Irving Stringham

Greenleaf's Mathematical Series.

UNIVERSITY ALGEBRA.

DESIGNED FOR THE USE OF SCHOOLS
AND COLLEGES.

PREPARED BY

WEBSTER WELLS, S. B.,
ASSISTANT PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY.

LEACH, SHEWELL, AND SANBORN,
BOSTON AND NEW YORK.

PREFACE.

THIS work was designed to take the place of Greenleaf's Higher Algebra, portions of which have been used in the preparation of the present volume. It contains the topics usually taught in High Schools and Colleges, and the author's aim has been to present the subject in a compact form and in clear and concise language. The principles have been developed with regard to logical accuracy, and care has been given to the selection of examples and practical illustrations which should exercise the student in all the common applications of the algebraic analysis. The full treatment given in the earlier chapters renders the previous study of a more elementary text-book unnecessary.

Attention is invited to the following chapters, including those in which the most important changes have been made in the Higher Algebra : —

Parentheses.
Factoring.
Zero and Infinity.
Theory of Exponents.
Simultaneous Equations involving Quadratics.
Binomial Theorem for Positive Integral Exponents.
Undetermined Coefficients.
Logarithms.

The answers have been put by themselves in the back part of the book, and those have been omitted which, if given, would destroy the utility of the problem. The examples are over eighteen hundred in number, and are progressive, commencing with simple applications of the rules, and passing gradually to those which require some thought for their solution.

The works of Todhunter and Hamblin Smith, and other standard volumes, have been consulted in the preparation of the work, and have furnished a number of examples and problems. The author has also received numerous suggestions from practical teachers, to whom he would here express his thanks.

WEBSTER WELLS.

Boston, 1884.

UNIVERSITY ALGEBRA.

CONTENTS.

ALGEBRA.

I.—DEFINITIONS AND NOTATION.

1. **Quantity** is anything that can be measured; as distance, time, weight, and number.

2. The **Measurement** of quantity is accomplished by finding the number of times it contains another quantity of the same kind, assumed as a standard. This standard is called the *unit of measure*.

3. **Mathematics** is the science of quantities and their relations.

4. **Algebra** is that branch of mathematics in which the relations of quantities are investigated, and the reasoning abridged and generalized, by means of symbols.

5. The **Symbols** employed in Algebra are of four kinds: symbols of quantity, symbols of operation, symbols of relation, and symbols of abbreviation.

SYMBOLS OF QUANTITY.

6. The **Symbols of Quantity** generally used are the *figures* of Arithmetic and the *letters* of the alphabet.

The *figures* are used to represent known quantities and determined values, and the *letters* any quantities whatever, known or unknown.

7. **Known Quantities**, or those whose values are given, when not expressed by figures, are usually represented by the first letters of the alphabet, as a, b, c.

8. Unknown Quantities, or those whose values are not given, are usually represented by the last letters of the alphabet, as x, y, z.

9. Zero, or the absence of quantity, is represented by the symbol 0.

10. Quantities occupying similar relations in different operations are often represented by the same letter, distinguished by different *accents*, as a', a'', a''', read "a prime," "a second," "a third," etc.; or by different *subscript figures*, as a_1, a_2, a_3, read "a one," "a two," "a three," etc.

SYMBOLS OF OPERATION.

11. The **Symbols of Operation** are certain *signs* or *characters* used to indicate algebraic operations.

12. The **Sign of Addition,** $+$, is called "*plus.*" Thus, $a + b$, read "a plus b," indicates that the quantity b is to be added to the quantity a.

13. The **Sign of Subtraction,** $-$, is called "*minus.*" Thus, $a - b$, read "a minus b," indicates that the quantity b is to be subtracted from the quantity a.

The sign $\sim$ indicates the *difference* of two quantities when it is not known which of them is the greater. Thus, $a \sim b$ indicates the difference of the two quantities a and b.

14. The **Sign of Multiplication,** $\times$, is read "*times,*" "*into,*" or "*multiplied by.*" Thus, $a \times b$ indicates that the quantity a is multiplied by the quantity b.

A simple point (.) is sometimes used in place of the sign $\times$. The sign of multiplication is, however, usually omitted, except between two arithmetical figures separated by no other sign; multiplication is therefore indicated by the absence of any sign. Thus, $2ab$ indicates the same as $2 \times a \times b$, or $2 . a . b$.

15. The quantities multiplied are called *factors*, and the result of the multiplication is called the *product*.

16. The **Sign of Division**, $\div$, is read "*divided by.*" Thus, $a \div b$ indicates that the quantity a is divided by the quantity b.

Division is otherwise often indicated by writing the dividend above, and the divisor below, a horizontal line. Thus, $\frac{a}{b}$ indicates the same as $a \div b$. Also, the sign of division may be replaced in an operation by a straight or curved line. Thus, $a \underline{|b}$, or $b\,)\,a$, indicates the same as $a \div b$.

17. The **Exponential Sign** is a figure or letter written at the right of and above a quantity, to indicate the number of times the quantity is taken as a factor. Thus, in x^3, the ^{3} indicates that x is taken three times as a factor; that is, x^3 is equivalent to xxx.

The product obtained by taking a factor two or more times is called a *power*. A single letter is also often called the *first* power of that letter. Thus,

a^2 is read "a to the *second* power," or "a square," and indicates aa;

a^3 is read "a to the *third* power," or "a cube," and indicates aaa;

a^4 is read "a to the *fourth* power," or "a fourth," and indicates $aaaa$;

a^n is read "a to the nth power," or "a nth," and indicates aaa etc., to n factors.

The figures or letters used to indicate powers are called *exponents*; and when no exponent is written, the *first* power is understood. Thus, a is equivalent to a^1.

The *root* of a quantity is one of its equal factors. Thus, the root of a^2, a^3, or a^4 is a.

18. The **Radical Sign**, $\sqrt{}$, when prefixed to a quantity, indicates that some root of the quantity is to be extracted.

Thus,

$\sqrt[2]{a}$ indicates the *second* or *square* root of a;

$\sqrt[3]{a}$ indicates the *third* or *cube* root of a;

$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The *index* of the root is the figure or letter written over the radical sign. Thus, ^{2} is the index of the square root, ^{3} of the cube root; and so on.

When the radical sign has no index written over it, the index ^{2} is understood. Thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

SYMBOLS OF RELATION.

19. The **Symbols of Relation** are signs used to indicate the relative magnitudes of quantities.

20. The **Sign of Equality,** $=$, read "*equals*," or "*equal to*," indicates that the quantities between which it is placed are equal. Thus, $x = y$ indicates that the quantity x is equal to the quantity y.

A statement that two quantities are equal is called an *equation*. Thus, $x + 4 = 2x - 1$ is an equation, and is read "x plus 4 equals $2x$ minus 1."

21. The **Sign of Ratio,** $:$, read "*to*," indicates that the two quantities between which it is placed are taken as the terms of a ratio. Thus, $a : b$ indicates the ratio of the quantity a to the quantity b, and is read "the ratio of a to b."

A *proportion*, or an *equality of ratios*, is expressed by writing the sign $=$, or the sign $::$, between equal ratios. Thus, $30 : 6 = 25 : 5$ indicates that the ratio of 30 to 6 is equal to the ratio of 25 to 5, and is read "30 is to 6 as 25 is to 5."

22. The **Sign of Inequality,** $>$ or $<$, read "*is greater than*," or "*is less than*," respectively, when placed between two quantities, indicates that the quantity toward which the opening of the sign turns is the greater. Thus, $x > y$ is read "x is greater than y"; $x - 6 < y$ is read "x minus 6 is less than y."

23. The **Sign of Variation,** $\propto$, read "*varies as,*" indicates that the two quantities between which it is placed increase or diminish together, in the same ratio. Thus, $a \propto \frac{c}{d}$ is read "a varies as c divided by d."

SYMBOLS OF ABBREVIATION.

24. The **Signs of Deduction,** $\therefore$ and $\because$, stand the one for *therefore* or *hence,* the other for *since* or *because.*

25. The **Signs of Aggregation,** the *vinculum* ——, the *bar* |, the *parenthesis* ( ), the *brackets* [ ], and the *braces* { }, indicate that the quantities connected or enclosed by them are to be subjected to the *same* operations. Thus,

$$\overline{a+b} \times x, \quad \left.\begin{matrix} a \\ b \end{matrix}\right| x, \quad (a+b)\,x, \quad [a+b]\,x, \quad \{a+b\}\,x,$$

all indicate that the quantity $a+b$ is to be multiplied by x.

26. The **Sign of Continuation,** ......, stands for *and so on,* or *continued by the same law.* Thus,

$$a,\ a+b,\ \ a+2\,b,\ \ a+3\,b,\ \ldots\ldots \text{ is read}$$

"a, a plus b, a plus $2\,b$, a plus $3\,b$, and so on."

ALGEBRAIC EXPRESSIONS.

27. An **Algebraic Expression** is any combination of algebraic symbols.

28. A **Coefficient** of a quantity is a figure or letter prefixed to it, to show how many times the quantity is to be taken. Thus, in $4\,a$, 4 is the coefficient of a, and indicates that a is taken four times, or $a+a+a+a$. Where any number of quantities are multiplied together, the product of any of them may be regarded as the coefficient of the product of the others; thus, in $a\,b\,c\,d$, $a\,b$ is the coefficient of $c\,d$, b of $a\,c\,d$, $a\,b\,d$ of c, and so on.

When no coefficient of a quantity is written, 1 is understood to be the coefficient. Thus, a is the same as $1\,a$, and $x\,y$ is the same as $1\,x\,y$.

29. The **Terms** of an algebraic expression are its parts connected by the signs + or −. Thus,

a and b are the terms of the expression $a+b$;

$2\,a$, b^2, and $-2\,a\,c$, of the expression $2\,a+b^2-2\,a\,c$.

30. The **Degree** of a term is the number of literal factors which it contains. Thus,

$2\,a$ is of the *first* degree, as it contains but *one* literal factor.

$a\,b$ is of the *second* degree, as it contains *two* literal factors.

$3\,a\,b^2$ is of the *third* degree, as it contains *three* literal factors.

The degree of any term is determined by adding the exponents of its several letters. Thus, $a\,b^2\,c^3$ is of the *sixth* degree.

31. **Positive Terms** are those preceded by a *plus* sign; as,

$$+2\,a, \text{ or } +a\,b^2.$$

When a term has no sign written, it is understood to be positive. Thus, a is the same as $+a$.

Negative Terms are those preceded by a *minus* sign; as,

$$-3\,a, \text{ or } -b\,c.$$

This sign can never be omitted.

32. In a positive term, the coefficient indicates how many times the quantity is taken *additively* (Art. 28); in a negative term, the coefficient indicates how many times the quantity is taken *subtractively*. Thus,

$+2\,x$ is the same as $+x+x$;

$-3\,a$ is the same as $-a-a-a$.

33. If the same quantity be both added to and subtracted from another, the value of the latter will not be changed; hence if any quantity b be added to any other quantity a, and b be subtracted from the result, the remainder will be a; that is,

$$a + b - b = a.$$

Consequently, equal terms affected by unlike signs, in an expression, neutralize each other, or *cancel*.

34. **Similar** or **Like Terms** are those which differ only in their numerical coefficients. Thus,

$2xy^2$ and $-7xy^2$ are similar terms.

Dissimilar or **Unlike Terms** are those which are not similar. Thus,

bx^2y and bxy^2 are dissimilar terms.

35. A **Monomial** is an algebraic expression consisting of only one term; as, $5a$, $7ab$, or $3b^2c$.

A monomial is sometimes called a *simple quantity*.

36. A **Polynomial** is an algebraic expression consisting of more than one term; as, $a + b$, or $3a^2 + b - 5b^3$.

A polynomial is sometimes called a *compound quantity*, or a *multinomial*.

37. A **Binomial** is a polynomial of two terms; as,

$$a - b,\ 2a + b^2, \text{ or } 3ac^2 - b.$$

A binomial whose second term is negative, as $a - b$, is sometimes called a *residual*.

38. A **Trinomial** is a polynomial of three terms; as,

$$a + b + c, \text{ or } ab + c^2 - b^3.$$

39. **Homogeneous Terms** are those of the same degree; as,

$$a^2,\ 3bc, \text{ and } -4x^2.$$

40. A polynomial is homogeneous when all its terms are homogeneous; as, $a^3 + 2abc - 3b^3$.

41. A polynomial is said to be *arranged* according to the *decreasing* powers of any letter, when the term having the highest exponent of that letter is placed first, that having the next lower immediately after, and so on. Thus,

$$a^3 + 3a^2b + 3ab^2 + b^3$$

is arranged according to the decreasing powers of a.

A polynomial is said to be *arranged* according to the *increasing* powers of any letter, when the term having the lowest exponent of that letter is placed first, that having the next higher immediately after, and so on. Thus,

$$a^3 + 3a^2b + 3ab^2 + b^3$$

is arranged according to the increasing powers of b.

42. The **Reciprocal** of a quantity is 1 divided by that quantity. Thus, the reciprocal of

$$a \text{ is } \frac{1}{a}, \text{ and of } x + y \text{ is } \frac{1}{x+y}.$$

43. The **Interpretation** of an algebraic expression consists in rendering it into an arithmetical quantity, by means of the numerical values assigned to its letters. The result is called the *numerical value* of the expression.

Thus, the numerical value of

$$4a + 3bc - d$$

when $a = 4$, $b = 3$, $c = 5$, and $d = 2$, is

$$4 \times 4 + 3 \times 3 \times 5 - 2 = 16 + 45 - 2 = 59.$$

AXIOMS.

44. An **Axiom** is a self-evident truth.

Algebraic operations are based upon definitions, and the following axioms:—

1. If the same quantity, or equal quantities, be *added* to equal quantities, the *sums* will be equal.

2. If the same quantity, or equal quantities, be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* by the same quantity, or by equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same quantity, or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be changed.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be changed.

7. Quantities which are equal to the same quantity are equal to each other.

8. Like powers and like roots of equal quantities are equal.

9. The whole of a quantity is equal to the sum of all its parts.

EXERCISES ON THE PRECEDING DEFINITIONS AND PRINCIPLES.

45. Translate the following algebraic expressions into ordinary language:

1. $3a^2 + bc - \frac{d}{3}$.

2. $4m \sim \frac{x}{y}$.

3. $\sqrt[3]{a+b} = \sqrt{a^2 - c}$.

4. $mn > pq$.

5. $cd : \frac{m}{n} = ab : \sqrt{x^3}$.

6. $(a-b)x = [c+d]y$.

7. $\{m + \overline{r-s}\}n = \frac{3a-d}{2c+b}$.

8. $\sqrt{\frac{3a}{x-y}} < (c-d)\left(h + \frac{k}{5}\right)$.

46. Put into the form of algebraic expressions the following:

1. Five times a, added to two times b.

2. Two times x, minus y to the second power.

3. The difference of x and y.

4. The product of a, b, c square, and d cube.

5. $x + y$ multiplied by $a - b$.

6. a square divided by the sum of b and c.

7. x divided by 3, increased by 2, equals three times y, diminished by 11.

8. The reciprocal of $a + b$, plus the square of a, minus the cube root of b, is equal to the square root of c.

9. The ratio of $5a$ divided by b, to d divided by c square, equals the ratio of x square y cube to y square z fourth.

10. The product of m and $a + b$ is less than the reciprocal of x cube.

11. The product of $x + y$ and $x - y$ is greater than the product of the square of $a - d$ into the cube of $a + b$.

12. The quotient of a divided by $3a - 2$ is equal to the square root of the quotient of $m + n$ divided by $2x - y^2$.

47. Find the numerical values of the following:—

When $a = 6$, $b = 5$, $c = 4$, and $d = 1$, of

1. $a^2 + 2ab - c + d$.

2. $2a^3 - 2a^2b + c^3$.

3. $2a^2 + 3bc - 5$.

4. $a^2(a + b) - 2abc$.

5. $5a^2b - 4ab^2 + 27c$.

6. $7a^2 + (a - b)(a - c)$.

When $a = 4$, $b = 2$, $c = 3$, and $d = 1$, of

7. $15a - 7(b + c) - d$.

8. $25a^2 - 7(b^2 + c^2) + d^2$.

9. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d}$.

10. $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{d^2}$.

11. $\frac{4}{3a - 3c} + \frac{8}{3}$.

12. $\frac{25a - 30c - d}{b + c}$.

When $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{5}$, and $x = 2$, of

13. $(2a + 3b + 5c)(8a + 3b - 5c)(2a - 3b + 15c)$.

14. $x^3 + \left(\frac{1}{a} + \frac{1}{b}\right) x^2 + \left(\frac{1}{b} - \frac{1}{a}\right) x + \frac{2}{b^2}.$

15. $x^4 - (2a + 3b)x^3 + (3a - 2b)x^2 - cx + bc.$

When $a = b$, and $b = \frac{1}{3}$, of

16. $\frac{5a + b - [3a - (2a - b)]}{a}.$

17. $\frac{13a + 3b + \{7(a + b) + [3a + 8(4a - b)]\}}{2a + 3b}.$

When $b = 3$, $c = 4$, $d = 6$, and $e = 2$, of

18. $\sqrt{27b} - \sqrt[3]{2c} + \sqrt{2e}.$ **19.** $\sqrt{3bc} + \sqrt[3]{9cd} - \sqrt[3]{2e^2}.$

When $a = 16$, $b = 10$, $x = 5$, and $y = 1$, of

20. $(b - x)(\sqrt{a} + b) + \sqrt{(a - b)(x + y)}.$

48. What is the coefficient of

1. x in $3n^2x$? **3.** xy in $-20m^2xyz^3$?

2. ac^3 in $ab^2c^3d^4$? **4.** m^2n^3 in $5a^3m^2xn^3$?

What is the degree of

5. $3ax$? **6.** $2m^2nx^4$? **7.** $a^2b^3c^2d^5$? **8.** $2mx^2y^3z$?

Arrange the following expressions according to the increasing powers of x:

9. $2x^2 - 3x + x^3 + 1 - 4x^4.$

10. $3xy^3 - 5x^3y + y^4 - x^4 - x^2y^2.$

Arrange the following expressions according to the decreasing powers of a:

11. $1 - a^2 - 2a + a^3 + 2a^4.$

12. $ab^3 - b^4 + a^4 - 4a^2b^2 - 3a^3b.$

NEGATIVE QUANTITIES.

49. The signs + and —, besides indicating the operations of addition and subtraction, are also used, in Algebra, to indicate the *nature* or *quality* of the quantities to which they are prefixed.

To illustrate, let us suppose a person, having a property of \$500, to lose \$150, then gain \$250, and finally to incur a debt of \$450; it is required to find the amount of his property.

Since gains have an *additive* effect on property, and debts or losses a *subtractive* effect, we may indicate these different qualities algebraically by prefixing the signs + and — to them, respectively; thus, we should represent the transactions as follows,

$$\$500 - \$150 + \$250 - \$450;$$

which reduces to \$150, the amount required.

But suppose, having a property of \$500, he incurs a debt of \$700; we should represent the transaction algebraically as follows,

$$\$500 - \$700;$$

or, as incurring a debt of \$700 is equivalent to incurring two debts, one of \$500 and the other of \$200, the transaction may be expressed thus,

$$\$500 - \$500 - \$200.$$

Now since, by Art. 33, \$500 and —\$500 neutralize each other, we have remaining the isolated negative quantity —\$200 as the algebraic representative of the required property. In Arithmetic, we should say that he owed or was in debt \$200; in Algebra, we make also the equivalent statement that his property amounts to —\$200.

In this way we can conceive the possibility of the independent existence of negative quantities; and as, in Arithmetic, losses may be added, subtracted, multiplied, etc., precisely as though they were gains, so, in Algebra, negative quantities may be added, subtracted, multiplied, etc., precisely as though they were positive.

The distinction of positive and negative quantities is applied in a great many cases in the language of every-day life and in the mathematical sciences. Thus, in the thermometer, we speak of a temperature above zero as $+$, and one below as $-$; for instance, $+25°$ means $25°$ above zero, and $-10°$ means $10°$ below zero. In navigation, north latitude is considered $+$, and south latitude $-$; longitude west of Greenwich is considered $+$, and longitude east of Greenwich $-$; for example, a place in latitude $-30°$, longitude $+95°$, would be in latitude $30°$ south of the equator, and in longitude $95°$ west of Greenwich. And, in general, when we have to consider quantities the exact reverse of each other in quality or condition, we may regard quantities of either quality or condition as positive, and those of the opposite quality or condition as negative. It is immaterial which quality we regard as positive; but having assumed at the commencement of an investigation a certain quality as positive, we must retain the same notation throughout.

The *absolute value* of a quantity is the number represented by that quantity, taken independently of the sign affecting it. Thus, 2 and -2 have the same absolute value.

But as we consider a person who owns \$2 as better off than one who owes \$2, so, in Algebra, we consider $+2$ as greater than -2; and, in general, *any positive quantity, however small, is considered greater than any negative quantity.*

Also, as we consider a person who owes \$2 as better off than one who owes \$3, so, in Algebra, we consider -2 as greater than -3; and, in general, *of two negative quantities, that is regarded as the greater which has the less number of units, or which has the smaller absolute value.*

Again, as we consider a person who has no property or debt as better off than one who is in debt, so, in Algebra, *zero is considered greater than any negative quantity.*

II. — ADDITION.

50. Addition, in Algebra, is the process of collecting two or more quantities into one equivalent expression, called the *sum.*

51. In Arithmetic, when a person incurs a debt of a certain amount, we regard his property as diminished by the amount of the debt. So, in Algebra, using the interpretation of negative quantities as given in Art. 49, *adding a negative quantity is equivalent to subtracting an equal positive quantity.* Thus, the sum of a and $-b$ is obtained by subtracting b from a, giving as a result $a-b$.

Hence, the addition of monomials is indicated by *uniting the quantities with their respective signs.* Thus, the sum of $a, -b, c, d, -e$, and $-f$, is

$$a-b+c+d-e-f.$$

The addition of polynomials is indicated by enclosing them in parentheses (Art. 25), and uniting the results with + signs. Thus, the sum of $a+b$ and $c-d$ is

$$(a+b)+(c-d).$$

52. Let it be required to add $c-d$ to $a+b$.

If we add c to $a+b$, the sum will be $a+b+c$. But we have to add to $a+b$ a quantity which is d less than c. Consequently our result is d too large. Hence the required sum will be $a+b+c$ diminished by d, or $a+b+c-d$.

Hence, the addition of polynomials may also be indicated by uniting their terms with their respective signs.

53. Let it be required to add $2a$ and $3a$.

By Art. 32, $2a=a+a$,

and $3a=a+a+a$.

Hence (Art. 52) the sum of $2a$ and $3a$ is indicated by

$$a + a + a + a + a,$$

which, by Art. 32, is equal to $5a$. Hence, $2a + 3a = 5a$.

54. Let it be required to add $-3a$ and $-2a$.

By Art. 32, $$-3a = -a - a - a,$$

and $$-2a = -a - a.$$

Hence (Art. 52), the sum of $-3a$ and $-2a$ is indicated by

$$-a - a - a - a - a,$$

or $-5a$ (Art. 32). Hence, $-3a - 2a = -5a$.

From our ideas of negative quantities (Art. 49), we may explain this result arithmetically as follows:

If a person has two debts, one of \$3 and the other of \$2, he may be considered to be in debt to the amount of \$5.

55. Let it be required to add $4a$ and $-2a$.

$$4a = a + a + a + a,$$

and $$-2a = -a - a.$$

Hence, the sum of $4a$ and $-2a$ is indicated by

$$a + a + a + a - a - a.$$

Now, by Art. 33, the third and fourth terms are neutralized by the fifth and sixth, leaving as the result $a + a$, or $2a$. Hence, $4a - 2a = 2a$.

We may explain this result arithmetically as follows:

If a person has \$4 in money, and incurs a debt of \$2, his property may be considered to amount to \$2.

56. Let it be required to add $-4a$ and $2a$.

$$-4a = -a - a - a - a,$$

and $$2a = a + a.$$

Hence, the sum of $-4a$ and $2a$ is indicated by

$$-a-a-a-a+a+a.$$

The third and fourth terms neutralize the fifth and sixth, leaving as the result $-a-a$ or $-2a$. Hence,

$$-4a+2a=-2a.$$

We may explain this result arithmetically as follows:

If a person has \$2 in money, and incurs a debt of \$4, he may be considered to be in debt to the amount of \$2.

57. From Arts. 55 and 56 we derive the following rule for the addition of two similar (Art. 34) terms of opposite sign:

To add two similar terms, the one positive and the other negative, subtract the smaller coefficient from the larger, affix to the result the common symbols, and prefix the sign of the larger.

For example, the sum of $7xy$ and $-3xy$ is $4xy$;

the sum of $3a^2b^3$ and $-11a^2b^3$ is $-8a^2b^3$.

58. In Arithmetic, when adding several quantities, it makes no difference in which order we add them; thus, $3+5+9$, $5+3+9$, $9+3+5$, etc., all give the same result, 17. So also in Algebra, it is immaterial in what order the terms are united, provided each has its proper sign. Thus, $-b+a$ is the same as $a-b$.

Hence, in adding together any number of similar terms, some positive and some negative, we may add the positive terms first, and then the negative, and finally combine these two results by the rule of Art. 57.

Thus, in finding the sum of $2a$, $-a$, $7a$, $6a$, $-4a$, and $-5a$, the sum of the positive terms $2a$, $7a$, and $6a$, is $15a$, and the sum of the negative terms $-a$, $-4a$, and $-5a$, is $-10a$; and the sum of $15a$ and $-10a$ is $5a$.

59. Let it be required to add $6a-7x$, $3x-2a+3y$, and $2x-a-mn$.

We might obtain the sum in accordance with Art. 52, by uniting the terms by their respective signs, and combining similar terms by the methods previously given. It is however customary in practice, and more convenient, to set the expressions down one underneath the other, similar terms being in the same vertical column; thus,

$$\begin{array}{rrrr} 6\,a & -\,7\,x & & \\ -\,2\,a & +\,3\,x & +\,3\,y & \\ -\,a & +\,2\,x & & -\,m\,n \\ \hline 3\,a & -\,2\,x & +\,3\,y & -\,m\,n. \end{array}$$

It should be remembered that *only similar terms can be combined by addition;* and that *the algebraic sum of dissimilar terms can only be indicated by uniting them by their respective signs.*

60. From the preceding principles and illustrations is derived the following

RULE.

To add together two or more expressions, set them down one underneath the other, similar terms being in the same vertical column. Find the sum of the similar terms, and to the result obtained unite the dissimilar terms, if any, by their respective signs.

EXAMPLES.

1.	2.	3.	4.	5.
$7\,a$	$-\,6\,m$	$13\,n$	$-\,4\,a\,x$	$2\,a^2\,b$
$3\,a$	m	n	$-\,3\,a\,x$	$-\,a^2\,b$
a	$-\,11\,m$	$-\,20\,n$	$a\,x$	$11\,a^2\,b$
$5\,a$	$-\,5\,m$	$6\,n$	$-\,7\,a\,x$	$-\,5\,a^2\,b$
$11\,a$	$-\,m$	$8\,n$	$-\,a\,x$	$4\,a^2\,b$
a	$20\,m$	$-\,n$	$12\,a\,x$	$-\,9\,a^2\,b$

6.	7.	8.
$7a - mp^2$	$2a - 3x$	$ab + cd$
$a + 6mp^2$	$-a + 4x$	$-ab + cd$
$-11a - 3mp^2$	$a + x$	$3ab - 2cd$
$8a + 11mp^2$	$5a - 7x$	$7ab - 5cd$
$-9a - 7mp^2$	$-4a - x$	$-4ab + 6cd$
$18a - 15mp^2$	$-3a + 7x$	$2ab - 5cd$

Find the sum of the following:

9. $4xyz$, $-3xyz$, $-5xyz$, $6xyz$, $-9xyz$, and $3xyz$.

10. $5mn^2 - 8x^2y$, $-mn^2 + x^2y$, $-6mn^2 - 3x^2y$, $4mn^2 + 7x^2y$, $2mn^2 + 3x^2y$, and $-mn^2 - 2x^2y$.

11. $3a^2 + 2ab + 4b^2$, $5a^2 - 8ab + b^2$, $-a^2 + 5ab - b^2$, $18a^2 - 20ab - 19b^2$, and $14a^2 - 3ab + 20b^2$.

12. $2a - 5b - c + 7$, $3b - 2 - 6a + 8c$, $c + 3a - 4$, and $1 + 2b - 5c$.

13. $6x - 3y + 7m$, $2n - x + y$, $2y - 4x - 5m$, and $m + n - y$.

14. $2a - 3b + 4d$, $2b - 3d + 4c$, $2d - 3c + 4a$, and $2c - 3a + 4b$.

15. $3x - 2y - z$, $3y - 5x - 7z$, $8z - y - x$, and $4x$.

16. $2m - 3n + 5r - t$, $2n - 6t - 3r - m$, $r + 3m - 5n + 4t$, and $3t - 2r + 7n - 4m$.

17. $4mn + 3ab - 4c$, $3x - 4ab + 2mn$, and $3m^2 - 4p$.

18. $3a + b - 10$, $c - d - a$, $-4c + 2a - 3b - 7$, and $4x^2 + 5 - 18m$.

19. $4x^3 - 5a^3 - 5ax^2 + 6a^2x$, $6a^3 + 3x^3 + 4ax^2 + 2a^2x$, $-17x^3 + 19ax^2 - 15a^2x$, and $10x^3 + 7a^2x + 5a^3 - 18ax^2$.

20. $7a - 5y^3$, $8\sqrt{x} + 2a$, $5y^3 - \sqrt{x}$, and $-9a + 7\sqrt{x}$.

21. $3ab + 3(a + b)$, $-ab + 2(a + b)$, $7ab - 4(a + b)$, and $-2ab + 6(a + b)$.

22. $7\sqrt{y} - 4(a - b)$, $6\sqrt{y} + 2(a - b)$, $2\sqrt{y} + (a - b)$, and $\sqrt{y} - 3(a - b)$.

III. — SUBTRACTION.

61. **Subtraction**, in Algebra, is the process of finding one of two quantities, when their sum and the other quantity are given.

Hence, Subtraction is the converse of Addition.

The *Minuend* is the sum of the quantities.

The *Subtrahend* is the given quantity.

The *Remainder* is the required quantity.

As the remainder is the difference between the minuend and subtrahend, subtraction may also be defined as the process of finding the difference between two quantities.

62. Subtraction may be indicated by writing the subtrahend after the minuend, with a — sign between them. Thus, the subtraction of b from a is indicated by

$$a - b.$$

In indicating subtraction in this way, the subtrahend, if a negative quantity or a polynomial, should be enclosed in a parenthesis. Thus, the subtraction of $-b$ from a is indicated by

$$a-(-b),$$

and the subtraction of $b-c$ from a by

$$a-(b-c).$$

63. Let it be required to subtract $b-c$ from a.

According to the definition of Art. 61, we are to find a quantity which when added to $b-c$ will produce a; this quantity is evidently $a-b+c$, which is the remainder required.

Now, if we had changed the sign of each term of the subtrahend, giving $-b+c$, and had *added* the resulting expression to a, we should have arrived at the same result, $a-b+c$.

Hence, *to subtract one quantity from another, we may change the sign of each term of the subtrahend, and add the result to the minuend.*

64. **1.** Let it be required to subtract $3\,a$ from $8\,a$.

According to Art. 63, the result may be obtained by adding $-3\,a$ to $8\,a$, giving $5\,a$ (Art. 55).

2. Subtract $8\,a$ from $3\,a$.

By Art. 63, the result is $3\,a - 8\,a$ or $-5\,a$ (Art. 56).

3. Subtract $-2\,a$ from $3\,a$.

Result, $3\,a + 2\,a$ or $5\,a$.

4. Subtract $3\,a$ from $-2\,a$.

Result, $-2\,a - 3\,a$ or $-5\,a$.

5. Subtract $-2\,a$ from $-5\,a$.

Result, $-5\,a + 2\,a$ or $-3\,a$.

6. Subtract $-5\,a$ from $-2\,a$.

Result, $-2\,a + 5\,a$ or $3\,a$.

65. In Arithmetic, addition always implies *augmentation*, and subtraction *diminution*. In Algebra this is not always the case; for example, in adding $-2\,a$ to $5\,a$ the sum is $3\,a$, which is smaller than $5\,a$; also, in subtracting $-2\,a$ from $5\,a$ the remainder is $7\,a$, which is larger than $5\,a$. Thus, the terms *Addition*, *Subtraction*, *Sum*, and *Remainder* have a much more general signification in Algebra than in Arithmetic.

66. From Art. 63 we derive the following

RULE.

To subtract one expression from another, set the subtrahend underneath the minuend, similar terms being in the same vertical column. Change the sign of each term of the subtrahend from + to −, or from − to +, and add the result to the minuend.

EXAMPLES.

1.	2.	3.	4.	5.
$27a$	$17x$	$-13y$	$-10mn$	$5a^2b$
$13a$	$-11x$	$4y$	$-18mn$	$14a^2b$

6.

$$\begin{array}{r} ab + cd - ax \\ 4ab - 3cd + 4ax \\ \hline \end{array}$$

7.

$$\begin{array}{r} 7x + 5y - 3a \\ x - 7y + 5a - 4 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 7abc - 11x + 5y - 48 \\ 11abc + 3x + 7y + 100 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 5\sqrt{a} - 3y^2 + 7a - 6 \\ 3\sqrt{a} + y^2 - 5a - 7 \\ \hline \end{array}$$

10. Subtract $-5b$ from $-12b$.

11. From $31x^2 - 3y^2 + ab$ take $17x^2 + 5y^2 - 4ab + 7$.

12. Subtract $a - b + c$ from $a + b - c$.

13. Subtract $6a - 3b - 5c$ from $6a + 3b - 5c + 1$.

14. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.

15. Take $4a - b + 2c - 5d$ from $d - 3b + a - c$.

16. From $m^2 + 3n^3$ take $-4m^2 - 6n^3 + 71x$.

17. From $a + b$ take $2a - 2b$ and $-a + b$.

18. From $a - b - c$ take $-a + b + c$ and $a - b + c$.

IV.—USE OF PARENTHESES.

67. The use of parentheses is very frequent in Algebra, and it is necessary to have rules for their removal or introduction.

68. Let it be required to indicate the *addition* of $3a$ and $5b-c+2d$; this we may do by placing the latter expression in a parenthesis, prefixing a + sign, and writing after the former quantity, thus:

$$3a+(5b-c+2d).$$

If the operation be performed, we obtain (Art. 60),

$$3a+5b-c+2d.$$

69. Again, let it be required to indicate the *subtraction* of $5b-c+2d$ from $3a$; this we may do by placing the former expression in a parenthesis, prefixing a − sign, and writing after the latter quantity, thus:

$$3a-(5b-c+2d).$$

If the operation be performed, we obtain (Art. 66),

$$3a-5b+c-2d.$$

70. It will be observed that in the former case the signs of the terms within the parenthesis are unchanged when the parenthesis is removed; while in the latter case the sign of each term within is changed, from + to −, or from − to +. Hence, we have the following rule for the removal of a parenthesis:

If the parenthesis is preceded by a + sign, it may be removed if the sign of every enclosed term be unchanged; and if the parenthesis is preceded by a − sign, it may be removed if the sign of every enclosed term be changed.

71. To enclose any number of terms in a parenthesis, we take the reverse of the preceding rule:

Any number of terms may be enclosed in a parenthesis, with a + sign prefixed, if the sign of every term enclosed be unchanged; and in a parenthesis, with a − sign prefixed, if the sign of every term enclosed be changed.

72. As the bracket, brace, and vinculum (Art. 25) have the same signification as the parenthesis, the rules for their removal or introduction are the same. It should be observed in the case of the vinculum, that the sign apparently prefixed to the first term underneath is in reality the sign of the vinculum; thus, $+\overline{a-b}$ signifies $+(a-b)$, and $-\overline{a-b}$ signifies $-(a-b)$.

73. Parentheses will often be found enclosing others; in this case they may be removed successively, by the preceding rule; and it is better to begin by removing the *inside* pair.

74. **1.** Remove the parentheses from $3a-(2a-5)-(-a+7)$.

Result, $3a-2a+5+a-7=2a-2.$

2. Remove the parentheses etc., from

$$6a-[3a+(2a-\{5a-[4a-\overline{a-2}]\})].$$

In accordance with Art. 73, we remove the vinculum first, and the others in succession. Thus,

$$\begin{aligned} &6a-[3a+(2a-\{5a-[4a-\overline{a-2}]\})] \\ &=6a-[3a+(2a-\{5a-[4a-a+2]\})] \\ &=6a-[3a+(2a-\{5a-4a+a-2\})] \\ &=6a-[3a+(2a-5a+4a-a+2)] \\ &=6a-[3a+2a-5a+4a-a+2] \\ &=6a-3a-2a+5a-4a+a-2=3a-2, \textit{ Ans.} \end{aligned}$$

3. Enclose the last three terms of $a-b-c+d+e-f$ in a parenthesis with a $-$ sign prefixed.

Result, $a-b-c-(-d-e+f).$

EXAMPLES.

Remove the parentheses, etc., from the following:

4. $a-(b-c)+(d-e)$.

5. $3a-(2a-\{a+2\})$.

6. $5x-(2x-3y)-(2x+4y)$.

7. $a-b+c-(a+b-c)-(c+b-a)$.

8. $m^2-2n+(a-n+3m^2)-(5a+3n-m^2)$.

9. $2m-[n-\{3m-(2n-m)\}]$.

10. $3x-(5x-[4x-\overline{y-x}])-(-x-3y)$.

11. $2a-[5b+\{3c-(a+[2b-\overline{3a+4c}])\}]$.

12. $3c+(2a-[5c-\{3a+\overline{c-4a}\}])$.

13. $6a-[5a-(4a-\{-3a-[2a-\overline{a-1}]\})]$.

14. $2m-[3m-(5m-2)-\{m-(2m-\overline{3m+4})\}]$.

75. As another application of the rule of Art. 70, we have the following four results:

$$+(+a) \text{ is equivalent to } +a;$$
$$+(-a) \text{ is equivalent to } -a;$$
$$-(+a) \text{ is equivalent to } -a;$$
$$-(-a) \text{ is equivalent to } +a.$$

V.—MULTIPLICATION.

76. Multiplication, in Algebra, is the process of taking one quantity as many times as there are units in another quantity.

The *Multiplicand* is the quantity to be multiplied or taken.

The *Multiplier* is the quantity by which we multiply.

The *Product* is the result of the operation.

The multiplicand and multiplier are often called *factors.*

77. *The product of the factors is the same, in whatever order they are taken.*

For we know, from Arithmetic, that the product of two numbers is the same, in whatever order they are taken; thus we have 3×4 or 4×3 each equal to 12. Similarly, in Algebra, where the symbols represent numbers, we have $a \times b$ or $b \times a$ each equal to $a\,b$ (Art. 14).

78. Let it be required to multiply $a - b$ by c.

By Art. 77, multiplying $a - b$ by c is the same as multiplying c by $a - b$. To multiply c by $a - b$, we multiply it first by a, and then by b, and subtract the second result from the first. c multiplied by a gives $a\,c$, and multiplied by b gives $b\,c$. Subtracting the second result from the first we have

$$a\,c - b\,c$$

the product required.

79. Let it be required to multiply $a - b$ by $c - d$.

To multiply $a - b$ by $c - d$, we multiply it first by c, and then by d, and subtract the second result from the first. By Art. 78, $a - b$ multiplied by c gives $a\,c - b\,c$, and multiplied by d gives $a\,d - b\,d$. Subtracting the second result from the first, we have

$$a\,c - b\,c - a\,d + b\,d$$

the product required.

80. We observe in the result of Art. 79,

1. The product of the positive term a by the positive term c gives the positive term $a\,c$.

2. The product of the negative term $-b$ by the positive term c gives the negative term $-b\,c$.

3. The product of the positive term a by the negative term $-d$ gives the negative term $-a\,d$.

4. The product of the negative term $-b$ by the negative term $-d$ gives the positive term $b\,d$.

From these considerations we can state what is known as the **Rule of Signs** in Multiplication, as follows:

+ multiplied by +, and − multiplied by −, produce +;
+ multiplied by −, and − multiplied by +, produce −.

Or, as may be enunciated for the sake of brevity with regard to the product of any two terms,

Like signs produce +, and unlike signs produce −.

81. Let it be required to multiply $7a$ by $2b$.

Since (Art. 77) the factors may be written in any order, we have $7a \times 2b = 7 \times 2 \times a \times b = 14ab$. Hence,

The coefficient of the product is equal to the product of the coefficients of the factors.

82. Let it be required to multiply a^3 by a^2.

By Art. 17, a^3 means $a \times a \times a$, and a^2 means $a \times a$; hence, $a^3 \times a^2 = a \times a \times a \times a \times a = a^5$. Hence,

The exponent of a letter in the product is equal to the sum of its exponents in the factors.

Or, in general, $a^m \times a^n = a^{m+n}$.

83. In Multiplication we may distinguish three cases.

CASE I.

84. *When both factors are monomials.*

From Arts. 80, 81, and 82 is derived the following rule for the product of any two monomials.

RULE.

Multiply the numerical coefficients together; annex to the result the letters of both monomials, giving to each letter an exponent equal to the sum of its exponents in the factors. Make the product + when the two factors have the same sign, and − when they have different signs.

EXAMPLES.

1. Multiply $2a^4$ by $3a^2$.

$$2a^4 \times 3a^2 = 6a^6, \textit{ Ans.}$$

2. Multiply $a^3 b^2 c$ by $-5a^2 b d$.

$$a^3 b^2 c \times -5a^2 b d = -5a^5 b^3 c d, \textit{ Ans.}$$

3. Multiply $-7x^m$ by $-5x^n$.

$$-7x^m \times -5x^n = 35x^{m+n}, \textit{ Ans.}$$

4. Multiply $3a(x-y)^2$ by $4a^3(x-y)$.

$$3a(x-y)^2 \times 4a^3(x-y) = 12a^4(x-y)^3, \textit{ Ans.}$$

Multiply the following:

5. $15m^5n^6$ by $3mn$.

6. $3ab$ by $2ac$.

7. $17abc$ by $-8abc$.

8. $-17a^4c^2$ by $-3a^2c^2$.

9. $11n^2y$ by $-5n^6z$.

10. $4a^6$ by $3aby^2$.

11. $-6ab^2c$ by a^3bm.

12. $-12a^2x$ by $-2x^2y$.

13. $3a^mb^n$ by $-5a^nb^r$.

14. $-4x^my^n$ by $-x^ny^nz^5$.

15. $2a^mb^n$ by $5a^3b$.

16. $-7m^nx^2$ by $m^nx^ry^2$.

17. $2m(a-b)^2$ by $m(a-b)$.

18. $7a(x-y)$ by $-3a^2b(x-y)$.

19. Find the continued product of $8ax^2$, $2a^3y$, and $4x^3y^4$.

20. Find the continued product of $2ac^2$, $-4ac^3$, and $-3ab^2$.

CASE II.

85. *When one of the factors is a polynomial.*

From Art. 78 we have the following

RULE.

Multiply each term of the multiplicand by the multiplier, remembering that like signs produce +, and unlike signs produce −.

EXAMPLES.

1. Multiply $3x - y$ by $2xy$.

$$\begin{array}{l} 3x - y \\ 2xy \\ \hline 6x^2y - 2xy^2, \text{ } Ans. \end{array}$$

2. Multiply $3a - 5x$ by $-4m$.

$$\begin{array}{l} \quad 3a - 5x \\ -4m \\ \hline -12am + 20mx, \text{ } Ans. \end{array}$$

Multiply the following:

3. $x^2 - 2x - 3$ by $4x$.

4. $8a^2bc - d$ by $5ad^2$.

5. $3x^2 + 6x - 7$ by $-2x^3$.

6. $3m^2 - 5mn - n^2$ by $-2m$.

7. $-x^4 - 10x^3 + 5$ by $-2x^3$.

8. $a^2 + 13ab - 6b^2$ by $4ab^2$.

9. $m^2 + mn + n^2$ by mn.

10. $5 - 6a - 8a^3$ by $-6a^2$.

11. $5x^3 - 4x^2 - 3x - 2$ by $-6x^5$.

12. $a^3 - 3a^2b + 3ab^2 - b^3$ by a^2b^2.

CASE III.

86. *When both of the factors are polynomials.*

In Art. 79 we showed that the product of $a - b$ and $c - d$ might be obtained by multiplying $a - b$ by c, and then by d, and subtracting the second result from the first. It would evidently be equally correct to multiply $a - b$ by c, and then by $-d$, and *add* the second result to the first. On this we base the following rule for finding the product of two polynomials.

RULE.

Multiply each term of the multiplicand by each term of the multiplier, remembering that like signs produce +, and unlike signs produce −, and add the partial products.

EXAMPLES.

1. Multiply $3a - 2b$ by $2a - 5b$.

$$\begin{array}{l} 3a - 2b \\ 2a - 5b \\ \hline 6a^2 - \ \ 4ab \\ \qquad - 15ab + 10b^2 \\ \hline 6a^2 - 19ab + 10b^2, \textit{ Ans.} \end{array}$$

The reason for shifting the second partial product one place to the right, is that in general it enables us to place like terms in the same vertical column, where they are more conveniently added.

2. Multiply $x^2 + 1 - x^3 - x$ by $x + 1$.

$$\begin{array}{l} 1 - x + x^2 - x^3 \\ 1 + x \\ \hline 1 - x + x^2 - x^3 \\ \quad + x - x^2 + x^3 - x^4 \\ \hline 1 \qquad\qquad\qquad\quad - x^4, \textit{ Ans.} \end{array}$$

It is convenient, though not essential, to have both multiplicand and multiplier arranged in the same order of powers (Art. 41), and to write the product in the same order.

Multiply the following:

3. $3x^2 - 2xy - y^2$ by $2x - 4y$.

4. $x^2 + 2x + 1$ by $x^2 - 2x + 3$.

5. $a + b - c$ by $a - b + c$.

6. $3a - 2b$ by $-2a + 4b$.

7. $a^2 + b^2 + ab$ by $b - a$.

8. $1 + x + x^3 + x^2$ by $ax - a$.

9. $5a^2 - 3ab + 4b^2$ by $6a - 5b$.

10. $3x^2 - 7x + 4$ by $2x^2 + 9x - 5$.

11. $6x - 2x^2 - 5 - x^3$ by $x^2 + 10 - 2x$.

12. $2x^3 + 5x^2 - 8x - 7$ by $4 - 5x - 3x^2$.

13. $a^3 b - a^2 b^2 - 4ab^3$ by $2a^2 b - ab^2$.

14. $x^{m+2} y - 3xy^{n-1}$ by $4x^{m+5} y^2 - 4x^4 y^n$.

15. $6x^4 - 3x^3 - x^2 + 6x - 2$ by $2x^2 + x + 2$.

16. $m^4 - m^3 n + m^2 n^2 - mn^3 + n^4$ by $m + n$.

17. $a^3 - 3a^2 b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

87. It is sometimes sufficient to *indicate* the product of polynomials, by enclosing each of the given factors in a parenthesis, and writing them one after the other, with or without the sign $\times$ between the parentheses. When the indicated multiplication is performed, the expression is said to be *expanded* or *developed*.

1. Indicate the product of $2x^2 - 3xy + 6$ by $3x^2 + 3xy - 5$.

Result, $(2x^2 - 3xy + 6)(3x^2 + 3xy - 5)$.

EXAMPLES.

2. Expand $(3a + 4b)(2a + b)$.

3. Expand $(a^4 - a^3 x + a^2 x^2 - ax^3 + x^4)(a + x)$.

4. Develop $(a^4 - x^4) \times (a^4 - x^4)$.

5. Develop $(a^m - a^n)(2a - a^n)$.

6. Expand $(1 + x)(1 + x^4)(1 - x + x^2 - x^3)$.

7. Find the value of $(a + 2x)(a - 3x)(a + 4x)$.

8. Expand $[a(a^2 - 3a + 3) - 1] \times [a(a - 2) + 1]$.

88. From the definition of Art. 76, $0 \times a$ means 0 taken a times. Since 0 taken any number of times produces 0, it follows that $0 \times a = 0$. That is,

If zero be multiplied by any quantity, the product is equal to zero.

89. Since $(+a) \times (+b) = ab$, and $(-a) \times (-b) = ab$, it follows that in the indicated product of two factors, *all the signs of both factors may be changed without altering the value of the expression.* Thus,

$$(x - y)\ (a - b) \text{ is equal to } (y - x)\ (b - a).$$

Similarly we may show that in the indicated product of any number of factors, *any even number of factors may have their signs changed without altering the value of the expression.*

Thus, $(x - y)\ (c - d)\ (e - f)\ (g - h)$ is equal to
$(y - x)\ (c - d)\ (f - e)\ (g - h)$, or to
$(y - x)\ (d - c)\ (f - e)\ (h - g)$, etc.; but is *not* equal to
$(y - x)\ (d - c)\ (f - e)\ (g - h)$.

VI. — DIVISION.

90. Division, in Algebra, is the process of finding one of two factors, when their product and the other factor are given.

Hence, Division is the converse of Multiplication.

The *Dividend* is the product of the two factors.

The *Divisor* is the given factor.

The *Quotient* is the required factor.

91. Since the quotient multiplied by the divisor produces the dividend, it follows, from Art. 80, that if the divisor and quotient have the same sign, the dividend is +; and if they have different signs, the dividend is −. Hence,

+ *divided by* +, *and* − *divided by* −, *produce* +;

+ *divided by* −, *and* − *divided by* +, *produce* −.

Hence, in division as in multiplication,

Like signs produce +, *and unlike signs produce* −.

92. Let it be required to find the quotient of $14\,a\,b$ divided by $7\,a$.

Since the quotient is such a quantity as when multiplied by the divisor produces the dividend, the quotient required must be such a quantity as when multiplied by $7\,a$ will produce $14\,a\,b$. That quantity is evidently $2\,b$. Hence,

The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.

93. Let it be required to find the quotient of a^5 divided by a^3.

The quotient required must be such a quantity as when multiplied by a^3 will produce a^5. That quantity is evidently a^2. Hence,

The exponent of a letter in the quotient is equal to its exponent in the dividend diminished by its exponent in the divisor.

Or, in general, $a^m \div a^n = a^{m-n}$.

94. If we apply the rule of Art. 93 to finding the quotient of a^m divided by a^m, we have $a^m \div a^m = a^{m-m} = a^0$.

Now, according to the previously given definition of an exponent (Art. 17), a^0 has no meaning, and we are therefore at liberty to give to it any definition we please. As $a^m \div a^m = 1$, we should naturally define a^0 as being equal to 1; and as a may represent any quantity whatever,

Any quantity whose exponent is 0 *is equal to* 1.

By this notation, the *trace* of a letter which has disappeared in the operation of division may be preserved. Thus, the quotient of $a^2\,b^3$ divided by $a^2\,b^2$, if important to indicate that a originally entered into the term, may be written $a^0\,b$.

95. In Division we may distinguish three cases.

CASE I.

96. *When both dividend and divisor are monomials.*

From the preceding articles is derived the following

RULE.

Divide the coefficient of the dividend by that of the divisor; and to the result annex the letters of the dividend, each with an exponent equal to its exponent in the dividend diminished by its exponent in the divisor; omitting all letters whose exponents become zero. Make the quotient + when the dividend and divisor have the same sign, and — when they have different signs.

EXAMPLES.

1. Divide $9\,a^2\,b\,c\,x\,y$ by $3\,a\,b\,c$.

$$9\,a^2\,b\,c\,x\,y \div 3\,a\,b\,c = 3\,a\,x\,y,\ \textit{Ans.}$$

2. Divide $24\,a^4\,m^3\,n^2$ by $-8\,a\,m^3\,n$.

$$24\,a^4\,m^3\,n^2 \div -8\,a\,m^3\,n = -3\,a^3\,n,\ \textit{Ans.}$$

3. Divide $-35\,x^m$ by $-7\,x^n$.

$$-35\,x^m \div -7\,x^n = 5\,x^{m-n},\ \textit{Ans.}$$

Divide the following:

4. $12\,a^5$ by $4\,a$.

5. $6\,a^2\,c$ by $6\,a\,c$.

6. $14\,m^3\,n^4$ by $-7\,m\,n^3$.

7. $-18\,x^2\,y^5\,z$ by $9\,x^2\,z$.

8. $-65\,a^3\,b^3\,c^3$ by $-5\,a\,b^2\,c^3$.

9. $72\,m^5\,n$ by $-12\,m^2$.

10. $-144\,c^5\,d^7\,e^6$ by $36\,c^2\,d^3\,e$.

11. $-91\,x^4\,y^3\,z^2$ by $-13\,x^3\,y^2$.

CASE II.

97. *When the dividend is a polynomial and the divisor is a monomial.*

The operation being just the reverse of that of Art. 85, we have the following

RULE.

Divide each term of the dividend by the divisor, remembering that like signs produce +, and unlike signs produce —.

EXAMPLES.

1. Divide $9\,a^3\,b + 6\,a^4\,c - 12\,a\,b$ by $3\,a$.

$$3\,a\overline{)9\,a^3\,b + 6\,a^4\,c - 12\,a\,b}$$
$$3\,a^2\,b + 2\,a^3\,c - 4\,b, \quad \textit{Ans.}$$

Divide the following:

2. $8\,a^3\,b\,c + 16\,a^5\,b\,c - 4\,a^2\,c^2$ by $4\,a^2\,c$.
3. $9\,a^5\,b\,c - 3\,a^2\,b + 18\,a^3\,b\,c$ by $3\,a\,b$.
4. $20\,a^4\,b\,c + 15\,a\,b\,d^3 - 10\,a^2\,b$ by $-5\,a\,b$.
5. $3\,a^3\,(a-b) + 9\,a\,(a+b)$ by $3\,a$.
6. $15\,(x+y)^2 - 5\,a\,(x+y) + 10\,b\,(x+y)$ by $-5\,(x+y)$.
7. $4\,x^7 - 8\,x^6 - 14\,x^5 + 2\,x^4 - 6\,x^3$ by $2\,x^3$.
8. $9\,x^4 + 27\,x^3 - 21\,x^2$ by $-3\,x^2$.
9. $-a^5\,b^6\,c^4 - a^4\,b^5\,c^3 + 3\,a^3\,b^4\,c^2$ by $-a^3\,b^2\,c^2$.
10. $-12\,a^p\,b^q - 30\,a^{12}\,b^3 + 108\,a^n\,b^n$ by $-6\,a^m\,b^m$.

CASE III.

98. *When the divisor is a polynomial.*

1. Let it be required to divide $12 + 10\,x^3 - 11\,x - 21\,x^2$ by $2\,x^2 - 4 - 3\,x$.

We are then to find a quantity which when multiplied by $2\,x^2 - 4 - 3\,x$ will produce $12 + 10\,x^3 - 11\,x - 21\,x^2$.

Now, in the product of two polynomials, the term containing the highest power of any letter in the multiplicand, multiplied by the term containing the highest power of the same letter in the multiplier, produces the term containing the highest power of that letter in the product. Hence, if the term containing the highest power of x in the dividend, $10\,x^3$, be divided by the term containing the highest power of x in the divisor, $2\,x^2$, the result, $5\,x$, will be the term containing the highest power of x in the quotient.

Multiplying the divisor by $5x$, the term of the quotient already found, and subtracting the result, $10x^3 - 20x - 15x^2$, from the dividend, the remainder, $12 + 9x - 6x^2$, may be regarded as the product of the divisor by the rest of the quotient.

Therefore, to find the rest of the quotient, we proceed as before, regarding $12 + 9x - 6x^2$ as a new dividend, and dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power of x in the divisor, $2x^2$, giving as a result -3, which is the term containing the highest power of x in the rest of the quotient.

Multiplying the divisor by -3, the term of the quotient just found, and subtracting the result, $-6x^2 + 12 + 9x$, from the second dividend, there is no remainder. Hence, $5x - 3$ is the quotient required.

99. It will be observed that in getting the terms of the quotient, we search for the terms containing the highest power of some letter in the dividend and divisor. These may be obtained most conveniently *by arranging both dividend and divisor in order of powers commencing with the highest* (Art. 41); this, too, facilitates the subsequent subtraction. We also should arrange each remainder or new dividend in the *same order*.

It is customary to arrange the work as follows:

$$\begin{array}{r|l}
10x^3 - 21x^2 - 11x + 12 & 2x^2 - 3x - 4, \textit{ Divisor.} \\ \cline{2-2}
10x^3 - 15x^2 - 20x & 5x - 3, \textit{ Quotient.} \\ \cline{1-1}
-6x^2 + 9x + 12 & \\
-6x^2 + 9x + 12 & \\ \cline{1-1}
\end{array}$$

100. We might have obtained the quotient by dividing the term containing the *lowest* power of x in the dividend, 12, by the term containing the *lowest* power of x in the divisor, -4, which would have given as a result -3, the term containing the *lowest* power of x in the quotient. In solving the problem in this way, we should first arrange both dividend and divisor in order of powers commencing with the *lowest*, and should afterwards bring down each remainder in the same order; remembering that a term which does not contain x at all contains a lower power of x than any term which contains x.

101. From the preceding principles we derive the following

RULE.

Arrange both dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the dividend, arranging the result in the same order of powers as the divisor and dividend.

Regard the remainder as a new dividend, and divide its first term by the first term of the divisor, giving the next term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the last remainder.

Continue in the same manner until the remainder becomes zero, or until the first term of the remainder will not contain the first term of the divisor.

When a remainder is found whose first term will not contain the first term of the divisor, the remainder may be written with the divisor under it in the form of a fraction, and added to the quotient.

2. Divide $a^3 - 3a^2b + 12b^3 + 5ab^2$ by $b + a$.

Arranging the dividend and divisor in order of powers,

$$\begin{array}{l}
a + b)\ a^3 - 3a^2b + 5ab^2 + 12b^3\ (a^2 - 4ab + 9b^2 \\
\qquad\ \ \underline{a^3 + a^2b\qquad\qquad} \\
\qquad\qquad - 4a^2b \\
\qquad\qquad \underline{- 4a^2b - 4ab^2\qquad} \\
\qquad\qquad\qquad\qquad 9ab^2 \\
\qquad\qquad\qquad\qquad \underline{9ab^2 + 9b^3} \\
\qquad\qquad\qquad\qquad\qquad 3b^3,\ \textit{Remainder.}
\end{array}$$

$$\textit{Ans,}\quad a^2 - 4ab + 9b^2 + \frac{3b^3}{a + b}.$$

EXAMPLES.

3. Divide $2a^2x^2 - 5ax + 2$ by $2ax - 1$.

4. Divide $3b^3 + 3ab^2 - 4a^2b - 4a^3$ by $a + b$.

5. Divide $8a^3 - 4a^2b - 6ab^2 + 3b^3$ by $2a - b$.

6. Divide $21a^5 - 21b^5$ by $7a - 7b$.

7. Divide $a^3 + 2x^3$ by $a + x$.

8. Divide $x^4 + y^4$ by $x + y$.

9. Divide $23x^2 - 48 + 6x^4 - 2x - 31x^3$ by $6 + 3x^2 - 5x$.

10. Divide $15x^4 - 32x^3 + 50x^2 - 32x + 15$ by $3x^2 - 4x + 5$.

11. Divide $2x^4 - 11x - 4x^2 - 12 - 3x^3$ by $4 + 2x^2 + x$.

12. Divide $x^5 - y^5$ by $x - y$.

13. Divide $35 - 17x + 16x^2 - 25x^3 + 6x^4$ by $2x - 7$.

14. Divide $3x^2 + 4x + 6x^5 - 11x^3 - 4$ by $3x^2 - 4$.

15. Divide $a^2 - b^2 + 2bc - c^2$ by $a + b - c$.

16. Divide $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.

17. Divide $x^{n+1} + x^n y + x y^n + y^{n+1}$ by $x^n + y^n$.

18. Divide $a^{2n} - b^{2m} + 2b^m c^r - c^{2r}$ by $a^n + b^m - c^r$.

19. Divide $1 + a$ by $1 - a$.

In examples of this kind the division *does not terminate*, there being a remainder however far the operation may be carried.

20. Divide a by $1 + x$.

21. Divide $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$
by $a^4 + a^3b + a^2b^2 + ab^3 + b^4$.

22. Divide $3a^3 + 2 - 4a^5 + 7a + 2a^6 - 5a^4 + 10a^2$
by $a^3 - 1 - a^2 - 2a$.

23. Divide $15x^2 - x^4 - 20 - 2x^5 + 6x + 2x^3$
by $5 - 3x^2 - 4x + 2x^3$.

24. Divide $2x^5 + 4x^2 - 14 + 7x - 7x^3 + x^6 - x^4$
by $2x^2 - 7 + x^3$.

25. Divide $12a^5 - 14a^4b + 10a^3b^2 - a^2b^3 - 8ab^4 + 4b^5$
by $6a^3 - 4a^2b - 3ab^2 + 2b^3$.

102. In Art. 88 we showed that $0 \times a = 0$. Since the product of the divisor and quotient equals the dividend, we may regard 0 as the quotient, a as the divisor, and 0 as the dividend. Therefore,

$$\frac{0}{a} = 0.$$

That is,

If zero be divided by any quantity the quotient is equal to zero.

VII.—FORMULÆ.

103. A **Formula** is an algebraic expression of a general rule.

The following formulæ will be found very useful in abridging algebraic operations.

104. By Art. 17, $(a + b)^2 = (a + b)(a + b)$; whence, by actual multiplication, we have

$$(a + b)^2 = a^2 + 2ab + b^2. \quad (1)$$

That is,

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

105. We may also show, by multiplication, that

$$(a - b)^2 = a^2 - 2ab + b^2. \quad (2)$$

That is,

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.

106. Again, by multiplication, we have

$$(a + b)(a - b) = a^2 - b^2. \quad (3)$$

That is,

The product of the sum and difference of two quantities is equal to the difference of their squares.

EXAMPLES.

107. 1. Square $3a + 2b$.

The square of the first term is $9a^2$, twice the product of the terms is $12ab$, and the square of the last term is $4b^2$. Hence, by formula (1),

$$(3a + 2b)^2 = 9a^2 + 12ab + 4b^2, \text{ } Ans.$$

Square the following:

2. $2m + 3n$. **4.** $3x + 11$. **6.** $2ab + 5ac$.

3. $x^2 + 4$. **5.** $4a + 5b$. **7.** $7x^3 + 3x$.

8. Square $4x - 5$.

The square of the first term is $16x^2$, twice the product of the terms is $40x$, and the square of the last term is 25. Hence, by formula (2),

$$(4x - 5)^2 = 16x^2 - 40x + 25, \text{ } Ans.$$

Square the following:

9. $3a^2 - b^3$. **11.** $1 - 2x^2$. **13.** $3 - a^n$.

10. $4ab - x$. **12.** $x^4 - y^2$. **14.** $2x^3 - 9x^2$.

15. Multiply $6a + b$ by $6a - b$.

The square of the first term is $36a^2$, and of the last term b^2. Hence, by formula (3),

$$(6a + b)(6a - b) = 36a^2 - b^2, \text{ } Ans.$$

Expand the following:

16. $(x + 3)(x - 3)$. **19.** $(a^m + a^n)(a^m - a^n)$.

17. $(2x + 1)(2x - 1)$. **20.** $(x^3 + 5x)(x^3 - 5x)$.

18. $(5a + 7b)(5a - 7b)$. **21.** $(4x^2 + 3)(4x^2 - 3)$.

22. Multiply $a + b + c$ by $a + b - c$.

$$(a + b + c)(a + b - c) = [(a + b) + c][(a + b) - c]$$
$$= \text{(Art. 106)}, (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2, \text{ } Ans.$$

Expand the following:

23. $[1+(a-b)][1-(a-b)]$. **25.** $(a-b+c)(a-b-c)$.

24. $(a+b+c)(a-b-c)$. **26.** $(c+a-b)(c-a+b)$.

27. $[(a+b)+(c-d)][(a+b)-(c-d)]$.

28. $(a-b+c-d)(a-b-c+d)$.

29. $(a+b+c+d)(a+b-c-d)$.

VIII. — FACTORING.

108. The **Factors** of a quantity are such quantities as will divide it without a remainder.

109. **Factoring** is the process of resolving a quantity into its factors.

110. A **Prime Quantity** is one that cannot be divided, without a remainder, by any integral quantity, except itself or unity.

Thus, a, b, and $a+c$ are prime quantities.

Quantities are said to be *prime to each other* when they have no common integral divisor except unity.

111. One quantity is said to be *divisible* by another when the latter will divide the former without a remainder.

Thus, $a\,b$ and $a\,b+a^2\,b^2$ are both divisible by a, b, and $a\,b$.

112. If a quantity can be resolved into two equal factors, it is said to be a *perfect square;* and one of the equal factors is called the *square root* of the quantity.

If a quantity can be resolved into three equal factors, it is said to be a *perfect cube;* and one of the equal factors is called the *cube root* of the quantity.

Thus, since $4\,a^2$ equals $2\,a \times 2\,a$, $4\,a^2$ is a perfect square and $2\,a$ is its square root; and since $27\,x^3$ equals $3\,x \times 3\,x \times 3\,x$, $27\,x^3$ is a perfect cube, and $3\,x$ is its cube root.

Note. $4\,a^2$ also equals $-2\,a \times -2\,a$, so that the square root of $4\,a^2$ may be either $2\,a$ or $-2\,a$. In the examples in this chapter we shall only consider the *positive* square root.

To find the square root of an algebraic quantity, extract the square root of the numerical coefficient, and divide the exponent of each letter by 2. Thus, the square root of $9\,a^6\,b^2$ is $3\,a^3\,b$.

To find the cube root, extract the cube root of the numerical coefficient, and divide the exponent of each letter by 3. Thus, the cube root of $8\,a^3\,b^6$ is $2\,a\,b^2$.

113. The factoring of monomials may be performed by inspection; thus,

$$12\,a^3\,b^2\,c = 2.\,2.\,3.\,a\,a\,a\,b\,b\,c.$$

But in the decomposition of polynomials we are governed by rules which may be derived from the laws of their formation. A polynomial is not always factorable; but in numerous cases we can always factor; and these cases, together with the rules for their solution, will be found in the succeeding articles.

CASE I.

114. *When the terms of a polynomial have a common monomial factor, it may be written as one of the factors of the polynomial, with the quotient obtained by dividing the given polynomial by this factor, as the other.*

1. Factor the expression $3\,x^3\,y^2 - 12\,x\,y^4 - 9\,x^2\,y^3$.

We observe that each term contains the factor $3\,x\,y^2$.

Dividing the given polynomial by $3\,x\,y^2$, we obtain as a quotient $x^2 - 4\,y^2 - 3\,x\,y$. Hence,

$$3\,x^3\,y^2 - 12\,x\,y^4 - 9\,x^2\,y^3 = 3\,x\,y^2\,(x^2 - 4\,y^2 - 3\,x\,y),\ \textit{Ans.}$$

EXAMPLES.

Factor the following expressions:

2. $a^3 + a$.

3. $16\,x^4 - 12\,x$.

4. $a^5 - 2\,a^4 + 3\,a^3 - a^2$.

5. $60m^4\,n^2 - 12\,n^3$.

6. $27\,c^4\,d^2 + 9\,c^3\,d$.

7. $36\,x^3\,y - 60\,x^2\,y^4 - 84\,x^4\,y^2$.

8. $a^5\,b - 3\,a^6\,b^4 - 2\,a^3\,b^4\,c + 6\,a^7\,b^5\,x$.

9. $84\,x^2\,y^3 - 140\,x^3\,y^4 + 56\,x^4\,y^5$.

10. $72\,a^4\,b^3\,c^3 + 126\,a^3\,c^2\,d + 162\,a^2\,c$.

11. $128\,c^4\,d^5 + 320\,c^2\,d^7 - 448\,c^5\,d^4$.

CASE II.

115. *When a polynomial consists of four terms, the first two and last two of which have a common binomial factor, it may be written as one of the factors of the polynomial, with the quotient obtained by dividing the given polynomial by this factor, as the other.*

1. Factor the expression $a\,m - b\,m + a\,n - b\,n$.

Factoring the first two and last two terms by the method of Case I, we obtain $m\,(a - b) + n\,(a - b)$. We observe that the first two and last two terms have the common binomial factor $a - b$. Dividing the expression by this, we obtain as a quotient $m + n$.

