Faculty Working Papers

UNEMPLOYMENT, INFLATION, AND INTEREST
IN MULTI-SECTOR NEOCLASSICAL GROWTH

Hans Brems

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College of Commerce and Business Administration
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<u>98-word Summary:</u>

The purpose of the present paper is to see if a multi-sector neoclassical model of steady-state imbalanced growth has room for familiar Keynesian and monetarist ideas on unemployment and inflation. It has. It can be shown to have infinitely many solutions, one for each value of the employment fraction. Solutions differ with respect to rate of inflation, nominal but not real interest rate, and rates of growth of money and national money income. Switching from one solution to another requires a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.

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UNEMPLOYMENT, INFLATION, AND INTEREST IN MULTI-SECTOR NEOCLASSICAL GROWTH

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I. PURPOSE

The purpose of the present paper is to see if a multi-sector neoclassical model of steady-state imbalanced growth has room for familiar Keynesian and monetarist ideas on unemployment and inflation. It has. It can be shown to have infinitely many solutions, one for each value of the employment fraction. Solutions differ with respect to rate of inflation, nominal but not real interest rate, and rates of growth of money and national money income. Switching from one solution to another requires a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.

Our neoclassical growth model will have two goods in it, each of which serves interchangeably as a consumers' or a producers' good. Three gains result. First, the model can allow for substitution both in consumption and production, hence let the price mechanism come into play. Second, the model can allow for realistic steady-state but imbalanced growth. Third, having heterogeneous physical capital stock, the model will be immune to Cambridge, England criticism. Despite the resulting complication the model is solvable——and solved. It will use the following notation.

II. NOTATION

Variables

C ≡ physical consumption

D ≡ demand for money

g_v ≡ proportionate rate of growth of variable v ≡ C, D, I, κ, L, M, P, r, ρ, S, w, X, and Y

I ≡ physical investment

k ≡ present gross worth of another physical unit of capital stock

κ ≡ physical marginal productivity of capital stock

L ≡ labor employed

λ ≡ proportion employed of available labor force

M ≡ supply of money

N ≡ present net worth of entire physical capital stock

n ≡ present net worth of another physical unit of capital stock

P ≡ price of good

r ≡ nominal rate of interest

ρ ≡ real rate of interest

S ≡ physical capital stock

U ≡ utility

w ≡ money wage rate

X ≡ physical output

Y ≡ money income

Parameters

a ≡ multiplicative factor of production function

α, β ≡ exponents of production function

c ≡ propensity to consume money income

F ≡ available labor force

g_p ≡ proportionate rate of growth of parameter p ≡ a and F

m ≡ multiplicative factor of demand for money function

μ ≡ exponent of demand for money function

p ≡ multiplicative factor of Phillips function

π ≡ exponent of Phillips function

u ≡ multiplicative factor of utility function

υ ≡ exponent of utility function

All parameters are stationary except a and F whose growth rates are. Time coordinates are t for general time and τ for specific time. Euler's number e is the base of natural logarithms. G, H, J, and K stand for agglomerations to be defined as we go along.

III. THE MODEL

1. *Definitions*

Define the proportionate rate of growth

(1) $$g_v \equiv \frac{dv}{dt} \frac{1}{v}$$

Define investment as the derivative of capital stock with respect to time:

(2) $$I_{ij} \equiv \frac{dS_{ij}}{dt}$$

2. *Production*

For its production each good needs capital stock of both goods: The output X_j of the jth good is produced from labor L_j and two immortal capital stocks S_{ij}, where i is the sector of origin and j the sector of installation. As a result, there are four distinct physical capital stocks S_{ij} in the model. Let every entrepreneur have access to a Cobb-Douglas production function

$$X_1 = a_1 L_1^{\alpha_1} S_{11}^{\beta_{11}} S_{21}^{\beta_{21}} \tag{3}$$

$$X_2 = a_2 L_2^{\alpha_2} S_{12}^{\beta_{12}} S_{22}^{\beta_{22}} \tag{4}$$

where $0 < \alpha_j < 1$; $0 < \beta_{ij} < 1$; $\alpha_1 + \beta_{11} + \beta_{21} = 1$; $\alpha_2 + \beta_{12} + \beta_{22} = 1$; and $a_j > 0$. Assume a fairly strong interindustry dependence: Let each industry be at least as dependent upon the capital stock supplied by the other as upon that supplied by itself, then $\beta_{11} \leq \beta_{21}$, $\beta_{22} \leq \beta_{12}$.

