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UICSM Newsletter

An occasional publication of the

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

1208 West Springfield

Urbana, Illinois

UICSM Newsletter No. 7 April 27, 1962

A BRITISH JOURNAL

Mathematics Teaching is the quarterly bulletin of the Association for Teaching Aids in Mathematics, a publication and sponsoring organization that are in some ways British parallels to our own Mathematics Teacher and National Council. Mathematics Teaching has received favorable attention from some on the UICSM staff for some time for its forthright and incisive comments on new developments in mathematics education in the United States, England, on the Continent, and elsewhere. The bulletin now appears (with the Spring 1962 issue--No. 18) in an attractive new format and includes a variety of features: a "puzzle page", research news, book reviews, and a wealth of clearly written articles that probe policy and practice, and are by no means confined to "gadgets".

Your editor recommends this journal to American mathematics teachers in grades K through 12 who feel they might benefit from the greater perspective on American problems it offers. Those interested may write to the ATAM Treasurer for a sample copy (75 cents) or an annual subscription (three dollars, including postage). Personal checks drawn on American banks are acceptable and should be made out and sent to:

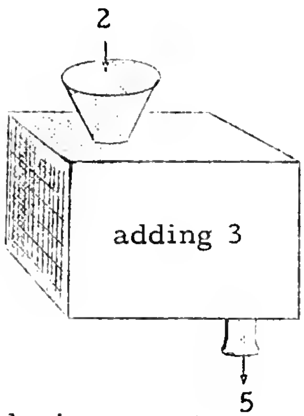
Mr. Ian Harris, Treasurer
Association for Teaching Aids in Mathematics
122 North Road
Dartford, Kent
England.

--R. S.

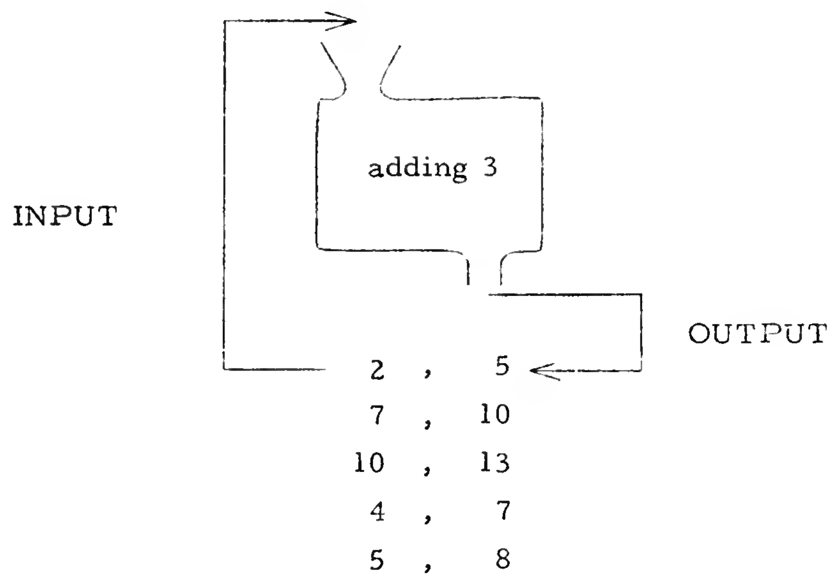
ANIMATED FUNCTIONAL NOTATION

While working on UICSM-PIP (Programed Instruction Project) materials, we came up with a device which seems useful for a variety of purposes. Its general usefulness resides in the fact that it suggests concepts and does so without the use of much talk. Thus, it can be used to stimulate lots of student discovery.

In essence, the device is nothing more than an animated functional notation. For example, consider the function adding 3. You can show that 5 is the value of this function for the argument 2 just by the picture:



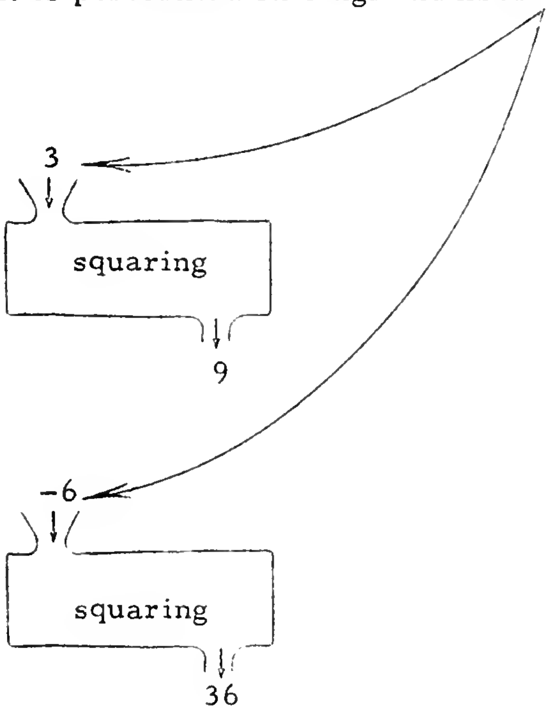
It is very easy to use this device to get across the idea that a function is nothing more than a set of ordered pairs. Just have students keep a record of the "input" numbers and the "output" numbers.



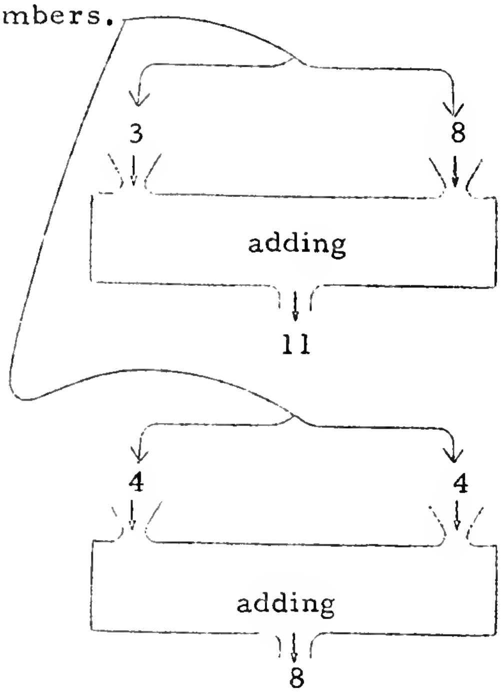
SINGULARY AND BINARY OPERATIONS

Just as this notation can be used to deal with singulary operations ("functions of one variable"), it can also be used for binary operations ("functions of two variables"). Thus, an apparent advantage of this notation is the way it emphasizes the difference between a singulary operation and a binary operation.

Squaring is a singulary operation. It is performed on single numbers.



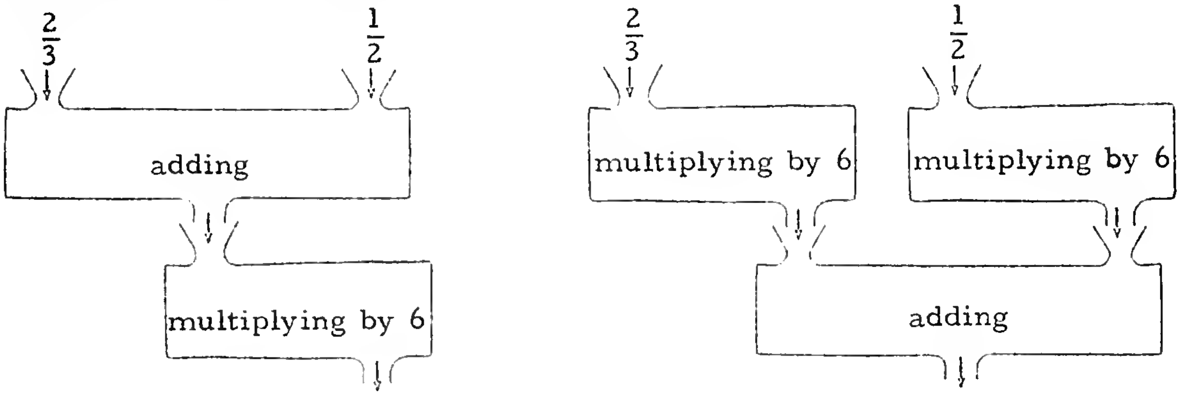
Adding is a binary operation. It is performed on ordered pairs of numbers.



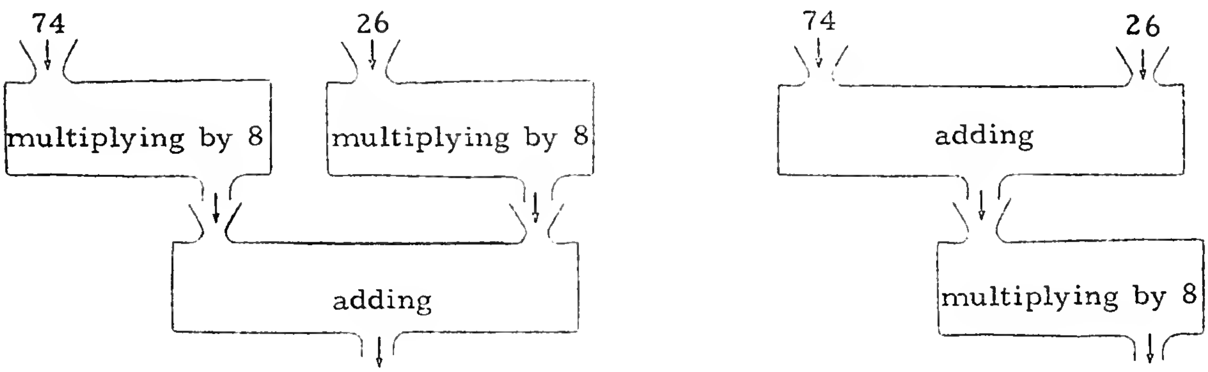
A notation such as '(7 + 3) - 3 = 7' does not distinguish the singulary operation adding 3 from the binary operation adding. Nor does it distinguish subtracting 3 from subtracting.

ILLUSTRATING SOME OF THE BASIC PRINCIPLES

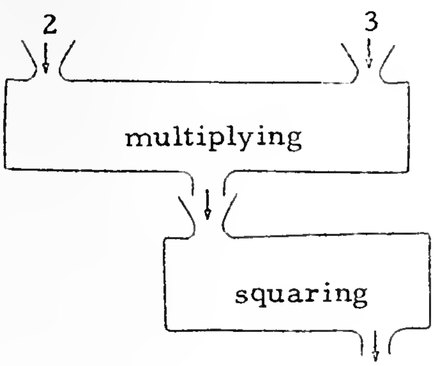
The distributive principle for multiplication over addition can be illustrated nicely using this notation.

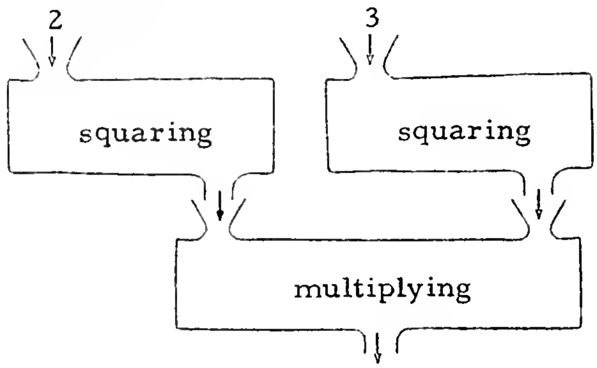


Notice how the pictures give meaning to: "distributing" multiplying by 6 "over" addition. Of course, the order of the two pictures may be reversed. In either case, the concept of a changed "principal operator" is made prominent (cf. UICSM Newsletters Nos. 1 and 3).



Other distributive principles could be investigated by using similar pictures. For instance, is squaring distributive over multiplication?





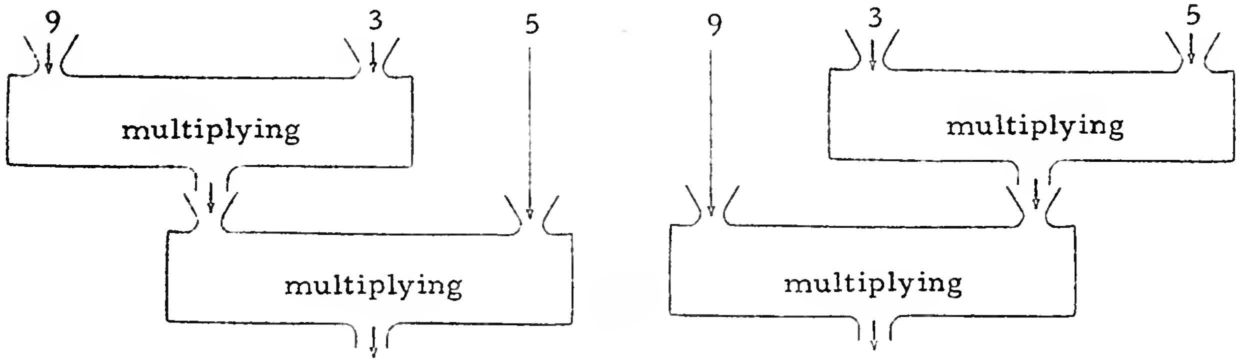
Is division distributive over subtraction?
Is reciprocating distributive over division?
Is oppositing distributive over addition?
Is absolute valuing distributive over multiplication?

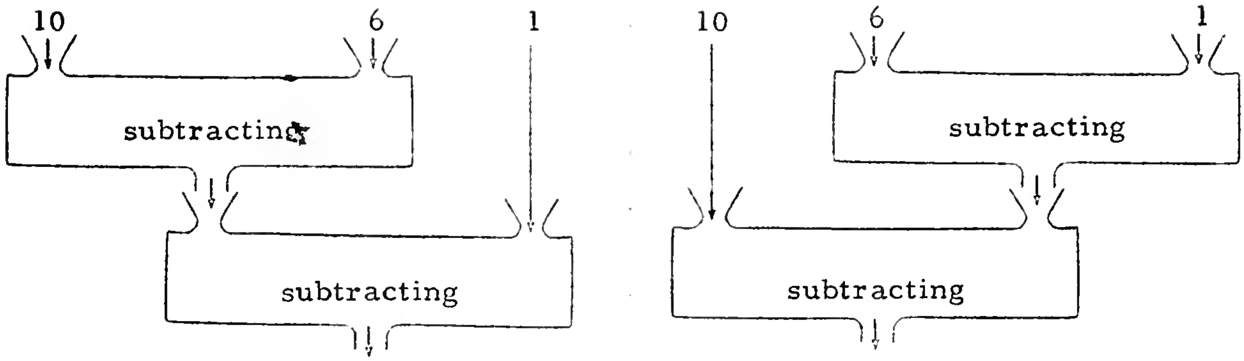
Also, one should probably consider such questions as:
Is addition distributive over multiplication?
Is absolute valuing distributive over addition?
Is oppositing distributive over multiplication?
Is squaring distributive over addition?

While investigating other distributive principles an interesting subtlety may come to light. That is: can a binary operation be distributive over a binary operation? The pictures definitely help resolve this. Even though the word 'multiplication' in the name of the dpma may suggest that the binary operation multiplication is distributive over the binary operation addition, a glance at one of our pictures will dispel such a misconception. In fact, one might properly think of that principle as the distributive principle for multiplying-by-a-number over addition. The notation '$(\frac{2}{3} + \frac{1}{2}) \times 6$' does not clearly distinguish between the binary operation multiplication and the singulary operation multiplying by 6. A study of the other distributive principles should

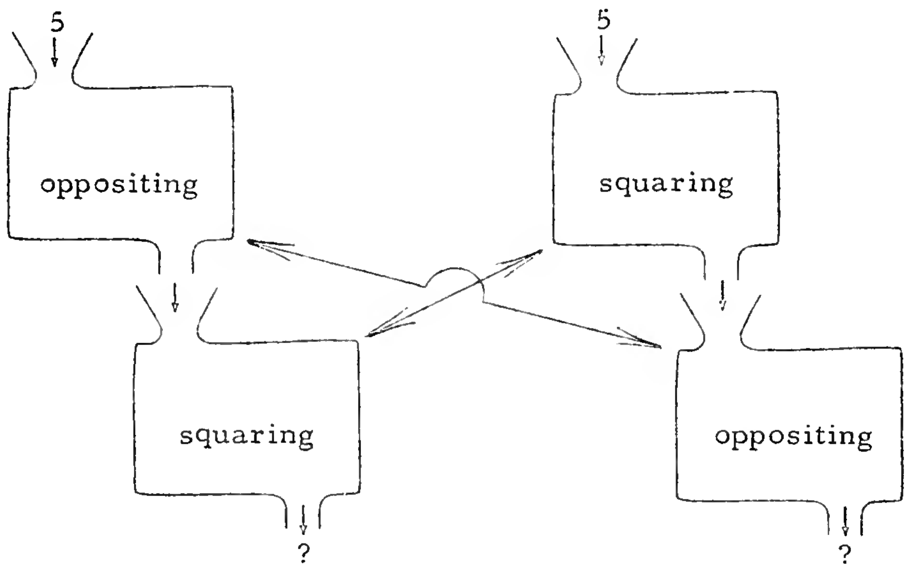
convince one that each refers to a singulary operation distributed over a binary operation. For instance, oppositing, a singulary operation, is distributive over addition, a binary operation.

You can use a pair of pictures to illustrate easily an instance of the associative principle for multiplication, or another pair to show that subtraction is not associative.





The usefulness of the device in eliminating problems dealing with order of operations has already been touched upon in the discussions of associativity and distributivity. A more general use, in connection with this topic, is in getting students to discover a need for an order-of-operation convention. Certainly, the ''output'' numbers for the figures below are different. And, students will have no difficulty in telling what the output numbers are.



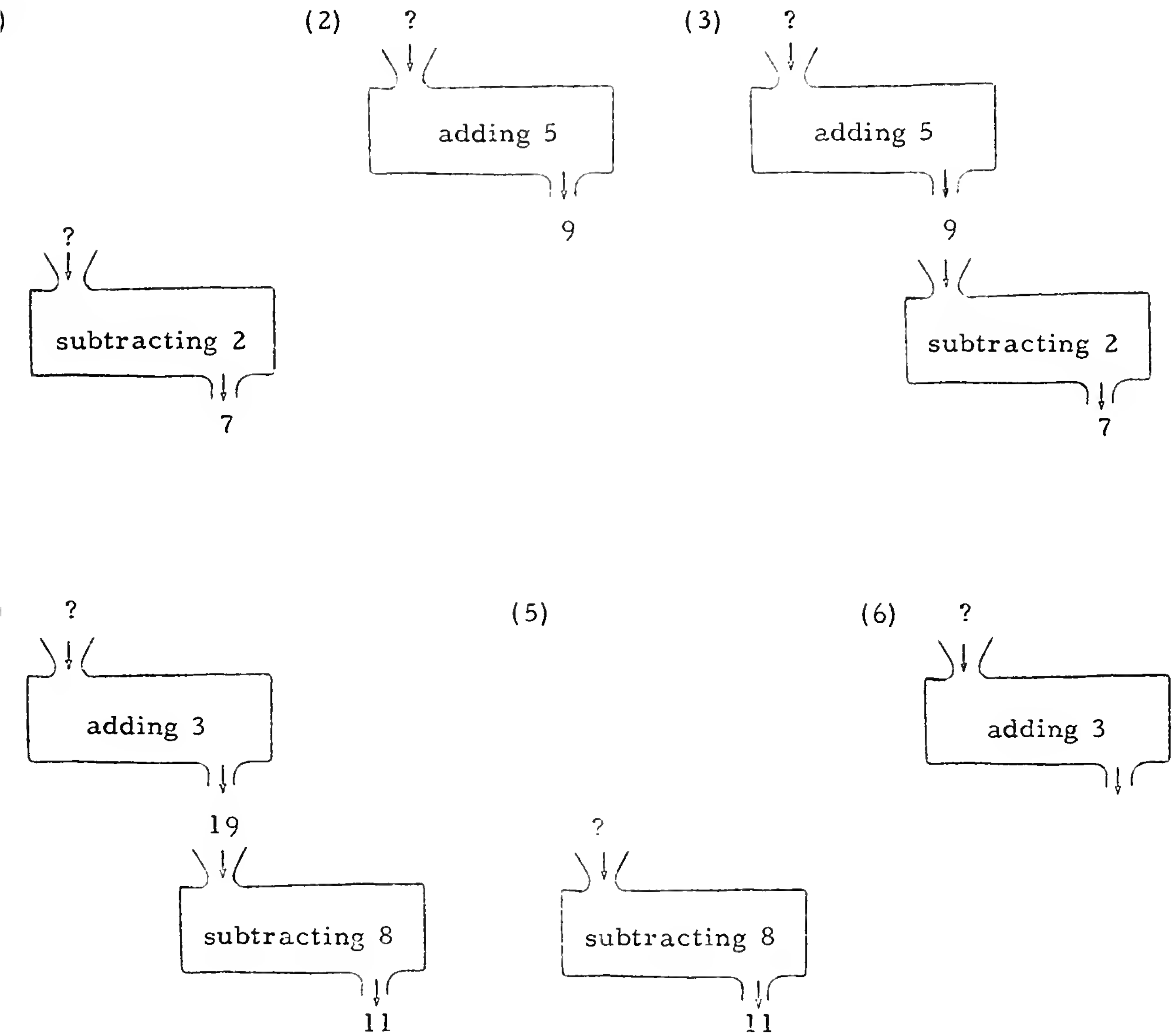
So, it is clear that we don't wish '-5^2' to name both output numbers. Which one, then, is it to name? This calls for a convention. There already is such a convention in use. Under this convention, $-5^2 = -(5^2)$. Hence, $-5^2 \neq (-5)^2$.

More generally, if the grouping is not specified then squaring is performed before oppositing.

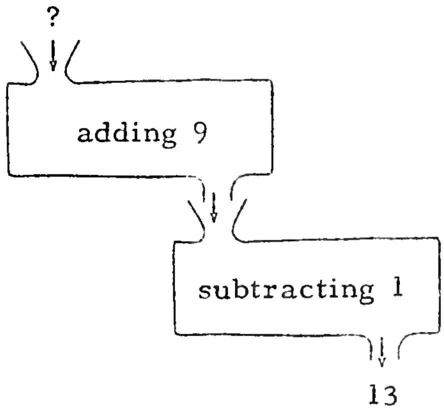
Similarly, one can consider conventions for expressions such as:

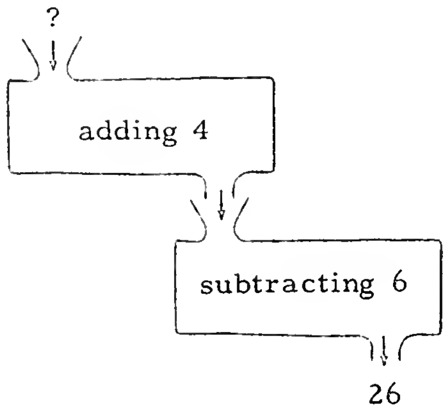
$7 + 3 \times 2$	$19 - 12 - 5$
3×5^2	$\log 1000^2$
$32 \div 8 \div 2$	4^{3^2}

This animated functional notation is useful in emphasizing that an inverse operation "gets you back to where you started". The word 'inverse' need not be introduced until after the concept has been developed. Also, it is an easy task to get students to discover this concept for themselves as a short cut. Consider the sequences of the exercises below and on the next two pages. Additional exercises could be used as needed.

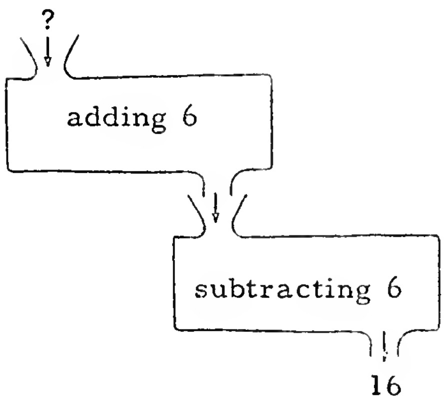


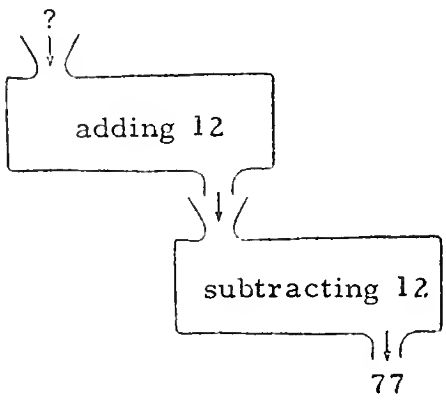
Exercises like the preceding ones should prepare students to do this kind:



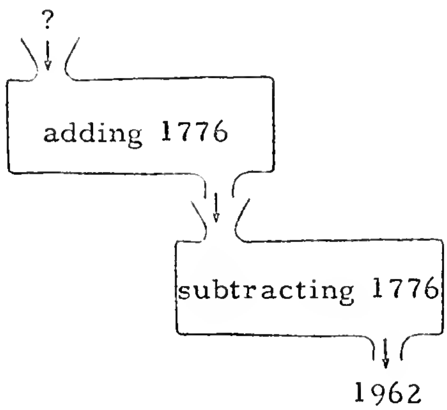


Then a teacher could present this type with increasing frequency:

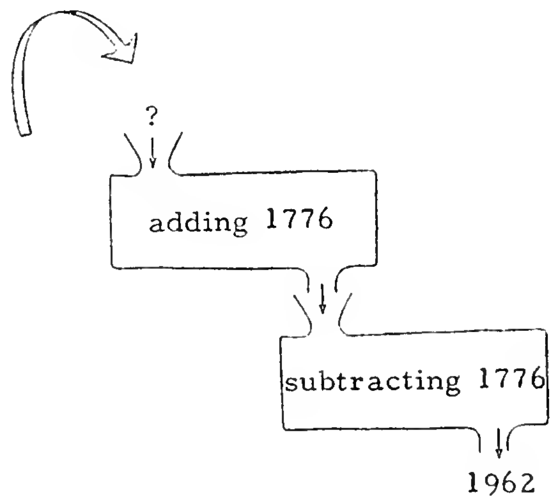




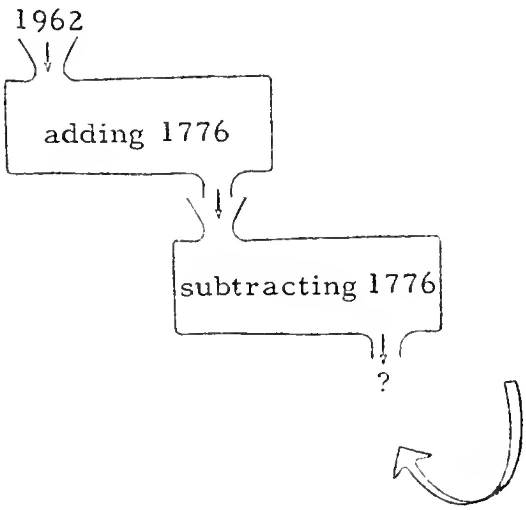
Soon he may insert:



to help force a student to look for a short cut. Note that this more difficult type of exercise is suggested:

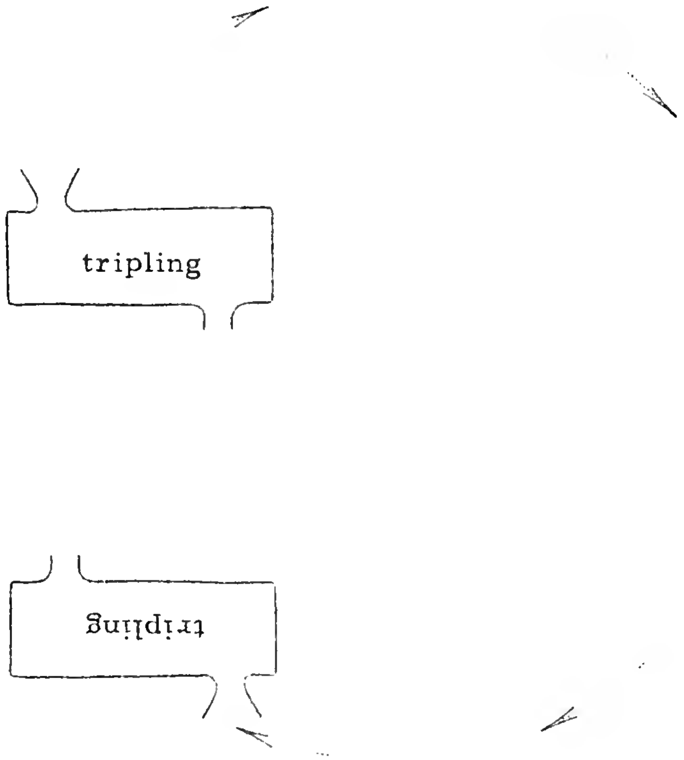


rather than one like:

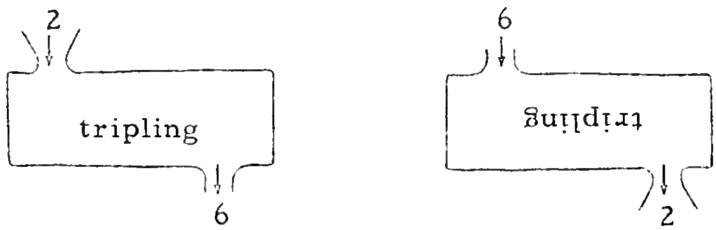


because discovering a short cut for a difficult task is more rewarding to a student.

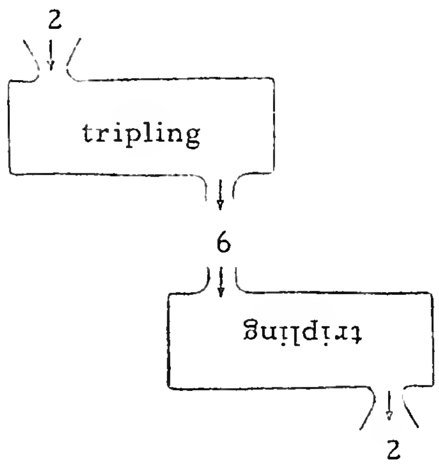
We have seen that this type of function notation clearly illustrates the concept of pairs of first and second numbers (ordered pairs). It announces in a picturesque manner those pairs which belong to a certain operation. By inverting a picture of the machine you get a vivid aid for studying inverse operations.



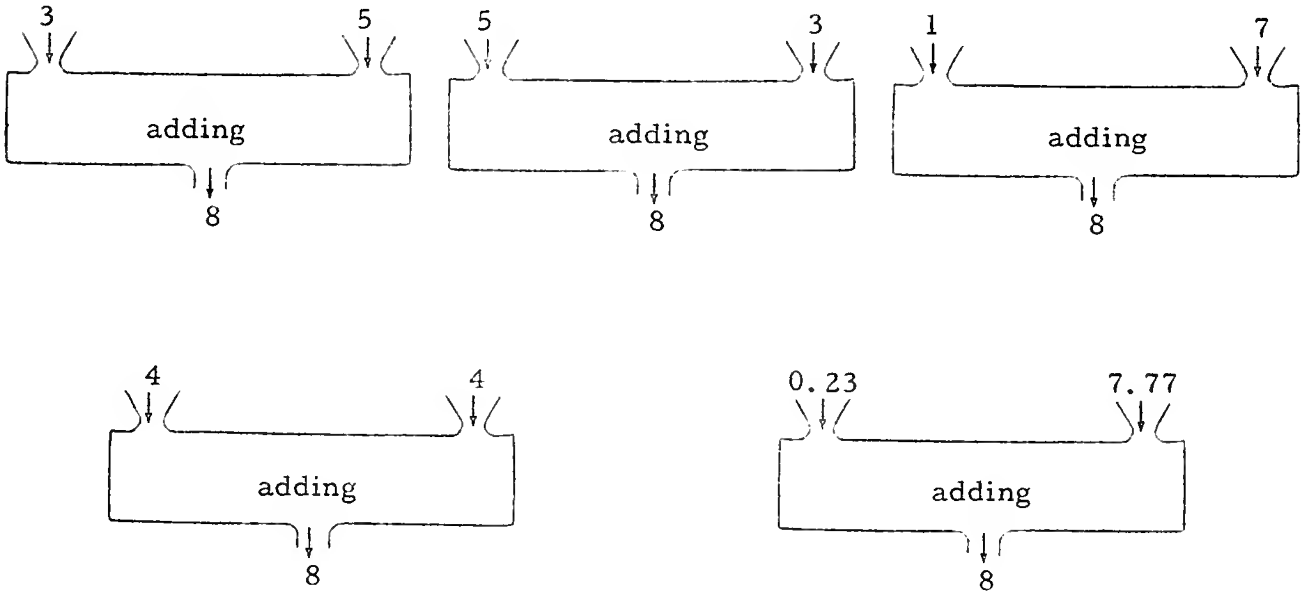
For instance, whereas the pair (2, 6) belongs to tripling, the pair (6, 2) belongs to the inverse of tripling.



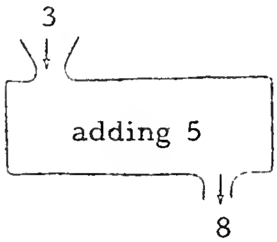
It is, of course, important to know that if a pair belongs to a operation having an inverse then the "reverse" of that pair belongs to the inverse of the operation. Note that the use of the inverted diagram is consistent with our previous meaning of 'inverse operation' -- getting back to where you started.



As was pointed out earlier, addition is an operation on pairs of numbers whereas adding 3 is an operation on single numbers. It is interesting to note that there are many pairs of numbers the sum of whose components is 8.



However, there is but one number which when 5 is added to it will give the sum 8.



This accounts for the fact that adding 3 has an inverse, whereas adding does not. The following two problems help emphasize this point.

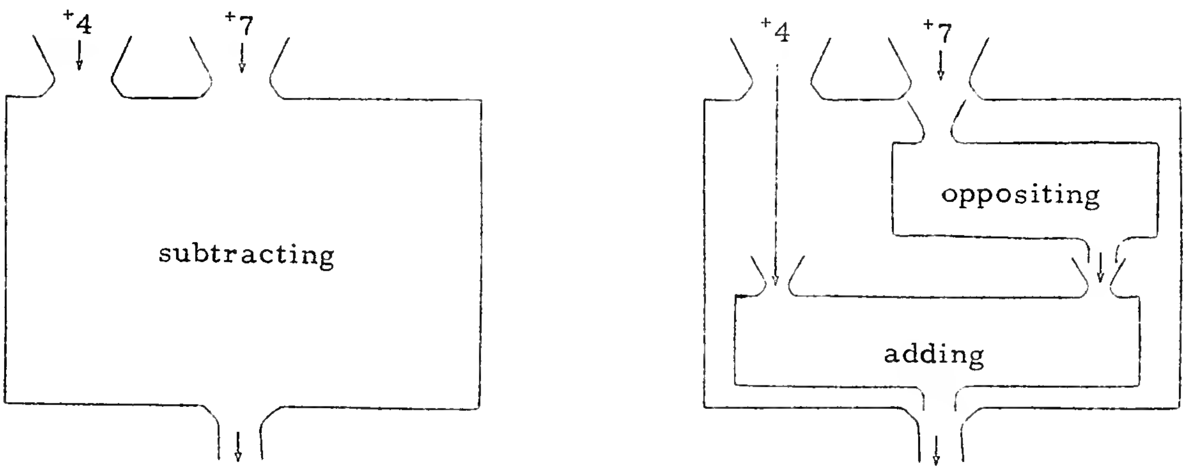
(a) Lon picked a number, added 3 to it, and got 10. What number did Lon pick?

(b) Don picked a number, then he added a second number to it and got 10. What was Don's first number and what was his second number?

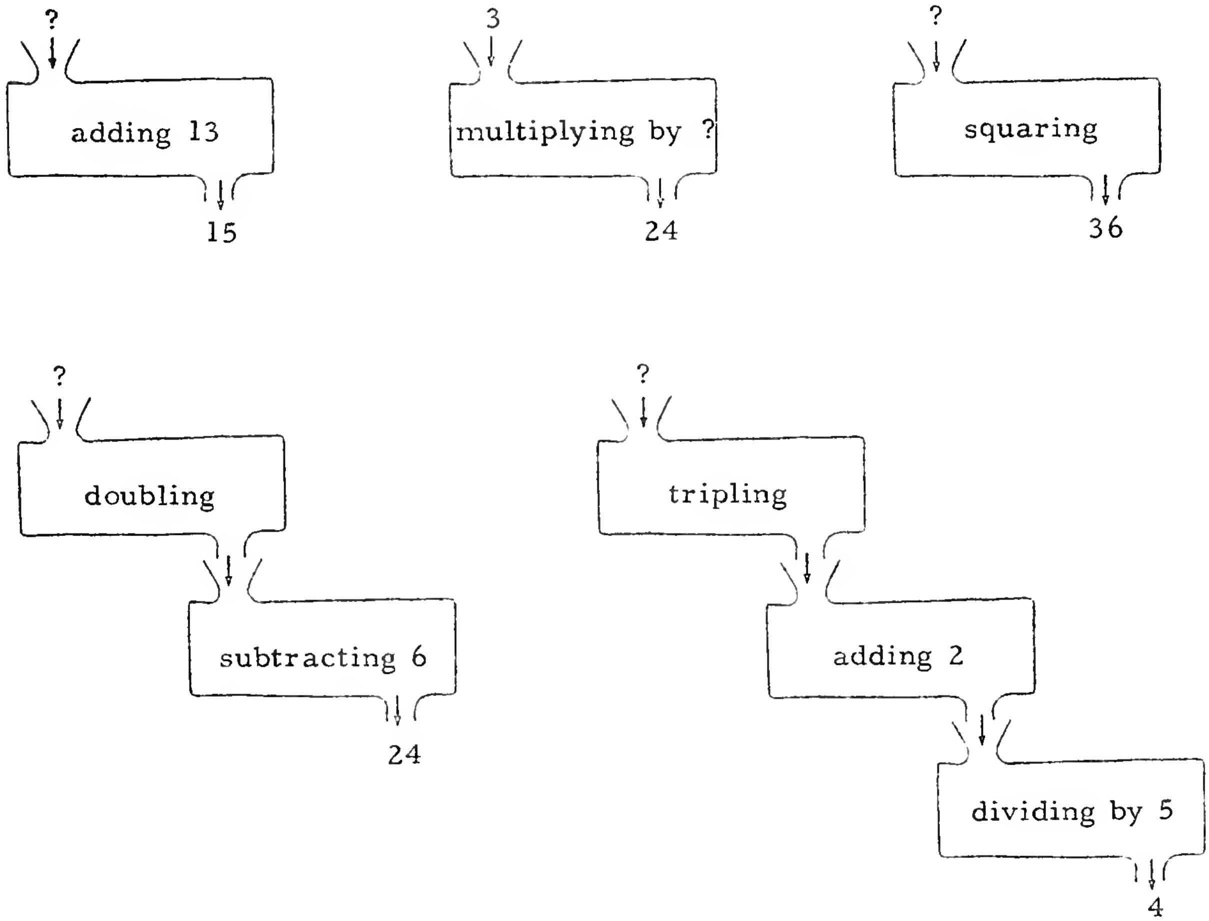
It should be easy for you to "puzzle out" (a). However, if you can do (b), you are probably a mind reader.

With this device it is easy to illustrate the concept of commutativity and show, for instance, that while addition is commutative, division is not.

This device can also be useful for illustrating the principle for subtraction.



Intuitive solution of equations may be introduced using this graphic notation. There are unlimited possibilities in exploring this use. One may vary the operations, the number of steps, or the location of a '?'. Here are just a few samples:

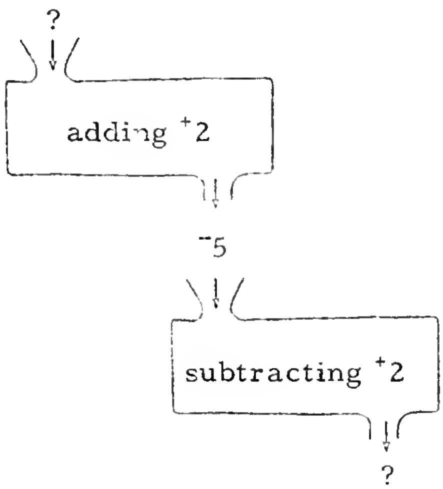


Mr. Howard Marston of the Principia Upper School in St. Louis suggests that a student could solve "equations" like these by merely inverting the picture and doing the inverse operations in order. The last figure above demonstrates effectively the complete freedom from grouping symbols enjoyed by this notation.

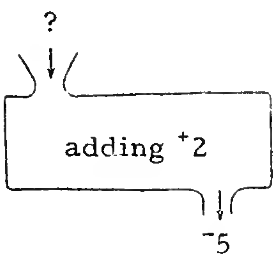
Perhaps one of the most striking advantages of this notation is revealed by a study of the fact that subtracting a real number is the inverse of adding that number. As you will see, this notation makes the concept clear and to the point. Consider this subtraction problem:

$$^{-}5 - {^{+}2} = ?$$

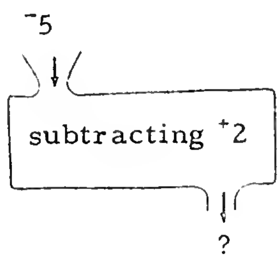
The figure below exhibits all of the needed information in a concise manner.



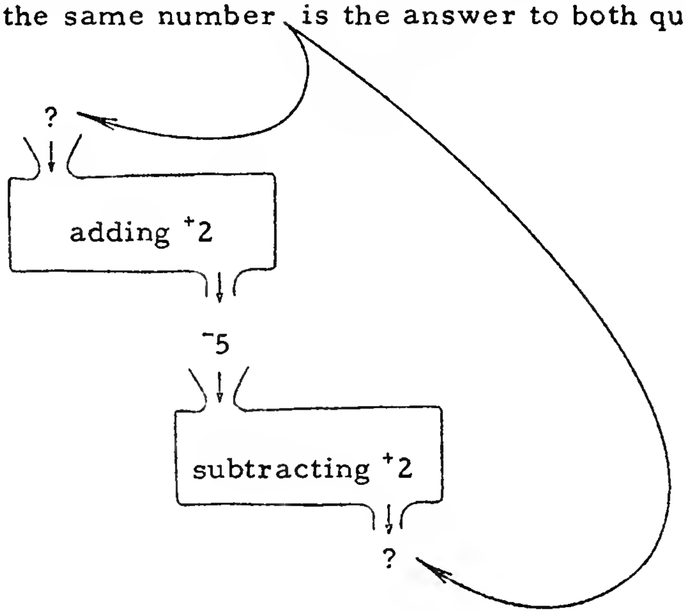
The figure presents this essential problem:



in a strategic position. It also displays the original subtraction problem:



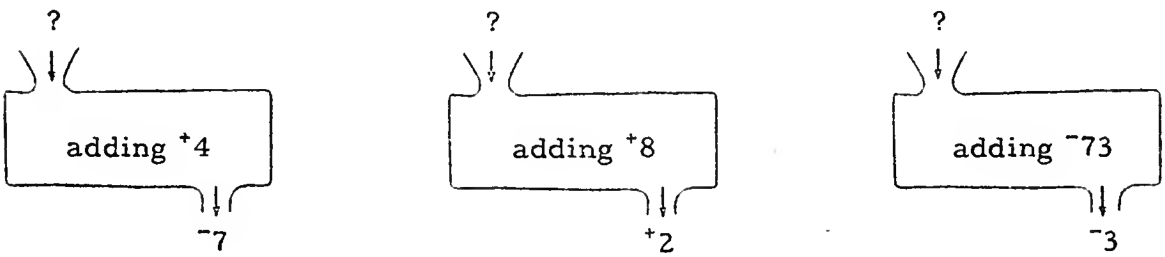
One can readily recognize subtracting ${}^{+}2$ as the inverse of adding ${}^{+}2$, and hence notice that the same number is the answer to both questions in:

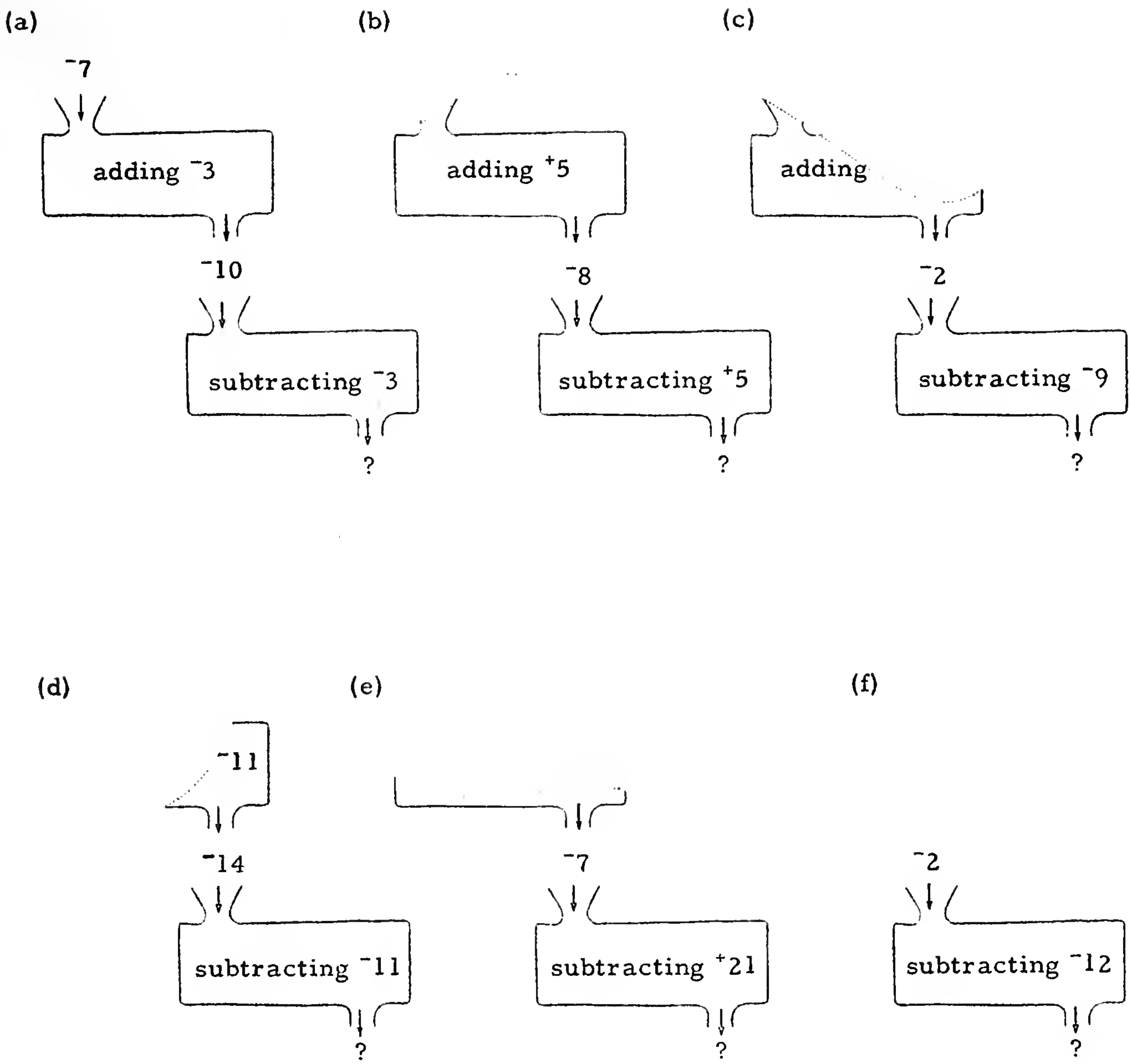


So to do this problem: ${}^{-}5 - {}^{+}2 = \ ?$

one could do this one: $? + {}^{+}2 = {}^{-}5$

Similar diagrams may be employed for rapidly leading students to do subtraction problems involving real numbers. The only prerequisites are the ability to add real numbers, and the knowledge that subtracting a real number is the inverse of adding that real number. A sample set of developmental exercises for this purpose appears on the next page. It would be helpful to precede this sequence of exercises with some "warm-up" on adding reals, and then sufficient exercises of this type:

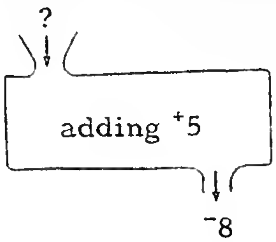




Here are some remarks on the above set of exercises.

(a) The student should recognize that the output number must be the same as the input number since subtracting ⁻3 is the inverse of adding ⁻3.

(b) As in (a), the output number has to be the same as the input number. Since the input number is not given, the student must find it. That is he must solve this problem:



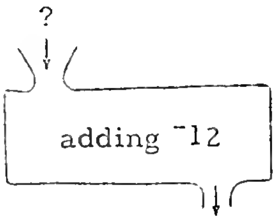
Once the problem is solved, he knows the input number. Hence, he also knows the output number.

(c) The top part of the diagram is ''fading''. If the operation named there were adding $^{-}9$, then the output number would be the same as the input number. So the student supposes that the first operation is adding $^{-}9$. Now in order to get the input number he must find the number to which $^{-}9$ has been added to get $^{-}2$. That number is $^{+}7$, so the output is also $^{+}7$.

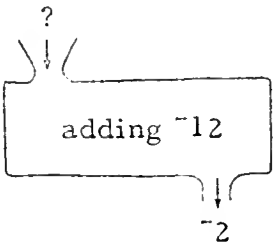
(d) This time the top part has faded so the word 'adding' is missing. However the student knows that subtracting $^{-}11$ is the inverse of adding $^{-}11$. So, if he knew the number to which $^{-}11$ was added to give $^{-}14$, he would have the answer to this exercise.

(e) The student is now almost on his own. He is, however, reminded that some operation was performed on an input number to give $^{-}7$. If that operation had subtracting $^{+}21$ as its inverse, then the input number would be the same as the output number. It is, therefore, more than convenient for him to assume that the first operation is adding $^{+}21$. Thus, the input number would have to be the number whose sum with $^{+}21$ is $^{-}7$. That number is $^{-}28$ which is also the output number.

(f) The first five exercises give a strong clue for finding the answer to this one. The student merely has to visualize a picture like this:



above the one given. Hence, the answer to the original problem is the answer to:



since subtracting $^{-}12$ is the inverse of adding $^{-}12$.

This experience can be extended to familiarity with the "additive method" of subtraction, which is nothing more than "checking" a subtraction problem before it has been solved. This subtraction tool is excellent for doing problems like:

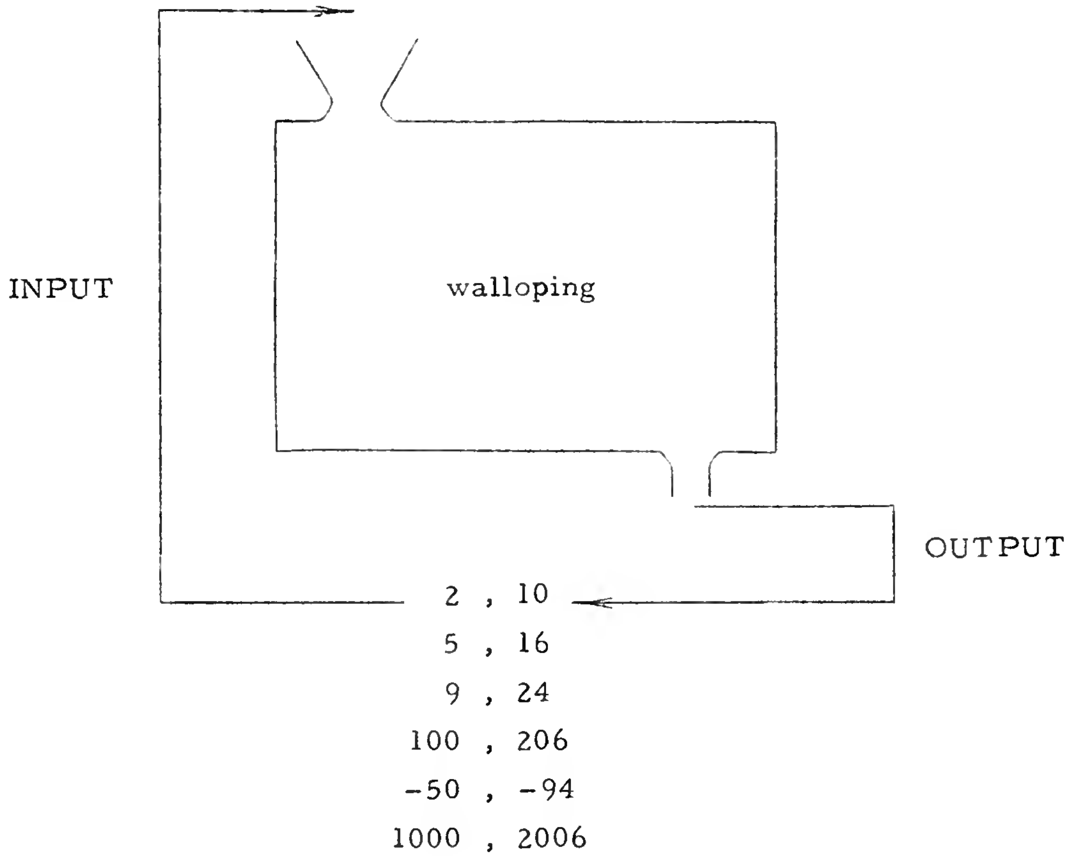
(a) ? - $^{-}2$ = $^{+}7$	(b) ? - $^{-}9$ = $^{+}5$	(c) ? - $^{+}37$ = $^{-}40$
(d) ? - $^{-}12$ = $^{+}83$	(e) ? - $^{-}4$ = $^{-}11$	(f) ? - $^{-}53$ = $^{-}28$

It also proves helpful for doing problems like: $^{+}9 - ? = {}^{-}3$.