Hence, $a\,m - b\,m + a\,n - b\,n = (a - b)\,(m + n)$, *Ans.*

2. Factor the expression $a\,m - b\,m - a\,n + b\,n$.

$$a\,m - b\,m - a\,n + b\,n = a\,m - b\,m - (a\,n - b\,n) = m\,(a - b) - n\,(a - b) = (a - b)\,(m - n), \textit{ Ans.}$$

Note. If the third term of the four is negative, as in Ex. 2, it is convenient to enclose the last two terms in a parenthesis with a – sign prefixed, before factoring.

EXAMPLES.

Factor the following expressions:

3. $a\,b + b\,x + a\,y + x\,y$.

4. $a\,c - c\,m + a\,d - d\,m$.

5. $x^2 + 2\,x - x\,y - 2\,y$.

6. $a^3 - a^2\,b + a\,b^2 - b^3$.

7. $m\,x^2 - m\,y^2 - n\,x^2 + n\,y^2$.

8. $x^3 + x^2 + x + 1$.

9. $6\,x^3 + 4\,x^2 - 9\,x - 6$.

10. $8\,c\,x - 12\,c\,y + 2\,d\,x - 3\,d\,y$.

11. $6\,n - 21\,m^2\,n - 8\,m + 28\,m^3$.

12. $a^2\,b\,c - a\,c^2\,d + a\,b^2\,d - b\,c\,d^2$.

13. $m^2\,n^2\,x^2 - n^3\,x\,y - m^3\,x\,y + m\,n\,y^2$.

14. $12\,a\,b\,m\,n - 21\,a\,b\,x\,y + 20\,c\,d\,m\,n - 35\,c\,d\,x\,y$.

CASE III.

116. *When the first and last terms of a trinomial are perfect squares and positive, and the second term is twice the product of their square roots.*

Comparing with Formulæ 1 and 2, Arts. 104 and 105, we observe that such expressions are produced by the product of two equal binomial factors. Reversing the rules of Arts. 104 and 105, we have the following rule for obtaining one of the equal factors:

Extract the square roots of the first and last terms, and connect the results by the sign of the second term.

1. Factor $a^2 + 2\,a\,b + b^2$.

The square root of the first term is a; of the last term, b; the sign of the second term is $+$. Hence, one of the equal factors is $a + b$.

Therefore, $a^2 + 2\,a\,b + b^2 = (a + b)\,(a + b)$ or $(a + b)^2$, *Ans.*

2. Factor $9\,a^2 - 12\,a\,b + 4\,b^2$.

The square root of the first term is $3\,a$; of the last term, $2\,b$; the sign of the second term is $-$. Hence, one of the equal factors is $3\,a - 2\,b$. Therefore,

$9\,a^2 - 12\,a\,b + 4\,b^2 = (3\,a - 2\,b)\,(3\,a - 2\,b)$ or $(3\,a - 2\,b)^2$, *Ans.*

Note. According to Art. 58, the given expression may be written $4\,b^2 - 12\,a\,b + 9\,a^2$. Applying the rule to this expression, we have

$$4\,b^2 - 12\,a\,b + 9\,a^2 = (2\,b - 3\,a)\,(2\,b - 3\,a) \text{ or } (2\,b - 3\,a)^2.$$

We should obtain this second form of the result in another way by applying the principles of Art. 89 to the first factors obtained.

EXAMPLES.

Factor the following expressions:

3. $x^2 - 14\,x + 49$.

4. $m^2 + 36\,m + 324$.

5. $y^2 + 20\,y + 100$.

6. $a^2 - 28\,a + 196$.

7. $n^6 - 26\,n^3 + 169$.

8. $x^2\,y^2 + 32\,x\,y + 256$.

9. $25 x^2 + 70 x y z + 49 y^2 z^2$.

10. $36 m^2 - 36 m n + 9 n^2$.

11. $16 m^2 - 8 a m + a^2$.

12. $4 a^2 + 44 a b + 121 b^2$.

13. $a^2 b^4 + 12 a b^2 c + 36 c^2$.

14. $9 a^4 + 60 a^2 b c^2 d + 100 b^2 c^4 d^2$.

15. $4 x^4 - 60 m n x^2 + 225 m^2 n^2$.

16. $64 x^6 - 160 x^5 + 100 x^4$.

CASE IV.

117. *When an expression is the difference between two perfect squares.*

Comparing with Formula 3, Art. 106, we observe that such expressions are the product of the sum and difference of two quantities. Reversing the rule of Art. 106, we have the following rule for obtaining the factors:

Extract the square roots of the first and last terms; add the results for one factor, and subtract the second result from the first for the other.

1. Factor $36 x^2 - 49 y^2$.

The square root of the first term is $6 x$; of the last, $7 y$. The sum of these is $6 x + 7 y$, and the second subtracted from the first is $6 x - 7 y$. Hence,

$$36 x^2 - 49 y^2 = (6 x + 7 y)(6 x - 7 y), \textit{ Ans.}$$

2. Factor $(a - b)^2 - (c - d)^2$.

The square root of the first term is $a - b$; of the last, $c - d$. The sum of these is $a - b + c - d$, and the second subtracted from the first is $a - b - c + d$. Hence,

$$(a - b)^2 - (c - d)^2 = (a - b + c - d)(a - b - c + d), \textit{ Ans.}$$

EXAMPLES.

Factor the following expressions:

3. $x^2 - 1$.

4. $4 x^2 - 9 y^2$.

5. $a^4 - b^4$.

6. $9 a^2 - 4$.

7. $4 x^4 - 225 m^2 n^2$.

8. $1 - 196 x^2 y^4 z^6$.

9. $(a+b)^2-(c+d)^2$.

10. $(a-c)^2-b^2$.

11. $m^2-(x-y)^2$.

12. $(x-m)^2-(y-n)^2$.

Many polynomials, consisting of four or six terms, may be expressed as the difference between two perfect squares, and hence may be factored by the rule of Case IV.

13. Factor $2mn+m^2-1+n^2$.

Arrange the expression as follows, $m^2+2mn+n^2-1$. Applying the method of Case III to the first three terms, we may write the expression $(m+n)^2-1$. The square root of the first term is $m+n$; of the last, 1. The sum of these is $m+n+1$, and the second subtracted from the first is $m+n-1$. Hence,

$$2mn+m^2-1+n^2=(m+n+1)\ (m+n-1), \textit{ Ans.}$$

14. Factor $2xy+1-x^2-y^2$.

$$2xy+1-x^2-y^2=1-x^2+2xy-y^2$$
$$=1-(x^2-2xy+y^2)=1-(x-y)^2, \text{ by Case III,}$$
$$=[1+(x-y)][1-(x-y)]=(1+x-y)\ (1-x+y), \textit{ Ans.}$$

15. Factor $2xy+b^2-x^2-2ab-y^2+a^2$.

$$2xy+b^2-x^2-2ab-y^2+a^2$$
$$=a^2-2ab+b^2-x^2+2xy-y^2$$
$$=a^2-2ab+b^2-(x^2-2xy+y^2)$$
$$=(a-b)^2-(x-y)^2, \text{ by Case III,}$$
$$=[(a-b)+(x-y)]\ [(a-b)-(x-y)]$$
$$=(a-b+x-y)\ (a-b-x+y), \textit{ Ans.}$$

Factor the following expressions:

16. $x^2+2xy+y^2-4$.

17. $a^2-b^2+2bc-c^2$.

18. $9c^2-1+d^2+6cd$.

19. $9-x^4-4y^2+4x^2y$.

20. $4a^2+b^2-9d^2-4ab$.

21. $4b-1-4b^2+4m^4$.

22. $a^2-2am+m^2-b^2-2bn-n^2$.

23. $2am-b^2+m^2+2bn+a^2-n^2$.

24. $x^2-y^2+c^2-d^2-2cx+2dy$.

CASE V.

118. *When an expression is a trinomial, of the form $x^2 + a x + b$; where the coefficient of x^2 is unity, and a and b represent any whole numbers, either positive or negative.*

To derive a rule for this case we will consider four examples in Multiplication:

I.

$$\begin{array}{l} x+5 \\ x+3 \\ \hline x^2+5x \\ \quad +3x+15 \\ \hline x^2+8x+15 \end{array}$$

II.

$$\begin{array}{l} x-5 \\ x-3 \\ \hline x^2-5x \\ \quad -3x+15 \\ \hline x^2-8x+15 \end{array}$$

III.

$$\begin{array}{l} x+5 \\ x-3 \\ \hline x^2+5x \\ \quad -3x-15 \\ \hline x^2+2x-15 \end{array}$$

IV.

$$\begin{array}{l} x-5 \\ x+3 \\ \hline x^2-5x \\ \quad +3x-15 \\ \hline x^2-2x-15 \end{array}$$

We observe in these results,

1. The coefficient of x is the algebraic sum of the numbers in the factors.

2. The last term is the product of the numbers.

Hence, in reversing the process, we have the following rule for obtaining the numbers:

RULE.

Find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term.

Note. We may shorten the work by considering the following points:

1. When the last term of the product is +, as in Examples I and II, the *sum* of the numbers is the coefficient of x; both numbers being + when the second term is +, and − when the second term is −.

2. When the last term is −, as in Examples III and IV, the *difference*

of the numbers (disregarding signs) is the coefficient of x; the larger number being + and the smaller − when the second term is +, and the larger number − and the smaller + when the second term is −.

We may embody these observations in the form of a rule which may be found more convenient than the preceding rule in the solution of examples.

I. *If the last term is +, find two numbers whose sum is the coefficient of x, and whose product is the last term; and give to both numbers the sign of the second term.*

II. *If the last term is −, find two numbers whose difference is the coefficient of x, and whose product is the last term; and give to the larger number the sign of the second term, and to the smaller number the opposite sign.*

1. Factor $x^2 + 14x + 45$.

Here we are to find two numbers whose $\begin{cases} \text{sum} = 14 \\ \text{product} = 45 \end{cases}$

The numbers are 9 and 5; and, the second term being +, both numbers are +. Hence,

$$x^2 + 14x + 45 = (x + 9)(x + 5), \text{ Ans.}$$

2. Factor $x^2 - 6x + 5$.

Here we are to find two numbers whose $\begin{cases} \text{sum} = 6 \\ \text{product} = 5 \end{cases}$

The numbers are 5 and 1; and, as the second term is −, both numbers are −. Hence,

$$x^2 - 6x + 5 = (x - 5)(x - 1), \text{ Ans.}$$

3. Factor $x^2 + 5x - 14$.

Here we are to find two numbers whose $\begin{cases} \text{difference} = 5 \\ \text{product} = 14 \end{cases}$

The numbers are 7 and 2; and as the second term is +, the larger number is +, and the smaller −. Hence,

$$x^2 + 5x - 14 = (x + 7)(x - 2), \text{ Ans.}$$

4. Factor $x^2 - 7x - 30$.

Here we are to find two numbers whose $\begin{cases} \text{difference} = 7 \\ \text{product} = 30 \end{cases}$

The numbers are 10 and 3; and as the second term is —, the larger number is —, and the smaller +. Hence,

$$x^2 - 7x - 30 = (x - 10)(x + 3), \textit{ Ans.}$$

Note. In case the numbers cannot be readily determined by inspection, the following method will always give them:

Required two numbers whose difference is 8 and product 48. Taking in order, beginning with the lowest, all possible pairs of integral factors of 48, we have

$$1 \times 48,$$
$$2 \times 24,$$
$$3 \times 16,$$
$$4 \times 12.$$

And, as 4 and 12 differ by 8, they are the numbers required.

Evidently this method will give the required numbers eventually, however large they may be, provided they exist.

EXAMPLES.

Factor the following expressions:

5. $x^2 + 5x + 6$.

6. $a^2 - 3a + 2$.

7. $y^2 + 2y - 8$.

8. $m^2 - 5m - 24$.

9. $x^2 - 11x + 18$.

10. $n^2 - n - 90$.

11. $x^2 + 13x + 36$.

12. $m^2 + 9m + 8$.

13. $m^2 + 2m - 80$.

14. $c^2 - 18c + 32$.

15. $x^2 + x - 42$.

16. $x^2 + 23x + 102$.

17. $y^2 - 9y - 90$.

18. $a^2 + 13a - 48$.

19. $x^2 - 9x - 70$.

20. Factor $15 - 2x - x^2$.

$$15 - 2x - x^2 = -(x^2 + 2x - 15)$$

By the rule of Case V, $x^2 + 2x - 15 = (x + 5)(x - 3)$.

Hence,

$$15 - 2x - x^2 = -(x + 5)(x - 3) = (x + 5)(3 - x), \textit{ Ans.}$$

Note. If the x^2 term is −, enclose the whole expression in a parenthesis with a − sign prefixed. Factor the quantity within the parenthesis, and change the signs of all the terms of *one factor*.

Factor the following expressions:

21. $20 - x - x^2$. **22.** $6 + x - x^2$. **23.** $84 - 5x - x^2$.

The method of Case V may be extended to the factoring of more complicated trinomials.

24. Factor $m^2 n^2 - 3 m n x + 2 x^2$.

Here we are to find two numbers whose $\begin{cases} \text{sum} = 3 \\ \text{product} = 2 \end{cases}$

The numbers are 2 and 1; and as the second term is $-$, both numbers are $-$. Hence,

$$m^2 n^2 - 3 m n x + 2 x^2 = (m n - 2 x)(m n - x), \textit{ Ans.}$$

Factor the following expressions:

25. $x^4 - 29 x^2 + 120$.

26. $c^6 + 12 c^3 + 11$.

27. $x^2 y^6 + 2 x y^3 - 120$.

28. $a^2 b^4 - 7 a b^2 - 144$.

29. $x^2 + 25 n x + 100 n^2$.

30. $m^4 + 5 m^2 n^2 - 66 n^4$.

31. $(a - b)^2 - 3 (a - b) - 4$.

32. $(x + y)^2 - 7 (x + y) + 10$.

33. $x^2 - 2 x y^2 z - 48 y^4 z^2$.

34. $(m + n)^2 + (m + n) - 2$.

CASE VI.

119. *When an expression is the sum or difference of two perfect cubes.*

By actual division, we may show that

$$\frac{a^3 + b^3}{a + b} = a^2 - a b + b^2, \text{ and } \frac{a^3 - b^3}{a - b} = a^2 + a b + b^2.$$

Whence,

$$(a^3 + b^3) = (a + b)(a^2 - a b + b^2), \text{ and}$$
$$(a^3 - b^3) = (a - b)(a^2 + a b + b^2).$$

These results may be enunciated as follows:

To factor the SUM *of two perfect cubes, write for the first factor the* SUM *of the cube roots of the quantities; and for the* *second, the square of the first term of the first factor,* MINUS *the product of the two terms, plus the square of the last term.*

To factor the DIFFERENCE *of two perfect cubes, write for the first factor the* DIFFERENCE *of the cube roots of the quantities; and for the second, the square of the first term of the first factor,* PLUS *the product of the two terms, plus the square of the last term.*

1. Factor $8a^3 + 1$.

The cube root of the first term is $2a$; of the last term, 1.

Hence, $8a^3 + 1 = (2a + 1)(4a^2 - 2a + 1)$, *Ans.*

2. Factor $27x^6 - 64y^3$.

The cube root of the first term is $3x^2$; of the last term, $4y$. Hence,

$$27x^6 - 64y^3 = (3x^2 - 4y)(9x^4 + 12x^2y + 16y^2), \textit{ Ans.}$$

EXAMPLES.

Factor the following expressions:

3. $x^3 - y^3$. **6.** $8c^6 - d^9$. **9.** $343 + 8a^3$.

4. $a^3 + 8$. **7.** $125a^3 - 216m^3$. **10.** $27x^3 - 125$.

5. $m^3 + 64n^6$. **8.** $729c^3d^9 + 512$. **11.** $1000 - 27a^3b^6$.

CASE VII.

120. *When an expression is the sum or difference of two like powers of two quantities.*

The following principles are useful to remember:

1. $a^n - b^n$ is always divisible by $a - b$, if n is an integer.
2. $a^n - b^n$ is always divisible by $a + b$, if n is an *even* integer.
3. $a^n + b^n$ is always divisible by $a + b$, if n is an *odd* integer.

We may prove the first principle as follows:

Commencing the division of $a^n - b^n$ by $a - b$, we have

$$\begin{array}{l|l} a^n - b^n & a - b \\ \underline{a^n - a^{n-1}b} & \overline{a^{n-1} + \ldots \textit{Quotient.}} \\ \quad a^{n-1}b - b^n & \textit{Remainder.} \end{array}$$

or, $$\frac{a^n - b^n}{a - b} = a^{n-1} + \frac{a^{n-1}b - b^n}{a - b} = a^{n-1} + \frac{b\,(a^{n-1} - b^{n-1})}{a - b}.$$

It is evident from this result that, if $a^{n-1} - b^{n-1}$ is exactly divisible by $a - b$, the dividend $a^n - b^n$ will be exactly divisible by $a - b$. That is, if the difference of two like powers of two quantities is divisible by the difference of those quantities, then the difference of the next higher powers of the same quantities is also divisible by the difference of the quantities. But $a^3 - b^3$ is divisible by $a - b$, hence $a^4 - b^4$ is; and since $a^4 - b^4$ is divisible by $a - b$, $a^5 - b^5$ is; and so on to any power. This proves the first principle.

Similarly the second and third principles may be proved.

By continuing the division, we should find,

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ldots + a\,b^{n-2} + b^{n-1} \quad (1)$$

$$\frac{a^n - b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots + a\,b^{n-2} - b^{n-1} \quad (2)$$

$$\frac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots\ldots - a\,b^{n-2} + b^{n-1} \quad (3)$$

It is useful to remember that when $a - b$ is the divisor, all the terms of the quotient are +; where $a + b$ is the divisor, the terms of the quotient are alternately + and —, the last term being + if n is odd, and — if n is even.

1. Factor $a^7 - b^7$.

Putting $n = 7$ in (1), we have

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + a\,b^5 + b^6.$$

Hence,

$$a^7 - b^7 = (a - b)\,(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + a\,b^5 + b^6),$$

Ans.

2. Factor $m^5 + x^5$.

Putting $a = m$, $b = x$, $n = 5$, in (3), we have

$$\frac{m^5 + x^5}{m + x} = m^4 - m^3 x + m^2 x^2 - m x^3 + x^4.$$

Hence,

$$m^5 + x^5 = (m + x)(m^4 - m^3 x + m^2 x^2 - m x^3 + x^4), \text{ } Ans.$$

3. Factor $x^6 - y^6$.

Putting $a = x$, $b = y$, $n = 6$, in (1), we have

$$\frac{x^6 - y^6}{x - y} = x^5 + x^4 y + x^3 y^2 + x^2 y^3 + x y^4 + y^5.$$

Hence,

$$x^6 - y^6 = (x - y)(x^5 + x^4 y + x^3 y^2 + x^2 y^3 + x y^4 + y^5), \text{ } Ans.$$

Or, putting $a = x$, $b = y$, $n = 6$, in (2), we have

$$\frac{x^6 - y^6}{x + y} = x^5 - x^4 y + x^3 y^2 - x^2 y^3 + x y^4 - y^5.$$

Hence,

$$x^6 - y^6 = (x + y)(x^5 - x^4 y + x^3 y^2 - x^2 y^3 + x y^4 - y^5), \text{ } Ans.$$

EXAMPLES.

Factor the following expressions:

4. $x^5 + y^5$. **6.** $n^6 - c^6$. **8.** $m^8 - n^8$. **10.** $a^4 - 16$.

5. $c^5 - d^5$. **7.** $a^7 + b^7$. **9.** $c^7 - 1$. **11.** $a^7 + 128$.

121. By the application of one or more of the given rules for factoring, a quantity may sometimes be separated into more than two factors.

1. Factor $2 a x^3 y^2 - 8 a x y^4$.

By Case I, $2 a x^3 y^2 - 8 a x y^4 = 2 a x y^2 (x^2 - 4 y^2)$.

Factoring the quantity in the parenthesis by Case IV,

$$2 a x^3 y^2 - 8 a x y^4 = 2 a x y^2 (x + 2 y)(x - 2 y), \text{ } Ans.$$

Note. If the method of Case I is to be used in connection with other cases, it should be applied first.

2. Resolve $a^6 - b^6$ into four factors.

By Case IV, $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$.

By Case VI, $a^3 + b^3 = (a + b)(a^2 - a\,b + b^2)$,

and $a^3 - b^3 = (a - b)(a^2 + a\,b + b^2)$.

Hence,

$$a^6 - b^6 = (a + b)(a - b)(a^2 - a\,b + b^2)(a^2 + a\,b + b^2), \textit{ Ans.}$$

EXAMPLES.

Factor the following expressions:

3. $3\,a^3\,b + 12\,a^2\,b + 12\,a\,b$.

4. $45\,x^3\,y^2 - 120\,x^2\,y^4 + 80\,x\,y^6$.

5. $18\,x^3\,y - 2\,x\,y^3$.

6. $x^3 + 8\,x^2 + 7\,x$.

7. $3\,a^4 - 21\,a^3 + 30\,a^2$.

8. $2\,c^3\,m + 8\,c^2\,m - 42\,c\,m$.

9. $m^2\,x\,y - 4\,m\,x\,y - 12\,x\,y$.

10. $32\,a^4\,b + 4\,a\,b^4$.

11. Resolve $n^9 - 1$ into three factors.

12. Resolve $x^4 - y^4$ into three factors.

13. Resolve $x^8 - m^8$ into four factors.

14. Resolve $m^6 - n^6$ into four factors.

15. Resolve $a^9 + c^9$ into three factors.

16. Resolve $64\,a^6 - 1$ into four factors.

Other methods for factoring will be given in Chapter XXIX.

IX.—GREATEST COMMON DIVISOR.

122. A **Common Divisor** or **Measure** of two or more quantities is a quantity that will divide each of them without a remainder.

Hence, *any factor common to two or more quantities is a common divisor of those quantities.*

Also, when quantities are prime to each other, they have no common measure greater than unity.

123. The **Greatest Common Divisor** of two or more quantities is the greatest quantity that will divide each of them without a remainder.

Hence, *the greatest common divisor of two or more quantities is the product of all the prime factors common to those quantities.*

By the *greatest* of two or more algebraic quantities, it may be remarked, is here meant the *highest,* with reference to the *coefficients* and *exponents* of the same letters.

In determining the greatest common divisor of algebraic quantities, it is convenient to distinguish three cases.

CASE I.

124. *When the quantities are monomials.*

1. Find the greatest common divisor of

$$42\,a^3\,b^2,\ 70\,a^2\,b\,c, \text{ and } 98\,a^4\,b^3\,d^2.$$

$$\begin{aligned} 42\,a^3\,b^2 &= 2 \times 3 \times 7\ \ a\,a\,a\ \ \ b\,b \\ 70\,a^2\,b\,c &= 2 \times 5 \times 7\ \ a\,a\ \ \ \ \ b\ \ \ c \\ 98\,a^4\,b^3\,d^2 &= 2 \times 7 \times 7\ \ a\,a\,a\,a\,b\,b\,b\ \ \ d\,d \end{aligned}$$

Hence, G. C. D. $= 2 \times 7\ a\,a\,b = 14\,a^2\,b$, *Ans.* (Art. 123).

RULE.

Resolve the quantities into their prime factors, and find the product of all the factors common to the several quantities.

Note. Any literal factor forming a part of the greatest common divisor will take the *lowest* exponent with which it occurs in either of the given quantities.

EXAMPLES.

Find the greatest common divisors of the following:

2. $a^3\,x^2$, $7\,a^4\,x$, and $3\,a\,b^2$.

3. $6\,c^5\,d^4$, $3\,c^3\,d^5$, and $9\,c^4\,d^3$.

4. $18\,m\,n^5$, $45\,m^2\,n$, and $72\,m^3\,n^2$.

5. $15\,c^2\,x$, $45\,c^3\,x^2$, and $60\,c^4\,x^3$.

6. $108\,y^2 z^7$, $144\,y^3 z^4$, and $120\,y^4 z^5$.

7. $96\,a^5 b^4$, $120\,a^3 b^5$, and $168\,a^4 b^6$.

8. $51\,m^4 n$, $85\,m^3 x$, and $119\,m^2 y^4$.

9. $84\,x^3 y^4 z^5$, $112\,x^4 y^5 z^6$, and $154\,x^7 y^6 z^5$.

CASE II.

125. *When the quantities are polynomials which can be readily factored by inspection.*

1. Find the greatest common divisor of

$5\,x\,y^3 - 15\,y^3$, $x^2 + 4\,x - 21$, and $m\,x - 3\,m - n\,x + 3\,n$.

$$5\,x\,y^3 - 15\,y^3 = 5\,y^3\,(x-3)$$
$$x^2 + 4\,x - 21 = (x+7)\,(x-3)$$
$$m\,x - 3\,m - n\,x + 3\,n = (m-n)\,(x-3)$$

Hence, by Art. 123, G. C. D. $= x - 3$, *Ans.*

2. Find the greatest common divisor of

$4\,x^2 - 4\,x + 1$, $4\,x^2 - 1$, and $8\,x^3 - 1$.

$$4\,x^2 - 4\,x + 1 = (2\,x - 1)\,(2\,x - 1)$$
$$4\,x^2 - 1 = (2\,x + 1)\,(2\,x - 1)$$
$$8\,x^3 - 1 = (2\,x - 1)\,(4\,x^2 + 2\,x + 1)$$

Hence, G. C. D. $= 2\,x - 1$, *Ans.*

The rule in this case is the same as in Case I.

EXAMPLES.

Find the greatest common divisors of the following:

3. $3\,a\,x^2 - 2\,a^2 x$, $a^2 x^2 - 3\,a\,b\,x$, and $5\,a^2 x^3 + 2\,a\,x^4 - 3\,a^3 x$.

4. $m^2 + 2\,m\,n + n^2$, $m^2 - n^2$, and $m^3 + n^3$.

5. $x^4 - 1$, $x^5 + x^3$, and $x^4 + 2\,x^2 + 1$.

6. $3\,a\,x\,y^2 + 21\,a\,y^2$, $3\,c\,x + 21\,c - 3\,d\,x - 21\,d$, and $x^2 - 3x - 70$.

7. $4\,x^2 - 12\,x + 9$, $4\,x^2 - 9$, and $4\,m^2 n\,x - 6\,m^2 n$.

8. $9\,x^2 - 16$, $3\,x\,y - 4\,y + 3\,x\,z - 4\,z$, and $27\,x^3 - 64$.

9. $x^3 - x$, $x^3 + 9x^2 - 10x$, and $x^6 - x$.

10. $a^3 - 8b^3$, $5ax + 2a - 10bx - 4b$, and $a^2 - 4ab + 4b^2$.

11. $x^2 - x - 42$, $x^2 - 4x - 60$, and $x^2 + 12x + 36$.

12. $8x^3 + 125$, $4x^2 - 25$, and $4x^2 + 20x + 25$.

13. $3ax^6 - 3ax^5$, $ax^3 - 9ax^2 + 8ax$, and $2ax^5 - 2ax$.

14. $12ax - 3a + 8cx - 2c$, $64x^3 - 1$, and $16x^2 - 8x + 1$.

CASE III.

126. *When the quantities are polynomials which cannot be readily factored by inspection.*

Let a and b be two expressions, arranged in order of powers of some common letter; and let the exponent of the highest power of that letter in b be either equal to or less than the exponent of the highest power of that letter in a. Suppose that b is contained in a, p times with a remainder c; suppose that c is contained in b, q times with a remainder d; and suppose that d is contained in c, r times with no remainder. The operation of division may be shown as follows:

$$\begin{array}{llll}
b)\ a\ (p & & & \\
\underline{\ p\,b\ } & & & \\
\quad c)\ b\ (q & & & \\
\quad\ \underline{\ q\,c\ } & & & \\
\qquad d)\ c\ (r & & & \\
\qquad\ \ \underline{\ r\,d\ } & & & \\
\qquad\quad 0 & & &
\end{array}$$

We will first show that d is a common divisor of a and b. From the nature of subtraction, the minuend equals the subtrahend plus the remainder; hence,

$$a = pb + c,\ b = qc + d, \text{ and } c = rd.$$

Substituting rd for c in the value of b, we have

$$b = qrd + d = d(qr + 1).$$

Substituting $q\,r\,d + d$ for b, and $r\,d$ for c in the value of a, we have

$$a = p\,q\,r\,d + p\,d + r\,d = d\,(p\,q\,r + p + r).$$

Hence, as d is a factor of a and also of b, it is a common divisor of a and b.

We will now show that every common divisor of a and b is a divisor of d. Let k be any common divisor of a and b, such that $a = m\,k$ and $b = n\,k$. From the nature of subtraction, the minuend minus the subtrahend equals the remainder; hence,

$$c = a - p\,b, \text{ and } d = b - q\,c.$$

Substituting $m\,k$ for a, and $n\,k$ for b in the value of c, we have

$$c = m\,k - p\,n\,k.$$

Substituting $m\,k - p\,n\,k$ for c, and $n\,k$ for b in the value of d, we have

$$\begin{aligned} d &= n\,k - q\,(m\,k - p\,n\,k) = n\,k - q\,m\,k + p\,q\,n\,k \\ &= k\,(n - q\,m + p\,q\,n). \end{aligned}$$

Hence, k is a factor or divisor of d.

Therefore, since every common divisor of a and b is a divisor of d, and no expression greater (Art. 123) than d can be a divisor of d, it follows that d is the greatest common divisor of a and b.

1. Find the greatest common divisor of $x^2 - 6\,x + 8$ and $4\,x^3 - 21\,x^2 + 15\,x + 20$.

$$\begin{array}{l} x^2 - 6\,x + 8)\; 4\,x^3 - 21\,x^2 + 15\,x + 20\;(4\,x + 3 \\ \qquad\qquad\quad\; \underline{4\,x^3 - 24\,x^2 + 32\,x} \\ \qquad\qquad\qquad\qquad\; 3\,x^2 - 17\,x + 20 \\ \qquad\qquad\qquad\qquad\; \underline{3\,x^2 - 18\,x + 24} \\ \qquad\qquad\qquad\qquad\qquad\qquad x - 4)\; x^2 - 6\,x + 8\,(x - 2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \underline{x^2 - 4\,x} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 2\,x + 8 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{- 2\,x + 8} \end{array}$$

Hence, $x - 4$ is the greatest common divisor, *Ans.*

RULE.

Divide the greater quantity (Art. 123) *by the less; and if there is no remainder, the less quantity will be the required greatest common divisor.*

If there is a remainder, divide the divisor by it, and continue thus to make the preceding divisor the dividend, and the remainder the divisor, until a divisor is obtained which leaves no remainder; the last divisor will be the greatest common divisor required.

Note 1. If there are three or more quantities, find the greatest common divisor of two of them; then of this result and the third of the quantities, and so on. The last divisor will be the greatest common divisor required.

Note 2. If a monomial factor is seen by inspection to be common to all the terms of one of the given quantities, and not of the other, it may be removed, as it evidently can form no part of the greatest common divisor; and, similarly, we may remove from a remainder any monomial factor which is not a common factor of the given quantities.

2. Find the greatest common divisor of

$$6ax^2 - 19ax + 10a \text{ and } 6x^3 - x^2 - 35x.$$

In the first quantity a is a common factor of all the terms, and is not a factor of the second quantity; in the second quantity x is a common factor of all the terms, and is not a factor of the first quantity. Hence we may remove a from each term of the first quantity, and x from each term of the second.

$$\begin{array}{r} 6x^2 - 19x + 10)\,6x^2 - x - 35\,(1 \\ \underline{6x^2 - 19x + 10} \\ 18x - 45 \end{array}$$

In this remainder 9 is a common factor of all the terms, and is not a common factor of the given quantities. Hence 9 may be removed from each term of the remainder.

$$\begin{array}{r} 2x - 5)\,6x^2 - 19x + 10\,(3x - 2 \\ \underline{6x^2 - 15x} \\ -4x + 10 \\ \underline{-4x + 10} \end{array}$$

Hence, $2x - 5$ is the greatest common divisor, *Ans.*

Note 3. If the first term of a remainder be negative, the sign of each term may be changed.

3. Find the greatest common divisor of $2x^2-3x-2$ and $2x^2-5x-3$.

$$\begin{array}{r|l} 2x^2-3x-2)\,2x^2-5x-3\,(1 \\ 2x^2-3x-2 \\ \hline -2x-1 \end{array}$$

The first term of this remainder being negative, we change the sign of each term, giving $2x+1$.

$$\begin{array}{r} 2x+1)\,2x^2-3x-2\,(x-2 \\ 2x^2+x \\ \hline -4x-2 \\ -4x-2 \\ \hline \end{array}$$

Hence, $2x+1$ is the greatest common divisor, *Ans.*

Note 4. The dividend or any remainder may be multiplied by any quantity which is not a common factor of all the terms of the divisor.

4. Find the greatest common divisor of $2x^3-7x^2+5x-6$ and $3x^3-7x^2-7x+3$.

To avoid a fraction as the first term of the quotient, we multiply each term of the second quantity by 2, giving $6x^3-14x^2-14x+6$.

$$\begin{array}{r} 2x^3-7x^2+5x-6)\,6x^3-14x^2-14x+6\,(3 \\ 6x^3-21x^2+15x-18 \\ \hline 7x^2-29x+24 \end{array}$$

To avoid a fraction as the first term of the next quotient, we multiply each term of the new dividend by 7, giving $14x^3-49x^2+35x-42$.

$$\begin{array}{r} 7x^2-29x+24)\,14x^3-49x^2+35x-42\,(2x \\ 14x^3-58x^2+48x \\ \hline 9x^2-13x-42 \end{array}$$

The first term of this remainder not being exactly divisible by the first term of the divisor, we multiply each term by 7, giving $63x^2 - 91x - 294$.

$$\begin{array}{r|l} 7x^2 - 29x + 24) & 63x^2 - 91x - 294\,(9 \\ & 63x^2 - 261x + 216 \\ \hline & 170x - 510 \end{array}$$

Dividing each term by 170, $x - 3)\,7x^2 - 29x + 24\,(7x - 8$

$$\begin{array}{r} 7x^2 - 21x \\ \hline -8x + 24 \\ -8x + 24 \\ \hline \end{array}$$

Hence, $x - 3$ is the greatest common divisor, *Ans.*

Note 5. When the two given quantities have a common monomial factor, it may be removed from each, and the greatest common divisor of the resulting expressions found. This result must be multiplied by the common monomial factor to give the greatest common divisor of the given quantities.

5. Find the greatest common divisor of $6x^3 - x^2 - 5x$ and $21x^3 - 26x^2 + 5x$.

The quantities have the common monomial factor x; removing it, we find the greatest common divisor of $6x^2 - x - 5$ and $21x^2 - 26x + 5$. We multiply the latter by 2, to avoid a fraction as the first term of the quotient, giving $42x^2 - 52x + 10$.

$$\begin{array}{r|l} 6x^2 - x - 5) & 42x^2 - 52x + 10\,(7 \\ & 42x^2 - 7x - 35 \\ \hline & -45x + 45 \end{array}$$

Dividing by -45, $x - 1)\,6x^2 - x - 5\,(6x + 5$

$$\begin{array}{r} 6x^2 - 6x \\ \hline 5x - 5 \\ 5x - 5 \\ \hline \end{array}$$

Hence, $x - 1$ is the greatest common divisor of $6x^2 - x - 5$ and $21x^2 - 26x + 5$. Multiplying by x, the common monomial factor, we obtain $x(x-1)$ or $x^2 - x$ as the required greatest common divisor, *Ans.*

EXAMPLES.

Find the greatest common divisors of the following:

6. $6x^2 - 7x - 24$ and $12x^2 + 8x - 15$.

7. $24x^2 + 11x - 28$ and $40x^2 - 51x + 14$.

8. $2x^3 - 2x^2 - 3x + 3$ and $2x^3 - 2x^2 - 5x + 5$.

9. $6x^2 - 13x - 28$ and $15x^2 + 23x + 4$.

10. $8x^2 - 22x + 5$ and $6x^2 - 23x + 20$.

11. $5x^2 + 58x + 33$ and $10x^2 + 41x + 21$.

12. $x^3 + 2x^2 + x + 2$ and $x^4 - 4x^2 - x - 2$.

13. $2x^3 - 3x^2 - x + 1$ and $6x^3 - x^2 + 3x - 2$.

14. $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$.

15. $a^2 - 5ax + 4x^2$ and $a^3 - a^2x + 3ax^2 - 3x^3$.

16. $x^4 - x^3 - 5x^2 + 2x + 6$ and $x^4 + x^3 - x^2 - 2x - 2$.

17. $6x^2y + 4xy^2 - 2y^3$ and $4x^3 + 2x^2y - 2xy^2$.

18. $2a^4 + 3a^3x - 9a^2x^2$ and $6a^3 - 17a^2x + 14ax^2 - 3x^3$.

19. $15a^2x^3 - 20a^2x^2 - 65a^2x - 30a^2$ and $12bx^3 + 20bx^2 - 16bx - 16b$.

X.—LEAST COMMON MULTIPLE.

127. A **Multiple** of a quantity is any quantity that can be divided by it without a remainder.

Hence, *a multiple of a quantity must contain all the prime factors of that quantity.*

128. A **Common Multiple** of two or more quantities is one that can be divided by each of them without a remainder.

Hence, *a common multiple of two or more quantities must contain all the prime factors of each of the quantities.*

129. The **Least Common Multiple** of two or more quantities is the least quantity that can be divided by each of them without a remainder.

Hence, *the least common multiple of two or more quantities must be the product of all their different prime factors, each taken only the greatest number of times it is found in any one of those quantities.*

By the *least* quantity, is here meant the *lowest* with reference to the *exponents* and *coefficients* of the same letters.

In determining the least common multiple of algebraic quantities, we may distinguish three cases.

CASE I.

130. *When the quantities are monomials.*

1. Find the least common multiple of $36\,a^3 x$, $60\,a^2 y^2$, and $84\,c\,x^3$.

$$36\,a^3 x = 2 \times 2 \times 3 \times 3 \;\; a\,a\,a\,x$$
$$60\,a^2 y^2 = 2 \times 2 \times 3 \times 5 \;\; a\,a \qquad y\,y$$
$$84\,c\,x^3 = 2 \times 2 \times 3 \times 7 \qquad x\,x\,x \quad c$$

$$\text{Hence, L. C. M.} = 2 \times 2 \times 3 \times 3 \times 5 \times 7\; a\,a\,a\,x\,x\,x\,y\,y\,c$$
$$= 1260\,a^3 x^3 y^2 c, \textit{ Ans.} \text{ (Art. 129).}$$

RULE.

Resolve the quantities into their prime factors; and the product of these, taking each factor only the greatest number of times it enters into any one of the quantities, will be the least common multiple.

Any literal factor forming a part of the least common multiple will take the highest exponent with which it occurs in either of the given quantities.

When quantities are prime to each other, their product is their least common multiple.

EXAMPLES.

Find the least common multiples of the following:

2. $8\,a^4\,c$, $10\,a^3\,b$, and $12\,a^2\,b^2$.
3. $5\,x^3\,y$, $10\,y^2\,z$, and $15\,x\,z^3$.
4. $a^5\,b^2$, $9\,a^3\,b^4$, and $12\,a^2\,b^3$.
5. $24\,m^3\,x^2$, $30\,n^2\,y$, and $32\,x\,y^2$.
6. $8\,c^2\,d^3$, $10\,a\,c$, and $42\,a^2\,d$.
7. $36\,x\,y^2\,z^3$, $63\,x^3\,y\,z^2$, and $28\,x^2\,y^3\,z$.
8. $40\,a^2\,b\,d^3$, $18\,a\,c^3\,d^4$, and $54\,b^2\,c\,d^2$.
9. $7\,m\,n^2$, $8\,x^3\,y^2$, and $84\,n\,x\,y^3$.

CASE II.

131. *When the quantities are polynomials which can be readily factored by inspection.*

1. Find the least common multiple of x^2+x-6, $x^2-6\,x+8$ and x^2-9.

$$\begin{aligned} x^2+\quad x-6 &= (x-2)\,(x+3) \\ x^2-6\,x+8 &= (x-2)\,(x-4) \\ x^2-9 \qquad &= (x-3)\,(x+3) \end{aligned}$$

Hence (Art. 129), L. C. M. $= (x-2)\,(x-3)\,(x+3)\,(x-4)$

or, $\quad x^4-6\,x^3-x^2+54\,x-72$, *Ans.*

The rule is the same as in Case I.

EXAMPLES.

Find the least common multiples of the following:

2. $a\,x^2+a^2\,x$, x^2-a^2, and x^3-a^3.
3. $2\,a^2+2\,a\,b$, $3\,a\,b-3\,b^2$, and $4\,a^2\,c-4\,b^2\,c$.

4. $x^2 + x$, $x^3 - x$, and $x^4 + x$.

5. $2 - 2x^2$, $4 - 4x$, $8 + 8x$, and $12 + 12x^2$.

6. $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

7. $x^3 - 10x^2 + 21x$, and $ax^2 + 5ax - 24a$.

8. $4x^2 - 4x + 1$, $4x^2 - 1$, and $8x^3 - 1$.

9. $ax - ay - bx + by$, $x^2 - 2xy + y^2$, and $3a^2b - 3ab^2$.

10. $9x^2 + 12x + 4$, $27x^3 + 8$, and $6ax^3 + 4ax^2$.

11. $x^2 - 4x + 3$, $x^2 + x - 12$, and $x^2 - x - 20$.

12. $x^2 - y^2 - z^2 + 2yz$ and $x^2 - y^2 + z^2 + 2xz$.

CASE III.

132. *When the quantities are polynomials which cannot be readily factored by inspection.*

Let a and b be two expressions; let d be their greatest common divisor, and m their least common multiple. Suppose that d is contained in a, x times, and in b, y times; then, from the nature of the greatest common divisor, x and y are prime to each other. Since the dividend is the product of the quotient and divisor, we have

$$a = dx \text{ and } b = dy.$$

Then (Art. 129) the least common multiple of a and b is dxy, or $m = dxy$; but $dx = a$, and $y = \frac{b}{d}$; substituting, we have $m = a \times \frac{b}{d}$.

In a similar manner we could show that $m = b \times \frac{a}{d}$.

Hence the following

RULE.

Find the greatest common divisor of the two quantities; divide one of the quantities by this, and multiply the quotient by the other quantity.

Note. If there are three or more quantities, find the least common multiple of two of them, and then of that result and the third quantity; and so on.

1. Find the least common multiple of $6x^2 - 17x + 12$ and $12x^2 - 4x - 21$.

$$
\begin{array}{l}
6x^2 - 17x + 12)\,12x^2 - 4x - 21\,(2 \\
\qquad\qquad\qquad\quad \underline{12x^2 - 34x + 24} \\
\qquad\qquad\qquad\qquad\qquad 30x - 45 \\
\qquad\qquad\qquad\qquad\quad 2x - 3)\,6x^2 - 17x + 12\,(3x - 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{6x^2 - 9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -8x + 12 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-8x + 12}
\end{array}
$$

Hence, $2x - 3$ is the greatest common divisor of the two quantities; dividing the first given quantity by this, we obtain, as a quotient, $3x - 4$; multiplying the second given quantity by this quotient, we have

$$(3x - 4)(12x^2 - 4x - 21), \text{ or } 36x^3 - 60x^2 - 47x + 84$$

as the required least common multiple, *Ans.*

EXAMPLES.

Find the least common multiples of the following:

2. $6x^2 + 13x - 28$ and $12x^2 - 31x + 20$.
3. $8x^2 + 30x + 7$ and $12x^2 - 29x - 8$.
4. $a^3 + a^2 - 8a - 6$ and $2a^3 - 5a^2 - 2a + 2$.
5. $2x^3 + x^2 - x + 3$ and $2x^3 + 5x^2 - x - 6$.
6. $a^3 - 2a^2b + 2ab^2 - b^3$ and $a^3 + a^2b - ab^2 - b^3$.
7. $x^4 + 2x^3 + 2x^2 + x$ and $ax^3 - 2ax - a$.
8. $2x^4 - 11x^3 + 3x^2 + 10x$ and $3x^4 - 14x^3 - 6x^2 + 5x$.

XI. — FRACTIONS.

133. A **Fraction** is an expression indicating a certain number of the equal parts into which a unit has been divided.

The *denominator* of a fraction shows into how many parts the unit has been divided, and the *numerator* how many parts are taken.

134. A fraction is expressed by writing the numerator above, and the denominator below, a horizontal line. Thus, $\frac{a}{b}$ is a fraction, signifying that the unit has been divided into b equal parts, and that a parts are taken.

The numerator and denominator are called the *terms* of a fraction.

Every integer may be considered as a fraction whose denominator is unity; thus, $a = \frac{a}{1}$.

135. An **Entire Quantity** is one which has no fractional part; as, $a\,b$, or $a - b$.

136. A **Mixed Quantity** is one having both entire and fractional parts; as, $a - \frac{b}{c}$, or $c + \frac{a}{x+y}$.

137. *If the numerator of a fraction be multiplied, or the denominator divided, by any quantity, the fraction is multiplied by that quantity.*

1. Let $\frac{a}{b}$ denote any fraction; multiplying its numerator by c, we have $\frac{a\,c}{b}$. Now, in $\frac{a}{b}$ and $\frac{a\,c}{b}$ the unit is divided into b equal parts, and a and $a\,c$ parts, respectively, are taken. Since

c times as many parts are taken in $\frac{a\,c}{b}$ as in $\frac{a}{b}$, it follows that $\frac{a\,c}{b}$ is c times $\frac{a}{b}$.

2. Let $\frac{a}{b\,c}$ denote any fraction; dividing its denominator by c, we have $\frac{a}{b}$. Now, in $\frac{a}{b\,c}$ and $\frac{a}{b}$, the same number of parts is taken; but, since in $\frac{a}{b\,c}$ the unit is divided into $b\,c$ equal parts, and in $\frac{a}{b}$ into only b equal parts, it follows that each part in $\frac{a}{b}$ is c times as large as each part in $\frac{a}{b\,c}$. Hence, $\frac{a}{b}$ is c times $\frac{a}{b\,c}$.

138. *If the numerator of a fraction be divided, or the denominator multiplied, by any quantity, the fraction is divided by that quantity.*

1. Let $\frac{a\,c}{b}$ denote any fraction; dividing its numerator by c, we have $\frac{a}{b}$. Now, in Art. 137, 1, we showed that $\frac{a\,c}{b}$ was c times $\frac{a}{b}$. Hence, $\frac{a}{b}$ is $\frac{a\,c}{b}$ divided by c.

2. Let $\frac{a}{b}$ denote any fraction; multiplying its denominator by c, we have $\frac{a}{b\,c}$. Now, in Art. 137, 2, we showed that $\frac{a}{b}$ was c times $\frac{a}{b\,c}$. Hence, $\frac{a}{b\,c}$ is $\frac{a}{b}$ divided by c.

139. *If the terms of a fraction be both multiplied, or both divided by the same quantity, the value of the fraction is not altered.*

For, multiplying the numerator by any quantity, multiplies the fraction by that quantity; and multiplying the denominator by the same quantity, divides the fraction by that quantity. And, by Art. 44, Ax. 6, if any quantity be both multiplied and divided by the same quantity, its value is not altered.

Similarly, we may show that if both terms are divided by the same quantity, the value of the fraction is not altered.

140. We may now show the propriety of the use of the fractional form to indicate division, as explained in Art. 16; that is, we shall show that $\frac{a}{b}$ represents the quotient of a divided by b.

For, let x denote the quotient of a divided by b.

Then, since the quotient, multiplied by the divisor, gives the dividend, we have $bx = a$.

But, by Art. 137, $b \times \frac{a}{b} = a$.

Therefore, $x = \frac{a}{b}$.

141. *A fraction is* POSITIVE *when its numerator and denominator have the same sign, and* NEGATIVE *when they have different signs.*

For, a fraction represents the quotient of its numerator divided by its denominator; consequently its proper sign can be determined as in division (Art. 91).

142. The **Sign** of a fraction is the sign prefixed to its dividing line, and indicates whether the fraction is to be added or subtracted.

Thus, in $x + \frac{-a}{b}$ the sign $+$ denotes that the fraction $\frac{-a}{b}$, although essentially *negative* (Art. 91), is to be added to x.

The sign written before the dividing line of a fraction is termed the *apparent* sign of the fraction; and that depending upon the value of the fraction itself is termed the *real* sign.

Thus, in $+\frac{-a}{b}$, the apparent sign is $+$, but the real sign is $-$.

Where no signs are prefixed, plus is understood.

143. From the principles of Arts. 140 and 141 we obtain,

$$\frac{ab}{b} = -\frac{-ab}{b} = -\frac{ab}{-b} = \frac{-ab}{-b} = +a;$$

$$-\frac{ab}{b} = \frac{ab}{-b} = \frac{-ab}{b} = -\frac{-ab}{-b} = -a.$$

From which it appears that,

Of the three signs prefixed to the numerator, denominator, and dividing line of a fraction, any two may be changed without altering the value of the fraction; but if any one, or all three are changed, the value of the fraction is changed from + to −, or from − to +.

144. If either the numerator or denominator of the fraction is a polynomial, we mean by its *sign* the sign of the *entire expression*, as distinguished from the sign of any one of its *individual terms;* and care must be taken, on changing signs in any such case, to change the sign before *each term.*

Thus, $$-\frac{a-b}{c-d} = \frac{-a+b}{c-d}, \text{ or } \frac{b-a}{c-d};$$

also, $$\frac{a-b}{c-d} = \frac{-a+b}{-c+d}, \text{ or } \frac{b-a}{d-c}.$$

145. From Art. 141 we have

$$\frac{abcd}{efgh} = \frac{(-a)\,b\,(-c)\,(-d)}{(-e)\,fgh} = \frac{a\,(-b)\,(-c)\,d}{e\,(-f)\,g\,(-h)}, \text{ etc.};$$

$$-\frac{abcd}{efgh} = \frac{(-a)\,bc\,(-d)}{(-e)\,fgh} = \frac{a\,(-b)\,(-c)\,d}{e\,(-f)\,(-g)\,(-h)}, \text{ etc.}$$

From which it appears that,

If the terms of a fraction are composed of any number of factors, any even number of factors may have their signs changed without altering the value of the fraction; but if any *odd number of factors have their signs changed, the value of the fraction is changed from + to —, or from — to +.*

Thus, $\frac{a-b}{(x-y)(x-z)} = \frac{a-b}{(y-x)(z-x)} = \frac{b-a}{(y-x)(x-z)}$

$= \frac{b-a}{(x-y)(z-x)}$; but does *not* equal $\frac{b-a}{(y-x)(z-x)}$.

REDUCTION OF FRACTIONS.

146. Reduction of Fractions is the process of changing their forms without altering their values.

TO REDUCE A FRACTION TO ITS SIMPLEST FORM.

147. A fraction is in its *simplest form,* when its terms are prime to each other.

CASE I.

148. *When the numerator and denominator can be readily factored by inspection.*

Since dividing both numerator and denominator by the same quantity, or cancelling equal factors in each, does not alter the value of the fraction (Art. 139), we have the following

RULE.

Resolve both numerator and denominator into their prime factors, and cancel all that are common to both.

EXAMPLES.

1. Reduce $\frac{18\,a^3\,b^2\,c}{45\,a^2\,b^2\,x}$ to its simplest form.

$$\frac{18\,a^3\,b^2\,c}{45\,a^2\,b^2\,x} = \frac{2\,.\,3\,.\,3\,.\,a\,.\,a\,.\,a\,.\,b\,.\,b\,.\,c}{5\,.\,3\,.\,3\,.\,a\,.\,a\,.\,b\,.\,b\,.\,x} = \frac{2\,a\,c}{5\,x}, \textit{ Ans.}$$

2. Reduce $\frac{x^2 + 2\,x - 15}{x^2 - 2\,x - 3}$ to its simplest form.

$$\frac{x^2 + 2\,x - 15}{x^2 - 2\,x - 3} = \frac{(x+5)(x-3)}{(x+1)(x-3)} = \frac{x+5}{x+1}, \textit{ Ans.}$$

3. Reduce $\frac{bc - ac - bd + ad}{am - bm - an + bn}$ to its simplest form.

$$\frac{bc - ac - bd + ad}{am - bm - an + bn} = \frac{(b-a)(c-d)}{(a-b)(m-n)}$$

$$= \text{(Art. 89)}\ \frac{(a-b)(d-c)}{(a-b)(m-n)} = \frac{d-c}{m-n},\ \textit{Ans.}$$

Note. If all the factors of the numerator be removed by cancellation, the number 1 (being a factor of all algebraic expressions) remains to form a numerator.

If all the factors of the denominator be removed, the result will be an entire quantity; this being a case of exact division.

Reduce the following to their simplest forms:

4. $\frac{2a^2b^5c}{5a^3bc^3}$.

5. $\frac{32mn}{56m^4n^3}$.

6. $\frac{65x^2y^3z^4}{26x^4y^3z^2}$.

7. $\frac{54a^3b^5c^2}{72a^2b^2c}$.

8. $\frac{15mxy^2}{75mx^2y^3}$.

9. $\frac{110c^3x^2y}{22c^2x^2}$.

10. $\frac{2a^2cd + 2abcd}{6a^2xy + 6abxy}$.

11. $\frac{3x^5 - 6x^4y}{6x^2y^2 - 12xy^3}$.

12. $\frac{x^2 - 2x - 15}{x^2 + 10x + 21}$.

13. $\frac{m^2 - 10m + 16}{m^2 + m - 72}$.

14. $\frac{4c^2 - 20c + 25}{25 - 4c^2}$.

15. $\frac{4a - 9an^2}{9bn^2 - 12bn + 4b}$.

16. $\frac{8x^3 + y^3}{4x^2 - y^2}$.

17. $\frac{27y^3 - 125}{25 - 30y + 9y^2}$.

18. $\frac{6x^2y - 2x^3y}{x^2 - 8x + 15}$.

19. $\frac{4 - x^2}{x^3 - 9x^2 + 14x}$.

20. $\frac{ac - bc - ad + bd}{ac + ad - bc - bd}$.

21. $\frac{2mx + 3my - 2n^2x - 3n^2y}{2m^2x + 3m^2y - 2nx - 3ny}$.

CASE II.

149. *When the numerator and denominator cannot be readily factored by inspection.*

Since the greatest common divisor of two quantities contains all the prime factors common to both, we have the following

RULE.

Divide both numerator and denominator by their greatest common divisor.

EXAMPLES.

1. Reduce $\frac{2a^2 - 5a + 3}{6a^2 - a - 12}$ to its simplest form.

By the rule of Art. 126, we find the greatest common divisor of the numerator and denominator to be $2a - 3$. Dividing the numerator by this, the quotient is $a - 1$. Dividing the denominator, the quotient is $3a + 4$. Therefore, the simplest form of the fraction is $\frac{a-1}{3a+4}$, *Ans.*

Reduce the following to their simplest forms:

2. $\frac{6x^2 + x - 35}{8x^2 + 22x + 5}$.

3. $\frac{10a^2 - a - 21}{2a^2 - 7a + 6}$.

4. $\frac{2m^2 - 5m + 3}{12m^2 - 28m + 15}$.

5. $\frac{x^3 + x^2 - 3x - 2}{x^3 - 4x^2 + 2x + 3}$.

6. $\frac{6x^3 - 7x^2 + 5x - 2}{2x^3 + 5x^2 - 2x + 3}$.

7. $\frac{6x^3 - 19x^2 + 7x + 12}{6x^3 - 25x^2 + 17x + 20}$.

8. $\frac{4x^3 + 14x^2 + 12x + 5}{4x^3 - 10x^2 - 12x - 7}$.

9. $\frac{12a^2 + 16a - 3}{10a^2 + a - 21}$.

10. $\frac{x^3 - 4x^2 + 4x - 1}{x^3 - 2x^2 + 4x - 3}$.

11. $\frac{6x^3 - x^2 - 7x - 2}{6x^3 + 11x^2 + 6x + 1}$.

TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

150. Since a fraction is an expression of division (Art. 140), we have the following

RULE.

Divide the numerator by the denominator, and the quotient will be the entire or mixed quantity required.

EXAMPLES.

1. Reduce $\dfrac{a\,x - a^2\,x^2}{a\,x}$ to an entire quantity.

$$(a\,x - a^2\,x^2) \div a\,x = 1 - a\,x, \text{ } Ans.$$

2. Reduce $\dfrac{a^3 - b^3 - x^3}{a - x}$ to a mixed quantity.

$$\begin{array}{l} a - x)\,a^3 - x^3 - b^3\,(a^2 + a\,x + x^2 - \dfrac{b^3}{a - x},\ Ans. \\ \qquad\quad a^3 - a^2\,x \\ \hline \qquad\quad a^2\,x - x^3 - b^3 \\ \qquad\quad a^2\,x - a\,x^2 \\ \hline \qquad\qquad a\,x^2 - x^3 - b^3 \\ \qquad\qquad a\,x^2 - x^3 \\ \hline \qquad\qquad\qquad - b^3 \end{array}$$

Reduce the following to entire or mixed quantities:

3. $\dfrac{a\,b - a^2}{b}$.

4. $\dfrac{x^3 + y^3}{x + y}$.

5. $\dfrac{2\,x^2 - 3\,x - 4}{5\,x}$.

6. $\dfrac{x^3 - x^2 + 7\,x - 6}{3\,x}$.

7. $\dfrac{a^2 - 3\,a\,b + 4\,b^2}{2\,a\,b}$.

8. $\dfrac{2\,x^2 + 5}{x - 3}$.

9. $\dfrac{x^3 - 1}{x - 1}$.

10. $\dfrac{4\,x^2 - 2\,x + 5}{2\,x^2 - x + 1}$.

11. $\dfrac{x^3 - x^2 - x - 2}{x^2 + x - 1}$.

12. $\dfrac{2\,x^3 - 3\,x^2 + 4\,x - 2}{2\,x^2 - 3\,x + 3}$.

TO REDUCE A MIXED QUANTITY TO A FRACTIONAL FORM.

151. This is the converse of Art. 150; hence we may proceed by the following

RULE.

Multiply the entire part by the denominator of the fraction; add the numerator to the product when the sign of the fraction is +, and subtract it when the sign is —; writing the result over the denominator.

EXAMPLES.

1. Reduce $a+b-\frac{a^2-b^2-5}{a-b}$ to a fractional form.

By the rule,

$$a+b-\frac{a^2-b^2-5}{a-b}=\frac{(a+b)(a-b)-(a^2-b^2-5)}{a-b}$$

$$=\frac{a^2-b^2-a^2+b^2+5}{a-b}=\frac{5}{a-b}, \textit{ Ans.}$$

Note. It will be found convenient to enclose the numerator in a parenthesis, when the sign before the fraction is —.

Reduce the following to fractional forms:

2. $x+1+\frac{4}{x-3}$.

3. $a+\frac{b^2-cd}{n}$.

4. $7x-\frac{4n^2+5a}{8}$.

5. $x+1+\frac{x+1}{x}$.

6. $a+b-\frac{a^2+b^2}{a+b}$.

7. $2a-\frac{3a^2-2b^2}{2a}$.

8. $a^2+ab+b^2-\frac{2b^3}{b-a}$.

9. $3x-2-\frac{3}{2x-1}$.

10. $a-b-\frac{a^2+b^2}{a+b}$.

11. $x^2-3x-\frac{3x(3-x)}{x-2}$.

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR.

152. **1.** Reduce $\frac{5\,c\,d}{3\,a^2\,b}$, $\frac{3\,m\,x}{2\,a\,b^2}$, and $\frac{3\,n\,y}{4\,a^3\,b}$ to a common denominator.

Since multiplying each term of a fraction by the same quantity does not alter the value of the fraction (Art. 139), we may multiply each term of the first fraction by $4\,a\,b$, giving $\frac{20\,a\,b\,c\,d}{12\,a^3\,b^2}$; each term of the second by $6\,a^2$, giving $\frac{18\,a^2\,m\,x}{12\,a^3\,b^2}$; and each term of the third by $3\,b$, giving $\frac{9\,b\,n\,y}{12\,a^3\,b^2}$.

It will be observed that the common denominator is the *least common multiple* of the given denominators, which is also called the *least common denominator;* and that each term of either fraction is multiplied by a quantity which is obtained by dividing the least common denominator by its own denominator. Hence the following

RULE.

Find the least common multiple of the given denominators. Divide this by each denominator, separately, and multiply the corresponding numerators by the quotients; writing the results over the common denominator.

Before applying the rule, each fraction should be in its simplest form; entire and mixed quantities should be changed to a fractional form (Arts. 134 and 151).

Note. The common denominator may be *any* common multiple of the given denominators. The product of all the denominators is evidently a common multiple, and the rule is sometimes given as follows: "Multiply each numerator by all the denominators except its own, and write the results over the product of all the denominators."

2. Reduce $\frac{a\,y}{1-x}$, $\frac{a\,x^2}{(1-x)^2}$, and $\frac{x\,y^3}{(1-x)^3}$ to a common denominator.

The least common multiple of the given denominators is $(1-x)^3$. Dividing this by the first denominator, the quotient is $(1-x)^2$; dividing it by the second denominator, the quotient is $(1-x)$; and dividing it by the third denominator, the quotient is 1. Multiplying the corresponding numerators by these quotients, we obtain $a\,y\,(1-x)^2$, $a\,x^2\,(1-x)$, and $x\,y^3$ as the new numerators. Hence the results are

$$\frac{a\,y\,(1-x)^2}{(1-x)^3},\ \frac{a\,x^2\,(1-x)}{(1-x)^3},\ \text{and}\ \frac{x\,y^3}{(1-x)^3},\ \textit{Ans.}$$

EXAMPLES.

Reduce the following fractions to a common denominator:

3. $\frac{3\,a\,b}{8},\ \frac{2\,a\,c}{9},$ and $\frac{5\,b\,c}{12}.$

4. $\frac{x^2\,y}{10},\ \frac{x\,y\,z}{15},$ and $\frac{7\,y\,z^2}{30}.$

5. $\frac{3\,y\,z}{2\,x},\ \frac{4\,x\,z}{3\,y},$ and $\frac{5\,x\,y}{4\,z}.$

6. $\frac{4\,c-1}{3\,a\,b},\ \frac{3\,b-2}{5\,a\,c},$ and $\frac{5\,a}{6\,b\,c}.$

7. $\frac{2}{a^3\,x^2},\ \frac{3}{a\,x^3},$ and $\frac{4}{a^2\,x}.$

8. $\frac{5\,a\,z}{6x^2y},\ \frac{3\,b\,x}{8\,y^2z},$ and $\frac{7\,c\,y-m}{10\,x\,z^2}.$

9. $\frac{1}{a-b},\ \frac{1}{a+b},$ and $\frac{1}{a^2+b^2}.$

10. $\frac{x+3}{x^2-3\,x+2},\ \frac{x+1}{x^2-5\,x+6},$ and $\frac{x+2}{x^2-4\,x+3}.$

11. $\frac{2\,a}{a^2+a-6},\ \frac{3\,b}{a^2+5\,a+6},$ and $\frac{4\,c}{a^2-4}.$

12. $\frac{1}{x-1},\ \frac{1}{x^2-1},$ and $\frac{1}{x^3-1}.$

13. $\frac{a\,b}{a\,m-b\,m+a\,n-b\,n},\ \frac{m-n}{2\,a^2-2\,a\,b},$ and $\frac{a+b}{3\,b\,m+3\,b\,n}.$

14. Reduce $\frac{1-a}{(a-b)\,(a-c)},\ \frac{1-b}{(b-a)\,(b-c)},$ and $\frac{1-c}{(c-a)\,(c-b)}$ to a common denominator.

The fractions may be written (Art. 145) as follows:

$$\frac{1-a}{(a-b)(a-c)},\ \frac{b-1}{(a-b)(b-c)},\ \text{and}\ \frac{1-c}{(a-c)(b-c)}.$$

The least common denominator is now $(a-b)(a-c)(b-c)$. Applying the rule, we have the results,

$$\frac{(1-a)(b-c)}{(a-b)(a-c)(b-c)},\ \frac{(b-1)(a-c)}{(a-b)(a-c)(b-c)},\ \text{and}$$

$$\frac{(1-c)(a-b)}{(a-b)(a-c)(b-c)},\ \textit{Ans.}$$

Reduce to a common denominator:

15. $\dfrac{3}{a-1},\ \dfrac{2}{a+1},\ \text{and}\ \dfrac{a-2}{1-a^2}.$

16. $\dfrac{1}{1+x},\ \dfrac{2-x}{x-1},\ \text{and}\ \dfrac{3}{1-x^2}.$

17. $\dfrac{c+d}{(a+b)(a-b)},\ \dfrac{1-x}{(b-a)(c-d)},\ \text{and}\ \dfrac{b-a}{(d-c)(a+b)}.$

153. A fraction may be reduced to an equivalent one having a *given* denominator, by *dividing the given denominator by the denominator of the fraction, and multiplying both terms by the quotient.*

1. Reduce $\dfrac{a-b}{a^2-ab+b^2}$ to an equivalent fraction having a^3+b^3 for its denominator.

$$(a^3+b^3) \div (a^2-ab+b^2) = a+b;$$

multiplying both terms by $a+b$,

$$\frac{a-b}{a^2-ab+b^2} = \frac{(a-b)(a+b)}{(a^2-ab+b^2)(a+b)} = \frac{a^2-b^2}{a^3+b^3},\ \textit{Ans.}$$

EXAMPLES.

2. Reduce $\dfrac{a-b}{a+b}$ to a fraction with a^2-b^2 for its denominator.

3. Reduce $\dfrac{x+1}{x-3}$ to a fraction with $x^2+5x-24$ for its denominator.

4. Reduce $\dfrac{3m+2}{2m-5}$ to a fraction with $6m^2-19m+10$ for its denominator.

5. Reduce $\dfrac{4}{a-b}$ to a fraction with a^3-b^3 for its denominator.

6. Reduce $1+x$ to a fraction with $1-x$ for its denominator.

ADDITION AND SUBTRACTION OF FRACTIONS.

154. 1. Let it be required to add $\dfrac{b}{c}$ to $\dfrac{a}{c}$.

In $\dfrac{a}{c}$ and $\dfrac{b}{c}$, the unit is divided into c equal parts, and a and b parts, respectively, are taken, or in all $a+b$ parts; that is $\dfrac{a+b}{c}$. Thus,

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

2. Let it be required to subtract $\dfrac{b}{c}$ from $\dfrac{a}{c}$.

The result must be such a quantity as when added to $\dfrac{b}{c}$ will produce $\dfrac{a}{c}$; that quantity is evidently $\dfrac{a-b}{c}$ (Art. 154, 1).

Thus,
$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

Hence the following

RULE.

To add or subtract fractions, reduce them, if necessary, to a common denominator. Add or subtract the numerators, and write the result over the common denominator.

The final result should be reduced to its simplest form, wherever such reduction is possible.

1. Add $\frac{3b-a}{3a}$, $\frac{b+a}{2b}$, and $\frac{1-4b^2}{4ab}$.

The least common multiple of the denominators is $12ab$. Then, by the rule of Art. 152,

$$\frac{3b-a}{3a}+\frac{b+a}{2b}+\frac{1-4b^2}{4ab}=\frac{12b^2-4ab}{12ab}+\frac{6ab+6a^2}{12ab}$$

$$+\frac{3-12b^2}{12ab}=\frac{12b^2-4ab+6ab+6a^2+3-12b^2}{12ab}$$

$$=\frac{6a^2+2ab+3}{12ab}, \textit{ Ans.}$$

2. Subtract $\frac{4x-1}{2x}$ from $\frac{6a-2}{3a}$.

The least common denominator is $6ax$.

Then, $$\frac{6a-2}{3a}-\frac{4x-1}{2x}=\frac{12ax-4x}{6ax}-\frac{12ax-3a}{6ax}$$

$$=\frac{12ax-4x-(12ax-3a)}{6ax}=\frac{12ax-4x-12ax+3a}{6ax}$$

$$=\frac{3a-4x}{6ax}, \textit{ Ans.}$$

Note. When a fraction whose numerator is not a monomial is preceded by a – sign, it will be found convenient to enclose its numerator in a parenthesis before combining with the other numerators. If this is not done, care must be taken to change the sign of each term in the numerator before combining.

3. Simplify $\frac{4a^2-1}{2ac}-\frac{3ab^2-2}{3b^2c}-\frac{5a^2c^2+3}{5ac^3}$.

The least common denominator is $30ab^2c^3$.