In each industry let profit maximization under pure competition equalize real wage rate and physical marginal productivity of labor:

(5) $$\frac{w}{P_i} = \frac{\partial X_i}{\partial L_i} = \alpha_i \frac{X_i}{L_i}$$

Physical marginal productivities of capital stock are

(6) $$\kappa_{ij} \equiv \frac{\partial X_j}{\partial S_{ij}} = \beta_{ij} \frac{X_j}{S_{ij}}$$

3. *Investment Demand*

Let N_j be the present net worth of new capital stock S_{ij} installed by an entrepreneur in the jth industry. Let his de-

To find desired capital stock, proceed as follows. Let entrepreneurs be purely competitive ones, hence price of output P_j is beyond their control. At time t, therefore, value marginal productivity of another physical unit of capital stock is $\kappa_{ij}(t)P_j(t)$. As seen from the present time τ, value marginal productivity at time t is $\kappa_{ij}(t)P_j(t)e^{-r(t - \tau)}$, where r is the stationary nominal rate of interest used as a discount rate. Define present gross worth of another physical unit of capital stock as the present worth of all its future value marginal productivities over its entire useful life

$$(9) \qquad k_{ij}(\tau) \equiv \int_{\tau}^{\infty} \kappa_{ij}(t)P_j(t)e^{-r(t - \tau)}dt$$

Let entrepreneurs expect physical marginal productivity of capital stock to be growing at the stationary rate $g_{\kappa ij}$:

(10) $$\kappa_{ij}(t) = \kappa_{ij}(\tau)e^{g_{\kappa ij}(t - \tau)}$$

and price of output to be growing at the stationary rate g_{Pj}:

(11) $$P_j(t) = P_j(\tau)e^{g_{Pj}(t - \tau)}$$

Insert (10) and (11) into (9), define

(12) $$\rho_{ij} \equiv r - (g_{\kappa ij} + g_{Pj})$$

and write the integral (9) as

$$k_{ij}(\tau) = \int_{\tau}^{\infty} \kappa_{ij}(\tau)P_j(\tau)e^{-\rho_{ij}(t - \tau)}dt$$

Neither $\kappa_{ij}(\tau)$ nor $P_j(\tau)$ are functions of t, hence may be taken outside the integral sign. Our $g_{\kappa ij}$, g_{Pj}, and r were all said to be stationary, hence the coefficient ρ_{ij} of t is

stationary, too. Assume $\rho_{ij} > 0$. As a result, find the integral to be

$$k_{ij} = \kappa_{ij}P_j/\rho_{ij}$$

Find present net worth of another physical unit of capital stock as its gross worth minus its price:

$$n_{ij} \equiv k_{ij} - P_i = \kappa_{ij}P_j/\rho_{ij} - P_i \tag{13}$$

Our appendix proves that (13) satisfies the second-order conditions for a maximum N_j. Applying the first-order condition (7) to our result (13) find equilibrium physical marginal productivity of capital stock

$$\kappa_{ij} = \rho_{ij}P_i/P_j \tag{14}$$

Take (14) and (6) together and find desired capital stock

(15) $$S_{ij} = \beta_{ij}P_jX_j/(\rho_{ij}P_i)$$

Apply the definitions (1) and (2) to (15) and find desired investment as the derivative of desired capital stock with respect to time:

(16) $$I_{ij} \equiv g_{Sij}S_{ij} = g_{Sij}\beta_{ij}P_jX_j/(\rho_{ij}P_i)$$

(15) and (16) are capital stock and investment desired by an individual entrepreneur in the jth industry. Except X_j everything on the right-hand sides of (15) and (16) is common to all entrepreneurs of the industry. Factor out all common factors, sum over all entrepreneurs of the industry, then X_j becomes industry output, and (15) and (16) become capital stock and investment desired by the industry.

So investment in the ith good by the industry producing the jth good is in direct proportion to, first, the rate of

growth g_{Sij} of desired capital stock; second, the elasticity β_{ij} of the output of the jth good with respect to the stock of the ith good; third, the relative price P_j/P_i of the jth and the ith good; and fourth, the output X_j of the jth good. Investment is in inverse proportion to what will turn out to be the real rate of interest ρ_{ij}.

