation to a conic, its directrices will be represented by the equation

$$f(\alpha, \beta, \gamma)+k\left\{\frac{df}{d\alpha},\ \frac{df}{d\beta},\ \frac{df}{d\gamma}\right\}^2=0,$$

where k must be so determined that the first member may resolve into two linear factors.

(376) Trace the cubic whose trilinear equation is

$$k^2\beta\,(\beta^2+\gamma^2+2\beta\gamma\cos A)=\gamma\,(a\alpha+b\beta+c\gamma)^2.$$

(377) Shew that the six points in which the cubic

$$(\beta+\gamma\cos A)\,(\gamma+\alpha\cos B)\,(\alpha+\beta\cos C)$$
$$=(\gamma+\beta\cos A)\,(\alpha+\gamma\cos B)\,(\beta+\alpha\cos C)$$

is cut by any circle concentric with that which circumscribes the triangle of reference lie two and two at the extremities of three diagonals of the circle.

Shew also that the centre of the circle is a point of inflexion of the cubic.

(378) If two cubics intersect in six points on a conic, their other three points of intersection are collinear.

(379) Through six points on a conic there are drawn three cubics. Shew that their other points of intersection lie by threes on three concurrent straight lines.

(380) Find the conic of five-pointic contact at any point of the cuspidal cubic $y^3 = x^2 z$.

THE NEXT EIGHTY EXERCISES ARE SELECTED FROM CAMBRIDGE COLLEGE EXAMINATION PAPERS.

(381) If the inscribed circle of a triangle ABC pass through the centre of the circumscribed, then

$$\cos A + \cos B + \cos C = \sqrt{2}.$$

(382) Determine the value of k that the equation $\alpha - k\beta = 0$ may represent a tangent to the circle described about the triangle of reference.

(383) Shew that the trilinear coordinates of the centre of the conic section $4\alpha\beta - \lambda\gamma^2 = 0$ are

$$\frac{\lambda abc \sin B}{2(c^2 - \lambda ab)}, \quad \frac{\lambda abc \sin A}{2(c^2 - \lambda ab)}, \quad \frac{abc \sin C}{\lambda ab - c^2}.$$

(384) Prove that if $u = 0$, $v = 0$, $w = 0$ be the equations of the sides of a triangle, the equation of a conic section circumscribed about the triangle will be

$$\frac{l}{u} + \frac{m}{v} + \frac{n}{w} = 0,$$

and that the equations of the tangents of the conic section at the three vertices of the triangle will be

$$\frac{m}{v}+\frac{n}{w}=0, \qquad \frac{n}{w}+\frac{l}{u}=0, \qquad \frac{l}{u}+\frac{m}{v}=0.$$

(385) The equation to the self-conjugate rectangular hyperbola passing through (f, g, h) is

$$(g^2-h^2)\alpha^2+(x^2-f^2)\beta^2+(f^2-g^2)\gamma^2=0.$$

(386) If ABC be a triangle such that the angular points are the poles of the opposite sides with respect to a conic, and abc be another triangle possessing the same properties with respect to the same conic, then that one conic will circumscribe the two triangles.

(387) If $\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0$, $\frac{l'}{\alpha}+\frac{m'}{\beta}+\frac{n'}{\gamma}=0$ be two conics, find the equations of the several lines joining the centre of the circle inscribed in the triangle of reference with the four points of intersection of the two conics.

(388) If A', B', C' be the middle points of the sides of a triangle ABC, and a parabola drawn through A', B', C' meet the sides again in A'', B'', C''—then will the lines AA'', BB'', CC'' be parallel to each other.

(389) Conics circumscribing a triangle have a common tangent at the vertex; through this point a straight line is drawn: shew that the tangents at the various points where it cuts the curves all intersect on the base.

(390) OA, OB are tangents to a conic section at the points A, B; and C is any point on the curve. If AC, BC be joined and OPQ be drawn to intersect AC, BC (or these lines produced) in P and Q, prove that BP, AQ intersect on the curve.

(391) AP, BP, CP are drawn to meet a conic circumscribing ABC in DEF. The tangents at DEF meet BC, AC, AB in $A'B'C'$. Prove that $A'B'C'$ lie on a straight line.

(392) A conic is described about a triangle so that the normals at the angular points bisect the angles. Shew that the

distances of the centre from the sides are inversely proportional to the radii of the escribed circles.

(393) Find the equation to the conic section circumscribing the triangle of reference and bisecting the exterior angles of the triangle.

(394) The diameter of a conic circumscribing ABC which bisects the chords parallel to AP, BP, CP where P is a given point, meet the tangents to the conic at A, B, C in DEF, prove that DEF lie on the polar of P.

(395) The tangents to a conic at ABC meet the opposite sides of the triangle produced in PQR. The other tangents from Q and R being drawn meet AB and CA respectively in q, r; prove that Pqr lie on a straight line.

(396) A conic section is inscribed in the triangle ABC and touches the sides opposite to A, B, C in A', B', C' respectively, any point P is taken in $B'C'$ and CP, BP meet AB, AC in c, b respectively; prove that bc is a tangent to the inscribed conic.

(397) If perpendiculars be drawn from the angular points of a triangle on the opposite sides, an ellipse can be drawn touching the sides at the feet of the perpendiculars; construct it.

(398) If a conic touch a triangle at the feet of the perpendiculars from the angular points, the distance of the centre from the feet is proportional to the length of the sides.

(399) Two conics touch each other in two points A, B. If O be any point in the straight line AB and if $OPP'Q'Q$ be any chord cutting the two conics in P, Q and P', Q' respectively, prove that

$$\frac{1}{OP} + \frac{1}{OQ} = \frac{1}{OP'} + \frac{1}{OQ'}.$$

(400) The four common tangents to two conics intersect two and two on the sides of their common conjugate triad.

(401) Shew that the general equation to a circle in trilinear coördinates is

$$S = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma) - (a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

and that the square of the tangents drawn to it from a point whose trilinear coordinates are α', β', γ' is $\frac{abc}{4\Delta^2} S'$: where a, b, c are the sides and Δ the area of the triangle of reference.

(402) The self conjugate, the nine-points', and the circumscribing circle of a triangle have a common radical axis, which is the polar of the centre of gravity with respect to the self-conjugate circle.

(403) The radical axes of the circles (areal)

$$(u, v, w, u', v', w')(\alpha\beta\gamma)^2 = 0,$$
$$(p, q, r, p', q', r')(\alpha\beta\gamma)^2 = 0,$$

will be represented by

$$\frac{u\alpha + v\beta + w\gamma}{u + v + w - u' - v' - w'} = \frac{p\alpha + q\beta + r\gamma}{p + q + r - p' - q' - r'}.$$

(404) Three circles described on the chords of a complete quadrilateral as diameter have a common radical axis.

(405) Shew that the equation to any circle that passes through the points B, C of the triangle of reference, may be expressed in the form

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C$$
$$+ k\alpha(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0;$$

and determine the value of the constant k in order that the circle may *touch* the side AB.

