[redacted]

21

$$\sum_{n=1} (\mathcal{NL})_n$$

COMPENDIUM NEWSLETTER

second edition

UICSM

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Why a Newsletter Compendium?

The UICSM Project Office has been publishing its Newsletter at periodic intervals since October, 1960. Originally this publication was thought of as a supplement to, and amplification of, the teacher commentary pages of the various units. Since then, this has been magnified somewhat. Now we feel that the Newsletter fits into our greatly expanded project as a sounding board for the new materials which are being discussed by our staff for use at a multiplicity of levels. Many of the later articles have no bearing on the materials currently in print, but they might be the basis of units of work in future materials. We hope that there will be some reader response to the types of things that are mentioned in Newsletters as new thoughts on the part of our creative staff. The response of teachers to our materials has always been a source of clarification of ideas for the UICSM Project.

Requests for back issues of Newsletters have become such a regular feature of our daily process at UICSM, that we have felt compelled to put the material contained in those Newsletters in a more convenient and up-to-date form. The result of this compulsion is this Compendium of Newsletter articles. Any future requests for back issues will result in the sending of this compendium.

We hope that this will be fulfilling a need on the part of many of our readers. If the reaction to this volume is favorable, we will try to keep it current by adding articles from future Newsletters to the Compendium on a regular basis.

The material in this compendium is from the following Newsletters:

Number	Date
1	October, 1960
2	November, 1960
3	January, 1961
4	March, 1961
5	May, 1961
6	October, 1961
7	April, 1962
8	May, 1962
9	November, 1962
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17	April, 1965
18	December, 1965
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20	November, 1966
21	May, 1967

On the Solution of a Special Trigonometric Equation

A problem of some interest to teachers of trigonometry (and to their students) is the solution of:

(1) $$a \cos x + b \sin x = c$$

[for a and b not both zero].

The purpose of this note is to give a method of solving (1) and to investigate the choices of a, b, and c which yield a solution for (1).

We are assuming that not both a and b are zero. Hence it follows that $\sqrt{a^2 + b^2} \neq 0$. Also since

$$\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1$$

it follows that $\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right)$ belongs to the unit circle.

For each point (c, d) of the unit circle, there is just one number θ such that

(2) $0 \leq \theta < 2\pi$, and

(3) $\cos \theta = \frac{c}{\sqrt{c^2 + d^2}}$ and $\sin \theta = \frac{d}{\sqrt{c^2 + d^2}}$.

In our development it will be convenient to have a more direct method of calculating θ. Recall that

(4) $$\theta = \begin{cases} \operatorname{Arccos} \frac{c}{\sqrt{c^2 + d^2}}, & d \geq 0 \\ 2\pi - \operatorname{Arccos} \frac{c}{\sqrt{c^2 + d^2}}, & d < 0. \end{cases}$$

Returning now to the solution of (1), we first transform it to:

(5) $$\sqrt{a^2 + b^2} \left[\cos x \cdot \frac{a}{\sqrt{a^2 + b^2}} + \sin x \cdot \frac{b}{\sqrt{a^2 + b^2}}\right] = c$$

We had previously established that for each pair of numbers a, b, not both zero, $\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right)$ belongs to the unit circle and that for this pair there is a unique θ such that, $0 \leq \theta < 2\pi$,

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos\theta, \text{ and } \frac{b}{\sqrt{a^2 + b^2}} = \sin\theta$$

Substituting in (5), we get:

(6) $\sqrt{a^2 + b^2}\ [\cos x \cos \theta + \sin x \sin \theta] = c$

The form of (6) suggests using the Subtraction Law for Cosines:

(7) $\forall_x \forall_y \cos(x - y) = \cos x \cos y + \sin x \sin y$,

to simplify the left member. Since $\sqrt{a^2 + b^2} \neq 0$, it follows from (6) and (7) that

(8) $\cos(x - \theta) = \dfrac{c}{\sqrt{a^2 + b^2}}$.

At this point we see that, if there are to be solutions to (1), we need a restriction on 'c'. If $|c| > \sqrt{a^2 + b^2}$ we would have $|\cos(x - \theta)| > 1$, which is impossible. Hence, we will have solutions of (1) only if $|c| \leq \sqrt{a^2 + b^2}$.

It is well known that, for $|y| \leq 1$, $\cos x = y$ if and only if

(9) $x = 2k\pi \pm \text{Arccos } y$, for some integer k.

(Equation (9) and its derivation can be found in UICSM Unit 10, page 125.)

The solution of (8) follows immediately from (9). We find that

(10) $x - \theta = 2k\pi \pm \text{Arccos } \dfrac{c}{\sqrt{a^2 + b^2}}$

and finally that

(11) $x = 2k\pi + \theta \pm \text{Arccos } \dfrac{c}{\sqrt{a^2 + b^2}}$

where θ is given by (4).

The solutions of (1) with the restrictions a and b not both 0 and $|c| \leq \sqrt{a^2 + b^2}$ are given by (11).

We end this note with two examples of the solution of (1).

Example 1. $\cos x + \sin x = 1$

Here $a = b = c = 1$ and $\sqrt{a^2 + b^2} = \sqrt{2}$.

Therefore, $c < \sqrt{a^2 + b^2}$.

We can write the original equation as:

$$\sqrt{2}\,[\cos x \cdot \frac{1}{\sqrt{2}} + \sin x \cdot \frac{1}{\sqrt{2}}] = 1$$

Here, $\theta = \frac{\pi}{4}$, and so this equation becomes:

$$\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

or:

$$\cos (x - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Therefore, $x - \frac{\pi}{4} = 2k\pi \pm \operatorname{Arccos} \frac{1}{\sqrt{2}} = 2k\pi \pm \frac{\pi}{4}$

and $x = 2k\pi + \frac{\pi}{4} \pm \frac{\pi}{4}$.

That is $x = 2k\pi$ or $x = 2k\pi + \frac{\pi}{2}$.

We may solve the same equation directly by using (4) and (11).

Since $a = b = 1$,

$$\theta = \operatorname{Arccos} \frac{a}{\sqrt{a^2 + b^2}} = \operatorname{Arccos} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

and $\operatorname{Arccos} \frac{c}{\sqrt{a^2 + b^2}} = \operatorname{Arccos} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

Therefore,

$$x = 2k\pi + \frac{\pi}{4} \pm \frac{\pi}{4} .$$

That is $x = 2k\pi$ or $x = 2k\pi + \frac{\pi}{2}$, k an integer.

Example 2. $\sqrt{3} \cos x - 1 \sin x = 2$

Here, $a = \sqrt{3}$, $b = -1$, $c = 2$. So, by (4),

$$\theta = 2\pi - \text{Arccos}\, \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + (-1)^2}}$$

$$= 2\pi - \text{Arccos}\, \frac{\sqrt{3}}{2}$$

$$= 2\pi - \frac{\pi}{6}$$

$$= \frac{11\pi}{6},$$

and $\text{Arccos}\, \frac{c}{\sqrt{a^2 + b^2}} = \text{Arccos}\, \frac{2}{\sqrt{(\sqrt{3})^2 + (-1)^2}}$

$$= \text{Arccos}\ 1$$

$$= 0.$$

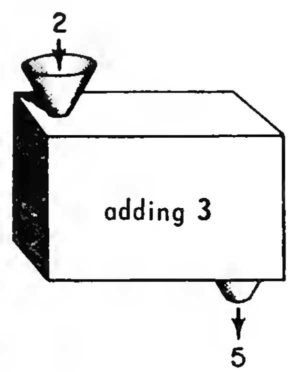
Therefore, by (11), $x = 2k\pi + \frac{11\pi}{6}$, k an integer.

J. Mueller and H. E. Vaughan
Newsletter 13

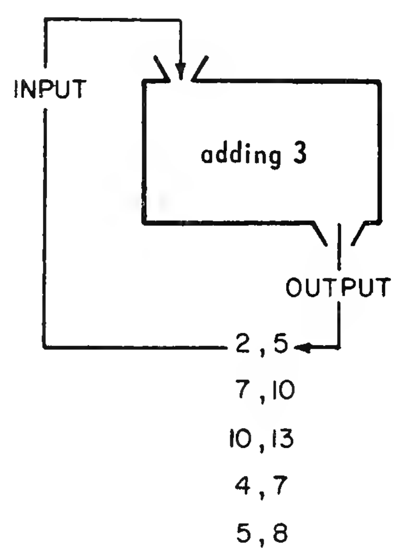
Animated Functional Notation

While working on UICSM-PIP (Programed Instruction Project) materials, we came up with a device which seems useful for a variety of purposes. Its general usefulness resides in the fact that it suggests concepts and does so without the use of much talk. Thus, it can be used to stimulate lots of student discovery.

In essence, the device is nothing more than an animated functional notation. For example, consider the function adding 3. You can show that 5 is the value of this function for the argument 2 just by the picture:



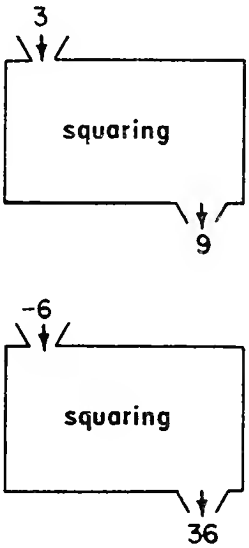
It is very easy to use this device to get across the idea that a function is nothing more than a set of ordered pairs. Just have students keep a record of the "input" numbers and the "output" numbers.



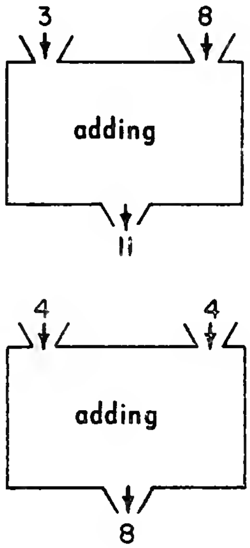
Singulary and Binary Operations

Just as this notation can be used to deal with singulary operations ("functions of one variable"), it can also be used for binary operations ("functions of two variables"). One obvious advantage of this notation is the way it emphasizes the difference between a singulary operation and a binary operation.

Squaring is a singulary operation. It is performed on single numbers.



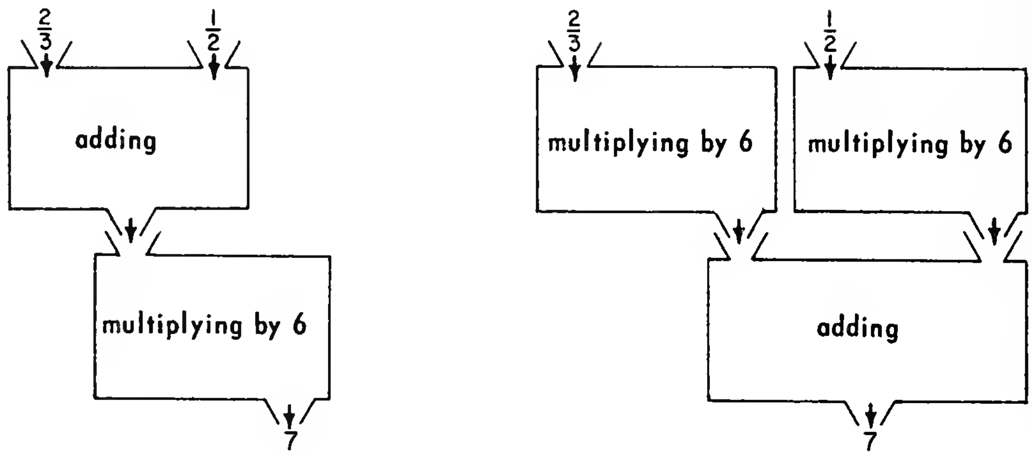
Adding is a binary operation. It is performed on pairs of numbers.



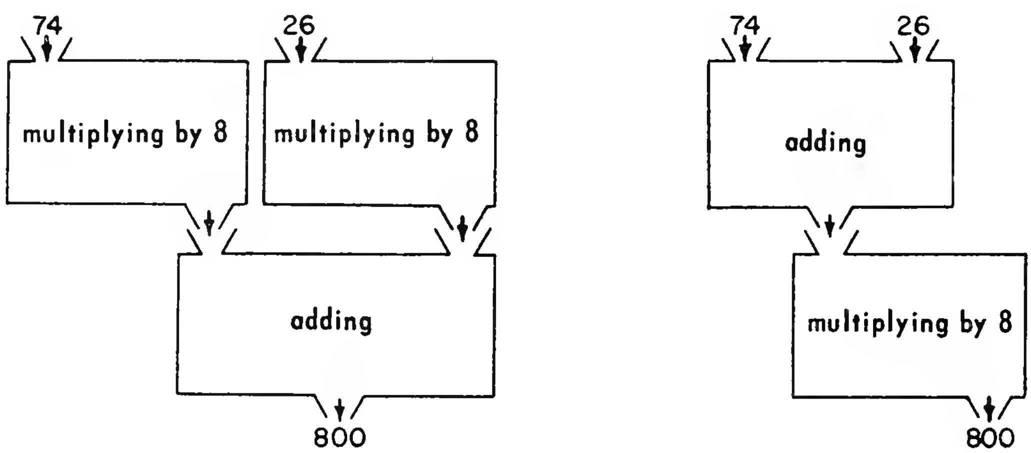
A notation such as '(7 + 3) - 3 = 7' does not distinguish the singulary operation adding 3 from the binary operation adding. Nor does it distinguish subtracting 3 from subtracting.

Illustrating Some of the Basic Principles

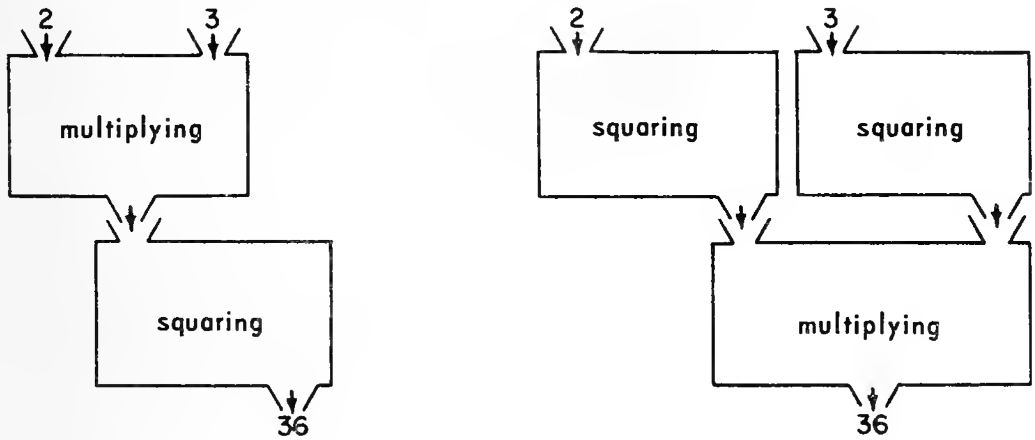
The distributive principle for multiplication over addition can be illustrated nicely using this notation.



Notice how the pictures give meaning to: "distributing" multiplying by 6 "over" addition. Of course, the order of the two pictures may be reversed. In either case, the concept of a changed "principal operator" is made prominent (see "Principal Operator" and "Another Use for Principal Operator" in this compendium).



Other distributive principles could be investigated by using similar pictures. For instance, is squaring distributive over multiplication?



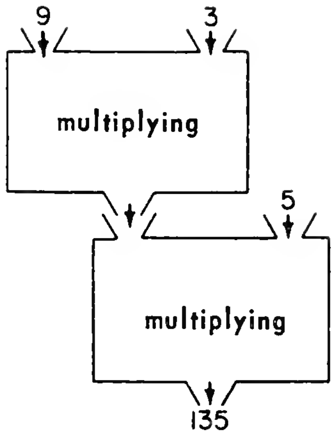
Is division distributive over subtraction?
Is reciprocating distributive over division?
Is oppositing distributive over addition?
Is absolute valuing distributive over multiplication?

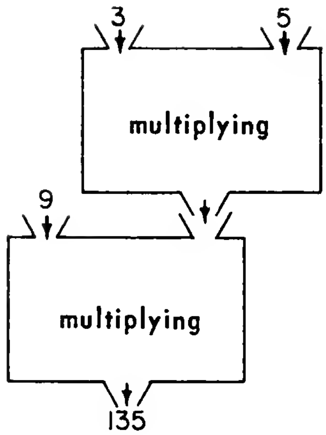
Also, one should probably consider such questions as:

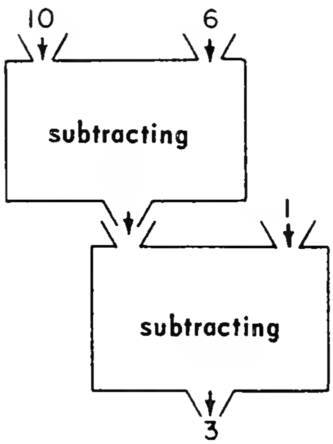
Is addition distributive over multiplication?
Is absolute valuing distributive over addition?
Is oppositing distributive over multiplication?
Is squaring distributive over addition?

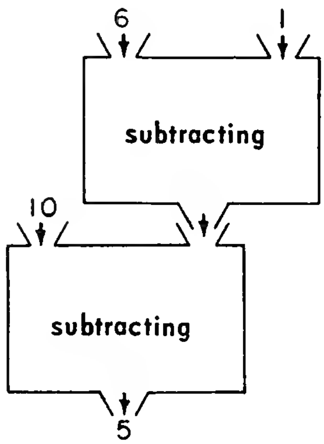
While investigating other distributive principles an interesting subtlety may come to light. That is: can a binary operation be distributive over a binary operation? The pictures definitely help resolve this. Even though the word 'multiplication' in the name of the dpma may suggest that the binary operation multiplication is distributive over the binary operation addition, a glance at one of our pictures will dispel such a misconception. In fact, one might properly think of that principle as the distributive principle for <u>multiplying-by-a-number</u> over addition. The notation '$(\frac{2}{3} + \frac{1}{2}) \times 6$' does not clearly distinguish between the binary operation multiplication and the singulary operation multiplying by 6. A study of the other distributive principles should convince one that each refers to a singulary operation distributed over a binary operation. For instance, oppositing, a singulary operation, is distributive over addition, a binary operation.

You can use a pair of pictures to illustrate easily an instance of the associative principle for multiplication, or another pair to show that subtraction is <u>not</u> associative.


9
3
multiplying
5
multiplying
135

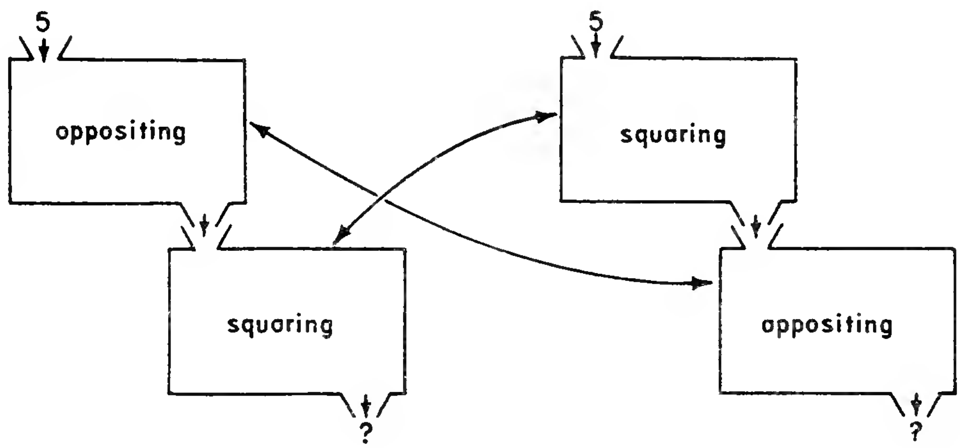

3
5
multiplying
9
multiplying
135


10
6
subtracting
1
subtracting
3


6
1
subtracting
10
subtracting
5

Order of Operations

The usefulness of the device in eliminating problems dealing with order of operations has already been touched upon in the discussions of associativity and distributivity. A more general use, in connection with this topic, is in getting students to discover a need for an order-of-operation convention. Certainly, the "output" numbers for the machines pictured below are different. And, students will have no difficulty in telling what the output numbers are.



So, it is clear that we don't wish '-5^2' to name both output numbers. Which one, then, is it to name? This calls for a convention. There already is such a convention in use. Under this convention, $-5^2 = -(5^2)$. Hence, $-5^2 \neq (-5)^2$.

More generally, if the grouping is not specified then squaring is performed before oppositing.

Similarly, one can consider conventions for expressions such as:

$$7 + 3 \times 2 \qquad 19 - 12 - 5$$

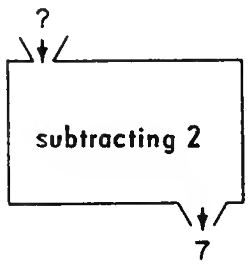
$$3 \times 5^2 \qquad \log 1000^2$$

$$32 \div 8 \div 2 \qquad 4^{3^2}$$

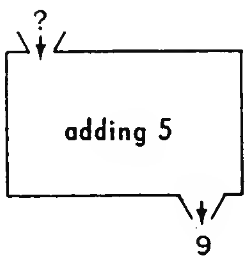
Inverse of A Singulary Operation

This animated functional notation is useful in emphasizing that an inverse operation "gets you back to where you started". The word 'inverse' need not be introduced until after the concept has been developed. Also, it is an easy task to get students to discover this concept for themselves as a short cut. Consider the sequences of the exercises below and on the next two pages. Additional exercises could be used as needed.

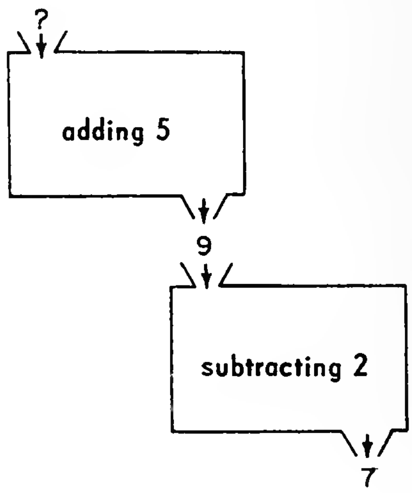
(1)



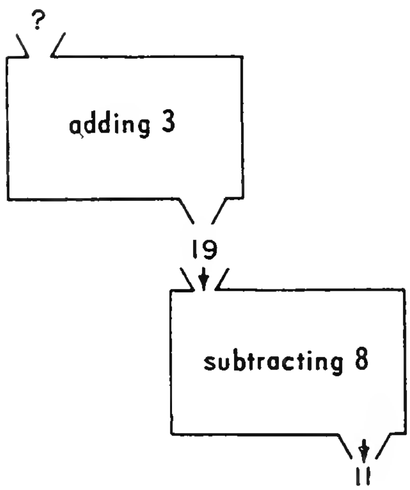
(2)



(3)

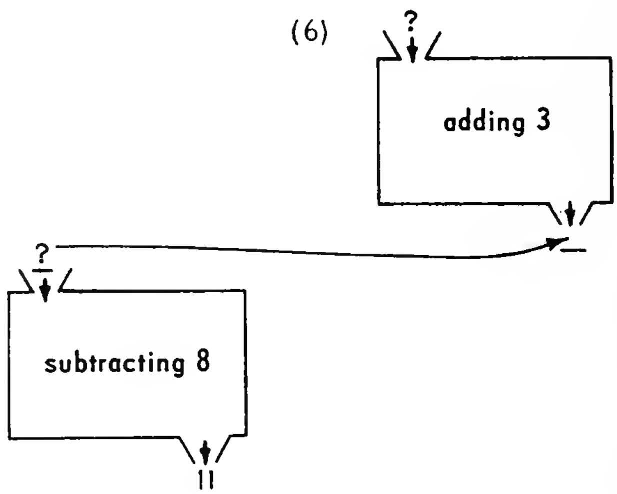


(4)

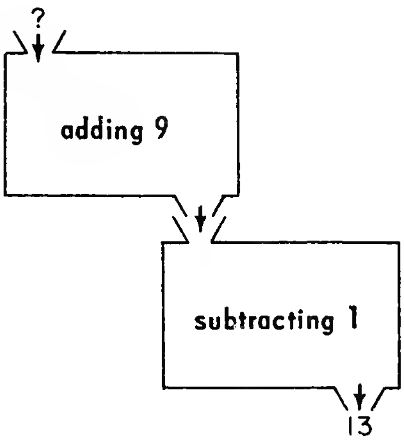


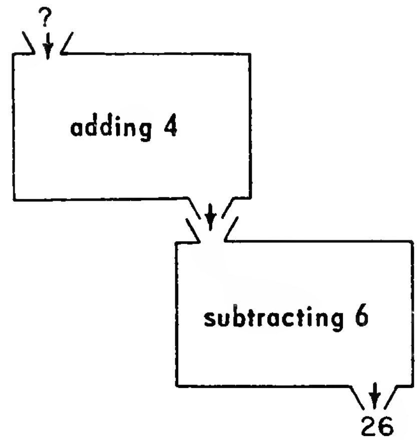
(5)

(6)

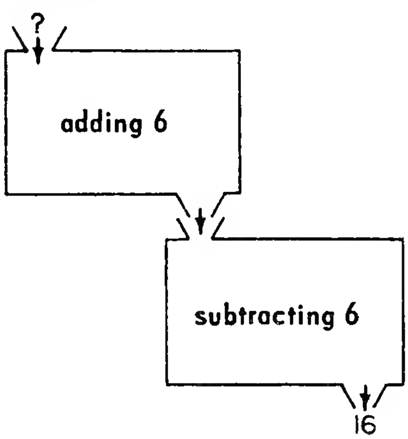


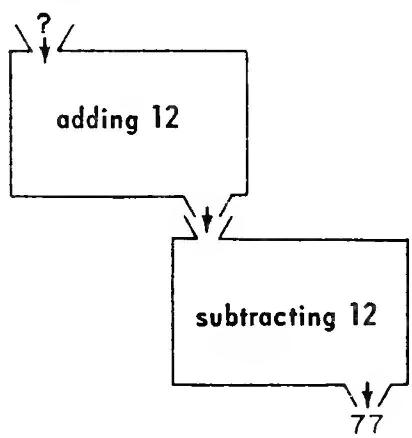
Exercises like the preceding ones should prepare students to do this kind:



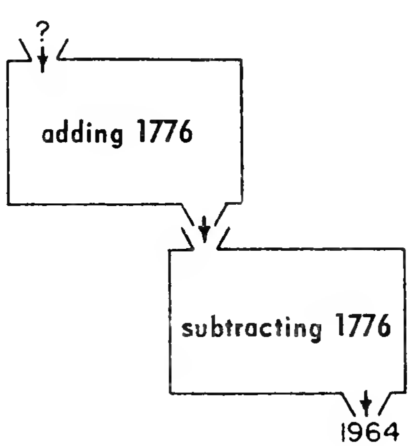


Then a teacher could present this type with increasing frequency:

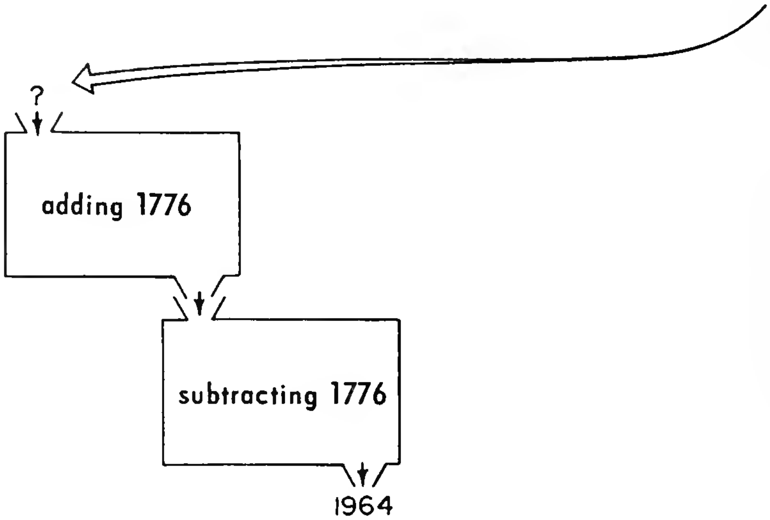




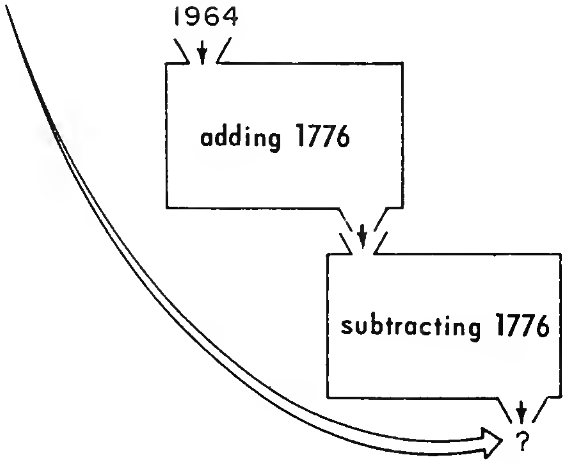
Soon he may insert:



to help force a student to look for a short cut. Note that this more difficult type of exercise is suggested:

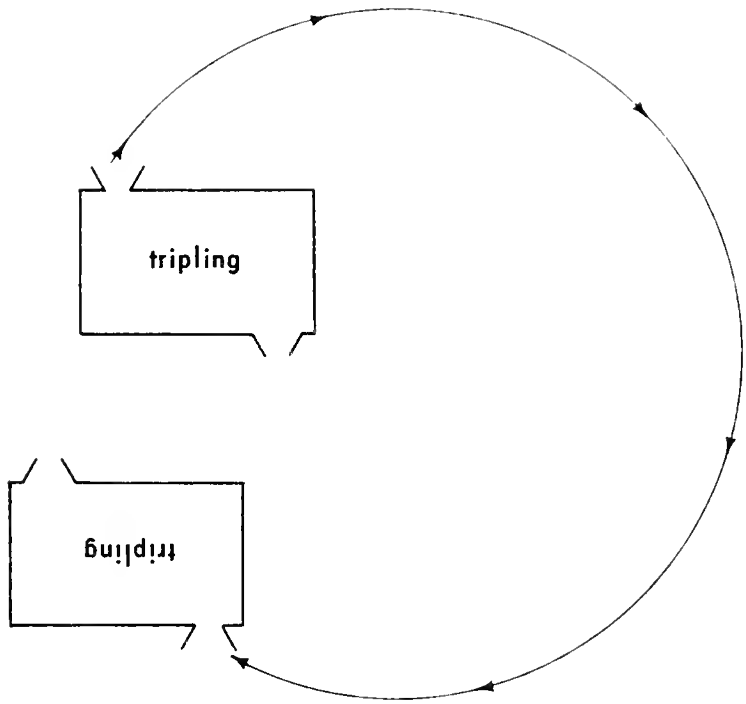


rather than one like:

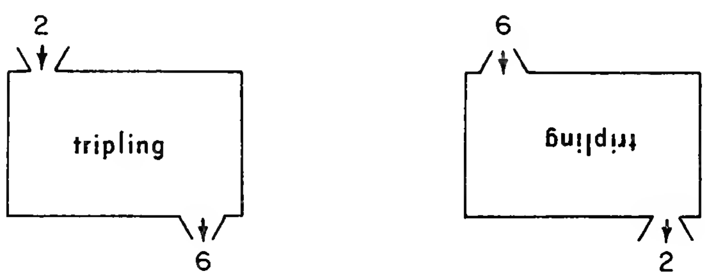


because discovering a short cut for a difficult task is more rewarding to a student.

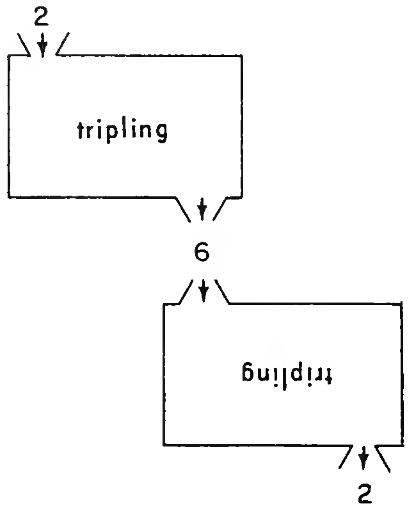
We have seen that this type of function notation clearly illustrates the concept of pairs of first and second numbers (ordered pairs). It announces in a picturesque manner those pairs which belong to a certain operation. By inverting a picture of the machine you get a vivid aid for studying inverse operations.



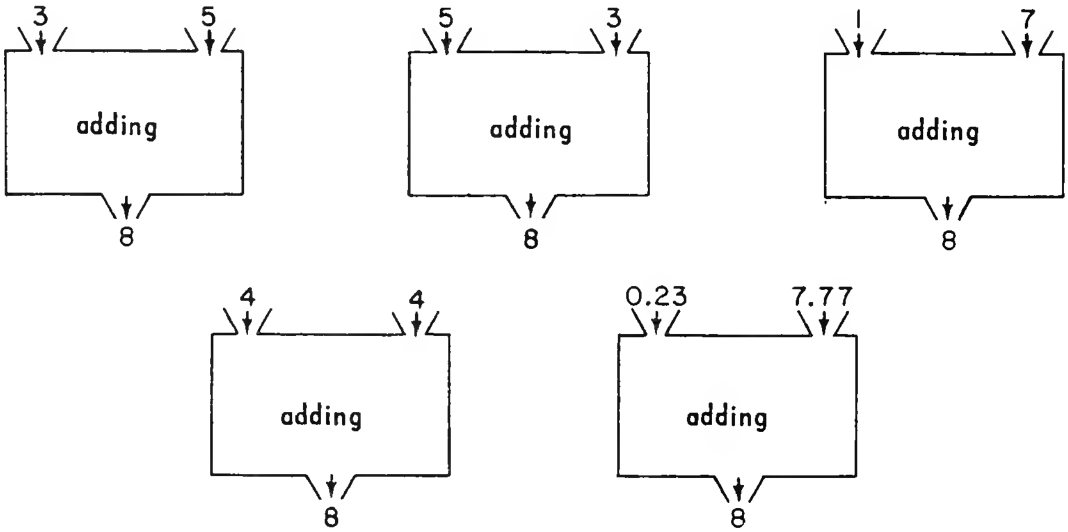
For instance, whereas the pair (2, 6) belongs to tripling, the pair (6, 2) belongs to the inverse of tripling.



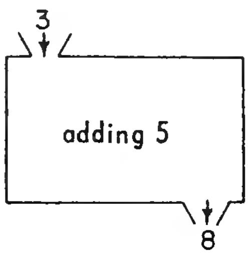
It is, of course, important to know that if a pair belongs to an operation having an inverse then the "reverse" of that pair belongs to the inverse of the operation. Note that the use of the inverted diagram is consistent with our previous meaning of 'inverse operation' — getting back to where you started.



As was pointed out earlier, addition is an operation on pairs of numbers whereas adding 3 is an operation on single numbers. It is interesting to note that there are many pairs of numbers the sum of whose components is 8.



However, there is but one number which when 5 is added to it will give the sum 8.



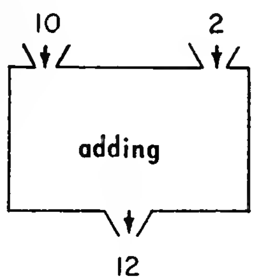
This accounts for the fact that adding 3 has an inverse, whereas adding does not. The following two problems help emphasize this point.

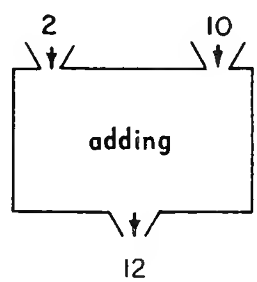
(a) Lon picked a number, added 3 to it, and got 10. What number did Lon pick?

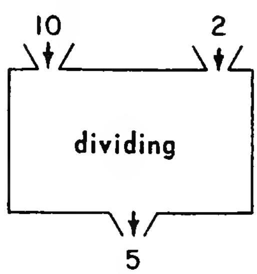
(b) Don picked a number, then he added a second number to it and got 10. What was Don's first number and what was his second number?

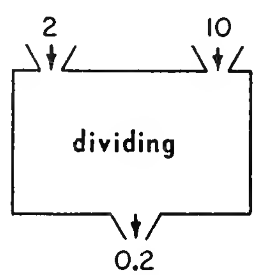
It should be easy for you to "puzzle out" (a). However, if you can do (b), you are probably a mind reader.

With this device it is easy to illustrate the concept of commutativity and show, for instance, that while addition is commutative, division is not.

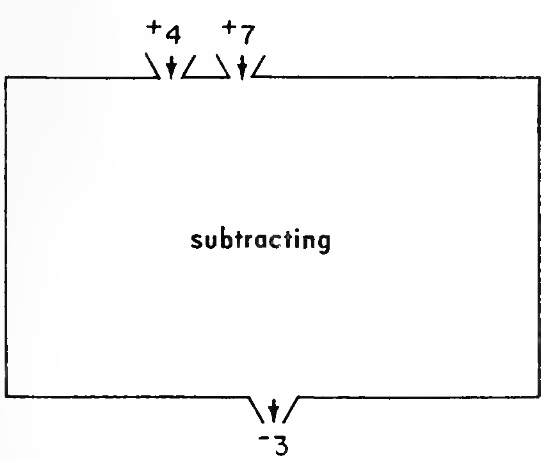


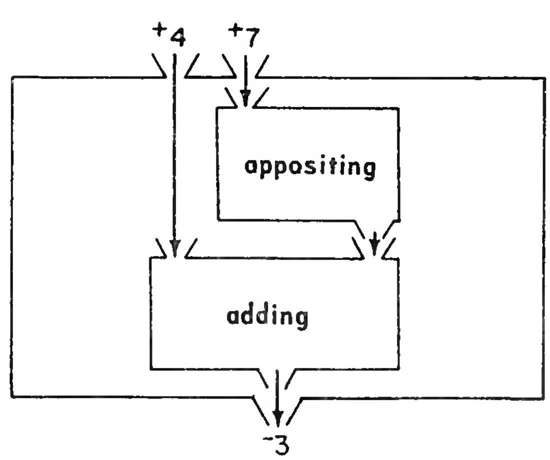






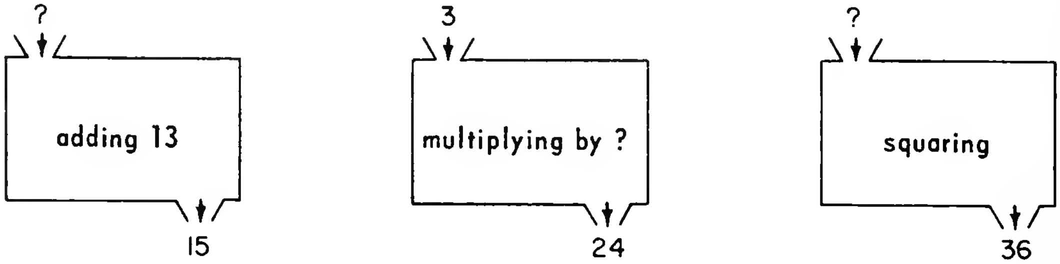
This device can also be useful for illustrating the principle for subtraction.

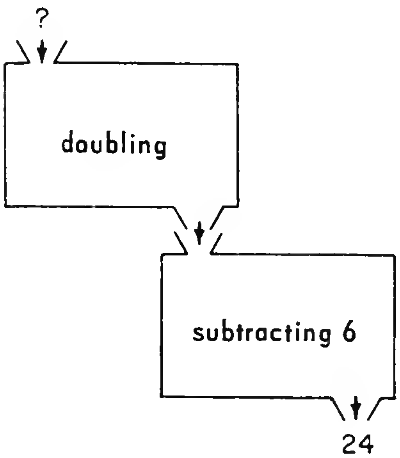


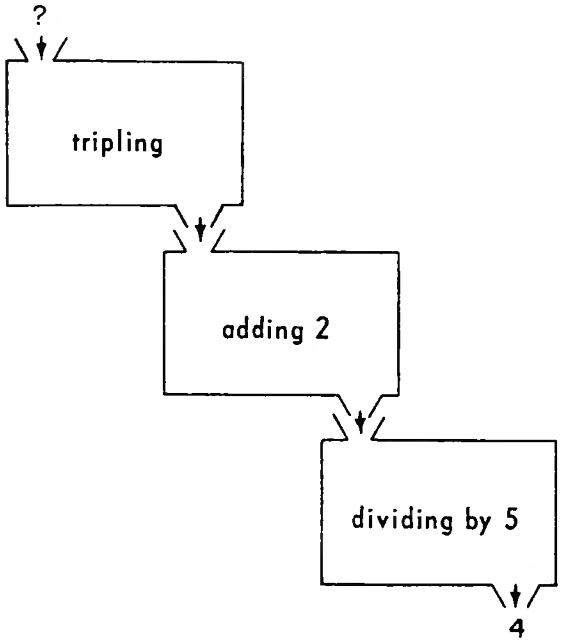


Informal Equation Solving

Intuitive solution of equations may be introduced using this graphic notation. There are unlimited possibilities in exploring this use. One may vary the operations, the number of steps, or the location of a '?'. Here are just a few samples:





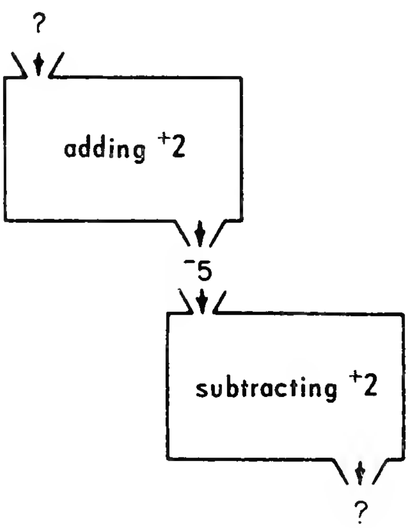


Mr. Howard Marston of the Principia Upper School in St. Louis suggests that a student could solve "equations" like these by merely inverting the picture and doing the inverse operations in order. The last figure above demonstrates effectively the complete freedom from grouping symbols enjoyed by this notation.

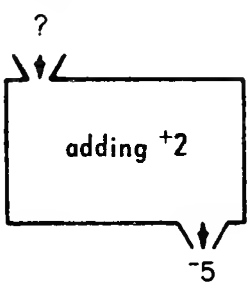
Perhaps one of the most striking advantages of this notation is revealed by a study of the fact that subtracting a real number is the inverse of adding that number. As you will see, this notation makes the concept clear and to the point. Consider this subtraction problem:

$$^{-}5 - {}^{+}2 = ?$$

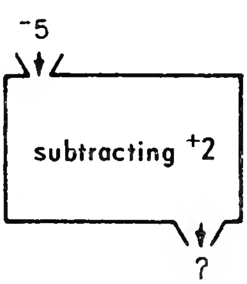
The figure below exhibits all of the needed information in a concise manner.



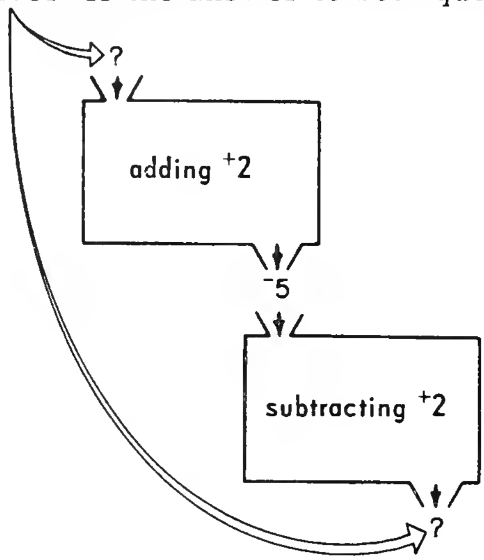
The figure presents this essential problem:



in a strategic position. It also displays the original subtraction problem:

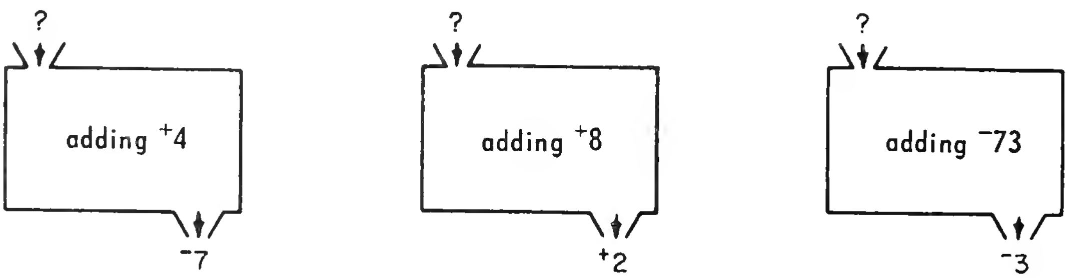


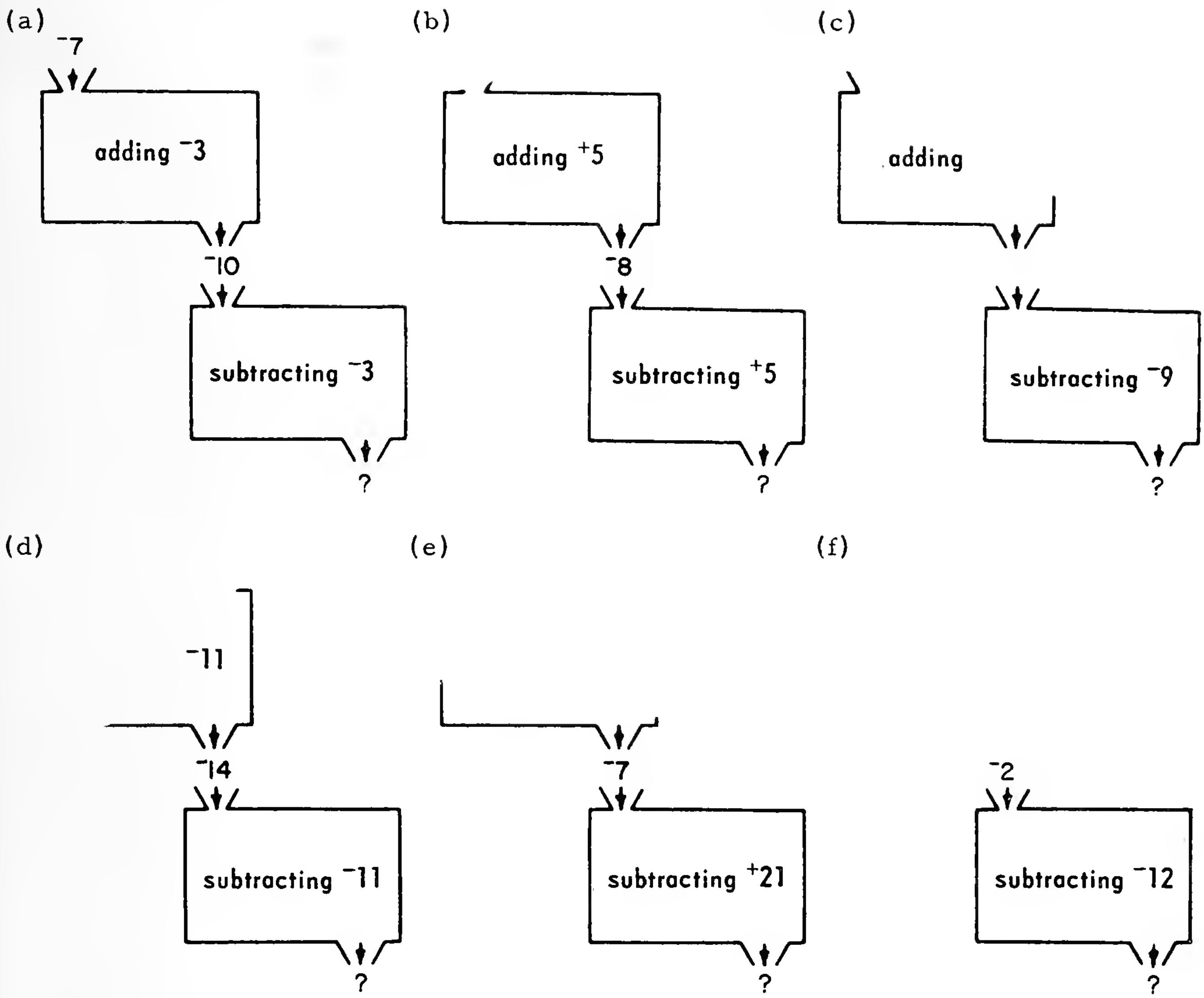
One can readily recognize subtracting $^{+}2$ as the inverse of adding $^{+}2$, and hence notice that the same number is the answer to both questions in:



So to do this problem: $^{-}5 - {}^{+}2 = ?$
one could do this one: $? + {}^{+}2 = {}^{-}5$

Similar diagrams may be employed for rapidly leading students to do subtraction problems involving real numbers. The only prerequisites are the ability to add real numbers, and the knowledge that subtracting a real number is the inverse of adding that real number. A sample set of developmental exercises for this purpose appears on the next page. It would be helpful to precede this sequence of exercises with some "warm-up" on adding reals, and then sufficient exercises of this type:

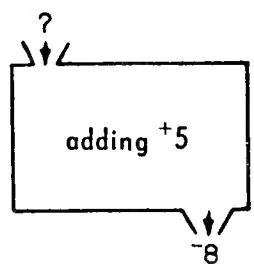




Here are some remarks on the above set of exercises.

(a) The student should recognize that the output number must be the same as the input number since subtracting ⁻3 is the inverse of adding ⁻3.

(b) As in (a), the output number has to be the same as the input number. Since the input number is not given, the student must find it. That is, he must solve this problem:



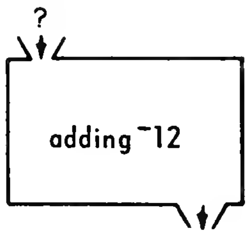
Once the problem is solved, he knows the input number. Hence, he also knows the output number.

(c) The top part of the diagram is "fading". If the operation named there were adding $^{-}9$, then the output number would be the same as the input number. So the student supposes that the first operation is adding $^{-}9$. Now in order to get the input number he must find the number to which $^{-}9$ has been added to get $^{-}2$. That number is $^{+}7$, so the output is also $^{+}7$.

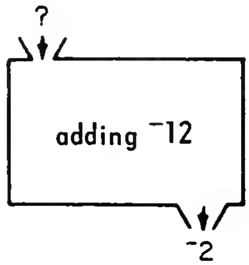
(d) This time the top part has faded so the word 'adding' is missing. However, the student knows that subtracting $^{-}11$ is the inverse of adding $^{-}11$. So, if he knew the number to which $^{-}11$ was added to give $^{-}14$, he would have the answer to this exercise.

(e) The student is now almost on his own. He is, however, reminded that some operation was performed on an input number to give $^{-}7$. If that operation had subtracting $^{+}21$ as its inverse, then the input number would be the same as the output number. It is, therefore, more than convenient for him to assume that the first operation is adding $^{+}21$. Thus, the input number would have to be the number whose sum with $^{+}21$ is $^{-}7$. That number is $^{-}28$ which is also the output number.

(f) The first five exercises give a strong clue for finding the answer to this one. The student merely has to visualize a picture like this:



above the one given. Hence, the answer to the original problem is the answer to:



since subtracting $^{-}12$ is the inverse of adding $^{-}12$.

This experience can be extended to familiarity with the "additive method" of subtraction, which is nothing more than "checking" a subtraction problem before it has been solved. This subtraction tool is excellent for doing problems like:

(a) ? - $^{-}2$ = $^{+}7$	(b) ? - $^{-}9$ = $^{+}5$	(c) ? - $^{+}37$ = $^{-}40$
(d) ? - $^{-}12$ = $^{+}83$	(e) ? - $^{-}4$ = $^{-}11$	(f) ? - $^{-}53$ = $^{-}28$

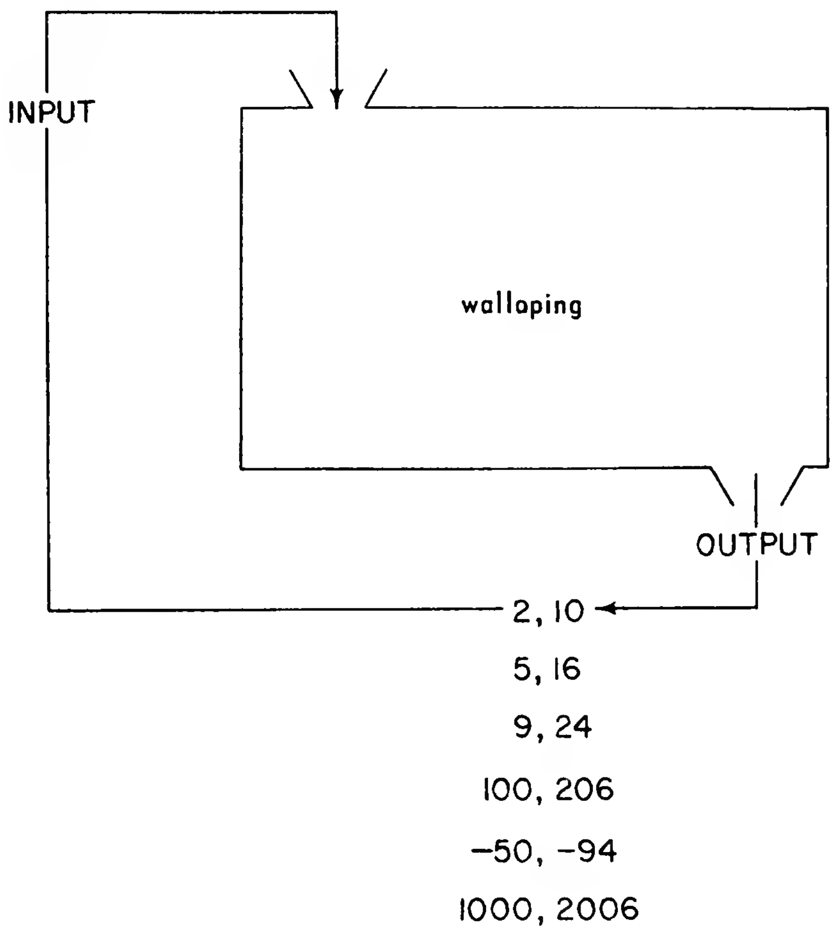
It also proves helpful for doing problems like:

$$^{+}9 - ? = {}^{-}3$$

Students may find this method useful as a supplement to the "adding the opposite" method, which has advantages along other lines and is a direct application of the principle for subtraction.

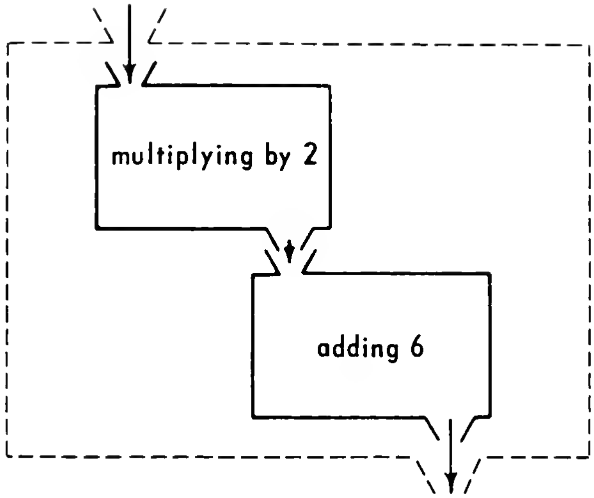
Finally, we leave with you the following problem which has proven to be of interest to students, and will undoubtedly suggest many more pedagogical applications to you — especially in the work on function-composition in Unit 5.

Problem:

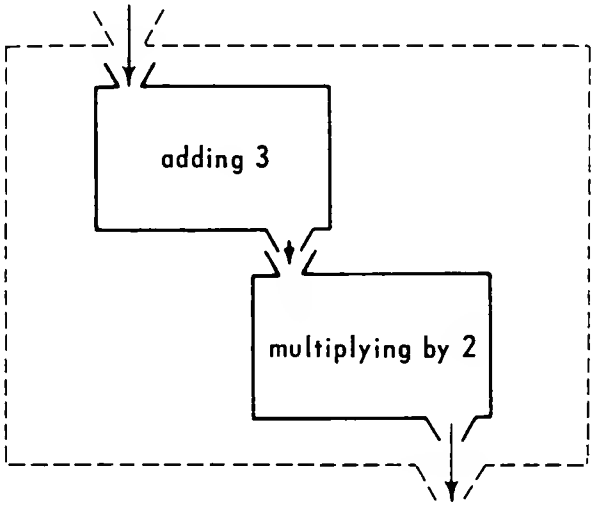


What's inside the walloping machine?

Does it look like this:



or does it look like this:



Those who say that each is correct should be prepared to prove it.

[The notation used here is similar to a notation used by E. G. Begle in Introductory Calculus (New York: Henry Holt & Co., 1954) pages 45, 56, and 79. I have recently seen this notation used to illustrate commutative and associative binary operations in J. B. Roberts' The Real Number System in an Algebraic Setting (San Francisco: W. H. Freeman & Co., 1962) pages 9 and 22.]

H. Wills
Newsletter 7

Operations as Functions, Part 1

Many of the concepts in High School Mathematics Course 3 by Beberman and Vaughan, can be readily anticpated in High School Mathematics Course 1 by Beberman and Vaughan. [And, this can be done without using the words 'relation' or 'function'.] Here are some ideas about how to do this. [We are assuming that the reader is familiar with Course 3.

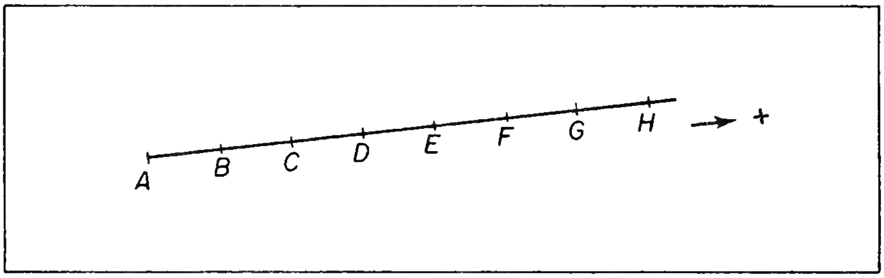
Let's start with Course 1. Near page 19, we can make our first contribution to a better understanding of:

(1) a relation is a set of ordered pairs;

(2) a function is a set of ordered pairs [relation] no two of which have the same first component.

*

Consider a road which begins at A.



Teacher: What is the starting point of a trip whose measure is $^{+}2$[or, 2 to the right, or $\vec{2}$]?

Student: A.

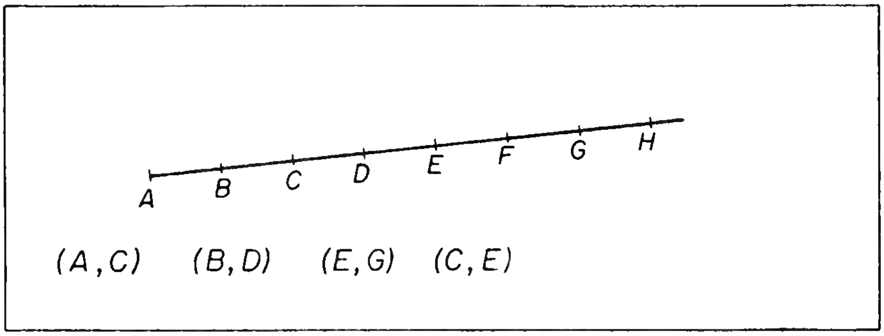
Teacher: What is the ending point?

Student: C.

Teacher: Is the trip from A to C the only trip whose measure is $^{+}2$?

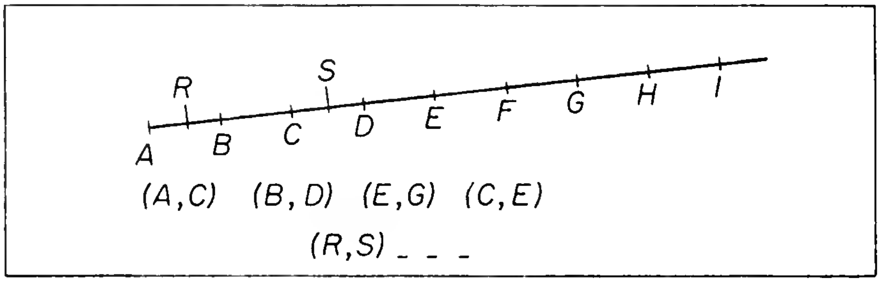
Student: No. The trip could begin at B and end at D.

Teacher: Any other trips of this kind?



Teacher: We're getting a lot of these now. To keep from getting them mixed up, let's use parentheses.

Student: A trip beginning midway between A and B and ending midway between C and D has measure $^{+}2$.

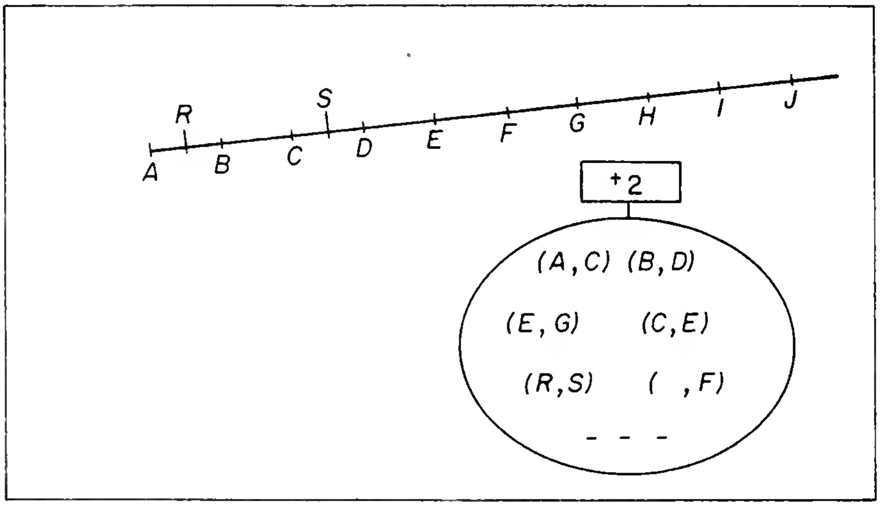


Teacher: How many trips are there with measure $^{+}2$?

Student: Lots of them.

Teacher: So all these trips and many, many more have measure $^{+}2$.

[Note the "balloon" notation introduced here. This anticipates page 23 of Course 1.]



Teacher: Is there a trip whose measure is $^{+}2$ and which ends at F?

Student: Yes, the trip from D to F.

Teacher: Does the trip from E to C belong to this set? [The child will understand the word 'set' here without any explanation.]

Student: No.

Teacher: Does the trip beginning at A and ending at D belong to this set?

Student: No.

Teacher: [Pointing to '(A, C)'. Is there any other trip besides this one that starts at A and has measure $^{+}2$?

*

[It is possible that some student may contend that the trip from A to C made today is different from the trip from A to C made yesterday. If this happens make clear that the word 'trip' is being used as the person who says, "I've made the trip from Urbana to Chicago many times", is using it

This development lays the groundwork for defining a real number as a certain set of ordered pairs of numbers of arithmetic. If we consider the road to be a number ray of arithmetic, we have defined $^{+}2$ to be the set of ordered pairs of numbers of arithmetic such that the second component of each ordered pair is 2 greater than its first component.]

Now, what can we do with section 2.04, Addition of real numbers?

*

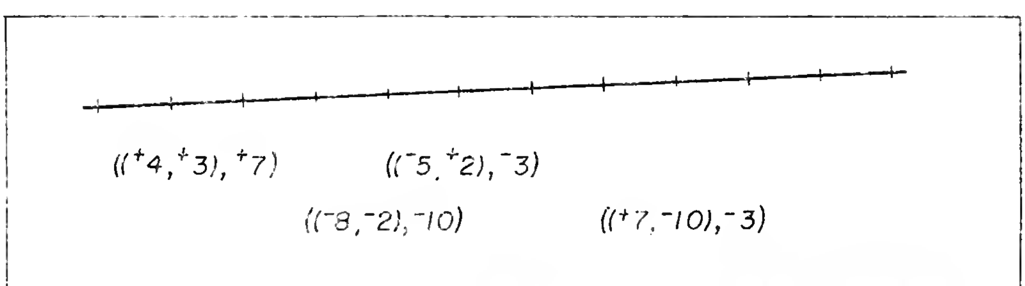
Teacher: Mary, think of a trip on this road. What real number measures the trip you are thinking about?

Student: $^{+}4$.

Teacher: Beginning at the end-point of that trip, take a trip whose measure is $^{+}3$. What is the measure of the single trip that would take you from the beginning point of the first trip to the ending point of the second trip?

[As each number is given, write a numeral for it in the proper place, putting in the commas and parentheses as you write. The stages by which you would arrive at '$((^{+}4, ^{+}3), ^{+}7)$' are:

$$^{+}4$$
$$(^{+}4, ^{+}3)$$
$$((^{+}4, ^{+}3), ^{+}7)$$
]



Student: ${}^{+}7$

Teacher: Correct.

*

Continue the questioning in order to obtain additional ordered pairs whose first components are themselves ordered pairs.

$$(({}^{+}4, {}^{+}3), {}^{+}7),\ (({}^{-}5, {}^{+}2), {}^{-}3),\ (({}^{-}8, {}^{-}2), {}^{-}10),\ (({}^{+}7, {}^{-}10), {}^{-}3),\ (({}^{+}3, {}^{+}4), {}^{+}7)$$

[Naturally, we are not suggesting that this ordered pair notation should replace the conventional '${}^{+}4 + {}^{+}3 = {}^{+}7$' type of listing of addition "facts".]

Again we have a set of ordered pairs, no two of which have the same first component. A function of this type in which the first component of each ordered pair is itself an ordered pair is often called a binary operation. Contrast this with:

$$\{({}^{+}4, {}^{+}7),\ ({}^{-}3, 0),\ ({}^{-}5, {}^{-}2),\ ({}^{+}8, {}^{+}11),\ ({}^{-}2, {}^{+}1),\ ---\}$$

This operation, adding ${}^{+}3$, is called a singulary operation on the set of real numbers.

Some mathematicians use the phrase 'operation on a set S' only if the set is closed with respect to the mapping. Thus, a singulary operation on S would be a mapping which takes you from a member of S to one and only one member of S. In the case of a binary operation on a set S, the mapping would take you from any member of $S \times S$ to one and only one member of S. Other mathematicians use the word 'operation' as synonymous with 'function'. So, for example, they would talk about the absolute value operation which takes you from the real numbers to the numbers of arithmetic. This is not a usage of 'operation' which conforms to the definition mentioned above.

* * *

Part of the following is a modified transcription of the questions and answers given in one of the classes at University High School. Some of you may recall it from one of our training films.

\- - -

The day before this discussion, the class worked exercises designed to create an awareness of the existence of the inverse of an operation. For example, among them were exercises such as "If you want to undo the result of adding 15, subtract ____ from the sum." Now the class is ready to find out that an operation is a set of ordered pairs and then to find how to form the inverse of an operation. The terminology 'converse of an operation' can also be introduced here, but this is optional.

Teacher: Suppose a third-grader said to you, "What do you mean by 'adding 9'?" What would you tell him?

Student: Well---add a number---

Teacher: That wouldn't be very helpful.

Student: Well, adding nine ones to whatever you're adding.

Teacher: I don't know whether that would help him or not.

Student: (Mary) Well, if he had one apple and you gave him nine more apples, he would have ten apples.

Teacher: I see. What would you say, Jack?

Student: Uh, have him a quantity-----so much more.

Teacher: Joan, what do you think?

Student: Well, he ought to know it.

Teacher: Let's see. Mary, you said to give him an example. He has one apple, give him nine apples. He now has ten apples.

$1 + 9 = 10$

What else would you do? Give him another example? Harry.

Student: You could say you have 12 apples. Then someone gives you nine more apples. Let him count them up to see how many he has.

$1 + 9 = 10$
$12 + 9 = 21$

Teacher: You could keep on giving him example after example. I think pretty soon he'd form some idea of what adding 9 is. Let's put some more examples down like these.

$1 + 9 = 10$	$200 + 9 = 209$
$12 + 9 = 21$	$2 + 9 = 11$
$31 + 9 = 40$	$3 + 9 = 12$
$7 + 9 = 16$	
$30 + 9 = 39$	
$10 + 9 = 19$	

Teacher: Imagine, for a moment, that we have all possible examples of adding 9. How many would there be?

Student: Lots of them. But he won't know how to add 8.

Teacher: Then we'll make him a specialist in adding 9. Imagine that we have all possible examples for adding 9. How do we solve a problem in adding 9? Imagine we had a book full of these examples of adding 9. Then we gave him a problem:

$$6 + 9 = ?$$

What would he do?

Student: Look in the six's column.

Teacher: Look for the example where a '6' appears in the first column. Suppose he finds it. What would he find in the last column?

Student: 15.

Teacher: So, he knows the answer to this problem must be 15. Let's pretend that we're actually going to have a book like this — all full of examples of adding 9. Now, I want to save some space in the book. Is there any way in which I can shorten these sentences so I can save space?

*

The students made various suggestions about how to shorten these sentences. Their suggestions amounted to something like the following:

"We don't need all those plus signs. Let's take them out." [The plus signs were all erased.] Someone else says, "Why not remove all the equal signs?" [Equal signs were erased.] Finally, someone says, "Why not take out all the '9's, since they are repeated in each sentence?" The teacher then pointed out that they needed something to separate the numerals for each pair; so, commas were decided upon. Finally, the teacher said, "Let's put the whole works in parentheses." The following shows different stages of the development.

1	9	=	10
12	9	=	21
31	9	=	40
7	9	=	16
30	9	=	39
10	9	=	19
2	9	=	11
3	9	=	12
6	9	=	15

1	9	10
12	9	21
31	9	40
7	9	16
30	9	39
10	9	19
2	9	11
3	9	12
6	9	15

1	10
12	21
31	40
7	16
30	39
10	19
2	11
3	12
6	15

Recopy to save space

1	,	10
12	,	21
31	,	40
7	,	16
30	,	39
10	,	19
2	,	11
3	,	12
6	,	15

(1	,	10)
(12	,	21)
(31	,	40)
(7	,	16)
(30	,	39)
(10	,	19)
(2	,	11)
(3	,	12)
(6	,	16)

(1, 10)
(12, 21)
(31, 40)
(7, 16)
(30, 39)
(10, 19)
(2, 11)
(3, 12)
(6, 15)
- - -

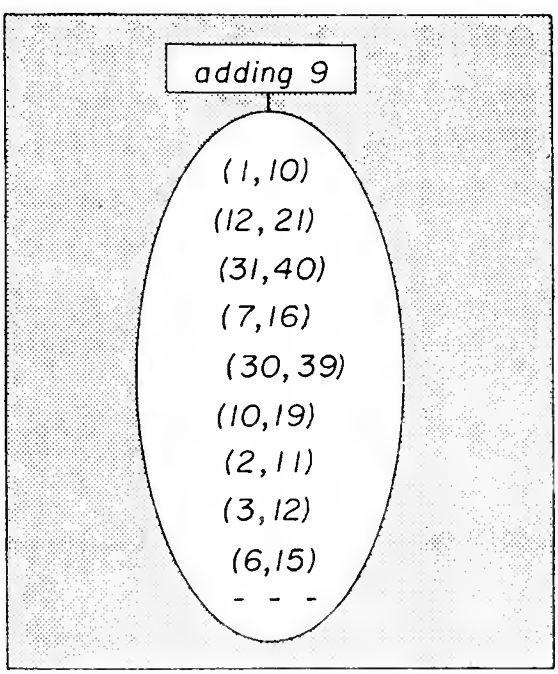
*

Teacher: Imagine then, these are just a few sample pairs from this book. What would be a good name for the book? Suppose we actually got real silly and printed a book like this. What would be a good name for the book? 'David Copperfield'? What would be a good name, John?

Student: Adding Nine.

[Other suggestions.]

Teacher: Let's pick that short title: Adding Nine.



Teacher: This book has lots more pairs. Let's get a few more samples to see that we've got the idea. A pair that begins with 15, ends with what?

Student: 24.

Teacher: A pair that begins with 40 ends with what?

Student: 49.

Teacher: I'm going to write a pair and I want you to tell me if it really could come from this book. Ready? (17, 26). Yes? No? Class?

Class: Yes.

Teacher: (39, 30) Yes? No? Class?

Class: No.

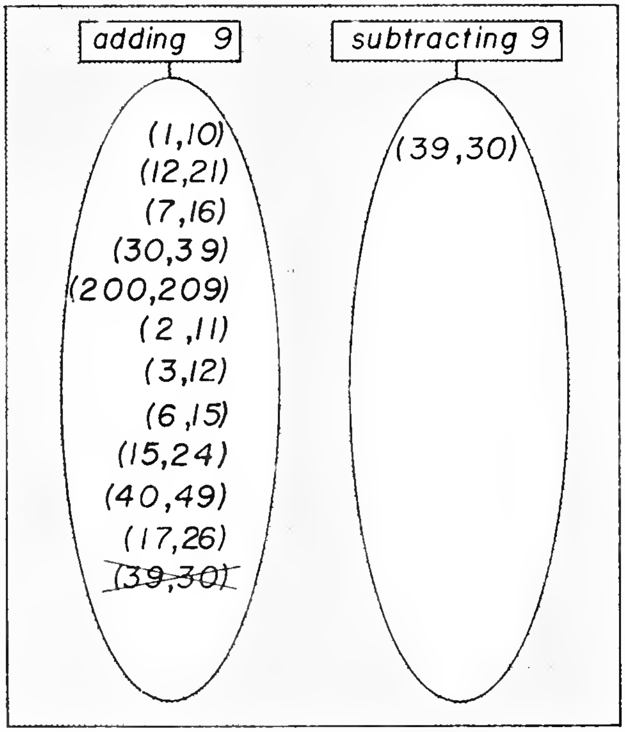
Teacher: What about this pair, (39, 30)? What book could that come from?

Student: The Book for Subtracting Nine.

Teacher: I think I like that.

Student: The Book for Adding Negative Nine.

Teacher: Good, but remember we were talking about numbers of arithmetic. [Third-grade.]



Teacher: Let's get some more pairs that would belong to the book, Subtracting Nine. I want someone very quickly — without doing much thinking — to give me a whole bunch of pairs that come from this book. Are you ready?

Student: (69, 60)

Teacher: Too much thinking.

Student: (10, 1)

Teacher: Faster.

Student: (21, 12), (16, 7), (39, 30).