Students may find this method useful as a supplement to the "adding the opposite" method, which has advantages along other lines and is a direct application of the principle for subtraction.

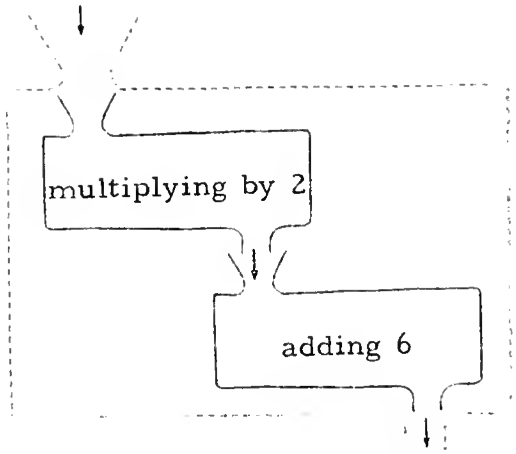
Finally, we leave with you the following problem which has proven to be of interest to students, and will undoubtedly suggest many more pedagogical applications to you--especially in the work on function-composition in Unit 5.

Problem:

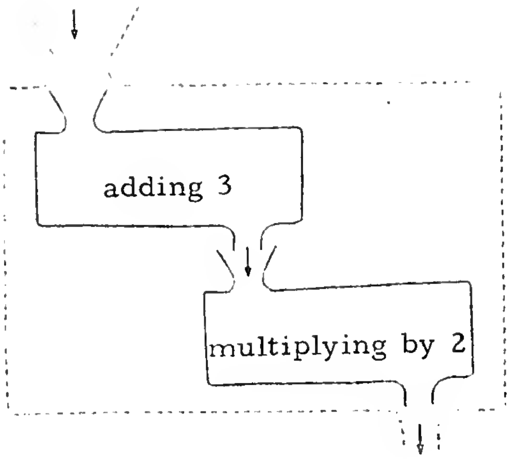


What's inside the walloping machine?

Does it look like this:



or does it look like this:



Those who say that each is correct should be prepared to prove it. --H. W.

[The notation used here is similar to a notation used by E. G. Begle in Introductory Calculus (New York: Henry Holt & Co., 1954) pages 45, 56, and 79. I have recently seen this notation used to illustrate commutative and associative binary operations in J. B. Roberts' The Real Number System in an Algebraic Setting (San Francisco: W. H. Freeman & Co., 1962) pages 9 and 22.]

* * *

UICSM-NETRC Tests Q and R

The last two tests in the film study series are reprinted below for the convenience of teachers who may wish to make use of the items they contain. Please note that item 12 of test P (Newsletter No. 6, p. 15) was defective; the correct answer should have been '(D) cannot tell'. UICSM teachers are urged to submit items they have devised for their own classes so that they may receive consideration for publication in future issues of this Newsletter. --R. S.

*

TEST Q[3-99]:

I. Choose the correct factor or expansion; if the correct answer is not given, mark (D).

1. $\forall_x x^2 - 64 = (x - 8)(\ \ ?\ \ )$

   (A) $8 + x$ (B) $8 - x$ (C) $x - 8$ (D) none of these

2. $\forall_x (x - 7)(x + 2) = \ ?$

   (A) $x^2 - 14$ (B) $x^2 - 9x - 14$ (C) $x^2 + 5x - 14$ (D) none of these

3. $\forall_x 2(\ \ ?\ \ )(x + 1) = 2x^2 - 6x + 8$

   (A) $4 - x$ (B) $4 + x$ (C) $x - 4$ (D) none of these

4. $\forall_x (9x - 1)(x - \frac{1}{9}) = \ ?$

   (A) $(3x - \frac{1}{3})^2$ (B) $9x^2 + \frac{1}{9}$ (C) $9x^2 - 2x - 1$ (D) none of these

5. $\forall_x\ 7x^2 - 42x + 63 = \ ?$

   (A) $7(x + 2)^2$ (B) $(7x + 3)(x + 3)$ (C) $(x + 3)(7x - 3)$ (D) none of these

6. $\forall_x \frac{4}{9}x^2 - \frac{2}{3}x + \frac{1}{4} \neq (\ \ ?\ \ )^2$ [Note the <u>inequality.</u>]

   (A) $\frac{2}{3}x + \frac{1}{2}$ (B) $\frac{1}{2} - \frac{2x}{3}$ (C) $\frac{4x - 3}{6}$ (D) none of these

II. For each of the following equations, choose the <u>sum</u> of its roots when it is solved for 'x'. If only one root is apparent, what is it? If the correct answer is not given, mark (D).

7. $x = 9 - \frac{8}{x}$

   (A) 8 (B) 9 (C) -9 (D) none of these

8. $x^2 + 9 = 6x$

(A) 0 (B) 6 (C) 3 (D) none of these

9. $3x^2 + 10x - 8 = 0$

(A) -10 (B) 10 (C) $\frac{10}{3}$ (D) none of these

10. $9x^2 + 25 = 30x$

(A) 30 (B) $3\frac{1}{3}$ (C) $\frac{5}{3}$ (D) none of these

11. $2x^2 + 13ax = 7a^2$

(A) 13a (B) $-13a$ (C) $\frac{13a}{2}$ (D) none of these

12. $x^2 - 5 = 0$

(A) 10 (B) 5 (C) 0 (D) none of these

III. Choose the correct answer to the question; if none are correct, mark (D).

13. Al is 5 years older than Bill. The product of their ages is 176. How old is Bill?

(A) 12 (B) 14 (C) 16 (D) none of these

14. The product of two consecutive negative whole numbers is 240. What is the sum of these numbers?

(A) -33 (B) -1 (C) 31 (D) none of these

15. One positive number exceeds another by 3. The sum of their squares is 149. What is the larger number?

(A) 7 (B) 10 (C) -7 (D) none of these

16. What is the sum of the roots of the equation '$2x^2 + 4x - 30 = 0$'?

(A) -4 (B) 4 (C) -2 (D) none of these

17. What is the product of the roots of the equation '$x^2 - 6x - 7 = 0$'?

(A) 7 (B) 6 (C) -6 (D) none of these

18. Which quadratic equation has the roots -3 and 5?

(A) $x^2 + 2x - 15 = 0$ (B) $x^2 - 2x + 15 = 0$

(C) $x^2 - 8x - 15 = 0$ (D) none of these

19. If one root of the quadratic equation '$x^2 - 6x + k = 0$' is 2, what is the other root?

(A) 8 (B) 4 (C) 3 (D) none of these

20. For what value of 'n' is the equation '$x^2 - 7x + n = 0$' satisfied by 2?

(A) 5 (B) 10 (C) 6 (D) none of these

Key for Text Q [3-99]:

1. A	2. D	3. D	4. A	5. D	6. A	7. B
8. C	9. D	10. C	11. D	12. C	13. D	14. D
15. B	16. C	17. D	18. D	19. B	20. B	

*

TEST R [3-123]:

I. Choose the correct solution set for each inequation. If none are correct, mark (D).

1. $2x - 7 > 1$

(A) $\{x: 4 < x\}$ (B) $\{x: x < 4\}$

(C) $\{x: 0 > x - 4\}$ (D) none of these

2. $-6x < 20$

(A) $\{x: x < \frac{10}{3}\}$ (B) $\{x: x < -\frac{10}{3}\}$

(C) $\{x: 3x < -10\}$ (D) none of these

3. $x^2 + 25 \geq 10x$

(A) $\{x: |x| \geq 5\}$ (B) $\{x: -5 \leq x \leq 5\}$

(C) $\{x: x = x\}$ (D) none of these

4. $2x^2 > 5x + 12$

(A) $\{x: x > \frac{3}{2} \text{ or } x > 4\}$ (B) $\{x: x < -\frac{3}{2} \text{ and } x > 4\}$

(C) $\{x: x > -\frac{3}{2} \text{ or } x < 4\}$ (D) none of these

5. $|x - 3| < 2$

(A) $\{x: x > 1 \text{ or } x < 5\}$ (B) $\{x: 1 < x < 5\}$

(C) $\{x: x < 1 \text{ or } x > 5\}$ (D) none of these

6. $|x - 5| \geq 2$

(A) $\{x: x \geq 7 \text{ or } x \leq -3\}$ (B) $\{x: 3 < x < 7\}$

(C) $\{x: x \geq 7 \text{ and } x \leq 3\}$ (D) none of these

II. Choose the expression which is equivalent to the given one. If none are correct, mark (D).

7. $\sqrt{147} =$

(A) $7\sqrt{3}$ (B) $10 + \sqrt{47}$ (C) 12.124 (D) none of these

8. $4\sqrt{3} - 5\sqrt{12} + 2\sqrt{75} =$

(A) $14\sqrt{3} - 30$ (B) $4\sqrt{3}$ (C) $16\sqrt{3}$ (D) none of these

9. $\sqrt{(-8)^2} =$

(A) $-\sqrt{-64}$ (B) -8 (C) 8 (D) none of these

10. $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3}) =$

(A) 4 (B) 40 (C) 46 (D) none of these

11. $(y - \sqrt{y})^2 =$

(A) $y^2 - y$ (B) $y^2 - 2\sqrt{y} + y$, $[y \geq 0]$

(C) $y^2 + y(1 - 2\sqrt{y})$, $[y \geq 0]$ (D) none of these

12. $(3a - 2\sqrt{b})^2 =$

(A) $9a^2 - 6a\sqrt{b} + 4b$, $[b \geq 0]$ (B) $9a^2 - 12a\sqrt{b} + 4b$, $[b \geq 0]$

(C) $9a - 12a\sqrt{b} + 4b$, $[b \geq 0]$ (D) none of these

III. Choose the correct answer; if none are correct, mark (D).

13. Given that $4.358 < \sqrt{19} < 4.359$, which of the following is justified?

(A) the approximation to $\sqrt{19}$ correct to the nearest 0.001 is 4.358

(B) the approximation to $\sqrt{19}$ correct to 3 decimal places is 4.358

(C) $\sqrt{19} - 4.358 \leq 0.0001$ (D) none of these

14. Given that $5.656 < \sqrt{32} < 5.657$, which of the following is <u>false</u>?

(A) the approximation to $\sqrt{32}$ correct to the nearest 0.01 is 5.66

(B) the approximation to $\sqrt{32}$ correct to 3 decimal places is 5.656

(C) $\sqrt{32} - 5.656 \leq \frac{0.001}{2}$ (D) none of these

15. What is the approximation to $\sqrt{8}$ correct to 2 decimal places?

(A) 2.82 (B) 2.83 (C) 2.8 (D) none of these

16. What is the approximation to $\sqrt{8}$ correct to the nearest 0.01?

(A) 2.82 (B) 2.83 (C) 2.80 (D) none of these

17. $\forall_x \forall_y \forall_{z>0}$ if $|x - y| < z$ then it is not the case that

(A) $x - z < y < x + z$ (B) $-z < x - y < z$

(C) $y - z < x < y + z$ (D) none of these

18. Which generalization is false?

(A) $\forall_x \forall_y$ if $x^2 = y^2$ then $|x| = |y|$ (B) $\forall_x \forall_y$ if $x > y$ then $x^2 > y^2$

(C) $\forall_{x>0} \forall_{y>0}$ if $x^2 > y^2$ then $x > y$ (D) none of these

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Key for Test R [3-123]:

1. A	2. D	3. C	4. D	5. B	6. D	7. A
8. B	9. C	10. A	11. C	12. B	13. A	14. B
15. A	16. B	17. D	18. B			

* * *

Primarily, mathematics is a method of inquiry known as postulational thinking. The method consists in carefully formulating definitions of the concepts to be discussed and in explicitly stating the assumptions that shall be the basis for reasoning. From these definitions and assumptions conclusions are deduced by the application of the most rigorous logic man is capable of using....

Mathematics is, also, a field for creative endeavor. In divining what can be proved, as well as in constructing methods of proof, mathematicians employ a high order of intuition and imagination. Kepler and Newton, for example, were men of wonderful imaginative powers, which enabled them not only to break away from age-long and rigid tradition but also to set up new and revolutionary concepts....

In its broadest aspect mathematics is a spirit, the spirit of rationality. It is this spirit that challenges, stimulates, invigorates, and drives human minds to exercise themselves to the fullest. It is this spirit that seeks to influence decisively the physical, moral, and social life of man, that seeks to answer the problems posed by our very existence, that strives to understand and control nature, and that exerts itself to explore and establish the deepest and utmost implications of knowledge already obtained.

--Morris Kline
MATHEMATICS IN WESTERN CULTURE
1953

The lengthy description of the first half year's work (Units 1 and 2, in a previous article) was deemed necessary as an attempt to formulate clearly the theoretical considerations that motivated UICSM's choice of material, as well as to outline the material itself. The descriptions of later units, which follow, are somewhat shorter.

*

Units 3 and 4--Equations and Inequations, Ordered Pairs and Graphs--complete the study of 9th grade algebra. Comparison with a traditional text will show that there is less practice with exponents in this part of the UICSM course, and that the treatment of quadratic equations stops with solution by factoring, while the traditional text may culminate with the quadratic formula. Test-scores indicate that these temporary deficiencies are more than made up for by the greater richness and depth of the UICSM development. Of course, both deficiencies are more than made up in later units.

Points especially worth noting in Unit 3 are: the roles played by graphing, by set concepts and notation, and finally, by the theorems of Unit 2, in explaining the ideas underlying the solution of equations; the procedure for obtaining an equation for solving a worded problem by testing a guessed-at solution [pp. 3-64 and 3-65]; the parallelism between methods for solving inequations and those for solving equations [pp. 3-100, 101, 104-107]; and the developing of some feeling for irrational numbers by obtaining approximations [see p. 3-117] to square roots by the dividing-and-averaging method.

Aside from the topics suggested by its title, Unit 4 contains work in factoring [''numerical'' and ''algebraic''] and on exponents and scientific notation.

*

After the experience students have had with set-notation in Units 3 and 4, and the treatment in Units 1 and 2 of some singulary operations as sets of ordered pairs, it is natural for them, in Unit 5, to construe relations as sets of ordered pairs and functions as relations no two of whose members have the same first component. Functions are also discussed as mappings

[see, for example, p. 5-71]. The two points of view--sets of ordered pairs and mappings--appear to be two sides of the same coin. Indeed, to quote Dieudonné [Foundations of Modern Analysis, New York (1960), p. 5]:

> It is customary, in the language, to talk of a mapping and a functional graph as if they were two different kinds of objects in one-to-one correspondence, and to speak therefore of "the graph of a mapping", but this is a mere psychological distinction (corresponding to whether one looks at F either "geometrically" or "analytically").

The notion that a function is a set of ordered pairs makes easily available many examples of functions whose domains are finite sets and, so, makes it particularly easy to develop, through examples and exercises, many concepts having to do with functions. Some of the topics which are thus made readily accessible at this level are function composition, inverse of a function, and functional dependence.

The more special study of numerical valued functions directs attention to another use of the word 'variable'. Such formulas as that for the area-measure of a square [$A = s^2$] are most readily interpreted as asserting a relationship of functional dependence between numerically valued functions [alternatively: between variable quantities]--in this case, the functions A and s. With this interpretation, '$A = s^2$' is a short way of saying that, for each square q, $A(q) = [s(q)]^2$. Not only is this simpler than is the interpretation:

> for each square q, for each number A, and for each number s, if A is the area-measure of q and s is the side-measure of q, then $A = s^2$,

but the simpler interpretation suggests that one define operations of addition, multiplication, subtraction, and division for numerical-valued functions [see p. 5-102]. It is then easy to see that, with respect to these operations, the set of all real-valued functions with a given domain has much in common with the system of real numbers. So much, indeed, that the real-number theorems of Unit 2 can be reinterpreted as statements about such a class of functions. In fact, if one replaced restrictions like '$x \neq 0$' [in theorems about division] by others like '0 is not a value of f', the Unit 2 proofs are still serviceable. This insight justifies one in manipulating the letters in formulas as if they were pronumerals--that is, as if

they were numerals.

Besides the analogy just mentioned between the algebra of real numbers and the algebra of real-valued functions with a given domain, students also investigate, earlier in the unit, some of the analogies between the algebra of real numbers and the algebra of subsets of a given set. After their experience in earlier units of proving theorems which they have guessed hold for real numbers, and their experience of guessing theorems about sets, students are often impatient to learn how to prove theorems about sets. [Rather than attempting to force on students an interest in so-called "modern mathematics", UICSM merely caters, minimally, and in passing, to this burgeoning interest in Boolean algebra.]

The remainder of Unit 5 [about half the unit] is devoted to a more detailed study of constant, of linear, and of quadratic functions. In addition to applying the general concepts previously developed to these special kinds of functions ["Each linear function has an inverse", "The composite of two linear functions is a linear function", etc.], due attention is paid to the geometric aspects of these functions and to their application ["ratio, proportion, variation"]. Moreover, in contrast to traditional treatments, the solution of quadratic equations [by completing the square, and by formula] is merely an application of the theory of quadratic functions. That there are other, perhaps more important, applications of this theory is suggested to students by exercises dealing with maxima and minima and with quadratic inequations. The final section takes up the solution of systems of [two or three] linear equations, and the application of this to the solution of worded problems.

Like the earlier units, Unit 5 ends with collections of Miscellaneous Exercises and Supplementary Exercises. In general, the former serve as reviews of the current and the preceding units. The Supplementary Exercises are intended to be used, as needed, for additional drill in techniques introduced in the current unit and, as time permits, as the basis for further exploration of topics treated in the current unit. This latter use of Supplementary Exercises is particularly well illustrated in Unit 5. Here there are, for example, exercises which introduce such properties of relations as transitivity and antisymmetry, a graphical procedure for composing functions, and a slight introduction to groups through compositions of rotations and reflections.

*

The content of Unit 6--Geometry-- is outlined in the Introduction to the commentary. Briefly, the deductive development of Euclidean plane geometry is based on 15 ''Introduction Axioms'' concerning incidence and order, and 10 ''Measure Axioms'' which deal with measures of segments, angles, and regions. Since geometric figures are construed as sets of points, and measures are numbers, the algebras of sets and of numbers are presupposed. Modulo these considerations, the set of 25 geometric axioms just referred to is adequate for a rigorous treatment of Euclidean plane geometry [but, the axioms are by no means independent], including a treatment of area-measures of polygonal regions. [Area-measures of circular regions, segments, and sectors is dealt with rather informally.] The details of such a rigorous development would, of course, be out of place in a secondary school text, but are carried out to a considerable extent in the commentary.

The experience which students have had, beginning with Unit 2, in using test-patterns to prove generalizations not only has made them aware of the role of proof in mathematics, but also has provided a firm foundation on which to add further knowledge of methods of proof. According to reports from teachers, the Appendix to Unit 6 in which basic methods of reasoning are developed from the viewpoint of natural inference [Gentzen, Jaskowski] is one of the most popular parts of the course. It is usually covered early in the study of the unit--as soon as students' attempts to give geometric proofs have convinced them of the need for more knowledge of methods of proof.

It may be recalled that the proofs in Unit 2 were given in what is, essentially, a two-column [''statement-reason''] form, reminiscent of the form of proofs in traditional geometry texts. This form is used in Unit 2 because the column of ''statements'' is essentially what a student might write in carrying out the simplification of an algebraic expression. The two-column form is not well adapted to more complicated kinds of proof [and its use in traditional texts may account for the traditional difficulty of teaching students to understand indirect proofs]. For this reason UICSM has adopted, in Unit 6 and in later units, a one-column form of proof which lends itself well to proofs of all kinds. It has the additional advantage that students pass rather easily from writing one-column proofs to writing understandable paragraph proofs. They are helped in this, as well as in developing the ability to formulate proofs, by practice in making tree-diagrams as outlines of proofs.

Although, in the earlier units, UICSM writers shied away from making what was thought to be a too explicit introduction of basic logical principles, the feeling now is that it would have been advantageous to introduce much of the content of the Unit 6 Appendix earlier in the course. This feeling is reinforced by the numerous requests from teachers for copies of the Appendix to be used by students who are studying earlier units.

As remarked above, it would be highly inappropriate, even if it were possible, to teach students at this level to give rigorous, complete proofs of geometrical theorems. What is desirable, and what turns out to be possible, is to teach them to give what mathematicians, generally, would accept as adequate proofs and to recognize where their proofs fail of completeness. The latter ability stems from their acquaintance with the Introduction Axioms and with examples of the sort of theorem which can be derived from them. The informal treatment of these matters in the Introduction to Unit 6 prepares students to recognize when, in later proofs, they make use of such facts as, say, that two points are on opposite sides of a given line; and, in such situations, they are able to judge whether this could be derived from earlier steps in the proof and Introduction Axioms. [See p. 6-16 and its commentary, and page 6-22.] Their attitude is that of the working mathematician who knows that his argument is sound and that he could, with sufficient labor, expand it--by the liberal use, say of 'ϵ's and 'δ's--to obtain a rigorous and complete proof. [As is to be expected, they, like the mathematician, are sometimes wrong--and the recognition of their errors is one way in which they learn to judge what steps in a proof can safely be omitted.]

In the actual development of the content of Unit 6, the proofs of several of the earlier theorems make explicit use of the Measure Axioms. However, once a sufficient stock of such theorems has been accumulated, proofs of most of the "standard" geometry theorems are along lines similar to those given by Euclid, and to be found in traditional texts. One exception to this is that, once the appropriate theorems have been proved, it is shown that rectangular coordinate systems can be set up. From this point on, students are free to use either "synthetic" or "analytic" proofs, as occasion warrants.

Feeling that students have little to gain from a deductive treatment of solid geometry, UICSM recommends that teachers using Unit 6 supplement

it with one of several pamphlets now available which acquaint students with the basic concepts, terminology, and theorems of solid geometry. One such treatment has been written by a UICSM teacher, and copies are furnished by the project to participating schools. It is probable that a revision of this will be included as an appendix to later editions of Unit 6.

*

Units 7 and 8, and Unit 9 [the last not yet published] introduce additional basic principles for real numbers sufficient to complete the description of the real number system as a complete ordered field and to characterize its subsets P [of positive numbers], I^+ [of positive integers], and I [of integers]. One notable aspect of these units is the inclusion of numerous sets of Miscellaneous Exercises, distributed throughout the text. Exercises at various levels of difficulty are included. Each unit has a set of Review Exercises [see pp. 7-133 through 7-144 and pp. 8-218 through 8-227], and, for reference, statements of the basic principles which have been adopted for real numbers and of the theorems which have been deduced from them, up to the end of the unit. In looking over such a list [see pp. 8-228 through 8-247], it should be kept in mind that UICSM students have, from the middle of the 9th grade, been accustomed to use quantifiers as the simplest way of stating the generalizations which they have discovered. Consequently, their reactions to what some readers may regard as "excessive formalism" might be expressed by "But, this is the easiest way to say just what I mean."

The first section, 7.01, of Unit 7 reviews the basic principles which were adopted in Unit 2, shows how "computing facts" can be established, and, after pointing out by the use of a model that '$1 \neq 0$' cannot be derived from these basic principles, introduces this sentence as a new basic principle. [It is important to realize that, for UICSM students, the basic principles are not abstract postulates, but, rather, statements of part of what they already know about the real numbers. Some theorems are also of this character, while others formulate new knowledge which has been guessed at on the basis of experience and tested by showing [by proof] that it could have been predicted from basic principles. So, although students have always known that 1 is different from 0, they see the need for including a statement of this among their basic principles. Similar remarks apply to the other basic principles adopted in Units 7 and 9.]

Consideration of whether '$2 \neq 0$' can be derived from basic principles leads to the adoption, in section 7.02, of four additional basic principles [see p. 8-232] concerning positive numbers. While choosing these it becomes clear that they make it possible to prove, for any of 2, 3, 4, etc., that it is not 0. However, there appears to be no way to prove, at this point, that each positive integer differs from 0. That this should not be surprising becomes clear on noting that none of the basic principles makes any explicit mention of the positive integers. So, there is a need for still more basic principles--a need which is satisfied in section 7.04.

In section 7.03, the relation > is defined [see (G) on p. 8-232], and the knowledge concerning this relation which students have acquired in earlier units--particularly in work with inequations in Unit 3--is systematized. Incidentally, it is pointed out that this could be done on the basis of the five statements which make up Theorem 86 [see p. 8-233], rather than on the basis of (P_1) - (P_4) and (G). New insights concerning > are obtained and formulated as theorems [see, for example, p. 7-39 and commentary, TC[7-39, 40]a and b] and students' knowledge of strategies of proof is increased.

In Unit 4 students did some work with positive integers [factoring with respect to I^+, even and odd integers, etc.]. In section 7.04 the assumptions on which this work was based are collected and proposed as theorems which should be derivable from basic principles [as a matter of fact, each is proved in Unit 7 or Unit 8]. The additional basic principles [see p. 8-234] which are needed have already been suggested by Exploration Exercises [see pp. 7-45, 7-46, and commentary, TC[7-45, 46]a, b, and c] which immediately precede this section. The technique of mathematical induction is developed, and much of the remainder of the unit is devoted to practice in the techniques of formulating recursive definitions and of using them as bases for proofs by mathematical induction. The traditional difficulty--that, in a poorly-written proof by mathematical induction, one seems to be assuming what one wishes to prove--seldom arises. Students' familiarity with test-patterns prepares them to understand what is going on. Such topics as figurate numbers, compound interest, and combinatorial problems yield interesting subjects for practice [see pp. 7-66, 7-70, 7-71 and commentary, TC[7-66], TC[7-70]a, b, and TC[7-71]]. Traditional treatments of induction at, say, the first-year college level concentrate on summation problems. The rather extensive UICSM treatment of such matters comes in Unit 8.

Among other matters relating to the order-type of the positive integers, section 7.05, after exploring the concepts of lower bound and least member [see p. 7-87 and commentary, TC[7-87]a, b], presents a proof of the least number theorem [see p. 7-88 and commentary, TC[7-88]]. Also, as a temporary expedient, a basic principle [see (C) on page 8-235], equivalent to the Archimedean property of >, is introduced. In Unit 9, (C) will be "reduced" to the status of a theorem on the adoption of a completeness principle. At present, (C) is needed to characterize the way in which I^+ is imbedded in the set of real numbers. In particular it is used, in section 7.06, to show that the domain of the greatest integer function (the greatest integer $\leq x$) is, as one naturally suspects, the set of all real numbers.

In section 7.06, some of the previous results concerning positive integers [closure, induction, least number theorem, etc.] are shown to hold [in appropriately modified forms] for all integers. In doing this, use is made of mappings--translations and the reflection through 0--of the real number system on itself. The section continues with a treatment of the greatest integer function. Students are well acquainted with this function from work in earlier units [see, for example, pp. 7-41 and 5-254]. They can now establish several theorems which they will have occasion to use later. Finally, a study is made of the divisibility relation, comparing it with $\leq$; pictorially representing the lattice of divisors of an integer; discovering the Euclidean algorithm; and solving Diophantine problems.

Unit 8--Sequences--comprises two sections, one centering on Σ-notation, the other on Π-notation. In the former, students have many opportunities to guess at summation theorems by inductive methods, and to verify their guesses by using mathematical induction and the recursive definition of Σ-notation. [The final form of the definition is reproduced on p. 8-237; the earlier form there referred to is:

$$\sum_{p=1}^{1} a_p = a_1, \quad \forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1}]$$

Students also learn short cuts--based, essentially, on finite-difference methods--for simultaneously discovering and proving summation theorems. In contrast to traditional texts, the "theory" of arithmetic progressions appears here, in something like proper perspective, as an almost trivial application of a general theory. In addition to dealing with special problems

[many of which are, by the way, geometrical in origin], students prove and apply general theorems [see pp. 8-238 through 8-240] on summation.

In the second section of Unit 8, the factorial sequence and the exponential sequences are introduced as important examples of sequences of the kind which arise when one considers continued products of the terms of other sequences. The domain of the exponential sequences is then extended to include all integers, and the usual "Laws of Exponents" are proved by mathematical induction. Appropriate exercises lead to the discovery [and consequent proof] of Bernoulli's inequality. This and an earlier theorem [Theorem 153 on p. 8-242] form the basis for discussing geometric progressions, both "finite" and "infinite". Incidentally, geometric progressions appear as a minor generalization of exponential sequences. There follows additional work on factoring, based in large part on the previously mentioned Theorem 153. There is, next, a rather extended discussion of combinatorial problems. Students have dealt with a few such problems in Units 5 and 7 ["How many reflexive (symmetric) relations are there whose field is a given set of 5 elements?", "How many shortest routes are there from one corner of a 3-by-5 block city neighborhood to the opposite corner?"]. Here they make a more organized study of combinatorial problems. This leads, finally, into the binomial theorem and, optionally, a recursion formula for computing sums of powers of consecutive positive integers and a formalization of their previous discoveries concerning the use of successive differences to obtain summation formulas.

The unit ends [except for Review Exercises] with eleven pages on prime numbers [sieve of Eratosthenes, infinitude of primes, occurrence of primes in arithmetic progressions, and proof of the existence and uniqueness of prime factorization].

Unit 9 continues the study of exponentiation. A completeness principle --the least upper bound principle--guarantees the existence of roots of nonnegative numbers. On the basis of appropriate theorems on roots it is easy to introduce rational exponents. Monotonicity of the exponential functions [restricted at first to rational arguments], and a certain amount of arm-waving, justifies the introduction of irrational exponents. The logarithm functions appear as inverses of the exponential functions. As a matter of course, this so-rapidly outlined theoretical structure is buttressed

and held together by much practice, leading to discoveries, and much application of such discoveries to problem-solving.

Like Unit 9, Units 10 and 11 exist only in preliminary [but already well-tested] versions. They deal with the circular functions, polynomial functions and complex numbers.

*

In conclusion, it seems appropriate to repeat that the relatively deep theoretical development, and the seeming emphasis on formality, are not evidences of an abstract approach to mathematics as a subject divorced from reality. On the contrary, the former is made possible only by allowing students ample opportunity to carry out the preliminary thinking and experimenting which they need as a basis for concept formation; and the latter [which is by no means as great as it has appeared to some who have been content with a superficial scanning of UICSM texts] is necessitated by the need which students feel to formulate their discoveries precisely and efficiently so that they can be used as a reliable basis for arriving at further discoveries. --H. E. V.

* * *

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

--George Polya
HOW TO SOLVE IT: A New Aspect
of Mathematical Method
1945

DICK ZILCH GOES TO PRISON

There are at least five state prison inmates in existence who know that half of '99' is '9', for UICSM mathematics has penetrated the walls of the Massachusetts State Correctional Institution at Norfolk. Two volunteers on the prisons committee of Harvard's Phillips Brooks House, the University instrument for mobilizing student social work, spend their Friday afternoons teaching squotes and the principles of arithmetic to a class of five at Norfolk. Joel Selig, a sophomore, and Ray Frieden, a freshman, the former a mathematics and the latter a physics major, and both past members of the first UICSM class in Newton, Massachusetts, are firmly convinced that the Illinois way is the only way to teach mathematics to adults.

Having become familiar with the philosophy of rehabilitation that obtains at Norfolk, where inmates live not in cells but in dormitories, move about freely within the walls that are prescribed by "careful custody", and spend a good part of their week in the Education Building, Mssrs. Selig and Frieden thought "What better way to interest these men in mathematics than to let them have a go at Units 1-4?" With the encouragement of Professor Beberman and Dr. W. Eugene Ferguson, head of the Math Department at Newton High School and their teacher for two consecutive years, they embarked, in this spirit, upon an unusual program. Each week one of them teaches and the other becomes a member of the class, and the emphasis is, of course, on student discovery and participation. At the pace of one class a week the group covered Unit 1 in the first semester, and the student-teachers plan to give their inmate-students a good taste of Units 2 and 3 at least during the remainder of the school year.

"Time for teaching, let alone testing, is scarce, and getting the men to do long assignments at home requires some doing, but we feel we can safely claim a high degree of success," say Mssrs. Selig and Frieden. "At this point three out of five do their homework very thoroughly, and the others spend a satisfactory amount of time on it. On the basis of three quizzes, two would be getting a B in a high school class, the other three a C. By the end of the year we hope to have all A and B scholars.

"More important than this, however, is the way in which the men respond to the UICSM method. For most of them, of course, it is a completely new way of thinking, and although only one has had an extensive background in mathematics, they all find it a bit difficult to change their old ways. They are all enthusiastic about learning modern mathematics, and they welcome us as they do all Harvard teachers as living proof that society has not completely rejected them. We have no idea what their crimes are, and we aren't interested. We look upon them as exactly what they are: mature adults who have the right to respect, patience, and understanding, and of whom we have the right to expect self-discipline.

"We try never to dismiss a question before we have gotten to its root, but we push the class as much as we can without losing them. Enrollment in prison classes tends, in general, to fluctuate; we have five steady, interested students. The question "What in the world is the book talking about?" is seldom asked; the men appreciate the ease and liveliness with which the textbooks are written, and they usually need only the clarification of a few points to complete the learning which they themselves have done at home. In short, when we can ask our students what they notice about the operation adding zero and get the answer "It is its own inverse", we are much happier than we would be could they automatically cancel out the 'x' in '$4x^2y/x$'. In our UICSM class the men have, for the first time, an opportunity to use their minds in the study of mathematics--and they enjoy doing so." --J. S.

* * *

It is only proper to realize that language is largely a historical accident. The basic human languages are traditionally transmitted to us in various forms, but their very multiplicity proves that there is nothing absolute and necessary about them. Just as languages like Greek or Sanskrit are historical facts and not absolute logical necessities, it is only reasonable to assume that logics and mathematics are similarly historical, accidental forms of expression. They may have essential variants, i. e. they may exist in other forms than the ones to which we are accustomed. Indeed, the nature of the central nervous system and of the message systems that it transmits indicate positively that this is so.

--John von Neumann
THE COMPUTER AND THE BRAIN
1958

In June, 1961, the School Board of Tucson (Arizona) School District No. 1 adopted a salary schedule credit in-service program. The first group to take advantage of this policy was the Mathematics In-Service Seminar. One group, under the direction of Mrs. Katharine J. S. Sassé, studied Units 1 and 2 of UICSM First Course and the NETRC films. This group is continuing second semester with the study of Units 3 and 4 and the NETRC films under the direction of Miss Ruby Matejka. The other group, directed by Mr. Ralph Futrell, studied Unit 6. This group is continuing second semester under the leadership of Mrs. Barbara Buchalter in the study of Unit 5. Evaluation returns for the first semester indicate that the program is very satisfactory--K. J. S. S.

Sister Mary of the Angels (St. Rosalia High School, Pittsburgh) spoke before the annual convention of the Catholic Educational Association of Pennsylvania last November. Her topic, "Experiences with a New Mathematics Program", was presented to the Secondary School Section on November 17. She reports covering "the history of our participation in UICSM, reactions of pupils, parents, and teachers, and a slight explanation of the presentation of equations and inequations."

"Walter Rucker and I would like to let you know of the progress of UICSM classes in Redlands [California], and to tell you of an in-service class which we are conducting for teachers in the district. There are about 15 teachers attending a 3-hour evening class which meets one night per week. It is presently scheduled for 12 sessions, in which we hope to complete Units 1 and 2, and possibly part of Unit 3. We have had a group of seventh graders in for a demonstration class, and we plan to have other demonstration classes at appropriate times."--Paul Krantz

At the December meeting of the Arctic Branch, American Association for the Advancement of Science, at the University of Alaska, the Rev. Fr. Charles A. Saalfeld, S.J., spoke on UICSM under the title "A New Approach to the Teaching of Mathematics." Father Saalfeld teaches at the Monroe High School in Fairbanks and has attended NSF summer institutes in Urbana.

Mr. Donald D. Hankins, Mathematics Chairman at the Crawford High School, San Diego, California, reports that about 70 secondary mathematics teachers attended his presentation on "The UICSM Program in Your School" at the Second Annual Math-Science Weekend in Santa Monica last December. Mr. Hankins writes: "The presentation included: 1. A brief description of the UICSM program; 2. Our experience with UICSM materials in our pilot program (San Diego City Schools). This consists presently of 35 classes and 17 teachers; 3. Problems in implementing a UICSM program--this included our in-service program, meeting with parents, etc. I am happy to report there was considerable interest."

Mrs. Mary S. Huzzard of the Cheltenham High School, Wyncote, Pa., spoke to the Association of Mathematics Teachers of the Independent Schools in the Baltimore area about UICSM last November. In succeeding weeks there were programs on the Maryland Program, Madison Plan, and SMSG. Mrs. Huzzard spoke for about an hour and a half, then answered questions.

Mr. Arnold Petersen, Head of the Mathematics Department at the Pascack Valley Regional High School, Hillsdale, N. J., and Miss Maureen Jordan of his staff shared the platform at a meeting of the Mathematics Committee of the Bergen County Education Association in October. Between 60 and 70 teachers from the northern New Jersey area heard Miss Jordan speak on "Proof and the Algebra of the Real Numbers;" Mr. Petersen spoke on "Logic and the UICSM Geometry." As a result of this presentation, the Committee will include in its roster of in-service training courses a class in UICSM First Course, Mr. Petersen appeared on a three-man panal sponsored by the N. J. Section of M. A. A. at St. Peter's College in Jersey City in November. The philosophy, content, and evidence for success of UICSM geometry were presented to an audience of about 100 college and secondary teachers at that time. Mr. Petersen arranged a number of meetings in his area at which Max Beberman spoke in December. In February, Mr. Petersen spoke on "Modern Mathematics: Lower Secondary Level" at the Don Bosco High School in Ramsey, N. J., and presented the entire UICSM program at a teachers' meeting in Chatham. The versatile Mr. Petersen led a discussion at a conference on language sponsored by the New Jersey English Teachers' Association at Montclair State College in March. He spoke there again at a conference on gifted children on April 5th and is to be consultant for a U. S. Office study on modern mathematics for academically talented students during two conferences held at Teachers College, Columbia University.

Newsletter No. 8 will be the last of the 1961-62 academic year, and will include several articles and letters on teaching aids and suggestions by UICSM teachers that have accumulated in the past several months. More are needed, however, and anything that reaches your editor by May 12th will receive careful consideration for publication. News items for the next issue must also be in the editor's hands by that date. Notes on the recent activities of the UICSM staff will be included in the next issue.

WANTED

A teacher trained in UICSM to fill an opening in the mathematics department at the Colegio Roosevelt in Lima, Peru, beginning August 1, 1962.

"The salary schedule--considering living costs-- is excellent (furnished housing thrown in), the students capable. ... I am leaving in August, and without a UICSM man to replace me, the program may collapse."

Interested parties should write to:

Mr. Richard E. Johnson
Mathematics Department
Colegio Roosevelt
Libertadores 500
San Isidro
Lima, PERU
South America

UICSM Newsletter

An occasional publication of the

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

1208 West Springfield

Urbana, Illinois

UICSM Newsletter No. 8 May 23, 1962

THE MATH WARS

Curriculum change is still very much "in the news" these days, though the mood of national self-searching following upon the October 1957 blow to our complacency seems to have faded somewhat. Within slightly over a decade, mathematics education in particular has been transformed from a rather settled and uniform scene into one of such apparent confusion and controversy that some teachers are understandably dismayed and feel wholly justified in "standing pat" until they see what comes of it all. Those who perceive an unsettled situation as an opportunity and a challenge rather than a threat, however, may profit from looking into what might well become a "controversial issue"--or whole pack of such issues--among mathematicians and mathematics teachers within the next few years. Have the recent reforms moved too far too fast, in wrong directions, and counter to some of the virtues of the traditional program? Something like this seems to be on the minds of the sixty-five mathematicians who signed the statement "On the Mathematics Curriculum of the High School", as published in the March 1962 issues of the Mathematics Teacher (pp. 191-195) and the American Mathematical Monthly (pp. 189-193). Mathematics teachers would do well to be aware of this statement, even if they should decide that it raises many more questions than it seeks to answer.

There is also a well-balanced and refreshingly literate article under the above martial title in the Spring 1962 American Scholar magazine, written by a professor of English at Amherst College (pp. 296-310, passim). Reading it may help clarify and bring into focus the situation in our somewhat untidy house for you. --R. S.

A BLACKBOARD LATTICE

Since this is my first year in teaching with UICSM materials, I have found it quite a challenge that no one, at least locally, has devised any classroom aids for this program. After some experimentation, I managed to make a lattice board for class demonstration that has become my favorite aid.

When I started teaching Units 4 and 5, the thought of having to draw or mimeograph lattices was dismaying enough without the added necessity of lattices on the blackboard! I tried punching holes in a piece of chart paper, and then in a window shade, pounding chalk over it whenever I needed a lattice (using the powdered chalk used for marking hemlines), but soon gave it up as too dusty and ineffective. The dots were not clear and rubbed off too easily.

A piece of pegboard proved to be better, using dowels and other pieces from a child's pegboard game. Even this had its drawbacks. The holes could not be seen easily and were too close together to allow much writing with chalk.

[This doesn't mean that I have discarded the pegboard! It is useful in Unit 6 in demonstrating the relations between the lengths of the sides of a triangle. Paper fasteners clipped through the holes support linkages easily made from stiff cardboard. Our math club members also use it for playing strategy games, such as Matrix (a trade name of a commercially-produced game).]

Finally, I used a portable blackboard, plastic Contac (a trade name), punched holes in the plastic shelving material (using light cardboard as a backing when punching the plastic since it was too soft to be cut cleanly by the punch), and used the punched-out circles of plastic, sticking them on the blackboard to make a permanent, washable latticeboard. The circles can be written on (as well as around) without falling off, can be purchased in a wide range of colors to contrast with the color of the board for easy viewing, and can be peeled off without damaging the board if they are no longer needed. It can withstand much use, even by the students. Once the major task of peeling the backing from the plastic (long fingernails are handy) and applying the circles is completed, it requires no more maintenance than the ordinary blackboard.

--Mrs. Dorothy Ono
Kaimuki High School
Honolulu, Hawaii

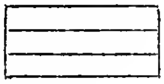
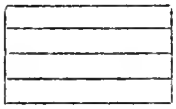
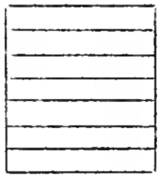
* * *

Recommended reading:

Edwin Moise, "The New Mathematics Programs."
THE SCHOOL REVIEW, Spring, 1962, pp. 82-101.
(Vol. 70, No. 1)

--R. S.

Then Miss Mills taught Peter to add and subtract and multiply and divide. She had once heard some lectures upon teaching arithmetic by graphic methods that had pleased her very much. They had seemed so clear. The lecturer had suggested that for a time easy sums might be shown in the concrete as well as in figures. You would draw an addition of 3 to 4, thus:

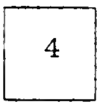
 added to  makes this heap 

And then when your pupil had counted it and verified it you would write it down:

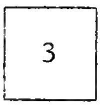
3 + 4 = 7

But Miss Mills, when she made her notes, had had no time to draw all the parallelograms; she had just put down one and a number over it in each case, and then her memory had muddled the idea. So she taught Joan and Peter thus: "See," she said, "I will make it perfectly plain to you. Perfectly plain. You take three--so," and she drew

"and then you take four--so," and she drew



"and then you see three plus four makes seven--so:



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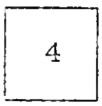
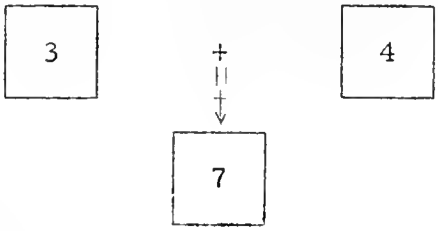


=


"Do you see now how it <u>must</u> be so, Peter?"

Peter tried to feel that he did.

Peter quite agreed that it was nice to draw frames about the figures in this way. Afterwards he tried a variation that looked like the face of old Chester Drawers:



But for some reason Miss Mills would not see the beauty of that. Instead of laughing, she said: "Oh, no, that's quite wrong!" which seemed to Peter just selfishly insisting on her own way.

Well, one had to let her have her own way. She was a grown-up. If it had been Joan, Peter would have had his way....

*

Peter was rather good at arithmetic, in spite of Miss Mills' instruction. He got sums right. It was held to be a gift. Joan was less fortunate. Like most people who have been badly taught, Miss Mills had one or two foggy places in her own arithmetical equipment. She was not clear about seven sevens and eight eights; she had a confused, irregular tendency to think that they might amount in either case to fifty-six, and also she had a trick of adding seven to nine as fifteen, although she always got from nine to seven correctly as sixteen. Every learner of arithmetic has a tendency to start little local flaws of this sort, standing sources of error, and every good, trained teacher looks out for them, knows how to test for them and set them right. Once they have been faced in a clear-headed way, such flaws can be cured in an hour or so. But few teachers in upper and middle-class schools in England, in those days, knew even the elements of their business; and it was the custom to let the baffling influence of such flaws develop into the persuasion that the pupil had not "the gift for mathematics." Very few women indeed of the English "educated" classes to this day can understand a fraction or do an ordinary multiplication sum. They think computation is a sort of fudging--in which some people are persistently lucky enough to guess right--"the gift for mathematics"--or impudent enough to carry their points. That was Miss Mills' secret and unformulated

conviction, a conviction with which she was infecting a large proportion of the youngsters committed to her care. Joan became a mathematical gambler of the wildest description. But there was a guiding light in Peter's little head that made him grip at last upon the conviction that seven sevens make always forty-nine, and eight eights always sixty-four, and that when this haunting fifty-six flapped about in the sums it was because Miss Mills, grown-up teacher though she was, was wrong.

--from a novel by H. G. Wells
JOAN AND PETER: The Story of an Education
New York: The Macmillan Company, 1918

* * *

A NEW TOOL FOR TEACHING MATHEMATICS: THE OVERHEAD PROJECTOR

The purpose of this paper is to acquaint the reader with a new teaching tool, the overhead projector. This tool provides him with a means of more readily visualizing concepts that may require more than just the spoken word for complete understanding. Unlike other types of projectors, it can be used in a fully lighted room in front of the class. This permits the teacher to face his class at all times. It also provides him with a means of presenting pre-drawn constructions, illustrations, and even fully detailed theorems or other written material without having to take class time to chalk them on the blackboard.

Mechanics of projection: the projector. A strong beam of light is cast up through the stage to a lens suspended about a foot above the stage. The light passes through this lens and is projected upon a screen several feet away.

The projectual. The projectual is the key factor to overhead projection. Just what is a projectual? Basically, it is nothing more than a colorless, transparent sheet of acetate, ranging in size from 12 by 12 inches to 8-1/2 by 11 inches. This sheet is placed upon the stage of the projector. Anything relatively opaque, such as the wax of a marking pencil or thick ink from a special pen, will stop passage of light through part of the projectual when applied to the acetate, thus causing a shadow to be cast upon the screen. For example, if the teacher uses a wax pencil to write his name, the name will appear in sharp detail on the screen many times the original size. If color is desired, translucent colored ink is used.

While facing the class the teacher can write on the projectual as he normally would, from left to right and from top to bottom.

Types of projectuals. Projectuals may be static, dynamic, or of the overlay type. I have discussed the static projectual, which is simply a pre-drawn sheet of acetate in one color. The "color" can be an opaque black shadow or some other color if translucent markings are made with colored ink.

The dynamic projectual is like this except that several colors are used on the same sheet or on several sheets which have been taped or stapled together. This produces a variety of color combinations in the image. However, care must be taken to avoid too great a thickness of these translucent inks or tinted sheets. This may cause the net result to be relatively opaque and cast a black shadow.

The overlay projectual is perhaps the most dramatic. It consists of a series of acetate sheets with hinges or slides. It permits the exposure of information at different times and buildup of a topic before the eyes of the audience.

Suggested use of projectuals. A combination of any or all of these types of projectuals can be used in the mathematics classroom to achieve the desired impact and illustrate an idea vividly to the student. In my own teaching I have found that projectuals which superimpose various types of geometric figures are particularly helpful. For example, in discussing regular polygons I use an overlay which has a regular pentagon as its basic projected image. Overlays are used to illustrate non-regular pentagons having some sides or angles congruent to those of the regular pentagon. By superposition of these images the material is more clearly presented.

Overlays of test patterns into which the teacher can write numerals or multicolored pronumerals are most enlightening to the student. Any geometric set which is the union or intersection of other sets can also be vividly illustrated by the use of the overlay. It is not easy to retrace steps or show parts independent of others when using chalk and a blackboard. It is very easy to accomplish this with the overlay, since any sheet may be removed or replaced, thus permitting a buildup or breakdown of the problem at any point in the lesson.

Other uses of the projector. Besides utilizing projectuals there are other ways to use the overhead projector. Opaque objects such as blocks, gears, etc., can be projected as black images by placing them directly on the stage of the projector. Slide rules, rulers, protractors, and other instruments made out of transparent materials can also be used for demonstration purposes with excellent results.

Making projectuals. A teacher can make over a hundred reusable static projectuals for less than $15.00. Clear sheets of 12 by 12 inch reprocessed X-ray film can be purchased for about 2 cents per sheet. With these sheets, a pen using special ink that lays an opaque line, and several felt tip ink markers of various colors, the ingenious teacher is well on his way to a personalized and very effective set of projectuals.

Several companies produce two types of felt markers. One type dispenses a permanent ink which produces a very effective color when applied to the acetate. The ink can be removed with a cutting agent such as duplicator fluid or alcohol. The second type dispenses a water base ink. It too produces effective color, although not as rich in tone as the permanent ink. However, this ink can be removed with just a water-dampened tissue.

When writing on the projectuals one should use letters at least 1/4 inch high to assure easy reading by the audience. The time spent in preparation of projectuals is a factor to consider--the time spent being in direct proportion to the degree of accuracy and professional-looking finish that the maker desires to achieve.

The Thermo-Fax Sales, Incorporated, produces a specially treated acetate which may be used with their standard copy machine to reproduce on acetate most printed material. These sheets cost about 20 cents each. A heat process is used in making such copies. If such a machine is available the teacher will find it easy to use for this process. The Technifax Corporation has a fairly easy but equally expensive process for doing the same thing. It is a light-sensitive process, which also requires the use of special equipment. While both these companies have good processes for developing the black or shadow image, I would not hesitate to recommend the Technifax process over the other when using color in the projectual. In addition, Technifax has a novel method of pin registration for perfect alighment when making overlays.

There are other ways for reproducing information on acetate which the reader may wish to investigate, but the mentioned ones seem to me to be the easiest and most economical. Several companies sell commercially prepared projectuals but as yet I have seen only a few in the field of mathematics worth purchasing.