4. Consumption Demand

Let every consumer have the utility function

$$U = uC_1^{\upsilon_1}C_2^{\upsilon_2}$$

where $0 < \upsilon_i < 1$, and $u > 0$. Let every consumer spend the fraction c, where $0 < c < 1$, of his money income Y. Then his budget constraint is

$$cY = \sum_{i=1}^{2} (P_i C_i)$$

Maximize the consumer's utility subject to his budget constraint and find his two demand functions

(17) $$C_i = c_i Y/P_i$$

where $c_i \equiv c\upsilon_i/(\upsilon_1 + \upsilon_2)$. (17) is consumption desired by an individual consumer. Except Y everything on the right-hand side of (17) is common to all consumers. Factor out all common factors, sum over all consumers, then Y becomes national money income, and (17) becomes national desired consumption. But with immortal capital stock, the entire value of national output represents value added, i. e., national money income

(18) $$Y \equiv \sum_{i=1}^{2} (P_i X_i)$$

Insert (18) into (17) and write national desired consump-

tion

$$C_1 = c_1(X_1 + P_2X_2/P_1) \tag{19}$$

$$C_2 = c_2(P_1X_1/P_2 + X_2) \tag{20}$$

5. *Goods-Market Equilibrium*

Goods-market equilibrium requires output to equal the sum of consumption and investment demand for it:

$$X_i = C_i + \sum_{j=1}^{2} I_{ij} \tag{21}$$

6. *Employment and the Phillips Function*

Let labor employed be the proportion λ of available labor force, where $0 < \lambda < 1$ and λ is not a function of time:

$$\sum_{i=1}^{2} L_i = \lambda F \tag{22}$$

Within their province let labor unions seek a relative and temporary gain by raising the money wage rate w. Knowing that the gain will be temporary will not keep them from seeking it; on the contrary, in anticipating inflation they are compelled to contribute to it. But let their compulsion be tempered by unemployment. Subtract employment (22) from available labor force F and find unemployment $(1 - \lambda)F$, and express a Phillips -curve relationship

$$g_w = p(1 - \lambda)^{\pi} \tag{23}$$

where $\pi < 0$, and $p > 0$.

7. Money

Let the demand for money be a function of national money income

and the nominal rate of interest:

(24) $$D = mYr^{\mu}$$

where $\mu < 0$, and $m > 0$.

Money-market equilibrium requires the supply of money to equal the demand for it:

(25) $$M = D$$

Let us now solve our model for growth rates as well as for levels at an instant of time.

IV. STEADY-STATE EQUILIBRIUM GROWTH-RATE SOLUTIONS

In our derivation of investment demand in Sec. III, 3 above, entrepreneurs were using a stationary nominal rate of interest r as a discount rate and expecting price and the physical marginal productivity of capital stock to be growing at stationary rates g_{Pj} and $g_{\kappa ij}$. Are such expectations self-fulfilling? In other words, may the system display steady-state equilibrium growth? It may. By taking derivatives with respect to time of all equations involving the variables C, D, I, κ, L, M, P, r, ρ, S, w, X, and Y the reader may convince himself that the system (1) through (25) is satisfied by the following steady-state equilibrium growth-rate solutions:

(26) $$g_{Ci} = g_{Xi}$$

(27) $$g_D = g_M$$

(28) $$g_{Iij} = g_{Xi}$$

(29) $$g_{\kappa ij} = g_{Xj} - g_{Sij}$$

(30) $$g_{Li} = g_F$$

(31) $$g_M = g_Y$$

(32) $$g_{P1} = g_w - \frac{(1 - \beta_{22})g_{a1} + \beta_{21}g_{a2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}$$

(33) $$g_{P2} = g_w - \frac{(1 - \beta_{11})g_{a2} + \beta_{12}g_{a1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}}$$

(34) $$g_r = 0$$

(35) $$g_{\rho ij} = 0$$

(36) $$g_{Sij} = g_{Xi}$$

(37) $$g_w = p(1 - \lambda)^{\pi}$$

(38) $$g_{X1} = \frac{(1 - \beta_{22})g_{a1} + \beta_{21}g_{a2}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_F$$

(39) $$g_{X2} = \frac{(1 - \beta_{11})g_{a2} + \beta_{12}g_{a1}}{(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}} + g_F$$

(40) $$g_Y = g_F + g_w$$

Are (26) through (40) meaningful? Use our assumptions $\alpha_1 + \beta_{11} + \beta_{21} = 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ to show that the denominator of (32), (33), (38), and (39) equals $\alpha_1\alpha_2 + \alpha_1\beta_{12} + \alpha_2\beta_{21}$. The denominator of (37) is $1 - \lambda$. Neither can be zero, hence all solutions are meaningful.

Is growth balanced? Use our assumptions $\alpha_1 + \beta_{11} + \beta_{21}$

$= 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ upon (38) and (39) to see that

(41) $$g_{X1} \gtreqless g_{X2} \text{ if } g_{a1}/g_{a2} \gtreqless \alpha_1/\alpha_2$$

Growth may be balanced, then, but only by an odd piece of luck. And now let us solve for levels at an instant of time.