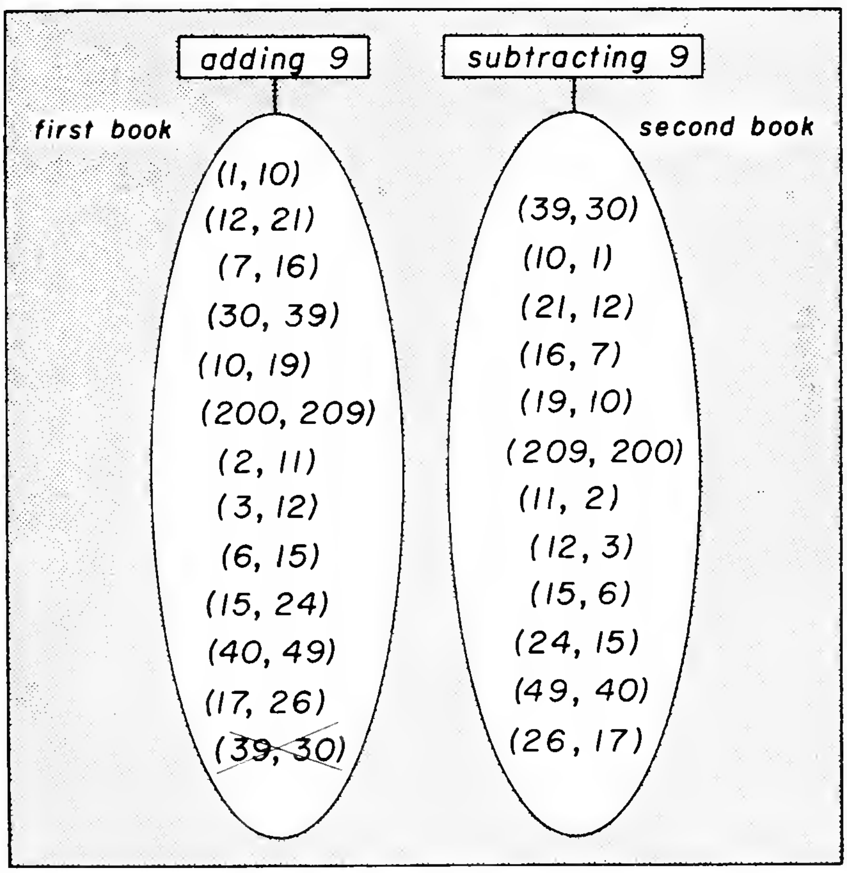
Teacher: You already have (39, 30). [(39, 30) was the first entry in this book.]

Student: (19, 10), (209, 200), (11, 2), (12, 3), (15, 6), (24, 15), (49, 40), (26, 17).

Teacher: Are there any more pairs that belong to Subtracting Nine?

Student: Lots.

Teacher: If you have the first book, do you need the second book?



Student: No.

Teacher: Right. It turns out that this second book is not really necessary. Any problem you want to do using the second book could have been done by using the first book.

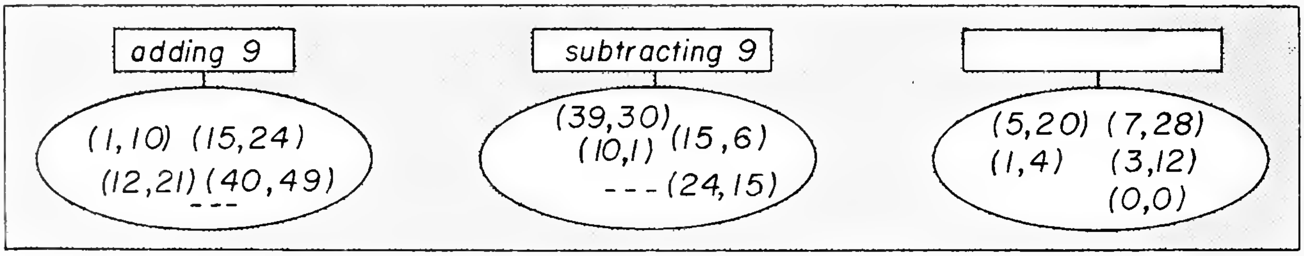
Student: For that matter, you could say that the Adding Nine book was not necessary. You could use the Subtracting Nine Book.

Teacher: Right. But, suppose that we all know how to add 9. does that automatically tell us how to subtract 9?

Student: Yes.

Student: Would there be books like these for multiplying and dividing?

Teacher: Well, I think so. Let's make up another book. Here are some of the pairs that belong to it.



Teacher: Give me some more pairs that belong to this book.

Student: (400, 1600), (80, 320), (9, 36)

Teacher: What is a good name for this book?

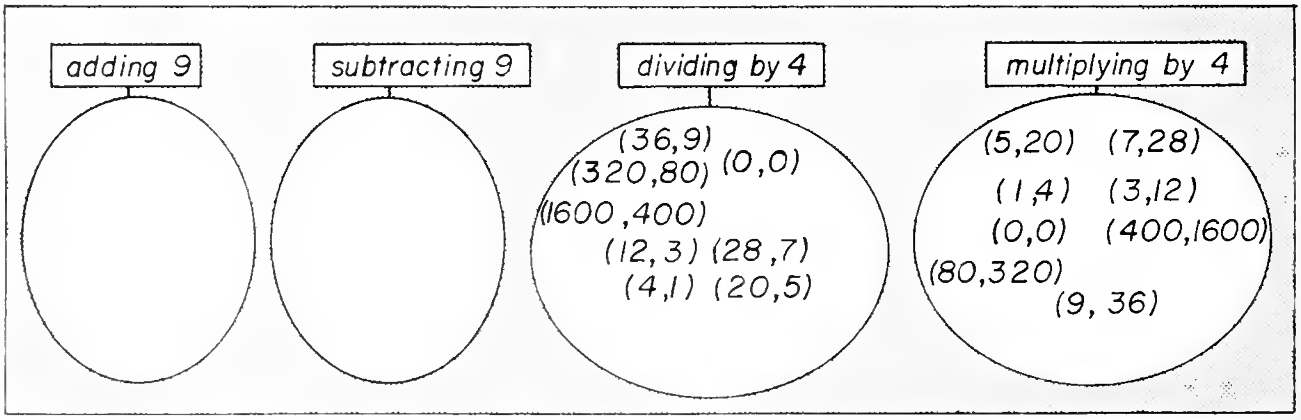
Student: Multiplying by 4.

Teacher: Remember that we could do without the Subtracting Nine book if we had the Adding Nine book? Is there any book we could do without if we had the Multiplying by 4 book?

Student: Dividing by 4.

Teacher: Tell me some of the pairs that would belong to that book.

Student: (36, 9), (320, 80), (1600, 400), (12, 3), (4, 1), (0, 0), (28, 7), (20, 5)



Teacher: Does anyone see how he gets the pairs that belong to this book?

Student: Well, he just flips them around.

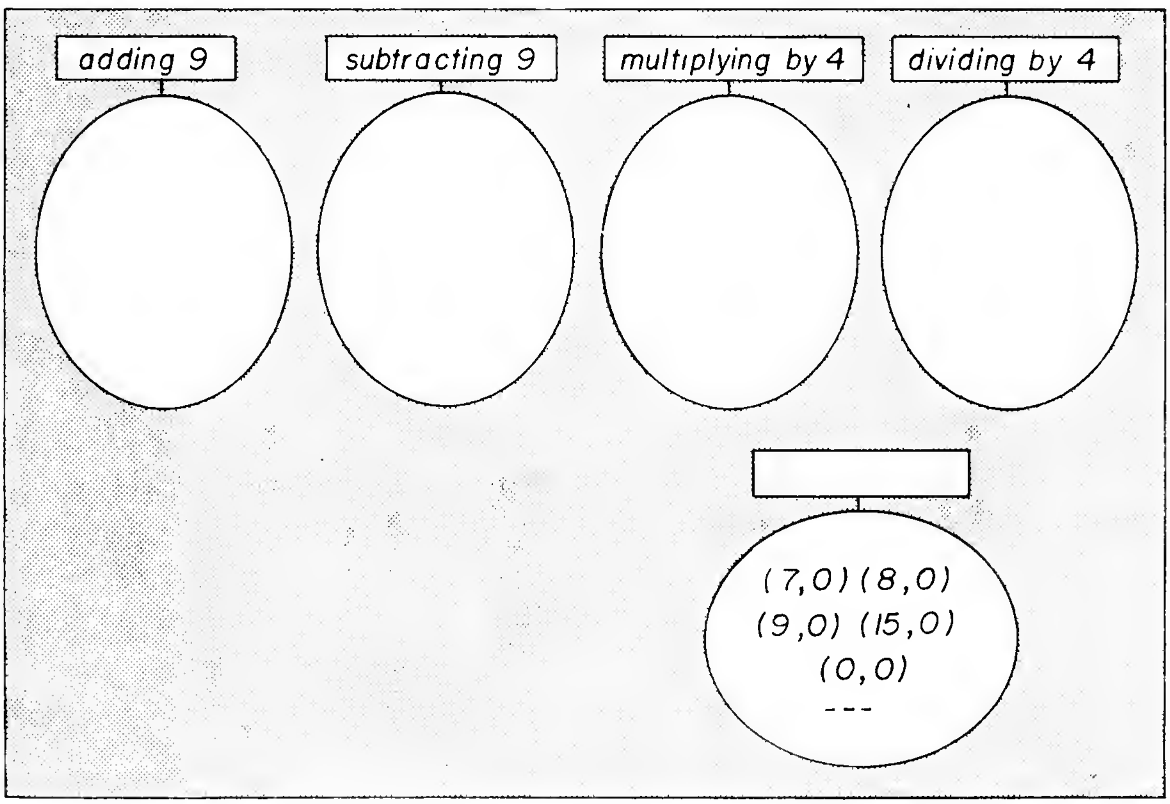
Teacher: Right, you reverse the pairs. That's the way it works. Reverse the pairs. Is there any pair that belongs to both of these last two books?

Student: (0, 0)

Teacher: Is there any pair that belongs to Adding 9 and Multiplying by 4?

Student: (3, 12)

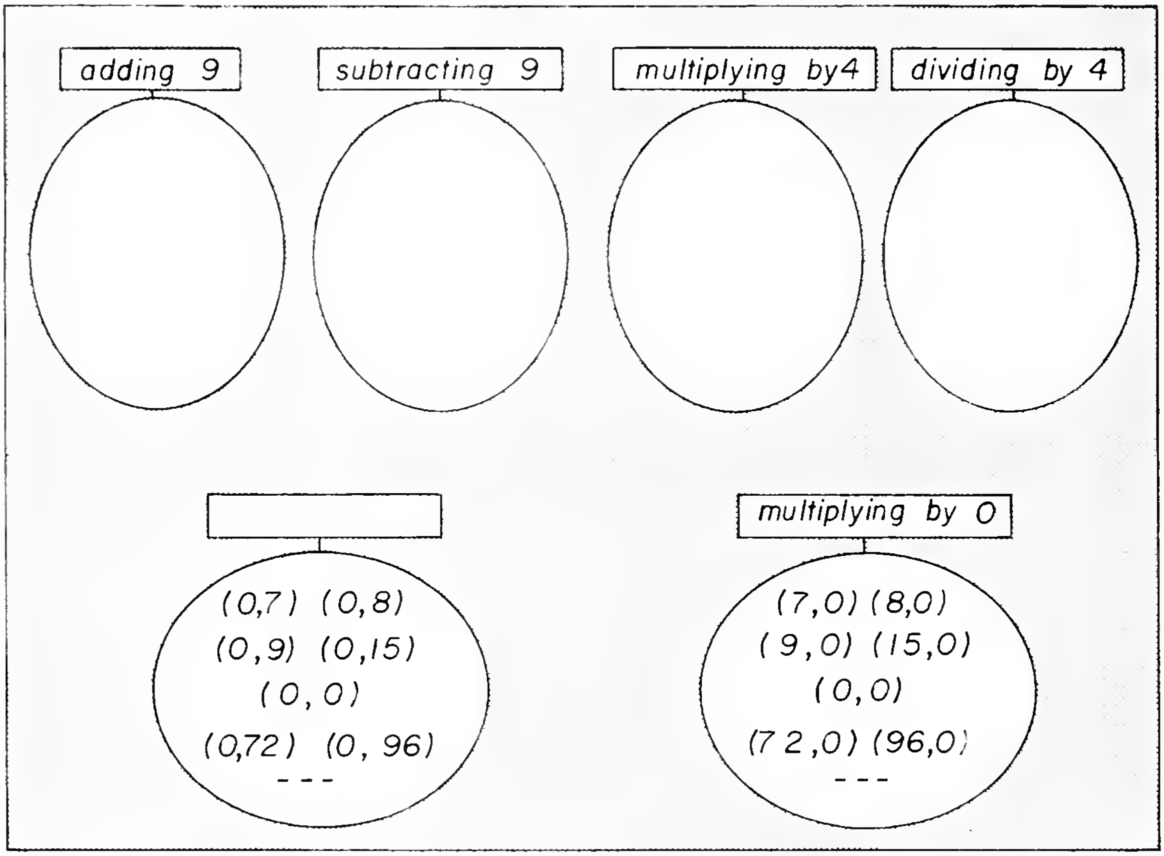
Teacher: Let's look at another book.



What are some other pairs that belong to this book?
What is a name for this book?

Student: Multiplying by 0.

Teacher: Let's reverse these pairs. What pairs will we have then?



Teacher: Which book would you use to do this problem?

$$74 + 9 = \_\_\_\_$$
$$98 - 9 = \_\_\_\_$$
$$15 \times 0 = \_\_\_\_$$

What kind of problem could you work using this book? [Pointing to the unlabeled balloon.]

Student: There isn't any. You'd never know which pair to pick because they all start with 0.

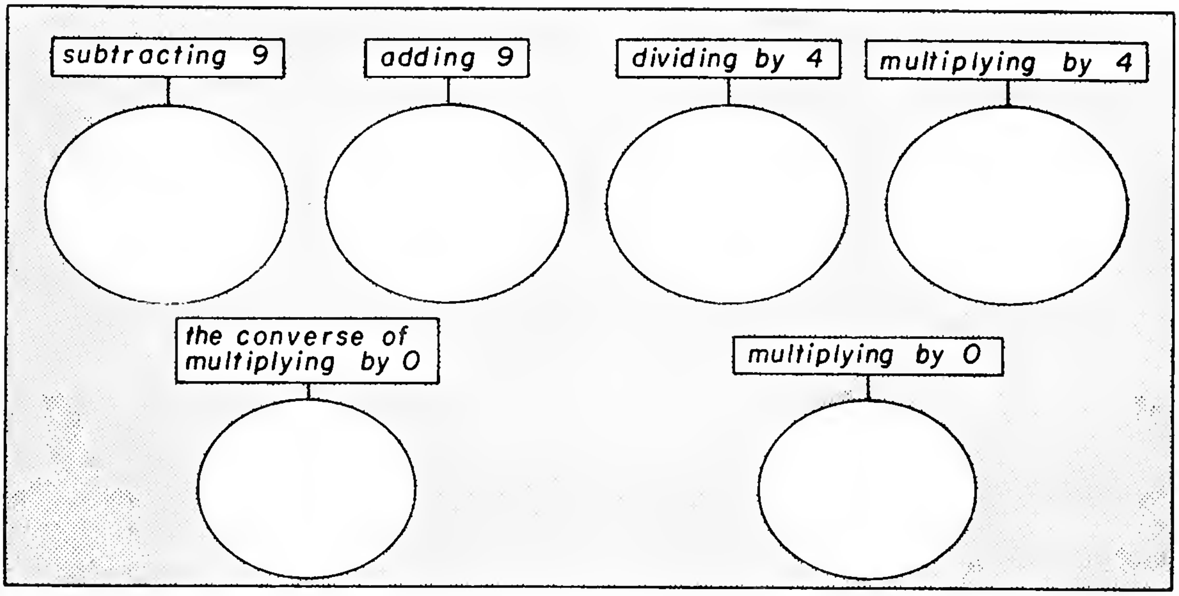
Teacher: Right. Then can you divide by 0?

Student: No.

Teacher: Can anyone divide by 0?

Student: No.

Teacher: However, there is a name for this last book. It's called 'The Converse of Multiplying by 0'. Converse. C-O-N-V-E-R-S-E. Can you give me another name for this book [pointing to first balloon]?



Student: The Converse of Adding 9.

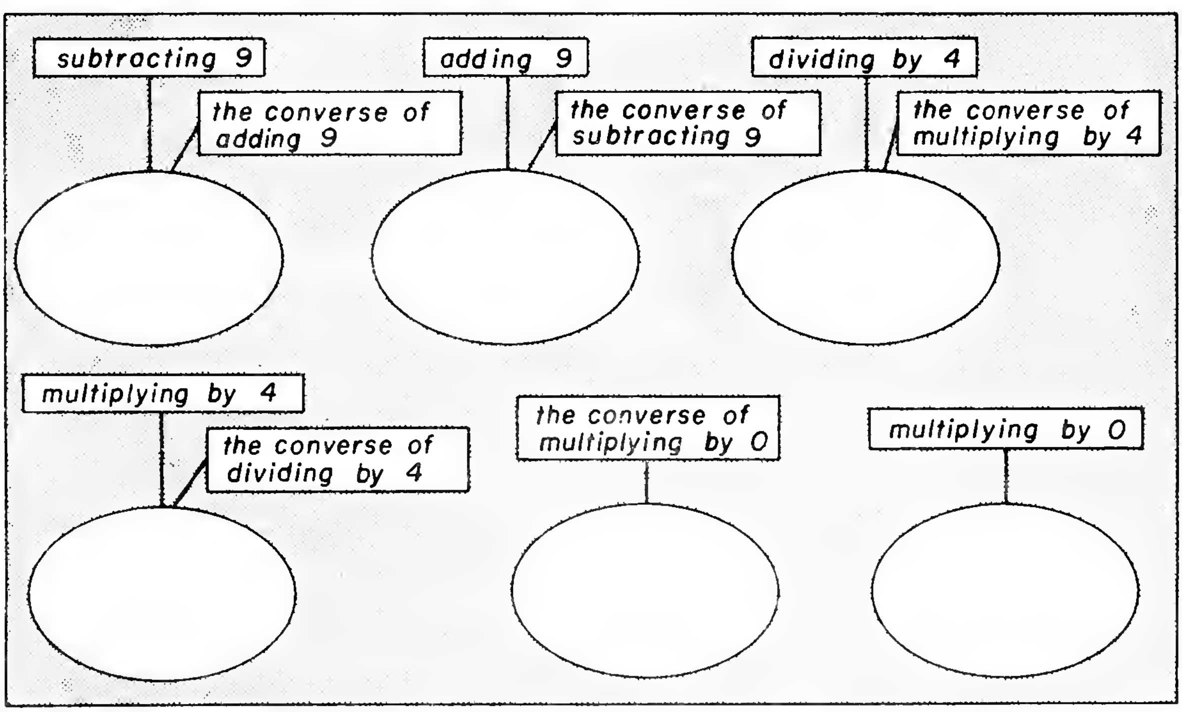
Teacher: Correct. Let's put that down.

Student: I think it ought to be called "The Reverse of Adding 9".

Teacher: Maybe that would be a good name, but that's not what mathematicians call it. We reverse the pairs and we get the converse. What is another name for the book Adding 9?

Student: The Converse of Subtracting 9.

Teacher: [Writes this in.] What is another name for the book Dividing by 4? Multiplying by 4?



[Students may want to give another name to Multiplying by 0. They may give 'The Converse of the Converse of Multiplying by 0.' This is acceptable. But, of course, 'The Converse of Dividing by 0' is meaningless and not acceptable, since 'Dividing by 0' has no referent.]

Teacher: Notice that in this book, Subtracting 9, if we start with 15, we go to 6 and nowhere else. In the book, Dividing by 5, if we start with 35, we go to 7 and nowhere else. Is this sort of thing true for the book we call the 'The Converse of Multiplying by 0'?

Student: No.

Teacher: Is it true for the book, Adding 9? Multiplying by 4? Multiplying by 0?

Student: Yes. Yes. Yes.

Teacher: Books like this, where you go nowhere else, we call 'operations'. Is Dividing by 5 an operation?

Student: Yes.

Teacher: Which of these books are operations? Which are not operations?

Student: All but one are operations. The Converse of Multiplying by 0 is not an operation.

Teacher: Each of you make up a book which is an operation. Draw a loop on your paper and list five pairs which belong to that operation. Give your operation a simple name.

[Check this work some way at this point.]

Now draw another loop. Reverse the pairs that are listed in your first loop and list these reversed pairs in the second loop. Is this set of pairs an operation? Give this set of ordered pairs a name.

[The word 'ordered' can be used here without special emphasis. If students question this, just point out that you are saying 'ordered' because it makes a difference about which number comes first.]

Teacher: We have a special name if both the book and its converse are operations. We use the name 'Inverse' instead of 'Converse'. Inverse. I-N-V-E-R-S-E. Now think about the book, Adding 9. Think about the book, The Converse of Adding 9. What other name have we already given this second book?

Student: Subtracting 9.

Teacher: What new name can we give this book?

Student: The Inverse of Adding 9. [Write this name in the proper place.]

Teacher: What other name can we give the book Adding 9?

Student: The Inverse of Subtracting 9. [Write this name in the proper place.]

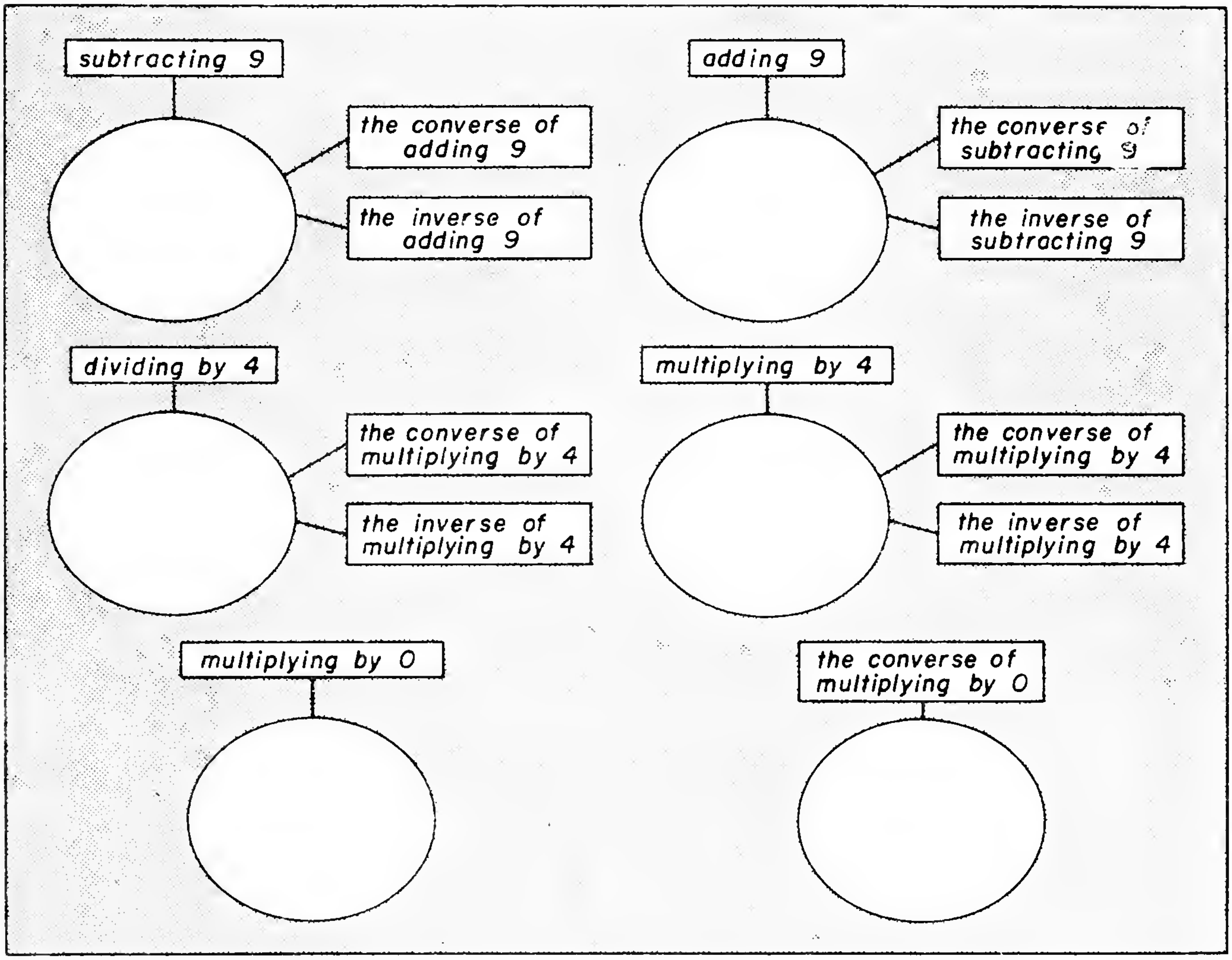
Teacher: Is there any one of these books for which we will not use the word 'inverse'?

Student: Yes. The Converse of Multiplying by 0.

Teacher: Will we use the word 'inverse' for any other of these books?

Student: Yes. Dividing by 4 is the Inverse of Multiplying by 4.

Teacher: Any others?



Teacher: If the word 'converse' appears in a name, can we always replace the word 'converse' by 'inverse' and get another correct name?

Student: No.

Teacher: If the word 'inverse' appears in a name can we replace the word 'inverse' by 'converse' and get another correct name?

Student: Yes.

Teacher: If we can correctly use the word 'inverse' we usually use it instead of using the word 'converse'. We would seldom say:

Subtracting 9 is the converse of adding 9.

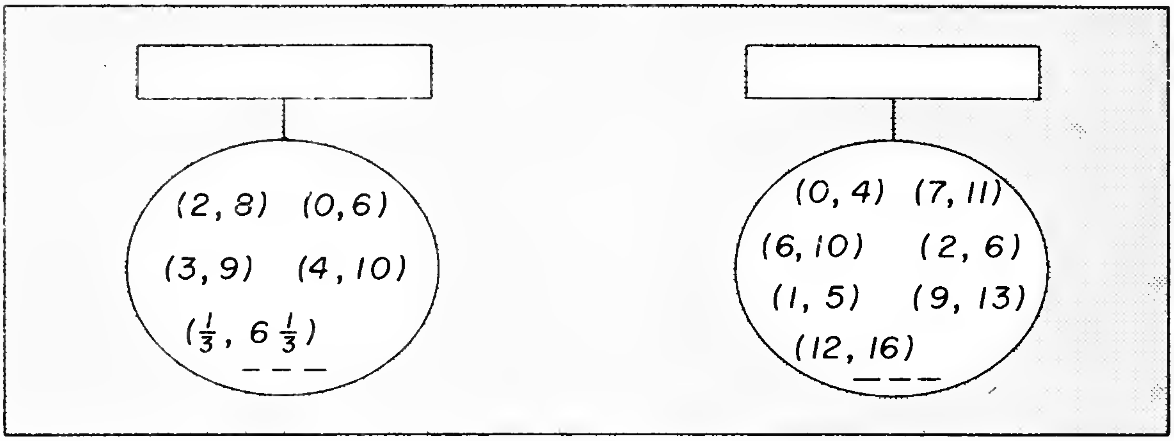
We would usually say:

Subtracting 9 is the inverse of adding 9.

Now look at your papers. Write another name for each of the sets of pairs.

[If the word 'converse' appears and if the word 'inverse' is applicable, ask them to write still another name.]

Here are two sets of pairs.



[Ask them to give more ordered pairs for each operation.]

What is a name for this first operation? For the second?

Student: Adding 6. The Inverse of Subtracting 6.

Student: Adding 4. The Inverse of Subtracting 4.

Teacher: Now. Jane, Pick a number. What one did you pick?

Student: 7.

Teacher: Let's go to the first operation. Adding 6.

[Put '(7,)' inside the proper loop.]

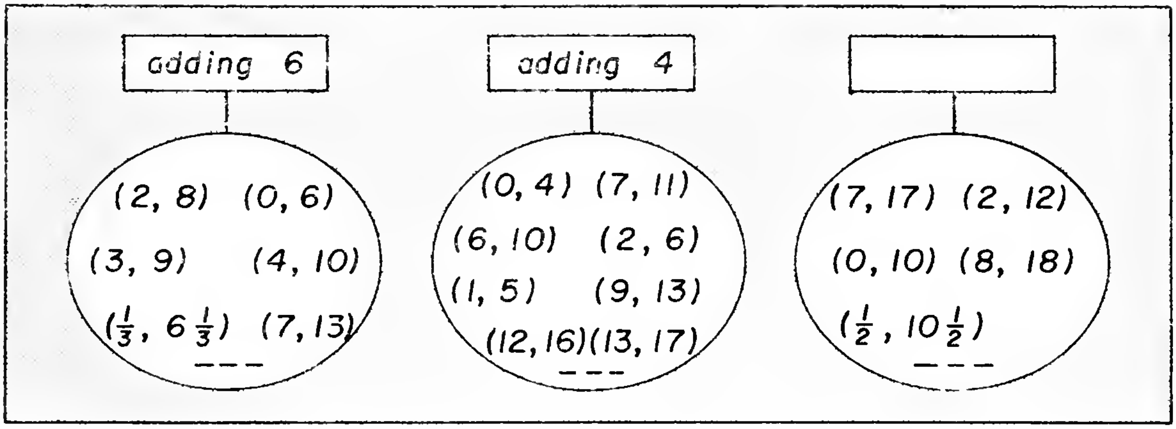
Teacher: Perform this operation on 7. What second number do we get?

Student: 13 [Write '13' in the proper place.]

Teacher: Now perform this second operation, adding 4, on the number 13. [Write '(13,)' inside the proper loop.] What shall I write?

Student: 17.

Teacher: So, if I start with 7 [write '(7,)'] and perform the operation, adding 6, then take that result and perform the operation adding 4 on it, I get 17. [Write '17' in the proper place.]



Teacher: Let's pick another number.

Student: 2.

Teacher: Perform the first operation. Now perform the second operation on that result. What do you have?

Student: 12.

[Now have each child go through this procedure. List the ordered pairs as shown above. Draw a loop around them.]

Teacher: How many such pairs are there? Imagine that we have all such pairs. Do we want to say that this set of pairs is an operation?

Student: Yes.

Teacher: Can you suggest a good name for this operation?

Student: Adding 10.

*

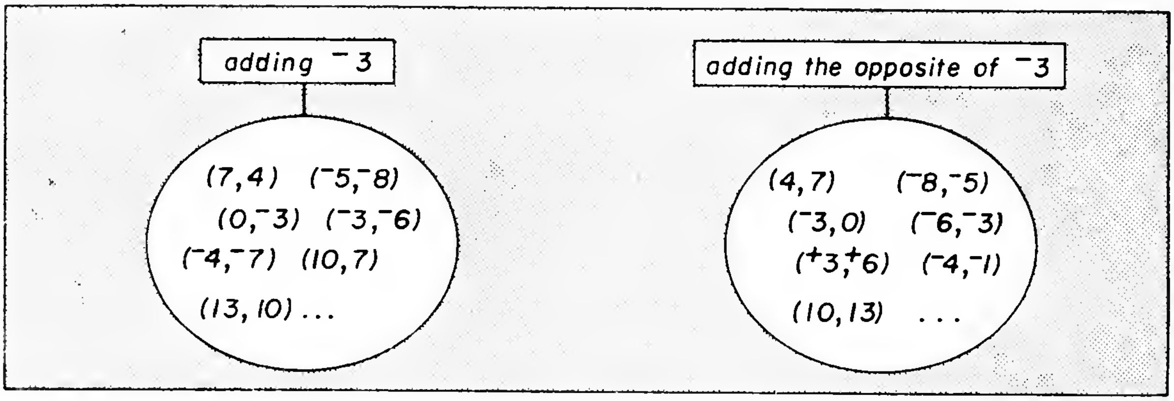
The examples given above have been based on the set of numbers of arithmetic. Examples using real numbers and the operations addition and multiplication on real numbers can also be used.

A. Hart
Newsletter 4

Operations as Functions, Part 2

This article is the sequel to the preceding article in this compendium. The first part dealt with operations as sets of ordered pairs, their uniqueness (since they are functions), and the question of converse and inverse sets. The operation equivalent to two successive operations was developed as a foreshadowing of the composition of functions.

When you are ready for page 33 of Course 1, the board work might "accidentally" result in the following:



[It might be well to list the ordered pairs for adding $^-3$ and ask for the name. Then, write the name 'adding the opposite of $^-3$' and ask for the pairs.]

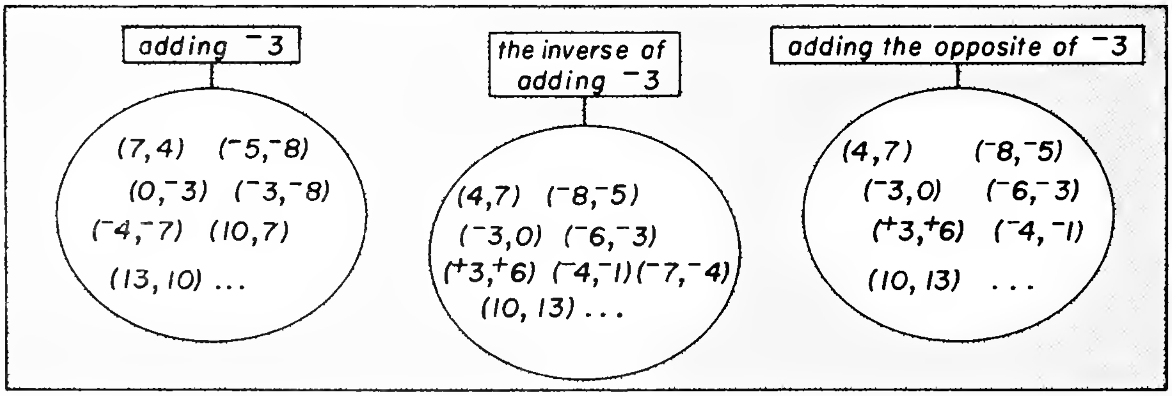
*

Teacher: Can you give me another name for the operation, adding the opposite of $^-3$?

Student: The inverse of adding $^-3$.

Student: Adding $^+3$.

Teacher: Both of you gave correct answers. Let's just check to see that we're clear on what we mean by 'inverse'. [Write 'the inverse of adding $^-3$' in the indicated position.]



Teacher: Let's start with $^-5$. If I apply the operation adding $^-3$ to $^-5$, what do I get?

Student: $^-8$.

Teacher: Now, I want to undo the result of the adding $^-3$, that is, I want to start out with $^-8$. What shall I get back to if I have properly "undone what adding $^-3$ did"?

Student: You should get back to $^-5$.

Teacher: Does the second operation get you back to where you started?

Student: Yes.

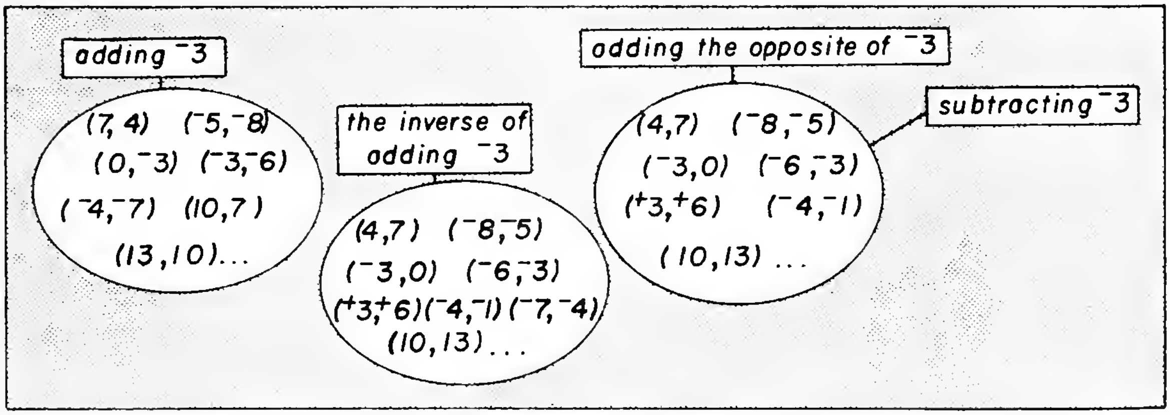
Teacher: What shows you that you get back to $^-5$?

Student: ($^-8$, $^-5$).

Teacher: Right. This operation (adding $^-3$) takes you from $^-5$ to $^-8$ and this one — the inverse of the first operation — takes you from $^-8$ back to $^-5$. That's why we call it 'the inverse'.

[More work like the above.]

Let's think back to the set of numbers of arithmetic. Just as we say that subtracting the number of arithmetic 3 is the inverse of adding the number of arithmetic 3, so we say that subtracting the real number $^-3$ is the inverse of adding the real number $^-3$.



*

Teacher: I am thinking of an operation one of whose names is 'the inverse of adding $^+8$'. What other name might we give that operation?

Student: Subtracting $^+8$.

Teacher: You are right. Give another name.

Student: Adding $^-8$. [It will be nicer if you don't get this one but you may. Just say, "OK" and go on.]

Teacher: Another name.

Student: Adding the opposite of $^+8$.

Teacher: Right. 'Subtracting $^{+}8$' and 'adding the opposite of $^{+}8$' are excellent names for this operation. Now I'm thinking of an operation, one of whose names is 'subtracting $^{-}2$'. What are other names?

And so on.

[After some work of this sort, we replace the written words 'the opposite of' by '−'. However, we still read '−' as 'the opposite of'.]

Teacher: John, pick a real number.

Student: $^{-}3$.

Teacher: Jane, what is the opposite of John's number?

Student: $^{+}3$.

[As these numbers are given, list them on the board as ordered pairs. Continue until you have several pairs. See that such pairs as $(^{-}3, ^{+}3)$ and $(^{+}3, ^{-}3)$ appear.]

Teacher: I'll pick a number: $^{+}3$. What is its opposite?

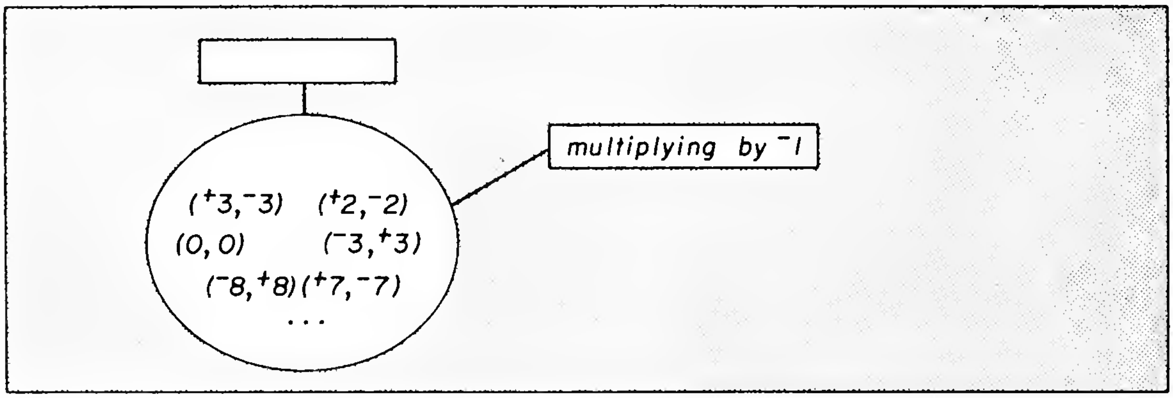
Student: $^{-}3$.

Teacher: Now imagine all such ordered pairs. Do you think we should call this an operation?

Student: Yes, because $^{+}3$ goes to $^{-}3$ and no where else.

Teacher: Right. Now, let's give this operation a name.

Student: Multiplying by $^{-}1$.



Teacher: Right. Now, try to think of a name that would make people remember that the second number in a pair is the opposite of the first number in that pair.

Student: Oppositing.

[You may have to tell them this name. This might be expected, since the choice of a name is an arbitrary matter.]

Teacher: Let's reverse these pairs. What pairs do we get?

Is this set of pairs an operation?

Student: Yes.

Teacher: Since this set is an operation whose pairs are obtained by reversing the pairs of the other operation, what shall we call it?

Student: The inverse of oppositing. [Teacher writes.]

Teacher: O. K. Now can you think of another name for it?

Student: Say, that's just the same as the first one!

Teacher: How about that? Do you mean that each pair in this first operation is in the second one?

Student: Yes.

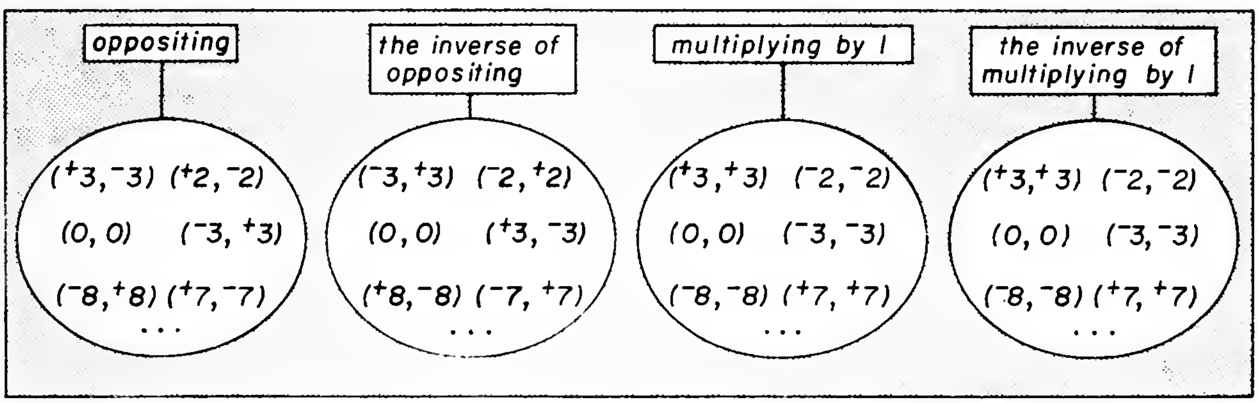
Teacher: But surely there's a pair in the second one that isn't in the first one?

Student: No!!

Teacher: So, the operation oppositing is its own inverse. Can you think of any other operation that is its own inverse? How about some of the operations involving multiplication? Is multiplying by 2 its own inverse? Multiplying by 3?

Student: Multiplying by 1 is its own inverse.

Teacher: Let's see if it is.



Student: Yes, it is.

Teacher: Let's try some of the addition operations. Is any one of these its own inverse?

Student: Yes. Adding 0.

Teacher: What are some pairs that belong to adding 0?

Student: (⁻8, ⁻8), (⁺2, ⁺2), (0, 0), (⁻3, ⁻3).

Teacher: I'm running out of room up here. Can you suggest a way that I can save room and still show that (⁺3, ⁺3), (⁺2, ⁺2), (0, 0), (⁻3, ⁻3), (⁻8, ⁻8) all belong to the operation adding 0?

Student: Just put the name 'adding 0' up there beside the loop that's named 'multiplying by 1'.

Teacher: Like this?

Student: Yes.

Teacher: Wait a minute. That looks to me like you believe that:

> The operation adding 0 is the same
> as the operation multiplying by 1.

Do you believe that?

Student: Yes — No — I don't know---

Teacher: Well, let's see. Certainly the names are different. Does that necessarily mean that the operations are different?

Student: No.

Teacher: Can you think of something that has two (or more) different names?

Student: The number 2 has lots of names.

Teacher: Right. So maybe these operations are the same. Is the operation the name or is it the set of pairs?

Student: It's not the name.

Student: I don't think it's the set of pairs either.

Teacher: That's too bad, because that's exactly what it is. The operation **is** the set of ordered pairs. You may not like it, but that's the way it is.

Student: Then the operation adding 0 is the same as the operation multiplying by 1???

Teacher: Well, let's see if they're the same. Are the pairs that belong to them exactly the same? Let's look at them and see. Each one of you think of a pair that belongs to adding 0. Does it belong to multiplying by 1? Now each of you think of a pair that belongs to multiplying by 1. Does it belong to adding 0? Do you think anyone can find a pair that belongs to one of these and does not belong to the other? The operation adding 0 is the same as the operation multiplying by 1.

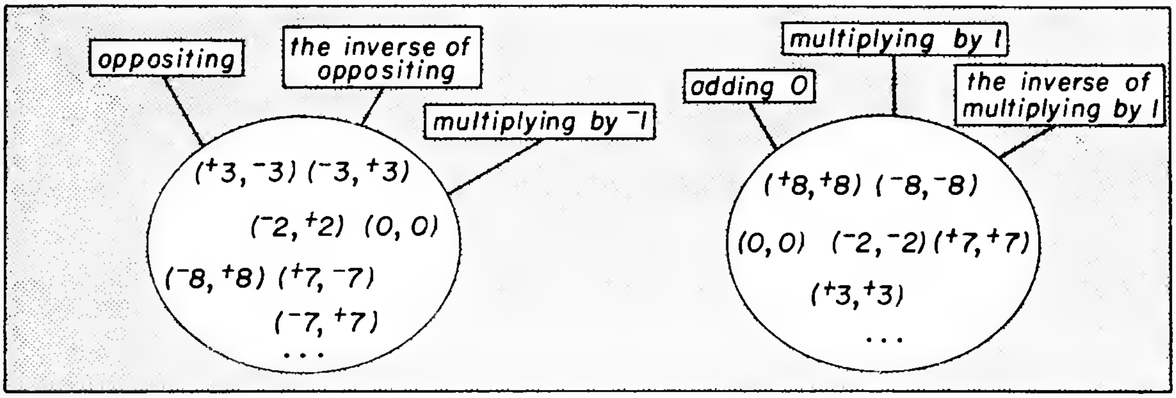
OK, let's go back to oppositing and its inverse. Mary pick a number. Now, oppositing takes you from the number $^{+}3$ to what number?

Student: $^{-}3$.

Teacher: Now, the inverse of oppositing [pointing to the proper name] takes you from $^{-}3$ to what number?

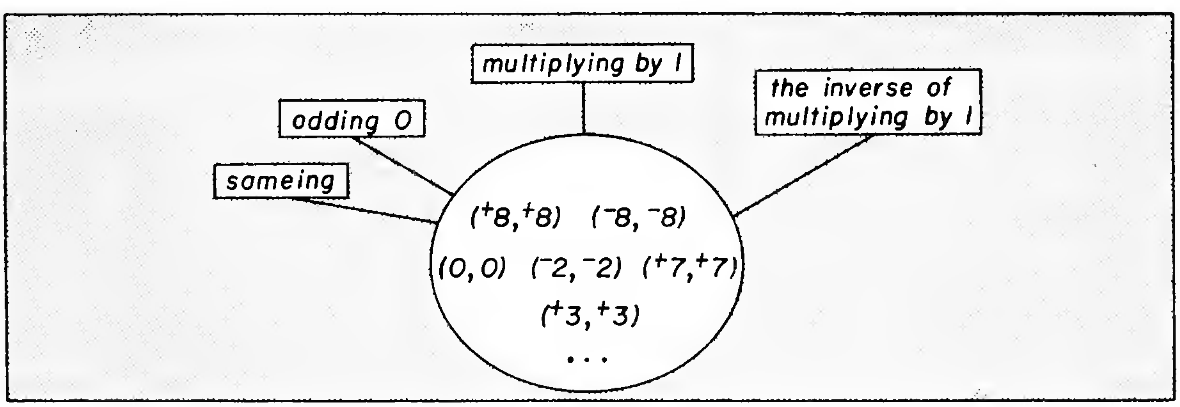
Student: $^{+}3$.

[Continue this way.]



Teacher:	Now, is this set of pairs an operation?
Student:	Yes! That's multiplying by 1.
Student:	It's adding 0.
Teacher:	Correct! So we already had this up here! Tell me some more pairs that belong to this operation.

Teacher:	Did you really go from $^-15$ to $^+15$ and then from $^+15$ to $^-15$ to get that pair $(^-15, ^-15)$?
Student:	No.
Teacher:	What did you do?
Student:	Well, the second number is just like the first.
Teacher:	Oh, the second number is <u>the same as</u> the first. Would that be true for each pair in this operation?
Student:	Yes.
Teacher:	Because that's true for each pair in this operation we sometimes call this operation: sameing.



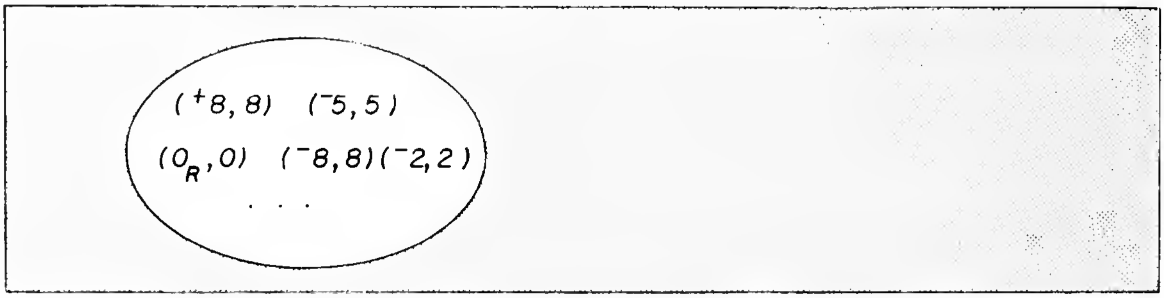
Student:	Is there any symbol for it like there is for oppositing?
Teacher:	Yes, we can use this sign '+'.

Let's consider absolute valuing. First, Course 1 uses the name 'arithmetic valuing' instead of 'absolute valuing' when considering the operation which maps the reals onto the numbers of arithmetic. I would also use the symbol '$||_A$' instead of '$||$'.

*

Teacher: Helen, pick a number. Tell me the number you picked and then tell me the arithmetic absolute value of the number. Remember, we are not using ambiguous names today.

Student: ($^+8$, 8).



Teacher: Now, I'll give you a number. What is its arithmetic absolute value? $^-5$.

Student: 5.

Teacher: 0.

Student: 0.

Teacher: Is the number 0 that I thought about the same as the number 0 that Jane thought about?

Student: No. Yours is a real number. Jane's is a number of arithmetic.

Teacher: To show that, let's write '0_R' today when we want a name for the real number zero and just '0' when we want a name for the number of arithmetic zero. (0_R, 0). Is this set of pairs an operation?

Student: Yes.

Teacher: What name do we give it?

Student: Sameing.

Teacher: Let's see. Is the number of arithmetic 8 the same as the real number $^+8$?

Student: No. So, this isn't sameing?

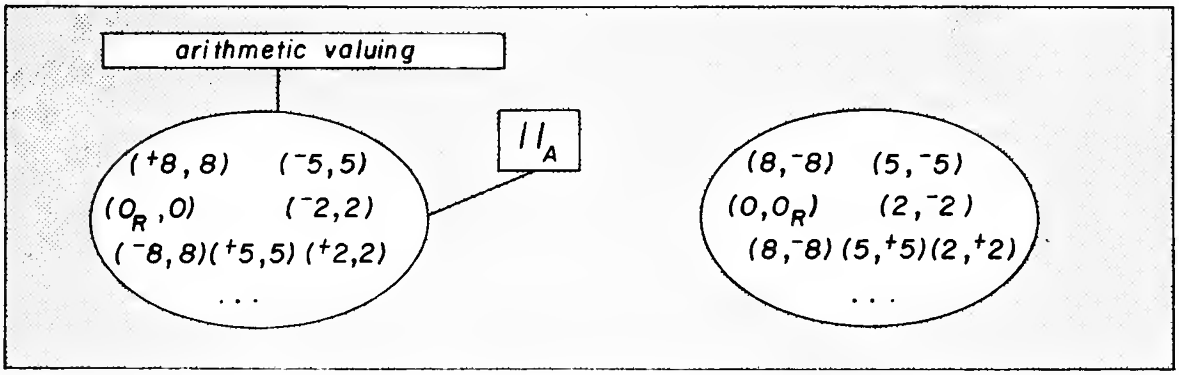
Student: We call it 'arithmetic valuing'.

Teacher: Good. So, arithmetic valuing takes you from a real number to a single number of arithmetic. By the way, have any of you ever carried out this operation before today?

Student: Well, yes. When we're doing some of these problems with real numbers, it's easier to multiply numbers of arithmetic.

Teacher: Yes. Every time you go from a real number to the corresponding number of arithmetic, you're actually performing the arithmetic valuing operation. So, you've been doing it for a long time even if didn't have a name for it.

Take another look at this operation. Now let's reverse the pairs.



Is this an operation?

Student: Yes.

Teacher: I'm going to pick a number of arithmetic. 5! Now, this set of pairs takes me from the number of arithmetic 5 to what?

Student: $^+5$.

Student: $^-5$.

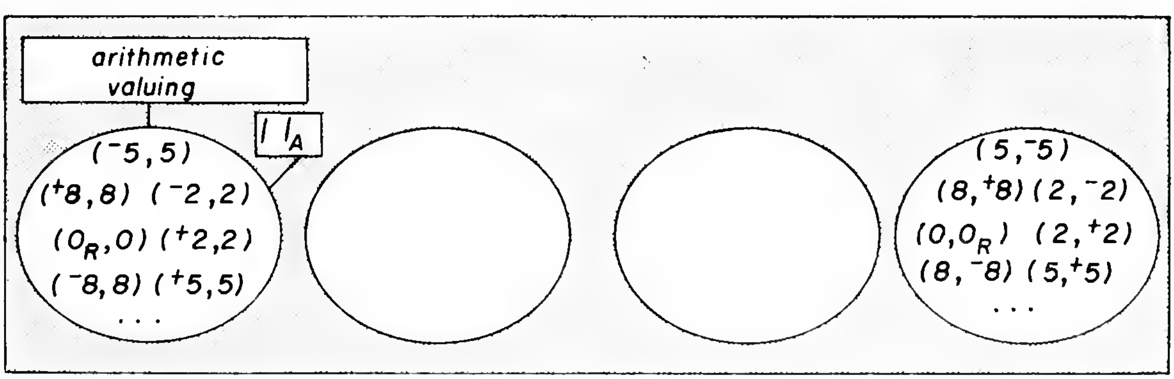
Teacher: Well, which is it? If this is an operation it <u>must</u> take you to a <u>single</u> number.

Student: There isn't just one single one. There are two.

Teacher: Then, is this an operation?

Student: No!

Teacher: Let's look at this set and sort of break it into two sets. What way of breaking it do you think I have in mind?



Student: Put '(8, $^+$8)', '(5, $^+$5)', '(6, $^+$6)' in the second loop.

Teacher: What shall I put in the third loop?

Student: '(5, $^-$5)', '(8, $^-$8)'.

Teacher: What are some others that would go in each one?

Does this take care of all the pairs you have in this set?

Student: No. (0, 0_R) is left out.

Teacher: Where does that go?

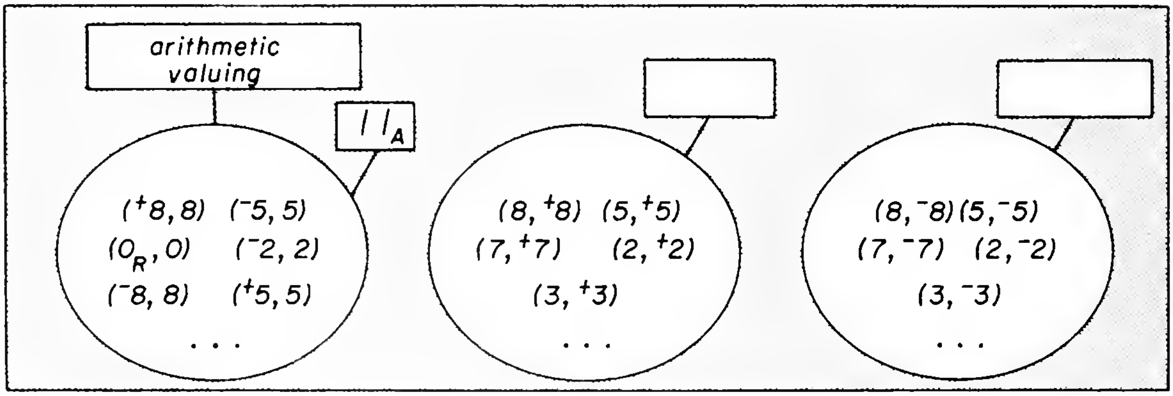
Student: Well, if you only want positives in the first one and negatives in the second one, it won't go any place.

Teacher: But we don't want to leave (0, 0_R) out in the cold? What shall we do?

Student: Put it in both. [If you don't get this answer, give it yourself.]

Teacher: Right. Are these operations?

Student: Yes.



Teacher: Now about names for them.

Student: Unabsolute valuing.

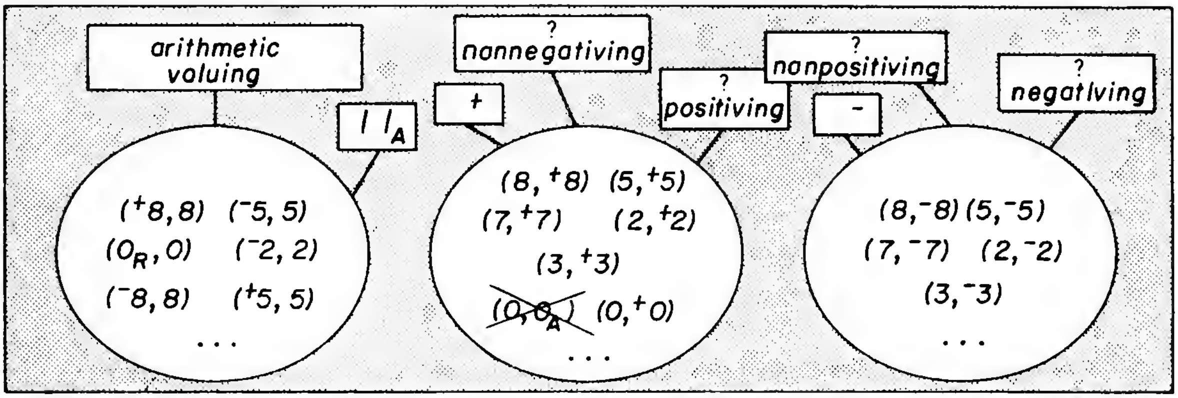
Student: The inverse of arithmetic valuing from the positive numbers.

Teacher: I don't think I would say it quite that way. I might say 'from the nonnegatives — '. Why would I use 'nonnegative' instead of 'positive'?

Student: To take care of 0.

Teacher: Correct. We might use the raised plus sign as a symbol for this operation. If we did that how would we write '(0, 0_R)'? Susan come up here and write it.

Susan: [writes] (0, $^+$0).



Teacher: Does that mean that the real number 0 is a positive number?

Student: No. It means it's nonnegative.

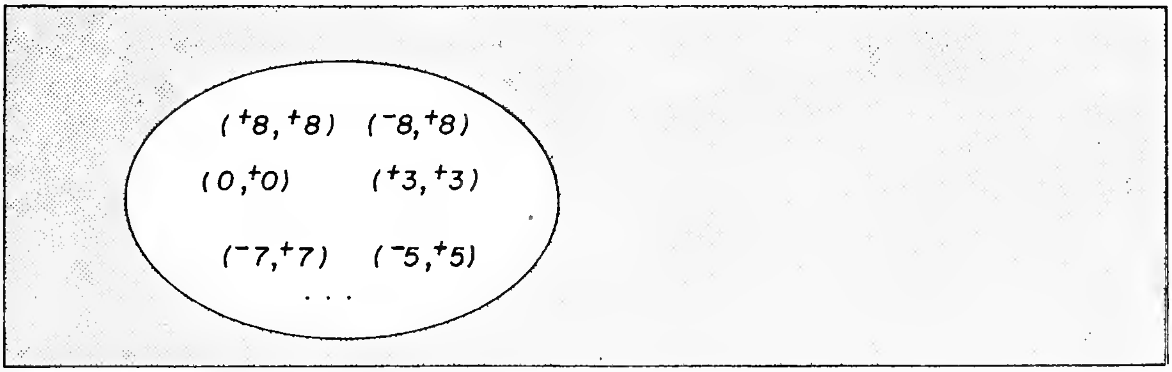
Teacher: Very good. Sometimes we call this operation 'positiving' but 'nonnegativing' might be a better name. How about names for this other operation? [And so on!]

*

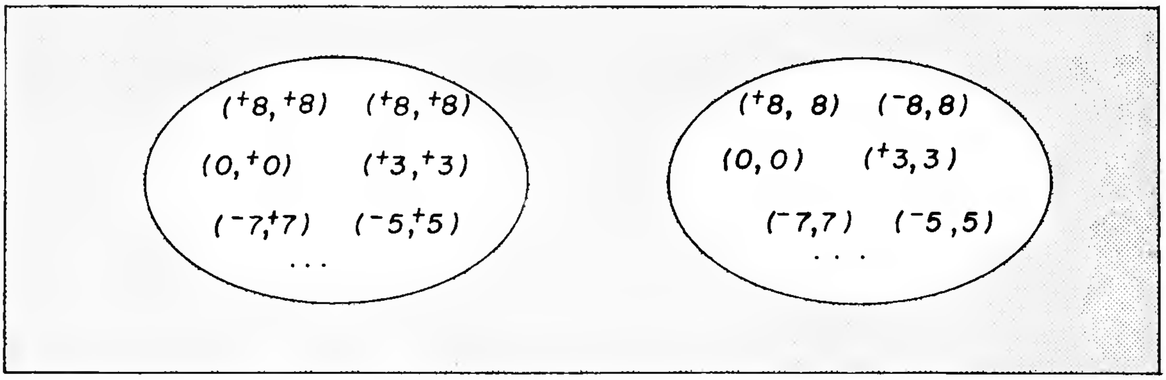
Now to develop the operation absolute valuing from real numbers to the nonnegative reals.

*

Teacher: Here are some pairs. Give me some more that are like them. [Usual questions about operation.]



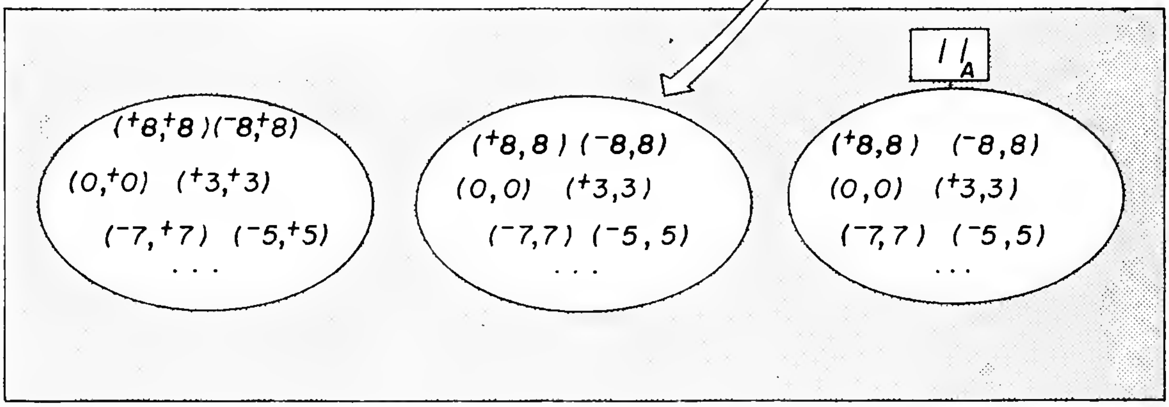
Now let's rewrite these names using the convention that a numeral for a number of arithmetic may be used to name a nonnegative real number.



Teacher: Suppose someone walked in the room and saw this second one. He might believe that we were thinking of what operation?

Student: Arithmetic valuing.

Teacher: Right. Let's look at some of the ordered pairs that belong to $||_A$.



So, if he saw this middle one, he wouldn't know which of the other two you had in mind. Actually people use the words 'absolute valuing' in naming both of the operations because the nonnegative reals behave like the numbers of arithmetic. The first is absolute valuing from the reals to the nonnegative reals. The second is absolute valuing from the reals to the numbers of arithmetic. We might use the symbol '$||_{R_0^+}$' and '$||_A$' to name them. However, most people give them the same name 'absolute valuing', and use the same symbol, '$||$'. Does that make them the same operation? [No.] From now on, in most of your work you will be using absolute valuing from the reals to the nonnegative reals. So, let's agree that:

Numerals which contain 'absolute value', '$||$', should be interpreted as numerals for real numbers — except in places where the context prohibits this interpretation.

A. Hart
Newsletter 10

A Mathematical Description of Course 1

As a starting point, we assume that the students are somewhat familiar with what we call the system of numbers of arithmetic. This system can be backhandedly described as the system of "unsigned" real numbers. It is isomorphic — with respect to ordering and the fundamental operations — to the system of nonnegative real numbers. The real number can be defined to be equivalence classes of ordered pairs of numbers of arithmetic. This is not done in the text. Thus, in terms of concepts with which students become acquainted later, each real number is a relation among the numbers of arithmetic; for example, $^+2$ is the relation of being 2 greater than, and $^-\sqrt{3}$ is the relation of being $\sqrt{3}$ less than.

Although the authors may think of a number of arithmetic as a set-theoretic entity of a certain kind — a set of sets of ordered pairs of finite cardinal numbers — and of a real number as a set of ordered pairs of numbers of arithmetic, the familiarity with the numbers of arithmetic which may be expected of students has, of course, a different basis. A student's feeling of being acquainted with the numbers of arithmetic probably has its origin in his experiences of using numerals for these numbers to formulate accepted answers to questions concerning measures of magnitudes such as lengths, areas, weights, etc., and of manipulating such symbols to obtain accepted answers to further questions of this kind. That these numerals, since they function as nouns, refer to entities of some kind, is an inference which, justifiably or not, he makes without much conscious thought. For him, numbers of arithmetic are things whose names occur in a characteristic way in sentences about measures of magnitudes.

This attitude toward the numbers of arithmetic makes a good starting point from which to develop a similar feeling for the real numbers. For example, this morning the dollar-measure of the money in my pocket was 3.59, and it is now 3.22. By a physical process analogous to subtraction the magnitude of my cash-on-hand has decreased by a magnitude whose dollar-measure is 0.37. Speaking somewhat loosely, the magnitude of my solvency has undergone a change which involves both a magnitude and a direction. This change can be described, as above, by giving its direction and a measure of its magnitude. Can it, itself, be measured? It seems reasonable that changes in magnitude should be measurable, but that their measures will be numbers of some kind other than the numbers of arithmetic, which are measures of magnitudes. That there are such other numbers — the real numbers — becomes an article of faith, on the same level as the student's belief in the existence of the numbers of arithmetic. In particular, the dollar-measure of the change in the magnitude of my cash-on-hand is the real number $^-0.32$.

It is to be noted that, on its own level, the above method of calling attention to the real numbers parallels the author's underlying conception of the real numbers as relations among the numbers of arithmetic. Magnitudes are measured by numbers of arithmetic, and changes in magnitude — which can be identified with relations among magnitudes (for example, the relation of being 0.37 dollars less than) — are measured by relations among the numbers of arithmetic.

Throughout the course students are led to build new mathematical concepts from earlier ones by ways which, like the foregoing, parallel the relationships which the former have to the latter in the underlying mathematical philosophy adopted by UICSM. As a result of such an approach, students see the parts of mathematics they study as parts of a single subject rather than as a collection of somewhat disparate topics. Consequently they have a better understanding of what mathematics is, and develop more power and freedom to make use of what they learn.

Returning to the students' observations of real numbers, it is now easy to discover how to compute the measure of the change in a magnitude which is the result of two successive changles whose measures are known. This discovery focuses attention on a certain operation on real numbers and motivates practice in finding the result of applying this operation to given real numbers. Thus students first discover the operation of addition of real numbers and become convinced of its utility before deciding that it is reasonable to call this operation 'addition'. Among the advantages of this approach is the by-passing of certain difficulties which may arise when addition of real numbers is introduced by a definition which, in part, tells the student that in order to find the <u>sum</u> of a positive and a negative number he should begin by finding the <u>difference</u> of the two corresponding numbers of arithmetic.

A similar device, dealing with rates of change, draws the students' attention to another operation — multiplication of real numbers — and stimulates him to practice it. The procedures which students develop for computing sums and products of real numbers lead easily to an awareness of the isomorphism — as far as concerns addition and multiplication — between the system of the numbers of arithmetic and that of the nonnegative real numbers. Quoting the text [page 322].

> addition and multiplication of nonnegative real numbers are "just like" addition and multiplication of the corresponding numbers of arithmetic,

The isomorphism just referred to suggests that it will often not be confusing to use the same symbol (for example, '3') either as a name for a number of arithmetic or as a name for the corresponding nonnegative real number (i.e., either as a name for the number 3 of arithmetic or as a name for the real number $^{+}3$).

Students' previous experience with the operations of addition and multiplication of numbers of arithmetic makes it easy for them to become aware of the commutative, associative, and distributive properties of these operations, and of the neutral character of the numbers 0 and 1. They see the value of being aware of these properties by discovering that they can be used to predict the successful outcome of common shortcuts. This is also preparation for discovering proofs. The isomorphism previously noted suggests that addition and multiplication of real numbers may have similar commutative, associative, and distributive properties, a suggestion which is strengthened by carrying out some computations. Such testing by computation prepares students to recognize, accept, and use instances of, for example, the associative principle for multiplication, such as:

$$^{+}385.7 \cdot (^{-}56.2 \cdot \frac{^{+}5}{2}) = (^{+}385.7 \cdot {}^{-}56.2) \cdot \frac{^{+}5}{2}$$

without testing them.

At this point students are still working with only the arithmetic of the real numbers. They have not yet been introduced to variables and so have no efficient way of stating the principles, for example:

$$\forall_x \forall_y \forall_z \; x(yz) = (xy)z,$$

whose instances they now recognize and accept. This introduction comes early in their study of Course 1. By this time they have gained sufficient familiarity with instances of the principles to have little trouble in formulating the principles for themselves once the appropriate lingusticdevices of variables and quantifiers have been introduced.

In Chapter 3 of Course 1 students use instances of these "basic principles for real numbers" at appropriate points to justify computational short cuts. This is valuable preparation for learning to derive theorems from the basic principles. The interplay between the process of dicovering real number properties through computation, and of formulating such discoveries and deriving the formulations from the basic principles, has much to do with the growth of the students' understanding of mathematics and of what mathematics is.

UICSM students are unlikely (to say the least) to adopt the usual lay viewpoint that the world's greatest mathematician is the idiot-savant who can perform the most astonishing feat of mental computation. The value of the interplay between discovery and proof is underlined by the first part of Hadamard's statement: "The object of mathematical rigor is to sanction and legitimatize the conquests of intuition, and there never was any other object for it."

In Course 1, students are led to conceive of a singulary operation, such as that of adding $^{-}5$ or of multiplying by $^{+}2$, as a set of ordered pairs. For example, one member of adding $^{-}5$ is the pair $(^{+}7, {}^{+}2)$. Defining subtraction as the inverse of addition — in particular, subtracting $^{-}5$ is the operation that undoes what adding $^{-}5$ does — leads rapidly to recognition of the fact that the pairs belonging to a subtracting operation are the converses of those which belong to the corresponding adding operation. Example: because $(^+7, ^+2)$ belongs to adding $^-5$, $(^+2, ^+7)$ belongs to subtracting $^-5$. Moreover, it turns out, on comparison of their members, that the operation of subtracting $^-5$ is the same as the operation of adding $^+5$. In general, subtracting a real number is the same as adding its *opposite*. (In *Course 1*, this is formulated as the *Definition principle for subtraction of real numbers*:

$$\forall_x \forall_y \; x - y = x + -y.$$

Together with the *principle of opposites*:

$$\forall_x \; x + -x = 0$$

and the previously mentioned principles concerning addition and multiplication it furnishes a basis from which to derive theorems about subtraction.)

Dividing by a nonzero real number is introduced as the inverse of multiplying by that number, and an exercise suggests that multiplying by 0 has no inverse. The further development of this insight is reserved for later in *Course 1*.

The introduction of the basic concepts involved in the structure of the real number system as an ordered field is completed by defining the relation less-than. After doing this there is some practice with the notations '$<$', '$>$', '$\not<$', '$\geq$', etc. Solution of inequations is taken up later in *Course 1* and the deductive organization of the theory of order is undertaken in *Course 3*. However, in *Course 1*, the use of '$<$', etc., significantly increases the variety of exercises which can be formulated. It also gives students an opportunity to become accustomed to considering inequations to be as "natural" as equations are.

Another important topic in *Course 1* is the arithmetic valuing operation. For pedagogical reasons this operation is defined, here, as an operation from the real numbers to the numbers of arithmetic. Its converse is the union of the two operations "positiving", which map the numbers of arithmetic on the "corresponding" nonnegative and nonpositive numbers, respectively. The three operations — arithmetic valuing, positiving, and negativing — make it possible, in *Course 1*, to formulate definitions of addition and multiplication of real numbers in terms of the corresponding operations on numbers of arithmetic and, in general, to pass readily back and forth from one system to the other. In our later work we introduce the more usual use of 'absolute value' to denote a mapping of the real numbers on the nonnegative real numbers. This second mapping is, of course, the composition of the positiving operation with the original arithmetic valuing operation.

*

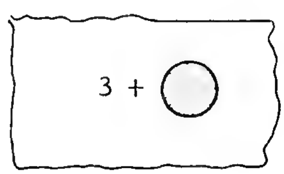
The purpose of *Course 1* is twofold: to help the students become proficient in the elementary techniques of symbol-pushing used in simplifying algebraic expressions, and to lead them to discover, prove, and use the theorems about numbers which justify these techniques. The attainment of the first of these goals — mastery of the skills whose practice makes up the bulk of a traditional secondary school mathematics program — is, of course, a *sine qua* *non* for further progress in mathematics. Efforts expended toward the second goal have turned out, perhaps rather unexpectedly, to provide strong motivation for attaining the first. UICSM students of all degrees of mathematical aptitude have found the discovery and proof of theorems to be an exciting and rewarding experience. But, they also find that errors in manipulation place barriers in the way of discovering provable theorems!