Physical setup of room. If the room is arranged so as to provide for use of the projector at a distance of nine feet from the front of the room, the image on the screen will be approximately 4 by 5 feet. Moving the projector closer to the screen will reduce the size of the image, moving it away will increase the size.

The projector is most effectively used at desk level. This allows the teacher to be seated and, in turn, makes the screen the natural center of attention. The image is seen to best advantage if it is projected onto a screen which has its lower edge above the upper edge of the blackboard. This arrangement also permits full use of the blackboard while the projector is in operation.

A screen which tilts forward at the top will prevent "keystoning". As the name suggests, keystoning gives the projected image a fanning out appearance from the bottom edge to the top. This is due to the top edge of the screen being farther from the light source than is the bottom edge. By using a tilted screen the image will appear in its intended rectangular shape. To achieve "professional-looking" projection a tilted screen is a must. But certainly, however, good results can be achieved by using a standard wall projection screen or even a large map turned to its blank side.

The size of the room and size of the class will dictate a procedure for best location on the projector. I have found the arrangement diagrammed on the next page to be quite satisfactory. Even the teacher who "floats" from room to room can use the overhead projector to advantage. I was in that situation myself for two years and found the overhead projector not only useful but almost a necessity.

<u>Building a tilted screen</u>. If a tilted screen is desired but funds are not available for purchasing one, such a screen 4 by 6 feet in size can be built for less than $9.00 in materials.

Materials and directions:

(a) One sheet of 1/8" by 4' by 8' masonite (pressed board similar to that used for pegboard but without the holes). Cut off a 2 foot width at the abvious end to make a 4 by 6 feet screen.

(b) 20 feet of soft pine pregrooved molding, and nails just a bit shorter than the molding is wide. Cut corners of molding at an angle of 45 degrees and attach to the screen. This molding is necessary for rigidity.

(c) 3 strong hinges and screws for attaching screen to wall (see diagram).

(d) 4 large screw eyes and about 20 feet of strong cord (venetian blind cord works well).

(e) 2/3 quart of flat white indoor wall paint to paint one side of the screen. Two or three coats will be necessary.

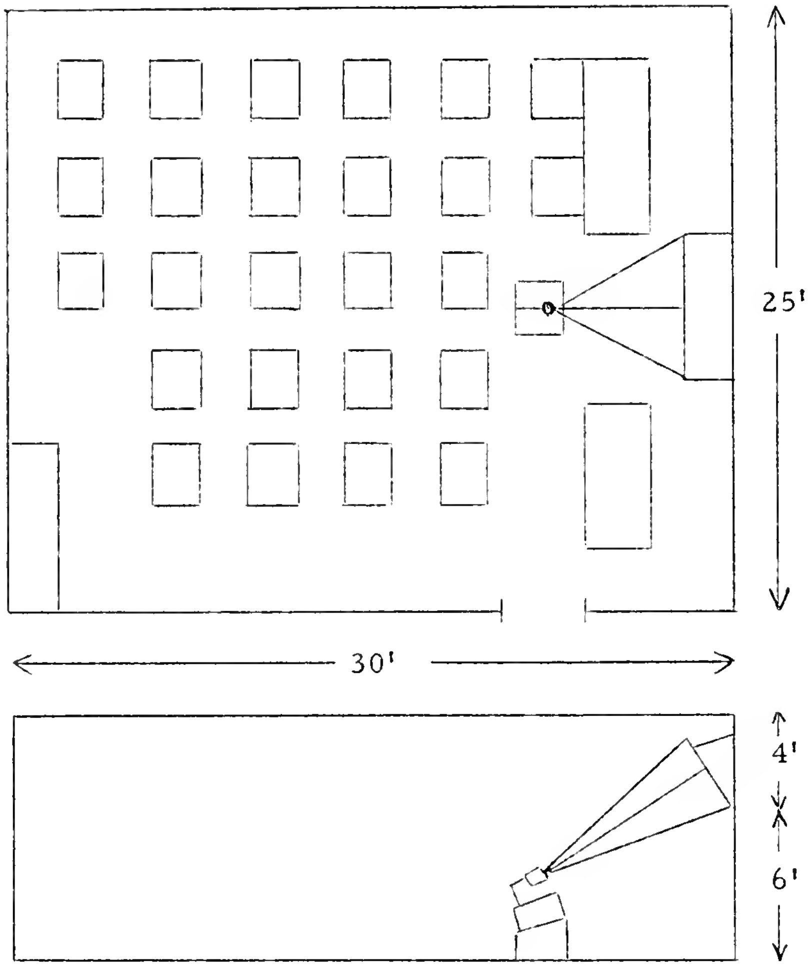
(f) 1/2 pint of flat black indoor paint to paint molding.

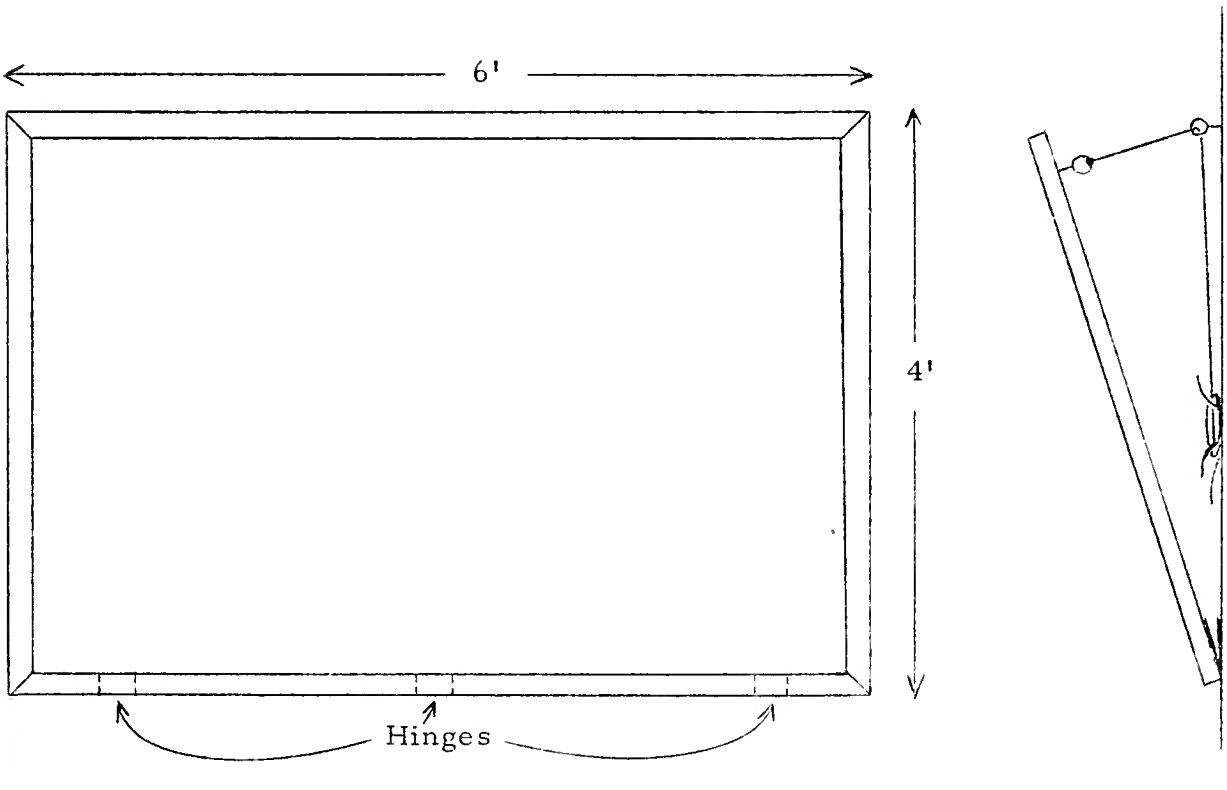
(g) 2 cleats to fasten cords.

I have made and am using a screen of this type and find it quite satisfactory.

<u>Where to begin</u>. I recommend the purchase of a supply of clear acetate, several ink markers, wax or grease type pencils, and a pen for using acetate-content ink which will adhere to the acetate. The Johnson Process Company distributes acetate reprocessed from old X-ray film for about 2 cents per sheet. New, clear acetate, prepunched for pin registration, as well as frame mounts to fit, can be purchased from Technifax. The acetate sells for about 10 cents per sheet and the mounts for about 20 cents each. The Charles Beseler Company also distributes these materials. The frames are handy, although shirt cardboard and a few staples will suffice.

Ink markers and pencils can be purchased from any stationery store. "China" type marking pencils work very well. A pen such as the Acetograph ($4.95), produced by the Koh-I-Noor Company, can be


25'
30'
4'
6'


6'
4'
Hinges

purchased at many stationery stores or from distributors such as Technifax. I suggest that the following concerns be contacted for more detailed information:

(a) The Charles Beseler Company, 219 South 18th Street, East Orange, New Jersey. Ask for "Price List: Vu-Graph materials and supplies".

(b) Johnson Process Company, 80 Front Street, Elizabeth, New Jersey.

(c) Technifax Corporation, Holyoke, Massachusetts. Ask for the "Diazochrome Slide Catalogue", Visucom (an occasional publication), and "Visucom Equipment and Materials".

(d) Thermo-Fax Sales, Incorporated, St. Paul 6, Minnesota. Ask for "Overhead Projection Tips", "A Study in School Communication", and "New Projection Transparencies for Modern Visual Communications".

--William J. Masalski
Greenwich Public Schools
Greenwich, Connecticut

* * *

I believe we shall discover that there is for the inquiring mind a hierarchy of significance, with a place for all reality, but a place in an ordered system. Perhaps the mind which gluts itself indiscriminately upon thousands of facts is itself a mind which loves reality but little. There is an intrinsic sense, or order and system, in the world of meaning, which--just because the world itself is ordered--can lead from one love of reality to a still more comprehensive love of reality. The role of the teacher will not be fulfilled by turning over a thousand stones, but by enabling the child of youth to see in the stone which arouses his interest the history of this world, the evolution of its waters, atmospheres, soils, and rocks, prying into deeper meanings "just because they are there."

--Gardner Murphy
FREEING INTELLIGENCE
THROUGH TEACHING
1961

A SUGGESTION ON CHECKING HOMEWORK

I would like to mention a procedure that has been helpful to me in checking homework assignments, and also in presenting new material to my algebra 9 class. Perhaps this is something that the teaching machine folks are working on or may be interested in.

I have been projecting the solution to homework assignments on the overhead projector at the beginning of the class period. In this way all of the students can check all of their work. This gives me an opportunity to walk around the room to observe the students' work. It also gives the students an opportunity to discover their own mistakes. When students question certain problems I can point to the parts of the problem or draw in the grouping symbols that are there by convention, etc.

If I had written all of this material on the plastic roll that comes with this machine I wouldn't have to do it all over again next year. I have been carrying four plastic sheets in my briefcase each day, erasing, and writing the next day's work at home.

--Richard S. Davis
York Community High School
Elmhurst, Illinois

* * *

New types of logic may help us eventually to understand how it is that electrons, the velocity of light, and other components of the subject matter of physics appear to behave illogically, or that phenomena which flout the sturdy common sense of yesteryear can nevertheless be true. Modern thinkers have long since pointed out that the so-called mechanistic way of thinking has come to an impasse before the great frontier problems of science. To rid ourselves of this way of thinking is exceedingly difficult when we have no linguistic experience of any other and when even our most advanced logicians and mathematicians do not provide any other--and obviously they cannot without the linguistic experience. For the mechanistic way of thinking is perhaps just a type of syntax natural to Mr. Everyman's daily use of the western Indo-European languages, rigidified and intensified by Aristotle and the latter's medieval and modern followers.

--Benjamin Lee Whorf
LANGUAGE, THOUGHT, AND REALITY
1956

CLOSURE UNDER A "MOVING RULE"

While covering the Miscellaneous Exercises A-E on pages 4-87 to 4-95, I had an interesting experience which might be of interest to others.

Although my classes (mixed 9th and 10th grade) had completed Ex. 11 on pages 89 and 90, I did not feel that they had really gained much "feeling" for closure, etc. We had progressed to Ex. C on page 4-93 and had finished number 2 when an idea flashed to mind and I posed this question:

"Suppose we have $\{(x, y): y = 5 + x\}$, part of the locus of which is shown by straight line "c" on page 92, can anyone give us a "moving rule" under which this set will be closed?"

One or two students were ready to give an answer immediately and I suggested that others might want to list some of the ordered pairs they could find on line "c", etc. Very shortly several students were ready with:

$$(x, y) \longrightarrow (x + 1, y + 1)$$

One boy then asked: "Couldn't you use the rule:

$$(x, y) \longrightarrow (x - 1, y - 1) \text{ ?"}$$

We continued this discussion with the remaining examples in this exercise. Many of the students solved for "y"; I felt that when we finished many more people had begun to have some idea of slope and also that exercise 11 on page 89 had more meaning for them.

--Miss Louise A. Brunell
Edwin O. Smith School
University of Connecticut, Storrs

* * *

Recommended reading:

Max Beberman, "The Old Mathematics in the New Curriculum." EDUCATIONAL LEADERSHIP, March, 1962, pp. 373-375. (Vol. 19, No. 6)

--R.S.

BULLETIN:
UNITS 9, 10, AND 11

The student edition of Unit 9 (Elementary Functions: Powers, Exponentials, and Logarithms) will be published on August 1, 1962, by the University of Illinois Press. The price of this edition will be two dollars per copy. The teacher edition of this unit will not be available from the Press until January 1, 1963, but teachers using this unit may obtain a mimeographed answer book without charge from the UICSM Project office after September 1, 1962. Orders for the published teacher edition may be placed at any time, but will not be filled until January. An overview of the contents of this unit is presented below.

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The publication dates for the new editions of Unit 10 (Circular Functions) and Unit 11 (Complex Numbers) have not yet been determined. It will probably turn out that they will not be published in time for classroom use in 1962-63. So, orders for the current (mimeographed) editions should be sent to the UICSM Project office and will be filled without charge. --M.B.

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Unit 9: An Overview of Contents

9.01 Definite description: Existence and uniqueness of principal square root and cube root; absolute value.

9.02 The need for a new basic principle: closure, bounds, greatest and least members, and motivation of the least upper bound principle.

9.03 The least upper bound principle: the ℓubp completes the list of assumptions needed to characterize the reals and permits derivation of the cofinality principle. Exploration exercises on inverses, increasing and decreasing functions, functions as monotonic and continuous.

9.04 Principal roots: operators, the principal nth root function, radical expressions, roots of negative numbers.

9.05 The rational numbers: using a definition for the set of rationals in proving theorems; rationality and roots; rational-linear combinations, density of the rationals, and reciprocation.

9.06 Rational exponents: building a definition; testing, extending, and using the definition.

9.07 The exponential functions: properties of exponential functions with rational arguments; irrational exponents; properties of exponential functions.

9.08 Computing with inverses of exponential functions: approximations and computations; linear interpolation.

9.09 The logarithm functions: the defining principle; theorems; scientific notation, characteristics, and mantissas; relating logarithms to two bases.

9.10 Some laws of nature: gas laws; growth and decay; the natural logarithm function.

Summary.

Review exercises.

Miscellaneous exercises.

Appendix A--The simplest functions.

Appendix B--Irrational numbers: countably and uncountably infinite sets.

Appendix C--The exponential functions: uniform continuity; theorems on continuity.

Appendix D--Volume-measures: points, lines, and planes; some simple solids; an axiom on volume-measure; volume formulas; prismatoids; frustums; solid spheres; surface area formulas; summary of mensuration formulas.

Appendix E--Some functional equations: homogeneous linear functions; theorems; an application.

Basic principles and theorems.

Table of squares and square roots.

Table of trigonometric ratios.

Table of common logarithms.

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NEWS AND NOTICES

Staff Notes

Mr. Beberman's professional calendar for 1961-62 included the following:

November--Demonstration class and speech at NSF Institute, Richmond, Indiana. Film and speech at Louisiana Polytechnic Institute, Ruston, and speech to Louisiana Teachers Association, Shreveport. Convention of Central Association of Science and Mathematics Teachers, Chicago. Visit UICSM schools in St. Louis and Webster Groves, Missouri. PTA meeting, Taylorville, Illinois.

December--Visit schools in New Jersey and Pennsylvania. Panel, American Association for the Advancement of Science, Denver.

January--Two weeks in and around Los Angeles County, California.

February--Meeting with New York State Regents Committee. Wheatley School PTA, Old Westbury, New York. Visit three high schools in Atlanta, Georgia; speeches at the University of Georgia (Athens) and Agnes Scott College (Decatur).

March--Visit schools in Andrews and Odessa, Texas. District Teachers Convention, Odessa. Spring Conference Panel, Havana (Illinois) High School. Demonstration class at Notre Dame (Indiana) University. Mathematics curriculum conference of the National Science Foundation Academic Year Institutes, Notre Dame University.

April--Speech, "Good Mathematics Is Not Enough", at annual meeting of the National Council, San Francisco. Mathematics symposium in Los Angeles. Manitoba Education Association meeting in Winnipeg (Canada).

May--Conference on mathematical learning, Berkeley, California. Visit schools in Tucson, Arizona.

Mr. Beberman announces that the UICSM Project has received a new grant of approximately $385, 000 from the National Science Foundation for the period April 1, 1962, through June 30, 1963.

Mr. O. Robert Brown, Jr., has been associated with two teaching machine projects this year: UICSM's Programed Instructional Project (PIP), for which he has programed, edited, and constructed achievement

tests; also the Programed Logic for Automatic Teaching Operations (PLATO) project, for which he has written a seven-chapter programed sequence teaching the rudiments of programing the Illiac computer, and about which a paper will appear this summer. "Bob" has taught a Unit 6 class at University High School during the second semester and has written revised tests for Units 5 and 6, for which norming statistics are now being computed. "Hopefully," he says, "the U. of I. Press will publish the two Unit 5 tests (one for each half of the unit) by September." Mr. Brown has been the Project Evaluator for the past two years.

Mr. William T. Hale, the Assistant Project Director, published an article on "UICSM's Decade of Experimentation" in the December 1961 Mathematics Teacher. Bill gave speeches at the NSF Summer Institute Directors meetings in Chicago and San Francisco and another at NCTM's Toronto meeting last summer. Most UICSM teachers know him as the Associate Director of the NSF Summer Institutes in Urbana. He has taught Units 7, 8, 9, and 10 at University High School this year.

Miss Gertrude Hendrix, Teacher Coordinator for the Project, wrote "The Psychological Appeal of Deductive Proof" and a review of The Child's Conception of Geometry (by Jean Piaget and his associates) for The Mathematics Teacher, both of which appeared in the November 1961 issue. Miss Hendrix supervises the distribution of the UICSM-NETRC teacher training films, and reports that "During the academic year 1961-62, the UICSM films in their present temporary form have been used by seventeen in-service institute and seminar classes, in which more than six hundred teachers have been studying Units 1, 2, 3, and 4 as a content course." She spoke on the use of the films in the Teacher Education Section of the NCTM annual meeting in San Francisco, and is engaged in revising Productions III and XII of this series.

Miss Hendrix spoke at an International Conference on Educational TV at Purdue University in October, was a leader of the faculty workshop on learning by discovery at the U. of I. College of Dentistry (Chicago) in December, and participated in a work-conference on kinesics and para-language sponsored by the U. S. Office of Education at Indiana University this month. She was a lecturer in the MAA-NSF Secondary School Lecture Program and has been Chairman of the Illinois Section MAA--Illinois

Council joint Committee on Strengthening the Teaching of Mathematics, which conducted a questionnaire study of college faculty opinion in Illinois on the teacher-training recommendations of MAA's Committee on the Undergraduate Program in Mathematics.

Miss Hendrix continues her interest in basic psycho-biological research on the origin of communication systems. She hopes eventually to obtain support for an extensive study of animal behavior aimed at clarifying the difference between learning by discovery and by communication of the thing learned, and identifying and clarifying the role of nonverbal communication in teaching.

Mr. Allen Holmes has been teaching Units 4, 5, 6, 7, and 8 to sophomores and juniors at University High School this year. He plans to get married next month and to obtain his M. A. in mathematics in June of 1963. Al will be a teaching assistant in the NSF Institute this summer.

Miss M. Eleanor McCoy, Project Associate Teacher Coordinator, authored an article on the UICSM Project for the AAAS publication Science Education News, which appeared last December in an issue devoted to reports on various curriculum improvement programs in science and mathematics. She was on the panel with Mr. Beberman at Havana, Illinois, in March and taught a demonstration class of 21 seventh graders at the NCTM April meeting in San Francisco. During the second semester of this year she has taught an extension class of 26 junior and senior high school mathematics teachers at Decatur, Illinois. She will teach a section of First Course during the Summer Institute in Urbana and be on the staff of a New England Institute meeting for one week in August at the Kent School for Girls, Kent, Connecticut, where she will deliver five 75-minute lectures on First Course. She expects to serve as Field Consultant on the staff of the new UICSM project during 1962-63.

Mr. Herbert E. Vaughan, Professor of Mathematics at the University of Illinois and mathematics editor of UICSM text materials, spoke on UICSM to a group of high school teachers in the seminar conducted by Professor Hans Zassenhaus at the University of Notre Dame in March. He was a member of the panel on "The Role of Vocabulary in Learning Mathematics" at the San Francisco NCTM meeting.

Mr. Herbert Wills, programer for UICSM-PIP, has written a brief description of the Programed Instruction Project which will appear in the "Mathematicsl Education Notes" department of the American Mathematical Monthly. The review of Unit 6 he wrote last year while teaching in Elmhurst, Illinois, was published in the May Mathematics Teacher, pp. 399-401. He spoke on "Prestidigitation or Pedagogy?" at the Kappa Mu Epsilon initiation banquet at Eastern Illinois University this spring and will speak at the summer meeting of the NCTM in Wisconsin on the topic "UICSM Programing Techniques for Live Learning." Herb served as program chairman for the Men's Mathematics Club of Chicago and Metropolitan Area this year and has been teaching the content of Units 1-6 in a University of Illinois extension class for teachers at Northbrook this year. This summer he will also program a segment of First Course to be used with the PLATO computer-based teaching machine project.

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This issue of the UICSM Newsletter is the last of the 1961-62 school year and the last to appear under the current editorship. Next year's editor, Mr. Clifford W. Tremblay, will be as grateful for your contributions as I have been, I'm sure, and will probably issue No. 9 sometime in the early fall. Please write to him at the project office when requesting copies of the Newsletter or submitting news items or articles for possible publication. --R. S.

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UICSM

NEWSLETTER

Number 9

1 Nov 62

An occasional publication of the

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

1208 West Springfield

Urbana, Illinois

TABLE OF CONTENTS

GREETINGS

A surprising number of letters has reached our desk, expressing alarm at the fact that no Newsletters have reached our correspondents. The main reason that no one has received a Newsletter yet is that there has not been any printed up until now. We seriously doubt that this one will receive a Pulitzer Prize, but we just want everyone to know that we have not disappeared or given up publishing. We hope that this will be the most meager offering of this type to reach you.

There has been a rather rapid sequence of events in and around 1208 W. Springfield, Urbana, Illinois. The enlarged staff of UICSM now occupies offices in four different buildings on the University of Illinois campus. This is gradually becoming a major problem in terms of communication between staff members. As the number of projects being undertaken by individuals increases, this will mean we will have to use a Newsletter just to inform each other of our present projects.

Since the close of the Summer NSF Institute, the major work of producing eighteen programed instruction books covering Unit One of "Traditional" UICSM texts has been accomplished. These programed texts are being used at present in eight schools in various parts of the country. About 560 students in ninth grade are involved in this experiment. A revision of these texts is now underway for next year.

A fire of undetermined origin did extensive damage to the 1208 W. Springfield building at a time when production was just beginning to get back into stride after having moved back into that building from the summer quarters in University High School. This fire managed to strike more or less at the core of production and shipping. It also meant that 1208 W. Springfield had to be vacated for about six weeks while some repair work was completed by the Physical Plant of the University of Illinois. The University High School Gymnasium became our home for that time. Unfortunately, it was a bit too crowded for us to play basketball during the coffee breaks.

In line with this story, we might note that several issues of the Newsletter were destroyed by the fire and the firemen. Some of the requests that we have received for past issues could not be honored because there were no copies available. The requests may still be filled in the future, however, when our production staff gets a chance to run some extra copies of these back issues. In the meantime, a file of requests will be kept by our office.

THE FUNCTION OF A FUNCTION THEOREM

> For each function h, for each function g,
> there is a function f such that $h = f \circ g$
> if and only if
> $\mathcal{D}_h \subseteq \mathcal{D}_g$ and, for all x_1 and x_2 in $\mathcal{D}_g$ such that $g(x_1) = g(x_2)$, if either x_1 or $x_2 \in \mathcal{D}_h$ then both belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$.

If you are a teacher who has been perplexed by how to teach this theorem [and its proof] in Unit 5 [1960 edition] on pages 5-90 and 5-91, I hope this article will be of some help.

GROUNDWORK

After thinking about this theorem for some time, I finally had a chance to try some of my ideas of teaching it to a class of students. Naturally, I tried to plan the groundwork carefully. This was actually begun the day we started functions.

It was necessary to make students aware that

R is a function

if and only if

$$\forall_a \forall_b \forall_c\, [((a, b) \in R \text{ and } (a, c) \in R) \Rightarrow b = c].$$

Of course, this is just another way of saying that a function is a set of ordered pairs no two of which have the same first component. Although students are given this definition on page 5-50 and make some applications of it at that point, they need to work with it in many ways in order to assimilate it to the point of being able to use it in a natural way. One of the things I did in this connection was to give them oral exercises like this [Handscript shows you what is written on the blackboard.]:

I am thinking of a function. Suppose that

(2, 8) belongs to the function.

Now, what can I put in place of the 'a' to make this:

(2, a) belongs to the function.

into a true sentence? Suppose that

(3, 15) belongs to the function.

Then, what replacements for the 'b' will change the sentence:

(3, b + 12) belongs to the function.

into a true sentence? Here is an open sentence:

{(2, 9), (5, 15), (8, 4), (2, a)} is a function.

What replacements for the 'a' will give us true sentences? What ones will give us false sentence?

A second groundwork-idea involves the notion of equality of functions. [**This notion is dealt with on what is now page 5-52 of the 1960 edition. As you will note in the commentary, the material was misplaced at this point because it uses notation not introduced until page 5-56.**] The idea is expressed **by the generalization:**

(**) For each function f, for each function g,

$$f = g$$

if and only if

f and g have the same domain--say $\mathfrak{D}$

--and $\forall_{x \in \mathfrak{D}}\ f(x) = g(x)$

Other ways of stating this theorem are:

(*) For each function f, for each function g,

$$f = g$$

if and only if

$\mathfrak{D}_f = \mathfrak{D}_g$ and $\forall_{x \in \mathfrak{D}_f \cap \mathfrak{D}_g}\ f(x) = g(x)$.

and

(***) For each function f, for each function g,

$$f = g$$

if and only if

f and g have a common domain and for, each of their arguments, they have the same function values.

Certainly, a formal proof of this need not be given. But, students should have a chance to look at the generalization and appreciate and understand the notation. This appreciation and understanding can be gained by a discussion of the if-part and the on-if-part of the generalization. The following proof for (*) is similar to the one developed by the class I taught.

Only-if-part

Suppose that $f = g$. Since each of f and g is a set of ordered pairs, it follows that the set of first components of f must be the same as the set of first conponents fg. That is, f and g have the same domain. Now, if $\mathcal{D}$ is the common domain gf and g then, for each $x \in \mathcal{D}$, the ordered pair in f whose first component is x must [serve $f = g$] be the same as the ordered pair in g whose first component is x. In particular, the first of these ordered pairs, $(x, f(x))$, must have the same second component as the ordered pair $(x, g(x))$. That is, $f(x) = g(x)$.

If-part

Suppose f and g have a common domain $\mathcal{D}$ and that, for each $x \in \mathcal{D}$, $f(x) = g(x)$. It follows that, for each $x \in \mathcal{D}$ the pair $(x, f(x))$ of f whose first component is x is the same as the pair $(x, g(x))$ of g whose first component is x. So, f and g are the same set of ordered pairs--that is, $f = g$.

COMPOSITION OF FUNCTIONS

The preceding developement on these two groundwork-ideas would take place during the early work on functions. Finally, we come to composition of functions. It seems to me that proof of the function-of-a-function theorem stated at the outset of this article has special value in getting students to assimilate the definition of the composition operation. For your convenience, here is the definition as it is given on page 5-74:

> For each function f, for each function g,
>
> $f \circ g$ is the function such that
>
> (1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$,
>
> and
>
> (2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g \colon g(x) \in \mathcal{D}_f\}$.

Condition (2) tells you what the arguments of f ∘ g are. Condition (1) tells us how to get the value of f ∘ g which corresponds with a given argument. Clearly, these things, together, tell you exactly all there is to know about a function.

It is important to understand this definition. As students work with composition, their understanding will increase. But let's make sure that we understand it now. Let's consider another definition of the same type:

> For each $x \geq 0$,
> $\sqrt{x}$ is the number such that
> (1) $(\sqrt{x})^2 = x$,
> and
> (2) $\sqrt{x} \geq 0$.

This definition tells me, in effect, what I must do to find the principal square root of a nonnegative number. For example, if I want to find the principle square root of the nonnegative number 4, it tells me to find a number whose square is 4 and which is not less than 0. If I do have such a number the definition assures me that my number is $\sqrt{4}$ because it tells me that there is only one number which will meet these conditions. [Of course, before such a definition can be used at all, one must establish an existence theorem and a uniqueness theorem. There will be a thorough discussion of these matters in the forthcoming edition of Unit 9.] So, the definition tells me that each nonnegative number has one and only one principle square root, and it gives me the guides I need in trying to find the principle square root. Both guides must be followed. For example, if I used only the first guide, I might think that $-2 = \sqrt{4}$ because $(-2)^2 = 4$.

Now, the definition of composition is used in precisely the same manner. Notice first that it is a reasonable definition. Condition (2) tells us what the domain of f ∘ g is, and condition (1) tells us how to find the values of f ∘ g for each of its arguments. Since f and g are functions, the procedure in (1) makes sense and works for each of the arguments prescribed by (2). So, there is a function which satisfies the definition and, by the theorem on equality of functions, there is only one. To see how to use the definition,

consider Exercise 1 of Part A on page 5-74.

$$g = \{(6, 2), (9, 4), (12, -1), (5, 3)\}$$
$$f = \{(x, y): y = x^2\}$$
$$f \circ g = \{ \underline{\qquad\qquad ? \qquad\qquad} \}$$

Since f and g are functions, the definition assures me that there is a function which is the composition of f with g, and that there is only one such function. This assurance is comforting in two respects--I know that I am not starting on a wild goose chase, and I know that when I find such a function, I can stop looking for any others. Now, the definition gives me two guides to follow in my search. The first guide tells me something about the ordered pairs of the function I am searching for.

(1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$

I am told that there is an ordered pair (a, b) which belongs to this function <u>if</u> $a \in \mathcal{D}_g$ and $g(a) \in \mathcal{D}_f$; and I am told how to find, for each such a, the second component b of this ordered pair. So, I have a way of getting some ordered pairs which belong to $f \circ g$. Here is the procedure.

> Consider each ordered pair in g. Take, for example, the ordered pair (6, 2). The first component 6 is a candidate for first-component-hood in the sought-for function. I'll be sure that it is if g(6)--that is, 2--is an argument of f. It turns out that 2 is an argument of f. So, I know that 6 is an argument of the sought-for function. Not only that, I also know that f(2) is the value of the function for this argument 6. Since $f(2) = 2^2 = 4$, I know that (6, 4) belongs to the sought-for function. [I also know that the sought-for function contains no other ordered pairs with 6 as first component. It wouldn't be a function if it did.]

By continuing this procedure, I can discover many ordered pairs which belon to the sought-for function. In fact, in the example question, I can collect a total of four ordered pairs in this manner.

Now, Now, shall I stop looking? If I did, I would be like the person who is trying to find the number which is $\sqrt{4}$ and stops when he learns that $(-2)^2 = 4$.

The second condition of the definition tells me what to do at this point.

(2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : \; g(x) \in \mathcal{D}_f\}$

This says that the first components of the ordered pairs in $f \circ g$ are just those arguments of g for which the corresponding g-values are arguments of f. Having found all the pairs of $f \circ g$ which have such arguments for first components, we are finished. We have all the ordered pairs in $f \circ g$. In exercise 1, it happens to be the case that, for each $x \in \mathcal{D}_g$, $g(x) \in \mathcal{D}_f$--and, as $\mathcal{D}_{f \circ g} = \mathcal{D}_g$.

Exercise 2 of Part A on page 5-74 points out the fact that this is not always the case [although by (2) $\mathcal{D}_{f \circ g}$ is always a subset of $\mathcal{D}_g$].

$$a = \{(\text{John}, 7), (\text{Bill}, 5), (\text{Emma}, 8)\}$$
$$b = \{(x, y), x > 6: \; y = 3x + 1\}$$
$$b \circ a = \{ \underline{\qquad\qquad ? \qquad\qquad} \}$$

Suppose that we follow the procedure suggested by condition (1). Doing so, we discover that the ordered pairs (John, b(7)) and (Emma, b(8)) belong to the sought-for function, But, condition (1) does not tell whether or not Bill is an argument of the sought-for function. [Nor does condition (1) rule out any other arguments.] It is condition (2) which helps us. For it says that even though Bill $\in \mathcal{D}_a$, since g(Bill) $\notin \mathcal{D}_b$, Bill $\notin \mathcal{D}_{b \circ a}$. [And, of course, condition (2) rules out all other arguments except John and Emma.]

Your understanding of the definition of composition of functions may be strengthened by considering:

$$g = \{(2, 5), (7, 8)\}$$
$$f = \{(5, 9), (8, 14), (3, 2)\}$$

What is $f \circ g$? Suppose someone says that $f \circ g$ is the function

$$\{(2, 9), (7, 14), (3, 2)\}$$

For convenience, let's name that function 'h_1'. So he believes that

$$h_1 = f \circ g.$$

Let's use our definition to see if he is correct. Condition (1) of the definition tells us that

for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$, $[f \circ g](x) = f(g(x))$.

So, if he is right, then

for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$, $h_1(x)$ must be $f(g(x))$.

Now, $\mathcal{D}_g = \{2, 7\}$. Also, $g(2) \in \mathcal{D}_f$ and $g(7) \in \mathcal{D}_f$. Examination shows that

$$f(g(2)) = f(5) = 9 = h_1(2)$$

and

$$f(g(7)) = f(\{) = 14 = h_1(7).$$

So, it is the case that

$$\text{for each } x \in \mathcal{D}_g \text{ such that } g(x) \in \mathcal{D}_f,\ h_1(x) = f(g(x)).$$

If condition (2) were ignored, we might well believe that

$$f \circ g = \{(2, 9), (7, 14), (3, 2)\}$$

But, do you see that we would also believe that

$$f \circ g = \{(2, 9), (7, 14), (\text{Al}, \text{Mary}), (10, 17)$$

that

$$f \circ g = \{(x, y):\ y = x + 7\}$$

and that

$$f \circ g = \{(2, 9), (7, 14)\}.$$

What does condition (2) of the definition tell us? Why, it tells us that the only one of these functions that is $f \circ g$ is that one whose domain is $\{x \in \mathcal{D}_g:\ g(x) \in \mathcal{D}_f\}$. In this case, $f \circ g$ is that one of these functions whose domain is $\{2, 7\}$. Hence, $f \circ g = \{(2, 9), (7, 14)\}$. In brief, condition (1) gives us a way to find certain ordered pairs which belong to $f \circ g$ and condition (2) tells us that the <u>only</u> ordered pairs which belong to $f \circ g$ are those which can be obtained in that way.

PRELUDE TO THE THEOREM

The Exploration Exercises on pages 5-86 ff. are designed to make the student aware of the theorem toward which we are moving. The task is to find, if possible, a function f such that $h = f \circ g$. Here are some of the things we did in class.

Exercise 4. $g = \{(0, 1), (1, 5), (2, 9)\}$ [page 5-86]

$f =$ ______?______

$h = \{(1, 5), (2, 8), (4, 8)\}$

There is no function f such that $h = f \circ g$ since there is no way of "getting" (4, 8) into h when there is no member of g which has first component 4. Out of this should come the requirement that

$$\mathcal{D}_h \subseteq \mathcal{D}_g.$$

[Naturally, this requirement can also come out of an examination of the definition of composition. If there is an f such that $h = f \circ g$ then

$$\mathcal{D}_h = \mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}.$$

This tells us that each element of $\mathcal{D}_h$ must belong to $\mathcal{D}_g$. So, if there is such an f, $\mathcal{D}_h \subseteq \mathcal{D}_g$.]

Exercise 2 shows that the converse is not true. Other ideas should also come from this exercise.

Exercise 2. $g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$ [page 5-86]

f = ______?______

$h = \{(2, 12), (3, 18), (6, 14), (5, 2)\}$

If there were such a function f then it would have to contain (5, 12), (8, 18), (8, 14), and (0, 2). A relation which contains both (8, 18) and (8, 14) is not a function. So, there is no function f such that $h = f \circ g$.

Now, can you change h so that there is such a function f? Let h(6) be 18 instead of 14 or let h(3) be 14 instead of 18. So, we put on the board:

if there is a function f such that $h = f \circ g$
then [if $3 \in \mathcal{D}_g$ and $6 \in \mathcal{D}_g$ and $3 \in \mathcal{D}_h$ and
$6 \in \mathcal{D}_h$ and $g(3) = g(6)$, h(3) must be h(6)]

Now, we considered a new problem. Suppose that the ordered pair (6, 14) is removed from h in Exercise 2.

$g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$

f = ______?______

$h_1 = \{(2, 12), (3, 18), (5, 2)\}$

Here, (5, 12), (8, 18) and (0, 2) must belong to f if there is an f. However,

$$\{(5, 12), (8, 18), (0, 2)\} \circ g = \{(2, 12), (3, 18), (6, 1\ ), (5, 2)\}.$$

Clearly, this is not h_1.

We also considered removing both (3, 18) and (6, 14) from h.

$$g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$$

$$f = \underline{\qquad\qquad ? \qquad\qquad}$$

$$h_2 = \{(2, 12), (5, 2)\}$$

Then, if there is an f, (5, 12), and (0, 2) must belong to it.

$$\{(5, 12), (0, 2)\} \circ g = \{(2, 12), (5, 2)\},$$

and this is h_2. The class decided that if $3 \in \mathcal{D}_g$ and $6 \in \mathcal{D}_g$ and $g(3) = g(6)$ and either 3 or 6 belongs to $\mathcal{D}_h$ then the other one must also belong to $\mathcal{D}_h$.

So, we change to:

if there is a function f such that $h = f \circ g$

then [if $3 \in \mathcal{D}_g$ and $6 \in \mathcal{D}_g$ and $3 \in \mathcal{D}_h$ and $g(3) = g(6)$

then $6 \in \mathcal{D}_h$ and $h(3) = h(6)$]

We were now ready to state the only-if part of the theorem:

if For each function h, for each function g, there is a function f such that $h = f \circ g$

$$\text{then} \begin{cases} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \text{and} \\ \text{if} \begin{Bmatrix} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{Bmatrix} \text{then} \begin{cases} x_2 \in \mathcal{D}_h \\ \text{and} \\ h(x_1) = h(x_2) \end{cases} \end{cases}$$

Notice that our exercises really led to the contrapositive of this. If the then-things did not happen, there was no such function f. Contrapositively, if there is such a function, the then-things must happen.

At this point, I suggested that we consider the converse of this theorem. This was stated and the class decided it was probably true also. We left this and finished the period by doing the rest of the exercises on page 5-87.

PROOF OF THE THEOREM

At this stage, the students are probably willing to accept the theorem on the basis of the exercises. I decided to try the proof of the theorem. We did the proof [questions and boardwork done by me] in fifty-six minutes. Obviously the students could not reproduce it but they had all contributed and they all stayed awake. Questions were of these types:

What will we need to establish in order to get ____________?

What does it mean to say ____________?

I had the students keep their books open at page 5-74 [definition of composition].

IF-PART OF THE THEOREM

Prove: For each function h, for each function g,

$$\text{if} \left\{ \begin{array}{l} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \text{and} \\ \text{if when} \end{array} \right. \left\{ \begin{array}{l} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{array} \right\} \text{then} \left\{ \begin{array}{l} x_2 \in \mathcal{D}_h \\ \\ h(x_1) = h(x_2) \end{array} \right.$$

then there is a function f such that $h = f \circ g$.

Suppose that, for a function h and a function g, $\mathcal{D}_h \subseteq \mathcal{D}_g$. If $x_1 \in \mathcal{D}_h$, what else can you say? [Answer: $x_1 \in \mathcal{D}_g$] If $x_1 \in \mathcal{D}_h$, what ordered pair belongs to h? Give a very, very inexpensive answer--no work at all. [Ans.: $(x_1, h(x_1))$] If $x_1 \in \mathcal{D}_g$, what ordered pair belongs to g? [Ans.: $(x_1, g(x_1))$]

So, on the supposition that $\mathcal{D}_h \subseteq \mathcal{D}_g$ and $x_1 \in \mathcal{D}_h$, we know that

$$g = \{(x_1, g(x_1)), \ldots\}$$

$$f = \underline{\qquad ? \qquad}$$

$$h = \{(x_1, h(x_1)), \ldots\}$$

Now, if there is a function f such that $h = f \circ g$, what ordered pair must belong to that function? [Ans.: $(g(x_1), h(x_1))$] OK, think of all the ordered pairs which must belong to f. Let's say that set of such ordered pairs is the relation k. I wrote the following on the board:

> $k = \{(z, w)$: there is an $x \in \mathcal{D}_h$ such that $z = g(x)$ and $w = h(x)\}$

Note that $k = \emptyset \qquad h = \emptyset$

The way to find the members of k is to pick arguments of h ["there is an $x \in \mathcal{D}_h$"] and find the corresponding g-values and h-values. Will each such argument have an h-value? [Ans.: Of course, because it <u>is</u> an argument of h.] Will each such argument have a g-value? [Ans.: Of course. By hypothesis, $\mathcal{D}_h \subseteq \mathcal{D}_g$.]

What must we show about k? Two things. First, that k is a function [the first groundwork idea]. And, second, that $h = k \circ g$ [the second groundwork idea].

Let's show that k is a function. How can we do this? Well, k is a set of ordered pairs. So, to show that k is a function, we'll suppose that $(a, b) \in k$ and $a, (c) \in k$ and prove that $b = c$.

Since $(a, b) \in k$, it follows that [according to the set selector in the description of k] that

there is an $x \in \mathcal{D}_h$ such that $a = g(x)$ and $b = h(x)$.

Since $(a, b) \in k$, it follows [according to the set selector in the description of k] that

there is an $x \in \mathcal{D}_h$ such that $a = g(x)$ and $b = h(x)$.

Let x be that number of $\mathcal{D}_h$. Hence, $a = g(x_1)$ and $b = h(x_1)$.
Also, since $\mathcal{D}_h \in \mathcal{D}_g$, $x_1 \in \mathcal{D}_g$.

Similarly, since $(a, c) \in k$, it follows that

there is an $x \in \mathcal{D}_h$ such that $a = g(x)$ and $c = h(x)$.

Let x_2 be that member of $\mathcal{D}_h$. Hence, [illegible] and $c = h(x_2)$.

Also, since $\mathcal{D}_h \subseteq \mathcal{D}_g$, $x_2 \in \mathcal{D}_g$.

So, from the blue sentences

$$g(x_1) = g(x_2).$$

Look back at all the green sentences. We have

$x_1 \in \mathcal{D}_g$

$x_2 \in \mathcal{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathcal{D}_h$

Now look at the conditional sentence in our hypothesis. We must conclude, by modus ponens, that $h(x_1) = h(x_2)$.

Now look at the red sentences. We conclude that $b = c$. So, k is a function.

Take another breath and we'll show that $h = k \circ g$.

Remember that to show this, all we need to show is that

(1) h and $k \circ g$ have the same domain [i.e., $\mathcal{D}_h = \mathcal{D}_{k \circ g}$]

and

(2) for each element in that domain, they have the same value. [i.e., $\forall_{x \in \mathcal{D}_h}\ h(x) = [k \circ g](x)$]

(1) $\mathcal{D}_h$ and $\mathcal{D}_{k \circ g}$ are both sets. So to show they are the same set we will show: (a) $\mathcal{D}_h \subseteq \mathcal{D}_{k \circ g}$ and (b) $\mathcal{D}_{k \circ g} \subseteq \mathcal{D}_h$.

(a) Suppose $x_1 \in \mathcal{D}_h$. Then (since $\mathcal{D}_h \subseteq \mathcal{D}_g$) $x_1 \in \mathcal{D}_g$. Now, look back at the definition of k. From it, it follows that

$$(g(x_1), h(x_1)) \in k.$$

Since that is the case, $g(x_1) \in \mathcal{D}_k$. So, $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_k$.

That is, $x_1 \in \{x \in \mathcal{D}_g : g(x_1) \in \mathcal{D}_h\}$. But, by definition

$\mathcal{D}_{k \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_k\}$. Hence, $x_1 \in \mathcal{D}_{k \circ g}$. So, if $x_1 \in \mathcal{D}_h$ then $x_1 \in \mathcal{D}_{k \circ g}$. Since this reasoning would hold for any $x \in \mathcal{D}_h$, it follows that

$$\mathcal{D}_h \subseteq \mathcal{D}_{k \circ g}$$

(b) Suppose $x_2 \in \mathcal{D}_{k \circ g}$. Then $x_2 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_k\}$. That is, $x_2 \in \mathcal{D}_g$ and $g(x_2) \in \mathcal{D}_k$. Now, $g(x_2) \in \mathcal{D}_k$ says that there is an ordered pair,

$(g(x_2), k(g(x_2))) \in k$.

Again, we use the definition of k. This ordered pair,

$$(g(x_2), k(g(x_2)))$$

belongs to k if and only if

there is an $x \in \mathcal{D}_h$ such that $g(x_2) = g(x)$ and $k(g(x_2)) = h(x)$.

[Maybe this x is x_2 but we can't make such an assumption].

Let's say that this x is x_1. So, $x_1 \in \mathcal{D}_h$ and $g(x_2) = g(x_1)$.

Furthermore, since $\mathcal{D}_h \subseteq \mathcal{D}_g$, $x_1 \in \mathcal{D}_g$.

Now, pick up the four green sentences and go back to the conditional sentence in the hypothesis. By modus ponens, we conclude that $x_2 \in \mathcal{D}_h$. So, if $x_2 \in \mathcal{D}_{k \circ g}$ then $x_2 \in \mathcal{D}_h$. Since the reasoning would hold for any $x \in \mathcal{D}_{k \circ g}$, it follows that

$$\mathcal{D}_{k \circ g} \subseteq \mathcal{D}_h$$

and hence, that $\mathcal{D}_h = \mathcal{D}_{k \circ g}$.

We now know that ha dn $k \circ g$ have the same domain. It remains to show that for each element x in that domain (we will call it '$\mathcal{D}_h$') $h(x) = [k \circ g](x)$.

Suppose $x_1 \in \mathcal{D}_h$. Then $x_1 \in \mathcal{D}_{k \circ g}$. Also, $x_1 \in \mathcal{D}_g$.

So, back to the definition of k:

$$(g(x_1), h(x_1)) \in k.$$

Since k is a function, the second component of an ordered pair whose first component is $g(x_1)$ and which belongs to k must be $k(g(x_1))$. So

$$(g(x_1),\ k(g(x_1))) \in k,$$

and

$$h(x_1) = k(g(x_1)).$$

However, $k(g(x_1))$ is precisely $[k \circ g](x_1)$, since $x_1 \in \mathcal{D}_{k \circ g}$.

Consequently, for each element x in the domain of h (and $k \circ g$)

$$h(x) = [k \circ g](x).$$

So, h and $k \circ g$ have the same domain and for each element in that domain, they have the same value. Hence, $h = k \circ g$.

So, there is a function f (we used the name 'k') such that $h = f \circ g$.

ONLY-IF PART OF THE THEOREM

Prove: For each function h, for each function

if there is a function f such that $h = f \circ g$

$$\text{then} \begin{cases} \mathcal{D}_h \subseteq \mathcal{D}_g \\ \quad \text{and} \\ \text{if} \begin{cases} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{cases} \text{then} \begin{cases} x_2 \in \mathcal{D}_h \\ h(x_1) = h(x_2). \end{cases} \end{cases}$$

Let's see what we have to start. Not very much. We have

f, g, and h are functions

$h = f \circ g$.

First, we want to show that $\mathcal{D}_h \subseteq \mathcal{D}_g$. So, we will show that each member of $\mathcal{D}_h$ belongs to $\mathcal{D}_g$. Suppose that $x_1 \in \mathcal{D}_h$. Then, since $h = f \circ g$, it follows [by substitution] that $x_1 \in \mathcal{D}_{f \circ g}$. But, by the definition of composition, $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g\colon\ g(x) \in \mathcal{D}_f\}$. Hence, $x_1 \in \{x \in \mathcal{D}_g\colon\ g(x) \in \mathcal{D}_f\}$--that is, $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_f$. So, if $x_1 \in \mathcal{D}_h$ then $x_1 \in \mathcal{D}_g$. Therefore, $\mathcal{D}_h \subseteq \mathcal{D}_g$. So far, so good.

Consider this correspondence:

there is a function f such that $h = f \circ g$	$\longrightarrow$	p
$\mathcal{D}_h \subseteq \mathcal{D}_g$	$\longrightarrow$	q
$x_1 \in \mathcal{D}_g$ and $x_2 \in \mathcal{D}_g$ and $g(x_1) = g(x_2)$ and $x_1 \in \mathcal{D}_h$	$\longrightarrow$	r
$x_2 \in \mathcal{D}_h$ and $h(x_1) = h(x_2)$	$\longrightarrow$	s

So, the pattern of our theorem is

if p then [q and (if r then s)].

We have already shown that

if there is a function f such that $h = f \circ g$ then $\mathcal{D}_h \subseteq \mathcal{D}_g$.

This corresponds to

if p then q.

Now, examine the following diagram:

```
                         †
                         p    if p then q
                      †  ----------------
                      p              q     if (p and q and r) then s
 †                    ------------------   -------------------------- (Exportation)
 p   if p then q      p        and q       if p and q then (if r then s)
 ---------------     ----------------------------------------------------
      q                               if r then s
     ----------------------------------------------
           q and (if r then s)
      ------------------------------        †
        if p then [q and (if r then s)]
```

So, using

if p then q

and

if (p and q and r) then s

there is a pattern by which we can arrive at

if p then [q and (if r then s)].

Now, prove

$$
\text{if} \begin{cases} \text{there is a function } f \text{ such that } h = f \circ g & \longrightarrow \quad p \\ \mathcal{D}_h \subseteq \mathcal{D}_g & \longrightarrow \quad q \\ \left.\begin{array}{l} x_1 \in \mathcal{D}_g \\ x_2 \in \mathcal{D}_g \\ g(x_1) = g(x_2) \\ x_1 \in \mathcal{D}_h \end{array}\right\} & \longrightarrow \quad r \end{cases}
$$

$$
\text{then} \left\{\begin{array}{l} x_2 \in \mathcal{D}_h \\ h(x_1) = h(x_2) \end{array}\right\} \longrightarrow \quad s
$$

This discussion of the logical background need not (in fact, probably should not) be done with the students. You might do something like this:

We started with

f, g, and h are functions

$h = f \circ g$

What other sentence can we now use whenever we like? [Ans: $\mathcal{D}_h \subseteq \mathcal{D}_g$]. Now, our hypothesis is

f, g, and h are functions

$h = f \circ g$

$\mathcal{D}_h \subseteq \mathcal{D}_g$

We want to show that

if $x_1 \in \mathcal{D}_g$, $x_2 \in \mathcal{D}_g$, $g(x_1) = g(x_2)$, $x_1 \in \mathcal{D}_h$ then $[x_2 \in \mathcal{D}_h$ and $h(x_1) = h(x_2)]$.