V. STEADY-STATE EQUILIBRIUM LEVEL SOLUTIONS

1. Rates of Interest

Into desired investment (16) insert the solution (36). Then insert desired investment (16) and desired consumption (19) and (20) into the good-market equilibrium condition (21), multiply the condition for the first good by P_1 and that for

the second by P_2, rearrange and find

(42) $(1 - c_1 - \beta_{11}g_{X1}/\rho_{11})P_1X_1 = (c_1 + \beta_{12}g_{X1}/\rho_{12})P_2X_2$

(43) $(c_2 + \beta_{21}g_{X2}/\rho_{21})P_1X_1 = (1 - c_2 - \beta_{22}g_{X2}/\rho_{22})P_2X_2$

Into the definition (12) insert (29) and (36). Use (32), (33), (38), and (39) to realize that $g_{P1} + g_{X1} = g_{P2} + g_{X2}$. Use that result to find

(44) $$\rho_{ij} = r - g_{Pi}$$

hence $\rho_{11} = \rho_{12} = r - g_{P1}$, and $\rho_{21} = \rho_{22} = r - g_{P2}$. So in (42) and (43) replace ρ_{12} by ρ_{11} and ρ_{21} by ρ_{22}, respectively. Divide (42) by (43), getting rid of P_jX_j, and find the nonlinear equation

(45) $$\rho_{11}\rho_{22} - J_1\rho_{11} - J_2\rho_{22} + K = 0$$

where

$$J_1 \equiv [\beta_{22}(1 - c_1) + \beta_{21}c_1]g_{X2}/(1 - c)$$

$$J_2 \equiv [\beta_{11}(1 - c_2) + \beta_{12}c_2]g_{X1}/(1 - c)$$

$$K \equiv (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})g_{X1}g_{X2}/(1 - c)$$

and where, according to (17), $c \equiv c_1 + c_2$. Use (44) to write

(46) $$\rho_{22} = \rho_{11} + g_{P1} - g_{P2}$$

Insert (46) into (45) and solve the latter for ρ_{11} and ρ_{22}:

$$(47) \qquad \rho_{11} = \frac{J_1 + J_2 - (g_{P1} - g_{P2})}{2}$$

$$\pm \{[\frac{J_1 + J_2 - (g_{P1} - g_{P2})}{2}]^2 + J_2(g_{P1} - g_{P2}) - K\}^{\frac{1}{2}}$$

$$(48) \qquad \rho_{22} = \frac{J_1 + J_2 + g_{P1} - g_{P2}}{2}$$

$$\pm \{[\frac{J_1 + J_2 + g_{P1} - g_{P2}}{2}]^2 - J_1(g_{P1} - g_{P2}) - K\}^{\frac{1}{2}}$$

where g_{P1} and g_{P2} stand for the solutions (32) and (33). The roots of a quadratic equation may be real or complex and may be positive and/or nonpositive. In the cases of (47) and (48) which will it be? Constraints upon the parameters β_{ij} and c

already imposed guarantee that $J_i > 0$ and $K \leq 0$. Consider four possibilities.

First, if $g_{P1} < g_{P2}$ then in (48) $- J_1(g_{P1} - g_{P2}) - K > 0$. Consequently the brace of (48) will be positive, and the absolute value of the square root in (48) will be greater than $(J_1 + J_2 + g_{P1} - g_{P2})/2$ regardless of the sign of the latter. As a result, (48) will have one positive and one negative root, and both will be real.

Second, if $g_{P1} = g_{P2}$ and $K < 0$ the same will be true.

Third, if $g_{P1} = g_{P2}$ and $K = 0$ then (47) and (48) will be identical and will have the positive root $J_1 + J_2$ and the root zero.

Fourth, if $g_{P1} > g_{P2}$ then in (47) $J_2(g_{P1} - g_{P2}) - K > 0$. Consequently the brace of (47) will be positive, and the absolute value of the square root in (47) will be greater than $[J_1 + J_2 - (g_{P1} - g_{P2})]/2$ regardless of the sign of the latter. As a result, (47) will have one positive and one negative root, and both will be real.

All roots (47) and (48) are meaningful: We are dividing by 2 and 1 - c only, and neither can be zero. Our four possibilities exhaust the universe. Each generates a nonpositive root to be rejected, because it violates the constraint $\rho_{ij} >$ 0 under which the integral (9) was taken. But if one side of (46) is single-valued, the other must be. Consequently, as constrained, the system has one and only one positive root for every ρ_{ij}. That root is real and meaningful. Once that root has been found, (44) determines the nominal rate of interest r.

Derivation with respect to time of (44), (47), and (48) will show that they are stationary, as (34) and (35) say.