Work toward either of these goals thus helps in attaining the other, and the attainment of either is easier for one who has a clear understanding of some of the purposes for which letters are used in mathematical language. Two of these are particularly relevant to the work of *Course 1*: the use of letters as real (or: free) variables, and their use as apparent (or: bound) variables. For example, in the expression:

$$\int_1^x (t + 2y)^2 \, dt$$

the 'x' and the 'y' are free ("*really* variables") and the 't's are bound ("only *apparently* variables"). The 'x' marks a position that is open to substitution; the 't's do not, but can, without changing the meaning of the expression, be replaced by occurrences of any other "dummy" symbol. (Since the above use of 'real variable' conflicts with its conventional meaning in, say, 'theory of functions of a real variable', it is not introduced into the text. *However, it will be convenient to continue this usage in what follows.*)

Since a real variable merely marks a place where substitutions "can" be made (i. e., an argument-place) and, once its domain is specified, delimits the class of expressions that can be substituted for it, it turns out to be enlightening to simulate the notion of free variable by using actual holes in the paper. For example:



The notion of substitution is conveyed by writing a numeral on a second sheet of paper and placing this sheet under the first so that the numeral appears in the hole. The next step is to use frames
(□, △, ▽, ▭ , etc.) in which numerals and other appropriate expressions can be written. (Such expressions may themselves contain frames and so allow for further "substitutions".) Finally, letters are introduced as real variables. The first expressions using letters as variables are those on pages 140ff.

An alternative concept often introduced in preference to the concept of a real variable is that of an "unknown". In line with this latter concept the 'x' in, say, '$x + 3 = 5$' is a numeral for a "definite, but unspecified" number. One advantage claimed for this concept is that it leads students to manipulate variables in the same way as they have learned to manipulate (other) numerals. This is, of course, a desirable end, and it should be obvious that the same result is attained when such occurrences of 'x' are construed as real variables. Clearly, a symbol for which numerals can be substituted fares, during manipulation, exactly as do numerals. While one is symbol-pushing he treats a real variable as if it were a numeral. The inadequacy of the concept of 'x' as an unknown is first apparent when one considers equations such as '$x + 1 = x$' and '$x^2 + x - 2 = 0$'. The "definite but unspecified" number which 'x' is supposed to represent fails, in the first case, to exist, and, in the second, to be unique. Students have some difficulty in extending their concept of number to include such queer entities. The concept of 'x' as an indeterminate is, of course, something entirely different. It seems unlikely that this concept, valuable as it is in modern algebra, would be helpful to students at this stage. A fourth concept, that of a random variable, i. e., a measurable function, is also beyond consideration at this level.

The role that apparent variables play in mathematical language is also one played in other languages by pronouns — and, more frequently, by common nouns. This role is that of linking operators with argument places. For example, the role of the 't's in:

$$\int_1^x (t + 2v)^2 \, dt$$

is to link the operator '$\int_1^x \ldots d\ldots$' with the argument-place occupied by 'u' in the expression '$(u + 2v)^2$'. [Note 3] Natural languages are quite irregular in the operations they use and the ways in which these are linked with argument-places. However, the following somewhat archaic-sounding sentence illustrates how the operator (more specifically: the quantifier) 'each' is linked by means of a common noun and two pronouns to two argument-places:

Each man, as he is honorable, so shall he prosper.

In the same vein, the following sentence illustrates the linking of two quantifiers, each to its appropriate argument-places:

Each man, and each woman, as he cherishes her,
so shall she cleave to him.

In mathematical language constructions similar to those in the two examples above are used in the statements of two of the basic principles for real numbers:

For each x, $x + 0 = x$

and:

For each x, for each y, $xy = yx$

In Chapter 4 of Course 1, these principles are written as:

$$\forall_x\ x + 0 = x \text{ and: } \forall_x \forall_y\ xy = yx.$$

The discussion and exercises leading up to such use of apparent variables occur on pages 116 and 117 of Course 1.

Having learned linguistic devices which make it possible to state their discoveries about real numbers concisely, it is natural for students to wonder how many such discoveries they need make. Is it possible that some can be justified, or even predicted, on the basis of others? It is easy to see that one can disprove a generalization by finding a counter-example; but, how can one prove a generalization? If one equates the property of being provable with that of following logically from the basic principles, it turns out that a generalization is provable if one has a uniform method for showing that each of its instances follows from appropriate instances of the basic principles. Such a method for showing that instances of, for example:

$$\forall_x\ 3(x2) = 6x$$

follow from basic principles is illustrated in the case of the instance '$3(5 \cdot 2) = 6 \cdot 5$' by:

	$3(5 \cdot 2) = 3(2 \cdot 5)$	[commutative principle for multiplication]
	$3(2 \cdot 5) = (3 \cdot 2)5$	[associative principle for multiplication]
	$(3 \cdot 2)5 = 6 \cdot 5$	[$3 \cdot 2 = 6$]
Hence,	$3(5 \cdot 2) = 6 \cdot 5$.	

(In Course 1, $3 \cdot 2 = 6$ is accepted as a "computing fact". The derivation of such computing facts from definitions, such as '$2 = 1 + 1$', and the basic principles is taken up in Course 3.)

Using, say 'a' as a real variable, the testing method illustrated above can be indicated by a test-pattern:

	$3(a \cdot 2) = 3(2 \cdot a)$	[cpm]
	$3(2 \cdot a) = (3 \cdot 2)a$	[apm]
	$(3 \cdot 2)a = 6a$	[$3 \cdot 2 = 6$]
Hence,	$3(a \cdot 2) = 6a$.	

The foregoing is conceived as a pattern which can be used to test any instance of '$\forall_x\ 3(x2) = 6x$'. For example, all that is needed to show that, say, '$3(7 \cdot 2) = 6 \cdot 7$' is a consequence of appropriate instances of the basic principles is to substitute '7' for 'a'. The test-pattern can be used to confound anyone who claims to have a counter-example to the generalization and, so, merits being admitted as a proof of the generalization. [Note 5]

If one compares the foregoing test-pattern with the "work" expected of a beginning student who is to "simplify" '3(x · 2)' to '6x' the connection between simplifying algebraic expressions and proving elementary theorems about fields becomes evident. Of more immediate import is the fact that students can learn to use their knowledge of basic principles both to discover computational shortcuts and to discover errors in procedures (such as simplifying '1 + 2x' to '3x') which they have adopted in hope, but which have proved to lead to disaster.

In connection with simplifying expressions, the notion of equivalent expressions is introduced.

After considerable practice in simplifying algebraic expressions — both through strict adherence to the basic principles and by the free-wheeling methods which successful application of the former procedure suggest — students are brought back to something nearer mathematics than symbol-pushing by a short discussion on theorems and basic principles and an opportunity to organize their knowledge of subtraction and division. In Course 1, students discover and prove a considerable number of theorems.

Examples:

$$\forall_x\ x \cdot 0 = 0$$

$$\forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y$$

$$\forall_x \forall_y\ -(x - y) = y - x$$

$$\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$$

$$\forall_x \forall_y \text{ if } xy = 0 \text{ then } x = 0 \text{ or } y = 0$$

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$$

$$\forall_x \forall_{y \neq 0}\ -\frac{x}{y} = \frac{-x}{y}$$

Interspersed with this "theoretical" development are numerous simplification exercises on which students can perfect the symbol-pushing techniques whose justification lies in the theorems they have proved.

*

Summarizing, it should be noted that students begin by becoming acquainted with the real numbers through comparing and contrasting the ways in which they and the more familiar numbers of arithmetic can be used in solving physical problems. On the basis of this acquaintance they accept some basic principles which describe the basic properties of the fundamental operations on real numbers. Having learned to use variables and quantifiers in order to give concise statements of these principles, they next learn how to derive theorems from them. Along the way, the procedures used in leading students to discover much of this theoretical structure for themselves give ample opportunity for them to develop the manipulating skills whose rationale is the basic structure and which are needed in furthering its development. At all times students feel that they are discussing a "real" subject matter — the real numbers — but their actual procedure is much like that of one who would abstract the notion of a field from examples like that furnished by the real number system, and, having done so, would proceed to develop the elementary theory of this kind of mathematical structure. This early introduction of proof pays off throughout the course, and particularly in the deductive development of Euclidean plane geometry in Course 2.

*

Notes

1. A real variable is a symbol and is no more subject to change than is any other physical object. Hence the word 'variable' tends to be misleading. On this subject I can't refrain from quoting from Professor E. J. McShane's Theory of Limits (MAA Film Manual, No. 2, p. 3). Discussing common misconceptions concerning the limit concept, Professor McShane writes:

> Incidentally, all these offside notions have one bad feature in common. They all involve the idea that b_j is "doing something". ... There is something alluring about the idea that j has a personality and "goes from 1 to 2 to 3" and so on, and that b_j "does" something like getting closer to 0. But this trick of personifying j and b_j is misleading even for sequences, and in more complex situations it is worse. ...
>
> In case you have been thinking of the b_j as a brisk little thing, jumping from b_1 to b_2 to b_3, and so on, don't blush too hard. Not so long ago that was a pretty customary way of thinking of it, and the custom died hard. ...

2. Here is an amusing example of a use of pronouns that strictly parallels the use of free variables in equation-solving problems:

Identify the person described by the following sentences:

> He was a president of the United States.
> He commanded American armed forces.
> He has (had) a name consisting of six letters.
> He has (had) another name consisting of ten letters.
> He died in the nineteenth century.

The word 'he' is, here, a real variable whose domain is the set of all male human beings (living or not). The problem is to find such a person who satisfies (is a solution of) all five sentences.

3. This linking function of variables is well expounded by Quine in his Mathematical Logic (Cambridge, 1958), pp. 67-71. As Quine points out, the concept goes back at least to Peano's Formulaire of 1897, and was also exploited by Moses Schonfinkel in his "Uber die Bausteine der Mathematischen Logik" (Math. Annalen, vol. 92 (1924), pp. 305-316). Schonfinkel's position is that

> ... the variable in a logical proposition serves only as a mark distinctive of certain argument-places and operators as mutually relevant. ...

4. The inadequacy of using real variables (instead of quantifiers and apparent variables) in stating generalizations should be apparent to anyone who has attempted to teach the distinction between, say, continuity at each point of a set and uniform continuity on that set. If the use of quantifiers and apparent variables turned out to be difficult for students to master, one might argue for reserving these concepts for a course in function theory. However, a great deal of experience shows that UICSM students have no particular difficulty with these concepts. Early familiarity with them should place those students who continue in mathematics in a good position to appreciate the pointwise-uniform distinction. Actually, UICSM is more interested in the great majority of students who will not develop into mathematicians. Attention to linguistic matters such as the various roles played by variables appears not only to make it easier for such students to make sense out of mathematics, but also sharpen their appreciation for correct use of language, generally.

5. It should perhaps be noted that, while '$3(5 \cdot 2) = 3(2 \cdot 5)$' is not an instance of the cpm, it is a consequence of the instance '$5 \cdot 2 = 2 \cdot 5$'. This is the case because '=' refers to the logical relation of identity and, so, multiplication having been accepted as an operation, '$\forall_x \forall_y \forall_z$ if $x = y$ then $zx = zy$' is a tautology.

H. E. Vaughan
Newsletter 6

A Note on Inverse Operations

The work on inverse operations in Course 1 culminates in the students' discovery that

(1) for each number of arithmetic x,

SUBTRACTING x

is the same thing as

THE INVERSE OF ADDING x

and (2) for each nonzero number of arithmetic x

DIVIDING BY x

is the same thing as

THE INVERSE OF MULTIPLYING BY x.

We capitalize on these agreements by defining, for each real number x:

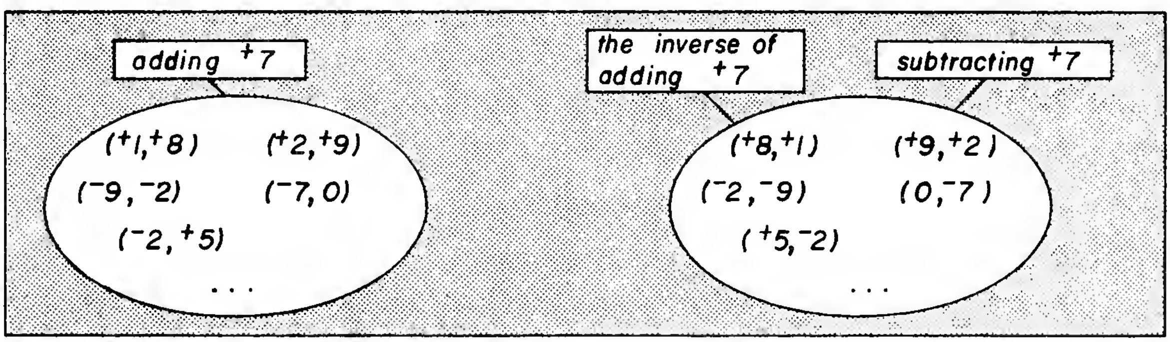
SUBTRACTING x

to mean:

THE INVERSE OF ADDING x.

Here is one way of using lists of ordered pairs to give the students meaningful practice in using this definition. Starting with a list of pairs belonging to ADDING $^+7$, we quickly get a corresponding list of pairs which belong to THE INVERSE OF ADDING $^+7$.

Since we can also think of the latter list as a list of pairs belonging to SUBTRACTING $^+7$, we can write:



Using the lists, the students solve problems such as these:

$$^+8 - {}^+7 = ?$$
$$^+9 - {}^+7 = ?$$
$$^-2 - {}^+7 = ?$$

Now consider the problem:

$$^-16 - {}^+7 = ?$$

When the students see that the problem can't be solved using the pairs we have listed, have someone <u>guess</u> at the answer. The discussion in my class went something like this:

Teacher: Someone guess at the answer to:

$$^{-}16 - {}^{+}7 = ?$$

Student: $^{-}9$.

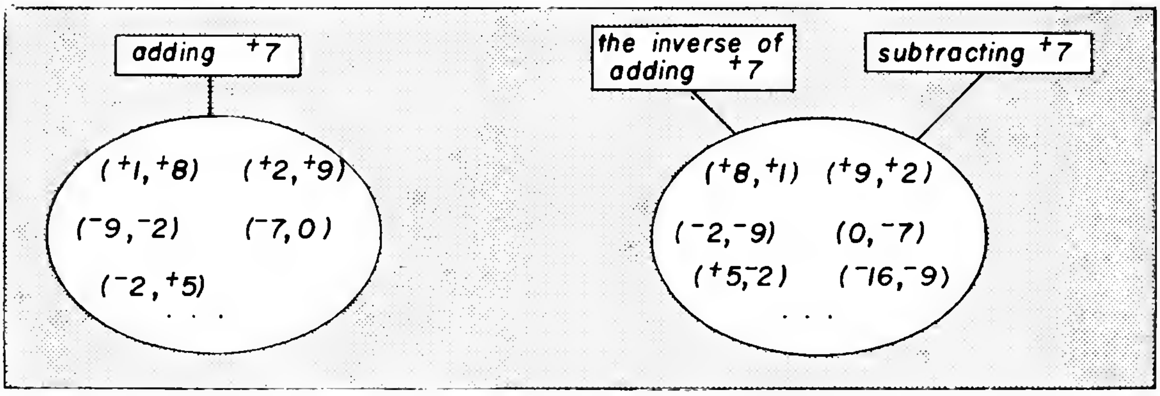
Teacher: In saying that:

$$^{-}16 - {}^{+}7 = {}^{-}9$$

we are saying that a certain ordered pair belongs to SUBTRACTING $^{+}7$. What pair is this?

Student: $(^{-}16, {}^{-}9)$

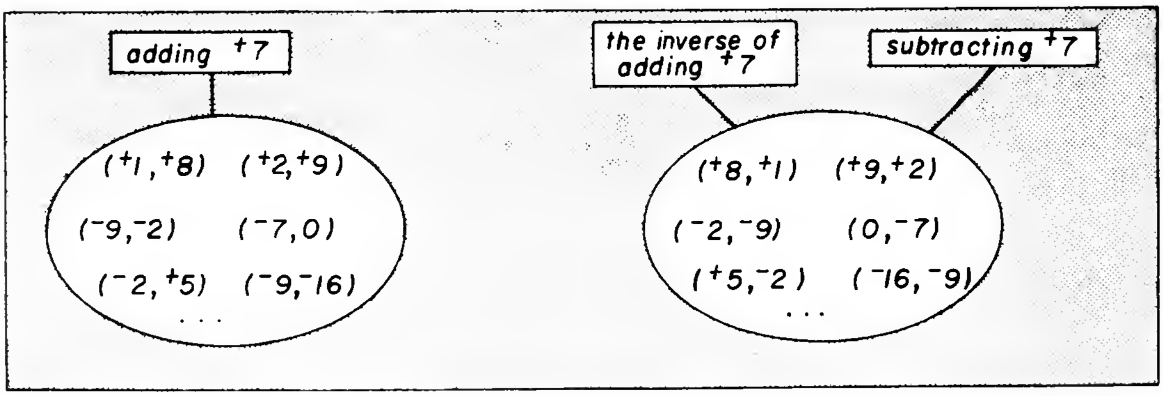
The teacher adds '$(^{-}16, {}^{-}9)$' to the list for SUBTRACTING $^{+}7$. The board now has these entries:



Teacher: Since SUBTRACTING $^{+}7$ is the same thing as THE INVERSE OF ADDING $^{+}7$, it follows that the pair $(^{-}16, {}^{-}9)$ belongs to SUBTRACTING $^{+}7$ just if [if and only if] <u>what</u> pair belongs to ADDING $^{+}7$?

Student: $(^{-}9, {}^{-}16)$

The teacher adds '$(^{-}9, {}^{-}16)$' to the list for ADDING $^{+}7$. The board now reads:



Teacher: Does $(^{-}9, {}^{-}16)$ belong to ADDING $^{+}7$?

Student(s): No!

Teacher: Is it possible, then, for $(^{-}16, ^{-}9)$ to belong to SUBTRACTING $^{+}7$?

Student(s): No!

Teacher: Hence it can't be the case that:

$$^{-}16 - ^{+}7 = ^{-}9$$

is a true statement.
Someone else take a guess at the answer.

Succeeding guesses are tested in the manner indicated until (hopefully) a student or (if necessary) the teacher submits the correct answer.

The reasoning involved is simply illustrated by:

$\forall_x \forall_y \forall_z \; x - y = z$
if and only if
(x, z) belongs to SUBTRACTING y
if and only if
(z, x) belongs to ADDING y

A similar approach can be used in discussing division of real numbers [Unit I, pages 92-94]. The reasoning in this case is illustrated by:

$\forall_x \forall_{y \neq 0} \forall_z \; x \div y = z$
if and only if
(x, z) belongs to DIVIDING BY y
if and only if
(z, x) belongs to MULTIPLYING BY y

In addition, we show that the converse of MULTIPLYING BY 0 is not a function and, hence, is not an operation.

R. Kansky
Newsletter 10

Principal Operator

Few people begin difficult undertakings without making arrangements to have the proper tool at hand. In addition to having the tool, they also wish to have practice in using it in a less complicated situation. In the work on unabbreviating expressions we can introduce a tool that can be used in more complex situations and we can give the students the necessary practice in using it. This tool is the notion of principal operator.

When all the grouping symbols have been restored in unabbreviating an expression, the principal operator is the one which corresponds with (or links) the pair of outermost grouping symbols. For example, the principal operator in '3 + 4 × 5' is '+' because unabbreviating this expression gives us:

$$\{3 + (4 \times 5)\}$$

The principal operator in:

$$\{([3 \times 3] + [5 \times 3]) - 10\}$$

is '-'. It may be helpful not to omit the outermost grouping symbols as quickly as we have been doing in the past.

[An expression whose principal operator is '+' is sometimes called 'an indicated sum', one whose principal operator is 'x' is called 'an indicated product'.] Before assigning exercises from the text, we might ask the class to name the principal operator in each of several exercises. A student who is able to do this immediately in exercises such as:

$$3 + 8 - 2 + 5,$$

$$5 \times 6 \div 3 \times 4,$$

and:

$$4 + [5 - 3 - 1] \times [7 + (2 \times 4)]$$

must be using the conventions correctly.

There are several places in the first four units where the tool, principal operator, may be used to advantage. One of these is discussed in the following pages. In other articles, we will bring other applications of the principal operator notion to your attention.

* * *

Pattern-Sentences, Instances, Consequences

Distinguishing between

> sentences which are instances of a principle
> [and, so, are also consequences of that principle]

and

> sentences which are consequences of a principle
> and are not instances of that principle

is often quite difficult.

We hope that the suggestions which follow will help resolve this difficulty.

Following page 64 of Course 1, arrange to have:

The commutative principle for addition

Instances

$$\begin{cases} 3 + 7 = 7 + 3 \\ 17 + 2\frac{1}{2} = 2\frac{1}{2} + 17 \\ 4.5 + 2.98 = 2.98 + 4.5 \end{cases}$$

on one section of your blackboard.

Now begin something like this:

Teacher: Write an instance of the commutative principle for addition on your paper. John, if Mary has actually written an instance of the commutative principle for addition what is one symbol which must be on her paper?

[From here on we will picture the blackboard after the question has been answered.]

=

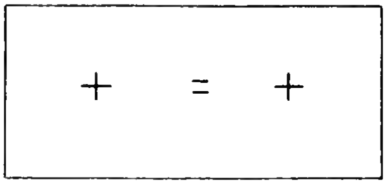
or

+

Teacher: What other symbols must she have?

= + +

[We hope you have to change the above.]



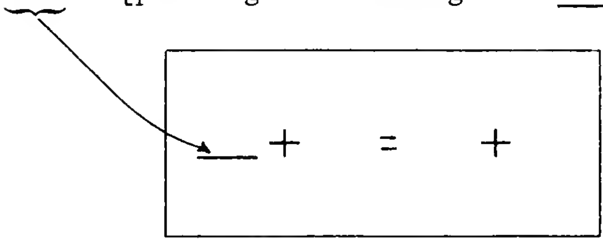
Teacher: What else does she have?

Student: Numerals.

Teacher: Where will one of these numerals be written?

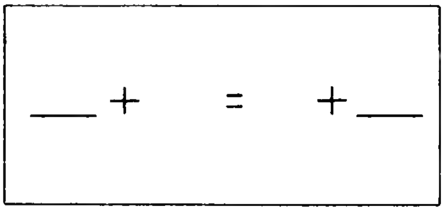
Student: Before the first '+'.

Teacher: Here? [pointing and drawing the '____'.]



Student: A copy of that numeral must be written after the second '+'.

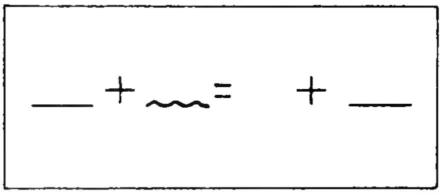
Teacher: Let's indicate that we want a copy of that numeral by using another '____'.



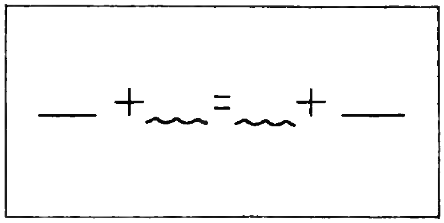
Teacher: What else must Mary have?

Student: A numeral after the first '+'.

Teacher: Does that have to be a copy of the numeral we placed before the first '+'? [No!] Let's indicate that by using a '   '.



[Continue until you have a pattern-sentence]



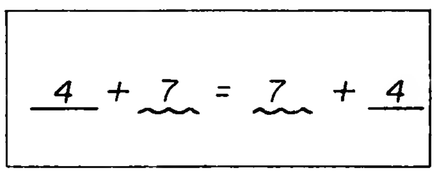
Teacher: We call this a pattern-sentence for the commutative principle for addition. If you have actually written an instance of the commutative principle for addition, then if we put the proper numerals in the blanks in this pattern-sentence we will have a copy of the sentence you wrote.

Mary, tell me how to fill in these blanks so that we will have a copy of your sentence.

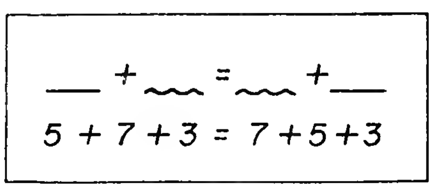
Student: Put a '4' above each '____' and a '7' above each '~~~'.

Teacher: Is this a copy of your sentence?

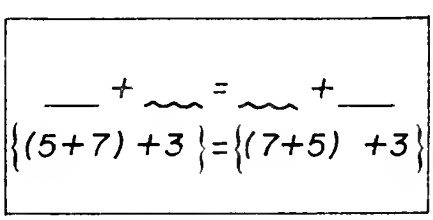
Student: Yes.



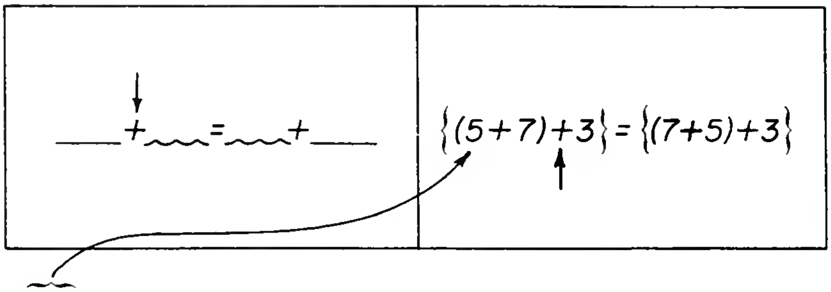
Teacher: Then your sentence is an instance of the commutative principle for addition. Look at this sentence. Is this an instance of the commutative principle for addition?



Let's unabbreviate it.



What is the principal operator on the left side of the pattern-sentence? What is the principal operator on the left side of the other sentence?



Teacher: If this is an instance, then we must be able to fill the blanks in the pattern sentence so that we get a copy. What shall we place above the '~~~' on the left side of the pattern-sentence?

Student: A '3'.

Teacher: Where else must a '3' go?

Student: Above the '~~~' on the right.

Teacher: What must go above the '____' on the left side?

Student: A '(5 + 7)' above each '____'.

(5+7) + 3 = 3 + (5+7)

Teacher: Is this a copy of our original sentence?

Student: No.

Teacher: Then our original sentence is not an instance of the commutative principle of addition. However, an instance of the commutative principle for addition can be used to help us decide if such a sentence as:

$$(759 + 78) + 95 = (78 + 759) + 95$$

is true.

Show: (759 + 78) + 95 = (78 + 759) + 95

____ + ~~~ = ~~~ + ____

What is the instance?

Student: '759 + 78 = 78 + 759'.

Show: $(759+78)+95=(78+759)+95$
$(759+78)=(78+759)$

Teacher: If we accept the commutative principle for addition then without doing any computation we must believe that '759 + 78' and '78 + 759' name the same number.

Show: $(759+78)+95=(78+759)+95$	
$(759+78)=(78+759)$	[cpa]
$(759+78)+95=$ ________	

What is the simplest way to complete this sentence so that we will have a true statement?

Student: '95 + (759 + 78)'.

Teacher: Too hard.

Student: '932'.

Teacher: Too hard.

Student: '(759 + 78) + 95'.

Teacher: Right!!!

Show:	$(759+78)+95=(78+759)+95$	
(*)	$(759+78)=(78+759)$	[cpa]
	$(759+78)+95=(759+78)+95$	

Teacher: Now how are we going to use (*) to show that the sentence at the top of the board is true?

[Deathly silence!!! But wait. If it doesn't come, go ahead.]

What does (*) tell you? (Mean to you?)

Student: (*) tells me that '(759 + 78)' and '(78 + 759)' are names for the same number.

Teacher: Do you have to use the name '(759 + 78)'?

Student: No.

Teacher: Do you <u>have</u> to use the name '(78 + 759)'?

Student: No.

Teacher: Can you use either name you decide you want to use?

Student: Yes.

Teacher: Let's look at:

$$(759 + 78) + 95 = (759 + 78) + 95$$

and decide whether we want to use the name '(759 + 78)' or the name '(78 + 759)'. Which one do you want to use?

Student: Keep a '(759 + 78)' on the left side and use a '(78 + 759)' on the right side.

Teacher: What shall I write?

Show: $(759 + 78) + 95 = (78 + 759) + 95$

$$(759 + 78) = (78 + 759) \quad [cpa]$$
$$(759 + 78) + 95 = (\underline{759 + 78}) + 95$$
$$(759 + 78) + 95 = (78 + 759) + 95$$

Is this what we wanted?

Student: Yes.

Teacher: We usually abbreviate all this as:

$$(759 + 78) + 95 = (78 + 759) + 95 \quad [cpa]$$

Is this sentence an instance of the cpa?

Student: No.

Teacher: Did we use an instance of the cpa?

Student: Yes.

Teacher: We say that

$$(759 + 78) = (78 + 759)$$

is <u>both</u> an instance <u>and</u> a consequence of the cpa.

Teacher: However,

(759 + 78) + 95 = (78 + 759) + 95

is a consequence but is not an instance of the cpa.

[Obviously, the dialogue above is intended only to indicate possible questions and answers.]

*

Now develop pattern sentences for the cpm, the apa, and the apm.

cpm: ____ × ~~~~ = ~~~~ × ____

apa: (____ + ~~~~) + ---- = ____ + (~~~~ + ----)

or ____ + (~~~~ + ----) = (____ + ~~~~) + ----

or ____ + ~~~~ + ---- = ____ + (~~~~ + ----)

or ____ + (~~~~ + ----) = ____ + ~~~~ + ----

apm: similar to those for the apa.

[Be certain that the students see that the four pattern-sentences for the apa are equivalent.]

* * *

[The following material is very much condensed.]

Teacher: Now let's examine this exercise.

72 + (45 + 63) + 85 = 85 + [72 + (45 + 63)]

We are told this is an instance of one of the four principles.

First, let's unabbreviate.

{[72 + (45 + 63)] + 85} = {85 + [72 + (45 + 63)]}
↑ ↑

What is the principal operator in the expression on the left side?

Student: The third '+'.

Teacher: In the expression on the right side?

Student: The fourth '+'. [The first '+' on the right side.]

Teacher: These '+' signs must match the principal operators in the two sides of one of the patterns. Which pattern do you want to try?

Student: ____ + 85 = 85 + ____

[72 + (45 + 63)] + 85 = 85 + [72 + (45 + 63)]

Teacher: Is this a copy of the unabbreviated sentence?

Student: Yes.

Teacher: Then the original sentence is an instance of the cpa.

Now look at this exercise, and unabbreviate. What is the principal operator on the first side? On the second side?

{(72 + 45) + (63 + 85)} = {72 + [45 + (63 + 85)]}

Let's examine the principles one at a time to see if this sentence fits one of the pattern sentences. We'll start with the cpa pattern sentence.

(72 + 45) + (63 + 85) = (63 + 85) + (72 + 45)

Teacher: Is this a copy?

Student: No.

Teacher: Let's try the pattern sentence for the apa

____ + (~~~ + ----) = (____ + ~~~) + ----

(72 + 45) + (~~~ + ----) = ((72 + 45) + ~~~) + ----

(72 + 45) + (63 + 85) = ((72 + 45) + 63) + 85

Is this a copy?

Student: No.

Teacher: Let's see if the second side of our sentence will fit the first side of the pattern:

72 + (~~~ + ----) = (72 + ~~~) + ----

72 + (45 + (63 + 85)) = (72 + ~~~) + (63 + 85)

Is this a copy of our sentence?

Student: Not quite.

[Now we can emphasize that if we believe that '72 + [45 + (63 + 85)]' and '(72 + 45) + (63 + 85)' name the same number, then we should believe that

'(72 + 45) + (63 + 85)' and '72 + [45 + (63 + 85)]' name the same number. Paying attention to the symmetric property of equality at this time will help in the proofs later.]

Teacher: So the sentence in this exercise is an instance of the apa..

We hope that this development using the tool, principal operator, will be helpful in the discussion of instances and consequences.

A. Hart
Newsletter 1

Proving the Division Theorem

Proving the division theorem may be difficult for some students. Mr. Jones and Mr. Edwards of Zabranchburg Junior High School had some ideas about this development. We thought you might find it interesting. If you duplicate and use it, we would like to know how it goes.

Mr. Edwards Teaches the

Zabranchburg High Math Class Again!

Remember Mr. Jones who teaches mathematics in Zabranchburg Junior High School and the principal, Mr. Edwards? Mr. Edwards took the class when Mr. Jones was ill. Mr. Edwards had not gone to the same college that Mr. Jones had. However, he knew that the class was studying basic principles and proving theorems. He couldn't stay in the room all hour but he wrote some homework, had his secretary duplicate it and told the students that they could have the hour to do the work. This is a copy of the assignment sheet.

Basic Principles:

(1) $\forall_a \forall_b\ a + b = b + a$ [cpa]

(2) $\forall_a \forall_b\ a \cdot b = b \cdot a$ [cpm]

(3) $\forall_a \forall_b \forall_c\ a + b + c = a + (b + c)$ [apa]

(4) $\forall_a \forall_b \forall_c\ abc = a \cdot (bc)$ [apm]

(5) $\forall_a \forall_b \forall_c\ (a + b) \cdot c = ac + bc$ [dpma]

(6) $\forall_a\ a + 0 = a$ [pa0]

(7) $\forall_a\ a \cdot 1 = a$ [pm1]

(8) $\forall_a \forall_b\ (a - b) + b = a$ [pr]

(9) $\forall_a \forall_{b \neq 0}\ (a \div b) \cdot b = a$ [pq]

Assignment. Using only these nine basic principles, and theorems you prove using them, prove:

$$\forall_x \forall_y \forall_z \text{ if } z + y = x, \text{ then } z = x - y.$$

* * *

Even Fred, "The Brain", was puzzled. He knew what to do if he could use the po and the ps. But Fred knew this wouldn't do because Mr. Edwards said use only the basic principles he had given them and theorems they could prove using these principles. Suddenly Fred thought — just maybe he could prove a theorem that would take the place of the po. He decided that 'pr' was an abbreviation for 'principle of remainders' and that he would have to use that.

Just what is the pr? It is given this way:

$$\forall_a \forall_b\ (a - b) + b = a$$

Now what does that mean? Fred decided to look at some instances of it. He can get an instance by replacing 'a' by the name of any real number he decides to choose. So he got:

$$\forall_b\ (1 - b) + b = 1$$

$$\forall_b\ (8 - b) + b = 8$$

$$\forall_b\ (-2 - b) + b = -2$$

$$\forall_b\ (0 - b) + b = 0$$

Suddenly Fred said, "Oh". That last instance looked mighty good. He decided he had found something he could use instead of the po, because

if $\forall_b\ (0 - b) + b = 0$

then $\forall_b\ b + (0 - b) = 0$ [cpa],

so anytime he wanted to use '$-b$' he could use '$0 - b$' in place of '$-b$'. He decided to call this theorem the 'ot' [Opposites Theorem].

Fred thought things were going pretty well, but he still was a little worried. Maybe he had better see just what he could do with the Opposites Theorem. What would he try it on? He could try it on the Uniqueness Principle for Addition, but that depends only on logical principles. Well, the cancellation theorem for addition was used a lot. Maybe he could prove it using his new theorem. Just how would he prove:

$$\forall_a \forall_b \forall_c \text{ if } a + b = c + b, \text{ then } a = c.$$

He thought about the way he would do it if he had the po. [Think about it].

Suppose that $a + b = c + b$.

$(a + b) + (0 - b) = (c + b) + (0 - b)$	[upa]
$a + [b + (0 - b)] = c + [b + (0 - b)]$	[apa]
$a + 0 = c + 0$	[ot]
$a = c$	[pa0]

Hence, if $a + b = c + b$, then $a = c$.

Fred was pleased. He could use his ot instead of using the old po. He decided to tackle the assignment. Prove:

$$\forall_x \forall_y \forall_z \text{ if } z + y = x, \text{ then } z = x - y.$$

"OK", Fred thought, "here goes".

Suppose that (1) $z + y = x$

Fred decided he had to get a subtraction sign into the proof. The only generalization that had a '−' in it was the pr.

(2) $$\forall_x \forall_y\ (x - y) + y = x$$

He sat and stared at (1) and (2). Then he realized that from the two of them, he could substitute '$(x - y) + y$' for 'x' and get:

$$z + y = (x - y) + y.$$

Since he had already proved the cancellation theorem for addition, he could get:

$$z = (x - y).$$

So, if $z + y = x$ then $z = x - y$.

He decided that Mr. Edwards knew some mathematics after all. Just then the bell rang.

Mr. Jones didn't have a chance to see Mr. Edwards before the next class meeting. When the class started, Mr. Jones told the students he wanted to introduce the new work on division first so that they could be sure to have enough time to set the stage for the next day's assignment. Then they'd have a discussion at the end of the period. Mr. Jones wrote the Principle of Quotients on the board.

$$\forall_a \forall_{b \neq 0}\ (a \div b) \cdot b = a$$

Then he asked them for some instances of it and wrote them on the board.

$$(2 \div 3) \cdot 3 = 2$$

$$(\tfrac{1}{2} \div 8) \cdot 8 = \tfrac{1}{2}$$

$$(0 \div 2) \cdot 2 = 0$$

$$(1 \div 7) \cdot 7 = 1$$

$$(-5 \div 3) \cdot 3 = -5$$

$$(4 \div -2) \cdot -2 = 4$$

$$(-6 \div -7) \cdot -7 = -6$$

About this time Fred started digging for his scratch paper from the previous day. Where were his instances of Mr. Edwards' pr? Here they were:

$$\forall_b \, (1 - b) + b = 1$$

$$\forall_b \, (8 - b) + b = 8$$

My, but these were like the ones on the board. He wondered if the succeeding proofs would also show a similarity. Just then Mr. Jones said, "Can anyone think of a generalization we might be able to prove immediately by using the pq?"

Fred had one. [What do you think it was?]

$$\forall_{b \neq 0} \, (1 \div b) \cdot b = 1$$

Even Sammy thought that was pretty cool because it was just an instance of the pq.

Mr. Jones asked for another generalization. By now Fred was sure that the cancellation theorem for multiplication could be proved using the pq.

$$\forall_a \forall_{b \neq 0} \forall_c \text{ if } ab = cb, \text{ then } a = c$$

[How would you prove it? Hint: Go back to Fred's proof of the cancellation theorem for addition. What will replace '0 - b'? '+'? '-'?]

The assignment for the next day was to prove:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x, \text{ then } z = x \div y \ (= \tfrac{x}{y}).$$

Fred decided this assignment wouldn't be very hard.

A. Hart
Newsletter 3

'Simplify'

The word 'simplify' is used in so many ways that it is impossible to decide that any particular expression is simpler than another equivalent expression. Whenever you read 'simpler' ask: "Simpler for what purpose?"

(1) How many pennies are in the box? If the number of pennies is fifty-seven, we would all feel that '57' is a better name than '3 • 19'.

But consider the question: If nineteen pencils cost fifty-seven cents, how much does each pencil cost? If we use the name '3 • 19' in our thoughts it is much easier to realize immediately that each pencil costs 3 cents. This is a case in which '3 • 19' is a simpler name for 57 than '57' is.

(2) Which of these two equivalent expressions is the simpler?

$$2(19x - 7)$$
$$38x - 14$$

If we are concerned with the number of operations involved after a numeral replaces 'x', we see that the expression:

$$2 \cdot (19 \cdot 3 - 7)$$

involves 19 • 3, 57 - 7, and 2 • 50; we have three operations here. On the other hand, the expression:

$$38 \cdot 3 - 14$$

involves 38 • 3 and 114 - 14 — only two operations.

If we replace 'x' by '3' and evaluate expressions such as:

(1) $$\frac{38x - 14}{19x - 7}$$

and:

(2) $$\frac{2(19x - 7)}{19x - 7}$$

we find that (1) involves five operations while (2) can be simplified quickly by using:

$$\forall_x \forall_{y \neq 0} \frac{xy}{y} = x$$

So, for this purpose, we would consider '2(19x - 7)' simpler than '38x - 14'.

We should begin (as soon as possible) to teach the child that the meaning of the word 'simplify' must be considered relative to the problem in which we are engaged. Try to do this by stressing the principal operation that is involved in an expression. The first time (and many other times, for a period of several days or even weeks) that we consider the theorem:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \frac{x \cdot z}{y \cdot z} = \frac{x}{y}$$

point out that in order to use this generalization we must have multiplication as the principal operation for both numerator and denominator.

Long before this, a discussion similar to the following should have occurred.

Consider the pattern sentence:

$$(1) \qquad \underline{a \cdot (b + c)} = \underline{\underline{a \cdot b + a \cdot c}}$$

What is the principal operation indicated in the expression which is underlined once? What is the principal operation in the expression which is underlined twice?* So (1) gives us a pattern which we can use to find an expression in which the principal operation is addition and which is equivalent to a given expression in which the principal operation is multiplication. Sentence (1) also gives us a pattern:

$$a \cdot b + a \cdot c = a \cdot (b + c)$$

which we can use to find an expression in which the principal operation is multiplication and which is equivalent to a given expression in which the principal operation is addition.

In some cases it is more convenient that the principal operation be multiplication. In others, we can do our work more rapidly if the principal operation is addition. Much of the practice that we are doing now will prepare us for more complicated problems later. When that time comes, we must decide whether it will be simpler to use an expression such as:

$$a \cdot (b + c)$$

or to use an expression such as:

$$ab + ac$$

We can only decide this after we have encountered the problem. At that time we must be able to change from one expression to an equivalent one rapidly.

To go back to:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \frac{x \cdot z}{y \cdot z} = \frac{x}{y}$$

If we try to 'reduce a fraction' we must first be certain that the principal operatic in both numerator and denominator is multiplication. You may find that this is a very effective way to help students avoid the following error:

$$\frac{ac + bc}{c} = ac + b$$

Of course, this work on principal operation starts back in Unit 1 when we are finding unstructured names for such numbers as $3 + 8 \cdot 7$ and $(3 + 8) \cdot 7$.

When we get to an exercise such as:

Simplify: $\frac{3}{7x} + \frac{2}{5y}$

we point out that for some purposes we consider an expression in which the principal operation is division simpler than one in which the principal operation is addition. So, such exercises are designed to give us practice in making such changes quickly and correctly.

*See earlier article on principal operators.

Some of the difficulty arising from the use of the word 'simplify' is due to the child's earlier training. We are all inclined to feel that we have not multiplied 3 by 19 unless we write (or say) '57'. When we see '3 • 19', we construe this as a command to find the equivalent standard decimal numeral (which is '57'). As arithmetic teachers we say

What is 3 times 19?

and we expect '57' for the answer to the question. We mean:

What is the standard decimal numeral for the product of 3 by 19?

We must begin our study of the word 'simplify' as soon as possible. If we can begin in elementary school, so much the better. If we have to start in the higher grades, let's start then.

A. Hart
Newsletter 3

$\sqrt{8}$: Rational or Irrational?

Students may have some difficulty understanding the proof that $\sqrt{8}$ is irrational unless there has been some special preparation. Here is a suggestion for such a development. We'll use .52 as an example.

Teacher: You have said that .52 is a rational number. Can you justify that statement?

Student: Yes. If there are two integers whose quotient is .52 then .52 is rational. One name for .52 is '$\frac{52}{100}$' or '52 ÷ 100'. So there are two integers, 52 and 100, whose quotient is .52.

.52
52 ÷ 100

$\frac{52}{100}$

Teacher: Can you give me another name for .52, in which the principal operator is '÷'?

Student: $\frac{26}{50}$, 26 ÷ 50.

Teacher: Another one. Another one. Another one.

.52
52 ÷ 100
26 ÷ 50
13 ÷ 25
104 ÷ 200
208 ÷ 400
-52 ÷ -100
-26 ÷ -50

Teacher: Let's throw out all negative divisors. Can you fill in the blanks so we will have more numerals for .52?

.52		
52	÷	100
26	÷	50
13	÷	25
104	÷	200
208	÷	400
~~-52~~	~~÷~~	~~-100~~
~~-26~~	~~÷~~	~~-50~~
___	÷	75
___	÷	125
___	÷	500
___	÷	10

Student: 39, 65, 260

Teacher: How about ______ ÷ 10.

Student: Well, $5\frac{1}{5} \div 10$ is .52, but $5\frac{1}{5}$ is not an integer.

Teacher: How many more numerals like these could we write?

Student: Many.

Teacher: Now really stretch your imaginations. Think of all the possible numerals for .52 that are of this kind. Remember: throw out all negative divisors. Now think of all the divisors.

.52		
52	÷	100
26	÷	50
13	÷	25
104	÷	200
208	÷	400
~~-52~~	~~÷~~	~~-100~~
~~-26~~	~~÷~~	~~-50~~
39	÷	75
65	÷	125
260	÷	500
	~~÷~~	~~10~~

Teacher: How many numbers are there in this set?

Student: Lots and lots.

Teacher: What is the largest number in that set?

Student: 500

Teacher: Stretch your imagination some more —

Student: There isn't a largest.

Teacher:	What is the smallest number in that set?
Student:	25
Teacher:	You mean that no one can find a name of this type for .52 such that the principal operator is '÷' and the divisor is less than 25?
Student:	That's right!
Teacher:	Now, let's fill in these blanks.

$.52 \times 100 =$ ____
$.52 \times 50 =$ ____
$.52 \times 25 =$ ____
$.52 \times 200 =$ ____
$.52 \times 400 =$ ____
$.52 \times 75 =$ ____
$.52 \times 125 =$ ____
$.52 \times 500 =$ ____

Teacher:	What kind of number did you get in each case?
Student:	Real number.
Teacher:	That's correct. What kind of real number?
Student:	Positive.
Teacher:	Right. What kind of positive real number?
Student:	Rational.
Teacher:	Right. What kind of positive rational number?
Student:	Integer.
	[We hope it won't take this long to get this answer.]
Teacher:	Do you think you can find a positive integer smaller than 25 such that if we multiply .52 by that integer the product will be an integer? How many positive integers are there that are smaller than 25?
Student:	24
Teacher:	Let's try them. Mary, you multiply .52 by 1. John, multiply .52 by 2. Kate, multiply .52 by 3. — Fred, multiply .52 × 24. Did anyone get an integer for the product?
Student:	No.
Teacher:	Then,

25 is the smallest positive integer such that the product of .52 by that integer is an integer.

Teacher: Do you think that we could do the same sort of thing for .125?

Student: Yes.

Teacher: How about .3? $\frac{25}{30}$? .6? $\frac{2}{3}$? Do you believe we could do this for any rational number?

Student: Yes.

Teacher: Let's think about $\sqrt{8}$. If you think that $\sqrt{8}$ is not rational, you should try to prove it. Suppose it were rational. Now, if it's really not rational, and we suppose it is, what is going to happen?

Student: We'll be in a mess.

Student: Everything will get fouled up.

These answers were actually given in a class where this was tried.

Teacher: Right.

Prove $\sqrt{8}$ is not rational.
Suppose $\sqrt{8}$ is rational.

[Now quickly review the development that led to

25 is the smallest positive integer such that the product of .52 by that integer is an integer.]

Teacher: If $\sqrt{8}$ is rational can you make a sentence like this one?

Student: Yes.

Teacher: What marks in this sentence will have to be replaced?

Student: '25' and '.52'.

Teacher: What shall we put in place of '.52'?

Student: '$\sqrt{8}$'.

Teacher: Do you know what to put in place of '25'?

Student: No.

Teacher: Let's use a pronumeral, 'q'.
Let's call that product integer 'p'.

q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer.
So, $\frac{p}{q}$ = ________.

How shall I fill in the blank?

Student: '$\sqrt{8}$'.

[Now, you are ready for line 5 on page 4-48.]

Teacher: I am thinking of a number that is a positive integer and it is less than q. Let's call it 'r'. What can you tell me about r?

Student: It's rational.

Teacher: That's true. What else? Look at this sentence on the board.

q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer.

What can you tell me about $\sqrt{8} \cdot r$?

Student: $\sqrt{8} \cdot r$ is not an integer.

Teacher: Right! If you multiply $\sqrt{8}$ by a positive integer which is less than q the product is not an integer. Let's put that up here.

q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer.
$\frac{p}{q} = \sqrt{8}$
$\sqrt{8}$ · (any positive integer less than q) is not an integer.

Now, let's think about $\sqrt{8}$. Can you tell me an integer that is larger than $\sqrt{8}$?

Student: 10, 9, 1,000,000, 3.

Teacher: One that is smaller than $\sqrt{8}$?

Student: −10, −1,000,000, 0, 1, 2.

Teacher: Let's use 2 and 3. So,

$$2 < \sqrt{8} < 3$$

Or if we use '$\frac{p}{q}$' instead of '$\sqrt{8}$' we write '$2 < \frac{p}{q} < 3$'.
Now let's use the transformation principles and get some equivalent inequations.

Student: $2q < p < 3q$ [$p > 0$ and the multiplication principle for inequations.

Teacher: Can you get another one that begins

$$0 <$$

Student: $0 < (p - 2q) < q$

Teacher: So what do we know about $p - 2q$? Can it be 0? Can it be negative? Can it be as large as q?

Student: $(p - 2q)$ is positive and is less than q.

Teacher: What kind of number is $(p - 2q)$?

Student: Integer. The set of integers is closed under multiplication and subtraction.

Teacher: So (p - 2q) is a positive integer that is less than q. Now finish this sentence:

$$\sqrt{8} \cdot (p - 2q) \text{ is } \_\_\_\_\_$$

Student: Not an integer!

Teacher: Let's use '$\frac{p}{q}$' instead of '$\sqrt{8}$'.

$\frac{p}{q}$ $(p-2q)$ *is not an integer*

Let's write this in some other form. What expression is equivalent to '$\frac{p}{q}(p - 2q)$'?

Student: $\frac{p^2}{q} - 2p$

$(\frac{p^2}{q} - 2p)$ *is not an integer.*

Teacher: Let's get really clever now and find another expression that is equivalent to

$$\frac{p^2}{q}$$

What is one principle that is very useful in getting an equivalent fraction?

Student: Principle of multiplying by 1.

Teacher: Now think of expressions that are equivalent to 1 and we'll try some of them in the blank.

$$\frac{p^2}{q} \_\_\_\_\_ - 2p \text{ is not an integer}$$

Student: '$\frac{p}{p}$', '$\frac{2}{2}$', '$\frac{q}{q}$'

Teacher: Let's use that last one.

$$(\frac{p^2}{q} \cdot \frac{q}{q} - 2p) \text{ is not an integer}$$

Student: So $[(\frac{p^2q}{q^2} - 2p)]$ is not an integer.

Teacher: I rather like that. Look at this part of it.

$(\frac{p^2q}{q^2} - 2p)$ *is not an integer.* (with $\frac{p^2}{q^2}$ circled)

How else could this be written?

Student: $(\frac{p}{q})^2$

Teacher: [So $(\frac{p}{q})^2$ q - 2p] is not an integer.

Say, just what was $\frac{p}{q}$ anyway?

Student: $\sqrt{8}$

Teacher: So,

$[(\sqrt{8})^2 \cdot q - 2p]$ is not an integer.

Student: But, — but — but $(\sqrt{8})^2$ is 8 and <u>8's</u> an integer. So 8q is an integer and 2q is an integer. So, (8q - 2p) is an integer.

Teacher: How about that? What's happened?

Student: We've made a mistake?

Teacher: Let's see if we did. What was our original problem?

Student: Show $\sqrt{8}$ is not rational.

Teacher: How were we going to do that?

Student: By supposing $\sqrt{8}$ were rational and that that would foul things up.

Teacher: Well, has that supposition "fouled things up"?

Student: Yes.

Teacher: Let's stop calling this a "mess" and say that:

"Our supposition has led to two contradictory statements".

Now what do you conclude?

Student: That $\sqrt{8}$ is not rational.

SUMMARY OF DEVELOPMENT

Show $\sqrt{8}$ is not rational

Suppose $\sqrt{8}$ is rational

q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer (p). So $\frac{p}{q} = \sqrt{8}$.

$\sqrt{8} \cdot$ (any positive integer less than q) is not an integer

$2 < \sqrt{8} < 3$

$2 < \frac{p}{q} < 3$

$2q < p < 3q$

$0 < (p - 2q) < q$

So $(p - 2q)$ is a positive integer less than q.

Hence $\sqrt{8}(p - 2q)$ is not an integer

$\frac{p}{q} \cdot (p - 2q)$ is not an integer

$\left(\frac{p^2}{q} - 2p\right)$ is not an integer

$\left(\frac{p^2}{q} \cdot \frac{q}{q} - 2p\right)$ is not an integer

$\left(\frac{p^2q}{q^2} - 2p\right)$ is not an integer

$[(\frac{p}{q})^2 \cdot q - 2p]$ is not an integer

$[(\sqrt{8})^2 \cdot q - 2p]$ is not an integer

$(8q - 2p)$ is not an integer
but it is an integer } Contradiction

So $\sqrt{8}$ is not rational.

A. Hart
Newsletter 3

The Function of a Function Theorem

> For each function h, for each function g,
> there is a function f such that $h = f \circ g$
> if and only if
> $\mathcal{D}_h \subseteq \mathcal{D}_g$ and, for all x_1 and x_2 in $\mathcal{D}_g$ such that $g(x_1) = g(x_2)$, if either x_1 or $x_2 \in \mathcal{D}_h$ then both belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$.

If you are a teacher who has been perplexed by how to teach this theorem in Course 3 I hope this article will be of some help.

Groundwork

After thinking about this theorem for some time, I finally had a chance to try some of my ideas of teaching it to a class of students. Naturally, I tried to plan the groundwork carefully. This was actually begun the day we started functions.

It was necessary to make students aware that

R is a function
if and only if

$$\forall_a \forall_b \forall_c \; [((a, b) \in R \text{ and } (a, c) \in R) \Longrightarrow b = c].$$

Of course, this is just another way of saying that a function is a set of ordered pairs no two of which have the same first component. Students need to work with this definition in many ways in order to assimilate it to the point of being able to use it in a natural way. One of the things I did in this connection was to give them oral exercises like this [Handscript shows you what is written on the blackboard.]:

I am thinking of a function. Suppose that

> *(2, 8) belongs to the function.*

Now, what can I put in place of the 'a' to make this:

> *(2, a) belongs to the function.*

into a true sentence? Suppose that

> *(3, 15) belongs to the function.*

Then, what replacements for the 'b' will change the sentence:

$(3, b + 12)$ *belongs to the function.*

into a true sentence? Here is an open sentence:

$\{(2, 9), (5, 15), (8, 4), (2, a)\}$ *is a function.*

What replacements for the 'a' will give us true sentences? What ones will give us false sentence?

A second groundwork-idea involves the notion of equality of functions. The idea is expressed by the generalization:

(**) For each function f, for each function g,

$$f = g$$

if and only if

f and g have the same domain — say $\mathcal{D}$ — and $\forall_{x \in \mathcal{D}}\ f(x) = g(x)$

Other ways of stating this theorem are:

(*) For each function f, for each function g,

$$f = g$$

if and only if

$$\mathcal{D}_f = \mathcal{D}_g \text{ and } \forall_{x \in \mathcal{D}_f \cap \mathcal{D}_g}\ f(x) = g(x).$$

and

(***) For each function f, for each function g,

$$f = g$$

if and only if

f and g have a common domain and for each of their arguments, they have the same function values.

Certainly, a formal proof of this need not be given. But, students should have a chance to look at the generalization and appreciate and understand the notation. This appreciation and understanding can be gained by a discussion of the if-part and the only if-part of the generalization. The following proof for (*) is similar to the one developed by the class I taught.

<u>Only-if part</u>

Suppose that f = g. Since f and g are sets of ordered pairs, it follows that the set of first components of f must be the same as the set of first components of g. That is, f and g have the same domain. Now, if $\mathcal{D}$ is the common domain of f and g then, for each $x \in \mathcal{D}$, the ordered pair in f whose first component is x must [since f = g] be the same as the ordered pair in g whose first component is x. In particular, the first of these ordered pairs, (x, f(x)) must have the same second component as the ordered pair (x, g(x)). That is f(x) = g(x).

<u>If-part</u>

Suppose f and g have a common domain $\mathcal{D}$ and that, for each $x \in \mathcal{D}$, f(x) = g(x). It follows that, for each $x \in \mathcal{D}$ the pair (x, f(x)) of f whose first component is x is the same as the pair (x, g(x)) of g whose first component is x. So, f and g are the same set of ordered pairs — that is, f = g.

<u>Composition of Functions</u>

The preceding development on these two groundwork-ideas would take place during the early work on functions. Finally, we come to composition of functions. It seems to me that the proof of the function-of-a-function theorem stated at the outset of this article has special value in getting students to assimilate the definition of the composition operation. For your convenience, here is a statement of the definition:

> For each function f, for each function g,
> $f \circ g$ is the function such that
> (1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$,
> and
> (2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$.

Condition (2) tells you what the arguments of $f \circ g$ are. Condition (1) tells us how to get the value of $f \circ g$ which corresponds with a given argument. Clearly, these things, together, tell you all there is to know about a function.

It is important to understand this definition. As students work with composition, their understanding will increase. But let's make sure that <u>we</u> understand it now. Let's consider another definition of the same type:

> For each $x \geq 0$,
> $\sqrt{x}$ is the number such that
> (1) $(\sqrt{x})^2 = x$,
> and
> (2) $\sqrt{x} \geq 0$.

This definition tells me, in effect, what I must do to find the principal square root of a nonnegative number. For example, if I want to find the principal square root of the nonnegative number 4, it tells me to find a number not less than 0, whose square is 4. If I do have such a number the definition assures me that my number is $\sqrt{4}$ because it tells me that there is only one number which will meet these conditions. [Of course, before such a definition can be used at all, one must establish an existence theorem and a uniqueness theorem.]

So, the definition tells me that each nonnegative number has one and only one principal square root, and it gives me the guides I need in trying to find the principal square root. Both guides must be followed. For example, if I used only the first guide, I might think that $-2 = \sqrt{4}$ because $(-2)^2 = 4$.

Now, the definition of composition is used in precisely the same manner. Notice first that it is a reasonable definition. Condition (2) tells us what the domain of $f \circ g$ is, and condition (1) tells us how to find the value of $f \circ g$ for each of its arguments. Since f and g are functions, the procedure in (1) makes sense and works for each of the arguments prescribed by (2). So, there is a function which satifies the definition and, by the theorem on equality of functions, there is only one. To see how to use the definition, consider this exercise.

$$g = \{(6, 2), (9, 4), (12, -1), (5, 3)\}$$

$$f = \{(x, y): y = x^2\}$$

$$f \circ g = \{ \underline{\qquad\qquad ? \qquad\qquad} \}$$

Since f and g are functions, the definition assures me that there is a function which is the composition of f with g, and that there is only one such function. This assurance is comforting in two respects — I know that I am not starting on a wild goose chase, and I know that when I find such a function, I can stop looking for any others. Now, the definition gives me two guides to follow in my search. The first guide tells me something about the ordered pairs of the function I am searching for.

(1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$

I am told that there is an ordered pair (a, b) which belongs to this function if $a \in \mathcal{D}_g$ and $g(a) \in \mathcal{D}_f$; and I am told how to find, for each such a, the second component b of this ordered pair. So, I have a way of getting some ordered pairs which belong to $f \circ g$. Here is the procedure.

> Consider each ordered pair in g. Take, for example, the ordered pair (6,2). The first component 6 is a candidate for first-componenthood in the sought-for function. I'll be sure that it is if g(6) — that is, 2 — is an argument of f. It turns out that 2 is an argument of f. So, I know that 6 is an argument of the sought-for function. Not only that, I also know that f(2) is the value of the function for this argument 6. Since $f(2) = 2^2 = 4$, I know that (6,4) belongs to the sought-for function. [I also know that the sought-for function contains no other ordered pairs with 6 as first component. It wouldn't be a function if it did.]

By continuing this procedure, I can discover many ordered pairs which belong to the sought-for function In fact, in this example, I can collect a total of four ordered pairs in this manner.

Now, shall I stop looking? If I did, I would be like the person who is trying to find the number which is $\sqrt{4}$ and stops when he learns that $(-2)^2 = 4$.

The second condition of the definition tells me what to do at this point.

(2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$

This says that the first components of the ordered pairs in $f \circ g$ are just those arguments of g for which the corresponding g-values are arguments of f. Having found all the pairs of $f \circ g$ which have such arguments for first components we are finished. We have all the ordered pairs of $f \circ g$. In this exercise it happens to be the case that, for each $x \in \mathcal{D}_g$, $g(x) \in \mathcal{D}_f$ — and, so $\mathcal{D}_{f \circ g} = \mathcal{D}_g$.

Here is an exercise which points out the fact that this is not always the case

$$a = \{(\text{John}, 7), (\text{Bill}, 5), (\text{Emma}, 8)\}$$

$$b = \{(x, y), x > 6: y = 3x + 1\}$$

$$b \circ a = \{ \underline{\qquad\qquad ? \qquad\qquad} \}$$

Suppose that we follow the procedure suggested by condition (1). Doing so, we discover that the ordered pairs (John, b(7)) and (Emma, b(8)) belong to the sought-for function, but, condition (1) does not tell whether or not Bill is an argument of the sought-for function. [Nor does condition (1) rule out any other arguments.] It is condition (2) which helps us. For it says that even though Bill $\in \mathcal{D}_a$, since g(Bill) $\notin \mathcal{D}_b$, Bill $\notin \mathcal{D}_{b \circ a}$. [And, of course, condition (2) rules out all other arguments except John and Emma.]

Your understanding of the definition of composition of functions may be strengthened by considering:

$$g = \{(2, 5), (7, 8)\}$$

$$f = \{(5, 9), (8, 14), (3, 2)\}$$

What is $f \circ g$? Suppose someone says that $f \circ g$ is the function

$$\{(2, 9), (7, 14), (3, 2)\}$$

For convenience, let's name that function 'h_1'. So he believes that

$$h_1 = f \circ g.$$

Let's use our definition to see if he is correct. Condition (1) of the definition tells us that for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$, $[f \circ g](x) = f(g(x))$.

So, if he is right, then

for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$, $h_1(x)$ must be $f(g(x))$.

Now, $\mathcal{D}_g = \{2, 7\}$. Also, $g(2) \in \mathcal{D}_f$ and $g(7) \in \mathcal{D}_f$. Examination shows that

$$f(g(2)) = f(5) = 9 = h_1(2)$$

and

$$f(g(7)) = f(8) = 14 = h_1(7).$$

So, it is the case that

for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$, $h_1(x) = f(g(x))$.

If condition (2) were ignored, we might well believe that

$$f \circ g = \{(2, 9), (7, 14), (3, 2)\}$$

But, do you see that we would also believe that

$$f \circ g = \{(2, 9), (7, 14), (\text{Al}, \text{Mary}), (10, 17)$$

that

$$f \circ g = \{(x, y): y = x + 7\}$$

and that

$$f \circ g = \{(2, 9), (7, 14)\}.$$

What does condition (2) of the definition tell us? Why, it tells us that the only one of these functions that is $f \circ g$ is that one whose domain is $\{x \in \mathcal{D}_g: g(x) \in \mathcal{D}_f\}$. In this case, $f \circ g$ is that one of these functions whose domain is $\{2, 7\}$. Hence $f \circ g = \{(2,9), (7, 14)\}$. In brief, condition (1) gives us a way to find certain ordered pairs which belong to $f \circ g$ and condition (2) tells us that the only ordered pairs which belong to $f \circ g$ are those which can be obtained in that way.

PRELUDE TO THE THEOREM

Exercises may be developed which are designed to make the student aware of the theorem toward which we are moving. The task is to find, if possible, a function f such that $h = f \circ g$. Here are some of the things we did in class.

Example. $g = \{(0, 1), (1, 5), (2, 9)\}$

$f =$ ______?______

$h = \{(1,5), (2, 8), (4, 8)\}$

There is no function f such that $h = f \circ g$ since there is no way of "getting" (4, 8) into h when there is no member of g which has first component 4. Out of this should come the requirement that

$$\mathscr{D}_h \subseteq \mathscr{D}_g .$$

[Naturally, this requirement can also come out of an examination of the definition of composition. If there is an f such that h = f∘g then

$$\mathscr{D}_h = \mathscr{D}_{f \circ g} = \{x \in \mathscr{D}_g : g(x) \in \mathscr{D}_f\}.$$

This tells us that each element of $\mathscr{D}_h$ must belong to $\mathscr{D}_g$. So, if there is such an f, $\mathscr{D}_h \subseteq \mathscr{D}_g$.]

This exercise shows that the converse is not true. Other ideas should also come from this exercise.

$$g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$$

$$f = \underline{\qquad\qquad ? \qquad\qquad}$$

$$h = \{(2, 12), (3, 18), (6, 14), (5, 2)\}$$

If there were such a function f then it would have to contain (5, 12), (8, 18), (8, 14), and (0, 2). A relation which contains both (8, 18) and (8, 14) is not a function. So, there is no function f such that h = f∘g.

Now, can you change h so that there is such a function f? Let h(6) be 18 instead of 14 or let h(3) be 14 instead of 18. So, we put on the board:

> if there is a function f such that h = f∘g
> then [if $3 \in \mathscr{D}_g$ and $6 \in \mathscr{D}_g$ and $3 \in \mathscr{D}_h$ and $6 \in \mathscr{D}_h$ and g(3) = g(6), h(3) must be h(6)]

Now, we considered a new problem. Suppose that the ordered pair (6, 14) is removed from h in the second example.

$$g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$$

$$f = \underline{\qquad\qquad ? \qquad\qquad}$$

$$h_1 = \{(2, 12), (3, 18), (5, 2)\}$$

Here, (5, 12), (8, 18) and (0, 2) must belong to f if there is an f. However,

$$\{(5, 12), (8, 18), (0, 2)\} \circ g = \{(2, 12), (3, 18), (6, 8), (5, 2)\}$$

Clearly, this not h_1.

We also considered removing both (3, 18) and (6, 14) from h.

$$g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$$

$$f = \underline{\qquad\qquad ? \qquad\qquad}$$

$$h_2 = \{(2, 12), (5, 2)\}$$

Then, if there is an f, (5, 12), and (0, 2) must belong to it.

$$\{(5, 12), (0, 2)\} \circ g = \{(2, 12), (5, 2)\},$$

and this is h_2. The class decided that if $3 \in \mathcal{D}_g$ and $6 \in \mathcal{D}_g$ and $g(3) = g(6)$ and either 3 or 6 belongs to $\mathcal{D}_h$ then the other one must also belong to $\mathcal{D}_h$.

So, we change to:

if there is a function f such that $h = f \circ g$

then [if $3 \in \mathcal{D}_g$ and $6 \in \mathcal{D}_g$ and $3 \in \mathcal{D}_h$ and $g(3) = g(6)$
then $6 \in \mathcal{D}_h$ and $h(3) = h(6)$]

We were now ready to state the only-if part of the theorem:

if For each function h, for each function g,
there is a function f such that $h = f \circ g$

then

$$\begin{cases} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \text{and} \\ \text{if } \begin{Bmatrix} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{Bmatrix} \text{ then } \begin{cases} x_2 \in \mathcal{D}_h \\ \text{and} \\ h(x_1) = h(x_2) \end{cases} \end{cases}$$

Notice that our exercises really led to the contrapositive of this. If the then-things did not happen, there was no such function f. Contrapositively, if there is such a function, the then-things must happen.

At this point, I suggested that we consider the converse of this theorem. This was stated and the class decided it was probably true also.

PROOF OF THE THEOREM

At this stage, the students are probably willing to accept the theorem on the basis of the exercises. I decided to try the proof of the theorem. We did the proof [questions and boardwork done by me] in fifty-six minutes. Obviously the students could not reproduce it but they had all contributed and they all had stayed awake. Questions were of these types:

What will we need to establish in order to get __________?

What does it mean to say __________?

I had the students keep their books open to the definition of composition.

<u>If-Part of the Theorem</u>

Prove: For each function h, for each function g,

$$\text{if } \begin{cases} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \text{and} \\ \text{if when } \left\{\begin{array}{l} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{array}\right\} \text{ then } \begin{cases} x_2 \in \mathcal{D}_h \\ h(x_1) = h(x_2) \end{cases} \end{cases}$$

then there is a function f such that $h = f \circ g$.

Suppose that, for a function h and a function g, $\mathcal{D}_h \subseteq \mathcal{D}_g$. If $x_1 \in \mathcal{D}_h$, what else can you say? [Answer: $x_1 \in \mathcal{D}_g$] If $x_1 \in \mathcal{D}_h$, what ordered pair belongs to h? Give a very, very inexpensive answer — no work at all. [Ans.: $(x_1, h(x_1))$]. If $x_1 \in \mathcal{D}_g$, what ordered pair belongs to g? [Ans.: $(x_1, g(x_1))$]

$$g = \{(x_1, g(x_1)), \ldots\}$$

$$f = \underline{\qquad ? \qquad}$$

$$h = (x_1, h(x_1)), \ldots\}$$

Now, if there is a function f such that $h = f \circ g$, what ordered pair must belong to that function? [Ans.: $(g(x_1), h(x_1))$] OK, think of all the ordered pairs which must belong to f. Let's say that set of such ordered pairs is the relation k. I wrote the following on the board:

$$\boxed{k = \{(z, w)\text{: There is an } x \in \mathcal{D}_h \text{ such that } z = g(x) \text{ and } w = h(x)\}}$$

Note that $k = \emptyset \iff h = \emptyset$

The way to find the members of K is to pick arguments of h ["there is an $x \in \mathcal{D}_h$"] and find the corresponding g-values and h-values. Will each such argument have an h-value? [Ans.: Of course, because it <u>is</u> an argument of h.] Will each such argument have a g-value? [Ans.: Of course. By hypothesis, $\mathcal{D}_h \subseteq \mathcal{D}_g$.]

What must we show about k? Two things. First, that k is a function [the first groundwork idea]. And, second, that $h = k \circ g$ [the second groundwork idea].

Let's show that k is a function. How can we do this? Well, k is a set of ordered pairs. So, to show that k is a function, we'll suppose that $(a, b) \in k$ and $(a, c) \in k$ and prove that $b = c$.

Since $(a, b) \in k$, it follows that [according to the set selector in the description of k] that

$$\text{there is an } x \in \mathcal{D}_h \text{ such that } a = g(x) \text{ and } b = h(x).$$

Since $(a, b) \in k$, it follows [according to the set selector in the description of k] that

$$\text{there is an } x \in \mathcal{D}_h \text{ such that } a = g(x) \text{ and } b = h(x).$$

$\underline{\text{Let } x_1 \text{ be that member of } \mathcal{D}_h}$. Hence, $\underline{\underline{a = g(x_1)}}$ and $\underline{\underline{\underline{b = h(x_1)}}}$.

Also, since $\mathcal{D}_h \subseteq \mathcal{D}_g$, $\underline{x_1 \in \mathcal{D}_g}$.

Similarly, since $(a, c) \in k$, it follows that $\underline{\underline{a = g(x_2)}}$

$$\text{there is an } x \in \mathcal{D}_h \text{ such that } a = g(x) \text{ and } c = h(x).$$

Let x_2 be that member of $\mathcal{D}_h$. Hence, $\underline{\underline{a = g(x_2)}}$ and $\underline{\underline{\underline{c = h(x_2)}}}$.

Also, since $\mathcal{D}_h \subseteq \mathcal{D}_g$, $\underline{x_2 \in \mathcal{D}_g}$.

So, from the sentences which are underlined twice

$$\underline{g(x_1) = g(x_2)}.$$

Look back at all the sentences which are underlined once. We have

$x_1 \in \mathcal{D}_g$

$x_2 \in \mathcal{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathcal{D}_h$

Now look at the conditional sentence in our hypothesis. We must conclude, by modus ponens, that $\underline{\underline{\underline{h(x_1) = h(x_2)}}}$.

Now look at the sentences which are underlined three times. We conclude that $b = c$. So, k is a function.

Take another breath and we'll show that $h = k \circ g$.

Remember that to show this, all we need to show is that

(1) h and $k \circ g$ have the same domain [i. e., $\mathcal{D}_h = \mathcal{D}_{k \circ g}$]

and

(2) for each element in that domain, they have the same value. [i. e., $\forall_{x \in \mathcal{D}_h}\ h(x) = [k \circ g](x)$]

(1) $\mathcal{D}_h$ and $\mathcal{D}_{k \circ g}$ are both sets. So to show they are the same set we will show: (a) $\mathcal{D}_h \subseteq \mathcal{D}_{k \circ g}$ and (b) $\mathcal{D}_{k \circ g} \subseteq \mathcal{D}_h$.

(a) Suppose $x_1 \in \mathcal{D}_h$. Then (since $\mathcal{D}_h \subseteq \mathcal{D}_g$) $x_1 \in \mathcal{D}_g$. Now, look back at the definition of k. From it, it follows that

$$(g(x_1),\ h(x_1)) \in k.$$

Since that is the case, $g(x_1) \in \mathcal{D}_k$. So, $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_k$. That is, $x_1 \in \{x \in \mathcal{D}_g\colon g(x_1) \in \mathcal{D}_h\}$. But, by definition $\mathcal{D}_{k \circ g} = \{x \in \mathcal{D}_g\colon g(x) \in \mathcal{D}_k\}$. Hence, $x_1 \in \mathcal{D}_{k \circ g}$. So, if $x_1 \in \mathcal{D}_h$ then $x_1 \in \mathcal{D}_{k \circ g}$. Since this reasoning would hold for any $x \in \mathcal{D}_h$, it follows that

$$\mathcal{D}_h \subseteq \mathcal{D}_{k \circ g}$$

(b) Suppose $x_2 \in \mathcal{D}_{k \circ g}$. Then $x_2 \in \{x \in \mathcal{D}_g\colon g(x) \in \mathcal{D}_k\}$. That is, $\underline{x_2 \in \mathcal{D}_g}$ and $g(x_2) \in \mathcal{D}_k$. Now, $g(x_2) \in \mathcal{D}_k$ says that there is an ordered pair, $(g(x_2),\ k(g(x_2))) \in k$.

Again, we use the definition of k. This ordered pair,

$$(g(x_2),\ k(g(x_2)))$$

belongs to k if and only if

there is an $x \in \mathcal{D}_h$ such that $g(x_2) = g(x)$ and $k(g(x_2)) = h(x)$.

[Maybe this x is x_2 but we can't make such an assumption].

Let's say that this x is x_1. So, $\underline{x_1 \in \mathcal{D}_h}$ and $\underline{g(x_2) = g(x_1)}$. Furthermore, since $\mathcal{D}_h \subseteq \mathcal{D}_g$, $\underline{x_1 \in \mathcal{D}_g}$.

Now, pick up the four sentences, which are underlined once, and go back to the conditional sentence in the hypothesis. By modus ponens, we conclude that $x_2 \in \mathcal{D}_h$. So, if $x_2 \in \mathcal{D}_{k \circ g}$ then $x_2 \in \mathcal{D}_h$. Since the reasoning would hold for any $x \in \mathcal{D}_{k \circ g}$, it follows that

$$\mathcal{D}_{k \circ g} \subseteq \mathcal{D}_h$$

and hence, that $\mathcal{D}_h = \mathcal{D}_{k \circ g}$.

We now know that h and $k \circ g$ have the same domain. It remains to show that for each element x in that domain (we will call it '$\mathcal{D}_h$') $h(x) = [k \circ g](x)$.

Suppose $x_1 \in \mathcal{D}_h$. Then $x_1 \in \mathcal{D}_{k \circ g}$. Also, $x_1 \in \mathcal{D}_g$.