How do we do this? [Ans. Suppose those four things and prove that $x_2 \in \mathscr{D}_h$ and $h(x_1) = h(x_2)$.] That means that we now have:

Hypothesis:

f, g, and h are functions

$h = f \circ g$

$x_1 \in \mathscr{D}_g$

$x_2 \in \mathscr{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathscr{D}_h$

Conclusion:

$x_2 \in \mathscr{D}_h$

$h(x_1) = h(x_2)$

We know very little about x_2. So let's ask our questions in reverse order. First: We want to end with

$$\begin{array}{c} | \\ | \\ | \\ | \\ x_2 \in \mathscr{D}_h \end{array}$$

Look at the hypothesis. Can you suggest a step from which it would be very easy to deduce $x_2 \in \mathscr{D}_h$? [Ans. $x_2 \in \mathscr{D}_{f \circ g}$ because $h = f \circ g$.] How about that? Do you think that the proof might end

$$\frac{\text{Given} \qquad h = f \circ g, \quad x_2 \in \mathscr{D}_{f \circ g}}{x_2 \in \mathscr{D}_h}$$

Now, what is $\mathscr{D}_{f \circ g}$? Look at the definition of $f \circ g$.

$$\mathscr{D}_{f \circ g} = \{x \in \mathscr{D}_g : \ g(x) \in \mathscr{D}_f\}$$

What can we show about x_2 in order to get $x_2 \in \mathscr{D}_{f \circ g}$

$$x_2 \in \mathscr{D}_g \text{ and } g(x_2) \in \mathscr{D}_f$$

Given $\quad x_2 \in \{x \in \mathscr{D}_g:\ g(x) \in \mathscr{D}_f\}$

$h = f \circ g, \quad x_2 \in \mathscr{D}_{f \circ g}$

$$x_2 \in \mathscr{D}_h$$

Now, what about this? Our hypothesis tells us that $x_2 \in \mathscr{D}_g$. So, all we need is to show that $g(x_2) \in \mathscr{D}_f$. Look again. Since we are given that $g(x_1) = g(x_2)$, we can show that $g(x_2) \in \mathscr{D}_f$ if we can show that $g(x_1) \in \mathscr{D}_f$.

Given

Given $\quad g(x_1) = g(x_2), \quad g(x_1) \in \mathscr{D}_f$

$$x_2 \in \mathscr{D}_g \text{ and } g(x_2) \in \mathscr{D}_f$$

Given $\quad x_2 \in \{x \in \mathscr{D}_g:\ g(x) \in \mathscr{D}_f$

$h = f \circ g, \quad x_2 \in \mathscr{D}_{f \circ g}$

$$x_2 \in \mathscr{D}_h$$

So, our problem is to show that $g(x_1) \in \mathscr{D}_f$. What do we know about x_1?
[Ans. $x_1 \in \mathscr{D}_h$, $x_1 \in \mathscr{D}_g$]

Let's examine these. Since $x_1 \in \mathscr{D}_h$, what else can you say about x_1?
[Ans. $x_1 \in \mathscr{D}_{f \circ g}$] What does that mean? Why that means that $x_1 \in \mathscr{D}_g$ and $g(x_1) \in \mathscr{D}_f$. Hey, that's what we wanted.

Given $\qquad$ Given

$h = f \circ g, \quad x_1 \in \mathscr{D}_h$

$$x_1 \in \mathscr{D}_{f \circ g}$$

$$x_1 \in \mathscr{D}_g \text{ and } g(x_1) \in \mathscr{D}_f$$

Given

$g(x_1) = g(x_2), \quad g(x_1) \in \mathscr{D}_f$

Given

$$x_2 \in \mathscr{D}_g \text{ and } g(x_2) \in \mathscr{D}_f$$

$$x_2 \in \{x \in \mathscr{D}_g:\ g(x) \in \mathscr{D}_f\}$$

Given

$h = f \circ g, \quad x_2 \in \mathscr{D}_{f \circ g}$

$$x_2 \in \mathscr{D}_h$$

Now, let's finish it off. Show that $h(x_1) = h(x_2)$.

We know that $x_1 \in \mathcal{D}_{f \circ g}$ and $x_2 \in \mathcal{D}_{f \circ g}$. So, by the definition of composition

$$[f \circ g](x_1) = f(g(x_1)) \text{ and } [f \circ g](x_2) = f(g(x_2))$$

But, $g(x_1) = g(x_2)$ and f is a funtion. So,

$$f(g(x_1)) = f(g(x_2)), \text{ and } [f \circ g](x_1) = [f \circ g](x_2).$$

Since $h = f \circ g$,

$$h(x_1) = h(x_2).$$

We did it.

* * *

The following is an abbreviated form of the proof of the only-if part of the theorem.

(1) Suppose $x_1 \in \mathcal{D}_h$. Since $h = f \circ g$, $x_1 \in \mathcal{D}_{f \circ g}$. But,

$$\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$$

It follows that $x_1 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$. Hence, $x_1 \in \mathcal{D}_g$. So, if $x_1 \in \mathcal{D}_h$ then $x_1 \in \mathcal{D}_g$.

Since this reasoning would hold for any $x \in \mathcal{D}_h$,

$$\mathcal{D}_h \subseteq \mathcal{D}_g.$$

Now we must prove a conditional sentence.

if	$x_1 \in \mathcal{D}_g$ $x_2 \in \mathcal{D}_g$ $g(x_1) = g(x_2)$ $x_1 \in \mathcal{D}_h$	then	$x_2 \in \mathcal{D}_h$ $h(x_1) = h(x_2)$

So, in addition to knowing that

f, g, and h are functions

$h = f \circ g$

$\mathcal{D}_h \subseteq \mathcal{D}_g$

we have the additional premisess

$x_1 \in \mathcal{D}_g$

$x_2 \in \mathcal{D}_g$

$g(x_1) = g(x_2)$

$x_1 \in \mathcal{D}_h$

Each use of any one of these will be in green.

Since $h = f \circ g$, it follows that

$$\mathcal{D}_h = \mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\} \text{ [by definition]}$$

Since $x_1 \in \mathcal{D}_h$ it follows that $x_1 \in \mathcal{D}_{f \circ g}$ and $x_1 \in \mathcal{D}_g$ and $g(x_1) \in \mathcal{D}_f$.

Since $g(x_1) = g(x_2)$ it follows from that $g(x_2) \in \mathcal{D}_f$.

Since $x_2 \in \mathcal{D}_g$ and $g(x_2) \in \mathcal{D}_f$ it follows that

$$x_2 \in \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}.$$

That is $x_2 \in \mathcal{D}_{f \circ g}$. Since $h = f \circ g$ $x_2 \in \mathcal{D}_h$.

Now to show that $h(x_1) = h(x_2)$

Now we have $x_1 \in \mathcal{D}_{f \circ g}$ and $x_2 \in \mathcal{D}_{f \circ g}$.

By definition of composition

$$[f \circ g](x_1) = f(g(x_1)) \text{ and } [f \circ g](x_2) = f(g(x_2)).$$

Since $g(x_1) = g(x_2)$ and f is a function it follows that

$$f(g(x_1)) = f(g(x_2))$$

So, $[f \circ g](x_1) = [f \circ g](x_2)$. Since $h = f \circ g$

$$h(x_1) = h(x_2)$$

- A. H. -

Is h a function of g?

Find your way thru this maze!

When you come to a **red sentence, stop**; you have finished the problems.

When you come to a green sentence, go on; you have not finished.

	YES	NO
(1) Is h a function?	Go to (2)	h is not a function of g
(2) Is g a function?	Go to (3)	h is not a function of g
(3) Is $\mathfrak{D}_h \subseteq \mathfrak{D}_g$?	Go to (4)	h is not a function of g
(4) Are there two elements in $\mathfrak{D}_g$ whose g-values are the same?	Go to (5)	h is a function of g
(5) Are there two elements in $\mathfrak{D}_g$ whose g-values are the same but exactly one of them belongs to $\mathfrak{D}_h$?	h is not a function of g	Go to (6)
(6) Are there two elements in $\mathfrak{D}_g$ whose g-values are the same and both of them belong to $\mathfrak{D}_h$ but whose h-values are different?	h is not a function of g	h is a function of g

PROGRAMED INSTRUCTION TRIAL RUN

The UICSM-PIP (Programed Instruction Project) materials covering the course content of Unit I have been undergoing a trial run in nine pilot schools. Test results and teacher comments from those pilot schools are being studied preparatory to the revision of the programed materials. Significant data arising from this study will be published in future newsletters.

In preparing Unit I in programed form, we have modified and augmented both mathematical and pedagogical content. We have made extensive use of function machines [see "Animated Functional Notation", Newsletter #7], used proofs more extensively, and expanded our discussion of isomorphisms. Before looking at specific differences between Unit I and the PIP materials, here is a list of the 18 programed parts which notes their content and length. The approximate student study time necessary to complete any part averages two school days.

Part #		# of Pages
101	Things and Names of Things	80
102	Trips on Roads	79
103	Addition of Real Numbers	96
104	Multiplication of Real Numbers	64
105	Isomorphisms	109
106	Conventions for Grouping	74
107	Conventions for Grouping	83
108	The cpm and the cpa	79
109	The apa and the apm	92
110	The dpma and the ℓdpma	85
110.5	Logical Consequence, pm1, pm0, pa0, and Principles for Real Numbers	64
111	Inverse Operations	81
112	Subtraction of Real Numbers	53
113	Principle of Opposites and the Zero-Sum Principle	65
114	Principle for Subtraction	59
114.5	Division of Real Numbers	75
115	The Greater-Than and Less-Than Relations for Numbers of Arithmetic and Real Numbers	57
116	Number Ray, Number Line and Absolute Valuing	80

There are three major differences between the mathematical content of Unit I and the mathematical content of the PIP materials. Each UICSM teacher should be aware of these differences since there is now the possibility that a PIP-trained student may come his way.

The first change occurs in Part 109 where the student accepts:

$$(\sim\sim + \ldots) + \_\_\_ = \sim\sim + (\ldots + \_\_\_)$$

as a pattern sentence for the associative principle for addition (for numbers of arithmetic). It follows that a sentence about numbers of arithmetic is an INSTANCE of the apa if and only if it fits the accepted pattern sentence. Hence:

$$(29 + 8) + 2 = 29 + (8 + 2)$$

is an <u>instance</u> of the apa whereas:

$$29 + (8 + 2) = (29 + 8) + 2$$

is <u>not</u> an instance of the apa.

We later point out [Part 110.5] that since:

$$29 + (8 + 2) = (29 + 8) + 2$$

is a LOGICAL CONSEQUENCE of the fact that:

$$(29 + 8) + 2 = 29 + (8 + 2)$$

and since:

$$(29 + 8) + 2 = 29 + (8 + 2)$$

is an instance of the apa, it follows that:

$$29 + (8 + 2) = (29 + 8) + 2$$

is a LOGICAL CONSEQUENCE of the apa.

This precise meaning of "instance" gives rise to a second mathematical difference in Part 110. In this part, the examples lead the student to accept:

$$(\sim\sim \times \ldots) + (\_\_\_ \times \ldots) = (\sim\sim + \_\_\_) \times \ldots$$

as a pattern sentence for the distributive principle for multiplication over addition (for numbers of arithmetic). Hence, in view of our previous discussion, we say that:

$$(12 \times 7) + (88 \times 7) = (12 + 88) \times 7$$

is an instance of the dpma whereas:

$$(12 + 88) \times 7 = (12 \times 7) + (88 \times 7)$$

is not. Because of the curious American habit of reading from left to right, an objection has been raised to the effect that it would be more appropriate to think of:

$$(\sim\sim \times \ldots\ldots) + (\_\_\_ \times \ldots\ldots) = (\sim\sim + \_\_\_) \times \ldots\ldots$$

as a pattern sentence for the "collecting principle for multiplication from over addition." For this reason, we shall most likely take the precaution of establishing:

$$(\sim\sim + \_\_\_) \times \ldots\ldots = (\sim\sim \times \ldots\ldots) + (\_\_\_ \times \ldots\ldots)$$

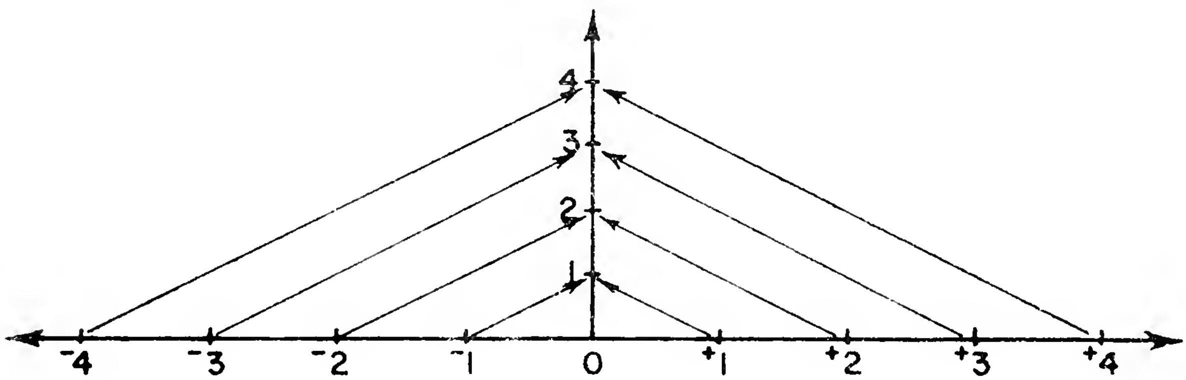
as a pattern sentence for the dpma when rewriting Part 110.

Our new approach to absolute valuing (Part 116) comprises the third major change in mathematics. Most texts declare absolute valuing to be a mapping of the real numbers onto the nonnegative real numbers which is defined by:

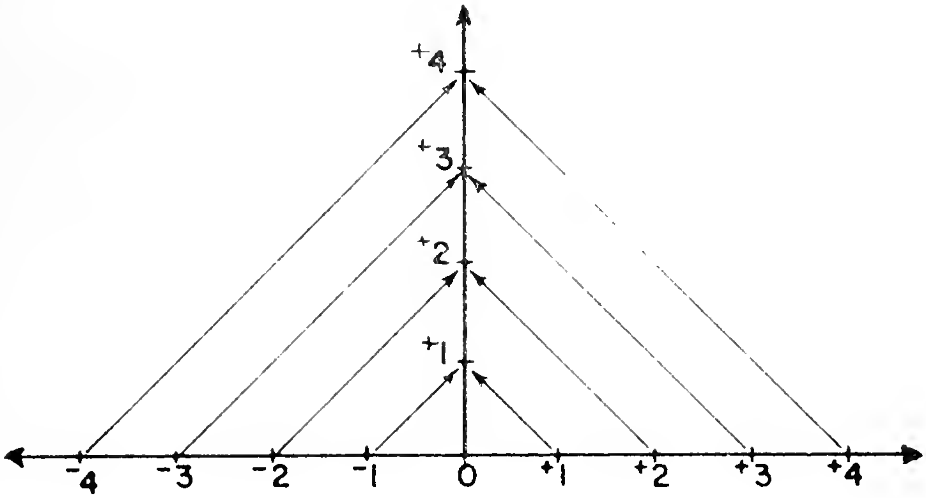
$$\forall_x \quad |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

We could not use the above sentence in introducing absolute valuing in Unit I, of course, since the student doesn't yet have the necessary language.

Many high school texts creep up on the idea of absolute valuing by discussing the "distance" between real numbers. This gives them a mapping which looks like:

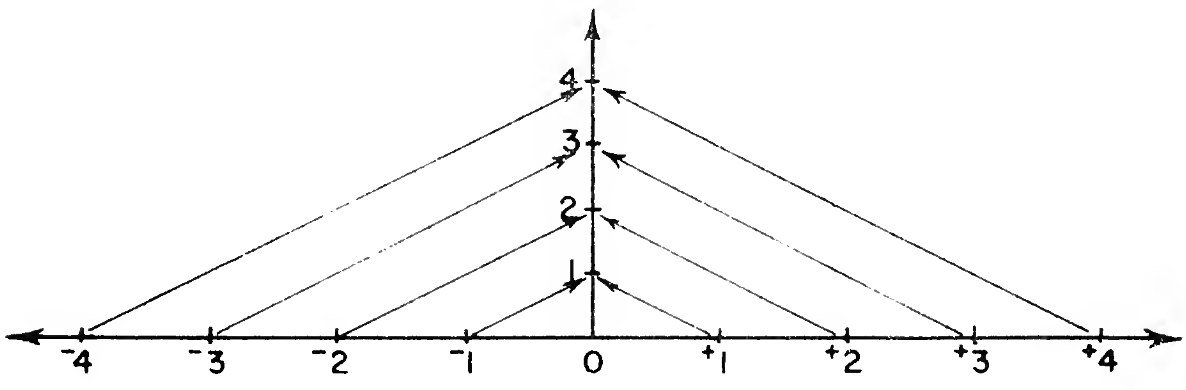


They follow this with a comment to the effect that the nonnegative real numbers "are" or "act like" the numbers of arithmetic and, hence, we may think of measuring the "distances" between real numbers with nonnegative real numbers. This allows them to replace the foregoing diagram with:



They then define absolute valuing to be the mapping suggested by the latter diagram.

Because UICSM finds it most convenient to maintain that the "distances" between real numbers are measured by numbers of arithmetic and that the nonnegative real numbers are merely isomorphic to the numbers of arithmetic with respect to certain operations, the treatment of absolute valuing just reviewed is not consistent with the rest of the UICSM texts. One possible solution to this problem is suggested in Unit I. There the discussion of the "distance" between real numbers leads us to define the absolute value of a real number as the number of arithmetic which corresponds to that real number. The particular correspondence we have in mind is visualized in the diagram:



Unfortunately defining the absolute value of a real number to be a number of arithmetic makes a sentence like:

$$|{}^{-}8| - {}^{+}10 = {}^{-}2$$

amount to so much nonsense since:

$$|{}^{-}8|$$

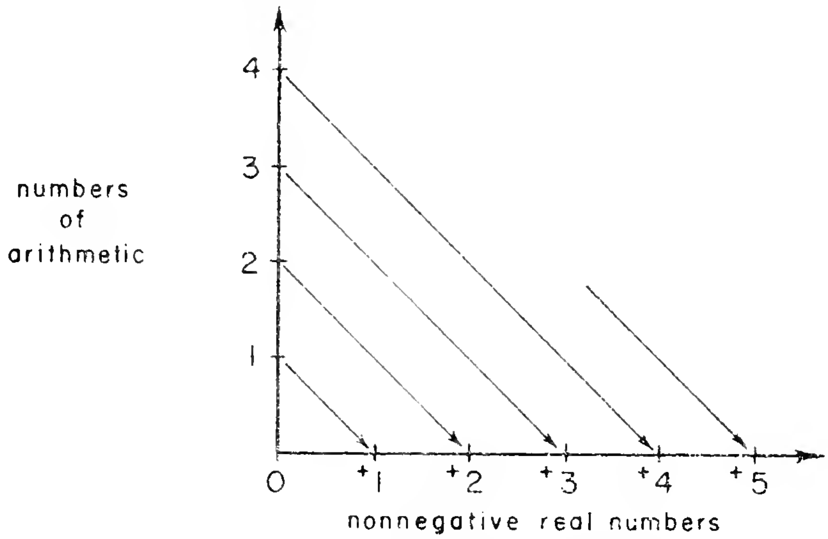
is, by definition, a name for the number of arithmetic 8. Since we will, in practice, want to make statements such as:

$$|{}^{-}8| - {}^{+}10 = {}^{-}2$$

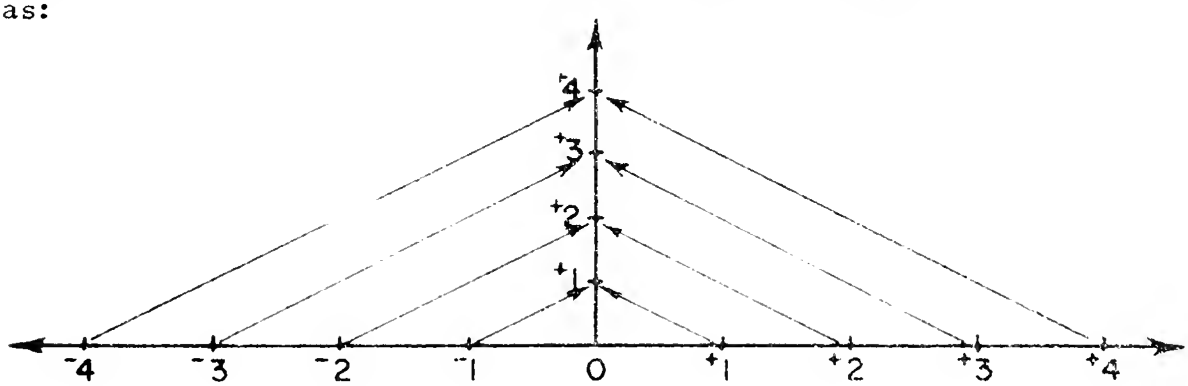
we must arrive at an agreement by which:

$$|{}^{-}8|$$

can be interpreted to be a name for the real number ${}^{+}8$. We solve this problem in Unit I by discussing an operation '+' which is defined to be a mapping of the numbers of arithmetic onto the nonnegative real numbers and which can be visualized as:



It is then possible to consider an operation which is the composition of '+' with absolute valuing and, hence, is a mapping of the real numbers onto the nonnegative real numbers. This final mapping can be visualized as:



We now find ourselves in a troublesome situation. We have the desired mapping, but we cannot name it 'absolute valuing' also. It might appear that, in Unit I, we fudge a bit on this point. We tell the student to read:

$$| \quad |$$

as:

the absolute value of

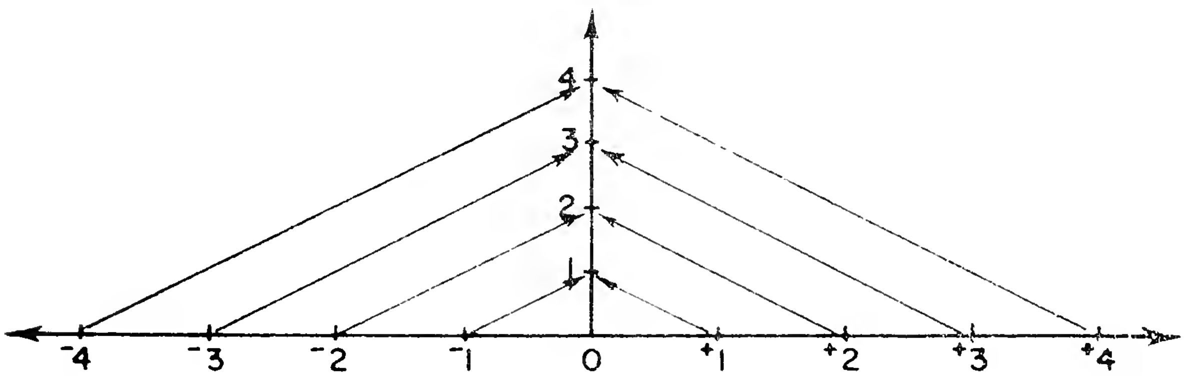
and then to intepret the resulting numeral in such a way that it will make sense in the given context. [A full explanation of what is really happening here can be found in the article entitled "Absolute Valuing" in Newsletter No. 2.] On the basis of this agreement, the student interprets '$|^{-}8|$' as a name for the real number $^{+}8$ in the sentence:

$$|^{-}8| - {}^{+}10 = {}^{-}2$$

and interprets '$|^{-}8|$' as a name for the number of arithmetic 8 in the sentence:

$|^{-}8| - 8$ is the number of arithmetic 0.

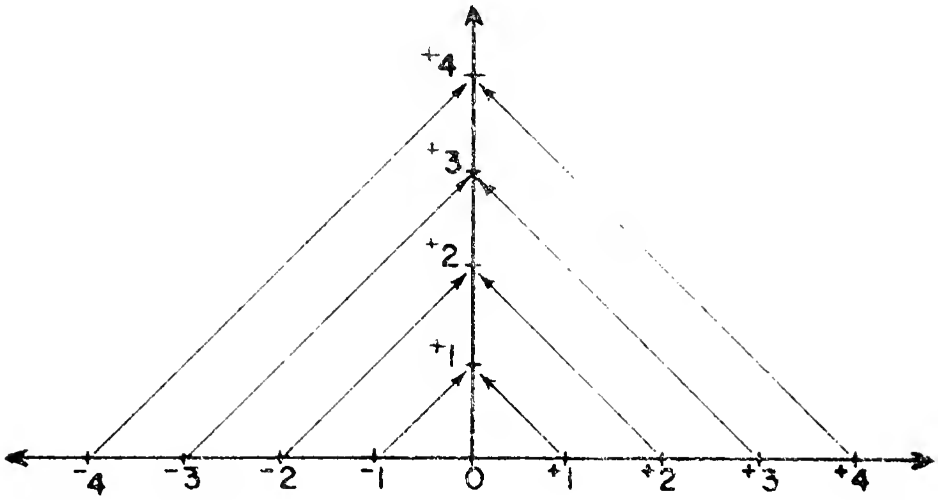
This agreement may suggest to the student that we are using 'absolute valuing' as a name for <u>two different mappings</u>! Perhaps we were too quick to use the name 'absolute valuing'. If, as in Part 116, we use a different name (say: arithmetic valuing) for the mapping suggested by:



We would be able to use:

absolute valuing

to name the mapping suggested by:



In Part 116, we denote:

the arithmetic value of $^-8$

by writing:

$\mathcal{A}\,(^-8)$

which is, therefore, a name for the number of arithmetic 8. We then denote:

the absolute value of $^-8$

by writing:

$|^-8|$

which is a name for the real number $^+8$.

Having developed these two operations with their corresponding notation, Part 116 comes to a close. At this point, the student would simplify '$|^-8|$' by writing '$^+8$'. He would not simplify '$|^-8|$' by writing '8'. Since we wish:

$$|^-8| = 8$$

and:

$$|8| - 9 = {}^-9$$

to be meaningful statements, some additional comments need to be made. Lists are handy for this purpose. Our job now is to introduce the agreement to use '| |' as an operation symbol for both arithmetic valuing and absolute valuing. In doing this, we wish to communicate the idea that we are using the <u>same symbol</u> to denote <u>two distinct operations</u>. This is not a new idea. For instance we use the symbol '-' to denote 4 operations at present. These operations are:

(1) subtraction of numbers of arithmetic $[10 - 8 = 2]$

(2) subtraction of real numbers $[{}^{+}10 - {}^{+}8 = {}^{+}2]$

(3) oppositing $[-{}^{-}6 = {}^{+}6]$

(4) direction $[-8 = \text{negative } 8]$

Here is one way you might introduce the dual use of the '| |'.

Look at the following list of ordered pairs which belong to an operation:

$({}^{+}6,\ 6)$ $({}^{-}9,\ 9)$
$({}^{-}6,\ 6)$ $({}^{+}3,\ 3)$
$({}^{+}9,\ 9)$
.
.
.

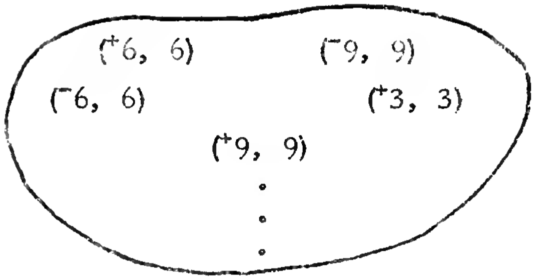
What would be a good name for this operation? Most students will recognize these pairs as belonging to arithmetic valuing. Recall, then, our agreement on writing shorter names for nonnegative numbers. By this agreement:

$$6 = {}^{+}6, \quad 9 = {}^{+}9, \quad 3 = {}^{+}3$$

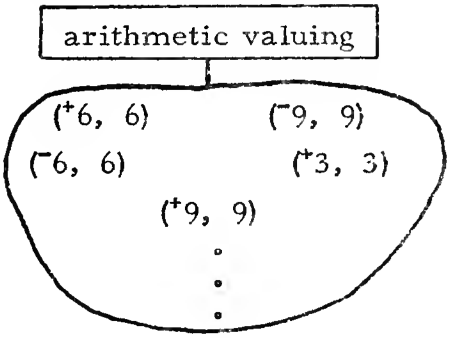
Hence, the foregoing list might be the list:

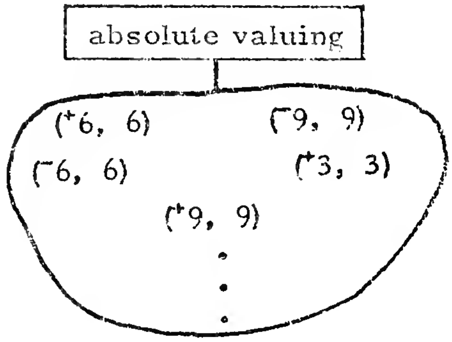
$({}^{+}6,\ {}^{+}6)$ $({}^{-}9,\ {}^{+}9)$
$({}^{-}6,\ {}^{+}6)$ $({}^{+}3,\ {}^{+}3)$
$({}^{+}9,\ {}^{+}9)$
.
.
.

in disguise. But this is a list of pairs belonging to absolute valuing. So, considering our agreement on shorter names for nonnegative numbers, the list:



might be a list of pairs belonging to absolute valuing. Hence, if it weren't for the flags, the pairs in the lists:



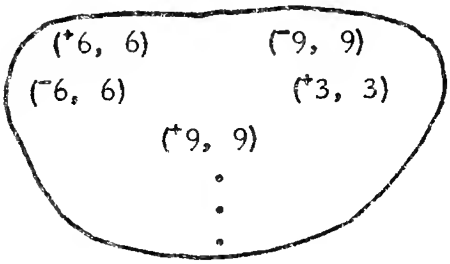


could not be distinguished as belonging to two distinct operations. Suppose then, that we wished to simplify:

$$\mathcal{A}\,(^{-}6)$$

by using the above lists. By definition of the '$\mathcal{A}(\ )$' we <u>know</u> that we should use the list on the left. Here's a copy of that list:

(☆)



Use (☆) to simplify '$\mathcal{A}(^{-}6)$'. Got the answer? Okay ... I lied. (☆) is really a copy of the list for absolute valuing. So, we can use (☆) to simplify '$\mathcal{A}(^{-}6)$' on the understanding that (☆) is a list for arithmetic valuing. We could, moreover, use (☆) to simplify '$|^{-}6|$' on the understanding that (☆) is a list for absolute valuing which was written using our shorter names for nonnegative real numbers in the right member of each ordered pair. Use (☆) to help you solve these problems:

(1) $|^{+}9| - 12 = ?$

(2) $\mathcal{A}(^{+}3) - 3 = ?$

(3) $\mathcal{A}(^{-}6) - 3 = ?$

(4) $|^{-}6| - 3 = ?$

Clearly, we can use (☆) to solve each of these four problems. In doing (1) and (4), we think of (☆) as a list of pairs belonging to absolute valuing. In doing (2) and (3), we think of (☆) as a list of pairs belonging to arithmetic valuing. In (1) and (4), it is the '$|\ |$' which sends us to (☆). In (2) and (3), it is the '$\mathcal{A}(\ )$' that sends us to (☆). Since both '$|\ |$' and '$\mathcal{A}(\ )$' simply serve as "cues" to use (☆), let us agree to abandon the use of:

$$\mathcal{A}(\ )$$

and use:

$$|\ \ |$$

as our cue to use (☆).

By this we are agreeing that, <u>in a given instance</u>, the numeral:

$$|^{-}6|$$

may be used as a name for the absolute value of $^{-}6$ or as a name for the arithmetic value of $^{-}6$ <u>but</u> <u>not</u> <u>both</u>. The context in which:

$$|^{-}6|$$

appears will usually make it clear which operation is referred to by the '$|\ |$'. For example, in the statement:

$$|^{-}6| - 13 = {}^{-}7$$

it is clear that:

$$|^{-}6|$$

is a name for the absolute value of $^{-}6$. On the other hand, in the statement:

$|^{-}6| - 4$ is a number of arithmetic

it is clear that:

$$|^{-}6|$$

refers to the number of arithmetic 6.

On page 27 we noted that we wished to arrive at some agreement that would make:

$$|8| - 9 = {}^{-}1$$

be a meaningful sentence. Clearly, the '| |' must be referring to the operation absolute valuing. But absolute valuing is an operation which is performed on real numbers. Hence:

$$|8|$$

must be an abbreviation for:

$$|{}^{+}8|$$

Consider the statement:

$$|6| + 9 = 15$$

The '| |' may be referring to the operation absolute valuing or the '| |' may be referring to the operation arithmetic valuing. Regardless of which operation is intended, we know that:

$$|6|$$

must be an abbreviation for:

$$|{}^{+}6|$$

since the domain of either operation is the set of real numbers.

—B. K. —

[Author's note: UICSM (and many others) define a singulary operation on a set S to be a mapping of S into S. Similarly a binary operation on a set S is a mapping of S × S into S. In view of this fact, absolute valuing and '+' (as defined in Unit I) and arithmetic valuing (as defined in Part 116) are NOT operations. They are merely functions. In rewriting these materials we hope to introduce the definitions of 'operation' and 'function' sooner, so that we may make this distinction for the student.]

UICSM TRAVELS

The following is an incomplete resumé of Mr. Beberman's engagements through December 16. We realize some of it is past history, but some people may like to know where he has been.

October 25 - Storrs, Connecticut

Speak at the joint meeting of the Connecticut Education Association and the Associated Teachers of Mathematics in Connecticut.

October 26, 27 and 28 - Gearhart, Oregon

October 29 and 30 - Portland, Oregon

Will visit these schools: Franklin High School
Central Catholic High School
Reed College

November 1 - Boulder City, Nevada

Will visit the Boulder City High School

November 3 and 4 - Stanford, California

Attend meeting of the SMSG Test Panel

November 4

Return to Champaign

Weekend of November 9, 10, and 11 Mr. Beberman and UICSM staff will consult with Professor Dick Wick Hall of Harpur College.

November 16 - Columbus, Ohio

Will speak to the University Symposium on Mathematics at Ohio State University

November 16 (evening), 17 and 18 - Washington, D.C.

Attend NSF Advisory Conference on Coordination of Curriculum Studies

December 6, 7 and 8 - Washington D.C.

Attend NCTM Board of Directors meeting

December 15, 16 - Chicago, Illinois

ADVISORY BOARD MEETS

The UICSM Advisory Board met on Oct. 6, 1962 to discuss various phases of the project and suggest some future points of exploration. Progress since receipt of the NSF grant on April 1, 1962 was related by Mr. Beberman. This included production of 18 volumes of approximately 1400 pages of programed material, sponsorship of a summer institute for 324 teachers, recruitment of the entire staff called for in the NSF proposal with two exceptions, and consultation with Professor Rosenbloom and Professor Dick Wick Hall. The Board suggested some work sessions which might prove fruitful. These are intended as outlining and brainstorming sessions to get some new course content started. The Board suggested two approaches to developing seventh grade material which might be used, but the Board was strongly in favor of experimenting for two or three years before producing a year's course for seventh grade.

Authors Wanted

One very important aim of the Newsletter is to provide a medium in which teachers of UICSM materials may exchange their ideas and suggestions on the teaching of these materials. Several such items of information have appeared in previous issues of the Newsletter. In particular, Newsletter Number Eight was primarily devoted to the publication of articles which were contributed by teachers in the field.

It is hoped that this request for contribution s will bring forth a heavy response from readers of the Newsletter. Your article can be a paragraph or several pages in length. It can concern itself with any phase of teaching mathematics, whether it be content or pedagogy. Suggestions for the use of visual aids such as charts, pegboards, overhead projectors, photographs, blocks, models and any other physical materials should be most welcome to other teachers. So would supplementary problems or test questions which have been useful to you in teaching the course content.

In brief, information is wanted from you on anything which might be of some interest to other readers of the UICSM Newsletter.

Please send this material (in any form) to:

UICSM Newsletter Editor
1208 West Springfield
Urbana, Illinois

OTHER AMBASSADORS

Gene Epperson of Talawanda High School in Oxford, Ohio spoke at the junior high division of the East Division of Indiana State Teachers Association on October 26. He illustrated the function machine in his talk entitled "You May Be Teaching High School Mathematics Sooner Than You Think."

In addition, the following UICSM staff members are teaching First Course in extension course classes this Fall: Eleanor McCoy - Springfield, Herb Wills - Decatur, Russ Zwoyer - Downers Grove.

If there are any teachers in your school who would like to receive the UICSM Newsletter, and they are not on the current list, you may submit their names for the mailing list by completing the following form---- a plain piece of paper will suffice if you have more than one name to send.

To: Editor, UICSM Newsletter
1208 W. Springfield
Urbana, Illinois

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UICSM

NEWSLETTER

Number 10

4 Jan 63

An Occasional Publication of the:

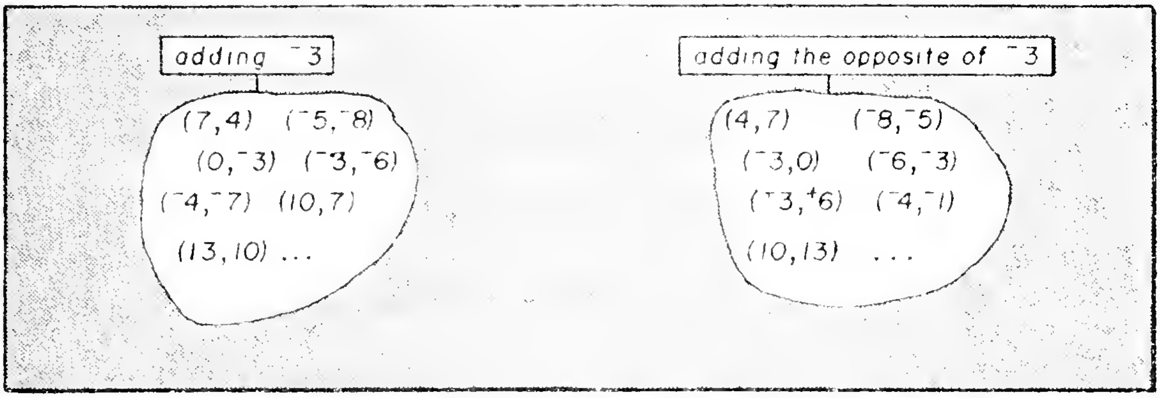
University of Illinois Committee on School Mathematics
1208 W. Springfield
Urbana, Illinois

Contents, [This Time]:

OPERATIONS AS FUNCTIONS, PART 2

This material is belated fulfillment of the promise made in the "Operations as Functions" article which appeared in Newsletter No. 4, pages 2-19. The first part dealt with operations as sets of ordered pairs, their uniqueness (since they are functions), and the question of converse and inverse sets. The operation equivalent to two successive operations was developed as a foreshadowing of the composition of functions.

When you are ready for page 1-75, the board work might "accidentally" result in the following:



[It might be well to list the ordered pairs for adding $^{-}3$ and ask for the name. Then, write the name 'adding the opposite of $^{-}3$' and ask for the pairs.]

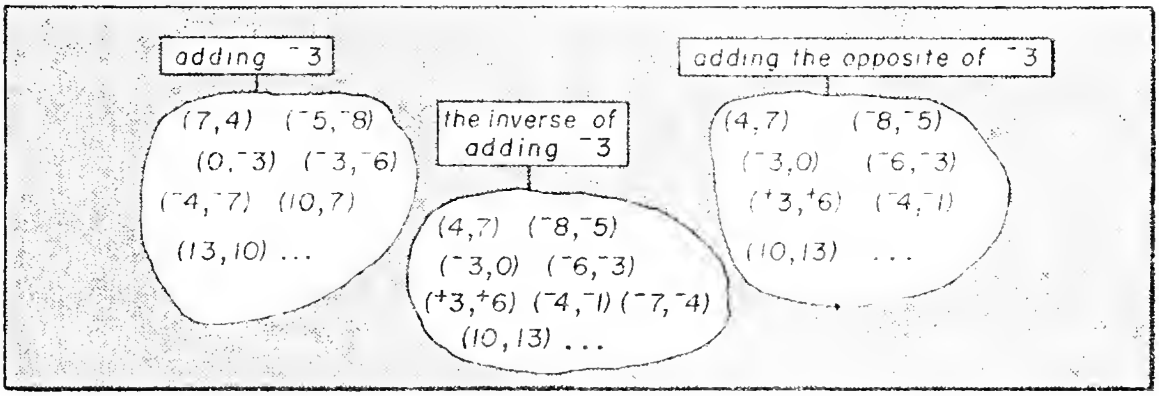
*

Teacher: Can you give me another name for the operation, adding the opposite of $^{-}3$?

Student: The inverse of adding $^{-}3$.

Student: Adding $^{+}3$.

Teacher: Both of you gave correct answers.
Let's just check to see that we're clear on what we mean by 'inverse'. [Write 'the inverse of adding $^{-}3$' in the indicated position.]



Teacher: Let's start with $^-5$. If I apply this operation to $^-5$, what do I get?

Student: $^-8$.

Teacher: Now, I want to undo the result of the adding $^-3$, that is, I want to start out with $^-8$. What shall I get back to if I have properly "undone what adding $^-3$ did"?

Student: You should get back to $^-5$.

Teacher: Does the second operation get you back to where you started?

Student: Yes.

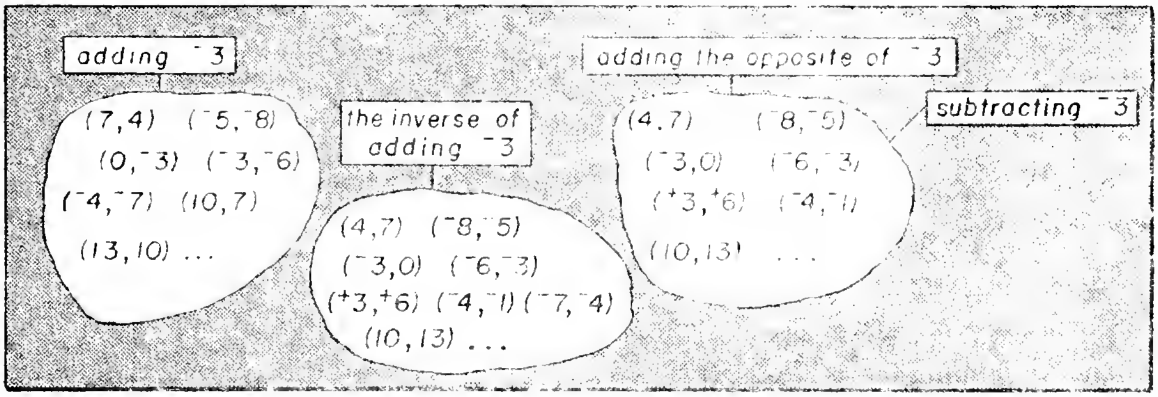
Teacher: What shows you that you get back to $^-5$?

Student: ($^-8$, $^-5$).

Teacher: Right. This operation (adding $^-3$) takes you from $^-5$ to $^-8$ and this one--the inverse of the first operation--takes you from $^-8$ back to $^-5$. That's why we call it 'the inverse'.

[More work like the above.]

Let's think back to the set of numbers of arithmetic. Just as we say that subtracting the number of arithmetic 3 is the inverse of adding the number of arithmetic 3, so we say that subtracting the real number $^-3$ is the inverse of adding the real number $^-3$.



*

Teacher: I am thinking of an operation one of whose names is 'the inverse of adding $^+8$'. What other name might we give that operation?

Student: Subtracting $^+8$.

Teacher: You are right. Give another name.

Student: Adding $^{-}8$. [It will be nicer if you don't get this one but you may. Just say, "OK" and go on.]

Teacher: Another name.

Student: Adding the opposite of $^{-}8$.

Teacher: Right. 'Subtracting $^{+}8$' and 'adding the opposite of $^{+}8$' are excellent names for this operation. Now I'm thinking of an operation, one of whose names is 'subtracting $^{-}2$'. What are other names?

And so on.

[After some work of this sort, we replace the written words 'the opposite of' by '−'. However, we still read '−' as 'the opposite of'.]

Teacher: John, pick a real number.

Student: $^{-}3$.

Teacher: Jane, what is the opposite of John's number?

Student: $^{+}3$.

[As these numbers are given, list them on the board as ordered pairs. Continue until you have several pairs. See that such pairs as ($^{-}3$, $^{+}3$) and ($^{+}3$, $^{-}3$) appear.]

Teacher: I'll pick a number: $^{+}3$. What is its opposite?

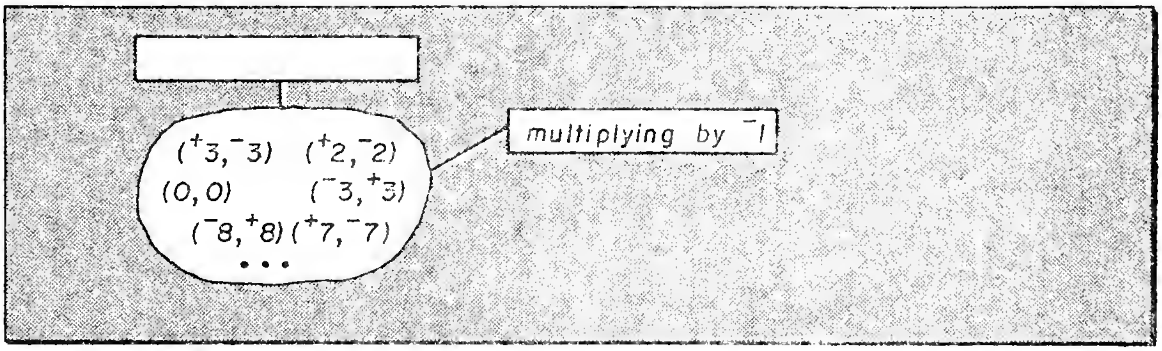
Student: $^{-}3$.

Teacher: Now imagine all such ordered pairs. Do you think we should call this an operation?

Student: Yes, because $^{+}3$ goes to $^{-}3$ and no where else.

Teacher: Right. Now, let's give this operation a name.

Student: Multiplying by $^{-}1$.



Teacher: Right. Now, try to think of a name that would make people remember that the second number in a pair is the opposite of the first number in that pair.

Student: Oppositing.

[You may have to tell them this name. This might be expected, since the choice of a name is an arbitrary matter.]

Teacher: Let's reverse these pairs. What pairs do we get?

Is this set of pairs an operation?

Student: Yes.

Teacher: Since this set is an operation whose pairs are obtained by reversing the pairs of the other operation, what shall we call it?

Student: The inverse of oppositing. [Teacher writes.]

Teacher: O.K. Now can you think of another name for it?

Student: Say, that's just the same as the first one!

Teacher: How about that? Do you mean that each pair in this first operation is in the second one?

Student: Yes.

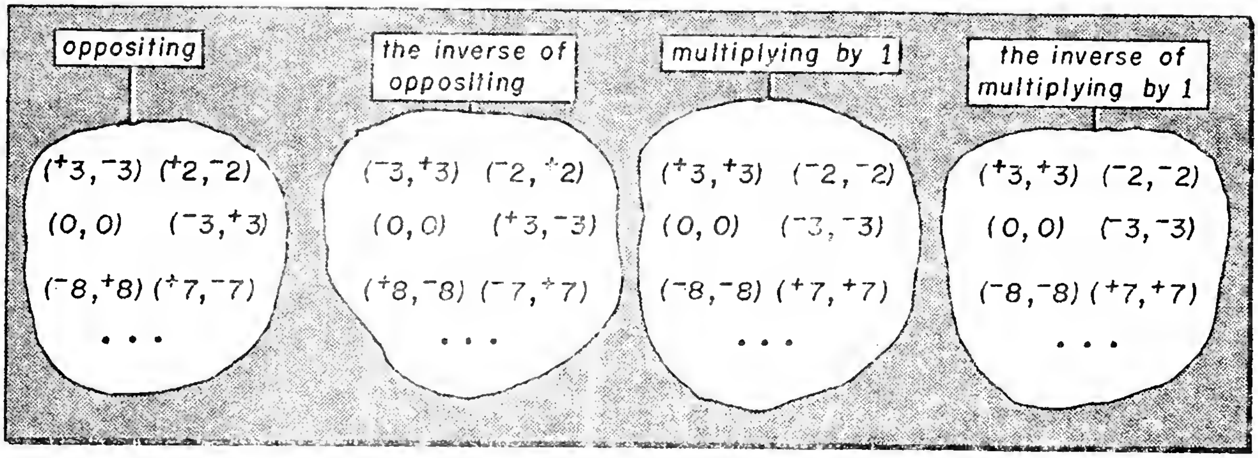
Teacher: But surely there's a pair in the second one that isn't in the first one?

Student: No!!

Teacher: So, the operation oppositing is its own inverse. Can you think of any other operation that is its own inverse? How about some of the operations involving multiplication? Is multiplying by 2 its own inverse? Multiplying by 3?

Student: Multiplying by 1 is its own inverse.

Teacher: Let's see if it is.



Student: Yes, it is.

Teacher: Let's try some of the addition operations. Is any one of these its own inverse?

Student: Yes. Adding 0.

Teacher: What are some pairs that belong to adding 0?

Student: ($^-$8, $^-$8), ($^+$2, $^+$2), (0, 0), ($^-$3, $^-$3).

Teacher: I'm running out of room up here. Can you suggest a way that I can save room and still show that ($^+$3, $^+$3), ($^+$2, $^+$2), (0, 0), ($^-$3, $^-$3), ($^-$8, $^-$8) all belong to the operation adding 0?

Student: Just put the name 'adding 0' up there beside the loop that's named 'multiplying by 1'.

Teacher: Like this?

Student: Yes.

Teacher: Wait a minute. That looks to me like you believe that:

> The operation adding 0 is the same
> as the operation multiplying by 1.

Do you believe that?

Student: Yes - No ---I don't know---

Teacher: Well, let's see. Certainly the <u>names</u> are different. Does that necessarily mean that the operations are different?

Student: No.

Teacher: Can you think of something that has two (or more) different names?

Student: The number 2 has lots of names.

Teacher: Right. So maybe these operations are the same. Is the operation the name or is it the set of pairs?

Student: It's not the name.

Student: I don't think it's the set of pairs either.

Teacher: That's too bad, because that's exactly what it is. The operation <u>is</u> the set of ordered pairs. You may not like it, but that's the way it is.

Student: Then the operation adding 0 is the same as the operation multiplying by 1 ? ? ?

Teacher: Well, let's see if they're the same. Are the pairs that belong to them exactly the same? Let's look at them and see. Each one of you think of a pair that belongs to adding 0. Does it belong to multiplying by 1? Now each of you think of a pair that belongs to multiplying by 1. Does it belong to adding 0? Do you think <u>anyone</u> can find a pair that belongs to one of these and does not belong to the other? The operation adding 0 is the same as the operation multiplying by 1.

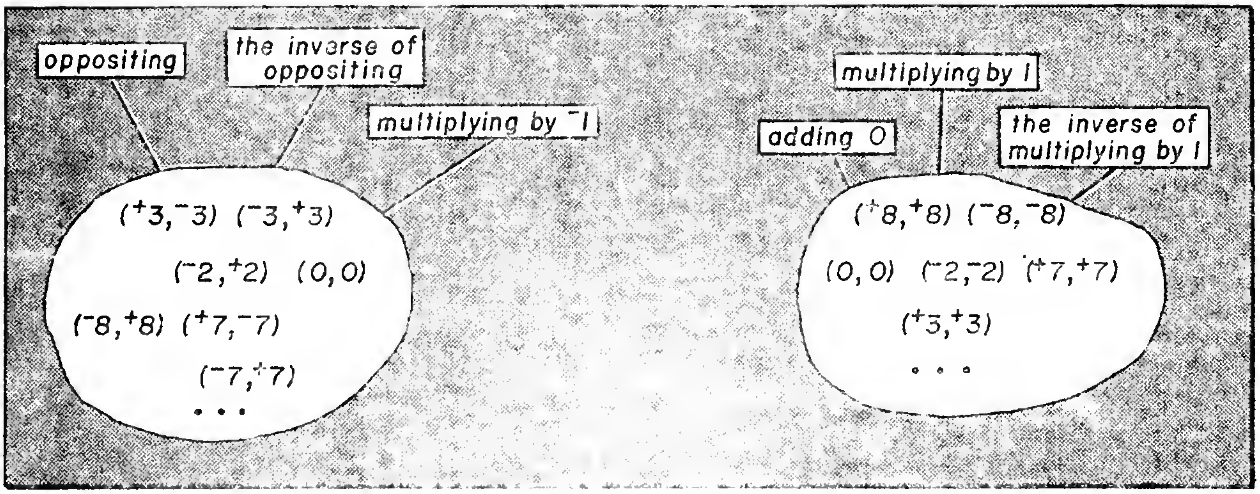
OK, let's go back to oppositing and its inverse. Mary pick a number. Now, oppositing takes you from the number $^{+}3$ to what number?