(406) Shew that the equation of the fourth tangent common to the circle inscribed in the triangle of reference, and to the escribed circle that touches BC externally is

$$\alpha \cos \frac{A}{2} + (\beta - \gamma) \sin \frac{B - C}{2} = 0.$$

(407) The two points at which the escribed circles of a triangle subtend equal angles, lie on the straight line whose equation in trilinear coordinates referred to the triangle is

$$\alpha \cos A\,(b-c)+\beta \cos B\,(c-a)+\gamma \cos C\,(a-b)=0.$$

(408) If T be the intersection of perpendiculars from A, B, C, on the opposite sides of the triangle ABC and L the middle point of BC, and if TL be produced to meet the circle circumscribing ABC in A'; shew that AA' is a diameter of the circle.

(409) Prove that four fixed points on a conic subtend at any other point on the curve a pencil of constant anharmonic ratio, which is harmonic if the line joining two of the points which are conjugate passes through the pole of the line joining the other two.

(410) The anharmonic ratio of the pencil formed by joining a point on one of two conics to their four points of intersection is equal to the anharmonic range formed on a tangent to the other by their four common tangents.

(411) Pp, Qq, Rr, Ss are four chords of a conic passing through the same point, shew that a conic can be drawn touching

$$SR,\ RQ,\ PQ,\ sr,\ rq,\ qp.$$

(412) Having given five tangents to a conic, shew how to determine their points of contact.

(413) The equation of the line passing through the feet of the perpendiculars from a point $(\alpha_1, \beta_1, \gamma_1)$ of the circle

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$$

on the sides of the fundamental triangle, may be put in the form

$$\frac{c\beta_1+b\gamma_1}{\beta_1\cos C-\gamma_1\cos B}a\alpha+\frac{a\gamma_1+c\alpha_1}{\gamma_1\cos A-\alpha_1\cos C}b\beta+\frac{b\alpha_1+a\beta_1}{\alpha_1\cos B-\beta_1\cos C}c\gamma=0.$$

(414) AP, BP, CP are drawn to meet a conic circumscribing the triangle ABC in D, E, F; EF, FD, DE meet BC, CA, AB in A_1, B_1, C_1 respectively. Shew that these three points are in a straight line, which is the polar of P with regard to the conic.

(415) One conic touches OA, OB in A and B, and a second conic touches OB, OC in B and C: prove that the other common tangents to the two conics intersect on AC.

(416) Two conics touch each other, and through the point of contact any chord is drawn: if the tangents to the conics at the other extremities of the chord meet on the common tangent, the common chord of the conics will pass through their intersection.

(417) Two rectangular hyperbolas intersect in four points, shew that each point is in the intersection of perpendiculars from the angles on the sides of the triangles formed by joining the other three.

(418) If three conics be drawn each touching two sides of a triangle and having the third for their chord of contact, shew that the three chords of intersection pass through a point.

(419) If three parabolas are drawn having two of the sides of a triangle for tangents and the third for their chord of contact, shew that their other three points of intersection form a triangle similar to the original one and of one-ninth its area.

(420) If a triangle is self-conjugate with respect to each of a series of parabolas, the lines joining the middle points of its sides will be tangents: all the directrices will pass through O the centre of the circumscribing circle: and the focal chords, which are the polars of O, will envelope an ellipse inscribed in the given triangle which has the nine-points' circle for its auxiliary circle.

(421) Shew that there are two points P, Q in the polar of O with respect to a conic, such that PO is perpendicular to the polar of P, and QO to the polar of Q and that then POQ is a right angle.

(422) Through a point P within the triangle ABC a line is drawn parallel to each side. Prove that the sum of the rectangles contained by the segments into which each of these lines is divided by the point P is equal to $R^2 - OP^2$, R being the radius of the circumscribed circle, O its centre.

(423) The diameter of the circumscribing circle of the triangle ABC

$$= \frac{\alpha'}{\sin 2A} + \frac{\beta'}{\sin 2B} + \frac{\gamma'}{\sin 2C},$$

where α', β', γ' are the perpendiculars on any tangent from A, B, C.

(424) Similar circular arcs are described on the sides of a triangle ABC, their convexities being towards the interior of the triangle; shew that the locus of the radical centres of these three circles is the rectangular hyperbola

$$\frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = 0,$$

α, β, γ being the trilinear coordinates of a point with respect to the sides of the triangle.

(425) The pole of a tangent to a fixed circle with respect to another fixed circle will have a conic section for its locus.

(426) A conic circumscribes a triangle ABC, the tangents at the angular points meeting the opposite sides on a straight line DEF. The lines joining any point P to A, B, and C meet the conic again in A', B', C': shew that the triangle $A'B'C'$ envelopes a fixed conic inscribed in ABC, and having double contact with the given conic at the points where they are met

by DEF. Also the tangents at A', B', C' to the original conic meet $B'C'$, $C'A'$, $A'B'$ in points lying on DEF.

(427) If straight lines be drawn from the angular points of a triangle ABC, through a point P, to meet the opposite sides in α, β, γ, shew that if P moves on a conic, the intersection of PA and $\beta\gamma$ traces out a conic, and that tangents to corresponding points of the conics intersect on BC.

(428) A tangent to a conic cuts two fixed tangents in T and T', R and R' are fixed points, shew that the locus of intersection of TR and $T'R'$ is a conic.

(429) A, B, C are three fixed points, and G such that $\tan BAG$ varies as $\tan BCG$; prove that the locus of G is a conic passing through A, B, and C.

(430) The locus of the centre of rectangular hyperbolas inscribed in the triangle of reference of trilinear coordinates is the circle

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0.$$

(431) A rectangular hyperbola circumscribes a triangle; shew that the loci of the poles of its sides are three straight lines forming another triangle, whose angular points lie on the sides of the first, where they are met by perpendiculars from the opposite angular points.

(432) Ellipses are described on AB as diameter, and touching BC; if tangents be drawn to them from C, shew that the locus of the points of contact is a straight line.

(433) The locus of the centres of conics inscribed in a triangle and such that the centres of the escribed circles form a conjugate triad with respect to them is a straight line parallel to $a\alpha + b\beta + c\gamma = 0$ in triangular coordinates.

(434) If P move so that a tangent to its path is always parallel to its polar with respect to an ellipse, then P traces out an ellipse similar and similarly situated to the former.

(435) The square of the distance of a point from the base of a triangle is equal to the sum of the squares of its distances from the sides. Prove that its locus is a conic, and will be an ellipse, parabola, or hyperbola, according as the vertical angle of the triangle is obtuse, right or acute.

In the case of the parabola find its focus and directrix.

(436) The section of a cone cannot be an equilateral hyperbola unless the angle of the cone is at least a right angle.

(437) The equation of the asymptotes of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \text{ is}$$

$$(l\alpha^2 + m\beta^2 + n\gamma^2)\left(\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n}\right) = (a\alpha + b\beta + c\gamma)^2.$$

(438) A conic is described about a given quadrilateral, prove that its centre always lies on a conic which passes through the middle points of the sides and diagonals of the quadrilateral and also through the three points of intersection of the diagonals.