So, back to the definition of k:

$$(g(x_1), h(x_1)) \in k.$$

Since k is a function, the second component of an ordered pair whose first component is $g(x_1)$ and which belongs to k must be $k(g(x_1))$. So

$$(g(x_1), k(g(x_1))) \in k,$$

and

$$h(x_1) = k(g(x_1)).$$

However, $k(g(x_1))$ is precisely $[k \circ g](x_1)$, since $x_1 \in \mathcal{D}_{k \circ g}$.

Consequently, for each element x in the domain of h (and $k \circ g$)

$$h(x) = [k \circ g](x).$$

So, h and $k \circ g$ have the same domain and for each element in that domain, they have the same value. Hence, $h = k \circ g$.

So, there is a function f (we used the name 'k') such that $h = f \circ g$.

<u>Only-If Part of the Theorem</u>

Prove: For each function h, for each function g

if there is a function f such that $h = f \circ g$

$$\text{then} \left\{ \begin{array}{l} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \quad \text{and} \\ \text{if} \left\{ \begin{array}{l} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{array} \right\} \text{then} \left\{ \begin{array}{l} x_2 \in \mathcal{D}_h \\ h(x_1) = h(x_2). \end{array} \right. \end{array} \right.$$

Let's see what we have to start. Not very much. We have

f, g, and h are functions

$h = f \circ g$.

First, we want to show that $\mathscr{D}_h \subseteq \mathscr{D}_g$. So, we will show that each member of $\mathscr{D}_h$ belongs to $\mathscr{D}_g$. Suppose that $x_1 \in \mathscr{D}_h$. Then, since $h = f \circ g$, it follows [by substitution] that $x_1 \in \mathscr{D}_{f \circ g}$. But, by the definition of composition, $\mathscr{D}_{f \circ g} = \{x \in \mathscr{D}_g : g(x) \in \mathscr{D}_f\}$. Hence, $x_1 \in \{x \in \mathscr{D}_g : g(x) \in \mathscr{D}_f\}$ — that is, $x_1 \in \mathscr{D}_g$ and $g(x_1) \in \mathscr{D}_f$. So, if $x_1 \in \mathscr{D}_h$ then $x_1 \in \mathscr{D}_g$. Therefore, $\mathscr{D}_h \subseteq \mathscr{D}_g$. So far, so good.

Consider this correspondence:

there is a function f such that $h = f \circ g$ ⟶ p

$\mathscr{D}_h \subseteq \mathscr{D}_g$ ⟶ q

$x_1 \in \mathscr{D}_g$ and $x_2 \in \mathscr{D}_g$ and $g(x_1) = g(x_2)$ and $x_1 \in \mathscr{D}_h$ ⟶ r

$x_2 \in \mathscr{D}_h$ and $h(x_1) = h(x_2)$ ⟶ s

So, the pattern of our theorem is

if p then [q and (if r then s)].

We have already shown that

if there is a function f such that $h = f \circ g$ then $\mathscr{D}_h \subseteq \mathscr{D}_g$.

This corresponds to

if p then q.

Now, examine the following diagram:

```
                          †
                          p   if p then q
                      †   _______________
                      p               q     if (p and q and r) then s
                      _________________     _________________________ (Exportation)
†
p   if p then q       p           and q    if p and q then (if r then s)
_______________       _________________    _____________________________
    q                                      if r then s
    ______________________________________________________
                 q and (if r then s)
    ___________________________________________ †
       if p then [q and (if r then s)]
```

2. Would you have gotten the same result if you had counted the objects in a different order?

3. If you touched each object as you counted it, did you touch any object more than once? How many objects had you touched when you said the number name 'tr'? Then, what does 'tr' tell you when you use it in counting? Could you make a similar statement about the other number names?

4. How many number names did you use? If you count correctly, do you have to use the number names in order? How do you know when to stop counting?

5. What can you do if you run out of number names before you have counted all the objects on your desk?

What the real Stone Agers did was use a word meaning "a lot" for any number larger than the largest number they could name. Their culture did not require names for large numbers. As someone once remarked, thirty was infinity to them.

In our culture, we need names for millions of numbers, very large numbers, very small numbers, and different kinds of numbers.

Let's reorient ourselves to the twentieth century and summarize what we have discussed so far. Fill in the blanks.

1. The number which tells how many members a set has is a _____.

2. The cardinal number of any set whose members could be paired, one-to-one, with the members of the set of wheels on a car is _____.

3. The cardinal number of any set whose members can be placed in a one-to-one correspondence with the members of the set of your eyes is _____.

4. A set with no members at all is called _____.

5. Before the cardinal numbers can be used for counting, they have to be ordered so that each number is _____ more than the number just before it.

6. In counting, we make a one-to-one correspondence between the members of the set of objects we are counting and the equivalent set, starting with 'one', of ordered number names. We use each number name in order exactly _____ (how many times) and count each object exactly _____.

7. The last number name we say in a counting sequence tells us the _____ number of the set whose members we are counting.

8. The smallest counting number is _____; the smallest cardinal number is _____; the largest cardinal number is _____.

J. Phillips
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7. Make a list of numbers, in order, from ug through ds. Are you now ready to count? If so, how far can you count? Why?

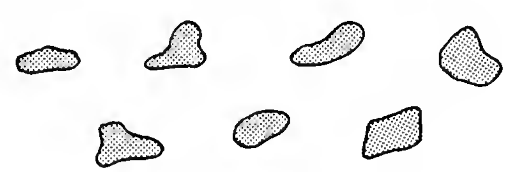
8. Under what circumstances would a Stone-Age father be able to use the list to tell someone else how many children he had?

Apparently, we need something more than some numbers lined up in order from smallest to largest before we can say we have "counting numbers." Let's go out and get a heap of pebbles so that we can duplicate the collections shown in our list. It is important that we realize that ug is the number of objects in each set which contains a single object. If we have any pebbles at all, ug is the smallest number we can have, so we'll call ug our first counting number. Now we have the idea of combining members of two collections. Let's start with two sets, each of which has ug members. Consider a new set built by putting the members of these two sets together. Which of the model collections does this new set match? Let's keep a record, twentieth-century style, and let's be orderly. Complete each of the following:

1. Ug and ug is _____.
2. Glm and ug is _____.
3. Tr and ug is _____.
4. Qut and ug is _____.
5. Kog and ug is _____.
6. Hk and ug is _____.
7. Okz and ug is _____.
8. Nn and ug is _____.
9. Ds and ug is _____.

It appears that we have a couple of problems to solve, difficult problems for Stone-Agers. What suggestions can you make?

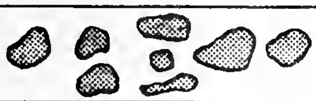
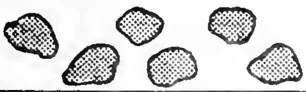
We need a number which is ug greater than hk. A picture of the members of a set which would have this number might be as follows:



Suppose we agree to name this number 'pr' (pronounce 'pr' as you would pronounce 'purr'). Now can we count up to ds? We can if each of our numbers, as far as they go, is ug greater than the number it follows. Do we have numbers like that? Let's see how we use them.

Lay out a few objects on your desk and count them, using our Stone-Age number names. Think about how you did it.

1. Did you say a number name for each object as you counted it? Did you make a one-to-one correspondence between the number names and objects you were counting?

Pictures of collections of pebbles	Number names
	okz (ox)
	tr (tor)
	nn (noon)
	hk (heck)

You find it hard to remember which name goes with which collection. You reason that, if there were some way of organizing the numbers, it would make it easier to remember their names. (We said you are smart.) Start with the first two numbers in the list. How will you decide which is greater, glm or kog? (So far, all you know how to do is match.) Match the appropriate collections of pebbles. What do you find out? If you decide to call the collection in which you had pebbles left over the larger collection, you would probably call the number of this collection the greater number. If so, you really know two things — kog is greater than glm, and glm is less than kog. Jump forward in time a few thousand years so that you know not only how to write but also how to use modern symbols, and you may write:

kog > glm and glm < kog

This may be read:

kog is greater than glm and glm is less than kog

Now jump back into the Stone Age (except that you can use modern English, an enormous "except") and answer the following questions:

1. Is the same number named twice in the list? How do you tell?
2. Which of these are true statements:

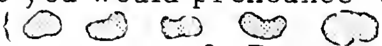
   nn > tr? qut > hk? How do you tell?

3. Is glm less than every other number? How many comparisons would you have to make to find this out?
4. What is the name of the smallest number? How could you find out which number is next smallest?
5. Which number names could you write in the blank to make true statements?

   glm < _____ tr > _____ tr < _____

   Is this all you need to know in order to tell which number is next larger than glm?

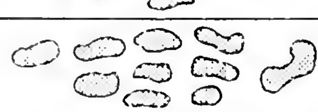
6. Could you extend the scheme described in question 5 so that you could line up the collections of pebbles in order from fewest to most? Once you have done this, you have also ordered the cardinal numbers ug through ds from least to greatest, and you can learn to say their names in order.

In this language, 'kog' (pronounce 'kog' as you would pronounce 'cog') is the word for the number of pebbles in the set {  }. How can we find other sets which have this same number property? Does a man have kog fingers on his right hand? Does he have kog fingers on his left hand? Where else might he find sets with kog members?

Would he have to know the word 'kog' in order to tell that he has the same number of fingers on each hand? To tell that he has the same number of fingers on a hand as he has toes on a foot? Would he need to know how to count, or how to add, in order to be sure he had the same number of fingers on both hands as he has toes on both feet (assuming that he was a normal Stone-Age baby and has not been damaged since)?

In order to tell whether sets have the same number or different numbers, all we need is a matching technique. Given the members of any two sets of objects, we can try to match each member of one set with a member of the other set. If the matching "comes out even" — each member of each set paired with exactly one member of the other set — we can be sure that the two sets have the same number of members. When this happens, we say there is a <u>one-to-one correspondence</u> between the members of the two sets. Try matching the members of the set {a, b, c} with those of the set {Tom, Dick, Harry}. Pair each letter with the name of one boy and different letters with different names of boys so that each member of each set belongs to exactly one pair. Can you match the members of these sets in another way? In how many ways? Does each one-to-one correspondence assure you that there is the same number of members in the set of letters as in the number of boys? If two sets have the same number of members, is it always possible to set up a one-to-one correspondence between their members? Is it always possible to do this in more than one way?

Our Stone-Age technique of pairing pebbles with other objects has given us a way of telling when two sets have the same number of members. You are a smart Stone-Age boy or girl. Your father has chosen you (out of the entire family group) to teach all the number names he knows so that you can pass this knowledge along to others and, eventually, to your own children. This is what he taught you:

Pictures of collections of pebbles	Number names
	glm (pronounced glum)
	kog (cog)
	ds (dees)
	ug (ug)
	qut (cat)

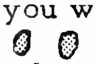
Stone Age Math

Do you think a small child may know what four is without knowing anything about one or two or three? Could he know that four tells how many wheels a car has and how many feet his dog has and how many legs a chair has, and still could he know nothing about any other numbers? Have you ever heard a child count to five by saying, "One, seven, four, nine, three, ten, five"? He knows some words, but he doesn't really know what all of them mean or how they are related to each other. How does anyone know that five is greater than three but less than nine, for example? First, he has to know what three is and what five is and what nine is. Then, he has to know what is meant by greater than and less than. In any kind of productive discussion, we need preliminary agreement about the meaning of some terms we shall use. Pay careful attention to the underlined words in the next paragraph.

Any specific collection of separate objects (where object can refer to anything: a person, a planet, an electron, or even an idea) is a set. Each object which belongs to a set is a member of the set. For each object and each set, either the object is a member of the set or it is not. In mathematics, the word 'set' is always used in this precise sense. To each set there corresponds a cardinal number which tells how many members the set has. Sets which have the same cardinal number are equivalent sets. Describe another set which has the same cardinal number as the set whose members are your eyes. Is the set whose members are your feet equivalent to the set you have just described? What is the cardinal number of the set whose only member is the chair on which you are now sitting? If you are not now sitting on a chair, how many members belong to the set of chairs you are sitting on. You see that it sometimes makes sense to think of a set which has no members. The set with no members is the empty set. What number tells how many members the empty set has? Then, the cardinal number of the empty set is zero. A set cannot have fewer than no members, so zero is the smallest cardinal number. Is there an upper limit to the number of members a set may have? Is there a largest cardinal number?

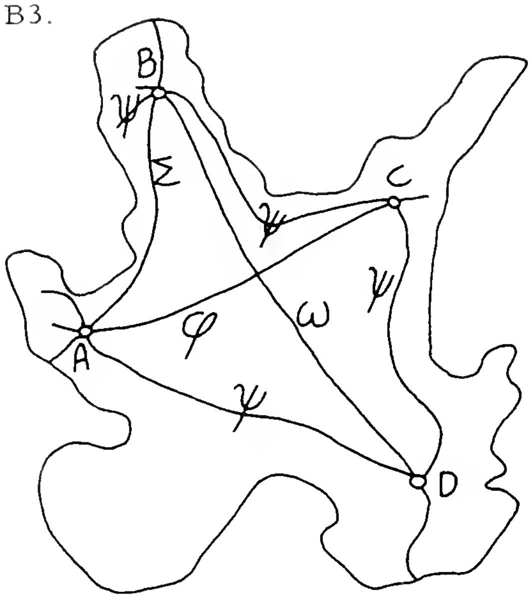
Cardinal numbers are the numbers primitive peoples used first in developing their cultures. Cardinal numbers are also the numbers very young children use first in learning to deal with the cultures into which they are born. It may help us to a better understanding of our own use of cardinal numbers if we take a make-believe excursion several thousand years back into history.

Suppose we pretend that we are living in the Stone Age. We have a simple language with which we can think and with which we can express thoughts to one another.

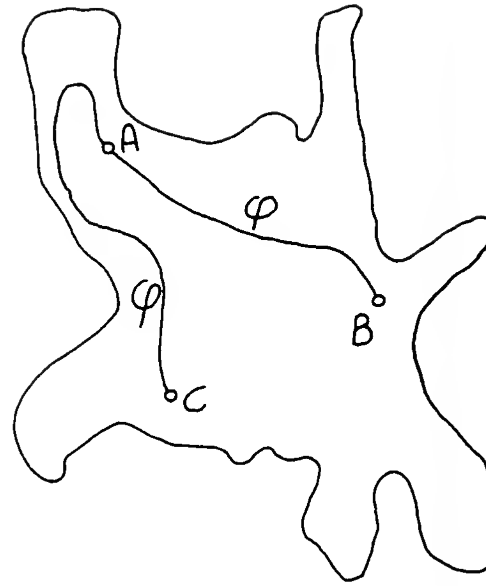
In this language, 'glm' (pronounce 'glm' as you would pronounce 'glum') is the word for the number of pebbles in the set{  }. The idea of counting has not occurred to us. How would we pick out other sets which have glm members?

One of us might pick up the pebbles, lay one pebble on his left foot and the other pebble on his right foot. He has run out of pebbles and also out of feet. Can he now say he has glm feet? Could he pick out other examples of "glmness" in the same way?

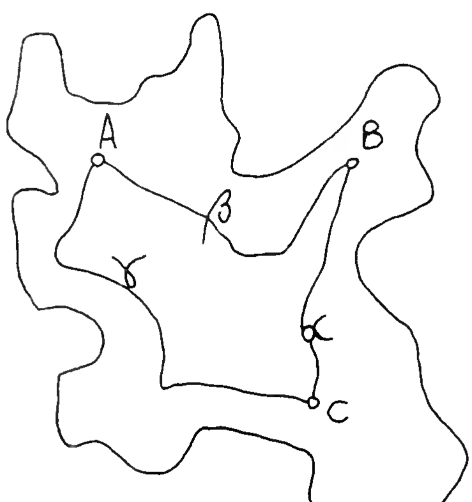
B3.



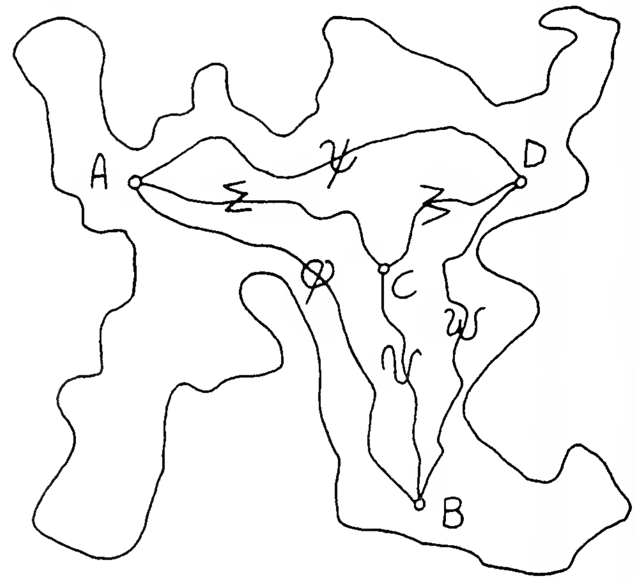
B4.



B5.

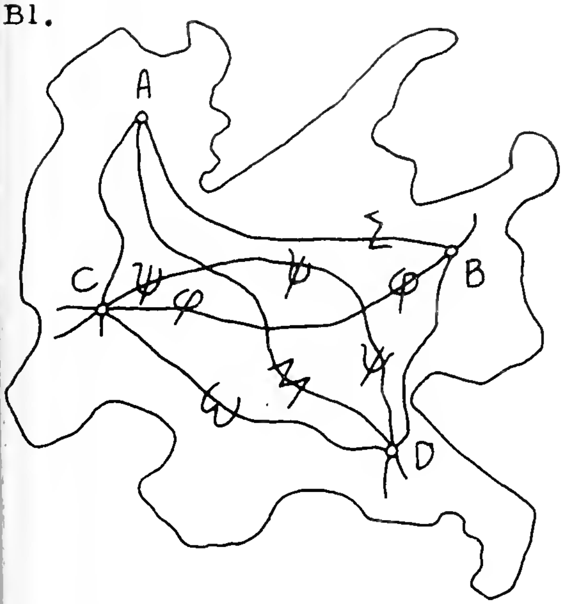


B6.

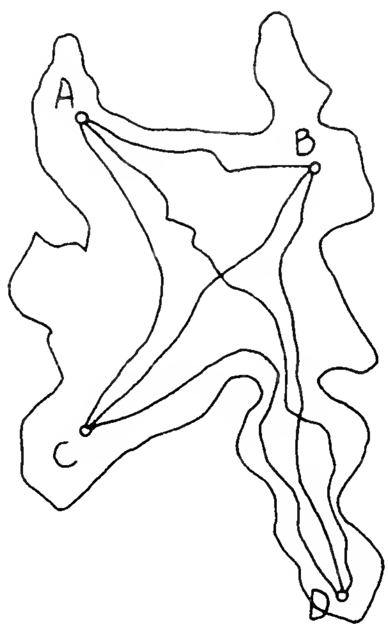


B1	No	Message 5	Two highways between C and D
B2	Yes		
B3	No	Message 3	
B4	No	Message 3	
B5	Yes		
B6	Yes		

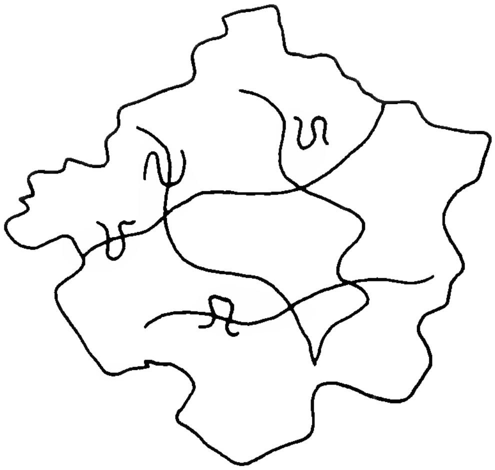
B1.



B2.

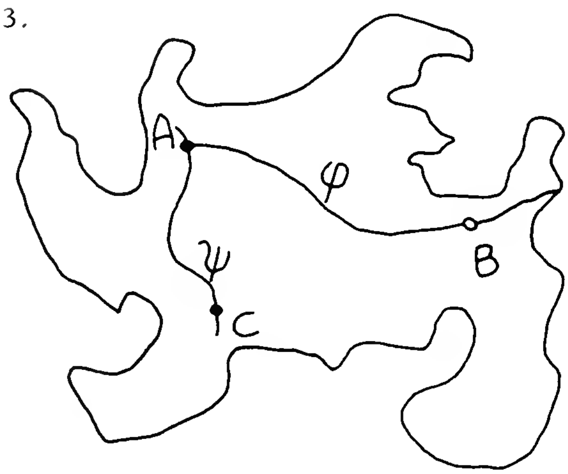


A1.

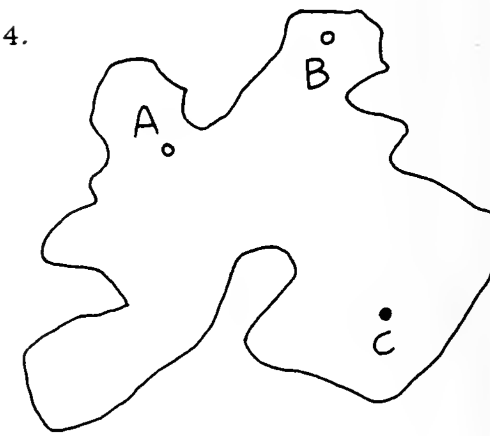


A2.

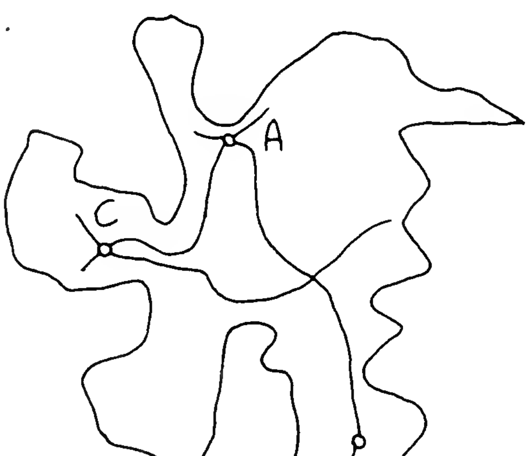
A3.



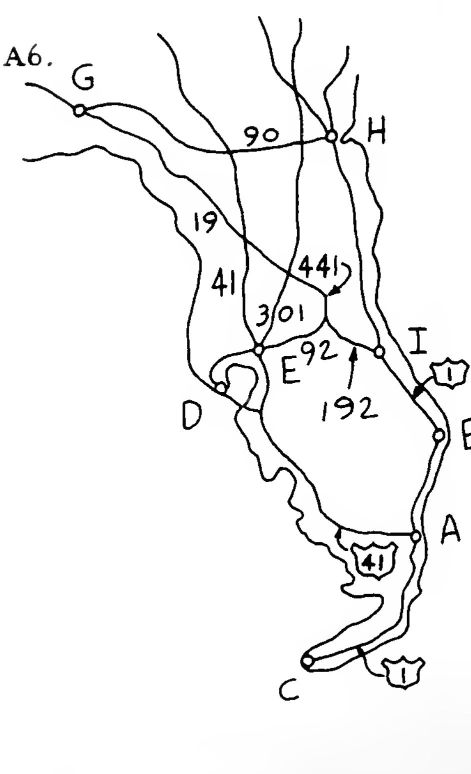
A4.



A5.



A6.



Questions like these should be asked, discussed, answered, and understood in every situation in which fractions appear.

J. Phillips
H. E. Vaughan
Newsletter 12

Calling All Cartographers

After one day spent on Glox, with a group of 13 year olds, an exercise on maps was developed. Sketches similar to the ones shown here were placed on the board. The students were asked: "Could this be a complete map of the region on Glox about which the spaceman sent messages?" This could be used as homework or adapted for a written exercise in class. Adaptations may need to be made for older children.

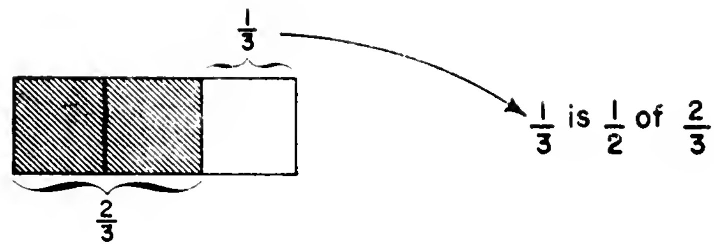
*

Long before the first space ship reached Glox, the dark side of the moon had been explored. Evidence had been found that intelligent beings had established a base there. Among the debris, papers had been discovered. Copies of these papers are shown on the next two pages. These pages have been labeled for easier identification. The drawing labeled A6 received a lot of attention. Everyone was certain that these were maps. But none of the others were recognizable.

When the messages started coming from the spaceman on Glox, the Commander set everyone the job of studying these maps to see if any of them were maps of the regions where the spaceman had landed on Glox. The first message made them discard A1 as a possible map. Why? Decide which of the others might be maps for that region of Glox. Justify your decisions.

Key

Map	Answer	Justification
A1	No	Message 1 or 3
A2	No	Message 3
A3	No	Message 5
A4	No	Message 2
A5	No	Message 4
A6	-	--------



In one whole candy bar, there are 1 two-thirds of the candy bar and 1/2 of another two-thirds of the candy bar. Another way to write '1 and 1/2' is '3/2'. There are 3/2 two-thirds of a candy bar in 1 whole candy bar. It is essential to realize that, in this situation, the 3/2 does not refer to 3/2 (or 1-1/2) candy bars, but to 3/2 (or 1-1/2) two-thirds of a candy bar. Once this idea is clear, it is easy to see that there would be 5 times as many 2/3 candy bars in 5 such bars as there are in 1, or 7/8 times as many in 7/8 candy bars, and so on.

The particular relevance of the foregoing paragraph to our discussion of division of fractional numbers comes in the interpretation of answers to "word problems".

> How many pieces of ribbon, each 27 inches long, can be cut from a 10-yard spool of ribbon?
>
> 27 inches $= \frac{3}{4}$ of a yard
>
> $10 \div \frac{3}{4}$ should give the answer
>
> $10 \div \frac{3}{4} = 10 \times \frac{4}{3} = \frac{40}{3} = 13\frac{1}{3}$
>
> The answer to the question is surely 13, but how long is the extra piece? Is it 1/3 of a yard (12 inches) or 1/3 of 27 inches (9 inches)? A sixth grader should be able to tell.

Larded through an entire development of the use of fractions should be a great deal of work on estimating answers.

How can one tell whether $1/2 + 2/3 > 1$ is a correct statement?

If '1/2' refers to an acre of land and '2/3' also refers to an acre of land, what does '1' refer to?

If '1/2' and '2/3' refer to fractional numbers, what does '1' refer to?

Suppose '1/2' refers to a mile and '2/3' refers to an hour?

$$\frac{3}{10} \div \frac{3}{5} = \frac{3 \div 3}{10 \div 5} = \frac{1}{2}$$ *correct answer?*

$$\frac{9}{11} \div \frac{3}{11} = \frac{9 \div 3}{11 \div 11} = \frac{3}{1}$$ *correct answer?*

$$\frac{9}{12} \div \frac{6}{12} = \frac{9 \div 6}{12 \div 12} = \frac{3/2}{1} = \frac{3}{2}$$ *correct answer?*

$$\frac{3}{5} \div \frac{7}{13} = \frac{3 \div 7}{5 \div 13} = \frac{3/7}{5/13}$$ *correct answer?*

(Now we are involved with another use for the fraction symbol: $a/b = a \div b$.) Pursue the last example a little bit further and see what happens.

$$\frac{3/7}{5/13} = \frac{3}{7} \div \frac{5}{13} = \frac{3 \div 5}{7 \div 13} = \frac{3}{5} \div \frac{7}{13}$$

The fact that division of cardinals is interspersed with division of rationals in this development makes it hard to explain.

After exploring a number of avenues which suggest ways of dealing with fractions in division examples, we are ready to introduce the standard algorithm. If we wish to verbalize the instructions, we need the word *reciprocal* (or the equivalent phrase, *multiplicative inverse*, which is such a tongue-twister that most people avoid it). Two numbers are reciprocals if their product is 1. [We follow the custom of writing '1/1' as '1', '3/1' as '3', and so on.] Being reciprocals is like being cousins; it takes two, and each bears the given relation to the other (except for 1 which is its own reciprocal and 0 which has no reciprocal. The number 2/3 is the reciprocal of the number 3/2, and 3/2 is the reciprocal of 2/3, because $3/2 \times 2/3 = 1$. It soon becomes apparent that the reciprocal of a fractional number can be named by the fraction obtained by interchanging the numerator and denominator of a fraction which names the given number. [The word *invert*, properly defined, should not be in ill repute. It does not mean "turn the fraction upside down." If it did, the reciprocal of $\frac{3}{4}$ would be $\frac{\rotatebox{180}{4}}{\rotatebox{180}{3}}$.]

In the standard algorithm for dividing fractional numbers, a division example becomes a multiplication example.

$$\frac{7}{8} \div \frac{2}{5} = \frac{7}{8} \times \frac{5}{2}$$

The quotient of a first fractional number by a second fractional number equals the product of the first by the reciprocal of the second.

It is helpful to note that, in a "real'world" context, the reciprocal tells how many of a given part of an object are contained in 1 whole of that object.

Now, consider the following example of the standard algorithm for multiplying fractional numbers:

$$\frac{3}{\not{5}_{1}} \times \frac{\overset{2}{\not{10}}}{17} = \frac{6}{17}$$

We could write this:

$$\frac{3}{5 \div 5} \times \frac{10 \div 5}{17} = \frac{3 \times 2}{1 \times 17} = \frac{6}{17}$$

Who understands what we have done, in terms of operations on numbers? Abou two years before we tackled an example like this, we should have started to develop an understanding of the principle that dividing one factor of a product by a given number divides the product by that number. I have found this principle difficult for children to grasp. They need a long time, and many experiences like the following, before any real understanding appears.

$$7 \times 6 = \_\_\_\_$$

$$7 \times (6 \div 2) = \_\_\_\_$$

$$7 \times (6 \div 2) = \_\_\_\_ \div 2$$

Eventually they get to something like this:

$$10 \times 12 = \_\_\_\_$$

$$(10 \div 5) \times 12 = \_\_\_\_$$

$$(10 \div 5) \times 12 = \_\_\_\_ \div 5$$

$$(10 \div \_\_) \times 12 = 120 \div 5$$

$$10 \times (12 \div 6) = \_\_\_\_$$

$$10 \times (12 \div 6) = \_\_\_\_ \div 6$$

$$(10 \div 5) \times (12 \div 6) = \_\_\_\_$$

$$(10 \div 5) \times (12 \div 6) = 120 \div \_\_\_\_$$

$$(10 \div 5) \times (12 \div 6) = 120 \div (\_\_\_\_ \times \_\_\_\_)$$

We are now ready to see the rationale of the example with the fractions.

$$\frac{3}{5} \times \frac{10}{17} = \frac{3 \times 10}{5 \times 17}$$ *by definition*

These products are not the same, so the fractions are not equivalent.

$$105 > 104 \qquad\qquad 104 < 105$$

$$\frac{5}{8} > \frac{13}{21} \qquad\qquad \frac{13}{21} < \frac{5}{8}$$

Again, this amounts to comparing numerators of fractions having a common denominator.

$$\frac{5}{8} \times \frac{21}{21} \; ? \; \frac{13}{21} \times \frac{8}{8} \quad \text{and} \quad \frac{13}{21} \times \frac{8}{8} \; ? \; \frac{5}{8} \times \frac{21}{21}$$

$$\frac{5 \times 21}{168} \; ? \; \frac{13 \times 8}{168} \qquad\qquad \frac{13 \times 8}{168} \; ? \; \frac{5 \times 21}{168}$$

$$\frac{105}{168} > \frac{104}{168} \qquad\qquad \frac{104}{168} < \frac{105}{168}$$

Stated formally, the necessary and sufficient condition that the fractional number a/b be grater than the fractional number c/d is that the product $a \times d$ be greater than the product $c \times b$. $a/b > c/d \iff ad > cb$. The "greater than" relation also establishes the "less than" relation:

$$a/b < c/d \iff ad < cb.$$

Once order relations are established, we have a way of telling whether or not an expression containing a minus sign is meaningful, and subtraction of fractional numbers can be treated in a manner analogous to the treatment of subtraction of cardinals. Everything which has been said about fractions in addition exercises applies to fractions in subtraction exercises.

105 one hundred sixty-eighths
- 104 one hundred sixty-eighths
1 one hundred sixty-eighths

$$\frac{105}{168} - \frac{104}{168} = \frac{1}{168}$$

"You have to get what you're subtracting." In this case, there is no "simpler" name for the difference.

In essence, multiplication of fractional numbers is defined as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$\frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8}.$$

The fifth grader can accept this definition, given the appropriate background. He knows that $1/2$ of $3/4$ of a strip of paper is $3/8$ of that strip of paper; with numbers, $1/2$ of $3/4 = 3/8$ and, since $1/2$ of $3/4 = 1/2 \times 3/4$ (it has been so defined), $1/2 \times 3/4 = 3/8$.

$$[(n \div n) \times a] \times b = a \times [(n \div n) \times b] \qquad [(5 \div 5) \times 3] \times 4 = 3 \times [(5 \div 5) \times 4]$$

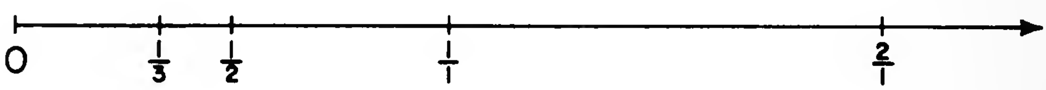
$$(1 \times a) \times b = a \times (1 \times b) \qquad (1 \times 3) \times 4 = 3 \times (1 \times 4)$$

$$a \times b = a \times b. \qquad 3 \times 4 = 3 \times 4.$$

Two fractions either name the same number or name different numbers. If the numbers are different, one is larger than the other. Once we have a way of determining which of two fractional numbers is the larger [and which is the smaller] we have a way of ordering the set of fractional numbers.

One widely-used definition of order among [fractional] numbers is the following:

A number whose graph is to the right of the graph of another number on a picture of a number-line like this:



is the larger number. It may be pedagogically better to start with the "opposite" definition, since we usually read from left to right: Of two numbers, graphed on this number-line picture, the one to the left is the smaller. '1/3' is to the left of '1/2'; $1/3 < 1/2$.

This definition has always seemed unsatisfactory to me. How does one tell what points on the scale should be associated with certain given numbers? We can take a strip of paper which we will call 1 unit, fold it into two equal parts and then into three equal parts, and conclude that 1/3 is less than 1/2 because the first fold for thirds "comes before" the fold for halves, perhaps; but suppose the two numbers we are considering are 5/8 and 13/21. It is not easy to divide some object, even a unit segment, into 21 congruent parts. Comparing 13 of 21 congruent parts of an object with 5 of 8 congruent parts of the same object is an extremely difficult mechanical task, and even if we could achieve this, we might still have the problem of associating the results with the numbers we are comparing. It is better to work with the numbers themselves.

The same test we used for equivalence of fractions can be expanded to determine which of the numbers named by the fractions is larger.

$$\frac{5}{8} \; ? \; \frac{13}{21} \qquad \frac{13}{21} \; ? \; \frac{5}{8}$$

$$5 \times 21 \quad ? \quad 13 \times 8 \qquad 13 \times 8 \quad ? \quad 5 \times 21$$

$\frac{21}{91} = \frac{21 \div 7}{91 \div 7} = \frac{3}{13}$	or	$\frac{21}{91} = \frac{3 \times 7}{13 \times 7} =$
$\frac{21}{91} = \frac{3}{13}$		$\frac{3 \times 7 \div 7}{13 \times 7 \div 7} = \frac{3 \times 1}{13 \times 1} = \frac{3}{13}$

We say that two fractions are equivalent (they name the same number or are different ways of thinking about the same number) if we can multiply or divide the numerator and denominator numbers of one of them by some [same] number and get the numerator and denominator numbers of the other. [After multiplication and division of fractional numbers have been introduced, it can be shown that this amounts to multiplying or dividing by 1/1, operations which always result in the same number we started with.] If two fractions are equivalent, then the corresponding ordered pairs of cardinals belong to the same fractional number. In general, $\{(p, q), (2p, 2q), \ldots, (np, nq), \ldots\} \in p/q \; (q \neq 0, \; n \neq 0)$.

Another way of telling whether two fractions are equivalent is to note the following illustrations of an important relation:

$\frac{1}{2} = \frac{2}{4}$	$\frac{5}{8} = \frac{10}{16}$	$\frac{18}{12} = \frac{3}{2}$	$\frac{14}{7} \neq \frac{3}{2}$
$1 \times 4 = 2 \times 2$	$5 \times 16 = 10 \times 8$	$18 \times 2 = 3 \times 12$	$14 \times 2 \neq 3 \times 7$

Note that this amounts to comparing numerators of fractions having a common denominator.

$\frac{1}{2} \times \frac{4}{4} = \frac{2}{4} \times \frac{2}{2}$	$\frac{14}{7} \times \frac{2}{2} \neq \frac{3}{2} \times \frac{7}{7}$
$\frac{1 \times 4}{8} = \frac{2 \times 2}{8}$	$\frac{14 \times 2}{14} \neq \frac{3 \times 7}{14}$
$\frac{4}{8} = \frac{4}{8}$	$\frac{28}{14} \neq \frac{21}{14}$

Stated formally, '$a/b = c/d$' is a necessary and sufficient condition for '$a \times d = c \times b$' — that is, '$a/b = c/d$ if and only if $ad = cb$' is a theorem.

We can use this relation to show that, if the numerator and denominator numbers of a fraction have a common factor, removing this common factor (by division) produces an equivalent fraction.

	$\frac{15}{20} = \frac{5 \times 3}{5 \times 4}$
$\frac{n \times a}{n \times b} = \frac{a}{b}$ because	$\frac{5 \times 3}{5 \times 4} = \frac{3}{4}$ because
$(n \times a) \times b = a \times (n \times b)$	$(5 \times 3) \times 4 = 3 \times (5 \times 4)$

The sum exists. In fact, we have just written a couple of its names. These names are probably not the most convenient ones for our purposes, so we need a way to find a more convenient name. If the denominators of the fractions were the same, we could add the numerator numbers to get a numerator for the "simple" name for the sum and use the same denominator. Can we find a name for $1/4$ whose denominator is '8'? (Can we think of 1 fourth as some number of eighths?) If a child knows without going through all of this that $1/4 + 3/8 = 5/8$, should we insist that he go through it anyway? [A kindergarten child once said to me, "A half and a third is five sixths." I said, "How do you know that?" He shrugged, "How do I know anything? It just is."] I would let a child who knows an answer write that answer and give him an example for which he needed the computation in teaching him how to do the computation.

Suppose someone does not know a simple name for $1/2 + 1/3$. The standard algorithm for finding this name is to choose, out of the endless list of names for $1/2$ and $1/3$, a pair of fractions with the same denominator. It is standard practice to choose that pair of names in which the denominator number is smallest. $1/2 = 3/6 = 6/12 = 9/18 = \ldots$ and $1/3 = 2/6 = 4/12 = 6/18 = \ldots$, but we usually choose $3/6$ and $2/6$; $3/6 + 2/6 = (3 + 2)/6 = 5/6$. There is no mathematical law which says we have to make this choice. In fact, a person who would like a picturesque way of going insane could spend the rest of his sane years writing different correct computations for the sum of $1/2$ and $1/3$. However, small numbers are usually easier to work with than large numbers, and $5/6$ sounds "simpler" than $65/78$. Discretion is not out of place.

Before introducing subtraction, we must establish order-relations among fractional numbers. The "function vs. number" distinction enters the picture here I think. Why will a child who knows without question that one-half of something is more than one-third of it very often state with equal conviction that $1/3 > 1/2$? When he is dealing with numbers, $3 > 2$, so why isn't $1/3 > 1/2$? If he has ever encountered some of the foolishness about "the Golden Rule for equations and inequalities" or "treating both sides alike", he can "prove" that, because $3 > 2$, $1/3 > 1/2$.

We must have established, intuitively at least, that each ordered pair of cardinals (second component $\neq 0$) belongs to one and only one fractional number.

Next, we need a way of determining whether or not two different such ordered pairs belong to the same fractional number. A usual way of doing this is to parlay the question into equivalence of fractions, somewhat as follows:

$$\frac{3}{13} \stackrel{?}{=} \frac{21}{91}$$

$$\frac{3}{13} = \frac{3 \times 7}{13 \times 7} = \frac{21}{91}$$

$$\frac{3}{13} = \frac{21}{91}$$

or

is the measure of this length in terms of the unit we are using. Thus, {(1, 1), (2, 2), (3, 3), . . . , (n, n), . . . } is the fractional number which we can name by using the name suggested by any one of its members.

Similarly, the length of segment b is 1 half-unit (1/2), 2 quarter-units (2/4), 4 eighth-units (4/8), and so on; and if we took the same unit scale and divided it differently, we could find the same length expressible by '3/6', '6/12', '5/10', and an endless list of equivalent fractions.

A great deal of work, preferably with a variety of materials, must be done to establish the idea that each fractional number, like each cardinal number, has an endless list of names. For some children, this idea becomes clearer if one speaks of different ways to think of a number rather than of different names for the number. "I can think of 99 as 100 - 1 if I want to." "Why would you want to do that?" "Well, the easy way to find the cost of 3 pounds of steak at 99¢ a pound, . . ." "Oh, I see. How can you tell how you want to think of a number?" "It depends upon what I want to do with it. Sometimes I want to think of 1/2 as 5/10. Sometimes I want to think of 1/2 as 2/4. The way my mother cuts pies makes me think of 1/2 as 3/6." Different ways of thinking about, or writing about, or talking about, or naming the same number — this makes sense. And aren't we lucky to have a choice?

The words applied to the two terms of a fraction give a clue to the concepts implied.

$$\begin{matrix} 3 \leftarrow \text{numerator} \\ 8 \leftarrow \text{denominator} \end{matrix} \Big\} \quad \text{number name}$$

The numerator number is used definitely in the cardinal sense; it tells how many of something. What is the "something"? The name of the "something" is suggested by the denominator.

In the introduction of addition of fractional numbers in its first abstract presentation, I have found it helpful to write the fractions like this:

1 eighth	$\frac{1}{8}$	1 eighth + 2 eighths = 3 eighths
+ 2 eighths	$+\frac{2}{8}$	$\frac{1}{8} + \frac{2}{8} = \frac{3}{8}$
3 eighths	$\frac{3}{8}$	

"You have to get what you're adding." If you are adding numbers of eighths, the sum is a number of eighths. Always this sum may be expressed in other ways (6/16, for instance, in this case) and sometimes it may be expressed in "simpler form", but these are matters of what you choose to call your answer.

Now, consider this:

1 fourth	$\frac{1}{4} + \frac{3}{8}$
+ 3 eighths	

2 of 3 apples is (2×3) apples.
2 disjoint sets with 3 members each make one set with 2×3 members.
1/4 of 1/3 of an apple equals $(1/4 \times 1/3)$ of an apple.
3/4 of an apple and 1/4 of an apple are 4/4 of an apple, which is equivalent to 1/1 (1 whole) of an apple.

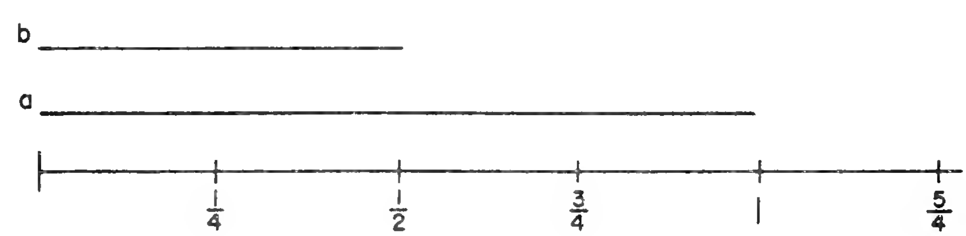
$$3/4 + 1/4 = 4/4 = 1/1$$

How big are the steps between successive lines in the preceding list? Are all the steps the same size? Does having taken one of these steps make subsequent steps easier? We need answers for these questions.

Having established the requisite concepts informally, we may proceed to a more formal treatment of fractional numbers and operations on fractional numbers.

What is a fractional number? It is an infinite set, an equivalence class, of ordered pairs of cardinals in which the second component of a pair cannot be zero. Thus, any ordered pair of cardinals of the form $(n, 2n)$, $n \neq 0$, belongs to the fractional number 1/2, and any pair like (2, 4), (50, 100), or (124, 248) which also belongs to that number suggests a name ('2/4', '50/100', '124/248') for the number. Are these ideas easy for children to grasp? Surely, a few simple declarative sentences will not suffice. It is not wise to be too glib too soon.

Where do we start? [Remember that the children have been working informally with fractions for several years. This is the "start" of the formal treatment.] Perhaps the best way to initiate the development of the meaning of fractional number is to work with equivalence relations among fractions. Suppose that we define a fraction as a name, or a symbol, for a number or a relation. We shall be concerned now with the fraction as a numeral, a name or symbol for a number. To show equivalence among fractions, we might use strips of paper which can be folded appropriately, a number scale which resembles a magnified version of a few inches on an architect's rule, a make-believe machine which will cut and join strips of wood according to the way certain buttons are pressed or other schemes for showing that certain sequences of maneuvers produce equivalent results in terms of the names which may be given to these results.



How long is segment a? It is 1 whole unit (1/1), 2 half-units (2/2), 4 fourth-units (4/4), and 8 eighth-units (8/8); and if we had more marks on the scale, we could see that any fraction for which the two components of the pair of numbers are the same could be used as a name for the number which written symbol. He can discover "equivalence relations", "order relations", "addition facts", "subtraction facts", "multiplication facts", and "division facts" for fractions and fractional numbers, but he never has been working with fractional numbers. In the more complex of his discovery exercises, he has been working with concrete representations of composition of functions.

Usually, the child's first written computation with fractions has to do with finding one half of a number. He does this by dividing the number by 2:

$1/2$ of $18 = 18 \div 2 = 9$.

He is not multiplying $1/2 \times 18$; he is performing the operation of halving on 18, and he finds the number which corresponds to 18 under this operation by dividing 18 by 2.

Somewhat later, he has no difficulty in seeing that two-thirds of 18 will be twice as much as one-third of 18:

$2/3$ of $18 = 2$ of $1/3$ of $18 = 2(18 \div 3) = 12$

The fraction symbol can be introduced initially as an abbreviation for the expressions '1 half', '2 thirds', and the like. In this symbolism, 2 wholes is written '2/1'. Manipulation of fraction cutouts show clearly that '1/2', '2/4', '3/6', '4/8', ... represent the same relation. It serves also to show the commutativity and associativity of certain particular function compositions and, eventually, to demonstrate concretely the analogous operations on numbers.

1/4 of 1/3 of (an object or a number) = 1/12 of (the object or the number)
1/3 of 1/4 of (an object or a number) = 1/12 of ( )
2/3 of 1/4 of ( ) = (2 of 1/3) of 1/4 of ( ) =
2 of (1/3 of 1/4) of ( ) = 2/12 of ( ) = 1/6 of ( )
1/4 of 2/3 of ( ) = 1/4 of (2 of 1/3) of ( ) =
(2 of 1/3) of 1/4 of ( ) = 2 of (1/3 of 1/4) of ( ) =
2/12 of ( ) = 1/6 of ( )

The step from functions to numbers seems to be a small and simple matter, but it may be quite the opposite. At any rate, recognizing that it is a problem is one necessary requirement for its solution.

If you have 2 apples and 3 apples, you have 5 apples.

$2 + 3 = 5$

If you have 2/8 of an apple and 3/8 of an apple, you have

5/8 of an apple.

$2/8 + 3/8 = 5/8$

in such activities may enable a child to discover some relationships which are analogous to relationships among fractional numbers, and this is both a blessir and a curse. It seems not impossible that the traumatic experiences which all too many people associate with mathematics begin with the introduction of '1/2 as a synonym for 'a half'. 'A half of a banana' makes sense, although, as we have seen, what sense it makes depends upon the context in which it occurs; but in '1/2 banana' the '1/2' surely is not the name for a number. Introducing '1/2 banana' as a rephrasing of 'a half of a banana' does not lead toward recognition of fractional numbers. The fraction symbol has several uses, only one of which is naming a fractional number.

A bright first grader who is asked, "Which is more, one half of an orange or one third of an orange?" may quite properly ask, "Is it the same orange?" If his teacher says, "No," he may ask, "Are the oranges both the same size? Would there be any need for his question if the conversation were about numbers? The number one-half is greater than the number one-third, period.

A large part of the difficulty we have been having in teaching the role of fractions in the computational algorithms may lie in our failure to recognize those situations in which the child is dealing with fractional numbers and those situations in which he is not. The "old-fashioned" teacher who kept his questions specific and concrete was probably exhibiting more wisdom in this regarc than some of the "modern" approaches we have seen recently. For instance, the confusion invited by an exercise like the following may be the result of the unfortunate formulation of the directions:

> Name the fractional number represented by the shaded portions in each of the figures below:

The child who answers "correctly" that the first figure shows the fractional number 2/4 and the second figure shows the fractional number 2/3 may note that the shaded portions of the two figures are the same size. An obvious outcome of this line of reasoning is the conclusion 2/4 = 2/3. But do those diagrams show fractional numbers? Two quarters of the first figure are shade; two thirds of the second figure are shaded. It makes no sense to compare the shaded portions unless the appropriate wholes (units) are taken into account. We can note that relatively more of the second figure is shaded than of the first An absolute comparison requires more information. Why? Because we are not dealing with numbers. The '2/4' and '2/3', if used at all, are names for the relation of the shaded portion of each figure to its whole.

Quite properly, a child spends a long time [at least two years] performing man experiments folding, cutting, separating, or joining pieces of paper, cloth, plastic or wood. He thereby discovers many relations which hold for fractiona numbers as well as for the objects he has been using before he ever sees a

Fractions

Consider the child's initial contact with the words we use with fractions. He learns to use the word 'half' in many kinds of contexts.

> Give me half your apple.
> Would you like half of this orange?
> I'll give you half of this egg for lunch.
> I'll let you have half of my candy bar.
> Which half of this banana do you want?
> To make a banana split, you put half of the banana on each side of a long dish.
> I like half of a banana on my cereal.
> Put half of the marbles in each bag.
> What is half of 10?

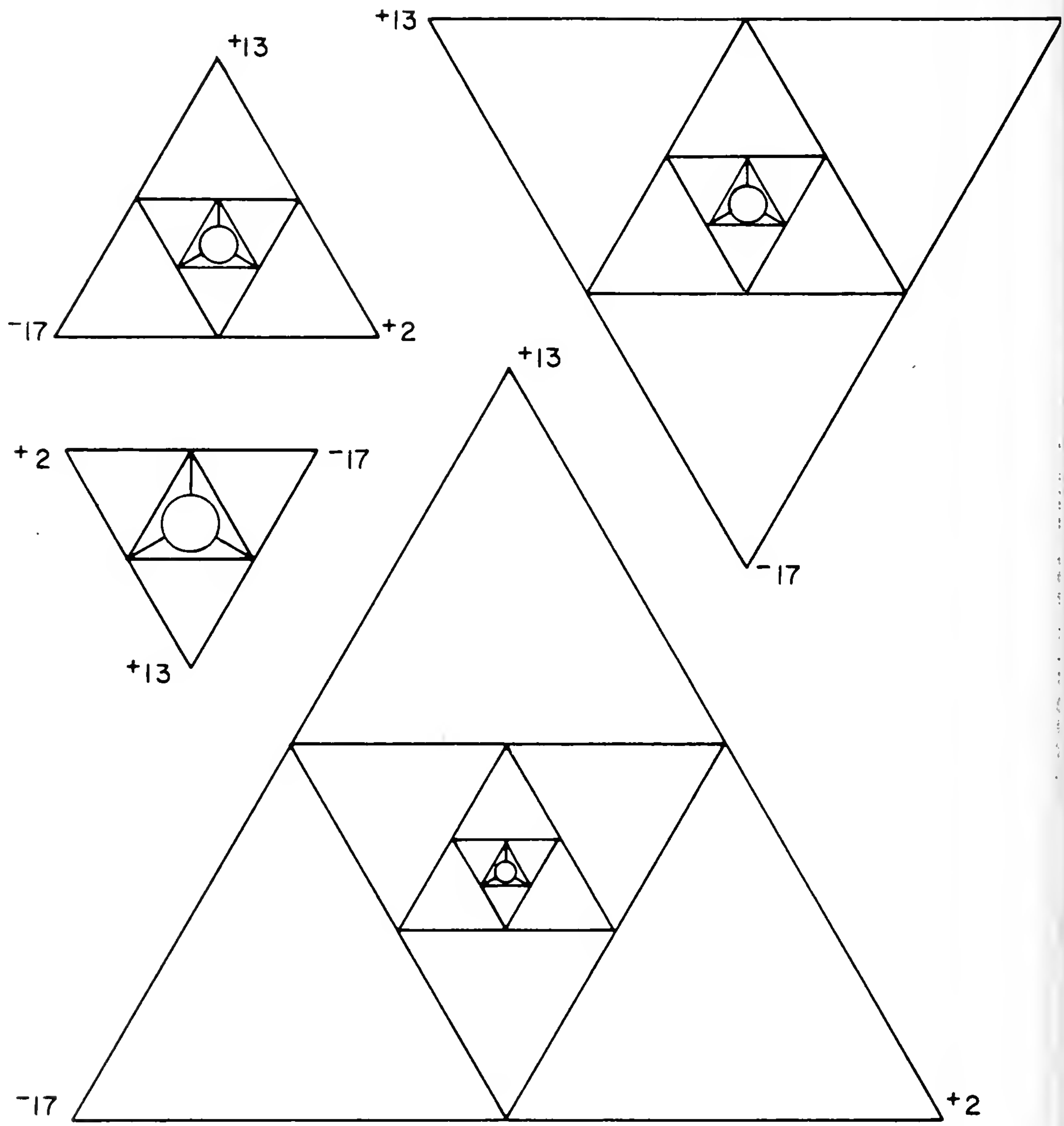
In each of these illustrations, 'half of' refers to a relation, but the relations are, in most cases, significantly different from one another. There are a large number of ways to cut a given apple in half. If half of an orange is thought of as similar to half of an apple, the orange may be cut in any direction; but if the half orange is to be considered an appropriate number of sections, the number of ways to get half of the orange is smaller. The ambiguity in 'half of this egg' is considerable. [Are we going to scramble it? Is it hard-boiled? If it it hard-boiled, shall we slice it? If so, shall we slice it the long way or the short way? Any more questions?] In each of the cases which concern a banana, there are ideally just two halves — but what passes for half of a banana in some instances will not do for a banana split; and the half of a banana on cereal is a quite different situation. Half of a collection of marbles is another collection of marbles, and if you have 10 marbles, all different, there are 252 ways you can divide them into two collections of 5 marbles each. Half of the number 10 is the number 5 — a cardinal number has at most one half [there is nothing which has this halving relation to the cardinal number 3].

There are contexts of other quite different kinds. These are not relevant to our present purposes, but — to compound the confusion — here are some examples:

> This recipe calls for half a cup of sugar.
> It's half a mile to school.
> I'd like just half a bowl of soup.
> There will be a half moon tonight.
> Half of me wants to go to the show; the other half wants to stay home.
> He is half-way committed.
> It's a half-baked idea.
> He only half did the job.
> Would you rather have a half dollar or a half of this dollar?
> He listened with half an ear.

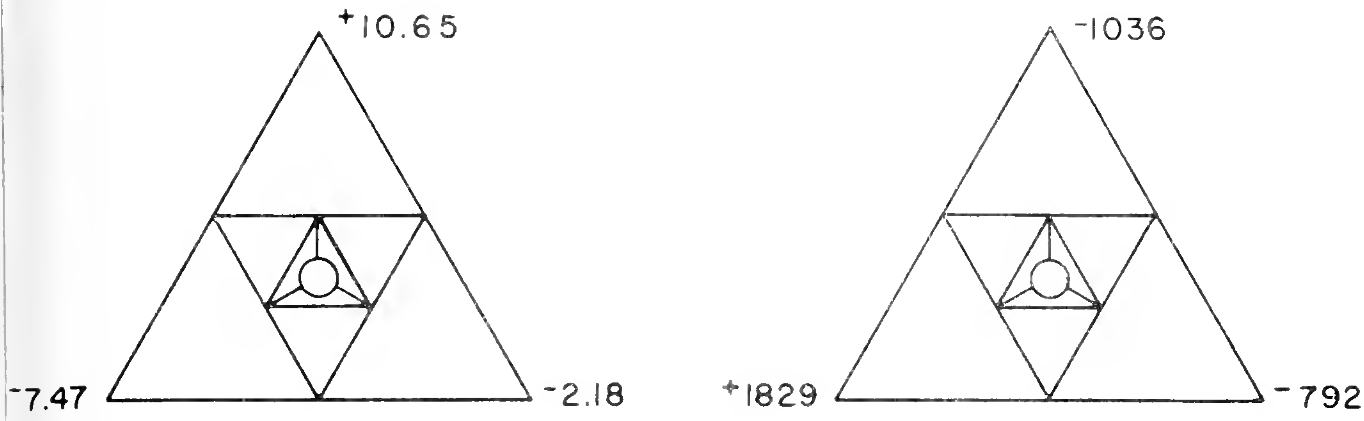
In none of these examples is there one dealing with the number $1/2$. Moreover, in none of the common teaching aids — folding or cutting paper or flannel, playing with blocks or other counters, pouring sand from one container into another, et al., — does a child make any use of fractional numbers. Engaging

(6) Give the inner-number for each of these pyramids.



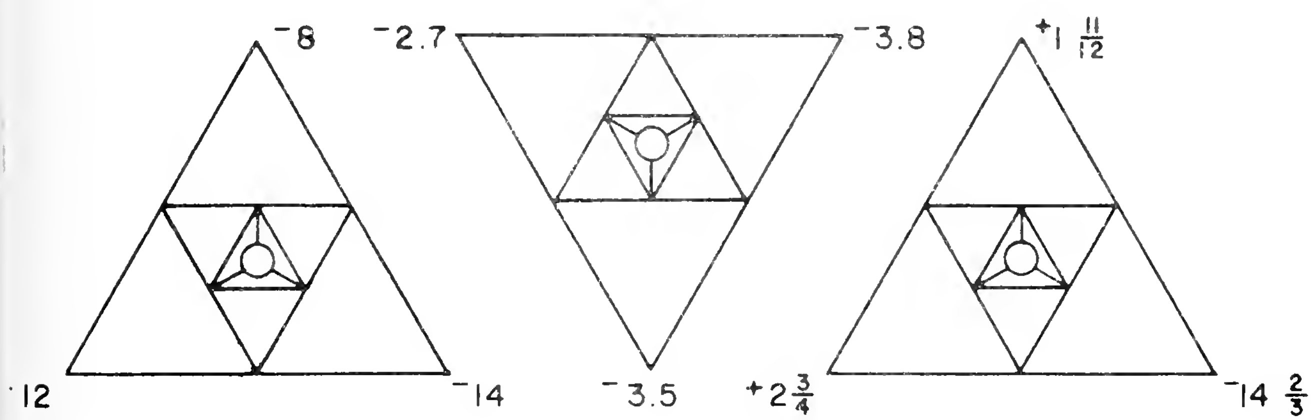
It's about time that you proved your conjectures, don't you think?

J. Hoffmann
Newsletter 13



The inner-number for each pyramid is ⁺4.

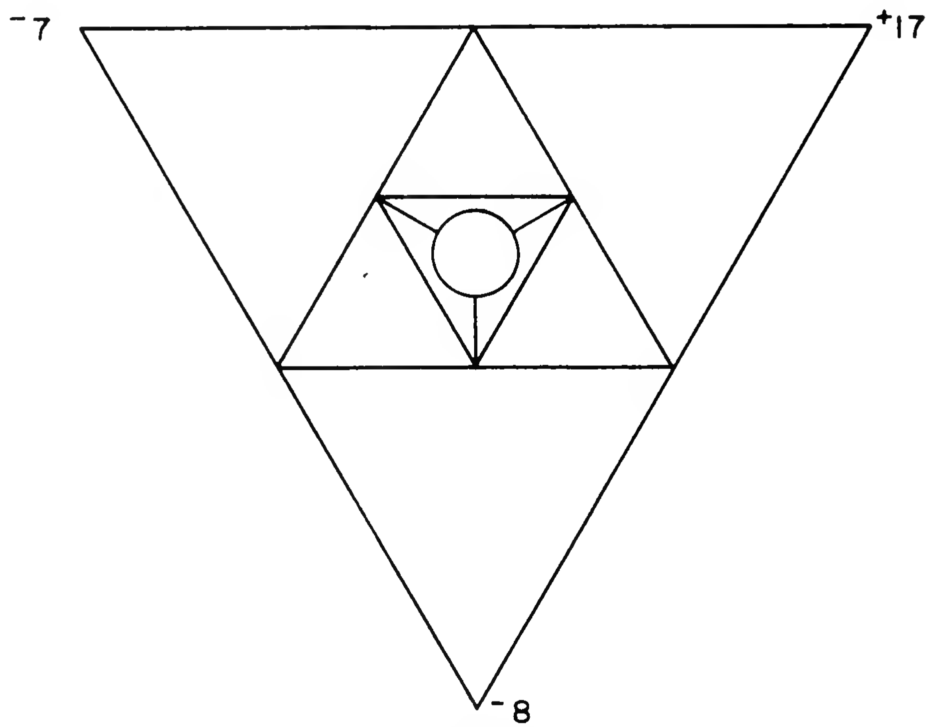
(5) As simply as possible, tell whether each of these pyramids has the same inner-number.



Yes, they do.

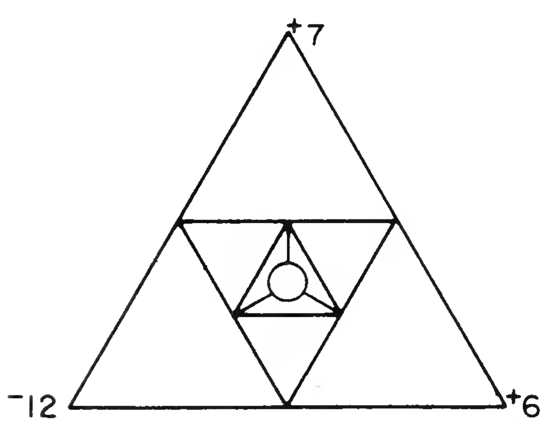
The inner-number is $^{-}16$. For more complicated pyramids, just extend th[e] process.

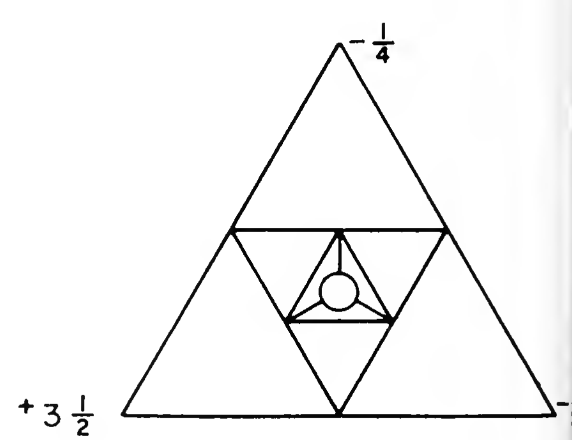
(3) Find the inner-number of this pyramid.



Did you get $^{+}8$? If not, you better check your computation. [I had to!] Now, it is time to make a discovery.

(4) Find the inner-number for each of these pyramids.

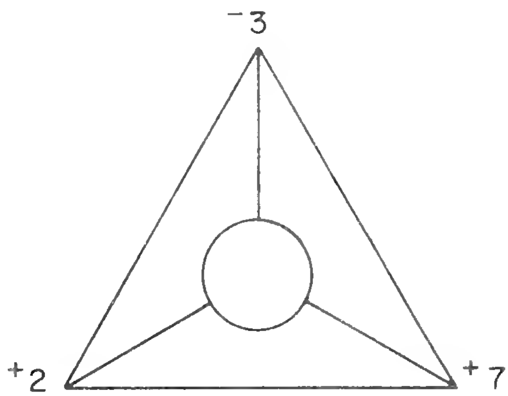




Non-Pharoahic Pyramids

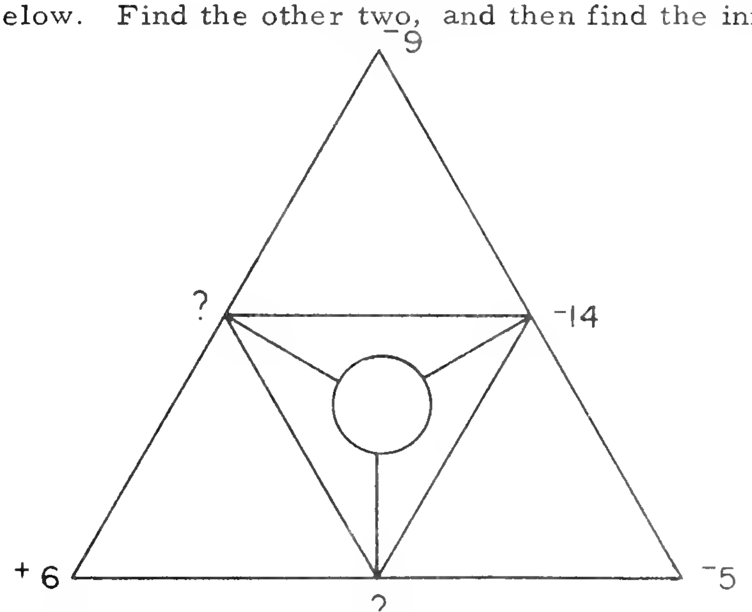
Here is a device which can be used as an interesting drill on addition of real numbers. It has the peculiar inherent property of providing more drill for the methodical students but less drill for the ingenious ones. A short-cut may be discovered by the student at any time the teacher wishes to allow him to make the discovery. On the other hand, the short-cut is very simple to see and still involves a little addition of real numbers. See if you can discover the short-cut to use in the following exercises.

(1) The inner-number is the sum of the 3 "corner-numbers". Give the inner-number of this pyramid.



Of course, the answer is ⁺6. Now, consider a slightly more complicated pyramid. To find the inner-number, one must find the corner-numbers for the inner-triangle. This is done by adding the given numbers at each end of one side.

(2) One corner-number of the inner-triangle has been found for you in the pyramid below. Find the other two, and then find the inner-number.



Since his lecture ended, you are supposed to mean something slightly different by 'complex number' than you were told it meant at the beginning of the lecture. Now, nothing this man has done is wrong — aside from a certain intolerance which he seems to have shown toward his student's lack of a previous understanding of his subject. His primary interest is in structure — often even to the exclusion of any interest in particular things which exhibit that structure. His elimination of the real-complex numbers in favor of the real numbers is an example of this. He doesn't — at least at the moment — see any "real" difference between them — when he was defining the complex numbers, the difference between them was relevant; but he's through with that phase. And, now, it will be much easier for him to say 'real number' than 'real-complex number'. It would be somewhat better for him to answer the question "Are the real numbers among the complex numbers?" by saying "It depends on what you mean by 'complex number'. We decided that it was simpler to use them in place of the real-complex numbers; so for us, they are. But, we didn't need to do this. If we hadn't, they wouldn't be". Of course, in the UICSM program, we didn't, and they aren't.

Tell your students that their problem is one they will probably meet many times. It's that of communicating with someone who uses words a little differently.

Hope this helps.

H. E. Vaughan

Newsletter 14

A Note on Real-Complex Numbers

[This article is an excerpt from a letter written by Professor H. E. Vaughan in answer to a letter from a UICSM-trained teacher. Since many will, no doubt, run across a similar situation, we are sharing this letter withour readers. The question which arose was that of the apparent inconsistency between the stand of UICSM that the set of real numbers is not a subset of the set of complex numbers, and of those who use 'real numbers' as a synonym for 'real-complex numbers,' i. e., numbers of the form 'a + 0i'.]

It seems to me that you have done very well in explaining our point of view. Perhaps the following will also be helpful.

The first thing to note is that people do — legitimately — use the same word with different meanings. In fact, the same person may use the same word with two meanings during the same quarter-hour. If his hearers share enough of his background, they will be able to supply the correct meaning each time. If not — and he wishes to communicate — he should do whatever is necessary to indicate what meanings he intends. However — perhaps because people tend to believe that others think as they themselves do — one often overlooks the need to clarify the meanings of one's words.

'Real numbers' is a term which may be used in many ways. For example, a mathematician may say that when he speaks of real numbers he means the members of any complete ordered field. [He doesn't care which such field, because everything he's going to say applies to any of them equally well — each two complete ordered fields are isomorphic.] He talks this way just because it's easier to say 'real numbers' than 'members of a complete ordered field'. However, in lecturing to another class, this same man may have defined 'real number' in any <u>one</u> of several ways and proved that <u>these</u> real numbers form a complete ordered field. Needless to say, his proof was not merely an application of the principle of identity!

Now, let's see how the same man handles complex numbers. He defines them as ordered pairs of real numbers. If you press him at this point, he says that he doesn't care which real numbers — by now you ought to know that one complete ordered field is just as good as another. [He may even tell you that elements of isomorphic systems differ in name only. If he does, tell <u>him</u> that this is likely to be the only characteristic in which they <u>don't</u> differ.] Next, he establishes an isomorphism between the sub-system of complex numbers with second component zero and the real number system. [When doing so, he carefully points out that these "real-complex numbers" aren't the same as the real numbers — if they were, there would be no point in establishing the isomorphism.] Finally, he says that, since we already have the real numbers, there is no need to keep the real-complex numbers. Let's just excise these from the complex number system and insert, in their places, the corresponding real numbers. From now on a complex number is either a real number or an ordered pair of real numbers with <u>non</u>zero second component. If, later, you ask him whether a real number is a complex number, he thinks you're a dope.

$$
\begin{aligned}
(\square + 2) \times (\square + 5) &= \square \times (\square + 5) + 2 \times (\square + 5) && [\text{dpma}] \\
&= (\square \times \square + \square \times 5) + (2 \times \square + 2 \times 5) && [\ell\text{dpma}] \\
&= [(\square \times \square + \square \times 5) + 2 \times \square] + 2 \times 5 && [\text{apa}] \\
&= [\square \times \square + (\square \times 5 + 2 \times \square)] + 2 \times 5 && [\text{apa}] \\
&= [\square \times \square + (5 \times \square + 2 \times \square)] + 2 \times 5 && [\text{cpm}] \\
&= [\square \times \square + (5 + 2) \times \square] + 2 \times 5 && [\text{dpma}] \\
&= (\square \times \square + 7 \times \square) + 10 && \left[\begin{array}{l} 5+2=7 \\ 2\times 5=10 \end{array}\right] \\
\text{Hence } (\square + 2) \times (\square + 5) &= (\square \times \square + 7 \times \square) + 10.
\end{aligned}
$$

In case one wishes to test an instance — say:

$$(3 + 2) \times (3 + 5) = (3 \times 3 + 7 \times 3) + 10$$

all he need do is write a '3' in each frame:

$$
\begin{aligned}
(\boxed{3} + 2) \times (\boxed{3} + 5) &= \boxed{3} \times (\boxed{3} + 5) + 2 \times (\boxed{3} + 5) && [\text{dpma}] \\
&= (\boxed{3} \times \boxed{3} + \boxed{3} \times 5) + (2 \times \boxed{3} + 2 \times 5) && [\ell\text{dpma}] \\
&= [(\boxed{3} \times \boxed{3} + \boxed{3} \times 5) + 2 \times \boxed{3}] + 2 \times 5 && [\text{apa}] \\
&= [\boxed{3} \times \boxed{3} + (\boxed{3} \times 5 + 2 \times \boxed{3})] + 2 \times 5 && [\text{apa}] \\
&= [\boxed{3} \times \boxed{3} + (5 \times \boxed{3} + 2 \times \boxed{3})] + 2 \times 5 && [\text{cpm}] \\
&= [\boxed{3} \times \boxed{3} + (5 + 2) \times \boxed{3}] + 2 \times 5 && [\text{dpma}] \\
&= (\boxed{3} \times \boxed{3} + 7 \times \boxed{3}) + 10 && \left[\begin{array}{l} 5+2=7 \\ 2\times 5=10 \end{array}\right] \\
\text{Hence } (\boxed{3} + 2) \times (\boxed{3} + 5) &= (\boxed{3} \times \boxed{3} + 7 \times \boxed{3}) + 10.
\end{aligned}
$$

The result is a derivation of the instance from accepted principles and facts. The test-pattern can be thought of as a "general proof" of the generalization.[4]

M. Beberman
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[4] This procedure for teaching deductive proof is used in UICSM High School Mathematics: First Course by M. Beberman and H. E. Vaughan [Boston, Mass.: D.C. Heath and Company, 1964]

$(9 + 2) \times (9 + 5) = 11 \times 14 = 154; \quad (9 \times 9 + 7 \times 9) + 10 = (81 + 63) + 10 = 154$

$(\frac{1}{2} + 2) \times (\frac{1}{2} + 5) = \frac{5}{2} \times \frac{11}{2} = \frac{55}{4}; \quad (\frac{1}{2} \times \frac{1}{2} + 7 \times \frac{1}{2}) + 10 = (\frac{1}{4} + \frac{7}{2}) + 10 = \frac{55}{4}$

This method of testing instances does not shed any light on other instances of the generalization. So, let us use another technique for testing the instances; let us try to derive them from principles which are already accepted. Here is such a derivation:

$$\begin{aligned}
(9 + 2) \times (9 + 5) &= 9 \times (9 + 5) + 2 \times (9 + 5) && [\text{dpma}]\\
&= (9 \times 9 + 9 \times 5) + (2 \times 9 + 2 \times 5) && [\ell\text{dpma}]\\
&= [(9 \times 9 + 9 \times 5) + 2 \times 9] + 2 \times 5 && [\text{apa}]\\
&= [9 \times 9 + (9 \times 5 + 2 \times 9)] + 2 \times 5 && [\text{apa}]\\
&= [9 \times 9 + (5 \times 9 + 2 \times 9)] + 2 \times 5 && [\text{cpm}]\\
&= [9 \times 9 + (5 + 2) \times 9] + 2 \times 5 && [\text{dpma}]\\
&= (9 \times 9 + 7 \times 9) + 10 && \left[\begin{smallmatrix} 5+2=7 \\ 2\times 5=10 \end{smallmatrix}\right]\\
\text{Hence } (9 + 2) \times (9 + 5) &= (9 \times 9 + 7 \times 9) + 10.
\end{aligned}$$

We have tested the first instance and have verified it by showing it to be a consequence of distributive, associative, and commutative principles together with two computing facts. How shall we test the second instance? The procedure is clear — just erase the '9's from the derivation of the first instance and replace them by '$\frac{1}{2}$'s:

$$\begin{aligned}
(\tfrac{1}{2} + 2) \times (\tfrac{1}{2} + 5) &= \tfrac{1}{2} \times (\tfrac{1}{2} + 5) + 2 \times (\tfrac{1}{2} + 5) && [\text{dpma}]\\
&= (\tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times 5) + (2 \times \tfrac{1}{2} + 2 \times 5) && [\ell\text{dpma}]\\
&= [(\tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times 5) + 2 \times \tfrac{1}{2}] + 2 \times 5 && [\text{apa}]\\
&= [\tfrac{1}{2} \times \tfrac{1}{2} + (\tfrac{1}{2} \times 5 + 2 \times \tfrac{1}{2})] + 2 \times 5 && [\text{apa}]\\
&= [\tfrac{1}{2} \times \tfrac{1}{2} + (5 \times \tfrac{1}{2} + 2 \times \tfrac{1}{2})] + 2 \times 5 && [\text{cpm}]\\
&= [\tfrac{1}{2} \times \tfrac{1}{2} + (5 + 2) \times \tfrac{1}{2}] + 2 \times 5 && [\text{dpma}]\\
&= (\tfrac{1}{2} \times \tfrac{1}{2} + 7 \times \tfrac{1}{2}) + 10 && \left[\begin{smallmatrix} 5+2=7 \\ 2\times 5=10 \end{smallmatrix}\right]\\
\text{Hence } (\tfrac{1}{2} + 2) \times (\tfrac{1}{2} + 5) &= (\tfrac{1}{2} \times \tfrac{1}{2} + 7 \times \tfrac{1}{2}) + 10.
\end{aligned}$$

This provides us with a derivation of the second instance from the same principles and computing facts as for the first instance. What we have here is a pattern for testing any instance of the given generalization. One can show the test-pattern most clearly by erasing the '$\frac{1}{2}$'s and replacing them by copies of a free variable:

turned into a proof for the multiplication generalization. [This is evidence of their ability to recognize patterns.] I then asked if the result would hold if one used subtraction instead of addition or multiplication. The class was unanimous in maintaining that it wouldn't hold. I suggested trying an example. The ensuing discussion resulted in a deductive proof of the subtraction generalization, and showed students the usefulness of a deductive proof in convincing one of the validity of a surprising result. It also showed them not to confuse sufficient conditions [commutativity and associativity] with necessary ones.[3]

Searching-for-patterns as preparation for proof

Students who have become accustomed to searching for patterns have very little difficulty in understanding the role that a free variable plays in mathematical language. They learn that one can exhibit a pattern for the members of a set of sentences by writing a single sentence in which free variables take the place of numerals. For example, they understand that the open sentence:

$$(\square - \triangle) - \bigcirc = (\square - \bigcirc) - \triangle$$

exhibits one of the patterns for the sentences:

$$(5 - 8) - 2 = (5 - 2) - 8$$
$$(3 - 7) - 4 = (3 - 4) - 7$$
$$(11 - 1) - 4 = (11 - 4) - 1$$

The frames in the open sentence are free variables. Each of these "numerical" sentences can be obtained from the open sentence by writing the same numeral in all frames of the same shape. The free variables are marks which show where numerals may be written in order to obtain sentences of the prescribed form. They do not "represent" unknown numbers. [The word 'pro-numeral' describes quite precisely the role of free numerical variables.] This understanding of the nature of a free variable makes it easy for students to grasp the idea of a proof in their study of elementary algebra. For example, consider the generalization:

$$\forall_x \ (x + 2) \times (x + 5) = (x \times x + 7 \times x) + 10$$

and the problem of finding out whether this generalization is true. A student might first deal with some instances:

$$(9 + 2) \times (9 + 5) = (9 \times 9 + 7 \times 9) + 10$$

$$(\frac{1}{2} + 2) \times (\frac{1}{2} + 5) = (\frac{1}{2} \times \frac{1}{2} + 7 \times \frac{1}{2}) + 10$$

He might test these instances by computation:

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[3] Such analogies between addition and subtraction have been discussed in the paper "Double Identities" by Professor Herbert E. Vaughan [UICSM Project, 1210 West Springfield, Urbana, Illinois; U.S.A.]