Student: $^{-}3$.

Teacher: Now, the inverse of oppositing [pointing to the proper name] takes you from $^{-}3$ to what number?

Student: $^{+}3$.

[Continue this way.]



Teacher: Now, is this set of pairs an operation?

Student: Yes! That's multiplying by 1.

Student: It's adding 0.

Teacher: Correct! So we already had this up here. Tell me some more pairs that belong to this operation.

Teacher: Did you really go from ⁻15 to ⁺15 and then from ⁺15 to ⁻15 to get that pair (⁻15, ⁻15)?

Student: No.

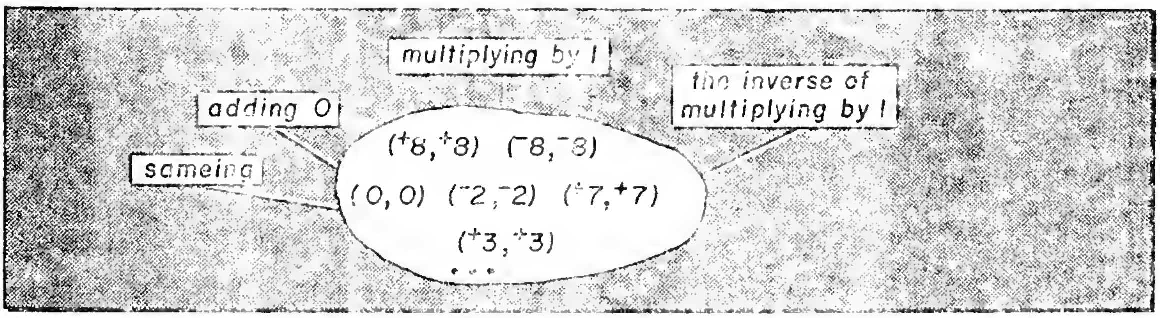
Teacher: What did you do?

Student: Well, the second number is just like the first.

Teacher: Oh, the second number is the same as the first. Would that be true for each pair in this operation?

Student: Yes.

Teacher: Because that's true for each pair in this operation we sometimes call this operation: sameing.



Student: Is there any symbol for it like there is for oppositing?

Teacher: Yes, we can use this sign '+'.

*

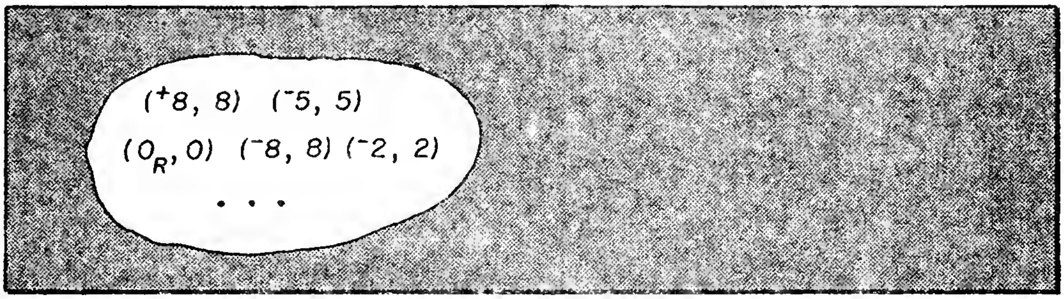
Let's consider absolute valuing. First, I would use the name 'arithmetic absolute valuing' instead of 'absolute valuing' when I consider the operation which maps the reals onto the numbers of arithmetic. I would also use the symbol '$|\,|_A$' instead of '$|\,|$'.

Now for pages 1-105 and 1-106.

*

Teacher: Helen, pick a number. Tell me the number you picked and then tell me the arithmetic absolute value of the number. Remember, we are not using ambiguous names today.

Student: $(^{+}8, 8)$.



Teacher: Now, I'll give you a number. What is its arithmetic absolute value? $^{-}5$.

Student: 5.

Teacher: 0.

Student: 0.

Teacher: Is the number 0 that I thought about the same as the number 0 that Jane thought about?

Student: No. Yours is a real number. Jane's is a number of arithmetic.

Teacher: To show that, let's write '0_R' today when we want a name for the real number zero and just '0' when we want a name for the number of arithmetic zero. $(0_R, 0)$. Is this set of pairs an operation?

Student: Yes.

Teacher: What name do we give it?

Student: Sameing.

Teacher: Let's see. Is the number of arithmetic 8 the same as the real number $^+8$?

Student: No. So, this isn't sameing?

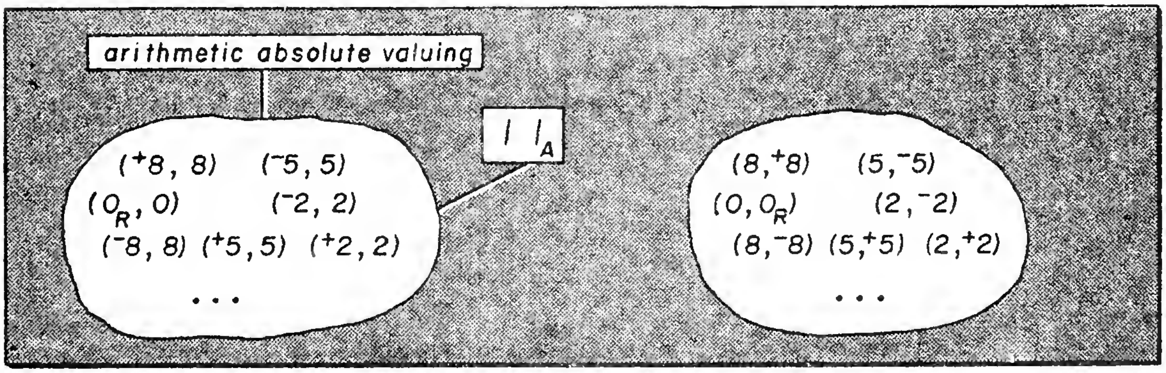
Student: We call it 'arithmetic absolute valuing'.

Teacher: Good. So, arithmetic absolute valuing takes you from a real number to a single number of arithmetic. By the way, have any of you ever carried out this operation before today?

Student: Well, yes. When we're doing some of these problems with real numbers, it's easier to multiply numbers of arithmetic.

Teacher: Yes. Every time you go from a real number to the corresponding number of arithmetic, you're actually performing the arithmetic absolute valuing operation. So, you've been doing it for a long time even if you didn't have a name for it.

Take another look at this operation. Now let's reverse the pairs.



Is this an operation?

Student: Yes.

Teacher: I'm going to pick a number of arithmetic. 5! Now, this set of pairs takes me from the number of arithmetic 5 to what?

Student: $^+5$.

Student: $^-5$.

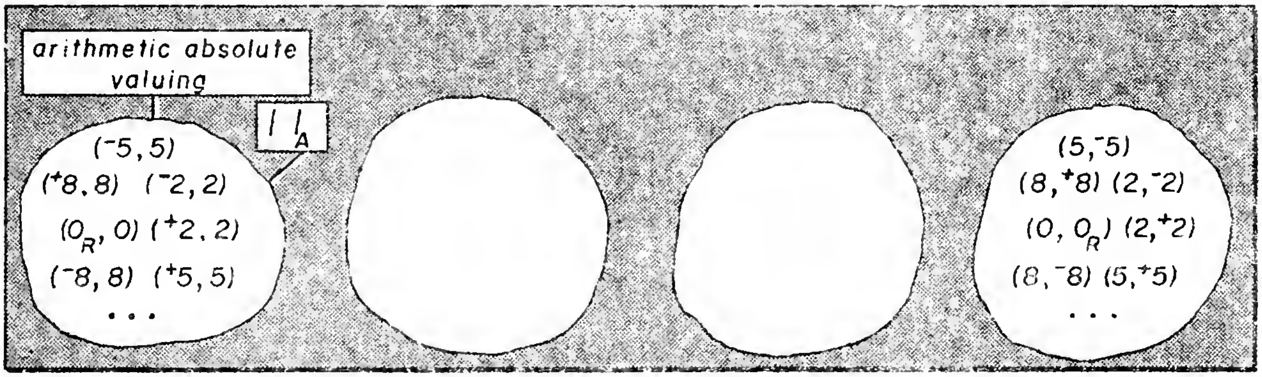
Teacher: Well, which is it? If this is an operation it must take you to a single number.

Student: There isn't just one single one. There are two.

Teacher: Then, is this an operation?

Student: No!

Teacher: Let's look at this set and sort of break it into two sets. What way of breaking it do you think I have in mind?



Student: Put '$(8, ^+8)$', '$(5, ^+5)$', '$(6, ^+6)$' in the second loop.

Teacher: What shall I put in the third loop?

Student: '$(5, ^-5)$', '$(8, ^-8)$'.

Teacher: What are some others that would go in each one?

Does this take care of all the pairs you have in this set?

Student: No. $(0, 0_R)$ is left out.

Teacher: Where does that go?

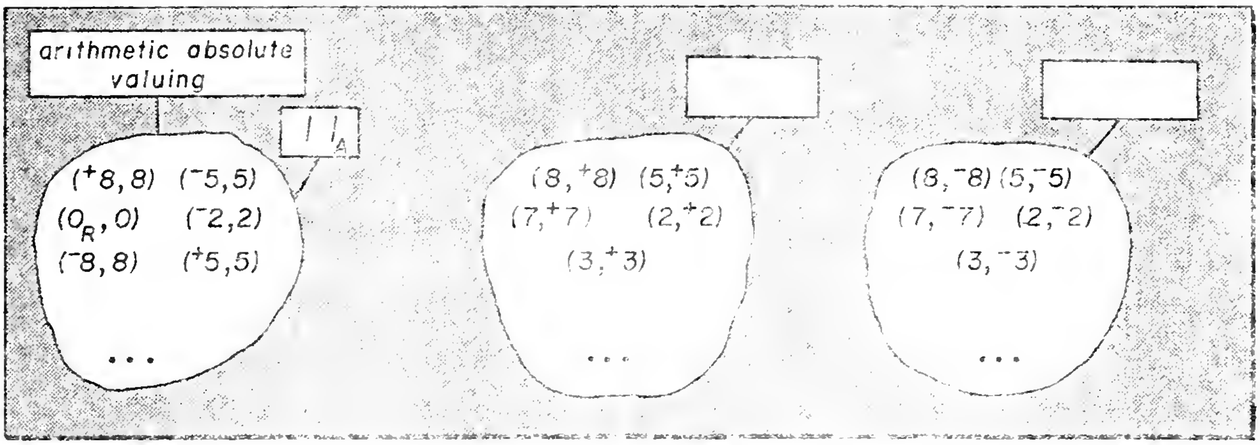
Student: Well, if you only want positives in the first one and negatives in the second one, it won't go any place.

Teacher: But we don't want to leave $(0, 0_R)$ out in the cold? What shall we do?

Student: Put it in both. [If you don't get this answer, give it yourself.].

Teacher: Right. Are these operations?

Student: Yes.



Teacher: Now about names for them.

Student: Unabsolute valuing.

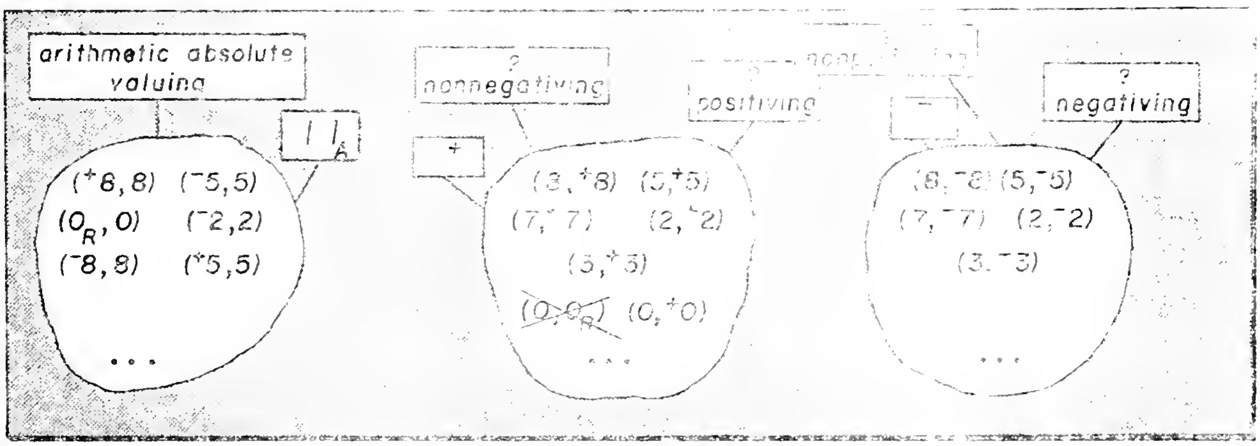
Student: The inverse of arithmetic absolute valuing from the positive numbers.

Teacher: I don't think I would say it quite that way. I might say 'from the nonnegatives--'. Why would I use 'nonnegative' instead of 'positive'?

Student: To take care of 0.

Teacher: Correct. We might use the raised plus sign as a symbol for this operation. If we did that how would we write '$(0, 0_R)$'? Susan, come up here and write it.

Susan: $(0, {}^{+}0)$.



Teacher: Does that mean that the real number 0 is a positive number?

Student: No. It means it's nonnegative.

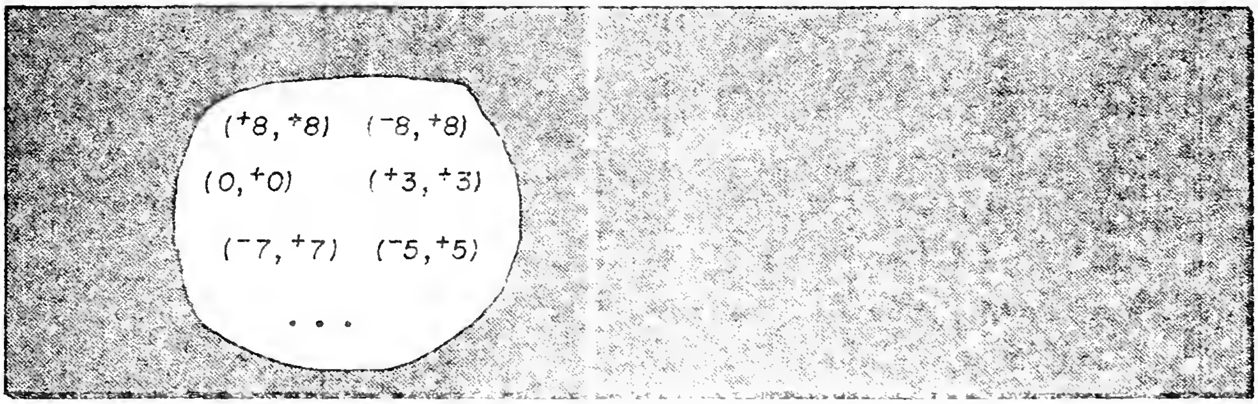
Teacher: Very good. Sometimes we call this operation 'positiving' but 'nonnegativing' might be a better name. How about names for this other operation? [And so on!]

*

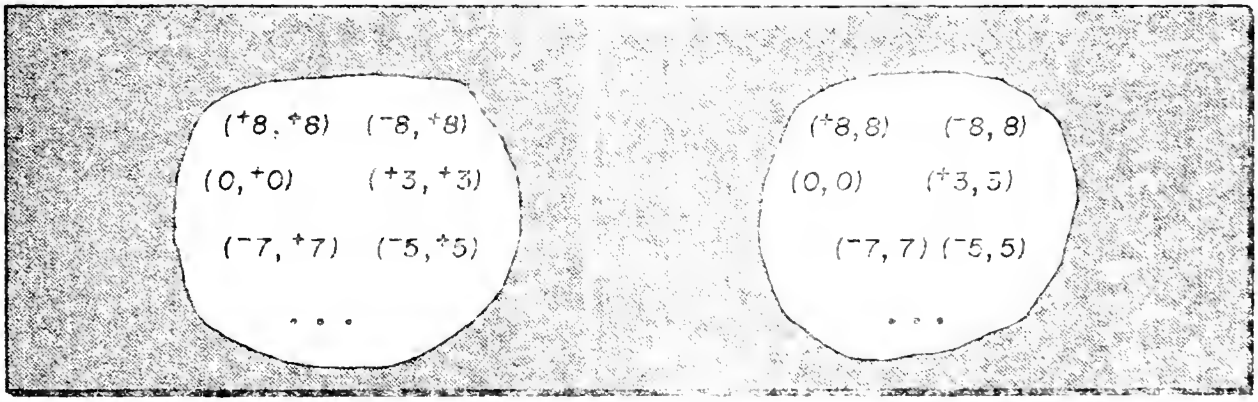
Now to develop the operation absolute valuing from real numbers to the nonnegative reals.

*

Teacher: Here are some pairs. Give me some more that are like them. [Usual questions about operation.]



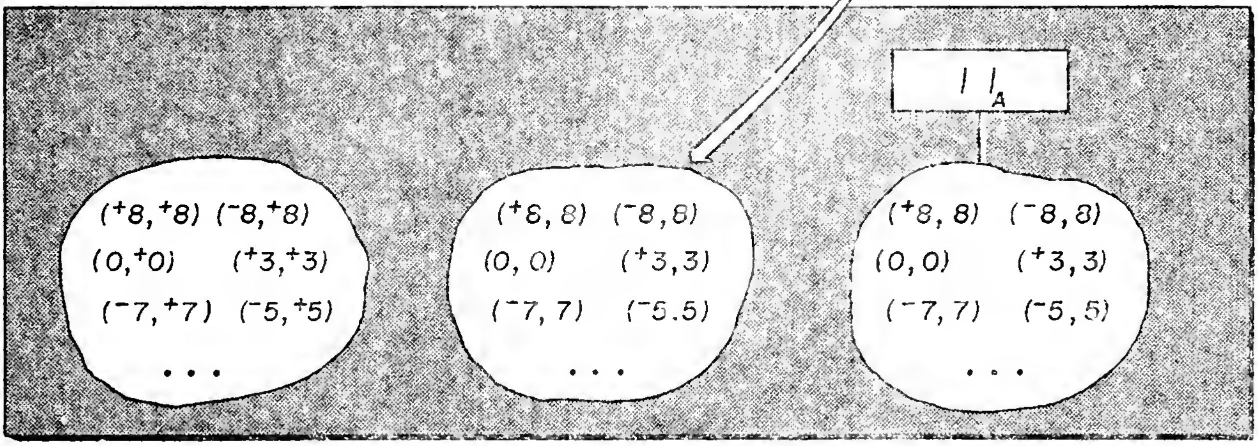
Now let's rewrite these names using the convention that a numeral for a number of arithmetic may be used to name a nonnegative real number.



Teacher: Suppose someone walked in the room and saw this second one. He might believe that we were thinking of what operation?

Student: Arithmetic absolute valuing.

Teacher: Right. Let's look at some of the ordered pairs that belong to $| \ |_A$.



So, if he saw this middle one, he wouldn't know which of the other two you had in mind. Actually people use the words 'absolute valuing' in naming both of the operations because the nonnegative reals behave like the numbers of arithmetic. The first is absolute valuing from the reals to the nonnegative reals. The second is absolute valuing from the reals to the numbers of arithmetic. We might use the symbol '$| \ |_{R_0^+}$' and '$| \ |_A$' to name them. However, most people give them the same name 'absolute valuing', and use the same symbol, '$| \ |$'. Does that make them the same operation? [No.] From now on, in most of your work you will be using absolute valuing from the reals to the nonnegative reals. So, let's agree that:

Numerals which contain 'absolute value', '$| \ |$', should be interpreted as numerals for real numbers--except in places where the context prohibits this interpretation.

A. H.

PROGRAMED INSTRUCTION REPORT

As a part of the UICSM Programed Instruction Project, we have prepared a report which summarizes the content and pedagogy of the programed Unit I. It consists of a great number of sample pages accompanied by explanations which tell the specific objectives and methods of our programers as they programed.

If you would like a copy of this report for your mathematics department, please complete the following form and return to

Clifford W. Tremblay
UICSM Project Office
1208 W. Springfield
Urbana, Illinois

______________________________, Mathematics Department Head

______________________________ (School)

______________________________ (Address of School)

______________________________ (City and State)

A NOTE ON INVERSE OPERATIONS

The work on inverse operations [Unit I, pages 66-70] culminates in the students' discovery that

(1) for each number of arithmetic x,

SUBTRACTING x

is the same thing as

THE INVERSE OF ADDING x

and (2) for each nonzero number of arithmetic x

DIVIDING BY x

is the same thing as

THE INVERSE OF MULTIPLYING BY x.

We capitalize on these agreements on page 75 of Unit I by defining, for each real number x:

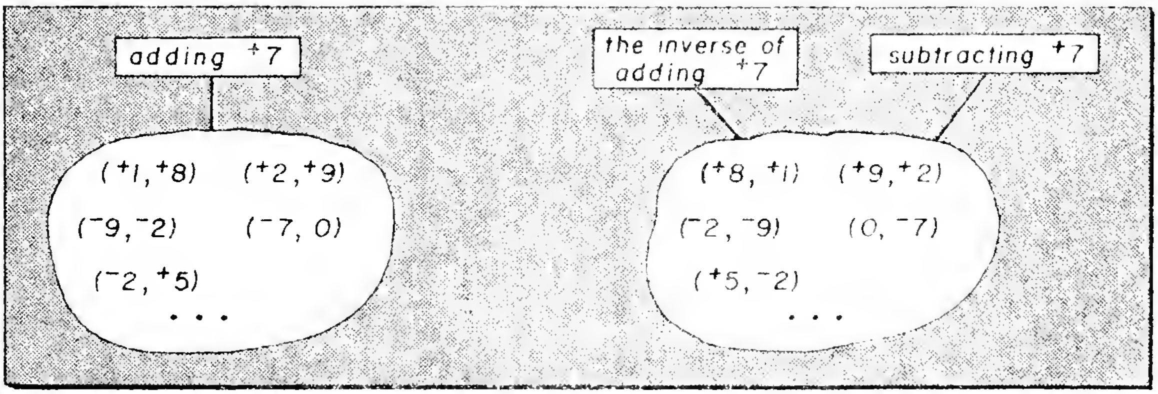
SUBTRACTING x

to mean:

THE INVERSE OF ADDING x

Here is one way of using lists of ordered pairs to give the students meaningful practice in using this definition. Starting with a list of pairs belonging to ADDING $^+7$, we quickly get a corresponding list of pairs which belong to THE INVERSE OF ADDING $^+7$.

Since we can also think of the latter list as a list of pairs belonging to SUBTRACTING $^+7$, we can write:



Using the lists, the students solve problems such as these:

$$^{+}8 - {}^{+}7 = ?$$
$$^{+}9 - {}^{+}7 = ?$$
$$^{-}2 - {}^{+}7 = ?$$

Now consider the problem:

$$^{-}16 - {}^{+}7 = ?$$

When the students see that the problem can't be solved using the pairs we have listed, have someone guess at the answer. The discussion in my class went something like this:

Teacher: Someone guess at the answer to:

$$^{-}16 - {}^{+}7 = ?$$

Student: $^{-}9$.

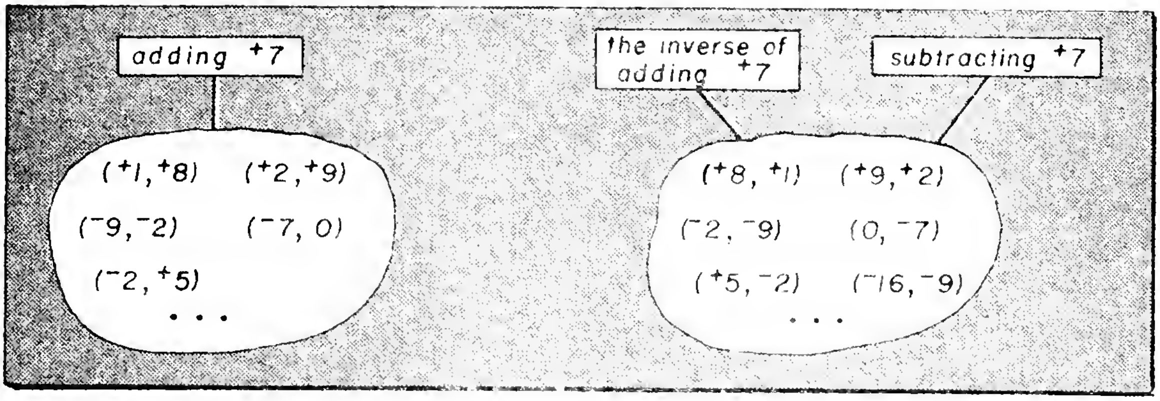
Teacher: In saying that:

$$^{-}16 - {}^{+}7 = {}^{-}9$$

we are saying that a certain ordered pair belongs to SUBTRACTING $^{+}7$. What pair is this?

Student: $(^{-}16, {}^{-}9)$.

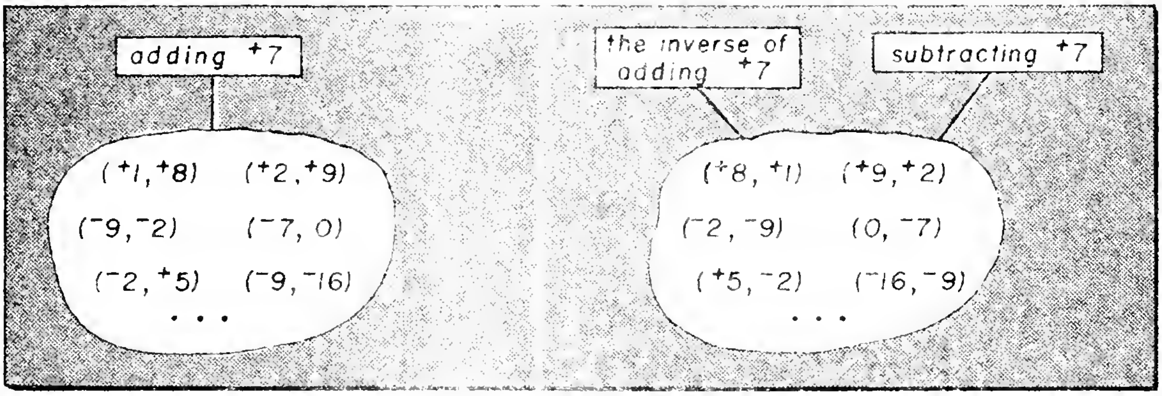
The teacher adds '$(^{-}16, {}^{-}9)$' to the list for SUBTRACTING $^{+}7$. The board now has these entries:



Teacher: Since SUBTRACTING $^+7$ is the same thing as THE INVERSE OF ADDING $^+7$, it follows that the pair ($^-16$, $^-9$) belongs to SUBTRACTING $^+7$ just if [if and only if] <u>what</u> pair belongs to ADDING $^+7$?

Student: ($^-9$, $^-16$)

The teacher adds '($^-9$, $^-16$)' to the list for ADDING $^+7$. The board now reads:



Teacher: Does ($^-9$, $^-16$) belong to ADDING $^+7$?

Student(s): No!

Teacher: Is it possible, then, for ($^-16$, $^-9$) to belong to SUBTRACTING $^+7$?

Student(s): No! [Also: Nix! Nyet!]

Teacher: Hence it can't be the case that:

$$^-16 - {}^+7 = {}^-9$$

is a true statement.
Someone else take a guess at the answer.

Succeeding guesses are tested in the manner indicated until (hopefully) a student or (if necessary) the teacher submits the correct answer.

The reasoning involved is simply illustrated by:

$\forall_x \forall_y \forall_z$ x - y = z
if and only if
(x, z) belongs to SUBTRACTING y
if and only if
(z, x) belongs to ADDING y

A similar approach can be used in discussing division of real numbers [Unit I, pages 92-94]. The reasoning in this case is illustrated by:

$\forall_x \forall_{y \neq 0} \forall_z$ x ÷ y = z
if and only if
(x, z) belongs to DIVIDING BY y
if and only if
(z, x) belongs to MULTIPLYING BY y

In addition, we show that the converse of MULTIPLYING BY 0 is not a function and, hence, is not an operation.

B. K.

Mathematical Overlap

Some ideas in one mathematics course overlap ideas developed in other mathematics courses. For example students in elementary algebra courses find that:

$$(1) \quad \{(x, y): (x - a)^2 + (y - b)^2 = r^2\}$$

is a circle. If we let:

$$u = x - a$$

$$v = y - b$$

and substitute in (1). The result is:

$$(2) \quad \{(u, v): u^2 + v^2 = r^2\}$$

what we obtain makes it clear that the original set was a circle.

But, we miss a good trick by not emphasizing the notiton of translations. Earlier (in most course sequences), we study quadratic functions

$$(3) \quad f = \{(x, y): y = ax^2 + bx + c, a \neq 0\}$$

and could employ the translation (if we mentioned it then)

$$u = x + \frac{b}{2a} \qquad \text{or} \qquad x = u - \frac{b}{2a}$$

$$v = y.$$

This translation is a particularly convenient one since it forces the axis of symmetry to be given by $u = 0$.

Using this translation in (3) we get:

$$(4) \quad f = \{(u, v): v = au^2 + \frac{4ac - b^2}{4a}, a \neq 0\}.$$

From (4), we can compute readily the usual quantities:

A. Axis of symmetry

$$\{(u, v): u = 0\} \quad \text{and} \quad \{(x, y): x = 0 - \frac{b}{2a}\}$$

B. Extreme Value

$$u = 0 \Rightarrow v = \frac{4ac - b^2}{4a} = y.$$

C. Roots

$$v = 0 \Rightarrow u = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a.}$$

These two examples show how the notion of translations overlaps elementary topics developing more sophisticated connections. Another example of this is found in Unit 7 where the students repeatedly acknowledge that "it's easy to get the recursive definition, but how do you find the explicit definition of a sequence quickly?" For the answer to this question they must wait unit Unit 8. In that unit we find the necessary language and techniques to do this job relatively easily.

A sample recursive definition might be:

$$\text{(i)} \quad b_1 = 8$$

$$\text{(ii)} \quad \forall_n \; b_{n+1} = b_n + 3n + 2$$

From (ii) $b_{q+1} = b_q + 3_q + 2$

so, $\quad b_{q+1} - b_q = 3q + 2.$ (*)

By our definition of a difference sequence:

$$\forall_p \; (\Delta a)_p = a_{p+1} - a_p$$

it follows that $(\Delta b)_q = b_{q+1} - b_q.$ (**)

From Theorem 140 $[\forall_n \; a_n = a_1 + \sum_{p=1}^{n-1} (\Delta a)_p]$

we obtain $\forall_n \; b_n = b_1 + \sum_{p=1}^{n-1} (\Delta b)_p.$

An instance of this is $b_q = b_1 + \sum_{p=1}^{q-1} (\Delta b)_p$ (***)

From (*) and (**) we get:

$$(\Delta b)q = 3q + 2$$

Substituting this in (***) we get:

$$b_q = b_1 + \sum_{p=1}^{q-1} (3q + 2)$$

By our summation theorems this may be written:

$$b_q = b_1 + \frac{3q(q-1)}{2} + 2(q-1)$$

$$= 8 + \frac{3q(q-1)}{2} + 2(q-1) \quad \text{from (i)}$$

$$b_q = \frac{3}{2}q^2 + \frac{1}{2}q + 6$$

So, we have a test pattern for the generalization:

$$\forall_n \, b_n = \frac{3}{2}n^2 + \frac{1}{2}n + 6.$$

This is the required explicit definition.

A. Holmes

Instructional Aids

Most of the teachers who have taught unit 4, have probably felt the need for a device to help them in the classroom when dealing with lattices.

One of the teachers who did something about this described his experiment to us in a letter. Our correspondent is Charles J. Searey from Sterling, Colorado.

He acquired a 4′ x 4′ pegboard and painted it. The pegboard had quarter inch holes spaced one inch apart. He used half inch long dowel segments to "plot points".

The graphs of linear functions were illustrated by ribbons, different colors distinguishing different graphs. It occurred to us that an arbitrary point set might be illustrated by inserting tacks of a particular color in the tips of those dowels which define the set.

If the axes are drawn with ribbon (rather than being fixed), such topics as adding constant functions can be simplified considerably.

Editor's Note:

If you happen to try this experiment, you will probably find (as our correspondent did) that the temptation will be to expand its uses constantly. Should you come up with a new (to you) use, please send us a note about it, and we will use the Newsletter as our medium for disseminating this information.

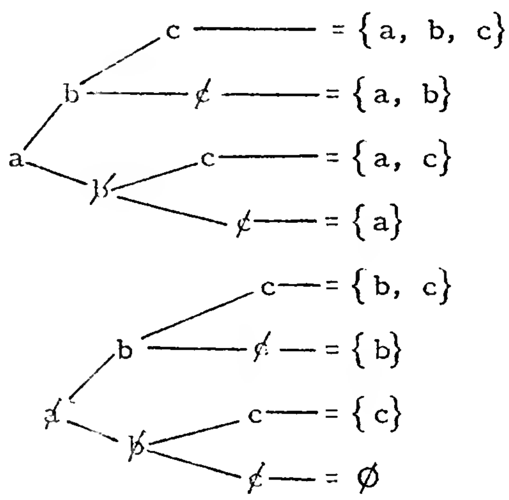
C. T.

This article is the work of another of our correspondents, Sister Mary Lucy, V. S. C., from Pittsburgh, Pennsylvania. We are indebted to our readers who want to share their ideas with us via this medium.

Relations -- Reflexive and Symmetric

How many subsets does a set of n elements have?

$S = \{a, b, c\}$



S has eight or 2^3 subsets. Each column of entries after the first has twice as many elements as the preceding column.

It is clear that this process could be extended to determine the number of subsets of any finite set. A set of n elements has 2^n subsets.

A relation R is reflexive $\Longleftrightarrow \forall_{x \in r}\; x R x$

Consider $S = \{a, b\}$, $n(s) = 2$

$$R = S \times S = \{(a, a), (a, b), (b, a), (b, b)\}$$

Question 1. How many relations are there among the members of a set of 2 elements? or, How many subsets does R have?

$R_1 = \{(a, a)\}$

$R_2 = \{(b, b)\}$

$R_3 = \{(a, b)\}$

$R_4 = \{(b, a)\}$

$R_5 = \{(a, a), (b, b)\}$

$R_6 = \{(a, a), (b, a)\}$

$R_7 = \{(a, a), (a, b)\}$

$R_8 = \{(a, b), (b, a)\}$

$R_9 = \{(b, b), (b, a)\}$

$R_{10} = \{(b, b), (a, b)\}$

$R_{11} = \{(a, a), (b, a), (a, b)\}$

$R_{12} = \{(b, b), (a, b), (b, a)\}$

$R_{13} = \{(a, a), (b, b); (a, b)\}$

$R_{14} = \{(a, a), (b, b), (b, a)\}$

$R_{15} = \{(a, a), (b, b), (b, a), (a, b)\}$

$R_{16} = \emptyset$

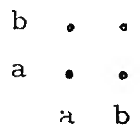
If S has n elements, then $R = S \times S$ has n^2 elements. So, the class of all subsets of R has 2^{n^2} elements.

In our case $n(S) = 2$, so R has $2^{2^2} = 16$ subsets.

There are 2^{n^2} relations among the members of a set of n elements.

Question 2. How many reflexive relations are there whose field is a given set of 2 elements.

Consider



A reflexive relation whose field is $\{a, b\}$ is the union of two sets; S_1 consisting of the "diagonal" ordered pairs united to S_2 whose members are the remaining ordered pairs.

S_1 has 2 elements

S_2 has $2^2 - 2$ elements

The reflexive relations are:

$\{(a, a), (b, b)\} \cup \{(a, b)\} = \{(a, a), (b, b), (a, b)\}$

$\{(a, a), (b, b)\} \cup \{(b, a)\} = \{(a, a), (b, b), (b, a)\}$

$\{(a, a), (b, b)\} \cup \{(a, b), (b, a)\} = \{(a, a), (b, b), (a, b), (b, a)\}$

$\{(a, a), (b, b)\} \cup \emptyset = \{(a, a), (b, b)\}$

Therefore, since S_1 is common to all of these reflexive relations, the number of reflexive relations is equal to the number of subsets of S_2--that is, $2^{(2^2 - 2)}$ or 4.

Thus, there are 4 reflexive relations whose field is a given set of 2 elements.

A relation R among the members of a set S is symmetric if and only if

$$\forall_{(x,\ y) \in S \times S} \text{ if } y\,Rx \text{ then } x\,Ry.$$

Question 1: How many symmetric relations are there among the members of a set of 2 elements?

Consider: $S = \{a, b\}$

b • •
a • •
  a b

Each symmetric relation among the members of S can be thought of as resulting from choosing members o $\{(x, y) \in S \times S : x \geq y\}$, and then choosing those members of $S \times S$ "above" the diagonal which are required by symmetry. Since $\{(x, y) \in S \times S: x \geq y\}$ has 3 members $\left[2 + \frac{2^2 - 2}{2} = 3\right]$, there are 3 choices to be made so there are $2^3 = 8$ symmetric relations among 2 members.

These are: R_1, R_2, R_5, R_8, R_{11}, R_{12}, R_{15}, and R_{16}.

When n (S) = 3, the number of symmetric relations among the members of the set of 3 elements is

$$2^{(\frac{3^2+3}{2})} = 2^6 = 64$$

Question 2: How many relations are there among the members of a set of 2 elements which are both reflexive and symmetric?

Consider:

I. Such relations which have a given 2-element set as field:

$R_5 = \{(a, a), (b, b)\}$
$R_{15} = \{(a, a), (b, b), (a, b), (b, a)\}$

II. Such relations which have a given 1-element set as field:

$R_1 = \{(a, a)\}$
$R_2 = \{(b, b)\}$

III. Such relations which have $\emptyset$ as field:

$R_{16} = \emptyset$

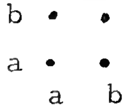
$$\binom{2}{2} \cdot 2^{(\frac{2^2-2}{2})} + \binom{2}{1} \cdot 2^{(\frac{1^2-1}{2})} + \binom{2}{0} \cdot 2^{(\frac{0^2-0}{2})}$$

$1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 = 5$ relations among 2 elements which are both reflexive and symmetric.

Question 3: How many reflexive relations are there among the members of a given set of 2 elements?

Consider:

I. Reflexive Relations which have a given 2-element set as field:



$R_{13}, R_{14}, R_{15}, R_5$

$\binom{2}{2} \cdot 2^{2^2-2}$

\+

II. Reflexive Relations which have a given 1-element set as field:

• (b, b)
• (a, a)

R_1, R_2

$\binom{2}{1} \cdot 2^{1^2-1}$

\+

III. Reflexive Relations which have $\emptyset$ as field:

$\emptyset = R_{16}$

$\binom{2}{0} \cdot 2^{0^2-0}$

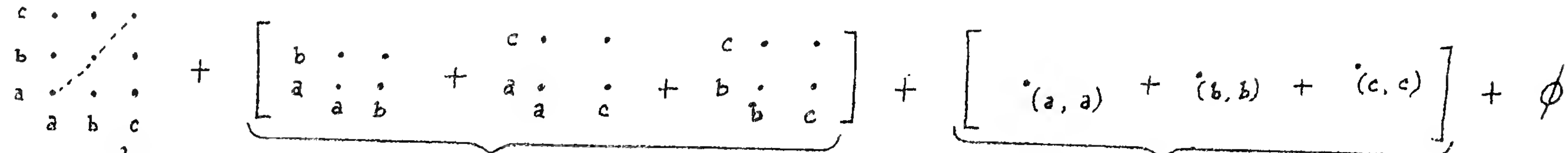
Total: $1 \cdot 4 + 2 \cdot 1 + 1 \cdot 1 = 7$ reflexive relations

---

$S_2 = \{a, b, c\}$ $R = S_2 \times S_2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

There are $2^9 = 512$ relations among the members of a set of 3 elements.

The number of reflexive relations among the members of a given set of 3 elements can be determined as follows:



$\binom{3}{3} \cdot 2^{3^2-3} + \binom{3}{2} \cdot 2^{2^2-2} + \binom{3}{1} \cdot 2^{1^2-1} + \binom{3}{0} \cdot 2^{0^2-0}$

$1 \cdot 2^6 + 3 \cdot 4 + 3 \cdot 1 + 1 \cdot 1 = 80$ reflexive relations among the members of a given set of 3 elements.

UICSM

NEWSLETTER

Number 11

17 April 63

An Occasional Publication of the

University of Illinois Committee on School Mathematics

1208 West Springfield

Urbana, Illinois

CONTENTS

A Preview of the UICSM Programed Text on Solid Geometry

This spring, a programed edition of a solid geometry course will be tested in several classes. This program will not be made available for general use until it has been through at least this initial trial run and has been revised on the basis of the information gathered in this trial run. Even though the solid geometry program will not be generally available, the thought occurred to us that our Newsletter readers might like to know something about it.

As you probably know, the importance of studying solid geometry, as a separate semester-long course, has been minimized in recent years. Many up-to-date geometry texts have taken to integrating the solid geometry topics with appropriate topics in plane geometry. In some of these texts, these topics are discussed toward the end of appropriate chapters. In some cases they are so far at the end that these discussions appear after the chapter summaries [of important things to remember], chapter tests, and other exercises. The placement of the present UICSM solid geometry course is Appendix D of Unit 9. This placement leads some teachers to believe the topic is secondary to other topics and may be omitted without loss.

While we do not advocate that a full semester be set aside to develop the topics of solid geometry and the student's spatial concepts, this development is important enough to consume a portion of one semester. By preparing a programed edition of a full course in solid geometry, we hope to limit the study to a concentrated 3- or 4-week course of self-instruction, without short-changing the students. Another goal we hope to achieve is that of developing important spatial concepts in a greater percentage of our students. This we feel may be done by giving the student a program in which to participate. The programed edition of solid geometry combines the development found in Appendix D of Unit 9 with a short unit of solid geometry prepared by Howard Marston for UICSM. The format of the program calls for little formal development of a postulational system. An appeal is made to the student's intuition to get a "feeling" for the basic concepts of a three-dimensional space. Good diagrams can be used to (a) give cues to properties of points, lines, and planes in space, (b) help develop a student's intuition and feeling for 3-space, and (c) help reinforce correct concepts

and generate counter-examples to false generalizations. The student's intuition and knowledge of 3-space is also developed by the program's built-in feature of reinforcement.

Included in the content of the programed edition are the topics which are normally covered in a full course in solid geometry. One could think of the course as being divided into several broad categories:

(a) properties of points, lines, and planes in space
(b) solids whose volumes are computed by '$V = Bh$'
(c) solids whose volumes are computed by '$V = \frac{1}{3}Bh$'
(d) prismatoids
(e) solid spheres
(f) locus problems

In the following sequence of pages, we have an example of the approach we use in this program to develop an elementary space concept involving lines determined by sets of points.

[Page 18] [Part 901]

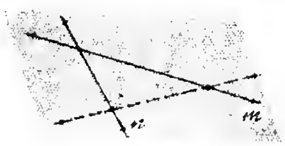
We have now discussed three conditions under which a plane is determined:

(1) Given three noncollinear points, they determine a plane.

(2) Given a line and a point not on the line, the line and the point determine a plane.

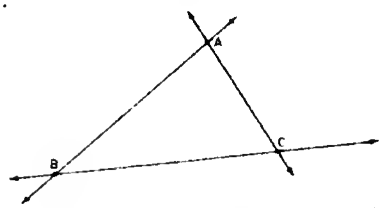
and:

(3) Given two intersecting lines, these lines determine a plane.



[Part 901] [Page 19]

Consider three noncollinear points. Each pair of these points determines a line.

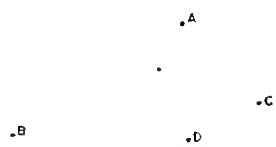


The noncollinear points A, B, and C determine lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.

Since C is not on $\overleftrightarrow{AB}$, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are different. Since B is not on $\overleftrightarrow{AC}$, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are different. Since A is not on $\overleftrightarrow{BC}$, lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AB}$ are different. Hence, lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ are all different.

So, we conclude that three noncollinear points determine three lines.

Now, consider 4 points, no three of which are collinear.



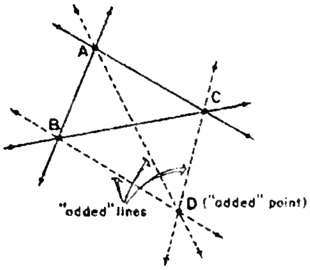
How many lines are determined by these 4 points?

Write your answer on your work sheet.

Check your answer.

4 points, no 3 collinear, determine *6 lines*.

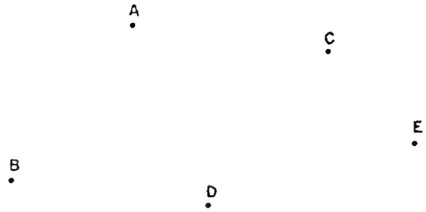
Here is one way to think about the last problem. [There are others.] Considering the problem of counting the lines determined by 4 points, no 3 collinear, we can solve this problem by finding the number of lines which are "added" to the original 3 lines when we "add" a point to 3 noncollinear points.



A, B, C are the "original" points.

D is the added point. D determines one line with each of the 3 original points. Hence, 3 lines are "added" to the original set of 3. Therefore, the four points determine 3 + 3, or 6, lines.

Now, add a fifth point, noncollinear with each pair of the points considered thus far.

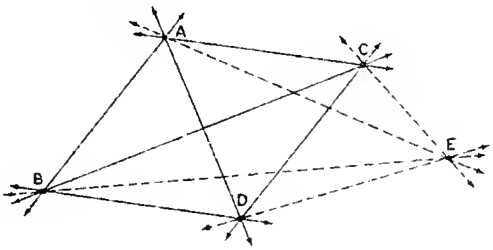


Answer these questions on your work sheet.

(1) How many lines are "added" to the collection of determined lines by the addition of this fifth point?

(2) How many lines are determined by 5 points, no 3 collinear?

Check your answers.



(1) The addition of the fifth point "added" *4 lines* to the set of lines determined by the 4 given points.

(2) Five points, no 3 collinear, determine 6 + 4, or *10 lines*.

Here is a table which has been filled in with the results of counting "determined" lines thus far. Answer on your work sheet.

Number of points, no three collinear	3	4	5	6	7	8	9	10	100	n
Number of lines each containing two of these points	3	6	10	?	?	?	?	?	?	?

[Look for a pattern in the first eight columns.]

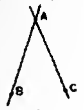
[Page 24] [Part 901]

Check your answers.

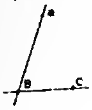
Number of points, no three collinear	3	4	5	6	7	8	9	10	100	n
Number of lines each containing two of these points	3	6	10	15	21	28	36	45	4950	$\frac{n(n-1)}{2}$

Here is another way to look at this problem. [There are others.]

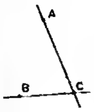
Consider 3 noncollinear points.



Start at A. Determine all lines through A and each remaining point. [2 such lines]



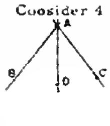
Start at B. Determine all lines through B and each remaining point. [2 such lines]



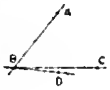
Start at C. Determine all lines through C and each remaining point. [2 such lines]

Note carefully that each line is counted twice. So, there are actually $\frac{3 \cdot 2}{2}$, or 3 lines determined.

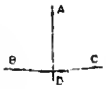
Consider 4 points, no 3 collinear



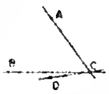
Starting at A, there are 3 lines [through B, C, D] determined.



Starting at B, there are 3 lines [through A, C, D] determined.



Starting at D, there are 3 lines [through A, B, C] determined.



Starting at C, there are 3 lines [through A, B, D] determined.

Since each line is counted twice, there are $\frac{4 \cdot 3}{2}$, or 6 lines determined.

[Part 901] [Page 25]

Now, consider the case of n points, no 3 of which are collinear $[n \geq 2]$. Following the same plan, we can count $(n - 1)$ lines through each of the n points. But, in doing this, we count each line twice. Hence, there are $\frac{n(n-1)}{2}$ lines determined by the n given points.

Thus, in the case of 5 points, no 3 collinear, there are $\frac{5 \cdot 4}{2}$ or 10 lines. In the case of 100 points, no 3 collinear, there are $\frac{100 \cdot 99}{2}$, or 4950 lines.

* * *

● A set of points is said to be a <u>coplanar set</u> if and only if the set is wholly contained in a plane.

[Say 'co-PLANE-er' for 'coplanar'.]

● Two or more sets are said to be <u>coplanar</u> if and only if, together, they form a coplanar set.

Thus, for example, every line is a coplanar set. So, each two points on a line are coplanar. In fact, each three points are coplanar.

Answer the following questions.

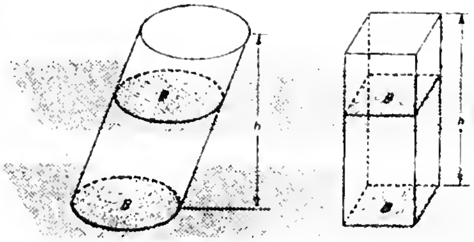
(1) We have discovered that a line and a point not in the line determine a plane. Hence, it follows that a line together with a point not in the line is a ___?___ set. We also say that a line and a point not in the line are ___?___.

(2) Each two intersecting lines determine a plane. So, a set formed by two intersecting lines is a ___?___ set. In other words, each two ___?___ lines are ___?___.

Later, the student develops a means for counting the number of planes determined by sets of concurrent lines, no three coplanar, and sets of parallel lines, no three coplanar. By working a considerable number of exercises, the student develops some notions about sufficient conditions for determining a plane. Out of this he evolves some of the basic ideas found in conventional solid geometry classes.

Later, we introduce the student to Cavalieri's Principle through a sequence of developmental pages. The student is lead to agree that the volume of a rectangular solid is the product of its length, width, and height. We then find it natural to develop the mensuration formulas for prisms and cylinders. On Page 199, we employ Cavalieri's Principle to obtain a formula for the volume of a cylinder.

Consider the problem of finding the volume of a cylinder whose base has area B and whose altitude is h. All that we need to find is a rectangular solid whose base has area B and whose altitude is h. Such a cylinder and rectangular solid are pictured below.



Since each section of a cylinder which is parallel to a base has the same area as that base, it follows, by Cavalieri's Principle that these two solids have the same volume. Now, the volume, V, of the rectangular solid pictured above is given by the formula:

$$V = Bh$$

Since the given cylinder has the same volume, it follows that the volume, V, of the given cylinder is also given by the formula:

$$V = Bh$$

Solve these problems.

(1) Find the volume of a right circular cylinder which has altitude of 14 and whose diameter of a base is 16.

(2) Find the radius of a circular cylinder with an altitude of 12 and whose volume is 75π.

The approach taken in developing a formula for computing the lateral area of a cylinder is shown on pages 201-203.

Consider the sequence of regular polygons inscribed in circles of radius 10, as shown here.

In this sequence of successive inscribed polygons, as the number of sides increases, the perimeter also increases. However, while there is no limit to the number of sides which can be attained by a polygon in this sequence, there is a limit to how large a perimeter which can be attained.

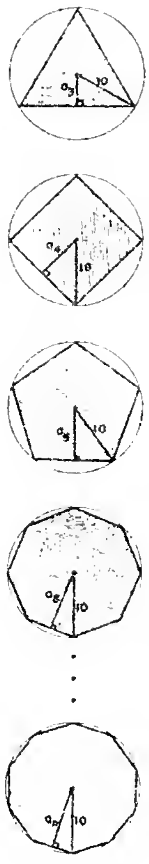
Many numbers which you can name are larger than each of the perimeters of the polygons in this sequence. For example, 68 and 71 are two such numbers. Each such number is an upper bound for the sequence. If there is a smallest number which is an upper bound, then this number is the least upper bound.