(439) Shew that a conic section can be described passing through the middle points of the four sides and of the two diagonals of any quadrilateral, and also through the intersections of the diagonals and of the two pairs of opposite sides, its centre being the centre of gravity of four equal particles placed at the angular points of the quadrilateral. Prove also that this conic is similar to each of the four conics which have their centres respectively at the four angular points of the quadrilateral and to which the triangle formed by joining the other three points is self-conjugate.

(440) Shew that the conic which touches the sides of a triangle and has its centre at the centre of the circle passing through the middle points of the sides, has one focus at the intersection of the perpendiculars from the angles on the opposite sides, and the other at the centre of the circle circumscribing the triangle.

(441) One focus of a conic inscribed in the triangle ABC lies in a conic touching AB, AC at B, C respectively; prove that the other focus lies on another conic touching as before.

If these two conics coincide, the major axis of the conic inscribed in the triangle passes through a fixed point.

(442) If a conic be inscribed in a triangle and its focus move along a given straight line, the locus of the other focus is a conic circumscribing the triangle.

(443) If an ellipse inscribed in a triangle has for one focus the point of intersection of the perpendiculars from the angular points of the triangle on the opposite sides, shew that (i) the other focus is the centre of the circle circumscribing the triangle, and (ii) the major axis of the elliptic is equal to the radius of this circle.

(444) If $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ be the equations to four straight lines all expressed in the form

$$x\cos\alpha+y\sin\alpha-p=0,$$

and if $a\alpha+b\beta+c\gamma+d\delta=0$ for all values of x and y, then the foci of all the conic sections which touch the four straight lines lie on the curve

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}+\frac{d}{\delta}=0.$$

(445) If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

(446) The poles of any fixed straight line with respect to a series of confocal conics lie on another straight line.

(447) Tangents are drawn from any point on an ellipse, to an interior confocal ellipse, and with the points of contact as foci a third ellipse is described passing through the given point on the first: prove that its latus rectum is constant.

(448) If a series of parabolas touch three straight lines their foci lie on a circle and their directrices are concurrent.

(449) Three parabolas are drawn touching the three sides of a triangle ABC. If D, E, F be the foci, prove that

$$\frac{ABC}{DEF}=\frac{a.b.c}{f.g.h},$$

where f, g, h are the sides of the triangle DEF.

(450) ABC is any triangle and P any point: four conic sections are described with a given focus touching the sides of the triangles ABC, PBC, PCA, PAB respectively, shew that they all have a common tangent.

(451) Tangents are drawn at two points P, P' on an ellipse. If any tangent be drawn meeting those at P, P' in R, R', shew that the line bisecting the angle RSR' intersects RR' on a fixed tangent to the ellipse. Find the point of contact of this tangent.

(452) Four conics can be described about a triangle having a given point as focus. If the sides of the triangle subtend equal angles at the given point, one of the conics will touch the other three.

(453) Circles are described on a system of parallel chords to an ellipse as diameters, shew that they will have double contact with an ellipse, having the extremities of the diameter of the chords as foci, and itself having double contact with the original ellipse.

(454) With the centre of the circumscribed circle as focus three hyperbolas can be described passing through ABC with excentricities cosec B cosec C, cosec C cosec A, cosec A cosec B, their directrices being the lines joining the middle points of the sides. The fourth point of intersection of any two lies on the line joining one of the angles to the middle point of the opposite side.

(455) An ellipse is described round a triangle, and one focus is the intersection of perpendiculars from the angular points on the opposite sides. Shew that the latus rectum

$$=\frac{2R\cos A\cos B\cos C}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}},$$

where R is the radius of the circumscribing circle.

(456) ABC is a triangle, S is any point, SA, SB, SC are joined, Sa, Sb, Sc are drawn perpendicular to SA, SB, SC, meeting the sides in abc. Straight lines Aa, Bb, Cc are drawn, forming a triangle PQR. If two conics with S as focus be inscribed in the two triangles ABC, PQR, shew that the latus rectum of one is half that of the other.

(457) The equation to the directrix of a parabola which touches the sides of the fundamental triangle and the straight line $l\alpha + m\beta + n\gamma = 0$ may be expressed

$$\alpha \cot A \left(\frac{1}{m} - \frac{1}{n}\right) + \beta \cot B \left(\frac{1}{n} - \frac{1}{l}\right) + \gamma \cot C \left(\frac{1}{l} - \frac{1}{m}\right) = 0.$$

(458) Three points ABC are taken on an ellipse. The circle about ABC meets the ellipse again in P, and $PP_{,}$ is a diameter. Prove that of all the ellipses passing through $ABCP'$, the given ellipse is the one of minimum excentricity.

(459) Shew that the reciprocal of a given conic A with respect to another conic B will be a rectangular hyperbola, if the centre of B lies on a certain circle.

(460) A series of equal similar and equally eccentric ellipses are reciprocated with respect to a circle, shew that, if one of the reciprocals be a rectangular hyperbola, they will all be so, and have double contact with a hyperbola whose eccentricity e' is given by

$$e^2 + e'^2 = 2,$$

e being the eccentricity of the ellipses.

THE FORTY EXERCISES WHICH FOLLOW ARE SELECTED FROM THE PAPERS OF THE CAMBRIDGE MATHEMATICAL TRIPOS EXAMINATIONS.

(461) Let four points be taken at random in a plane, join them two and two in every possible way, the joining lines being produced, if necessary, to intersect. Join these points of inter-

section two and two, in every possible way, producing as before the joining lines. Every line in the figure so formed is divided harmonically.

(462) Prove the following method of drawing a tangent to any curve of the second order from a given point P without it. From P draw any two lines, each cutting the curve in two points. Join the points of intersection two and two, and let the points in which the joining lines (produced if necessary) cross each other be joined by a line which will, in general, cut the curve in two points A, B. PA, PB are tangents at A and B.

(463) Two points are taken within a triangle: each is joined with an angular point, and the line produced to intersect the opposite side. Prove that the six points of intersection so formed will lie on the same conic section, and find its equation.

(464) An ellipse is described so as to touch the three sides of a triangle; prove that if one of its foci move along the circumference of a circle passing through two of the angular points of the triangle, the other will move along the circumference of another circle, passing through the same two angular points. Prove also that if one of these circles pass through the centre of the circle inscribed in the triangle, the two circles will coincide.

(465) Shew that all conic sections, which have the same focus, have two imaginary common tangents passing through that focus; and hence derive a general definition of foci.

(466) Prove that the locus of the centre of a conic section, passing through four given points, is a conic section; and shew, (1) that when the straight line joining each pair of the given points is perpendicular to the straight line joining the other pair, this locus will be a circle, (2) that when the four given points lie in the circumference of a circle, this locus will be a rectangular hyperbola.

(467) Find a point the distances of which from three given points, not in the same straight line, are proportional to p, q, and r respectively, the four points being in the same plane.

(468) OA, OB are common tangents to two conics having a common focus S; CA, CB are tangents at one of their points of intersection; BD, AE tangents intersecting CA, CB in D, E. Prove that SDE is a straight line.

(469) Two tangents OA, OB are drawn to a conic, and are cut in P and Q by a variable tangent; prove that the locus of the centres of all circles described about the triangle OPQ is a hyperbola.