Promoting transfer of discovered generalizations

A student who has discovered a generalization as a result of a search for a pattern among related problems may be able to use it only in problems which are much like those from which he formed the original pattern. He may fail to use his discovery in settings which do not preserve the symbolic arrangements of the original pattern. The teacher must provide a variety of "transfer" situations so that the student can "rediscover" the generalization. For example, in connection with the foregoing lesson on the distributive principle, the equation-solving sequence should be followed by computation exercises like:

$$7 \times 8 + 3 \times 8 = ?$$

$$7 \times 19 + 3 \times 19 = ?$$

$$6 \times 83 + 4 \times 83 = ?$$

$$74 \times 98 + 26 \times 98 = ?$$

$$6.5 \times 7 + 2.5 \times 7 = ?$$

and with worded problems like:

> Mrs. A buys 7 cans of peaches at 37 cents per can and 3 cans of pears at 37 cents per can. What is the total bill for the peaches and pears?

I have found students who had discovered commutativity and associativity as a result of equation-solving and computation exercises such as those described above but who were surprised and puzzled by the following situation: The numbers 8, 3, 9, and 5 are listed in the four cells of a 2-by-2 table:

8	3	→ 11
9	5	→ 14
↓ 17	↓ 8	↓ → 25

Row-sums [11 and 14] are computed and then column-sums [17 and 8] are computed. It turns out that the sum of the row-sums is the same as the sum of the column-sums. "Will this last result hold no matter what numbers are selected?" Many students were not at all sure of the answer to this question. Some even accused me of using very special numbers. When all of the class finally agreed that the result would hold generally, I asked if the result would hold if one used multiplication instead of addition. Again, some doubt was expressed. The issue was settled when commutativity and associativity were mentioned by name and proofs exhibited. Students were very quick in pointing out that the proof for the addition generalization needed only slight modifications to be

Just drop the zero.

Instead he offers the following equation:

$$7 \times \square + 3 \times \square = 291$$

This is greeted with groans and cries of "Not fair." One or two students will offer the right solution after just a little contemplation. The teacher can stimulate activity by asking for "approximate" solutions. In any case, once the correct answer has been exhibited <u>and</u> <u>checked</u> by the teacher, rapid responses are obtained for equations of the form:

$$7 \times \square + 3 \times \square = n$$

in which the 'n' is replaced by '79', '122', '15608', etc.

The teacher may now move to equations of the form:

$$6 \times \square + 4 \times \square = n$$

or of the form:

$$78 \times \square + 22 \times \square = n$$

Each alternative either reinforces the generalization a student has been forming or causes him to modify his generalization.

The next step is quite important because many students have become aware of a generalization which is based on the fact that it is easy to divide by a power of 10. Although the distributive principle for multiplication over addition does play a role in the student's thinking at this stage, it is probably not present in the kind of generality the teacher is seeking. That this is the case is borne out by the answers the teacher gets for the next equation:

$$7 \times \square + 5 \times \square = 48$$

In fact, the teacher should try to find a student who will say '4.8' for this equation. Many students are amazed when the check reveals that 4.8 is not a solution. When the correct solution is obtained, the teacher is ready to bring the students closer to discovering the distributive principle.

Other principles can be discovered through a search-for-a-pattern approach. Here are examples of equations each of which can be used to generate such a discovery sequence:

$$9 \times n = \square \times 9$$

$$76 + \square = n + 76$$

$$(5 \times \square) \times (x \times \square) = n$$

$$\frac{8 \times \square}{n} = 8$$

$$(17 + \square) - n = 17$$

$$\frac{9 \times n}{\square} = 9$$

(13) $\frac{\square+\square}{2} = 29$ (14) $\frac{\square+\square}{2} = 33$ (15) $\frac{\square+\square}{2} = 317$

(16) $\frac{\square+\square}{2} = 973$ (17) $\frac{\square+\square}{2} = 6789$

At some point during the sequence, the students begin giving answers very quickly. The teacher expresses mild surprise at their speed. Most students enjoy this pretense of fooling the teacher. The teacher then shocks students by giving them the equation:

$$\frac{\square+\square}{2} = \frac{1}{2}$$

Some children are hesitant about applying their discovery. A brave one offers $\frac{1}{2}$ as the solution and the teacher checks it. The students have now developed the attitude that short cuts are to be sought and may be used with confidence. The teacher proceeds to the following equation:

$$7 \times \square + 3 \times \square = 20$$

The students must now experiment to find a solution. The teacher does not allow the solution to be announced publicly until most students have found it. [The teacher gives individual recognition to students who have found the solution by inspecting their written answers at their desks and announcing 'Right!'.] Finally, he asks a student for the solution and checks it:

$$\underbrace{7 \times \boxed{2}} + \underbrace{3 \times \boxed{2}} = 20$$

$$\begin{array}{r}7\\ \times 2\\ \hline 14\end{array} \qquad \begin{array}{r}3\\ \times 2\\ \hline 6\end{array} \qquad \begin{array}{r}14\\ +6\\ \hline 20\end{array}$$

The teacher now modifies the equation by erasing the '2's from the frames and replacing the '20' by a '50'. After some experimenting, several students will produce the correct solution. The teacher writes a '5' in each frame and carries out the check:

$$7 \times \boxed{5} + 3 \times \boxed{5} = 50$$

$$\begin{array}{r}7\\ \times 5\\ \hline 35\end{array} \qquad \begin{array}{r}3\\ \times 5\\ \hline 15\end{array} \qquad \begin{array}{r}35\\ +15\\ \hline 50\end{array}$$

The teacher then proceeds with the following sequence of equations:

(1) $7 \times \square + 3 \times \square = 30$ (2) $7 \times \square + 3 \times \square = 60$ (3) $7 \times \square + 3 \times \square = 80$

(4) $7 \times \square + 3 \times \square = 130$ (5) $7 \times \square + 3 \times \square = 120$ (6) $7 \times \square + 3 \times \square = 170$

(7) $7 \times \square + 3 \times \square = 380$ (8) $7 \times \square + 3 \times \square = 480$ (9) $7 \times \square + 3 \times \square = 5720$

The correct solutions are listed in the frames each time but are checked only occasionally. Most children have now become aware of a generalization which produces correct solutions. The teacher does not ask for a statement of any such generalization. If he did, he would probably get something like:

Avoiding the too-narrow generalization

A teacher who wishes a student to discover a generalization as a result of a search for a pattern must be careful to provide exercises which prevent students from settling on generalizations of limited scope. Suppose that a teacher wishes a class of students to discover that multiplication distributes over addition. We shall assume that his students are reasonably skilled [five minutes of practice required] in finding numbers which satisfy equations such as:

$$\square + 2 = 7 \qquad \frac{\square + 1}{3} = 2 \qquad 3 \times \square + 4 = 19 \qquad 2 + 4 \times \square = 26$$

The students solve such equations intuitively, using what they call 'common sense'. The teacher then introduces equations which contain two occurrences of the variable, and he establishes the rule that, when the variable is replaced by a numeral, both of its occurrences must be replaced by copies of the same numeral. Thus:

allowed	not allowed
$\boxed{6} + \boxed{6} = 12$	~~$\boxed{8} + \boxed{4} = 12$~~

Next the teacher presents a sequence of problems to make students aware of the fact that searching for a pattern and using their discovery are praiseworthy activities. [Many children who have been "educated" to do only as the teacher and textbook demand are fearful of finding shortcuts.] Here is one such sequence [The teacher should start with (1). Then, by erasing '18' and writing '24', he obtains (2). Similarly, he converts (2) into (3), and so on, keeping frames, plus sign, and fraction bar intact. The teacher writes each answer in the frame occasionally checks, and clears the frames for the next equation.]

(1) $\square + \square = 18$ (2) $\square + \square = 24$ (3) $\frac{\square + \square}{6} = 4$

(4) $\frac{\square + \square}{5} = 4$ (5) $\frac{\square + \square}{4} = 4$ (6) $\frac{\square + \square}{3} = 4$

(7) $\frac{\square + \square}{2} = 4$ (8) $\frac{\square + \square}{2} = 5$ (9) $\frac{\square + \square}{2} = 7$

(10) $\frac{\square + \square}{2} = 11$ (11) $\frac{\square + \square}{2} = 15$ (12) $\frac{\square + \square}{2} = 18$

Children can be asked to list the multiples of 2 in one counting table and the multiples of 3 in a copy of the same table. If the tables are drawn on sheets of translucent paper and one table is superposed on the other then, when the sheets are held up to the light, the intersection reveals the multiples of 6. The Sieve of Eratosthenes is close at hand.[1]

<u>Asking the right question first</u>[2]

Here is an example of a sequence of exercises for which the discovery of a pattern enables the student to answer an otherwise difficult question:

1. Compute the sum of the first 2 odd numbers. $1 + 3 = ?$
2. Compute the sum of the first 3 odd numbers. $1 + 3 + 5 = ?$
3. Compute the sum of the first 4 odd numbers. $1 + 3 + 5 + 7 = ?$
4. Compute the sum of the first 5 odd numbers. $1 + 3 + 5 + 7 + 9 = ?$
5. Compute the sum of the first 90 odd numbers.

The teacher hopes that a student will discover a generalization as he works the first four exercises and that he will apply this generalization in solving the fifth exercise. This is an example of a misuse of the search-for-a-pattern technique. As the student works Exercises 1-4, there is little incentive to relate the sums to the questions posed. Moreover, it is difficult to predict just how many easier questions are needed to promote recognition of the patterns.

Contrast the foregoing sequence of questions with the following sequence in which the difficult question is posed first:

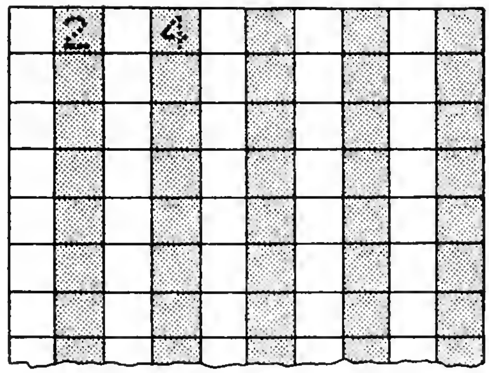
1. Compute the sum of the first 90 odd numbers.
   $1 + 3 + 5 + 7 + 9 + \ldots =$ ______

   If you think this question is too hard to answer because it involves too much computing, try answering the following questions, and return to this question later.
2. Compute the sum of the first 2 odd numbers. $1 + 3 =$ ______
3. Compute the sum of the first 3 odd numbers. $1 + 3 + 5 =$ ______
4. Compute the sum of the first 4 odd numbers. $1 + 3 + 5 + 7 =$ ______
5. Compute the sum of the first 5 odd numbers. $1 + 3 + 5 + 7 + 9 =$ ______
6. Complete the following table to summarize the results of Exercises 2-5.
7. Continue the table for as long as you need to in order to find the sum of the first 90 odd numbers. As soon as you think you know what that sum is, write the answer to Exercise 1.

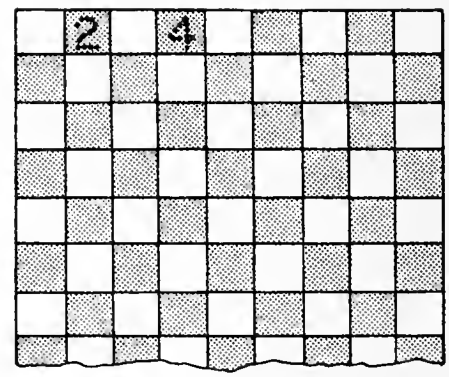
This last activity was first suggested to me by Mrs. Lore Rasmussen of The Miquon School in Miquon, Pennsylvania; U.S.A. Mrs. Rasmussen's students cut out the filled cells in each counting table, and then superpose. Professor Robert Wirtz of UICSM suggested the use of transparencies.

See "Learning By Discovery" by Gertrude Hendrix in <u>The Mathematics Teacher</u>, Vol. LIV, No. 5, May, 1961, pp. 294, 296.

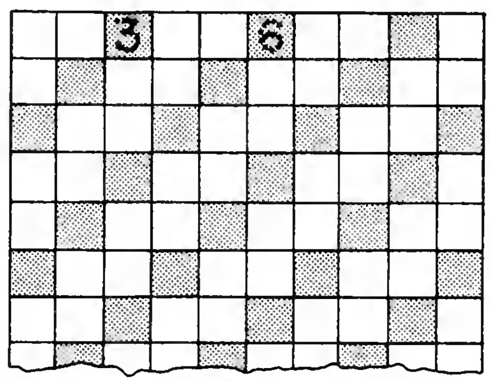
1. (a)



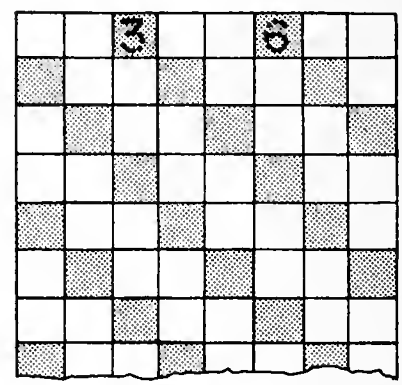
(b)



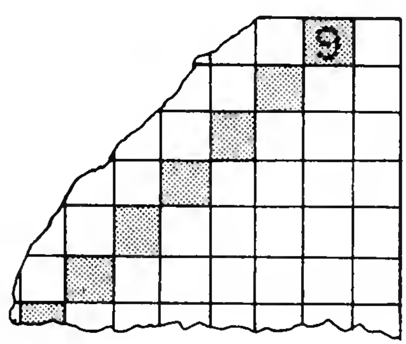
2. (a)



(b)



3.



By formulating counting exercises in a way which stresses patterns, the teache may provide himself with a bonus, a source of additional questions which are suggested by the patterns. For example, in connection with Exercises 1(a) and 1(b), the teacher may ask students to suggest the size [number of cells] of the top row for another counting table in which counting by twos gives a checker board pattern. Similarly, for counting by threes, how many cells should there be in the top row so that the shaded cells are arranged in vertical columns?

Searching for Patterns*

Many children are stimulated to learn mathematics by searching for patterns. In order that I not be accused of fanatical devotion to this mode of learning, I state at the outset that I regard the searching-for-patterns activity as just one of many worthwhile things a child may do in learning mathematics. Moreover, as I shall point out later, this technique has pedagogical disadvantages.

Searching for a pattern is a worthwile activity because it provides a student with an opportunity to make conjectures and to test them. Also, it may demand a considerable amount of self-motivated computational practice. A student who has found a pattern has discovered a generalization. At this point, he may wish to state his generalization, or he may wish to "explain" it, or he may wish to search for further generalizations. In any case, the teacher now has several opportunities for additional teaching, all of which arise out of an interest generated by the act of discovery.

Some children have discovered patterns as a result of their own explorations and without direction from their teachers. They have used the discoveries in organizing their knowledge of mathematics. These children are confident in their understanding of mathematics because they have discovered that mathematics is a logical and orderly subject. Each teacher should make such explorations an integral part of his teaching so that all of his students may develop this confidence.

In this paper I shall present several examples of how teachers may use the student's interest in searching for patterns to promote and stimulate learning. I shall also call attention to pedagogical errors and to some of the dangers inherent in the use of this teaching tool.

<u>Modifying conventional arrangements of exercises</u>

One of the easiest things a teacher can do to promote the search for patterns is to depart from the direct line-by-line and left-to-right arrangements of exercises customarily found in textbooks. For example, suppose that a teacher wishes to give students practice in counting by twos, or by threes, or by fours, etc. One way to present such exercises is to give them directly in textbook style:

1. Count by twos starting with 2.
2. Count by threes starting with 3.
3. Count by nines starting with 9.

An alternative presentation which reveals striking patterns is one in which students are asked to list numbers in only the shaded cells of the counting tables:

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* A paper prepared by Professor Max Beberman [Director, University of Illinois Committee on School Mathematics] for the O.E.C.D. International Working Session on New Methods in the Teaching of Mathematics; Athens, Greece, November 19, 1963.

The solution of (6) is not similar to the solution of the quadratic equation. The technique of completing the square has no analogue for this type of equation. In the same vein, the angle's vertex is (0, c), while the vertex of the graph of the quadratic function is

$$\left(\frac{-b}{2a}, \frac{-b^2 + 4ac}{4a}\right)$$

The vertex of the former is restricted to the y-axis. The vertex of the latter may be any point on the plane.

This suggests that we should look for a more general angle function. Let A be the function, for some $a \neq 0$, b, c, and h, such that

(9) $$A = \{(x, y): y = a|x - h| + bx + c\}.$$

Let's use a technique analogous to completing the square to analyze A. We see that

$$bx + c = b(x - h) + (c + hb)$$

so that

(10) $$A = \{(x, y): y = a|x - h| + b(x - h) + (c + hb)\}.$$

Using the definition of $|x - h|$, and abbreviating as in (5),

(11) $$y = \begin{cases} (b + a)(x - h) + (c + hb) & \text{if } x - h \geq 0 \\ (b - a)(x - h) + (c + hb) & \text{if } x - h \leq 0 \end{cases}$$

This certainly has an angle as its graph.

Finally, we shall find the roots of:

$$a|x - h| + b(x - h) + (c + hb) = 0$$

just as we solved (6), retaining the same assumptions.

(12) $$\left(x - h \geq 0 \text{ and } x - h = \frac{-(c + hb)}{b + a}\right)$$

or

$$\left(x - h \leq 0 \text{ and } x - h = \frac{-(c + hb)}{b - a}\right)$$

Rearranging, we see that

(13) $$\left(x = \frac{-(bc + ha^2) + a(c + hb)}{b^2 - a^2} \text{ and } x \geq h\right)$$

or

$$\left(x = \frac{-(bc + ha^2) - a(c + hb)}{b^2 - a^2} \text{ and } x \leq h\right).$$

So, we have a result which is analogous to the familiar quadratic formula.

A. Holmes
Newsletter 11

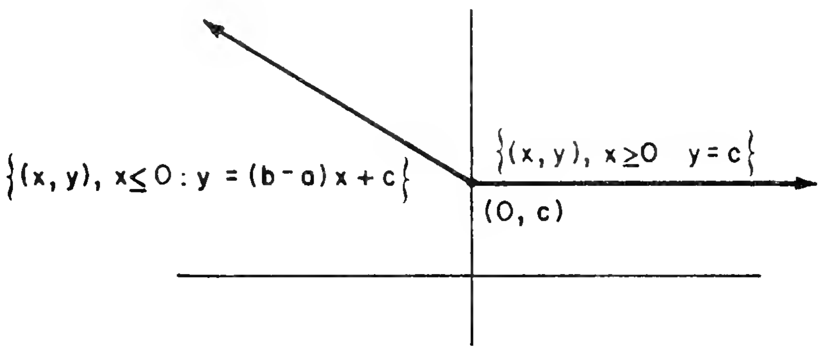
We may abbreviate this by writing

(5) $$y = \begin{cases} (a + b)x + c & \text{if } x \geq 0 \\ (b - a)x + c & \text{if } x \leq 0. \end{cases}$$

Thus g is the union of two functions, the graph of each is a ray, with common end point (0, c). Now, if the two rays are collinear then $b + a = b - a$ — that is, $a = 0$. However, $a \neq 0$, so the rays are noncollinear. Hence, the graph of g is <u>an angle</u>, as our example suggested.

Let's investigate the graph in more detail. The side of the angle is parallel to or contained in the x-axis if and only if $b + a = 0$ or $b - a = 0$. For example, let $b + a = 0$. Substituting in (5), we see that

$$y = \begin{cases} 0 \quad x + c & \text{if } x \geq 0 \\ (b - a)x + c & \text{if } x \leq 0. \quad [b - a \neq 0] \end{cases}$$



On the other hand, if $b - a = 0$, one side of the angle is either the negative half of the horizontal axis [$c = 0$] or parallel to it [$c \neq 0$]. So, by assuming that $b^2 - a^2 \neq 0$, we can require that both sides of the angle be oblique to the x-axis.

$$y = \begin{cases} (b - a)x + c, \ x \geq 0 & [b - a \neq 0] \\ (b + a)x + c, \ x \leq 0 & [b + a \neq 0] \end{cases}$$

and both sides of the angle are subsets of some linear function which cannot be parallel to the x-axis.

Following the development of quadratic functions, it is natural to ask next about roots of the equation when $y = 0$. Let's assume that $b^2 - a^2 \neq 0$ and that $a \neq 0$. We wish to solve:

(6) $$a|x| + bx + c = 0$$

This equation is equivalent to the sentence:

(7) $$[x \geq 0 \text{ and } (b + a)x + c = 0] \text{ or } [x \leq 0 \text{ and } (b - a)x + c = 0]$$

which, under the above assumptions, is equivalent to:

(8) $$\left[x \geq 0 \text{ and } x = \frac{-c}{b + a}\right] \text{ or } \left[x \leq 0 \text{ and } x = \frac{-c}{b - a}\right]$$

Angle Functions

Your students may enjoy comparing quadratic functions with what I have called angle functions. This kind of activity will be quite appropriate when quadratic functions are studied.

Let's begin by graphing the function

$$f = \{(x,y):\ y = 3|x| + 2x + 4\}.$$

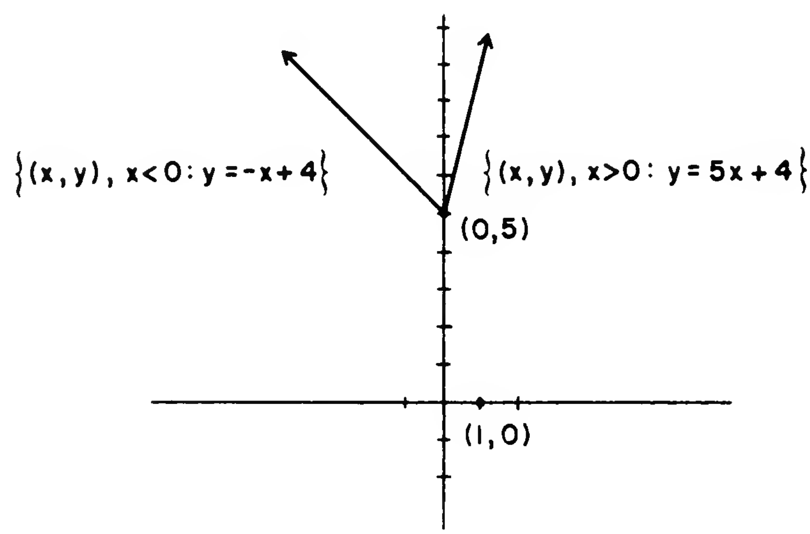
One way is to obtain ordered pairs that belong to f by substitution and computation. A more convenient way is to recall a definition of absolute value:

(1) $$\forall_x\ |x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

Using this definition, we find that f is the union of three sets.

$$\begin{aligned} f = &\ \{(x,y):\ x > 0 \text{ and } y = (3+2)x + 4\} \\ \cup &\ \{(x,y):\ x = 0 \text{ and } y = 4\} \\ \cup &\ \{(x,y):\ x < 0 \text{ and } y = (-3+2)x + 4\} \end{aligned}$$

So, the graph of f is an angle,



Recalling that the graph of the quadratic function

(2) $$q = \{(x,y):\ y = ax^2 + bx + c\},\ [a \neq 0]$$

is a parabola whose orientation and shape depend on the values of a, b, and c we can draw an analogy here by noticing that the position and orientation of the angle were dependent on the numbers 3, 2, and 4.

Let's investigate the function, g, such that for some $a \neq 0$, b, and c,

(3) $$g = \{(x,y):\ y = a|x| + bx + c\}$$

In view of the definition (1),

(4) $$y = \begin{cases} (a+b)x + c \text{ if } x > 0 \\ c \text{ if } x = 0 \\ (b-a)x + c \text{ if } x < 0. \end{cases}$$

<u>Solution</u>. Let x be the number of cents increase in the rate per mile. Then the gross income is $(25 + x)(1000 - 25x)$. This factors to:

$$25(25 + x)(40 - x)$$

This will be maximum when

$$\sqrt{(25 + x)(40 - x)}$$

is maximum.

By the AM-GM inequality

$$\frac{(25 + x) + (40 - x)}{2} \geq \sqrt{(25 + x)(40 - x)}$$

[with equality if and only if $25 + x = 40 - x$].

Since $\frac{25 + x + 40 - x}{2}$ is constant, $\sqrt{(25 + x)(40 - x)}$ is maximum when

$$\frac{25 + x + 40 - x}{2} = \sqrt{(25 + x)(40 - x)}$$

That is when $25 + x = 40 - x$

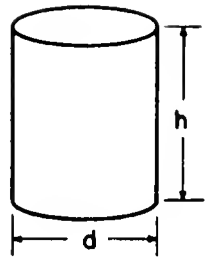
or $x = \frac{15}{2}$.

So the taxi company will have maximum gross income when the rate per mile is $32\frac{1}{2}$ cents.

C. Dilley

Newsletter 15

Solution.



Recall that the volume-measure of a cylinder is given in terms of its altitude h and the diameter d of its base by the formula:

$$V = \frac{\pi}{4}d^2h$$

Now we want $d + h$ to be minimum for constant d^2h [$\frac{\pi}{4}d^2h$ will be minimum when d^2h is minimum]. The problem is to write $d + h$ as the sum of terms whose product is a constant multiple of d^2h. This is easily done by:

$$\frac{d}{2} + \frac{d}{2} + h$$

Now by the AM-GM inequality

$$\frac{\frac{d}{2} + \frac{d}{2} + h}{3} \geq \sqrt[3]{\frac{d^2h}{4}}$$

[with equality if and only if $\frac{d}{2} = \frac{d}{2} = h$].

Since $\sqrt[3]{\frac{d^2h}{4}}$ is constant, $\frac{\frac{d}{2} + \frac{d}{2} + h}{3}$ is minimum when $\frac{\frac{d}{2} + \frac{d}{2} + h}{3} = \sqrt[3]{\frac{d^2h}{4}}$.

That is, when $\frac{d}{2} = h$ or when $d = 2h$.

Now

$$V = \frac{\pi}{4}d^2h = \frac{\pi}{4}(4h^3) = \pi h^3$$

and

$$h = \sqrt[3]{\frac{V}{\pi}}$$

and

$$d = 2\sqrt[3]{\frac{V}{\pi}}$$

Sample 5. A taxi company charges 25 cents a mile and logs 1000 passenger miles a day. 25 fewer passenger miles a day would be logged for each cent increase in the rate per mile. What rate yields the greatest gross income?

From the AM-GM inequality we get

$$\frac{b + 4a}{2} \geq \sqrt{b \cdot 4a}$$

[with equality if and only if $b = 4a$].

Since $b + 4a$ is constant, $\sqrt{b \cdot 4a}$ will be maximum when

$$\frac{b + 4a}{2} = \sqrt{b \cdot 4a}.$$

That is, when $b = 4a$. Since the area $2ab$ is maximum when $\sqrt{4ab}$ is maximum, the altitude of the rectangle with maximum area is 4 times the half-length or two times the length. The altitude of the rectangle with maximum area is 6 and the length is 3.

Sample 3. We want to make a rectangular box with volume measure 200 so that the sum of the length measure, width measure, and height measure is as small as possible. What are the dimensions of the box?

Solution. Let ℓ, w, and h be the length measure, width measure, and height measure respectively. The volume is ℓwh and we want $\ell + w + h$ to be as small as possible. By the AM-GM inequality

$$\frac{\ell + w + h}{3} \geq \sqrt[3]{\ell wh}$$

[with equality if and only if $\ell = w = h$].

Since $\ell wh = 200$, $\frac{\ell + w + h}{3}$ will be minimum when

$$\frac{\ell + w + h}{3} = \sqrt[3]{\ell wh}.$$

That is, when

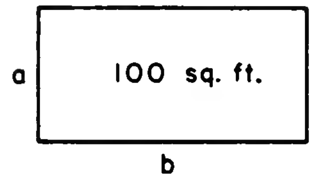
$$\ell = w = h$$

and the box is cubical with each edge $\sqrt[3]{200}$ units long.

Sample 4. A lard manufacturer wishes to package his product in cylindrical cans which hold a given amount of lard and such that the sum of the diameter and altitude will be as small as possible. What should be the dimensions of the can?

Sample 1. What is the least amount of fencing which can be used to enclose a rectangular field whose area is 100 sq ft.?

Solution.



If the measures of the width and the length of the rectangular field are a and b respectively, then the perimeter is $2(a + b)$ and the area-measure is ab.

Now according to the theorem:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

[with equality if and only if $a = b$]

Now since $\sqrt{ab}$ is constant $\frac{a+b}{2}$ will be minimum when

$$\frac{a+b}{2} = \sqrt{ab}.$$

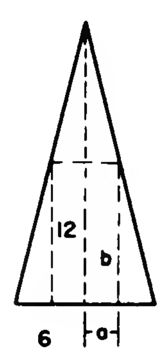
This is the case if and only if

$$a = b.$$

Of course, the perimeter, $2(a + b)$, will be minimum when $\frac{a+b}{2}$ is minimum. Hence the field is square with perimeter 40.

Sample 2. An isosceles triangle has a base 6 units long and altitude 12. Rectangles are inscribed with one side contained in the base of the triangle. Which of the rectangles has greatest area?

Solution.



Let a be $\frac{1}{2}$ the length of the rectangle and b be its altitude. From similar triangles we get

$$\frac{3-a}{b} = \frac{3}{12}$$

or:

$$b + 4a = 12.$$

Solving Maximum-Minimum Problems by the Arithmetic Mean—Geometric Mean Inequality

An interesting topic to consider is that of applications of the arithmetic mean-geometric mean inequality to maximum-minimum problems. This article is written to acquaint the reader with this application of an important inequality. We would be interested in receiving comments from any reader who tries teaching these ideas to his students.

<u>Two words of caution</u>:

(1) The teacher will have to acquaint the students with the meaning of '$\sqrt[3]{\ }$', '$\sqrt[4]{\ }$', etc.

(2) While the teacher can find many maximum-minimum problems in any good calculus text, he should try working them himself before assigning them to students, because <u>not all</u> maximum-minimum problems can be solved using the arithmetic mean-geometric mean inequality.

In Course 3 the students are in a position to prove the theorem:

$$\forall_{x \geq 0} \forall_{y \geq 0} \frac{x+y}{2} \geq \sqrt{xy}$$

[with equality if and only if $x = y$]

Another way of stating this theorem is: The arithmetic mean of two nonnegative numbers is greater than or equal to their geometric mean.

This is a special case of the arithmetic mean-geometric mean inequality:

For any sequence a of nonnegative numbers, for each n,

$$\frac{\sum_{p=1}^{n} a_p}{n} \geq \sqrt[n]{\prod_{p=1}^{n} a_p}$$

[with equality if and only if $\forall_{p \leq n} \forall_{q \leq n} a_p = a_q$]

As stated above, this theorem can be used to solve maximum-minimum problems of a typical calculus text. The use of the theorem is most easily demonstrated by examples.

Is h a function of g?

Find your way thru this maze!

When you come to a red sentence, stop, you have finished the problems.

When you come to a green sentence, go on, you have not finished.

	YES	NO
(1) Is h a function?	Go to (2)	h <u>is not</u> a function of g
(2) Is g a function?	Go to (3)	h is not a function of g
(3) Is $\mathcal{D}_h \subset \mathcal{D}_g$?	Go to (4)	h <u>is not</u> a function of g
(4) Are there two elements in $\mathcal{D}_g$ whose g-values are the same?	Go to (5)	h <u>is</u> a function of g
(5) Are there two elements in $\mathcal{D}_g$ whose g-values are the same but <u>exactly</u> one of them belongs to $\mathcal{D}_h$?	h <u>is not</u> a function of g	Go to (6)
(6) Are there two elements in $\mathcal{D}_g$ whose g-values are the same and both of them belong to $\mathcal{D}_h$ but whose h-values are different?	h <u>is not</u> a function of g	h <u>is</u> a function of g

Since $\underline{x_1 \in \mathcal{D}_h}$ it follows that $x_1 \in \mathcal{D}_{f \circ g}$ and $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_f$.

Since $\underline{g(x_1) = g(x_2)}$ it follows from ⟶ that $g(x_2) \in \mathcal{D}_f$.

Since $\underline{x_2 \in \mathcal{D}_g}$ and $g(x_2) \in \mathcal{D}_f$ it follows that

$$x_2 \in \{x \in \mathcal{D}_g \colon g(x) \in \mathcal{D}_f\}.$$

That is $x_2 \in \mathcal{D}_{f \circ g}$. Since $\underline{h = f \circ g}$ $x_2 \in \mathcal{D}_h$.

Now to show that $h(x_1) = h(x_2)$

Now we have $x_1 \in \mathcal{D}_{f \circ g}$ and $x_2 \in \mathcal{D}_{f \circ g}$.

By definition of composition

$$[f \circ g](x_1) = f(g(x_1)) \text{ and } [f \circ g](x_2) = f(g(x_2)).$$

Since $\underline{g(x_1) = g(x_2)}$ and <u>f is a function,</u> it follows that

$$f(g(x_1)) = f(g(x_2))$$

So, $[f \circ g](x_1) = [f \circ g](x_2)$. Since $\underline{h = f \circ g}$

$$h(x_1) = h(x_2)$$

A. Hart
Newsletter 9

(1) Suppose $x_1 \in \mathcal{D}_h$. Since $h = f \circ g$, $x_1 \in \mathcal{D}_{f\circ g}$. But,

$$\mathcal{D}_{f\circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$$

It follows that $x_1 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$. Hence, $x_1 \in \mathcal{D}_g$. So, if $x_1 \in \mathcal{D}_h$ then $x_1 \in \mathcal{D}_g$.

Since this reasoning would hold for any $x \in \mathcal{D}_h$,

$$\mathcal{D}_h \subseteq \mathcal{D}_g.$$

Now we must prove a conditional sentence.

$$\text{if} \quad \left\{\begin{array}{l} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{array}\right\} \quad \text{then} \quad \left\{\begin{array}{l} x_2 \in \mathcal{D}_h \\ h(x_1) = h(x_2) \end{array}\right.$$

So, in addition to knowing that

f, g, and h are functions

$h = f \circ g$

$\mathcal{D}_h \subseteq \mathcal{D}_g$

we have the additional premisses

$x_1 \in \mathcal{D}_g$

$x_2 \in \mathcal{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathcal{D}_h$

} Each use of any one of these will be underlined once.

Since $\underline{h = f \circ g}$, it follows that

$$\mathcal{D}_h = \mathcal{D}_{f\circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\} \text{ [by definition]}$$

Given $\qquad\qquad$ Given

$$h = f \circ g \quad , \quad x_1 \in \mathscr{D}_h$$

$$x_1 \in \mathscr{D}_{f \circ g}$$

Given $\quad x_1 \in \mathscr{D}_g$ and $g(x_1) \in \mathscr{D}_f$

$$g(x_1) = g(x_2) \quad , \quad g(x_1) \in \mathscr{D}_f$$

Given

$$x_2 \in \mathscr{D}_g \quad \text{and} \quad g(x_2) \in \mathscr{D}_f$$

$$x_2 \in \{x \in \mathscr{D}_g : \; g(x) \in \mathscr{D}_f\}$$

Given

$$h = f \circ g \quad , \quad x_2 \in \mathscr{D}_{f \circ g}$$

$$x_2 \in \mathscr{D}_h$$

Now, let's finish it off. Show that $h(x_1) = h(x_2)$.

We know that $x_1 \in \mathscr{D}_{f \circ g}$ and $x_2 \in \mathscr{D}_{f \circ g}$. So, by the definition of composition

$$[f \circ g](x_1) = f(g(x_1)) \quad \text{and} \quad [f \circ g](x_2) = f(g(x_2))$$

But, $g(x_1) = g(x_2)$ and f is a function. So,

$$f(g(x_1)) = f(g(x_2)), \text{ and } [f \circ g](x_1) = [f \circ g](x_2).$$

Since $h = f \circ g$,

$$h(x_1) = h(x_2).$$

We did it.

* * *

The following is an abbreviated form of the proof of the only-if part of the theorem.

Now, what is $\mathcal{D}_{f \circ g}$? Look at the definition of $f \circ g$.

$$\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$$

What can we show about x_2 in order to get $x_2 \in \mathcal{D}_{f \circ g}$

$x_2 \in \mathcal{D}_g$ and $g(x_2) \in \mathcal{D}_f$

Given $\quad x_2 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$

$h = f \circ g, \quad x_2 \in \mathcal{D}_{f \circ g}$

$x_2 \in \mathcal{D}_h$

Now, what about this? Our hypothesis tells us that $x_2 \in \mathcal{D}_g$. So, all we need is to show that $g(x_2) \in \mathcal{D}_f$. Look again. Since we are given that $g(x_1) = g(x_2)$, we can show that $g(x_2) \in \mathcal{D}_f$ if we can show that $g(x_1) \in \mathcal{D}_f$.

Given

Given $\quad g(x_1) = g(x_2), \; g(x_1) \in \mathcal{D}_f$

$x_2 \in \mathcal{D}_g$ and $g(x_2) \in \mathcal{D}_f$

Given $\quad x_2 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$

$h = f \circ g, \quad x_2 \in \mathcal{D}_{f \circ g}$

$x_2 \in \mathcal{D}_h$

So, our problem is to show that $g(x_1) \in \mathcal{D}_f$. What do we know about x_1?
[Ans. $x_1 \in \mathcal{D}_h$, $x_1 \in \mathcal{D}_g$]

Let's examine these. Since $x_1 \in \mathcal{D}_h$, what else can you say about x_1?
[Ans. $x_1 \in \mathcal{D}_{f \circ g}$] What does that mean? Why that means that $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_f$. Hey, that's what we wanted.

We want to show that

if $[x_1 \in \mathcal{D}_g,\ x_2 \in \mathcal{D}_g,\ g(x_1) = g(x_2),\ x_1 \in \mathcal{D}_h]$ then $[x_2 \in \mathcal{D}_h$ and $h(x_1) = h(x_2)]$.

How do we do this? [Ans. Suppose those four things and prove that $x_2 \in \mathcal{D}_h$ and $h(x_1) = h(x_2)$.] That means that we now have:

Hypothesis:

f, g, and h are functions

$h = f \circ g$

$x_1 \in \mathcal{D}_g$

$x_2 \in \mathcal{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathcal{D}_h$

Conclusion:

$x_2 \in \mathcal{D}_h$

$h(x_1) = h(x_2)$

We know very little about x_2. So let's ask our questions in reverse order. First: We want to end with

$$\vdots$$

$$x_2 \in \mathcal{D}_h$$

Look at the hypothesis. Can you suggest a step from which it would be very easy to deduce $x_2 \in \mathcal{D}_h$? [Ans. $x_2 \in \mathcal{D}_{f \circ g}$ because $h = f \circ g$.] How about that? Do you think that the proof might end

Given

$h = f \circ g$, $x_2 \in \mathcal{D}_{f \circ g}$

$$x_2 \in \mathcal{D}_h$$

So, using

if p then q

and

if (p and q and r) then s

there is a pattern by which we can arrive at

if p then [q and (if r then s)].

Now, prove

if
- there is a function f such that $h = f \circ g$ ⟶ p
- $\mathcal{D}_h \subseteq \mathcal{D}_g$ ⟶ q
- $x_1 \in \mathcal{D}_g$, $x_2 \in \mathcal{D}_g$, $g(x_1) = g(x_2)$, $x_1 \in \mathcal{D}_h$ ⟶ r

then
- $x_2 \in \mathcal{D}_h$, $h(x_1) = h(x_2)$ ⟶ s

This discussion of the logical background need not (in fact, probably should not) be done with the students. You might do something like this:

We started with

f, g, and h are functions

$h = f \circ g$

What other sentence can we now use whenever we like? [Ans: $\mathcal{D}_h \subseteq \mathcal{D}_g$].

Now, our hypothesis is

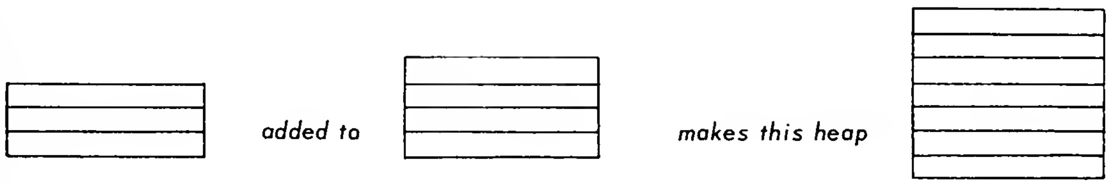
f, g, and h are functions

$h = f \circ g$

$\mathcal{D}_h \subseteq \mathcal{D}_g$

Arithmetic With Frames

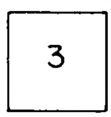
Then Miss Mills taught Peter to add and subtract and multiply and divide. She had once heard some lectures upon teaching arithmetic by graphic methods that had pleased her very much. They had seemed so clear. The lecturer had suggested that for a time easy sums might be shown in the concrete as well as in figures. You would draw an addition of 3 to 4, thus:



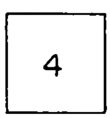
And then when your pupil had counted it and verified it you would write it down:

$$3 \quad + \quad 4 \quad = \quad 7$$

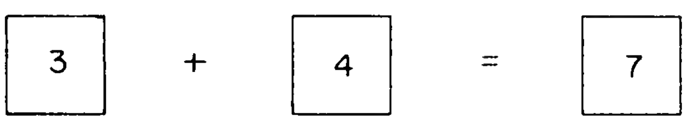
But Miss Mills, when she made her notes, had had no time to draw all the parallelograms; she had just put down one and a number over it in each case, and then her memory had muddled the idea. So she taught Joan and Peter thus: "See," she said, "I will make it perfectly plain to you. Perfectly plain. You take three — so," and she drew



"and then you take four — so," and she drew

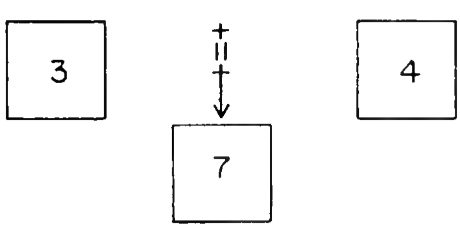


"and then you see three plus four makes seven — so:



"Do you see now how it must be so, Peter?" Peter tried to feel that he did.

Peter quite agreed that it was nice to draw frames about the figures in this way. Afterwards he tried a variation that looked like the face of old Chester Drawers:



But for some reason Miss Mills would not see the beauty of that. Instead of laughing, she said: "Oh, no, that's quite wrong!" which seemed to Peter just selfishly insisting on her own way.

Well, one had to let her have her own way. She was a grown-up. If it had been Joan, Peter would have had his way. . . .

*

Peter was rather good at arithmetic, in spite of Miss Mills' instruction. He got sums right. It was held to be a gift. Joan was less fortunate. Like most people who have been badly taught, Miss Mills had one or two foggy places in her own arithmetical equipment. She was not clear about seven sevens and eight eights; she had a confused, irregular tendency to think that they might amount in either case to fifty-six, and also she had a trick of adding seven to nine as fifteen, although she always added nine to seven correctly as sixteen. Every learner of arithmetic has a tendency to start little local flaws of this sort, standing sources of error, and every good, trained teacher looks out for them, knows how to test for them and set them right. Once they have been faced in a clear-headed way, such flaws can be cured in an hour or so. But few teachers in upper and middle-class schools in England, in those days, knew even the elements of their business; and it was the custom to let the baffling influence of such flaws develop into the persuasion that the pupil had not "the gift for mathematics." Very few women indeed of the English "educated" classes to this day can understand a fraction or do an ordinary multiplication problem. They think computation is a sort of fudging — in which some people are persistently lucky enough to guess right — "the gift for mathematics" — or impudent enough to carry their points. That was Miss Mills' secret and unformulated conviction, a conviction with which she was infecting a large proportion of the youngsters committed to her care. Joan became a mathematical gambler of the wildest description. But there was a guiding light in Peter's little head that made him grip at last upon the conviction that seven sevens make always forty-nine, and eight eights always sixty-four, and that when this haunting fifty-six flapped about in the products it was because Miss Mills, grown-up teacher though she was, was wrong.

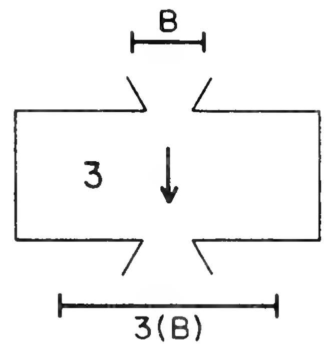
--from a novel by H. G. Wells
JOAN AND PETER: The Story of an Education, New York: The Macmillan Company, 1918
Newsletter 8

The Seventh Grade Project

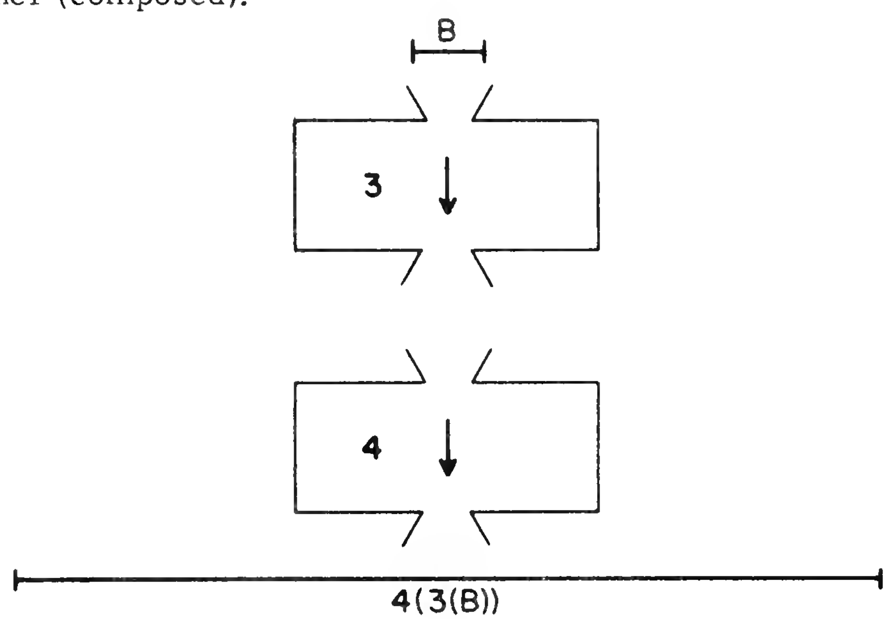
One of the current major UICSM projects is the writing and testing of a new seventh grade course. This course has been taught in each of the three Champaign junior high schools, Urbana Junior High School, and University High School. In each junior high school two seventh grade classes, one of average ability students and one of below average ability students, were taught and observed by UICSM staff members or they were also observed by a local teacher from the junior high school. In Uni High School, two above average subfreshmen classes were taught by UICSM staff members. Written reports submitted each day by staff members included lesson plans, student reaction, tests, suggestions for text changes, supplimentary materials, etc. Since some classes were ahead of others, it was possible to find trouble spots and revise the materials in time to try the revision with the slower classes later.

The authors, Peter Braunfeld and Walter Rucker, attempted to write a course for low achievers, especially the culturally deprived. Since such students don't understand and can't manipulate fractions, decimals and percents, it was decided to reteach these concepts from the beginning. For the sake of student interest and confidence which has not been developed and for the sake of understanding which has apparently not been taught by the usual techniques used by elementary text books and teachers, it was decided that a completely new vehicle was needed. The vehicle used at the suggestion of Dr. Bernard Friedman is to consider rational numbers as operators on length.

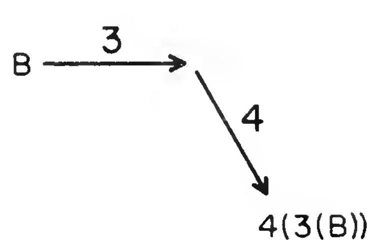
To do this, the positive integers are introduced as machines (operators) which multiply lengths.



They are called stretching machines. A 1-machine is an identity machine but is included among the stretchers (non-shrinkers?). These machines can be hooked together (composed).



The diagrams are simplified to:

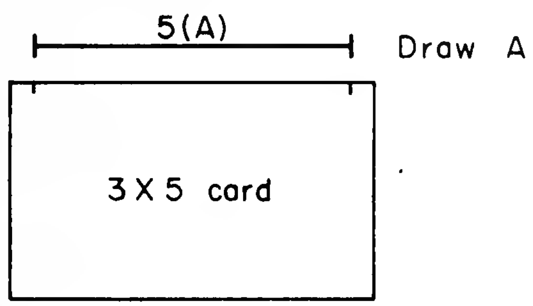


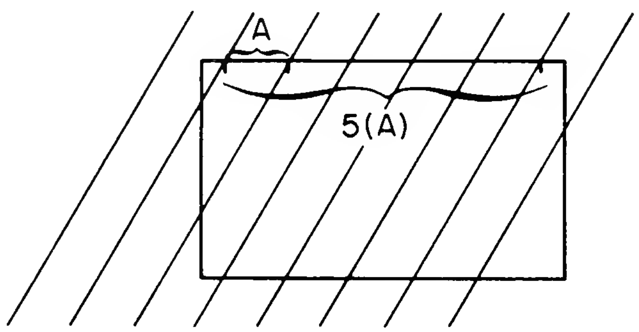
and the hook-ups are usually written:

$$4 \circ 3$$

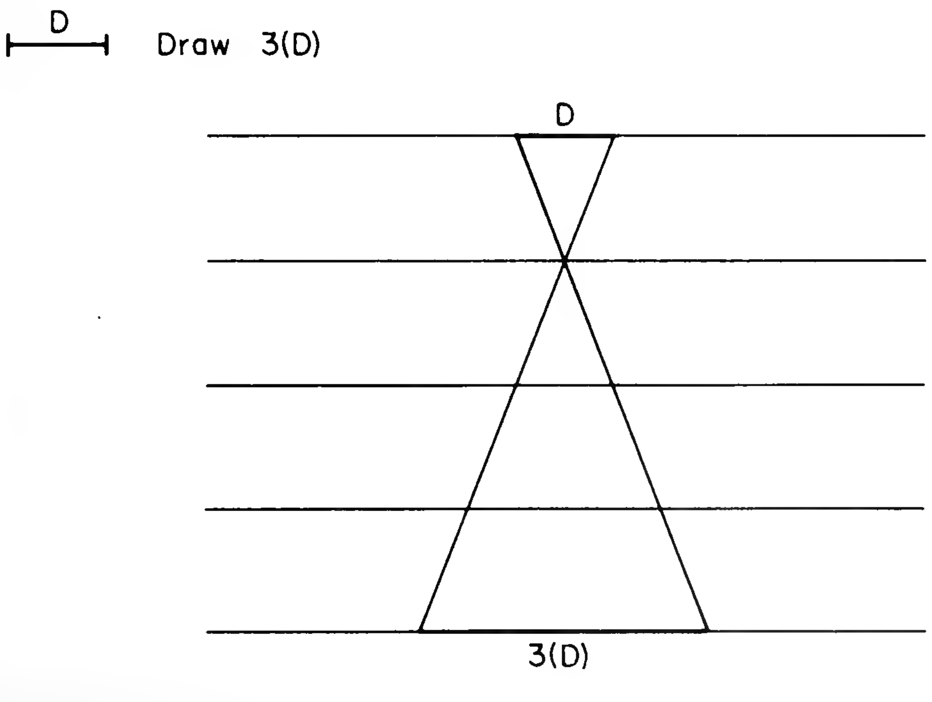
The students are required to make many drawings of lengths. For example, they are given a length B and asked to draw 3(B), or they are given a length 5(A) and asked to draw length A. In order to make accurate drawings, especially in the latter case, the students are taught two methods using ruling sheets (pages of equidistant parallel lines).

1. The parallel lines cut off congruent segments on a transversal.





2. Similar triangles with sides in a given ratio.

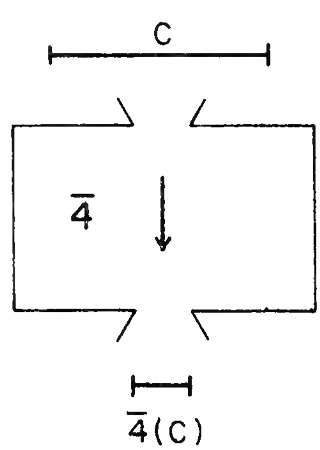


From these drawings the students learn that they can find, by multiplication, a single machine that is equivalent to a hook-up. That is, a $3 \circ 4$-hook-up is equivalent to a 12-machine. No point is made that composition of these operators is cummutative and associative, but a general principle is developed.

> <u>Rearrangement Principle For Stretching Machines</u>
>
> All rearrangements of the machines in a hook-up are equivalent.

The students finish their work on stretching machines with factoring, prime machines, and prime factorizations.

The reciprocals of the positive integers are introduced as <u>shrinking machines</u>. For example, the input of a $\overline{4}$-machine is 4 times the output.



These operators are treated in much the same way as the stretching machines were. The students learn a multiplication principle [$\overline{3} \circ \overline{5} = \overline{15}$] and a rearrangement principle [$\overline{4} \circ \overline{7} \circ \overline{3} = \overline{7} \circ \overline{4} \circ \overline{3}$]. They also learn that a 4-machine and a $\overline{4}$-machine, for example, are inverses.

Next, the students consider hook-ups which contain both stretchers and shrinkers. Some such hook-ups are equivalent to single machines.

$$2\circ\overline{8} = \overline{4}$$

$$2\circ 6\circ\overline{3} = 4$$

But many such as $3\circ\overline{2}$ are not equivalent to a single machine.

They discover a general rearrangement principle for stretchers and shrinkers so they can transform:

$$3\circ\overline{2}\circ 7\circ\overline{5}$$

into:

$$3\circ 7\circ\overline{2}\circ\overline{5}$$

Now using multiplication, this is transformed into:

$$21\circ\overline{10}$$

In this way each hook-up can be transformed into a <u>standard</u> <u>hook-up</u>. That is, a hook-up of two machines — a shrinker followed by a stretcher.

Hook-ups can be transformed into equivalent hook-ups by eliminating pairs of inverses.

$$\cancel{8}\circ\overline{7}\circ 3\circ\overline{\cancel{8}} = \overline{7}\circ 3$$

Some hook-ups contain "hidden" pairs of inverses which also can be eliminated.

$$8\circ\overline{6} = 4\circ\cancel{2}\circ\overline{\cancel{2}}\circ\overline{3} = 4\circ\overline{3}$$

In this way all hook-ups can be transformed into standard hook-ups in lowest terms. That is, into standard hook-ups which contain no "hidden" inverses [except, of course, 1 and $\bar{1}$].

The idea of standard hook-up in lowest terms is motivated by assuming that the machines cost money. They are given a partial price list:

Machine	Cost
$1, \bar{1}$	\$1
$2, \bar{2}$	\$2
$3, \bar{3}$	\$3
$4, \bar{4}$	\$4

and (among other things) asked to find the cheapest standard hook-up which is equivalent to a $12 \circ \overline{24}$-hook-up.

Now fraction machines are introduced. Each fraction machine is just a standard hook-up. For example:

$$\frac{2}{3} = 2 \circ \bar{3}$$

In the work with standard hook-ups, students then have learned to reduce fractions and to multiply fractions. This need only be gone over quickly in fractional notation.

Text materials have been completely developed only up to this point. Further articles will describe division, addition, and multiplication of fractions, percents and decimals after these chapters have been written.

Besides the text materials developed, supplementary units are being written. One unit reviews in clever ways the arithmetic of whole numbers. Another suggests kinds of mental arithmetic that can be used for a few minutes each class period. Another teaches graphing of ordered pairs from the stretching machine idea. These units will be expanded, and others will be written as they are needed.

Clyde Dilley

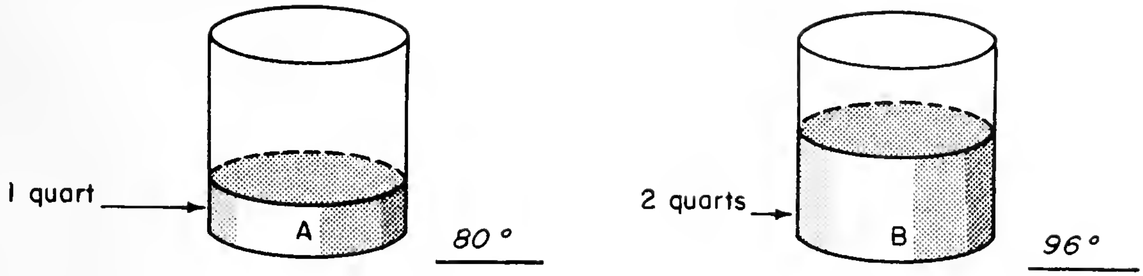
Walter Rucker

Newsletter 16

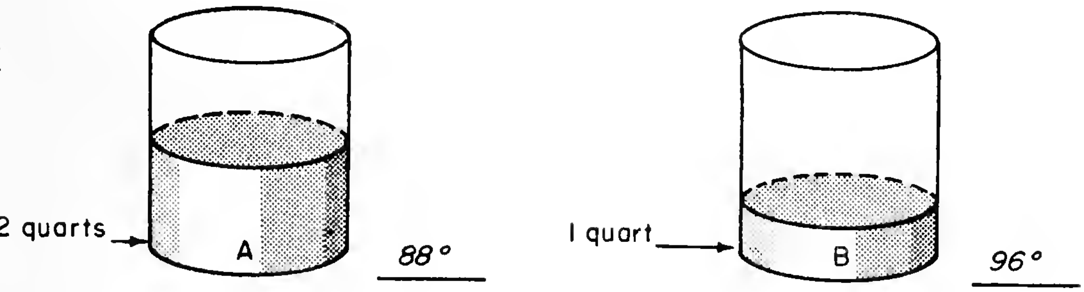
Limits and Nested Intervals—An Elementary Approach

This article is an attempt by the author to clarify some of the troublesome ideas in the elementary theory of limits. He has tried to program the material in such a way that the reader will see the development through the eyes of the student. We believe strongly that one of the goals of the high school mathematics teacher is to develop carefully the students' understanding of concepts that are largely intuitive and are usually brushed over rather rapidly in a college course. Thus, we feel that this article is worthy of your attention.

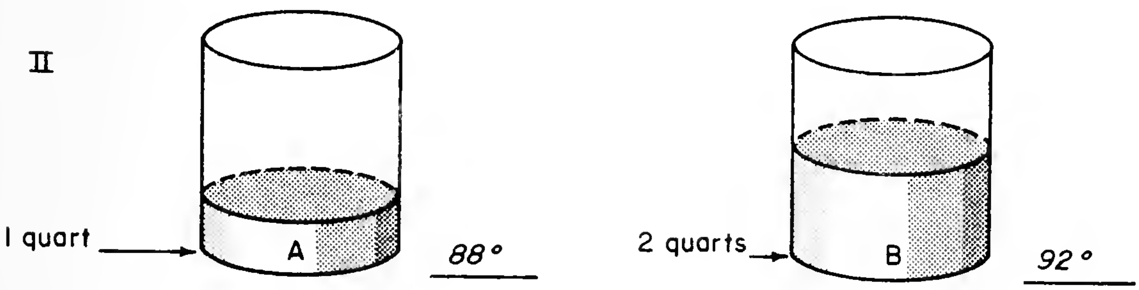
In introducing limits, we have decided to construct a situation which is readily imagined, that of pouring water from one jar to another. One jar contains one quart of water at a certain temperature, while the other jar contains two quarts of water at a different temperature. One quart of water is poured from the two quart jar into the one quart jar. This is done in as many successive moves as are necessary to achieve a certain temperature. The following pages picture the jars of water at the outset and after several successive moves. The particular situation pictured starts with the two-quart jar containing water at a higher temperature than the water in the one-quart jar. The first pouring will elevate the temperature of the water in the one-quart jar. The second pouring will lower the temperature of the water in the jar that initially held the warmer water.


I quart
A
80°
2 quarts
B
96°

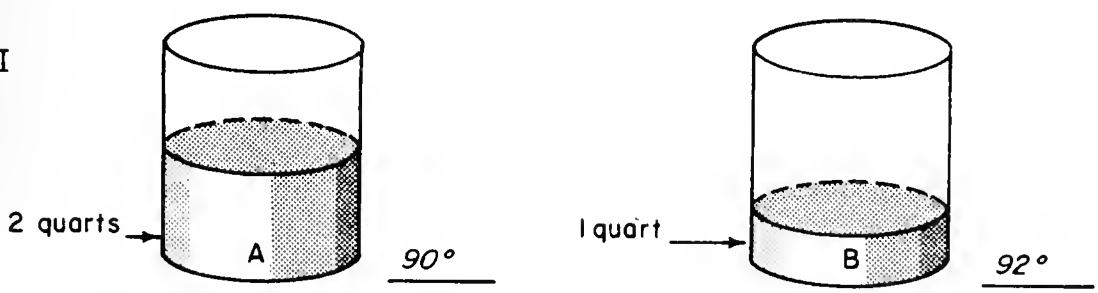
I


2 quarts
A
88°
I quart
B
96°

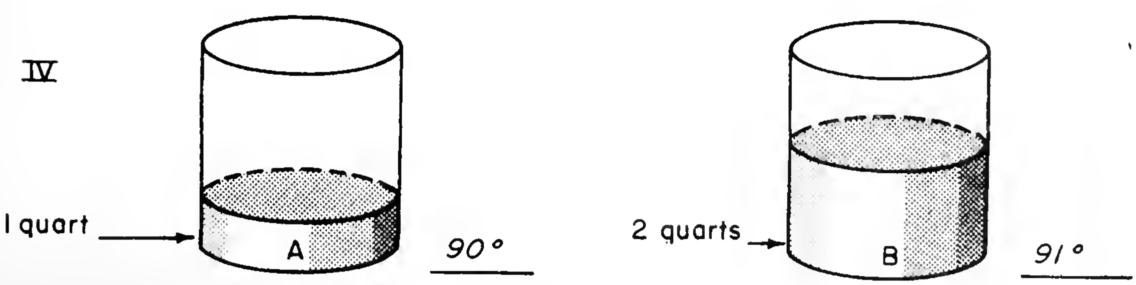
II


I quart
A
88°
2 quarts
B
92°

III

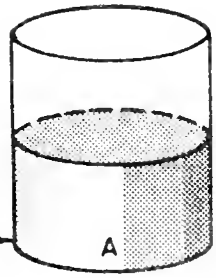

2 quarts
A
90°
I quart
B
92°

IV


I quart
A
90°
2 quarts
B
91°

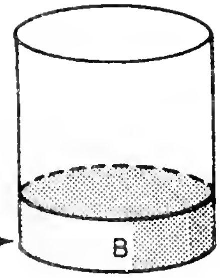
V

2 quarts



$90\frac{1}{2}^\circ$

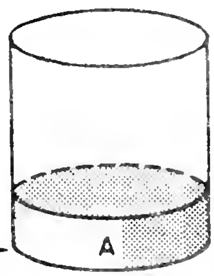
1 quart



91°

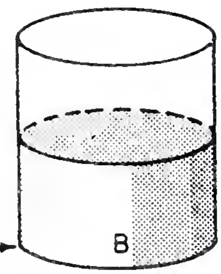
VI

1 quart



$90\frac{1}{2}^\circ$

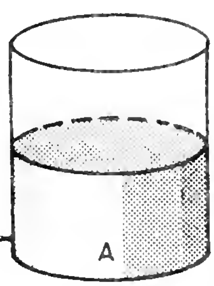
2 quarts



$90\frac{3}{4}^\circ$

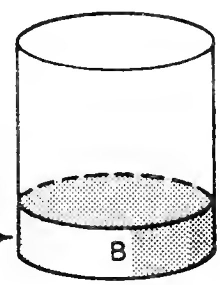
VII

2 quarts



$90\frac{5}{8}^\circ$

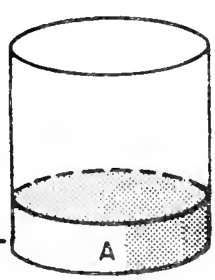
1 quart



$90\frac{3}{4}^\circ$

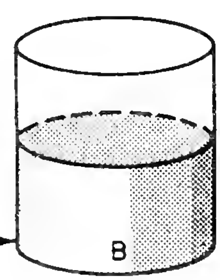
VIII

1 quart



$90\frac{5}{8}^\circ$

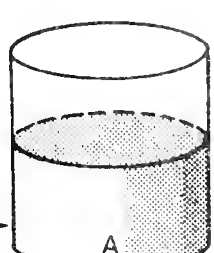
2 quarts



$90\frac{11}{16}^\circ$

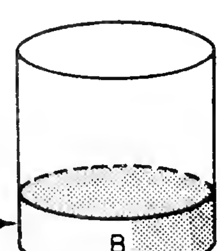
IX

2 quarts



$90\frac{21}{32}^\circ$

1 quart



$90\frac{11}{16}^\circ$

From the preceding information complete the following table.

Number of times the pouring process is completed	Temperature of water in Container A	Temperature of water in Container B	Difference of Temperature
I	88	96	8
II	88	92	4
III	90	92	2
IV	90	91	1
V	$90\frac{1}{2}$	91	$\frac{1}{2}$
VI	.	.	.
VII	.	.	.
VIII	.	.	.
IX			
X			

By referring to the above table you should see that to ensure a degree difference of 8 one must complete the pouring process 1 time, to ensure a degree difference of 4 one must complete the pouring process 2 times, to ensure a degree difference of 2 one must complete the pouring process 3 times, etc.

From the table we have the function:

$$\{(8, 1), (4, 2), (2, 3), (1, 4), (\tfrac{1}{2}, 5), (\tfrac{1}{4}, 6), (\tfrac{1}{8}, 7), (\tfrac{1}{16}, 8), (\tfrac{1}{32}, 9)\}$$

or

$$\{(2^3, 1), (2^2, 2), (2^1, 3), (2^0, 4), (2^{-1}, 5), (2^{-2}, 6), (2^{-3}, 7), (2^{-4}, 8), (2^{-5}, 9)\}$$

Of course, for each ordered pair, the first component represents the degree difference and the corresponding second component is the number of pourings that must be made to obtain the difference.

Suppose the degree difference is less than 2^{-5}. Does the above function give the required number of pourings? Of course what is needed is a function that does not have such a finite restriction. By examining the function given above, the function:

$$\{(x, y)y \in I^{+}: x = 2^{4-y}\}$$

can be obtained. Now regardless what integral power of 2 you pick for the degree difference, the number of pourings is easily determined from this new function. For example, suppose the desired degree difference is 2^{-17}. From the new function, 21 pourings will be needed, since

$$2^{-17} = 2^{4-21}.$$

How many pourings will be needed to ensure a degree difference of 2^{-37}? Of 2^{-329}?

In the preceding discussion the only degree differences considered were those expressible as a power of 2. In each case a number was determined such that if the pouring process were carried out that number of times the desired degree difference could be obtained <u>exactly</u>. Now let's consider any degree difference and only require that the number of pourings made will ensure a degree difference which is less than or equal to the one desired. For example, suppose you wanted the degree difference to be less than or equal to $\frac{1}{25}$. Since

$$\frac{1}{25} > \frac{1}{32}$$

$$\text{and} \quad \frac{1}{32} = 2^{-5} = 2^{4-9},$$

it follows that if the pouring process is carried out 9 times, the degree difference would be $\frac{1}{32}$ which is less than the desired degree difference, $\frac{1}{25}$. Is 9 the <u>only</u> number of pours that will ensure a degree difference which is less than or equal to $\frac{1}{25}$? Since a degree difference of $\frac{1}{64}$, which is less than or equal to $\frac{1}{25}$, can be obtained by 10 pourings you should see that 10 pourings will also "do the job". In fact, if the pouring process is carried out 9 times or any number of times greater than 9 the degree difference will be less than $\frac{1}{25}$.

EXERCISES

In the following exercises certain degree differences are listed. For each degree difference, determine a number of pourings that will ensure a degree difference which is less than or equal to the one desired.

<u>Sample</u>. $\frac{367}{1000}$

<u>Solution</u>. $\frac{367}{1000} > \frac{256}{1000} > \frac{256}{1024} = \frac{2^8}{2^{10}} = 2^{-2} = 2^{4-6}$

Hence, if the pouring process is completed 6 times the degree difference will be less than $\frac{367}{1000}$.

1. $\frac{1}{500}$
2. $\frac{37}{1503}$
3. $\frac{129}{143}$
4. $\frac{29}{991}$
5. $\frac{2}{4007}$
6. $\frac{67}{8132}$
7. $\frac{1}{1,000}$
8. $\frac{1}{1,000,000}$
9. $\frac{5}{3,685,934}$

If you encountered some difficulty in determining a number of pourings for any of the degree differences listed above, the following technique may be helpful.

Since $$10^3 = 1000 < 1024 = 2^{10},$$

it follows that $$\frac{1}{10^3} > \frac{1}{2^{10}}.$$

This inequality will now be used in solving some of the above exercises.

Exercise 7. $$\frac{1}{1000} = \frac{1}{10^3} > \frac{1}{2^{10}} = 2^{-10} = 2^{4-14}$$

Hence if the pouring process is completed 14 or more times the degree difference will be less than or equal to $\frac{1}{1000}$.

Exercise 8. $$\frac{1}{1,000,000} = \frac{1}{10^3\,10^3} > \frac{1}{2^{10}\,2^{10}} = 2^{-20} = 2^{4-24}$$

Hence if the pouring process is completed 24 or more times the degree difference will be less than or equal to $\frac{1}{1,000,000}$.

Exercise 9. $$\frac{5}{3,685,934} > \frac{1}{3,685,934} > \frac{1}{10,000,000} = \frac{1}{10^7} > \frac{1}{10^9}$$

$$\frac{1}{10^9} = \frac{1}{10^3\,10^3\,10^3} > \frac{1}{2^{10}\,2^{10}\,2^{10}} = 2^{-30} = 2^{4-34}.$$

Hence if the pouring process is completed 34 or more times the degree difference will be less than or equal to $\frac{5}{3,685,934}$. Suppose that the desired degree difference is $\frac{1}{1,000,000,000,000}$. Determine a corresponding number of pours.

In summary, container A contains 1 quart of water at a temperature of 80° and container B contains 2 quarts of water at a temperature of 96°. If a quart is poured "back and forth" the degree difference of the water in the two containers becomes less and less. In fact a number of pourings was determined which would ensure a degree difference of only $\frac{1}{1,000,000,000,000}$, a "very small" degree difference.