Answer these questions on your work sheet.

(1) For the sequence of perimeters, the least upper bound is the ___?___ of a ___?___ with ___?___ 10.

(2) The circumference of a circle with a radius 10 is ___?___.

(3) For the sequence of measures of apothems, a_3, a_4, a_5, a_6, ..., a_n, shown in the figures, the least upper bound is the ___?___ of the ___?___, which is ___?___.



Check your answers.

(1) For the sequence of perimeters, the least upper bound is the circumference of a circle with radius 10.

(2) The circumference of a circle with radius 10 is 20π. [To find the circumference, c, of a circle with radius r, use the formula: $c = 2\pi r$]

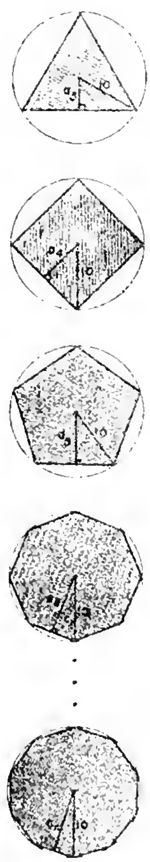
(3) For the sequence of measures of apothems, a_3, a_4, a_5, a_6, ..., a_n, shown in the figures, the least upper bound is the radius of the circle, which is 10.

* * *

Recall that the area of a regular polygon is the product of the measure of an apothem and one-half the perimeter. If we think of the sequence of areas of inscribed polygons as successive approximations for the area of a circle with radius 10, then we may approximate the area of the given circle as closely as we wish. If we use the limiting values for the apothem and perimeter, the area can be written:

$$10 \times \frac{1}{2} \times (2\pi \times 10)$$

or simply: $10^2 \cdot \pi$

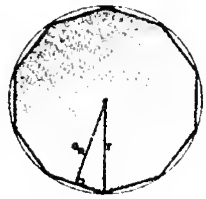


[Part 901] [Page 203]

Of course, we didn't have to choose 10 as the radius of our given circle. Any other measure number would have served just as well. In fact, if r is the radius of the circle, then it follows that:

(a) the least upper bound of the sequence of measures of apothems of the set of all inscribed regular polygons is r, and

(b) the least upper bound of the sequence of perimeters of the set of all inscribed regular polygons is $2\pi r$.



So, using these limiting values to compute the area of the given circle of radius r we obtain:

$$r \cdot \frac{1}{2} \cdot (2\pi r)$$

or, more simply:

$$\pi r^2$$

Similarly, we can consider a sequence of prisms with bases inscribed in the bases of a circular cylinder. If each prism has successively more lateral faces, then the least upper bound for the sequence of perimeters of successive right sections is the perimeter of a right section of the cylinder. So, we shall agree that:

- <u>The lateral area of a circular cylinder</u> is the product of the measure of an element and the perimeter of a right section of the cylinder.

Solve these problems.

(1) Find the lateral area and total area of a right circular cylinder with diameter of 6 feet and an altitude of 14 feet.

(2) Find the measure of an element of a circular cylinder which has a right section with perimeter of 9 inches and lateral area of 126 square inches.

We ask the student many times in the course of the program to employ his intuition. He has also been shown how one can use his intuition and reasoning to develop a new idea from familiar ideas. To avoid leaving you with the impression that the program is composed entirely of developmental treatments as the samples show, we should mention that we ask the student to solve many problems involving each of the space concepts which have been discussed and developed. The examples shown on pages 303-304 prove to be useful because they afford the student some review of graphing and inequalities. In addition to giving him a review, we ask the student to generate solids, and compute their volumes.

[Part 901] [Page 303]

You may recall having been asked to graph the solution sets of sentences like:

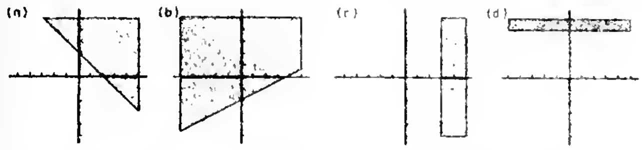
(a) $x + y \geq 2$

(b) $x - 2y \leq 4$

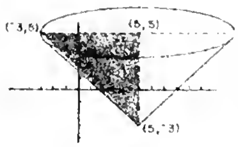
(c) $3 \leq x \leq 5$

(d) $y \geq 4$

If the restriction is made that $^-5 \leq x \leq 5$ and $^-5 \leq y \leq 5$, then the graphs of the solution sets of each of the above sentences are:



Consider the graph of sentence (a). Suppose that the triangular region is revolved about the line which is the graph of '$x = 5$' as an axis. The resulting cone of revolution is pictured at the right. To compute the volume of this cone, notice that the radius of the base is 8 [since one of the radii has endpoints ($^-3$, 5) and (5, 5)] and the altitude is 8 [measure of the segment whose endpoints are (5, 5) and (5, $^-3$)]. So, it follows that



$$V = \frac{1}{3}Bh$$
$$= \frac{1}{3}(\pi \cdot 8^2) \cdot 8$$
$$= \frac{512\pi}{3}.$$

Hence, the volume of the given cone of revolution is $\frac{512\pi}{3}$.

Solve these problems.

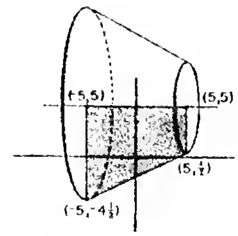
(1) Compute the volume of the solid generated by revolving the quadrangular region (b), above, about the graph of '$y = 5$' as an axis.

(2) Compute the volume of the solid generated by revolving the rectangular region (c), above, about the y-axis as an axis.

[Page 304] [Part 901]

Check your answers.

(1) Compute the volume of the solid generated by revolving the quadrangular region (b) about the graph of '$y = 5$' as an axis.



The solid generated is the frustum of a cone. The radius of the "upper" base is $4\frac{1}{2}$, or $\frac{9}{2}$, and the radius of the "lower" base is $9\frac{1}{2}$, or $\frac{19}{2}$. The altitude of the frustum is 10. To find the volume V:

$$V = \frac{1}{3}\cdot\left\{\pi\left(\frac{9}{2}\right)^2 + \pi\left(\frac{19}{2}\right)^2 + \sqrt{\pi\cdot\left(\frac{9}{2}\right)^2\pi\left(\frac{19}{2}\right)^2}\right\}\cdot 10$$
$$= \frac{1}{3}\cdot\left\{\frac{81\pi}{4} + \frac{361\pi}{4} + \frac{171\pi}{4}\right\}\cdot 10$$
$$= \frac{10\pi}{12}(81 + 361 + 171)$$
$$= \frac{10\pi \cdot 612}{12} = 510\pi$$

So, the volume of the given solid of revolution is 510 π.

(2) Compute the volume of the solid generated by revolving the rectangular region (c) about the y-axis as an axis. The solid generated is sometimes called a "cylindrical shell". The radius of the "hole" is 3, and the radius of the "outer" surface is 5. The altitude in each case is 10. To find the volume V:

$$\pi \cdot 5^2 \cdot 10 - \pi \cdot 3^2 \cdot 10$$
$$10\pi(25 - 9)$$
$$160\pi$$

Hence, the volume of the cylindrical shell is 160π.

The entire program is about 400 pages in length. The program is written in linear form. There is one optional section on spherical polygons which we hope will prove to be of interest to those students who elect to try this portion of the program. We have included 5 self-administered quizzes in this program. The student periodically checks his progress through this material by taking these quizzes. The answers are readily available, so he has immediate reinforcement to aid his learning.

S. Szabo

Test for Pages 4-A Through 4-42

The next several pages of this Newsletter contain a test which is meant to be given in Unit 4 when the students have reached page 42. It is designed to take 50 minutes to administer. We have not attempted to standardize this test, but we do feel that it tests important outcomes of these pages.

Math. 1 Test [pp. 4-A through 4-42]

A. Consider the cartesian product

$$\{-2, 0, 2\} \times \{0, 1, 2, 3\}.$$

How many ordered pairs in the product have:

_____ 1. first component 0?

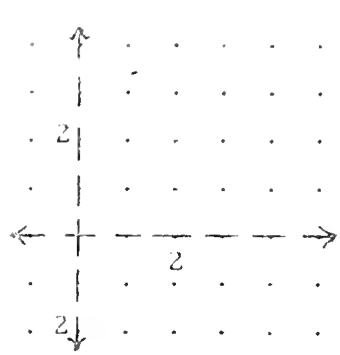
_____ 2. second component 2?

_____ 3. first component 2 less than second component?

_____ 4. second component 3 more than first component?

_____ 5. first component 2 less than second component and second component 3 more than first component?

B. On the picture below, graph the set of all ordered pairs of integers such that the first component is greater than 0 and the second component is less than 2.



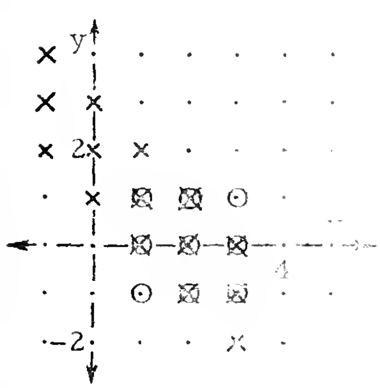
2. Suppose that

$R = \{(x, y), \text{ x and y integers: } 1 < x < 5 \text{ and } 2 < y < 5\}$

$S = \{(x, y), \text{ x and y integers: } 0 < x < 3 \text{ and } 1 < y < 4\}$.

Use the lattice picture below to show graphs of R and S [you will need to draw dashed lines to indicate the axes]. Draw a loop around each dot which corresponds to a point in set R; show the points in set S by drawing small cross-marks over the appropriate dots [X].

3. In the picture below, a graph of a set G is indicated by loops, and a graph of a set H by cross-marks.



(a) Fill these blanks.

n(G) = ____

n(H) = ____

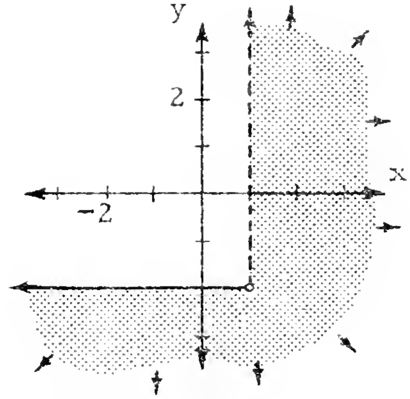
n(G ∩ H) = ____

n(G ∪ H) = ____

(b) Write a brace-notation description of the set G.

____

4. Here is a graph of a certain set.

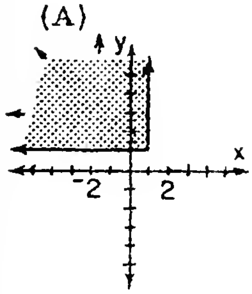


Write a brace-notation description of the set pictured.

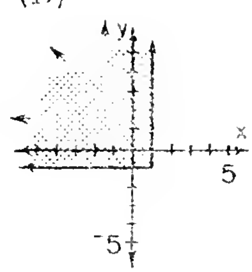
____

C. 1. Which of these is a graph of '$x \leq 1$ and $y \geq 1$'?

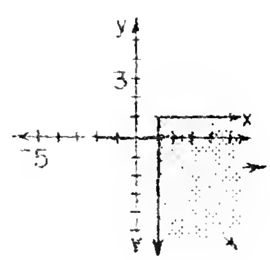
(A)



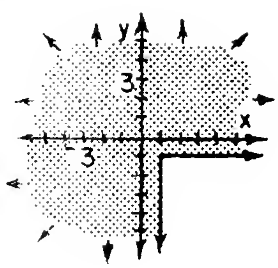
(B)



(C)

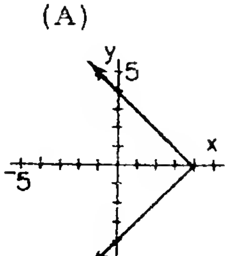


(D)

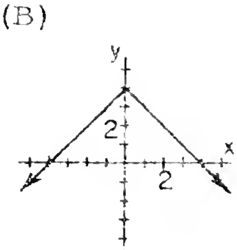


2. Which of these is a graph of '$x = 4 - |y|$'?

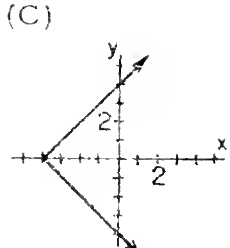
(A)



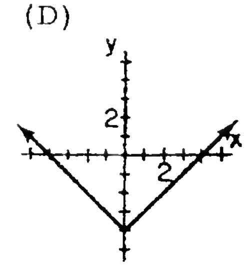
(B)



(C)



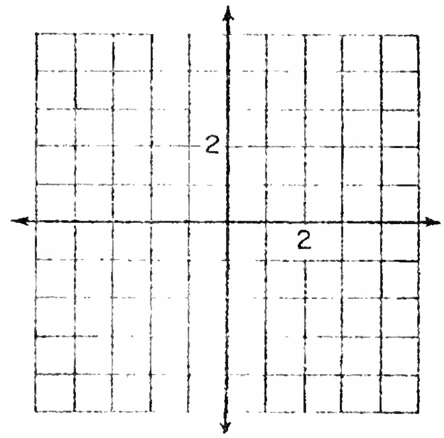
(D)



D. Each exercise contains a pair of equations. Graph each of the two equations on the appropriate picture, and give the point(s) in the intersection of their solution sets.

1. (a) $y - 3x = 3$

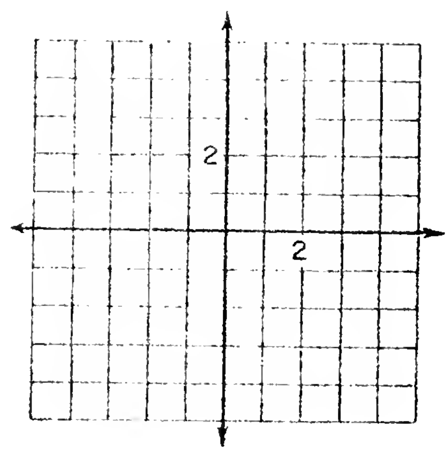
   (b) $y + x = -1$



   (c) $\{(x, y): y - 3x = 3\} \cap \{(x, y): y + x = -1\} = \{$__________$\}$

2. (a) $|x| = 3$

   (b) $2y = 4 - x$



   (c) $\{(x, y): |x| = 3\} \cap \{(x, y): 2y = 4 - x\} = \{$__________$\}$

E. Fill in the blanks.

1. Quadrant I is $\{(x, y):$ __________________$\}$.
2. Quadrant ____ is $\{(x, y): x < 0 \text{ and } y < 0\}$.
3. The x-axis is $\{x, y):$ __________$\}$.
4. The intersection of Quadrant II and Quadrant IV is ________.
5. The intersection of the x-axis and the y-axis is $\{$________$\}$.

F. For each equation, tell which quadrants contain points in its solution set. [Use 'Q_1', 'Q_2', 'Q_3', and 'Q_4' as names of the quadrants.]

1. (a) $y = -5x$ ________________

   (b) $y = -5x + 2$ ________________

   (c) $y = -5x - 2$ ________________

2. (a) $x = 3$ ________________

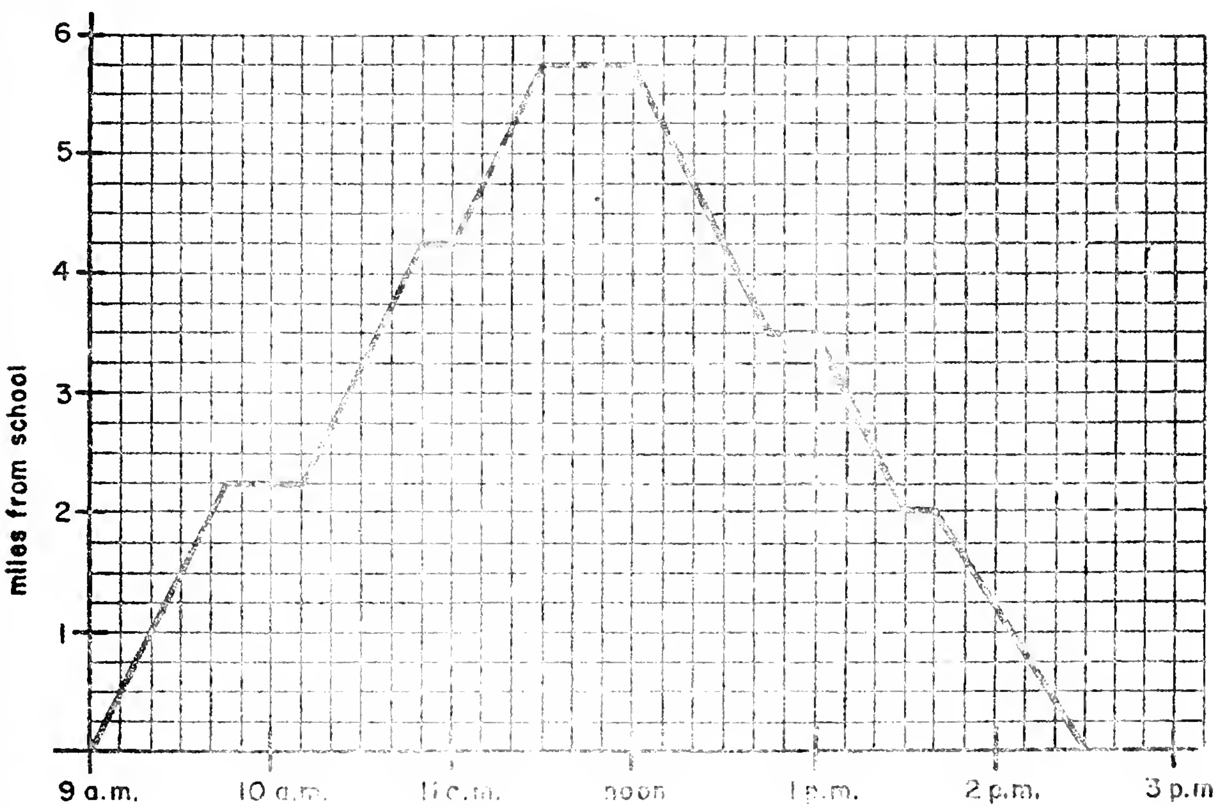
   (b) $x = y + 3$ ________________

   (c) $x = -y + 3$ ________________

3. (a) $x^2 + y^2 = 36$ ________________

   (b) $|x| = y - 6$ ________________

G. Here is a graph which is an approximate record of a hike taken on a Saturday morning by members of a biology class at Zabranchburg High.



They left the school building at 9:00 a.m., and kept track of their distance from school at various times [chiefly when they stopped to

gather specimens, or to rest]. For example, their first stop was made at 9:45 a.m., and they were $2\frac{1}{4}$ miles from school. [They arrived back at school at 2:28 p.m.]

_______ (a) How far from school were the hikers at 11:30 a.m.?

_______ (b) At what time were they $3\frac{1}{4}$ miles from school?

_______ (c) By what time had they traveled a total of 7 miles?

_______ (d) What was the average speed of the hikers from 10:10 a.m. to 10:50 a.m.? From 1:00 p.m. to 1:30 p.m.?
____

_______ (e) During which period was the group hiking faster--from noon to 12:45 p.m., or from 1:40 p.m. to 2:28 p.m.?

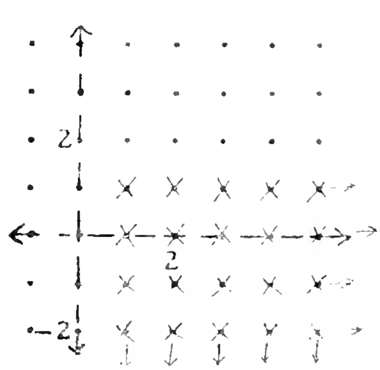
_______ (f) What was the average hiking speed during the last 48 minutes of their trip?

E. McCoy

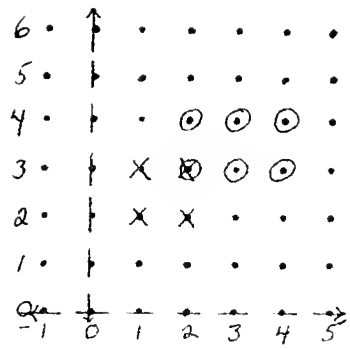
Answers for Test for pages 4-A through 4-42

A. 1. 4 2. 3 3. 2 4. 2 5. 0

B. 1.



2.



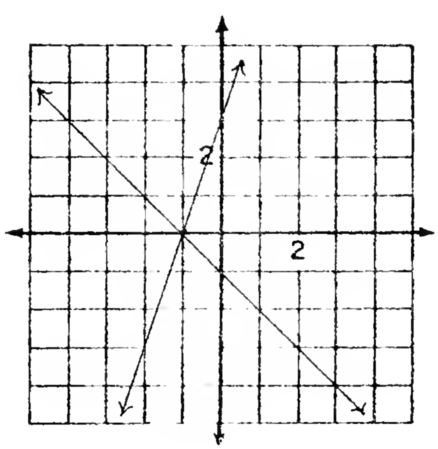
3. (a) $n(G) = 9$, $n(H) = 15$, $n(G \cap H) = 7$, $n(G \cup H) = 17$

(b) $\{(x, y)$, x and y integers: $0 < x < 4$ and $-2 < y < 2\}$

4. $\{(x, y) : x > 1 \text{ or } y \leq -2\}$

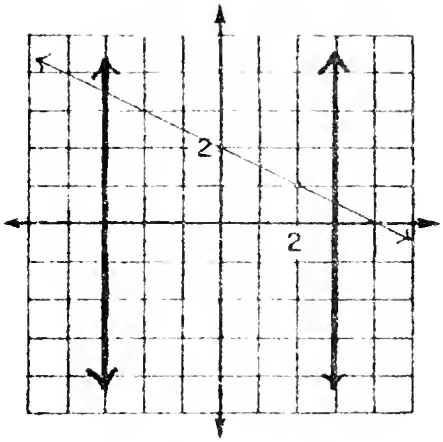
C. 1. (A) 2. (A)

D. 1.



(c) $\{(-1, 0)\}$

2.



(c) $\{(-3, 3\frac{1}{2}), (3, \frac{1}{2})\}$

E. 1. $x > 0$ and $y > 0$

2. III (or: 3)

3. $y = 0$

4. $\emptyset$

5. (0, 0)

F. 1. (a) Q_2 and Q_4 (b) Q_1, Q_2, Q_4 (c) Q_2, Q_3, Q_4.

2. (a) Q_1 and Q_4 (b) Q_1, Q_3, Q_4 (c) Q_1, Q_2, Q_4.

3. (a) Q_1, Q_2, Q_3, Q_4 (b) Q_1 and Q_2.

G. (a) $5\frac{3}{4}$ mi. (b) 10:30 A. M., and 1:05 P. M.

(c) 12:25 (d) 3 m. p. h., 3 m. p. h.

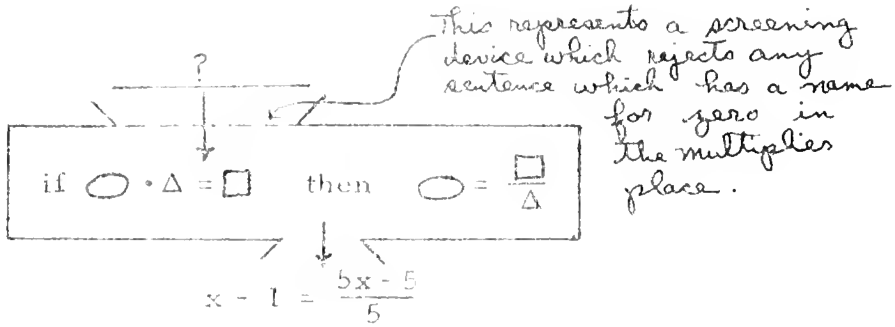
(e) noon to 12:45 P. M. (f) $2\frac{1}{2}$ m. p. h.

Supplementary Program — Division Theorem

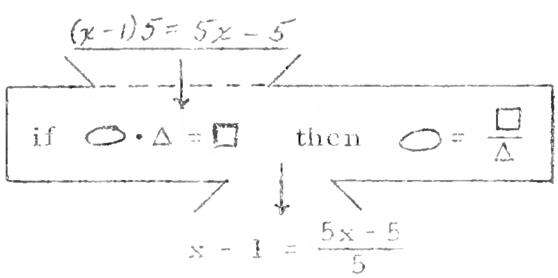
The use of the division theorem $[\forall_x \forall_{y \neq 0} \ \forall_z$ if $z \cdot y = x$ then $z = \frac{x}{y}]$ is often difficult for a student to grasp. Much class time is consumed clarifying the idea for the student. For instance, a student might be able to state and prove the theorem handily; but, have no idea where to begin when he is asked to derive:

$$\forall_x \ x - 1 = \frac{5x - 5}{5}$$

A student should see that the sentence can be generated as a conclusion of an instance of the division theorem where 'x' is replaced by '5x - 5', 'y' by '5', and 'z' by 'x - 1'. The following programed supplement develops a facility for seeing these replacements by using a machine like this:



The student should answer the question this way:



The program was written to be used as a homework assignment for use with pages 86-92 of Unit 2, UICSM First Course. This was done to cut down the time used in class. Most teachers do not have facilities available for producing a program such as the supplement. This does not mean that it has no value for them. Many of the ideas and "gimmicks" in this program may be employed in a normal classroom presentation to great advantage.

In your earlier work, you learned about operations which "undo" what other operations "do". We shall now put this knowledge to work.

* * *

Fill the blanks.

Multiplying by 2

(3, 6)	($^{-}2$, $^{-}4$)
(5, 10)	($^{-}6$, $^{-}12$)
($^{-}1$, $^{-}2$)	(0, 0)
($^{-}5$, __)	($^{-}8$, __)
($^{+}4$, __)	⋮

Check your answers.

Multiplying by 2

(3, 6)	($^{-}2$, $^{-}4$)
(5, 10)	($^{-}6$, $^{-}12$)
($^{-}1$, $^{-}2$)	(0, 0)
($^{-}5$, $^{-}10$)	($^{-}8$, $^{-}16$)
($^{+}4$, $^{+}8$)	⋮

* * *

Name the following lists.

?

(1, 8)	($^{-}3$, $^{-}24$)
($^{+}5$, $^{+}40$)	($^{+}6$, $^{+}48$)
($^{-}5$, $^{-}40$)	($^{-}2$, $^{-}16$)
(0, 0)	⋮

?

($^{+}1$, $^{-}6$)	($^{-}2$, $^{+}12$)
($^{+}5$, $^{-}30$)	($^{-}8$, $^{+}48$)
(0, 0)	⋮

Check your answers.

Multiplying by $^{+}8$

(1, 8)	($^{-}3$, $^{-}24$)
($^{+}5$, $^{+}40$)	($^{+}6$, $^{+}48$)
($^{-}5$, $^{-}40$)	($^{-}2$, $^{-}16$)
(0, 0)	⋮

Multiplying by $^{-}6$

($^{+}1$, $^{-}6$)	($^{-}2$, $^{+}12$)
($^{+}5$, $^{-}30$)	($^{-}8$, $^{+}48$)
(0, 0)	⋮

* * *

Complete the following sentences.

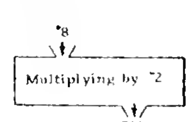
1. (2, __) belongs to multiplying by $^{+}3$ because $2 \cdot {}^{+}3$ = __ .
2. ($^{-}4$, __) belongs to multiplying by $^{-}5$ because $^{-}4 \cdot {}^{-}5$ = __ .
3. ($^{+}5$, __) belongs to multiplying by 0 because $^{+}5 \cdot 0$ = __ .
4. ($^{-}8$, __) belongs to multiplying by $\frac{1}{4}$ because $^{-}8 \cdot \frac{1}{4}$ = __ .

Check your answers.

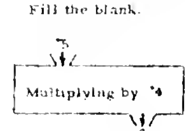
1. (2, 6) belongs to multiplying by $^{+}3$ because $2 \cdot {}^{+}3 = 6$.
2. ($^{-}4$, 20) belongs to multiplying by $^{-}5$ because $^{-}4 \cdot {}^{-}5 = 20$.
3. ($^{+}5$, 0) belongs to multiplying by 0 because $^{+}5 \cdot 0 = 0$.
4. ($^{-}8$, $^{-}2$) belongs to multiplying by $\frac{1}{4}$ because $^{-}8 \cdot \frac{1}{4} = {}^{-}2$.

* * *

The following "machines" help us do problems like the ones above. Here is a sample of how a machine works.



Here is another machine.
Fill the blank.



Check your answers.

+5
Multiplying by -4
-20

* * *

Fill in the blanks in the following machines.

1. -6 → Multiplying by +7 → ?

2. +8 → Multiplying by ? → -24

3. ? → Multiplying by -4 → +40

4. -8 → Multiplying by ? → 0

Check your answers.

1. -6 → Multiplying by +7 → -42

2. +8 → Multiplying by -3 → -24

3. -10 → Multiplying by -4 → +40

4. -8 → Multiplying by 0 → 0

* * *

Here are some complex machines. Fill in the blanks in these machines.

1. -2 → Multiplying by +3 → -6 → "undoing" multiplying by +3 → ?

2. ? → Multiplying by -2 → -10 → "undoing" multiplying by -2 → ?

Check your answers.

1. -2 → Multiplying by +3 → -6 → "undoing" multiplying by +3 → -2

2. +5 → Multiplying by -2 → -10 → "undoing" multiplying by -2 → +5

* * *

Fill the blanks.

? → Multiplying by +8 → ? → "undoing" multiplying by +8 → -6

+6 → Multiplying by 0 → ? → "undoing" multiplying by 0 → ?

Check your answers.

-6 → Multiplying by +8 → -48 → "undoing" multiplying by +8 → -6

+6 → Multiplying by 0 → 0 → "undoing" multiplying by 0 → ____

If you wrote a numeral in this blank, turn to page 9. If you did not write a numeral in this blank, turn to page 14.

Fill the blanks.

Multiplying by $^{+}5$

($^{-}2$, ?) (?, $^{+}10$)
(3, ?)
(0, ?) ($^{-}8$, ?)
(?, $^{-}25$)
(?, 0)
...

"Undoing" multiplying by $^{+}5$

(?, $^{-}2$) ($^{+}10$, ?)
(?, 3) (?, 0)
(?, $^{-}8$) ($^{-}25$, ?)
(0, ?)
...

Check your answers.

Multiplying by $^{+}5$

($^{-}2$, $^{-}10$) ($^{+}2$, $^{+}10$)
(3, 15) (0, 0)
($^{-}8$, $^{-}40$) ($^{-}5$, $^{-}25$)
(0, 0)
...

"Undoing" Multiplying by $^{+}5$

($^{-}10$, $^{-}2$) ($^{+}10$, $^{+}2$)
(15, 3) (0, 0)
($^{-}40$, $^{-}8$) ($^{-}25$, $^{-}5$)
(0, 0)
...

* * *

Fill the blanks.

Multiplying by 0

(5, ?) (0, ?)
($^{-}3$, ?) (2, ?)
($^{-}8$, ?) ($\frac{1}{2}$, ?)
...

Check your answers.

Multiplying by 0

(5, 0) (0, 0)
($^{-}3$, 0) (2, 0)
($^{-}8$, 0) ($\frac{1}{2}$, 0)
...

* * *

If we wanted to build an "undoing" multiplying by 0 list using the above list as our guide, it would look like this:

? "Undoing" multiplying by 0

(0, 5) (0, 0)
(0, $^{-}3$) (0, 2)
(0, $^{-}8$) (0, $\frac{1}{2}$)
...

Could we use this new list to find out what "undoing" multiplying by 0 applied to $^{+}6$ is?

Check your answer.

No. $^{+}6$ does not appear as a first entry in any pair in this list, so the list does not help us answer the question.

* * *

Now look again at the list and answer the following question.

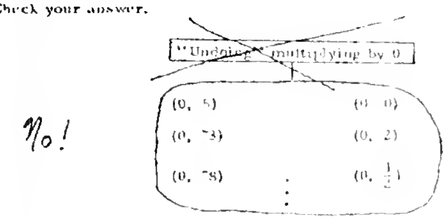
? "Undoing" multiplying by 0

(0, 5) (0, 0)
(0, $^{-}3$) (0, 2)
(0, $^{-}8$) (0, $\frac{1}{2}$)
...

Could we use this list to find out what "undoing" multiplying by 0 applied to 0 is?

[D - 1] [13]

Check your answer.



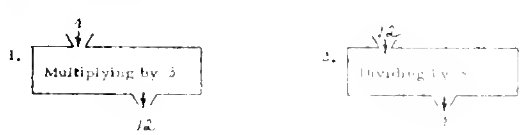
According to the list, any real number would qualify as an answer. This is a very undesirable feature. Therefore, we say "you can't undo multiplying by 0". [Neither can we.]

[D - 1] [14]

Fill the blanks.

[D - 1] [15]

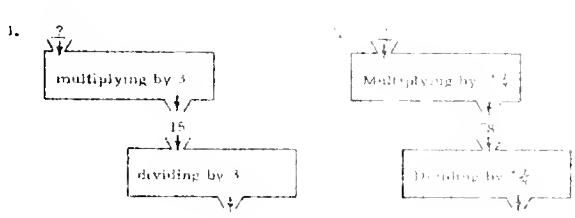
Check your answers.



You have probably noticed that the "dividing by" machine is the same as the "undoing multiplying by" machine.

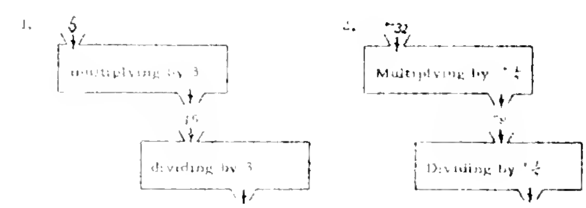
* * *

Fill the blanks.



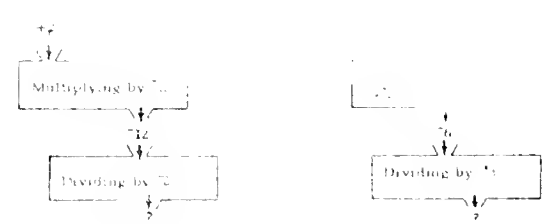
[D - 1] [16]

Check your answers.

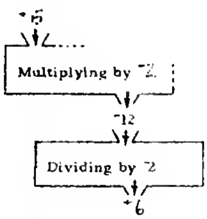


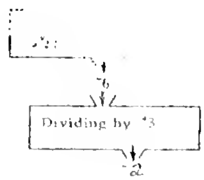
* * *

Fill the blanks.



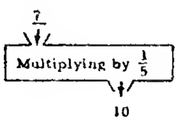
Check your answers.

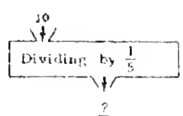


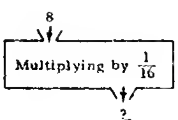


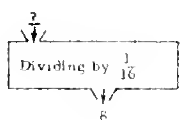
* * *

Fill the blanks.

1\.


2\.


3\.


4\.


Check your answers.

1. 50.
2. 50. In number 1 we found that $50 \cdot \frac{1}{5} = 10$, therefore, we know that $10 \div \frac{1}{5} = 50$.
3. $\frac{1}{2}$.
4. $\frac{1}{2}$. In number 3 we found that $8 \cdot \frac{1}{16} = \frac{1}{2}$, therefore, we know that $\frac{1}{2} \div \frac{1}{16} = 8$.

* * *

Sample: $^{-}32 \div ^{-}4 = ?$

Solution: $^{-}32 \div ^{-}4 = ^{+}8$ because $^{+}8 \cdot ^{-}4 = ^{-}32$

Fill the blanks.

1. $^{+}15 \div ^{-}3 =$ __?__ because __?__ $\cdot ^{-}3 = ^{+}15$.
2. $^{-}20 \div ^{+}5 =$ __?__ because __?__ $\cdot$ __?__ = __?__.
3. $^{+}30 \div$ __?__ $= ^{-}10$ because __?__ $\cdot$ __?__ = __?__.
4. __?__ $\div \frac{^{-}1}{2} = ^{+}6$ because __?__ $\cdot$ __?__ = __?__.
5. __?__ $\div \frac{^{+}1}{15} = ^{-}45$ because __?__ $\cdot$ __?__ = __?__.

Check your answers.

1. $^{+}15 \div ^{-}3 = ^{-}5$ because $^{-}5 \cdot ^{-}3 = ^{+}15$.
2. $^{-}20 \div ^{+}5 = ^{-}4$ because $^{-}4 \cdot ^{+}5 = ^{-}20$.
3. $^{+}30 \div ^{-}3 = ^{-}10$ because $^{-}10 \cdot ^{-}3 = ^{+}30$.
4. $^{-}3 \div \frac{^{-}1}{2} = ^{+}6$ because $^{+}6 \cdot \frac{^{-}1}{2} = ^{-}3$.
5. $^{-}3 \div \frac{^{+}1}{15} = ^{-}45$ because $^{-}45 \cdot \frac{^{+}1}{15} = ^{-}3$.

* * *

Recall that '$6 \div 0$' is meaningless. Using your answers above, complete the following pattern sentence.

$\square \div \triangle = \bigcirc$ because __?__ $\cdot$ __?__ = __?__, provided $\triangle \neq 0$.

Check your answers.

$\square \div \triangle = \bigcirc$ because $\bigcirc \cdot \triangle = \square$, provided $\triangle \neq 0$.

* * *

In the pattern above, let us make the replacements, '4' for '$\square$', '12' for '$\triangle$' and '5' for '$\bigcirc$'.

This gives us the instance:

$4 \div 12 = 5$ because $5 \cdot 12 = 4$.

You probably do not believe that $4 \div 12 = 5$. Nor do you believe that $5 \cdot 12 = 4$. But, you would believe that $4 \div 12 = 5$ *if* you believed that $5 \cdot 12 = 4$.

This suggests the following pattern:

if $\bigcirc \cdot \triangle = \square$ then $\square \div \triangle = \bigcirc$ provided $\triangle \neq 0$.

This pattern suggests a generalization. Complete the sentence:

$\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$ then __?__ $\div$ __?__ = __?__.

Check your answers.

$\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$ then $x \div y = z$

* * *

Is this a theorem? If it is--we should be able to derive it. If it is not, we should be able to give a counter-instance. Don't try too hard to get a counter-instance. Let's derive it, instead. Before we write a derivation we need to recall one of our basic principles:

$\forall_x \forall_{y \neq 0}$ $\frac{x}{y} \cdot y = x$ [principle of quotients]

('$\frac{x}{y}$' is an abbreviation for '$x \div y$')

This principle tells us for instance, that, since $5 \neq 0$,

$$(6 \div 5) \cdot 5 = 6$$

This principle will make it possible for us to derive the cancellation principle for multiplication. Recall the cancellation principle for addition:

$\forall_x \forall_y \forall_z$ if $x + y = z + y$ then $x = z$

1. Write the cancellation principle for multiplication.

2. Is 'If $3 \cdot 0 = 4 \cdot 0$ then $3 = 4$' an instance of the generalization you wrote for problem 1?

Check your answers.

1. $\forall_x \forall_{y \neq 0} \forall_z$ if $xy = zy$ then $x = z$

2. If you answered 'Yes' to this question (and your answer was correct) then your generalization was not the one we want to call:

The cancellation principle for multiplication

because it would allow us to reach a false conclusion ["3 = 4"] from a true premise ["3.0 = 4.0"]. We fix this by not allowing '0' as a replacement for 'y' in an instance of the generalization.

* * *

Let's derive:

$\forall_x \forall_{y \neq 0} \forall_z$ if $xy = zy$ then $x = z$

Fill the blanks.

Suppose that $ab = cb$.

It follows that $(ab)\frac{1}{b} = (cb)\frac{1}{b}$. [$b \neq 0$]

$a(b \cdot \frac{1}{b}) = c(b \cdot \frac{1}{b})$. [apm, apm]

$a(\frac{1}{b} \cdot b) = c(\frac{1}{b} \cdot b)$. [? , ?]

$a \cdot \underline{?} = c \cdot \underline{?}$. [principle of quotients $b \neq 0$]

$a = c$. [pm1, pm1]

Hence, [for $b \neq 0$], if $ab = cb$ then $a = c$.

Therefore, '$\forall_x \forall_{y \neq 0} \forall_z$ if $xy = zy$ then $x = z$' is a theorem.

Check your answers.

$\forall_x \forall_{y \neq 0} \forall_z$ if $xy = zy$ then $x = z$

Suppose that $ab = cb$.

It follows that $(ab)\frac{1}{b} = (cb)\frac{1}{b}$. [$b \neq 0$]

$a(b \cdot \frac{1}{b}) = c(b \cdot \frac{1}{b})$. [apm, apm]

$a(\frac{1}{b} \cdot b) = c(\frac{1}{b} \cdot b)$. [cpm, cpm]

$a \cdot \underline{1} = c \cdot \underline{1}$. [principle of quotients $b \neq 0$]

$a = c$. [pm1, pm1]

Hence, [for $b \neq 0$], if $ab = cb$ then $a = c$.

Therefore, '$\forall_x \forall_{y \neq 0} \forall_z$ if $xy = zy$ then $x = z$' is a theorem.

* * *

Now we are ready to complete the derivation of:

$\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$ then $z = \frac{x}{y}$

Suppose that $c \cdot b = a$.

It follows that $c \cdot b = \frac{a}{b} \cdot \underline{?}$. [principle of quotients $b \neq 0$]

and $c = \frac{a}{b}$. ____?____

Hence, [for $b \neq 0$] if $c \cdot b = a$ then $\underline{?} = \underline{?}$.

Therefore, '$\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$' is a ___?___.

Check your answers.

(3) $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$

Suppose that $c \cdot b = a$.

It follows that $c \cdot b = \frac{a}{b} \cdot b$, [principle of quotients, $b \neq 0$]

and $c = \frac{a}{b}$ [cancellation principle for multiplic., $b \neq 0$]

Hence, [for $b \neq 0$] if $cb = a$ then $c = \frac{a}{b}$.

Therefore, '$\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$' is a theorem.

* * *

Since '$z = \frac{x}{y}$' is equivalent to '$\frac{x}{y} = z$' we now know

(3a) $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $\frac{x}{y} = z$

is also a theorem.

[D - 1] [24b]

We can use (☆☆) to help us derive many other theorems. For example, we can derive '$\forall_x \ \frac{x}{1} = x$' in this manner.

We wish to write a test-pattern which will test any instance of '$\forall_x \ \frac{x}{1} = x$'. Let's look at this instance:

(1) $\frac{a}{1} = a$

The corresponding instance of (☆☆) is:

(2) if $a \cdot 1 = a$ then $\frac{a}{1} = a$ $[1 \neq 0]$

Notice that the then-part in (2) is a copy of (1). Therefore, if we can show that $a \cdot 1 = a$ then, since $1 \neq 0$, we can use (☆☆) to conclude that $\frac{a}{1} = a$.

$$\frac{\forall_x \ \frac{x}{1} = x}{a \cdot 1 = a} \quad [\text{pm}]$$

Hence, by (☆☆) [since $1 \neq 0$], $\frac{a}{1} = a$.

Therefore, '$\forall_x \ \frac{x}{1} = x$' is a theorem.

Answer this question.

If we know that $1 \cdot c = c$ then, for $c \neq 0$, we can use (☆☆) to conclude that ___?___ = 1.

[D - 1] [25]

Check your answers.

(☆☆) $\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$ then $\frac{x}{y} = z$

If we know that $1 \cdot c = c$ then, for $c \neq 0$, we can use (☆☆) to conclude that $\frac{c}{c}$ = 1.

✲ ✲ ✲

Answer these questions.

1. Since $-1 \neq 0$, we can use (☆☆) to show that $\frac{a}{-1} = -a$ if we can show that ___?___ · ___?___ = ___?___.

2. For ___?___ $\neq 0$, we can use (☆☆) to show that $\frac{0}{a} = 0$ if we can show that ___?___ · ___?___ = ___?___.

[D - 1] [26]

(☆☆) $\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$ then $\frac{x}{y} = z$

1. Since $-1 \neq 0$, we can use (☆☆) to show that $\frac{a}{-1} = -a$, if we can show that $-a$ · -1 = a.

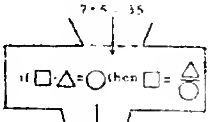
2. For a $\neq 0$, we can use (☆☆) to show that $\frac{0}{a} = 0$, if we can show that 0 · a = 0.

✲ ✲ ✲

Here are some pictures of a machine of a slightly different type. The samples tell you something about how the machine works.

Sample 1:

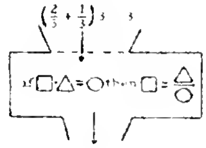
$7 \cdot 5 = 35$



$7 = \frac{35}{5}$

Sample 2:

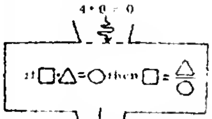
$\left(\frac{2}{3} + \frac{1}{3}\right) 3 = 3$



$\frac{2}{3} + \frac{1}{3} = \frac{3}{3}$

Sample 3:

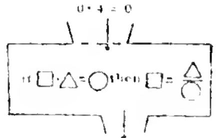
$4 \cdot 0 = 0$



This machine will not accept sentences with a name for 0 in the multiplier position.

Sample 4:

$0 \cdot 4 = 0$

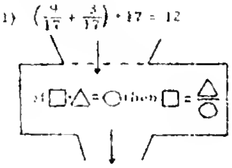


$0 = \frac{0}{4}$

[D-1] [27]

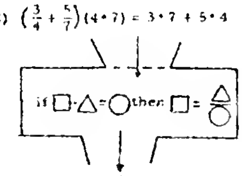
Fill the blanks.

1) $\left(\frac{9}{17} + \frac{3}{17}\right) \cdot 17 = 12$



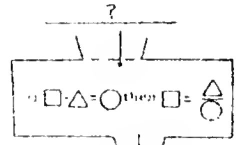
___?___

2) $\left(\frac{3}{4} + \frac{5}{7}\right)(4 \cdot 7) = 3 \cdot 7 + 5 \cdot 4$



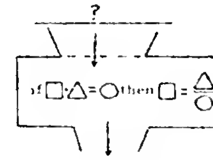
___?___

3) ___?___



$\frac{7}{9} + \frac{3}{5} = \frac{7 \cdot 5 + 3 \cdot 9}{9 \cdot 5}$

4) ___?___



$\frac{3}{4} + \frac{7}{8} = \frac{3 \cdot 8 + 7 \cdot 4}{4 \cdot 8}$

Check your answers.

1) $(\frac{9}{17} + \frac{3}{17}) \cdot 17 = 12$

if □·△=○ then □=$\frac{△}{○}$

$\frac{9}{17} + \frac{3}{17} = \frac{12}{17}$

2) $(\frac{3}{4} + \frac{5}{7})(4 \cdot 7) = 3 \cdot 7 + 5 \cdot 4$

if □·△=○ then □=$\frac{△}{○}$

$\frac{3}{4} + \frac{5}{7} = \frac{3 \cdot 7 + 5 \cdot 4}{4 \cdot 7}$

3) $(\frac{7}{9} + \frac{3}{5})(9 \cdot 5) = 7 \cdot 5 + 3 \cdot 9$

if □·△=○ then □=$\frac{△}{○}$

$\frac{7}{9} + \frac{3}{5} = \frac{7 \cdot 5 + 3 \cdot 9}{9 \cdot 5}$

4) $(\frac{3}{4} + \frac{7}{8})(4 \cdot 8) = 3 \cdot 8 + 7 \cdot 4$

if □·△=○ then □=$\frac{△}{○}$

$\frac{3}{4} + \frac{7}{8} = \frac{3 \cdot 8 + 7 \cdot 4}{4 \cdot 8}$

* * *

Fill in the blanks.

1. If $(\frac{2}{a} + \frac{5}{7}) \cdot (a \cdot \underline{?}) = 2 \cdot 7 + 5 \cdot a$ then $\frac{2}{a} + \frac{5}{7} = \frac{2 \cdot 7 + 5 \cdot \underline{?}}{a \cdot \underline{?}}$ $[\underline{?} \neq 0]$

2. If $(\frac{3}{a} + \frac{b}{6}) \cdot (? \cdot 6) = 3 \cdot 6 + b \cdot \underline{?}$ then $\frac{3}{a} + \frac{b}{6} = \frac{3 \cdot 6 + b \cdot \underline{?}}{\underline{?} \cdot 6}$ $[\underline{?} \neq 0]$

3. If $(\frac{5}{a} + \frac{b}{c}) \cdot (a \cdot c) = 5 \cdot \underline{?} + b \cdot \underline{?}$ then $\frac{5}{a} + \frac{b}{c} = \frac{5 \cdot c + b \cdot a}{\underline{?}}$ $[\underline{?} \neq 0]$

4. If $(\frac{a}{b} + \frac{c}{d}) \cdot (\underline{?}) = a \cdot d + c \cdot \underline{?}$ then $\frac{a}{b} + \frac{c}{d} = \frac{?}{b \cdot d}$ $[\underline{?} \neq 0]$

Check your answers.

1. If $(\frac{2}{a} + \frac{5}{7}) \cdot (a \cdot 7) = 2 \cdot 7 + 5 \cdot a$ then $\frac{2}{a} + \frac{5}{7} = \frac{2 \cdot 7 + 5 \cdot a}{a \cdot 7}$ $[a \cdot 7 \neq 0]$

2. If $(\frac{3}{a} + \frac{b}{6}) \cdot (a \cdot 6) = 3 \cdot 6 + b \cdot a$ then $\frac{3}{a} + \frac{b}{6} = \frac{3 \cdot 6 + b \cdot a}{a \cdot 6}$ $[a \cdot 6 \neq 0]$

3. If $(\frac{5}{a} + \frac{b}{c}) \cdot (a \cdot c) = 5 \cdot c + b \cdot a$ then $\frac{5}{a} + \frac{b}{c} = \frac{5 \cdot c + b \cdot a}{a \cdot c}$ $[a \cdot c \neq 0]$

4. If $(\frac{a}{b} + \frac{c}{d}) \cdot (b \cdot d) = a \cdot d + c \cdot b$ then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{b \cdot d}$ $[b \cdot d \neq 0]$

This is the end of this part. Return your book to your teacher tomorrow.

Angle Functions

Your students may enjoy comparing quadratic functions with what I have called angle functions. This kind of activity will be quite appropriate when quadratic functions are studied in Unit 5.

Let's begin by graphing the function

$$f = \{(x, y): \ y = 3|x| + 2x + 4\}.$$

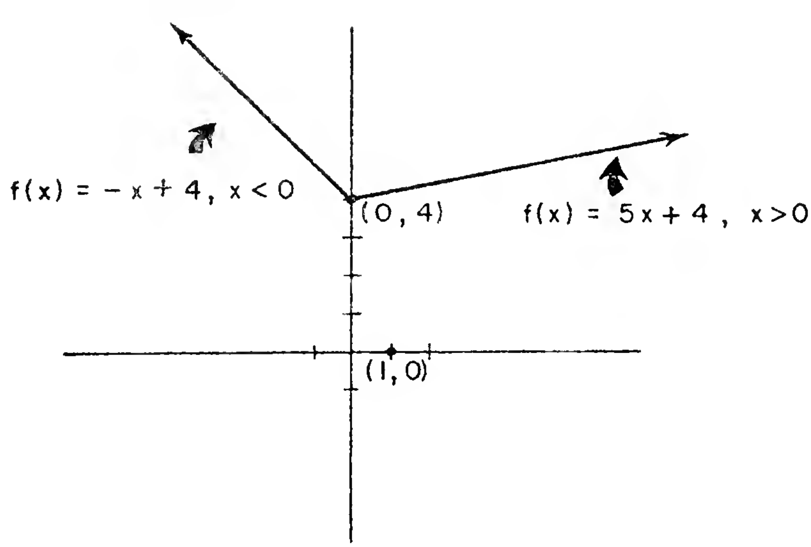
One way is to obtain ordered pairs that belong to f by substitution and computation. A more convenient way is to recall a definition of absolute value:

$$(1) \quad \forall_x \ |x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

Using this definition, we find that f is the union of three sets.

$$\begin{aligned} f = \big(&\{(x, y): \ x > 0 \text{ and } y = (3 + 2)x + 4\} \\ \cup &\{(x, y): \ x = 0 \text{ and } y = 4\} \\ \cup &\{(x, y): \ x < 0 \text{ and } y = (-3 + 2)x + 4\}\big) \end{aligned}$$

So, the graph of f is an angle,



Recalling that the graph of the quadratic function

$$(2) \qquad q(x) = ax^2 + bx + c \qquad [a \neq 0]$$

is a parabola whose orientation and shape depend on the values of a, b, and c, we can draw an analogy here by noticing that the position and orientation of the angle depended on the numbers 3, 2, and 4.