(470) The circles which touch the sides AC, BC of a triangle at C, and pass through B, A respectively, intersect AB in E and F. Lines drawn from the centres of the circles inscribed in the triangles ACF, BCE parallel to CE, CF respectively, meet AC, BC in P, Q. Prove that CP is equal to CQ.

(471) If ABC be a triangle whose sides touch a parabola, and p, q, r be the perpendiculars from A, B, C on the directrix, prove that

$$p \tan A + q \tan B + r \tan C = 0.$$

(472) A, P and B, Q are points taken respectively in two parallel straight lines, A, B being fixed and P, Q variable. Prove that if the rectangle AP, BQ be constant, the line PQ will always touch a fixed ellipse or a fixed hyperbola according as P and Q are on the same or opposite sides of AB.

(473) Three hyperbolas are drawn whose asymptotes are the sides of a triangle ABC taken two and two, prove that the directions of their three common chords pass through the angular points A, B, C and meet in a point,—which will be the centre of gravity of the triangle, if the hyperbolas touch one another.

(474) Prove that the straight lines represented by the equation

$$(\alpha^2 - \beta^2) \sin C + k (\alpha \sin A + \beta \sin B)(\beta \cos B - \alpha \cos A) = 0,$$

are parallel to the axes of the conic section $2\alpha\beta = k\gamma^2$.

(475) If the lines which bisect the angles between pairs of tangents to an ellipse be parallel to a fixed straight line, prove that the locus of the points of intersection of the tangents will be a rectangular hyperbola.

(476) Tangents to an ellipse are drawn from any point in a circle through the foci, prove that the lines bisecting the angle between the tangents all pass through a fixed point.

(477) P is a point within a triangle ABC, and AP, BP, CP meet the opposite sides in A', B', C' respectively; if Pa, Pb, Pc be measured along PA, PB, PC so that these last are harmonic means between PA', Pa; PB', Pb; PC', Pc respectively, prove that a, b, c lie on a straight line.

(478) Prove that the envelope of the polar of a given point, with respect to a system of confocal conics, is a parabola the directrix of which passes through the given point.

(479) ABC is a given triangle, P any point on the circumscribing circle, through P are drawn PA', PB', PC' at right angles to PA, PB, PC to meet BC, CA, AB respectively; shew that A', B', C' lie on one straight line that passes through the centre of the circumscribing circle.

(480) If tangents be drawn to the circle bisecting the sides of a triangle, at the points where it has contact with the four circles which touch the sides, these tangents will form a quadrilateral whose diagonals pass one through each angular point of the triangle.

(481) If POP', QOQ', ROR', SOS' be four chords of an ellipse, the conic sections passing through O, P, Q, R, S and O, P', Q', R', S' will have a common tangent at O.

(482) Four circles are described, each self-conjugate with respect to one of the triangles formed by four straight lines in the same plane; prove that the four circles have a common chord.

(483) A conic always touches four given straight lines; prove that the chord of intersection of the circle, described about any one of the triangles formed by three of these straight lines, with the circle which is the locus of the intersection of two tangents to the conic at right angles to each other, always passes through a fixed point.

(484) A triangle is circumscribed about a given conic, and two of its angular points lie on another given conic; prove that the locus of the third angular point is another conic, and that the three conics have a common conjugate triad.

(485) Two triangles, ABC, $A'B'C'$, are described about an ellipse, the side BC being parallel to $B'C'$, CA to $C'A'$, AB to $A'B'$. If $B'C'$, $C'A'$, $A'B'$ be cut by any tangent in P, Q, R respectively, prove that AP, BQ, CR will be parallel to one another.

(486) If a point be taken, such that each of the three diagonals of a given quadrilateral subtends a right angle at it, prove that the director circle of every conic which touches the four sides of the quadrilateral will pass through this point.

Prove also that the polars of this point with respect to all the conics will touch a conic of which the point is a focus.

(487) If the perpendiculars Aa, Bb, Cc be let fall from A, B, C the angular points of a triangle upon the opposite sides, prove that the intersections of BC and bc, of CA and ca, of AB and ab will lie on the radical axis of the circles circumscribing the triangles ABC, and abc.

(488) A series of conics are circumscribed about a triangle ABC, having a common tangent at A. Prove that the locus of the intersection of the normals at B and C is a conic passing through B and C,—and also through A if the given tangent form a harmonic pencil with AB, AC and the diameter of the circumscribing circle through A.

If in this case the corresponding locus be found for B and C, prove that the three conics will have a fourth point in common.

(489) Prove that if a rectangular hyperbola be reciprocated with respect to a circle, the tangents drawn to the reciprocal conic from the centre of the circle will be at right angles to one another.

(490) If $f(\alpha, \beta, \gamma) = 0$ be the trilinear equation to a plane curve, and $\phi(l, m, n) = 0$ the condition that the line

$$l\alpha + m\beta + n\gamma = 0$$

may be a tangent to it; prove that $f(l, m, n) = 0$ is the condition that the straight line $l\alpha + m\beta + n\gamma = 0$ may be a tangent to the curve $\phi(\alpha, \beta, \gamma) = 0$.

(491) If a triangle circumscribe a circle, and p_1, p_2, p_3 be the algebraical perpendiculars let fall from any point in the plane of the triangle upon the line joining its angular points to the centre of the circle, prove that

$$p_1 \cos\frac{A}{2} + p_2 \cos\frac{B}{2} + p_3 \cos\frac{C}{2} = 0,$$

A, B, C being the angles of the triangle.

(492) If an ellipse of given area be circumscribed about a given triangle, the locus of the centre, referred to the same triangle, will be represented by the equation

$$(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma) = C\alpha^2\beta^2\gamma^2,$$

C being a constant depending on the length of the sides at the triangle.

(493) A rectangular hyperbola passes through the angular points, and a parabola touches the sides of a given triangle: shew that the tangents drawn to the parabola, from one of the points where the hyperbola cuts the directrix of the parabola, are parallel to the asymptotes of the hyperbola. Which of the two points on the directrix is to be taken? When the two points coincide, shew that one curve is the polar reciprocal of the other with regard to the coincident points.

(494) Five straight lines are drawn in a plane thus forming five quadrilaterals: shew that the straight lines joining the middle points at the diagonals of these quadrilaterals meet in a point.

(495) A parabola is drawn so as to touch three given straight lines, shew that the chords of contact pass each through a fixed point.

(496) With any one of four given points as centre, a conic is described self-conjugate with regard to the other three; prove that its asymptotes are parallel to the axes of the two parabolas which pass through the four given points.

(497) If a triangle be self-conjugate with respect to a parabola, shew that its nine-points' circle passes through the focus.