Although the degree difference was made "very small", we still do not know that for each degree difference, no matter how small, there exists a number such that if the pouring process is completed that number of times the degree difference will be less than or equal to the desired degree difference. Or, equivalently, for each number greater than 0, is there an integral power of 2 which is less than the number?

To answer the question posed in the preceding paragraph consider the following two cases.

Case 1: $a \geq 1$

$\forall_x$ if $x \geq 1$ then $x > \frac{1}{2}$.

Since $\frac{1}{2} = 2^{-1} = 2^{4-5}$,

it follows that if $x \geq 1$ then the degree difference will be less than x if the pouring process is completed 5 times.

Case 2: $0 < a < 1$

$\forall_x$ if $0 < x < 1$ then $\frac{1}{x} > 1$

Now consider the closed interval:

$$[\frac{1}{x}, \frac{1}{x} + 1]$$

Without much strain on the imagination you should see that there exists an N such that

$$\frac{1}{x} \leq N < \frac{1}{x} + 1.$$

Since $N \geq \frac{1}{x}$ and $\frac{1}{x} > 1$ it follows that $N > 1$. Also, by induction

$$\forall_n \text{ if } N \in I^+ \text{ then } N < 2^N$$

Therefore,

$$\forall_x \exists_N \; 0 < x < 1 \text{ and } N \in I^+ \text{ and } \frac{1}{x} \leq N < 2^N.$$

Consequently,

$$\forall_x \exists_N \; 0 < x < 1 \text{ and } N \in I^+ \text{ and } x > 2^{-N}$$

$$\text{since } 2^{-N} = 2^{4-(N+4)}$$

$$x > 2^{4-(N+4)}$$

So if $0 < x < 1$ then the degree difference will be less than x if the pouring process is completed N + 4 times.

Hence, from the above two cases we have the generalization:

For each degree difference, <u>no matter how small</u>, there exists a number such that if the pouring process is completed that number of times, or any number of times greater than that number, then a degree difference will be obtained which is less than or equal to the one desired.

W. Rucker

Newsletter 16

Counting Problems

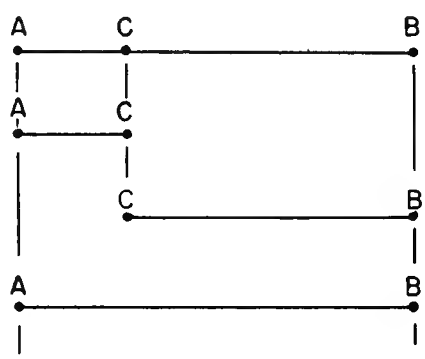
Below is a picture of a line segment.

The line segment AB is the set [of points] consisting of the two points, labeled A and labeled B, and all the points between A and B. The two points A and B are called <u>end-points</u> of the line segment AB. Of course, to identify a particular line segment you need only to identify its two end-points.

Suppose you "pick out" one of the points between A and B and label it C. [See picture below.]

From the preceding picture, how many line segments can you identify? ______

To check your answer, consider the following:



<u>or</u>

{A , C , B}

{A , C}

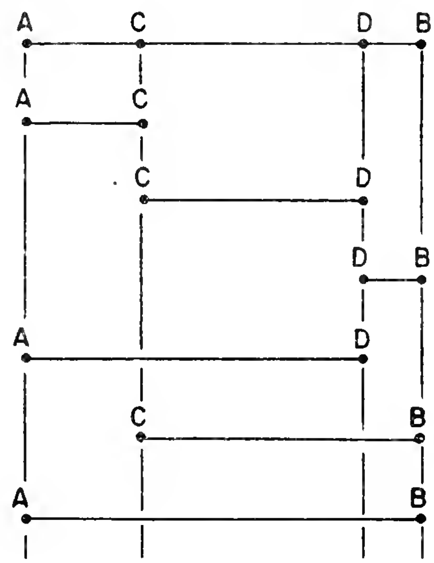
{C , B}

{A , B}

Suppose you "pick out" a fourth point of the line segment AB and label it D. [See picture below.]

From the preceding picture, how many line segments can you identify? ______

To check your answer, consider the following:



<u>or</u>

$\{A, C, D, B\}$

$\{A, C\}$

$\{C, D\}$

$\{D, B\}$

$\{A, D\}$

$\{C, B\}$

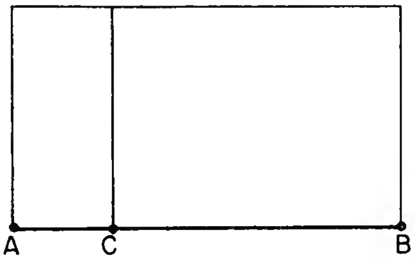
$\{A, B\}$

From the preceding examples, _____ line segment(s) can be identified given two points, _____ line segment(s) can be identified given three points, and _____ line segment(s) can be identified given four points. This information is recorded in the following table. Complete the table.

numbers of points	numbers of segments
2	1
3	3
4	6
5	
6	
7	
8	
9	
10	

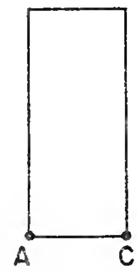
From the above exercises, you should see that the number of segments is just the number of combinations of the number of points taken two at a time, since any set of two points identifies a segment.

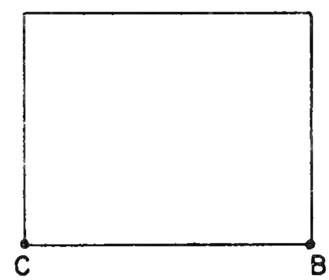
Now consider the following picture.

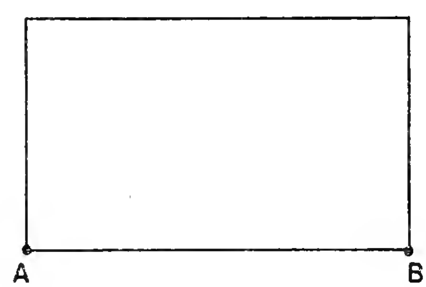


From the preceding picture, how many rectangles can you identify? _____

To check your answer, consider the following:







<u>or</u>

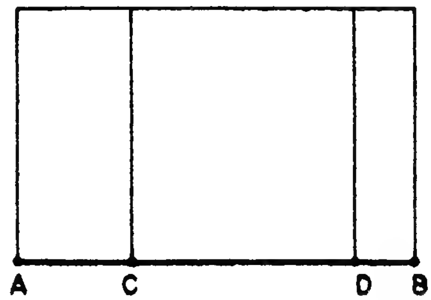
$\{A, C, B\}$

$\{A, C\}$

$\{C, B\}$

$\{A, B\}$

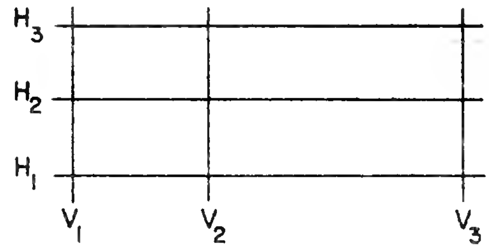
How many rectangles are identifiable in the picture below? ______



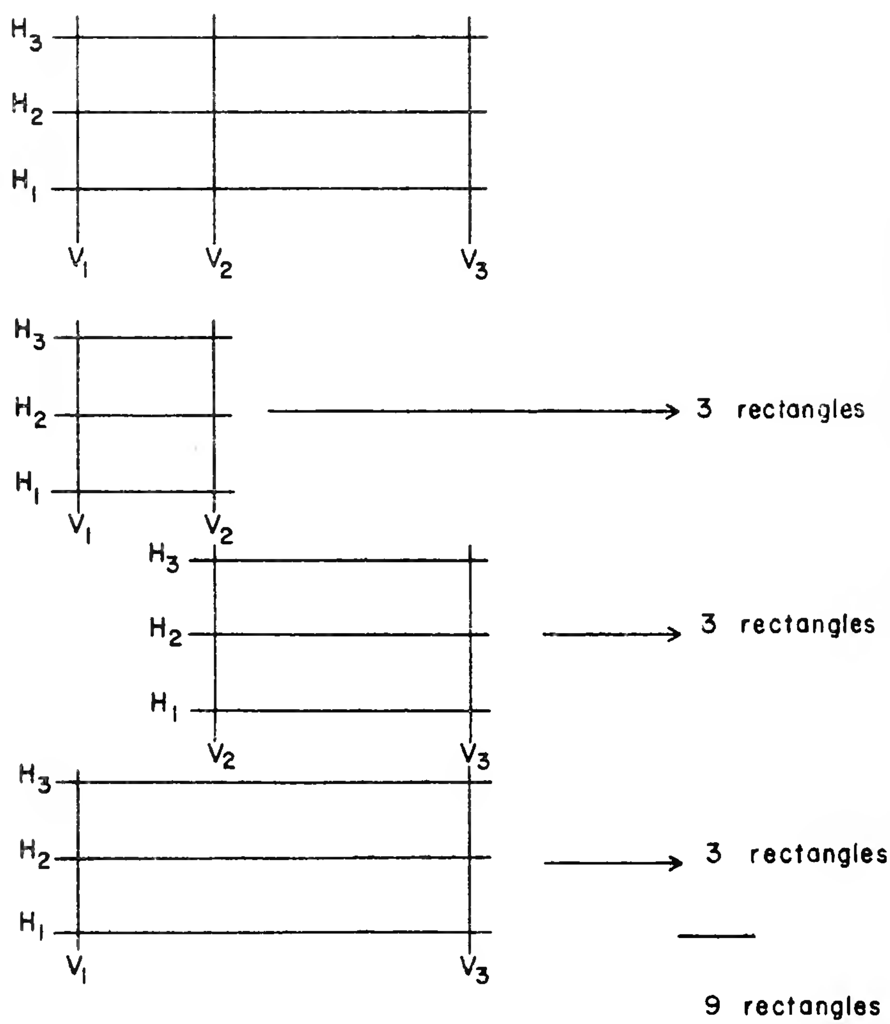
From the two preceding examples you should see that the "number of rectangles" problem is like "the number of line segments" problem. Complete the following table:

Number of "vertical" segments	Number of Rectangles
2	1
3	3
4	6
5	
6	
7	
8	
9	
10	

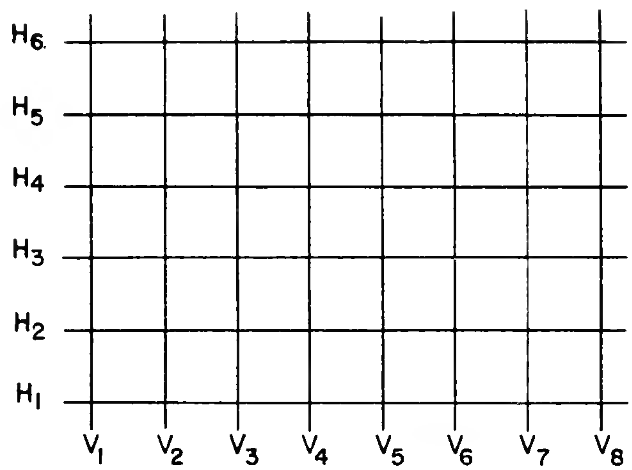
Now study the picture below. How many rectangles can you identify? _____



Perhaps you found the number of rectangles by "taking the above drawing apart".

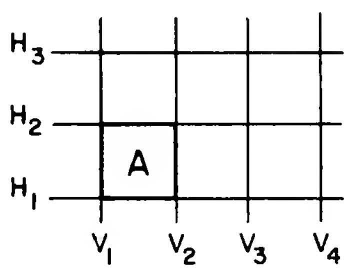


Of course the above method is a way to determine the number of rectangles. However the "taking apart method" would be quite difficult in a situation as pictured below.



Let's leave this problem and try to discover an easier method, for determining the number of rectangles, than the "taking apart method."

Let's first take a "corner" of the above picture.



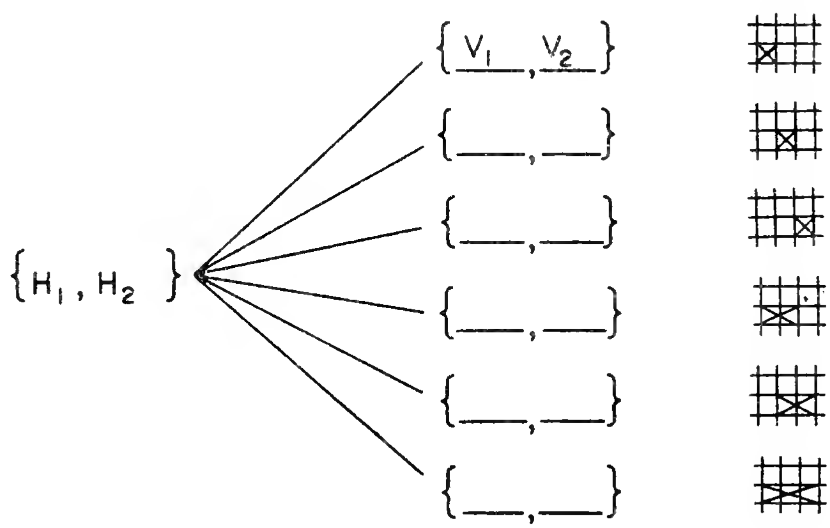
Note that rectangle A could be described by:

$$\{H_1, H_2\} - \{V_1, V_2\}$$

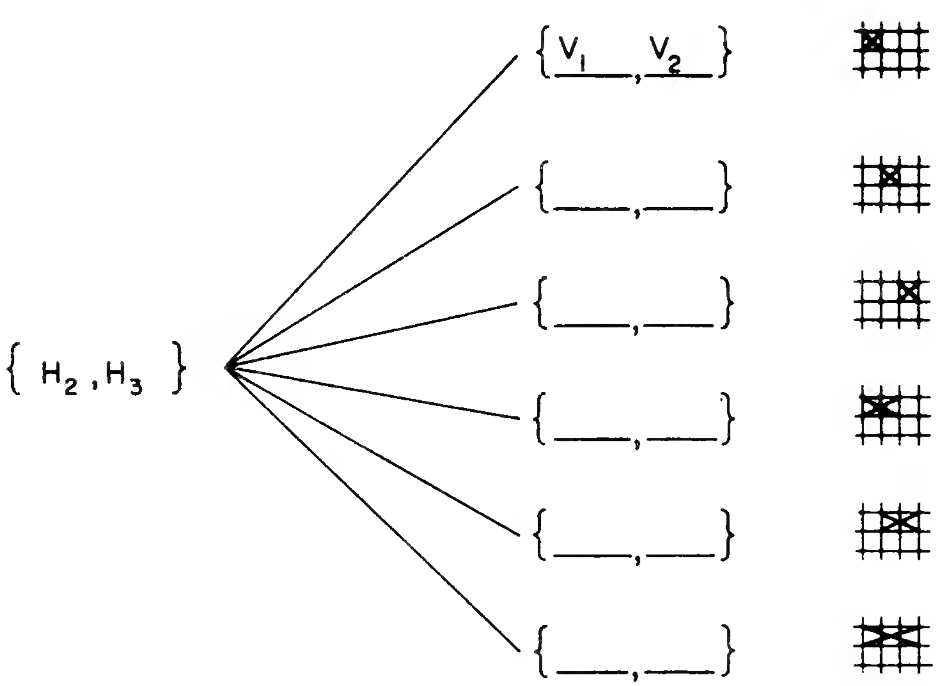
that is, by denoting the two horizontal segments and the two vertical segments that "make up" rectangle A.

Exercises

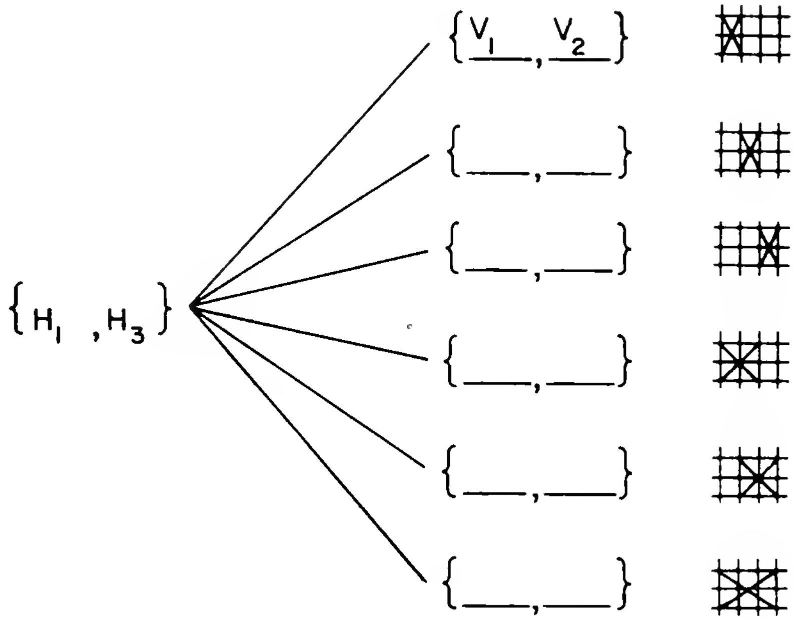
1. By studying the above pictures and the rectangles indicated below, determine the appropriate set of vertical segments and fill in the blanks.



2. By studying the above picture and the rectangles indicated below, determine the appropriate set of vertical segments and fill in the blanks.



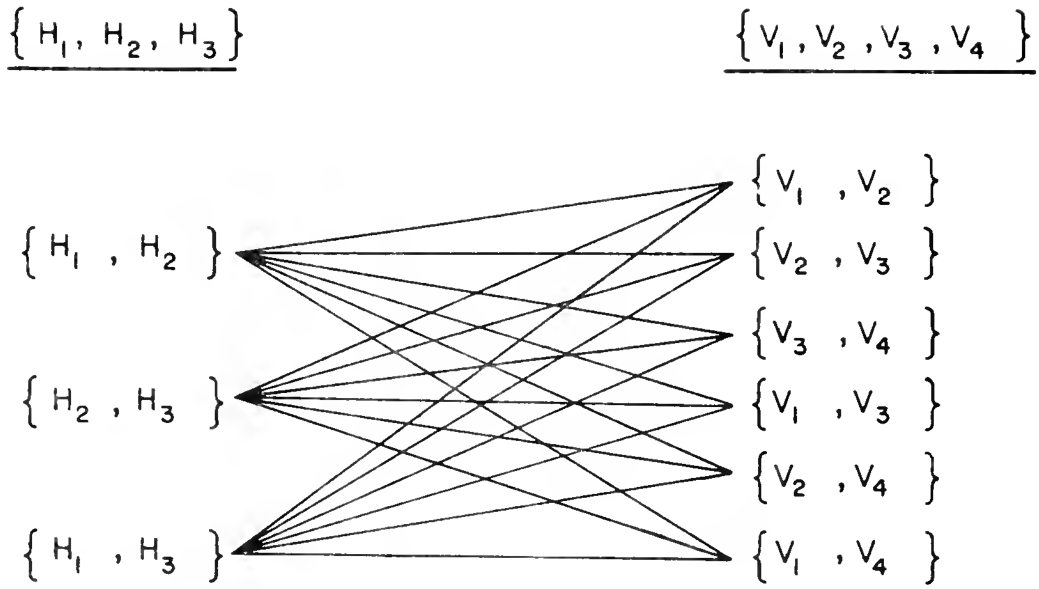
3. By studying the above picture and the rectangles indicated below, determine the appropriate set of vertical segments and fill in the blanks.



4. By considering the results of the three exercises above, what is the total number of rectangles? ______

* * *

Perhaps we should summarize the procedure employed in the preceding problem. Two horizontal segments were selected. Since these two horizontal segments together with each pair of vertical segments identifies a rectangle, the two horizontal segments were matched with the six pairs of vertical segments. This procedure was repeated until all possible pairs of horizontal segments had been matched with the six pairs of vertical segments. This procedure could have been indicated by the following diagram.

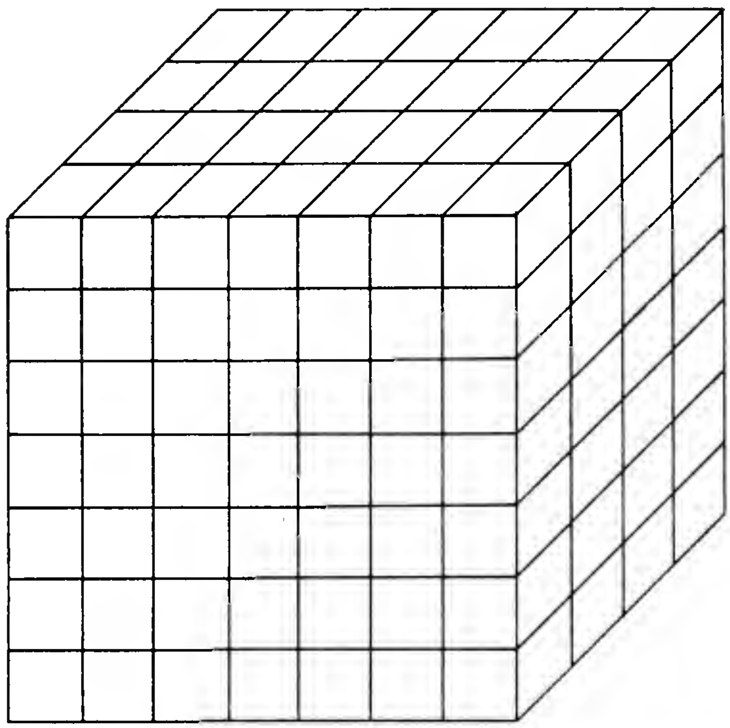


From the above diagram you should see that the number of identifiable rectangles is just the number of two-membered subsets of the set of horizontal segments multiplied by the number of two-membered subsets of the set of vertical segments.

* * *

5. Determine the number of identifiable rectangles in the problem pictured on Page 5. _____

☆6. Determine the number of rectangular solids identifiable in the following picture.



W. Rucker

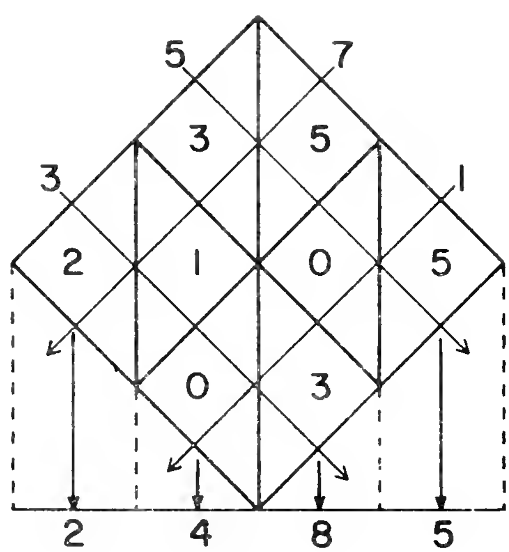
Newsletter 17

Diamond Multiplication

One of the chief difficulties youngsters seem to have in doing a multiplication problem is keeping the columns lined up properly. As we developed a remedial arithmetic unit to accompany the experimental seventh grade text, we experimented with several formats before we found one which seemed to make sense to try. The idea is probably not completely original, but we believe that our format and procedure is novel and interesting to youngsters.

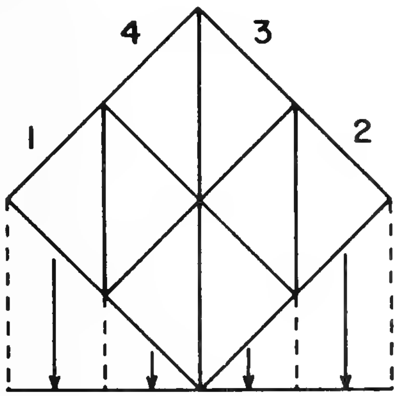
These exercises are what we call "Diamond Multiplication". A very interesting result occurs when we add down the vertical columns. This result was observed many years ago — see if you recognize it.

Example:

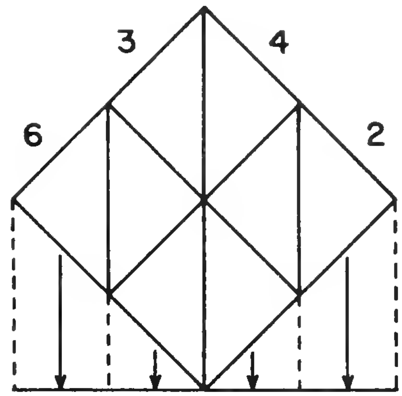


The following problems are given immediately after the sample given above.

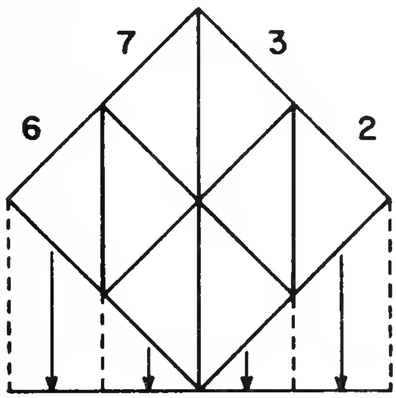
1.



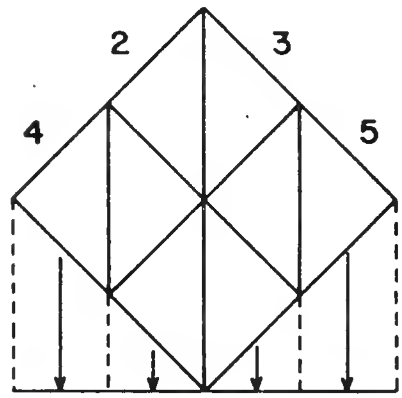
2.



3.

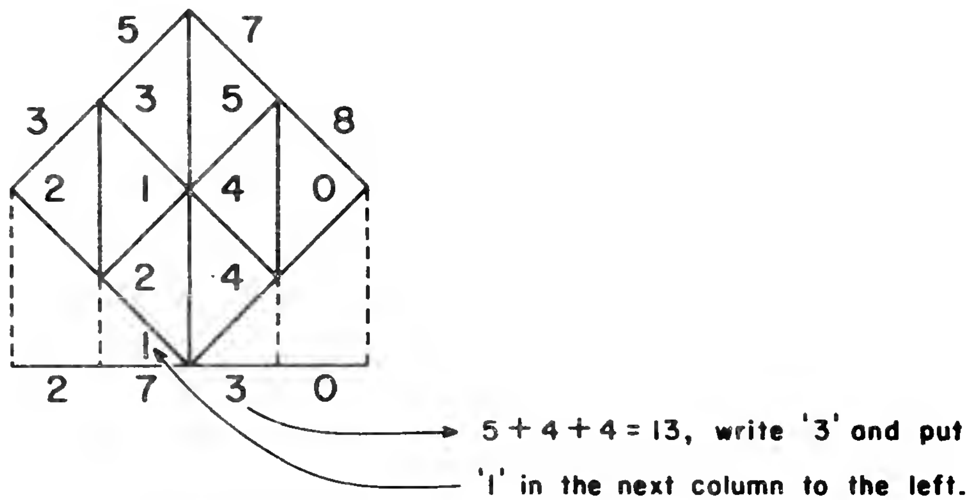


4.



So far, all of the problems have been ones which do not involve carrying. This format is, fortunately, one which lends itself to a carrying notation quite well. The sample below will illustrate our point.

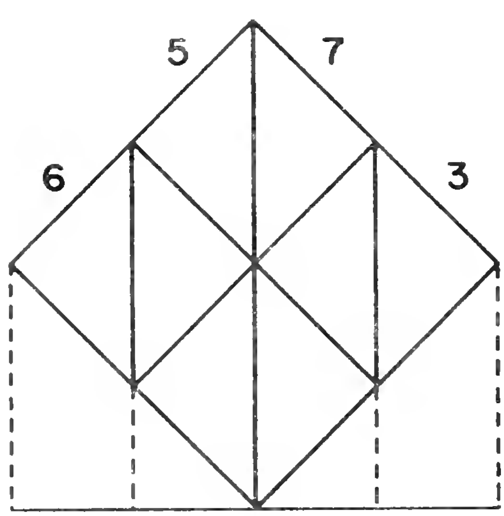
In case the sum for a column is 10 or more, your answer will contain two digits. Write the unit's digit below the column, and write the ten's digit in the next column. Then include it in the addition of the next column. Look at this example carefully:



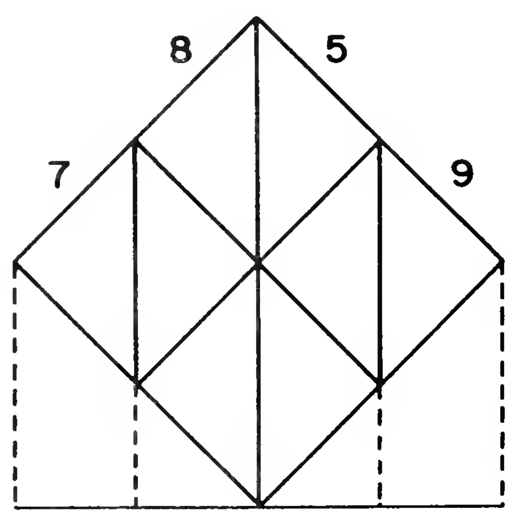
The following problems continue in the same manner.

Now try these:

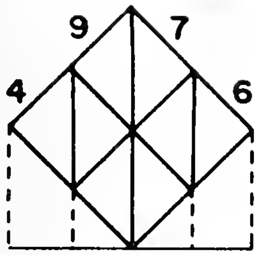
1.



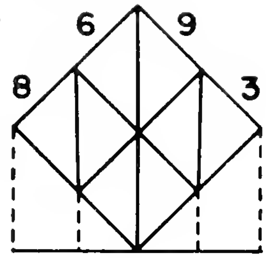
2.



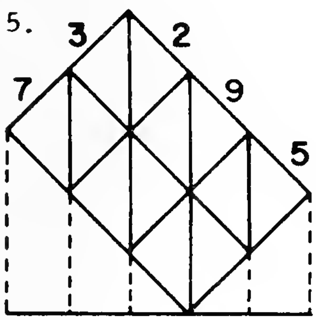
3.



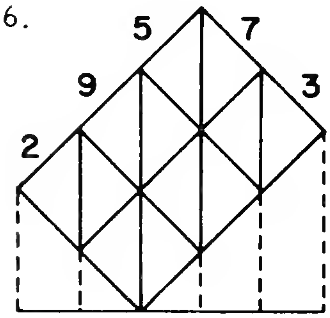
4.



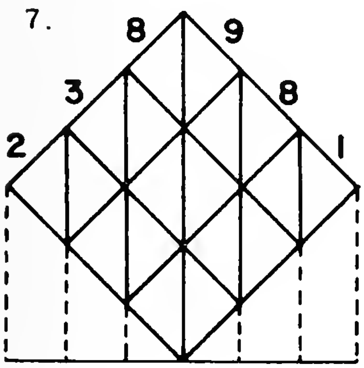
5.



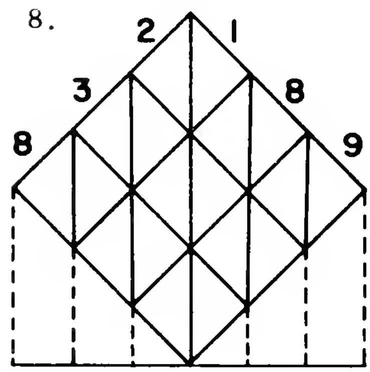
6.



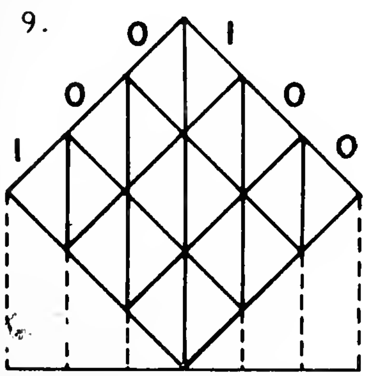
7.



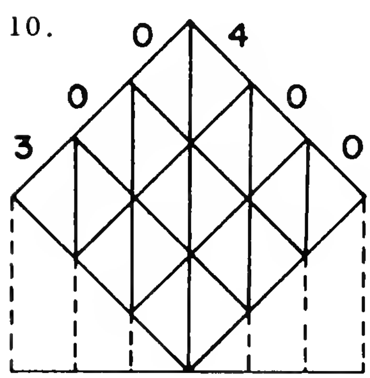
8.



9.



10.

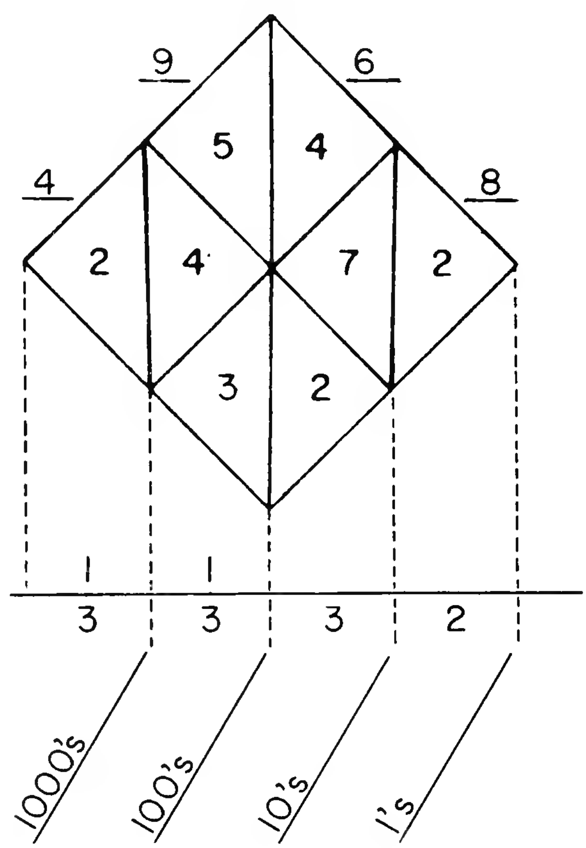


There are several payoffs in this approach to multiplication which could possibly straighten out some of the difficulties encountered by students.

First of all, once the pattern has been set, there is nothing more difficult here than keeping a record of products of two whole numbers between 0 and 9. Hence, it is possible to do this with youngsters who either know their multiplication facts or have a table at hand from which they may find them.

A second advantage, as we mentioned before, is that the "lining up" problem just does not exist in this array. Since the book (or the teacher) takes care of the mechanical part of arranging the addends, one common source of error is largely eliminated.

A third payoff is that with a little more exploration and detailed analysis of the picture it is possible to see place value and the distributive principles at work in the figure. Let's look at this figure, for example:



The problem given above is:

$$49 \times 68$$

of course, we have chosen to think of it as:

$$(40 + 9) \times (60 + 8)$$

And so, since $40 \times 60 = 2400$, we have written a '2' in the 1000's column and a '4' in the 100's column.

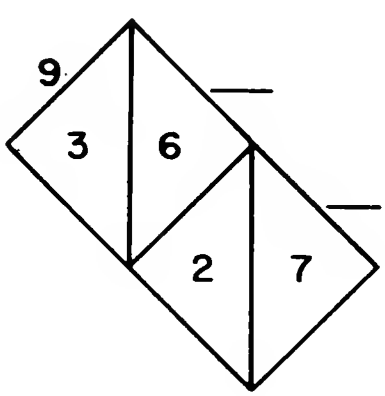
We could continue analysing the rest of the entries in a similar fashion, but we will leave that as an exercise for your students.

If you try this with your classes (at any level), we would suggest that you have great amounts of this available in duplicated form since it seems to generate interest and it moves very quickly.

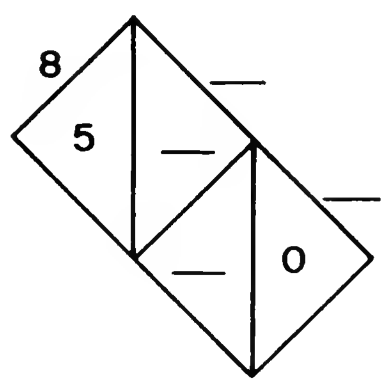
Finally, when your students have finished doing simple multiplication problems this way, you might try these exercises which are really intuitive simultaneous equations.

In each of the following boxes, we have filled some of the spaces. You are to fill in each of the blanks so that the boxes are properly completed.

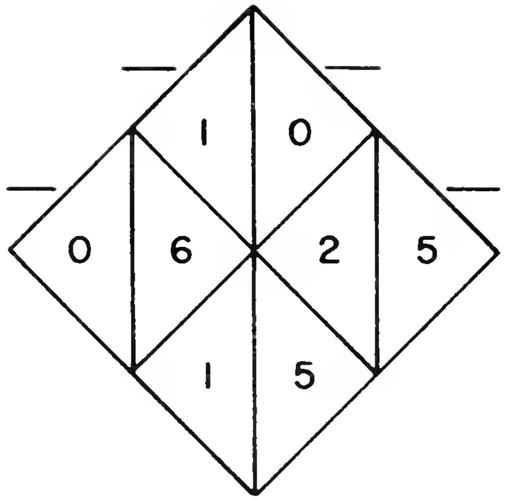
5.



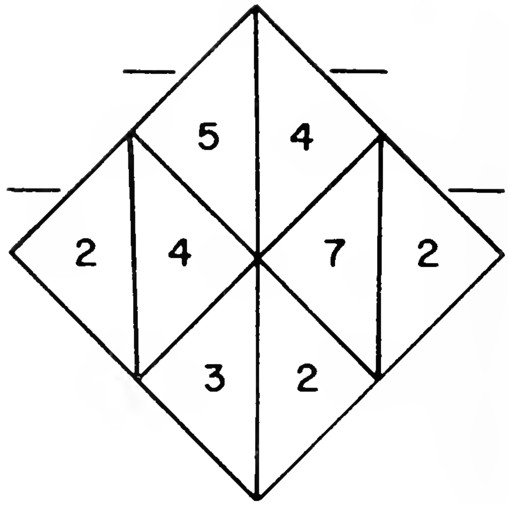
6.



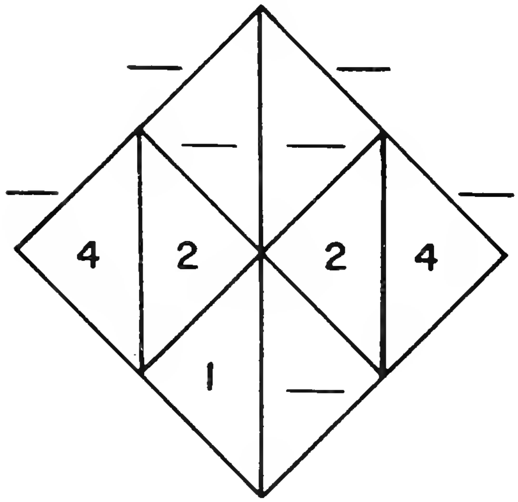
7.



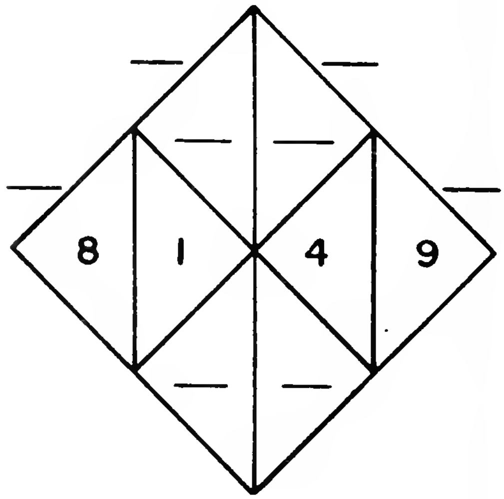
8.



9.



10.



We regret that we cannot provide classroom sets of this material, but much of it is in the revision stage now. So if any of our readers adapt this to their classroom situation, we would be interested in their reactions.

C. Tremblay

Newsletter 17

Overheard

Editor's Note: The following excerpt from a conversation was passed on to us by a staff member who received a letter from another part of the country where UICSM is taught. The conversation was overheard in a beauty parlor. We think teachers of "the new math" can take some heart from this exchange.

Mrs. A: My husband can't understand the new math the children are learning; how about yours?

Mrs. B: My husband doesn't understand it either. My neighbor who is an accountant went and bought the new math books and he doesn't understand it.

Mrs. A: But tell me, do your children like it?

Mrs. B: My children love and understand it. How about yours?

Mrs. A: Mine love and understand it! I guess that's all that matters.

Newsletter 18

Off Limits

Editor's Note: The two people who coauthored this article were participants in the August Institute at Urbana. In discussing the Computational Supplement to the seventh grade program, they became interested in the following problem:

> One way of getting upper and lower limits for the product of two whole numbers is as follows:
>
> 1. Lower limit:
>
>   Round off each number to the next lower multiple of 10 and multiply the results.
>
> 2. Upper limit:
>
>   Round off each number to the next higher multiple of 10 and multiply the results.
>
> Under what conditions will the actual product of the original pair of whole numbers be exactly the average of the upper and lower limits?

Our thanks and compliments to Mrs. Davidson and Mr. O'Donnell for their work on this problem.

Given the problem of finding the product of two 2-digit numbers, neither a multiple of 10, one can find a lower limit of the product by finding the product of the factors after each has been rounded to the next lower multiple of 10, and an upper limit of the product by finding the product of the factors after each has been rounded to the next upper multiple of 10. The original product is greater than this lower limit and less than this upper limit. For example,

$$30 \cdot 80 < 37 \cdot 85 < 40 \cdot 90$$

Consideration of this problem leads to the more interesting case of asking when the product of the original numbers is equal to the average of these two limits. Let's solve the second problem:

Let the two numbers be $10a + b$ and $10c + d$, where

$$a, b, c, \text{ and } d \text{ are integers between } 0 \text{ and } 10. \tag{1}$$

As a result the original problem is now

$$10a \cdot 10c < (10a + b)(10c + d) < 10(a + 1) \cdot 10(c + 1). \tag{2}$$

Now we'll examine the special case where

$$(10a + b)(10c + d) = \frac{(10a \cdot 10c) + 10(a + 1) \cdot 10(c + 1)}{2} \tag{3}$$

. . .

$$10ad + 10bc + bd = 50a + 50c + 50 \tag{4}$$

$$bd = 10(5a + 5c + 5 - ad - bc). \tag{5}$$

Since 5 divides the right-hand side, 5 divides bd and hence either b or d. [Why?]. Without loss of generality, suppose 5 divides b.

Then $5 = b$. [Why?] (6)

Substituting 5 for b in (4) we get

$$10ad + 50c + 5d = 50a + 50c + 50 \tag{7}$$

. . .

$$d(2a + 1) = 10(a + 1). \tag{8}$$

By (5) and (6), d is even and by (1) $d < 10$ so d is not divisible by 5. Since 5 divides the left-hand side of (8), then 5 divides $2a + 1$. [Why?] Since $a < 10$ and $2a + 1 < 19$, then $2a + 1$ is either 5 or 15. Hence $a = 2$ or $a = 7$.

Solving (8) for d, we get

$$d = \frac{10(a + 1)}{2a + 1} \qquad (9)$$

Substituting '7' for 'a' in (9), we get $d = \frac{80}{15}$, not an integer. Therefore $a = 2$, since substituting '2' for 'a' in (9), we get $d = 6$.

We now have that $a = 2$, $b = 5$, and $d = 6$. Note that the solution is independent of c since the terms containing 'c' vanish in (7).

Hence there are exactly 9 pairs of two-digit numbers whose product is the average of their lower and upper limits, namely 25 and ___6.

Thomas M. O'Donnell

Patricia Davidson

Newsletter 18

A Coordinate System

The introduction of the notion of coordinates in 3-space is often a troublesome period in teaching secondary school mathematics. The following is an approach that could be used with students at almost any level. In this approach, we try to appeal to a visual sense based on actual objects that students could construct and manipulate.

Suppose we had a large block made up of 27 smaller blocks. The blocks are arranged alphabetically in the order implied in Figure 1.

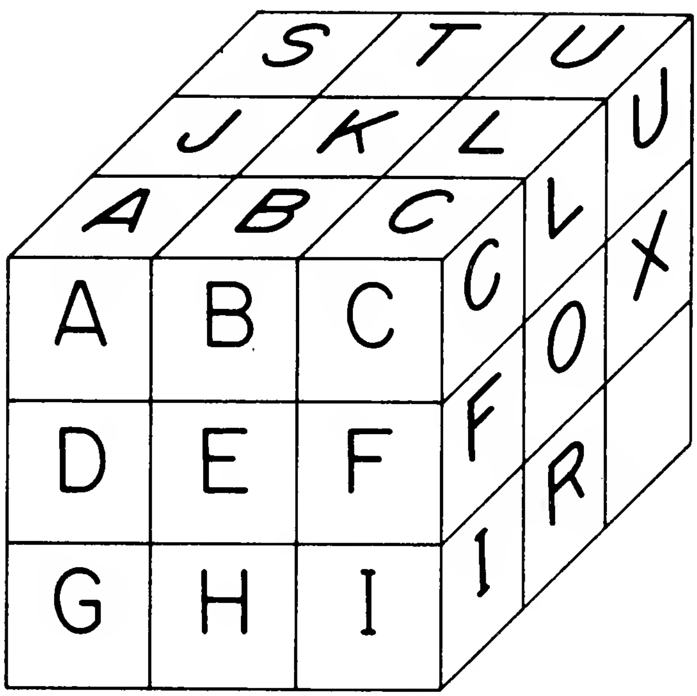


FIGURE 1

From Figure 1, we may assume that the M-block would be directly below the J-block and behind the D-block. Since there are 27 blocks and only 26 letters in our alphabet, the "last" block is left blank.

A first step that might be taken in introducing the coordinate system would be to get students to answer exercises such as the following.

1. The _____ block is directly above the Z-block.
2. The O-block is directly to the right of the _____ block.
3. The P-block is directly behind the _____ block.
4. The _____ block is directly above the N-block.

The next step that might reasonably be taken is to coordinatize the blocks after some scheme. How much should be said about the system is certainly a matter of personal choice based on the class with which one is dealing. It might be reasonable in some classes to give them a few blocks labeled as we have done these in Figure 2 and ask the class to suggest how the rest of the blocks should be labeled.

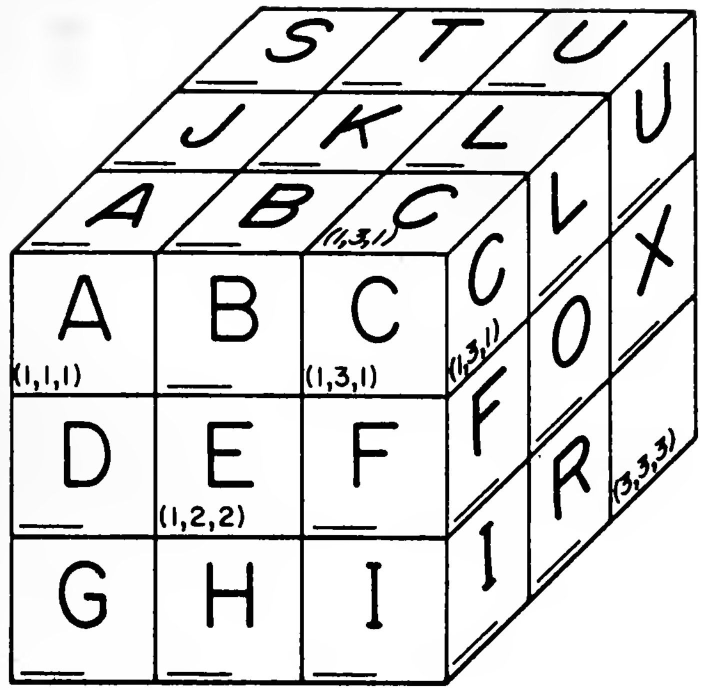


FIGURE 2

In class discussions, such references as 'layer', 'row', and 'column' might be useful. The teacher could lead the "discovery" by asking what letter is found in layer 1, column 2, row 1. Some classes might be able to devise their own coordinate system according to the way the students feel more comfortable in locating blocks. A very natural sort of exercise to follow the introduction of the ordered triple is to have the class give the ordered triple for each block that is visible. Then they should be asked to decide what block corresponds to a particular ordered triple. At all times in the discussions of these blocks, the students should have access to the picture. A demonstration size picture of this set of blocks could prove useful.

Figure 3 could prove useful to students who have trouble visualizing the break-down by layers, columns, and rows.

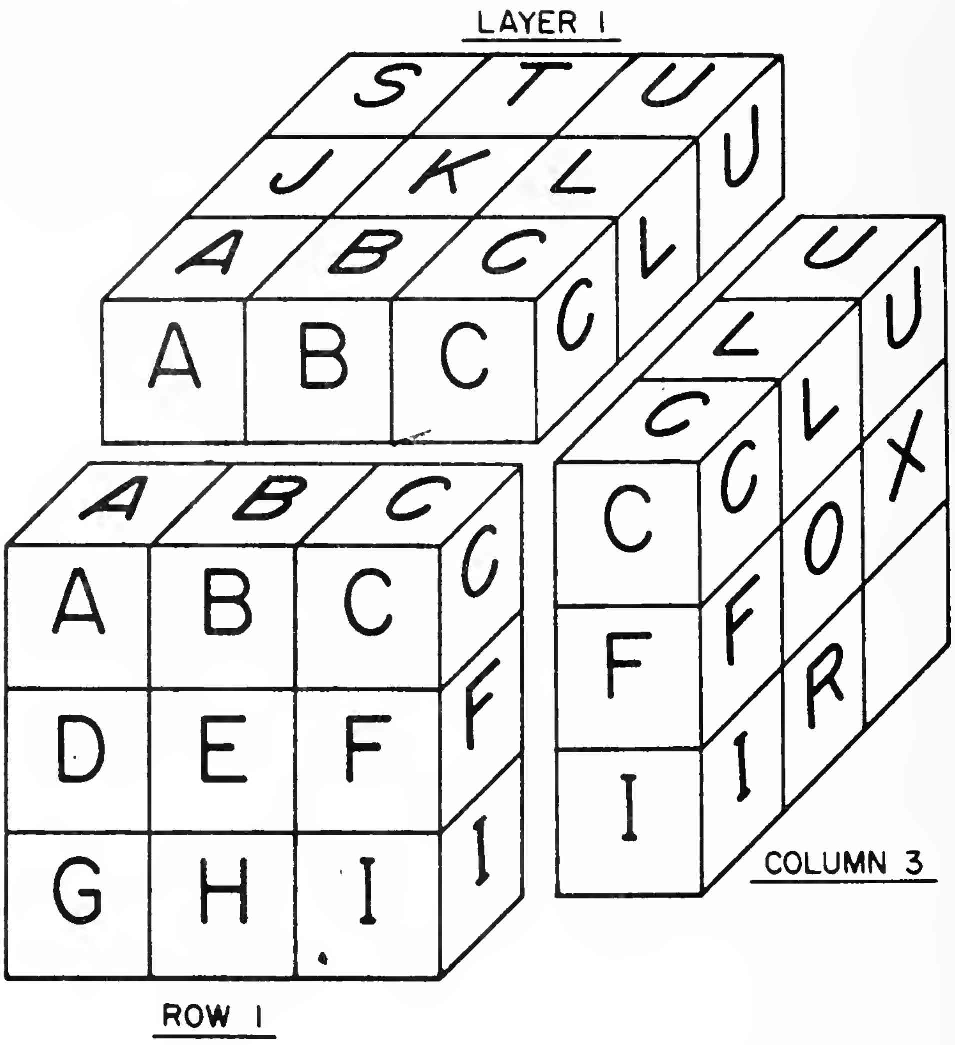
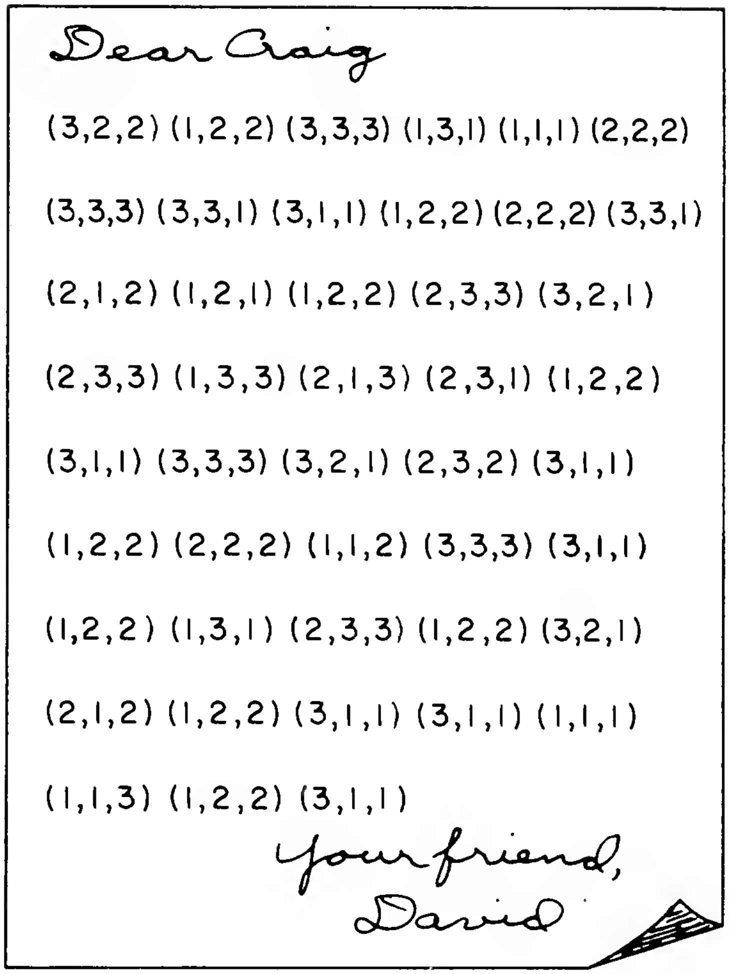


FIGURE 3

A simple motivation technique might be to use these ordered triples as a code. Once an agreement has been made in what order to name the blocks with ordered triples, the code is instantly determined. The teacher, for example, could devise a series of sentences or phrases that might interest the youngsters. The following "letter" is an example of the sort of exercise that might interest youngsters. For this "letter", we have used the order indicated in Figure 2.



With a little prodding (and in some cases, none), students will frequently volunteer to alter the code so others will not be able to decode their message. This behavior ought to be commended since any practice with ordered triples as names of points is going to be beneficial. Figure 4 illustrates one such change that might be made. Students ought to be encouraged to change their codes with reference to the picture.

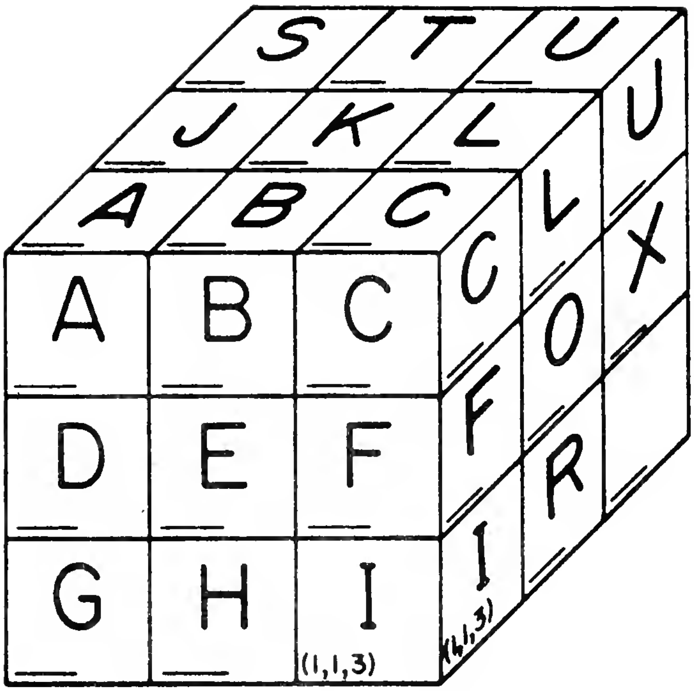
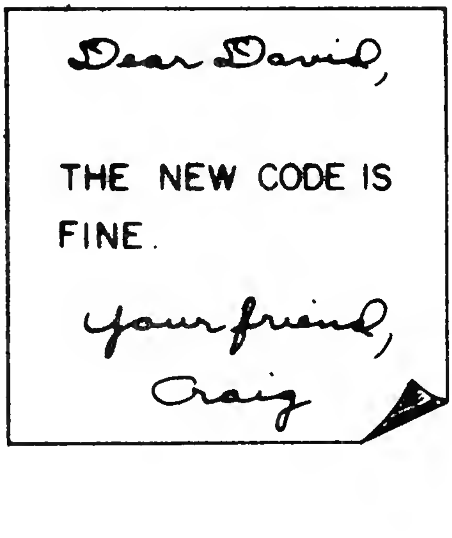
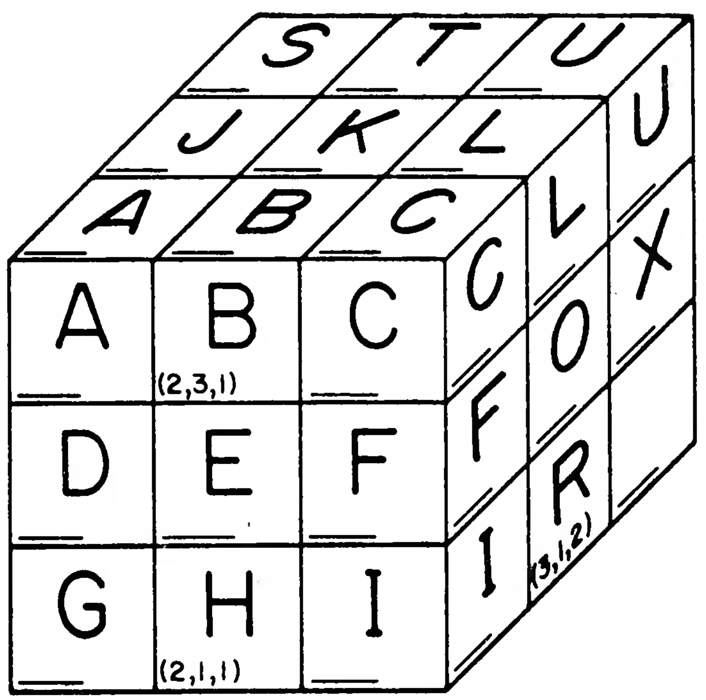


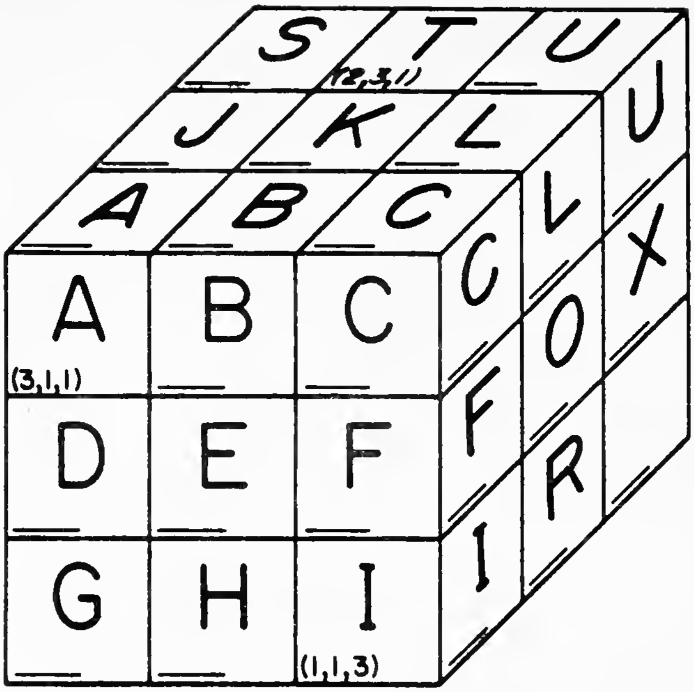
FIGURE 4

In order to have a quick check on understanding, the teacher could structure this exercise a little by providing a simple message similar to the one we gave for decoding earlier. One such as the following might prove easy to work with.



As a check on understanding, the following exercises could be used. In each case, one could ask students to name each visible block using ordered triples according to the indicated pattern.




S
T
U
(2,3,1)
J
K
L
U
A
B
C
L
A
B
C
C
X
(3,1,1)
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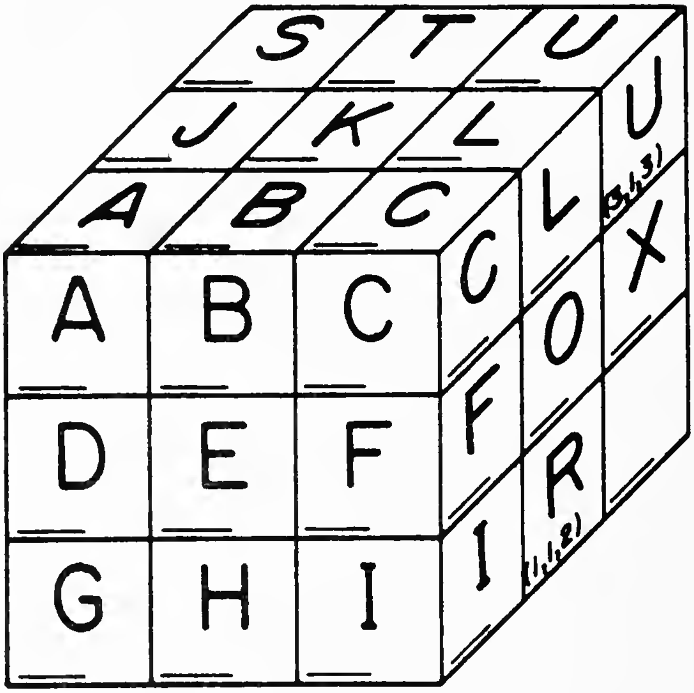
D
E
F
F
R
G
H
I
I
(1,1,3)


S
T
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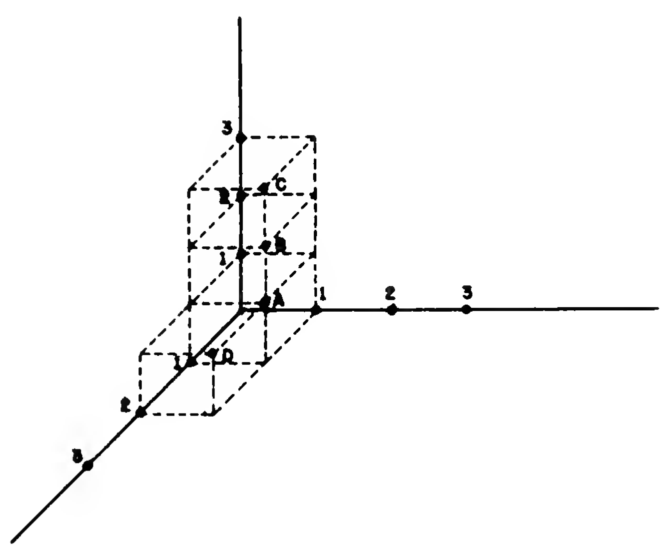
A
B
C
L
(3,1,3)
C
X
A
B
C
O
F
D
E
F
R
I
(1,1,2)
G
H
I

There are some questions that might be posed once the class has become comfortable with this notion. One such question is to decide how many blocks have to be named before the code is determined. Is it enough to know one block? 2 blocks? 3 blocks? Speculation on this question could shed some light on the traditional coordinate system.

The final transition to the regular coordinate system will depend on the reception the preceding exercises have had. One approach might be to draw a diagram somewhat as follows.



Given these four points, the students ought to be able to continue the pattern and name each letter of the alphabet using an ordered triple. Refinements of this could include a little work with negative numbers in the ordered triples. If any teachers try this approach to a 3-space coordinate system with their classes, we would appreciate comments or suggestions.

W. Rucker

Newsletter 18

Fractions

After a year and a half of teaching the present UICSM seventh grade program, several of us now believe in the stretching and shrinking machines that abound in the book and cannot seem to remember the words 'numerator' and 'denominator'. Some of the faith that the youngsters develop in these machines tends to rub off on their teachers. We find ourselves asking and answering questions about fractions by saying things like, "Which one has the larger stretcher?"

This new vocabulary has been brought about by the effort on the part of UICSM to develop a seventh and eighth grade program for the so-called "culturally disadvantaged" youngsters, who are found in great numbers in each large city. In this third year of writing and teaching the materials of our seventh grade program, we now find that we have a reasonable approach to teaching rational numbers and all the operations normally performed on them. It may seem to most seventh grade teachers that a year's concentrated effort on rational numbers is less than efficient. It is not our purpose to debate this issue in this present article.

Since much of the future work in this course depends on the students' having some proficiency in dealing with labels on "sticks," we start with exercises that help develop such a proficiency. Figure 1 shows such an exercise.

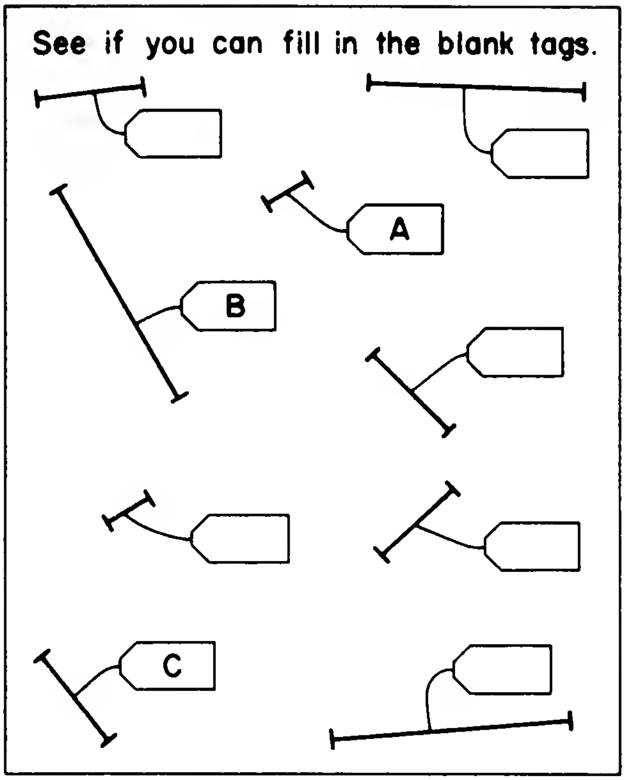


Figure 1

Some of the sticks the students are asked to label are clearly seen to be multiples of ones that have already been labelled. Figure 2 shows how we help the students see such multiples and how to label them.

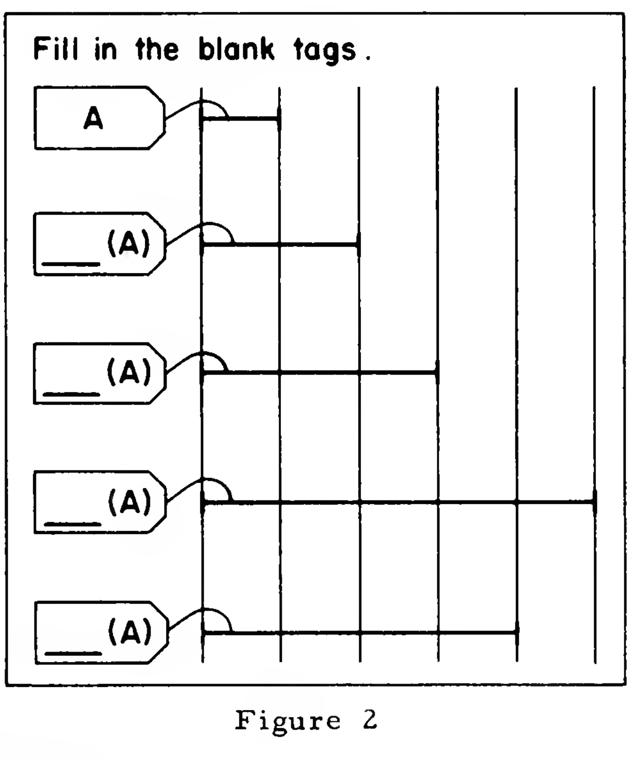


Figure 2

After this fairly short introduction to labeling sticks of varied lengths, these sticks are put into stretching machines. The stretching machines we use are from the world of fantasy, but they soon become quite real to the students who find them extremely helpful in doing exercises which are usually guessing games. Our stretching machines are merely function machines which map lengths onto lengths. Thus, in Figure 3, the 3cm. stick goes through a 2-machine and the result is a 6cm. stick.

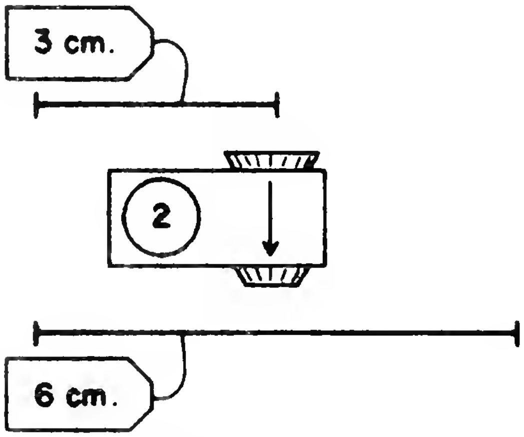


Figure 3

We do many exercises with rulings as seen in Figure 2. These rulings enable students to tell almost at a glance (at most, by counting spaces) what sort of stretching job has been done or what the input or output stick should be. For students whose manipulative skills are not well developed, the facility with which they can handle the exercises using rulings is a definite asset in getting more complex issues solved without getting bogged down in unnecessary device manipulation details.

In order to develop one of the fundamental concepts of the course, we introduce the notion of hookups of machines. Figure 4 shows such a hookup.

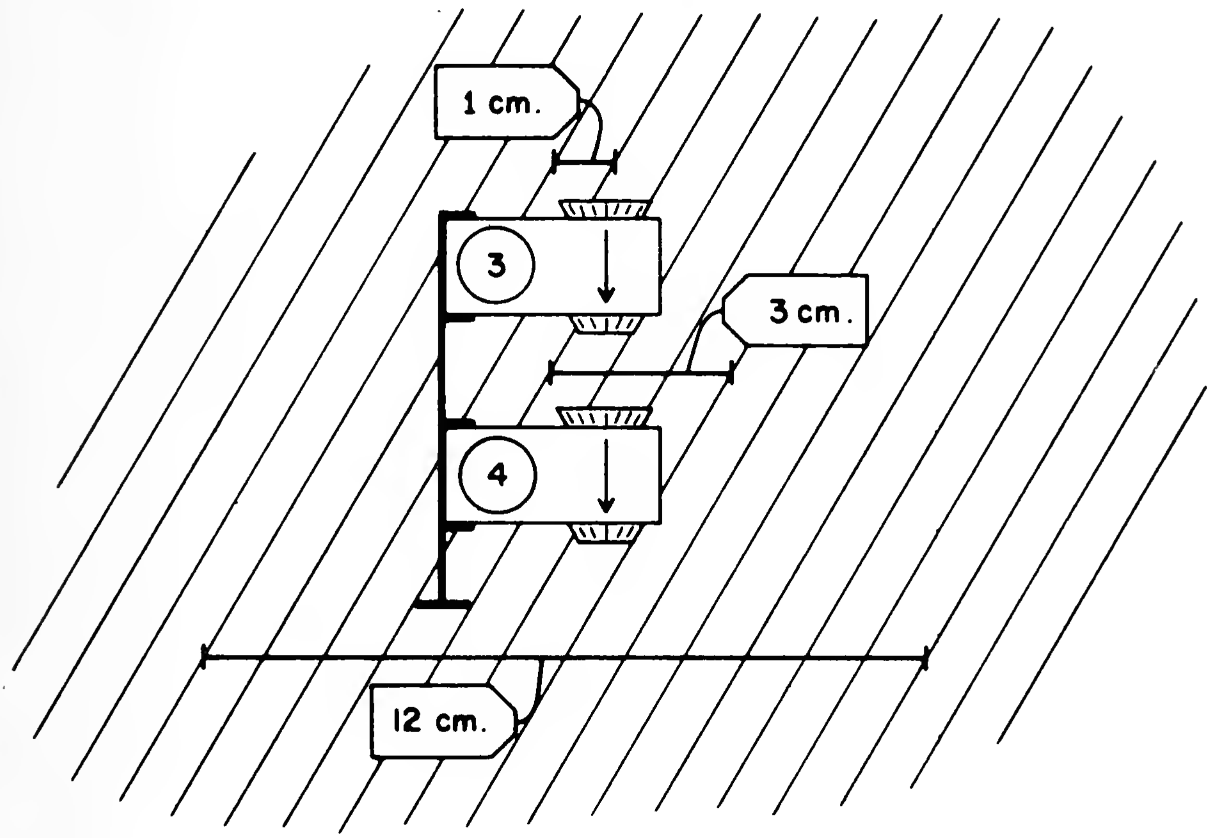


Figure 4

We frame questions about hookups in various ways in order to acquaint the student with many ways of doing and saying the same thing. The example in Figure 4 might be done as we have shown or it might be in a tabular array as in Figure 5.

Final Output	2nd. Machine	Intermediate Output	1st. Machine	Input
	4		3	1 cm.
28 ft.			2	2 ft.
	3		13	17 in.
50 mi.	2			5 mi.
3 in.				3 yd.

Figure 5

Since hookups are, essentially, compositions, we introduce a composition notation:

$$[3 \circ 2] \quad (1 \text{ inch})$$

to tell the student in abbreviated fashion that this is the output of a hookup of a 2-machine followed by a 3-machine when the input is a 1-inch stick. In telling the machines to be used in a hookup to get a certain job done, the student has been factoring. We lead him through several pages of exercises designed to develop the notion of factoring before he finally sees the word and learns that it is a fancy way of describing a familiar (and easy) concept. It is a fairly small step from the notion of factoring to the notion of getting prime factorizations. Punched cards as in Figure 6 are used to give students practice in dealing with factoring and prime factorizations.