2. *Employment*

Write (42) and (43) as

$$(49) \quad \frac{P_1 X_1}{P_2 X_2} = H \equiv \frac{c_1 + \beta_{12} g_{X1}/\rho_{12}}{1 - c_1 - \beta_{11} g_{X1}/\rho_{11}} = \frac{1 - c_2 - \beta_{22} g_{X2}/\rho_{22}}{c_2 + \beta_{21} g_{X2}/\rho_{21}}$$

where g_{Xi} stands for (38) and (39) and ρ_{ij} for (47) and (48). Use (5) and (49) to write $L_1/L_2 = H\alpha_1/\alpha_2$, insert that into (22), and find the solutions for employment:

(50) $$L_1 = H\alpha_1\lambda F/(H\alpha_1 + \alpha_2)$$

(51) $$L_2 = \alpha_2\lambda F/(H\alpha_1 + \alpha_2)$$

Derivation with respect to time of (50) and (51) will show that the latter are indeed growing at the proportionate rate (30).

3. *Output*

Insert (49) into desired capital stock (15) and find

(15a) $$S_{11} = \beta_{11}X_1/\rho_{11}$$

(15b) $$S_{12} = \beta_{12}X_1/(H\rho_{12})$$

(15c) $$S_{21} = \beta_{21}HX_2/\rho_{21}$$

(15d) $$S_{22} = \beta_{22}X_2/\rho_{22}$$

which are indeed growing the the proportionate rate (36). Insert (15a) through (15d) into the production functions (3) and (4) and find the solutions for output

(52) $$X_1 = (G_1^{1 - \beta_{22}} G_2^{\beta_{21}})^{1/[(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}]}$$

(53) $$X_2 = (G_1^{\beta_{12}} G_2^{1 - \beta_{11}})^{1/[(1 - \beta_{11})(1 - \beta_{22}) - \beta_{12}\beta_{21}]}$$

where

$$G_1 \equiv a_1 L_1^{\alpha_1} (\beta_{11}/\rho_{11})^{\beta_{11}} (\beta_{21} H/\rho_{21})^{\beta_{21}}$$

$$G_2 \equiv a_2 L_2^{\alpha_2} (\beta_{22}/\rho_{22})^{\beta_{22}} [\beta_{12}/(H\rho_{12})]^{\beta_{12}}$$

and where H stands for (49), L_i stands for (50) and (51), and ρ_{ij} for (47) and (48). Derivation with respect to time of (52) and (53) will show that the latter are indeed growing at the proportionate rates (38) and (39), respectively.

4. *Relative Prices*

Write (49) as a solution for relative price

(54) $$P_1/P_2 = HX_2/X_1$$

where X_i stands for (52) and (53). Derivation with respect to

time of (54) will show that the latter is consistent with (32), (33), (38), and (39).

5. *Real Wage Rate*

Equation (5) will be a solution for the real wage rate w/P_i if L_i stands for (50) and (51) and X_i for (52) and (53). Derivation with respect to time of (5) will show that the latter is consistent with (30), (32), (33), (38), and (39).

6. *National Money Income*

Write (5) as $P_i X_i = wL_i/\alpha_i$, insert (50) and (51), insert result into (18), and find national money income

(55) $$Y = w\lambda F(1 + H)/(H\alpha_1 + \alpha_2)$$

Derivation with respect to time of (55) will show that it is growing at the proportionate rate (40).

7. Required Money Supply

To the monetary authorities the money supply is a decision variable. Solving for it means finding the value of it required to uphold steady-state equilibrium growth at a given money wage rate w and a given employment fraction λ. Insert (24) into (25) and find required money supply

$$M = mYr^{\mu} \tag{56}$$

where r stands for (44) and Y for (55). Derivation with respect to time of (56) will show that it is growing at the proportionate rate (31).

Equation (56) concludes our solving. We have solved our model for growth rates and levels alike. We are now ready to study its properties.

VI. PROPERTIES OF SOLUTIONS

1. *Underemployment Steady-State Equilibrium Growth——A Dynamic Analogy to Keynes*

Since no right-hand side of our growth-rate solutions (26) through (40) is a function of time, our solutions are steady -state growth solutions. Since goods-market and money-market equilibrium conditions (21) and (25) are satisfied, our solutions (26) through (56) are equilibrium solutions.

But our steady-state equilibrium growth solutions are underemployment solutions, found under the assumption that $0 < \lambda < 1$, and λ was not a function of time. We have not solved for λ and cannot: The system has infinitely many solutions, one for each value of λ. In other words, nothing keeps employment from being less than full and staying so. The expect-

ations of entrepreneurs and consumers are still self-fulfilling. We have a neoclassical dynamic analogy to the celebrated Keynesian static underemployment equilibrium.