(498) A triangle is described about the conic

$$\alpha^2 + \beta^2 + \gamma^2 = 0;$$

two of its vertices moving along the lines

$$l\alpha + m\beta + n\gamma,\quad l'\alpha + m'\beta + n'\gamma = 0:$$

prove that the locus of the third vertex will be the conic

$$(ll' + mm' + nn')^2 (\alpha^2 + \beta^2 + \gamma^2) + \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \alpha, & \beta, & \gamma \end{vmatrix}^2 = 0.$$

(499) On the sides BC, CA, AB of a triangle ABC are taken points A', B', C' respectively, each of the angles $C'A'B'$, $A'B'C'$, $B'C'A'$ being of given magnitude; prove that, if the area of the triangle $A'B'C'$ be a minimum subject to these conditions, and K be the area of the triangle ABC, R the radius of the circle $A'B'C'$,

$$2R^2 \left\{\frac{\sin A' \sin (A + A')}{\sin A} + \frac{\sin B' \sin (B + B')}{\sin B} + \frac{\sin C' \sin (C + C')}{\sin C}\right\} = K.$$

(500) If $f(\alpha, \beta, \gamma) = 0$ be the equation of any curve referred to trilinear coordinates, and if

$$f_\alpha(\alpha, \beta, \gamma), \quad f_\beta(\alpha, \beta, \gamma), \quad f_\gamma(\alpha, \beta, \gamma)$$

be the partial differential coefficients of $f(\alpha, \beta, \gamma)$ with respect to α, β, γ respectively, shew that the line

$$\alpha f_\alpha(cx_1, c, -ax_1 - b) + \beta f_\beta(cx_1, c, -ax_1 - b) + \gamma f_\gamma(cx_1, c, -ax_1 - b) = 0$$

is in general an asymptote of the curve, x_1 being a root of the equation

$$f(cx, c, -ax - b) = 0,$$

and a, b, c being the sides of the triangle of reference.

Hence find the asymptotes of

$$\alpha(\beta - \gamma)^2 + \beta(\gamma - \alpha)^2 + \gamma(\alpha - \beta)^2 = 0,$$

and trace the curve, the triangle of reference being equilateral.

NOTES ON THE EXERCISES.

RESULTS AND OCCASIONAL HINTS.

(2) They all lie on the straight line joining (α, β) to the origin.

(3) $\alpha + \beta \cos C = 0.$ (4) $a\alpha = b\beta.$ Area $= ab \sin C.$

(5) $\rho^2 \sin^2 C = (\alpha - \alpha')^2 + (\beta - \beta')^2 + 2(\alpha - \alpha')(\beta - \beta') \cos C.$

(6) $\frac{1}{2}(\alpha\beta' \sim \alpha'\beta) \operatorname{cosec} C.$ (7) $\frac{1}{2}\beta d.$

(8) $\frac{\alpha + \alpha'}{2}, \ \frac{\beta + \beta'}{2}.$ (9) $b\alpha - a\beta = (a\alpha - b\beta) \cos C.$

(10) The points of trisection of BC are

$$\left(0, \frac{4\Delta}{3b}, \frac{2\Delta}{3c}\right) \text{ and } \left(0, \frac{2\Delta}{3b}, \frac{4\Delta}{3c}\right).$$

(11) $\frac{2\Delta}{3a}, \ \frac{2\Delta}{3b}, \ \frac{2\Delta}{3c}.$

(12) $\frac{a \cos B \cos C}{\sin A}, \ \frac{b \cos C \cos A}{\sin B}, \ \frac{c \cos A \cos B}{\sin C}.$

(13) The centre of the escribed circle opposite to A is given by

$$-\alpha = \beta = \gamma = \frac{2\Delta}{b + c - a}.$$

(14) $(mb^2 + nc^2)\alpha^2 + l(b\beta + c\gamma)^2 = (nc^2 + la^2)\beta^2 + m(c\gamma + a\alpha)^2$

$$= (la^2 + mb^2)\gamma^2 + n(a\alpha + b\beta)^2.$$

(15) $\frac{\Delta}{4}$. (16) $2\Delta \cos A \cos B \cos C$.

(17)
$$-\alpha \cos A + \beta \cos B + \gamma \cos C = 0,$$
$$\alpha \cos A - \beta \cos B + \gamma \cos C = 0,$$
$$\alpha \cos A + \beta \cos B - \gamma \cos C = 0.$$

(18) $\frac{2\Delta abc\, lmn}{(bn + cm)(cl + an)(am + bl)}$. } *

(19) They all lie on the line $l\alpha + m\beta + n\gamma = 0$.

(20) See (10). (21) $a\alpha - 2b\beta - 2c\gamma = 0$.

(22) The equation to PQR will be $a\alpha + 4b\beta - 2c\gamma = 0$.

(23) $a\alpha - (m-1)\, b\beta - (n-1)\, c\gamma = 0$. The coordinates are
$$0, \quad \frac{2\Delta}{b} \cdot \frac{n-1}{n-m}, \quad \frac{2\Delta}{c} \cdot \frac{m-1}{m-n}.$$

(24) The straight line is $\alpha \cos A = \beta \cos B$.

(25) [In each of the given equations for $-$ read $+$].

The equation to AP is $m\beta - n\gamma = 0$. To AQ, $m\beta + 3n\gamma = 0$. The other equations may be written down by symmetry.

(26) They are respectively parallel to the lines $\alpha + \beta = 0$, $\alpha - \beta = 0$, which are known to be at right angles.

(29) $\frac{m}{b} = \frac{n}{c}$. (30) $\frac{l}{a} = \frac{m+n}{b+c}$.

(32) $\tan^{-1} \frac{2 \sin C}{3 \cos A \cos B - \cos C}$.

(34)
$$m\beta + n\gamma - l\alpha = 0,$$
$$n\gamma + l\alpha - m\beta = 0,$$
$$l\alpha + m\beta - n\gamma = 0.$$

(35)
$$m\alpha + mn\beta + \gamma = 0,$$
$$n\beta + nl\gamma + \alpha = 0,$$
$$l\gamma + lm\alpha + \beta = 0.$$

(36) If BC, $B'C'$ intersect in P; CA, $C'A'$ in Q; AB, $A'B'$ in R; P, Q, R will be found to lie on the fourth straight line required.

(37) A particular case of (40), when AP, BQ, CR are perpendicular to the sides of the triangle of reference. See (40).

(38) A particular case of (40), when AP, BQ, CR are the bisectors of the angles of the triangle.

(39) A particular case of (40), when P, Q, R are the middle points of the sides of the triangle of reference. In this case the fourth straight line is the line at infinity.

(40) To construct the lines, let O be the point given by $l\alpha = m\beta = n\gamma$, join OA, OB, OC and produce them to meet BC, CA, AB respectively in P, Q, R; also let BC, QR intersect in P'; CA, RP in Q'; AB, PQ in R'; then three of the straight lines required will be the sides of the triangle PQR, and the fourth will pass through P', Q', R'. The coordinates of the middle points of PP' will be

$$0, \quad \frac{2\Delta \frac{b}{m}}{\frac{b^2}{m^2} - \frac{c^2}{n^2}}, \quad \frac{2\Delta \frac{c}{n}}{\frac{c^2}{n^2} - \frac{b^2}{m^2}},$$

and the coordinates of the middle points of QQ', RR' can be written down by symmetry. These three points lie on the straight line

$$\frac{l\alpha}{a} + \frac{m\beta}{b} + \frac{n\gamma}{c} = 0.$$

(42) The equations to the straight lines joining the point of reference A to the two given points at infinity can be written down, and the condition that they should be at right angles can be reduced to the given form.