Then, $$\frac{4a^2-1}{2ac}-\frac{3ab^2-2}{3b^2c}-\frac{5a^2c^2+3}{5ac^3}$$

$$=\frac{60a^2b^2c^2-15b^2c^2}{30ab^2c^3}-\frac{30a^2b^2c^2-20ac^2}{30ab^2c^3}-\frac{30a^2b^2c^2+18b^2}{30ab^2c^3}$$

$$=\frac{60a^2b^2c^2-15b^2c^2-(30a^2b^2c^2-20ac^2)-(30a^2b^2c^2+18b^2)}{30ab^2c^3}$$

$$=\frac{60a^2b^2c^2-15b^2c^2-30a^2b^2c^2+20ac^2-30a^2b^2c^2-18b^2}{30ab^2c^3}$$

$$=\frac{20ac^2-15b^2c^2-18b^2}{30ab^2c^3}, \text{ Ans.}$$

EXAMPLES.

Simplify the following:

4. $\frac{2x-5}{12}+\frac{3x+11}{18}$.

5. $\frac{3}{5ab^2}+\frac{1}{2a^2b}$.

6. $\frac{2a+3}{6}-\frac{3a+5}{8}$.

7. $\frac{m-2}{2mn}-\frac{2-3mn^2}{3m^2n^3}$.

8. $\frac{b-4a}{24a}+\frac{a+5b}{30b}$.

9. $\frac{a-b}{4}+\frac{2a+b}{6}+\frac{b-3a}{8}$.

10. $\frac{a^2+1}{3a^2}-\frac{6a^3+1}{12a^3}+\frac{b-2}{6b}$.

11. $\frac{2x-1}{12}+\frac{2x+3}{15}-\frac{6x+1}{20}$.

12. $\frac{m+2}{7}-\frac{m+2}{14}-\frac{m+3}{21}$.

13. $\frac{2}{3}-\frac{2x-1}{6x}-\frac{3x^2+1}{9x^2}$.

14. $\frac{x-2}{2}+\frac{3x+1}{3}-\frac{6x-5}{4}-\frac{3}{5}$.

15. $\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$.

16. Simplify $\frac{2x+1}{2x(x-1)} - \frac{3x-1}{3x(x+1)} - \frac{11}{4(x^2-1)}$.

The least common denominator is $12x(x^2-1)$.

Then, $$\frac{2x+1}{2x(x-1)} - \frac{3x-1}{3x(x+1)} - \frac{11}{4(x^2-1)}$$

$$= \frac{6(x+1)(2x+1)}{12x(x^2-1)} - \frac{4(x-1)(3x-1)}{12x(x^2-1)} - \frac{33x}{12x(x^2-1)}$$

$$= \frac{12x^2+18x+6}{12x(x^2-1)} - \frac{12x^2-16x+4}{12x(x^2-1)} - \frac{33x}{12x(x^2-1)}$$

$$= \frac{12x^2+18x+6-(12x^2-16x+4)-33x}{12x(x^2-1)}$$

$$= \frac{x+2}{12x(x^2-1)}, \textit{ Ans.}$$

Simplify the following:

17. $\frac{1}{x+2} + \frac{1}{3-x}$.

18. $\frac{1}{x+7} - \frac{1}{x+8}$.

19. $\frac{a+b}{a-b} + \frac{a-b}{a+b}$.

20. $\frac{1+x}{1-x} - \frac{1-x}{1+x}$.

21. $\frac{a}{a+b} + \frac{b}{a-b} + \frac{2ab}{a^2-b^2}$.

22. $\frac{1}{x+y} + \frac{1}{x-y} - \frac{2x}{x^2+y^2}$.

23. $\frac{1}{x-1} - \frac{x}{x^2-1} + \frac{3}{x^3-1}$.

24. $\frac{2x-6}{x^2+3x+2} - \frac{x+2}{x^2-2x-3} - \frac{x+1}{x^2-x-6}$.

25. Simplify $\frac{x}{x+1} + \frac{x}{1-x} + \frac{2x}{x^2-1}$.

The expression may be written (Art. 143) as follows:

$$\frac{x}{x+1}-\frac{x}{x-1}+\frac{2x}{x^2-1}.$$

The least common denominator is x^2-1.

Then, $\frac{x}{x+1}-\frac{x}{x-1}+\frac{2x}{x^2-1}=\frac{x^2-x}{x^2-1}-\frac{x^2+x}{x^2-1}+\frac{2x}{x^2-1}$

$$=\frac{x^2-x-(x^2+x)+2x}{x^2-1}=\frac{0}{x^2-1}=0, \textit{ Ans. } \text{(Art. 102)}.$$

Simplify the following:

26. $\frac{3}{a-b}+\frac{4}{b-a}.$

27. $\frac{5a+1}{3a+3}+\frac{3a-1}{2-2a}.$

28. $\frac{1}{3x-x^2}+\frac{1}{x^2-9}.$

29. $\frac{x}{1+x}-\frac{x}{1-x}+\frac{x^2}{x^2-1}.$

30. $\frac{1}{(a-b)(b-c)}+\frac{1}{(b-a)(a-c)}-\frac{1}{(c-a)(c-b)}.$

31. $\frac{2}{(x-2)(x-3)}-\frac{3}{(3-x)(4-x)}-\frac{1}{(x-4)(2-x)}.$

MULTIPLICATION OF FRACTIONS.

155. We showed, in Art. 137, that a fraction could be multiplied by an integer either by multiplying its numerator or by dividing its denominator by that integer. We will now show how to multiply one fraction by another.

Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$$\text{Let } \frac{a}{b}=x, \text{ and } \frac{c}{d}=y;$$

where x and y may be either integral or fractional.

Since the dividend equals the product of the divisor and quotient,

$$a = b\,x, \text{ and } c = d\,y.$$

Therefore, by Art. 44, Ax. 3, $a\,c = b\,d\,x\,y$.

Regarding $a\,c$ as the dividend, $b\,d$ as the divisor, and $x\,y$ as the quotient, we have

$$x\,y = \frac{a\,c}{b\,d}.$$

Therefore, putting for x and y their values,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a\,c}{b\,d}.$$

Hence the following

RULE.

Multiply the numerators together for the numerator of the resulting fraction, and the denominators for its denominator.

Mixed quantities should be reduced to a fractional form before applying the rule.

When there are common factors in the numerators and denominators, they should be cancelled before performing the multiplication.

EXAMPLES.

1. Multiply together $\frac{6\,x^2\,y}{5\,a^3\,b^2}$, $\frac{10\,a^2\,y}{9\,b\,x}$, and $\frac{3\,b^4\,x^3}{4\,a\,y^2}$.

$$\frac{6\,x^2\,y}{5\,a^3\,b^2} \times \frac{10\,a^2\,y}{9\,b\,x} \times \frac{3\,b^4\,x^3}{4\,a\,y^2} = \frac{6 \times 10 \times 3\,a^2\,b^4\,x^5\,y^2}{5 \times 9 \times 4\,a^4\,b^3\,x\,y^2} = \frac{b\,x^4}{a^2}, \quad \textit{Ans.}$$

Multiply together the following:

2. $\frac{a^2\,b\,c}{m\,n^2}$ and $\frac{a^3\,b^2}{m^3\,n\,d}$.

3. $\frac{3\,a^3\,x}{7\,h^4}$ and $\frac{4\,a\,b}{5\,h\,m}$.

4. $\frac{3\,a\,b\,x^2}{5\,a\,y^2}$ and $\frac{5\,x\,y^2}{3\,a\,b\,x^3}$.

5. $\frac{m\,y^n}{4\,a\,x}$ and $\frac{a\,x}{m\,y^n}$.

6. $\frac{2a}{3b}, \frac{6c}{5a}$, and $\frac{5b}{8c}$.

7. $\frac{8x^2}{9y^3}, \frac{15y^2}{16z^3}$, and $\frac{3z^4}{10x^3}$.

8. $\frac{3ab^2}{4cd}, \frac{3ac^2}{2bd}$, and $\frac{8ad^2}{9bc}$.

9. $\frac{3m^3}{2x^2}, \frac{2n^4}{3m}$, and $\frac{11x^2}{4n^2}$.

10. Multiply together

$$\frac{x^2-2x}{x^2-2x-3}, \frac{x^2-9}{x^2-x}, \text{ and } \frac{x^2+x}{x^2+x-6}.$$

$$\frac{x^2-2x}{x^2-2x-3} \times \frac{x^2-9}{x^2-x} \times \frac{x^2+x}{x^2+x-6}$$

$$= \frac{x(x-2)(x+3)(x-3)x(x+1)}{(x-3)(x+1)x(x-1)(x+3)(x-2)} = \frac{x}{x-1}, \text{ Ans.}$$

Multiply together the following:

11. $\frac{3x^2-x}{5}$ and $\frac{10}{2x^2-4x}$.

12. $\frac{4x+2}{3}$ and $\frac{5x}{2x+1}$.

13. $\frac{a^2-2ab+b^2}{a+b}$ and $\frac{b}{ax-bx}$.

14. $\frac{a-b}{a^2+ab}$ and $\frac{a^2-b^2}{a^2-ab}$.

15. $\frac{1-x^2}{1+y}, \frac{1-y^2}{x+x^2}$, and $\frac{1}{1-x}$.

16. $\frac{x^2-16}{x^2+5x}$ and $\frac{x^2-25}{x^2-4x}$.

17. $\frac{a^3-a^2+a}{x^2+2x+4}$ and $\frac{x^3-8}{a^3+1}$.

18. $\frac{x^2+5x+6}{x^2-4x-21}$ and $\frac{x^2-7x}{x^2-4}$.

19. $1+\frac{4}{x}-\frac{5}{x^2}$ and $\frac{x-7}{x^2-8x+7}$.

20. $\frac{9}{x^2}-1$ and $\frac{4-x^2}{x^2-5x+6}$.

21. $\frac{x^2-3x+2}{x^2-8x+15}$, $\frac{x^2-7x+12}{x^2-5x+4}$, and $\frac{x^3-5x^2}{x^2-4}$.

22. $\frac{x^3-y^3}{x^2-xy+y^2}$, $\frac{x^3+y^3}{x^2+xy+y^2}$, and $1+\frac{y}{x-y}$.

23. $\frac{a^2-b^2-c^2+2bc}{a^2+c^2-b^2+2ac}$ and $\frac{a^2-b^2-c^2-2bc}{a^2+c^2-b^2-2ac}$.

24. $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4b^2}{a^2-b^2}$ and $\frac{a+b}{2b}$.

25. $\frac{2x+y}{x+y}-1-\frac{y}{y-x}-\frac{x^2}{x^2-y^2}$ and $\frac{x^2-y^2}{x^2+y^2}$.

DIVISION OF FRACTIONS.

156. We showed, in Art. 138, that a fraction could be divided by an integer either by dividing its numerator or by multiplying its denominator by that integer. We will now show how to divide one fraction by another.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

Let x denote the quotient of $\frac{a}{b}\div\frac{c}{d}$.

Then, since the quotient multiplied by the divisor gives the dividend, we have

$$x\times\frac{c}{d}=\frac{a}{b};\text{ or, }\frac{xc}{d}=\frac{a}{b}.$$

Multiplying each of these equals by $\frac{d}{c}$ (Art. 44, Ax. 3),

$$x = \frac{a\,d}{b\,c}.$$

Therefore, $$\frac{a}{b} \div \frac{c}{d} = \frac{a\,d}{b\,c}.$$

Hence the following

RULE.

Invert the divisor, and proceed as in multiplication.

Mixed quantities should be reduced to a fractional form, before applying the rule.

EXAMPLES.

1. Divide $\frac{6\,a^2\,b}{5\,x^3\,y^4}$ by $\frac{9\,a\,b^3}{10\,x^2\,y^5}$.

$$\frac{6\,a^2\,b}{5\,x^3\,y^4} \div \frac{9\,a\,b^3}{10\,x^2\,y^5} = \frac{6\,a^2\,b}{5\,x^3\,y^4} \times \frac{10\,x^2\,y^5}{9\,a\,b^3} = \frac{4\,a\,y}{3\,b^2\,x}, \textit{ Ans.}$$

2. Divide $\frac{x^2 - 9}{15}$ by $\frac{x+3}{5}$.

$$\frac{x^2 - 9}{15} \div \frac{x+3}{5} = \frac{(x+3)\,(x-3)}{15} \times \frac{5}{x+3} = \frac{x-3}{3}, \textit{ Ans.}$$

Divide the following:

3. $\frac{7\,m^2}{2}$ by $\frac{3\,n^2}{13}$.

4. $\frac{7\,a^3\,b}{12\,m^2\,n^3}$ by $\frac{14\,a\,b^4}{3\,m\,n}$.

5. $\frac{18\,m\,x^3}{25\,n\,y^2}$ by $\frac{6\,m^2\,x^4}{5\,n^2\,y^5}$.

6. $\frac{1}{4} - \frac{4}{x^2}$ by $\frac{x^2}{12} + \frac{x}{3}$.

7. $\frac{x^2 - y^2}{x^2 - 2\,x\,y + y^2}$ by $\frac{x^2 + x\,y}{x - y}$.

8. $9 + \frac{5\,y^2}{x^2 - y^2}$ by $3 + \frac{5\,y}{x - y}$.

9. $\frac{1}{a^2 + 2\,a - 15}$ by $\frac{1}{a^2 - 2\,a - 3}$.

10. $\frac{x^3 - 4\,x}{x^2 + 5\,x + 6}$ by $\frac{x^2 - 3\,x + 2}{x^2 + 2\,x - 3}$.

COMPLEX FRACTIONS.

157. **A Complex Fraction** is one having a fraction in its numerator, or denominator, or both. It may be regarded as a case in division, since its numerator answers to the dividend, and its denominator to the divisor.

However, since multiplying a fraction by any multiple of its denominator must cancel that denominator, to simplify a complex fraction, we may *multiply both of its terms by the least common multiple of their denominators.*

EXAMPLES.

1. Reduce $\dfrac{\frac{a}{c}}{\frac{d}{b}}$ to its simplest form.

FIRST METHOD.

Proceeding as in division,

$$\frac{\frac{a}{c}}{\frac{d}{b}} = \frac{a}{c} \times \frac{b}{d} = \frac{a\,b}{c\,d}, \textit{ Ans.}$$

SECOND METHOD.

Multiplying both terms by the least common multiple of their denominators,

$$\frac{\frac{a}{c}}{\frac{d}{b}} = \frac{\frac{a}{c} \times b\,c}{\frac{d}{b} \times b\,c} = \frac{a\,b}{c\,d}, \textit{ Ans.}$$

2. Reduce $\dfrac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} + \frac{a}{a+b}}$ to its simplest form.

The least common multiple of the denominators is $a^2 - b^2$. Multiplying each term by $a^2 - b^2$, we have

$$\frac{a(a+b) - a(a-b)}{b(a+b) + a(a-b)} = \frac{a^2 + ab - a^2 + ab}{ab + b^2 + a^2 - ab} = \frac{2ab}{a^2 + b^2}, \textit{ Ans.}$$

3. Reduce $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}}$ to its simplest form.

$$\frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}} = \frac{1}{1 + \dfrac{x}{x+1}} = \frac{x+1}{x+1+x} = \frac{x+1}{2x+1}, \textit{ Ans.}$$

Reduce the following to their simplest forms:

4. $\dfrac{\dfrac{a}{b}}{m+n}$.

5. $\dfrac{a + \dfrac{b}{c}}{x - \dfrac{m}{n}}$.

6. $\dfrac{m - \dfrac{n}{3}}{x}$.

7. $\dfrac{y - x + \dfrac{a}{2}}{\dfrac{31}{4}}$.

8. $\dfrac{x - \dfrac{1}{x}}{1 + \dfrac{1}{x}}$.

9. $\dfrac{x^2 + \dfrac{1}{x}}{1 + \dfrac{1}{x}}$.

10. $\dfrac{\dfrac{a}{b} - \dfrac{b}{a}}{\dfrac{1}{b} - \dfrac{1}{a}}$.

11. $\dfrac{x^2 + \dfrac{1}{y^3}}{x - \dfrac{2}{y}}$.

12. $\dfrac{\dfrac{a}{b} - \dfrac{b^2}{a^2}}{a^2 + ab + b^2}$.

13. $\dfrac{x - 7 + \dfrac{12}{x}}{x + 3 - \dfrac{18}{x}}$.

14. $\dfrac{\dfrac{1}{1-x} - \dfrac{1}{1+x}}{\dfrac{1}{1-x} + \dfrac{1}{1+x}}$.

15. $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+1}{3-x}}}$.

16. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

18. $\dfrac{\dfrac{m-n}{m+n}-\dfrac{m^3-n^3}{m^3+n^3}}{\dfrac{m+n}{m-n}+\dfrac{m^2+n^2}{m^2-n^2}}$.

17. $\dfrac{\dfrac{x+2y}{x+y}+\dfrac{x}{y}}{\dfrac{x+2y}{y}-\dfrac{x}{x+y}}$.

19. $\dfrac{x-3a+\dfrac{4a^2}{a+x}}{x-\dfrac{2a^2}{a+x}}$.

158. In Art. 42, we defined the reciprocal of a quantity as being 1 divided by that quantity. Therefore the reciprocal of $\dfrac{m}{n}=\dfrac{1}{\frac{m}{n}}=\dfrac{n}{m}$; or, *the reciprocal of a fraction is the fraction inverted.*

XII.—SIMPLE EQUATIONS.

159. An **Equation** is an expression of equality between two quantities. Thus,

$$x+4=16$$

is an equation, expressing the equality of the quantities $x+4$ and 16.

160. The **First Member** of an equation is the quantity on the left of the sign of equality. The **Second Member** is the quantity on the right of that sign. Thus, in the equation $x+4=16$, $x+4$ is the first member, and 16 is the second member.

The *sides* of an equation are its two members.

161. An **Identical Equation** is one in which the two members are equal, whatever values are given to the letters involved, if the same value be given to the same letter in every part of the equation; as,

$$x - y = x - y,$$

$$2a + 2bc = 2(a + bc).$$

162. Equations usually consist of known and unknown quantities. Unknown quantities are generally represented by the last letters of the alphabet, x, y, z; but *any* letter may stand for an unknown quantity. Known quantities are represented by numbers, or by any except the last letters of the alphabet.

163. A **Numerical Equation** is one in which all the known quantities are represented by numbers; as,

$$2x - 17 = x - 5.$$

A **Literal Equation** is one in which some or all the known quantities are expressed by letters; as,

$$2x + a = bx^2 - 10.$$

164. The **Degree** of an equation containing but one unknown quantity is denoted by the highest power of that unknown quantity in the equation. Thus,

$$\left.\begin{array}{l} x + 14 = 3x - 4 \\ \text{and} \quad cx = a^2 + bd \end{array}\right\} \text{are equations of the } \textit{first degree.}$$

$3x^2 - 2x = 65$ is an equation of the *second degree.*

In like manner we have equations of the *third degree, fourth degree,* and so on.

When an equation contains more than one unknown quantity, its degree is determined by the greatest *sum* of the exponents of the unknown quantities in any term. Thus,

$x + xy = 25$ is an equation of the second degree.

$x^2 - y^2z = ab^3$ is an equation of the third degree.

Note. These definitions of degree require that the equation shall not contain unknown quantities in the denominators of fractions, or under radical signs, or affected with fractional or negative exponents.

165. A **Simple Equation** is an equation of the first degree.

166. The **Root** of an equation containing but one unknown quantity is the value of that unknown quantity; or it is the value which, being put in place of the unknown quantity, makes the equation identical. Thus, in the equation

$$3x - 7 = x + 9,$$

if 8 is put in place of x, the equation becomes

$$24 - 7 = 8 + 9,$$

which is identical; hence the root of the equation is 8.

Note. An equation may have more than one root. For example, in the equation

$$x^2 = 7x - 12,$$

if 3 is put in place of x, the equation becomes $9 = 21 - 12$; and if 4 is put in place of x, it becomes $16 = 28 - 12$. Each of these results being identical, it follows that either 3 or 4 is a root of the equation.

167 It will be shown hereafter that a simple equation has but *one* root; an equation of the second degree, *two* roots; and, in general, that the degree of the equation and the number of its roots correspond.

168. The *solution* of an equation is the process of finding its roots. A root is *verified*, or the equation *satisfied*, when, the root being substituted for its symbol, the equation becomes identical.

TRANSFORMATION OF EQUATIONS.

169. To **Transform** an equation is to change its form without destroying the equality.

170. The operations required in the transformation are based upon the general principle deduced directly from the axioms (Art. 44):

If the same operations are performed upon equal quantities, the results will be equal.

Hence,

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

TRANSPOSITION.

171. To **Transpose** a term of an equation is to change it from one member to the other without destroying the equality.

172. Consider the equation $x - a = b$.

Adding a to each member (Art. 170), we have

$$x - a + a = b + a$$

$$\text{or,} \quad x = b + a,$$

where $-a$ has been transposed to the second member by changing its sign.

173. Again, consider the equation $x + a = b$.

Subtracting a from each member (Art. 170), we have

$$x + a - a = b - a$$

$$\text{or,} \quad x = b - a,$$

where a has been transposed to the second member by changing its sign.

174. Hence the following

RULE.

Any term may be transposed from one member of an equation to the other, provided its sign be changed.

Also, *if the same term appear in both members of an equation affected with the same sign, it may be suppressed.*

1. In the equation $2x - 12 + 3 = x - 5x + 9$, transpose the unknown terms to the first member, and the known terms to the second.

Result, $2x - x + 5x = 12 - 3 + 9$.

EXAMPLES.

Transpose the unknown terms to the first member, and the known terms to the second, in the following:

2. $3x - 2a = 45 + 2x$.

3. $4x + 9 = 25 - 12x$.

4. $4a^2x + b^2 = -4abx + 4ac + b^2$.

5. $ac + cx - ad = 2a - 7x$.

6. $bc + a^2x - mn^2 = bx + ad - 5$.

7. $3 - b - x = c - 3x$.

8. $2a - 3c = 5x - b - dx$.

9. $10x - 312 = 32x + 21 - 52x$.

CLEARING OF FRACTIONS.

175. **1.** Clear the equation $\frac{2x}{3} - \frac{5}{4} = \frac{5x}{6} + \frac{3}{8}$ of fractions.

The least common multiple of 3, 4, 6, and 8 is 24. Multiplying each term of the equation by 24 (Art. 170), we have

$$16x - 30 = 20x + 9,$$

where the denominators have been removed. Hence the following

RULE.

Multiply each term of the equation by the least common multiple of the denominators.

Note. The operation of clearing of fractions may be performed by multiplying each term of the equation by *any* common multiple of the denominators. The product of all the denominators is obviously a common multiple, and the rule is sometimes given as follows: "Multiply each term of the equation by the product of all the denominators."

EXAMPLES.

Clear the following equations of fractions:

2. $\frac{ax}{b} - c = \frac{dx}{e} - \frac{m}{n}$.

3. $\frac{x}{2a} - \frac{2a}{3b} = \frac{1}{4ab} - \frac{x}{6}$.

4. $x - \frac{ax}{b} + \frac{cx}{d} - \frac{a}{e} = 0$.

5. $\frac{x}{5} + \frac{x}{12} = \frac{x}{10} - 22$.

6. $\frac{3x}{4} - a = \frac{5x}{6} + 2 - \frac{b}{3}$.

7. $x - \frac{x}{7} + 20 = \frac{x}{2} + \frac{x}{4} + 26$.

8. $\frac{2x}{a^2} - \frac{3c}{a^3} - \frac{5x}{2} + bd = 0$.

9. $\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}$.

10. Clear the equation $21 - \frac{5x - 5}{8} = \frac{11 - 3x}{16} - \frac{97 - 7x}{2}$ of fractions.

The least common denominator is 16; multiplying each term by 16, we have

$$336 - (10x - 10) = 11 - 3x - (776 - 56x)$$

or, $$336 - 10x + 10 = 11 - 3x - 776 + 56x, \textit{ Ans.}$$

Note. When a fraction, whose numerator is not a monomial, is preceded by a − sign, it will be found convenient, on clearing of fractions, to enclose the numerator in a parenthesis. If this is not done, care must be taken to change the sign of each term in the numerator.

Clear the following equations of fractions:

11. $\frac{x}{2} - \frac{a + x}{3} = \frac{15}{2}$.

12. $\frac{ax + b}{c} - \frac{cx + d}{bc} = \frac{a}{b}$.

13. $\frac{3}{1+x}-\frac{2}{1-x}=0.$

14. $\frac{x}{2}-\frac{x^2-3}{2x+1}-\frac{1}{3}=0.$

15. $\frac{3}{x+1}-\frac{2}{x-1}-\frac{5x}{x^2-1}=0.$

16. $\frac{x+1}{5}-\frac{x-3}{2}-\frac{2x+1}{3}=0.$

CHANGING SIGNS.

176. *The signs of all the terms of an equation may be changed without destroying the equality.*

For, in the equation $a-x=b-c$, let all the terms be multiplied by -1 (Art. 170). Then,

$$-a+x=-b+c$$

$$\text{or, } x-a=c-b.$$

For example, the equation $-5x-a=3x-b$, by changing the signs of all the terms, may be written

$$5x+a=b-3x.$$

SOLUTION OF SIMPLE EQUATIONS.

177. To solve a simple equation containing but one unknown quantity.

1. Solve the equation $5x-7=x+9$.

Transposing the unknown terms to the first member, and the known terms to the second,

$$5x-x=7+9$$

Uniting similar terms, $4x=16$

Dividing each member by 4 (Art. 170),

$$x=4, \textit{ Ans.}$$

This value of x we may verify (Art. 168). Thus, substituting 4 for x in the given equation, it becomes

$$20-7=4+9,$$

which is identical; hence the value of x is verified.

2. Solve the equation $8x + 19 = 25x - 32$.

Transposing, $8x - 25x = -19 - 32$

Uniting terms, $-17x = -51$

Dividing by -17, $x = 3$, *Ans.*

To verify the result, put 3 for x in the given equation.

Then, $24 + 19 = 75 - 32$

or, $43 = 43$.

3. Solve the equation $\frac{3x}{4} + \frac{5}{6} = \frac{2x}{3} - \frac{x}{2}$.

Clearing of fractions, by multiplying each term of the equation by 12, the least common multiple of the denominators,

$$9x + 10 = 8x - 6x$$

Transposing, $9x - 8x + 6x = -10$

Uniting terms, $7x = -10$

Dividing by 7, $x = -\frac{10}{7}$, *Ans.*

To verify this result, put $x = -\frac{10}{7}$ in the given equation.

Then, $$-\frac{30}{28} + \frac{5}{6} = -\frac{20}{21} + \frac{10}{14}$$

or, $$\frac{-90 + 70}{84} = \frac{-80 + 60}{84}$$

or, $$-\frac{20}{84} = -\frac{20}{84}.$$

RULE.

Clear the equation of fractions if it has any. Transpose the unknown terms to the first member, and the known terms to the second, and reduce each member to its simplest form. Divide both members of the resulting equation by the coefficient of the unknown quantity.

EXAMPLES.

Solve the following equations:

4. $3x + 5 = x + 11$.
5. $3x - 2 = 5x - 16$.
6. $2 - 2x = 3 - x$.
7. $3x + 2 - 5x = x - 7 + 3$.
8. $18 - 5x - 2x = 3 + x + 7x$.
9. $5x - 3 + 17 = 19 - 2x - 2$.

10. Solve the equation

$$5(7 + 3x) - (2x - 3)(1 - 2x) - (2x - 3)^2 - (5 + x) = 0.$$

Performing the operations indicated, we have

$$35 + 15x + 4x^2 - 8x + 3 - 4x^2 + 12x - 9 - 5 - x = 0$$

Transposing, and suppressing the terms $4x^2$ and $-4x^2$,

$$15x - 8x + 12x - x = -35 - 3 + 9 + 5$$

$$18x = -24$$

$$x = -\frac{24}{18} = -\frac{4}{3}, \text{ Ans.}$$

Solve the following equations:

11. $3 + 2(2x + 3) = 2x - 3(2x + 1)$.
12. $2x - (4x - 1) = 5x - (x - 1)$.
13. $7(x - 2) - 5(x + 3) = 3(2x - 5) - 6(4x - 1)$.
14. $3(3x + 5) - 2(5x - 3) = 13 - (5x - 16)$.
15. $(2x - 1)(3x + 2) = (3x - 5)(2x + 20)$.
16. $(5 - 6x)(2x - 1) = (3x + 3)(13 - 4x)$.
17. $(x - 3)^2 - (5 - x)^2 = -4x$.
18. $(2x - 1)^2 - 3(x - 2) + 5(3x - 2) - (5 - 2x)^2 = 0$.

19. Solve the equation $\frac{3}{x} - \frac{7}{2x} = \frac{7}{12} - \frac{5}{3x}$.

Clearing of fractions, by multiplying each term by $12x$, the least common multiple of the denominators,

$$\begin{aligned} 36 - 42 &= 7x - 20 \\ -7x &= -36 + 42 - 20 \\ -7x &= -14 \\ x &= 2, \ \textit{Ans.} \end{aligned}$$

Solve the following equations:

20. $\frac{3x}{4} - 7 = \frac{5x}{3} - \frac{13x}{4}.$

21. $\frac{1}{6} + \frac{1}{2x} = \frac{1}{4} + \frac{1}{12x}.$

22. $\frac{x}{3} - \frac{x}{4} + \frac{x}{6} = 18.$

23. $\frac{2}{3} - \frac{3}{4} - \frac{4}{5} = \frac{7}{x} - 1.$

24. $\frac{2x}{5} - x = 2x - \frac{3x}{2} - 11.$

25. $\frac{x}{2} + \frac{11}{6} - \frac{x}{3} = \frac{x}{6} - \frac{3x}{4}.$

26. $x - \frac{x}{7} + 20 = \frac{x}{2} + \frac{x}{4} + 26.$

27. $\frac{3}{x} - \frac{5}{2x} = 7 - \frac{3}{2x}.$

28. Solve the equation $\frac{3x-1}{4} - \frac{2x+1}{3} - \frac{4x-5}{5} = 4.$

Multiplying each term by 60,

$$\begin{aligned} 45x - 15 - (40x + 20) - (48x - 60) &= 240 \\ 45x - 15 - 40x - 20 - 48x + 60 &= 240 \\ 45x - 40x - 48x &= 15 + 20 - 60 + 240 \\ -43x &= 215 \\ x &= -5, \ \textit{Ans.} \end{aligned}$$

Solve the following equations:

29. $3x + \frac{5x+3}{7} = \frac{7x}{2}.$

30. $x - \frac{2x+1}{5} = 5x - \frac{5}{3}.$

31. $7x - \frac{11x-3}{4} = 3x + 7.$

32. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}$.

33. $\frac{5x-2}{3}-\frac{3x+4}{4}-\frac{7x+2}{6}=\frac{x-10}{2}$.

34. $\frac{x+1}{2}-\frac{2x-5}{5}=\frac{11x+5}{10}-\frac{x-13}{3}$.

35. $\frac{5x+1}{3}+\frac{17x+7}{9}-\frac{3x-1}{2}=\frac{7x-1}{6}$.

36. $\frac{4+x}{7}=\frac{3x-2}{2}-\frac{11x+2}{14}-\frac{2-9x}{3}$.

37. $\frac{2x+1}{3}=\frac{4x+5}{4}-\frac{8+x}{6}+\frac{2x+5}{8}$.

38. Solve the equation $\frac{2}{x-1}-\frac{3}{x+1}=\frac{1}{x^2-1}$.

Clearing of fractions, by multiplying each term by x^2-1,

$$2(x+1)-3(x-1)=1$$

$$2x+2-3x+3=1$$

$$2x-3x=-2-3+1$$

$$-x=-4$$

$$x=4, \textit{ Ans.}$$

39. Solve the equation $\frac{4x+3}{10}-\frac{12x-5}{5x-1}=\frac{2x-1}{5}$.

Clearing of fractions partially, by multiplying each term by 10,

$$4x+3-\frac{120x-50}{5x-1}=4x-2$$

$$4x+3-4x+2=\frac{120x-50}{5x-1}$$

$$5=\frac{120x-50}{5x-1}$$

Clearing of fractions, by multiplying each term by $5\,x - 1$,

$$25\,x - 5 = 120\,x - 50$$

$$25\,x - 120\,x = 5 - 50$$

$$-95\,x = -45$$

$$x = \frac{45}{95} = \frac{9}{19}, \textit{ Ans.}$$

Note. If the denominators are partly monomial, and partly polynomial, clear of fractions at first partially, multiplying by such a quantity as will remove the *monomial* denominators.

Solve the following equations:

40. $\frac{x}{3} - \frac{x^2 - 5\,x}{3\,x - 7} = \frac{2}{3}.$

41. $\frac{2\,x - 1}{3\,x + 4} = \frac{2\,x + 7}{3\,x + 2}.$

42. $\frac{5 - 2\,x}{x + 1} = \frac{3 - 2\,x}{x + 4}.$

43. $\frac{6\,x^2 - 3\,x + 2}{2\,x^2 + 5\,x - 7} = 3.$

44. $\frac{3}{1 - x} - \frac{2}{1 + x} - \frac{1}{1 - x^2} = 0.$

45. $\frac{x - 1}{x - 2} - \frac{x + 1}{x + 2} = \frac{3}{x^2 - 4}.$

46. $\frac{x}{9} = \frac{x + 1}{3} - \frac{7 - 2\,x^2}{1 - 9\,x}.$

47. $\frac{2\,x^2 + 3\,x}{2\,x + 1} + \frac{1}{3\,x} = x + 1.$

48. Solve the equation $2\,a\,x - 3\,b = x + c - 3\,a\,x.$

Transposing and uniting terms, $5\,a\,x - x = 3\,b + c$

Factoring the first member, $x\,(5\,a - 1) = 3\,b + c$

Dividing by $5\,a - 1$, $x = \frac{3\,b + c}{5\,a - 1}$, *Ans.*

49. Solve the equation $(b - c\,x)^2 - (a - c\,x)^2 = b\,(b - a).$

Performing the operations indicated,

$$b^2 - 2\,b\,c\,x + c^2\,x^2 - a^2 + 2\,a\,c\,x - c^2\,x^2 = b^2 - a\,b$$

Suppressing the term b^2 in both members, and the terms $c^2\,x^2$ and $-c^2\,x^2$ in the first member,

$$-2bcx - a^2 + 2acx = -ab$$

$$2acx - 2bcx = a^2 - ab$$

Factoring both members, $2cx(a-b) = a(a-b)$

Dividing by $2c(a-b)$, $x = \dfrac{a(a-b)}{2c(a-b)} = \dfrac{a}{2c}$, *Ans.*

Solve the following equations:

50. $2ax + d = 3c - bx$.

51. $2x - 4a = 3ax + a^2 - a^2x$.

52. $2ax + 6b^2 = 3bx + 4ab$.

53. $6bmx - 5an = 15am - 2bnx$.

54. $(a^2 - 2x)^2 = (4x - b)(x + 4c)$.

55. $(2a - 3x)(2a + 3x) = b^2 - (3x - b)^2$.

56. $(3a - x)(a + 2x) = (5a + x)(a - 2x)$.

57. $\dfrac{3bx}{a} - \dfrac{2}{c} = \dfrac{3}{a} - \dfrac{2bx}{c}$.

58. $\dfrac{x}{2a} - 3 + \dfrac{x}{4a^3} = \dfrac{x}{3a^2} - 2a(2 - 3a)$.

59. $\dfrac{x}{2} = \dfrac{1 + 2ax}{2a} - \dfrac{2x + 1}{a^2}$.

60. $\dfrac{x}{ab} - \dfrac{x + ab}{3b} = \dfrac{x}{3b} - (a - 1)$.

61. $\dfrac{x}{2} - \dfrac{a - bcx}{2bc} = \dfrac{x}{6c} - \dfrac{ac - 4bx}{3bc}$.

62. Solve the equation $.2x - .01 - .03x = .113x + .161$.

FIRST METHOD.

Changing the decimals into common fractions,

$$\frac{2x}{10} - \frac{1}{100} - \frac{3x}{100} = \frac{113x}{1000} + \frac{161}{1000}.$$

Multiplying each term by 1000,

$$200\,x - 10 - 30\,x = 113\,x + 161$$
$$57\,x = 171$$
$$x = 3, \textit{ Ans.}$$

SECOND METHOD.

Transposing, $.2\,x - .03\,x - .113\,x = .01 + .161$

Uniting terms, $.057\,x = .171$

Dividing by .057, $x = 3,$ *Ans.*

Solve the following equations:

63. $.3\,x - .02 - .003\,x = .7 - .06\,x - .006.$

64. $.001\,x - .32 = .09\,x - .2\,x - .653.$

65. $.3\,(1.2\,x - 5) = 14 + .05\,x.$

66. $.7\,(x + .13) = .03\,(4\,x - .1) + .5.$

67. $3.3\,x - \dfrac{.72\,x - .55}{.5} = .1\,x + 9.9.$

68. $\dfrac{2 - 3\,x}{1.5} + \dfrac{5\,x}{1.25} - \dfrac{2\,x - 3}{9} = \dfrac{x - 2}{1.8} + 2\dfrac{7}{9}.$

178. *To prove that a simple equation can have but one root.*

We have seen that every simple equation can be reduced to the form $x = a$.

Suppose, if possible, that a simple equation can have two roots, and that r_1 and r_2 are the roots of the equation $x = a$. Then (Art. 168),

$$r_1 = a,$$
$$r_2 = a.$$

Hence, $r_1 = r_2$; that is, the two supposed roots are identical. Therefore a simple equation can have but one root.

XIII. — PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

179. A **Problem** is a question proposed for solution.

180. The **Solution** of a problem by Algebra consists of two distinct parts:

1. The **Statement**, or the process of expressing the conditions of the problem in algebraic language, by one or more equations.

2. The **Solution** of the resulting equation or equations, or the process of determining from them the values of the unknown quantities.

The statement of a problem often includes a consideration of *ratio* and *proportion* (Art. 21).

181. **Ratio** is the relation, with respect to magnitude, which one quantity bears to another of the same kind, and is the result arising from the division of one quantity by the other.

A **Proportion** is an equality of ratios.

Thus,

$a : b$, or $\frac{a}{b}$, indicates the ratio of a to b.

$a : b = c : d$, is a proportion, indicating that the ratio of a to b, is equal to the ratio of c to d.

In a proportion the relation of the terms is such that the product of the first and fourth is equal to the product of the second and third.

For, $a : b = c : d$ is the same as $\frac{a}{b} = \frac{c}{d}$, which, by clearing of fractions, gives $a\,d = b\,c$.

182. For the statement of a problem no general rule can be given; much must depend on the skill and ingenuity of the operator. We will give a few suggestions, however, which will be found useful:

1. *Express the unknown quantity, or one of the unknown quantities, by one of the final letters of the alphabet.*

2. *From the given conditions, find expressions for the other unknown quantities, if any, in the problem.*

3. *Form an equation, by indicating the operations necessary to verify the values of the unknown quantities, were they already known.*

4. *Determine the value of the unknown quantity in the equation thus formed.*

Note. Problems which involve several unknown quantities may often be solved by representing one of them only by a single unknown letter.

1. What number is that to which if four sevenths of itself be added, the sum will equal twice the number, diminished by 27?

Let $x =$ the number.

Then $\frac{4x}{7} =$ four sevenths of it,

and $2x =$ twice it.

By the conditions, $x + \frac{4x}{7} = 2x - 27$

Solving this equation, $x = 63$, the number required.

2. Divide 144 into two parts whose difference is 30.

Let $x =$ one part.

Then, $144 - x =$ the other part.

By the conditions, $x - (144 - x) = 30$

Solving this equation, $x = 87$, one part.

$144 - x = 57$, the other part.

3. A is three times as old as B; and eight years ago he was seven times as old as B. What are their ages at present?

Let $x = $ B's age.

Then, $3x = $ A's age.

Now, $x - 8 = $ B's age, eight years ago,

and $3x - 8 = $ A's age, eight years ago.

By the conditions, $3x - 8 = 7(x - 8)$

Whence, $x = 12$, B's age,

and, $3x = 36$, A's age.

4. A can do a piece of work in 8 days, which B can perform in 10 days. In how many days can it be done by both working together?

Let $x = $ the number of days required.

Then, $\frac{1}{x} = $ what both can do in one day.

Also, $\frac{1}{8} = $ what A can do in one day,

and $\frac{1}{10} = $ what B can do in one day.

Since the sum of what each separately can do in one day is equal to what both can do together in one day,

$$\frac{1}{8} + \frac{1}{10} = \frac{1}{x}$$

Whence, $x = 4\frac{4}{9}$, number of days required.

5. A man has \$3.64 in dimes, half-dimes, and cents. He has 7 times as many cents as half-dimes, and one fourth as many half-dimes as dimes. How many has he of each?

Let $x =$ the number of dimes.

Then, $\frac{x}{4} =$ the number of half-dimes,

and $\frac{7x}{4} =$ the number of cents.

Now, $10x =$ the value of the dimes in *cents*,

and $\frac{5x}{4} =$ the value of the half-dimes in *cents*.

By the conditions, $10x + \frac{5x}{4} + \frac{7x}{4} = 364$

Whence, $x = 28$, number of dimes,

$\frac{x}{4} = 7$, number of half-dimes,

$\frac{7x}{4} = 49$, number of cents.

6. Two pieces of cloth were purchased at the same price per yard; but as they were of different lengths, the one cost \$5 and the other \$6.50. If each had been 10 yards longer, their lengths would have been as 5 to 6. Required the length of each piece.

Since the price of each per yard is the same, the lengths of the two pieces must be in the ratio of their prices, that is, as 5 to $6\frac{1}{2}$, or as 10 to 13. Therefore,

Let $10x =$ the length of the first piece in yards,

and $13x =$ the length of the second piece in yards.

By the conditions, $10x + 10 : 13x + 10 = 5 : 6$

or (Art. 181), $6(10x + 10) = 5(13x + 10)$

Whence, $x = 2$.

Then, $10x = 20$, length of first piece,

and $13x = 26$, length of second piece.

7. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of the digits, the quotient is 5. Required the number.

Let $x =$ the first digit.

Then, $x + 2 =$ the second.

Since the number is equal to 10 times the first digit, plus the second,

$$10x + x + 2, \text{ or } 11x + 2 = \text{the number.}$$

By the conditions, $\dfrac{11x + 2 + 6}{x + x + 2} = 5$

Whence, $x = 2,$ the first digit,

and $x + 2 = 4,$ the second digit.

Therefore the number is 24.

8. Two persons, A and B, 63 miles apart, set out at the same time and travel towards each other. A travels 4 miles an hour, and B 3 miles. What distance will each have travelled when they meet?

Let $x =$ the distance A travels.

Then, $63 - x =$ the distance B travels.

$\dfrac{x}{4} =$ the time A takes to travel x miles,

and $\dfrac{63 - x}{3} =$ the time B takes to travel $63 - x$ miles.

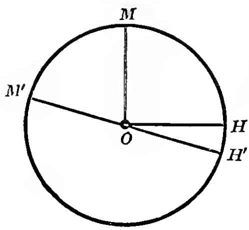
By the conditions of the problem, these times are equal;

or, $$\frac{x}{4} = \frac{63 - x}{3}$$

Whence, $x = 36,$ A's distance,

and $63 - x = 27,$ B's distance.

9. At what time between 3 and 4 o'clock are the hands of a watch opposite to each other?



Let OM represent the position of the minute-hand at 3 o'clock, and OH the position of the hour-hand at the same time.

Let OM' represent the position of the minute-hand when it is opposite to the hour-hand, and OH' the position of the hour-hand at the same time.

Let $x =$ the arc $MHH'M'$, the space over which the minute-hand has moved since 3 o'clock.

Then, $\frac{x}{12} =$ the arc HH', the space over which the hour-hand has moved since 3 o'clock.

Also, the arc $MH = 15$ minute spaces,

and the arc $H'M' = 30$ minute spaces.

Now, arc $MHH'M' =$ arc $MH +$ arc $HH' +$ arc $H'M'$,

or,
$$x = 15 + \frac{x}{12} + 30$$

Solving this equation, $x = 49\frac{1}{11}$ minute spaces.

That is, the time is $49\frac{1}{11}$ minutes after 3 o'clock.

PROBLEMS.

10. My horse and chaise are worth \$336; but the horse is worth twice as much as the chaise. Required the value of each.

11. What number is that from which if 7 be subtracted, one sixth of the remainder will be 5?

12. What two numbers are those whose difference is 3, and the difference of whose squares is 51?

13. Divide 20 into two such parts that 3 times one part may be equal to one third of the other.

14. Divide 100 into two parts whose difference is 17.

15. A is twice as old as B, and 10 years ago he was 3 times as old. What are their ages?

16. A is four times as old as B; in thirty years he will be only twice as old as B. What are their ages?

17. A can do a piece of work in 3 days, and B can do the same in 5 days. In how many days can it be done by both working together?

18. A can do a piece of work in $3\frac{2}{3}$ hours, which B can do in $2\frac{3}{4}$ hours, and C in $2\frac{1}{5}$ hours. In how many hours can it be done by all working together?

19. A and B can do a piece of work together in 7 days, which A alone can do in 10 days. In what time could B alone do it?

20. The first digit of a certain number exceeds the second by 4; and when the number is divided by the sum of the digits, the quotient is 7. What is the number?

21. The second digit of a certain number exceeds the first by 3; and if the number, diminished by 9, be divided by the difference of the digits, the quotient is 9. What is the number?

22. A drover has a lot of oxen and cows, for which he gave $1428. For the oxen he gave $55 each, and for the cows $32 each; and he had twice as many cows as oxen. Required the number of each.

23. A gentleman, at his decease, left an estate of $1872 for his wife, three sons, and two daughters. His wife was to receive three times as much as either of her daughters, and each son to receive one half as much as each of the daughters. Required the sum that each received.

24. A laborer agreed to serve for 36 days on these conditions, that for every day he worked he was to receive $1.25, but for every day he was absent he was to forfeit $0.50. At the end of the time he received $17. It is required to find how many days he labored, and how many days he was absent.

25. A man, being asked the value of his horse and saddle, replied that his horse was worth $114 more than his saddle, and that $\frac{2}{3}$ the value of the horse was 7 times the value of the saddle. What was the value of each?

26. In a garrison of 2744 men, there are 2 cavalry soldiers to 25 infantry, and half as many artillery as cavalry. Required the number of each.

27. The stones which pave a square court would just cover a rectangular area, whose length is 6 yards longer, and breadth 4 yards shorter, than the side of the square. Find the area of the court.

28. A person has travelled altogether 3036 miles, of which he has gone 7 miles by water to 4 on foot, and 5 by water to 2 on horseback. How many miles did he travel in each manner?

29. A certain man added to his estate $\frac{1}{4}$ its value, and then lost $760; but afterwards, having gained $600, his property then amounted to $2000. What was the value of his estate at first?

30. A capitalist invested $\frac{3}{8}$ of a certain sum of money in government bonds paying 5 per cent interest, and the remainder in bonds paying 6 per cent; and found the interest of the whole per annum to be $180. Required the amount of each kind of bonds.

31. A woman sells half an egg more than half her eggs. Again she sells half an egg more than half her remaining eggs. A third time she does the same; and now she has sold all her eggs. How many had she at first?

32. What number is that, the treble of which, increased by 12, shall as much exceed 54, as that treble is less than 144?

33. A asked B how much money he had. He replied, "If I had 5 times the sum I now possess, I could lend you $ 60, and then $\frac{1}{5}$ of the remainder would be equal to $\frac{1}{2}$ the dollars I now have." Required the sum B had.

34. A, B, and C found a purse of money, and it was mutually agreed that A should receive $ 15 less than one half, that B should have $ 13 more than one quarter, and that C should have the remainder, which was $ 27. How many dollars did the purse contain?

35. A number consists of 6 digits, of which the last to the left hand is 1. If this number is altered by removing the 1 and putting it in the units' place, the new number is three times as great as the original one. Find the number.

36. A prize of $ 1000 is to be divided between A and B, so that their shares may be in the ratio of 7 to 8. Required the share of each.

37. A man has $ 4.04 in dollars, dimes, and cents. He has one fifth as many cents as dimes, and twice as many cents as dollars. How many has he of each?

38. I bought a picture at a certain price, and paid the same price for a frame; if the frame had cost $ 1.00 less, and the picture $ 0.75 more, the price of the frame would have been only half that of the picture. Required the cost of the picture.

39. A gentleman gave in charity $ 46; a part in equal portions to 5 men, and the rest in equal portions to 7 women. Now, a man and a woman had between them $ 8. What was given to the men, and what to the women?

40. Separate 41 into two such parts, that one divided by the other may give 1 as a quotient and 5 as a remainder.

41. A vessel can be emptied by three taps; by the first alone it could be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. In what time will it be emptied if all the taps be opened?

42. A general arranging his troops in the form of a solid square, finds he has 21 men over; but, attempting to add 1 man to each side of the square, finds he wants 200 men to fill up the square. Required the number of men on a side at first, and the whole number of troops.

43. At what time between 7 and 8 are the hands of a watch opposite to each other?

44. At what time between 2 and 3 are the hands of a watch opposite to each other?

45. At what time between 5 and 6 are the hands of a watch together?

46. Divide 43 into two such parts that one of them shall be 3 times as much above 20 as the other wants of 17.

47. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. What proportions of gold and silver does it contain?

48. A gentleman let a certain sum of money for 3 years at 5 per cent compound interest; that is, at the end of each year there was added $\frac{1}{20}$ to the sum due. At the end of the third year there was due him $2315.25. Required the sum let.

49. A merchant has grain worth 9 shillings per bushel, and other grain worth 13 shillings per bushel. In what proportion must he mix 40 bushels, so that he may sell the mixture at 10 shillings per bushel?

50. A alone could perform a piece of work in 12 hours; A and C together could do it in 5 hours; and C's work is $\frac{2}{3}$ of B's. Now, the work has to be completed by noon. A begins work at 5 o'clock in the morning; at what hour can he be relieved by B and C, and the work be just finished in time?

51. A merchant possesses \$5120, but at the beginning of each year he sets aside a fixed sum for family expenses. His business increases his capital employed therein annually at the rate of 25 per cent. At the end of four years he finds that his capital is reduced to \$3275. What are his annual expenses?

52. At what times between 7 and 8 o'clock are the hands of a watch at right angles to each other?

53. At what time between 4 and 5 o'clock is the minute-hand of a watch exactly five minutes in advance of the hour-hand?

54. A person has $11\frac{1}{2}$ hours at his disposal; how far may he ride in a coach which travels 5 miles an hour, so as to return home in time, walking back at the rate of $3\frac{1}{2}$ miles an hour?

55. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox makes 9 leaps while the greyhound makes but 6; but the latter in 3 leaps goes as far as the former in 7. How many leaps does each make before the greyhound catches the fox?

56. A clock has an hour-hand, a minute-hand, and a second-hand, all turning on the same centre. At 12 o'clock all the hands are together, and point at 12. How long will it be before the minute-hand will be between the other two hands, and equally distant from each?

XIV. — SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

183. If we have a simple equation containing two unknown quantities, as $3x - 4y = 2$, we cannot determine *definitely* the values of x and y; because, for every value which we give to one of the unknown quantities, we can find a corresponding value for the other, and thus find *any number of pairs of values* which will satisfy the given equation.

Thus, if we put $x = 6$, then $18 - 4y = 2$, or $y = 4$;

if we put $x = -2$, then $-6 - 4y = 2$, or $y = -2$;

if we put $x = 1$, then $3 - 4y = 2$, or $y = \frac{1}{4}$; etc.

And any of the pairs of values $\begin{Bmatrix} x = 6 \\ y = 4 \end{Bmatrix}$, $\begin{Bmatrix} x = -2 \\ y = -2 \end{Bmatrix}$, $\begin{Bmatrix} x = 1 \\ y = \frac{1}{4} \end{Bmatrix}$, etc., will satisfy the given equation.

If we have another equation of the same kind, as $5x + 7y = 17$, we can find any number of pairs of values which will satisfy this equation also.

Now suppose we are required to determine a pair of values which will satisfy both equations. We shall find but *one* pair of values in this case. For, multiply the first equation by 5; thus,

$$15x - 20y = 10;$$

and multiply the second equation by 3; thus,

$$15x + 21y = 51.$$

Subtracting the first of these equations from the second (Art. 44), we have

$$41y = 41,$$

or,

$$y = 1.$$

In the first given equation put $y = 1$; then $3x - 4 = 2$, or $3x = 6$; whence, $x = 2$. The pair of values $\begin{Bmatrix} x = 2 \\ y = 1 \end{Bmatrix}$ satisfies both the given equations; and no other pair of values can be found which will satisfy both.

184. Simultaneous Equations are such as are satisfied by the *same values* of their unknown quantities.

185. Independent Equations are such as cannot be made to assume the same form.

186. It is evident, from Art. 183, that two unknown quantities require for their determination *two* independent, simultaneous equations. When two such equations are given, it is our object to obtain from them a *single* equation containing but *one* unknown quantity. The value of that unknown quantity may then be found; and by substituting it in either of the given equations we can find, as in Art. 183, the value of the other.

ELIMINATION.

187. Elimination is the process of combining simultaneous equations so as to obtain from them a single equation containing but one unknown quantity.

There are four principal methods of elimination: by *Addition or Subtraction*, by *Substitution*, by *Comparison*, and by *Undetermined Multipliers*.

CASE I.

188. *Elimination by Addition or Subtraction.*

1. Given $5x - 3y = 19$, and $7x + 4y = 2$, to find the values of x and y.

Multiplying the first equation by 4, $20x - 12y = 76$

Multiplying the second equation by 3, $21x + 12y = 6$

Adding these equations, $41x = 82$

Whence, $x = 2$.

Substituting this value in the first given equation,

$$10 - 3y = 19$$
$$-3y = 9$$
$$y = -3.$$

We might have solved the equations as follows:

Multiplying the first by 7, $35x - 21y = 133$ (1)

Multiplying the second by 5, $35x + 20y = 10$ (2)

Subtracting (2) from (1), $-41y = 123$

$$y = -3.$$

Substituting this value of y in the first given equation,

$$5x + 9 = 19$$
$$5x = 10$$
$$x = 2.$$

The first of these methods is elimination by *addition;* the second, elimination by *subtraction.*

RULE.

Multiply the given equations, if necessary, by such numbers or quantities as will make the coefficient of one of the unknown quantities the same in the two resulting equations. Then, if the signs of the terms having the same coefficient are alike, subtract one equation from the other; if unlike, add the two equations.

This method of elimination is usually the best in practice.

CASE II.

189. *Elimination by Substitution.*

Taking the same equations as before,

$$5x - 3y = 19 \quad (1)$$
$$7x + 4y = 2 \quad (2)$$

Transposing the term $7x$ in (2), $4y = 2 - 7x$

Dividing by 4, $y = \frac{2 - 7x}{4}$ (3)

Substituting this value of y in (1),

$$5x - 3\left(\frac{2 - 7x}{4}\right) = 19$$

Performing the operations indicated,

$$5x - \frac{6 - 21x}{4} = 19$$

Clearing of fractions, $20x - (6 - 21x) = 76$

or, $20x - 6 + 21x = 76$

Transposing, and uniting terms, $41x = 82$

Whence, $x = 2.$

Substituting this value in (3), $y = \frac{2 - 14}{4} = -3.$

RULE.

Find the value of one of the unknown quantities in terms of the other, from either of the given equations; and substitute this value for that quantity in the other equation.

This method is advantageous when either of the unknown quantities has 1 for its coefficient.

CASE III.

190. *Elimination by Comparison.*

Taking the same equations as before,

$$5x - 3y = 19 \quad (1)$$

$$7x + 4y = 2 \quad (2)$$

Transposing the term $-3y$ in (1), $5x = 3y + 19$

or, $x = \frac{3y + 19}{5}$ (3)

Transposing the term $4y$ in (2), $7x = 2 - 4y$

or, $x = \frac{2 - 4y}{7}$

Placing these two values of x equal to each other (Art. 44),

$$\frac{3y + 19}{5} = \frac{2 - 4y}{7}$$

Clearing of fractions, $21y + 133 = 10 - 20y$

Transposing, and uniting terms, $41y = -123$

Whence, $y = -3.$

Substituting this value in (3), $x = \frac{-9 + 19}{5} = 2.$

RULE.

Find the value of the same unknown quantity in terms of the other, from each of the given equations; and form a new equation by placing these values equal to each other.

CASE IV.

191. *Elimination by Undetermined Multipliers.*

An **Undetermined Multiplier** is a factor, at first undetermined, but to which a convenient value is assigned in the course of the operation.

Taking the same equations as before,

$$5x - 3y = 19 \qquad (1)$$

$$7x + 4y = 2 \qquad (2)$$

Multiplying (1) by m, $5mx - 3my = 19m$ (3)

Subtracting (3) from (2),

$$7x - 5mx + 4y + 3my = 2 - 19m$$

Factoring, $x(7 - 5m) + y(4 + 3m) = 2 - 19m$ (4)

Now, let the coefficient of y, $4 + 3m = 0$; then $3m = -4$, or $m = -\frac{4}{3}$; substituting this value of m in (4),

$$x\left(7 + \frac{20}{3}\right) = 2 + \frac{76}{3}$$

Clearing of fractions, $x(21 + 20) = 6 + 76$

$$41x = 82$$

$$x = 2.$$

Substituting this value in (2), $14 + 4y = 2$

$$4y = -12$$

$$y = -3.$$

We might have let the coefficient of x in (4), $7 - 5m = 0$; then m would have been $\frac{7}{5}$; substituting this value of m in (4),

$$y\left(4 + \frac{21}{5}\right) = 2 - \frac{133}{5}$$

Clearing of fractions, $y(20 + 21) = 10 - 133$

$$41y = -123$$

$$y = -3.$$

Instead of subtracting (3) from (2), we might have added them and obtained the same results. Also, in the first place, we might have multiplied (2) by m, and either added the result to, or subtracted it from, (1).

RULE.

Multiply one of the given equations by the undetermined quantity, m; and add the result to, or subtract it from, the other given equation.

In the resulting equation, factored with reference to the unknown quantities, place the coefficient of one of the unknown quantities equal to zero, and find the value of m. Substitute this value of m in the equation, and the result will be a simple equation containing but one unknown quantity.

This method is advantageous in the solution of literal equations.

2. Solve the equations.

$$ax + by = c \qquad (1)$$

$$a'x + b'y = c' \qquad (2)$$

Multiplying (1) by m, $$a\,m\,x + b\,m\,y = c\,m \qquad (3)$$

Add (2) and (3), $$a'\,x + a\,m\,x + b'\,y + b\,m\,y = c' + c\,m$$

Factoring, $$x\,(a' + a\,m) + y\,(b' + b\,m) = c' + c\,m \qquad (4)$$

In (4), put the coefficient of y, $b' + b\,m$, equal to zero.

Then, $$b\,m = -\,b'\,;\ \text{whence, } m = -\frac{b'}{b}.$$

Substituting this value of m in (4),

$$x\left(a' - \frac{a\,b'}{b}\right) = c' - \frac{c\,b'}{b}$$

Clearing of fractions, $$x\,(a'\,b - a\,b') = b\,c' - b'\,c$$

Whence, $$x = \frac{b\,c' - b'\,c}{a'\,b - a\,b'}.$$

In (4), put the coefficient of x, $a' + a\,m$, equal to zero.

Then, $$a\,m = -\,a'\,;\ \text{whence, } m = -\frac{a'}{a}.$$

Substituting this value of m in (4),

$$y\left(b' - \frac{a'\,b}{a}\right) = c' - \frac{a'\,c}{a}$$

Clearing of fractions, $$y\,(a\,b' - a'\,b) = a\,c' - a'\,c$$

Whence, $$y = \frac{a\,c' - a'\,c}{a\,b' - a'\,b}.$$

Before applying either of the preceding methods of elimination, the given equations should be reduced to their simplest forms.

EXAMPLES.

192. Solve, by whichever method may be most advantageous, the following equations:

3. $3\,x + 7\,y = 33\,;\ 2\,x + 4\,y = 20.$

4. $7\,x + 2\,y = 31\,;\ 3\,x - 4\,y = 23.$

5. $6\,x - 3\,y = 27\,;\ 4\,x - 6\,y = -\,2.$

6. $7x + 3y = -50$; $2y - 5x = 44$.

7. $8y + 12x = 116$; $2x - y = 3$.

8. $11x + 3y = -124$; $2x - 6y = 56$.

9. $9x + 4y = 22$; $2y + 3x = 14$.

10. $\frac{3x}{7} + \frac{5y}{14} = 8$; $-8x + 2y = -80$.

11. $7x - 2y = 6$; $2x + 2y = -24$.

12. $11y + 6x = 115$; $\frac{2x}{3} - \frac{11y}{6} = -\frac{5}{2}$.

13. $\frac{1}{3}x + \frac{1}{2}y = \frac{47}{6}$; $10x - 12y = -62$.

14. $-7x + 4y = -113$; $x + \frac{5}{6}y = \frac{7}{2}$.

15. $\frac{x}{2} - \frac{y}{3} = 0$; $\frac{x}{4} + \frac{y}{6} = 6$.

16. $\frac{3x}{5} - y = 31$; $x + \frac{y}{5} = 33$.

17. $\frac{4x}{7} - \frac{2y}{3} = -30$; $x + 7y = 119$.

18. $x + 2y = .6$; $1.7x - y = .58$.

19. $\dfrac{\frac{3x}{4} - \frac{y}{3}}{\frac{1}{2}} - \dfrac{\frac{x}{2} + \frac{2y}{5}}{\frac{13}{4}} = -\frac{7}{6}$; $4y - 3x = 11$.

20. $\frac{x + 3y}{2x - y} = -\frac{3}{8}$; $\frac{7y - x}{2 + x + 2y} = -17$.

21. $ax + by = m$; $cx + dy = n$.

22. $mx + ny = r$; $m'x - n'y = r'$.

23. $\frac{x}{a} - \frac{y}{b} = m$; $\frac{y}{d} + \frac{x}{c} = n$.

24. $\frac{x}{a+b}+\frac{y}{a-b}=\frac{1}{a^2-b^2}$; $\frac{x}{a-b}+\frac{y}{a+b}=\frac{1}{a^2-b^2}$.

25. $\frac{x}{2}-12=\frac{y}{4}+8$; $\frac{x}{3}-8-\frac{2y-x}{4}=27-\frac{x+y}{5}$.

26. $\frac{2}{x+y}+\frac{2}{x-y}=1$; $\frac{3}{x+y}-\frac{2}{x-y}=0$.

27. $x-\frac{2x+y}{3}=\frac{17}{12}-\frac{2y+x}{4}$; $\frac{4}{3}-\frac{2x-y}{4}=y-\frac{2y-x}{3}$.

28. $\frac{2x}{3}-\frac{3y}{5}-\frac{x+2y}{4}=3-\frac{5x-6y}{4}$;

$\frac{x}{2}+y-\frac{3x-y}{5}=-5+\frac{x}{15}$.

29. Solve the equations,

$$\frac{6}{x}-\frac{3}{y}=4$$
$$\frac{8}{x}+\frac{15}{y}=-1$$

Multiplying the first equation by 5,

$$\frac{30}{x}-\frac{15}{y}=20$$

Adding this to the second given equation,

$$\frac{38}{x}=19$$

Clearing of fractions, $38=19x$

Whence, $x=2$.

Substituting this value in the first given equation,

$$3-\frac{3}{y}=4$$

Transposing, $-\frac{3}{y}=1$

Whence, $y=-3$.

Solve the following equations:

30. $\frac{3}{x}+\frac{1}{y}=\frac{5}{4};\ \frac{2}{x}-\frac{3}{y}=-1.$

31. $\frac{12}{x}-\frac{18}{y}=-\frac{42}{5};\ \frac{15}{x}-\frac{8}{y}=-\frac{17}{3}.$

32. $\frac{11}{x}-\frac{7}{y}=\frac{3}{2};\ \frac{4}{x}+\frac{8}{y}=-10.$

33. $\frac{a}{x}+\frac{b}{y}=m;\ \frac{c}{x}+\frac{d}{y}=n.$

34. $\frac{6}{a\,x}+\frac{4}{b\,y}=4\,a\,b;\ \frac{9}{b\,x}-\frac{8}{a\,y}=3\,a^2-4\,b^2.$

35. $\frac{m}{n\,x}+\frac{n}{m\,y}=m+n;\ \frac{n}{x}+\frac{m}{y}=m^2+n^2.$

XV.—SIMPLE EQUATIONS

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

193. If we have given *three* independent, simultaneous equations, containing *three* unknown quantities, we may combine two of them by the methods of elimination explained in the last chapter, so as to obtain an equation containing only two unknown quantities; we may combine the third equation with either of the two former in the same way, so as to obtain another equation containing the same two unknown quantities. Then from these two equations containing two unknown quantities we may derive, as in the last chapter, the values of those unknown quantities. These values being substituted in either of the given equations, the value of the third unknown quantity may be determined from the resulting equation.

The method of elimination by addition or subtraction is usually the most convenient.

194. **1.** Solve the equations,

$$\begin{aligned} 8x - 9y - 7z &= -36 \\ 12x - y - 3z &= 36 \\ 6x - 2y - z &= 10 \end{aligned}$$

Multiplying the first by 3, $24x - 27y - 21z = -108$ (1)

Multiplying the second by 2, $24x - 2y - 6z = 72$ (2)

Multiplying the third by 4, $24x - 8y - 4z = 40$ (3)

Subtracting (1) from (2), $25y + 15z = 180$

or, $5y + 3z = 36$ (4)

Subtracting (3) from (2), $6y - 2z = 32$

or, $3y - z = 16$ (5)

Multiplying (5) by 3, $9y - 3z = 48$ (6)

Adding (4) and (6), $14y = 84$

$y = 6.$

Substituting this value in (5), $z = 2.$

Substituting the values of y and z in the third given equation,

$x = 4.$

In the same manner, if we have given n independent, simultaneous equations, containing n unknown quantities, we may combine them so as to form $n - 1$ equations, containing $n - 1$ unknown quantities. These, again, may be combined so as to form $n - 2$ equations, containing $n - 2$ unknown quantities; and so on: the operation being continued until we finally obtain one equation containing one unknown quantity.

RULE.

Multiply the given equations, if necessary, by such numbers or quantities as will make the coefficient of one of the unknown quantities the same in the resulting equations. Combine these equations by addition or subtraction, so as to form *a new set of equations, one less in number than before, and containing one less unknown quantity. Continue the operation with these new equations; and so on, until an equation is obtained containing one unknown quantity.*

Find the value of this unknown quantity. By substituting it in either of the equations containing only two unknown quantities, find the value of a second unknown quantity. By substituting these values in either of the equations containing three unknown quantities, find the value of a third unknown quantity; and so on, until the values of all are found.

Note. This rule corresponds only with the method of elimination by addition or subtraction; which, however, as we have observed before, is the best in practice.

2. Solve the equations,

$$u + x + y = 6$$
$$u + x + z = 9$$
$$u + y + z = 8$$
$$x + y + z = 7$$

The solution may here be abridged by the artifice of assuming the sum of the four unknown quantities to equal an auxiliary quantity, s. Thus,

Let $$u + x + y + z = s.$$

Then we may write the four given equations as follows:

$$s - z = 6 \quad (1)$$
$$s - y = 9 \quad (2)$$
$$s - x = 8 \quad (3)$$
$$s - u = 7 \quad (4)$$

Adding, $$4s - s = 30$$

Whence, $$s = 10.$$

Substituting the value of s in (1), (2), (3), and (4), we obtain

$$z = 4, \ y = 1, \ x = 2, \text{ and } u = 3.$$

EXAMPLES.

Solve the following equations:

3. $x+y+z=53$; $x+2y+3z=107$; $x+3y+4z=137$.

4. $3x-y-2z=-23$; $6x+2y+3z=15$;
$4x+3y-z=-6$.

5. $5x-3y+2z=41$; $2x+y-z=17$; $5x+4y-2z=36$.

6. $7x+4y-z=-50$; $4x-5y-3z=20$;
$x-3y-4z=30$.

7. $3u+x+2y-z=22$; $4x-y+3z=35$;
$4u+3x-2y=19$; $2u+4y+2z=46$.

8. $x+y=2$; $x+z=3$; $y+z=-1$.

9. $y+z=a$; $x+z=b$; $x+y=c$.

10. $4x-4y=a+4z$; $6y-2x=a+2z$; $7z-y=a+x$.

11. $\frac{x}{2}+\frac{y}{3}-\frac{z}{4}=-43$; $\frac{x}{3}-\frac{y}{4}+\frac{z}{2}=34$; $\frac{x}{4}+\frac{y}{2}-\frac{z}{3}=-50$.