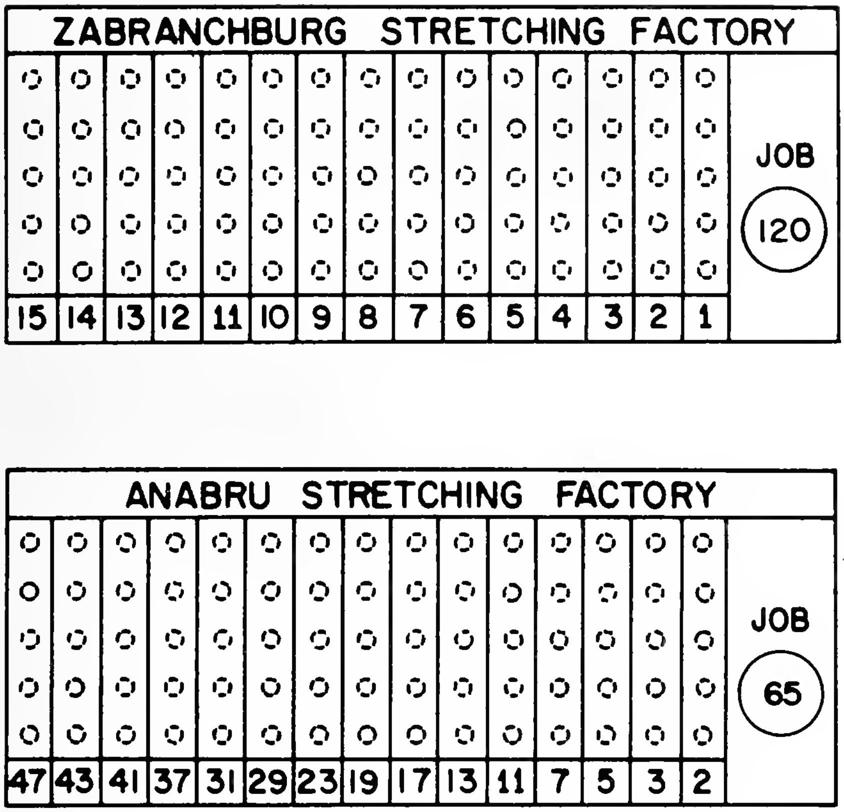


Figure 6

The students are to figure out what "punches" to make in each card so that the indicated job is done. On the first card in Figure 6, it is possible to make different kinds of punches and still be correct. On the second card, the response turns out to be unique since we have indicated prime machines are the only ones available.

Exponents are a natural outgrowth of this development. In order to describe a job order as punched on one of these cards, it frequently is the case that something such as this:

$$5 \circ 5 \circ 5 \circ 3 \circ 3 \circ 3 \circ 3 \circ 2 \circ 2$$

would have to be written. To avoid such long expressions, the student is lead to writing expressions such as this one:

$$5^3 \circ 3^4 \circ 2^2$$

Because of the context in which these exponents are learned, the usual sorts of confusion about exponents do not have a chance to develop. If a student does have a momentary lapse of memory, the teacher can get him back on the track readily by asking him what kinds of machines are being used.

Before we get finished with the chapter dealing with stretching machines, many youngsters are speculating on or asking about the possibility of shrinking machines. The second chapter of our text introduces shrinking machines. Once again, we go through a complete development to help the student gain a reasonable proficiency in handling this new machine. The only new notation in this chapter is that used in naming the shrinking machines. So, we say that '$\overline{4}$' names a machine that shrinks a stick to $\frac{1}{4}$ of its original size. Mostly, this is shown by example, without the need to resort to words that could cloud more than they clarify. The bar over the numeral is a foreshadowing of the fraction bar — a fact that occurs to very few youngsters. There are exercises in the chapter on shrinking machines that test the students' understanding of the inequality relationships and of the effect of hookups of shrinking machines. Once shrinking machines have been introduced and the students have had a chance to work with them for a while, we introduce the notion of inverses. They soon find out that a 3-machine and a $\overline{3}$-machine are inverses since the $\overline{3}$-machine "undoes" what the 3-machine does.

The next chapter prepares the students for everything they will encounter in working with fractions in the usual notation. We do this in terms of Standard Job Orders. These are orders such as:

$$3 \circ \overline{5}$$

In each case, a standard job order has a stretcher named first, followed by the circle followed by a name for a shrinker. Within this chapter, we get the notion of reducing Standard Job Orders to lowest terms by eliminating inverses. Figure 7 shows such an exercise.

Standard Job Order		Factor and Eliminate Hidden Inverses		Equivalent Job Order
$12 \circ \overline{8}$	=	$3 \circ \cancel{4} \circ \cancel{\overline{4}} \circ \overline{2}$	=	$3 \circ \overline{2}$
$6 \circ \overline{21}$	=	$\_\_\_ \circ 3 \circ \overline{3} \circ \_\_\_$	=	$\_\_\_ \circ \_\_\_$

Figure 7

There are also exercises involving inserting inverses that foreshadow the process of getting common denominators for addition of fractions and comparing fractions.

The next chapter (Chapter 4) finally lets the youngsters in on the fact that Standard Job Orders are simply fractions in disguise. This is done, as usual, in terms of our story line. Figure 8 shows the summary page that leads the students into exercises that make the transfer complete.

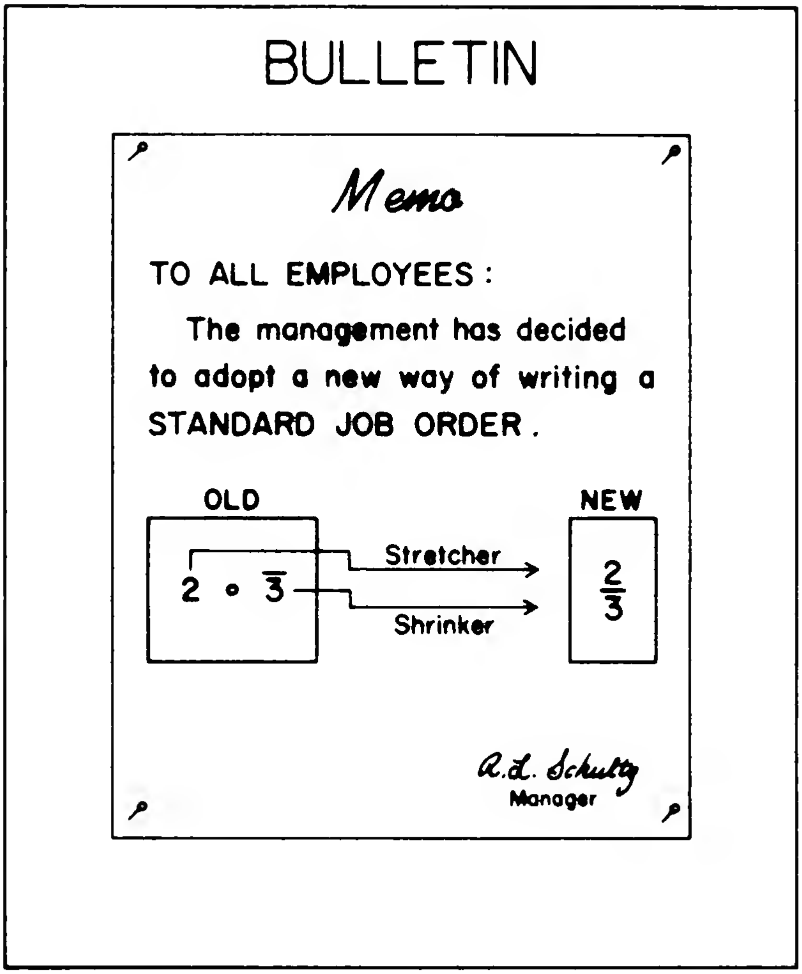


Figure 8

Now the groundwork has been laid, and we may revert to job orders in terms of shrinkers and stretchers when necessary to clarify the issue for students who find fractions a bit confusing. 'Hookups,' 'inserting inverses,' 'eliminating inverses,' and 'Standard Job Orders in lowest terms' become classroom aids that eliminate the need to resort to words that frequently rattle around mathematics classrooms and fall on deaf ears. Since these new words were introduced in a simple (and intuitively evident) context, they are reasonable to the youngsters and they, themselves, have a fairly high proficiency in using these words properly.

The remaining chapters on fractions continue to utilize the hookup of machines notion in order to help students understand how to compare fractions, how to multiply and divide fractions, and how to add and subtract fractions. Many of the exercises in the first eight chapters involve improper fractions, but it is not until Chapter 9 that mixed notation is introduced. In this chapter, we translate from mixed to fraction notation and from fraction to mixed notation as well as perform the four operations on numbers in mixed notation.

Later chapters in the text contain a careful treatment of decimals and percents. We should mention that applications of fractions are carefully interspersed with the purely manipulative exercises as they fit into the material being taught.

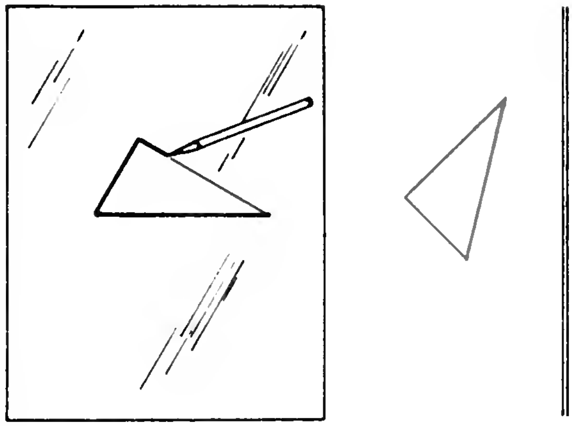
It is the hope and expectation of this project that the materials in the seventh grade program will be published commercially after the revision team uses the results of this year's trial to bring the text to a more polished state. In the meantime, we have to express our regrets to anyone who would like a copy of these materials since there are not enough copies at the present time to go around. We'll keep our readers posted on progress made in preparing this course for publication. We believe it will be a valuable course for youngsters at many levels. It should be possible for elementary school children to learn fractions from the start through these materials and avoid many of the traumatic experiences often encountered in learning about fractions.

C. Tremblay
Newsletter 19

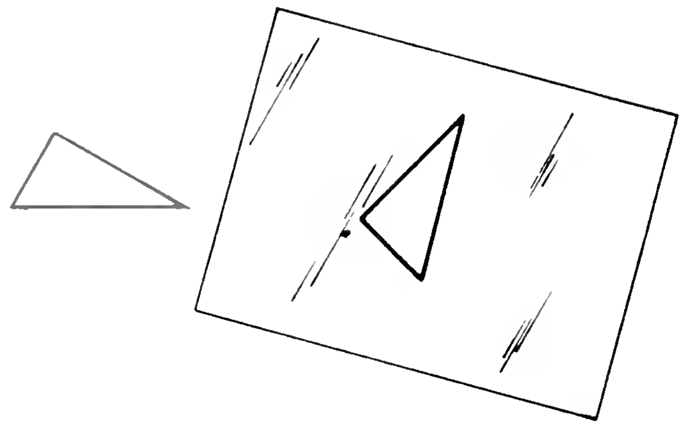
Geometry Au-Go-Go

Translations, Rotations, and Reflections? In eighth grade? UICSM, in its usual spirit of trying to teach good mathematics at various levels, is engaged in the preparation of an informal geometry for eighth graders which is based on the three isometries — translations, rotations, and reflections. The materials developed so far are being tested in the three junior high schools in Champaign-Urbana. The classes are those which studied our seventh grade program in the 1964-65 school year. The challenge has been to seek ways and devices for making this geometry reasonable to youngsters who have had little previous success in mathematics. The experience gained this year in teaching the present geometry program will be drawn on in revising the materials for the 1966-67 school year trial. Copies of the material will not be available except to the centers which have already been chosen to help in the development of the materials.

A major portion of plane geometry is concerned with congruence. Intuitively, two plane figures are congruent just if a tracing of one matches the other exactly. So, we adopt comparison by tracing as the fundamental method of testing for congruence.



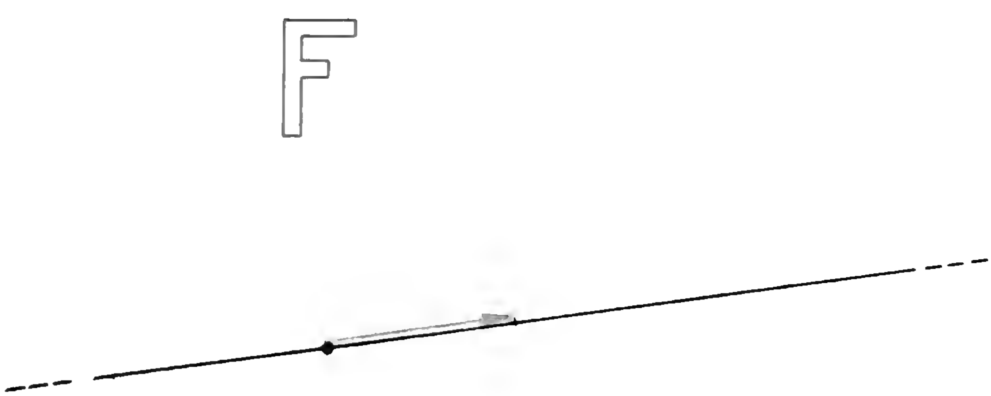
(a) trace



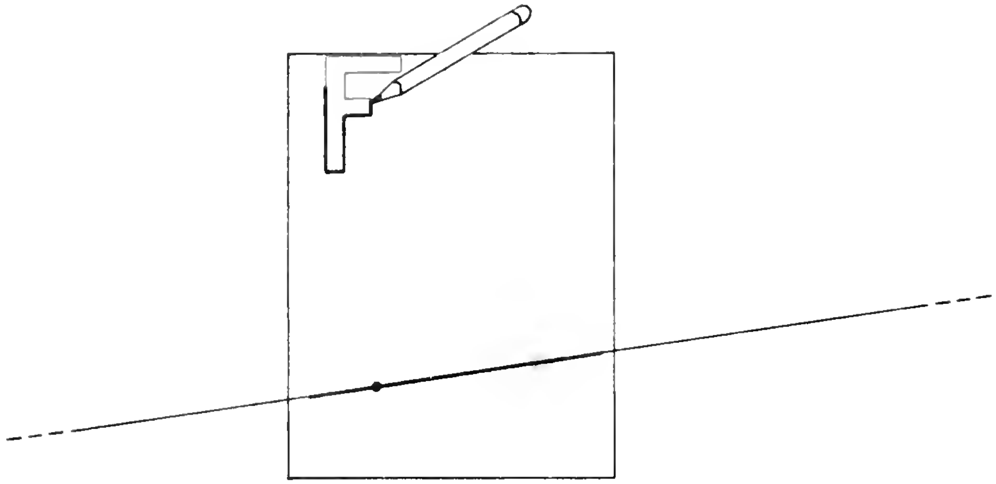
(b) match

Tracings can be used to derive geometric properties of plane figures. To do this, special motions of the tracing material are studied. Specifically, one is interested in the properties common to a pair of figures in the case where these special motions carry a tracing of one figure onto the other.

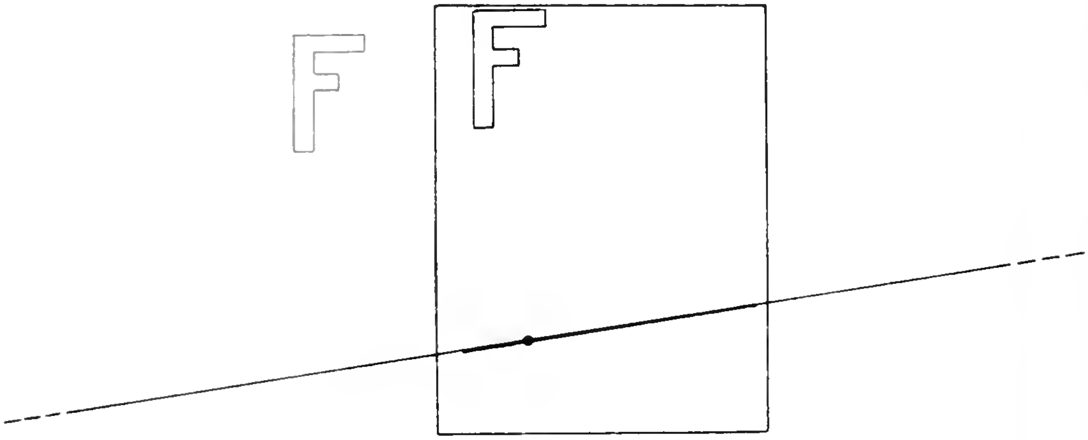
The first motion investigated is the slide. Slides are made with tracing material by using a guide line and an arrow as shown.



(a) figure, slide line and arrow given



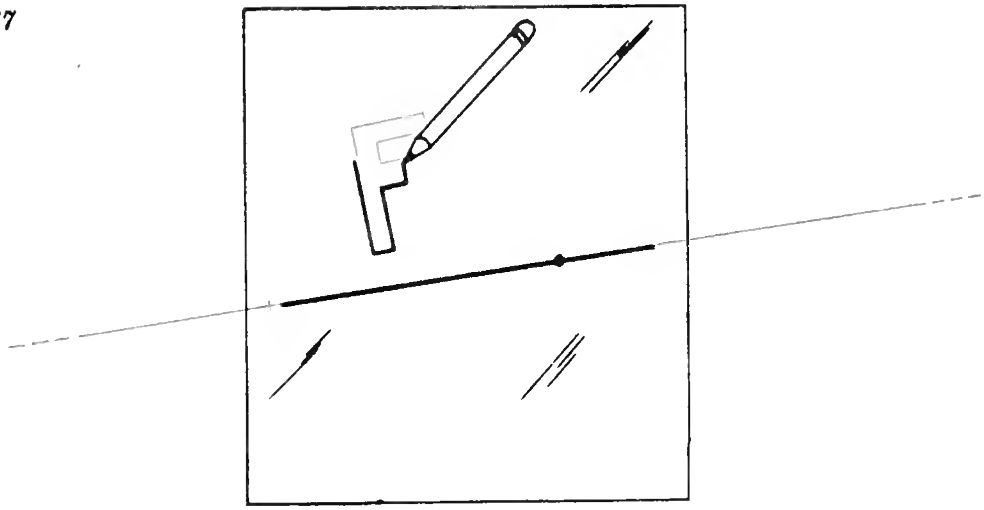
(b) trace figure, slide line, and mark tail of arrow



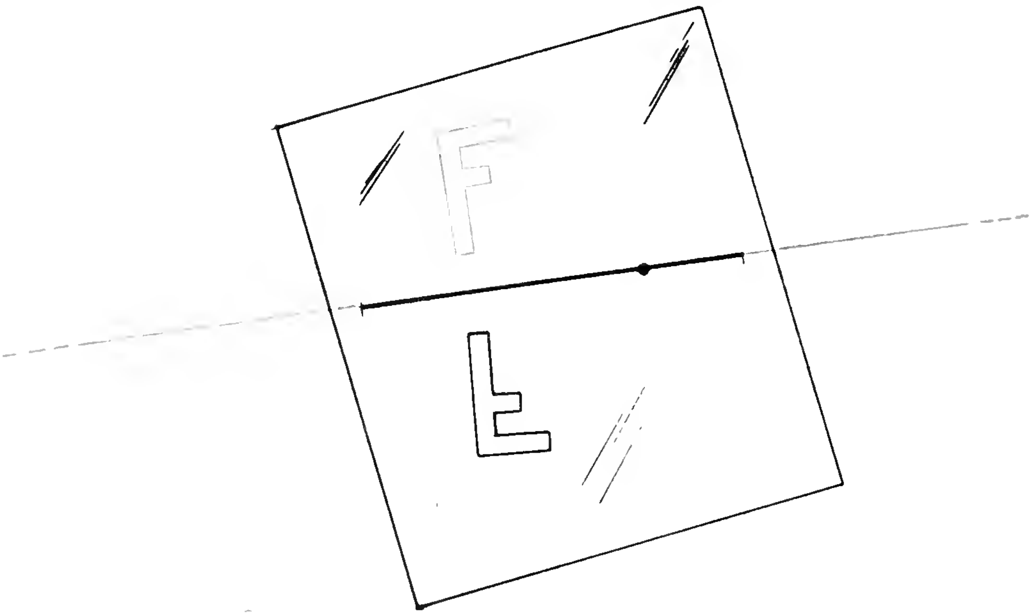
(c) slide so that the tracing of the slide line lines up on the slide line, and the mark made at the tail of the arrow is at the tip of the arrow

A second motion is the flip. Flips are made with tracing material by using a guide line with a point marked on it as shown.

(a) figure and flip line with point marked

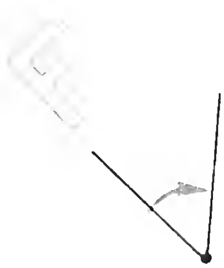


(b) trace figure, flip line and point

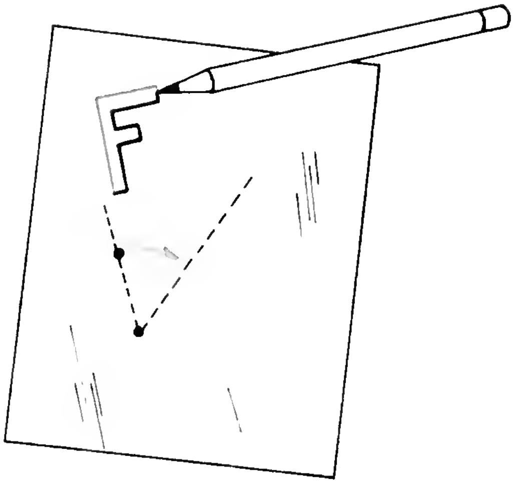


(c) flip the tracing over onto its other side and
line up flip line and marked point

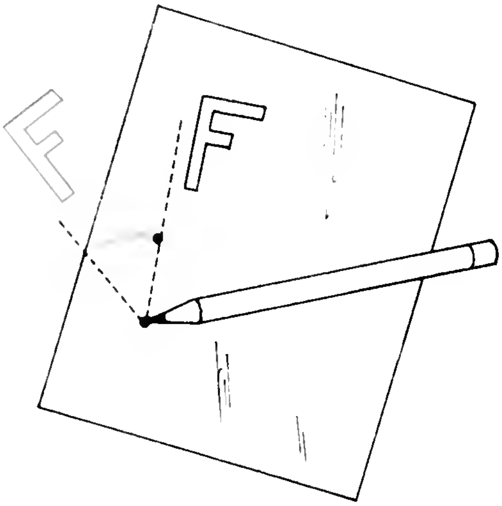
The third special motion of a tracing sheet is the turn. Turns are made with a tracing using a directed arc of a circle about a point called the turn center.



(a) figure, turn center and directed arc

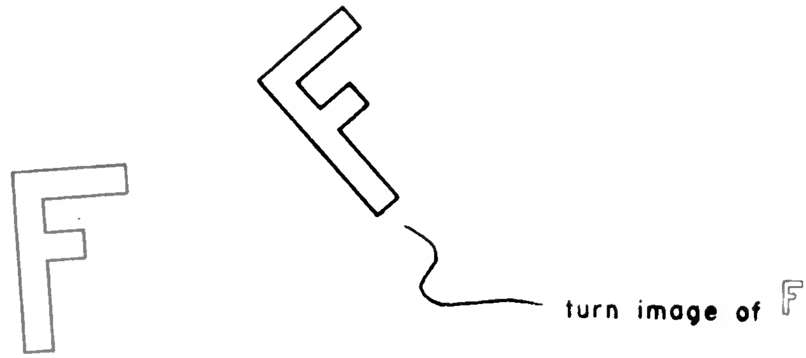


(b) trace figure, turn center and mark a dot at the tail of the arc

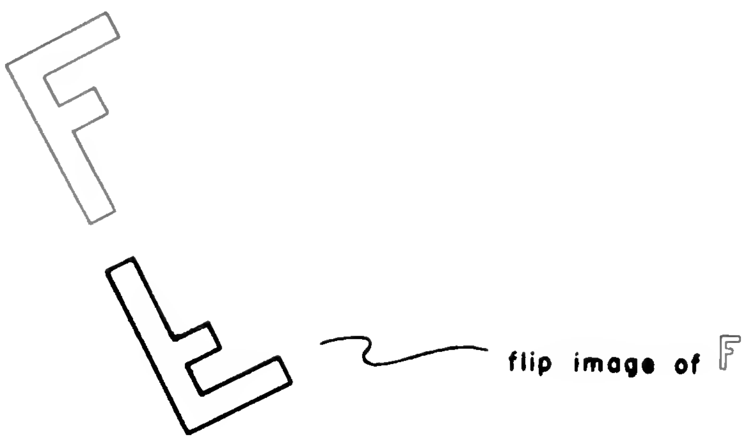


(c) holding pencil point on turn center, turn tracing so that dot is at tip of arc

Given a pair of congruent figures, it may be possible to slide a tracing of one onto the other. In such a case either figure is called a slide image of the other. Similarly, there may be a flip that carries a tracing of one figure onto a second. The second is then a flip image of the first. Finally, if a turn will carry a tracing of a first figure onto a second, the second is a turn image of the first.



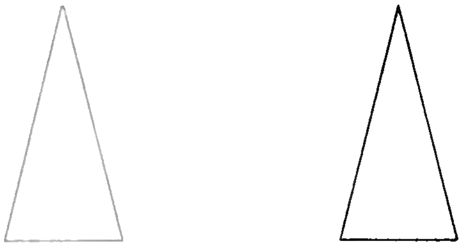
(a)



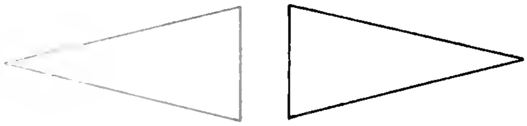
(b)

(c)

Sometimes a figure is a slide image and also a flip image of another figure. For example,  is a slide image of  and also a flip image.

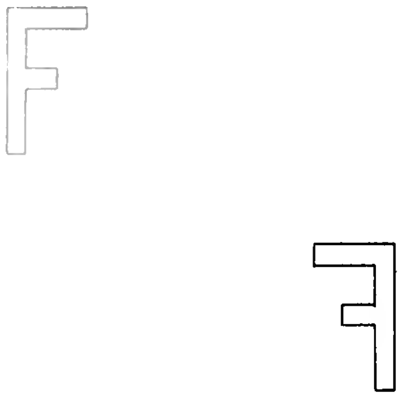


 is a flip image of  and also a turn image of  .



There may be slide, flip, and turn motions that carry a tracing of one figure onto another as for these two rectangles:

In some cases no single motion will carry a tracing of one figure onto a congruent figure.



Some questions that are raised in the UICSM eighth grade course are:

(1) Given a figure and its image under some motion, how does one construct a slide arrow, arc and turn center, or a flip line?

(2) Given a figure and an arrow, arc and turn center, or flip line, how does one construct the appropriate image figure?

(3) How does one construct a figure given its image under a given motion?

In answering these questions, students are encouraged to predict an approximate answer, then check with tracing material [and at a later stage, to use auxiliary equipment to construct the figure or motion indicator].

In a somewhat simplified sense, slides form a background for the geometry of parallel lines. In the same sense, flips provide background for perpendicularity, while turns are closely related to properties of angles and angle measure.

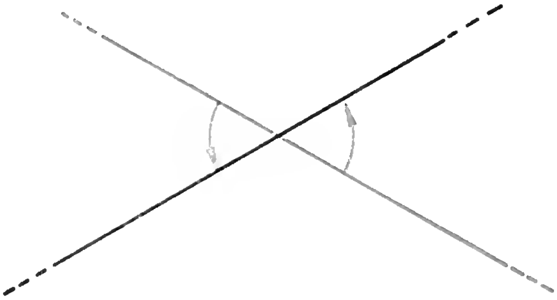
The reason for studying the motions described is that properties that are a consequence of these motions may be used to determine usual geometric theorems. As an example, a property of turns is:

> A turn rotates each point of the plane through the same angle about the turn center.

An immediate consequence of this turn property is the geometry theorem:

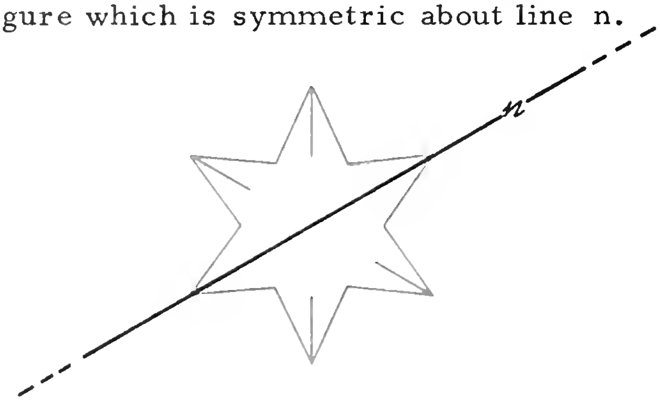
Vertical angles are congruent.

To see this, think of a pair of vertical angles as a line (shown in red) and its image under a turn (shown in black).



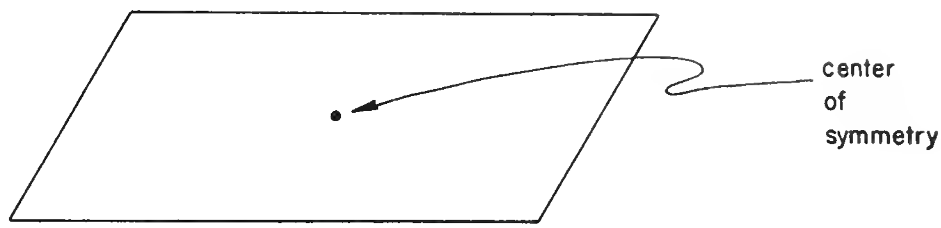
Other geometry theorems are approached in a similar manner. Again, students become intuitively aware of basic properties of the three motions so that they can draw upon these properties as needed in studying geometry.

The study of flips is easily extended to a study of line symmetry. A figure is symmetric about a line whenever it is its own image under the flip about that line. Here is a figure which is symmetric about line n.



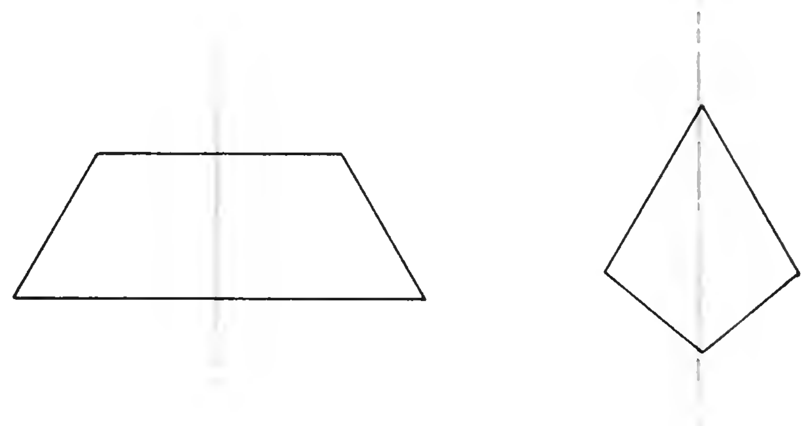
In addition to line symmetries, sometimes a figure is also its own image under a 180 degree turn about some point. In this case we say the figure has a point of symmetry or central symmetry and the point is referred to as the center of symmetry.

Here is a figure which has point symmetry.

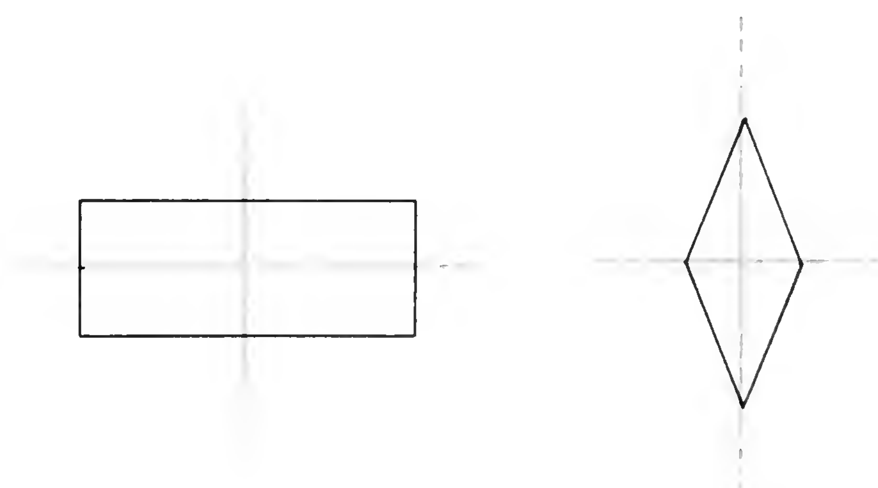


With these ideas of symmetry, one can classify triangles, quadrilaterals, and other polygons according to the types of symmetry possessed.

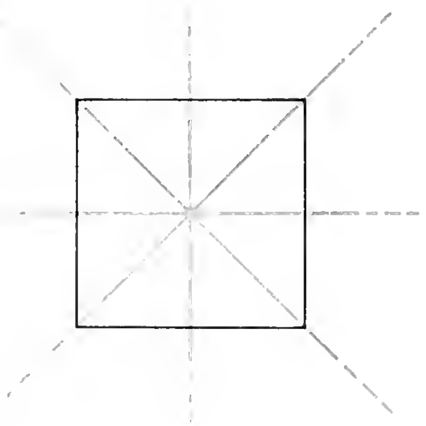
one line symmetry



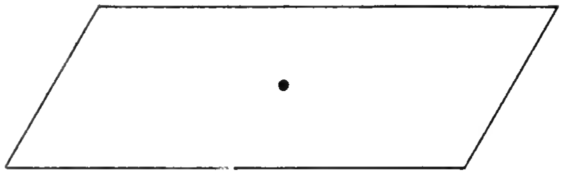
two line symmetries



four line symmetries



point symmetry



Any quadrilateral with two diagonal symmetries will have all sides congruent, opposite sides parallel and opposite angles congruent. These are the standard properties of a rhombus.

The work outlined above is intended to give a one semester treatment of geometry which is informal, interesting to the student, and does not require as much verbal skill as usual treatments of geometry.

The second semester is concerned with the arithmetic of positive and negative numbers. Again, the usual computation rules are motivated through a study of motions.

J. R. Dennis

W. Sanders

Newsletter 19

Some Comments On Vector Geometry

This short note is intended to convey some impressions of the new vector geometry course. This article has no logical thread, no reasonable ordering of thoughts. For a detailed logical outline of the course, see "An Approach to Euclidean Geometry through Vectors" by Steve Szabo, in the March, 1966 issue of The Mathematics Teacher.

Vectors have been studied rather intensively since the middle of the 18th Century. Many authors tended to distinguish between "line vectors" and "free vectors". The concepts tended to center on two key ideas:

(a) Vectors are directed line segments

(b) Vectors are quantities possessing both magnitude and direction

A common motivation for "addition of vectors" was the concept of two forces acting as a single force on an object. Another was that like complex numbers, ordered n-tuples could be added.

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

(This, along with "scalar" multiplication; $(a_1, a_2, \ldots, a_n)x = (a_1x, a_2x, \ldots, a_nx)$ yields many useful abstract results.)

But mathematics generally doesn't develop logically. The logical structure is often an afterthought. (For example, Cauchy and Weierstrass formalized calculus some 150 years after its invention by Newton and Leibniz.)

We mention the historical background so that we might note that Vaughan and Szabo's vector geometry put vector addition (along with various other vector concepts) on a firm foundation. Their approach has been to consider vectors as translations, i.e., functions whose domain and range are the set of points, and which map each point a fixed distance in a given (fixed) sense. The addition of vectors turns out to be function composition. Compare figures 1a and 1b.

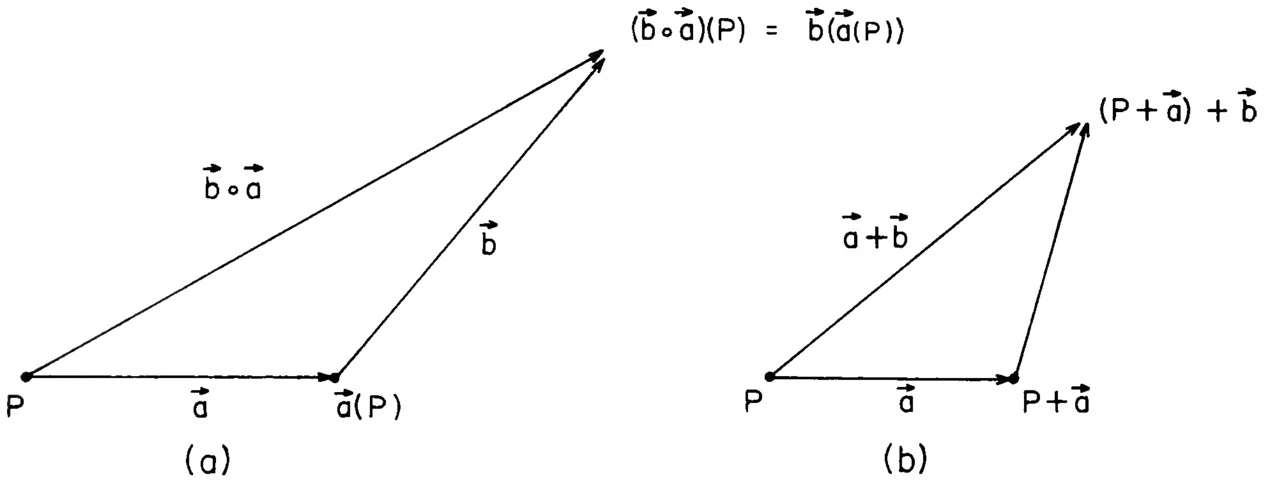


Figure 1

This leads to considerations of how to represent vectors. Since they are functions, one might be tempted to believe this an impossible task. But, on the other hand, the value of each point is obtained identically the same way. Thus, we use an arrow to picture what the vector does to each point in space. One arrow will suffice, and we are free to draw this whenever we choose (hence the notion of "free" vectors).

One of the key concepts in our courses is that of linear dependence. Vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly dependent if and only if there are numbers $x_1, x_2, \ldots, x_n$, not all zero, such that $\vec{a}_1x_1 + \vec{a}_2x_2 + \ldots + \vec{a}_nx_n = \vec{0}$. Vectors are linearly independent if and only if they are not linearly dependent.

For conciseness, suppose there are three linearly dependent vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, for which $x \neq 0$ in the equation

$$\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$$

It is easy to see that

$$\vec{a} = \vec{b}\left(\frac{-y}{x}\right) + \vec{c}\left(\frac{-z}{x}\right)$$

Conversely, suppose

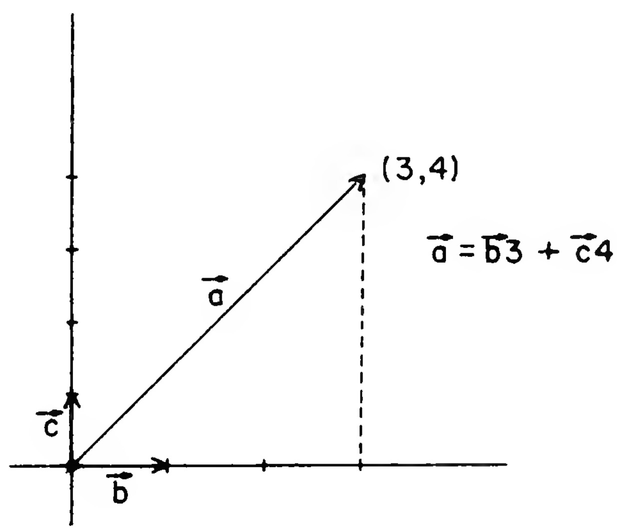
$$\vec{a} = \vec{b}u + \vec{c}v$$

then $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent, since

$$\vec{a}1 + \vec{b}(-u) + \vec{c}(-v) = \vec{0} \text{ and } 1 \neq 0.$$

Now, in particular, let $\vec{b}$ and $\vec{c}$ be linearly <u>independent.</u>

Here is a well-known model for the above discussion which proved to be very useful in strengthening these concepts. Draw the standard Cartesian axes. Let a picture of $\vec{b}$ start at the origin and end at the point B(1, 0). Let a picture of $\vec{c}$ start at O(0, 0) and end at C(0, 1). Let $\vec{a} = \vec{b}u + \vec{c}v$ with $u = 3$, $v = 4$. Do you recognize the numbers 3 and 4 as coming up in another context? Of course! (3, 4) is the graph of the image of O when you apply $\vec{a}$.

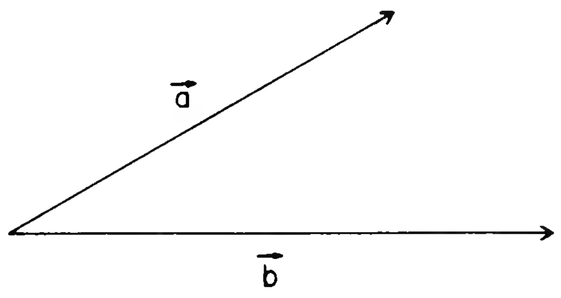


This model is extremely useful in conveying the notion that $\vec{a}$ depends on $\vec{b}$ and $\vec{c}$. Many problems can be posed analytically. If $\vec{d}$ maps (5, 1) onto (3, 7) (remember--it's a translation!) find (4, 3) + $\vec{d}$, (0, 0) + $\vec{d}$, and (2, 1) + $\vec{d}$. Find $\vec{e} = \vec{b}x + \vec{c}y$, if (2, 1) + $\vec{e}$ = (3, 8). By now, the connection of all this to the work done in algebra on points and slopes should be making itself obvious to the reader. Describe $\vec{f}$ which is not dependent on $\vec{b}$ and $\vec{c}$. Suppose $\vec{g} = \vec{a}5 + \vec{c}3$. Find the image of the origin under the mapping $\vec{g}$. Express $\vec{g}$ as $\vec{b}x + \vec{c}y$.

A direct analogue of this work is the three-dimensional Cartesian coordinate system. The class should realize that $\vec{f}$(above) should map points on the Cartesian plane into points not on the plane. One usually chooses $\vec{f}$ to have its picture perpendicular to $\vec{b}$ and $\vec{c}$ to generate 3-space coordinates. Note that this is not a necessary condition.

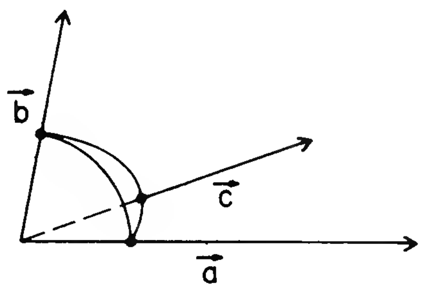
Here then is the physical limitation of our thinking. We can't visualize four independent vectors, since each vector in space appears to be a linear combination of $\vec{b}$, $\vec{c}$, and $\vec{f}$. Later in the course we formalize this intuition.

Here are two models to illustrate 3-space ideas. Draw $\vec{a}$ and $\vec{b}$ on the blackboard as follows.



Use your window opener, blackboard pointer, meter stick, or any other such object to represent $\vec{c}$ which does not depend on $\vec{a}$ and $\vec{b}$.

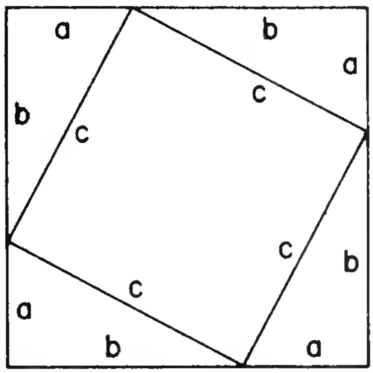
To represent four independent vectors, draw the following figure.



Now, use the window opener to represent a fourth vector, not dependent on these three.

All of this work proves very useful when we get to parametric equations of lines and planes. It also foreshadows the introduction of coordinates which comes later in the course.

The Pythagorean Theorem doesn't come until relatively late in our course. (Nor does length, for that matter.) But we found no harm in talking about length and the Pythagorean theorem long before we could do so formally. Since we did not have congruence, we used the area argument as an intuitive "proof" which would help justify the truth of the Pythagorean Theorem.



Since the total area of four triangles is $4(\frac{1}{2}ab)$, and the inner square, c^2, it follows that $c^2 + 2ab = (a + b)^2$, or that $a^2 + b^2 = c^2$.

There were two reasons for introducing this. One is that in the section on applications, it is a useful numerical tool. More important, however, is the fact that this gives the students a significant theorem which they eventually prove by vector methods. The beauty of the vector approach lies partly in new proofs of old theorems. The students should be made aware of some of this beauty.

Many books have claimed that the theory of real numbers could be developed in a step-by-step postulate-theorem approach. The Beberman-Vaughan series is probably the first to carry out <u>all</u> the details. Similarly, many authors have contended that Euclidean Geometry could be formalized by vector methods. (Some authors even published "proofs" of geometry theorems using vector techniques--not admitting that their vector structure was postulated on the basis of Euclidean Geometry.) But again, the UICSM vector geometry course has carried out the details of this task. It turns out to be a remarkably exciting and teachable approach to geometry.

A. Holmes

Newsletter 20

A "Little" Number System

Now and then each of us has an hour or so to use in front of a class or a group and do not want to go on with regular text material. This article may help you figure out a way of using the time to your own benefit as well as that of your listeners.

How many of you know how to do arithmetic with whole numbers? You know, adding, subtracting, multiplying, dividing and so on. For example, I'm sure you know that

$$2 + 2 = 4$$

and

$$2 + 1 = 3$$

and

$$3 + 3 = 1.$$

What? You didn't know that? Well, how about

$$2 \times 1 = 2$$

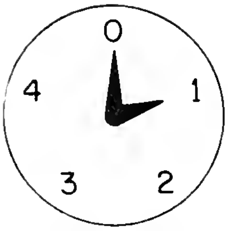
and

$$3 \times 3 = 4$$

What? You didn't know that either? I know your trouble. You have been thinking about arithmetic with whole numbers where there is no end to the list of whole numbers. Today, we are going to take a look at a simplified arithmetic.

Use your imagination now and imagine that the only numbers in the world are 0, 1, 2, 3, and 4. Also, imagine that there is no such thing as subtraction or division — only addition and multiplication.

The students at Zabranchburg High have to do this because their school clock looks like this!



For example, suppose it is now 1 o'clock. What time will it be in 2 "hours"? (3 o'clock) That is,

$$1 + 2 = 3$$

How about 2 hours after 3 o'clock? (0 o'clock)

$$3 + 2 = ?$$

4 hours after 2 o'clock?

$$2 + 4 = ?$$

So we won't have to bother figuring by the clock, let's make a table of all possible problems like the ones we have been doing. (We might have to use the clock in making the table, but after the table is completed, we can forget the clock.)

+	0	1	2	3	4
0					
1					
2					
3					
4					

Here is how the completed table should look:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

I told you they have multiplication at Z.H.S., so let's see how their multiplication works. Suppose school starts at 0 o'clock and periods are 2 hours long. What time is it at the end of 3 periods? This is like saying what time is it 2 × 3 hours after 0 o'clock? (1 o'clock)

Now, let's make a multiplication table.

×	0	1	2	3	4
0					
1					
2					
3					
4					

Here is how the completed table should look.

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

[Practice using tables. E.g. $3 + 4 = ?$, $4 + 2 = ?$, $3 \times 4 = ?$, $0 \times 3 = ?$]

They even do algebra at Z.H.S. using only the numbers 0, 1, 2, 3, and 4 and doing only addition or multiplication.

Let's solve some equations as they would.

$x + 1 = 4$	[Using the addition table we see that <u>3</u> + 1 = 4. So, 3 is the root.]
$x + 4 = 1$	[2 is the root]
$3x = 2$	[Using the multiplication table we see that 3 × <u>4</u> = 2. So, 4 is the root.]
$2x = 4$	[2 is the root]
$2x + 1 = 3$	[1 is the root]
$2x + 3 = 1$	[4 is the root]
$3x + 1 = 2x + 4$	[3 is the root]
$x + 4 = 3x + 2$	[1 is the root]
$x + 4 = 3x + 1$	[4 is the root]

Caution. If you make up more equations remember that 0, 1, 2, 3, 4 are the only numbers allowed as coefficients as well as roots.

How many know what a quadratic equation is?

<u>In your school</u> (not Z.H.S.), what would the roots of this equation be?

$$(x + 3)(x - 7) = 0$$

To find out, you would probably proceed something like this:

$$x + 3 = 0 \text{ or } x - 7 = 0$$

So the roots are −3 and 7.

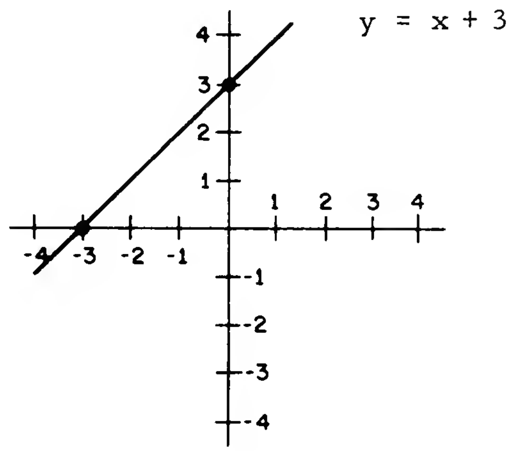
How about $x^2 + 5x + 6 = 0$?

$(\quad)(\quad)$ etc.

Now, let's try some like they do at Z. H. S.

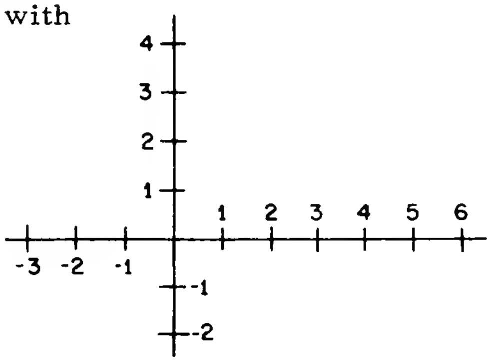
$x^2 + 3x + 2 = 0$	
$(x + 2)(x + 1) = 0$	[3 and 4 are the roots left to reader to verify]
$x^2 + x + 4 = 0$	[2 is the root]
$x^2 + x + 3 = 0$	[1 and 3 are the roots]
$x^2 + 4x + 3 = 0$	[left to reader as exercise]

Do you know how to graph an equation? How about this one?

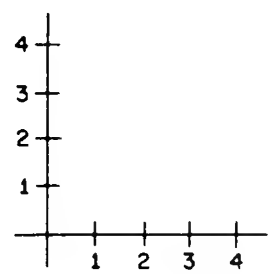


But, what happens at Z. H. S. ? Graph $y = x$ as they would at Z. H. S.

[Start with

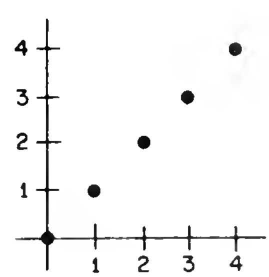


but back off to



because we only have 0, 1, 2, 3, and 4.

Then graph the ordered pairs which satisfy the equation.]



Graph:

$y = x + 2$ [The ordered pairs are (0, 2), (1, 3), (2, 4) (3, 0), and (4, 1).]

$y = 2x$

$y = 2x + 1$

Now, let's try to solve a system of equations, Z. H. S. style.

$$\begin{cases} y = 2x + 1 \\ y = x + 3 \end{cases}$$ (2, 0) satisfies this system.

$$\begin{cases} x + 2y = 3 \\ 2x + 3y = 4 \end{cases}$$ (4, 2) satisfies this system.

This "little" number system which we have been investigating is called:

arithmetic modulo 5

usually abbreviated to:

arithmetic mod 5

We used only 5 whole numbers, beginning with 0.

Why don't you investigate

arithmetic mod 3 ?

or even mod 2

or mod 7

But beware of mod 6 and mod 8 — they are different!

M. Wolfe

A New Teacher Guide for a New Course

Since 1964 the UICSM Project has been constructing a new course on the arithmetic of the rational numbers, i.e., "fractions." The course has many unusual features, some of which are discussed in an article in the March, 1967, Mathematics Teacher by Braunfeld, et al. These new features are accompanied by innovations in the materials for teachers. The materials in the new course's Teacher Guide are presented in a form which grew out of the usual UICSM Teacher Commentary. The Teacher Guide discusses the student text and gives planned classroom activities which will help teachers make the best use of the student text.

The UICSM Project consistently emphasizes the need for special training for teachers of its new materials. The Teacher Guide is an on-the-spot training program. It first divides the student text into lesson blocks. The first page of each Lesson is like that shown in Figure 1. Each Lesson presents a complete development of at least one topic from the basic student text. There are about fifteen basic text pages per lesson. In addition to the student pages, each Lesson includes notes and activities which apply the educational principles which have been selected for specific emphasis in that Lesson.

As they appear in the Teacher Guide, the student pages are annotated, including answers, and provide the teacher with pertinent notes to use while teaching. All the other materials listed in Figure 1 are collected together following the student pages for the Lesson. The Guide also makes immediately available the supplementary pages [with annotations] for the Lesson. In the student text the supplementary pages are in the back of the book.

LESSON 1

BOOK 1 CHAPTER 1

Pages 1 through 14

Teacher Guide Contents

Figure 1

On-the-spot training for busy teachers must be brief yet to the point. Consider the first thing that appears in the Lesson: the outline of the content of the student materials. It spotlights what the student should learn in the Lesson. Here is a sample from one of the early Lessons:

CONTENT OF STUDENT TEXT

Outline

	student page
Labeling sticks according to length	2
Comparing lengths of sticks and drawing sticks from given lengths	5
Using stretching machines on sticks	8
Establishing the relationship between input, machine, and output	10

Figure 2

The next part of the Lesson is the annotated student pages. On each page are references to relevant materials in the Guide, answers to exercises, and succinct notes on the mathematics and pedagogy for the page. These notes point out significant interrelationships between exercises, mathematical points to be emphasized, purposes of exercise sets, format considerations, and places where students may need extra help. They also suggest how to help. A sample annotated page is found in Figure 3.

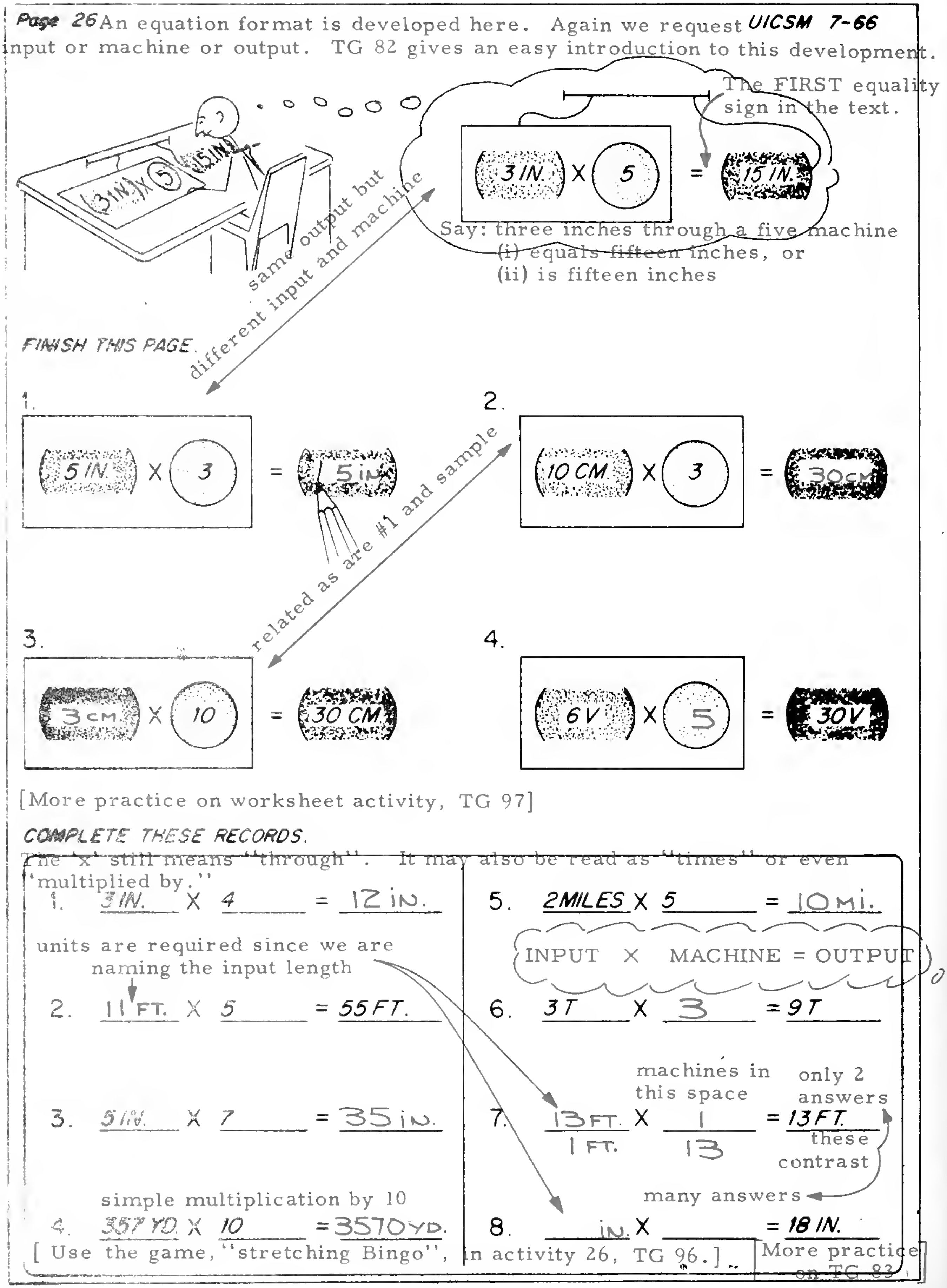

Page 26 An equation format is developed here. Again we request input or machine or output. TG 82 gives an easy introduction to this development.
UICSM 7-66
The FIRST equality sign in the text.
3 IN. X 5 = 15 IN.
Say: three inches through a five machine
(i) equals fifteen inches, or
(ii) is fifteen inches
same output but different input and machine
FINISH THIS PAGE.
1. 5 IN. X 3 = 5 in.
2. 10 CM. X 3 = 30 cm
related as are #1 and sample
3. 3 cm. X 10 = 30 CM.
4. 6 V X 5 = 30 V
[More practice on worksheet activity, TG 97]
COMPLETE THESE RECORDS.
The 'x' still means "through". It may also be read as "times" or even 'multiplied by.''
1. 3 IN. X 4 = 12 in.
units are required since we are naming the input length
2. 11 FT. X 5 = 55 FT.
3. 5 IN. X 7 = 35 in.
simple multiplication by 10
4. 357 YD. X 10 = 3570 yd.
5. 2 MILES X 5 = 10 mi.
INPUT X MACHINE = OUTPUT
6. 3 T X 3 = 9 T
machines in this space
only 2 answers
7. 13 FT. / 1 FT. X 1 / 13 = 13 FT.
these contrast
many answers
8. in. X = 18 IN.
[Use the game, "stretching Bingo", in activity 26, TG 96.]
More practice on TG 83

Figure 3

There are about six supplementary pages for each Lesson, the last of which is designed to serve as a twelve-item quiz for the Lesson. The Guide explains how to use the quiz as a useful instructional device or as a measure of each student's progress and understanding. The quiz items have been carefully constructed to test important concepts and required skills. Attention has been given to the difficulties of the items so that average scores will be fairly high. This should help students overcome a long history of repeated failures. Cumulative tests covering six or seven Lessons are also included in the Guide.

The Notes on Content summarize and extend the annotated comments which appear on the student pages. They deal with the mathematics and the various presentation devices in the student text. They are similar to the familiar UICSM commentary, except that extensive pedagogical suggestions have been omitted — the Teacher Guide groups pedagogical suggestions in its section on Classroom Presentation.

Each Classroom Presentation section has three parts — a Check List of Teaching Precepts, a Day-by-Day Plan, and a set of Activities. First, the Check List gives guidelines which have been gleaned from current as well as classic research reports, articles, and books. The precepts attempt to help teachers avoid classroom practices which are educationally unsound. Figure 4 . shows precepts from the first few Lessons. New precepts are stated in each Lesson. Previously mentioned precepts are occasionally reviewed.

Some Teaching Precepts

1. Make extensive use of student-centered activities rather than lectures.
2. Change activities at least every twenty minutes.
3. Alternate between quiet and lively activities.
4. Keep several special activities ready to us if interest lags.
5. Make assignments on a level at which students are sure to succeed.
6. Do not use low grades to punish or threaten students.
7. Adapt class assignments to fit the individual strengths of each student.
8. Encourage students to suggest and work on assignments of their own making.

Figure 4

In the section which includes the Day-by-Day Plan and Activities, teachers are given help in applying these precepts in their own classrooms. The Plan gives the teacher a practical schedule with suggestions for conducting the class through the entire Lesson. The schedule and activities help teachers treat the course with a light touch.

Each activity is referred to in the Day-by-Day Plan and is described fully in the Activities section. The materials which a teacher must prepare in advance are listed; practical step-by-step instructions for making each activity effective are given; and, forms to be used for worksheets and projectuals are included.

Since it is best to vary activities frequently, there are more activities than will be used in any one class during the year. Each class will have favorites which can be used and reused as needed. Because of the abundance of activities, ways to select among the activities are given. Activities include such things as "Take-Five" [5-minute] quizzes, demonstrations, dramatizations, puzzles and games, chalkboard work, and exercises in mental arithmetic. Figure 5 gives a sample of one of the worksheet forms teachers may use to make copies for their classes.

The Activities section is the heart of the Guide's teacher-training work. It is through the use of well-planned activities that a teacher can sustain student interest while skills are being systematically developed.

At present, the Teacher Guide is still expanding as new ideas arise and experiences enlighten. Because of a new idea which occurred to us only recently, the Guide will include bulletin board posters to illustrate important concepts in an interesting way. A sample which is self-explanatory is found in Figure 6.

O.R. Brown, Jr.

J.R. Hoffmann, Jr.

Worksheet

name ______________________

date ______________________

Centimeter Maze

First, trace the shortest route from START to FINISH. Second, measure this route in centimeters; write the length in the center square.

1.

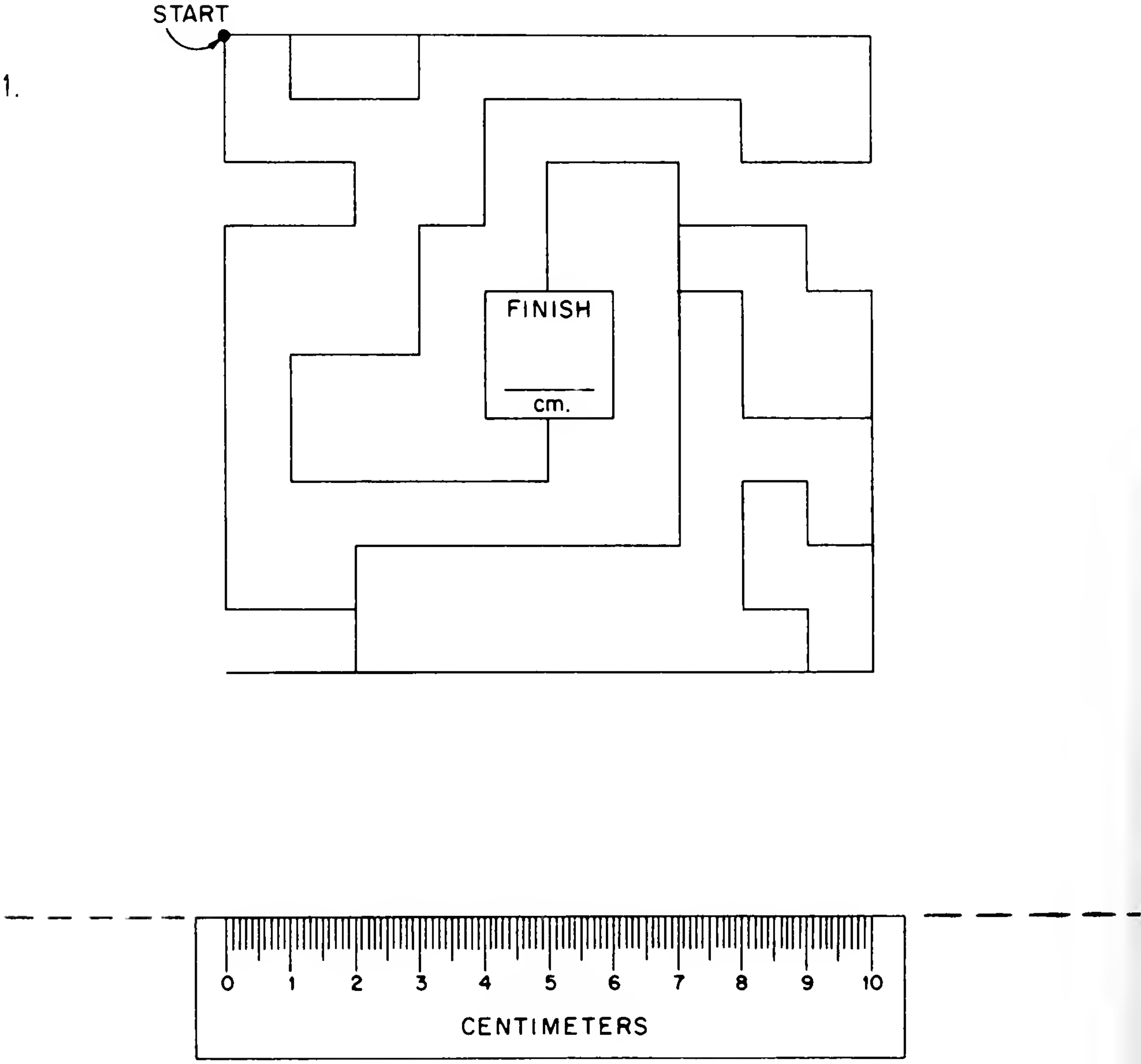
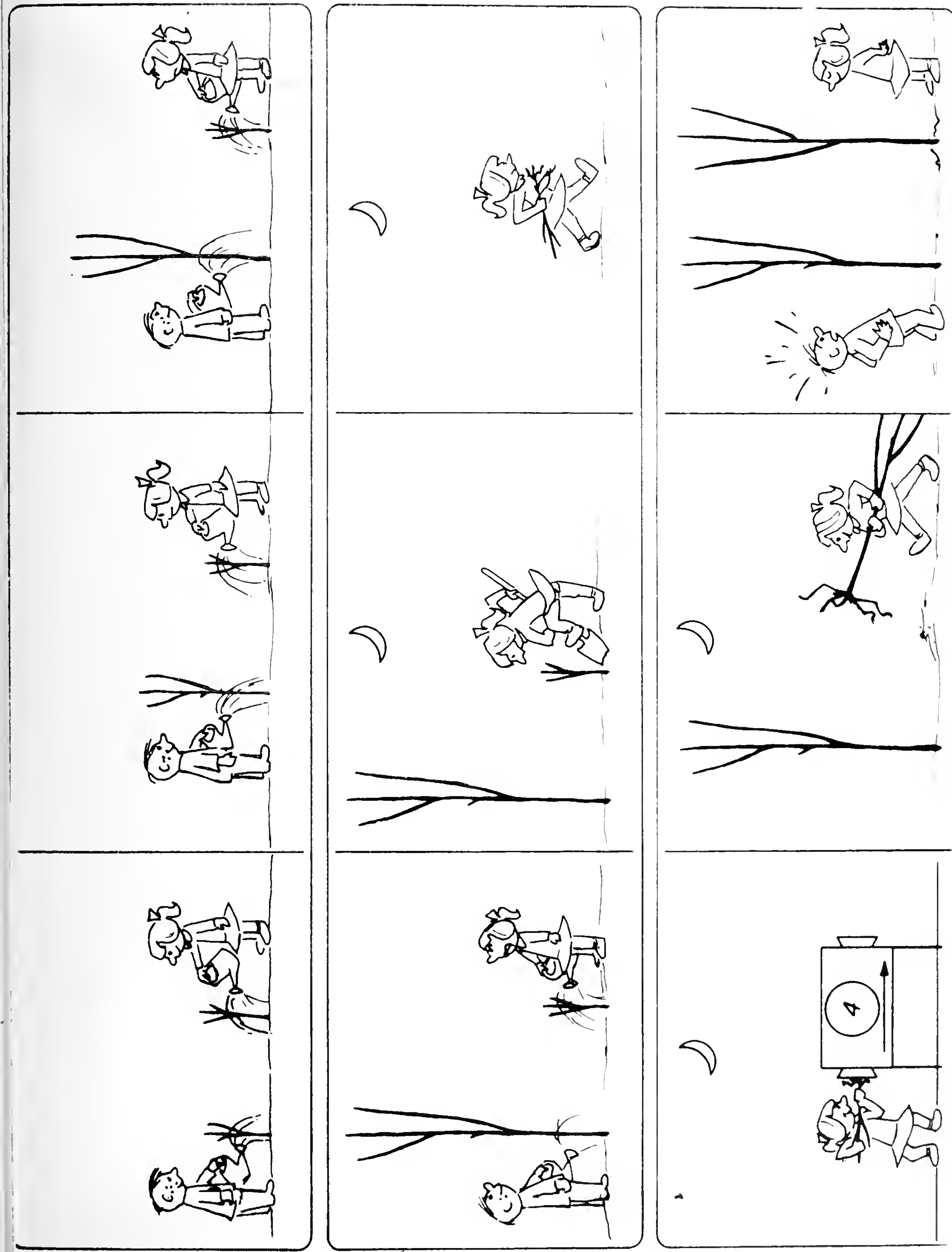


Figure 5



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61801

Newsletter Editor· Clifford Tremblay

TABLE OF CONTENTS

Editor's Page

As we note the approach of winter, it occurs to us that we have not kept up our correspondence with our growing UICSM Newsletter mailing list. Much of the work currently in progress at UICSM is a final revision of courses on which we have reported at length in the last two years. This makes it increasingly difficult for your editor to needle people into producing articles that make suitable Newsletter material. In spite of these difficulties, we have collected three articles that we believe to be worth the time that even our busiest readers would have to devote to reading them.

In the "pedagogy department", we have two articles that are devoted to classroom activities. The first article contains some samples of activities that the seventh grade project team is recommending for use in conjunction with their course. The second article shows what youngsters may do with a sheet of tracing paper and a modicum of knowledge about angles.

Finally, in the "For Your Information" department, we have the text of a speech delivered by a staff member, Mr. Russell E Zwoyer, at the Illinois Council of Teachers of Mathematics meeting in Edwardsville, Ill. This article should serve to keep our readers abreast of current events in one area of endeavor at UICSM.

While we do not usually "plug" commercially available materials, we regard a recent production of Eye Gate House as an important addition to each school's mathematics library. One of our staff members, Jo Phillips, is author of a series of filmstrips that are quite reminiscent of algebra as it is taught in UICSM texts. For complete information, we would recommend writing to:

Eye Gate House, Inc.
Jamaica, N.Y. 11435

This office will soon have approximately 300 copies of the Self-Instructional Solid Geometry text available for sale. Copies of this text may be ordered from:

Univ. of Illinois Committee on School Mathematics
1210 W. Springfield
Urbana, Illinois 61801

SUMMER CONFERENCE

The UICSM will conduct a conference in the summer of 1968 for supervisors or potential supervisors of 7th and 8th grade mathematics. The purpose of the 7-week conference [which carries an NSF stipend, allowances, and 6 semester hours of graduate credit] is to acquaint supervisors with the content and pedagogy of the UICSM courses for 7th and 8th grade low achievers. Each supervisor will be required to teach both courses in 1968-69 in his own school system and will be visited throughout the year by UICSM staff. This year of experience should enable the supervisor to become expert in our program and thus be able to train teachers in his system in subsequent years.

School systems interested in joining this program should write to

Professor Max Beberman
Director, UICSM
1210 West Springfield
Urbana, Illinois 61801

Priority will be given to those systems which have substantial numbers of low achieving junior high students. There will be a limited number of places in the training conference for nonstipend participants — that is, for those people from school systems with just a few classes of this type of student.

Seventh Grade Course
Teacher's Guide-Activities

In preparing the seventh grade UICSM course for publication, the writers have devised a series of activities to accompany the course. Since the students for whom the course is intended are not academically inclined, it is essential that the teacher have a ready-made bag-of-tricks that will foster involvement in the learning experience. While the use of these activity exercises is not required for understanding the course, most teachers who try them will soon consider them an indispensable part of the curriculum.

This article presents a variety of activities for teaching the first chapter of the course. The "masters" for making transparencies and worksheets are displayed at one-fourth size on the last page of this article.

Activity A

Classifying sticks according to length.

Purpose:

Determine congruence of line segments by superposition.

Preparation:

Write, on the board, the names of the characters in the story on pupil pages 1, 2, and 3:

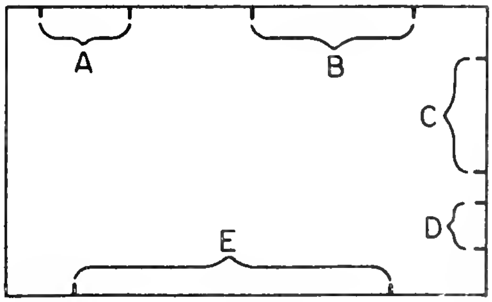
Mr. Davis
Jim
Narrator

Have one 3 by 5 card for each member of the class.

Procedure:

Choose one student for each role; write his name beside the name of the character he is to play in the story. Have the students read their parts, stopping long enough when they come to a blank for the rest of the class to decide how to fill it in.

When the page is completed, ask for several volunteers to read the last frame of their charts. Look for possible disagreements; then distribute the 3 by 5 cards and let the class settle the disagreement. Do not act as the "final authority". Suggest that the lengths of the sticks be marked along the edge of the 3 by 5 card as illustrated below.



* * *

Activity B

Introducing the term 'centimeter'.

Purpose:

Develop ability to estimate lengths in centimeter units.

Preparation:

Make copies of the worksheet for all the students. [Use Master 1-B]

Procedure:

Distribute the worksheets and set a laboratory atmosphere. Collect the papers and post some of the better ones on a bulletin board.

* * *

Further suggestions for activities:

Devise a game for a small group to play with a pair of dice, or the like, of different colors. One die tells the player how many centimeters he can move, the other how many inches he can move. The object would be to see who can complete a lap around a race track first.

* * *

Activity C

Using comparison terminology

Purpose:

Introduce equality and inequality relations.

Preparation:

None.

Procedure:

Have each student draw one stick either 1 cm., 2 cm., 3 cm., 4 cm., or 5 cm. long on a sheet of paper. Read the following questions aloud.

Question	Alternate Questions	
1. Is the stick you drew longer than 2 cm.?	2 cm.?	4 cm.?
2. Is the stick you drew shorter than 4 cm.?	5 cm.?	2 cm.?
3. Is the stick you drew the same length as 4 cm.?	3 cm.?	5 cm.?
4. Is the stick you drew longer than 1 cm.?	1 cm.?	3 cm.?
5. Is the stick you drew shorter than 3 cm.?	4 cm.?	3 cm.?
6. Is the stick you drew the same length as 2 cm.?	2 cm.?	1 cm.?

Each student should look at the stick he has drawn. If the answer to the question is "yes", he draws a slash through his stick. If the answer is "no", he doesn't do anything. The winner is the one with the most slashes. [The winning order of the possible choices is given following each set of questions.]