(43) The line through A will have the equation

$$b\beta(q-p) + c\gamma(r-p) = 0.$$

(44) $a\alpha(2p-q-r) + b\beta(2q-r-p) + c\gamma(2r-p-q) = 0.$

(45) $(a\alpha - b\beta)\cos A = \gamma(b + a\cos C).$

(46) $2\Delta\alpha + d(a\alpha + b\beta + c\gamma) = 0.$ (47) $4\Delta.$

(48) $\dfrac{4\Delta l^2m^2n^2}{(nl+lm-mn)(lm+mn-nl)(mn+nl-lm)}.$ (49) $\dfrac{\Delta}{4}.$

(53) The straight line is the perpendicular from C on BA.

(55) [For $r=0$, read $r=d$.] Two straight lines will satisfy the conditions, and their equations are

$$d(a\alpha+b\beta+c\gamma)=\pm(a\alpha+b\beta)\,b\sin A.$$

(56) Let s be the altitude required. Then the equation to RQ' is $a\alpha(p+s)+b\beta(q+s)+c\gamma(r+s)=0$. Hence at R the value of β is $\frac{2\Delta}{b}\cdot\frac{p+s}{p-q}$. But at P the value of β is $\frac{2\Delta}{b}\cdot\frac{r}{r-q}$, and the middle point of the diagonal PR lies on the locus of $\beta=0$, therefore $\frac{p+s}{p-q}+\frac{r}{r-q}=0$, which gives

$$s=\frac{2pr-pq-qr}{q-r}.$$

(58) The two paragraphs must be read as separate questions. In the second paragraph, for 'this point,' read

'the point $\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}$.'

(59) $$\frac{l\alpha'+m\beta'+n\gamma'}{l\lambda+m\mu+n\nu}\sim\frac{l'\alpha'+m'\beta'+n'\gamma'}{l'\lambda+m'\mu+n'\nu}.$$

(60) The coordinates of P are $\alpha'+\lambda\rho$, $\beta'+\mu\rho$, $\gamma'+\nu\rho$. So for the other points.

(62) Apply the result of (59).

(63)

$$\tan^{-1}\frac{2\Delta\{(m-n)^2+(n-l)^2+(l-m)^2\}}{(b^2-c^2)(l-m)(l-n)+(c^2-a^2)(m-n)(m-l)+(a^2-b^2)(n-l)(n-m)}.$$

(64) Apply Art. 27. All the straight lines are parallel to

$$x\cos^2\alpha+y\cos^2\beta+z\cos^2\gamma=0.$$

(69) $$\Delta\frac{l+m}{l-m}.$$

(70) $$\frac{\Delta(lmn+1)}{(l+1)(m+1)(n+1)}.$$

(74) $$\begin{vmatrix} u, & v, & w \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0.$$

(75)
$$\begin{vmatrix} -u, & v, & w \\ a, & b, & c \\ b, & a, & c \end{vmatrix} = 0.$$

(81) See Art. 331.

(85) The straight line
$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ f, & g, & h \\ f', & g', & h' \end{vmatrix} = 0.$$

(86) The equation may be written
$$(\alpha \cos A + \beta \cos B + \gamma \cos C)^2 + (\alpha \sin A + \beta \sin B + \gamma \sin C)^2 = 0.$$

(87) If $\frac{\pi}{2} - \theta = B - C$, $\frac{\pi}{2} - \phi = C - A$,

then will
$$\frac{\pi}{2} - \psi = A - B,$$

and the equation may be written
$$(u \cos A + v \cos B + w \cos C)^2 + (u \sin A + v \sin B + w \sin C)^2 = 0.$$

(88) The equation may be written
$$(x^2 + y^2 + z^2 + 2yz)(x^2 + y^2 + z^2 + 2zx)(x^2 + y^2 + z^2 + 2xy) = 0,$$
each factor of which, when equated to zero, represents a pair of imaginary straight lines parallel to a line of reference.

(90) If $l\alpha = m\beta = n\gamma$ be the point O, the points of intersection lie on the straight line
$$l\alpha + m\beta + n\gamma = 0.$$

(91) See Art. 129.

(92) Use one of the equations of Art. 108.

(93), (94) Apply Art. 130.

(95), (96), (97) Apply Art. 125.

(100) One system of lines satisfying the required conditions has the equations
$$\alpha = 0, \quad \frac{\beta}{\mu} = \frac{\gamma}{\nu}, \quad \frac{2\alpha}{\lambda} + \frac{\beta}{\mu} + \frac{\gamma}{\nu} = 0.$$

(101) Form the determinant as in Art. 149. The sum of three rows will be identically equal to the sum of the other three. Hence the determinant vanishes and the condition is fulfilled.

(102) See Art. 97.

(103) $$3(x-y)^2 - 4z(x+y) + z^2 = 0.$$

(104) Apply Art. 149. The conic is the circle of Art. 307.

(105) The straight lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0,$$

are asymptotes and their point of intersection the centre.

(106) Three parabolas.

(107) The four points in which the two straight lines $x = \pm y$ cut the two straight lines $x \pm 3y = -1$.

(110) See Arts. 161 and 129.

(114) If $(\alpha', \beta', \gamma')$ be the coordinates of P, the three tangents have the equations

$$\frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = \frac{2\alpha}{\alpha'}, \quad \frac{\gamma}{\gamma'} + \frac{\alpha}{\alpha'} = \frac{2\beta}{\beta'}, \quad \frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} = \frac{2\gamma}{\gamma'}.$$

(116) With the notation of Art. 91, the three conics have the equations

$$u^2 = v^2 + w^2, \quad v^2 = w^2 + u^2, \quad w^2 = u^2 + v^2.$$ See (119).

(117) A conic with respect to which two particular triangles are self-conjugate.

(120) Two imaginary straight lines dividing the right angle harmonically.

(123) The resulting equation should be

$$(l\kappa\alpha + m\beta)^2 + (l\kappa^2 + m)\, n\gamma^2 = 0.$$

(124) $$\left(\frac{f^2}{l} + \frac{g^2}{m} + \frac{h^2}{n}\right)(l\alpha^2 + m\beta^2 + n\gamma^2) - (f\alpha + g\beta + h\gamma)^2 = 0.$$

(126) $$a^2yz + b^2zx + c^2xy = \Delta\,(x^2 \cot A + y^2 \cot B + z^2 \cot C).$$

(130) Either condition is $\sqrt{l\lambda}+\sqrt{m\mu}+\sqrt{n\nu}=0$.

(132) If $$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0 \text{ be the conic,}$$

$$\frac{\alpha}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=0 \text{ is the straight line.}$$

(134) Take the triangle as triangle of reference, and let (x', y', z') be the fixed point. The centre will lie on the conic

$$\frac{x'^2}{x}+\frac{y'^2}{y}+\frac{z'^2}{z}=0.$$

(147) The polar of the point $(\alpha', \beta', \gamma')$ with respect to

$$(l\alpha+m\beta+n\gamma)(l'\alpha+m'\beta+n'\gamma)=0,$$

is given by

$$(l\alpha'+m\beta'+n\gamma')(l'\alpha+m'\beta+n'\gamma)$$
$$+(l'\alpha'+m'\beta'+n'\gamma')(l\alpha+m\beta+n\gamma)=0,$$

which proves the proposition (Art. 88).