12. $2x+2y+z=-17-2u$; $y+3z=-2$; $4x+z=13$;
$\frac{x}{3}+3y=-14$.

13. $ay+bx=c$; $cx+az=b$; $bz+cy=a$.

14. $\frac{4}{x}-\frac{9}{y}-\frac{6}{z}=\frac{81}{2}$; $\frac{2}{3x}+\frac{3}{2y}-\frac{10}{7z}=\frac{7}{2}$;
$\frac{8}{9x}-\frac{6}{y}+\frac{4}{7z}=11$.

15. $\frac{3}{4x}-\frac{2}{3y}=1$; $-\frac{2}{3y}+\frac{1}{2z}=1$; $\frac{1}{2z}+\frac{3}{4x}=1$.

16. $x-ay+a^2z=a^3$; $x-by+b^2z=b^3$; $x-cy+c^2z=c^3$.

17. $\frac{y-z}{2}-\frac{x+z}{4}=\frac{1}{2}$; $\frac{x-y}{5}-\frac{x-z}{6}=0$;

$$\frac{y+z}{4}-\frac{x+y}{2}=-4.$$

18. $\frac{cx+y}{a}-(2-z)=0$; $\frac{y+a^2z}{c}=2a-cx$;

$$(a+c)^2-ac(2+x+z)=-y.$$

XVI. — PROBLEMS

LEADING TO SIMPLE EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

195. In the solution of problems in which we represent more than one of the unknown quantities by letters, we must obtain, from the conditions of the problem, *as many independent equations as there are unknown quantities.*

1. If 3 be added to both numerator and denominator of a certain fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$. Required the fraction.

Let $x=$ the numerator,

and $y=$ the denominator.

By the conditions, $\frac{x+3}{y+3}=\frac{2}{3}$

$$\frac{x-2}{y-2}=\frac{1}{2}$$

Solving these equations, $x=7,\ y=12.$

That is, the fraction is $\frac{7}{12}$.

2. The sum of the digits of a number of three figures is 13; if the number, decreased by 8, be divided by the sum of the second and third digits, the result is 25; and if 99 be added to the number, the digits will be inverted. Find the number.

Let $x =$ the first digit,

$y =$ the second,

and $z =$ the third.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with its digits inverted.

By the conditions, $x + y + z = 13$

$$\frac{100x + 10y + z - 8}{y + z} = 25$$

$$100x + 10y + z + 99 = 100z + 10y + x$$

Solving these equations, $x = 2$, the first digit,

$y = 8$, the second,

$z = 3$, the third.

That is, the number is 283.

3. A crew can row 20 miles in 2 hours down stream, and 12 miles in 3 hours against the stream. Required the rate per hour of the current, and the rate per hour of the crew in still water.

Let $x =$ rate per hour of the crew in still water,

and $y =$ rate per hour of the current.

Then, $x + y =$ rate per hour rowing down stream,

and $x - y =$ rate per hour rowing up stream.

Since the distance divided by the rate gives the time, we have by the conditions,

$$\frac{20}{x + y} = 2$$

$$\frac{12}{x - y} = 3$$

Solving these equations, $x = 7$, and $y = 3$.

PROBLEMS.

4. A says to B, "If $\frac{1}{5}$ of my age were added to $\frac{2}{3}$ of yours, the sum would be $19\frac{1}{3}$ years." "But," says B, "if $\frac{2}{5}$ of mine were subtracted from $\frac{7}{8}$ of yours, the remainder would be $18\frac{1}{4}$ years." Required their ages.

5. If 1 be added to the numerator of a certain fraction, its value is $\frac{1}{3}$; but if 1 be added to its denominator, its value is $\frac{1}{4}$. What is the fraction?

6. A farmer has 89 oxen and cows; but, having sold 4 oxen and 20 cows, found he then had 7 more oxen than cows. Required the number of each at first.

7. A says to B, "If 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$990." B replied, "If 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be \$510." Required the property of each.

8. If $\frac{1}{7}$ of A's age were subtracted from B's age, and 5 years added to the remainder, the sum would be 6 years; and if 4 years were added to $\frac{1}{5}$ of B's age, it would be equal to $\frac{1}{14}$ of A's age. Required their ages.

9. Divide 50 into two such parts that $\frac{3}{8}$ of the larger shall be equal to $\frac{2}{3}$ of the smaller.

10. A gentleman, at the time of his marriage, found that his wife's age was to his as 3 to 4; but, after they had been married 12 years, her age was to his as 5 to 6. Required their ages at the time of their marriage.

11. A farmer hired a laborer for 10 days, and agreed to pay him \$12 for every day he labored, and he was to forfeit \$8 for every day he was absent. He received at the end of his time \$40. How many days did he labor, and how many days was he absent?

12. A gentleman bought a horse and chaise for \$208, and $\frac{4}{7}$ of the cost of the chaise was equal to $\frac{2}{3}$ the price of the horse. What was the price of each?

13. A and B engaged in trade, A with \$240, and B with \$96. A lost twice as much as B; and, upon settling their accounts, it appeared that A had three times as much remaining as B. How much did each lose?

14. Two men, A and B, agreed to dig a well in 10 days; but, having labored together 4 days, B agreed to finish the job, which he did in 16 days. How long would it have taken A to dig the whole well?

15. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each kind must he use?

16. A farmer has a box filled with wheat and rye; seven times the bushels of wheat are 3 bushels more than four times the bushels of rye; and the quantity of wheat is to the quantity of rye as 3 to 5. Required the number of bushels of each.

17. My income and assessed taxes together amount to \$50. But if the income tax be increased 50 per cent, and the assessed tax diminished 25 per cent, the taxes will together amount to \$52.50. Required the amount of each tax.

18. A and B entered into partnership, and gained \$200. Now 6 times A's accumulated stock (capital and profit) was equal to 5 times B's original stock; and 6 times B's profit exceeded A's original stock by \$200. Required the original stock of each.

19. A boy at a fair spent his money for oranges. If he had got five more for his money, they would have averaged a half-cent less; and if three less, a half-cent each more. How many cents did he spend, and how many oranges did he get?

20. A merchant has three kinds of sugar. He can sell 3 lbs. of the first quality, 4 lbs. of the second, and 2 lbs. of the third, for 60 cents; or, he can sell 4 lbs. of the first quality, 1 lb. of the second, and 5 lbs. of the third, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lbs. of the second, and 3 lbs. of the third, for 90 cents. Required the price per lb. of each quality.

21. A gentleman's two horses, with their harness, cost him $120. The value of the poorer horse, with the harness, was double that of the better horse; and the value of the better horse, with the harness, was triple that of the poorer horse. What was the value of each?

22. Find three numbers, so that the first with half the other two, the second with one third the other two, and the third with one fourth the other two, shall each be equal to 34.

23. Find a number of three places, of which the digits have equal differences in their order; and, if the number be divided by half the sum of the digits, the quotient will be 41; and, if 396 be added to the number, the digits will be inverted.

24. There are four men, A, B, C, and D, the value of whose estates is $14,000; twice A's, three times B's, half of C's, and one fifth of D's, is $16,000; A's, twice B's, twice C's, and two fifths of D's, is $18,000; and half of A's, with one third of B's, one fourth of C's, and one fifth of D's, is $4000. Required the property of each.

25. A and B are driving their turkeys to market. A says to B, "Give me 5 of your turkeys, and I shall have as many as you." B replies, "Give me 15 of yours, and then yours will be $\frac{3}{7}$ of mine." How many had each?

26. A says to B and C, "Give me half of your money and I shall have $55." B replies, "If you two will give me one third of yours, I shall have $50." But C says to A and B, "If I had one fifth of your money I should have $50." Required the sum that each possessed.

27. A gentleman left a sum of money to be divided among his four sons, so that the share of the eldest was $\frac{1}{2}$ of the sum of the shares of the other three, the share of the second $\frac{1}{3}$ of the sum of the other three, and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was found that the share of the eldest exceeded that of the youngest by \$14. What was the whole sum, and what was the share of each person?

28. If I were to enlarge my field by making it 5 rods longer and 4 rods wider, its area would be increased by 240 square rods; but if I were to make its length 4 rods less, and its width 5 rods less, its area would be diminished by 210 square rods. Required the present length, width, and area.

29. A boatman can row down stream, a distance of 20 miles, and back again in 10 hours; and he finds that he can row 2 miles against the current in the same time that he rows 3 miles with it. Required the time in going and in returning.

30. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In how many days can each of them alone perform it?

31. A person possesses a capital of \$30,000, on which he gains a certain rate of interest; but he owes \$20,000, for which he pays interest at another rate. The interest which he receives is greater than that which he pays by \$800. A second person has \$35,000, on which he gains the second rate of interest; but he owes \$24,000, for which he pays the first rate of interest. The sum which he receives is greater than that which he pays by \$310. What are the two rates of interest?

32. A man rows down a stream, which runs at the rate of $3\frac{1}{2}$ miles per hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 6 hours and 30 minutes to arrive at a point 2 miles short of his starting-place. Find the distance he pulled down the stream, and the rate of his pulling.

33. A train running from Boston to New York meets with an accident which causes its speed to be reduced to $\frac{1}{3}$ of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer New York, the train would have been only one hour late. What was the rate of the train before the accident?

34. A and B run a mile. A gives B a start of 44 yards and beats him by 51 seconds, and afterwards gives him a start of 1 minute 15 seconds and is beaten by 88 yards. In how many minutes can each run a mile?

35. A merchant has two casks, each containing a certain quantity of wine. In order to have an equal quantity in each, he pours out of the first cask into the second as much as the second contained at first; then he pours from the second into the first as much as was left in the first; and then again from the first into the second as much as was left in the second, when there are found to be 16 gallons in each cask. How many gallons did each cask contain at first?

36. A and B are building a fence 126 feet long; after three hours A leaves off, and B finishes the work in 14 hours. If seven hours had occurred before A left off, B would have finished the work in $4\frac{2}{3}$ hours. How many feet does each build in one hour?

GENERALIZATION OF PROBLEMS.

196. A problem is said to be *generalized* when letters are used to represent its known quantities, as well as unknown.

The *unknown quantities* thus found in terms of the *known* are general expressions, or *formulæ*, which may be used for the solution of any similar problem.

197. The algebraic solution of a generalized problem discloses many interesting truths and useful practical rules, as may be seen from the consideration of the following:

1. The sum of two numbers is a, and their difference is b; what are the two numbers?

Let $x =$ the greater number.
and $y =$ the less.

By the conditions,
$$x + y = a$$
$$x - y = b$$

Solving these equations, $x = \frac{a+b}{2}$, the greater number,

and $y = \frac{a-b}{2}$, the less.

Hence, since a and b may have any value whatever, the values of x and y are general, and may be expressed as rules for the numerical calculations in any like case; thus,

To find two numbers when their sum and difference are given, — *Add the sum and difference, and divide by 2, for the greater of the two numbers; and subtract the difference from the sum, and divide by 2, for the less number.*

For example, if the sum of two numbers is 35, and their difference 13,

$$\text{the greater} = \frac{35+13}{2} = 24,$$

$$\text{and the less} = \frac{35-13}{2} = 11.$$

2. A can do a piece of work in a days, which it requires b days for B to perform. In how many days can it be done if A and B work together?

Let $x =$ the number of days required.

Then $\frac{1}{x} =$ what both together can do in one day.

Also, $\frac{1}{a} =$ what A can do in one day,

and $\frac{1}{b} =$ what B can do in one day.

By the conditions, $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$

Whence, $x = \frac{ab}{a+b}$, number of days required.

Hence, to find the time for two agencies conjointly to accomplish a certain result, when the times are given in which each separately can accomplish the same, — *Divide the product of the given times by their sum.*

For example, if A can do a piece of work in 5 days, and B in 4 days, the time it will take them both working together will be $\frac{5 \times 4}{5+4} = \frac{20}{9} = 2\frac{2}{9}$ days.

3. Three men, A, B, and C, enter into partnership for a certain time. Of the capital stock, A furnishes m dollars; B, n dollars; and C, p dollars. They gain a dollars. What is each man's share of the gain?

Let $x =$ A's share.

Then, since the shares are proportional to the stocks,

$$\frac{nx}{m} = \text{B's share,}$$

and $$\frac{px}{m} = \text{C's share.}$$

By the conditions, $x + \frac{nx}{m} + \frac{px}{m} = a$

Whence, $$x = \frac{ma}{m+n+p}, \text{ A's share.}$$

Then, $$\frac{nx}{m} = \frac{na}{m+n+p}, \text{ B's share,}$$

and $$\frac{px}{m} = \frac{pa}{m+n+p}, \text{ C's share.}$$

Hence, to find each man's gain, when each man's stock and the whole gain are given, — *Multiply the whole gain by each man's stock, and divide the product by the whole stock.*

For example, suppose A's stock \$300, B's \$500, and C's \$800, and the whole gain \$320.

Then, $$\text{A's share} = \frac{320 \times 300}{300 + 500 + 800} = \frac{96000}{1600} = \$60,$$

$$\text{B's share} = \frac{320 \times 500}{300 + 500 + 800} = \frac{160000}{1600} = \$100,$$

$$\text{and C's share} = \frac{320 \times 800}{300 + 500 + 800} = \frac{256000}{1600} = \$160.$$

PROBLEMS.

4. A cistern can be filled by three pipes; by the first in a hours, by the second in b hours, and by the third in c hours. In what time can it be filled by all the pipes running together?

5. Using the result of the previous problem, suppose that the first pipe fills the cistern in 2 hours, the second in 5 hours, and the third in 10 hours. In what time can it be filled by all the pipes running together?

6. Divide the number a into two parts which shall have to each other the ratio of m to n.

7. Using the result of the previous problem, divide the number 20 into two parts which shall have to each other the ratio of 3 to 2.

8. A courier left this place n days ago, and goes a miles each day. He is pursued by another, starting to-day and going b miles daily. How many days will the second require to overtake the first?

9. In the last example, if $n = 3$, $a = 40$, and $b = 50$, how many days will be required?

10. Required what principal, at interest at r per cent, will amount to the sum a, in t years?

11. Using the result of the previous problem, what principal, at 6 per cent interest, will amount to \$3108 in 8 years?

12. Required the number of years in which p dollars, at r per cent interest, will amount to a dollars.

13. Using the result of the previous problem, in how many years will \$262, at 7 per cent interest, amount to \$472.91?

14. A banker has two kinds of money. It takes a pieces of the first to make a dollar, and b pieces of the second to make the same sum. If he is offered a dollar for c pieces, how many of each kind must he give?

15. In the last example, if $a = 10$, $b = 20$, and $c = 15$, how many of each kind must he give?

16. A gentleman, distributing some money among beggars, found that in order to give them a cents each he should want b cents more; he therefore gave them c cents each, and had d cents left. Required the number of beggars.

17. A mixture is made of a pounds of coffee at m cents a pound, b pounds at n cents, and c pounds at p cents. Required the cost per pound of the mixture.

18. A, B, and C hire a pasture together for a dollars. A puts in m horses for t months, B puts in n horses for t' months, and C puts in p horses for t'' months. What part of the expense should each pay?

XVII. — DISCUSSION OF PROBLEMS

LEADING TO SIMPLE EQUATIONS.

198. The **Discussion** of a problem, or of an equation, is the process of attributing any reasonable values and relations to the arbitrary quantities which enter the equation, and interpreting the results.

199. An **Arbitrary Quantity** is one to which any reasonable value may be given at pleasure.

200. A **Determinate Problem** is one in which the given conditions furnish the means of finding the required quantities.

A determinate problem leads to as many independent equations as there are required quantities (Art. 195).

201. An **Indeterminate Problem** is one in which there are fewer imposed conditions than there are required quantities, and, consequently, an insufficient number of independent equations to determine definitely the values of the required quantities.

202. An **Impossible Problem** is one in which the conditions are incompatible or contradictory, and consequently cannot be fulfilled.

203. A determinate problem, leading to a simple equation involving only one unknown quantity, can be satisfied by but *one* value of that unknown quantity (Art. 178).

An indeterminate problem, or one leading to a less number of independent equations than it has unknown quantities, may be satisfied by any number of values.

For example, suppose a problem involving three unknown quantities leads to only two equations, which, on combining, give

$$x - z = 10,$$

or,

$$x = 10 + z.$$

Now, if we make $z = 1$, then $x = 11$;
$z = 2$, then $x = 12$;
$z = 3$, then $x = 13$.

Thus, we may find sets of values without limit that will satisfy the equation. Hence,

An indeterminate equation may have any number of solutions.

204. When a problem leads to more independent equations than it has unknown quantities, it is impossible.

For, suppose we have a problem furnishing three independent equations, as,

$$x = y + 1$$
$$y = 7 - x$$
$$xy = 16$$

From the first two we find $x = 4$ and $y = 3$. But the third requires their product to be 16; hence the problem is impossible.

If, however, the third equation had not been independent, but derived from the other two, as,

$$x\,y = 12,$$

then the problem would have been possible; but the last equation, not being required for the solution, would have been *redundant*.

INTERPRETATION OF NEGATIVE RESULTS.

205. In a **Negative Result**, or a result preceded by a $-$ sign, the negative sign is regarded as a symbol of *interpretation*.

Its significance when thus used it is now proposed to investigate.

1. Let it be required to find what number must be added to the number a that the sum may be b.

Let $\quad x =$ the required number.

Then, $\quad a + x = b$

Whence, $\quad x = b - a.$

Here, the value of x corresponds with any assigned values of a and b. Thus, for example,

Let $\quad a = 12$, and $b = 25$.

Then $\quad x = 25 - 12 = 13,$

which satisfies the conditions of the problem; for if 13 be added to 12, or a, the sum will be 25, or b.

But, suppose $\quad a = 30$, and $b = 24$.

Then, $\quad x = 24 - 30 = -6,$

which indicates that, under the latter hypothesis, the problem is impossible in an *arithmetical sense*, though it is possible in the *algebraic sense* of the words "number," "added," and "sum."

The negative result, -6, points out, therefore, in the *arithmetical sense*, either *an error* or *an impossibility*.

But, taking the value of x with a contrary sign, we see that it will satisfy the enunciation of the problem, in an arithmetical sense, when modified so as to read:

What number must be *taken from* 30, that *the remainder* may be 24?

2. Let it be required to determine the epoch at which A's age is twice as great as B's; A's age at present being 35 years, and B's 20 years.

Let us suppose the required epoch to be *after* the present date.

Let $x =$ the number of years *after* the present date.

Then, $$35 + x = 2\ (20 + x)$$

Whence, $$x = -5, \text{ a negative result.}$$

On recurring to the problem, we find it so worded as to admit also of the supposition that the epoch is *before* the present date; and taking the value of x obtained, with the contrary sign, we find it will satisfy that enunciation.

Hence, a negative result here indicates that a wrong choice was made of two possible suppositions which the problem allowed.

From the discussion of these problems we infer:

1. *That negative results indicate either an erroneous enunciation of a problem, or a wrong supposition respecting the* QUALITY *of some quantity belonging to it.*

2. *That we may form, when attainable, a possible problem analogous to that which involved the impossibility, or correct*

the wrong supposition, by attributing to the unknown quantity in the equation a QUALITY DIRECTLY OPPOSITE *to that which had been attributed to it.*

In general, it is not necessary to form a *new* equation, but simply to change in the *old* one the sign of each quantity which is to have its quality changed.

Interpret the negative results obtained, and modify the enunciation accordingly, in the following

PROBLEMS.

3. If the length of a field be 10 rods, and the breadth 8 rods, what quantity must be added to its breadth so that the contents may be 60 square rods?

4. If 1 be added to the numerator of a certain fraction, its value becomes $\frac{4}{9}$; but if 1 be added to the denominator, it becomes $\frac{5}{8}$. What is the fraction?

5. The sum of two numbers is 90, and their difference is 120; what are the numbers?

6. A is 50 years old, and B 40; required the time when A will be twice as old as B.

7. A and B were in partnership, and A had 3 times as much capital as B. When A had gained \$2000, and B \$750, A had twice as much capital as B. What was the capital of each at first?

8. A man worked 14 days, his son being with him 6 days, and received \$39, besides the subsistence of himself and son while at work. At another time he worked 10 days, and had his son with him 4 days, and received \$28. What were the daily wages of each?

XVIII. — ZERO AND INFINITY.

206. A *variable quantity*, or simply a *variable*, is a quantity to which we may give, in the same discussion, any value within certain limits determined by the nature of the problem; a *constant* is a quantity which remains unchanged throughout the same discussion.

207. The *limit* of a variable quantity is a constant value to which it may be brought as near as we please, but which it can never reach.

Thus, if 3 be halved, the quotient $\frac{3}{2}$ again halved, and so on indefinitely, the limit to which the result may be brought as near as we please, but which it can never reach, is *zero*. And, in general, if any quantity be indefinitely diminished by division, its limiting value is zero.

208. If any quantity be indefinitely increased by multiplication or otherwise, its limiting value is called **Infinity**, and is denoted by the symbol ∞.

209. It is evident, from the definition of Art. 207, that if two variable quantities are always equal, their limiting values will be equal.

210. We will now show how to interpret certain forms which may be obtained in the course of mathematical operations.

Let us consider the fraction $\frac{a}{b}$; and let $\frac{a}{b} = x$.

1. Interpretation of $\frac{a}{0}$.

Let the numerator of $\frac{a}{b}$ remain constant, and the denominator be indefinitely diminished by division. By Art. 137, if the denominator is divided by any quantity, the value of the fraction is multiplied by that quantity; hence the value of the fraction, x, increases indefinitely as b is diminished indefinitely. The limiting value of b being 0 (Art. 207), the limiting value of $\frac{a}{b}$ will be $\frac{a}{0}$; and the limiting value of x is ∞ (Art. 208). Now $\frac{a}{b}$ and x being two variable quantities always equal, by Art. 209 their limiting values are equal; or,

$$\frac{a}{0}=\infty.$$

2. Interpretation of $\frac{a}{\infty}$.

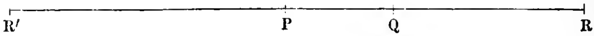
Let the numerator remain constant, and the denominator be indefinitely increased by multiplication. By Art. 138, if the denominator is multiplied by any quantity, the value of the fraction is divided by that quantity; hence x is diminished indefinitely by division as the denominator increases indefinitely. The limiting value of b being ∞, the limiting value of $\frac{a}{b}$ will be $\frac{a}{\infty}$; and the limiting value of x is 0. By Art. 209 these limiting values are equal; or,

$$\frac{a}{\infty}=0.$$

Problem of the Couriers.

211. The discussion of the following problem, commonly known as that of Clairaut, will serve to further illustrate the form $\frac{a}{0}$, besides furnishing us with an interpretation of the form $\frac{0}{0}$.

Two couriers, A and B, are travelling along the same road, in the same direction, R′ R, at the rates of m and n miles per hour respectively. If at any time, say 12 o'clock, A is at the point P, and B a miles from him at Q, *when* and *where* are they together?



Let $t =$ the required time in hours;
and $x =$ the distance A travels in the time t, or the distance from P to the place of meeting.
Then $x - a =$ the distance B travels in the time t, or the distance from Q to the place of meeting.

Since the distance equals the rate multiplied by the time,

$$x = m\,t$$

$$x - a = n\,t$$

Solving these equations with reference to t and x,

$$t = \frac{a}{m - n}$$

$$x = \frac{m\,a}{m - n}.$$

It is proposed now to discuss these values on different suppositions.

1. $m > n$.

This hypothesis makes the denominator $m - n$ positive; hence the values of both t and x are positive. That is, the couriers are together *after* 12 o'clock, and to the *right* of P.

This interpretation corresponds with the supposition made. For, if A travels faster than B, he will eventually overtake him, and in advance of their positions at 12 o'clock.

2. $m < n$.

This hypothesis makes the denominator $m - n$ negative; hence the values of both t and x are negative. Now, from what we have observed in regard to negative results (Art. 205), these values of t and x indicate that the couriers were together *before* 12 o'clock, and to the *left* of P.

This interpretation corresponds with the supposition made. For, if A travels more slowly than B, he will never overtake him; but as they are travelling along the same road, they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $m = n$.

This hypothesis makes the denominator $m - n$ equal to zero; so that the values of t and x become $\frac{a}{0}$ and $\frac{m\,a}{0}$, respectively; or, by Art. 210, $t = \infty$ and $x = \infty$. Since from its nature (Art. 208), ∞ is a value which we can never reach, the values of t and x may be regarded as indicating that the problem is impossible under the assumed hypothesis.

This interpretation corresponds with the supposition made. For, if the couriers were a miles apart at 12 o'clock, and were travelling at the same rate, they *never had been* and *never would be* together.

Thus, *infinite results indicate the impossibility of a problem.*

4. $a = 0$, and $m > n$ or $m < n$.

By this hypothesis, the values of t and x each become $\frac{0}{m-n}$; or (Art. 102), $t = 0$ and $x = 0$. That is, the couriers are together at 12 o'clock, at the point P, and at no other time and place.

This interpretation corresponds with the supposition made; for, if the distance between them at 12 o'clock is nothing, they are together at P; but as their rates are unequal, they cannot be together after 12 o'clock, nor could they have been together before that time.

5. $a = 0$, and $m = n$.

By this hypothesis, the values of t and x each take the form $\frac{0}{0}$.

Referring to the enunciation of the problem, we see that if the couriers were together at 12 o'clock, and were travelling at the same rate, they *always had been*, and *always would be*, together. There is, then, no single answer, or finite number of answers, to the problem in this case; and results of this form are therefore called *indeterminate*.

Thus, *a result* $\frac{0}{0}$ *indicates indetermination.*

212. The symbol $\frac{0}{0}$, however, does not always represent an indeterminate quantity which may have *any finite value*. Now, in the preceding problem the result $\frac{0}{0}$ was obtained in consequence of *two independent suppositions*, one causing the numerator to become zero, and the other the denominator. We say *independent*, because the quantity $m - n$ can be equal to 0 without necessarily causing the quantity a to become 0. And in all similar cases, we should find the result $\frac{0}{0}$ susceptible of the same interpretation.

But if the symbol $\frac{0}{0}$ is obtained in consequence of the *same supposition* causing both numerator and denominator to become zero, it will be found to have a *single definite limiting value*.

Take, for example, the fraction $\frac{a^2 - b^2}{a^2 - a\,b}$; if $b = a$, this single supposition causes both numerator and denominator to become zero, and the fraction takes the form $\frac{0}{0}$.

Now, dividing both terms by $a - b$, we have

$$\frac{a^2 - b^2}{a^2 - a\,b} = \frac{a + b}{a}, \qquad (1)$$

which equation is true so long as b is not equal to a. It is not necessarily true when b is equal to a, because the second member was obtained by dividing both terms of the first member by $a-b$ (which divisor becomes 0 when $b=a$), as we cannot speak of dividing a quantity by nothing.

In (1), as b approaches a, the limiting value of the first member is $\frac{0}{0}$, and the limiting value of the second member is 2. Thus we have (Art. 209), $\frac{0}{0}=2$.

Hence the limiting value of the fraction, as b approaches a, is 2.

213. A proper understanding of the theory of indetermination, and of the relation of zero to finite quantities, will lead to the detection of the fallacy in some *apparently* remarkable results.

For example, let	$a=b$	
Then	$a^2=ab$	
Subtracting b^2,	$a^2-b^2=ab-b^2$	
Factoring,	$(a+b)(a-b)=b(a-b)$	(1)
Dividing by $a-b$,	$a+b=b$	(2)
But $b=a$; hence	$a+a=a$	
then	$2a=a$	
or,	$2=1$	

The error was made in passing from (1) to (2). Equation (1) may be written

$$\frac{a+b}{b}=\frac{a-b}{a-b}$$

Now, as $b=a$, the second member is an expression of the form $\frac{0}{0}$. But we assumed in going from (1) to (2) that $\frac{a-b}{a-b}=1$, or that $\frac{0}{0}=1$; which we have seen in Arts. 211 and 212 is not necessarily the case, as it may have any value whatever.

XIX. — INEQUALITIES.

214. An **Inequality** is an expression indicating that one of two quantities is greater or less than the other; as,

$$a > b, \text{ and } m < n.$$

The quantity on the left of the sign is called the *first member*, and that on the right, the *second member* of the inequality.

215. Two inequalities are said to subsist in *the same sense* when the first member is the greater or less in both.

Thus,

$$a > b, \text{ and } c > d; \text{ or } 3 < 4, \text{ and } 2 < 3,$$

are inequalities which subsist in the same sense.

216. Two inequalities are said to subsist in *a contrary sense*, when the first member is the greater in the one, and the second in the other. Thus,

$$a > b, \text{ and } c < d; \text{ or } x < y, \text{ and } u > z,$$

are inequalities which subsist in a contrary sense.

217. In the discussion of inequalities, the terms *greater* and *less* must be taken as having an algebraic meaning. That is,

Of any two quantities, a and b, a is the greater when $a - b$ is positive, and a is the less when $a - b$ is negative.

Hence, a negative quantity must be considered as less than nothing; and, of two negative quantities, that is the greater which has the least number of units (Art. 49). Thus,

$$0 > -2, \text{ and } -2 > -3.$$

218. *An inequality will continue in the same sense after the same quantity has been added to, or subtracted from, each member.*

For, suppose $a > b$;

then, by Art. 217, $a - b$ is positive; consequently,

$$(a + c) - (b + c) \text{ and } (a - c) - (b - c)$$

are positive, since each equals $a - b$. Therefore,

$$a + c > b + c, \text{ and } a - c > b - c.$$

Hence, it follows that *a term may be transposed from one member of an inequality to the other, if its sign be changed.*

219. *If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

For, to change all the signs, is equivalent to transposing each term of the first member to the second, and each term of the second member to the first.

220. *If two or more inequalities, subsisting in the same sense, be added, member to member, the resulting inequality will also subsist in the same sense.*

For, let

$$a > b,\ a' > b',\ a'' > b'',\ \ldots\ldots$$

then, by Art. 217, $a - b$, $a' - b'$, $a'' - b''$, ... are all positive; and consequently their sum

$$a + a' + a'' + \ldots\ldots - b - b' - b'' - \ldots\ldots$$

$$\text{or, } (a + a' + a'' + \ldots\ldots) - (b + b' + b'' + \ldots\ldots)$$

is positive. Hence,

$$a + a' + a'' + \ldots\ldots > b + b' + b'' + \ldots\ldots$$

221. *If two inequalities, subsisting in the same sense, be subtracted, member from member, the resulting inequality will not always subsist in the same sense.*

For, let

$$a > b, \text{ and } a' > b';$$

then $a - b$ and $a' - b'$ are positive; but $a - b - (a' - b')$, or $(a - a') - (b - b')$, may be either positive, negative, or 0.

That is,

$$a - a' > b - b', \ a - a' < b - b', \text{ or } a - a' = b - b'.$$

222. *An inequality will continue in the same sense after each member has been multiplied or divided by the same positive quantity.*

For, suppose $a > b$;

then, since $a - b$ is positive, if m is positive,

$$m(a - b) \text{ and } \frac{1}{m}(a - b)$$

are positive. That is, $m a - m b$ and $\frac{a}{m} - \frac{b}{m}$ are positive.

Hence,

$$m a > m b, \text{ and } \frac{a}{m} > \frac{b}{m}.$$

223. *If each member of an inequality be multiplied or divided by the same negative quantity, the sign of inequality must be reversed.*

For, since multiplying or dividing by a negative quantity must change the signs of all the terms, the sign of inequality must be reversed (Art. 219).

224. The *solution* of an inequality consists in determining the limit to the value of its unknown quantity.

This may be done by the application of the preceding principles.

When, however, an inequality and an equation are given, containing two unknown quantities, the process of elimination will be required in the solution.

In verifying an inequality, if the symbols of the unknown quantities be taken equal to their respective limits, the inequality becomes an equation.

EXAMPLES.

225. 1. Find the limit of x in the inequality

$$7x - \frac{23}{3} > \frac{2x}{3} + 5.$$

Clearing of fractions, $21x - 23 > 2x + 15$

Transposing, and uniting, $19x > 38$

Whence, $x > 2$, *Ans.*

2. Find the limits of x in the inequalities,

$$ax + 5bx - 5ab > a^2 \quad (1)$$
$$bx - 7ax + 7ab < b^2 \quad (2)$$

From (1), $ax + 5bx > a^2 + 5ab$

$$x(a + 5b) > a(a + 5b)$$
$$x > a.$$

From (2), $bx - 7ax < b^2 - 7ab$

$$x(b - 7a) < b(b - 7a)$$
$$x < b.$$

Hence, x is greater than a, and less than b, *Ans.*

3. Find the limits of x and y in the following inequality and equation:

$$4x + 6y > 52 \quad (1)$$
$$4x + 2y = 32 \quad (2)$$

Subtracting (2) from (1), $4y > 20$

$$y > 5. \quad (3)$$

From (2), we have $y = 16 - 2x$

Substituting in (3), $16 - 2x > 5$

$$-2x > -11$$

$$-x > -\frac{11}{2}$$

or (Art. 219), $x < \frac{11}{2}$

Hence, $y > 5$, and $x < \frac{11}{2}$, *Ans.*

4. Given $5x - 6 > 19$. Find the limit of x.

5. Given $2x - 5 > 25$; $3x - 7 < 2x + 13$. Find the limits of x.

6. Given $3x + 1 > 13 - x$; $4x - 7 < 2x + 3$. Find an integral value of x.

7. Given $5x + 3y > 46 - y$; $y - x = -4$. Find the limits of x and y.

8. Given $\frac{cx}{3} + dx - cd > \frac{c^2}{3}$; $\frac{dx}{8} - cx + cd < \frac{d^2}{8}$. Find the limits of x.

9. Given $2x + 3y < 57$; $2x + y = 32$. Find the limits of x and y.

10. A teacher being asked the number of his pupils, replied that twice their number diminished by 7 was greater than 29; and that three times their number diminished by 5 was less than twice their number increased by 16. Required the number of his pupils.

11. Three times a certain number, plus 16, is greater than twice that number, plus 24; and two fifths of the number, plus 5, is less than 11. Required the number.

12. A shepherd has a number of sheep such that three times the number, increased by 2, exceeds twice the number, increased by 61; and 5 times the number, diminished by 70, is less than 4 times the number, diminished by 9. How many sheep has he?

XX. — INVOLUTION.

226. **Involution** is the process of raising a quantity to any required power.

This may be effected, as is evident from the definition of a power (Art. 17), by taking the given quantity as a factor as many times as there are units in the exponent of the required power.

227. If the quantity to be involved is positive, the signs of all its powers will evidently be positive; but if the quantity is negative, all its even powers will be positive, and all its odd powers negative. Thus,

$$(-a)^2 = (-a) \times (-a) = +a^2,$$

$$(-a)^3 = (-a) \times (-a) \times (-a) = +a^2 \times (-a) = -a^3,$$

$$(-a)^4 = (-a) \times (-a) \times (-a) \times (-a) = (-a^3) \times (-a) = +a^4,$$

and so on.

Hence,

Every EVEN *power is positive, and every* ODD *power has the same sign as its root.*

INVOLUTION OF MONOMIALS.

228. **1.** Let it be required to raise $5a^2bc^3$ to the fourth power.

$$5a^2bc^3 \times 5a^2bc^3 \times 5a^2bc^3 \times 5a^2bc^3 = 625a^8b^4c^{12}, \textit{ Ans.}$$

2. Raise $-3mn^3$ to the third power.

$$(-3mn^3) \times (-3mn^3) \times (-3mn^3) = -27m^3n^9, \textit{ Ans.}$$

RULE.

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power; making the sign of every even power positive, and the sign of every odd power the same as that of its root.

EXAMPLES.

Find the values of the following:

3. $(a^2 x)^2$.

4. $(-3 a^2 b)^3$.

5. $(-a b^2 c^3)^4$.

6. $(a^n b)^m$.

7. $(2 x^m)^4$.

8. $(2 a b^2 x^3)^5$.

9. $(a^2 b^2)^n$.

10. $(-a^2 c^3)^3$.

11. $(-2 a b^n x)^5$.

12. $(-7 m^3 n)^4$.

13. $(5 a^2 b^3 c^4)^3$.

14. $(-6 x^3 y^7)^3$.

A fraction is raised to any required power by *raising both numerator and denominator to the required power.*

Thus,

$$\left(-\frac{2 x^2}{3 y^3}\right)^3 = \left(-\frac{2 x^2}{3 y^3}\right) \times \left(-\frac{2 x^2}{3 y^3}\right) \times \left(-\frac{2 x^2}{3 y^3}\right) = -\frac{8 x^6}{27 y^9}.$$

Find the values of the following:

15. $\left(\frac{a c}{b}\right)^2$.

16. $\left(\frac{3 a^2 b^3}{4 x y^4}\right)^3$.

17. $\left(-\frac{2 a x^2}{3 b}\right)^2$.

18. $\left(\frac{2}{3} a^3 x^2\right)^6$.

19. $\left(-\frac{2 x y^2}{3 b}\right)^3$.

20. $\left(-\frac{b c x^n}{4 a^2}\right)^5$.

INVOLUTION OF POLYNOMIALS.

229. Polynomials may be raised to any power, as is obvious from Art. 226, by the process of successive multiplications. Thus,

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2 a b + b^2,$$

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3 a^2 b + 3 a b^2 + b^3,$$

and so on. Hence the following

RULE.

Multiply the polynomial by itself, until it has been taken as a factor as many times as there are units in the exponent of the required power.

EXAMPLES.

Find the values of the following:

1. $(a - b)^3$. **3.** $(1 + a^2 + b^2)^3$. **5.** $(a^m - a^n)^4$.

2. $\left(\frac{a}{b} - \frac{b}{a}\right)^2$. **4.** $(a + m - n)^2$. **6.** $(a + b)^5$.

In Chapter XXXVII will be given a method for raising a binomial to any required power, without going through with the process of actual multiplication.

SQUARE OF A POLYNOMIAL.

230. It has been shown (Arts. 104 and 105) that the *square* of any *binomial* expression can be written down, without recourse to formal multiplication, by application of the formulæ

$$(a + b)^2 = a^2 + 2\,a\,b + b^2,$$

$$(a - b)^2 = a^2 - 2\,a\,b + b^2.$$

We may also show, by actual multiplication, that

$$(a + b + c)^2 = a^2 + 2\,a\,b + 2\,a\,c + b^2 + 2\,b\,c + c^2,$$

$$(a + b + c + d)^2 = a^2 + 2\,a\,b + 2\,a\,c + 2\,a\,d + b^2 + 2\,b\,c + 2\,b\,d + c^2 + 2\,c\,d + d^2,$$

and so on.

These results, for convenience of enunciation, may be written in another form,

$$(a + b)^2 = a^2 + b^2 + 2\,a\,b,$$

$$(a - b)^2 = a^2 + b^2 - 2\,a\,b,$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,a\,b + 2\,a\,c + 2\,b\,c,$$

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2\,a\,b + 2\,a\,c + 2\,a\,d + 2\,b\,c + 2\,b\,d + 2\,c\,d,$$

and so on. Hence, the following

RULE.

Write the square of each term, together with twice its product by each of the terms following it.

1. Square $x^2 - 2x - 3$.

Square of each term,	x^4		$+ 4x^2$		$+ 9$
Twice $x^2 \times$ the terms following,		$- 4x^3$	$- 6x^2$		
Twice $- 2x \times$ the term following,				$+ 12x$	
Adding, the result is	x^4	$- 4x^3$	$- 2x^2$	$+ 12x$	$+ 9.$

EXAMPLES.

Square the following expressions:

2. $a - b + c.$
3. $2x^2 + 3x + 4.$
4. $2x^2 - 3x + \frac{1}{2}.$
5. $a - b - c + d.$
6. $x^3 + 2x^2 + x + 2.$
7. $1 - 2x + 3x^2.$
8. $1 + x + x^2 + x^3.$
9. $x^3 - 4x^2 - 2x - 3.$
10. $2x^3 + x^2 + 7x - 1.$
11. $x^3 + 5x^2 - x + 2.$
12. $3x^3 - 2x^2 - x + 4.$
13. $a + b - c - d + e.$

CUBE OF A BINOMIAL.

231. By actual multiplication we may show,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Hence, for finding the cube of a binomial, the following

RULE.

Write the cube of the first term, plus three times the square of the first term times the second, plus three times the first term times the square of the second, plus the cube of the second term.

EXAMPLES.

1. Find the cube of $2x^2 - 3y^3$.

$$(2x^2)^3 + 3(2x^2)^2(-3y^3) + 3(2x^2)(-3y^3)^2 + (-3y^3)^3$$
$$= 8x^6 + 3(4x^4)(-3y^3) + 3(2x^2)(9y^6) + (-27y^9)$$
$$= 8x^6 - 36x^4y^3 + 54x^2y^6 - 27y^9, \textit{ Ans.}$$

Cube the following:

2. $a^2 + 2b$.
3. $2m + 5n$.
4. $3x - 4$.
5. $2x^3 - 3$.
6. $4x^2 - xy$.
7. $3xy + 5ab^2$.

CUBE OF A POLYNOMIAL.

232. By actual multiplication we may show,

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2a + 3b^2c + 3c^2a + 3c^2b + 6abc,$$

$$(a+b+c+d)^3 = a^3 + b^3 + c^3 + d^3 + 3a^2b + 3a^2c + 3a^2d + 3b^2a + 3b^2c + 3b^2d + 3c^2a + 3c^2b + 3c^2d + 3d^2a + 3d^2b + 3d^2c + 6abc + 6abd + 6acd + 6bcd,$$

and so on. Hence, for finding the cube of a polynomial, the following

RULE.

Write the cube of each term, together with three times the product of its square by each of the other terms, and also six times the product of every three different terms.

EXAMPLES.

1. Find the cube of $2x^2 - 3x - 1$.

$$\begin{array}{l} 8x^6 \qquad\qquad\qquad\qquad -27x^3 \qquad\qquad\qquad -1 \\ \quad -36x^5 - 12x^4 \\ \qquad\qquad +54x^4 \qquad\qquad -27x^2 \\ \qquad\qquad\qquad\qquad\qquad +6x^2 - 9x \\ \qquad\qquad\qquad +36x^3 \\ \hline 8x^6 - 36x^5 + 42x^4 + 9x^3 - 21x^2 - 9x - 1, \textit{ Ans.} \end{array}$$

Find the cubes of the following:

2. $a+b-c$.
3. x^2-x-1.
4. $a-b+1$.
5. $2-2x+x^2$.
6. $1+x+x^2+x^3$.
7. $2x^3-x^2+2x-3$.

XXI. — EVOLUTION.

233. **Evolution** is the process of extracting any required root of a quantity.

This may be effected, as is evident from the definition of a root (Art. 17), by determining a quantity which, when raised to the proposed power, will produce the given quantity. It is, therefore, the reverse of involution.

234. Any quantity whose root can be extracted is called a *perfect power;* and any quantity whose root cannot be extracted is called an *imperfect power.*

A quantity may be a perfect power of one degree, and not of another. Thus, 8 is a perfect cube, but not a perfect square.

235. To extract any root of a simple quantity, the exponent of that quantity must be divided by the index of the root.

For, since the nth power of a^m is a^{mn} (Art. 228), it follows that the nth root of a^{mn} is a^m.

236. Any root of the product of two or more factors is equal to the product of the same root of each of the factors.

For, we have seen in Art. 228, in raising a quantity composed of factors to any required power, that each of the factors is raised to the same power.

237. From the relation of a root to its corresponding power, it follows, from Art. 227, that

1. *The odd roots of any quantity have the same sign as the quantity.*

$$\text{Thus, } \sqrt[3]{a^3} = a\text{; and } \sqrt[5]{-a^5} = -a.$$

2. *The even roots of a positive quantity are either positive or negative.*

For either a positive or negative quantity raised to an even power is positive. Thus,

$$\sqrt[4]{a^4} = a \text{ or } -a\text{; or, } \sqrt[4]{a^4} = \pm a.$$

Note. The sign $\pm$, called the *double sign*, is prefixed to a quantity when we wish to indicate that it is either + or −.

3. *Even roots of a negative quantity are not possible.*

For no quantity raised to an even power can produce a negative result. Such roots are called *impossible* or *imaginary* quantities.

EVOLUTION OF MONOMIALS.

238. From the principles contained in Arts. 235 to 237, we obtain the following

RULE.

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root; making the sign of every even root of a positive quantity $\pm$, and the sign of every odd root of any quantity the same as that of the quantity.

If the given quantity is a fraction, it follows from Art. 228 that we may take the required root of both of its terms.

EXAMPLES.

1. Find the square root of $9\,a^4\,b^2\,c^6$.

$$\sqrt{9\,a^4\,b^2\,c^6} = \pm 3\,a^2\,b\,c^3, \textit{ Ans.}$$

2. Find the cube root of $-64\,a^9\,x^3\,y^6$.

$$\sqrt[3]{-64\,a^9\,x^3\,y^6} = -4\,a^3\,x\,y^2, \textit{ Ans.}$$

3. Find the cube root of $\dfrac{8\,x^3\,m^{12}}{27\,a^6\,b^9}$.

$$\sqrt[3]{\left(\frac{8\,x^3\,m^{12}}{27\,a^6\,b^9}\right)} = \frac{2\,x\,m^4}{3\,a^2\,b^3},\ \textit{Ans.}$$

Find the values of the following:

4. $\sqrt[3]{-125\,x^3\,y^6}$. **9.** $\sqrt[m]{a^{mn}\,b^{mp}}$. **14.** $\sqrt[6]{729\,a^{18}\,b^{24}\,c^6}$.

5. $\sqrt[4]{81\,a^4\,b^8}$. **10.** $\sqrt[3]{-8\,a^3\,b^6\,x^9}$. **15.** $\sqrt[5]{-32\,c^{5n}\,d^{10m}}$.

6. $\sqrt[5]{\left(\dfrac{32\,m^5\,n^{10}}{243}\right)}$. **11.** $\sqrt{16\,x^{2m+2}\,a^{2n}}$. **16.** $\sqrt[5]{243\,m^{15}\,n^{20}}$.

7. $\sqrt{9\,a^4\,b^2\,c^{12}}$. **12.** $\sqrt{\left(\dfrac{9\,x^2\,y^{14}}{100\,c^4\,d^{10}}\right)}$. **17.** $\sqrt{(a+x)^2\,b^2\,y^4}$.

8. $\sqrt{625\,a^{12}\,c^2}$. **13.** $\sqrt[n]{3^{2n}\,b^{3n}\,a^n}$. **18.** $\sqrt[3]{x^{3n+3}\,y^{9m-6}}$.

SQUARE ROOT OF POLYNOMIALS.

239. In Art. 116 we explained a method of extracting the square root of a trinomial, provided it was a perfect square. We will now give a method of extracting the square root of any polynomial which is an exact square.

Since the square of $a + b$ is $a^2 + 2\,a\,b + b^2$, we know that the square root of $a^2 + 2\,a\,b + b^2$ is $a + b$. If we can discover an operation by which we can derive $a + b$ from $a^2 + 2\,a\,b + b^2$, we can give a rule for the extraction of the square root.

$$\begin{array}{l|l|l}
 & a^2 + 2\,a\,b + b^2 & a + b \\
 & a^2 & \\
\hline
2\,a + b & 2\,a\,b + b^2 & \\
 & 2\,a\,b + b^2 & \\
\hline
\end{array}$$

Arranging the terms of the square according to the descending powers of a, we observe that the square root of the first term, a^2, is a, which is the first term of the required root. Subtract its square, a^2, from the given polynomial, and bring down the remainder, $2\,a\,b + b^2$ or $(2\,a + b)\,b$. Dividing the first term of the remainder by $2\,a$, that is, by twice the first term of the root, we obtain b, the other term. This, added to $2\,a$, completes the divisor, $2\,a + b$;

which, multiplied by b, and the product, $2\,a\,b + b^2$, subtracted from the remainder, completes the operation.

By a similar process, a root consisting of more than two terms may be found from its square. Thus, by Art. 230, we know that $(a + b + c)^2 = a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + c^2$. Hence, the square root of $a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + c^2$ is $a + b + c$.

$$
\begin{array}{l|l|l}
 & a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + c^2 & a + b + c \\
 & a^2 & \\
\hline
2\,a + b & 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + c^2 & \\
 & 2\,a\,b + b^2 & \\
\hline
2\,a + 2\,b + c & 2\,a\,c + 2\,b\,c + c^2 & \\
 & 2\,a\,c + 2\,b\,c + c^2 & \\
\hline
\end{array}
$$

The square root of the first term, a^2, is a, which is the first term of the required root. Subtracting a^2 from the given polynomial, we obtain $2\,a\,b$ as the first term of the remainder. Dividing this by twice the first term of the root, $2\,a$, we obtain the second term of the root, b, which, added to $2\,a$, completes the divisor, $2\,a + b$. Multiplying this divisor by b, and subtracting the product, $2\,a\,b + b^2$, from the first remainder, we obtain $2\,a\,c$ as the first term of the next remainder.

Doubling the root already found, giving $2\,a + 2\,b$, and dividing the first term of the second remainder, $2\,a\,c$, by the first term of the result, $2\,a$, we obtain the last term of the root, c. This, added to $2\,a + 2\,b$, completes the divisor, $2\,a + 2\,b + c$; which, multiplied by the last term of the root, c, and subtracted from the second remainder, leaves no remainder.

From these operations we derive the following

RULE.

Arrange the terms according to the powers of some letter.

Find the square root of the first term, write it as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by double the root already found, and add the result to the root, and also to the divisor.

Multiply the divisor as it now stands by the term of the root last obtained, and subtract the product from the remainder.

If there are other terms remaining, continue the operation in the same manner as before.

Note. Since all even roots have the double sign $\pm$ (Art. 237), all the terms of the result may have their signs changed. In the examples, however, we shall consider only the positive sign of the result.

EXAMPLES.

1. Find the square root of $9x^4 - 12x^3 + 16x^2 - 8x + 4$.

$$\begin{array}{l|l|l}
 & 9x^4 - 12x^3 + 16x^2 - 8x + 4 & 3x^2 - 2x + 2 \\
 & 9x^4 & \\
\hline
6x^2 - 2x & -12x^3 & \\
 & -12x^3 + 4x^2 & \\
\hline
6x^2 - 4x + 2 & 12x^2 - 8x + 4 & \\
 & 12x^2 - 8x + 4 & \\
\hline
\end{array}$$

Ans. $3x^2 - 2x + 2$.

Find the square roots of the following:

2. $4x^4 - 4x^3 - 3x^2 + 2x + 1$.

3. $4a^4 - 16a^3 + 24a^2 - 16a + 4$.

4. $m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}$.

5. $9 - 12x + 10x^2 - 4x^3 + x^4$.

6. $19x^2 + 6x^3 + 25 + x^4 + 30x$.

7. $28x^3 + 4x^4 - 14x + 1 + 45x^2$.

8. $40x + 25 - 14x^2 + 9x^4 - 24x^3$.

9. $4x^4 + 64 - 20x^3 - 80x + 57x^2$.

10. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

11. $x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz$.

No *rational binomial* is an exact square; but, by the rule, the *approximate root* may be found.

Find, to four terms, the approximate square roots of the following:

12. $1 + x$. **13.** $a^2 + b$. **14.** $1 - 2x$. **15.** $a^2 + x^2$.

The square root of a perfect trinomial square may be obtained by the rule of Art. 116,

Find the square roots of the first and last terms, and connect the results by the sign of the second term.

Extract the square roots of the following:

16. $x^4 + 8x^2 + 16$.

17. $9x^2 - 6xy^3 + y^6$.

18. $a^2 - ax + \frac{x^2}{4}$.

19. $a^{2m} - 4a^{m+n} + 4a^{2n}$.

20. $\frac{a^2}{b^2} - \frac{4a}{3c} + \frac{4b^2}{9c^2}$.

21. $\frac{4x^2}{9y^4} + 2 + \frac{9y^4}{4x^2}$.

SQUARE ROOT OF NUMBERS.

240. The method of Art. 239 may be used to extract the square roots of numbers.

The square root of 100 is 10; of 10000 is 100; of 1000000, is 1000; and so on. Hence, the square root of a number less than 100 is less than 10; the square root of a number between 10000 and 100 is between 100 and 10; the square root of a number between 1000000 and 10000 is between 1000 and 100; and so on.

Or, in other words, the integral part of the square root of a number of one or two figures, contains *one* figure; of a number of three or four figures, contains *two* figures; of a number of five or six figures, contains *three* figures; and so on. Hence,

If a point is placed over every second figure in any integral number, beginning with the units' place, the number of points will show the number of figures in the integral part of its square root.

241. Let it be required to find the square root of 4356.

$$\begin{array}{r|l|l} & \dot{4}3\dot{5}6 & 60+6 \\ & 3600 & \\ \hline 120+6 & 756 & \\ & 756 & \\ \cline{2-2} \end{array}$$

Pointing the number according to the rule of Art. 240, it appears that there are two figures in the integral part of the square root. Let a denote the figure in the tens' place in the root, and b that in the units' place. Then a must be the greatest multiple of 10 whose square is less than 4356; this we find to be 60. Subtracting a^2, that is, the square of 60, or 3600, from the given number, we have the remainder 756. Dividing this remainder by $2a$, or 120, gives 6, which is the value of b. Adding this to 120, multiplying the result by 6, and subtracting the product, 756, there is no remainder. Therefore we conclude that $60 + 6$, or 66, is the required square root.

The zeros being omitted for the sake of brevity, we may arrange the work in the following form:

$$\begin{array}{r|l|l} & \dot{4}3\dot{5}6 & 66 \\ & 36 & \\ \hline 126 & 756 & \\ & 756 & \\ \cline{2-2} \end{array}$$

RULE.

Separate the given number into periods, by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and place its root on the right; subtract the square of this root from the first period, and to the remainder bring down the next period for a dividend.

Divide this dividend, omitting the last figure, by double the root already found, and annex the result to the root and also to the divisor.

Multiply the divisor, as it now stands, by the figure of the root last obtained, and subtract the product from the dividend.

If there are more periods to be brought down, continue the operation in the same manner as before.

If there be a final remainder, the given number has not an exact square root; and, since the rule applies equally to decimals, we may continue the operation, by annexing periods of zeros to the given number, and thus obtain a decimal part to be added to the integral part already found.

It will be observed that decimals require to be pointed to the *right;* and if they have no exact root, we may continue to form periods of zeros, and obtain decimal figures in the root to any desirable extent.

As the trial divisor is necessarily an *incomplete* divisor, it is sometimes found that after completion it gives a product larger than the dividend. In such a case, the last root figure is too large, and one less must be substituted for it.

The root of a common fraction may be obtained, as in Art. 238, by taking the root of both numerator and denominator, when they are perfect squares. If the denominator only is a perfect square, take the approximate square root of the numerator, and divide it by the square root of the denominator. If the denominator is not a perfect square, either reduce the fraction to an equivalent fraction whose denominator is a perfect square, or reduce the fraction to a decimal.

EXAMPLES.

1. Extract the square root of 49.434961.

	49.434961	7.031
	49	
1403	4349	
	4209	
14061	14061	
	14061	

Ans. 7.031.

Here it will be observed that, in consequence of the zero in the root, we annex one zero to the trial divisor, 14, and bring down to the corresponding dividend another period.

Extract the square roots of the following:

2. 273529. **6.** .9409. **10.** .006889.

3. 45796. **7.** $\frac{6561}{9025}$. **11.** .0000107584.

4. 106929. **8.** 1.170724. **12.** 811440.64.

5. 33.1776. **9.** 446.0544. **13.** .17015625.

Extract the square roots of the following to the fifth decimal place:

14. 2. **16.** 31. **18.** $\frac{7}{9}$. **20.** $\frac{1}{3}$.

15. 5. **17.** 173. **19.** $\frac{3}{16}$. **21.** $\frac{2}{7}$.

242. *When $n + 1$ figures of a square root have been obtained by the ordinary method, n more may be obtained by simple division only, supposing $2n + 1$ to be the whole number.*

Let N represent the number whose square root is required, a the part of the root already obtained, x the rest of the root; then

$$\sqrt{N} = a + x,$$

whence, $$N = a^2 + 2ax + x^2;$$

therefore, $$N - a^2 = 2ax + x^2,$$

or, $$\frac{N - a^2}{2a} = x + \frac{x^2}{2a}.$$

Then $N - a^2$ divided by $2a$ will give the rest of the square root required, or x, *increased by* $\frac{x^2}{2a}$; and we shall show that $\frac{x^2}{2a}$ is a *proper fraction*, less than $\frac{1}{2}$, so that by neglecting the remainder arising from the division, we obtain the part required. For, x by supposition contains n figures, so that x^2 cannot contain more than $2n$ figures; but a contains $2n + 1$

figures; and hence $\frac{x^2}{a}$ is a proper fraction. Therefore $\frac{x^2}{2a}$ is a proper fraction, and less than $\frac{1}{2}$.

In the demonstration we supposed N an integer with an exact square root; but the result may be extended to other cases.

From the examples in Art. 241, we observe that each remainder brought down is the given expression minus the square of the root already obtained; and is therefore in the form $N - a^2$. If, then, any remainder be divided by twice the root already found, we can obtain by the division as many more figures of the root as we already have, less one.

We will apply these principles to calculating the square root of 12 to the sixth decimal place. We will obtain the first four figures of the result by the ordinary method:

```
       12.000000 | 3.464
        9        |
64   | 300
     | 256
686  | 4400
     | 4116
6924 | 28400
     | 27696
         704
```

The remainder now is .000704; and twice the root already found is 6.928. Then, by dividing .000704 by 6.928, we can obtain the next three figures of the root. Thus,

```
6.928).0007040(.000102
       .0006928
           11200
```

That is, the square root of 12 to the nearest sixth decimal place is 3.464102.

The following rule will be found to save trouble in obtaining approximate square roots by this method:

Divide the remainder by twice the root already found (omitting the decimal point), and annex all of the quotient, except the decimal point, to the part of the root already found.

In practice the work would be arranged thus:

```
         12. | 3.464
          9  |
       ------
       64 | 300
          | 256
       --------
       686 | 4400
           | 4116
       ----------
       6924 | 28400
            | 27696
            ---------
         6928)704.000(.102
                6928
              ------
                11200
```

Ans. 3.464102.

EXAMPLES.

1. Extract the square root of 11 to the 4th decimal place.

2. Extract the square root of 3 to the 6th decimal place.

3. Extract the square root of 61 to the 8th decimal place.

4. Extract the square root of 131 to the 3d decimal place.

5. Extract the square root of 781 to the 5th decimal place.

6. Extract the square root of 12933 to the 4th decimal place.

CUBE ROOT OF POLYNOMIALS.

243. Since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we know that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$.

$$\begin{array}{r|l} \multicolumn{1}{r}{} & a^3 + 3a^2b + 3ab^2 + b^3 \;\big|\; a + b \\ \multicolumn{1}{r}{} & a^3 \\ \hline 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ & 3a^2b + 3ab^2 + b^3 \\ \cline{2-2} \end{array}$$

Arranging the terms of the cube according to the descending powers of a, we observe that the cube root of the first term, a^3, is a, which is the first term of the required root. Subtract its cube, a^3, from the given polynomial, and bring down the remainder, $3a^2b + 3ab^2 + b^3$ or $(3a^2 + 3ab + b^2)b$. Dividing the first term of the remainder by $3a^2$, that is, by three times the square of the first term of the root, we obtain b, the other term of the root. Adding to the trial divisor $3ab$, that is, three times the product of the first term of the root by the last, and b^2, that is, the square of the last term of the root, completes the divisor, $3a^2 + 3ab + b^2$; which, multiplied by b, and the product, $3a^2b + 3ab^2 + b^3$, subtracted from the remainder, completes the operation.

If there were more terms, we should proceed with $a + b$ exactly as previously with a; regarding it as one term, and dividing the first term of the remainder by three times its square; and so on. Hence, the following

RULE.

Arrange the terms according to the powers of some letter. Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the given polynomial.

Take three times the square of the root already found for a trial divisor, divide the first term of the remainder by it, and write the quotient for the next term of the root.

Add to the trial divisor three times the product of the first term by the second, and the square of the second term.

Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.

If there are other terms remaining, consider the root already found as one term, and proceed as before.

EXAMPLES.

1. Find the cube root of $x^6 - 6x^5 + 40x^3 - 96x - 64$.

$$
\begin{array}{l|l|l}
 & x^6 - 6x^5 + 40x^3 - 96x - 64 & x^2 - 2x - 4 \\
 & x^6 & \\
\hline
3x^4 - 6x^3 + 4x^2 & -6x^5 & \\
 & -6x^5 + 12x^4 - 8x^3 & \\
\hline
3x^4 - 12x^3 + 12x^2 & -12x^4 + 48x^3 & \\
\qquad -12x^2 + 24x + 16 & & \\
\hline
3x^4 - 12x^3 \qquad + 24x + 16 & -12x^4 + 48x^3 - 96x - 64 & \\
\hline
\end{array}
$$

Ans. $x^2 - 2x - 4$.

The formation of the second divisor may be explained thus:

Regarding the root already obtained, $x^2 - 2x$, as one term, three times its square gives $3x^4 - 12x^3 + 12x^2$; three times $x^2 - 2x$ times -4, gives $-12x^2 + 24x$; and the square of the last root term is 16. Adding these results, we have for the complete divisor, $3x^4 - 12x^3 + 24x + 16$.

Find the cube roots of the following:

2. $1 - 6y + 12y^2 - 8y^3$.

3. $8x^6 + 36x^4 + 54x^2 + 27$.

4. $64x^3 - 144abx^2 + 108a^2b^2x - 27a^3b^3$.

5. $x^6 + 6x^5 - 40x^3 + 96x - 64$.

6. $y^6 - 1 + 5y^3 - 3y^5 - 3y$.

7. $x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$.

8. $15x^4 - 6x - 6x^5 + 15x^2 + 1 + x^6 - 20x^3$.

9. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$.

10. $9x^3 - 21x^2 - 36x^5 + 8x^6 - 9x + 42x^4 - 1$.

No *rational binomial* is an exact cube; but, by the rule, the *approximate root* may be found.

Find, to four terms, the approximate cube roots of the following:

11. $x^3 + 1$. **12.** $x^3 - a^3$. **13.** $8x^6 - 3$.

CUBE ROOT OF NUMBERS.

244. The method of Art. 243 may be used to extract the cube roots of numbers.

The cube root of 1000 is 10; of 1000000, is 100; of 1000000000, is 1000; and so on. Hence, the cube root of a number less than 1000 is less than 10; the cube root of a number between 1000000 and 1000 is between 100 and 10; the cube root of a number between 1000000000 and 1000000 is between 1000 and 100; and so on.

Or, in other words, the integral part of the cube root of a number of one, two, or three figures, contains *one* figure; of a number of four, five, or six figures, contains *two* figures; of a number of seven, eight, or nine figures, contains *three* figures; and so on. Hence,

If a point is placed over every third figure in any integral number, beginning with the units' place, the number of points will show the number of figures in the integral part of its cube root.

245. Let it be required to find the cube root of 405224.

$$\begin{array}{r|r|l} & 40\dot{5}22\dot{4} & 70 + 4 \\ & 343000 & \\ \hline 14700 & 62224 & \\ 840 & & \\ 16 & & \\ \hline 15556 & 62224 & \\ \end{array}$$

Pointing the number according to the rule of Art. 244, it appears that there are two figures in the integral part of the cube root. Let a denote the figure in the tens' place in the root, and b that in the units' place. Then a must be the greatest multiple of 10 whose cube is less than 405224; this we find to be 70. Subtracting a^3, that is, the cube of 70, or 343000, from the given number, we have the remainder 62224. Dividing this remainder by $3a^2$, or 14700, gives 4, which is the value of b. Adding to the trial divisor $3\,a\,b$, which is 840, and b^2, which is 16, completes the divisor, 15556. Multiplying the result by 4, and subtracting the product, 62224, there is no remainder. Therefore we conclude that $70 + 4$, or 74, is the required cube root.

The work is usually arranged thus:

	405224	74
	343	
14700	62224	
840		
16		
15556	62224	

RULE.

Separate the given number into periods, by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and place its root on the right; subtract the cube of this root from the left-hand period, and to the remainder bring down the next period for a dividend.

Divide this dividend, omitting the last two figures, by three times the square of the root already found, and annex the quotient to the root.

Add together the trial divisor, with two zeros annexed; three times the product of the last root figure by the rest of the root, with one zero annexed; and the square of the last root figure.

Multiply the divisor, as it now stands, by the figure of the root last obtained, and subtract the product from the dividend.

If there are more periods to be brought down, continue the operation in the same manner as before, regarding the root already obtained as one term.

The observations made after the rule for the extraction of the square root (Art. 241) are equally applicable to the extraction of the cube root.

EXAMPLES.

1. Extract the cube root of 8.144865728.

	8.144865728	2.012
	8	
120000	144865	
600		
1		
120601	120601	
12120300	24264728	
12060		
4		
12132364	24264728	

Ans. 2.012.

Here it will be observed that, in consequence of the 0 in the root, we annex two additional zeros to the trial divisor, 1200, and bring down to the corresponding dividend another period.

Extract the cube roots of the following:

2. 1860867.
3. .724150792.
4. 1481.544.
5. $\frac{29791}{681472}$.
6. 51.478848.
7. .000517781627.

Extract the cube roots of the following to the third decimal place:

8. 3.
9. 7.
10. 212.
11. $\frac{5}{8}$.
12. $\frac{2}{27}$.
13. $\frac{3}{17}$.

246. *When $n+2$ figures of a cube root have been obtained by the ordinary method, n more may be obtained by division only, supposing $2n+2$ to be the whole number.*

Let N represent the number whose cube root is required, a the part of the root already obtained, x the rest of the root; then

$$\sqrt[3]{N} = a + x,$$

whence, $$N = a^3 + 3\,a^2\,x + 3\,a\,x^2 + x^3;$$

therefore, $$N - a^3 = 3\,a^2\,x + 3\,a\,x^2 + x^3,$$

or, $$\frac{N - a^3}{3\,a^2} = x + \frac{x^2}{a} + \frac{x^3}{3\,a^2}.$$

Then $N - a^3$ divided by $3\,a^2$ will give the rest of the cube root required, or x, *increased by* $\frac{x^2}{a} + \frac{x^3}{3\,a^2}$; and we shall show that the latter is a *proper fraction*, less than $\frac{1}{2}$, so that by neglecting the remainder arising from the division, we obtain the part required. For, x by supposition contains n figures, so that x^2 cannot contain more than $2\,n$ figures. But a contains $2\,n + 2$ figures; and hence $\frac{x^2}{a}$ is less than $\frac{1}{10}$. And as $\frac{x^3}{3\,a^2} = \frac{x^2}{a} \times \frac{x}{3\,a}$, and $\frac{x}{3\,a}$ is less than 1, $\frac{x^3}{3\,a^2}$ must also be less than $\frac{1}{10}$. Therefore, $\frac{x^2}{a} + \frac{x^3}{3\,a^2}$ is a proper fraction, less than $\frac{1}{2}$.

Remarks similar to those in the last part of Art. 242 apply here.

ANY ROOT OF POLYNOMIALS.

247. In order to establish a general rule for the extraction of roots, it will be necessary to notice the formation of the nth power of a polynomial, n being any entire number whatever. Thus,

$$(a + b)^n = a^n + n\,a^{n-1}\,b + \ldots\ldots$$

Therefore,

$$\sqrt[n]{a^n + n\,a^{n-1}\,b + \ldots\ldots} = a + b.$$

The first term of the root, a, is the nth root of a^n, the first term of the power; and the second term of the root, b, may be obtained by dividing the second term of the power by $n\,a^{n-1}$, or by n times the $(n-1)$th power of the first term of the root.

If the root now found be raised to the nth power, and subtracted from the given polynomial, it will be seen that two terms of the required root have been determined.

It will be observed that the process is essentially that of the preceding Articles, simplified by dispensing with completed divisors, and generalized. Hence the

RULE.

Arrange the terms according to the powers of some letter.

Find the required root of the first term for the first term of the root, and subtract its power from the given polynomial.

Divide the first term of the remainder by n times the $(n-1)$*th power of this root, for the second term of the root, and subtract the nth power of the root now found from the given polynomial.*

If other terms of the root require to be determined, use the same divisor as before, and proceed in like manner till the nth power of the root becomes equal to the given polynomial.

This rule is, also, applicable to numbers, by taking n figures in each period.

EXAMPLES.