Let's investigate the function, where for some $a \neq 0$, b, and c,

$$(3) \qquad g(x) = a|x| + bx + c.$$

In view of the definition (1),

$$(4) \qquad g(x) = \begin{cases} (a + b)x + c & \text{if } x > 0 \\ c & \text{if } x = 0 \\ (b - a)x + c & \text{if } x < 0. \end{cases}$$

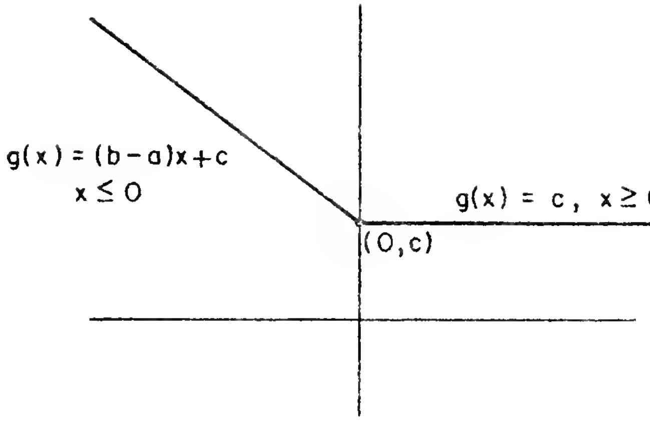
We may abbreviate this by writing

$$(5) \qquad g(x) = \begin{cases} (a + b)x + c & \text{if } x \geq 0 \\ (b - a)x + c & \text{if } x \leq 0. \end{cases}$$

Thus g is the union of two functions, the graph of each is a ray, with common end point (0, c). Now, if the two rays are collinear then $b + a = b - a$ -- that is, $a = 0$. Since $a \neq 0$, the rays are noncollinear. Hence, the graph of g is <u>an angle</u>, as our example suggested.

Let's investigate the graph in more detail. The side of the angle is parallel to or contained in the x-axis if and only if $b + a = 0$ or $b - a = 0$. For example, let $b + a = 0$. Substituting in (5), we see that

$$g(x) = \begin{cases} 0 \, x + c & \text{if } x \geq 0 \\ (b - a)x + c & \text{if } x \leq 0. \end{cases} \qquad [b - a \neq 0]$$



On the other hand, if $b - a = 0$, one side of the angle is either the negative half of the horizontal axis [$c = 0$] or parallel to it [$c \neq 0$]. So, by assuming that $b^2 - a^2 \neq 0$, we can require that both sides of the angle be oblique to the x-axis.

$$g(x) = \begin{cases} (b - a)x + c, & x \geq 0 \quad [b - a \neq 0] \\ (b + a)x + c, & x \leq 0 \quad [b + a \neq 0] \end{cases}$$

and both sides of the angle are subsets of some linear function which cannot be parallel to the x-axis.

Following the development of quadratic functions, it is natural to ask next about roots of the equation '$g(x) = 0$'. Let's assume that $b^2 - a^2 \neq 0$ and that $a \neq 0$. We wish to solve:

$$(6) \quad a|x| + bx + c = 0$$

This equation is equivalent to the sentence:

$$(7) \quad [x \geq 0 \text{ and } (b + a)x + c = 0] \text{ or }$$
$$[x \leq 0 \text{ and } (b - a)x + c = 0]$$

which, under the above assumptions, is equivalent to:

$$(8) \left[x \geq 0 \text{ and } x = \frac{-c}{b + a}\right] \text{ or } \left[x \leq 0 \text{ and } x = \frac{-c}{b - a}\right]$$

The solution of (6) is not similar to the solution of the quadratic equation. The technique of completing the square has no analogue. In the same vein, the angle's vertex is (0, c), while the vertex of the graph of the quadratic function is

$$\left(\frac{-b}{2a},\ \frac{-b^2 + 4ac}{4a}\right)$$

The vertex of the former is restricted to the y-axis. The vertex of the latter may be any point on the plane.

This suggests that we look for a more general angle function. Such a function turns out to be, for some $a \neq 0$, b, c, and h,

$$(9)\quad A(x) = a|x - h| + bx + c\ .$$

Let's use a technique analagous to completing the square to analyze A. We see that

$$bx + c = b(x - h) + (c + hb)$$

so that

$$(10)\quad A(x) = a|x - h| + b(x - h) + (c + hb)\ .$$

Using the definition of $|x - h|$, and abbreviating as in (5),

$$(11)\quad A(x) = \begin{cases} (b + a)(x - h) + (c + hb) & \text{if } x - h \geq 0 \\ (b - a)(x - h) + (c + hb) & \text{if } x - h \leq 0 \end{cases}$$

This certainly has for its graph an angle.

Finally, we shall find the roots of:

$$a\,|x - h| + b(x - h) + (c + hb) = 0$$

just as we solved (6), retaining the same assumptions.

(12) $$\left(x - h \geq 0 \text{ and } x - h = \frac{-(c + hb)}{b + a}\right)$$

or

$$\left(x - h \leq 0 \text{ and } x - h = \frac{-(c + hb)}{b - a}\right)$$

Rearranging, we see that

(13) $$\left(x = \frac{-(bc + ha^2) + a(c + hb)}{b^2 - a^2} \text{ and } x \geq h\right)$$

or

$$\left(x = \frac{-(bc + ha^2) - a(c + hb)}{b^2 - a^2} \text{ and } x \leq h\right).$$

So, we have a result which is analagous to the familiar quadratic formula.

The next step is to consider the absolute value relations in two variables. But this is material for a future article.

A. Holmes

UICSM

NEWSLETTER

Number 12

21 Oct 63

An Occasional Publication of the
University of Illinois Committee on School Mathematics
1210 West Springfield
Urbana, Illinois

Table of Contents

Fractions

Consider the child's initial contact with the words we use with fractions. He learns to use the word 'half' in many kinds of contexts.

> Give me half of your apple.
>
> Would you like half of this orange?
>
> I'll give you half of this egg for lunch.
>
> I'll let you have half of my candy bar.
>
> Which half of this banana do you want?
>
> To make a banana split, you put half of a banana on each side of a long dish.
>
> I like half of a banana on my cereal.
>
> Put half of the marbles in each bag.
>
> What is half of 10?

In each of these illustrations, 'half of' refers to a relation, but the relations are, in most cases, significantly different from one another. There are a large number of ways to cut a given apple in half. If half of an orange is thought of as similar to half of an apple, the orange may be cut in any direction; but if the half orange is to be considered an appropriate number of sections, the number of ways to get half of the orange is smaller. The ambiguity in 'half of this egg' is considerable. [Are we going to scramble it? Is it hard-boiled? If it is hard-boiled, shall we slice it? If so, shall we slice it the long way or the short way? Any more questions?] In each of the cases which concern a banana, there are ideally just two halves--but what passes for half of a banana in some instances will not do for a banana split; and the half of a banana on cereal is a quite different situation. Half of a collection of marbles is another collection of marbles, and if you have 10 marbles, all different, there are 252 ways you can divide them into two collections of 5 marbles each. Half of the number 10 is the number 5--a cardinal number has at most one half [there is nothing which has this halving relation to the cardinal number 3].

There are contexts of other quite different kinds. These are not relevant to our present purposes, but--to compound the confusion--here are some examples:

This recipe calls for half a cup of sugar.

It's half a mile to school.

I'd like just half a bowl of soup.

There will be a half moon tonight.

Half of me wants to go to the show; the other half wants to stay home.

He is half-way committed.

It's a half-baked idea.

He only half did the job.

Would you rather have a half dollar or a half of this dollar?

He listened with half an ear.

In none of these examples is there one dealing with the number 1/2. Moreover, in none of the common teaching aids--folding or cutting paper or flannel, playing with blocks or other counters, pouring sand from one container into another, et al.--does a child make any use of fractional numbers. Engaging in such activities may enable a child to discover some relationships which are analogous to relationships among fractional numbers, and this is both a blessing and a curse. It seems not impossible that the traumatic experiences which all too many people associate with mathematics begin with the introduction of '1/2' as a synonym for 'a half'. 'A half of a banana' makes sense, although, as we have seen, what sense it makes depends upon the context in which it occurs; but in '1/2 banana' the '1/2' surely is not the name for a number. Introducing '1/2 banana' as a rephrasing of 'a half of a banana' does not lead toward recognition of fractional numbers. The fraction symbol has several uses, only one of which is naming a fractional number.

A bright first grader who is asked, "Which is more, one half of an orange or one third of an orange?" may quite properly ask, "Is it the same orange?" If his teacher says, "No," he may ask, "Are the oranges both the same size?" Would there be any need for his question if the conversation were about numbers? The number one-half is greater than the number one-third, period.

A large part of the difficulty we have been having in teaching the role of fractions in the computational algorithms may lie in our failure to recognize those situations in which the child is dealing with fractional numbers and those situations in which he is not. The "old-fashioned" teacher who kept his questions specific and concrete was probably exhibiting more wisdom in this regard than some of the "modern" approaches we have seen recently. For instance, the confusion invited by an exercise like the following may be the result of the unfortunate formulation of the directions:

> Name the fractional number represented by the shaded portions in each of the figures below:

The child who answers "correctly" that the first figure shows the fractional number 2/4 and the second figure shows the fractional number 2/3 may note that the shaded portions of the two figures are the same size. An obvious outcome of this line of reasoning is the conclusion 2/4 = 2/3. But do those diagrams show fractional numbers? Two quarters of the first figure are shaded; two thirds of the second figure are shaded. It makes no sense to compare the shaded portions unless the appropriate wholes (units) are taken into account. We can note that relatively more of the second figure is shaded than of the first. An absolute comparison requires more information. Why? Because we are not dealing with numbers. The '2/4' and '2/3', if used at all, are names for the relation of the shaded portion of each figure to its whole.

Quite properly, a child spends a long time [at least two years] and performs many experiments folding, cutting, separating, or joining pieces of paper, cloth, plastic, or wood and discovers many relations which hold for fractional numbers as well as for the objects he has been using before he ever sees a written symbol. He can discover ''equivalence relations'', ''order relations'', ''addition facts'', ''subtraction facts'', ''multiplication facts'', and ''division facts'' for fractions and fractional numbers, but he never has been working with fractional numbers. In the more complex of his discovery exercises, he has been working with concrete representations of composition of functions.

Usually, the child's first written computation with fractions has to do with finding one half of a number. He does this by dividing the number by 2:

$$1/2 \text{ of } 18 = 18 \div 2 = 9.$$

He is not multiplying $1/2 \times 18$; he is performing the operation of halving on 18, and he finds the number which corresponds to 18 under this operation by dividing 18 by 2.

Somewhat later, he has no difficulty in seeing that two-thirds of 18 will be twice as much as one-third of 18:

$$2/3 \text{ of } 18 = 2 \text{ of } 1/3 \text{ of } 18 = 2(18 \div 3) = 12$$

The fraction symbol can be introduced initially as an abbreviation for the expressions '1 half', '2 thirds', and the like. In this symbolism, 2 wholes is written '2/1'. Manipulation of fraction cutouts show clearly that '1/2', '2/4', '3/6', '4/8', ... represent the same relation. It serves also to show the commutativity and associativity of certain particular function compositions and, eventually, to demonstrate concretely the analogous operations on numbers.

1/4 of 1/3 of (an object or a number) = 1/12 of (the object or the number)

1/3 of 1/4 of (an object or a number) = 1/12 of ( )

2/3 of 1/4 of ( ) = (2 of 1/3) of 1/4 of ( ) =

2 of (1/3 of 1/4) of ( ) = 2/12 of ( ) = 1/6 of ( )

1/4 of 2/3 of ( ) = 1/4 of (2 of 1/3) of ( ) =

(2 of 1/3) of 1/4 of ( ) = 2 of (1/3 of 1/4) of ( ) =

2/12 of ( ) = 1/6 of ( )

The step from functions to numbers seems to be a small and simple matter, but it may be quite the opposite. At any rate, recognizing that it is a problem is one necessary requirement for its solution.

If you have 2 apples and 3 apples, you have 5 apples.

$2 + 3 = 5$

If you have 2/8 of an apple and 3/8 of an apple, you have

5/8 of an apple.

$2/8 + 3/8 = 5/8$

2 of 3 apples is (2 × 3) apples.

2 disjoint sets with 3 members each make one set with 2 × 3 members.

1/4 of 1/3 of an apple equals (1/4 × 1/3) of an apple.

3/4 of an apple and 1/4 of an apple are 4/4 of an apple,

which is equivalent to 1/1 (1 whole) of an apple.

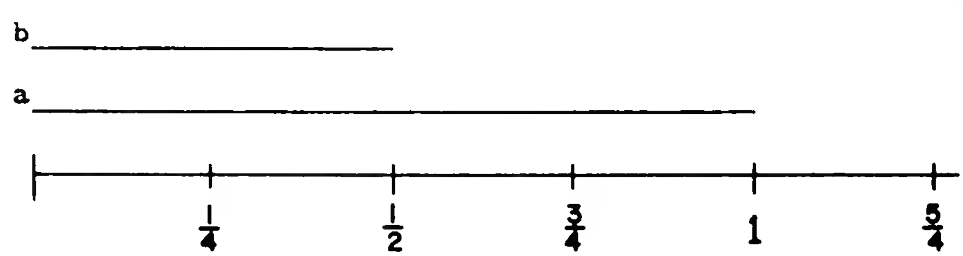
$3/4 + 1/4 = 4/4 = 1/1$

How big are the steps between successive lines in the preceding list? Are all the steps the same size? Does having taken one of these steps make subsequent steps easier? We need answers for these questions.

Having established the requisite concepts informally, we may proceed to a more formal treatment of fractional numbers and operations on fractional numbers.

What is a fractional number? It is an infinite set, an equivalence class, of ordered pairs of cardinals in which the second component of a pair cannot be zero. Thus, any ordered pair of cardinals of the form (n, 2n), $n \neq 0$, belongs to the fractional number 1/2, and any pair like (2, 4), (50, 100), or (124, 248) which also belongs to that number suggests a name (2/4, 50/100, 124/248) for the number. Are these ideas easy for children to grasp? Surely, a few simple declarative sentences will not suffice. It is not wise to be too glib too soon.

Where do we start? [Remember that the children have been working informally with fractions for several years. This is the ''start'' of the formal treatment.] Perhaps the best way to initiate the development of the meaning of fractional number is to work with equivalence relations among fractions. Suppose that we define a fraction as a name, or a symbol, for a number or a relation. We shall be concerned now with the fraction as a numeral, a name or symbol for a number. To show equivalence among fractions, we might use strips of paper which can be folded appropriately, a number scale which resembles a magnified version of a few inches on an architect's rule, a make-believe machine which will cut and join strips of wood according to the way certain buttons are pressed, or other schemes for showing that certain sequences of maneuvers produce equivalent results in terms of the names which may be given to these results.



How long is segment a? It is 1 whole unit ($\frac{1}{1}$), 2 half-units ($\frac{2}{2}$), 4 fourth-units ($\frac{4}{4}$), and 8 eighth-units ($\frac{8}{8}$); and if we had more marks on the scale, we could see that any fraction for which the two components of the pair of numbers are the same could be used as a name for the number which is the measure of this length in terms of the unit we are using. Thus, $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n), \ldots\}$ is the fractional number which we can name by using the name suggested by any one of its members.

Similarly, the length of segment b is 1 half-unit ($\frac{1}{2}$), 2 quarter-units ($\frac{2}{4}$), 4 eighth-units ($\frac{4}{8}$), and so on; and if we took the same unit scale and divided it differently, we could find the same length expressible by '$\frac{3}{6}$', '$\frac{6}{12}$', '$\frac{5}{10}$', and an endless list of equivalent fractions.

A great deal of work, preferably with a variety of materials, must be done to establish the idea that each fractional number, like each cardinal number, has an endless list of names. For some children, this idea becomes clearer if one speaks of different ways to think of a number rather than of different names for the number. "I can think of 99 as 100 - 1 if I want to." "Why would you want to do that?" "Well, the easy way to find the cost of 3 pounds of steak at 99¢ a pound, ..." "Oh, I see. How can you tell how you want to think of a number?" "It depends upon what I want to do with it. Sometimes I want to think of $\frac{1}{2}$ as $\frac{5}{10}$. Sometimes I want to think of $\frac{1}{2}$ as $\frac{2}{4}$. The way my mother cuts pies makes me think of $\frac{1}{2}$ as $\frac{3}{6}$." Different ways of thinking about, or writing about, or talking about, or naming the same number--this makes sense. And aren't we lucky to have a choice?

The words applied to the two terms of a fraction give a clue to the concepts implied.

$$\begin{matrix} 3 \leftarrow \text{numerator} \\ \overline{8} \leftarrow \text{denominator} \end{matrix} \Bigg\} \quad \text{number name}$$

The numerator number is used definitely in the cardinal sense; it tells how many of something. What is this "something"? The name of the "something" is suggested by the denominator.

In the introduction of addition of fractional numbers in its first abstract presentation, I have found it helpful to write the fractions like this:

1 eighth + 2 eighths ——— 3 eighths	$\frac{1}{8}$ $+\frac{2}{8}$ ——— $\frac{3}{8}$	1 eighth + 2 eighths = 3 eighths $\frac{1}{8} + \frac{2}{8} = \frac{3}{8}$

"You have to get what you're adding." If you are adding numbers of eighths, the sum is a number of eighths. Always this sum may be expressed in other ways ($\frac{6}{16}$, for instance, in this case) and sometimes it may be expressed in "simpler form", but these are matters of what you choose to call your answer.

Now, consider this:

$$\begin{array}{r} 1 \text{ fourth} \\ +\ 3 \text{ eighths} \\ \hline \end{array} \qquad \frac{1}{4} + \frac{3}{8}$$

The sum exists. In fact, we have just written a couple of its names. These names are probably not the most convenient ones for our purposes, so we need a way to find a more convenient name. If the denominators of the fractions were the same, we could add the numerator numbers to get a numerator for the "simple" name for the sum and use the same denominator. Can we find a name for $\frac{1}{4}$ whose denominator is '8'? (Can we think of 1 fourth as some number of eighths?) If a child knows without going through all of this that $\frac{1}{4} + \frac{3}{8} = \frac{5}{8}$, should we insist that he go through it anyway? [A kindergarten child once said to me, "A half and a third is five sixths." I said, "How do you know that?" He shrugged, "How do I know anything? It just is."] I would let a child who knows an answer write that answer and give him an example for which he needed the computation in teaching him how to do the computation.

Suppose someone does not know a simple name for $\frac{1}{2} + \frac{1}{3}$. The standard algorism for finding this name is to choose, out of the endless list of names for $\frac{1}{2}$ and $\frac{1}{3}$, a pair of fractions with the same denominator. It is standard practice to choose that pair of names in which the denominator number is smallest. $\frac{1}{2} = \frac{3}{6} = \frac{6}{12} = \frac{9}{18} = \ldots$ and $\frac{1}{3} = \frac{2}{6} = \frac{4}{12} = \frac{6}{18} = \ldots$, but we usually choose $\frac{3}{6}$ and $\frac{2}{6}$; $\frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}$. There is no mathematical law which says we have to make this choice. In fact, a person who would like a picturesque way of going insane could spend the rest of his sane years writing different correct computations for the sum of $\frac{1}{2}$ and $\frac{1}{3}$. However, small numbers are usually easier to work with than large numbers, and $\frac{5}{6}$ sounds "simpler" than $\frac{65}{78}$. Discretion is not out of place.

Before introducing subtraction, we must establish order-relations among fractional numbers. The "function vs. number" distinction enters the picture here, I think. Why will a child who knows without question that one-half of something is more than one-third of it very often state with equal conviction that $\frac{1}{3} > \frac{1}{2}$? When he is dealing with numbers, $3 > 2$, so why isn't $\frac{1}{3} > \frac{1}{2}$? If he has ever encountered some of the foolishness about "the Golden Rule for equations and inequalities" or "treating both sides alike", he can "prove" that, because $3 > 2$, $\frac{1}{3} > \frac{1}{2}$.

We must have established, intuitively at least, that each ordered pair of cardinals (second component $\neq 0$) belongs to one and only one fractional number.

Next, we need a way of determining whether or not two different such ordered pairs belong to the same fractional number. A usual way of doing this is to parlay the question into equivalence of fractions, somewhat as follows:

$$\frac{3}{13} \stackrel{?}{=} \frac{21}{91}$$

$$\frac{3}{13} = \frac{3 \times 7}{13 \times 7} = \frac{21}{91}$$

$$\frac{3}{13} = \frac{21}{91}$$

or

$$\frac{21}{91} = \frac{21 \div 7}{91 \div 7} = \frac{3}{13}$$

$$\frac{21}{91} = \frac{3}{13}$$

or

$$\frac{21}{91} = \frac{3 \times 7}{13 \times 7} =$$

$$\frac{3 \times 7 \div 7}{13 \times 7 \div 7} = \frac{3 \times 1}{13 \times 1} = \frac{3}{13}$$

We say that two fractions are equivalent (they name the same number or are different ways of thinking about the same number) if we can multiply or divide the numerator and denominator numbers of one of them by some [same] number and get the numerator and denominator numbers of the other. [After multiplication and division of fractional numbers have been introduced, it can be shown that this amounts to multiplying or dividing by $\frac{1}{1}$, operations which always result in the same number we started with.] If two fractions are equivalent, then the corresponding ordered pairs of cardinals belong to the same fractional number. In general, $\{(p, q), (2p, 2q), \ldots, (np, nq), \ldots\} \in \frac{p}{q}$ $(q \neq 0, n \neq 0)$.

Another way of telling whether two fractions are equivalent is to note the following illustrations of an important relation:

$\frac{1}{2} = \frac{2}{4}$	$\frac{5}{8} = \frac{10}{16}$	$\frac{18}{12} = \frac{3}{2}$	$\frac{14}{7} \neq \frac{3}{2}$
$1 \times 4 = 2 \times 2$	$5 \times 16 = 10 \times 8$	$18 \times 2 = 3 \times 12$	$14 \times 2 \neq 3 \times 7$

Note that this amounts to comparing numerators of fractions having a common denominator.

$$\frac{1}{2} \times \frac{4}{4} = \frac{2}{4} \times \frac{2}{2} \qquad \frac{14}{7} \times \frac{2}{2} \neq \frac{3}{2} \times \frac{7}{7}$$

$$\frac{1 \times 4}{8} = \frac{2 \times 2}{8} \qquad \frac{14 \times 2}{14} \neq \frac{3 \times 7}{14}$$

$$\frac{4}{8} = \frac{4}{8} \qquad \frac{28}{14} \neq \frac{21}{14}$$

Stated formally, '$\frac{a}{b} = \frac{c}{d}$' is a necessary and sufficient condition for '$a \times d = c \times b$' --that is, '$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = cb$' is a theorem.

We can use this relation to show that, if the numerator and denominator numbers of a fraction have a common factor, removing this common factor (by division) produces an equivalent fraction.

$$\frac{15}{20} = \frac{5 \times 3}{5 \times 4}$$

$\frac{n \times a}{n \times b} = \frac{a}{b}$ because | $\frac{5 \times 3}{5 \times 4} = \frac{3}{4}$ because

$$(n \times a) \times b = a \times (n \times b) \qquad (5 \times 3) \times 4 = 3 \times (5 \times 4)$$

$$[(n \div n) \times a] \times b = a \times [(n \div n) \times b] \qquad [(5 \div 5) \times 3] \times 4 = 3 \times [(5 \div 5) \times 4]$$

$$(1 \times a) \times b = a \times (1 \times b) \qquad (1 \times 3) \times 4 = 3 \times (1 \times 4)$$

$$a \times b = a \times b. \qquad 3 \times 4 = 3 \times 4.$$

Two fractions either name the same number or name different numbers. If the numbers are different, one is larger than the other. Once we have a way of determining which of two fractional numbers is the larger [and which is the smaller] we have a way of ordering the set of fractional numbers.

One widely-used definition of order among [fractional] numbers is the following:

> *A number whose graph is to the right of the graph of another number on a picture of a number-line like this:*

is the larger number. It may be pedagogically better to start with the "opposite" definition, since we usually read from left to right: Of two numbers, graphed on this number-line picture, the one to the left is the smaller. '$\frac{1}{3}$' is to the left of '$\frac{1}{2}$'; $\frac{1}{3} < \frac{1}{2}$.

This definition has always seemed unsatisfactory to me. How does one tell what points on the scale should be associated with certain given numbers? We can take a strip of paper which we will call 1 unit, fold it into two equal parts and then into three equal parts, and conclude that $\frac{1}{3}$ is less than $\frac{1}{2}$ because the first fold for thirds "comes before" the fold for halves, perhaps; but suppose the two numbers we are considering are $\frac{5}{8}$ and $\frac{13}{21}$. It is not easy to divide some object, even a unit segment, into 21 congruent parts. Comparing 13 of 21 congruent parts of an object with 5 of 8 congruent parts of the same object is an extremely difficult mechanical task, and even if we could achieve this, we might still have the problem of associating the results with the numbers we are comparing. It is better to work with the numbers themselves.

The same test we used for equivalence of fractions can be expanded to determine which of the numbers named by the fractions is larger.

$\frac{5}{8} \;?\; \frac{13}{21}$	$\frac{13}{21} \;?\; \frac{5}{8}$
$5 \times 21 \quad ? \quad 13 \times 8$	$13 \times 8 \quad ? \quad 5 \times 21$

These products are not the same, so the fractions are not equivalent.

$$105 > 104 \qquad\qquad 104 < 105$$

$$\frac{5}{8} > \frac{13}{21} \qquad\qquad \frac{13}{21} < \frac{5}{8}$$

Again, this amounts to comparing numerators of fractions having a common denominator.

$$\frac{5}{8} \times \frac{21}{21} \ ? \ \frac{13}{21} \times \frac{8}{8} \quad \text{and} \quad \frac{13}{21} \times \frac{8}{8} \ ? \ \frac{5}{8} \times \frac{21}{21}$$

$$\frac{5 \times 21}{168} \ ? \ \frac{13 \times 8}{168} \qquad\qquad \frac{13 \times 8}{168} \ ? \ \frac{5 \times 21}{168}$$

$$\frac{105}{168} > \frac{104}{168} \qquad\qquad \frac{104}{168} < \frac{105}{168}$$

Stated formally, the necessary and sufficient condition that the fractional number $\frac{a}{b}$ be greater than the fractional number $\frac{c}{d}$ is that the product $a \times d$ be greater than the product $c \times b$. $\frac{a}{b} > \frac{c}{d} \Longleftrightarrow ad > cb$. The "greater than" relation also establishes the "less than" relation:

$$\frac{a}{b} < \frac{c}{d} \Longleftrightarrow ad < cb.$$

Once order relations are established, we have a way of telling whether or not an expression containing a minus sign is meaningful, and subtraction of fractional numbers can be treated in a manner analogous to the treatment of subtraction of cardinals. Everything which has been said about fractions in addition exercises applies to fractions in subtraction exercises.

105 one hundred sixty-eighths	
- 104 one hundred sixty-eighths	$\frac{105}{168} - \frac{104}{168} = \frac{1}{168}$
1 one hundred sixty-eighths	

"You have to get what you're subtracting." In this case, there is no "simpler" name for the difference.

In essence, multiplication of fractional numbers is defined as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$\frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8}.$$

The fifth grader can accept this definition, given the appropriate background. He knows that $\frac{1}{2}$ of $\frac{3}{4}$ of a strip of paper is $\frac{3}{8}$ of that strip of paper; with numbers, $\frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$ and, since $\frac{1}{2}$ of $\frac{3}{4} = \frac{1}{2} \times \frac{3}{4}$ (it has been so defined), $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.

Now, consider the following example of the standard algorithm for multiplying fractional numbers:

$$\frac{3}{\underset{1}{\cancel{5}}} \times \frac{\overset{2}{\cancel{10}}}{17} = \frac{6}{17}$$

We could write this:

$$\frac{3}{5 \div 5} \times \frac{10 \div 5}{17} = \frac{3 \times 2}{1 \times 17} = \frac{6}{17}$$

Who understands what we have done, in terms of operations on numbers? About two years before we tackled an example like this, we should have started to develop an understanding of the principle that dividing one factor of a product by a given number divides the product by that number. I have found this principle difficult for children to grasp. They need a long time, and many experiences like the following, before any real understanding appears.

$$7 \times 6 = \_\_\_\_\_$$

$$7 \times (6 \div 2) = \_\_\_\_\_$$

$$7 \times (6 \div 2) = \_\_\_\_\_ \div 2$$

Eventually they get to something like this:

$$10 \times 12 = \_\_\_\_\_$$

$$(10 \div 5) \times 12 = \_\_\_\_\_$$

$$(10 \div 5) \times 12 = \_\_\_\_\_ \div 5$$

$$(10 \div \quad) \times 12 = 120 \div 5$$

$$10 \times (12 \div 6) = \_\_\_\_\_$$

$$10 \times (12 \div 6) = \_\_\_\_\_ \div 6$$

$$(10 \div 5) \times (12 \div 6) = \_\_\_\_\_$$

$$(10 \div 5) \times (12 \div 6) = 120 \div \_\_\_\_\_$$

$$(10 \div 5) \times (12 \div 6) = 120 \div (\_\_\_\_\_ \times \_\_\_\_\_)$$

We are now ready to see the rationale of the example with the fractions.

$$\frac{3}{5} \times \frac{10}{17} = \frac{3 \times 10}{5 \times 17}$$

by definition

$$3 \times (10 \div 5) = (3 \times 10) \div 5$$

and

$$(5 \div 5) \times 17 = (5 \times 17) \div 5$$

dividing one factor of a product by a given number divides the product by that number.

$$\frac{3}{5} \times \frac{10}{17} = \frac{3 \times 10}{5 \times 17} = \frac{3 \times (5 \times 2)}{5 \times 17} =$$

$$\frac{3 \times 2 \times 5}{17 \times 5} = \frac{3 \times 2 \times 5 \div 5}{17 \times 5 \div 5} = \frac{6}{17}$$

There is a number, 5, which is a factor of both the numerator and denominator numbers of this product. Removing (by division) this common factor produces a "simpler" name for the product.

$$\frac{3}{\not{5}_{1}} \times \frac{\not{10}^{2}}{17} = \frac{3 \times 2}{1 \times 17} = \frac{6}{17}$$

is a handy way to record the mechanics of this procedure.

At some later date, the child should be able to interpret a complicated example somewhat as follows:

$$\frac{3}{5} \times \frac{5}{6} \times \frac{10}{11} = \frac{\overset{1}{3}}{\underset{1}{\cancel{5}}} \times \frac{\overset{1}{\cancel{5}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{6}}} \times \frac{\overset{5}{\cancel{10}}}{11} = \frac{5}{11}$$

because

$$\frac{3}{5} \times \frac{5}{6} \times \frac{10}{11} = \frac{3 \times 5 \times 10}{5 \times 6 \times 11} =$$

$$\frac{3 \times 5 \times (2 \times 5)}{5 \times (3 \times 2) \times 11} = \frac{5 \times 3 \times 2 \times 5}{5 \times 3 \times 2 \times 11} =$$

$$\frac{5 \times (5 \times 3 \times 2) \div (5 \times 3 \times 2)}{11 \times (5 \times 3 \times 2) \div (5 \times 3 \times 2)} = \frac{5}{11}$$

and the top line in this example is a handy way to record the mechanics of this procedure. It is sad that we cannot point out, at this stage, that what we have done amounts to dividing the product by 1, but we are just getting ready to introduce division of fractional numbers.

In any division example,

$$\frac{3}{4} \div \frac{1}{2} = \square$$

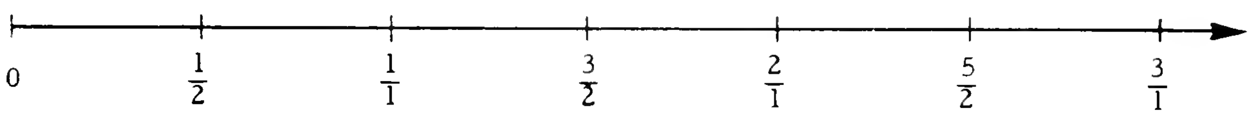
because

$$\square \times \frac{1}{2} = \frac{3}{4}$$

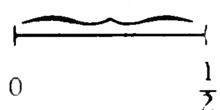
We do not introduce division examples involving fractions with one like the foregoing. We begin by recalling what we know about division of whole numbers. We note, pointedly, that if $\square \times \triangle = \bigcirc$, then $\bigcirc \div \triangle = \square$ and $\bigcirc \div \square = \triangle$, where $\square$, $\triangle$, and $\bigcirc$ are used as placeholders for numerals for appropriate whole numbers. We review this generalization about whole numbers at this time, because we wish to show, eventually, that the same generalization holds for fractional numbers. We have found that fractional numbers whose names have the form $\frac{\text{'n'}}{1}$ behave like whole numbers. This gives us immediately a class of instances of the same generalization with fractional numbers. We probably proceed next to examples like this:

$$\frac{3}{1} \div \frac{1}{2} = \square$$

We ask questions, such as "How many halves in 1 grapefruit? Then how many halves in 3 grapefruit?" "Starting with 3, how many times could you subtract $\frac{1}{2}$ (from the successive differences) to arrive at 0?" "$\frac{1}{2}$ × what number = $\frac{3}{1}$?" "Refer to this number scale:"



"Starting at '0' how many times will you have to add a segment this long:



to arrive at '$\frac{3}{1}$'?" "Does $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1}$? Why?" All but two of these questions suggest physical interpretations of a division situation.

There are other useful tactics. For instance: "Does multiplying and dividing both dividend and divisor of a division example by the same number change the quotient? By what number can we multiply $\frac{1}{2}$ to make the result easy to divide by? Then, $\frac{3}{1} \times \frac{2}{1} \div \frac{1}{2} \times \frac{2}{1} = \frac{3}{1} \div \frac{1}{2}$. But $\frac{1}{2} \times \frac{2}{1}$ is equivalent to 1, and dividing by 1 produces a quotient equal to the dividend. [Eventually we get tired of writing the expression which has no effect on our answer, so we leave it out.] So $\frac{3}{1} \div \frac{1}{2} = \frac{3}{1} \times \frac{2}{1} = \frac{6}{1}$. Check: $\frac{1}{2} \times \frac{6}{1} = \frac{3}{1}$."

Another tactic, which probably should not be taken except to provide enrichment for high achievers, employs the following relation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}.$$

This does not appear in any textbook, to my knowledge, although it is implied in the "common denominator method" for division of fractional numbers. The trouble with it, of course, is that it frequently produces an expression just as complicated as the one in the original example; in other words, it frequently gets you nowhere. But notice:

$$\frac{3}{10} \div \frac{3}{5} = \frac{3 \div 3}{10 \div 5} = \frac{1}{2}$$ correct answer?

$$\frac{9}{11} \div \frac{3}{11} = \frac{9 \div 3}{11 \div 11} = \frac{3}{1}$$ correct answer?

$$\frac{9}{12} \div \frac{6}{12} = \frac{9 \div 6}{12 \div 12} = \frac{3/2}{1} = \frac{3}{2}$$ correct answer?

$$\frac{3}{5} \div \frac{7}{13} = \frac{3 \div 7}{5 \div 13} = \frac{3/7}{5/13}$$ correct answer?

(Now we are involved with another use for the fraction symbol: $\frac{a}{b} = a \div b$.) Pursue the last example a little bit further and see what happens.

$$\frac{3/7}{5/13} = \frac{3}{7} \div \frac{5}{13} = \frac{3 \div 5}{7 \div 13} = \frac{3}{5} \div \frac{7}{13}$$

The fact that division of cardinals is interspersed with division of rationals in this development makes it hard to explain.

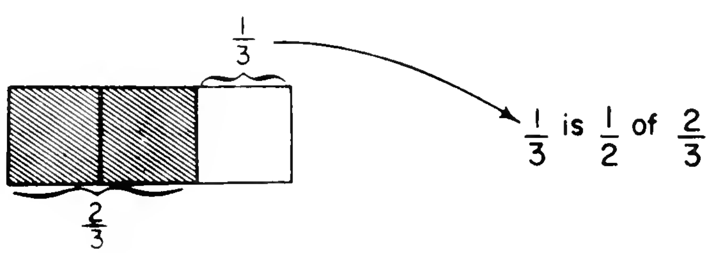
After exploring a number of avenues which suggest ways of dealing with fractions in division examples, we are ready to introduce the standard algorithm. If we wish to verbalize the instructions, we need the word reciprocal (or the equivalent phrase, multiplicative inverse, which is such a tongue-twister that most people avoid it). Two numbers are reciprocals if their product is 1. [We follow the custom of writing '$\frac{1}{1}$' as '1', '$\frac{3}{1}$' as '3', and so on.] Being reciprocals is like being cousins; it takes two, and each bears the given relation to the other (except for 1 which is its own reciprocal and 0 which has no reciprocal). The number $\frac{2}{3}$ is the reciprocal of the number $\frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$, because $\frac{3}{2} \times \frac{2}{3} = 1$. It soon becomes apparent that the reciprocal of a fractional number can be named by the fraction obtained by interchanging the numerator and denominator of a fraction which names the given number. [The word <u>invert</u>, properly defined, should not be in ill repute. It does not mean "turn the fraction upside down." If it did, the reciprocal of $\frac{3}{4}$ would be ᔭ/Ɛ.]

In the standard algorithm for dividing fractional numbers, a division example becomes a multiplication example.

$$\frac{7}{8} \div \frac{2}{5} = \frac{7}{8} \times \frac{5}{2}$$

The quotient of a first fractional number by a second fractional number equals the product of the first by the reciprocal of the second.

It is helpful to note that, in a "real-world" context, the reciprocal tells how many of a given part of an object are contained in 1 whole of that object.



In one whole candy bar, there are 1 two-thirds of the candy bar and $\frac{1}{2}$ of another two-thirds of the candy bar. Another way to write '1 and $\frac{1}{2}$' is '$\frac{3}{2}$'. There are $\frac{3}{2}$ two-thirds of a candy bar in 1 whole candy bar. It is essential to realize that, in this situation, the $\frac{3}{2}$ does not refer to $\frac{3}{2}$ (or $1\frac{1}{2}$) candy bars, but to $\frac{3}{2}$ (or $1\frac{1}{2}$) <u>two-thirds of a candy bar</u>. Once this idea is clear, it is easy to see that there would be 5 times as many $\frac{2}{3}$ candy bars in 5 such bars as there are in 1, or $\frac{7}{8}$ times as many in $\frac{7}{8}$ candy bars, and so on.

The particular relevance of the foregoing paragraph to our discussion of division of fractional numbers comes in the interpretation of answers to "word problems".

> How many pieces of ribbon, each 27 inches long, can be cut from a 10-yard spool of ribbon?
>
> $$27 \text{ inches} = \frac{3}{4} \text{ of a yard}$$
>
> $$10 \div \frac{3}{4} \text{ should give the answer}$$
>
> $$10 \div \frac{3}{4} = 10 \times \frac{4}{3} = \frac{40}{3} = 13\frac{1}{3}$$
>
> The answer to the question is surely 13, but how long is the extra piece? Is it $\frac{1}{3}$ of a yard (12 inches) or $\frac{1}{3}$ of 27 inches (9 inches)? A sixth grader should be able to tell.

Larded through an entire development of the use of fractions should be a great deal of work on estimating answers.

How can one tell whether $\frac{1}{2} + \frac{2}{3} > 1$ is a correct statement?

If '$\frac{1}{2}$' refers to an acre of land and '$\frac{2}{3}$' also refers to an acre of land, what does '1' refer to?

If '$\frac{1}{2}$' and '$\frac{2}{3}$' refer to fractional numbers, what does '1' refer to?

Suppose '$\frac{1}{2}$' refers to a mile and '$\frac{2}{3}$' refers to an hour?

Questions like these should be asked, discussed, answered, and understood in every situation in which fractions appear.

J. Phillips
H. Vaughan

Activities in the Field of Self-Instruction

General

Since early in 1961, the University of Illinois Committee on School Mathematics has been preparing and evaluating self-instruction materials. Our experimental programed texts have been used in research studies in over fifteen schools with the co-operation of more than twenty teachers. Topics treated have come from Units 1, 3, 4, 6, and 9 of UICSM High School Mathematics [University of Illinois Press, Urbana, Illinois].

The materials that the UICSM Programed Instruction Project has produced have been made subject to circulation restraints, following the general rule that, until the results obtained in our evaluations have stabilized at an acceptable level, more modifications are necessary and general circulation is not possible.

Trials of Unit 1 materials during the 1961-62 and 1962-63 school years have resulted in the design of a final research study on those materials. The study will be conducted during the 1963-64 school year. Upon its completion, a definitive statement about the Unit 1 materials and their availability will be made. Also during the 1963-64 school year, studies will be conducted involving the materials derived from Units 6 and 9.

Description of materials

Our self-instruction materials are printed as programed texts. Most texts are accompanied by a work sheet which provides a place for the student to write his response to each problem. Sometimes an entire problem is repeated on the work sheet page, especially when the student is to fill in blanks to complete a sentence. The work sheets are obviously not re-usable, but the programed texts are re-usable.

Perhaps the most basic characteristic of all the UICSM-produced self-instruction materials is the flexibility of presentation. Most of the time we employ a linear technique, which requires each student to understand each discussion and work each problem. There are times when we use branch schemes, which allow any two students to follow different discussions and problems. Some branches provide enrichment work; some provide remedial work.

Much flexibility derives from the fact that our use of a linear technique has not been restricted to the presentation of small bits of information at scattered intervals. Instead, a full-page format is usually used, and there are sometimes lengthy discussions of previous problems or of new ideas. There are frequent illustrations. With occasional exceptions, we have up to ten problems on a page, all to be done before checking any answers. In a certain sense, the variety of situations and problems encountered by each student approximates what he encounters in a carefully conducted UICSM class.

Achievement tests explicitly covering the material treated in the programed texts have been our basic evaluation instruments.

Research -- school year 1961-62

Our research work began officially in June, 1961. At that time a project co-directed by Max Beberman, Director of UICSM and Professor in the College of Education, and Lawrence M. Stolurow, Director of the Training

Research Laboratory of the Department of Psychology and Professor in the Colleges of Education and Liberal Arts and Sciences, was granted support under the National Defense Education Act, Title VII, through the Educational Media Branch of the United States Office of Education, Department of Health, Education and Welfare.

The project, entitled "Comparative Studies of Principles for Programing Mathematics in Automated Instruction", has December 31, 1963, as its terminal date. The research has been oriented so that, in putting various sections of the UICSM curriculum materials into self-instruction form, we could learn principles for programing which would have a generality beyond their use in secondary school mathematics materials.

For one of our studies during the 1961-62 school year we prepared eight programed texts, with accompanying work sheet materials and achievement tests. The content was based upon certain portions of the UICSM Unit 1. In preparing these texts, we selected topics from the Unit which needed different pedagogical approaches so that we would become familiar with problems of re-creating, in self-instruction form, many classroom situations.

The nine teachers from seven schools who co-operated in the trial of these early materials were chosen on the basis of their experience with the regular UICSM materials, including Unit 1. As we had not prepared programed texts embodying all of the Unit, a student received instruction both from self-instruction materials and from his teacher.

The results of this study are presented in reference (1), as listed on page 29. In brief, the study assured us of the feasibility of instruction employing only our programed texts with their flexible teaching strategies and flexible format.

Weekly reports submitted by our co-operating teachers, together with analyses of test data and other response data from the students, gave us this assurance.

Also during the 1961-62 school year, we produced five programed texts which covered selected topics from UICSM Unit 3 and three which covered Unit 4 topics. A preliminary survey of the results achieved by students using the Unit 3 self-instruction materials led to the conclusion that the texts should be extensively revised before another trial. The programing skills developed in the preparation of those texts had application in the preparation of additional Unit 1 materials, and no Unit 3 sequence was included in later studies.

The materials treating Unit 4 topics were used in a separate study. A significant aspect of the study was the use of self-instruction materials to control many variables present in typical teacher-class interactions. With such control, data relevant to a specific pedagogical question could be gathered without the introduction of unwanted sources of variance.

One finding of the Unit 4 study was that students who had been taught by discovery [inductive] techniques for a full year had acceptable achievement on new materials which were taught with deductive techniques.

The work of the 1961-62 school year resulted in the formulation of some preliminary conclusions about the construction and classroom use of self-instruction materials. In brief, the actual writing of the materials contributed significantly to our understanding of the pedagogy of classroom presentation, and from their classroom trial we immediately received useful feedback for revision both of the materials and of our teaching technique.

Research -- school year 1962-63

Another series of studies commenced in July, 1962, when a set of eighteen programed texts was prepared covering the whole of UICSM Unit 1. Eight of these texts were revisions of the first Unit 1 self-instruction materials, while ten were new.

A brief statement about the pedagogy, which cannot be fully descriptive, is that we have included many developmental sequences of exercises to implement discovery techniques and many introductory "real-life" situations to lead to discussions of new mathematical ideas. Personal mannerisms which are often part of classroom pedagogy are impossible to put in a book. We have used frequent changes of pace to help the student learn and stay interested in mathematics. Both the appearance and content of the pages contribute to such changes.

Nine teachers from eight schools and a total of twelve classes co-operated in a trial of the eighteen programed texts in the first semester of the school year 1962-63. Data from the students is currently being analyzed.

Being aware that there were many ways a teacher could use self-instruction materials with students, we defined three modes of use for our study. We could then relate student achievement to the properties of each mode. Following are descriptions of the modes:

Mode 1 -- "Pure"

The only instruction students receive, except for unusual circumstances, is by means of the programed texts. Students work in the materials throughout each class meeting, except when taking appropriate achievement tests. Homework assignments, when given, are to do further work in the programed texts.

Mode 2 -- "Anticipating"

The students receive instruction both from the programed texts and from their regular classroom teacher. Assignments either for homework or for in-class work are given in such a way that every topic the teacher discusses has been anticipated by its treatment in a programed text. As these texts give the introduction to topics, the teacher's discussion includes elaboration and clarification of the new topics. In addition, the teacher is encouraged to give a different perspective on the topic and help clear up general or individual difficulties of the students.

Mode 3 -- "Following"

The students receive instruction both from their regular classroom teacher and the programed texts. Assignments either for homework or for in-class work are given in such a way that every topic the teacher discusses is followed by its treatment in a programed text. As the teacher gives the introduction to topics, the programed texts give the additional examples and a different perspective.

Although complete quantitative analyses of the student data collected in the 1962-63 study are not ready, some preliminary statements of conclusions can be given as follows:

(1) The mode most suited to controlled research in which instructional variables are carefully assessed is Mode 1. In that mode, teacher-student interaction is severely minimized and the self-instruction materials are the exclusive vehicles of instruction. There is some indication, however, that student attitudes are still strongly influenced by the classroom teacher.

(2) It is extremely difficult to write self-instruction materials that alone are as effective as a well-trained and experienced teacher. When control and "pure" experimental groups are compared, the control groups often outperform the experimental groups, but not always to a statistically significant degree.

(3) Materials which do an excellent job under Mode 2 are unlikely to do as well under Mode 3. Mode 2 conditions tend to produce results superior to Mode 1 conditions, and the results are more similar to those from control groups. The co-operating teachers who had participated in both the 1961-62 and 1962-63 studies indicated that they preferred either Mode 2 or Mode 3 to the Mode 1 restrictions of the 1961-62 trial.

(4) Special measures must be taken to ensure that discovery sequences function as intended. When a teacher leads a class through a discovery sequence, the discovery of the generalization by some of the students strongly motivates

> other students to find it. Students studying self-instruction materials work independently and at their own pace. Therefore, the class as a functioning unit does not exist and the beneficial interaction effects are not present.

On the basis of these preliminary observations, we feel that self-instruction materials written to be the exclusive medium of instruction [as under Mode 1] will do their best job under flexible adaptations of the conditions of Mode 2. On the other hand, none of our materials have been written to provide review only. We think that there is a place for self-instruction review materials and that these could be successfully prepared for use under conditions like those of Mode 3. In addition, we are experimenting with new techniques to solve the problem regarding discovery sequences.

The production and experimental use of the eighteen texts was supported in part by the Course Content Improvement Section of the National Science Foundation. This support will also make possible our continuing efforts to use self-instruction both as a means of developing text materials, not necessarily self-instructional in nature, and as a research tool.

A programed text treating topics from elementary symbolic logic, based on the appendix to UICSM Unit 6, is being prepared and tested using students at various stages of their mathematical development. A knowledge of rules of reasoning is useful to students wherever proofs are studied. This can occur in almost any high school mathematics course.

One of the appendices to Unit 9 treats the mensuration formulas of solid geometry. We have written a programed text based on this and other UICSM-produced solid geometry materials. This text was intended for use by students with either a modern or a traditional background. Student data gathered from a trial in May, 1963, indicates that the test is quite difficult and does not communicate well enough with students having no previous UICSM training. The results of our trial indicate that students with UICSM training do significantly better in this particular text.

Future plans

As has been indicated above, the UICSM project has an extensive research program planned for the 1963-64 school year in the use of self-instruction materials. Both the logic and solid geometry materials will undergo extensive revision based on the results of their recent trials, and new evaluation studies will be conducted.

Questions regarding teaching strategies, methods of presentation within a programed text, and methods of use of programed texts are being investigated. In addition, we plan to investigate relationships between use of self-instruction materials and such variables as ability and time.

An entirely different dimension of the use of self-instruction materials will be investigated as we begin to include the step of programing into the writing of new curriculum materials. We hope that close co-ordination between the authors of our new curriculum materials and experienced programers will help produce new insights and perspectives. Since putting materials into self-instruction form yields both content and pedagogical ideas, this interaction should contribute both to the student and teacher editions of materials.

O. Robert Brown, Jr.

General Interest Publications

(1) Wills, Herbert, "The UICSM Programed Instruction Project." American Mathematical Monthly, Vol. 69, No. 8, October, 1962.

(2) UICSM Staff, A Description of UICSM Materials for Self-Instruction. February, 1963. [Includes a reprint of (1)]

Technical Reports

(1) Brown, O. Robert, Jr., "A Comparison of Test Scores of Students Using Programed-Instructional Materials with those of Students Not Using Programed-Instructional Materials." Technical Report No. 3, July, 1962.

(2) Wolfe, Martin, "Effects of Expository Instruction in Mathematics on Students Accustomed to Discovery Methods." Technical Report No. 4, January, 1963.

Test Item File Project

One of the most interesting and most challenging projects which is presently being worked on by the UICSM research section is the development of a file of test items whose validity and level of difficulty have been determined but which are, as much as possible, neutral with respect to the course of instruction.

Item analysis is being carried out on a variety of test items which have been used in the past, and work is in progress on constructing new test items for further analysis. We believe that teachers can help us greatly by sending in interesting test ideas, and we shall try to reciprocate by publishing ideas teachers might like to use. We intend to include sample items from the pool in the UICSM newsletters and invite any of you who can to try these items in your classes and send us the results. When you send us items, we would appreciate knowing the following:

1. What per cent of your classes got it right?
2. What were some of the common erroneous responses?
3. What per cent made each of the erroneous responses?
4. How many students were in the classes that took the test?
5. Grade level(s) of classes taking the test?

Here is a test item which shows something about understanding the decimal numeral system. Do not be surprised if many of your students fail to get it correct.