(148) Apply Arts. 46 and 146.

(149) Let $\alpha=0$ be one of the straight lines: let the others be parallel to $\beta=0$, and let $\gamma=0$ be the straight line at infinity. Then the equation to the locus can be written

$$l\alpha^2+m(\beta+k\gamma)^2+m'(\beta+k'\gamma)^2+\ldots=n\gamma^2,$$

or $$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma=0,$$

which shews that $\alpha=0$ is the chord of contact of tangents from $\beta=0$, $\gamma=0$, i.e. of tangents parallel to $\beta=0$. Which proves the proposition.

(150) Apply a method similar to that of (149).

(151) Let $\alpha=0$ be the chord of contact, $\beta=0$ the tangent to the first conic, $\gamma=0$ the straight line joining the point of contact of this tangent to one of the points of contact of the conics. Then the equation to the first conic may be written

$$l\gamma^2+m\alpha\beta+n\beta\gamma=0,$$

and the equation to the other will be

$$k\alpha^2 + l\gamma^2 + m\alpha\beta + n\beta\gamma = 0,$$

which meets $\beta = 0$ in two points given by

$$k\alpha^2 + l\gamma^2 = 0,$$

which therefore divide harmonically the line joining the point of contact of the tangent with the point where the tangent meets the common chord.

(152) A conic circumscribing the triangle which is self-conjugate with respect to both the conics.

(156) The equation to the locus is

$$\{f(\alpha, \beta, \gamma)\}^2 = k\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\}^2.$$

(157) If $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ be the first conic, the fixed point is given by $l\alpha + m\beta = 0$, $\gamma = 0$.

(161) See Art. 417 (first column).

(164) Apply Art. 65. (165) Apply Art. 65.

(166) If λ, μ, ν be the direction sines of the chords, and h^2 the constant area, the equation to the locus is

$$f(\alpha, \beta, \gamma) + h^2 f(\lambda, \mu, \nu) = 0,$$

which may be rendered homogeneous.

(167) If the constant be $\frac{1}{h}$, the equation to the locus will be

$$f(\alpha, \beta, \gamma) + h\left(\alpha\frac{df}{d\lambda} + \beta\frac{df}{d\mu} + \gamma\frac{df}{d\nu}\right) = 0,$$

which may be rendered homogeneous.

(168) Refer the conic to a self-conjugate triangle having one vertex at the focus.

(172) Tan $\frac{\theta}{2} = \mathfrak{B} : \mathfrak{A}$. Apply Art. 285.

(174) To deduce Euclid III. 31. Apply Art. 159.

(177) Take the given centres as points of reference.

(181) In trilinear coordinates:

(i) $\alpha \sin A (\beta \sin B + \gamma \sin C) = (\beta \cos B - \gamma \cos C^2)$,

(ii) $2\alpha \cos^2 \frac{A}{2} (\beta + \gamma) = (\beta - \gamma) (\beta \cos B - \gamma \cos C)$,

(iii) $\alpha (c\beta + b\gamma) + \alpha \cos A (b\beta + c\gamma)$
$= (b\beta - c\gamma) (\beta \cos B - \gamma \cos C)$,

and similar equations.

(182) If the area of the given rectangle be 2μ times that of the triangle, the equation to the locus may be written

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C$$
$$= \mu (\alpha \sin A + \beta \sin B + \gamma \sin C)^2,$$

shewing a circle concentric with the circle circumscribing the triangle.

(183) If the given constant be 2μ times the area of the triangle the locus will be represented by the equation

$$(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) \sin A \sin B \sin C$$
$$= \mu (\alpha \sin A + \beta \sin B + \gamma \sin C)^2.$$

(184) Apply Arts. 210, 289, 314.

(185) In general the equation

$$\left\{\frac{df}{d\alpha}, \frac{df}{d\beta}, \frac{df}{d\gamma}\right\}^2 = 0$$

represents the polars of the circular points with respect to the conic $f(\alpha, \beta, \gamma) = 0$. If the latter be a circle, the polars are tangents and pass through the circular points. Hence, Art. 318, the equation represents an indefinitely small circle. But since the polars of points at infinity intersect at the centre of the curve, this indefinitely small circle is at the centre, or is *concentric* with the given circle.

(186) Apply a method analogous to that of Art. 273.

(188) Any conic touching the sides $\alpha=0$, $\beta=0$, $\gamma=0$ is known to have the equation

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2-2mn\beta\gamma-2nl\gamma\alpha-2lm\alpha\beta=0.$$

The straight line $\delta=0$ or $\alpha+\beta+\gamma=0$ will be a tangent (Art. 210) provided $l+m+n=0$. Eliminate the terms involving α^2, β^2, γ^2 (Art. 327) and we have

$$(m+n)^2\beta\gamma+(n+l)^2\gamma\alpha+(l+m)^2\alpha\beta+l^2\alpha\delta+m^2\beta\delta+n^2\gamma\delta=0,$$

or in virtue of

$$l+m+n=0,\quad l^2(\beta\gamma+\alpha\delta)+m^2(\gamma\alpha+\beta\delta)+n^2(\alpha\beta+\gamma\delta)=0.$$

And we may write

$$l=\mu-\nu,\quad m=\nu-\lambda,\quad n=\lambda-\mu.$$

(189) Apply the condition that the line at infinity should be a tangent. A solution is given at length in Vol. I. of the *Messenger of Mathematics*, p. 201.

(190) This may be deduced from (189) by writing

$$\lambda=1,\quad \lambda'=k,\quad \mu=\mu'=\nu=\nu'=0.$$

(191) The two roots of the quadratic in (190) cannot be equal unless either $A=B$ and $C=D$, or else $A=C$ and $B=D$. In either of these cases the parabola would be altogether at infinity. In any other case there can therefore be two parabolas drawn through four fixed points, one of which will however degenerate into two parallel straight lines if the points lie on two such lines.

(192) In virtue of the relation

$$\alpha+\beta+\gamma+\delta\equiv 0,$$

the equation may be written

$$\lambda(\alpha+\beta)(\alpha+\gamma)+\mu(\beta+\gamma)(\beta+\alpha)+\nu(\gamma+\alpha)(\gamma+\beta)=0,$$

which shews that its locus circumscribes the triangle whose sides are $\beta+\gamma=0$, $\gamma+\alpha=0$, $\alpha+\beta=0$.