2. *What Difference Does the Employment Fraction λ Make?*

The employment fraction λ makes a difference, both for certain levels and certain growth rates.

Beginning with levels, we find five physical quantities being neatly in direct proportion to λ. First, no λ enters into the definition (49) of H, consequently our solutions (50) and (51) for employment L_i are in direct proportion to λ. Second, insert those solutions as well as our assumptions $\alpha_1 + \beta_{11} + \beta_{21} = 1$ and $\alpha_2 + \beta_{12} + \beta_{22} = 1$ into our solutions (52) and (53) for output and find the employment fraction λ entering as a factor raised to the power one. Consequently physical output X_i is in direct proportion to λ.

If employment L_i and physical output X_i are both in direct proportion to λ, then in our solutions (5) for the real wage rate w/P_i and (54) for relative price P_1/P_2 λ cancels in numerator and denominator. According to (32) and (33) g_w and with it λ cancels in the difference $g_{P1} - g_{P2}$ entering our solutions (47) and (48) for the real rate of interest ρ_{ij}. Real wage rate, relative price, and the real rate of interest are independent of λ! But then, third, fourth, and fifth, according to (15), (16), (19), and (20), desired physical capital stock S_{ij}, investment I_{ij}, and consumption C_i are in direct proportion to X_i, hence to λ.

Turning to growth rates we find g_w and with it λ to be absent from the growth-rate solutions for the five physical quantities C_i, I_{ij}, L_i, S_{ij}, and X_i. They are also absent from the growth rate $g_w - g_{Pi}$ of the real wage rate w/P_i and from the growth rate $g_{P1} - g_{P2}$ of relative price P_1/P_2: Our solutions (32) and (33) can express those differences in terms containing no λ.[1] But our growth-rate solutions for money supply M, prices P_i, and national money income Y do include g_w which, according to

(37), is the higher, the higher is λ. Then according to (44), (47), and (48) the nominal but not the real rate of interest is the higher, the higher is λ.

VII. SWITCHING FROM ONE STEADY-STATE TRACK TO ANOTHER

1. *Nonsteady-State Nonequilibrium Growth*

Once settled on a steady-state equilibrium growth track, the economy will stay on it. Could it be switched from a low-λ steady-state equilibrium growth track to a high-λ one? Let the monetary authorities try to switch it as follows. Fig. 1 shows the time curves of the nominal and the real rate of interest r and ρ_{ij}. In accordance with (44) the vertical distance between the two curves is the rate of growth g_{Pi} of price. Before Time 1 and after Time 2 let there be steady-state