1. Extract the cube root of $x^6+6\,x^5-40\,x^3+96\,x-64$.

$$
\begin{array}{rl|l}
 & x^6+6\,x^5-40\,x^3+96\,x-64 & x^2+2\,x-4 \\
(x^2)^3 = & x^6 & \\
\hline
 & 3\,x^4 \mid 6\,x^5 & \\
(x^2+2\,x)^3 = & x^6+6\,x^5+12\,x^4+8\,x^3 & \\
\hline
 & 3\,x^4 \mid -12\,x^4 & \\
(x^2+2x-4)^3 = & x^6+6\,x^5-40\,x^3+96\,x-64 &
\end{array}
$$

Ans. $x^2+2\,x-4$.

2. Extract the cube root of $m^6-6\,m^5+40\,m^3-96\,m-64$.

3. Extract the square root of $a^4-2\,a^3x+3\,a^2x^2-2\,ax^3+x^4$.

4. Extract the fifth root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.

5. Extract the fourth root of $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1$.

248. When the index of the required root is a multiple of two or more numbers, we may obtain the root *by successive extractions of the simpler roots.*

For, since (Art. 17), $(\sqrt[mn]{a})^{mn} = a$,

taking the nth root of both members, we have (Art. 235),

$$(\sqrt[mn]{a})^m = \sqrt[n]{a}.$$

Taking the mth root of both members,

$$\sqrt[mn]{a} = \sqrt[m]{(\sqrt[n]{a})}.$$

Or, *the mnth root of a quantity is equal to the mth root of the nth root of that quantity.*

EXAMPLES.

1. Extract the fourth root of $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$.

2. Extract the sixth root of $a^{12} - 6a^{10} + 15a^8 - 20a^6 + 15a^4 - 6a^2 + 1$.

3. Extract the fourth root of $m^8 - 8m^7 + 12m^6 + 40m^5 - 74m^4 - 120m^3 + 108m^2 + 216m + 81$.

XXII. — THE THEORY OF EXPONENTS.

249. In Art. 17, we defined an exponent as indicating how many times a quantity was taken as a factor; thus

a^m means $a \times a \times a$ ...... to m factors.

Obviously this definition has no meaning unless the exponent is a positive integer; and as fractional and negative exponents have not been previously defined, we may give to them any definition we please.

250. We found (Arts. 82, 93, and 228) that when m and n were positive integers,

$$\text{I.}\quad a^m \times a^n = a^{m+n}.$$

$$\text{II.}\quad \frac{a^m}{a^n} = a^{m-n}.$$

$$\text{III.}\quad (a^m)^n = a^{mn}.$$

As it is convenient to have all exponents follow the same laws, as regards multiplication, division, and involution, we shall define fractional and negative exponents in such a way as to make Rule I hold for *any* values of m and n. We shall now find what meanings must be assigned to them in consequence.

251. To find the meaning of $a^{\frac{3}{2}}$.

As Rule I is to hold universally, it follows that

$$a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^{\frac{6}{2}} = a^3.$$

Hence, $a^{\frac{3}{2}}$ is such a quantity as when multiplied by itself produces a^3. Then, by the definition of *root* (Art. 17), $a^{\frac{3}{2}}$ must be the square root of a^3; or, $a^{\frac{3}{2}} = \sqrt{a^3}$.

Again, to find the meaning of $a^{\frac{1}{3}}$.

By Rule I, $$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^{\frac{3}{3}} = a.$$

Hence, $a^{\frac{1}{3}}$ is such a quantity as when taken 3 times as a factor produces a; or, $a^{\frac{1}{3}} = \sqrt[3]{a}$.

252. We will now consider the general case.

To find the meaning of $a^{\frac{p}{q}}$, *p and q being positive integers.*

By Rule I, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \ldots\ldots$ to q factors

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\ldots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Hence, $a^{\frac{p}{q}}$ is such a quantity as when taken q times as a factor produces a^p. Then $a^{\frac{p}{q}}$ must be the qth root of a^p; or, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $c^{\frac{5}{2}} = \sqrt{c^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.,

and, conversely, $\sqrt[4]{a^5} = a^{\frac{5}{4}}$; $\sqrt{x} = x^{\frac{1}{2}}$; $\sqrt[3]{m^5} = m^{\frac{5}{3}}$; etc.

EXAMPLES.

253. Express the following with radical signs instead of fractional exponents:

1. $a^{\frac{1}{4}}$. **3.** $2\,c^{\frac{1}{2}}$. **5.** $x^{\frac{3}{4}}\,y^{\frac{2}{3}}$. **7.** $4\,a^{\frac{1}{5}}\,b^{\frac{1}{6}}$. **9.** $5\,y^{\frac{3}{10}}\,z^{\frac{7}{10}}$.

2. $b^{\frac{3}{7}}$. **4.** $3\,a\,m^{\frac{5}{4}}$. **6.** $m^{\frac{3}{5}}\,n^{\frac{5}{3}}$. **8.** $2\,c^{\frac{3}{8}}\,d^{\frac{2}{5}}$. **10.** $3\,a\,b^{\frac{1}{3}}\,c^{\frac{4}{5}}\,d^{\frac{7}{2}}$.

Express the following with fractional exponents instead of radical signs:

11. $\sqrt[5]{x^6}$. **13.** $\sqrt{n}$. **15.** $3\sqrt{m^5}$. **17.** $\sqrt[3]{a^7}\,\sqrt[4]{a^3}$.

12. $\sqrt[8]{y^2}$. **14.** $\sqrt[3]{c^4}$. **16.** $4\sqrt[7]{a^{10}}$. **18.** $\sqrt{x^5}\,\sqrt[5]{y^2}$.

19. $5\sqrt{m^9}\,\sqrt[3]{n^7}$. **20.** $2\,a\sqrt[4]{x^7}\,\sqrt[5]{y}$.

254. To find the meaning of a^{-3}.

By Rule I, $a^{-3} \times a^3 = a^0 = 1$, by Art. 94.

Hence, $$a^{-3} = \frac{1}{a^3}.$$

To find the meaning of $a^{-\frac{1}{2}}$.

By Rule I, $a^{-\frac{1}{2}} \times a^{\frac{1}{2}} = a^0 = 1$.

Hence, $$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}.$$

255. We will now consider the general case.

To find the meaning of a^{-s}, s being integral or fractional.

By Rule I, $$a^{-s} \times a^{s} = a^{0} = 1.$$

Hence, $$a^{-s} = \frac{1}{a^{s}}.$$

For example, $a^{-1} = \frac{1}{a}$; $a^{-4} = \frac{1}{a^4}$; $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$; etc.,

and, conversely, $\frac{1}{a^2} = a^{-2}$; $\frac{x^2}{a^3} = x^2 a^{-3}$; $\frac{2}{a^{\frac{3}{4}}} = 2\,a^{-\frac{3}{4}}$; etc.

We observe, in this connection, the following important principle:

A quantity may be changed from the denominator of a fraction to the numerator, or from the numerator to the denominator, if the sign of its exponent be changed.

EXAMPLES.

256. Remove all *powers* from the denominators to the numerators in the following:

1. $\frac{2}{x^2} - \frac{3\,c}{5\,x^3} + \frac{5}{2\,x^{-1}}$.

2. $\frac{1}{x} + \frac{1}{x^2} - \frac{3}{x^{-2}} - \frac{4}{x^{-3}}$.

3. $\frac{a^2}{x^{\frac{2}{3}}} + \frac{a^3 - 1}{x^{\frac{3}{4}}} + \frac{a^4}{x^{\frac{1}{2}}} - \frac{a^5 - b}{x^{-\frac{2}{5}}}$.

4. $\frac{7\,m}{6\,c^{-1}} - \frac{3\,m}{7\,c^{\frac{3}{8}}} - \frac{4\,m^2 - 1}{5\,c^{\frac{1}{6}}} + \frac{3\,m^3 + 2\,n}{2\,c^{-\frac{3}{4}}}$.

Remove all *powers* from the numerators to the denominators in the following:

5. $\frac{2\,x}{3} + \frac{3\,x^{\frac{1}{2}}}{4\,a} - \frac{a}{x - 3}$.

6. $\frac{x}{2} - \frac{x^3}{3} + \frac{x^{-2}}{4} - \frac{2\,x^{-1}}{5}$.

7. $\dfrac{a^{\frac{2}{3}}}{x+2}-\dfrac{3\,a^{-\frac{3}{4}}}{5\,b}-\dfrac{5\,a^{-2}}{2\,c^{\frac{1}{3}}}+\dfrac{a^3}{7-b\,c}.$

8. $\dfrac{m^{-1}}{1-x^2}-\dfrac{m^{\frac{2}{3}}}{3\,x}-\dfrac{n^{-\frac{3}{5}}}{5\,x^{-1}}-\dfrac{2\,p}{7\,x^{-3}}.$

Express the following with *positive* exponents:

9. $2\,x^2\,y^{\frac{1}{2}}-3\,x^{-1}\,y^{\frac{2}{3}}-x^{-4}\,y^{-\frac{3}{7}}.$

10. $a^{-1}\,b^{-2}+2\,a^{-3}\,b^{-4}-3\,a^{\frac{1}{2}}\,b^{-\frac{2}{5}}.$

11. $3\,x^{-\frac{1}{3}}\,y^{-\frac{2}{7}}-4\,x\,y^{-\frac{1}{3}}+x^3\,y^{-5}.$

12. $a^{-1}\,b^{-2}\,c^3+a^{-2}\,b^{-\frac{3}{4}}\,c^{-\frac{1}{3}}+a^3\,b^{-2}\,c.$

257. We obtained the meanings of fractional and negative exponents on the supposition that Rule I, Art. 250, was to hold universally. Hence, for any values of m and n,

$$a^m \times a^n = a^{m+n}$$

For example, $a^2 \times a^{-5} = a^{2-5} = a^{-3}$; $a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}}$;

$a^{-4} \times a^{\frac{5}{2}} = a^{-4+\frac{5}{2}} = a^{-\frac{3}{2}}$; $a^{\frac{4}{3}} \times a^{\frac{1}{5}} = a^{\frac{4}{3}+\frac{1}{5}} = a^{\frac{23}{15}}$; etc.

EXAMPLES.

Multiply together the following:

1. a^3 and a^{-1}.

2. a^2 and a^{-2}.

3. a^{-1} and a^{-5}.

4. c^3 and $\sqrt[8]{c^2}$.

5. x^{-1} and $\sqrt[4]{x^{-3}}$.

6. m^2 and $\dfrac{1}{\sqrt[5]{m}}$.

7. n and $n^{-\frac{2}{3}}$.

8. $x^{\frac{5}{2}}$ and $x^{-\frac{3}{2}}$.

9. $2\,c^{-\frac{2}{7}}$ and $-3\,a\sqrt[5]{c^3}$.

10. Multiply $a^{\frac{2}{3}}b^{-\frac{1}{2}}+2a^{\frac{1}{3}}-3b^{\frac{1}{2}}$ by $2b^{-\frac{1}{2}}-4a^{-\frac{1}{3}}-6a^{-\frac{2}{3}}b^{\frac{1}{2}}$.

$$\begin{array}{llll}
a^{\frac{2}{3}}b^{-\frac{1}{2}}+2a^{\frac{1}{3}}-3b^{\frac{1}{2}} & & & \\
2b^{-\frac{1}{2}}-4a^{-\frac{1}{3}}-6a^{-\frac{2}{3}}b^{\frac{1}{2}} & & & \\
\hline
2a^{\frac{2}{3}}b^{-1}+4a^{\frac{1}{3}}b^{-\frac{1}{2}}-6 & & & \\
\quad -4a^{\frac{1}{3}}b^{-\frac{1}{2}}-8+12a^{-\frac{1}{3}}b^{\frac{1}{2}} & & & \\
\qquad\qquad -6-12a^{-\frac{1}{3}}b^{\frac{1}{2}}+18a^{-\frac{2}{3}}b & & & \\
\hline
2a^{\frac{2}{3}}b^{-1} \qquad -20 \qquad +18a^{-\frac{2}{3}}b, \textit{ Ans.} & & &
\end{array}$$

Note. It should be carefully remembered, in performing examples like the above, that any quantity whose exponent is 0 is equal to 1 (Art. 94).

Multiply together the following:

11. $a^2b^{-2}-2+a^{-2}b^2$ and $a^2b^{-2}+2+a^{-2}b^2$.

12. $a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}-b^{\frac{3}{4}}$ and $a^{\frac{1}{4}}+b^{\frac{1}{4}}$.

13. $a^{-2}-2a^{-1}b+b^2-ab^3$ and $a^{-3}+2a^{-2}b$.

14. $3a^{-1}-a^{-2}b^{-1}+a^{-3}b^{-2}$ and $6a^3b^2+2a^2b+2a$.

15. $x^{-3}y^2-x^{-2}y-2x^{-1}$ and $2x^2y^{-1}+2x^3y^{-2}-4x^4y^{-3}$.

16. $x^{\frac{2}{3}}y^{-\frac{3}{4}}+2+x^{-\frac{2}{3}}y^{\frac{3}{4}}$ and $2x^{-\frac{2}{3}}y^{\frac{3}{4}}-4x^{-\frac{4}{3}}y^{\frac{3}{2}}+2x^{-2}y^{\frac{9}{4}}$.

17. $2x^{\frac{2}{3}}-3x^{\frac{1}{3}}-4+x^{-\frac{1}{3}}$ and $3x^{\frac{4}{3}}+x-2x^{\frac{2}{3}}$.

18. $4a^{\frac{3}{4}}b^{-1}+a^{\frac{1}{4}}-3a^{-\frac{1}{4}}b$ and $8a^{\frac{1}{4}}b^{-1}-2a^{-\frac{1}{4}}-6a^{-\frac{3}{4}}b$.

258. *To prove that Rule II holds for all values of m and n.*

By Rule I, $\quad a^{m-n}\times a^n=a^{m-n+n}=a^m$.

Inverting the equation, and dividing by a^n, we have

$$\frac{a^m}{a^n}=a^{m-n}.$$

For example, $\dfrac{a^3}{a} = a^{3-1} = a^2$; $\dfrac{a^{-2}}{a^3} = a^{-2-3} = a^{-5}$;

$$\frac{a^{-\frac{3}{4}}}{a^{-2}} = a^{-\frac{3}{4}+2} = a^{\frac{5}{4}};\quad \frac{a^3}{a^{-\frac{2}{5}}} = a^{3+\frac{2}{5}} = a^{\frac{17}{5}};\ \text{etc.}$$

EXAMPLES.

Divide the following:

1. a^3 by a^{-1}. **4.** $a^{-\frac{1}{2}}$ by $a^{-\frac{4}{7}}$. **7.** $x^{\frac{1}{3}}$ by $\dfrac{1}{\sqrt[4]{x^3}}$.

2. a by a^3. **5.** c^{-1} by $\sqrt[4]{c^5}$. **8.** $5\,n$ by $2\,a^{-1}\sqrt[3]{b}$.

3. $a^{\frac{3}{7}}$ by $a^{\frac{4}{5}}$. **6.** m^2 by $\sqrt[5]{m^{-2}}$. **9.** $6\,a^{-1}b^{\frac{2}{3}}$ by $-3\,a\,b^{-\frac{1}{5}}$.

10. Divide $2\,a^{\frac{2}{3}}b^{-1} - 20 + 18\,a^{-\frac{2}{3}}b$ by $a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2\,a^{\frac{1}{3}} - 3\,b^{\frac{1}{2}}$.

$$\begin{array}{l|l}
2\,a^{\frac{2}{3}}b^{-1} - 20 + 18\,a^{-\frac{2}{3}}b & a^{\frac{2}{3}}b^{-\frac{1}{2}} + 2\,a^{\frac{1}{3}} - 3\,b^{\frac{1}{2}} \\ \cline{2-2}
\underline{2\,a^{\frac{2}{3}}b^{-1} + 4\,a^{\frac{1}{3}}b^{-\frac{1}{2}} - 6} & 2\,b^{-\frac{1}{2}} - 4\,a^{-\frac{1}{3}} - 6\,a^{-\frac{2}{3}}b^{\frac{1}{2}},\ \textit{Ans.} \\
\qquad -4\,a^{\frac{1}{3}}b^{-\frac{1}{2}} - 14 + 18\,a^{-\frac{2}{3}}b & \\
\qquad \underline{-4\,a^{\frac{1}{3}}b^{-\frac{1}{2}} - \ \ 8 + 12\,a^{-\frac{1}{3}}b^{\frac{1}{2}}} & \\
\qquad\qquad -6 - 12\,a^{-\frac{1}{3}}b^{\frac{1}{2}} + 18\,a^{-\frac{2}{3}}b & \\
\qquad\qquad \underline{-6 - 12\,a^{-\frac{1}{3}}b^{\frac{1}{2}} + 18\,a^{-\frac{2}{3}}b} &
\end{array}$$

Note 1. Particular attention must be given to seeing that the dividend and divisor are arranged in the same order of powers, and that each remainder is brought down in the same order. It must be remembered that a zero exponent is greater than any negative exponent; and that negative exponents are the smaller, the greater their absolute value.

Note 2. In dividing the first term of the dividend or remainder by the first term of the divisor, it will be found convenient to write the quotient at first in a fractional form; reducing the result by the principles of Art. 258. Thus, in getting the first term of the quotient in Ex. 10, we divide $2\,a^{\frac{2}{3}}b^{-1}$ by $a^{\frac{2}{3}}b^{-\frac{1}{2}}$. Then, the result $= \dfrac{2\,a^{\frac{2}{3}}b^{-1}}{a^{\frac{2}{3}}b^{-\frac{1}{2}}} = 2\,a^{\frac{2}{3}-\frac{2}{3}}b^{-1+\frac{1}{2}} = 2\,b^{-\frac{1}{2}}$.

Divide the following:

11. $a - b$ by $a^{\frac{1}{5}} - b^{\frac{1}{5}}$.

12. $a^{-4} + a^{-2} b^{-2} + b^{-4}$ by $a^{-2} + a^{-1} b^{-1} + b^{-2}$.

13. $2 x^{-2} y^2 + 6 + 8 x^2 y^{-2}$ by $2 x + 2 x^2 y^{-1} + 4 x^3 y^{-2}$.

14. $2 x^{\frac{2}{3}} y^{-1} - 2 x^{-\frac{2}{3}} y + 32 x^{-2} y^3$ by $2 + 6 x^{-\frac{2}{3}} y + 8 x^{-\frac{4}{3}} y^2$.

15. $x^{-3} y^{-5} - 3 x^{-5} y^{-7} + x^{-7} y^{-9}$ by $x^{-2} y^{-3} + x^{-3} y^{-4} - x^{-4} y^{-5}$.

16. $8 - 10 x^{-2} y^{\frac{10}{3}} + 2 x^{-4} y^{\frac{20}{3}}$ by $4 x^{-\frac{1}{2}} y^{\frac{2}{3}} + 2 x^{-\frac{3}{2}} y^{\frac{7}{3}} - 2 x^{-\frac{5}{2}} y^4$.

259. *To prove that Rule III holds for all values of m and n.*

We will consider three cases.

Case I. Let m have *any* value, and n be a positive integer.

Then, from the definition of a positive integral exponent,

$$(a^m)^n = a^m \times a^m \times a^m \ldots\ldots \text{ to } n \text{ factors}$$

$$= a^{m + m + m \ldots\ldots \text{ to } n \text{ terms}} = a^{mn}.$$

Case II. Let m have *any* value, and n be a positive fraction, which we will denote by $\frac{p}{q}$.

Then, $(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$, by the definition of Art. 252.

$$= \sqrt[q]{a^{mp}}, \quad \text{by Case I, Art. 259,}$$

$$= a^{\frac{mp}{q}}, \quad \text{by Art. 252,}$$

$$= a^{m \times \frac{p}{q}} = a^{mn}.$$

Case III. Let m have *any* value, and n be a negative quantity, integral or fractional, which we will denote by $-s$.

Then, $(a^m)^n = (a^m)^{-s} = \frac{1}{(a^m)^s}$, by the definition of Art. 255,

$$= \frac{1}{a^{ms}}, \text{ by Cases I and II, Art. 259,}$$

$$= a^{-ms} = a^{m(-s)} = a^{mn}.$$

Thus, we have proved Rule III to hold for all values of m and n.

For example, $(a^2)^3 = a^6$; $(a^{-1})^5 = a^{-5}$; $(a^{-\frac{2}{3}})^{\frac{1}{2}} = a^{-\frac{1}{3}}$; $(a^{\frac{3}{4}})^{\frac{4}{3}} = a$; $(a^{\frac{2}{5}})^{-\frac{3}{7}} = a^{-\frac{6}{35}}$; $(a^2)^{-\frac{3}{2}} = a^{-3}$; etc.

EXAMPLES.

260. Find the values of the following:

1. $(a^2)^{-3}$.
2. $(a^{-2})^3$.
3. $(a^3)^{\frac{5}{2}}$.
4. $(a^{-1})^{-\frac{3}{8}}$.
5. $(c^{-\frac{2}{5}})^{-\frac{10}{3}}$.
6. $(\sqrt{x})^{\frac{1}{3}}$.
7. $\sqrt[5]{(c^{-\frac{1}{2}})^2}$.
8. $(\sqrt[4]{m^3})^{-\frac{4}{3}}$.
9. $(\sqrt[5]{y^3})^{-5}$.
10. $\left(\frac{1}{a^5}\right)^{\frac{3}{5}}$.
11. $\left(\frac{1}{\sqrt[4]{n^3}}\right)^{\frac{4}{3}}$.
12. $\{(x^{-\frac{1}{2}})^{-1}\}^{-\frac{2}{3}}$.

261. *To prove that* $(ab)^n = a^n b^n$ *for any value of* n.

In Art. 228 we showed the truth of the theorem when n was a positive integer.

Case I. Let n be a positive fraction, which we will denote by $\frac{p}{q}$. We have then to show that $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$.

$$[(ab)^{\frac{p}{q}}]^q = (ab)^p, \text{ by Art. 259.}$$

$$[a^{\frac{p}{q}}\, b^{\frac{p}{q}}]^q = (a^{\frac{p}{q}})^q\, (b^{\frac{p}{q}})^q = a^p\, b^p = (a\, b)^p, \text{ by Art. 228.}$$

Hence, $$[(a\,b)^{\frac{p}{q}}]^q = [a^{\frac{p}{q}}\, b^{\frac{p}{q}}]^q.$$

Therefore, $$(a\,b)^{\frac{p}{q}} = a^{\frac{p}{q}}\, b^{\frac{p}{q}}.$$

Case II. Let n be a negative quantity, which we will denote by $-s$. We have then to show that $(a\,b)^{-s} = a^{-s}\, b^{-s}$.

$$(a\,b)^{-s} = \frac{1}{(a\,b)^s} = \frac{1}{a^s\, b^s}, \text{ by Art. 228 and Case I,}$$
$$= a^{-s}\, b^{-s}.$$

262. *To find the value of a numerical quantity affected with a fractional exponent.*

1. Find the value of $8^{\frac{5}{3}}$.

From Art. 252, we should have $8^{\frac{5}{3}} = \sqrt[3]{8^5}$; and to find the value in this way, we should raise 8 to the fifth power, and take the cube root of the result.

A better method, however, is as follows:

$$8^{\frac{5}{3}} = (8^{\frac{1}{3}})^5, \text{ by Art. 259,}$$
$$= (\sqrt[3]{8})^5 = 2^5 = 32, \textit{ Ans.}$$

Note. Place the numerator of the fractional exponent as the exponent of the parenthesis, and 1 divided by the denominator as the exponent of the quantity within.

2. Find the value of $16^{-\frac{5}{4}}$.

$$16^{-\frac{5}{4}} = \frac{1}{16^{\frac{5}{4}}} = \frac{1}{(16^{\frac{1}{4}})^5} = \frac{1}{(\sqrt[4]{16})^5} = \frac{1}{(\pm 2)^5} = \pm\frac{1}{32}, \textit{ Ans.}$$

EXAMPLES.

Find the values of the following:

3. $27^{\frac{2}{3}}$. 5. $1000^{-\frac{4}{3}}$. 7. $(-8)^{\frac{2}{3}}$. 9. $\dfrac{(-27)^{\frac{1}{3}} \times 25^{\frac{5}{2}}}{36^{\frac{3}{2}} \times 16^{-\frac{5}{4}}}$.

4. $36^{\frac{3}{2}}$. 6. $9^{-\frac{7}{2}}$. 8. $(-27)^{\frac{5}{3}}$. 10. $\dfrac{4^{\frac{3}{2}} \times 9^{-2}}{81^{-\frac{3}{2}} \times 16^{\frac{7}{4}}}$.

If the numerical quantity is not a perfect power of the degree indicated by the denominator of the fractional exponent, the first method explained in Ex. 1, Art. 262, is the best.

For example, to find the value of $7^{\frac{3}{2}}$, we write it $\sqrt{7^3}$, or $\sqrt{343}$; and find the square root of 343 to any desired degree of accuracy.

MISCELLANEOUS EXAMPLES.

263. Extract the square roots of the following:

1. $a^{-2}x^{\frac{3}{4}}$. 2. $9\,m\,n^{\frac{1}{3}}$. 3. $\dfrac{c^{\frac{2}{3}}\,d^{-\frac{5}{4}}}{4\,x\,y^3}$. 4. $\dfrac{2\,a^{-\frac{2}{3}}\,b^{-1}}{c^4\,d\,e^{\frac{1}{2}}}$.

5. $9\,x^{-4}y^2 - 12\,x^{-3}y - 2\,x^{-2} + 4\,x^{-1}y^{-1} + y^{-2}$.

6. $4\,x^{\frac{4}{3}} + 4\,x^{\frac{5}{3}}\,y^{-\frac{1}{4}} - 15\,x^2\,y^{-\frac{1}{2}} - 8\,x^{\frac{7}{3}}\,y^{-\frac{3}{4}} + 16\,x^{\frac{8}{3}}\,y^{-1}$.

7. $x^3\,y^{-\frac{2}{3}} + 6 - 4\,x^{-\frac{3}{2}}\,y^{\frac{1}{3}} + x^{-3}\,y^{\frac{2}{3}} - 4\,x^{\frac{3}{2}}\,y^{-\frac{1}{3}}$.

Extract the cube roots of the following:

8. $a\,b^2$. 9. $-8\,x^{-4}\,y^{\frac{2}{3}}$. 10. $\dfrac{3\,m^2\,n^{-\frac{2}{7}}}{a\,x^5}$.

11. $8\,y^2 - 12\,y^{\frac{11}{6}}\,x^{-1} + 6\,y^{\frac{5}{3}}\,x^{-2} - y^{\frac{3}{2}}\,x^{-3}$.

Reduce the following to their simplest forms:

12. $\dfrac{x^{m+n}\,x^{m+r}\,x^{r-m}}{x^{n+2m+r}}$.

15. $a^{x-y+2z}\,a^{2x+y-3z}\,a^{z}$.

13. $(x^{a})^{-b} \div (x^{-a})^{-b}$.

16. $\left(a^{1\frac{1}{3}} \times a^{-2} \times \sqrt{a^{-1}}\right)^{-\frac{3}{7}}$.

14. $\left(\dfrac{a^{x+y}}{a^{y}}\right)^{x} \div \left(\dfrac{a^{y}}{a^{y-x}}\right)^{x-y}$.

17. $\left[\left(x^{\frac{1}{a-b}}\right)^{a-\frac{b^2}{a}}\right]^{\frac{a}{a+b}}$.

Change the following to the form of entire quantities:

18. $\dfrac{15\,a\,b\,x^2}{5\,a\,b^2\,m^2}$.

19. $\dfrac{x^3\,y^2}{a^{-2}\,b^{-1}\,x^{-\frac{1}{2}}\,y^{\frac{2}{3}}}$.

20. $\dfrac{x^2}{(a-b)^{-2}(a+b)^{-1}}$.

Reduce the following to their simplest forms:

21. $\dfrac{a^{\frac{2}{3}} + a^{-1}}{a^{-\frac{5}{2}} - 3\,a^{\frac{1}{2}}}$.

22. $\dfrac{a^{-1} - b^{-2}}{c^{-3} + d^{-4}}$.

23. $\dfrac{5\,x\,(x^2-1)^{-1}}{3\,a x^{-2}\,(x^2-1)^{-2}}$.

Factor the following expressions:

24. $9\,x^{\frac{1}{2}} - 12\,x^{\frac{1}{4}} + 4$.

25. $a^{\frac{4}{3}} - 3\,a^{\frac{2}{3}} - 88$.

26. $a^{-2}\,b + 5\,a^{-1}\,b^{\frac{1}{2}} - 66$.

Factor by the method of Art. 117:

27. $a - b$.

28. $a^{\frac{4}{3}} - b^{-\frac{1}{5}}$.

29. $x^{-3}\,y - 4\,m^{\frac{5}{2}}$.

Factor by the method of Art. 119:

30. $a - b$.

31. $a + b$.

32. $x^{-3} + 8\,c\,m^{\frac{1}{4}}$.

XXIII. — RADICALS.

264. A **Radical** is a root of a quantity, indicated by a radical sign; as, $\sqrt{a}$, $\sqrt[3]{x+1}$, $\sqrt{m^2-2n+3}$.

When the root indicated can be exactly obtained, it is called a *rational* quantity; and when it cannot be exactly obtained, it is called an *irrational* or *surd* quantity.

265. The **Degree** of a radical is denoted by the index of the radical sign; thus, $\sqrt{a}$ is of the *second* degree; $\sqrt[3]{x+1}$ of the *third* degree.

Similar Radicals are those of the same degree, with the same quantity under the radical sign; as, $\sqrt[5]{ax}$ and $7\sqrt[5]{ax}$.

266. Most problems in radicals depend for their solution on the following important principle:

For any values of n, a, and b, by Art. 236,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}.$$

REDUCTION OF RADICALS.

TO REDUCE RADICALS OF DIFFERENT DEGREES TO EQUIVALENT RADICALS OF THE SAME DEGREE.

267. 1. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same degree.

By Art. 252, $\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64}$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4} = \sqrt[12]{81}$$

$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}$$

RULE.

Express the radicals with fractional exponents; reduce these fractions to a common denominator; express the resulting fractional exponents with radical signs; and, finally, reduce the quantities under the radical signs to their simplest forms.

Note. This method affords a means of comparison of the relative magnitudes of two or more radicals; thus, in Example 1, as $\sqrt[12]{125}$ is evidently greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$, hence $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXAMPLES.

Reduce the following to equivalent radicals of the same degree:

2. $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[3]{5}$.

3. $\sqrt[3]{5}$, $\sqrt[4]{6}$, and $\sqrt[6]{7}$.

4. $\sqrt{xy}$, $\sqrt[3]{xz}$, and $\sqrt[4]{yz}$.

5. $\sqrt[3]{2a}$, $\sqrt[5]{3b}$, and $\sqrt[5]{4c}$.

6. $\sqrt[6]{a+b}$ and $\sqrt[4]{a-b}$.

7. $\sqrt{a^2-x^2}$ and $\sqrt[3]{a^3-x^3}$.

8. Which is the greater, $\sqrt{3}$ or $\sqrt[4]{5}$?

9. Which is the greater, $\sqrt[3]{2}$ or $\sqrt[5]{3}$?

10. Which is the greater, $\sqrt[4]{4}$ or $\sqrt[5]{5}$?

TO REDUCE RADICALS TO THEIR SIMPLEST FORMS.

268. A radical is in its *simplest form* when the quantity under the radical sign is not a perfect power of the degree denoted by any factor of the index of the radical, and has no factor which is a perfect power of the same degree as the radical.

CASE I.

269. *When the quantity under the radical sign is a perfect power of the degree denoted by some factor of the index of the radical.*

1. Reduce $\sqrt[6]{8}$ to its simplest form.

$$\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2},\ \textit{Ans.}$$

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt[6]{9}$. **4.** $\sqrt[9]{27}$. **6.** $\sqrt[mn]{a^n b^{2n}}$.

3. $\sqrt[8]{25 x^2}$. **5.** $\sqrt[6]{125 a^3 b^9}$. **7.** $\sqrt[4]{\frac{25 a^2}{36 b^6}}$.

CASE II.

270. *When the quantity under the radical sign has a factor which is a perfect power of the same degree as the radical.*

1. Reduce $\sqrt{32}$ to its simplest form.

$$\sqrt{32} = \sqrt{16 \times 2} = \text{(Art. 266)}\ \sqrt{16} \times \sqrt{2} = 4\sqrt{2},\ \textit{Ans.}$$

2. Reduce $\sqrt[3]{54 a^4 x}$ to its simplest form.

$$\sqrt[3]{54 a^4 x} = \sqrt[3]{27 a^3 \times 2 a x} = \sqrt[3]{27 a^3} \times \sqrt[3]{2 a x} = 3 a \sqrt[3]{2 a x},\ \textit{Ans.}$$

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect power of the same degree as the radical. Extract the required root of this factor, and prefix the result to the indicated root of the other.

Note. In case the greatest perfect power in the numerical part of the quantity cannot be readily determined by inspection, it may always be obtained by resolving the numerical quantity into its prime factors. Let it be required, for example, to reduce $\sqrt{1944}$ to its simplest form. $1944 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 = 2^3 \times 3^5$. Hence,

$$\sqrt{1944} = \sqrt{2^3 \times 3^5} = \sqrt{2^2 \times 3^4} \times \sqrt{6} = 18\sqrt{6}.$$

EXAMPLES.

Reduce the following to their simplest forms:

3. $\sqrt{50}$. **6.** $\sqrt[3]{320}$. **9.** $7\sqrt{63a^4b^5c^6}$.

4. $3\sqrt{24}$. **7.** $2\sqrt[4]{80}$. **10.** $\sqrt[3]{250x^3y^6z^8}$.

5. $\sqrt{72}$. **8.** $\sqrt{98a^3b^2}$. **11.** $\sqrt{18x^3y^4-27x^4y^3}$.

12. $\sqrt{ax^2-6ax+9a}$. **14.** $\sqrt{20ax^2+60a^2x+45a^3}$.

13. $\sqrt{(x^2-y^2)(x+y)}$. **15.** $\sqrt[3]{192a^4b^5+320a^3b^4}$.

When the quantity under the radical sign is a fraction, *multiply both terms by such a quantity as will make the denominator a perfect power of the same degree as the radical.* Then proceed as before.

16. Reduce $\sqrt{\frac{2}{3}}$ to its simplest form.

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\left(\frac{1}{9}\times 6\right)}=\sqrt{\frac{1}{9}}\times\sqrt{6}=\frac{1}{3}\sqrt{6},\ \textit{Ans.}$$

17. Reduce $\sqrt{\frac{9}{8}}$ to its simplest form.

$$\sqrt{\frac{9}{8}}=\sqrt{\frac{18}{16}}=\sqrt{\left(\frac{9}{16}\times 2\right)}=\sqrt{\frac{9}{16}}\times\sqrt{2}=\frac{3}{4}\sqrt{2},\ \textit{Ans.}$$

Reduce the following to their simplest forms:

18. $\sqrt{\frac{3}{2}}$. **21.** $\sqrt{\frac{4a^2}{27}}$. **24.** $\sqrt{\left(\frac{9a^2b^3}{10cd}\right)}$.

19. $\sqrt{\frac{5}{6}}$. **22.** $\sqrt[3]{\frac{3x}{4}}$. **25.** $\sqrt{\left(\frac{7xy^2}{8a^3}\right)}$.

20. $\sqrt{\frac{7}{12}}$. **23.** $\sqrt[3]{\frac{5}{9}}$. **26.** $\frac{3}{11}\sqrt{\frac{4}{7}}$.

27. $\sqrt{\left(\frac{ab^2}{4(a+x)}\right)}$. **28.** $\frac{a}{a^2-b^2}\sqrt{\left(\frac{a^3c-2a^2bc+ab^2c}{b^3}\right)}$.

TO REDUCE A RATIONAL QUANTITY TO A RADICAL FORM.

271. 1. Reduce $3x^2$ to a radical of the third degree.

$$3x^2 = \sqrt[3]{(3x^2)^3} = \sqrt[3]{27x^6}, \text{ } Ans.$$

RULE.

Raise the quantity to the power indicated by the given root, and write it under the corresponding radical sign.

EXAMPLES.

Reduce the following to radicals of the second degree:

2. $7a$. **3.** $\frac{3x}{5}$. **4.** $a + 2x$. **5.** $\frac{x-3}{x-2}$.

6. Reduce $\frac{2a}{3}$ to a radical of the fourth degree.

TO INTRODUCE THE COEFFICIENT OF A RADICAL UNDER THE RADICAL SIGN.

272. 1. Introduce the coefficient of $2a\sqrt[3]{3x^2}$ under the radical sign.

$$2a\sqrt[3]{3x^2} = \sqrt[3]{8a^3} \times \sqrt[3]{3x^2} = \text{(Art. 266) } \sqrt[3]{8a^3 \times 3x^2} = \sqrt[3]{24a^3x^2},$$

Ans.

RULE.

Reduce the coefficient to the form of a radical of the given degree; multiply together the quantities under the radical signs, and write the product under the given radical sign.

EXAMPLES.

Introduce the coefficients of the following under the radical signs:

2. $3\sqrt{5}$. **4.** $4a^2\sqrt{4a}$. **6.** $5c\sqrt[3]{2a}$.

3. $2\sqrt[3]{7}$. **5.** $3\sqrt[3]{1+x}$. **7.** $(x-1)\sqrt{\left(\frac{x+1}{x-1}\right)}$.

8. $(a-b)\sqrt[3]{a-b}$. **9.** $\frac{1+a}{1-a}\sqrt{\left(\frac{1-a}{1+a}\right)}$.

ADDITION AND SUBTRACTION OF RADICALS.

273. 1. Find the sum of $\sqrt{18}$, $\sqrt{27}$, $\sqrt{\frac{1}{2}}$, and $12\sqrt{\frac{1}{18}}$.

By Art. 270,

$$\begin{aligned}\sqrt{18} &= 3\sqrt{2}\\ \sqrt{27} &= \qquad\quad 3\sqrt{3}\\ \sqrt{\tfrac{1}{2}} &= \tfrac{1}{2}\sqrt{2}\\ 12\sqrt{\tfrac{1}{18}} &= 2\sqrt{2}\\ \hline &\tfrac{11}{2}\sqrt{2} + 3\sqrt{3},\ \textit{Ans.}\end{aligned}$$

2. Subtract $\sqrt[3]{48}$ from $\sqrt[3]{162}$.

By Art. 270,

$$\begin{aligned}\sqrt[3]{162} &= 3\sqrt[3]{6}\\ \sqrt[3]{48} &= 2\sqrt[3]{6}\\ \hline &\sqrt[3]{6},\ \textit{Ans.}\end{aligned}$$

RULE.

Reduce each radical to its simplest form. Combine the similar radicals, and indicate the addition or subtraction of the dissimilar.

EXAMPLES.

Add together the following radicals:

3. $\sqrt{8}$, $\sqrt{18}$, and $\sqrt{50}$.

4. $\sqrt{12}$, $\sqrt{48}$, and $\sqrt{108}$.

5. $\sqrt[3]{16}$, $\sqrt[3]{54}$, and $\sqrt[3]{128}$.

6. $\sqrt{20}$, $\sqrt{\frac{1}{5}}$, and $\sqrt{\frac{5}{9}}$.

7. $\sqrt{\frac{3}{8}}$, $\sqrt{\frac{1}{6}}$, and $\sqrt{\frac{2}{27}}$.

8. $\sqrt[3]{\frac{1}{4}}$, $\sqrt[3]{\frac{1}{32}}$, and $\sqrt[3]{\frac{2}{3}}$.

Subtract the following:

9. $\sqrt{45}$ from $\sqrt{245}$. **10.** $\sqrt{\frac{3}{5}}$ from $\sqrt{\frac{16}{15}}$.

Simplify the following:

11. $\sqrt{243\,a\,b^2}+\sqrt{75\,a^3}+\sqrt{3\,a^3-54\,a^2\,b+243\,a\,b^2}$.

12. $7\sqrt{27}-\sqrt{75}-\sqrt{\frac{1}{3}}+\sqrt{12}-\sqrt{\frac{1}{12}}-\sqrt{\frac{1}{27}}$.

13. $\sqrt[3]{16}+5\sqrt[3]{54}-\sqrt[3]{250}-\sqrt[3]{\frac{2}{27}}+\sqrt[3]{\frac{1}{9}}+\sqrt[3]{\frac{1}{72}}$.

14. $\sqrt{\left(\frac{x^2\,(x-y)}{x+y}\right)}+\sqrt{\left(\frac{y^2\,(x+y)}{x-y}\right)}-(3\,y^2-x^2)\sqrt{\left(\frac{1}{x^2-y^2}\right)}$.

15. $\sqrt{63\,a^2x-84\,a\,b\,x+28\,b^2x}-\sqrt{7\,a^2x+42\,a\,b\,x+63\,b^2x}$.

MULTIPLICATION OF RADICALS.

274. 1. Multiply $\sqrt{2}$ by $\sqrt{5}$.

$$\sqrt{2}\times\sqrt{5}=\text{(Art. 266)}\ \sqrt{2\times 5}=\sqrt{10},\ \textit{Ans.}$$

2. Multiply $\sqrt{2}$ by $\sqrt[3]{3}$.

Reducing to equivalent radicals of the same degree (Art. 267), we have

$$\sqrt{2}\times\sqrt[3]{3}=\sqrt[6]{8}\times\sqrt[6]{9}=\sqrt[6]{72},\ \textit{Ans.}$$

RULE.

Reduce the radicals, if necessary, to equivalent ones of the same degree. Multiply together the quantities under the radical signs, and write the product under the common radical sign.

EXAMPLES.

Multiply together the following:

3. $\sqrt{12}$ and $\sqrt{3}$.

4. $\sqrt[3]{2}$ and $\sqrt[3]{4a}$.

5. $\sqrt{ax}$ and $\sqrt[3]{bx}$.

6. $\sqrt[3]{6a^2}$ and $\sqrt{5a^3}$.

7. $\sqrt[4]{3x^3}$, $\sqrt[3]{2x^4}$, and $\sqrt{\left(\frac{1}{4x^6}\right)}$.

8. $\sqrt[5]{2}$, $\sqrt{5}$, and $\sqrt{\frac{1}{2}}$.

9. Multiply $2\sqrt{x}-3\sqrt{y}$ by $4\sqrt{x}+\sqrt{y}$.

$$\begin{array}{l} 2\sqrt{x}-3\sqrt{y} \\ 4\sqrt{x}+\ \ \sqrt{y} \\ \hline 8x-12\sqrt{xy} \\ \qquad +\ 2\sqrt{xy}-3y \\ \hline 8x-10\sqrt{xy}-3y,\ \textit{Ans.} \end{array}$$

Note. It should be remembered that to multiply a radical of the second degree by itself is simply to remove the radical sign; thus,

$$\sqrt{x}\times\sqrt{x}=x.$$

Multiply together the following:

10. $\sqrt{x}-2$ and $\sqrt{x}+3$.

11. $3\sqrt{x}-5$ and $7\sqrt{x}-1$.

12. $\sqrt{x+1}-\sqrt{x-1}$ and $\sqrt{x+1}+\sqrt{x-1}$ (Art. 106).

13. $\sqrt{a^2-1}-a$ and $\sqrt{a^2-1}+a$.

14. $\sqrt{x}-\sqrt{y}+\sqrt{z}$ and $\sqrt{x}+\sqrt{y}-\sqrt{z}$.

15. $\sqrt{2}-\sqrt{3}+\sqrt{5}$ and $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

16. $3\sqrt{5}-2\sqrt{6}+\sqrt{7}$ and $6\sqrt{5}+4\sqrt{6}+2\sqrt{7}$.

17. $4\sqrt{3}-5\sqrt{2}-2\sqrt{5}$ and $8\sqrt{3}+10\sqrt{2}-4\sqrt{5}$.

Simplify the following:

18. $\sqrt{\left(\frac{a x}{a+x}\right)} \times \sqrt{\left(\frac{a^2-x^2}{a}\right)} \times \sqrt{a x - x^2}.$

19. $\sqrt{\left(\frac{(m+n)^2}{m-n}\right)} \times \sqrt{\left(\frac{m^2+n^2}{m+n}\right)} \times \sqrt{\left(\frac{m^2-n^2}{m^2+n^2}\right)}.$

Square the following (Arts. 104 and 105):

20. $2\sqrt{3}-\sqrt{2}.$

21. $3\sqrt{8}+5\sqrt{3}.$

22. $\sqrt{1-a^2}+a.$

23. $\sqrt{a-b}-\sqrt{a+b}.$

DIVISION OF RADICALS.

275. Since (Art. 266), $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a b}$, it follows that

$$\sqrt[n]{a b} \div \sqrt[n]{a} = \sqrt[n]{b}.$$

RULE.

Reduce the radicals, if necessary, to equivalent ones of the same degree. Divide the quantities under the radical signs, and write the quotient under the common radical sign.

EXAMPLES.

1. Divide $\sqrt[3]{15}$ by $\sqrt{5}$.

Reducing to equivalent radicals of the same degree, we have

$$\sqrt[3]{15} \div \sqrt{5} = \sqrt[6]{225} \div \sqrt[6]{125} = \sqrt[6]{\frac{225}{125}} = \sqrt[6]{\frac{9}{5}}, \textit{ Ans.}$$

Divide the following:

2. $\sqrt{108}$ by $\sqrt{18}$.

3. $\sqrt{50 c^3}$ by $\sqrt{2 c}$.

4. $\sqrt{54}$ by $\sqrt{6}$.

5. $\sqrt[3]{9 a^4}$ by $\sqrt[3]{3 a}$.

6. $\sqrt{2}$ by $\sqrt[3]{3}$.

7. $\sqrt[5]{2}$ by $\sqrt[4]{3}$.

8. $\sqrt[3]{12}$ by $\sqrt{2}$.

9. $\sqrt[3]{4 a}$ by $\sqrt[4]{2 a}$.

INVOLUTION OF RADICALS.

276. **1.** Raise $\sqrt[3]{2}$ to the fourth power.

$$(\sqrt[3]{2})^4 = (2^{\frac{1}{3}})^4 = 2^{\frac{4}{3}} = \sqrt[3]{2^4} = \sqrt[3]{16},\ \textit{Ans.}$$

2. Raise $\sqrt[6]{3}$ to the third power.

$$(\sqrt[6]{3})^3 = (3^{\frac{1}{6}})^3 = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3},\ \textit{Ans.}$$

We observe that in the first example the quantity under the radical sign is raised to the required power; while in the second, the index of the radical is divided by the exponent of the required power. Hence the following

RULE.

If possible, divide the index of the radical by the exponent of the required power. Otherwise, raise the quantity under the radical sign to the required power.

Note. If the radical has a coefficient, it may be involved separately. The final result should be reduced to its simplest form.

EXAMPLES.

3. Raise $\sqrt[5]{5}$ to the third power.

4. Square $\sqrt[4]{7}$.

5. Find the fourth power of $4\sqrt{3x}$.

6. Find the sixth power of $\sqrt[3]{a^2 x}$.

7. Raise $\sqrt[8]{a-b}$ to the fourth power.

8. Raise $3a\sqrt[3]{bx}$ to the fourth power.

9. Find the value of $(\sqrt{x+1})^4$.

10. Find the square of $4\sqrt{x^2-3}$.

EVOLUTION OF RADICALS.

277. **1.** Extract the square root of $\sqrt[8]{6x^2}$.

$$\sqrt{(\sqrt[3]{6x^2})} = (\sqrt[3]{6x^2})^{\frac{1}{2}} = \{(6x^2)^{\frac{1}{3}}\}^{\frac{1}{2}} = (6x^2)^{\frac{1}{6}} = \sqrt[6]{6x^2}, \textit{ Ans.}$$

2. Extract the cube root of $\sqrt{27x^3}$.

$$\sqrt[3]{(\sqrt{27x^3})} = (\sqrt{27x^3})^{\frac{1}{3}} = \{\sqrt{(3x)^3}\}^{\frac{1}{3}} = \{(3x)^{\frac{3}{2}}\}^{\frac{1}{3}} = (3x)^{\frac{1}{2}}$$
$$= \sqrt{3x}, \textit{ Ans.}$$

We observe that in the first example the index of the radical is multiplied by the index of the required root; while in the second, the required root is taken of the quantity under the radical sign. Hence the following

RULE.

If possible, extract the required root of the quantity under the radical sign. Otherwise, multiply the index of the radical by the index of the required root.

Note. If the radical has a coefficient, which is not a perfect power of the same degree as the required root, it should be introduced under the radical sign before applying the rule. Thus,

$$\sqrt[3]{(4\sqrt{ax})} = \sqrt[3]{(\sqrt{16ax})} = \sqrt[6]{16ax}.$$

The final result should be reduced to its simplest form.

EXAMPLES.

3. Extract the square root of $\sqrt{2}$.

4. Find the cube root of $\sqrt{8}$.

5. Find the cube root of $\sqrt[4]{a+b}$.

6. Find the square root of $\sqrt[3]{x^2-2x+1}$.

7. Extract the fifth root of $\sqrt{32}$.

8. Extract the cube root of $\sqrt[5]{27}$.

9. Find the value of $\sqrt[3]{(3\sqrt{3})}$.

10. Find the fourth root of $\sqrt[5]{x^8 y^{12}}$.

11. Find the value of $\sqrt[5]{(4\sqrt{2})}$.

TO REDUCE A FRACTION HAVING AN IRRATIONAL DENOMINATOR TO AN EQUIVALENT ONE WHOSE DENOMINATOR IS RATIONAL.

CASE I.

278. *When the denominator is a monomial.*

1. Reduce $\frac{2b}{\sqrt{a}}$ to an equivalent fraction whose denominator is rational.

Multiplying both terms by $\sqrt{a}$,

$$\frac{2b}{\sqrt{a}} = \frac{2b\sqrt{a}}{\sqrt{a}\sqrt{a}} = \frac{2b\sqrt{a}}{a}, \text{ Ans.}$$

2. Reduce $\frac{5}{\sqrt[3]{3}}$ to an equivalent fraction whose denominator is rational.

Multiplying both terms by $\sqrt[3]{9}$,

$$\frac{5}{\sqrt[3]{3}} = \frac{5\sqrt[3]{9}}{\sqrt[3]{3}\sqrt[3]{9}} = \frac{5\sqrt[3]{9}}{\sqrt[3]{27}} = \frac{5\sqrt[3]{9}}{3}, \text{ Ans.}$$

RULE.

Multiply both terms of the fraction by a radical of the same degree as the denominator, with such a quantity under the radical sign as will make the denominator of the resulting fraction rational.

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

3. $\frac{3}{\sqrt{2}}$. **4.** $\frac{1}{\sqrt[3]{2a}}$. **5.** $\frac{5}{\sqrt[3]{4}}$. **6.** $\frac{2c}{\sqrt[4]{27a^2}}$.

CASE II.

279. *When the denominator is a binomial, containing only radicals of the second degree.*

1. Reduce $\frac{10}{3+\sqrt{2}}$ to an equivalent fraction whose denominator is rational.

Multiplying both terms by $3-\sqrt{2}$,

$$\frac{10}{3+\sqrt{2}}=\frac{10(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}=\text{(Art. 106)}\ \frac{30-10\sqrt{2}}{7},\ \textit{Ans.}$$

2. Reduce $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction whose denominator is rational.

Multiplying both terms by $\sqrt{5}+\sqrt{2}$,

$$\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}=\frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}=\frac{7+2\sqrt{10}}{3},\ \textit{Ans.}$$

RULE.

Multiply both terms of the fraction by the denominator with the sign between its terms changed.

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

3. $\frac{4}{3+\sqrt{2}}$.

4. $\frac{4-\sqrt{3}}{2-\sqrt{3}}$.

5. $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$.

6. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

7. $\frac{2\sqrt{5}+\sqrt{2}}{\sqrt{5}-3\sqrt{2}}$.

8. $\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}$.

9. $\frac{2+\sqrt{a+1}}{1-\sqrt{a+1}}$.

10. $\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

11. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

12. $\frac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}$.

13. $\frac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}$.

14. $\frac{\sqrt{x}-4\sqrt{x-2}}{2\sqrt{x}+3\sqrt{x-2}}$.

280. If the denominator is a trinomial, containing only radicals of the second degree, by multiplying both terms of the fraction by the denominator with *one* of its signs changed, we shall obtain a fraction which can be reduced to an equivalent fraction with a rational denominator by the method of Case II. Thus, to reduce the fraction

$$\frac{\sqrt{2}-\sqrt{3}-\sqrt{7}}{\sqrt{2}+\sqrt{3}+\sqrt{7}}.$$

Multiplying both terms by $\sqrt{2}+\sqrt{3}-\sqrt{7}$,

$$\frac{\sqrt{2}-\sqrt{3}-\sqrt{7}}{\sqrt{2}+\sqrt{3}+\sqrt{7}}=\frac{(\sqrt{2}-\sqrt{3}-\sqrt{7})\,(\sqrt{2}+\sqrt{3}-\sqrt{7})}{(\sqrt{2}+\sqrt{3}+\sqrt{7})\,(\sqrt{2}+\sqrt{3}-\sqrt{7})}=\frac{6-2\sqrt{14}}{2\sqrt{6}-2}$$
$$=\frac{3-\sqrt{14}}{\sqrt{6}-1}.$$

Multiplying both terms by $\sqrt{6}+1$, we have

$$\frac{(3-\sqrt{14})\,(\sqrt{6}+1)}{(\sqrt{6}-1)\,(\sqrt{6}+1)}=\frac{3-\sqrt{14}+3\sqrt{6}-\sqrt{84}}{5}, \textit{ Ans.}$$

If the denominator is a binomial, containing radicals of any degrees whatever, it is possible to reduce the fraction to an equivalent form with a rational denominator; but the process is more complicated than the preceding and rarely necessary.

281. To find the approximate value of a fraction whose denominator is irrational, reduce it to an equivalent fraction whose denominator is rational.

1. Find the value of $\frac{1}{2-\sqrt{2}}$ to three decimal places.

$$\frac{1}{2-\sqrt{2}} = (\text{Art. } 279)\ \frac{2+\sqrt{2}}{2} = \frac{2+1.414}{2} = 1.707, \textit{ Ans.}$$

It will be seen that the value of the fraction is obtained in this way more easily than by dividing 1 by $2-\sqrt{2}$, or its value .586.

EXAMPLES.

Find the values to three decimal places of the following:

2. $\frac{2}{\sqrt{5}}$. **3.** $\frac{3}{\sqrt{2}-1}$. **4.** $\frac{7}{\sqrt[3]{9}}$. **5.** $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

IMAGINARY QUANTITIES.

282. An **Imaginary Quantity** is an indicated even root of a negative quantity; as, $\sqrt{-4}$, $\sqrt[4]{-a^2}$.

In contradistinction, all other quantities, rational or irrational, are called *real* quantities.

283. All imaginary quantities may be expressed in one common form, which is, a real quantity multiplied by $\sqrt{-1}$.

For example,

$$\sqrt{-a^2} = \sqrt{a^2 \times (-1)} = (\text{Art. } 266)\ \sqrt{a^2} \times \sqrt{-1} = a\sqrt{-1};$$

also, $\sqrt{-2} = \sqrt{2 \times (-1)} = \sqrt{2}\sqrt{-1}$.

Hence, we may regard $\sqrt{-1}$ as a universal factor of every imaginary quantity, and use it in our investigations as the only symbol of such a quantity.

284. Imaginary quantities may be added, subtracted, and divided the same as other radicals; but with regard to multiplication, the usual rule requires some modification.

285. By Art. 17, $\sqrt{-1}$ means such an expression as when multiplied by itself produces -1;

or, $(\sqrt{-1})^2 = -1$;

also, $(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -1\sqrt{-1}$;

and, $(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1.$

By continuing the multiplication, we should find

$$(\sqrt{-1})^5 = \sqrt{-1};\ (\sqrt{-1})^6 = -1;\ (\sqrt{-1})^7 = -\sqrt{-1};\ (\sqrt{-1})^8 = 1;\ \text{etc.}$$

Or, in general, where n is any positive integer,

$$(\sqrt{-1})^{4n+1} = \sqrt{-1};\ (\sqrt{-1})^{4n+2} = -1;\ (\sqrt{-1})^{4n+3} = -\sqrt{-1};\ (\sqrt{-1})^{4n+4} = 1.$$

MULTIPLICATION OF IMAGINARY QUANTITIES.

286. 1. Multiply $\sqrt{-a^2}$ by $\sqrt{-b^2}$.

$$\sqrt{-a^2} \times \sqrt{-b^2} = (\text{Art. } 283)\ a\sqrt{-1} \times b\sqrt{-1} = ab(\sqrt{-1})^2 = -ab, \textit{ Ans.}$$

2. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

$$\sqrt{-2} \times \sqrt{-3} = \sqrt{2} \times \sqrt{3} \times (\sqrt{-1})^2 = -\sqrt{6}, \textit{ Ans.}$$

3. Multiply together $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

$$\sqrt{-4} \times \sqrt{-9} \times \sqrt{-16} \times \sqrt{-25} = 2 \times 3 \times 4 \times 5 \times (\sqrt{-1})^4 = 120(\sqrt{-1})^4 = 120, \textit{ Ans.}$$

RULE.

Reduce all the imaginary quantities to the form of a real quantity multiplied by $\sqrt{-1}$. *Multiply together the real quantities, and multiply the result by the required power of* $\sqrt{-1}$.

EXAMPLES.

Multiply the following:

4. $4\sqrt{-3}$ and $2\sqrt{-2}$.

5. $-3\sqrt{-a}$ and $4\sqrt{-b}$.

6. $4+\sqrt{-7}$ and $8-2\sqrt{-7}$.

7. $1+\sqrt{-1}$ and $1-\sqrt{-1}$.

8. $\sqrt{-a^2}$, $\sqrt{-b^2}$, and $\sqrt{-c^2}$.

9. $a+\sqrt{-b}$ and $a-\sqrt{-b}$.

10. $2\sqrt{-3}-3\sqrt{-2}$ and $4\sqrt{-3}+6\sqrt{-2}$.

11. Divide $\sqrt{-a}$ by $\sqrt{-b}$.

$$\frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}\sqrt{-1}}=\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}},\ \textit{Ans.}$$

We should obtain the same result by using the rule of Art. 275; hence, that rule applies to the division of all radicals, whether real or imaginary.

Divide the following:

12. $\sqrt{-6}$ by $\sqrt{-2}$.

13. $\sqrt[4]{-12}$ by $\sqrt[4]{-3}$.

14. $\sqrt{-5}$ by $\sqrt{-1}$.

15. $\sqrt[6]{-54}$ by $\sqrt[6]{-2}$.

Simplify the following:

16. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. (Art. 279).

17. $\dfrac{4+\sqrt{-2}}{2-\sqrt{-2}}$.

18. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}+\dfrac{a-\sqrt{-b}}{a+\sqrt{-b}}$. (Art. 154).

19. Expand $(2-\sqrt{-3})^2$.

20. Expand $(2+3\sqrt{-2})^3$.

QUADRATIC SURDS.

287. A **Quadratic Surd** is the indicated square root of an imperfect square; as, $\sqrt{3}$, $\sqrt{x+1}$.

288. A **Binomial Surd** is a binomial in which one or both of the terms are irrational.

289. *The square root of a rational quantity cannot be equal to a rational quantity plus a quadratic surd.*

If possible, let $\sqrt{a} = b + \sqrt{c}$

Squaring the equation, $a = b^2 + 2b\sqrt{c} + c$

or, $2b\sqrt{c} = a - b^2 - c$

or, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$

that is, a surd equal to a rational quantity, which is impossible.

290. *If the sum of a rational quantity and a quadratic surd be equal to the sum of another rational quantity and another quadratic surd, the two rational quantities will be equal, also the two quadratic surds.*

That is, if $a + \sqrt{b} = c + \sqrt{d}$

then $a = c$ and $\sqrt{b} = \sqrt{d}$

For, if a is not equal to c, suppose $a = c + x$

then $c + x + \sqrt{b} = c + \sqrt{d}$

or, $x + \sqrt{b} = \sqrt{d}$

which is impossible by Art. 289. Hence, a must equal c, and consequently $\sqrt{b}$ must equal $\sqrt{d}$.

291. *To prove that if* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Squaring the equation $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

we have $a + \sqrt{b} = x + 2\sqrt{xy} + y$

Whence, by Art. 290, $a = x + y$ and $\sqrt{b} = 2\sqrt{xy}$.

Subtracting these two equations, we have

$$a-\sqrt{b}=x-2\sqrt{xy}+y$$

Extracting the square root, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

292. *To extract the square root of a binomial surd whose first term is rational.*

For example, to extract the square root of $a+\sqrt{b}$.

Assume $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ (1)

then (Art. 291), $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$ (2)

Multiplying (1) by (2), $\sqrt{a^2-b}=x-y$ (3)

Squaring (1), $a+\sqrt{b}=x+2\sqrt{xy}+y$

Whence (Art. 290), $a=x+y$. (4)

Adding (3) and (4), $a+\sqrt{a^2-b}=2x$, or $x=\dfrac{a+\sqrt{a^2-b}}{2}$.

Subtracting (3) from (4), $a-\sqrt{a^2-b}=2y$, or $y=\dfrac{a-\sqrt{a^2-b}}{2}$.

Substituting these values of x and y in (1) and (2),

$$\sqrt{a+\sqrt{b}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}. \quad (5)$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}. \quad (6)$$

EXAMPLES.

1. Find the square root of $3+2\sqrt{2}$ or $3+\sqrt{8}$.

Here $a=3$ and $b=8$. Substituting in (5), we have

$$\sqrt{3+\sqrt{8}}=\sqrt{\left(\frac{3+\sqrt{9-8}}{2}\right)}+\sqrt{\left(\frac{3-\sqrt{9-8}}{2}\right)}$$

$$=\sqrt{\left(\frac{3+1}{2}\right)}+\sqrt{\left(\frac{3-1}{2}\right)}=\sqrt{2}+1, \text{ Ans.}$$

2. Find the square root of $6-\sqrt{20}$.

Here $a=6$ and $b=20$. Substituting in (6), we have

$$\sqrt{6-\sqrt{20}}=\sqrt{\left(\frac{6+\sqrt{36-20}}{2}\right)}-\sqrt{\left(\frac{6-\sqrt{36-20}}{2}\right)}$$
$$=\sqrt{\left(\frac{6+4}{2}\right)}-\sqrt{\left(\frac{6-4}{2}\right)}=\sqrt{5}-1, \textit{ Ans.}$$

293. Examples of this kind may always be solved by the following method:

3. Extract the square root of $14-4\sqrt{6}$.

$$\sqrt{14-4\sqrt{6}}=\sqrt{14-2\sqrt{24}}=\sqrt{12-2\sqrt{24}+2}$$
$$=(\text{Art. } 116)\ \sqrt{12}-\sqrt{2}=2\sqrt{3}-\sqrt{2}, \textit{ Ans.}$$

4. Extract the square root of $43+15\sqrt{8}$.

$$\sqrt{43+15\sqrt{8}}=\sqrt{43+\sqrt{1800}}=\sqrt{43+2\sqrt{450}}$$
$$=\sqrt{25+2\sqrt{450}+18}=\sqrt{25}+\sqrt{18}=5+3\sqrt{2}, \textit{ Ans.}$$

RULE.

Reduce the surd term so that its coefficient may be 2. Separate the rational term into two parts whose product shall be the quantity under the radical sign (see first note on page 48), *writing one part before the surd term and the other part after it. Extract the square roots of these parts, and connect them by the sign of the surd term.*

The advantage of this method is that it does not require the memorizing of formulæ (5) and (6).

EXAMPLES.

Extract the square roots of the following:

5. $12+2\sqrt{35}$. **8.** $35+10\sqrt{10}$. **11.** $20-5\sqrt{12}$.

6. $24-2\sqrt{63}$. **9.** $12-\sqrt{108}$. **12.** $14+3\sqrt{20}$.

7. $16+6\sqrt{7}$. **10.** $8-\sqrt{60}$. **13.** $67-7\sqrt{72}$.

Extract the square roots of the following, using formulæ (5) and (6), Art. 292:

14. $1 - 12\sqrt{-2}$. **15.** $7 + 30\sqrt{-2}$. **16.** $35 - 3\sqrt{-16}$.

17. $2m - 2\sqrt{m^2 - n^2}$. **18.** $x^2 + ax - 2\sqrt{ax^3}$.

Extract the fourth roots of the following:

19. $193 + 22\sqrt{72}$. **20.** $17 - 12\sqrt{2}$. **21.** $97 - 56\sqrt{3}$.

SOLUTION OF EQUATIONS CONTAINING RADICALS.

CASE I.

294. *When there is only one radical term in the equation.*

1. Solve the equation $\sqrt{x^2 - 5} - x = -1$.

Transposing, $\sqrt{x^2 - 5} = x - 1$

Squaring, $x^2 - 5 = x^2 - 2x + 1$

Whence, $x = 3$, *Ans.*

CASE II.

295. *When there are two radical terms in the equation.*

2. Solve the equation $\sqrt{x} - \sqrt{x - 3} = 1$.

Transposing, $\sqrt{x} - 1 = \sqrt{x - 3}$

Squaring, $x - 2\sqrt{x} + 1 = x - 3$

Transposing and uniting, $-2\sqrt{x} = -4$

or, $\sqrt{x} = 2$

Whence, $x = 4$, *Ans.*

CASE III.

296. *When there are three radical terms in the equation.*

3. Solve the equation $\sqrt{x + 6} + \sqrt{x + 13} - \sqrt{4x + 37} = 0$.

Transposing, $\sqrt{x+6}+\sqrt{x+13}=\sqrt{4x+37}$

Squaring, $x+6+2\sqrt{x^2+19x+78}+x+13=4x+37$

Transposing and uniting, $2\sqrt{x^2+19x+78}=2x+18$

or, $\sqrt{x^2+19x+78}=x+9$

Squaring, $x^2+19x+78=x^2+18x+81$

Whence, $x=3$, *Ans.*

RULE.

297. *Transpose the terms of the given equation so that a radical term may stand alone in one member; then raise each member to a power of the same degree as the radical.*

If there is still a radical term remaining, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

Note. All the examples in this chapter reduce to simple equations; radical equations, however, may reduce to equations of the second degree, for the solution of which see Chapter XXIV.

EXAMPLES.

Solve the following equations:

4. $\sqrt{x-8}=3.$ **6.** $\sqrt[3]{3x+4}+3=6.$ **8.** $8-2\sqrt{x}=4.$

5. $\sqrt[4]{x-3}=2.$ **7.** $\sqrt{5x-1}-2=1.$ **9.** $5-\sqrt[3]{2x}=3.$

10. $\sqrt{4x^2-19}-2x=-1.$ **14.** $6+\sqrt{x}=\sqrt{12+x}.$

11. $\sqrt{x^2-3x+6}-1=1-x.$ **15.** $\sqrt{x-32}+\sqrt{x}=16.$

12. $\sqrt[3]{x^3-6x^2}+2=x.$ **16.** $\sqrt{x-3}-\sqrt{x+12}=-3.$

13. $\sqrt{x}+\sqrt{x+5}=5.$ **17.** $\sqrt{2x-7}+\sqrt{2x+9}=8.$

18. $\sqrt{3x+10}-\sqrt{3x+25}=-3.$

19. $\sqrt{x^2-3x+5}-\sqrt{x^2-5x-2}=1.$

20. $\sqrt{x^2 + 4x + 12} + \sqrt{x^2 - 12x - 20} = 8.$

21. $\sqrt{x} - \sqrt{x - 3} = \frac{2}{\sqrt{x}}.$

22. $\sqrt{3x} + \sqrt{3x + 13} = \frac{91}{\sqrt{3x + 13}}.$

23. $\frac{\sqrt{x} - 3}{\sqrt{x} + 7} = \frac{\sqrt{x} - 4}{\sqrt{x} + 1}.$

24. $\frac{\sqrt{x} + 38}{\sqrt{x} + 6} = \frac{\sqrt{x} + 28}{\sqrt{x} + 4}.$

25. $\sqrt{x - 1} + \sqrt{x + 4} = \sqrt{4x + 5}.$

26. $\sqrt{x + 1} + \sqrt{x - 2} - \sqrt{4x - 3} = 0.$

27. $\sqrt{2x - 3} - \sqrt{8x + 1} + \sqrt{18x - 92} = 0.$

28. $\sqrt{x - 3} - \sqrt{x - 14} - \sqrt{4x - 155} = 0.$

29. $x - \sqrt{(9 + x\sqrt{x^2 - 3})} = 3.$

30. $x + 1 = \sqrt{(1 + x\sqrt{x^2 + 16})}.$

31. $\frac{\sqrt{3x} - \sqrt{3}}{\sqrt{2x} - \sqrt{2}} = \frac{\sqrt{x} + 3}{\sqrt{x} + 2}.$

32. $\sqrt[3]{(a^3 - 3a^2x + x^2\sqrt{3a - x})} = a - x.$

XXIV. — QUADRATIC EQUATIONS.

298. A **Quadratic Equation**, or an equation of the *second degree* (Art. 164), is one in which the *square* is the highest power of the unknown quantity; as,

$$a x^2 = b, \text{ and } x^2 + 8x = 20.$$

299. A **Pure Quadratic Equation** is one which contains only the square of the unknown quantity; as,

$$a x^2 = b; \text{ and } x^2 = 400.$$

Equations of this kind are sometimes called *incomplete equations* of the second degree.