(195) Apply the result of (194). If $(\alpha', \beta', \gamma', \delta')$ be the point 0, the two tangents are given by

$$\frac{\alpha}{\alpha'} - \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} - \frac{\delta}{\delta'} = 0,$$

and

$$\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} - \frac{\delta}{\delta'} = 0,$$

which form a harmonic pencil (Art. 124) with the lines

$$\frac{\alpha}{\alpha'} - \frac{\delta}{\delta'} = 0 \text{ and } \frac{\beta}{\beta'} - \frac{\gamma}{\gamma'} = 0.$$

(196) Let both conics circumscribe the quadrilateral of reference, and let $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ be the two points of contact of a common tangent. Then the equations to the two conics are $\frac{\beta\gamma}{\beta_1\gamma_1} = \frac{\alpha\delta}{\alpha_1\delta_1}$ and $\frac{\beta\gamma}{\beta_2\gamma_2} = \frac{\alpha\delta}{\alpha_2\delta_2}$, and the equations to the straight lines can be readily formed.

(197) Apply a method analogous to that of (196).

(200) One of the points is given by $\alpha = \beta = \gamma$. The other two are the points in which $\delta = 0$ meets the fourteen-points' conic.

(215) If $p = 0$, $q = 0$, $r = 0$, $lp + mq + nr = 0$ are the equations to the angular points in order, the middle points of the diagonals lie upon the straight line given by

$$(n - l)\,p = (l + 2m + n)\,q = (l - n)\,r.$$

(217) By Art. 393 the centre is at the point

$$p \tan A + q \tan B + r \tan C = 0,$$

which by Ex. (211) is the centre of the circle with respect to which the triangle is self-conjugate (Art. 179, Cor. 2).

(218), (219), (220) Apply a method similar to that of Ex. (217).

(221) See Ex. (214) and apply Art. 380. The second form of the equation shews that the circle circumscribes the triangle whose angular points are $q + r = 0$, $r + p = 0$, $p + q = 0$.

(227) The angular points of a triangle co-polar with the triangle of reference may be expressed by the equations

$$\lambda p = mq + nr, \quad \mu q = nr + lp, \quad \nu r = lp + mq;$$

the equation to the conic is then

$$\lambda lp^2 + \mu mq^2 + \nu nr^2 + (\mu\nu + mn)\, qr + (\nu\lambda + nl)\, rp + (\lambda\mu + lm)\, pq = 0.$$

(236) See Euclid III. 32.

(237) See Euclid III. 20 and 22.

(239) Any tangent to the interior of two concentric circles is bisected at the point of contact.

(240) A circle can be described touching the escribed and inscribed circle of a triangle.

(241) Reciprocate with respect to the focus.

(242) Reciprocate with respect to S.

(243) Tangents drawn from the point of reciprocation are at right angles.

(245) Reciprocate 244.

(246), (247) A series of confocal conics may be reciprocated into a system of circles with the same radical axis.

(248) Reciprocate with respect to O.

(249) The locus of the intersection of tangents to a conic which are at right angles is a circle.

(250) See Euclid III. 10.

(251) Reciprocate with respect to S.

(252) Reciprocate (110).

(253) Reciprocate with respect to any point.

(254) Extend Exercise (132), applying Art. 95. Then reciprocate.

(266) That the centre lies on a given straight line is equivalent to saying that the given straight line and the straight line at infinity are conjugate. Hence this is a line condition.

(269) The double point $(u=0,\ v=0)$.

(270) The double tangent $(u=0,\ v=0)$.

(271) At the point $(u=0,\ v=0)$.

(272) The six points $(l:m:n)$, $(m:n:l)$, $(n:l:m)$, $(n:m:l)$, $(l:n:m)$, $(m:l:n)$.

(275) If $S=0$ be the equation to one cubic, and $u=0$ the straight line joining the points of contact, the equation to the cubic will be $S+u^2v=0$, where $v=0$ represents a straight line on which the other points of intersection lie.

(276) Form the equations as in (275).

(277) See Art. 481. (279) Apply (275).

(280) The equation to the curve is

$$\alpha\,(P\beta^2+Q\beta\gamma+R\gamma^2)=\beta\gamma\,(N\beta+M\gamma).$$

(282) The equation to the cubic is

$$(l\alpha+m\beta+n\gamma)\,(m\alpha+n\beta+l\gamma)\,(n\alpha+l\beta+m\gamma)$$
$$=lmn\,(\lambda\alpha+m\beta+\nu\gamma)\left(\frac{\alpha}{\lambda}+\frac{\beta}{\mu}+\frac{\gamma}{\nu}\right)(\alpha+\beta+\gamma).$$

(283) The equation to the cubic is

$$(l^2\alpha-\beta-\gamma)\,(m^2\beta-\gamma-\alpha)\,(n^2\gamma-\alpha-\beta)$$
$$=(\alpha+\beta+\gamma)\,(l\alpha+m\beta+n\gamma)^2.$$

(286) The six points

$(a:b:c)$, $(b:c:a)$, $(c:a:b)$, $(c:b:a)$, $(a:c:b)$, $(b:a:c)$

on a conic, and the three points in which the straight line $x+y+z=0$ cuts the lines of reference.

(288) A particular case of the next exercise.

(289) If $\alpha = 0$ be the asymptote, the cubic will have the equation $\alpha . f(\alpha, \beta, \gamma) = (l\alpha + m\beta + n\gamma)(\alpha + \beta + \gamma)$ in triangular coordinates, where $f(\alpha, \beta, \gamma) = 0$ is the equation to a conic. The equation to the required locus will be

$$\alpha\left(\frac{df}{d\beta} - \frac{df}{d\gamma}\right) = (m - n)(\alpha + \beta + \gamma)^2,$$

which represents a hyperbola. In the particular case when $l = m = n$, the locus reduces to the straight line

$$\frac{df}{d\beta} = \frac{df}{d\gamma}.$$

(290) If $x = 0$, $y = 0$, $z = 0$ be the tangents, and $u = 0$ the line of contact, the equation must take the form of Art. 483.

(292) Using the equation of Art. 472, the chords are represented by

$$n_2\beta + m_3\gamma = 0, \quad l_3\gamma + n_1\alpha = 0, \quad m_1\alpha + l_2\beta = 0,$$

and the tangents by

$$m_1\beta + n_1\gamma = 0, \quad n_2\gamma + l_2\alpha = 0, \quad l_3\alpha + m_3\beta = 0.$$

The condition that either system should be concurrent is

$$l_2 m_3 n_1 + l_3 m_1 n_2 = 0.$$

(304) Apply the last result.

(306) The coefficients λ, μ, ν cannot be all of one sign. The real or imaginary tangents at the points of reference have the equations

$$\frac{\beta^2}{\mu} + \frac{\gamma^2}{\nu} = 0, \quad \frac{\gamma^2}{\nu} + \frac{\alpha^2}{\lambda} = 0, \quad \frac{\alpha^2}{\lambda} + \frac{\beta^2}{\mu} = 0.$$

(307), (308) The equation may be written

$$\frac{1}{\frac{l}{\alpha}-\frac{m}{\beta}-\frac{n}{\gamma}}+\frac{1}{\frac{m}{\beta}-\frac{n}{\gamma}-\frac{l}{\alpha}}+\frac{1}{\frac{n}{\gamma}-\frac{l}{\alpha}-\frac{m}{\beta}}=0.$$

(311) Take the equation of (309) and apply the condition that each equation of (310) may represent a pair of coincident straight lines.

INDEX.

[The numerals refer to pages.]

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pages 390
through
425

if	possible