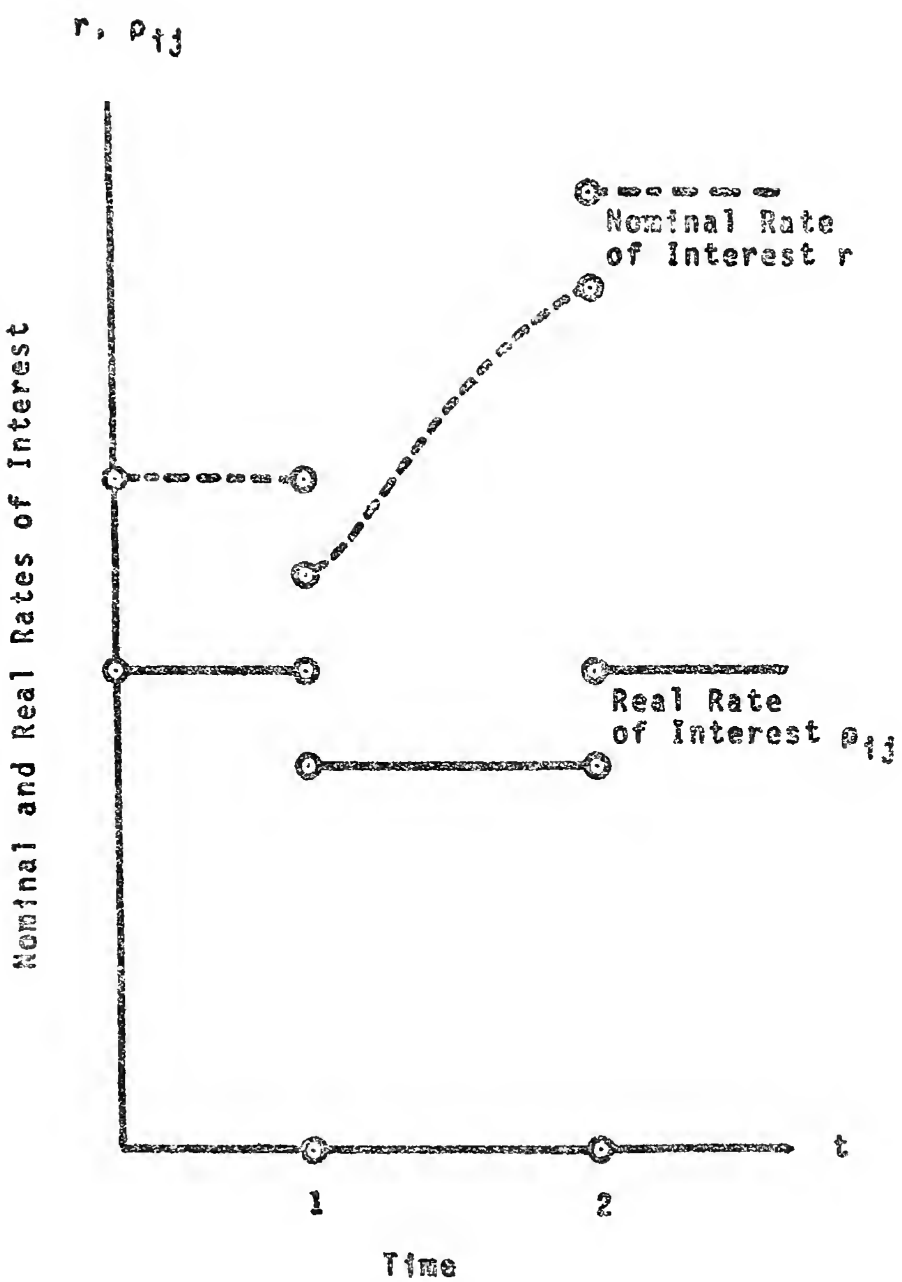

Fig. 1. Time Paths of Nominal and Real Rates of Interest

equilibrium growth. Before Time 1 λ is low, after Time 2 λ is higher, hence g_{Pi} is higher, and the gap between r and ρ_{ij} higher. At Time 1 let the monetary authorities raise the money supply M beyond its equilibrium level (56), At a so far unchanged inflationary expectation g_{Pi}, the nominal and real rate of interest r and ρ_{ij} will then fall below their equilibrium levels (44), (47), and (48).

On the resulting nonsteady-state nonequilibrium growth our model is silent and must be: Our integral (9) can be taken only if entrepreneurs are using a stationary nominal rate of interest r as a discount rate and expecting price and physical marginal productivity of capital stock to be growing at stationary rates g_{Pj} and g_{Kij}.

Little may be said, then, about what will happen once the economy has been derailed, as it were. But five things may be said about the forces released by the derailment.

2. *Excess Demand Will Emerge*

The first thing we may say is that reducing ρ_{ij} will generate excess demand: In (16) we found an entrepreneur's desired investment I_{ij} to be in inverse proportion to ρ_{ij}. As a result, in the equilibrium condition (21) the right-hand side, demand, is up while the left-hand side, supply, remains the same: There is excess demand.

3. *Excess Demand Will Stimulate Output and Employment*

Excess demand is a signal to expand output, and entrepreneurs would like to heed the signal. Can they? Write (5) as $L_i = \alpha_i P_i X_i / w$. The second thing we may say is that expanding physical output X_i will expand employment L_i in direct proportion—which is feasible because of unemployment.

4. *Expanding Output Will Not Eliminate Excess Demand*

Let entrepreneurs heed the signal. According to (16), (19), and (20) desired physical investment I_{ij} and consumption C_i are in direct proportion to physical output X_i. The third thing we may say, then, is that in the equilibrium condition (21) the right-hand side, demand, and the left-hand side, supply, are expanding in the same proportion, hence their difference, excess demand, is expanding in the same proportion.

5. *Price Adjustments Will Not Eliminate Excess Demand*

But could not appropriate price changes eliminate excess demand? Take a closer look at (16), (19), and (20) and write them out as follows:

(16a) $$I_{11} = g_{S11}\beta_{11}X_1/\rho_{11}$$

(16b) $$I_{12} = g_{S12}\beta_{12}P_2X_2/(\rho_{12}P_1)$$

(16c) $$I_{21} = g_{S21}\beta_{21}P_1X_1/(\rho_{21}P_2)$$

(16d) $$I_{22} = g_{S22}\beta_{22}X_2/\rho_{22}$$

(19) $$C_1 = c_1(X_1 + P_2X_2/P_1)$$

(20) $$C_2 = c_2(P_1X_1/P_2 + X_2)$$

No price change could pare down I_{11} and I_{22}, for no prices appear in them. The only price change which could pare down I_{12} and C_1 would be a reduction of the ratio P_2/P_1. But that ratio cannot be reduced without increasing its reciprocal P_1/P_2, and such an increase would expand I_{21} and C_2. The fourth thing we may say is that no change in relative prices will eliminate excess demand. If absolute prices changed in the same proportion, relative prices would remain unchanged. The fifth thing

we may say is that no change in absolute prices will eliminate excess demand.