Arrange the following from smallest to largest:

.00501 .017 .000008 .0030 .010005 .001 .007 .0081
.0015060 .0000001 .00003 .01 .0057 .196 .0400000
.001963 .0002 .0000063 .001060 .00069

A common erroneous response is as follows:

.0000001
.0000063
.0015060
.0400000
.000008
.001060
.001963
.010005
.00003
.00069
.00501
.0002
.0030
.0057
.0081
.001
.007
.017
.196
.01

Please send any correspondence regarding the test item pool to:

Mr. J. A. Easley, Jr.
Research Coordinator
UICSM Math Project
1210 W. Springfield
Urbana, Illinois

Follow-Up Study

The UICSM research section is presently conducting a Follow-up Study on students who have completed three or more years of UICSM materials. It is especially hoped that this study will provide us with some knowledge of the effects of the UICSM mathematics curriculum on college work. Following is a brief outline of the work being done on this study.

In December, 1962, the first phase of our Follow-up Study was begun. It was decided to limit the initial sample to those who had graduated from high school in 1962, who had completed at least three years of UICSM mathematics courses, and who were enrolled in a college or university. This limitation was made in the light of several factors — first, this was the first group to complete (or nearly complete) the current UICSM mathematics courses; second, these students would provide us with the most easily accessible sample; and third, we would be able to follow the students throughout their four years of undergraduate study.

We wrote to the high schools which we knew had graduates with the above qualifications and, from 26 high schools throughout the country, received the names of approximately 600 students. Each of these students was contacted and asked to fill out a questionnaire pertaining to his high school and college work. We also asked the students for permission to obtain their high school and college transcripts. The co-operation we have received from the students, the high schools, and the colleges has been exceptionally good. At the present time, we have had about 75 per cent of the questionnaires completed and returned from students in over 200 different colleges and universities, and 95 per cent of those returned have included the permission for the transcripts.

We are now processing the data received from the questionnaires and transcripts and hope to have soon a summary report which will include data on the following:

1. The mathematics and science training received in high school
2. The mathematics courses being studied by these students in college
3. Grades received in both high school and college
4. Majors in college and planned careers of these students
5. Comments from the students about their high school mathematics training

Work is in progress on the second phase of this study involving the high school graduates of 1963. In this second sample we hope to ask approximately 2000 students to fill out a similar questionnaire. Students of the second phase sample will also be followed as they progress through their college careers in order to obtain more data indicative of the effects of the UICSM curriculum on college work.

Those who would like to receive a copy of the summary report of the first phase of our Follow-up Study should send their names and addresses to:

Mrs. Judith E. Boyle
UICSM Mathematics Project
1210 West Springfield
Urbana, Illinois

Your names will be added to our mailing list and you will receive a copy of the report as soon as it is completed.

UICSM

NEWSLETTER

Number 13

December 1963

An Occasional Publication of the

University of Illinois Committee on School Mathematics
1210 West Springfield
Urbana, Illinois

Table of Contents

Editor's Page

Since the activities of UICSM have become so diverse, it is natural that our Newsletters should reflect this. The information in the articles appearing in Newsletters will not apply equally to all the classes taught by UICSM-trained teachers. On the editor's page, there will be a short discussion of each article and how it fits into the educational picture as we see it. The editor's page will also contain any other messages which we believe fit better there than in separate articles. If you have any comments on our new format, we would welcome them, and we will try to use any suggestions you might have for improving the Newsletters.

The UICSM staff is not unlike the staff of most schools. Each year sees many changes in personnel. In recent years, we have had many additions and very few subtractions. Thus, it is possible that some of you are still corresponding with people who are known to you. Most of our readers, however, are probably seeing, on articles and correspondence from this office, names that are completely new to them. It is with the thought in mind that you might like to know something about the people behind the names that, in this and further Newsletters, we will provide a short biographical sketch of people on our staff. Some of the sketches will be about people who have begun this year, and some will be about people who joined the staff in recent years.

Mr. Hoffmann has provided us with an interesting article that provides drill on real numbers for average (or below average) students and gives better students a challenge in discovering patterns.

"Stone-Age Math" by Dr. Phillips is essentially a short classroom unit on the development of an elementary number system. She uses devices that provide their own motivation. The vocabulary she develops is one which can interest children in grades six, seven, or eight. Some students might even want to enlarge on the system she has developed.

We are most fortunate in being permitted to use some letters received by our staff members from former students. If any readers have letters dealing with the experiences of former UICSM students, we would be pleased to print any parts of the letters you would care to designate. These reactions are good motivation for continuing with contemporary mathematics.

Mr. Mueller and Mr. Vaughan have some notes they felt should be shared with teachers of Unit 10. The notes on the "Solution of a Special Trigonometric Equation" will be found most useful to these teachers.

Mr. Gabai has summarized the work currently being done in vector geometry by our staff. We hope that this material will prove useful in the third year of high school mathematics. We are sure that it will at least throw a new light on the ancient subject of geometry.

Stone-Age Math

Do you think a small child may know what four is without knowing anything about one or two or three? Could he know that four tells how many wheels a car has and how many feet his dog has and how many legs a chair has, and still could he know nothing about any other numbers?

Have you ever heard a child count to five by saying, "One, seven, four, nine, three, ten, five"? He knows some words, but he doesn't really know what all of them mean or how they are related to each other.

How does anyone know that five is greater than three but less than nine, for example? First, he has to know what three is and what five is and what nine is. Then, he has to know what is meant by greater than and less than.

In any kind of productive discussion, we need preliminary agreement about the meaning of some terms we shall use. Pay careful attention to the underlined words in the next two paragraphs.

Any specific collection of separate objects (where object can refer to anything: a person, a planet, an electron, or even an idea) is a set. Each object which belongs to a set is a member of the set. For each object and each set, either the object is a member of the set or it is not. In mathematics, the word 'set' is always used in this precise sense.

To each set there corresponds a cardinal number which tells how many members the set has. Sets which have the same cardinal number are equivalent sets. Describe another set which has the same cardinal number as the set whose members are your eyes. Is the set whose members are your feet equivalent to the set you have just described? What is the cardinal number of the set whose only member is the chair on which you are now sitting? If you are not now sitting on a chair, how many members belong to the set of chairs you are sitting on? You see that it sometimes makes sense to think of a set which has no members. The set with no members is the empty set. What number tells how many members the empty set has? Then, the cardinal number of the empty set is zero. A set cannot have fewer than no members, so zero is the smallest cardinal number. Is there an upper limit to the number of members a set may have? Is there a largest cardinal number?

Cardinal numbers are the numbers primitive peoples used first in developing their cultures. Cardinal numbers are also the numbers very young children use first in learning to deal with the cultures into which they are born. It may help us to a better understanding of our own use of cardinal numbers if we take a make-believe excursion several thousand years back into history.

Suppose we pretend that we are living in the Stone Age. We have a simple language with which we can think and with which we can express thoughts to one another.

In this language, 'glm' (pronounce 'glm' as you would pronounce 'glum') is the word for the number of pebbles in the set {  }. The idea of counting has not occurred to us. How would we pick out other sets which have glm members?

One of us might pick up the pebbles, lay one pebble on his left foot and the other pebble on his right foot. He has run out of pebbles and also out of feet.

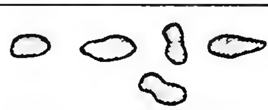
Can he now say he has glm feet? Could he pick out other examples of "glmness" in the same way?

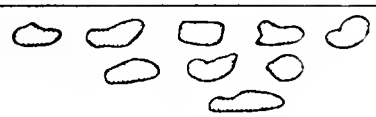
In this language, 'kog' (pronounce 'kog' as you would pronounce 'cog') is the word for the number of pebbles in the set {  }. How can we find other sets which have this same number property? Does a man have kog fingers on his right hand? Does he have kog fingers on his left hand? Where else might he find sets with kog members?

Would he have to know the word 'kog' in order to tell that he has the same number of fingers on each hand? To tell that he has the same number of fingers on a hand as he has toes on a foot? Would he need to know how to count, or how to add, in order to be sure he had the same number of fingers on both hands as he has toes on both feet (assuming that he was a normal Stone-Age baby and has not been damaged since)?

In order to tell whether sets have the same number or different numbers, all we need is a matching technique. Given the members of any two sets of objects, we can try to match each member of one set with a member of the other set. If the matching "comes out even" — each member of each set paired with exactly one member of the other set — we can be sure that the two sets have the same number of members. When this happens, we say there is a one-to-one correspondence between the members of the two sets. Try matching the members of the set {a, b, c} with those of the set {Tom, Dick, Harry}. Pair each letter with the name of one boy and different letters with different names of boys so that each member of each set belongs to exactly one pair. Can you match the members of these sets in another way? In how many ways? Does each one-to-one correspondence assure you that there is the same number of members in the set of letters as in the number of boys? If two sets have the same number of members, is it always possible to set up a one-to-one correspondence between their members? Is it always possible to do this in more than one way?

Our Stone-Age technique of pairing pebbles with other objects has given us a way of telling when two sets have the same number of members. You are a smart Stone-Age boy or girl. Your father has chosen you (out of the entire family group) to teach all the number names he knows so that you can pass this knowledge along to others and, eventually, to your own children. This is what he taught you:

Pictures of collections of pebbles	Number names
	glm (pronounced glum)
	kog (cog)
	ds (dees)
	ug (ug)
	qut (cat)
	okz (ox)
	tr (tor)

Pictures of collections of pebbles	Number names
	nn (noon)
	hk (heck)

You find it hard to remember which name goes with which collection. You reason that, if there were some way of organizing the numbers, it would make it easier to remember their names. (We said you are smart.) Start with the first two numbers in the list. How will you decide which is greater, glm or kog? (So far, all you know how to do is match.) Match the appropriate collections of pebbles. What do you find out? If you decide to call the collection in which you had pebbles left over the larger collection, you would probably call the number of this collection the greater number. If so, you really know two things — kog is greater than glm, and glm is less than kog. Jump forward in time a few thousand years so that you know not only how to write but also how to use modern symbols, and you may write:

$$\text{kog} > \text{glm} \quad \text{and} \quad \text{glm} < \text{kog}$$

This may be read:

kog is greater than glm and glm is less than kog

Now jump back into the Stone Age (except that you can use modern English, an enormous "except") and answer the following questions:

1. Is the same number named twice in the list? How do you tell?
2. Which of these are true statements:

   nn > tr? qut > hk? How do you tell?

3. Is glm less than every other number? How many comparisons would you have to make to find this out?

4. What is the name of the smallest number? How could you find out which number is next smallest?

5. Which number names could you write in the blank to make true statements?

   glm $<$ _____ tr $>$ _____ tr $<$ _____

   Is this all you need to know in order to tell which number is next larger than glm?

6. Could you extend the scheme described in question 5 so that you could line up the collections of pebbles in order from fewest to most? Once you have done this, you have also ordered the cardinal numbers ug through ds from least to greatest, and you can learn to say their names in order.

7. Make a list of numbers, in order, from ug through ds. Are you now ready to count? If so, how far can you count? Why?

8. Under what circumstances would a Stone-Age father be able to use the list to tell someone else how many children he had?

Apparently, we need something more than some numbers lined up in order from smallest to largest before we can say we have "counting numbers." Let's go out and get a heap of pebbles so that we can duplicate the collections shown in our list. It is important that we realize that ug is the number of objects in each set which contains a single object. If we have any pebbles at all, ug is the smallest number we can have, so we'll call ug our first counting number. Now we have the idea of combining members of two collections. Let's start with two sets, each of which has ug members. Consider a new set built by putting the members of these two sets together. Which of the model collections does this new set match? Let's keep a record, twentieth-century style, and let's be orderly. Complete each of the following:

1. Ug and ug is _____.
2. Glm and ug is _____.
3. Tr and ug is _____.
4. Qut and ug is _____.
5. Kog and ug is _____.
6. Hk and ug is _____.
7. Okz and ug is _____.
8. Nn and ug is _____.
9. Ds and ug is _____.

It appears that we have a couple of problems to solve, difficult problems for Stone-Agers. What suggestions can you make?

We need a number which is ug greater than hk. A picture of the members of a set which would have this number might be as follows:

Suppose we agree to name this number 'pr' (pronounce 'pr' as you would pronounce 'purr'). Now can we count up to ds? We can if each of our numbers, as far as they go, is ug greater than the number it follows. Do we have numbers like that? Let's see how we use them.

Lay out a few objects on your desk and count them, using our Stone-Age number names. Think about how you did it.

1. Did you say a number name for each object as you counted it? Did you make a one-to-one correspondence between the number names and objects you were counting?
2. Would you have gotten the same result if you had counted the objects in a different order?
3. If you touched each object as you counted it, did you touch any object more than once? How many objects had you touched when you said the number name 'tr'? Then, what does 'tr' tell you when you use it in counting? Could you make a similar statement about the other number names?
4. How many number names did you use? If you count correctly, do you have to use the number names in order? How do you know when to stop counting?
5. What can you do if you run out of number names before you have counted all the objects on your desk?

What the real Stone Agers did was use a word meaning "a lot" for any number larger than the largest number they could name. Their culture did not require names for large numbers. As someone once remarked, thirty was infinity to them.

In our culture, we need names for millions of numbers, very large numbers, very small numbers, and different kinds of numbers.

Let's reorient ourselves to the twentieth century and summarize what we have discussed so far. Fill in the blanks.

1. The number which tells how many members a set has is a _____.
2. The cardinal number of any set whose members could be paired, one-to-one, with the members of the set of wheels on a car is _____.

3. The cardinal number of any set whose members can be placed in a one-to-one correspondence with the members of the set of your eyes is _____.
4. A set with no members at all is called _____.
5. Before the cardinal numbers can be used for counting, they have to be ordered so that each number is _____ more than the number just before it.
6. In counting, we make a one-to-one correspondence between the members of the set of objects we are counting and the equivalent set, starting with 'one', of ordered number names. We use each number name in order exactly _____ (how many times) and count each object exactly _____.
7. The last number name we say in a counting sequence tells us the _____ number of the set whose members we are counting.
8. The smallest counting number is _____; the smallest cardinal number is _____; the largest cardinal number is _____.

J. Phillips

Staff Biographies

Dr. Josephine M. Phillips

We have chosen to start with Dr. Phillips since she is an author of some of the material in this Newsletter. Prior to joining the UICSM staff last year, Dr. Phillips was an arithmetic editor for D. C. Heath and Company. Her experience before she went into the publishing field was in the teaching of mathematics and in the training of teachers. She has taught in public schools in New Jersey, at Longwood College, at Montclair State College, and at Boston University. Her vast experience also includes some time as a Lt. (j. g.) in the U. S. Coast Guard Women's Reserve.

Dr. Phillips has a productive background in writing. Her publications include articles in *The Arithmetic Teacher, School Science and Mathematics*, and *Book Production*. She has also produced sixty filmstrips under the general title "Seeing the Use of Numbers". These filmstrips are produced by Eye Gate House Inc., Jamaica, New York. Dr. Phillips is also co-author of a revised edition of *Learning to Use Arithmetic,* published by D. C. Heath and Company, Boston.

At present, Dr. Phillips is engaged in writing materials to be used by the seventh grade classes at University High School on campus. The emphasis of this course is chiefly on the physical applications of mathematics. From time to time, we will be able to present some of these topics for your consideration in this Newsletter. Many of the topics which she has developed will not be used in the course because of time allotments, but their importance in general education is recognized, and we hope to share them with our readers.

Dr. John A. Easley Jr.

Dr. Easley left the salubrious climate of beautiful Hawaii last year to be a Visiting Associate Professor at the University of Illinois. His job on the UICSM staff is to coordinate the objective studies which we are constantly undertaking on our materials. He brings a varied and interesting background to that particular position. At one time or another, he has been a teacher, a principal, a Peace Corps consultant, and a radio engineer.

Dr. Easley's stint as a radio engineer for the Carnegie Institute in Washington, D. C., took him to Baffin Island and to Hawaii. He then moved to California to take a job as a physics instructor at Vallejo Junior College. From there, he went to Magino in the Marshall Islands, where he was Principal of the Marshall Islands Intermediate School. Then he became an instructor in Science Education at the University of Hawaii. After a Teaching Fellowship at Harvard, Dr. Easley went back to Hawaii as an Associate Professor of Science. Finally, he made his way back to the mainland to join the UICSM staff.

The list of publications by Dr. Easley is very nearly as diverse as his background. His Ph.D. dissertation at Harvard is entitled "A Study of Scientific Method as an Educational Objective". An article he had printed in the *Proceedings of the Hawaiian Academy of Science* was entitled "A Pedagogical Device for Clarifying the Concept of Drift". An article printed in *Philosophy and Education* (Israel Scheffler, editor) was entitled "Is Scientific Method a Significant Educational Objective?" The Winter, 1959, *Harvard Educational Review* carries his article "The Physical Science Study Committee and Educational Theory". He has also written a college level general science text being published by Wadsworth Publishing Company, called *Introduction to Scientific Thought in the Physical Sciences*. In addition to these publications, Dr. Easley has had his speeches before some academic bodies printed in the proceedings.

Mr. Howard Marston

Howard Marston is here on leave of absence from Principia Upper School in St. Louis. He has probably taught more UICSM classes than any other person. He started with one UICSM class at Principia in 1955 and since then has taught a total of 34 UICSM classes there. In the past six summers he has also taught eight institute classes the ways of UICSM. The classes were held at the University of Arizona and the University of Illinois. Prior to joining the staff at Principia, Mr. Marston taught in public and private schools in Connecticut and New York. He received his B.A. degree from Wesleyan University in Connecticut and his M.A. degree from Teachers College, Columbia University. He has had one publication, *Worktext in Modern Mathematics* published by Harper & Row. UICSM has run several printings of a short text on Solid Geometry written by Mr. Marston. This text was made available at cost to UICSM teachers for several years. Mr. Marston is currently engaged in revising and rewriting the programed edition of solid geometry which had its first trial run last spring. Mr. Marston has a wife teaching at Principia and two children attending Principia, with whom he gets reacquainted each weekend by commuting to St. Louis.

Test Score Comparison Report

The first report produced by the newly-organized research section of UICSM is entitled "Comparison of UICSM vs. Traditional Algebra Classes on Co-op Algebra Test Scores." This report describes the results of an evaluation study involving approximately 1700 eighth and ninth grade UICSM students, called the experimental sample, and nearly 700 non-UICSM algebra students, called the control sample. All students in these samples began the study of algebra in either 1958 or 1959. Theii achievement in algebra was measured by the Co-operative Algebra Test (Elementary) Forms T, S, and Y. We do not claim that this standardized traditional algebra test provides a complete description of achievement by UICSM students. However, we are willing to consider the results of this test as a rough basis for evaluation of the UICSM curriculum because the skills and knowledge expected from the traditional curriculum are included in the objectives of the UICSM program.

The experimental sample was divided into six groups depending on grade, year of first course, and duration of study prior to testing. The control sample was divided into two groups depending on the year. Further subdivision of the control sample was unnecessary because non-UICSM classes were uniform with respect to grade and time of testing. Table 1, below, contains a description of the groups and the means and standard deviations of the scores on the Co-op Algebra Test and the Differential Aptitude Tests of Numerical Reasoning and Verbal Ability. Table 1 gives this information for the subgroups of both the experimental and control samples.

Table 1. Descriptions of Experimental and Control Samples and the Subgroups Thereof. Means and Standard Deviations of the Co-op Alg Test, DAT-VR, and DAT-NA for Each.

Sample	Group	Grade	Year Course Was Begun	Time of Testing	N	Co-op Alg Mean	Co-op Alg S. D.	DAT-NA Mean	DAT-NA S. D.	DAT-VR Mean	DAT-VR S. D.
EXPERIMENTAL SAMPLE	E_1	8	1959	May '60 - - Mar '61	226	66.9	10.0	23.7	7.9	27.4	7.6
	E_2	9	1958	May '59	118	66.6	7.9	26.1	5.6	28.4	7.5
	E_3	9	1958	Sept - Dec '59	270	63.7	8.5	25.7	6.7	28.6	7.9
	E_4	9	1959	May 60	574	59.6	11.2	23.0	7.7	24.2	8.3
	E_5	9	1959	Nov - Dec '60	382	65.8	8.6	26.0	6.4	28.6	7.7
	E_6	9	1959	Jan '61	135	63.2	9.5	25.8	5.5	29.0	7.0
	$E_{2...6}$	9	1958 and 1959	May '59 - - Jan '61	1479	62.8	10.1	24.8	7.0	26.9	8.2
	E_{Total}	8 and 9	1958 and 1959	May '59 - - Mar '61	1705	63.4	10.2	24.6	7.2	27.0	8.1
CONTROL SAMPLE	C_1	9	1958	Mav '59	515	58.8	11.3	21.7	7.4	23.5	8.2
	C_2	9	1959	May '60	161	56.6	10.1	23.1	6.5	24.2	7.3
	C_{Total}	9	1958 and 1959	May '59 and May '60	676	58.3	11.1	21.3	7.2	23.6	8.0

A direct comparison of the means is not appropriate because the UICSM classes had a higher mean on the DAT-tests as shown in Table 1. Therefore, adjustments for these inequalities between groups were made by using the analysis of covariance in comparing group means on the Co-op algebra test. With these adjustments, the experimental sample as a whole showed significantly greater achievement than the control sample. This was also true of four of the six experimental groups taken separately, while the remaining two were not significantly different from the control sample. Various other comparisons between groups within the experimental or control samples were also made. Table 2 lists all the comparisons between various groups that were made. An inequality sign in this table indicates whether the adjusted Co-op Algebra mean for the group named on the left margin was significantly greater than (>), significantly less than (<), or not significantly different from (≐) that for the other group.

Table 2. Comparisons of adjusted group means on Co-op Algebra

Statistical test used: Standard Covariance Analysis

Groups Compared	Adjusted Group Means	F-ratio	P	Conclusions
E_{total} C_{total}	62.50 60.45	31.19	$< .001$	$E_{total} > C_{total}$
E_1 $E_{(2...6)}$	67.33 62.76	34.79	$< .001$	$E_1 > E_{(2...6)}$
$E_{(2...6)}$ C_{total}	61.85 60.04	14.90	$< .001$	$E_{(2...6)} > C_{total}$

Table 2. (continued)

E_2 C_{total}	62.41 58.99	17.57	< .001	$E_2 > C_{total}$
E_3 C_{total}	60.48 59.56	2.53	> .05	$E_3 \doteq C_{total}$
E_4 C_{total}	59.07 58.68	.67	> .05	$E_4 \doteq C_{total}$
E_2 E_4	63.49 60.20	15.63	< .001	$E_2 > E_4$
E_3 E_5	63.21 65.75	11.86	< .01	$E_5 > E_3$

Statistical test used: Unrestricted Linear Hypothesis Model

Compared	Differences Between Adjusted Means	Standard Error of Estimate	t-ratio	P	Conclusions: Computer Program due to Watson	Conclusions: Johnson-Neyman Technique
C_1 C_2	1.11	2.93	.38	> .05	$C_1 \doteq C_2$	
E_5 C_{total}	3.00	.538	5.57	< .001	$E_5 > C_{total}$	$E_5 > C_{total}$ ($P < .05$)
E_3 C_1	6.92	2.33	2.97	< .01	$E_3 > C_1$	$E_3 \doteq C_1$ ($P > .05$)

Possible reasons for these results as well as the specific details of the statistical analysis are discussed in full in the report. Readers may obtain a copy of this report by writing Mrs. Ann Perkins, 1210 W. Springfield, Urbana, Illinois, and asking for a copy of "Comparison of UICSM vs. 'Traditional' Algebra Classes on Co-op Algebra Test Scores".

J. Boyle

A Pat on the Back

Many UICSM teachers have, no doubt, received messages of one kind or another from former students who have started work on college mathematics and who have realized that their UICSM background is helping them. [Your editor is no exception to this statement.] One of our present staff members received such a letter recently. He had been teaching UICSM math in an Illinois high school prior to joining our staff. One of his students in a class that he had for four years is now attending a major university and felt moved to communicate with his former teacher. The following is an excerpt from this letter.

> ...I never believed that (name of high school) was such a good school til (sic) I got here. Many, many people could not pass the placement tests to get into freshman calculus. Also, this first week for me has been purely review. Many others are already having trouble. Inequality and absolute value (also the empty set) are new concepts to them. I am frankly surprised.
>
> What I really want to say is that I apologize for every time I criticized the exactness of U. of I. math. The only way I'll get through this course is to listen to what the professor means rather than to what he says. (Underlining is ours.) ... [There follows a discussion of changes in notation which are minor nuisances to the student.]
>
> ...Set theory is given here, but it is very lax. A set $\{x: x > 1\}$ can be shown as $\{x| x > 1\}$ or $\{x > 1\}$. This second one is silly! Also $g = \{(x, y): y = x^2\}$ can be written $g = \{x, x^2\}$!! No set selector, no nuthin!

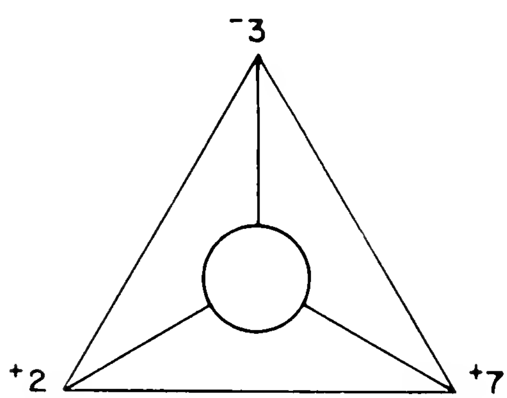
Another letter reached a member of the UICSM staff from a former student of his who now attends Wesleyan University. The following is an excerpt from this letter.

> A month of college life has already passed, but I can say it has been the most rewarding month of my life. My courses are all great, and the professors are marvelous. There is nothing that can compare with college life.
>
> My favorite subject is calculus mainly because I got into an honors group for students who want to treat the calculus from the viewpoint of theory or the "UICSM" philosophy. We are using a mimeographed text written by Einar Hille of Yale. Ducas is using the same text at Yale. I can't tell you how valuable the UICSM program has proved to be for me. While most of the guys in my class had already had a lot of calculus before, I am way ahead of them due to my UICSM background. We have been studying set theory, math induction (Peano's Postulates), the development of theorems for rational and real numbers, and associated material. This is pure review for me, but most of the students are really having a tough time. I do find that I have to spend almost 7 hours a week on calculus, but it is really interesting. It is quite a challenge also. I never had to spend more than 7 hours a month in high school. There are some bright kids up here, needless to say....

Non-Pharoahic Pyramids

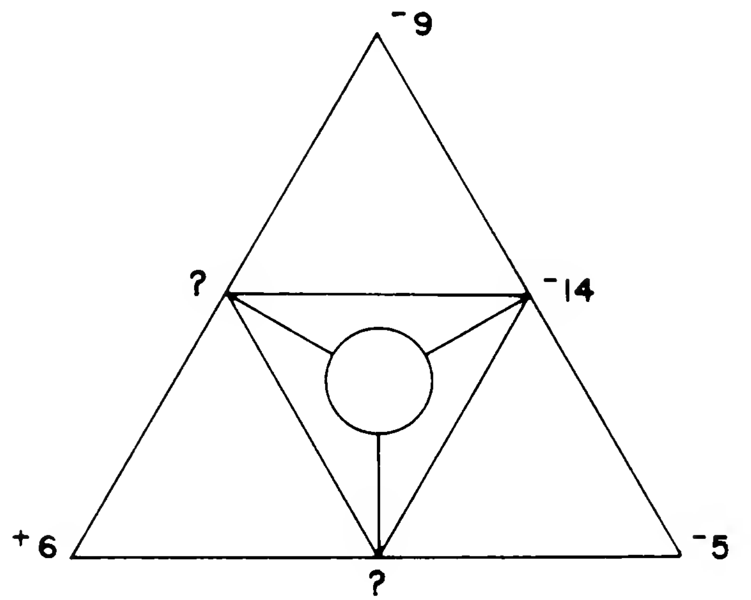
Here is a device which can be used as an interesting drill on addition of real numbers. It has the peculiar inherent property of providing more drill for the methodical students but less drill for the ingenious ones. A short-cut may be discovered by the student at any time the teacher wishes to allow him to make the discovery. On the other hand, the short-cut is very simple to see and still involves a little addition of real numbers. See if you can discover the short-cut to use in the following exercises.

(1) The inner-number is the sum of the 3 "corner-numbers". Give the inner-number of this pyramid.



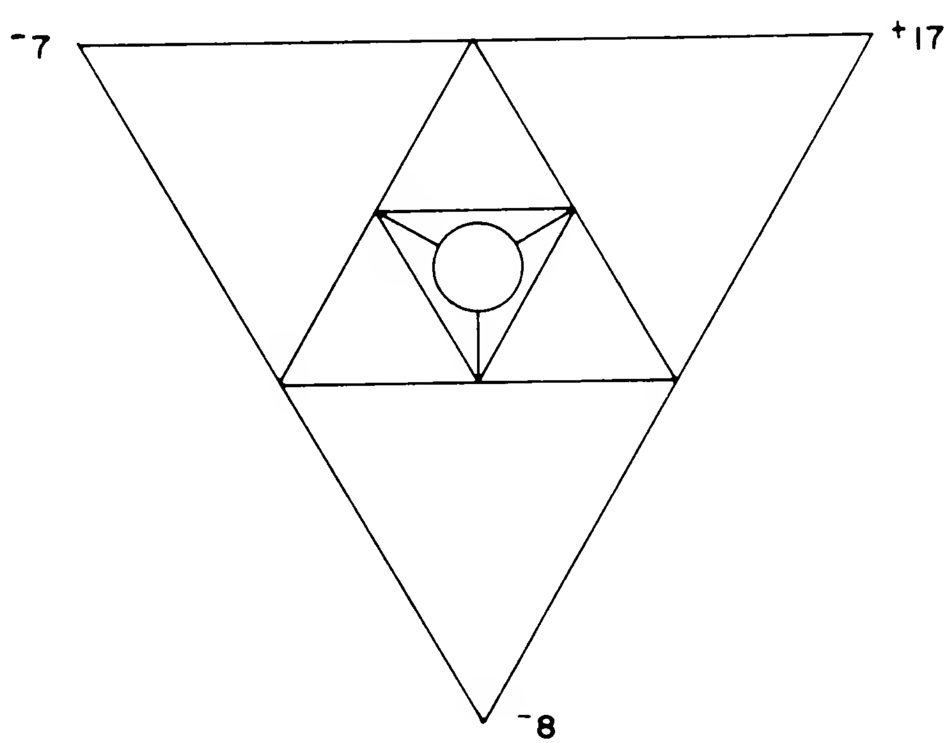
Of course, the answer is $^+6$. Now, consider a slightly more complicated pyramid. To find the inner-number, one must find the corner-numbers for the inner-triangle. This is done by adding the given numbers at each end of one side.

(2) One corner-number of the inner-triangle has been found for you in the pyramid below. Find the other two, and then find the inner-number.



The inner-number is ⁻16. For more complicated pyramids, just extend the process.

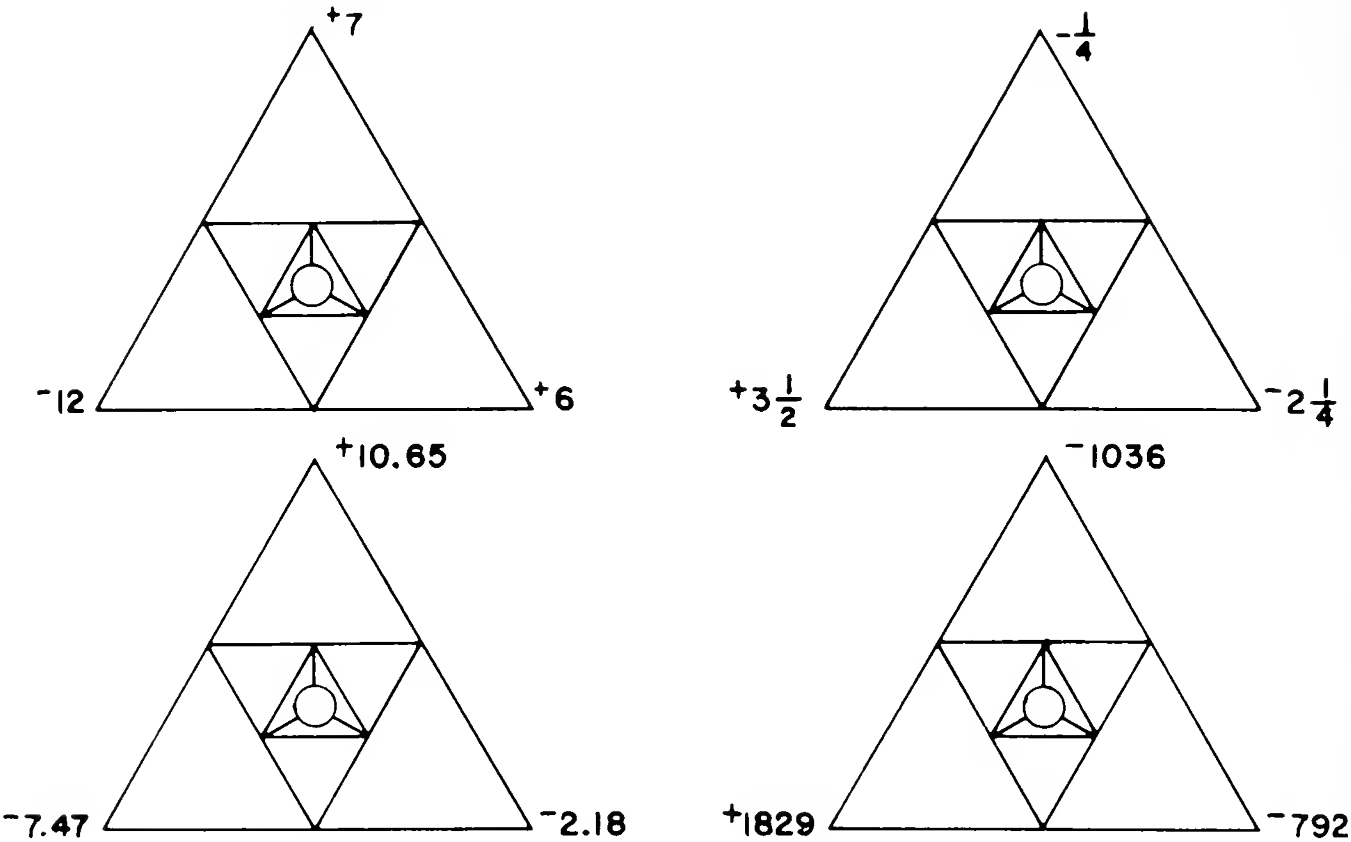
(3) Find the inner-number of this pyramid.



Did you get $^+8$? If not, you better check your computation. [I had to!]

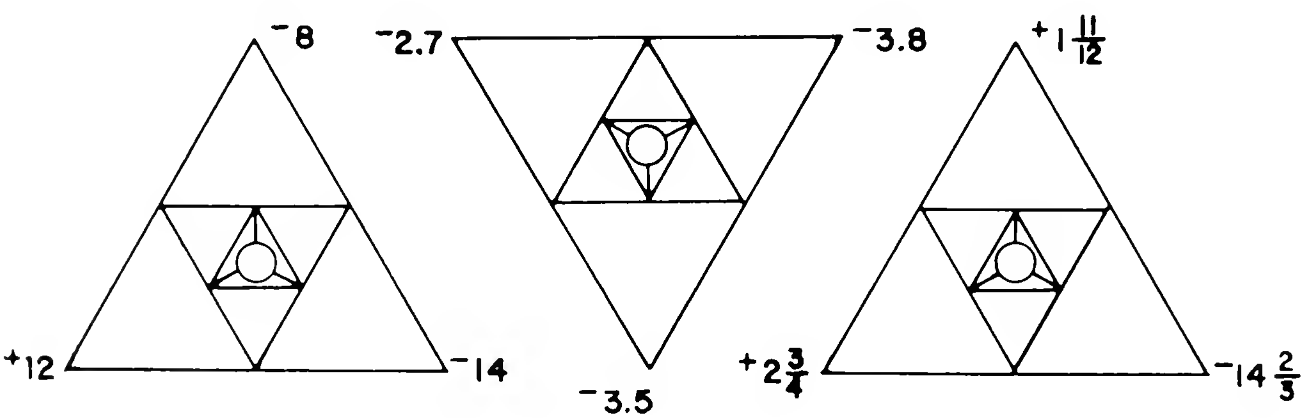
Now, it is time to make a discovery.

(4) Find the inner-number for each of these pyramids.



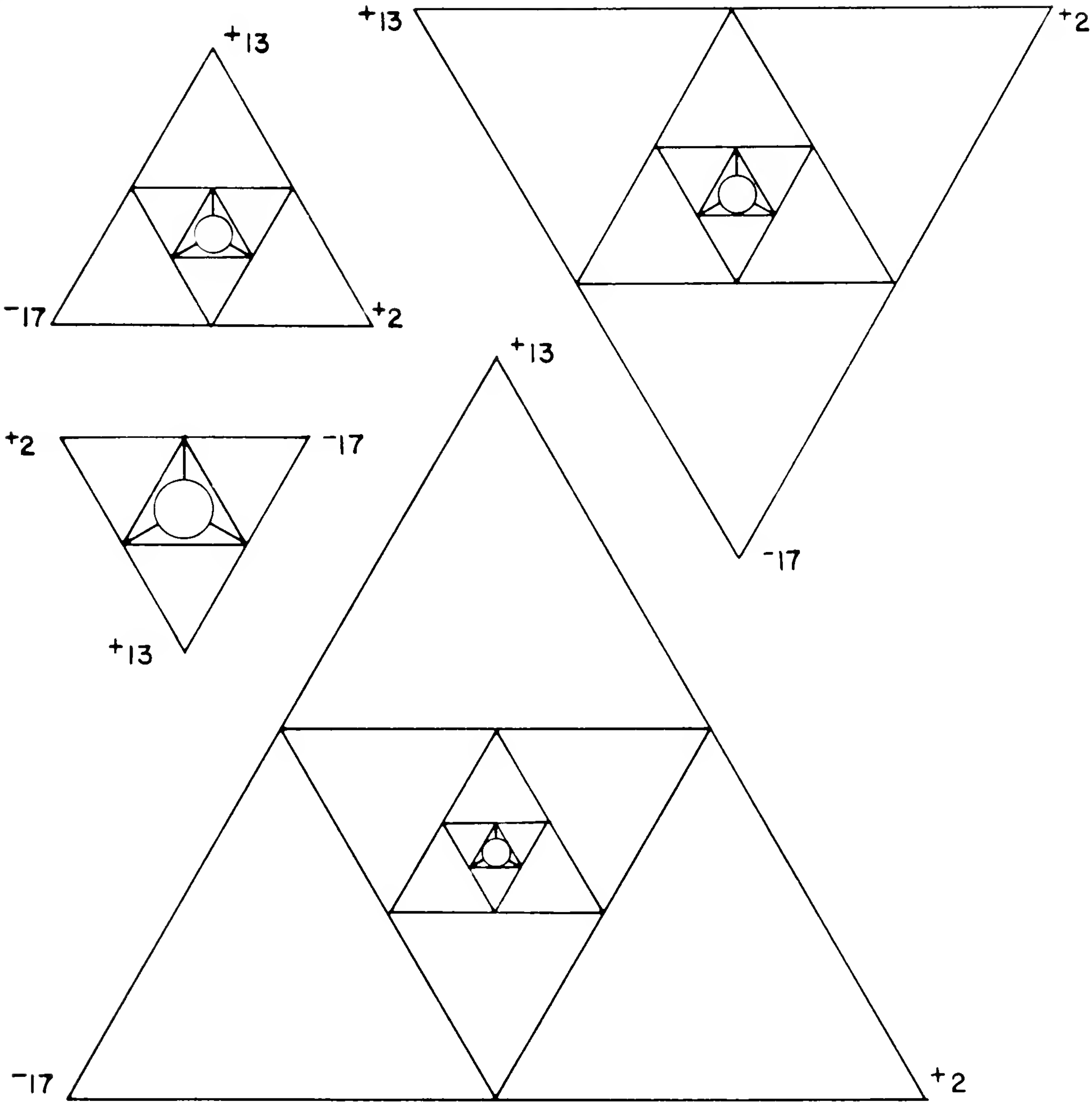
The inner-number for each pyramid is $^+4$.

(5) As simply as possible, tell whether each of these pyramids has the same inner-number.



Yes, they do.

(b) Give the inner-number for each of these pyramids.



It's about time that you proved your conjectures, don't you think?

Vector Geometry—Experimental Edition

This year at UICSM, one of the exciting projects underway is that of developing a course in vector geometry. This course was designed by Herbert E. Vaughan. It is being taught to two experimental classes in the University High School by Max Beberman and Steven Szabo, with the assistance of Dick Dennis. The classes are being critically observed and analyzed by a team of staff members which includes the experienced observers Harvey Gelder, Alice Hart, Gertrude Hendrix, and Eleanor McCoy. Hyman Gabai, Steven Szabo, and Dick Dennis are assisting in the preparation of the text materials for the course, under the supervision and guidance of Max Beberman. It is, of course, still too early to make any definite statements about how well the two classes are working out, but so far they seem to be off to a good start.

The course itself is a unique and interesting development of geometry. Translations (that is, vectors) are introduced as mappings of ordinary 3-dimensional euclidean space on itself. These mappings satisfy certain postulates which are introduced in small doses. The introduction of postulates is always preceded by discussions based on the student's intuitive feelings and knowledge about the physical world in which he lives. Thus motivated, the postulates appear as clear and natural statements which are reasonable to assume in the development of our geometry.

The first few postulates relate translations with euclidean space, $\mathcal{E}$. For instance, one postulate states, in effect, that given any points A, B of $\mathcal{E}$, there is a translation which maps A on B.

The remaining postulates refer to properties of translations. For example, it is postulated that the composition of translations is itself a translation. These postulates are eventually stated as a single postulate: the set of all translations of $\mathcal{E}$ onto itself is an abelian group.

Later, more postulates are introduced until we are able to make the statement that the set of all translations of $\mathcal{E}$ is, itself, a three-dimensional vector space over the real numbers. Still later, we introduce other postulates and arrive at our final statement that the set of all translations is a three-dimensional inner product space.

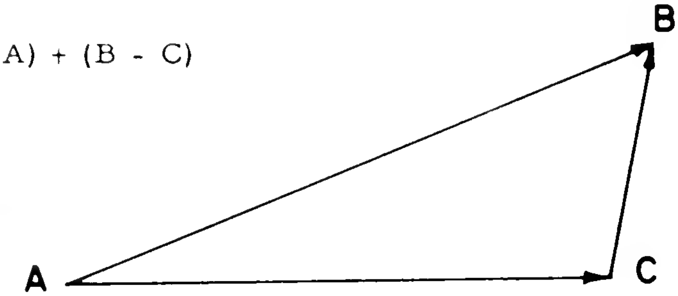
Along the way, as these postulates are being introduced,

1. the usual geometrical entities, such as lines and planes, are introduced as subsets of $\mathcal{E}$ having certain properties;
2. quite a number of "traditional" geometrical theorems are proved;
3. many important mathematical concepts are studied. Among these are:
   (a) The concepts of groups, abelian groups, vector spaces, and inner product spaces. (A distance function is defined and shown to satisfy the usual axioms of a metric space.)
   (b) The concepts of linear dependence and linear independence.
   (c) Dimension and basis of a vector space. (Including the concept of orthonormal bases.)
   (d) Schwartz's inequality.

One especially interesting feature in the development is the notation which is introduced. The notation permits us to operate on points and translations in a manner which closely parallels the ordinary rules of algebra. (We even speak of our "algebra of points and translations".) For instance, we write sentences like this:

$$B - A = (C - A) + (B - C)$$

This stotes thot, for points A, B, C of $\mathcal{E}$, the sum of the vectors from A to C and C to B is equol to the vector from A to B



Some of the advantages of this approach to geometry, as developed by Professor Vaughan, are:

1. It is a very rigorous and strict deductive development of geometry, but the many motivational and intuitive discussions and examples will permit the student to see it unfold in a natural manner. Even if we strip it of the motivational and intuitive discussions, the rigor, logic, and novelty of the development will appeal to a mature mathematical mind.
2. The student will gain experiences and develop a broad background covering many important mathematical concepts which otherwise would not be touched upon in his high school courses.
3. The student will become acquainted with much more than geometry; he will study vectors, mathematical structures, and the development of a deductive system.
4. The student will work with many interrelated concepts and develop an awareness that all of these concepts (including geometry, algebra, and arithmetic) are locked together in the foundations of the structure of mathematics.

From time to time you will be brought up to date on the progress of the course through this Newsletter.

H. Gabai

On the Solution of a Special Trigonometric Equation

A problem of some interest to teachers of trigonometry (and to their students) is the solution of:

(1) $$a \cos x + b \sin x = c$$

[for a and b not both zero].

The purpose of this note is to give a method of solving (1) and to investigate the choices of a, b, and c which yield a solution for (1).

We are assuming that not both a and b are zero. Hence, it follows that $\sqrt{a^2+b^2} \neq 0$ and that $\frac{1}{\sqrt{a^2+b^2}} \neq 0$. Also since

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

it follows that $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$ belongs to the unit circle.

For each point (c, d) of the unit circle, there is just one number θ such that

(2) $0 \leq \theta < 2\pi$, and

(3) $\cos\theta = \frac{c}{\sqrt{c^2+d^2}}$ and $\sin\theta = \frac{d}{\sqrt{c^2+d^2}}$.

In our development it will be convenient to have a more direct method of calculating θ. It is easily seen that

(4) $$\theta = \begin{cases} \text{Arccos} \dfrac{c}{\sqrt{c^2+d^2}}, & d \geq 0 \\ 2\pi - \text{Arccos} \dfrac{c}{\sqrt{c^2+d^2}}, & d < 0. \end{cases}$$

Returning now to the solution of (1), we first transform it to:

(5) $\sqrt{a^2 + b^2}\left[\cos x \cdot \frac{a}{\sqrt{a^2 + b^2}} + \sin x \cdot \frac{b}{\sqrt{a^2 + b^2}}\right] = c$

We have previously established that for each pair of numbers a, b, not both zero, $\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right)$ belongs to the unit circle and that for this pair there is a unique θ such that, $0 \leq \theta < 2\pi$, $\frac{a}{\sqrt{a^2 + b^2}} = \cos\theta$, and $\frac{b}{\sqrt{a^2 + b^2}} = \sin\theta$. Substituting in (5), we get:

(6) $\sqrt{a^2 + b^2}\,[\cos x \cos\theta + \sin x \sin\theta] = c$

The form of (6) suggests using the Subtraction Law for Cosines:

(7) $\forall_x \forall_y \cos(x - y) = \cos x \cos y + \sin x \sin y$,

to simplify the left member. Since $\sqrt{a^2 + b^2} \neq 0$, it follows from (6) and (7) that

(8) $\cos(x - \theta) = \frac{c}{\sqrt{a^2 + b^2}}$.

At this point we see that, if there are to be solutions to (1), we need a restriction on 'c'. If $|c| > \sqrt{a^2 + b^2}$ we would have $|\cos(x - \theta)| > 1$, which is impossible. Hence, we will have solutions of (1) only if $|c| \leq \sqrt{a^2 + b^2}$.

It is well known that, for $|y| \leq 1$, $\cos x = y$ if and only if

(9) $x = 2k\pi \pm \text{Arccos } y$, for some integer k.

(Equation (9) and its derivation can be found in UICSM Unit 10, page 125.)

The solution of (8) follows immediately from (9). We find that

(10) $x - \theta = 2k\pi \pm \text{Arccos } \frac{c}{\sqrt{a^2 + b^2}}$

and finally that

(11) $x = 2k\pi + \theta \pm \text{Arccos}\, \frac{c}{\sqrt{a^2 + b^2}}$

where θ is given by (4).

The solutions of (1) with the restrictions a and b not with 0 and $|c| \leq \sqrt{a^2 + b^2}$ are given by (11).

We end this note with two examples of the solution of (1).

Example 1. $\cos x + \sin x = 1$

Here $a = b = c = 1$ and $\sqrt{a^2 + b^2} = \sqrt{2}$.

Therefore, $c < \sqrt{a^2 + b^2}$.

We can write the original equation as:

$$\sqrt{2}\, [\cos x \cdot \frac{1}{\sqrt{2}} + \sin x \cdot \frac{1}{\sqrt{2}}] = 1$$

Here, $\theta = \frac{\pi}{4}$, and so this equation becomes:

$$\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

or:

$$\cos (x - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Therefore, $x - \frac{\pi}{4} = 2k\pi \pm \text{Arccos}\frac{1}{\sqrt{2}} = 2k\pi \pm \frac{\pi}{4}$

and $x = 2k\pi + \frac{\pi}{4} \pm \frac{\pi}{4}$.

That is $x = 2k\pi$ or $x = 2k\pi + \frac{\pi}{2}$.

We may solve the same equation directly by using (4) and (11).

Since $a = b = 1$,

$$\theta = \text{Arccos}\, \frac{a}{\sqrt{a^2 + b^2}} = \text{Arccos}\, \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

and $\text{Arccos}\ \frac{c}{\sqrt{a^2 + b^2}} = \text{Arccos}\ \frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

Therefore,

$$x = 2k\pi + \frac{\pi}{4} \pm \frac{\pi}{4}.$$

That is $x = 2k\pi$ or $x = 2k\pi + \frac{\pi}{2}$, k an integer.

Example 2. $\sqrt{3}\cos x - 1\sin x = 2$

Here, $a = \sqrt{3}$, $b = -1$, $c = 2$. So, by (4),

$$\begin{aligned}\theta &= 2\pi - \text{Arccos}\ \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + (-1)^2}} \\ &= 2\pi - \text{Arccos}\ \frac{\sqrt{3}}{2} \\ &= 2\pi - \frac{\pi}{6} \\ &= \frac{11\pi}{6},\end{aligned}$$

and $\text{Arccos}\ \frac{c}{\sqrt{a^2 + b^2}} = \text{Arccos}\ \frac{2}{\sqrt{(\sqrt{3})^2 + (-1)^2}}$

$$\begin{aligned}&= \text{Arccos}\ 1 \\ &= 0.\end{aligned}$$

Therefore, by (11), $x = 2k\pi + \frac{11\pi}{6}$, k an integer.

J. Mueller and H. E. Vaughan

UICSM News and Notes

We are including a brief News and Notes Column in this newsletter in order to reestablish a line of communication which has been missing in the last few newsletters. Our hope is that we will hear from many of our readers who will tell us of their activities in the area of public relations for the UICSM program.

Mr. Arnold Petersen spoke at the October meeting of the Association of Mathematics Teachers of New Jersey. His topic dealt with the training of teachers for modern mathematics programs. Mr. Petersen was a Teacher Associate on the UICSM staff in the school year 1960-61. He is the chairman of the mathematics department at Pascack Valley Regional High School in Hillsdale, New Jersey.

Professor Max Beberman was one of three representatives from the United States to speak in Athens, Greece. He spoke before the International Working Session on Modern Teaching of Mathematics. At that meeting, he delivered a paper entitled "Searching for Patterns". He then went on to the Sixth Annual Conference of Overseas Schools, held in Rome. At that conference, Professor Beberman participated in a panel discussion on Elementary and Secondary School Mathematics.

Alice Hart spoke at the Fall meeting of the Eastern Division of the Illinois Education Association. Her topic at this meeting was "Mathematics in the Schools Today".

Herbert Wills spoke at the Thanksgiving weekend meeting of the Central Association of Science and Mathematics Teachers. His topic for this meeting was "UICSM — 1964 Model".

Robert Wirtz and Alice Hart were among the speakers at the Idaho Centennial Mathematics Conference which was held in Boise, Idaho. Mrs. Hart's contribution consisted of four speeches. These included a speech at the breakfast meeting of Phi Delta Kappa, and three speeches entitled "UICSM Program", "New Mathematics?", and "The Next Step". Professor Wirtz' speeches were entitled "An Introduction to Sets" and "Numbers have Many Names".

William Hale spoke on "The Evolution in Secondary School Mathematics Program" at a Mathematics teachers conference at Westmar College in Le Mars, Iowa.