Order of the most slashes:

Question	Alternate Questions	
First: 5 cm. (TIE with Second)	5 cm. (TIE with next)	3 cm.
Second: 1 cm. (TIE with First)	1 cm. (TIE with above)	4 cm. (TIE with next)
Third: 4 cm. (TIE with Fourth)	4 cm.	2 cm. (TIE with above)
Fourth: 3 cm. (TIE with Third)	2 cm.	5 cm. (TIE with next)
Fifth: 2 cm.	3 cm.	1 cm. (TIE with above)

Note: Be sure to stop before interest lags. Save it for another day.

* * *

Activity E

Getting acquainted with the new machines.

Purpose:

Develop skill in making three-dimensional drawings.

Preparation:

Colored chalk

Procedure:

Have volunteers draw some views of a stretching machine on the chalkboard — for example, top view, bottom view, side view, front view, back view, and perhaps a perspective view. Permit others to do the drawings at their seats.

* * *

Activity F

Finding uses for stretching machines

Purpose:

Introduce function idea

Preparation:

Select a suitable student on the day prior to using this activity. Ask him to prepare a stretching machine sales pitch to present to the class.

Procedure:

Tell the class to pretend that the student you have selected is a salesman from the factory that makes stretching machines. As the class listens to the "salesman", have each one pick a machine he would like to buy and write (or picture) a use he would have for it. Bring up questions about cost, discount, weight, size, delivery date, freight costs, etc. Have each student compute the cost of the machine he has chosen.

* * *

Activity G

Getting practice in visualizing what stretching machines do.

Purpose:

Understanding the domain and range of a function.

Further suggestions for activities:

1. Use the same projectual to demonstrate that the rulings only work when the sticks are parallel.

2. Have students make a page of exercises on a Ditto Master using selected pages from the text as a guide.

* * *

Activity I

Quiz over unit of work.

Preparation:

Make copies from the master provided.

Procedure:

Work through the quiz with the whole class, one problem at a time, to be sure each student understands each instruction. Ask students who understand to help those who do not.

Note: The students should have little difficulty with this quiz. It will encourage those students who usually fail and help break the "failure-hibit".

* * *

Activity J

Working with labels only.

Purpose:

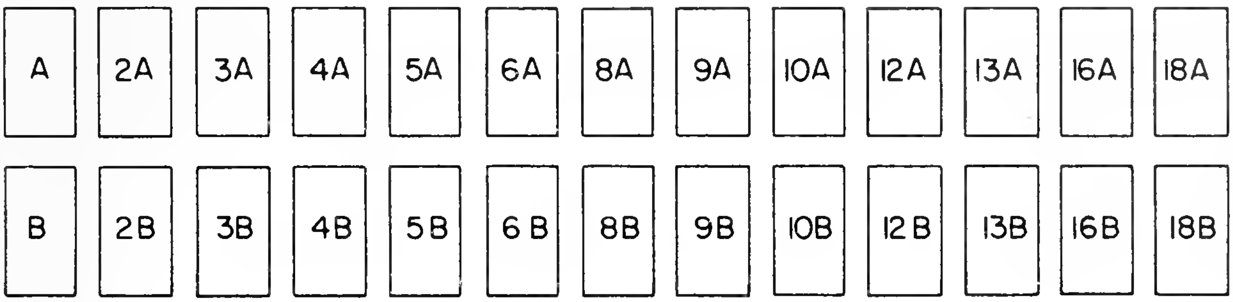
Reviewing basic multiplication facts.

Identifying members of the domain and range of a function.

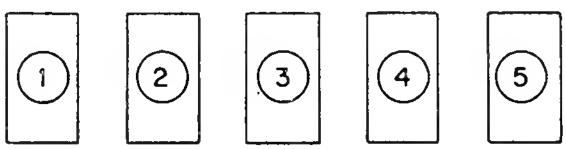
Preparation:

Make the 3 by 5 cards below - stick labels and machine labels - so that each student has two stick-label cards and one machine-label card. [For a class of 35, make three each of the stick labels, seven each of the machine-labels.] Student help could be of value in getting this job done.

Stick labels:

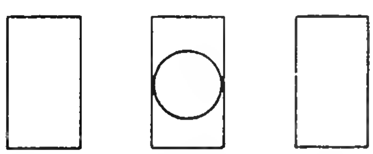


Stretching machine labels:



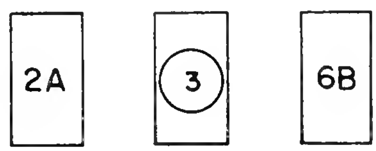
Procedure:

Near the end of the class period, distribute the cards without regard to correct combinations of input, machine, and output. Ask each student to arrange his cards on his desk like this:



As you walk through the class, look for a correct combination of cards; that is, see if anyone has the correct output for the given input and machine. If there is one such combination, show it to the class. Now, pick a student who

has an incorrect combination. Have him tell you which card he would trade off in order to get a correct combination. Ask if there is anyone who will trade with him. Assign the class the job of trading cards among themselves until everyone has a correct combination. Have them do their trading outside of class. Since not everyone will be successful, have some blank cards ready the next day to "trade" with those who could not get a correct combination. Have each student tell what combination of sticks and machine he has. Let the class decide if he is correct or not. Be sure to point out incorrect combinations such as the following:



[A's don't change to B's]

* * *

Further suggestions for activities:

Hold up two of the cards described in Activity J, and ask who has a card to make a correct combination. Stand the three on the chalk tray for everyone to see. Pull some tricks:

Input	Machine	Output	
9 A	?	3 A	(Can't do)
?	3	7 B	(Can't do)
etc.			

Give a demonstration to help students get the "feel" of what machines do. Here is how you might present it: "Now suppose that I have a 6 stretching machine. I need a stick to put in this 6 machine so that the output will just fit across the room. Someone come up and help me hold the input stick."

(Have a volunteer pretend to hold the end of the stick while you hold the other end. Let another volunteer check to see if the "stick" is the right length.)

* * *

Activity L

Matching history cards with the outputs.

Purpose:

Horizontal multiplication notation.

Introduction to equation format.

Preparation:

Select two students to hand-copy Master 1-L onto a duplicating master, instead of using a direct process. If colored masters are available, have them use color for the tags. If possible, show them how to run the duplicating machine and let them make enough copies for the class.

Procedure:

Let two assistants distribute the worksheets at the end of the class period. On the following day, have them collect, mark, grade, and return the worksheets to the students.

* * *

Further suggestion for activities:

Let the pupils work in pairs when there is a 2-character dialogue in the text. They should take turns reading the words of the characters, filling in the blanks as they read. An assistant might help by going around the classroom at the same time you do and checking the answers.

* * *

Activity N

Solving stretching machine problems.

Purpose:

Solving informal equations.

Strengthening multiplication skills.

Preparation:

None

Procedure:

Have students make one or more copies of the "IMO" chart shown below. 'IMO' stands for 'input, machine, and output.'

I	M	O

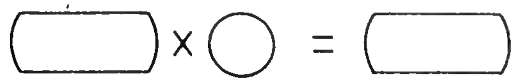
Let each one put the digits for 1 through 9 in the boxes, any way he chooses. Make up and read questions whose answers use the numbers from 1 through 9, e.g.:

(For input column) I wish to get a 10-cm. stick from a 2-machine. How many centimeters in the input? Answer: 5 under I.

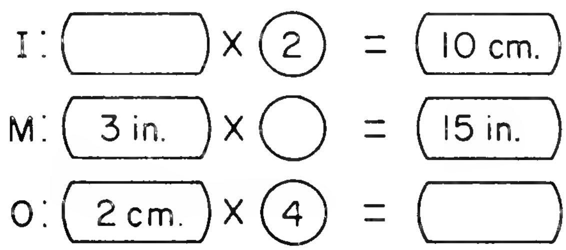
(For machine column) I wish to stretch a 3-inch stick to a 15-inch stick. What machine should I use? Answer: 5 under M

(For output column) I've got a 2-cm. stick. I put it into a 4-machine. How many centimeters in the output? Answer: 8 under O

First student getting three in a row, column, or diagonal, yells "IMO". Start over again. Record the statements in this form:



Have each student devise a series of problems such that all possibilities are covered. In problems for the nine possible answers, there should be nine for Input, nine more for Machine, and nine more for Output. Have them write the problems this way:



Assign as homework. Collect and use the next day in class to play IMO again.

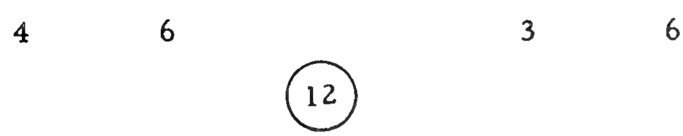
* * *

Further suggestions for activities:

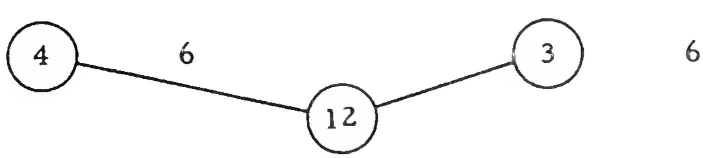
For practice in factoring make up a worksheet with problems such as the following:

Choose the numbers whose product is the given number. Draw a lasso around your choice.

Sample:



Solution:



For practice in addition, the same type of exercises may be used. Change 'product' to 'sum'.

* * *

Give students 100 seconds to use addition, subtraction, multiplication, or division on some number to get the final total of 100 in as many ways as they can. Here are some ways, for example, if you start with 60:

1) Add 40
2) Subtract 10, multiply by 2
3) Add 10, add 30
4) Divide by 6, multiply by 10
5) Multiply by 0, add 100

Use teams. Abbreviate the operations: +, -, ×, ÷.

J. Hoffmann

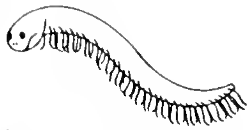
Book I Chapter I — Master 1-B

Worksheet

Name ______________

Date ______________

CENTIMETERS



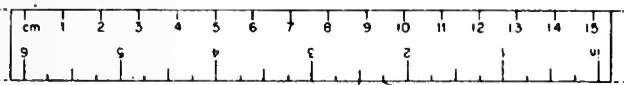
Centimeters are not bugs!

Did you ever hear about centipedes?

What does the CENTI part mean?

Centimeters are used in measuring lengths. On this page there is a paper ruler with centimeters marked on one side and inches on the other. By folding this worksheet along the dotted lines, you can use this ruler to measure things.

FOLD HERE FOR CENTIMETERS

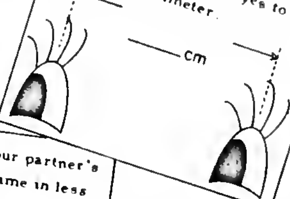


FOLD HERE FOR INCHES

Choose a partner and do these problems.

Is your partner's shoe more than 8 cm. wide? ______ yes/no

Have your partner find out the length between your eyes to the nearest centimeter.



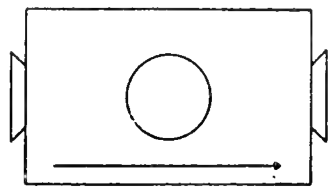
Try to write your partner's first and last name in less than 4 cm.

Can your partner stick his tongue out more than 4 cm.? ______ yes/no

A 4-inch stick is about ______ centimeters long

Mark a one-meter length. How long is it in centimeters? ______

Book I Chapter I — Master 1-G/1



Book I Chapter I — Master 1-G/2

MASTER 1 - L

Name ______________

Date ______________

Match the history tags and the output lengths.

2 ft × 6	6 ft
4 ft × 4	8 ft
3 ft × 5	9 ft
8 ft × 1	12 ft
3 ft × 3	15 ft
1 ft × 6	16 ft

Suppose that you get 10¢ for each correct answer. After your paper has been scored, draw a loop around the amount of money you would get.

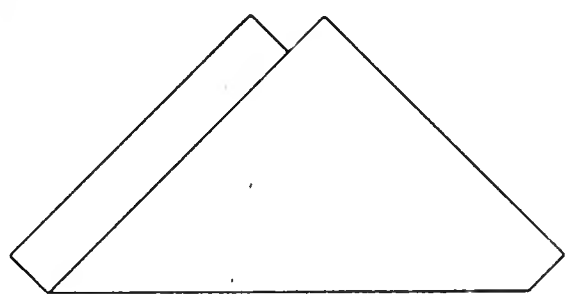
25¢ 5¢ 10¢ 25¢ 5¢

Primitive Protractor

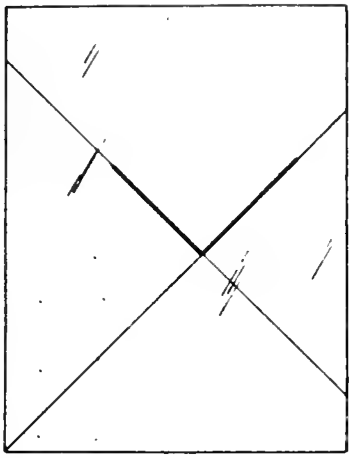
In searching for some way to develop an awareness of the value and use of a protractor, the eighth grade project team found a "laboratory" technique that would appear to have some relevance at many levels. We expect to include this as an optional activity in the eighth grade course for underachievers, but believe that it has a place in an elementary school introduction to geometric notions and tools.

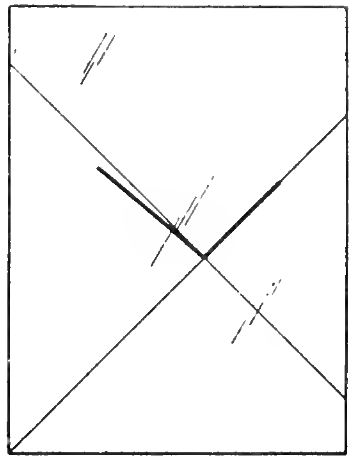
To help the students understand the ideas upon which the use of a protractor is based, a useful activity may be to construct a very simple instrument for measuring angles. Each pupil will need tracing paper and scissors.

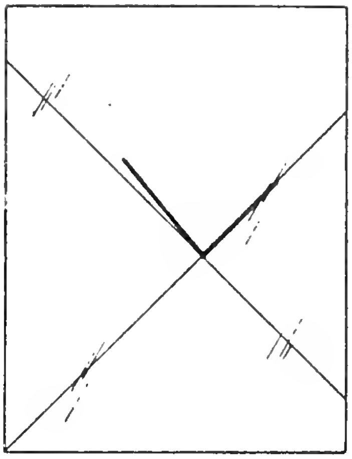
1) Take a piece of tracing paper and fold it once, just any old way (preferably not straight across).



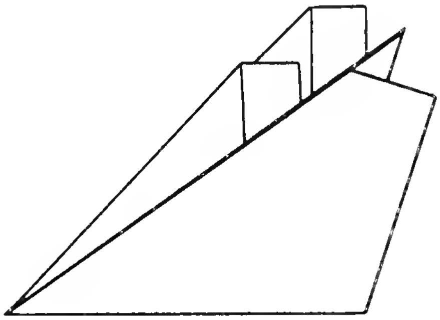
2) Fold it again, but this time, make the first fold fall along itself. If you open the paper now, you see four right angles. The vertex of each angle is the place where the folds intersect (make a cross, in this case). So far, you have a tool to use for telling whether a given angle (a) is a right angle, (b) is greater than a right angle, or (c) is less than a right angle. Your pupils should practice doing this, making sure that the vertex of the angle on the tracing paper is matched with the vertex of the angle they are measuring and that one line of fold in the tracing paper matches one side of the angle.

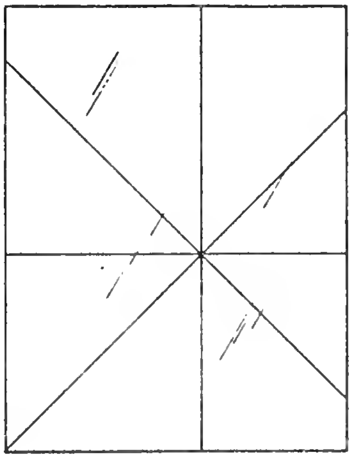




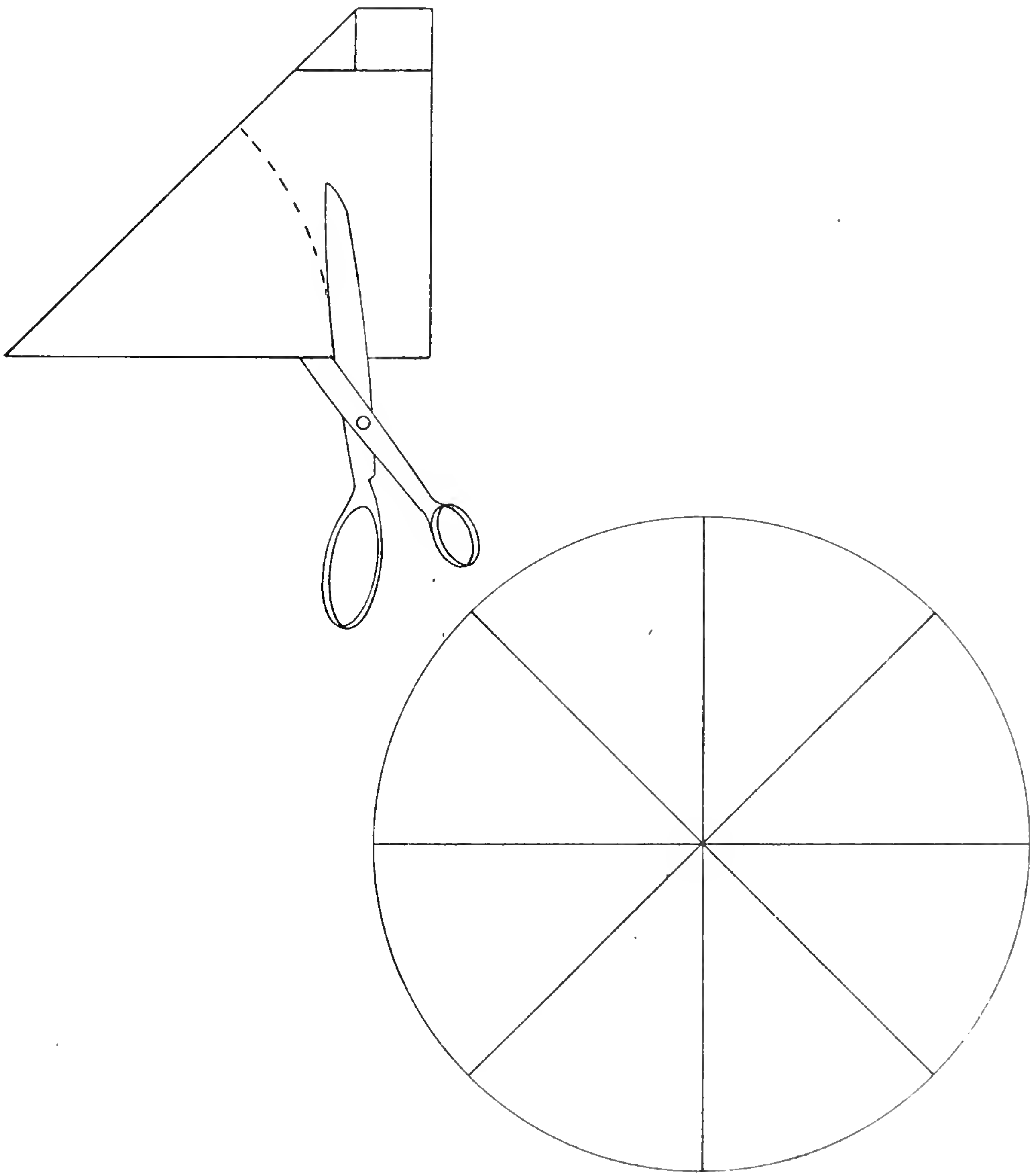


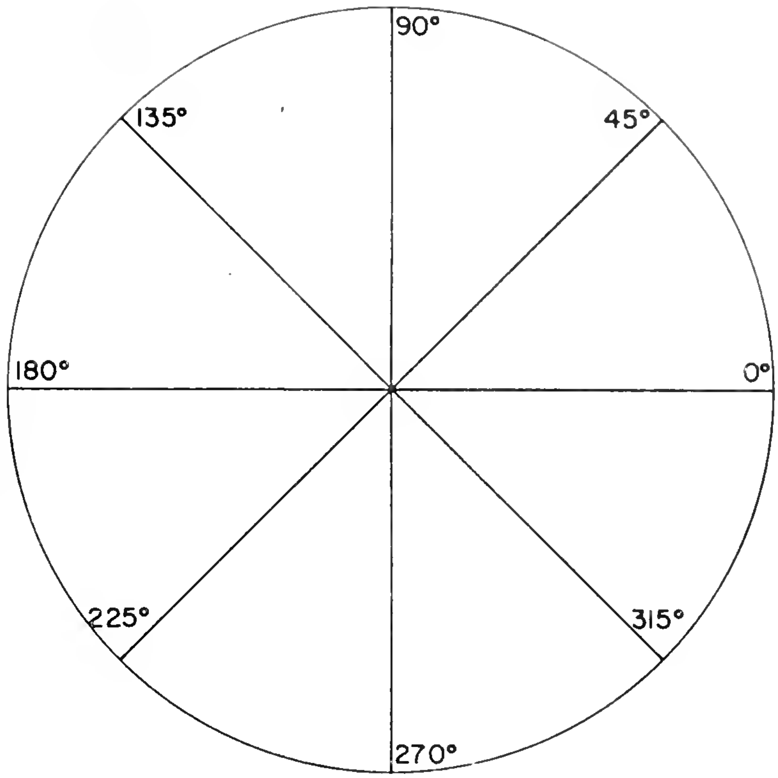
3) Starting where Step 2 left off, fold the paper again, very carefully, so that the old folds fall along each other and the new fold goes right through the point.





The pupils should see that this new fold cuts each 90° angle in half, so if they open the paper now, they will have eight 45° angles. Maybe it would be a good idea to number the folds now. Just to make the thing easier to handle, trim off the jagged edges of the paper.





Discuss the advantages of numbering clockwise vs. numbering counterclockwise for easy reading of angle measures. You do not have to resolve this argument. Most commercial protractors are numbered both ways, and eventually you may want these primitive versions numbered both ways. (Pupils are prone to be confused about which scale to read, and having scaled a protractor for themselves might help clear this up.)

Now the pupils may practice comparing the angles on their protractors with given angles. They can tell whether an angle is a 45° angle, is between 0° and 45° (they might estimate - a little less than half judged to be about 20°, etc.), between 45° and 90°, and so on. They should practice drawing angles which they judge to be about 45° and then test with their protractors to see how close they can come. The paper folding should have given them the "feel" of a 45° angle, and they should train their eyes to recognize angles close to 90°, 45°, 30°, and 60°.

4) To get 30° angles on this hand-made protractor, fold the trimmed-off paper back up to where Step 2 left off (90°) angles. This is a fussy job, but it is possible to fold these 90° angles into three congruent angles. Make a teepee with three sides just alike. Be sure all folds go right through the point. Now, multiples of 30° may be recorded on appropriate folds, and the same kinds of activities suggested previously should be done with these new, more refined angle measures.

See if your pupils can find rays (folds) on their protractors to use for comparing a given angle with a 15° angle. Can they figure out how to draw a 15° angle using this protractor? Can they use this 15° angle as a model for scaling the protractor in 15° intervals all the way around?

When you introduce a commercial protractor for the first time, have the pupils compare it with those they have made. Let them "fool around" with it; encourage them to decide how this commercial protractor should be placed on an angle they are measuring with it, which scale to read, how to use it to draw an angle of stated size, and the like.

J. Phillips

Intuitive Geometry Via Motions

Based on the assumption that some knowledge of geometry is useful in almost all walks of life, it seems not only unfortunate but also unreasonable that in the past two decades we have not given more effort to the development of a curriculum for pre-tenth grade geometry. Since our project is involved in curriculum development we are constantly kept aware of the areas in which change is taking place. This awareness is fostered not only during the professional part of our day, but also during many of the hours which should be classified as leisure time. For example, in a typical introduction to someone we have never met before, it frequently will be mentioned that we write modern math books. The new acquaintance then starts in on such topics as sets of numbers, commutative principles, associative principles, number-numeral distinction, bases other than 10, open sentences, clock arithmetics, etc. etc. We are not complaining about the interest people have in our work, but we notice that the only way the conversation will turn to geometry is for us to initiate it. Most lay people have little to contribute to this topic. A moment's reflection on this phenomenon brings these reasons to mind.

1) Modern math articles in newspapers and periodicals deal with topics in arithmetic and algebra.
2) Young children, those who still report enthusiastically on school activities, report on topics other than geometry.
3) Curriculum innovators have given more time to the development of other topics in mathematics.

Once again there is a movement afoot at the secondary school level toward integrated courses in mathematics. This movement has the obvious advantage that by tieing together the various branches of mathematics, students gain greater understanding. However, there are also certain dangers which must be considered One of these is that integrating a particular branch may result in curtailing some

of the time and effort spent on it. This is not necessarily bad, but it may be. (Whatever happened to solid geometry?)

Another danger, involved in any curriculum change, is the danger of misinterpretation by those responsible for implementing the change. This includes not only those who have the direct responsibility of teaching the new course, but also those who have the responsibility of preparing students to enter such a course. It is entirely possible that an administrator or teacher would get the impression that new mathematics courses emphasize algebra rather than geometry. Obvious conclusion? Beef up topics in their mathematics program which will prepare students for algebra. Where to get the time for this? Easy— de-emphasize topics in geometry. Unfortunately, this reaction to such a proposed change at the secondary level would be disastrous. For any geometry course at the tenth grade level to be meaningful to a student, he must bring to it some feeling and understanding of the concepts which he will be trying to organize deductively. This may be even more important in courses relying on the use of algebra in their organization. The task to be performed by such a teacher or administrator is to establish some reasonable balance between strengthening the students' background for algebra and anticipating the formal geometry through an intuitive approach to the subject.

So far, we have been pressing the point that curriculum innovators must consider all of the ramifications of a new program and make it their business to help avoid confusion and misinterpretation of the program. This should obtain at all levels, not just the particular level at which the curriculum is aimed.

What follows is a brief description of the history and content of an informal geometry course under development by the UICSM. It is part of a two year program for culturally disadvantaged junior high school students. Much of the

excitement about this course lies in the fact that the approach may be used to start acquainting very young students with topics from geometry. They may continue to develop their knowledge as they mature so that they will be ready to organize this knowledge deductively by the time they reach the tenth grade. The course is intended to provide the background necessary for success in a formal treatment of geometry. It acquaints students with some of the terminology and relationships among sets of points and subsets of such sets - facts they will be expected to know when they reach tenth grade. It places a heavy emphasis on:

1) verifying conclusions
2) predicting results
3) giving simple arguments based on experimentation

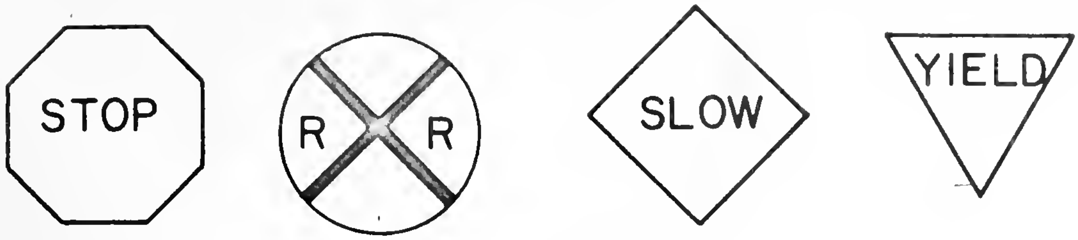
The four years used in developing and refining this course seem to be paying dividends in terms of the quality of the final product. This project was initiated in the 1963-1964 school year when some of our staff members discussed the possibility of developing such a course. The following year, two members of the staff, J. Richard Dennis and Walter Sanders tried some material, on an informal basis, at one of the junior high schools in the Champaign-Urbana area. They were encouraged by this initial trial to expand their material to a year's work. This material was taught on a formal basis at all four junior high schools in Champaign-Urbana, by UICSM staff members, during the 1965-1966 school year. On the basis of this trial we revised the material, held a 3 week training conference for teachers <u>and</u> principals in August, 1966, and ran a full-fledged pilot study during the 1966-1967 school year. These pilot centers were located in the Boston area, New York City, Philadelphia, Memphis, Los Angeles, Honolulu, and in Illinois, Springfield and Urbana. There were 27 teachers and 2,500 students involved. We are currently in the process of revising

the material on the basis of this trial and expect that the course will be published in 1969 by Harper and Row. In marketing the materials, emphasis will be placed on teacher training in the content and pedagogy of the course.

A vital aspect of this course is that it acquaints students with terminology. That is, we expect all students finishing this course to react when they hear such words as rectangle. That is, we expect them to get a mental picture of a 4-sided, closed figure (i.e., a quadrilateral). In fact, we go even further than that - we hope that this 4-sided closed figure will have two pairs of parallel sides - and that it will have a right angle. We also hope that the student realizes that the opposite sides are congruent and that all four angles are congruent (right angles). Furthermore, they should recall that the diagonals are congruent and bisect each other. It is also very possible that on completing this course, a student will know that a rectangle together with one of its diagonals "makes" two congruent triangles. He should also know that associated with each rectangle there is an area and a perimeter - we are even hopeful that he will be able to compute these. Our ultimate goal in this respect is to get the student to know what information he needs to have in order to compute areas and perimeters. It is also possible that some students who finish this course will have still more information about rectangles at their command.

At this point we should say that the course is not a course in rectangles. It is our intention that this course will be a "sort of" laboratory course. That is, it is student activity oriented. The activities are designed to provide interesting - not too difficult - not too easy - experiences from which students can draw simple conclusions regarding geometric relationships.

Even very young students develop an awareness of differences among shapes. Frequently they can tell what these signs mean before they can read.

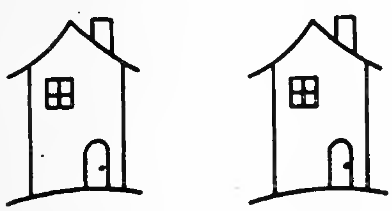


The course is divided into two parts. The first part is designed to acquaint students with techniques and experiences which they will use to discover geometric relationships. These relationships are strengthened and summarized in the second half of the course.

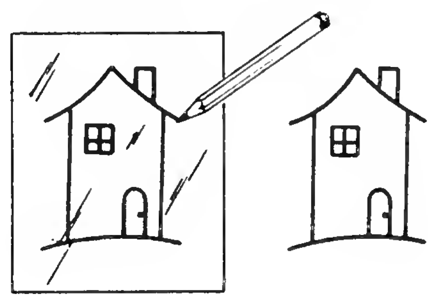
To start with we have activities designed to provide a test for congruence.

When a tracing of a figure just matches a second figure, the figures are congruent.

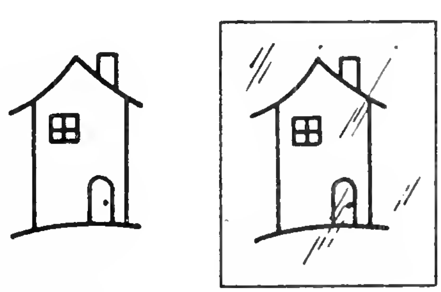
Are these two figures congruent?



You can show that they are congruent by tracing one of the figures.



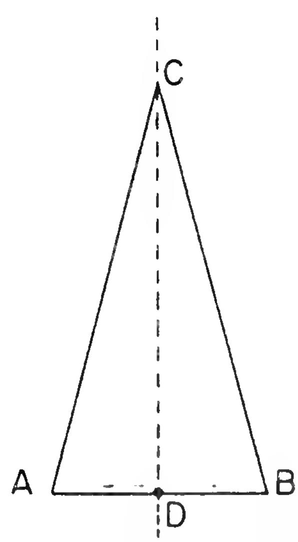
then matching the tracing with the other figure:



Since it matches, the figures are congruent.

Once we have developed this test for congruence we turn our attention to the particular motions of the tracing paper that are made in matching the tracing of one figure with a congruent figure. These motions include straight slides, turns, and flips, or combinations of these motions. These motions of the tracing paper are used to focus attention on the particular way in which congruent figures "match up". It is in this way that we develop an awareness of corresponding parts. Such a development makes it obvious that corresponding parts of congruent figures are themselves congruent.

Every figure is trivially congruent to itself. When a figure is congruent to another figure in two distinctly different ways then each of the figures is congruent to itself in some nontrivial way. That is, the figures have a nontrivial symmetry. We say that such a figure is invariant under the motion that maps it to itself. One of the simplest examples of this is the isosceles triangle, which is congruent to itself under a flip about the line containing the bisector of the angle between a pair of congruent sides.

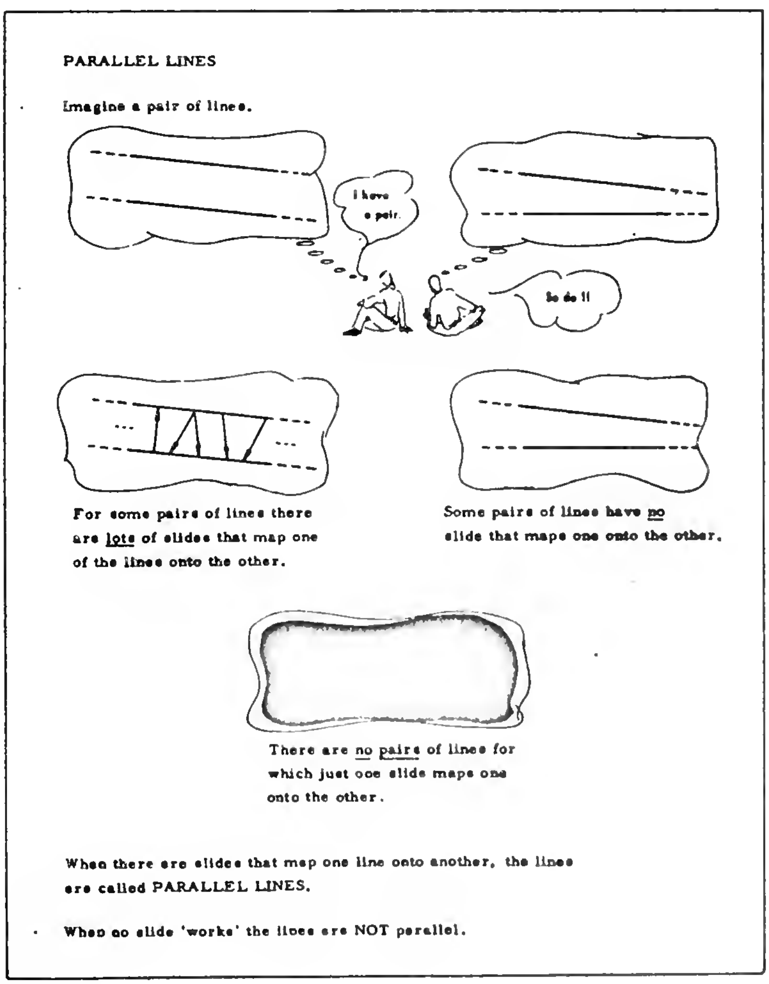


The flip about $\overleftrightarrow{CD}$ maps A to B, B to A, and C to C, and $\triangle ABC$ is mapped to $\triangle ABC$.

After establishing tracing paper as our test for congruence, making slides, flips, and turns, and discussing invariant sets, we introduce some of the standard geometric figures:

Lines
Segments
Rays
Angles

After spending some time learning to measure turns and angles, we are ready to move on to the second half of the course. In this part, we focus on more of the standard topics of geometry. We start with parallel lines.


PARALLEL LINES

Imagine a pair of lines.

For some pairs of lines there are lots of slides that map one of the lines onto the other.

Some pairs of lines have no slide that maps one onto the other.

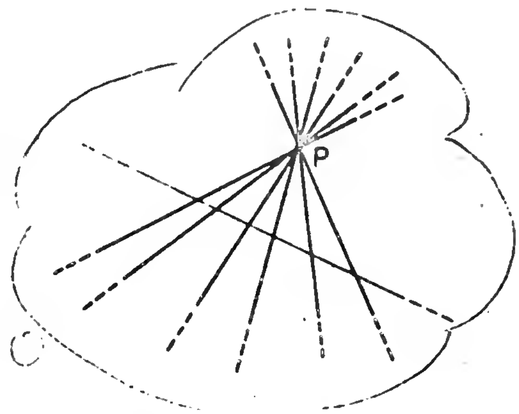
There are no pairs of lines for which just one slide maps one onto the other.

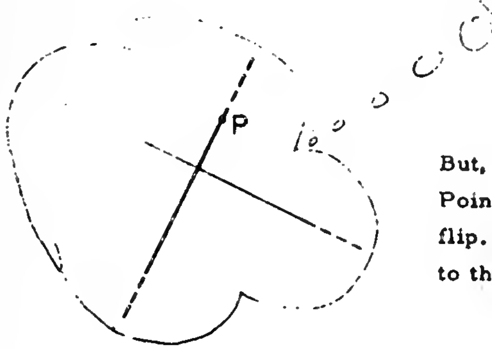
When there are slides that map one line onto another, the lines are called PARALLEL LINES.

When no slide 'works' the lines are NOT parallel.

We use the notion of invariance to introduce perpendicular lines.

You can draw many lines through Point P which are <u>not</u> invariant under the flip.

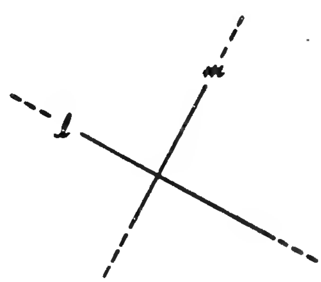




But, there is <u>only</u> <u>one</u> line through Point P which <u>is</u> invariant under the flip. This line is PERPENDICULAR to the flip line.

EXERCISES

<u>Part A</u>



1. Line *l* has been drawn so that it is invariant under a flip about Line *m*. So Line *l* is ____________ to Line *m*.

2. Is Line *m* invariant under a flip about Line *l* ?_____

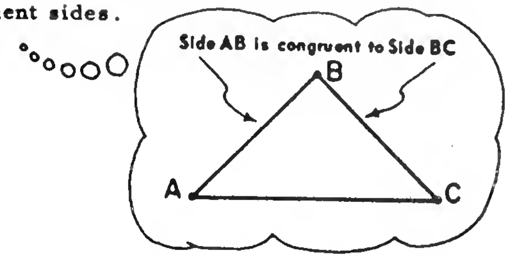
In Exercise 2 you should have found that Line *m* is invariant under a flip about Line *l*. So Line *m* is perpendicular to Line *l*.

In general,

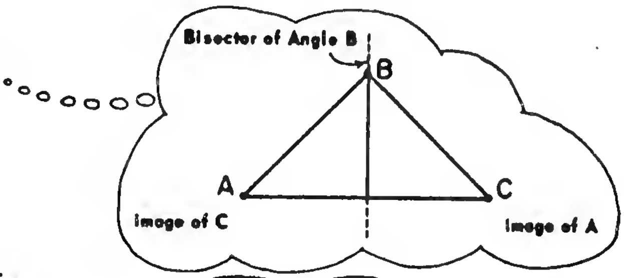
Two (different) lines are PERPENDICULAR to each other whenever one of them is invariant under a flip about the other.

We continue with the notion of invariance to classify triangles according to their symmetries.

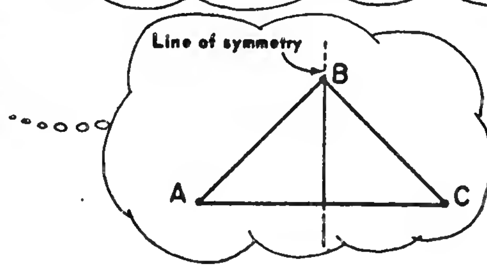
Imagine a triangle with two congruent sides.



Under a flip about the bisector of Angle ABC, Point A is the image of Point C and Point C is the image of Point A.

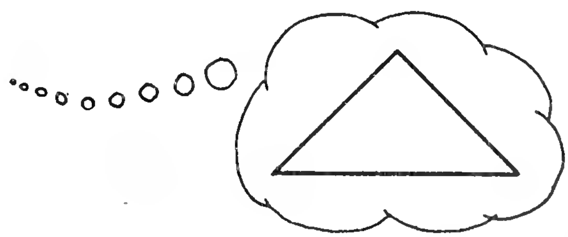


Under that same flip, Point B is its own image. So, Triangle ABC is invariant under this flip. This means that Triangle ABC is line-symmetric. [The bisector of Angle B is a line of symmetry.]

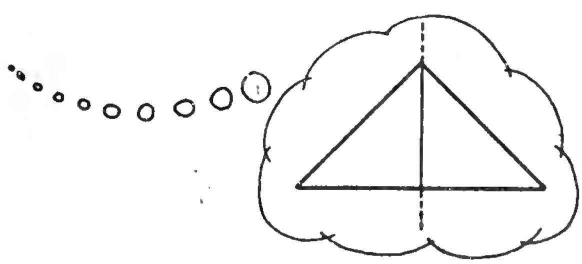


In general,

If a triangle has two congruent sides

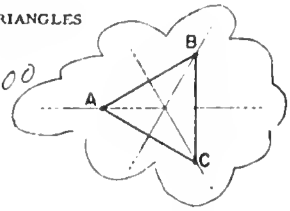


then the triangle is line-symmetric.



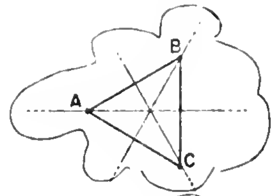
EQUILATERAL AND EQUIANGULAR TRIANGLES

Imagine a triangle with three lines of symmetry.

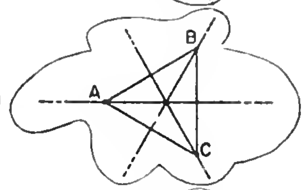


When a triangle has three lines of symmetry, it is invariant under the flips about each of the lines.

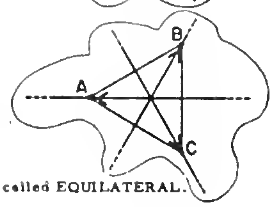
If a triangle has three lines of symmetry,



then it has three congruent sides



and it has three congruent angles



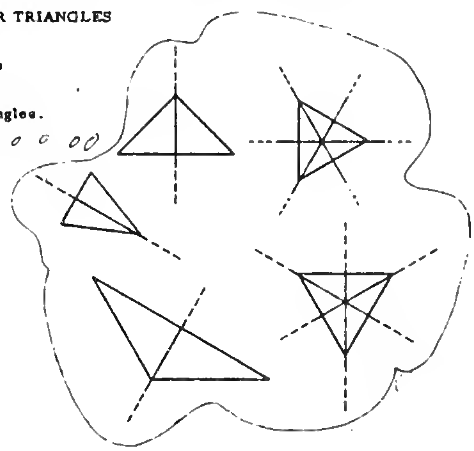
Triangles with three congruent sides are called EQUILATERAL.

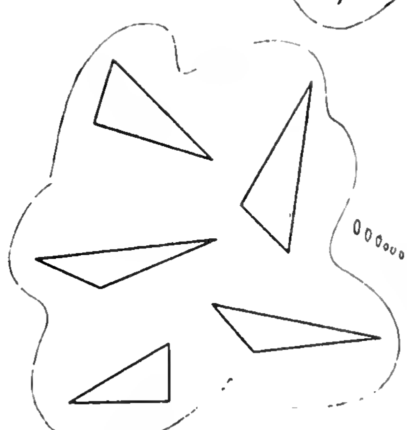
Triangles with three congruent angles are called EQUIANGULAR.

Triangles with three lines of symmetry are both equilateral and equiangular.

LINE SYMMETRIES FOR TRIANGLES

Triangles that have lines of symmetry are called LINE-SYMMETRIC triangles.



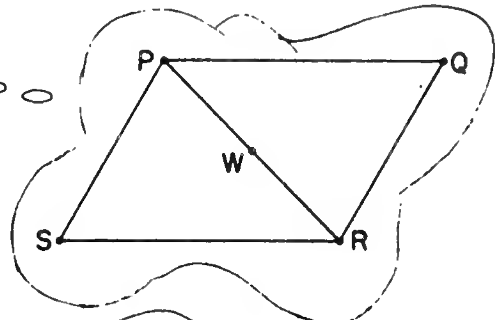


Triangles that have no lines of symmetry are called SCALENE triangles.

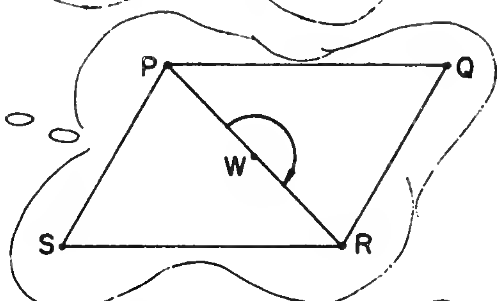
Next we can consider the symmetries of 4-sided figures and classify them accordingly.

PROPERTIES OF PARALLELOGRAMS

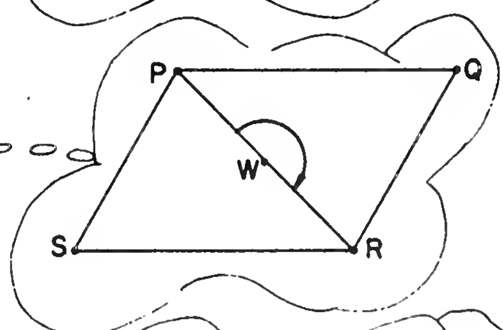
Quadrilateral PQRS is a parallelogram and Point W is the midpoint of Diagonal PR.



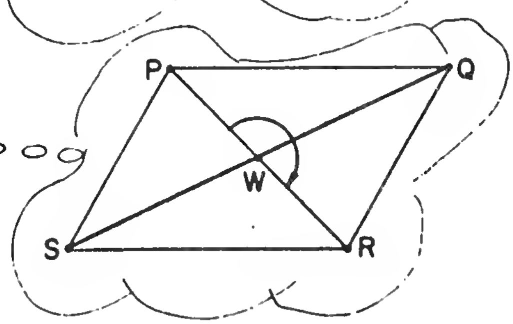
So, PQRS is invariant under a 180° turn about Point W.



Since P and R are images of each other and the figure is invariant, Q and S must also be images of each other.



This means that Segment QS is invariant under a 180° turn about W. So, W must also be the midpoint of Segment QS.

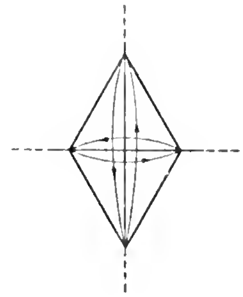


In general,

the diagonals of a parallelogram bisect each other at the point of symmetry for the parallelogram.

What are some other properties a quadrilateral with two diagonal symmetry lines have?

All four sides will be congruent <u>to</u> <u>each</u> <u>other</u>.

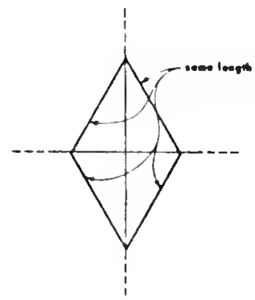


So,

A quadrilateral with two diagonal symmetry lines is a parallelogram <u>with</u> <u>four</u> <u>congruent</u> <u>sides</u>.

A parallelogram with four congruent sides is called a RHOMBUS.

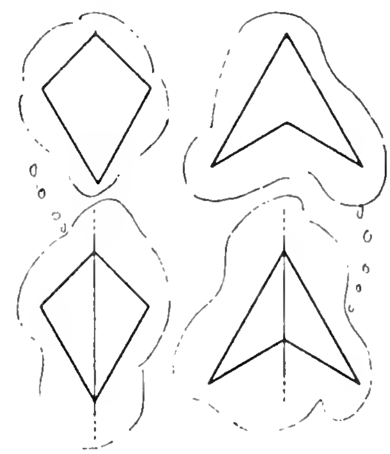
A quadrilateral with two diagonal symmetry lines is a <u>rhombus</u>.



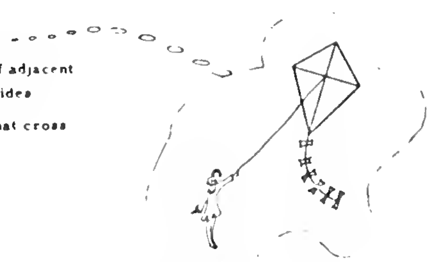
TWO KINDS OF QUADRILATERALS

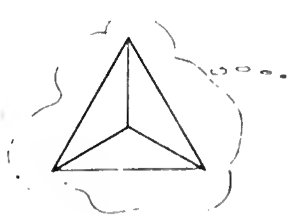
A quadrilateral with two pairs of adjacent congruent sides,

has a diagonal symmetry line.



A quadrilateral with
(a) two pairs of adjacent congruent sides
and (b) diagonals that cross
is called a KITE.





A quadrilateral with
(a) two pairs of adjacent congruent sides
and (b) diagonals that do not cross
is called a CHEVRON.

R. E. Zwoyer

UICSM NEWSLETTER

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Newsletter Editor: Clifford Tremblay

Table of Contents

Editor's Page

After an extremely busy summer in Urbana, we find that it is relatively easy to keep the wheels rolling. The result is that we have the earliest Newsletter on record for a school year. We trust that each reader will find something of interest in this issue.

As you will note from the first two picture stories, there continues to be a summer migration to Urbana for training sessions in "Traditional UICSM" courses as well as in newly-developed UICSM courses. It is gratifying to realize that the years of curriculum development by the UICSM staff have had a lasting appeal to teachers on an international scale.

The article entitled "Another Model for Teaching Operations on Directed Numbers" is an extremely abbreviated description of some supplementary materials that will be published by Harper and Row to accompany the Motion Geometry course. This article will be followed up in later issues of the Newsletter with the remaining operations on directed numbers, and the models that we use to help youngsters gain proficiency in these operations.

Finally, let us put forth our annual plea for suggestions you, the reader, might have for improving the Newsletter, or articles or comments you might like to offer for publication. Your daily encounter with the consumers of our product (the students) qualifies you as an expert on what makes the materials we produce work better in the classroom.

C. Tremblay

New Publications Available

UICSM announces the publication of three documents of interest to those who teach Course 1.

(1) Schor, Harry. Memoranda and Notes on Teaching HIGH SCHOOL MATHEMATICS, Course 1. [Urbana: University of Illinois Committee on School Mathematics, 1967] 229 pages

(2) Schor, Harry and Meng, Gloria. Teaching Guide for Elementary Algebra. [Urbana: University of Illinois Committee on School Mathematics, 1968] 668 pages

(3) Schor, Harry and Meng, Gloria. Homework Exercises for Elementary Algebra. [Urbana: University of Illinois Committee on School Mathematics, 1968] 96 pages

The first of these items consists of memoranda dealing with lesson plans and pedagogical matters of interest to a group of New York City teachers who taught Course 1 in 1966-67 after a summer institute at the University of Illinois. The lesson plans show how to teach HIGH SCHOOL MATHEMATICS, Course 1, in one year. The price for this document is $5.00.

Item 2 consists of 120 highly detailed lesson plans for teaching a one-year course in elementary algebra and appendices on pedagogical and mathematical questions including an extensive appendix on enrichment activities. The course in question was developed by Mr. Schor, with the assistance of Miss Meng, as a result of their work with New York City teachers who were trained in Urbana to teach Course 1. Modifications in Course 1 were introduced to make the course accessible to a wide range of students and to meet the special requirements of New York City [Regents Examination, short class periods, etc.]. Although the course is based on Course 1, it departs from it in many ways. For example, there is less emphasis on proof. Also, work on equation solving and problem solving is introduced early in the course and is maintained throughout it. Teachers who have taught Course 1 will be completely familiar with this adaptation, and might find it useful for students of average and below average ability who are eligible to take elementary algebra. The guide can be used without any textbook at all since each of the 120 lessons contains suggested homework exercises. Also, the appendix on enrichment provides enrichment exercises as well as references to such exercises in Course 1. The price of this document is $3.75

In order to increase the usefulness of Item 2 for teachers who wish to use it without a textbook, we have extracted the homework exercises and printed them in a separate booklet for students. The price of Item 3 is $.75.

UICSM Summer Institute

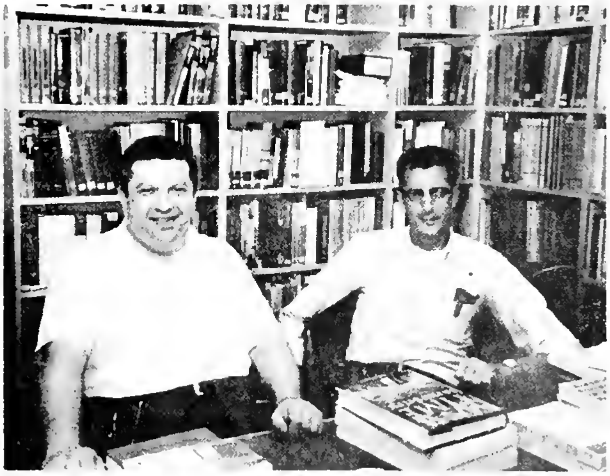
A grand total of 231 teachers participated in the UICSM-NSF Summer Institute this year, held July 1 - August 9. For the first time in the twelve summers that this program has been offered at this university, one class of the institute was not held on campus. Richard Dennis and Aileen Aizawa, UICSM staff members, were the instructors of a Course One class of 37 participants held at Maui Community College, Hawaii.

Among the participants at Urbana were approximately 50 teachers from high schools in Philadelphia who studied Course One and the Teaching Guide for Elementary Algebra developed by Harry Schor and Gloria Meng. These teachers will introduce UICSM materials in their schools this fall and will supplement their summer study with an inservice course during the winter. Harry Schor and Gloria Meng, instructors for this group, were assisted by Nick Grant and George Slogoff.

Other Course One instructors were Eleanor McCoy and Robert Drechsel. They were assisted by Maureen Demes and Andy May. The remaining sections of the institute were taught by Woodrow Fildes and Dale Yingst, Course Two; Arnold Petersen and Janice Flake, Course Three; James Nelson and Allen Holmes, Vector Geometry I; and Steven Szabo, Vector Geometry II. Participants in previous institutes will recognize many names among this roster of faculty.

UICSM has already submitted a proposal for an institute for summer 1969. Inquiries about that institute should be directed to Mr. Russell Zwoyer, Associate Director, Summer Institute, 1210 W. Springfield, Urbana, Illinois, 61801.

The following picture story highlights some of the activities of this institute.

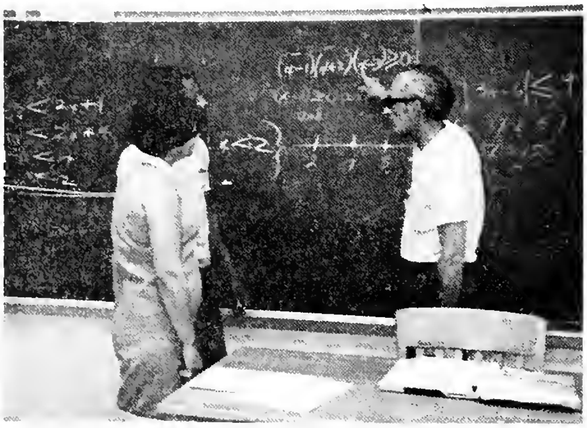


Director Max Beberman and Associate Director Harvey Gelder

First Course Instructor, Robert Drechsel, with Phyllis Evanenko (Zillah, Washington), Bob Riley (Fremont, Calif.), and Marlene La Mont (Elkins Park, Penna.) illustrates a point on the overhead projector.

Another First Course Instructor, M. Eleanor McCoy, discusses a problem with Mahtash Esfandiari, Tehran, Iran; and Joshua Joseph, Morrow, Louisiana.

First Course Instructors Harry Schor and Nicholas Grant watch as participant Mrs. Erslia Hawkins, Philadelphia, Pa., works with the overhead projector.



Beatrice Lanno and Jesse Newton, both of Philadelphia, Pennsylvania, discuss solutions of inequalities with Instructor George Slogoff.

Third Course Instructor, Arnold Petersen, discusses logarithms with Sister Cornelius, Lancaster, Pennsylvania; and Sister Patricia, Spokane, Washington.

Instructor Steve Szabo looks on while participant Bob Rundus, Urbana, Ill., explains a vector geometry problem to Leatrice Shiroma, Los Angeles, California; Sister Elaine, Chicago, Illinois; and Kenji Inouye, Honolulu, Hawaii.

In addition to the summer activities in which they were involved, several staff members will be involved in teacher training during this academic year. Dr. Dennis will work with the teachers in Hawaii throughout the year under a UICSM- Hawaii Coop Project supported by a grant from NSF. Mr. Grant and Mr. Slogoff will work with the Philadelphia teachers this year under a UICSM-Philadelphia Coop Project.

H. Gelder

Stretcher-Shrinker and Motion Geometry Institute

A second institute directed by the UICSM staff took place from July 8 through August 22 at Urbana. This institute involved 62 junior high school mathematics teachers and supervisory personnel from schools throughout the country. The majority of the participants were from inner city schools in densely populated areas. The curriculum studied by this group was the seventh and eighth grade materials currently being published by Harper and Row. These materials are being developed primarily for use by underachievers in urban areas. The content of these courses has been described in some detail in past issues of the UICSM Newsletter.

During the first three weeks of this institute, the participants discussed the Stretcher-Shrinker course with Max Beberman, and broadened their insights into the teaching process during discussion periods led by Joseph Hoffmann. Wednesday evening seminars provided an opportunity for informal open-ended discussions of problems that are peculiar to teaching in the inner city. The last four weeks were devoted to a discussion of the Motion Geometry course as described by Russell Zwoyer. The accompanying materials on directed number and nondirected number arithmetic were described by Clifford Tremblay. Since these materials continue to be under development, the suggestions of the participants were carefully noted, and will be used as the texts are prepared for publication. During the last week, Robert Wirtz introduced the program "Developing Insights Into Elementary Mathematics" [developed by Wirtz, Botel, and Beberman] parts of which appear to have considerable promise for remedial work in whole-number arithmetic.

Here are a few scenes from this institute.

The class met in a lecture hall for six and one-half hours each day for the seven weeks

Note the television cameras in the background. The lectures and subsequent discussions were videotaped by the Associated Colleges of the Midwest Videotape Project, Carleton College, Northfield, Minn.

Two staff members of ACM, Jacques Maynard and Norbert Welage, at the console during a taping session. The videotaping recorded over 100 hours of the institute sessions. [Watch for a later announcement concerning the availability of these tapes.]

School spirit (Zabranchburg) was strong — as evidenced by these sweatshirts.

The Wednesday evening seminar provided opportunities for group interaction not possible in the lecture hall.

Guest speaker at one seminar was Morris Janowitz, whose topic for presentation to the group was "The Problems of Education as seen through the Eyes of a Sociologist."

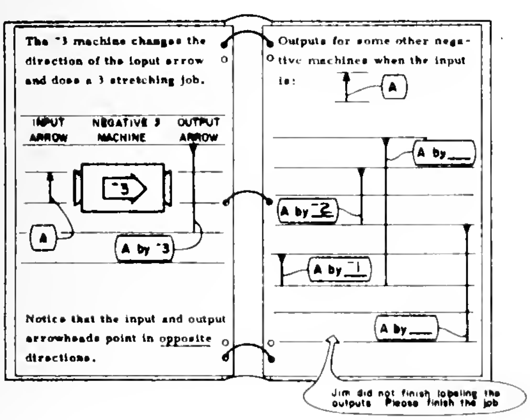
Staff members and guest speakers huddled in informal groups with participants in the institute.

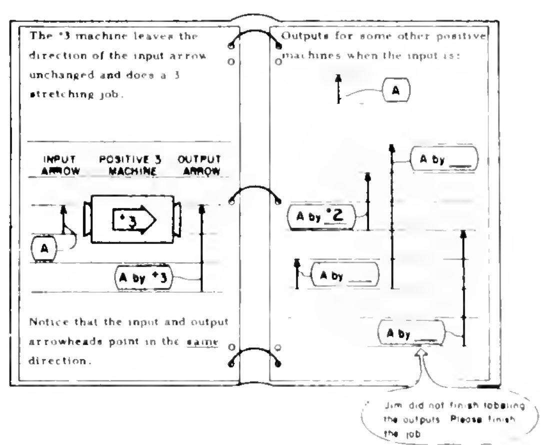
C. Tremblay

Another Model for Teaching Operations on Directed Numbers

In our search for a model for operations on directed numbers that will be mathematically sound and pedagogically palatable for youngsters in grade eight, we found an extension of the function machine approach which produced the desired results. This article will attempt merely to describe the approach taken in developing the multiplication and division models. In a later issue, we will outline the addition and subtraction models developed for this course. Since this article is necessarily shorter than the materials that will be used in the classroom, we will not be able to show the full development, much of which is a direct outcome of working sets of exercises.

We begin by considering a stretching machine that operates on directed sticks (arrows). As you will see from these two samples, these machines stretch the sticks, and either leave the direction of the arrowhead unchanged, or change it to the opposite direction.

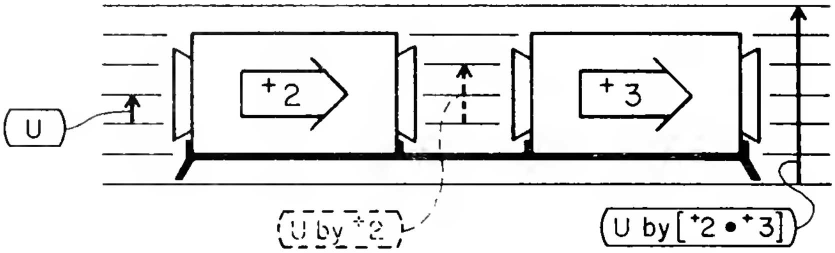




After a sequence of exercises that are meant to familiarize students with how the machines operate on directed lengths, we introduce the following hookup of two such machines.

HOOKUPS

The machines can be put together into hookups. For example, here is how a hookup of a $^{+}2$ machine and a $^{+}3$ machine works.



This hookup is described by writing:

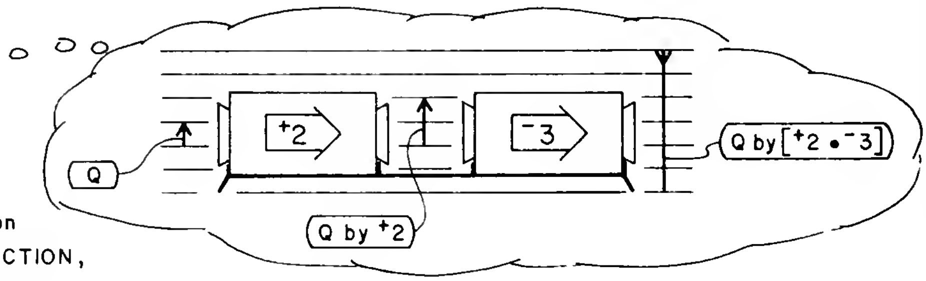
$$^{+}2 \bullet {}^{+}3$$

By observing the results of a hookup's operating on a directed stick, we are able to move one step closer to multiplication. We consider the amount of stretching that took place between original input and final output, and compare the directions in which the sticks are pointed. The single machine that could be used to replace the hookup is the answer we would expect to a multiplication problem.

HOOKUPS AND SINGLE MACHINES

The hookup

$$^{+}2 \bullet {}^{-}3$$

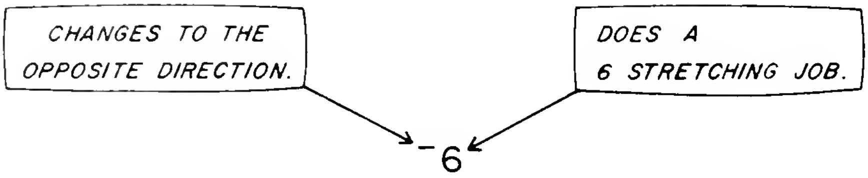


does two things to an input arrow:

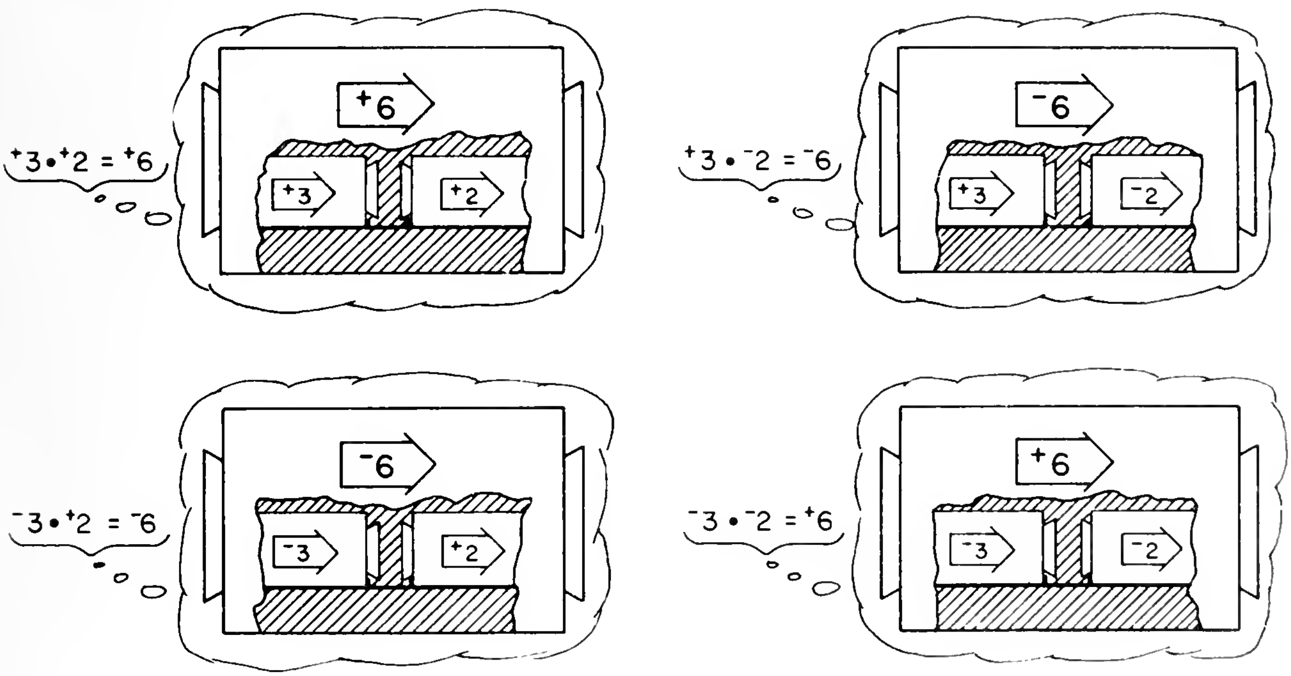
(1) it changes the direction to the OPPOSITE DIRECTION,

(2) it does a 6 STRETCHING job.

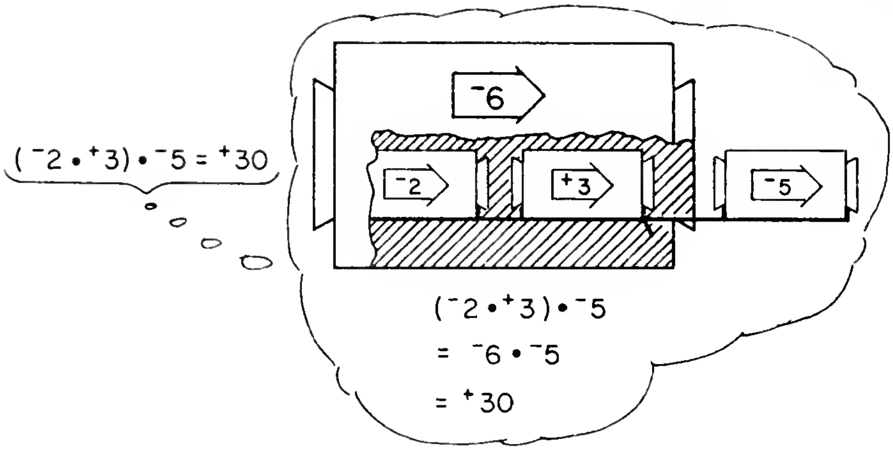
Is there a single machine that will do both of these things? Of course, the answer is 'yes'. A $^{-}6$ machine (by itself) will do both things.



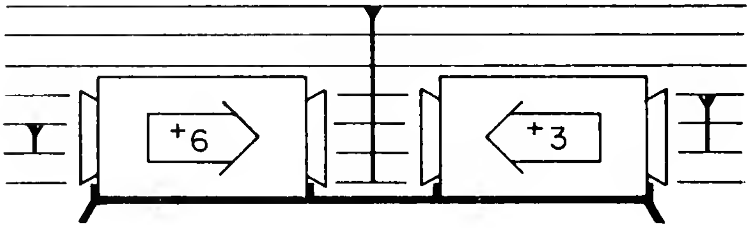
After a number of pages of practice using the ' • ' notation without direct reference to multiplication, we finally state that finding a single machine that may replace a hookup is what most people call multiplying directed numbers. The remaining problems in multiplication are stated in terms of finding products, but the model is still available for any students who feel safer using it or who want to verify answers found by using some self-expressed rules of operations. We do not verbalize any of the usual ''rules'' of operating on directed numbers. The familiarity students gain with directed numbers by working the exercises will more than likely convince them of the validity of their self-expressed rules. Furthermore, it is likely that sample exercises such as the following will focus the student's attention on the practicality of some summary rules.



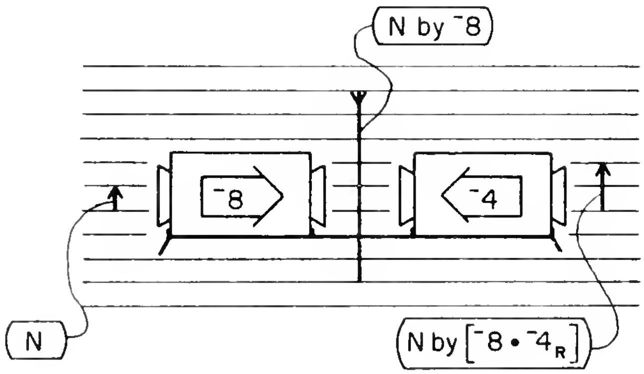
A natural extension of the work on multiplication is to consider three or more factors. The machine model lends itself to this quite well. Here is how one such problem would be pictured.

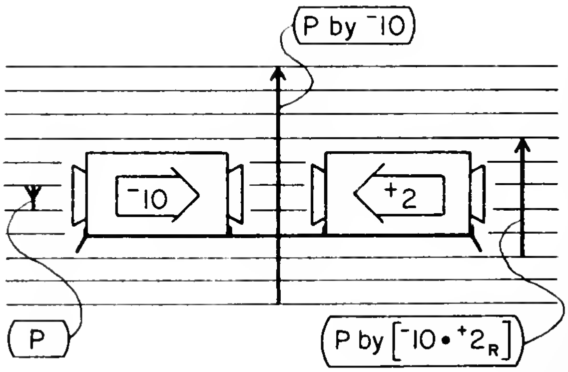


The work on division is a direct outgrowth of that on multiplication. We consider the output of a hookup in which the second machine has been placed in reverse. The effect of such an event is readily seen to be that of using a shrinker. Here is one such hookup.

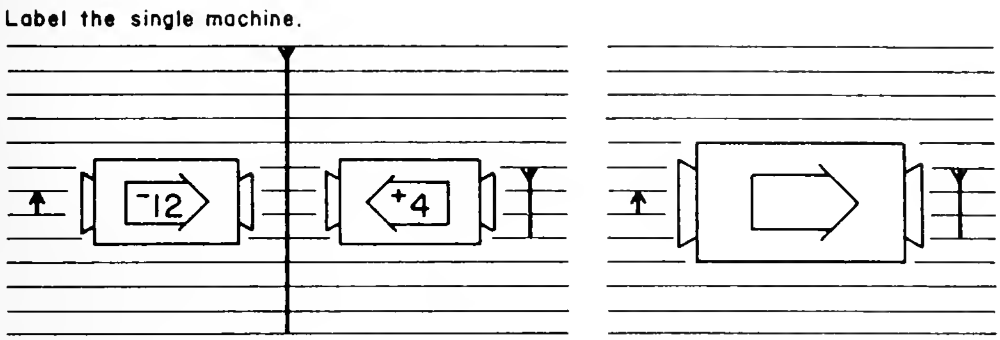


Next, we have another series of exercises designed to make the student comfortable with the notion of division before we ever name the operation. In fact, they have gotten all the usual division results without ever having seen the '÷' symbol of division. For example, note the tags on the directed sticks in the following hookups.

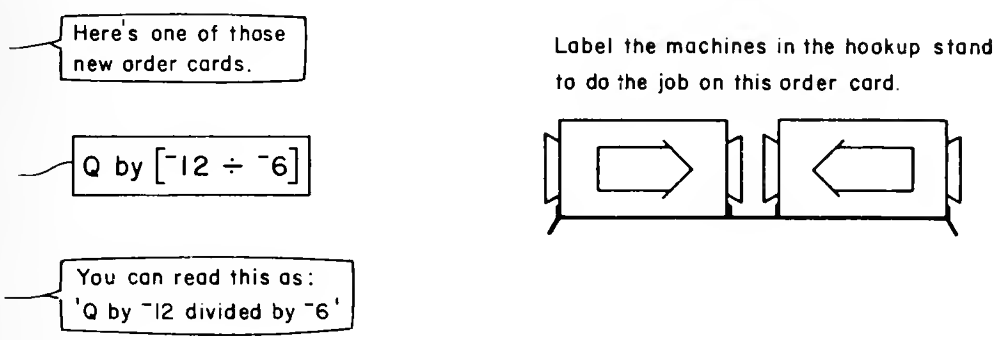




After considerable practice, the student is finally ready to consider the single machine that could replace a hookup of this type, and to practice finding such single machines.



Now that the students have become proficient in handling the notion of division, we introduce the '÷' symbol and repeat some of the types of exercises done earlier in the other notation.



In order to give the student a ready frame of reference, the '÷' symbol is related to the reversed machine on the hookup stand.

Finally, we introduce the term 'reciprocal', and complete the topic by relating division to multiplication by a reciprocal. This is done without direct reference to the machines, but they are still available, and could be used by classes in discussing the topic. The fraction form for division is introduced as a subtopic which certainly could be expanded upon in class discussions.

Teachers who have used this unit with classes of disadvantaged youngsters generally agree that results have been favorable. The youngsters have gained proficiency in operating on real numbers, and have been highly motivated by the format.

C. Tremblay

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