300. An **Affected Quadratic Equation** is one which contains both the square and first power of the unknown quantity; as,

$$x^2 + 8x = 20; \text{ and } ax^2 + bx - c = bx^2 - ax + d.$$

Equations of this kind, containing every power of the unknown quantity from the first to the highest given, are sometimes called *complete equations*.

PURE QUADRATIC EQUATIONS.

301. A pure quadratic equation can always be reduced to the form

$$x^2 = a,$$

in which a may represent any quantity, positive or negative, integral or fractional. Thus, in the equation

$$\frac{20x^2}{3} - (5x^2 + 4) = \frac{41}{12} - \frac{3 - 5x^2}{4}$$

Clearing of fractions, $80x^2 - 12(5x^2 + 4) = 41 - (9 - 15x^2)$

or, $80x^2 - 60x^2 - 48 = 41 - 9 + 15x^2$

Transposing and uniting terms, $5x^2 = 80$

$$x^2 = 16$$

which is in the form $x^2 = a$.

Equations of this kind have, therefore, sometimes been denominated *binomial*, or those of *two terms*.

302. An equation of the form

$$x^2 = a$$

may be readily solved by taking the square root of each member. Thus,

$$x = \pm\sqrt{a},$$

where the double sign is used, because the square root of a quantity may be either positive or negative (Art. 237).

Note. It may seem at first as though we ought to write the double sign before the square root of each member, as follows:

$$\pm x = \pm\sqrt{a}.$$

We do not omit the double sign before the square root of the first member because it is incorrect, but because we obtain no new results by considering it. The equation $\pm x = \pm\sqrt{a}$ can be written in four different ways, thus,

$$x = \sqrt{a}$$
$$x = -\sqrt{a}$$
$$-x = \sqrt{a}$$
$$-x = -\sqrt{a}$$

where the last two forms are equivalent to the first two, and become identical with them on changing all the signs. Hence it is sufficient, in extracting the square root of both members of an equation, to place the double sign before *one* member only.

303. 1. Solve the equation $3x^2 + 7 = \frac{5x^2}{4} + 35.$

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140$

Transposing and uniting terms, $7x^2 = 112$

$$x^2 = 16$$

Extracting the square root of both members,

$$x = \pm 4, \textit{Ans.}$$

RULE.

Reduce the given equation to the form $x^2 = a$, and then extract the square root of both members.

EXAMPLES.

Solve the following equations:

2. $4x^2 - 7 = 29.$
3. $5x^2 + 5 = 3x^2 + 55.$
4. $7x^2 - 5 = 3x^2 - 11.$
5. $\frac{5}{4+x} = \frac{8}{3} - \frac{5}{4-x}.$

6. $\dfrac{245}{x} = 5x.$ **7.** $13 - \sqrt{3x^2 + 16} = 6.$

8. $x + \sqrt{x^2 + 3} = \dfrac{6}{\sqrt{x^2 + 3}}.$

9. $\dfrac{1}{1 - \sqrt{1 - x^2}} - \dfrac{1}{1 + \sqrt{1 - x^2}} = \dfrac{\sqrt{3}}{x^2}.$

10. $\dfrac{x^2}{2} - 3 + \dfrac{5x^2}{12} = \dfrac{7}{24} - x^2 + \dfrac{335}{24}.$

11. $2(x - 3)(x + 3) = (x + 1)^2 - 2x.$

12. $ax^2 + b = c.$ **13.** $\dfrac{a}{x^2 - b} = \dfrac{b}{x^2 - a}.$

AFFECTED QUADRATIC EQUATIONS.

304. An affected quadratic equation may always be reduced to the form

$$x^2 + px = q,$$

where p and q represent any quantities, positive or negative, integral or fractional. Thus, in the equation

$$5x - \frac{3x - 3}{x - 3} = 2x + \frac{3x - 6}{2}$$

Clearing of fractions,

$$10x(x - 3) - (6x - 6) = 4x(x - 3) + (3x - 6)(x - 3)$$

or, $$10x^2 - 30x - 6x + 6 = 4x^2 - 12x + 3x^2 - 15x + 18$$

Transposing and uniting terms, $3x^2 - 9x = 12$

Dividing by 3, $x^2 - 3x = 4$

which is in the form $x^2 + px = q$.

Equations of this kind have, therefore, sometimes been denominated *trinomial*, or those of *three terms*.

305. Let it be required to solve the equation

$$x^2 + p\,x = q.$$

Equations of this kind are solved by adding to both members such a quantity as will make the first member a perfect square, and taking the square root of the resulting equation. The process of adding such a quantity to both sides as will make the first member a perfect square, is termed *Completing the Square.*

In any trinomial square (Arts. 104 and 105), the middle term is twice the product of the square roots of the extreme terms; therefore the square root of the last term must be equal to half the second term divided by the square root of the first. Hence the *square root* of the quantity which must be added to $x^2 + p\,x$ to render it a perfect square, is $\frac{p\,x}{2} \div x$, or $\frac{p}{2}$. Adding to both members the square of $\frac{p}{2}$, or $\frac{p^2}{4}$, we have

$$x^2 + p\,x + \frac{p^2}{4} = q + \frac{p^2}{4} = \frac{4\,q + p^2}{4}$$

Extracting the square root of both members,

$$x + \frac{p}{2} = \pm \frac{\sqrt{4\,q + p^2}}{2}$$

or,

$$x = -\frac{p}{2} \pm \frac{\sqrt{4\,q + p^2}}{2}.$$

Thus, there are *two* values of x,

$$x = -\frac{p}{2} + \frac{\sqrt{4\,q + p^2}}{2}, \text{ or } -\frac{p}{2} - \frac{\sqrt{4\,q + p^2}}{2}.$$

We observe from the preceding investigation that the quantity to be added to complete the square is found by taking half the coefficient of x, and squaring the result.

Hence, for solving affected quadratic equations, we have the following

RULE.

Reduce the equation to the form $x^2 + p\,x = q$.

Complete the square by adding to both members the square of half the coefficient of x. *Extract the square root of both members, and solve the simple equation thus found.*

1. Solve the equation $x^2 - 3\,x = 4$.

Completing the square, by adding to both members the square of $\frac{3}{2}$, or $\frac{9}{4}$,

$$x^2 - 3\,x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}$$

Extracting the square root, $x - \frac{3}{2} = \pm \frac{5}{2}$

Transposing, $x = \frac{3}{2} \pm \frac{5}{2}$

Taking the upper sign, $x = \frac{3}{2} + \frac{5}{2} = \frac{8}{2} = 4.$

Taking the lower sign, $x = \frac{3}{2} - \frac{5}{2} = -\frac{2}{2} = -1.$

Ans. $x = 4$ or -1.

We may verify these values as follows:

Putting $x = 4$ in the given equation, $16 - 12 = 4$.

Putting $x = -1$, $1 + 3 = 4$.

These results being identical, the values of x are verified.

2. Solve the equation $3\,x^2 + 8\,x = -4$.

Dividing through by 3 $x^2 + \frac{8\,x}{3} = -\frac{4}{3}$

Completing the square, by adding to both members the square of $\frac{4}{3}$, or $\frac{16}{9}$,

$$x^2 + \frac{8x}{3} + \frac{16}{9} = -\frac{4}{3} + \frac{16}{9} = \frac{4}{9}$$

Extracting the square root, $x + \frac{4}{3} = \pm\frac{2}{3}$

Transposing, $$x = -\frac{4}{3} \pm \frac{2}{3}$$

Taking the upper sign, $$x = -\frac{4}{3} + \frac{2}{3} = -\frac{2}{3}.$$

Taking the lower sign, $$x = -\frac{4}{3} - \frac{2}{3} = -\frac{6}{3} = -2.$$

Ans. $x = -\frac{2}{3}$ or -2.

3. Solve the equation $-3x^2 - 7x = \frac{10}{3}$.

Dividing through by -3, $$x^2 + \frac{7x}{3} = -\frac{10}{9}$$

Completing the square, by adding to both members the square of $\frac{7}{6}$, or $\frac{49}{36}$,

$$x^2 + \frac{7x}{3} + \frac{49}{36} = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}$$

Extracting the square root, $$x + \frac{7}{6} = \pm\frac{3}{6}$$

Transposing, $$x = -\frac{7}{6} \pm \frac{3}{6}$$

Whence, $x = -\frac{2}{3}$ or $-\frac{5}{3}$, *Ans.*

A SECOND METHOD OF COMPLETING THE SQUARE.

306. Although any affected quadratic equation may be solved by the method of Art. 305, since its rule is general, still it is sometimes more convenient to employ a second method of completing the square, known as the "Hindoo Method."

An affected quadratic, reduced to three terms, and cleared of all fractions, may be reduced to the form

$$a x^2 + b x = c.$$

Multiplying each term by $4a$, we have

$$4 a^2 x^2 + 4 a b x = 4 a c$$

By an operation similar to that of Art. 305, we may show that b^2 must be added to both members, in order that the first member may be a perfect square. Thus,

$$4 a^2 x^2 + 4 a b x + b^2 = b^2 + 4 a c$$

Extracting the square root, $2 a x + b = \pm \sqrt{b^2 + 4 a c}$

Transposing, $2 a x = -b \pm \sqrt{b^2 + 4 a c}$

Dividing by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 + 4 a c}}{2a}.$

It will be observed that the quantity necessary to complete the square, is the square of the coefficient of x in the given equation. Hence the following

RULE.

Reduce the equation to the form $a x^2 + b x = c$.

Multiply both members of the equation by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

Extract the square root of both members, and solve the simple equation thus produced.

Note. The only advantage of this method over the preceding is in avoiding fractions in completing the square.

4. Solve the equation $2x^2 - 7x = -3$.

Multiplying both members by four times 2, or 8,

$$16\,x^2 - 56\,x = -24$$

Adding to each member the square of 7, or 49,

$$16\,x^2 - 56\,x + 49 = -24 + 49 = 25$$

Extracting the square root, $4\,x - 7 = \pm 5$

Transposing, $4\,x = 7 \pm 5 = 12$ or 2

Dividing by 4, $x = 3$ or $\frac{1}{2}$, *Ans.*

307. This method is usually to be preferred in solving literal equations.

5. Solve the equation $x^2 + (a-1)\,x = a.$

Multiplying both members by four times 1, or 4,

$$4\,x^2 + 4\,(a-1)\,x = 4\,a$$

Adding to each member the square of $a - 1$, or $(a-1)^2$,

$$4\,x^2 + 4\,(a-1)\,x + (a-1)^2 = 4\,a + (a-1)^2$$

$$= a^2 + 2\,a + 1 = (a+1)^2$$

Extracting the square root,

$$2\,x + (a-1) = \pm (a+1)$$

Transposing, $2\,x = -(a-1) \pm (a+1)$

Taking the upper sign, $2\,x = -(a-1) + (a+1)$

$$= -a + 1 + a + 1 = 2$$

or, $x = 1.$

Taking the lower sign, $2\,x = -(a-1) - (a+1)$

$$= -a + 1 - a - 1 = -2\,a$$

or, $x = -a.$

Ans. $x = 1$ or $-a.$

308. In case the coefficient of x in the given equation is an *even* number, the rule may be modified as follows:

Multiply both members of the equation by the coefficient of x^2, and add to each the square of half the coefficient of x in the given equation.

6. Solve the equation $7x^2 + 4x = 51.$

Multiplying both members by 7, $49x^2 + 28x = 357$

Adding to each member the square of 2, or 4,

$$49x^2 + 28x + 4 = 361$$

Extracting the square root, $7x + 2 = \pm 19$

Transposing, $7x = -2 \pm 19 = 17 \text{ or } -21$

Dividing by 7, $x = \frac{17}{7} \text{ or } -3$, *Ans.*

SOLUTION OF QUADRATIC EQUATIONS BY A FORMULA.

309. In Art. 306, we showed that if $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}. \qquad (1)$$

We may use this as a formula for the solution of quadratic equations as follows:

7. Solve the equation $3x^2 + 5x = 42.$

Here $a = 3$, $b = 5$, $c = 42$; substituting these values in (1),

$$x = \frac{-5 \pm \sqrt{25 + 504}}{6}$$

$$= \frac{-5 \pm \sqrt{529}}{6} = \frac{-5 \pm 23}{6} = 3 \text{ or } -\frac{14}{3}, \textit{ Ans.}$$

8. Solve the equation $110x^2 - 21x = -1.$

Here $a = 110$, $b = -21$, $c = -1$; substituting in (1),

$$x = \frac{21 \pm \sqrt{441 - 440}}{220} = \frac{21 \pm 1}{220} = \frac{1}{10} \text{ or } \frac{1}{11}, \textit{ Ans.}$$

Note. Particular attention must be paid to the *signs* of the coefficients in substituting.

9. Solve the equation, $-x^2 - 6x = 8$.

Here $a = -1$, $b = -6$, $c = 8$; substituting in (1),

$$x = \frac{6 \pm \sqrt{36 - 32}}{-2} = \frac{6 \pm 2}{-2} = -4 \text{ or } -2, \textit{ Ans.}$$

RULE.

Reduce the equation to the form $ax^2 + bx = c$.

The value of x *is then a fraction, whose numerator is the coefficient of* x *with its sign changed, plus or minus the square root of the sum of the square of said coefficient, and four times the product of the second member by the coefficient of* x^2; *and whose denominator is twice the coefficient of* x^2.

310. The following equations may be solved by either of the preceding methods, preference being given to the one best adapted to the example considered. Special methods and devices may also be employed whenever any advantage can thereby be gained.

EXAMPLES.

Solve the following equations:

10. $x^2 + 2x + 7 = 42$.

11. $x^2 - 9x - 22 = 0$.

12. $x^2 - 8x = -15$.

13. $x^2 + 18x = -65$.

14. $6x^2 + 7x - 3 = 0$.

15. $13x - 14 = 3x^2$.

16. $26x + 15x^2 = -7$.

17. $-40 + x = 6x^2$.

18. $17x = 2x^2 - 6$.

19. $\frac{x^2}{2} + \frac{x}{3} = -\frac{1}{24}$.

20. $\frac{x}{2} = \frac{7}{6} - \frac{2x^2}{3}$.

21. $\frac{3x^2}{5} - \frac{22}{5} = x$.

22. $\frac{4x^2}{3} - \frac{17}{2} - \frac{x}{3} = 0.$

23. $\frac{2x^2}{5} - \frac{5x}{2} = -\frac{15}{4}.$

24. $(x-3)(2x+1) = 4.$

25. $(x+5)(x-5)-(11x+1)=0.$

26. $4x(18x-1) = (10x-1)^2.$

27. $(3x-5)^2 - (x+2)^2 = -5.$

28. $(x-1)^2 - (3x+8)^2 = (2x+5)^2.$

29. $\frac{2}{x} + \frac{x}{2} = -\frac{5}{2}.$

30. $\frac{x}{x-1} - \frac{x-1}{x} = \frac{3}{2}.$

31. $\frac{x}{5-x} - \frac{5-x}{x} = \frac{15}{4}.$

32. $\frac{5}{x} - \frac{3x+1}{x^2} = \frac{1}{4}.$

33. $\frac{x}{3x+4} = \frac{3}{4x+1}.$

34. $\frac{x}{3x+4} - \frac{2}{7x-4} = 0.$

35. $6x + \frac{35-3x}{x} = 44.$

36. $4x - \frac{14-x}{x+1} = 14.$

37. $\frac{21}{5-x} - \frac{x}{7} = 3\frac{4}{7}.$

38. $\frac{x+1}{x+2} - \frac{x+3}{x+4} = \frac{8}{3}.$

39. $\frac{3x^2}{x-7} - \frac{1-8x}{10} = \frac{x}{5}.$

40. $\frac{2x-1}{x} - \frac{3x}{3x-1} + \frac{1}{2} = 0.$

41. $\sqrt{20+x-x^2} = 2(x-5).$

42. $x + \sqrt{5x+10} = 8.$

43. $\frac{x^3-x^2+7}{x^2+3x-1} = x + \frac{11}{3}.$

44. $\frac{7}{x^2-4} - \frac{3}{x+2} = \frac{22}{5}.$

45. $\frac{1}{x^2-1} + \frac{1}{3} = \frac{1}{3(x-1)} + \frac{1}{x+1}.$

46. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$.

47. $\frac{x+2}{x-1}+\frac{x-2}{x+1}=\frac{2x+16}{x+5}$.

48. $\frac{12+5x}{12-5x}+\frac{2+x}{x}=\frac{1}{1-5x}$.

49. $\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\frac{x}{2}$.

50. $x+\sqrt{2(5x+3)}=9$.

51. $\sqrt{3x-5}=\frac{\sqrt{7x^2+36x}}{x}$.

52. $acx^2-bcx+adx=bd$.

53. $x^2-2ax+a^2-b^2=0$.

54. $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}$.

55. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$.

56. $(3x-2)(x+5)-(x-6)(5x-16)=301$.

57. $(2x+3)(3x+4)=(8+x)(2x+9)$.

58. $(2x-5)^2-(2x-1)^2=8x-5x^2-5$.

59. $x^2+bx+cx=(a+c)(a-b)$.

60. $abx^2+\frac{3a^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}-\frac{b^2x}{c}$.

61. $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1)$.

XXV. — PROBLEMS

LEADING TO PURE OR AFFECTED QUADRATIC EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.

311. **1.** I bought a lot of flour for \$175; and the number of dollars per barrel was to the number of barrels, as 4 to 7. How many barrels were purchased, and what was the price of each?

Let $x =$ the number of dollars per barrel,

then $\frac{7x}{4} =$ the number of barrels.

By the conditions, $\frac{7x^2}{4} = 175$

Whence, $x = \pm 10.$

Only the positive value is applicable, as the negative value does not answer to the conditions of the problem.

That is, $x = 10$, the number of dollars per barrel,

and $\frac{7x}{4} = 17\frac{1}{2}$, the number of barrels.

2. There is a certain number, whose square increased by 30, is equal to 11 times the number itself. Required the number.

Let $x =$ the number.

By the conditions, $x^2 + 30 = 11x$

Solving this equation, $x = 5$ or 6.

That is, the number is either 5 or 6, for each of these values satisfies the conditions of the problem.

3. I bought a watch, which I sold for \$56, and thereby gained as much per cent as the watch cost me. Required the amount paid for it.

Let $x =$ the amount paid, in dollars.

Then $x =$ the gain per cent,

and $\frac{x}{100} \times x = \frac{x^2}{100} =$ the whole gain in dollars.

By the conditions, $\frac{x^2}{100} = 56 - x$

Solving this equation, $x = 40$ or -140.

Only the positive value of x is here admissible, as the negative result does not answer to the conditions of the problem. The cost, therefore, was \$40.

Note. When two answers are found to a problem, they should be examined to see whether they answer to the conditions of the problem or not. Only those which answer to the conditions should be retained.

PROBLEMS.

4. I have three square house-lots, of equal size. If I were to add 193 square rods to their contents, they would be equivalent to a square lot whose sides would each measure 25 rods. Required the length of each side of the three lots.

5. There are two square fields, the larger of which contains 25,600 square rods more than the other, and the ratio of their sides is as 5 to 3. Required the contents of each.

6. Find two numbers whose sum shall be 15, and the sum of their squares 117.

7. A person cut and piled two ranges of wood, whose united contents were 26 cords, for 356 dimes; and the labor on each of them cost as many dimes per cord as there were cords in its range. Required the number of cords in each range.

8. A grazier bought a certain number of oxen for \$240, and having lost 3, he sold the remainder at \$8 a head more than they cost him, and gained \$59. How many did he buy?

9. The plate of a rectangular looking-glass is 18 inches by 12, and is to be framed with a frame all parts of which are of equal width, and whose area is to be equal to that of the glass. Required the width of the frame.

10. A merchant sold a quantity of flour for \$39, and gained as much per cent as the flour cost him. What was the cost of the flour?

11. There are two numbers whose difference is 9, and whose sum multiplied by the greater is 266. What are the numbers?

12. A and B gained by trade $1800. A's money was in the firm 12 months, and he received for his principal and gain $2600. B's money, which was $3000, was in the firm 16 months. What money did A put into the firm?

13. A merchant bought a quantity of flour for $72, and found that if he had bought 6 barrels more for the same money, he would have paid $1 less for each barrel. How many barrels did he buy, and what was the price of each?

14. A square courtyard has a gravel-walk around it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 164. Required the area of the court.

15. My gross income is $1000. After deducting a percentage for income tax, and then a percentage, less by one than that of the income tax, from the remainder, the income is reduced to $912. Required the rate per cent at which the income tax is charged.

16. The sum of the squares of two consecutive numbers is 113. What are the numbers?

17. Find three consecutive numbers such that twice the product of the first and third is equal to the square of the second, increased by 62.

18. I have a rectangular field of corn which consists of 6250 hills; and the number of hills in the length exceeds the number in the breadth by 75. How many hills are there in the length and breadth?

19. A certain company agreed to build a vessel for $6300; but, two of their number having died, those that survived had each to advance $200 more than they otherwise would have done. Of how many persons did the company at first consist?

20. A detachment from an army was marching in regular column, with 6 men more in depth than in front; but when the enemy came in sight, the front was increased by 870 men, and the whole was thus drawn up in 4 lines. Required the number of men.

21. A has two square gardens, and the side of the one exceeds that of the other by 4 rods, while the contents of both are 208 square rods. How many square rods does the larger garden contain more than the smaller?

22. A certain farm is a rectangle, whose length is twice its breadth; but should it be enlarged 20 rods in length and 24 rods in breadth, its contents would be doubled. Of how many acres does the farm consist?

23. A square courtyard has a rectangular gravel-walk around it. The side of the court wants one yard of being six times the breadth of the gravel-walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 340. What is the area of the court and width of the walk?

24. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many dimes per bushel as there were bushels of barley, and for the latter 4 dimes per bushel less. He sold the mixture at $1 per bushel, and lost $57.60 by his bargain. Required the quantity of barley, and its price per bushel.

25. A lady wishes to purchase a carpet for each of her square parlors; the side of one of them is 1 yard longer than the other, and it will require 85 square yards for both rooms. What will it cost the lady to carpet each of the rooms with carpeting 40 inches wide, at $1.75 per yard?

26. A man has two square lots of unequal dimensions, containing together 15,025 square feet. If the lots were contiguous to each other, it would require 530 feet of fence to embrace them in a single enclosure of six sides. Required the area of each lot.

27. A certain number consists of two digits, the left-hand digit being twice the right-hand; and if the digits are inverted, the product of the number thus formed, increased by 11, and the original number, is 4956. Find the number.

28. A man travelled 108 miles. If he had gone 3 miles more an hour, he would have performed the journey in 6 hours less time. How many miles an hour did he go?

29. A cistern can be filled by two pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller by 2 hours. What time will each pipe separately take to fill it?

30. A set out from C towards D, and travelled 3 miles an hour. After he had gone 28 miles, B set out from D towards C, and went every hour $\frac{1}{10}$ of the entire distance; and after he had travelled as many hours as he went miles in an hour, he met A. Required the distance from C to D.

31. A courier proceeds from P to Q in 14 hours; a second courier starts at the same time from a place 10 miles behind P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 20 miles. Find the distance from P to Q.

XXVI.—EQUATIONS IN THE QUADRATIC FORM.

312. An equation is in the *quadratic form* when it is expressed in three terms, two of which contain the unknown quantity; and of these two, *one has an exponent twice as great as the other.* As,

$$x^6 - 6x^3 = 16,$$

$$x^3 + x^{\frac{3}{2}} = 72,$$

$$(x^2 - 1)^2 + 3(x^2 - 1) = 18, \text{ etc.}$$

313. The rules already given for the solution of quadratics will apply to equations having the same form. For, in the equation

$$a x^{2n} + b x^n = c,$$

let $x^n = y$; then $x^{2n} = y^2$. Substituting,

$$a y^2 + b y = c$$

Whence, by Art. 309, we have

$$y = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

or,
$$x^n = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

from which equation x may be found by extracting the nth root of both members.

314. **1.** Solve the equation $x^4 - 5x^2 = -4$.

The equation may be solved as in Art. 313, by representing x^2 by y. A better method, however, is the following:

Completing the square, $x^4 - 5x^2 + \frac{25}{4} = -4 + \frac{25}{4} = \frac{9}{4}$

Extracting the square root, $x^2 - \frac{5}{2} = \pm \frac{3}{2}$

Transposing, $x^2 = \frac{5}{2} \pm \frac{3}{2} = 4 \text{ or } 1$

Whence, $x = \pm 2 \text{ or } \pm 1$, *Ans.*

2. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square, $x^6 - 6x^3 + 9 = 16 + 9 = 25$

Extracting the square root, $x^3 - 3 = \pm 5$

Transposing, $x^3 = 3 \pm 5 = 8 \text{ or } -2$

Whence, $x = 2 \text{ or } -\sqrt[3]{2}$, *Ans.*

Here, although the equation is of the sixth degree, we find but two roots. The equation in reality has six roots, but this method fails to give more than two. It will be shown hereafter how to obtain the other four.

3. Solve the equation $x + 4\sqrt{x} = 21.$

Writing the radical with a fractional exponent,

$$x + 4x^{\frac{1}{2}} = 21$$

which is in the quadratic form.

Completing the square, $x + 4\sqrt{x} + 4 = 21 + 4 = 25$

Extracting the square root, $\sqrt{x} + 2 = \pm 5$

Transposing, $\sqrt{x} = -2 \pm 5 = 3$ or -7

Whence, squaring, $x = 9$ or 49, *Ans.*

4. Solve the equation $3x^2 + x^{\frac{7}{6}} = 3104x^{\frac{1}{3}}.$

Dividing by $x^{\frac{1}{3}}$, $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$

which is in the quadratic form.

Multiplying by four times 3, or 12,

$$36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} = 37248$$

Completing the square, $36x^{\frac{5}{3}} + 12x^{\frac{5}{6}} + 1 = 37249$

Extracting the square root, $6x^{\frac{5}{6}} + 1 = \pm 193$

Transposing, $6x^{\frac{5}{6}} = -1 \pm 193 = 192$ or -194

Dividing by 6, $x^{\frac{5}{6}} = 32$ or $-\frac{97}{3}$

Extracting the fifth root, $x^{\frac{1}{6}} = 2$ or $-\left(\frac{97}{3}\right)^{\frac{1}{5}}$

Raising both members to the sixth power,

$$x = 64 \text{ or } \left(\frac{97}{3}\right)^{\frac{6}{5}}, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

5. $x^4 + 4x^2 = 117$.

6. $x^{-4} - 9x^{-2} + 20 = 0$.

7. $x^{10} + 31x^5 - 10 = 22$.

8. $81x^2 + \frac{1}{x^2} = 82$.

9. $x^2 + \frac{1225}{x^2} - 14 = 60$.

10. $x^6 + 20x^3 - 10 = 59$.

11. $3x^{\frac{4}{3}} - \frac{5x^{\frac{8}{3}}}{2} = -592$.

12. $x^3 - x^{\frac{3}{2}} = 56$.

13. $x - 2 - \sqrt{x} = 0$.

14. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$.

15. $\frac{\sqrt{4x} + 2}{4 + \sqrt{x}} = \frac{4 - \sqrt{x}}{\sqrt{x}}$.

16. $\frac{\frac{3\sqrt{x}}{5} - 2}{x - 5} = \frac{1}{20}$.

17. Solve the equation $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40$.

Completing the square, $(x-5)^3 - 3(x-5)^{\frac{3}{2}} + \frac{9}{4} = 40 + \frac{9}{4} = \frac{169}{4}$

Extracting the square root, $(x-5)^{\frac{3}{2}} - \frac{3}{2} = \pm\frac{13}{2}$

Transposing, $(x-5)^{\frac{3}{2}} = \frac{3}{2} \pm \frac{13}{2} = 8$ or -5

Squaring both members, $(x-5)^3 = 64$ or 25

Extracting the cube root, $x - 5 = 4$ or $\sqrt[3]{25}$

Whence, $x = 9$ or $5 + \sqrt[3]{25}$, *Ans.*

Solve the following equations:

18. $(x^2 - 5x)^2 - 8(x^2 - 5x) = 84.$

19. $(2x - 1)^2 - 2(2x - 1) = 15.$

20. $(3x^2 - 2)^2 - 11(3x^2 - 2) + 10 = 0.$

21. $(x^3 - 5)^2 + 29(x^3 - 5) = 96.$

22. Solve the equation $x^4 + 10x^3 + 17x^2 - 40x - 84 = 0.$

We may write the equation in the form

$$x^4 + 10x^3 + 25x^2 - 8x^2 - 40x = 84$$

or, $$(x^2 + 5x)^2 - 8(x^2 + 5x) = 84$$

Completing the square, $(x^2 + 5x)^2 - 8(x^2 + 5x) + 16 = 100$

Extracting the square root, $(x^2 + 5x) - 4 = \pm 10$

Transposing, $(x^2 + 5x) = 4 \pm 10 = 14$ or -6

Taking the first value, we have $x^2 + 5x = 14$

Whence (Art. 309), $x = \dfrac{-5 \pm \sqrt{25 + 56}}{2} = \dfrac{-5 \pm 9}{2} = 2$ or $-7.$

Taking the second value, we have $x^2 + 5x = -6$

Whence, $x = \dfrac{-5 \pm \sqrt{25 - 24}}{2} = \dfrac{-5 \pm 1}{2} = -2$ or $-3.$

Ans. $x = 2, -7, -2,$ or $-3.$

Note. In solving equations of this form, our object is to form a perfect trinomial square with the x^4 and x^3 terms, and a portion of the x^2 term. By Art. 305, we may effect this by separating the x^2 term into two parts, one of which shall be the square of the quotient obtained by dividing the x^3 term by twice the square root of the x^4 term.

Solve the following equations:

23. $x^4 - 12x^3 + 34x^2 + 12x = 35.$

24. $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0.$

25. $x^4 - 6x^3 - 29x^2 + 114x = 80.$

26. $x^4 + 14x^3 + 47x^2 - 14x - 48 = 0.$

27. Solve the equation $2x^2 + \sqrt{2x^2+1} = 11.$

We may write the equation, $(2x^2 + 1) + \sqrt{2x^2+1} = 12$

Completing the square, $(2x^2 + 1) + \sqrt{2x^2+1} + \frac{1}{4} = \frac{49}{4}$

Extracting the square root, $\sqrt{2x^2+1} + \frac{1}{2} = \pm\frac{7}{2}$

Transposing, $\sqrt{2x^2+1} = -\frac{1}{2} \pm \frac{7}{2} = 3 \text{ or } -4$

Squaring, $2x^2 + 1 = 9 \text{ or } 16$

Transposing, $2x^2 = 8 \text{ or } 15$

Dividing by 2, $x^2 = 4 \text{ or } \frac{15}{2}$

Whence, $x = \pm 2 \text{ or } \pm\sqrt{\frac{15}{2}}$, *Ans.*

Note. In solving equations of this form, add such quantities to both members, that the expression without the radical in the first member may be the same as that within, or some multiple of it.

Solve the following equations:

28. $2x^2 + 3x - 5\sqrt{2x^2+3x+9} = -3.$

29. $x^2 - 6x + 5\sqrt{x^2-6x+20} = 46.$

30. $4x^2 + 6\sqrt{4x^2+12x-2} = -3(1+4x).$

31. $x^2 - 10x - 2\sqrt{x^2-10x+18} + 15 = 0.$

32. $3x^2 + 15x - 2\sqrt{x^2+5x+1} = 2.$

XXVII.—SIMULTANEOUS EQUATIONS

INVOLVING QUADRATICS.

315. The most general form of an equation of the second degree containing two unknown quantities, is

$$a x^2 + b x y + c y^2 + d x + e y + f = 0,$$

where the coefficients a, b, c, etc. represent any quantities, positive or negative, integral or fractional.

316. Two equations of the second degree containing two unknown quantities will generally produce, by elimination, an equation of the fourth degree containing one unknown quantity. Thus, if the equations are

$$x^2 + y = a$$
$$x + y^2 = b$$

From the first, by transposition, $y = a - x^2$; substituting in the second,

$$x + (a - x^2)^2 = b$$

or, $$x^4 - 2 a x^2 + x + a^2 - b = 0$$

an equation of the fourth degree. The rules for quadratics are, therefore, not sufficient to solve *all* simultaneous equations of the second degree.

In several cases, however, their solution may be effected by means of the ordinary rules.

CASE I.

317. *When each equation is of the form* $a x^2 + b y^2 = c$.

1. Solve the equations,

$$3 x^2 + 4 y^2 = 76$$
$$3 y^2 - 11 x^2 = 4$$

Multiplying the first equation by 3, and the second by 4,

$$9x^2 + 12y^2 = 228$$
$$12y^2 - 44x^2 = 16$$

Subtracting, $53x^2 = 212$

$$x^2 = 4,\ x = \pm 2.$$

Substituting these values in either given equation,

When $x = 2,\ y = \pm 4.$

When $x = -2,\ y = \pm 4.$

Ans. $x = 2,\ y = \pm 4$; or, $x = -2,\ y = \pm 4.$

EXAMPLES.

Solve the following equations:

2. $2x^2 + y^2 = 9;\ 5x^2 + 6y^2 = 26.$

3. $4x^2 - 3y^2 = -11;\ 11x^2 + 5y^2 = 301.$

4. $9x^2 + 24y^2 = 7;\ 72x^2 - 180y^2 = -37.$

5. $20x^2 - 16y^2 = 179;\ 5x^2 - 336y^2 = 24.$

CASE II.

318. *When one equation is of the first degree.*

1. Solve the equations,

$$x^2 + y^2 = 13$$
$$x + y = 1$$

From the second, by transposition, $y = 1 - x$ (1)

Substituting in the first, $x^2 + 1 - 2x + x^2 = 13$

or, $x^2 - x = 6$

Whence (Art. 309), $x = \dfrac{1 \pm \sqrt{1 + 24}}{2} = \dfrac{1 \pm 5}{2} = 3 \text{ or } -2.$

Substituting these values in (1),

When $x = 3,\ y = 1 - 3 = -2.$

When $x = -2,\ y = 1 + 2 = 3.$

Ans. $x = 3,\ y = -2$; or, $x = -2,\ y = 3.$

In solving examples under Case II, we find an expression for the value of one of the unknown quantities in terms of the other from the simple equation, which we substitute for that quantity in the other equation, thus producing a quadratic containing only one unknown quantity, by means of which the values of the unknown quantities are readily obtained.

Note. Although some examples, in which one equation is of the first degree (Ex. 1 for instance), may be solved by the methods of the next case, yet the method of Case II will be found in general the simplest.

EXAMPLES.

Solve the following equations:

2. $x+y=-1;\ xy=-56.$

3. $x+y=3;\ x^2+y^2=29.$

4. $x^3-y^3=-37;\ x-y=-1.$

5. $x-y=\dfrac{11}{2};\ xy=20.$

6. $10x+y=3xy;\ y-x=2.$

7. $x-y=5;\ xy=-6.$

8. $x^3+y^3=9;\ x+y=3.$

9. $3x^2-2xy=15;\ 2x+3y=12.$

10. $x-y=3;\ x^2+y^2=117.$

11. $x+y=11;\ xy=18.$

12. $x-y=6;\ x^2+y^2=90.$

13. $x^3+y^3=152;\ x+y=2.$

14. $x^2+3xy-y^2=23;\ x+2y=7.$

15. $x^3-y^3=98;\ x-y=2.$

16. $x+y=-4;\ x^2+y^2=58.$

CASE III.

319. *When the given equations are symmetrical with respect to x and y.*

1. Solve the equations,

$$x^2 + y^2 = 68$$

$$x y = 16$$

Multiplying the second by 2, $\qquad 2 x y = 32$

Adding this to the first equation, $\quad x^2 + 2 x y + y^2 = 100 \quad (1)$

Subtracting it from the first equation,

$$x^2 - 2 x y + y^2 = 36 \quad (2)$$

Extracting the square root of (1), $\qquad x + y = \pm 10 \quad (3)$

Extracting the square root of (2), $\qquad x - y = \pm 6 \quad (4)$

Equations (3) and (4) furnish four pairs of simple equations,

$x + y = 10$	$x + y = 10$	$x + y = -10$	$x + y = -10$
$x - y = 6$	$x - y = -6$	$x - y = 6$	$x - y = -6$
$2 x = 16$	$2 x = 4$	$2 x = -4$	$2 x = -16$
$x = 8.$	$x = 2.$	$x = -2.$	$x = -8.$
$y = 2.$	$y = 8.$	$y = -8.$	$y = -2.$

Ans. $x = 8, y = 2$; $x = 2, y = 8$;
$x = -2, y = -8$; or, $x = -8, y = -2$.

2. Solve the equations,

$$x^3 + y^3 = 133$$

$$x^2 - x y + y^2 = 19$$

Dividing the first equation by the second,

$$x + y = 7 \quad (1)$$

Squaring (1), $\qquad x^2 + 2 x y + y^2 = 49 \quad (2)$

Subtracting the second given equation from (2),

$$3 x y = 30; \text{ or, } 4 x y = 40 \quad (3)$$

Subtracting (3) from (2), $\qquad x^2 - 2 x y + y^2 = 9$

Whence, $\qquad x - y = \pm 3 \quad (4)$

Adding (1) and (4), $2x = 10$ or 4

Whence, $x = 5$ or 2.

Substituting these values in (1),

When $x = 5,\ y = 2$

$x = 2,\ y = 5.$

Ans. $x = 5,\ y = 2$; or, $x = 2,\ y = 5$.

The example might have been solved by substituting the value of y derived from (1) in either of the given equations, as in Case II.

The student will notice the difference between Examples 1 and 2 as regards the arrangement of the last portion of the work.

3. Solve the equations,

$$x^2 + y^2 = 208$$
$$x + y = 20$$

Multiplying the first equation by 2, $2x^2 + 2y^2 = 416$ (1)

Squaring the second equation, $x^2 + 2xy + y^2 = 400$ (2)

Subtracting (2) from (1), $x^2 - 2xy + y^2 = 16$

Whence, $x - y = \pm 4$ (3)

Adding the second given equation and (3),

$$2x = 24 \text{ or } 16$$

Whence, $x = 12$ or 8.

Substituting these values in (3),

When $x = 12,\ y = 8$

$x = 8,\ y = 12.$

Ans. $x = 12,\ y = 8$; or, $x = 8,\ y = 12$.

This example is solved more readily by the method of Case II; we solve it by Case III merely to show how equations may be solved symmetrically, when one is of the first degree.

EXAMPLES.

Solve the following equations:

4. $x^2 + y^2 = 25$; $xy = 12$.

5. $x^2 + y^2 = 85$; $xy = 42$.

6. $x^3 + y^3 = -19$; $x^2 - xy + y^2 = 19$.

7. $x^3 - y^3 = -65$; $x^2 + xy + y^2 = 13$.

8. $x + y = 1$; $xy = -6$.

9. $x^2 + y^2 = 65$; $x - y = 11$.

10. $x^2 + y^2 = 61$; $x + y = 11$.

11. $x^3 - y^3 = 117$; $x - y = 3$.

Note. Exs. 8, 9, 10, and 11 are to be solved like Ex. 3, and not by the method of Case II. In solving Ex. 11, begin by dividing the first equation by the second.

CASE IV.

320. *When the equations are of the second degree, and homogeneous.*

Note. Some examples, in which both equations are of the second degree and homogeneous, are solved more easily by the methods of Cases I and III, than by that of Case IV. The method of Case IV is to be used *only* when the example can be solved in no other way.

1. Solve the equations,

$$x^2 - xy = 35$$

$$xy + y^2 = 18$$

Letting $y = vx$, we have

$$x^2 - vx^2 = 35, \text{ or } x^2(1 - v) = 35; \text{ whence, } x^2 = \frac{35}{1 - v} \quad (1)$$

$$vx^2 + v^2x^2 = 18, \text{ or } x^2(v + v^2) = 18; \text{ whence, } x^2 = \frac{18}{v + v^2}$$

Equating the values of x^2, $\frac{35}{1-v}=\frac{18}{v+v^2}$

Clearing of fractions, $35v+35v^2=18-18v$

Transposing and uniting, $35v^2+53v=18$

Whence (Art. 309),

$$v=\frac{-53\pm\sqrt{2809+2520}}{70}=\frac{-53\pm 73}{70}=\frac{2}{7}\text{ or }-\frac{9}{5}$$

If $v=\frac{2}{7}$, substituting in (1), $x^2=49$, or $x=\pm 7$

Substituting in the equation $y=vx$,

When $x=7,\ y=\frac{2}{7}\times 7=2$

$x=-7,\ y=\frac{2}{7}\times -7=-2.$

If $v=-\frac{9}{5}$, substituting in (1), $x^2=\frac{25}{2}$, or $x=\pm\frac{5}{\sqrt{2}}$

Substituting in the equation $y=vx$,

When $x=\frac{5}{\sqrt{2}},\ y=-\frac{9}{5}\times\frac{5}{\sqrt{2}}=-\frac{9}{\sqrt{2}}$

$x=-\frac{5}{\sqrt{2}},\ y=-\frac{9}{5}\times-\frac{5}{\sqrt{2}}=\frac{9}{\sqrt{2}}.$

Ans. $x=7,\ y=2;\ x=-7,\ y=-2;$

$x=\frac{5}{\sqrt{2}},\ y=-\frac{9}{\sqrt{2}};$ or, $x=-\frac{5}{\sqrt{2}},\ y=\frac{9}{\sqrt{2}}.$

Note. In using the equation $y=vx$, to calculate the value of y when x has been found, care should be taken to *use that value of v which was used in getting the particular value of x.*

EXAMPLES.

Solve the following equations:

2. $x^2+xy+4y^2=6;\ 3x^2+8y^2=14.$

3. $6x^2-5xy+2y^2=12;\ 3x^2+2xy-3y^2=-3.$

4. $x^2 + xy = 12$; $xy - y^2 = 2$.

5. $2y^2 - 4xy + 3x^2 = 17$; $y^2 - x^2 = 16$.

6. $x^2 + xy - y^2 = 1$; $x^2 - xy + 2y^2 = 8$.

7. $2x^2 - 2xy - y^2 = 3$; $x^2 + 3xy + y^2 = 11$.

321. We append a few miscellaneous examples, for the solution of which no general rules can be given. Various artifices are used; familiarity with which can only be obtained by experience.

1. Solve the equations,

$$x^3 - y^3 = 19$$

$$x^2 y - x y^2 = 6$$

Multiplying the second by 3, $3x^2 y - 3xy^2 = 18$ (1)

Subtracting (1) from the first given equation,

$$x^3 - 3x^2 y + 3xy^2 - y^3 = 1$$

Extracting the cube root, $x - y = 1$ (2)

Transposing, $x = 1 + y$ (3)

Dividing the second given equation by (2), $xy = 6$ (4)

Substituting from (3) in (4), $y(1 + y) = 6$

or, $y^2 + y = 6$

Whence, $$y = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = 2 \text{ or } -3.$$

Substituting in (3),

When $y = 2$, $x = 3$

$y = -3$, $x = -2$.

Ans. $x = 3$, $y = 2$; or, $x = -2$, $y = -3$.

2. Solve the equations,

$$\frac{x^2}{y} + \frac{y^2}{x} = 18$$
$$x + y = 12$$

Let $x = u + v$, and $y = u - v$.

Then $x + y = 2u$; whence, $2u = 12$, or $u = 6$.

From the first given equation, $x^3 + y^3 = 18xy$

Substituting $x = 6 + v$, and $y = 6 - v$, we have

$$(6 + v)^3 + (6 - v)^3 = 18(6 + v)(6 - v)$$

Reducing, $432 + 36v^2 = 648 - 18v^2$

Whence, $54v^2 = 216$

$$v^2 = 4, \ v = \pm 2$$

Then $x = 6 + v = 6 \pm 2 = 8$ or 4.

Substituting these values in the second given equation,

When $x = 8, \ y = 4$

$x = 4, \ y = 8$, *Ans.*

3. Solve the equations,

$$x^2 + y^2 + x + y = 18$$
$$xy = 6$$

Adding twice the second equation to the first,

$$x^2 + 2xy + y^2 + x + y = 30$$

or, $(x + y)^2 + (x + y) = 30$

Whence, $(x + y) = \dfrac{-1 \pm \sqrt{1 + 120}}{2} = \dfrac{-1 \pm 11}{2} = 5$ or -6.

Taking the first value, $x + y = 5$ (1)

and the second given equation, $xy = 6$ (2)

From (1), $y = 5 - x$; substituting in (2), $x^2 - 5x = -6$

Whence, $x = \dfrac{5 \pm \sqrt{25 - 24}}{2} = \dfrac{5 \pm 1}{2} = 3$ or 2.

Substituting in (1),

When, $x = 3, \ y = 2$

$x = 2, \ y = 3$.

Taking the second value, $x + y = -6$ (3)

and the second given equation, $xy = 6$ (4)

From (3), $y = -6 - x$; substituting in (4),

$$x^2 + 6x = -6$$

Whence, $x = \frac{-6 \pm \sqrt{36 - 24}}{2} = \frac{-6 \pm 2\sqrt{3}}{2} = -3 \pm \sqrt{3}.$

Substituting in (3),

When $x = -3 + \sqrt{3}$, $y = -3 - \sqrt{3}$.

$x = -3 - \sqrt{3}$, $y = -3 + \sqrt{3}$.

Ans. $x = 3$, $y = 2$; $x = 2$, $y = 3$;
$x = -3 + \sqrt{3}$, $y = -3 - \sqrt{3}$; or, $x = -3 - \sqrt{3}$, $y = -3 + \sqrt{3}$

4. Solve the equations,

$$x^4 + y^4 = 97 \quad (1)$$
$$x + y = -1 \quad (2)$$

Raising (2) to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 1 \quad (3)$$

Subtracting (1) from (3), $4x^3y + 6x^2y^2 + 4xy^3 = -96$

or, $3x^2y^2 + 2xy(x^2 + y^2) = -48$ (4)

But from (2), $x^2 + y^2 = 1 - 2xy$

Substituting in (4), $3x^2y^2 + 2xy(1 - 2xy) = -48$

or, $x^2y^2 - 2xy = 48$

Whence, $xy = \frac{2 \pm \sqrt{4 + 192}}{2} = \frac{2 \pm 14}{2} = -6$ or 8.

Taking the first value, $xy = -6$

From (2), $y = -1 - x$; substituting, $x^2 + x = 6$

Whence, $x = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm 5}{2} = 2$ or -3.

Substituting in (2),

When $x = 2$, $y = -3$.

$x = -3$, $y = 2$.

Taking the second value, $xy = 8$

From (2), $y = -1 - x$; substituting, $x^2 + x = -8$

Whence, $$x = \frac{-1 \pm \sqrt{1-32}}{2} = \frac{-1 \pm \sqrt{-31}}{2}.$$

Substituting in (2),

When $$x = \frac{-1 + \sqrt{-31}}{2},\ y = \frac{-1 - \sqrt{-31}}{2}.$$

$$x = \frac{-1 - \sqrt{-31}}{2},\ y = \frac{-1 + \sqrt{-31}}{2}.$$

Ans. $x = 2$, $y = -3$; $x = -3$, $y = 2$; $x = \frac{-1 + \sqrt{-31}}{2}$,

$y = \frac{-1-\sqrt{-31}}{2}$; or, $x = \frac{-1-\sqrt{-31}}{2}$, $y = \frac{-1 + \sqrt{-31}}{2}$.

EXAMPLES.

Solve the following equations:

5. $x + y = 9$; $\sqrt[3]{x} + \sqrt[3]{y} = 3$.

6. $x + \sqrt{x y} + y = 19$; $x^2 + x y + y^2 = 133$.

7. $x^2 y + x y^2 = 30$; $x^4 y^2 + x^2 y^4 = 468$.

8. $x^2 + y^2 - x - y = 18$; $x y + x + y = 19$.

9. $x^2 + 3x + y = 73 - 2 x y$; $y^2 + 3y + x = 44$.

10. $x^2 + y^2 = \frac{5 x y}{2}$; $x - y = \frac{x y}{4}$.

11. $\frac{x}{y} + \frac{4\sqrt{x}}{\sqrt{y}} = \frac{33}{4}$; $x - y = 5$.

12. $\frac{x}{2} + \frac{y}{3} = 1$; $\frac{2}{x} + \frac{3}{y} = 4$.

13. $x^2 y + x y^2 = 30$; $x^3 + y^3 = 35$.

14. $x + \sqrt{x y} = 3$; $y + \sqrt{x y} = -2$

15. $x^2 y + y^2 x = 6$; $\frac{1}{x} + \frac{1}{y} = \frac{2}{3}$.

16. $x^4 + y^4 = 17$; $x - y = 3$.

17. $x^5 - y^5 = -211$; $x - y = -1$.

18. $x^2 + y^2 = 7 + x y$; $x^3 + y^3 = 6 x y - 1$.

19. $2x^2 - 7 x y - 2y^2 = 5$; $3 x y - x^2 + 6 y^2 = 44$.

20. $\frac{3x}{y+3}+\frac{2y}{x+2}=\frac{5}{2};\ \frac{x}{2}+\frac{y}{3}=2.$

21. $x+z=7;\ 2y-3z=-5;\ x^2+y^2-z^2=11.$

22. $xz=y^2;\ (x+y)(z-x-y)=3;\ (x+y+z)(z-x-y)=7.$

XXVIII. — PROBLEMS

LEADING TO SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

322. **1.** What two quantities are those, the sum of whose squares is 130, and the difference of whose squares is 32?

Let $x=$ one number,

and $y=$ the other.

By the conditions, $x^2+y^2=130$

$$x^2-y^2=32$$

Solving these equations, as in Case I, Art. 317,

$$x=9,\ y=\pm 7;$$

or, $x=-9,\ y=\pm 7.$

This indicates *four* answers to the problem:

9 and 7,

9 and -7,

-9 and 7,

-9 and -7.

Any one of these pairs of values will satisfy the conditions of the problem.

2. A says to B, "The sum of our money is \$18." B replies, "If twice the number of your dollars were multiplied by mine, the product would be \$154." How many dollars had each?

Let $\qquad x =$ A's dollars,

and $\qquad y =$ B's.

By the conditions, $\qquad x + y = 18$

$$2xy = 154$$

Solving these equations, as in Case II, Art. 318,

$$x = 7,\ y = 11;$$

or, $\qquad x = 11,\ y = 7.$

That is, either A has \$7, and B \$11, or A has \$11, and B \$7.

3. The price of two coats and one vest is \$38. And the price of a coat less that of a vest, is to \$23, as \$7 is to the sum of the prices of a coat and vest. What is the price of a coat, and what of a vest?

Let $\qquad x =$ the price of a coat in dollars,

and $\qquad y =$ the price of a vest.

By the conditions, $2x + y = 38$

and $\qquad x - y : 23 = 7 : x + y$

or (Art. 181), $\qquad x^2 - y^2 = 161$

Solving these equations, as in Case II, Art. 318,

$$x = 15,\ y = 8;$$

or, $\qquad x = \frac{107}{3},\ y = -\frac{100}{3}.$

Only the first answer is admissible, as a negative value of either unknown quantity does not answer to the conditions of the problem. Hence, the price of a coat is \$15, and of a vest, \$8.

Note. The note after Ex. 3, Art. 311, applies with equal force to the problems in this chapter.

PROBLEMS.

4. The difference of two quantities is 5, and the sum of their squares is 193. What are the quantities?

5. There are two quantities whose product is 77, and the difference of whose squares is to the square of their difference as 9 to 2. Required the quantities.

6. A and B have each a field, in the shape of an exact square, and it requires 200 rods of fence to enclose both. The contents of these fields are 1300 square rods. What is the value of each at $2.25 per square rod?

7. Two gentlemen, A and B, were speaking of their ages. A said that the product of their ages was 750. B replied, that if his age were increased 7 years, and A's were diminished 2 years, their product would be 851. Required their ages.

8. A certain garden is a rectangle, and contains 15,000 square yards, exclusive of a walk, 7 yards wide, which surrounds it, and contains 3696 square yards. Required the length and breadth of the garden.

9. What two numbers are those whose difference multiplied by the less produces 42, and by their sum, 133?

10. A and B lay out money on speculation. The amount of A's stock and gain is $27, and he gains as much per cent on his stock as B lays out. B's gain is $32; and it appears that A gains twice as much per cent as B. Required the capital of each.

11. I bought sugar at such a rate, that the price of a pound was to the number of pounds as 4 to 5. If the cost of the whole had been 45 cents more, the number of pounds would have been to the price of a pound as 4 to 5. How many pounds were bought, and what was the price per pound?

12. A and B engage in speculation. A disposes of his share for $11, and gains as many per cent as B invested dollars.

B's gain was \$ 36, and the gain upon A's investment was 4 times as many per cent as upon B's. How much did each invest?

13. A man bought 10 ducks and 12 turkeys for \$ 22.50. He bought 4 more ducks for \$ 6, than turkeys for \$ 5. What was the price of each?

14. A man purchased a farm in the form of a rectangle, whose length was 4 times its breadth. It cost $\frac{1}{4}$ as many dollars per acre as the field was rods in length, and the number of dollars paid for the farm was 4 times the number of rods round it. Required the price of the farm, and its length and breadth.

15. I have two cubic blocks of marble, whose united lengths are 20 inches, and contents 2240 cubic inches. Required the surface of each.

16. A's and B's shares in a speculation altogether amount to \$ 500. They sell out at *par*, A at the end of 2 years, B of 8, and each receives in capital and profits \$ 297. How much did each embark?

17. A person has \$ 1300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest, it would have produced \$ 36; and if the second portion had been loaned at the first rate of interest, it would have produced \$ 49. Required the rates of interest.

18. Two men, A and B, bought a farm of 104 acres, for which they paid \$ 320 each. On dividing the land, A says to B, "If you will let me have my portion in the situation which I shall choose, you shall have so much more land than I, that mine shall cost \$ 3 per acre more than yours." B accepted the proposal. How much land did each have, and what was the price of each per acre?

19. A and B start at the same time from two distant towns. At the end of 7 days, A is nearer to the half-way house than B is, by 5 miles more than A's day's journey. At the end of 10 days they have passed the half-way house, and are distant from each other 100 miles. Now it will take B 3 days longer to perform the whole journey than it will A. Required the distance of the towns, and the rate of walking of A and B.

20. Divide the number 4 into two such parts that the product of their squares shall be 9.

21. The fore-wheel of a carriage makes 15 revolutions more than the hind-wheel in going 180 yards; but if the circumference of each wheel were increased by 3 feet, the fore-wheel would only make 9 revolutions more than the hind-wheel in going the same distance. Find the circumference of each wheel.

22. A ladder, whose foot rests in a given position, just reaches a window on one side of a street, and when turned about its foot, just reaches a window on the other side. If the two positions of the ladder are at right angles to each other, and the heights of the windows are 36 and 27 feet respectively, find the width of the street and the length of the ladder.

23. A and B engaged to reap a field for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them the last two days, in consequence of which B received 3*s.* 9*d.* less than he otherwise would have done. In what time could B and C each reap the field?

24. Cloth, being wetted, shrinks $\frac{1}{8}$ in its length and $\frac{1}{16}$ in its width. If the surface of a piece of cloth is diminished by $5\frac{3}{4}$ square yards, and the length of the four sides by $4\frac{1}{4}$ yards, what was the length and width of the cloth originally?

XXIX.—THEORY OF QUADRATIC EQUATIONS.

323. *A quadratic equation cannot have more than two roots.*

We have seen (Art. 304) that every complete quadratic equation can be reduced to the form

$$x^2 + p\,x = q.$$

Suppose, if possible, that a quadratic equation can have three roots, and that r_1, r_2, and r_3 are the roots of the equation $x^2 + p\,x = q$. Then (Art. 166),

$$r_1^2 + p\,r_1 = q \qquad (1)$$

$$r_2^2 + p\,r_2 = q \qquad (2)$$

$$r_3^2 + p\,r_3 = q \qquad (3)$$

Subtracting (2) from (1), $(r_1^2 - r_2^2) + p\,(r_1 - r_2) = 0$

Dividing through by $r_1 - r_2$, which by supposition is not zero, as the roots are not equal,

$$r_1 + r_2 + p = 0$$

Similarly, by subtracting (3) from (1), we have

$$r_1 + r_3 + p = 0$$

Hence, $r_1 + r_2 + p = r_1 + r_3 + p$

or, $r_2 = r_3.$

That is, two of the roots are identical. Therefore, a quadratic equation cannot have more than two roots.

DISCUSSION OF THE GENERAL EQUATION.

324. By Art. 305, the roots of the equation $x^2 + p\,x = q$ are

$$\frac{-p + \sqrt{p^2 + 4\,q}}{2}, \text{ and } \frac{-p - \sqrt{p^2 + 4\,q}}{2}.$$

1. *Suppose q positive.*

Since p^2 is essentially positive (Art. 227), the quantity under the radical sign is positive and greater than p^2; so that the value of the radical is greater than p. Hence, one root is positive, and the other negative.

If p is positive, the negative root is numerically the larger; if p is zero, the roots are numerically equal; and if p is negative, the positive root is numerically the larger.

2. *Suppose q equal to zero.*

The quantity under the radical sign is now equal to p^2; so that the value of the radical is p. Hence, one of the roots is equal to 0. The other root is positive when p is negative, and negative when p is positive.

3. *Suppose q negative, and* $4q < p^2$.

The quantity under the radical sign is now positive and less than p^2; so that the value of the radical is less than p.

If p is positive, both roots are negative; and if p is negative, both roots are positive.

4. *Suppose q negative, and* $4q = p^2$.

The quantity under the radical sign is now equal to zero; so that the two roots are equal; being positive if p is negative, and negative if p is positive.

5. *Suppose q negative, and* $4q > p^2$.

The quantity under the radical sign is now negative; hence, by Art. 282, both roots are imaginary.

325. All these cases may be readily verified by examples.

Thus, in the equation $x^2 - 3x = 70$, as p is negative and q positive, we should expect to find one root positive and the other negative, and the positive root numerically the larger And this is actually the case, for on solving the equation, we find $x = 10$ or -7.

326. From the quadratic equation $x^2 + px = q$, denoting the roots by r_1 and r_2, we have

$$r_1 = \frac{-p + \sqrt{p^2 + 4q}}{2}, \text{ and } r_2 = \frac{-p - \sqrt{p^2 + 4q}}{2}.$$

Adding these together, we have

$$r_1 + r_2 = -\frac{2p}{2} = -p.$$

Multiplying them together, we have

$$r_1 r_2 = \frac{p^2 - (p^2 + 4q)}{4} \text{ (Art. 106)} = -\frac{4q}{4} = -q.$$

That is, *if a quadratic equation be reduced to the form* $x^2 + px = q$, *the algebraic sum of the roots is equal to the coefficient of the second term, with its sign changed; and the product of the roots is equal to the second member, with its sign changed.*

327. The equation $ax^2 + bx + c = 0$, by transposing c, and dividing each term by a, becomes

$$x^2 + \frac{bx}{a} = -\frac{c}{a}.$$

Denoting the roots of the equation by x_1 and x_2, we have, by the previous article,

$$x_1 + x_2 = -\frac{b}{a}, \text{ and } x_1 x_2 = \frac{c}{a}.$$

328. A **Quadratic Expression** is a trinomial expression of the form $ax^2 + bx + c$. The principles of the preceding article enable us to resolve any quadratic expression into two binomial factors.

The expression $ax^2 + bx + c$ may be written

$$a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right).$$

By the previous article, $\frac{b}{a} = -(x_1 + x_2)$, and $\frac{c}{a} = x_1 x_2$, where x_1 and x_2 are the roots of the equation $a x^2 + b x + c = 0$; which, we observe, may be obtained by placing the given expression equal to 0. Hence,

$$a x^2 + b x + c = a[x^2 - (x_1 + x_2) x + x_1 x_2].$$

The expression in the bracket may be written

$$x^2 - x x_1 - x x_2 + x_1 x_2,$$

which, by Case II, Chap. VIII, is equal to $(x - x_1)(x - x_2)$.

Therefore, $a x^2 + b x + c = a(x - x_1)(x - x_2)$.

1. Factor $6 x^2 + 11 x + 3$.

Placing the expression equal to 0, and solving the equation thus formed, we find

$$x = \frac{-11 \pm \sqrt{121 - 72}}{12} = \frac{-11 \pm 7}{12} = -\frac{3}{2}, \text{ or } -\frac{1}{3}.$$

Then, $\quad a = 6,\ x_1 = -\frac{3}{2},\ x_2 = -\frac{1}{3}.$

$$\text{Therefore, } 6 x^2 + 11 x + 3 = 6\left(x + \frac{3}{2}\right)\left(x + \frac{1}{3}\right)$$

$$= 2\left(x + \frac{3}{2}\right) 3\left(x + \frac{1}{3}\right)$$

$$= (2 x + 3)(3 x + 1), \textit{ Ans.}$$

2. Factor $4 + 13 x - 12 x^2$.

Placing the expression equal to 0, and solving the equation formed, we have

$$x = \frac{-13 \pm \sqrt{169 + 192}}{-24} = \frac{-13 \pm 19}{-24} = \frac{4}{3}, \text{ or } -\frac{1}{4}.$$

Then, $\quad a = -12,\ x_1 = \frac{4}{3},\ x_2 = -\frac{1}{4}.$

Therefore, $4 + 13x - 12x^2 = -12\left(x - \frac{4}{3}\right)\left(x + \frac{1}{4}\right)$

$$= -3\left(x - \frac{4}{3}\right) 4\left(x + \frac{1}{4}\right)$$

$$= (4 - 3x)(4x + 1), \textit{ Ans.}$$

Note. It should be remembered, in using the formula $a(x - x_1)(x - x_2)$, that a represents the coefficient of x^2 in the *given expression*; hence, in Example 2, we made $a = -12$.

EXAMPLES.

Factor the following expressions:

3. $x^2 + 73x + 780$.

4. $x^2 - 11x + 18$.

5. $x^2 - 4x - 60$.

6. $x^2 + 10x - 39$.

7. $2x^2 - 7x - 15$.

8. $21x^2 + 58x + 21$.

9. $8x^2 + 18x - 5$.

10. $4x^2 - 15x + 9$.

11. $2x^2 + x - 6$.

12. $9x^2 - 12x + 1$.

13. $1 - 8x - x^2$.

14. $49x^2 + 14x - 19$.

329. The principles of Art. 328 furnish a method of forming a quadratic equation which shall have any required roots.

For, the equation $ax^2 + bx + c = 0$, if its roots be denoted by x_1 and x_2, may be written, by Art. 328,

$$a(x - x_1)(x - x_2) = 0, \text{ or } (x - x_1)(x - x_2) = 0.$$

Hence, to form an equation whose roots shall be x_1 and x_2, *we subtract each of the two roots from x, and place the product of the resulting binomials equal to zero.*

1. Required the equation whose roots are 4 and $-\frac{7}{4}$.

By the rule, $(x - 4)\left(x + \frac{7}{4}\right) = 0$

or,
$$x^2 - \frac{9x}{4} - 7 = 0$$

Clearing of fractions,
$$4x^2 - 9x - 28 = 0, \textit{ Ans.}$$

EXAMPLES.

Form the equations whose roots are

2. 1 and -2.
3. 4 and 5.
4. 3 and $-\frac{3}{5}$.
5. 7 and $-6\frac{1}{3}$.
6. $-\frac{8}{3}$ and $\frac{4}{7}$.
7. $-2\frac{1}{3}$ and $-3\frac{1}{2}$.
8. $-\frac{17}{3}$ and 0.
9. $1 + \sqrt{5}$ and $1 - \sqrt{5}$.
10. $m + \sqrt{n}$ and $m - \sqrt{n}$.

330. By Art. 328, the equation $ax^2 + bx + c = 0$ may be written $(x - x_1)(x - x_2) = 0$, if x_1 and x_2 are its roots; we observe that the roots may be obtained by *placing the factors of the first member separately equal to zero, and solving the simple equations thus formed.*

This principle is often useful in solving equations.

1. Solve the equation $(2x - 3)(3x + 5) = 0$.

Placing the first factor equal to zero, $2x - 3 = 0$, or $x = \frac{3}{2}$.

Placing the second factor equal to zero, $3x + 5 = 0$, or $x = -\frac{5}{3}$.

Ans. $x = \frac{3}{2}$ or $-\frac{5}{3}$.

2. Solve the equation $x^2 + 5x = 0$.

The equation may be written $x(x + 5) = 0$

Placing the first factor equal to zero, $x = 0$.

Placing the second factor equal to zero, $x + 5 = 0$, or $x = -5$.

Ans. $x = 0$ or -5.

EXAMPLES.

Solve the following equations:

3. $\left(x-\frac{3}{4}\right)(x-2)=0.$

4. $(x+5)(x-1)=0.$

5. $\left(x-\frac{3}{5}\right)\left(x+\frac{2}{7}\right)=0.$

6. $(x+8)\left(x+\frac{1}{8}\right)=0.$

7. $2x^2-13x=0.$

8. $3x^3+12x^2=0.$

9. $2x^3-18x=0.$

10. $(2x+5)(3x-1)=0.$

11. $(ax+b)(cx-d)=0.$

12. $(x^2-4)(x^2-9)=0.$

13. $(3x+1)(4x^2-25)=0.$

14. $(x^2-a)(x^2-ax-b)=0.$

15. $x(2x+5)(3x-7)(4x+1)=0.$

16. $(x^2-5x+6)(x^2+7x+12)(2x^2+9x-5)=0.$

331. Many expressions may be factored by the artifice of completing the square, used in connection with the method of Case IV, Chapter VIII.

1. Factor x^4+a^4.

$$x^4+a^4=x^4+2x^2a^2+a^4-2x^2a^2$$
$$=(x^2+a^2)^2-(ax\sqrt{2})^2$$
$$=\text{(Art. 117)}\ (x^2+ax\sqrt{2}+a^2)(x^2-ax\sqrt{2}+a^2),\ \textit{Ans.}$$

2. Factor x^2-ax+a^2.

$$x^2-ax+a^2=x^2+2ax+a^2-3ax.$$
$$=(x+a)^2-\left(\sqrt{3ax}\right)^2$$
$$=(x+\sqrt{3ax}+a)(x-\sqrt{3ax}+a),\ \textit{Ans.}$$

EXAMPLES.

Factor the following expressions:

3. x^2+1.
4. x^2+x+1.
5. $a^2-3ab+b^2$.
6. $x^4-7x^2y^2+y^4$.
7. x^2-x-1.
8. $m^4+m^2n^2+n^4$.

332. We have seen (Art. 330) that any equation whose first member can be factored, and whose second member is zero, may be solved by placing the factors separately equal to zero and solving the equations thus formed. This method of solution is frequently the only one which will give all the roots of the equation.

1. Solve the equation $x^3=1$.

The equation may be written $x^3-1=0$, or (Art. 119),

$$(x-1)(x^2+x+1)=0.$$

Placing the first factor equal to zero,

$$x-1=0, \text{ or } x=1.$$

Placing the second factor equal to zero,

$$x^2+x+1=0, \text{ or } x^2+x=-1$$

Whence (Art. 309), $$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

Hence, $$x=1 \text{ or } \frac{-1\pm\sqrt{-3}}{2}, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

2. $x^4=-1$.
3. $x^3=-1$.
4. $x^4+a^4=0$.
5. $x^4-x^2+1=0$.
6. $x^6=1$.
7. $x^4-\frac{3x^2}{2}+1=0$.

These examples afford an illustration of the statement made in Art. 167 that the degree of an equation indicates the number of its roots.

XXX. — DISCUSSION OF PROBLEMS
LEADING TO QUADRATIC EQUATIONS.

333. In the discussion of problems leading to quadratic equations, we find involved the same general principles which have been established in connection with simple equations (Arts. 205 – 212), but with certain peculiarities.