6. Restoring Steady-State Growth Equilibrium?

The one thing which could eliminate excess demand would be the restoration of the equilibrium levels (47) and (48) of the real rate of interest ρ_{ij}. At Time 2 in Fig. 1 let the monetary authorities restore them. Won't the economy then relapse into its old low-λ track? Doesn't its capital stock then look too large? It doesn't. The higher desired capital coefficient S_{ij}/X_j——encouraged according to (15) by the low ρ_{ij}——hasn't materialized! Every individual entrepreneur wanted to raise his capital coefficient but, under excess demand, was able to do so only at the expense of others.

The economy may find its way, then, into a high-λ steady-state equilibrium growth track. Will it? Rigorously we don't know: Our model is silent on anything else than steady-state equilibrium growth.

VIII. CONCLUSION

We have built a multi-sector neoclassical model of steady-state imbalanced growth equilibria. Our model has infinitely many solutions, one for each value of the employment fraction λ.

Our model is Wicksellian——but only in the sense that it determines an interest rate at which saving equals investment. Unlike a Wicksellian model it allows for unemployment and distinguishes between a nominal and a real rate of interest. Its infinitely many solutions, one for each value of the employment fraction λ, all have the same real but a different nominal rate of interest.

Our model is Keynesian——but only in the sense that it may generate underemployment equilibria. Unlike the static Keynesian model it traces the time paths of such equilibria.

Our model is monetarist——but only in the sense that it distinguishes between a nominal and a real rate of interest. Its infinitely many solutions, one for each value of the employment fraction λ, all have the same real but a different money wage rate. Indeed, all have the same <u>level</u> w/P_i of the real wage rate as well as

the same <u>rate of growth</u> $g_w - g_{P_i}$ of it. In that sense any value of the employment fraction λ is a Friedmanian "natural rate of unemployment". But if so, the model has scope for employment policy. Raising employment might require a monetary policy permitting both interest rates to deviate temporarily from their steady-state equilibrium solutions.

APPENDIX

SECOND-ORDER CONDITIONS FOR A MAXIMUM N_j ARE SATISFIED

Insert (6) into (13), take the derivatives of n_{ij} defined by (13) as ordered by the Hessian (8), and write the latter as

$$\begin{vmatrix} \beta_{1j}(\beta_{1j} - 1)\dfrac{P_jX_j}{\rho_{1j}S_{1j}^2} & \beta_{1j}\beta_{2j}\dfrac{P_jX_j}{\rho_{1j}S_{1j}S_{2j}} \\ \beta_{1j}\beta_{2j}\dfrac{P_jX_j}{\rho_{2j}S_{1j}S_{2j}} & \beta_{2j}(\beta_{2j} - 1)\dfrac{P_jX_j}{\rho_{2j}S_{2j}^2} \end{vmatrix}$$

$$= \beta_{1j}\beta_{2j}\frac{P_j^2X_j^2}{\rho_{1j}\rho_{2j}S_{1j}^2S_{2j}^2}\begin{vmatrix} \beta_{1j} - 1 & \beta_{2j} \\ \beta_{1j} & \beta_{2j} - 1 \end{vmatrix}$$

Use our assumptions $\alpha_j + \beta_{1j} + \beta_{2j} = 1$ to see that the value of the last determinant is α_j. So the Hessian is positive. Since $\beta_{1j} - 1 < 0$, the principal minor of the Hessian is negative.

F O O T N O T E S

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[1] In his 1967 presidential address, delivered at the eightieth annual meeting of the American Economic Association, Friedman defined a "natural rate of unemployment" as one at which "real wage rates are tending on the average to rise at a 'normal' secular rate, i. e., at a rate that can be indefinitely maintained so long as capital formation, technological improvements, etc., remain on their long-run trends."

In our steady-state equilibrium growth model the real wage rate is always rising like that. But it can rise like that for any value of the employment fraction λ: We have found λ cancelling in the level w/P_i of the real wage rate as well as in its rate of growth $g_w - g_{Pi}$. Any value of λ, then, is a Friedmanian "natural rate of unemployment". Friedman's rate is not unique!

END

Including equations, diagrams, spaces the length of the paper is the equivalent of 6,400 words.