These peculiarities will be now considered. They arise from two facts:

1. That every quadratic equation has *two roots;* and

2. That these roots are sometimes imaginary.

334. In the solution of problems involving quadratics, it has been observed that the positive root of the equation is usually the true answer; and that, when both roots are positive, there may be two answers, either of which conforms to the given conditions.

The reason why results are sometimes obtained which do not apply to the problem under consideration, and are therefore not admissible, is that the algebraic mode of expression is more general than ordinary language; and thus the equation which conforms properly to the conditions of the problem will also apply to other conditions.

1. Find a number such that twice its square added to three times the number may be 65.

Let $x =$ the number.

Then $$2x^2 + 3x = 65 \quad (1)$$

Whence, $$x = 5 \text{ or } -\frac{13}{2}.$$

The positive value alone gives a solution to the problem in the sense in which it is proposed.

To interpret the negative value, we observe that if we change x to $-x$, in equation (1), the term $3x$, only, changes its sign, giving as a result the equation $2x^2 - 3x = 65$. Solving this eqnation, we shall find $x = \frac{13}{2}$ or -5, which values only differ from the others in their signs. We therefore may consider the negative solution, $-\frac{13}{2}$, taken independently of its sign, the proper answer to the analogous problem (Art. 205):

"Find a number such that twice its square *diminished by* three times the number may be 65."

2. A farmer bought some sheep for \$72, and found that if he had bought 6 more for the same money, he would have paid \$1 less for each. How many sheep did he buy?

Let $x =$ the number of sheep bought.

Then $\frac{72}{x} =$ the price paid for one,

and $\frac{72}{x+6} =$ the price paid, if 6 more.

By the conditions, $$\frac{72}{x} = \frac{72}{x+6} + 1$$

Whence, $x = 18$ or -24.

Here the negative result is not admissible as a solution of the problem in its present form; the number of sheep, therefore, was 18.

If, in the given problem, "6 more" be changed to "6 *fewer*," and "\$1 less" to "\$1 *more*," 24 will be the true answer.

Hence, we infer that

A negative result, obtained as one of the answers to a problem, is sometimes the answer to another analogous problem, formed by attributing to the unknown quantity a quality directly opposite to that which has been attributed to it.

INTERPRETATION OF IMAGINARY RESULTS.

335. It has been shown (Art. 324) under what circumstances a quadratic equation will be in form to produce imaginary roots. It is now proposed to interpret such results.

Let it be required to divide 10 into two such parts that their product shall be 26.

Let $x =$ one of the parts.

Then $10 - x =$ the other.

By the conditions, $x(10 - x) = 26$

Whence, $x = 5 \pm \sqrt{-1}.$

Thus, we obtain an imaginary result. We therefore conclude that the problem cannot be solved numerically; in fact, if we call one of the parts $5 + y$, the other must be $5 - y$, and their product will be $25 - y^2$, which, so long as y is numerical, is less than 25. But we are required to find two *numbers* whose sum is 10 and product 26; there are, then, no such *numbers*.

Had it been required to find two *expressions*, whose sum is 10 and product 25, the answer $5 + \sqrt{-1}$ and $5 - \sqrt{-1}$ would have satisfied the conditions.

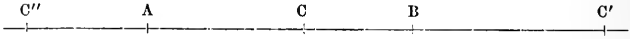
The given problem, however, expresses conditions incompatible with each other, and, consequently, is impossible. Hence,

Imaginary results indicate that the problem is impossible.

PROBLEM OF THE LIGHTS.

336. The principles of interpretation will be further illustrated in the discussion of the following general problem.

Find upon the line which joins two lights, A and B, the point which is equally illuminated by them; admitting that the intensity of a light, at a given distance, is equal to its intensity at the distance 1, divided by the square of the given distance.



Assume A as the origin of distances, and regard all distances estimated to the right as positive.

Let a denote the intensity of the light A, at the distance 1; b the intensity of the light B, at the distance 1; and c the distance A B, between the two lights.

Suppose C the point of equal illumination, and let x represent the distance from it to A, or the distance A C. Then, $c - x$ will represent the distance B C.

By the conditions of the problem, since the intensity of the light A, at the distance 1, is a, at the distance x it is $\frac{a}{x^2}$; and since the intensity of the light B, at the distance 1, is b, at the distance $c - x$ it is $\frac{b}{(c-x)^2}$. But, by supposition, at C these intensities are equal; hence,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}; \text{ or } \frac{(c-x)^2}{x^2} = \frac{b}{a}$$

Whence,
$$\frac{c-x}{x} = \pm \frac{\sqrt{b}}{\sqrt{a}}$$

From this equation we obtain as the values of x:

$$x = \frac{c\sqrt{a}}{\sqrt{a} + \sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}\right),$$

or,
$$\frac{c\sqrt{a}}{\sqrt{a} - \sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a} - \sqrt{b}}\right).$$

Since both a and b are positive, the two values of x are both real. Hence,

There are two points of equal illumination on the line of the lights.

Since there are *two* lights, c must always be greater than 0; consequently neither a, b, nor c can be 0. The problem, then, admits properly of only these three different suppositions:

1. $a > b$. 2. $a < b$. 3. $a = b$.

We shall now discuss the values of x under each of these suppositions.

1. $a > b$.

In this case, the first value of x is less than c; because $\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}$, being a proper fraction, is less than 1. This value of x is also greater than $\frac{c}{2}$; because, the denominator being less than twice the numerator, as b is less than a, the fraction is greater than $\frac{1}{2}$. Hence, the first point of equal illumination is at C, between the two lights, but nearer the lesser one.

The second value of x is greater than c; because $\frac{\sqrt{a}}{\sqrt{a} - \sqrt{b}}$, being an improper fraction, is greater than 1. Hence, the second point is at C′, in the prolongation of the line A B, beyond the lesser light.

These results agree with the supposition. For, if a is greater than b, then B evidently is the lesser light. Hence, both points of equal illumination will be nearer B than A; and since the two lights emit rays in all directions, one of the points must be in the prolongation of A B beyond both lights.

2. $a < b$.

In this case, the first value of x is positive. It is also less than $\frac{c}{2}$; because $\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}$, having the denominator greater than twice the numerator, b being greater than a, is less than $\frac{1}{2}$. Hence, the first point of equal illumination is between the lights, but nearer A, the lesser light.

The second value of x is negative, because the denominator $\sqrt{a} - \sqrt{b}$ is negative; which must be interpreted as measuring distance from A towards the left (Art. 205). Hence, the second point of equal illumination is at C″, in the prolongation of the line, at the left of the lesser light, A.

These results correspond with the supposition; the case being the same as the preceding one, except that A is now the lesser light.

3. $a = b$.

In this case, the first value of x is positive, and equal to $\frac{c}{2}$. Hence, the first point of equal illumination is midway between the two lights.

The second value of x is not finite; because $\frac{\sqrt{a}}{\sqrt{a} - \sqrt{b}}$, if $a = b$, reduces to $\frac{\sqrt{a}}{0} = \infty$ (Art. 210), which indicates that no finite value can be assigned to x. Hence, there is no second point of equal illumination in the line A B, or its prolongation.

These results agree with the supposition. For, since the lights are of equal intensity, a point of equal illumination will obviously be midway between them; and it is evident that there can be no other like point in their line.

The preceding discussion illustrates the precision with which algebraic processes will conform to every allowable interpretation of the enunciation of a problem.

XXXI.—RATIO AND PROPORTION.

337. The **Ratio** of one quantity to another of the same kind is the quotient arising from dividing the first quantity by the second (Art. 181).

Thus, the ratio of a to b is $\frac{a}{b}$, or $a : b$.

338. The **Terms** of a ratio are the two quantities required to form it. Of these, the first is called the *antecedent*, and the second the *consequent*.

Thus, in the ratio $a : b$, a and b are the terms, a the antecedent, and b the consequent.

339. A **Proportion** is an equality of ratios (Art. 181).

Thus, if the ratios $a : b$ and $c : d$ are equal, they form a proportion, which may be written

$$a : b = c : d, \text{ or } a : b :: c : d.$$

340. The **Terms** of a proportion are the four terms of its two ratios. The first and third terms are called the *antecedents;* the second and fourth, the *consequents;* the first and last, the *extremes;* the second and third, the *means;* and the terms of each ratio constitute a *couplet.*

Thus, in $a : b = c : d$, a and c are antecedents; b and d, consequents; a and d, extremes; b and c, means; a and b, the first couplet; and c and d, the second couplet.

341. A **Proportional** is any one of the terms of a proportion; a **Mean Proportional** between two quantities is either of the two means, when they are equal; a **Third Proportional** to two quantities is the fourth term of a proportion, in which the first term is the first of the quantities, and the second and third terms each equal to the second quantity; a **Fourth Proportional** to three quantities is the fourth term of a proportion whose first, second, and third terms are the three quantities taken in their order.

Thus, in $a : b = b : c$, b is a mean proportional between a and c; and c is a third proportional to a and b. In $a : b = c : d$, d is a fourth proportional to a, b, and c.

342. A **Continued Proportion** is one in which each consequent is the same as the next antecedent; as,

$$a : b = b : c = c : d = d : e.$$

PROPERTIES OF PROPORTIONS.

343. *When four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be

$$a : b = c : d.$$

This may be written (Art. 337),

$$\frac{a}{b} = \frac{c}{d}$$

Whence, $$a\,d = b\,c.$$

Hence, *if three quantities be in continued proportion, the product of the extremes is equal to the square of the means.*

Thus, if

$$a : b = b : c$$

then, $$a\,c = b^2.$$

By this theorem, if three terms of a proportion are given, the fourth may be found. Thus, if

$$a : b = c : x$$

then, $$a\,x = b\,c$$

Whence, $$x = \frac{b\,c}{a}.$$

344. *If the product of two quantities be equal to the product of two others, one pair of them may be made the extremes, and the other pair the means, of a proportion.*

Thus, if

$$a\,d = b\,c$$

Dividing by $b\,d$, $$\frac{a\,d}{b\,d} = \frac{b\,c}{b\,d}, \text{ or } \frac{a}{b} = \frac{c}{d}$$

Whence (Art. 337), $$a : b = c : d.$$

In a similar manner, we might derive from the equation $a\,d = b\,c$, the following proportions:

$$a : c = b : d,$$

$$b : d = a : c,$$

$$c : d = a : b,$$

$$d : b = c : a, \text{ etc.}$$

345. *If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the antecedents will have to each other the same ratio as the consequents.*

Thus, if $\qquad a : b = c : d$

then (Art. 343), $\qquad a\,d = b\,c$

Whence (Art. 344), $\qquad a : c = b : d.$

346. *If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second term will be to the first, as the fourth is to the third.*

Thus, if $\qquad a : b = c : d$

then, $\qquad a\,d = b\,c$

Whence, $\qquad b : a = d : c.$

347. *If four quantities are in proportion, they will be in proportion by* COMPOSITION; *that is, the sum of the first two terms will be to the first term, as the sum of the last two terms is to the third term.*

Thus, if $\qquad a : b = c : d$

then, $\qquad a\,d = b\,c$

Adding both members to $a\,c$,

$$a\,c + a\,d = a\,c + b\,c, \text{ or } a\,(c + d) = c\,(a + b)$$

Whence, $\qquad a + b : a = c + d : c$ (Art. 344).

Similarly, we may show that

$$a + b : b = c + d : d.$$

348. *If four quantities are in proportion, they will be in proportion by* Division; *that is, the difference of the first two terms will be to the first term, as the difference of the last two terms is to the third term.*

Thus, if $a : b = c : d$

then, $a d = b c$

Subtracting both members from $a c$,

$$a c - a d = a c - b c, \text{ or } a(c - d) = c(a - b)$$

Whence, $a - b : a = c - d : c.$

Similarly, we may prove that

$$a - b : b = c - d : d.$$

349. *If four quantities are in proportion, they will be in proportion by* Composition and Division; *that is, the sum of the first two terms will be to their difference, as the sum of the last two terms is to their difference.*

Thus, if $a : b = c : d$

by Art. 347, $$\frac{a+b}{a} = \frac{c+d}{c} \quad (1)$$

and, by Art. 348, $$\frac{a-b}{a} = \frac{c-d}{c} \quad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

or, $a + b : a - b = c + d : c - d.$

350. *Quantities which are proportional to the same quantities, are proportional to each other.*

Thus, if $a : b = e : f$

and $c : d = e : f$

then, $\frac{a}{b} = \frac{e}{f}$ and $\frac{c}{d} = \frac{e}{f}$

Therefore, $\frac{a}{b} = \frac{c}{d}$

Whence, $a : b = c : d.$

351. *If any number of quantities are proportional, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Thus, if

$$a : b = c : d = e : f$$

then (Art. 343), $a d = b c$

and $a f = b e$

also, $a b = a b$

Adding, $a(b + d + f) = b(a + c + e)$

Whence (Art. 344), $a : b = a + c + e : b + d + f.$

352. *If four quantities are in proportion, if the first and second be multiplied or divided by any quantity, as also the third and fourth, the resulting quantities will be in proportion.*

Thus, if

$$a : b = c : d$$

then, $\frac{a}{b} = \frac{c}{d}$

Therefore, $\frac{m a}{m b} = \frac{n c}{n d}$

Whence, $m a : m b = n c : n d.$

In a similar manner we could prove

$$\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}.$$

Either m or n may be made equal to unity. That is, either couplet may be multiplied or divided, without multiplying or dividing the other.

353. *If four quantities are in proportion, and the first and third be multiplied or divided by any quantity, as also the second and fourth, the resulting quantities will be in proportion.*

Thus, if $$a : b = c : d$$

then, $$\frac{a}{b} = \frac{c}{d}$$

Therefore, $$\frac{m\,a}{n\,b} = \frac{m\,c}{n\,d}$$

Whence, $$m\,a : n\,b = m\,c : n\,d.$$

In a similar manner we could prove

$$\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}.$$

Either m or n may be made equal to unity.

354. *If there be two sets of proportional quantities, the products of the corresponding terms will be in proportion.*

Thus, if $$a : b = c : d$$

and $$e : f = g : h$$

then, $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h}$$

Therefore, $$\frac{a\,e}{b\,f} = \frac{c\,g}{d\,h}$$

Whence, $$a\,e : b\,f = c\,g : d\,h.$$

355. *If four quantities are in proportion, like powers or like roots of these quantities will be in proportion.*

Thus, if $$a : b = c : d$$

then, $$\frac{a}{b} = \frac{c}{d};\ \text{therefore, } \frac{a^n}{b^n} = \frac{c^n}{d^n}$$

Whence, $$a^n : b^n = c^n : d^n.$$

In a similar manner we could prove

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

356. *If three quantities are in continued proportion, the first is to the third, as the square of the first is to the square of the second.*

Thus, if $a : b = b : c$

then, $$\frac{a}{b} = \frac{b}{c}$$

Multiplying by $\frac{a}{b}$, $$\frac{a^2}{b^2} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c}$$

Whence, $a : c = a^2 : b^2$.

In a similar manner we could prove that if

$$a : b = b : c = c : d, \text{ then } a : d = a^3 : b^3.$$

Note. The ratio $a^2 : b^2$ is called the *duplicate ratio*, and the ratio $a^3 : b^3$ the *triplicate ratio*, of $a : b$.

PROBLEMS.

357. 1. The last three terms of a proportion being 18, 6, and 27, what is the first term?

2. The first, third, and fourth terms of a proportion being 4, 20, and 55, respectively, what is the second term?

3. Find a fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

4. Find a third proportional to 5 and 3.

5. Find a mean proportional between 2 and 8.

6. Find a mean proportional between 6 and 24.

7. Find a mean proportional between 49 and 4.

8. Find two numbers in the ratio of $2\frac{1}{2}$ to 2, such that when each is diminished by 5, they shall be in the ratio of $1\frac{1}{3}$ to 1.

9. Divide 50 into two such parts that the greater increased by 3, shall be to the less diminished by 3, as 3 to 2.

10. Divide 27 into two such parts that their product shall be to the sum of their squares as 20 to 41.

11. There are two numbers which are to each other as 4 to 9, and 12 is a mean proportional between them. What are the numbers?

12. The sum of two numbers is to their difference as 10 to 3, and their product is 364. What are the numbers?

13. Find two numbers such that if 3 be added to each, they will be in the ratio of 4 to 3; and if 8 be subtracted from each, they will be in the ratio of 9 to 4.

14. There are two numbers whose product is 96, and the difference of their cubes is to the cube of their difference as 19 to 1. What are the numbers?

15. Each of two vessels contains a mixture of wine and water; a mixture consisting of equal measures from the two vessels, contains as much wine as water; and another mixture consisting of four measures from the first vessel and one from the second, is composed of wine and water in the ratio of 2 to 3. Find the ratio of wine to water in each vessel.

16. If the increase in the number of male and female criminals be 1.8 per cent, while the decrease in the number of males alone is 4.6 per cent, and the increase in the number of females alone is 9.8 per cent, compare the number of male and female criminals, respectively, at the first time mentioned.

17. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; whereas, if it had been moving in the same direction with him, it would have passed him in 30 seconds. Compare the rates of the two trains.

XXXII.—VARIATION.

358. Variation, or general proportion, is an abridged method of expressing common proportion.

Thus, if A and B be two sums of money loaned for equal times, at the same rate of interest, then

$$A : B = A\text{'s interest} : B\text{'s interest}$$

or, in an abridged form, by expressing only *two* terms, the interest varies as the principal; thus (Art. 23),

The interest $\propto$ the principal.

359. One quantity varies *directly* as another, when the two increase or decrease together in the same ratio.

Sometimes, for the sake of brevity, we say simply one quantity varies as another, omitting the word "directly."

Thus, if a man works for a certain sum per day, the amount of his wages varies as the number of days during which he works. For, as the number of days increases or decreases, the amount of his wages will increase or decrease, and in the *same ratio.*

360. One quantity varies inversely as another, when the first varies as the reciprocal of the second.

Thus, if a courier has a fixed route, the time in which he will pass over it varies inversely as his speed. That is, if he *double* his speed, he will go in *half* the time; and so on.

361. One quantity varies as two others *jointly*, when it has a constant ratio to the product of the other two.

Thus, the wages of a workman will vary as the number of days he has worked, and the wages per day, *jointly.*

362. One quantity varies directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, in physics, the attraction of a planetary body varies directly as the quantity of matter, and inversely as the square of the distance.

363. *If A varies as B, then A is equal to B multiplied by some constant quantity.*

Let a and b denote one pair of corresponding values of two quantities, and A and B any other pair. Then, by Art. 358,

$$A : a = B : b$$

Whence (Art. 343), $Ab = aB$, or $A = \frac{a}{b} B$

Denoting the constant ratio $\frac{a}{b}$ by m,

$$A = m B.$$

Hence, also, *when any quantity varies as another, if any two pairs of values of the quantities be taken, the four will be proportional.*

For, if $A \propto B$, and A' and B' be any pair of values of A and B, and A'' and B'' any other pair, by Art. 363,

$$A' = m B', \text{ and } A'' = m B''$$

Whence, $\frac{A'}{B'} = m$, and $\frac{A''}{B''} = m$

Therefore, $\frac{A'}{B'} = \frac{A''}{B''}$

or (Art. 337), $A' : B' = A'' : B''$.

364. The terms used in Variation may now be distinguished as follows:

1. *If* $A = m B$, *A varies directly as B.*

2. *If* $A = \frac{m}{B}$, *A varies inversely as B.*

3. *If* $A = m B C$, *A varies jointly as B and C.*

4. *If* $A = \frac{m B}{C}$, *A varies directly as B, and inversely as C.*

Problems in variation, in general, are readily solved by converting the variation into an equation, by the aid of Art. 364.

EXAMPLES.

365. **1.** Given that $y \propto x$, and when $x = 2$, $y = 10$. Required the value of y in terms of x.

If $y \propto x$, by Art. 364, $y = mx$

Substituting $x = 2$ and $y = 10$, $10 = 2m$, whence $m = 5$.

Hence, the required value is $y = 5x$.

2. Given that $y \propto x$, and that $y = 2$ when $x = 1$. What will be the value of y when $x = 2$?

3. If $y \propto z$, and $y = 24$ when $z = 3$, find the value of y in terms of z.

4. If x varies inversely as y, and $x = 4$ when $y = 2$, what is the value of x when $y = 6$?

5. Given that z varies jointly as x and y, and that $z = 1$ when $x = 1$ and $y = 1$. Find the value of z when $x = 2$ and $y = 2$.

6. If y equals the sum of two quantities, of which one is constant, and the other varies as xy; and when $x = 2$, $y = -2\frac{1}{3}$, but when $x = -2$, $y = 1$; express y in terms of x.

7. Two circular plates of gold, each an inch thick, the diameters of which are 6 inches and 8 inches, respectively, are melted and formed into a single circular plate one inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

8. Given that the illumination from a source of light varies inversely as the square of the distance; how much farther from a candle must a book, which is now 3 inches off, be removed, so as to receive just half as much light?

9. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of cars attached. With four cars its speed is 20 miles an hour. Find the greatest number of cars which the engine can move.

XXXIII. — ARITHMETICAL PROGRESSION.

366. An **Arithmetical Progression** is a series of quantities, in which each term is derived from the preceding term by adding a constant quantity, called the *common difference.*

367. When the series is *increasing*, as, for example,

$$1, 3, 5, 7, 9, 11, \ldots\ldots$$

each term is derived from the preceding term by adding a *positive* quantity; consequently the common difference is positive.

When the series is *decreasing*, as, for example,

$$19, 17, 15, 13, 11, 9, \ldots\ldots$$

each term is derived from the preceding term by adding a *negative* quantity; consequently the common difference is negative.

368. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression will be

$$a,\ a + d,\ a + 2\,d,\ a + 3\,d,\ \ldots\ldots$$

We observe that these terms differ only in the coefficient of d, which is 1 in the second term, 2 in the third term, 3 in the fourth term, etc. Consequently in the nth term, the coefficient of d will be $n - 1$. Hence, the nth term of the series, or the *last* term, as the number of the terms is n, will be

$$l = a + (n - 1)\,d \qquad (1)$$

369. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the series, S.*

$$S = a + (a + d) + (a + 2\,d) + \ldots\ldots + (l - 2\,d) + (l - d) + l$$

Writing the terms of the second member in the reverse order,

$$S = l + (l - d) + (l - 2\,d) + \ldots\ldots + (a + 2\,d) + (a + d) + a$$

Adding these equations, term by term, we have

$$2S=(a+l)+(a+l)+(a+l)+\ldots\ldots+(a+l)+(a+l)+(a+l)$$

In this result, $(a+l)$ is taken as many times as there are terms, or n times; hence

$$2S=n(a+l), \text{ or } S=\frac{n}{2}(a+l) \qquad (2)$$

Using the value of l given in (1), Art. 368, this may be written

$$S=\frac{n}{2}[2a+(n-1)d]$$

370. **1.** In the series 5, 8, 11, ... to 18 terms, find the last term and the sum of the series.

Here $a=5$, $n=18$; the common difference is always found by *subtracting the first term from the second;* hence $d=8-5=3$.

Substituting these values in (1) and (2), we have

$$l=5+(18-1)3=5+17\times 3=5+51=56.$$

$$S=\frac{18}{2}(5+56)=9\times 61=549.$$

2. In the series $2, -1, -4$, ... to 27 terms, find the last term and the sum of the series.

Here $a=2$, $n=27$, $d=$ the second term minus the first $=-1-2=-3$. Substituting these values in (1) and (2), we have

$$l=2+(27-1)(-3)=2+26(-3)=2-78=-76.$$

$$S=\frac{27}{2}(2-76)=\frac{27}{2}(-74)=27(-37)=-999.$$

EXAMPLES.

Find the last term and the sum of the series in the following:

3. 1, 6, 11, ... to 15 terms.

4. 7, 3, −1, ... to 20 terms.

5. −9, −6, −3, ... to 23 terms.

6. −5, −10, −15, ... to 29 terms.

7. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6},$ ...... to 16 terms.

8. $\frac{3}{5}, \frac{8}{15},$ ...... to 19 terms.

9. $\frac{1}{2}, \frac{5}{11},$ ...... to 22 terms.

10. $-\frac{2}{5}, \frac{1}{3},$ ...... to 14 terms.

11. $-3, -\frac{5}{2},$ ...... to 17 terms.

12. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4},$ ...... to 35 terms.

371. Formulæ (1) and (2) constitute two independent equations, together containing all the five elements of an arithmetical progression; hence, when any three of the five elements are given, we may readily deduce the values of the other two, as by substituting the three known values we shall have two equations with only two unknown quantities, which may be solved by methods previously given.

1. The first term of an arithmetical progression is 3, the number of terms 20, and the sum of the terms 440. Find the last term and the common difference.

Here $a = 3$, $n = 20$, $S = 440$; substituting in (1) and (2), we have

$$l = 3 + 19\,d$$

$$440 = 10\,(3 + l), \text{ or } 44 = 3 + l$$

From the second equation, $l = 41$; substitute in the first,

$$41 = 3 + 19\,d;\ 19\,d = 38;\ d = 2.$$

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting the given quantities in (1) and (2),

$$-39 = a + (n-1)(-3), \text{ or } 3n - a = 42$$

$$-264 = \frac{n}{2}(a - 39), \text{ or } an - 39n = -528$$

From the first of these equations, $a = 3n - 42$; substitute in the second,

$$(3n - 42)n - 39n = -528, \text{ or } n^2 - 27n = -176$$

$$\text{Whence, } n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11$$

Substituting in the equation $a = 3n - 42$,

When $n = 16$, $a = 6$

$n = 11$, $a = -9$, *Ans.*

The signification of the two answers is as follows:

If $n = 16$, and $a = 6$, the series will be

$6, 3, 0, -3, -6, -9, -12, -15, -18, -21, -24, -27,$
$-30, -33, -36, -39.$

If $n = 11$, and $a = -9$, the series will be

$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$

In either of which the last term is -39 and the sum -264.

3. Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$; find l and n.

Substituting the given quantities in (1) and (2), we have

$$l = \frac{1}{3} + (n-1)\left(-\frac{1}{12}\right), \text{ or } 12l + n = 5$$

$$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + l\right), \text{ or } n + 3ln = -9$$

From the first of these, $n=5-12\,l$; substitute in the second,

$$5-12\,l+3\,l(5-12\,l)=-9, \text{ or } 36\,l^2-3\,l=14$$

Whence, $l=\dfrac{3\pm\sqrt{9+2016}}{72}=\dfrac{3\pm 45}{72}=\dfrac{2}{3}$ or $-\dfrac{7}{12}$

Substituting in the equation $n=5-12\,l$,

When $\quad l=\dfrac{2}{3},\ n=-3$

$l=-\dfrac{7}{12},\ n=12$, *Ans.*

The first answer is inapplicable, as a negative number of terms has no meaning. Hence the only answer is,

$$l=-\frac{7}{12},\quad n=12.$$

Note. A negative or fractional value of n is always inapplicable, and should be neglected, together with all other values dependent on it.

EXAMPLES.

4. Given $d=4$, $l=75$, $n=19$; find a and S.

5. Given $d=-1$, $n=15$, $S=-\dfrac{165}{2}$; find a and l.

6. Given $a=-\dfrac{2}{3}$, $n=18$, $l=5$; find d and S.

7. Given $a=-\dfrac{3}{4}$, $n=7$, $S=-7$; find d and l.

8. Given $l=-31$, $n=13$, $S=-169$; find a and d.

9. Given $a=3$, $l=42\frac{2}{3}$, $d=2\frac{1}{3}$; find n and S.

10. Given $a=7$, $l=-73$, $S=-363$; find n and d.

11. Given $a=\dfrac{15}{2}$, $d=\dfrac{5}{2}$, $S=\dfrac{2625}{2}$; find n and l.

12. Given $l = -47$, $d = -1$, $S = -1118$; find a and n.

13. Given $d = -3$, $S = -328$, $a = 2$; find l and n.

372. *To insert any number of arithmetical means between two given terms.*

1. Insert 5 arithmetical means between 3 and -5.

This may be performed in the same manner as the examples in the previous article; we have given the first term $a = 3$; the last term $l = -5$; the number of terms $n = 7$; as there are 5 means and two extremes, or in all 7 terms. Substituting in (1), Art. 368, we have

$$-5 = 3 + 6d; \text{ or, } 6d = -8; \text{ whence, } d = -\frac{4}{3}.$$

Hence the terms are obtained by subtracting $\frac{4}{3}$ from 3 for the first, $\frac{4}{3}$ from that result for the second, and so on; or,

$$3, \frac{5}{3}, \frac{1}{3}, -1, -\frac{7}{3}, -\frac{11}{3}, -5, \textit{Ans.}$$

EXAMPLES.

2. Insert 5 arithmetical means between 2 and 4.

3. Insert 7 arithmetical means between 3 and -1.

4. Insert 4 arithmetical means between -1 and -6.

5. Insert 6 arithmetical means between -8 and -4.

6. Insert 4 arithmetical means between -2 and 6.

7. Insert m arithmetical means between a and b.

PROBLEMS.

373. **1.** The 6th term of an arithmetical progression is 19, and the 14th term is 67. Find the first term.

By Art. 368, the 6th term is $a + 5d$, and the 14th term is $a + 13d$. Hence,

$$a + 5d = 19$$
$$a + 13d = 67$$

Whence, $8d = 48$, or $d = 6$

Therefore, $a = -11$, *Ans.*

2. Find four quantities in arithmetical progression, such that the product of the extremes shall be 45, and the product of the means 77.

Let a, $a + d$, $a + 2d$, and $a + 3d$ be the quantities.

Then, by the conditions,
$$a^2 + 3ad = 45 \qquad (1)$$
$$a^2 + 3ad + 2d^2 = 77 \qquad (2)$$

Subtracting (1) from (2),
$$2d^2 = 32$$
$$d^2 = 16$$
$$d = \pm 4.$$

If $d = 4$, substituting in (1), we have

$$a^2 + 12a = 45$$

Whence, $a = \dfrac{-12 \pm \sqrt{144 + 180}}{2} = \dfrac{-12 \pm 18}{2} = 3$ or -15.

This indicates two answers,

3, 7, 11, and 15, or, -15, -11, -7, and -3.

If $d = -4$, substituting in (1), we have

$$a^2 - 12a = 45$$

Whence, $a = \dfrac{12 \pm \sqrt{144 + 180}}{2} = \dfrac{12 \pm 18}{2} = 15$ or -3.

This also indicates two answers,

15, 11, 7, and 3, or, -3, -7, -11, and -15.

But these two answers are the same as those obtained with the other value of d. Hence, the two answers to the problem are

3, 7, 11, and 15, or, -3, -7, -11, and -15.

3. Find the sum of the odd numbers from 1 to 100.

4. A debt can be discharged in a year by paying \$1 the first week, \$3 the second, \$5 the third, and so on. Required the last payment, and the amount of the debt.

5. A person saves \$270 the first year, \$210 the second, and so on. In how many years will a person who saves every year \$180 have saved as much as he?

6. Two persons start together. One travels ten leagues a day, the other eight leagues the first day, which he augments daily by half a league. After how many days, and at what distance from the point of departure, will they come together?

7. Find four numbers in arithmetical progression, such that the sum of the first and third shall be 22, and the sum of the second and fourth 36.

8. The 7th term of an arithmetical progression is 27; and the 13th term is 51. Find the first term.

9. A gentleman set out from Boston to New York. He travelled 25 miles the first day, 20 miles the second day, each day travelling 5 miles less than on the preceding. How far was he from Boston at the end of the eleventh day?

10. If a man travel 20 miles the first day, 15 miles the second, and so continue to travel 5 miles less each day, how far will he have advanced on his journey at the end of the 8th day?

11. The sum of the squares of the extremes of four quantities in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the quantities?

12. After A had travelled for $2\frac{3}{4}$ hours, at the rate of 4 miles an hour, B set out to overtake him, and went $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, 5 the third, and so on, increasing his speed a quarter of a mile every hour. In how many hours would he overtake A?

XXXIV. — GEOMETRICAL PROGRESSION.

374. A **Geometrical Progression** is a series in which each term is derived from the preceding term by multiplying by a constant quantity, called the *ratio.*

375. When the series is *increasing,* as, for example,

$$2, 6, 18, 54, 162, \ldots\ldots$$

each term is derived from the preceding term by multiplying by a quantity *greater than* 1; consequently the ratio is a quantity greater than 1.

When the series is *decreasing,* as, for example,

$$9, 3, 1, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, \ldots\ldots$$

each term is derived from the preceding term by multiplying by a quantity *less than* 1; consequently the ratio is a quantity less than 1.

Negative values of the ratio are admissible; for example,

$$-3, 6, -12, 24, -48, \ldots\ldots$$

is a progression in which the ratio is -2.

376. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression will be

$$a, ar, ar^2, ar^3, \ldots\ldots$$

We observe that the terms differ only in the exponent of r, which is 1 in the second term, 2 in the third term, 3 in the fourth term, etc. Consequently in the nth or last term, the exponent of r will be $n-1$, or

$$l = a\,r^{n-1} \qquad (1)$$

377. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the series, S.*

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-3} + ar^{n-2} + ar^{n-1}$$

Multiplying each term by r,

$$rS = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$$

Subtracting the first equation from the second, we have

$$rS - S = ar^n - a, \text{ or } S(r-1) = ar^n - a, \text{ or } S = \frac{ar^n - a}{r-1}$$

But from (1), Art. 376, by multiplying each term by r,

$$rl = ar^n$$

Substituting this value of ar^n in the value of S, we have

$$S = \frac{rl - a}{r - 1} \qquad (2)$$

378. **1.** In the series 2, 4, 8, ... to 11 terms, find the last term and the sum of the series.

Here $a = 2$, $n = 11$; the ratio is always found by *dividing the second term by the first*; hence, $r = \frac{4}{2} = 2$.

Substituting these values in (1) and (2), we have

$$l = 2\,(2)^{11-1} = 2 \times 2^{10} = 2 \times 1024 = 2048.$$

$$S = \frac{(2 \times 2048) - 2}{2 - 1} = 4096 - 2 = 4094.$$

2. In the series 3, 1, $\frac{1}{3}$, ... to 7 terms, find the last term and the sum of the series.

Here $a = 3$, $n = 7$, $r =$ second term divided by first term $= \frac{1}{3}$.

Substituting these values in (1) and (2), we have

$$l = 3\left(\frac{1}{3}\right)^{7-1} = 3\left(\frac{1}{3}\right)^6 = \frac{3}{3^6} = \frac{1}{3^5} = \frac{1}{243}.$$

$$S=\frac{\left(\frac{1}{3}\times\frac{1}{243}\right)-3}{\frac{1}{3}-1}=\frac{\frac{1}{729}-3}{\frac{1}{3}-1}=\frac{-\frac{2186}{729}}{-\frac{2}{3}}=\frac{2186}{729}\times\frac{3}{2}=\frac{1093}{243}.$$

3. In the series $-2, 6, -18, \ldots\ldots$ to 8 terms, find the last term and the sum of the series.

Here $a=-2$, $n=8$, $r=\frac{6}{-2}=-3$. Hence,

$$l=(-2)(-3)^{8-1}=(-2)(-3)^7=(-2)(-2187)=4374.$$

$$S=\frac{(-3\times 4374)-(-2)}{(-3)-1}=\frac{-13122+2}{-4}=\frac{-13120}{-4}=3280.$$

EXAMPLES.

Find the last term and the sum of the series in the following:

4. $1, 2, 4, \ldots\ldots$ to 12 terms.

5. $3, 2, \frac{4}{3}, \ldots\ldots$ to 7 terms.

6. $-2, 8, -32, \ldots\ldots$ to 6 terms.

7. $2, -1, \frac{1}{2}, \ldots\ldots$ to 10 terms.

8. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots$ to 11 terms.

9. $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots$ to 8 terms.

10. $8, 4, 2, \ldots\ldots$ to 9 terms.

11. $\frac{3}{4}, -\frac{1}{4}, \frac{1}{12}, \ldots\ldots$ to 6 terms.

12. $-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{6}, \ldots\ldots$ to 10 terms.

13. $3, -6, 12, \ldots\ldots$ to 7 terms.

379. Formulæ (1) and (2) together contain all the five elements of a geometrical progression; hence, if any three of the five are given, we may find the other two, exactly as in arithmetical progression. But in certain cases the operation involves the solution of an equation of a higher degree than the second, for which rules have not been given; and in other cases the unknown quantity appears as an exponent, the solution of which equations can usually only be effected by the use of logarithms; although in certain simple cases they may be solved by inspection.

1. Given $l = 6561$, $r = 3$, $n = 9$; find a and S.

Substituting these values in (1) and (2), Arts. 376 and 377, we have

$$6561 = a\,(3)^8;\ \text{or}\ 6561 = 6561\,a;\ \text{or}\ a = 1.$$

$$S = \frac{(3 \times 6561) - 1}{3 - 1} = \frac{19683 - 1}{2} = \frac{19682}{2} = 9841.$$

2. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting these values in (1) and (2), we have

$$-32 = (-2)\,(r)^{5-1};\ \text{or}\ -32 = -2\,r^4;\ r^4 = 16;\ r = \pm 2.$$

If $r = 2$, $S = \dfrac{(2 \times -32) - (-2)}{2 - 1} = -64 + 2 = -62.$

If $r = -2$, $S = \dfrac{(-2 \times -32) - (-2)}{-2 - 1} = \dfrac{64 + 2}{-3} = \dfrac{66}{-3} = -22.$

The signification of the two answers is as follows:

If $r = 2$, the series will be -2, -4, -8, -16, -32, in which the sum is -62.

If $r = -2$, the series will be -2, 4, -8, 16, -32, in which the sum is -22.

3. Given $a = 3$, $r = -\dfrac{1}{3}$, $S = \dfrac{1640}{729}$; find n and l.

Substituting these values in (1) and (2), we have

$$l=3\left(-\frac{1}{3}\right)^{n-1};\text{ or }\frac{1}{(-3)^{n-1}}=\frac{l}{3};\text{ or }(-3)^{n-1}=\frac{3}{l}.$$

$$\frac{1640}{729}=\frac{-\frac{1}{3}l-3}{-\frac{1}{3}-1}=\frac{l+9}{4};\text{ or }l+9=\frac{6560}{729};\text{ or }l=-\frac{1}{729}.$$

Substituting this value of l in the equation $(-3)^{n-1}=\frac{3}{l}$, we have $(-3)^{n-1}=-\frac{3}{\frac{1}{729}}=-2187$; whence, by inspection, $n-1=7$, or $n=8$.

EXAMPLES.

4. Given $l=-256$, $r=-2$, $n=10$; find a and S.

5. Given $r=\frac{1}{3}$, $n=8$, $S=\frac{6560}{6561}$; find a and l.

6. Given $a=2$, $n=7$, $l=1458$; find r and S.

7. Given $a=3$, $n=6$, $l=-\frac{3}{1024}$; find r and S.

8. Given $a=1$, $r=3$, $l=81$; find n and S.

9. Given $a=2$, $l=\frac{1}{32}$, $S=\frac{127}{32}$; find n and r.

10. Given $a=\frac{1}{2}$, $r=-3$, $S=-91$; find n and l.

11. Given $l=-128$, $r=2$, $S=-255$; find n and a.

380. The limit to which the sum of the terms of a *decreasing* geometrical progression approaches, as the number of terms becomes larger and larger, is called the *sum of the series to infinity*. We may write the value of S obtained in Art. 377 as follows:

$$S = \frac{a - rl}{1 - r}$$

In a decreasing geometrical progression, the larger the number of terms taken the smaller will be the value of the last term; hence, by taking terms enough, the last term may be made as small as we please. Then (Art. 207), the limiting value of l is 0. Consequently the limit to which the value of S approaches, as the number of terms becomes larger and larger, is $\frac{a}{1-r}$.

Therefore the sum of a decreasing geometrical progression to infinity is given by the formula

$$S = \frac{a}{1 - r}. \qquad (3)$$

1. Find the sum of the series $3, 1, \frac{1}{3}, \ldots\ldots$ to infinity.

Here $a = 3$, $r = \frac{1}{3}$; substituting in (3), we have

$$S = \frac{3}{1 - \frac{1}{3}} = \frac{9}{3 - 1} = \frac{9}{2}, \text{ Ans.}$$

2. Find the sum of the series $4, -\frac{8}{3}, \frac{16}{9}, \ldots\ldots$ to infinity.

Here $a = 4$, $r = \frac{-\frac{8}{3}}{4} = -\frac{2}{3}$; substituting in (3), we have

$$S = \frac{4}{1 + \frac{2}{3}} = \frac{12}{3 + 2} = \frac{12}{5}, \text{ Ans.}$$

EXAMPLES.

Find the sum of the following to infinity:

3. $2, 1, \frac{1}{2}, \ldots\ldots$

4. $4, -2, 1, \ldots\ldots$

5. $-1, \frac{1}{3}, -\frac{1}{9}, \ldots\ldots$

6. $-3, -\frac{3}{5}, -\frac{3}{25}, \ldots\ldots$

7. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$

8. $\frac{2}{5}, -\frac{6}{35}, \frac{18}{245}, \ldots\ldots$

9. $8, \frac{2}{5}, \frac{1}{50}, \ldots\ldots$

10. $-4, \frac{4}{5}, -\frac{4}{25}, \ldots\ldots$

381. *To find the value of a repeating decimal.*

This is a case of finding the sum of a geometrical progression to infinity, and may be solved by the formula of Art. 380.

1. Find the value of .363636……

$$.363636\ldots\ldots = .36 + .0036 + .000036 + \ldots\ldots$$

Here $a = .36$, and $r = \frac{.0036}{.36} = .01$; substituting in (3),

$$S = \frac{.36}{1 - .01} = \frac{.36}{.99} = \frac{36}{99} = \frac{4}{11}, \textit{ Ans.}$$

2. Find the value of .285151 ……

$$.285151\ldots\ldots = .28 + .0051 + .000051 + \ldots\ldots$$

To find the sum of all the terms except the first, we have $a = .0051$, $r = .01$; substituting in (3),

$$S = \frac{.0051}{1 - .01} = \frac{.0051}{.99} = \frac{51}{9900} = \frac{17}{3300}.$$

Adding the first term to this, the value of the given decimal

$$= \frac{28}{100} + \frac{17}{3300} = \frac{941}{3300}, \textit{ Ans.}$$

EXAMPLES.

Find the values of the following:

3. .074074……
5. .7333……
7. .113333……
4. .481481……
6. .52121……
8. .215454……

382. *To insert any number of geometrical means between two given terms.*

1. Insert 4 geometrical means between 2 and $\frac{64}{243}$.

This may be performed in the same manner as the examples in Art. 379. We have $a=2$, $l=\frac{64}{243}$, and $n=6$, or two more than the number of means.

Substituting these values in (1), Art. 376, we have

$$\frac{64}{243}=2r^5; \text{ or } r^5=\frac{32}{243}; \text{ or } r=\frac{2}{3}.$$

Hence the terms are obtained by multiplying 2 by $\frac{2}{3}$ for the first, that result by $\frac{2}{3}$ for the second, and so on; or,

$$2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}, \textit{ Ans.}$$

2. Insert 5 geometrical means between -2 and -128.

Here $a=-2$, $l=-128$, $n=7$. Substituting in (1), Art. 376, we have

$$-128=-2r^6; \text{ or } r^6=64; \text{ whence, } r=\pm 2.$$

If $r=2$, the series will be

$$-2, -4, -8, -16, -32, -64, -128.$$

If $r=-2$, the series will be

$$-2, 4, -8, 16, -32, 64, -128.$$

EXAMPLES.

3. Insert 6 geometrical means between 3 and $\frac{128}{729}$.

4. Insert 5 geometrical means between $\frac{1}{2}$ and $364\frac{1}{2}$.

5. Insert 6 geometrical means between -2 and -4374.

6. Insert 4 geometrical means between 3 and $-\frac{729}{1024}$.

7. Insert 7 geometrical means between $\frac{3}{2}$ and $\frac{3}{512}$.

PROBLEMS.

383. **1.** What is the first term of a geometrical progression, when the 5th term is 48, and the 8th term is -384?

By Art. 376, the 5th term is ar^4, and the 8th term is ar^7. Hence,

$$ar^4 = 48, \text{ and } ar^7 = -384.$$

Dividing the second of these equations by the first,

$$r^3 = -8; \text{ whence, } r = -2.$$

Then, $$a = \frac{48}{r^4} = \frac{48}{16} = 3, \textit{ Ans.}$$

2. Find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let a, ar, and ar^2 denote the numbers. Then, by the conditions,

$$a + ar + ar^2 = 14 \quad (1)$$

$$a^2 + a^2r^2 + a^2r^4 = 84 \quad (2)$$

Dividing (2) by (1), $$a - ar + ar^2 = 6 \quad (3)$$

Adding (1) and (3), $$a + ar^2 = 10 \quad (4)$$

Subtracting (3) from (1), $$ar = 4, \text{ or } r = \frac{4}{a} \quad (5)$$

Substituting from (5) in (4), $$a + \frac{16}{a} = 10$$

or, $$a^2 - 10a = -16$$

Whence (Art. 309), $$a = \frac{10 \pm \sqrt{100 - 64}}{2} = \frac{10 \pm 6}{2} = 8 \text{ or } 2.$$

If $a = 8$, $r = \frac{4}{8} = \frac{1}{2}$, and the numbers are 8, 4, and 2.

If $a = 2$, $r = \frac{4}{2} = 2$, and the numbers are 2, 4, and 8.

Therefore, the numbers are 2, 4, and 8, *Ans.*

3. A person who saved every year half as much again as he saved the previous year, had in seven years saved $2059. How much did he save the first year?

4. A gentleman boarded 9 days, paying 3 cents for the first day, 9 cents for the second day, 27 cents for the third day, and so on. Required the cost.

5. Suppose the elastic power of a ball that falls from a height of 100 feet, to be such as to cause it to rise 0.9375 of the height from which it fell, and to continue in this way diminishing the height to which it will rise, in geometrical progression, till it comes to rest. How far will it have moved?

6. The sum of the first and second of four quantities in geometrical progression is 15, and the sum of the third and fourth is 60. Required the quantities.

7. The fifth term of a geometrical progression is -324, and the 9th term is -26244. What is the first term?

8. The third term of a geometrical progression is $\frac{1}{24}$, and the sixth term is $\frac{9}{512}$. What is the second term?

XXXV. — HARMONICAL PROGRESSION.

384. Quantities are said to be in **Harmonical Progression** when their reciprocals form an arithmetical progression.

For example, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$

are in harmonical progression, because their reciprocals,

$$1,\ 3,\ 5,\ 7,\ \ldots\ldots$$

form an arithmetical progression.

385. From the preceding it follows that all problems in harmonical progression, which are susceptible of solution, may be solved by inverting the terms and applying the rules of the arithmetical progression. There will be found, however, no general expression for the sum of a harmonical series.

386. *To find the last term of a given harmonical series.*

1. In the series $2, \frac{2}{3}, \frac{2}{5}, \ldots\ldots$ to 36 terms, find the last term.

Inverting the series, we have the arithmetical progression

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\ldots \text{ to 36 terms.}$$

Here $a = \frac{1}{2}$, $d = 1$, $n = 36$; hence, by (1), Art. 368,

$$l = \frac{1}{2} + (36 - 1)\, 1 = \frac{1}{2} + 35 = \frac{71}{2}.$$

Inverting this, we obtain $\frac{2}{71}$ as the last term of the given series.

EXAMPLES.

Find the last terms of the following:

2. $\frac{5}{3}, \frac{3}{2}, \ldots\ldots$ to 23 terms.

3. $\frac{1}{2}, -\frac{1}{3}, -\frac{1}{8}, \ldots\ldots$ to 17 terms.

4. $\frac{4}{3}, \frac{3}{2}, \frac{12}{7}, \ldots\ldots$ to 26 terms.

5. $a, b, \ldots\ldots$ to n terms.

387. *To insert any number of harmonical means between two given terms.*

1. Insert 5 harmonical means between 2 and -3.

Inverting, we have to insert 5 arithmetical means between $\frac{1}{2}$ and $-\frac{1}{3}$.

Here $a=\frac{1}{2}$, $l=-\frac{1}{3}$, $n=7$; substituting in (1), Art. 368, we have

$$-\frac{1}{3}=\frac{1}{2}+6\,d\,;\ \text{or}\ 6\,d=-\frac{5}{6};\ \text{whence,}\ d=-\frac{5}{36}.$$

Hence, the arithmetical means are

$$\frac{13}{36},\ \frac{2}{9},\ \frac{1}{12},\ -\frac{1}{18},\ -\frac{7}{36}.$$

Then, the harmonical means will be

$$\frac{36}{13},\ \frac{9}{2},\ 12,\ -18,\ -\frac{36}{7},\ \textit{Ans.}$$

EXAMPLES.

2. Insert 7 harmonical means between $\frac{2}{5}$ and $\frac{3}{10}$.

3. Insert 3 harmonical means between -1 and -5.

4. Insert 6 harmonical means between 3 and -1.

5. Insert m harmonical means between a and b.

388. *If three consecutive terms of a harmonical progression be taken, the first has the same ratio to the third, that the first minus the second has to the second minus the third.*

Let a, b, c be in harmonical progression; then their reciprocals $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ will be in arithmetical progression. Hence,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}.$$

Clearing of fractions, $ab-ac=ac-bc$

or, $a\,(b-c)=c\,(a-b)$

Dividing through by $c\,(b-c)$, we have

$$\frac{a}{c}=\frac{a-b}{b-c},$$

which was to be proved.

389. Let a and c be any two quantities; b their harmonical mean. Then, by the previous theorem, $\frac{a}{c}=\frac{a-b}{b-c}$.

Clearing of fractions, $ab-ac=ac-bc$; then, $ab+bc=2ac$

or,
$$b=\frac{2ac}{a+c}.$$

390. We may note the following results: if a and c are any two quantities, their arithmetical mean $=\frac{a+c}{2}$; their geometrical mean $=\sqrt{ac}$; and their harmonical mean $=\frac{2ac}{a+c}$.

Since $\frac{2ac}{a+c}\times\frac{a+c}{2}=(\sqrt{ac})^2$, it follows that the product of the harmonical and arithmetical means of two quantities is equal to the square of their geometrical mean.

Consequently the geometrical mean must be intermediate in value between the harmonical and the arithmetical mean. But the harmonical mean is less than the arithmetical mean, because $\frac{a+c}{2}-\frac{2ac}{a+c}=\frac{(a+c)^2-4ac}{2(a+c)}=\frac{a^2+2ac+c^2-4ac}{2(a+c)}$
$=\frac{a^2-2ac+c^2}{2(a+c)}=\frac{(a-c)^2}{2(a+c)}$, a positive quantity.

Hence of the three quantities, the arithmetical mean is the greatest, the geometrical mean next, and the harmonical mean the least.

XXXVI. — PERMUTATIONS AND COMBINATIONS.

391. The different orders in which quantities can be arranged are called their **Permutations.**

Thus, the *permutations* of the quantities a, b, c, taken *two* at a time, are

$$a\,b,\ b\,a;\ a\,c,\ c\,a;\ b\,c,\ c\,b;$$

and taken *three* at a time, are

$$a\,b\,c,\ a\,c\,b;\ b\,a\,c,\ b\,c\,a;\ c\,a\,b,\ c\,b\,a.$$

392. The **Combinations** of quantities are the different collections that can be formed out of them, without regard to the order in which they are placed.

Thus, the *combinations* of the quantities a, b, c, taken two at a time, are

$$a\,b,\ a\,c,\ b\,c;$$

$a\,b$, and $b\,a$, though different permutations, forming the same combination.

393. *To find the number of permutations of n quantities, taken r at a time.*

Let P denote the number of permutations of n quantities, taken r at a time. By placing before each of these the other $n-r$ quantities one at a time, we shall evidently form $P\,(n-r)$ permutations of the n quantities, taken $r+1$ at a time. That is, the number of permutations of n quantities, taken r at a time, multiplied by $n-r$, gives the number of permutations of the n quantities, taken $r+1$ at a time.

But the number of permutations of n quantities, taken *one* at a time, is obviously n. Hence, the number of permutations taken *two* at a time, is the number taken *one* at a time, multiplied by $n-1$, or $n\,(n-1)$. The number of permutations, taken *three* at a time, is the number taken *two* at a time, multiplied by $n-2$, or $n\,(n-1)\,(n-2)$; and so on. We observe that the last factor in the number of permutations is n, minus a number 1 less than the number of quantities taken at a time. Hence, the number of permutations of n quantities, taken r at a time, is

$$n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-(r-1))$$

or, $$n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-r+1). \qquad (1)$$

394. If all the quantities are taken together, $r = n$ and Formula (1) becomes

$$n(n-1)(n-2) \ldots\ldots 1;$$

or, by inverting the order of the factors,

$$1 \times 2 \times 3 \ldots\ldots (n-1)\, n. \qquad (2)$$

That is,

The number of permutations of n quantities, taken n at a time, is equal to the product of the natural numbers from 1 up to n.

For the sake of brevity, this result is often denoted by $\underline{|n}$, read "*factorial n*"; thus $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive.

395. *To find the number of combinations of n quantities, taken r at a time.*

The number of permutations of n quantities, taken r at a time, is (Art. 393),

$$n(n-1)(n-2) \ldots\ldots (n-r+1).$$

By Art. 394, each combination of r quantities produces $\underline{|r}$ permutations. Hence, the number of combinations must equal the number of permutations divided by $\underline{|r}$, or

$$\frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{\underline{|r}}. \qquad (3)$$

396. *The number of combinations of n quantities, taken r at a time is the same as the number of combinations of n quantities taken n − r at a time.*

For, it is evident that for every combination of r quantities which we take out of n quantities, we leave one combination of $n - r$ quantities, which contains the remaining quantities.

EXAMPLES.

397. 1. How many changes can be rung with 10 bells, taking 7 at a time?

Here, $n=10$, $r=7$; then $n-r+1=4$.

Then, by Formula (1),

$$10\times 9\times 8\times 7\times 6\times 5\times 4=604800, \textit{ Ans.}$$

2. How many different combinations can be made with 5 letters out of 8?

Here, $n=8$, $r=5$; then $n-r+1=4$.

Then, by Formula (3),

$$\frac{8\times 7\times 6\times 5\times 4}{1\times 2\times 3\times 4\times 5}=56, \textit{ Ans.}$$

3. In how many different orders may 7 persons be seated at table?

Here $n=7$; then, by Formula (2),

$$1\times 2\times 3\times 4\times 5\times 6\times 7=5040, \textit{ Ans.}$$

4. How many different words of 4 letters each can be made with 6 letters? How many words of 3 letters each? How many of 6 letters each? How many in all possible ways?

5. How often can 4 students change their places in a class of 8, so as not to preserve the same order?

6. From a company of 40 soldiers, how many different pickets of 6 men can be taken?

7. How many permutations can be formed of the 26 letters of the alphabet, taken 4 at a time?

8. How many different numbers can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, taking 5 at a time, each digit occurring not more than once in any number?

9. How many different permutations may be formed of the letters in the word *since*, taken all together?

10. How many different combinations may be formed of the letters in the word *forming*, taken three at a time?

11. How many different combinations may be formed of 20 letters, taken 5 at a time?

12. How many different combinations may be formed of 18 letters, taken 11 at a time?

13. How many different committees, consisting of 7 persons each, can be formed out of a corporation of 20 persons?

14. How many different numbers, of three different figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0?

XXXVII. — BINOMIAL THEOREM.

POSITIVE INTEGRAL EXPONENT.

398. The **Binomial Theorem,** discovered by Newton, is a formula, by means of which any binomial may be raised to any required power, without going through the process of involution.

399. *To prove the Theorem for a positive integral exponent.*

By actual multiplication we may show that

$$(a+x)^2 = a^2 + 2\,a\,x + x^2$$
$$(a+x)^3 = a^3 + 3\,a^2\,x + 3\,a\,x^2 + x^3$$
$$(a+x)^4 = a^4 + 4\,a^3\,x + 6\,a^2\,x^2 + 4\,a\,x^3 + x^4$$
....

In these results we observe the following laws:

1. *The number of terms is one more than the exponent of the binomial.*

2. *The exponent of a in the first term is the same as the exponent of the binomial, and decreases by one in each succeeding term.*

3. *The exponent of x in the second term is unity, and increases by one in each succeeding term.*

4. *The coefficient of the first term is unity; and of the second term, is the exponent of the binomial.*

5. *If the coefficient of any term be multiplied by the exponent of a in that term, and the product divided by the number of the term, beginning at the left, the result will be the coefficient of the next term.*

Assuming that the laws hold for any positive integral exponent, n, we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3+\ldots\ldots$$

This result is called the *Binomial Theorem.*

400. To prove that it holds for any positive integral exponent, we multiply both members by $a+x$, thus

$$(a+x)^{n+1}=a^{n+1}+na^n x+\frac{n(n-1)}{1.2}a^{n-1}x^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-2}x^3$$
$$+\ldots\ldots+a^n x+na^{n-1}x^2+\frac{n(n-1)}{1.2}a^{n-2}x^3+\ldots\ldots$$

$$=a^{n+1}+(n+1)a^n x+\left[\frac{n(n-1)}{1.2}+n\right]a^{n-1}x^2$$
$$+\left[\frac{n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)}{1.2}\right]a^{n-2}x^3+\ldots\ldots$$

$$=a^{n+1}+(n+1)a^n x+\frac{n}{1.2}\Big[n-1+2\Big]a^{n-1}x^2$$
$$+\frac{n(n-1)}{1.2.3}\Big[n-2+3\Big]a^{n-2}x^3+\ldots\ldots$$

$$=a^{n+1}+(n+1)a^n x+\frac{(n+1)n}{1.2}a^{n-1}x^2$$
$$+\frac{(n+1)n(n-1)}{1.2.3}a^{n-2}x^3+\ldots\ldots$$

where it is evident that every term except the first will contain the factor $n+1$.

We observe that the expansion is of the same form as the value of $(a+x)^n$, having $n+1$ in the place of n.

Hence, if the laws of Art. 399 hold for any positive integral exponent, n, they also hold when that exponent is increased by 1. But we have shown them to hold for $(a+x)^4$, hence they hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on. Hence they hold for any positive integral exponent.

401. Since $1.2 = \underline{|2}$, $1.2.3 = \underline{|3}$, etc. (Art. 394), the Binomial Theorem is usually written as follows:

$$(a+x)^n = a^n + n\,a^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \ldots\ldots$$

402. If $a=1$, then, since any power of 1 equals 1, we have

$$(1+x)^n = 1 + n\,x + \frac{n\,(n-1)}{\underline{|2}}x^2 + \frac{n\,(n-1)\,(n-2)}{\underline{|3}}x^3 + \ldots\ldots$$

403. In performing examples by the aid of the Binomial Theorem, we may use the laws of Art. 399 to find the exponents and coefficients of the terms.

1. Expand $(a+x)^6$ by the Binomial Theorem.

The number of terms is 7.

The exponent of a in the first term is 6, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, 6; if the coefficient of the second term, 6, be multiplied by 5, the exponent of a in that term, and the product, 30, be divided by the number of the term, 2, the result, 15, will be the coefficient of the third term; etc.

Result, $a^6 + 6\,a^5x + 15\,a^4x^2 + 20\,a^3x^3 + 15\,a^2x^4 + 6\,a\,x^5 + x^6$.

Note. It will be observed that the coefficients of any two terms taken equidistant from the beginning and end of the expansion are the same. The reason for this will be obvious if, in Art. 401, x and a be interchanged, which is equivalent to inverting the series in the second member. Thus, the coefficients of the latter half of an expansion may be written out from the first half.

2. Expand $(1+x)^7$ by the Binomial Theorem.

Result, $1^7 + 7.1^6.\,x + 21.1^5.\,x^2 + 35.1^4.\,x^3 + 35.1^3.\,x^4 + 21.1^2.\,x^5 + 7.1^1.\,x^6 + x^7$;

or, $1 + 7\,x + 21\,x^2 + 35\,x^3 + 35\,x^4 + 21\,x^5 + 7\,x^6 + x^7$.

Note. If the first term of the binomial is a numerical quantity, it will be found convenient, in applying the laws, to retain the exponents at first without reduction, as then the laws for coefficients may be used. The result should afterwards be reduced to its simplest form.

3. Expand $(2\,a + 3\,b)^5$ by the Binomial Theorem.

$$(2\,a + 3\,b)^5 = [(2\,a) + (3\,b)]^5$$

$$= (2\,a)^5 + 5\,(2\,a)^4\,(3\,b) + 10\,(2\,a)^3\,(3\,b)^2 + 10\,(2\,a)^2\,(3\,b)^3 + 5\,(2\,a)\,(3\,b)^4 + (3\,b)^5$$

$$= 32\,a^5 + 240\,a^4\,b + 720\,a^3\,b^2 + 1080\,a^2\,b^3 + 810\,a\,b^4 + 243\,b^5,$$ *Ans.*

4. Expand $(m^{-\frac{1}{2}} - n^{-1})^6$ by the Binomial Theorem.

$$(m^{-\frac{1}{2}} - n^{-1})^6 = [(m^{-\frac{1}{2}}) + (-n^{-1})]^6$$

$$= (m^{-\frac{1}{2}})^6 + 6(m^{-\frac{1}{2}})^5(-n^{-1}) + 15(m^{-\frac{1}{2}})^4(-n^{-1})^2 + 20(m^{-\frac{1}{2}})^3(-n^{-1})^3 + 15(m^{-\frac{1}{2}})^2(-n^{-1})^4 + 6\,(m^{-\frac{1}{2}})\,(-n^{-1})^5 + (-n^{-1})^6$$

$$= m^{-3} + 6\,m^{-\frac{5}{2}}\,(-n^{-1}) + 15\,m^{-2}\,(n^{-2}) + 20\,m^{-\frac{3}{2}}\,(-n^{-3}) + 15\,m^{-1}\,(n^{-4}) + 6\,m^{-\frac{1}{2}}\,(-n^{-5}) + (n^{-6})$$

$$= m^{-3} - 6\,m^{-\frac{5}{2}}\,n^{-1} + 15\,m^{-2}\,n^{-2} - 20\,m^{-\frac{3}{2}}\,n^{-3} + 15\,m^{-1}\,n^{-4} - 6\,m^{-\frac{1}{2}}\,n^{-5} + n^{-6},$$ *Ans.*

Note. If either term of the binomial is not a single letter, with unity as its coefficient and exponent, or if either term is preceded by a minus sign, it will be found convenient to enclose the term, sign and all, in a parenthesis, when the usual laws for exponents and coefficients may be applied. In reducing, care must be taken to apply the principles of Arts. 227 and 259.

EXAMPLES.

Expand the following by the Binomial Theorem:

5. $(1 + c)^5$. **8.** $(ab - cd)^7$. **11.** $(c^{\frac{2}{3}} + d^{\frac{3}{4}})^8$.

6. $(a + x^3)^6$. **9.** $(m^2 + 3n^2)^6$. **12.** $(m^{-\frac{3}{5}} + 2n^3)^7$.

7. $(x^2 - 2y)^4$. **10.** $(a^{-2} - 4x^{\frac{1}{2}})^5$. **13.** $(a^{-1} - b^2 x^{\frac{1}{3}})^4$.

404. *To find the rth or general term of the expansion of* $(a + x)^n$.

We have now to determine, from the observed laws of the expansion, three things; the exponent of a in the term, the exponent of x in the term, and the coefficient of the term.

The exponent of x in the second term is 1; in the third term, 2; etc. Hence, in the rth term it will be $r - 1$.

In any term the sum of the exponents of a and x is n. Hence, in the rth term, the exponent of a will be such a quantity as when added to $r - 1$, the exponent of x, will produce n; or, the exponent of a will be $n - r + 1$.

The coefficient of the term will be a fraction, of the form $\frac{n(n-1)(n-2)\ \ldots\ldots}{1 \cdot 2 \cdot 3 \cdot \ldots\ldots}$; in which we must determine the *last* factors of the numerator and denominator.

We observe that the last factor of the numerator of any term is 1 more than the exponent of a in that term; hence the last factor of the numerator of the rth term will be $n - r + 2$.

Also, the last factor of the denominator of any term is the same as the exponent of x in that term; hence the last factor of the denominator of the rth term will be $r - 1$.

Therefore the

$$r\text{th term} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1.2.3.\ldots\ldots(r-1)} a^{n-r+1} x^{r-1}.$$

1. Find the 8th term of $(3a^{\frac{1}{2}} - 2b^{-1})^{11}$.

Here $r = 8$, $n = 11$; hence, the

$$8\text{th term} = \frac{11.10.9.8.7.6.5}{1.2.3.4.5.6.7}(3a^{\frac{1}{2}})^4(-2b^{-1})^7$$

$$= 330(81a^2)(-128b^{-7}) = -3421440a^2b^{-7}, \textit{ Ans.}$$

Note. The note to Ex. 4, Art. 403, applies with equal force to examples in this article.

EXAMPLES.

Find the

2. 10th term of $(a+x)^{15}$.

3. 6th term of $(1+m)^{14}$.

4. 8th term of $(c-d)^{17}$.

5. 5th term of $(1-a^2)^{12}$.

6. 9th term of $(x^{-1} - 2y^{\frac{1}{2}})^{11}$.

7. 8th term of $(a^{\frac{2}{3}} + 3x^{-1})^{10}$.

405. A trinomial may be raised to any power by the Binomial Theorem, if two of its terms be enclosed in a parenthesis and regarded as a single term; the operations indicated being performed after the expansion by the Theorem has been effected.

1. Expand $(2a - b + c^2)^3$ by the Binomial Theorem.

$$(2a - b + c^2)^3 = [(2a-b) + (c^2)]^3$$

$$= (2a-b)^3 + 3(2a-b)^2(c^2) + 3(2a-b)(c^2)^2 + (c^2)^3$$

$$= 8a^3 - 12a^2b + 6ab^2 - b^3 + 3c^2(4a^2 - 4ab + b^2) + 3c^4(2a-b) + c^6$$

$$= 8a^3 - 12a^2b + 6ab^2 - b^3 + 12a^2c^2 - 12abc^2 + 3b^2c^2$$

$$+ 6ac^4 - 3bc^4 + c^6, \textit{ Ans.}$$

The same method will apply to the expansion of any polynomial by the Binomial Theorem.

EXAMPLES.

Expand the following by the Binomial Theorem:

2. $(1-x-x^2)^4$.

3. $(x^2+3x+1)^3$.

4. $(1-2x-2x^2)^3$.

5. $(1+x-x^2)^5$.

XXXVIII.—UNDETERMINED COEFFICIENTS.

406. A **Series** is a succession of terms, so related that each may be derived from one or more of the others, in accordance with some fixed law.

The simpler forms of series have already been exhibited in the progressions.

407. A **Finite Series** is one having a finite number of terms.

408. An **Infinite Series** is one whose number of terms is unlimited.

The progressions, in general, are examples of finite series; but, in Art. 380, we considered infinite Geometrical series.

409. An infinite series is said to be *convergent* when the sum of the first n terms cannot numerically exceed some finite quantity, however large n may be; and it is said to be *divergent* when the sum of the first n terms can numerically exceed any finite quantity by taking n large enough.

For example, consider the infinite series

$$1+x+x^2+x^3+\ldots\ldots$$

The sum of the first n terms

$$1+x+x^2+x^3+\ldots\ldots+x^{n-1}=\frac{1-x^n}{1-x} \text{ (Art. 120).}$$

If x is less than 1, x^n is less than x, however large n may be; consequently the numerator and denominator of the fraction are each less than 1, and positive; and the numerator is larger than the denominator; hence the fraction is equal to some finite quantity greater than 1. The series is therefore *convergent* if x is less than 1.

If x is equal to 1, each term of the series equals 1, consequently the sum of the first n terms is n; and this can numerically exceed any finite quantity by taking n large enough. The series is therefore *divergent* if $x = 1$.

If x is greater than 1, each term of the series after the first is greater than 1, consequently the sum of the first n terms is greater than n; and this sum can numerically exceed any finite quantity by taking n large enough. The series is therefore *divergent* if x is greater than 1.

410. Every infinite literal series, arranged in order of powers of some letter, is *convergent* for *some* values of that letter, and *divergent* for other values.

We will now show that it is *convergent* when that letter equals zero.

Let the series be

$$a + bx + cx^2 + dx^3 + \dots\dots + kx^{n-1} + \dots\dots$$

The sum of the first n terms is

$$a + bx + cx^2 + dx^3 + \dots\dots + kx^{n-1},$$

which is equal to a, if x is made equal to 0.

Hence, however large n may be, the sum of the first n terms is equal to a, if x is equal to 0. Consequently the series is convergent if $x = 0$.

411. Infinite series may be developed by the common processes of *Division*, as in Art. 101, Exs. 19 and 20, and *Extraction of Roots*, as in Arts. 239 and 243; and by other methods which it will now be our object to elucidate.

UNDETERMINED COEFFICIENTS.

412. A method of expanding algebraic expressions into series, simple in its principles, and general in its application, is based on the following theorem, known as the

THEOREM OF UNDETERMINED COEFFICIENTS.

413. *If the series $A + Bx + Cx^2 + Dx^3 + \ldots\ldots$ is always equal to the series $A' + B'x + C'x^2 + D'x^3 + \ldots\ldots$, for any value of x which makes both series convergent, the coefficients of like powers of x in the two series will be equal.*

For, since the equation

$$A + Bx + Cx^2 + Dx^3 + \ldots\ldots = A' + B'x + C'x^2 + D'x^3 + \ldots\ldots$$

is satisfied for any value of x which makes both members convergent; and since by Art. 410, if x is equal to 0, both members are convergent; it follows that the equation is satisfied if $x = 0$. Making $x = 0$, the equation becomes

$$A = A'.$$

Subtracting A from the first member of the equation, and its equal, A', from the second member, we have

$$Bx + Cx^2 + Dx^3 + \ldots\ldots = B'x + C'x^2 + D'x^3 + \ldots\ldots$$

Dividing through by x,

$$B + Cx + Dx^2 + \ldots\ldots = B' + C'x + D'x^2 + \ldots\ldots$$

This equation is also satisfied for any value of x which makes both members convergent; hence it is satisfied if $x = 0$. Making $x = 0$, we have

$$B = B'.$$

Proceeding in this way, we may show $C = C'$, $D = D'$, etc.

Note. The necessity for the limitation of the theorem to values of x which make both series convergent, is that a convergent series evidently cannot be equal to a divergent series; and two divergent series cannot be equal, as two quantities which numerically exceed any finite quantity cannot be said to be equal.

Hence, in all applications of the theorem, the results are only true when both members are convergent.

APPLICATION TO THE EXPANSION OF FRACTIONS INTO SERIES.

414. 1. Expand $\frac{2+5x}{1-3x}$ into a series.

We have seen (Art. 101), that any fraction may be expanded into a series by dividing the numerator by the denominator; consequently, we know that the proposed development is possible. Assume then,

$$\frac{2+5x}{1-3x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots \quad (1)$$

where A, B, C, D, E, ... are quantities independent of x.

Clearing of fractions, and collecting together in the second member the terms containing like powers of x, we have

$$2 + 5x = A + \left.\begin{matrix} B \\ -3A \end{matrix}\right| x + \left.\begin{matrix} C \\ -3B \end{matrix}\right| x^2 + \left.\begin{matrix} D \\ -3C \end{matrix}\right| x^3 + \left.\begin{matrix} E \\ -3D \end{matrix}\right| x^4 + \ldots\ldots$$

Equation (1), and also the preceding equation, are evidently to be satisfied by all values of x which make the second member a convergent series. Hence, applying the Theorem of Undetermined Coefficients to the latter, we have

$$A = 2.$$

$$B - 3A = 5;\ \text{whence, } B = 3A + 5 = 11.$$

$$C - 3B = 0;\ \text{whence, } C = 3B \quad = 33.$$

$$D - 3C = 0;\ \text{whence, } D = 3C \quad = 99.$$

$$E - 3D = 0;\ \text{whence, } E = 3D \quad = 297.$$

Substituting these values of A, B, C, D, E, ... in **(1)**, we have

$$\frac{2+5x}{1-3x}=2+11x+33x^2+99x^3+297x^4+\dots\dots,$$

which may be readily verified by division.

This result, in accordance with the last part of the Note to Art. 413, only expresses the value of the fraction for such values of x as make the second member a convergent series.

2. Expand $\frac{1-3x-x^2}{1-2x-x^2}$ into a series.

Assume $\frac{1-3x-x^2}{1-2x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\dots\dots$

Clearing of fractions, and collecting terms,

$$1-3x-x^2=A+\left.\begin{matrix}B\\-2A\\\end{matrix}\right|x+\left.\begin{matrix}C\\-2B\\-A\end{matrix}\right|x^2+\left.\begin{matrix}D\\-2C\\-B\end{matrix}\right|x^3+\left.\begin{matrix}E\\-2D\\-C\end{matrix}\right|x^4\dots\dots$$

Equating the coefficients of like powers of x,

$$A=1.$$

$$B-2A=-3;\ \text{whence, } B=2A-3=-1.$$

$$C-2B-A=-1;\ \text{whence, } C=2B+A-1=-2.$$

$$D-2C-B=\ \ 0;\ \text{whence, } D=2C+B=-5.$$

$$E-2D-C=\ \ 0;\ \text{whence, } E=2D+C=-12.$$

Substituting these values,

$$\frac{1-3x-x^2}{1-2x-x^2}=1-x-2x^2-5x^3-12x^4-\dots\dots,\ \textit{Ans.}$$

Note. This method enables us to find the law of the coefficients in any expansion. For instance, in Example 1, we obtained the equations $C=3B$, $D=3C$, $E=3D$, etc.; or, in general, any coefficient, after the second, is three times the preceding. In Example 2, we obtained the equations $D=2C+B$, $E=2D+C$, etc.; or, in general, any coefficient, after the third, is twice the preceding plus the next but one preceding. After the law of the coefficients of any expansion has been found, we may write out the subsequent terms to any desired extent by its aid.

EXAMPLES.

Expand the following to five terms:

3. $\dfrac{1-x}{1+x}$.

4. $\dfrac{3+4x}{1-5x}$.

5. $\dfrac{2-x+x^2}{1-x^2}$.

6. $\dfrac{1-x-x^2}{1+x+x^2}$.

7. $\dfrac{1-2x^2}{1+2x-3x^2}$.

8. $\dfrac{1+2x}{2-x-x^2}$.

9. $\dfrac{2-3x+4x^2}{1+2x-5x^2}$.

10. $\dfrac{3+x-2x^2}{3-x+x^2}$.

11. $\dfrac{1-3x^2}{2-3x-2x^2}$.

415. If the lowest power of x in the denominator is higher than the lowest power of x in the numerator, the method of the preceding article will be found inapplicable. We may, however, determine by actual division what will be the exponent of x in the first term of the expansion, and assume the fraction equal to a series whose first term contains that power of x; the exponents afterwards increasing by unity each term as usual.

1. Expand $\dfrac{1}{3x-x^2}$ into a series.

Proceeding in the usual way, we should assume

$$\frac{1}{3x-x^2}=A+Bx+\ldots\ldots$$

Clearing of fractions, $\quad 1=3Ax+(3B-A)x^2+\ldots\ldots$

Equating the coefficients of like powers of x, we have $1=0$; a result manifestly absurd, and showing that the usual method is inapplicable.

But, dividing 1 by $3x-x^2$, we obtain $\dfrac{x^{-1}}{3}$ as the first term of the quotient; hence we assume the fraction equal to a series whose first term contains x^{-1}; next term x^0, or 1; next term x; etc. Or,

$$\frac{1}{3x-x^2}=Ax^{-1}+B+Cx+Dx^2+Ex^3+\ldots\ldots$$

Clearing of fractions, and collecting terms,

$$1 = 3A + \left.\begin{matrix}3B \\ -A\end{matrix}\right| x + \left.\begin{matrix}3C \\ -B\end{matrix}\right| x^2 + \left.\begin{matrix}3D \\ -C\end{matrix}\right| x^3 + \left.\begin{matrix}3E \\ -D\end{matrix}\right| x^4 + \dots\dots$$

Equating the coefficients of like powers of x,

$$3A = 1; \text{ whence, } A = \frac{1}{3}.$$

$$3B - A = 0; \text{ whence, } B = \frac{A}{3} = \frac{1}{9}.$$

$$3C - B = 0; \text{ whence, } C = \frac{B}{3} = \frac{1}{27}.$$

$$3D - C = 0; \text{ whence, } D = \frac{C}{3} = \frac{1}{81}.$$

$$3E - D = 0; \text{ whence, } E = \frac{D}{3} = \frac{1}{243}.$$

Substituting these values,

$$\frac{1}{3x - x^2} = \frac{1}{3}x^{-1} + \frac{1}{9} + \frac{1}{27}x + \frac{1}{81}x^2 + \frac{1}{243}x^3 + \dots\dots, \textit{ Ans.}$$

EXAMPLES.

Expand the following to five terms:

2. $\dfrac{2}{3x^2 - 2x^3}$. 3. $\dfrac{1 + x - x^2}{x - 2x^2 + 3x^3}$. 4. $\dfrac{1 - 2x^2 - x^3}{x^2 + x^3 - x^4}$.

APPLICATION TO THE EXPANSION OF RADICALS INTO SERIES.

416. As any root of any expression consisting of two or more terms can be obtained by the method of Art. 247, we know that the development is possible.

1. Expand $\sqrt{1+x^2}$ into a series by the Theorem of Undetermined Coefficients.

Assume $\sqrt{1 + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots\dots$

Squaring both members, we have (Art. 230),

$$1+x^2=\begin{array}{l|l|l|l|l} A^2 & x+\quad B^2 & x^2 & x^3+\quad C^2 & x^4\ldots\ldots \\ +2AB & +2AC & +2AD & +2AE & \\ & & +2BC & +2BD & \end{array}$$

Equating the coefficients of like powers of x,

$$A^2=1\,;\ \text{whence, } A=1.$$

$$2AB=0\,;\ \text{whence, } B=\frac{0}{2A}=0.$$

$$B^2+2AC=1\,;\ \text{whence, } C=\frac{1-B^2}{2A}=\frac{1}{2}.$$

$$2AD+2BC=0\,;\ \text{whence, } D=-\frac{BC}{A}=0.$$

$$C^2+2AE+2BD=0\,;\ \text{whence, } E=-\frac{2BD+C^2}{2A}=-\frac{1}{8}.$$

Substituting these values,

$$\sqrt{1+x^2}=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots\ldots,$$

which may be verified by the method of Art. 239.

Note. From the equation $A^2=1$, we may have $A=\pm 1$; and taking $A=-1$, we should find $C=-\frac{1}{2}$, $E=\frac{1}{8}$, ..., so that the expansion might be as follows:

$$\sqrt{1+x^2}=-1-\frac{1}{2}x^2+\frac{1}{8}x^4\ldots\ldots$$

This agrees with the remark made after the rule in Art. 239.

EXAMPLES.

Expand the following to five terms:

2. $\sqrt{1+x}$. **4.** $\sqrt{1-2x+3x^2}$. **6.** $\sqrt[3]{1-x}$.

3. $\sqrt{1-2x}$. **5.** $\sqrt{1+x+x^2}$. **7.** $\sqrt[3]{1+x+x^2}$.

APPLICATION TO THE DECOMPOSITION OF RATIONAL FRACTIONS.

417. When the denominator of a fraction can be resolved into factors, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions*, whose denominators are the factors of the given denominator.

We shall consider only those cases in which the factors of the denominator are all of the first degree.

CASE I.

418. *When the factors of the denominator are all unequal.*

Let $\frac{x+7}{(3x-1)(5x+2)}$ be a fraction, whose denominator is composed of two unequal first degree factors. We wish to prove that it can be decomposed into two fractions, whose denominators are $3x-1$ and $5x+2$, and whose numerators are independent of x. To prove this, assume

$$\frac{x+7}{(3x-1)(5x+2)}=\frac{A}{3x-1}+\frac{B}{5x+2}.$$

We will now show that such values, independent of x, may be given to A and B, as will make the above equation identical, or true for all values of x. Clearing of fractions,

$$x+7=A(5x+2)+B(3x-1)$$

or,

$$x+7=(5A+3B)x+2A-B,$$

which is to be true for all values of x. Then, by Art. 413, the coefficients of like powers of x in the two members must be equal; or,

$$5A+3B=1$$

$$2A-B=7$$

From these two equations we obtain $A=2$, and $B=-3$. Hence, the proposed decomposition is possible, and we have

$$\frac{x+7}{(3x-1)(5x+2)}=\frac{2}{3x-1}+\frac{-3}{5x+2}$$
$$=\frac{2}{3x-1}-\frac{3}{5x+2}.$$

This result may be readily verified by finding the sum of the fractions.

In a similar manner we can prove that any fraction, whose denominator is composed of unequal first degree factors, can be decomposed into as many fractions as there are factors, having these factors for their denominators, and for their numerators quantities independent of x.

EXAMPLES.

1. Decompose $\frac{3x-5}{x^2-13x+40}$ into its partial fractions.

The factors of the denominator are $x-8$ and $x-5$ (Art. 118).

Assume, then, $$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-8}+\frac{B}{x-5} \qquad (1)$$

Clearing of fractions, and uniting terms,

$$3x-5=A(x-5)+B(x-8)$$

Putting $x=8$, $19=3A$, or $A=\frac{19}{3}$.

Putting $x=5$, $10=-3B$, or $B=-\frac{10}{3}$.

Note. The student may compare the above method of finding A and B with that used on page 312.

Substituting these values in (1),

$$\frac{3x-5}{x^2-13x+40}=\frac{\frac{19}{3}}{x-8}+\frac{-\frac{10}{3}}{x-5}=\frac{19}{3(x-8)}-\frac{10}{3(x-5)},\ \textit{Ans.}$$

EXAMPLES.

Decompose the following into their partial fractions:

2. $\frac{5x-2}{x^2-4}$.

3. $\frac{x+9}{x^2+3x}$.

4. $\frac{3x+2}{x^2-2x}$.

5. $\frac{2x-3}{x^2-3x-4}$.

6. $\frac{x}{x^2-13x+42}$.

7. $\frac{17}{6x^2-13x-5}$.

8. $\frac{7x+9}{9+9x-4x^2}$.

9. $\frac{x^2}{(x^2-1)(x-2)}$.

CASE II.

419. *When the factors of the denominator are all equal.*

1. Separate $\frac{x^2-11x+26}{(x-3)^3}$ into its partial fractions.

If we attempt to perform the example by the method of Case I, we should assume

$$\frac{x^2-11x+26}{(x-3)^3}=\frac{A}{x-3}+\frac{B}{x-3}+\frac{C}{x-3}.$$

This would evidently be impossible, as the sum of the fractions in the second member is $\frac{A+B+C}{x-3}$; which, as A, B, and C are, by supposition, independent of x, cannot be equal to $\frac{x^2-11x+26}{(x-3)^3}$.

The method to be used in Case II depends on the following:

Consider the fraction $\frac{a x^{n-1}+b x^{n-2}+c x^{n-3}+\dots\dots+k}{(x+h)^n}$.

Putting $x=y-h$, the fraction becomes

$$\frac{a(y-h)^{n-1}+b(y-h)^{n-2}+c(y-h)^{n-3}+\dots\dots+k}{y^n}$$

If the terms of the numerator are expanded by the binomial theorem, and the terms containing like powers of y collected together, we shall have a fraction of the form

$$\frac{a_1 y^{n-1} + b_1 y^{n-2} + c_1 y^{n-3} + \ldots\ldots + k_1}{y^n}$$

Dividing each term of the numerator by y^n, we have

$$\frac{a_1}{y} + \frac{b_1}{y^2} + \frac{c_1}{y^3} + \ldots\ldots + \frac{k_1}{y^n}.$$

Changing back y to $x + h$, this becomes

$$\frac{a_1}{x+h} + \frac{b_1}{(x+h)^2} + \frac{c_1}{(x+h)^3} + \ldots\ldots + \frac{k_1}{(x+h)^n}.$$

This shows that the assumed fraction can be expressed as the sum of n partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x + h$, beginning with the first, and ending with the nth.

In accordance with this, we assume

$$\frac{x^2 - 11x + 26}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}.$$

Clearing of fractions,

$$\begin{aligned} x^2 - 11x + 26 &= A(x-3)^2 + B(x-3) + C \\ &= A(x^2 - 6x + 9) + B(x-3) + C \\ &= Ax^2 + (B - 6A)x + 9A - 3B + C. \end{aligned}$$

Equating the coefficients of like powers of x,

$$A = 1, \quad B - 6A = -11, \quad \text{and } 9A - 3B + C = 26$$

Whence, $\quad A = 1, \quad B = -5, \quad \text{and } C = 2.$

Substituting these values,

$$\frac{x^2 - 11x + 26}{(x-3)^3} = \frac{1}{x-3} - \frac{5}{(x-3)^2} + \frac{2}{(x-3)^3}, \ \textit{Ans.}$$

EXAMPLES.

Separate the following into their partial fractions:

2. $\frac{x^2+3x+3}{(x+1)^3}$.

3. $\frac{2x-13}{(x-5)^2}$.

4. $\frac{x^2}{(x-2)^3}$.

5. $\frac{3x^2-4}{(x+1)^3}$.

6. $\frac{3x-10}{(2x-5)^2}$.

7. $\frac{18x^2+12x-3}{(3x+2)^3}$.

CASE III.

420. *When some of the factors of the denominator are equal.*

1. Separate $\frac{3x+2}{x(x+1)^3}$ into its partial fractions.

The method in this case is a combination of the methods of Cases I and II. We assume

$$\frac{3x+2}{x(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{x}.$$

Clearing of fractions,

$$3x+2=Ax(x+1)^2+Bx(x+1)+Cx+D(x+1)^3$$
$$=(A+D)x^3+(2A+B+3D)x^2+(A+B+C+3D)x+D.$$

Equating the coefficients of like powers of x,

$$D=2,\ A+B+C+3D=3,\ 2A+B+3D=0,\ \text{and}\ A+D=0$$

Whence, $A=-2,\ B=-2,\ C=1,$ and $D=2.$

Substituting these values,

$$\frac{3x+2}{x(x+1)^3}=-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}+\frac{2}{x},\ \textit{Ans.}$$

It is impossible to give an example to illustrate every possible case; but no difficulty will be found in assuming the proper partial fractions, if attention be given to the following general case. A fraction of the form

$$\frac{X}{(x+a)\,(x+b)\,\ldots\ldots\,(x+m)^r\,(x+n)^s\,\ldots\ldots}$$

should be put equal to

$$\frac{A}{x+a}+\frac{B}{x+b}+\ldots\ldots+\frac{E}{x+m}+\frac{F}{(x+m)^2}+\ldots\ldots+\frac{K}{(x+m)^r}$$
$$+\frac{L}{x+n}+\frac{M}{(x+n)^2}+\ldots\ldots+\frac{R}{(x+n)^s}+\ldots\ldots$$

Single factors, like $x+a$ and $x+b$, having single fractions like $\frac{A}{x+a}$ and $\frac{B}{x+b}$, corresponding; and repeated factors, like $(x+m)^r$, having r partial fractions corresponding, arranged as in Case II.

EXAMPLES.

Separate the following into their partial fractions:

2. $\dfrac{8-3x-x^2}{x(x+2)^2}$.

3. $\dfrac{3x^3-11x^2+13x-4}{x(x-1)(x-2)^2}$.

4. $\dfrac{3x-1}{x^2(x+1)^2}$.

5. $\dfrac{15-7x+3x^2-3x^3}{x^3(x+5)}$.

6. $\dfrac{6x^2-14x+6}{(x-2)(2x-3)^2}$.

7. $\dfrac{5x^2+3x+2}{x^3(x+1)^2}$.

421. Unless the numerator is of a lower degree than the denominator, the preceding methods are inapplicable.

For example, let it be required to separate $\frac{2x^2+1}{x^2-x}$ into its partial fractions. Proceeding in the usual way, we assume

$$\frac{2x^2+1}{x^2-x}=\frac{A}{x}+\frac{B}{x-1}$$

Clearing of fractions,

$$2x^2+1=A(x-1)+Bx=(A+B)x-A.$$

Equating the coefficients of like powers of x, we have $2=0$; an absurd result, and showing that the usual method is inapplicable.

But by actual division, as in Art. 150, we have

$$\frac{2x^2+1}{x^2-x}=2+\frac{2x+1}{x^2-x}.$$

We may now separate $\frac{2x+1}{x^2-x}$ into its partial fractions by the usual method, obtaining

$$\frac{2x+1}{x^2-x}=-\frac{1}{x}+\frac{3}{x-1}.$$

Hence, $$\frac{2x^2+1}{x^2-x}=2+\frac{2x+1}{x^2-x}=2-\frac{1}{x}+\frac{3}{x-1}, \textit{ Ans.}$$

APPLICATION TO THE REVERSION OF SERIES.

422. **1.** Given $y=2x+x^2-2x^3-3x^4+\ldots\ldots$, to revert the series, or to express x in terms of y.

Assume $x=Ay+By^2+Cy^3+Dy^4+\ldots\ldots$ (1)

Substituting in this the given value of y, we have

$$x=A(2x+x^2-2x^3-3x^4+\ldots)+B(4x^2+x^4+4x^3-8x^4+\ldots)$$
$$+C(8x^3+12x^4+\ldots)+D(16x^4+\ldots)+\ldots\ldots$$

or, $$x=2Ax+\begin{array}{r|}A\\+4B\\ \\ \\ \end{array}\,x^2\begin{array}{r|}-2A\\+4B\\+8C\\ \\ \end{array}\,x^3\begin{array}{r|}-3A\\-7B\\+12C\\+16D\end{array}\,x^4+\ldots\ldots$$

Equating the coefficients of like powers of x,

$$2A=1;\text{ whence, }A=\frac{1}{2}.$$

$$A+4B=0;\text{ whence, }B=-\frac{A}{4}=-\frac{1}{8}.$$

$$-2A+4B+8C=0;\text{ whence, }C=\frac{3}{16}.$$

$$-3A-7B+12C+16D=0;\text{ whence, }D=-\frac{13}{128}.$$

Substituting these values in (1),

$$x = \frac{y}{2} - \frac{y^2}{8} + \frac{3\,y^3}{16} - \frac{13\,y^4}{128} + \ldots\ldots, \textit{ Ans.}$$

If the even powers of x are wanting in the given series, we may abridge the operation by assuming x equal to a series containing only the odd powers of y.

Thus, to revert the series $y = x - x^3 + x^5 - x^7 + \ldots\ldots$, we assume $x = A\,y + B\,y^3 + C\,y^5 + D\,y^7 + \ldots\ldots$

If the odd powers of x are wanting in the given series, the reversion of the series is impossible by the method previously given. But by substituting another letter, say t, for x^2, we may revert the series and obtain a value of t, or of x^2, in terms of y; and by taking the square root of the result, express x itself in terms of y.

If the first term of the series is independent of x, we cannot, by the method previously given, express x definitely in terms of y; though we can express it in the form of a series in which y is the only unknown quantity.

2. Revert the series $y = 2 + 2\,x - x^2 - x^3 + 2\,x^4 + \ldots\ldots$

We may write the series,

$$y - 2 = 2\,x - x^2 - x^3 + 2\,x^4 + \ldots\ldots \qquad (1)$$

Assume $x = A\,(y-2) + B(y-2)^2 + C(y-2)^3 + D(y-2)^4 + \ldots$ (2)

Substituting in this the value of $y - 2$ given in (1), we have

$$x = A\,(2x - x^2 - x^3 + 2x^4 + \ldots) + B\,(4\,x^2 + x^4 - 4\,x^3 - 4\,x^4 + \ldots)$$
$$+ C\,(8\,x^3 - 12\,x^4 + \ldots) + D\,(16\,x^4 + \ldots) + \ldots\ldots$$

or,

$$x = 2\,A\,x \left.\begin{matrix} - A \\ + 4\,B \\ \\ \\ \end{matrix}\right| x^2 \left.\begin{matrix} - A \\ - 4\,B \\ + 8\,C \\ \\ \end{matrix}\right| x^3 \left.\begin{matrix} + 2\,A \\ - 3\,B \\ - 12\,C \\ + 16\,D \end{matrix}\right| x^4 + \ldots\ldots$$

Equating the coefficients of like powers of x,

$$2A = 1;\ \text{whence, } A = \frac{1}{2}.$$

$$-A + 4B = 0;\ \text{whence, } B = \frac{1}{8}.$$

$$-A - 4B + 8C = 0;\ \text{whence, } C = \frac{1}{8}.$$

$$2A - 3B - 12C + 16D = 0;\ \text{whence, } D = \frac{7}{128}.$$

Substituting in (2),

$$x = \frac{1}{2}(y-2) + \frac{1}{8}(y-2)^2 + \frac{1}{8}(y-2)^3 + \frac{7}{128}(y-2)^4 + \ldots\ldots,\ \textit{Ans.}$$

EXAMPLES.

Revert the following series to four terms:

3. $y = x + x^2 + x^3 + x^4 + \ldots\ldots$

4. $y = 2x + 3x^3 + 4x^5 + 5x^7 + \ldots\ldots$

5. $y = x - x^3 + x^5 - x^7 + \ldots\ldots$

6. $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$

7. $y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

8. $y = 3x - 2x^2 + 3x^3 - 4x^4 + \ldots\ldots$

Note. This method may sometimes be used to find, approximately, the root of an equation of higher degree than the second. Thus, to solve the equation

$$2x + x^2 - 2x^3 - 3x^4 = .1$$

we may put $.1 = y$, and revert the series; giving, as in Ex. 1, Art. 422,

$$x = \frac{1}{2}y - \frac{1}{8}y^2 + \frac{3}{16}y^3 - \frac{13}{128}y^4 + \ldots\ldots$$

Putting back $y = .1$, we have

$$x = \frac{.1}{2} - \frac{.01}{8} + \frac{.003}{16} - \frac{.0013}{128} + \ldots\ldots$$

$$= .05 - .00125 + .00019 - .00001 + \ldots\ldots = .04893+,\ \textit{Ans.}$$

This method can, of course, only be used when the series in the second member is convergent.

XXXIX.—BINOMIAL THEOREM.

ANY EXPONENT.

423. We have seen (Art. 402) that when n is a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots\ldots$$

We shall now prove that this formula is true when n is a positive fraction, a negative integer, or a negative fraction.

1. Let n be a positive fraction, which we will denote by $\frac{p}{q}$; p and q being positive integers.

Now (Art. 252), $(1+x)^{\frac{p}{q}} = \sqrt[q]{(1+x)^p}$

$$= \sqrt[q]{1 + px + \ldots\ldots}, \text{ (Art. 402).}$$

Extracting the qth root of this expression by the method of Art. 247,

$$\begin{array}{r|l} 1 + px + \ldots\ldots & 1 + \frac{px}{q} + \ldots\ldots \\ 1^q = 1 & \\ \hline q \mid px & \end{array}$$

That is, $$(1+x)^{\frac{p}{q}} = 1 + \frac{px}{q} + \ldots\ldots \qquad (1)$$

2. Let n be a negative quantity, either integer or fraction, which we will denote by $-s$.

Then (Art. 255),

$$(1+x)^{-s} = \frac{1}{(1+x)^s}$$

$$= \frac{1}{1 + sx + \ldots\ldots}, \text{ (by Arts. 402, and 423, 1).}$$

From which, by actual division, we have

$$(1+x)^{-s} = 1 - sx + \ldots\ldots \qquad (2)$$

From (1), (2), and Art. 402, we observe that whether n is positive or negative, integral or fractional, the form of the expansion is

$$(1+x)^n = 1 + n\,x + A\,x^2 + B\,x^3 + C\,x^4 + \ldots\ldots \qquad (3)$$

Writing $\frac{x}{a}$ in place of x, we have

$$\left(1+\frac{x}{a}\right)^n = 1 + n\,\frac{x}{a} + A\,\frac{x^2}{a^2} + B\,\frac{x^3}{a^3} + C\,\frac{x^4}{a^4} + \ldots\ldots$$

Multiplying this through by a^n, and remembering that

$$a^n\left(1+\frac{x}{a}\right)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = (a+x)^n, \text{ we have}$$

$$(a+x)^n = a^n + n\,a^{n-1}x + A\,a^{n-2}x^2 + B\,a^{n-3}x^3 + \ldots\ldots \qquad (4)$$

To find the values of A, B, etc., we put $x+z$ for x in (3), and regarding $(x+z)$ as one term, we shall have

$$\begin{aligned}[1+(x+z)]^n &= 1 + n\,(x+z) + A\,(x+z)^2 + B\,(x+z)^3 + \ldots\ldots \\ &= 1 + n\,x + A\,x^2 + B\,x^3 + \ldots\ldots \\ &\quad + (n + 2\,A\,x + 3\,Bx^2 + \ldots\ldots)\,z + \ldots\ldots \qquad (5)\end{aligned}$$

Regarding $(1+x)$ as one term, we shall have, by (4),

$$[(1+x)+z]^n = (1+x)^n + n\,(1+x)^{n-1}z + \ldots\ldots \qquad (6)$$

Since $[1+(x+z)]^n = [(1+x)+z]^n$, identically, we have from (5) and (6),

$$\begin{aligned}1+nx+Ax^2+Bx^3+\ldots\ldots+(n+2\,Ax+3\,Bx^2+\ldots\ldots)z+\ldots\ldots \\ = (1+x)^n + n\,(1+x)^{n-1}z + \ldots\ldots\end{aligned}$$

which is true for all values of z which make both members of the equation convergent. Hence, by Art. 413, the coefficients of z in the two series must be equal; or,

$$n\,(1+x)^{n-1} = n + 2\,A\,x + 3\,B\,x^2 + \ldots\ldots$$

Multiplying both members by $1 + x$,

$$n(1+x)^n = n + (2A + n)x + (3B + 2A)x^2 + \ldots\ldots$$

or, by (3),

$$n + n^2 x + nAx^2 + nBx^3 + \ldots\ldots = n + (2A + n)x + (3B + 2A)x^2 + \ldots\ldots$$

which is true for all values of x which make both members of the equation convergent; hence, equating the coefficients of like powers of x,

$$2A + n = n^2; \text{ whence, } 2A = n^2 - n, \text{ or } A = \frac{n(n-1)}{\underline{|2}}$$

$$3B + 2A = nA; \text{ whence, } 3B = nA - 2A = A(n-2)$$

or,
$$B = \frac{A(n-2)}{3} = \frac{n(n-1)(n-2)}{\underline{|3}}$$

Substituting in (4),

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \ldots\ldots$$

which has thus been proved to hold for *all* values of n, positive or negative, integral or fractional. Hence, the Binomial Theorem has been proved in its most general form. The result, however, only expresses the value of $(a + x)^n$ for such values of x as make the second member convergent (Art. 413).

424. When n is a positive integer, the number of terms in the expansion is $n + 1$ (Art. 399). When n is a fraction or negative quantity, the expansion never terminates, as no one of the quantities $n - 1$, $n - 2$, etc., can become equal to zero. The development in that case furnishes an infinite series.

425. The method and notes of Art. 403 apply to the expansion of expressions by the Binomial Theorem when the exponent is a fractional or negative quantity.

1. Expand $(a+x)^{\frac{2}{3}}$ to five terms.

The exponent of a in the first term of the expansion is $\frac{2}{3}$, and decreases by one in each succeeding term.

The exponent of x in the second term of the expansion is 1, and increases by one in each succeeding term.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$; multiplying the coefficient of the second term, $\frac{2}{3}$, by the exponent of a in that term, $-\frac{1}{3}$, and dividing the product, $-\frac{2}{9}$, by the number of the term, 2, we obtain $-\frac{1}{9}$ as the coefficient of the third term; etc.

Result, $a^{\frac{2}{3}}+\frac{2}{3}a^{-\frac{1}{3}}x-\frac{1}{9}a^{-\frac{4}{3}}x^2+\frac{4}{81}a^{-\frac{7}{3}}x^3-\frac{7}{243}a^{-\frac{10}{3}}x^4+\ldots\ldots$

2. Expand $(1+2x^{\frac{1}{2}})^{-2}$ to five terms.

$$(1+2x^{\frac{1}{2}})^{-2}=[1+(2x^{\frac{1}{2}})]^{-2}$$

$$=1^{-2}-2.1^{-3}.(2x^{\frac{1}{2}})+3.1^{-4}.(2x^{\frac{1}{2}})^2-4.1^{-5}.(2x^{\frac{1}{2}})^3$$
$$+5.1^{-6}.(2x^{\frac{1}{2}})^4\ldots\ldots$$

$$=1-2(2x^{\frac{1}{2}})+3(4x)-4(8x^{\frac{3}{2}})+5(16x^2)-\ldots\ldots$$

$$=1-4x^{\frac{1}{2}}+12x-32x^{\frac{3}{2}}+80x^2-\ldots\ldots,\ \textit{Ans.}$$

3. Expand $(a^{-1}-3x^{-\frac{1}{2}})^{-\frac{4}{3}}$ to five terms.

$$(a^{-1}-3x^{-\frac{1}{2}})^{-\frac{4}{3}}=[(a^{-1})+(-3x^{-\frac{1}{2}})]^{-\frac{4}{3}}$$

$$= (a^{-1})^{-\frac{4}{3}} - \frac{4}{3}(a^{-1})^{-\frac{7}{3}}(-3x^{-\frac{1}{2}}) + \frac{14}{9}(a^{-1})^{-\frac{10}{3}}(-3x^{-\frac{1}{2}})^2$$

$$-\frac{140}{81}(a^{-1})^{-\frac{13}{3}}(-3x^{-\frac{1}{2}})^3 + \frac{455}{243}(a^{-1})^{-\frac{16}{3}}(-3x^{-\frac{1}{2}})^4 - \ldots\ldots$$

$$= a^{\frac{4}{3}} - \frac{4}{3}a^{\frac{7}{3}}(-3x^{-\frac{1}{2}}) + \frac{14}{9}a^{\frac{10}{3}}(9x^{-1}) - \frac{140}{81}a^{\frac{13}{3}}(-27x^{-\frac{3}{2}})$$

$$+ \frac{455}{243}a^{\frac{16}{3}}(81x^{-2}) - \ldots\ldots$$

$$= a^{\frac{4}{3}} + 4a^{\frac{7}{3}}x^{-\frac{1}{2}} + 14a^{\frac{10}{3}}x^{-1} + \frac{140}{3}a^{\frac{13}{3}}x^{-\frac{3}{2}} + \frac{455}{3}a^{\frac{16}{3}}x^{-2}\ldots\ldots,$$

Ans.

EXAMPLES.

Expand the following to five terms:

4. $(a+x)^{\frac{5}{2}}$

5. $(1+x)^{-6}$.

6. $(1-x)^{-\frac{3}{5}}$.

7. $\sqrt{a-x}$.

8. $\dfrac{1}{\sqrt[3]{1+x}}$.

9. $\dfrac{1}{(a-x)^3}$.

10. $\dfrac{1}{c^{\frac{3}{2}}+d}$.

11. $(x^{-\frac{1}{2}}-3y)^{\frac{2}{3}}$.

12. $(m^{-\frac{2}{3}}-2n^{\frac{3}{2}})^{-\frac{3}{2}}$.

13. $(1+6xy^{-1})^{-\frac{5}{3}}$.

14. $(x^4+4ab)^{\frac{3}{4}}$.

15. $\dfrac{1}{(a^{-1}-3y^{-2})^4}$.

426. The expression for the rth term, derived in Art. 404, holds for any value of n, as it was deduced from the expansion which has been proved to hold universally.

1. Find the 7th term of $(1-x)^{-\frac{1}{3}}$.

Here $r=7$, $n=-\frac{1}{3}$; hence, the

$$\text{7th term} = \frac{-\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}\cdot-\frac{10}{3}\cdot-\frac{13}{3}\cdot-\frac{16}{3}}{1.2.3.4.5.6}(-x)^6 = \frac{728x^6}{6561},$$

Ans.

2. Find the 8th term of $(a^{\frac{1}{2}} + x^{-\frac{2}{3}})^{-3}$.

Here $r = 8$, $n = -3$; hence, the

$$\text{8th term} = \frac{-3 \,.\, -4 \,.\, -5 \,.\, -6 \,.\, -7 \,.\, -8 \,.\, -9}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} (a^{\frac{1}{2}})^{-10} (x^{-\frac{2}{3}})^7$$

$$= -36\, a^{-5} x^{-\frac{14}{3}}, \textit{ Ans.}$$

EXAMPLES.

Find the

3. 8th term of $\sqrt{a + x}$.

4. 7th term of $(1 + m)^{-4}$.

5. 5th term of $(1 - a^2)^{-\frac{3}{2}}$.

6. 6th term of $\dfrac{1}{\sqrt[4]{x^2 + y^3}}$.

7. 7th term of $(x^{-1} - y^{\frac{1}{2}})^{\frac{4}{3}}$.

8. 5th term of $\dfrac{1}{(n^{-\frac{2}{3}} - c^{-2})^7}$.

9. 6th term of $(a^{\frac{2}{3}} + 3x^{-1})^{-\frac{2}{3}}$.

10. 8th term of $(x^3 y - z^{-\frac{2}{3}})^{-3}$.

427. *To find any root of a number approximately by the Binomial Theorem.*

1. Find the approximate square root of 10.

$$\sqrt{10} = 10^{\frac{1}{2}} = (9 + 1)^{\frac{1}{2}} = (3^2 + 1)^{\frac{1}{2}}$$

Expanding this by the Binomial Theorem,

$$(3^2 + 1)^{\frac{1}{2}} = (3^2)^{\frac{1}{2}} + \frac{1}{2}(3^2)^{-\frac{1}{2}} - \frac{1}{8}(3^2)^{-\frac{3}{2}} + \frac{1}{16}(3^2)^{-\frac{5}{2}}$$

$$- \frac{5}{128}(3^2)^{-\frac{7}{2}} + \ldots\ldots$$

$$= 3 + \frac{1}{2} \,.\, 3^{-1} - \frac{1}{8} \,.\, 3^{-3} + \frac{1}{16} \,.\, 3^{-5} - \frac{5}{128} \,.\, 3^{-7} + \ldots\ldots$$

$$= 3 + \frac{1}{2 \,.\, 3} - \frac{1}{8 \,.\, 3^3} + \frac{1}{16 \,.\, 3^5} - \frac{5}{128 \,.\, 3^7} + \ldots\ldots$$

$$= 3 + .16667 - .00463 + .00026 - .00002 + \ldots\ldots$$

$$= 3.16228 +,$$

which is the approximate square root of 10 to the fifth decimal place, as may be verified by evolution.

2. Find the approximate cube root of 26.

$$\sqrt[3]{26} = 26^{\frac{1}{3}} = (27 - 1)^{\frac{1}{3}} = (3^3 - 1)^{\frac{1}{3}}$$

Expanding this by the Binomial Theorem,

$$(3^3 - 1)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} + \frac{1}{3}(3^3)^{-\frac{2}{3}}(-1) - \frac{1}{9}(3^3)^{-\frac{5}{3}}(-1)^2$$
$$+ \frac{5}{81}(3^3)^{-\frac{8}{3}}(-1)^3 - \ldots\ldots$$

$$= 3 - \frac{1}{3} . 3^{-2} - \frac{1}{9} . 3^{-5} - \frac{5}{81} . 3^{-8} - \ldots\ldots$$

$$= 3 - \frac{1}{3 . 3^2} - \frac{1}{9 . 3^5} - \frac{5}{81 . 3^8} - \ldots\ldots$$

$$= 3 - .037037 - .000457 - .000009 - \ldots\ldots$$

$$= 2.962497 +, \textit{Ans.}$$

RULE.

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root. Expand the result by the Binomial Theorem.

Note. If the second term of the binomial is small, the terms in the expansion converge rapidly, and we obtain an approximate value of the required root by taking the sum of a few terms of the development. But if the second term is large, the terms converge slowly, and it requires the sum of many terms to insure a considerable degree of accuracy.

EXAMPLES.

Find the approximate values of the following to five decimal places:

3. $\sqrt[3]{31}$.

4. $\sqrt[3]{9}$.

5. $\sqrt{99}$.

6. $\sqrt[5]{29}$.

7. $\sqrt[4]{17}$.

8. $\sqrt[4]{78}$.

XL. — SUMMATION OF INFINITE SERIES.

428. The **Summation of a Series** is the process of finding a finite expression equivalent to the series.

Different series require different methods of summation, according to the nature of the series, or the law of its formation. Methods of summing arithmetical and geometrical series have already been given (Arts. 369, 377, and 380). Methods applicable to other series will now be treated.

RECURRING SERIES.

429. A **Recurring Series** is one in which each term, after some fixed term, bears a uniform relation to a fixed number of the preceding terms. Thus

$$1 + 2x + 3x^2 + 4x^3 + \dots\dots$$

is a recurring series, in which each term, after the second, is equal to the product of the preceding term by $2x$, plus the product of the next term but one preceding by $-x^2$.

The sum of these constant multipliers is called the *scale of relation* of the series, and their coefficients constitute the scale of relation of the coefficients of the series. For example, in the series $1 + 2x + 3x^2 + 4x^3 + \dots\dots$, the scale of relation is $2x - x^2$, and the scale of relation of the coefficients is $2 - 1$.

430. A recurring series is said to be of the *first order* when each term, commencing with the second, depends on the *one* immediately preceding; of the *second order*, when each term, commencing with the third, depends upon the *two* immediately preceding; and so on.

If the series is of the first order, the scale of relation will consist of one term; if of the second order, it will consist of two terms; and, in general, the order and the number of terms in the scale of relation will correspond.

431. *To find the scale of relation of the coefficients of a recurring series.*

1. If the series is of the first order, it is a simple geometrical progression, and the scale of relation of the coefficients is found by dividing the coefficient of any term by the coefficient of the preceding term.

2. If the series is of the second order, let $a, b, c, d, \ldots\ldots$ represent the consecutive coefficients of the series, and $p + q$ their scale of relation. Then,

$$\left.\begin{array}{l} c = p\,b + q\,a \\ d = p\,c + q\,b \end{array}\right\} (A)$$

to determine p and q; solving, we obtain

$$p = \frac{a\,d - b\,c}{a\,c - b^2}, \text{ and } q = \frac{c^2 - b\,d}{a\,c - b^2}.$$

3. If the series is of the third order, let $a, b, c, d, e, f, \ldots\ldots$ represent the consecutive coefficients of the series, and $p + q + r$ their scale of relation. Then,

$$d = p\,c + q\,b + r\,a$$
$$e = p\,d + q\,c + r\,b$$
$$f = p\,e + q\,d + r\,c$$

from which we can find p, q, and r.

432. To ascertain the order of a series, we may first make trial of a scale of two terms, and if the result does not correspond with the series, we may try three terms, four terms, and so on, till the true scale of relation is found. If we assume the series to be of too high an order, the terms of the scale will take the form $\frac{0}{0}$.

433. *To find the sum of a recurring series, when the scale of relation of its coefficients is known.*

Let

$$a + bx + cx^2 + dx^3 + \ldots\ldots + jx^{n-3} + kx^{n-2} + lx^{n-1} + \ldots\ldots$$

be a recurring series of the second order. Let S denote the sum of n terms of the series; and let $p+q$ be the scale of relation of the coefficients. Then,

$$S=a+b\,x+c\,x^2+d\,x^3+\ldots\ldots+l\,x^{n-1}$$

$$p\,Sx=p\,a\,x+p\,b\,x^2+p\,c\,x^3+\ldots\ldots+p\,k\,x^{n-1}+p\,l\,x^n$$

$$q\,Sx^2=q\,a\,x^2+q\,b\,x^3+\ldots\ldots+q\,j\,x^{n-1}+q\,k\,x^n+q\,l\,x^{n+1}$$

Subtracting the last two equations from the first,

$$S-p\,Sx-q\,Sx^2=a+b\,x-p\,a\,x-p\,l\,x^n-q\,k\,x^n-q\,l\,x^{n+1}$$

the rest of the terms of the second member disappearing, because, since $p+q$ is the scale of relation of the coefficients,

$$c=p\,b+q\,a,\; d=p\,c+q\,b,\;\ldots\ldots\; l=p\,k+q\,j.$$

Therefore we have

$$S=\frac{a+(b-p\,a)\,x-(p\,l+q\,k)\,x^n-q\,l\,x^{n+1}}{1-p\,x-q\,x^2}$$

the formula for finding the sum of n terms of a recurring series of the second order.

But if n becomes indefinitely great, and the series is convergent, then the limiting values of the terms which involve x^n and x^{n+1} must become 0, and we have at the limit

$$S=\frac{a+(b-p\,a)\,x}{1-p\,x-q\,x^2} \qquad (1)$$

the formula for finding the sum of an infinite recurring series of the second order.

If $q=0$, then the series is of the first order, and consequently $b=p\,a$; then,

$$S=\frac{a}{1-p\,x} \qquad (2)$$

the formula for finding the sum of an infinite recurring series of the first order. (Compare Art. 380.)

In like manner, we should obtain

$$S = \frac{a + (b - p\,a)\,x + (c - p\,b - q\,a)\,x^2}{1 - p\,x - q\,x^2 - r\,x^3} \tag{3}$$

the formula for the summation of an infinite recurring series of the third order.

434. A recurring series, like other infinite series, originates from an irreducible fraction, called the *generating fraction.* The summation of the series, therefore, reproduces the fraction; the operation being, in fact, the exact reverse of that in Art. 414.

435. **1.** Find the sum of $1 + 2x + 8x^2 + 28x^3 + 100x^4 + \ldots\ldots$

We must first determine the scale of relation of the coefficients. In accordance with Art. 432, we first assume the series to be of the second order. We have $a = 1$, $b = 2$, $c = 8$, $d = 28$. Substituting in the values of p and q derived from (A), Art. 431, we have $p = 3$ and $q = 2$. To ascertain if this is the proper scale of relation, consider the fifth term, $100\,x^4$; this should be $3\,x$ times the preceding term, plus $2\,x^2$ times the next preceding term but one, or, $84\,x^4 + 16\,x^4$. This shows that the series is of the second order.

Substituting in (1) the values of a, b, p, and q, we have

$$S = \frac{1 + (2 - 3)\,x}{1 - 3\,x - 2\,x^2} = \frac{1 - x}{1 - 3\,x - 2\,x^2},\ Ans.$$

EXAMPLES.

Find the sum of the following series:

2. $1 + 2\,x + 3\,x^2 + 5\,x^3 + 8\,x^4 + \ldots\ldots$

3. $\frac{a}{b} - \frac{ac}{b^2}\,x + \frac{a\,c^2}{b^3}\,x^2 - \frac{a\,c^3}{b^4}\,x^3 + \ldots\ldots$

4. $4 + 9\,x + 21\,x^2 + 51\,x^3 + \ldots\ldots$

5. $1 + 3\,x + 5\,x^2 + 7\,x^3 + \ldots\ldots$

6. $2 - a + 2a^2 - 5a^3 + 10a^4 - 17a^5 + \ldots\ldots$

7. $3 + 5x + 7x^2 + 13x^3 + 23x^4 + 45x^5 + \ldots\ldots$

8. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + \ldots\ldots$

9. $2 + 4x - x^2 - 3x^3 + 2x^4 + 4x^5 + \ldots\ldots$

DIFFERENTIAL METHOD.

436. The **Differential Method** is the process of finding any term, or the sum of any number of terms, of a regular series, by means of the successive differences of its terms.

437. If, in any series, we take the first term from the second, the second from the third, the third from the fourth, and so on, the remainders will form a new series called the *first order of differences.*

If the differences be taken in this new series in like manner, we obtain a series called the *second order of differences;* and so on.

Thus, if the given series is

$$1,\quad 8,\quad 27,\quad 64,\quad 125,\quad 216,\ \ldots\ldots$$

the successive orders of differences will be as follows:

1st order,	7, 19, 37, 61, 91, ...
2d order,	12, 18, 24, 30, ...
3d order,	6, 6, 6, ...
4th order,	0, 0, ...

Hence, in this case there are only three orders of differences.

438. *To find any term of a series.*

Let the series be

$$a_1,\quad a_2,\quad a_3,\quad a_4,\quad a_5,\ \ldots\ldots\ a_n,\quad a_{n+1},\ \ldots\ldots$$

Then the first order of differences will be

$$a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,\ a_5 - a_4,\ \ldots\ldots\ a_{n+1} - a_n,\ \ldots\ldots,$$

the second order of differences will be

$$a_3 - 2\,a_2 + a_1,\ a_4 - 2\,a_3 + a_2,\ a_5 - 2\,a_4 + a_3,\ \ldots\ldots,$$

the third order of differences will be

$$a_4 - 3\,a_3 + 3\,a_2 - a_1,\ a_5 - 3\,a_4 + 3\,a_3 - a_2,\ \ldots\ldots,$$

the fourth order of differences will be

$$a_5 - 4\,a_4 + 6\,a_3 - 4\,a_2 + a_1,\ \ldots\ldots,$$

and so on; where each difference, although a compound quantity, is called a *term*.

Let now d_1, d_2, d_3, d_4, ... represent the first terms of the several orders of differences. Then,

$d_1 = a_2 - a_1$; whence, $a_2 = a_1 + d_1$.

$d_2 = a_3 - 2\,a_2 + a_1$; whence, $a_3 = 2\,a_2 - a_1 + d_2 = 2\,a_1 + 2\,d_1 - a_1 + d_2 = a_1 + 2\,d_1 + d_2$.

$d_3 = a_4 - 3\,a_3 + 3\,a_2 - a_1$; whence, $a_4 = a_1 + 3\,d_1 + 3\,d_2 + d_3$.

$d_4 = a_5 - 4\,a_4 + 6\,a_3 - 4\,a_2 + a_1$; whence, $a_5 = a_1 + 4\,d_1 + 6\,d_2 + 4\,d_3 + d_4$.

....

We observe that the coefficients of the value of a_2 are the same as the coefficients of the *first* power of a binomial; the coefficients of the value of a_3 are the same as the coefficients of the *second* power of a binomial; and so on. Assume that this law holds for the nth term; that is, that the coefficients of the value of a_n are the same as the coefficients of the $(n-1)$th power of a binomial; then,

$$a_n = a_1 + (n-1)\,d_1 + \frac{(n-1)(n-2)}{\underline{|2}}\,d_2 + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}\,d_3 + \ldots\ldots \qquad (1)$$

If the law holds for the nth term in the given series, it will also hold for the nth term in the first order of differences; or,

$$a_{n+1} - a_n = d_1 + (n-1)\,d_2 + \frac{(n-1)(n-2)}{\underline{|2}}\,d_3 + \ldots\ldots \qquad (2)$$

Adding (1) and (2), we have

$$a_{n+1} = a_1 + [1 + (n-1)]\,d_1 + \left[(n-1) + \frac{(n-1)(n-2)}{\underline{|2}}\right] d_2$$

$$+ \left[\frac{(n-1)(n-2)}{\underline{|2}} + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}\right] d_3 + \ldots\ldots$$

$$= a_1 + n\,d_1 + \frac{n-1}{\underline{|2}}[2 + n - 2]\,d_2$$

$$+ \frac{(n-1)(n-2)}{\underline{|3}}[3 + n - 3]\,d_3 + \ldots\ldots$$

$$= a_1 + n\,d_1 + \frac{n(n-1)}{\underline{|2}}\,d_2 + \frac{n(n-1)(n-2)}{\underline{|3}}\,d_3 + \ldots\ldots \qquad (3)$$

where the coefficients are the same as the coefficients of the nth power of a binomial. Hence, if the law holds for the nth term, it also holds for the $(n+1)$th term; but we have shown it to hold for the fifth term, a_5; hence it holds for the sixth term; and so on. That is, Formula (1) holds for any term in the series.

When the differences finally become 0, the value of the nth term can be obtained exactly; but, in other cases, the result is merely an approximate value.

439. *To find the sum of any number of terms of a series.*

Let the series be

$$a, \quad b, \quad c, \quad d, \quad e, \ \ldots\ldots \qquad (1)$$

Let S denote the sum of the first n terms. Assume the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \ldots\ldots \qquad (2)$$

in which the $(n+1)$th term is obviously equal to the sum of n terms of the given series; that is, S is the $(n+1)$th term of series (2). Now the first order of differences of series (2) is the same as series (1); hence, the second order of differences of series (2) is the same as the first order of (1); the third order of (2) is the same as the second order of (1); and so on. Then, letting a', d'_1, d'_2, d'_3, ... represent the first term, and the first terms of the several orders of differences of (2), we have $a'=0$, $d'_1=a$, $d'_2=d_1$, $d'_3=d_2$,...... where a, d_1, d_2,...... are the first term, and the first terms of the several orders of differences of (1). But, by (3), Art. 438, the $(n+1)$th term of series (2) will be

$$a'+n\,d'_1+\frac{n\,(n-1)}{\underline{|2}}\,d'_2+\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\,d'_3+\ldots\ldots$$

In this put for a', d'_1, d'_2, d'_3, ... their values; then

$$S=n\,a+\frac{n\,(n-1)}{\underline{|2}}\,d_1+\frac{n\,(n-1)\,(n-2)}{\underline{|3}}\,d_2+\ldots\ldots \quad (3)$$

440. **1.** Find the 12th term of the series 2, 6, 12, 20, 30, ...

The successive orders of differences will be as follows:

1st order,	4,	6,	8,	10,	......
2d order,		2,	2,	2,	......
3d order,			0,	0,	......

Then $a_1=2$, $d_1=4$, $d_2=2$, d_3, d_4, ...$=0$, and $n=12$.

Substituting in (1), Art. 438, the 12th term

$$=2+(12-1)\,4+\frac{(12-1)\,(12-2)}{\underline{|2}}\,2=2+44+110=156, \textit{Ans.}$$

2. Find the sum of 8 terms of the series 2, 5, 10, 17, ...

1st order of differences,	3,	5,	7,	......
2d order of differences,		2,	2,	......
3d order of differences,			0,	......

Then $a=2,\ d_1=3,\ d_2=2,\ n=8.$

Substituting these values in (3), Art. 439, we have

$$S = 8 \times 2 + \frac{8(8-1)}{\lfloor 2} 3 + \frac{8(8-1)(8-2)}{\lfloor 3} 2$$

$$= 16 + 84 + 112 = 212, \ \textit{Ans.}$$

EXAMPLES.

3. Find the first term of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, ...

4. Find the first term of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, ...

5. Find the seventh term of the series 3, 5, 8, 12, 17, ...

6. Sum the first twelve terms of the series 1, 4, 10, 20, 35, ...

7. Sum the first hundred terms of the series 1, 2, 3, 4, 5, ...

8. Find the 15th term of the series 1^2, 2^2, 3^2, 4^2, ...

9. Sum the first n terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, ...

10. Sum the first n terms of the series 1, 2^4, 3^4, 4^4, 5^4, 6^4, ...

11. If shot be piled in the shape of a pyramid, with a triangular base, each side of which exhibits 9 shot, find the number contained in the pile.

12. If shot be piled in the shape of a pyramid, with a square base, each side of which exhibits 25 shot, find the number contained in the pile.

INTERPOLATION.

441. Interpolation is the process of introducing between terms of a series other terms conforming to the law of the series.

Its usual application is in finding *intermediate* numbers between those given in Mathematical Tables, which may be regarded as a series of equidistant terms.

442. The interpolation of any intermediate term in a series, is essentially finding the nth term of the series, by the differential method (Art. 438). Thus,

Let t represent the term to be interpolated in a series of equidistant terms, and p the distance the term t is removed from the first term, a, expressed in intervals and fractions of an interval; that is, p being the distance to the nth term, $p = n - 1$ intervals.

In Formula (1), Art. 438, putting p for $n-1$, the nth term

$$t = a + p\,d_1 + \frac{p\,(p-1)}{\underline{|2}}\,d_2 + \frac{p\,(p-1)\,(p-2)}{\underline{|3}}\,d_3 + \ldots\ldots$$

443. 1. In the series $\frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \ldots\ldots$, find the middle term between $\frac{1}{15}$ and $\frac{1}{16}$.

Here, the first differences of the denominators are

$$1, \quad 1, \quad 1, \quad 1, \quad \ldots\ldots$$

The second differences are

$$0, \quad 0, \quad 0, \quad \ldots\ldots$$

Whence, $\qquad d_1 = 1$, and $d_2 = 0$.

The distance to the required term is $2\frac{1}{2}$ intervals, or $p = \frac{5}{2}$. Make $a = 13$, the denominator of the first term; then by the preceding formula, the denominator of the required term,

$$t = 13 + \frac{5}{2} \times 1 = 13 + \frac{5}{2} = \frac{31}{2}.$$

Therefore the required term is $\dfrac{1}{\frac{31}{2}}$ or $\frac{2}{31}$, *Ans.*

2. Given $\sqrt{94} = 9.69536$, $\sqrt{95} = 9.74679$, $\sqrt{96} = 9.79796$; to find $\sqrt{94\frac{1}{4}}$.

Here, the first differences are

$$.05143, \quad .05117, \quad \ldots\ldots$$

and the second differences are

$$-.00026, \quad \ldots\ldots$$

Whence, $\quad d_1 = .05143, \; d_2 = -.00026, \; \ldots\ldots$

The distance of the required term is $\frac{1}{4}$ interval, or $p = \frac{1}{4}$.

Then the required term,

$$t = 9.69536 + \frac{1}{4} \times .05143 + \frac{\frac{1}{4}\left(\frac{1}{4} - 1\right)}{\lfloor 2}(-.00026) + \ldots\ldots$$

$$= 9.69536 + .01286 - \frac{3}{32}(-.00026) + \ldots\ldots$$

$$= 9.69536 + .01286 + .00002 + \ldots\ldots$$

$$= \text{(approximately) } 9.70824, \; \textit{Ans.}$$

EXAMPLES.

3. Given $\sqrt[3]{64} = 4$, $\sqrt[3]{65} = 4.0207$, $\sqrt[3]{66} = 4.0412$, $\sqrt[3]{67} = 4.0615$; find $\sqrt[3]{66.5}$.

4. Given $\sqrt[3]{45} = 3.556893$, $\sqrt[3]{47} = 3.608826$, $\sqrt[3]{49} = 3.659306$, $\sqrt[3]{51} = 3.708430$; find $\sqrt[3]{48}$.

5. Given $\sqrt{5} = 2.23607$, $\sqrt{6} = 2.44949$, $\sqrt{7} = 2.64575$, $\sqrt{8} = 2.82843$; find $\sqrt{5.01}$.

6. Given the length of a degree of longitude in latitude $41° = 45.28$ miles; in latitude $42° = 44.59$ miles; in latitude $43° = 43.88$ miles; in latitude $44° = 43.16$ miles. Find the length of a degree of longitude in latitude $42°\,30'$.

7. If the amount of \$1 at 7 per cent compound interest for 2 years is \$1.145, for 3 years \$1.225, for 4 years \$1.311, and for 5 years \$1.403, what is the amount for 4 years and 6 months?

XLI. — LOGARITHMS.

444. *The logarithm of a quantity to any given base, is the exponent of the power to which the base must be raised to equal the quantity.*

For example, if $a^x = m$, x is the exponent of the power to which the base, a, must be raised to equal the quantity, m; or, x is the logarithm of m to the base a; which is briefly expressed thus:

$$x = \log_a m.$$

445. If a remain fixed, and m receive different values, a certain value of x will correspond to each value of m; and these values of x taken together constitute a *System of Logarithms.* And as the base, a, may have any value whatever, the number of possible systems is unlimited.

For example, suppose $a = 3$.

Then, since $3^0 = 1$, by Art. 444, $0 = \log_3 1$

" $3^1 = 3$, " " $1 = \log_3 3$

" $3^2 = 9$, " " $2 = \log_3 9$

Hence, in the system whose base is 3, $\log 1 = 0$, $\log 3 = 1$, $\log 9 = 2$, etc.

Again, suppose $a = 12$.

Then, since $12^1 = 12$, $1 = \log_{12} 12$

" $12^2 = 144$, $2 = \log_{12} 144$

Hence, in the system whose base is 12, $\log 12 = 1$, $\log 144 = 2$, etc.

446. The only system in extensive use for numerical computations is the Common System or Briggs' System, whose base is 10. Therefore the definition of the common logarithm of a quantity is *the exponent of that power of* 10 *which equals the quantity.* Hence,

Since $10^0 = 1,$ $\qquad \log_{10} 1 = 0$

" $10^1 = 10,$ $\qquad \log_{10} 10 = 1$

" $10^2 = 100,$ $\qquad \log_{10} 100 = 2$

" $10^3 = 1000,$ $\qquad \log_{10} 1000 = 3$

" $10^{-1} = \frac{1}{10} = .1,$ $\qquad \log_{10} .1 = -1$

" $10^{-2} = \frac{1}{10^2} = .01,$ $\qquad \log_{10} .01 = -2$

" $10^{-3} = \frac{1}{10^3} = .001,$ $\qquad \log_{10} .001 = -3,$ etc.

447. It is customary in using common logarithms to omit the subscript 10 which denotes the base; hence, we may write the results of Art. 446 as follows:

$\log 1 = 0$ $\qquad \log .1 = -1 = 9 - 10$

$\log 10 = 1$ $\qquad \log .01 = -2 = 8 - 10$

$\log 100 = 2$ $\qquad \log .001 = -3 = 7 - 10$

$\log 1000 = 3$ $\qquad$ etc.

The second form of the results in the second column will be found less complicated in the solution of examples.

448. We infer the following from the first column of Art. 447:

The logarithm of any number between 1 and 10, lies between 0 and 1.

The logarithm of any number between 10 and 100, lies between 1 and 2.

The logarithm of any number between 100 and 1000, lies between 2 and 3, etc.

Or, in other words,

The logarithm of any number with *one* figure to the left of its decimal point, is equal to 0 plus some decimal.

The logarithm of any number with *two* figures to the left of its decimal point, is equal to 1 plus some decimal.

The logarithm of any number with *three* figures to the left of its decimal point, is equal to 2 plus some decimal, etc.

449. Reasoning in the same way from the second column of Art. 447,

The logarithm of any number between 1 and .1, lies between 0 and 9 — 10, or between 10 — 10 and 9 — 10.

The logarithm of any number between .1 and .01, lies between 9 — 10 and 8 — 10.

The logarithm of any number between .01 and .001, lies between 8 — 10 and 7 — 10, etc.

Or, in other words,

The logarithm of any decimal with *no* zeros between its point and first figure, is equal to 9 plus some decimal — 10.

The logarithm of any decimal with *one* zero between its point and first figure, is equal to 8 plus some decimal — 10.

The logarithm of any decimal with *two* zeros between its point and first figure, is equal to 7 plus some decimal — 10, etc.

450. It will be seen from the two preceding articles that in general the logarithm of a number consists of two parts, one integral, the other decimal. The integral part is called the *characteristic*; the decimal part, the *mantissa*. For reasons which will be given hereafter, only the mantissa of the logarithm is given in the tables; the characteristic must be supplied by the reader. The rules for characteristic are based on the results obtained in the last parts of Arts. 448 and 449.

451. I. *If the number is greater than* 1, *the characteristic is* 1 *less than the number of figures to the left of the decimal point.*

For example, characteristic of log 354.89 = 2,

characteristic of log 906328.3 = 5, etc.

II. *If the number is less than* 1, *the characteristic is found by subtracting the number of zeros between the decimal point and first significant figure from* 9; *writing* — 10 *after the mantissa.*

For example, characteristic of log .00792 = 7, with — 10 after the mantissa; characteristic of log .2583 = 9, with — 10 after the mantissa; etc.

It is customary in ordinary computation to omit the — 10 after the mantissa; it should be remembered, however, that it is really a part of the logarithm, and should be allowed for, and subjected to precisely the same operations as the rest of the logarithm. Beginners will find it useful to write it in all cases; and in some problems it cannot conveniently be omitted.

Note. Many writers, in dealing with the characteristics of the logarithms of numbers less than 1, combine the two portions of the characteristic, writing the result as a *negative characteristic* before the mantissa. Thus, instead of such an expression as 7.603582 − 10, the student will frequently find $\bar{3}.603582$; a minus sign being written *over* the characteristic, to denote that it alone is negative, the mantissa being always positive. The objection to this notation is the inconvenience of using numbers partly positive and partly negative.

PROPERTIES OF LOGARITHMS.

452. *In any system the logarithm of unity is zero.*

For, since $a^0 = 1$, for any value of a, $0 = \log_a 1$.

453. *In any system the logarithm of the base itself is unity.*

For, since $a^1 = a$, for any value of a, $1 = \log_a a$.

454. *In any system, whose base is greater than unity, the logarithm of zero is minus infinity.*

$$\text{For, since } a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0, \ -\infty = \log_a 0.$$

If the base is less than unity, the logarithm of 0 is $+\infty$.

455. *In any system the logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.*

Assume the equations,

$$\left.\begin{matrix} a^x = m \\ a^y = n \end{matrix}\right\} \text{whence, by Art. 444, } \left\{\begin{matrix} x = \log_a m \\ y = \log_a n \end{matrix}\right.$$

Multiplying, $a^x \times a^y = m\,n$, or $a^{x+y} = m\,n$

Whence, $x + y = \log_a m\,n$

Substituting values of x and y,

$$\log_a m\,n = \log_a m + \log_a n.$$

If there are three factors, m, n, and p,

$$\log_a m\,n\,p = \log_a (m\,n \times p) = (\text{Art. 455}) \log_a m\,n + \log_a p$$

$$= \log_a m + \log_a n + \log_a p.$$

An extension of this method will prove the theorem for any number of factors.

By the application of this theorem, we may find the logarithm of a number, provided we know the logarithm of each of its factors. For example, given $\log 2 = 0.301030$, $\log 3 = 0.477121$, required $\log 72$.

$$\begin{aligned} \log 72 &= \log (2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \\ &= 3 \times \log 2 + 2 \times \log 3 \\ &= 0.903090 + 0.954242 = 1.857332, \ \textit{Ans.} \end{aligned}$$

EXAMPLES.

Given $\log 2 = 0.301030$, $\log 3 = 0.477121$, $\log 7 = 0.845098$, calculate :

1. log 48. **4.** log 98. **7.** log 168. **10.** log 3087.

2. log 441. **5.** log 84. **8.** log 7056. **11.** log 15552.

3. log 56. **6.** log 567. **9.** log 504. **12.** log 14406.

456. *In any system the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations,

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\} \text{ whence, } \left\{\begin{array}{l} x = \log_a m \\ y = \log_a n \end{array}\right.$$

Dividing, $$\frac{a^x}{a^y} = \frac{m}{n}, \text{ or } a^{x-y} = \frac{m}{n}$$

Whence, $$x - y = \log_a \frac{m}{n}$$

Substituting values of x and y,

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

By this theorem, a logarithm being given, we may derive certain others from it. For instance, if we know $\log 2 = 0.301030$, then

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1. - 0.301030 = 0.698970.$$

EXAMPLES.

Given $\log 2 = 0.301030$, $\log 3 = 0.477121$, $\log 7 = 0.845098$, calculate :

1. log 15.	**4.** log 175.	**7.** log $7\frac{1}{7}$.
2. log 125.	**5.** log $3\frac{1}{3}$.	**8.** log $\frac{35}{3}$.
3. log $\frac{10}{7}$.	**6.** log $11\frac{1}{9}$.	**9.** log $5\frac{4}{9}$.

457. *In any system the logarithm of any power of a quantity is equal to the logarithm of the quantity, multiplied by the exponent of the power.*

Assume the equation,

$$a^x = m, \text{ whence, } x = \log_a m$$

Raising both members of the assumed equation to the pth power,

$$(a^x)^p = m^p, \text{ or } a^{px} = m^p$$

Whence, $$px = \log_a m^p$$

Substituting the value of x,

$$\log_a m^p = p \log_a m.$$

458. *In any system the logarithm of any root of a quantity is equal to the logarithm of the quantity, divided by the index of the root.*

For, $$\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = (\text{Art. } 457)\ \frac{1}{r} \log_a m.$$

459. *In the common system, the mantissæ of the logarithms of all numbers having the same sequence of figures will be the same.*

For example, suppose we know that $\log 3.053 = .484727$.

Then, $\log 30.53 = \log(3.053 \times 10) = \log 3.053 + \log 10 = .484727 + 1 = 1.484727$.

Also, $\log 30530 = \log(3.053 \times 10000) = \log 3.053 + \log 10000 = .484727 + 4 = 4.484727$.

Again, $\log .03053 = \log\left(\frac{3.053}{100}\right) = \log 3.053 - \log 100 = .484727 - 2 = .484727 + 8 - 10 = 8.484727 - 10.$

It is clear, then, that if a number be multiplied or divided by any integral power of 10, thereby producing another number having the same sequence of figures, the mantissæ of their logarithms will be the same.

Or, to illustrate, if $\log 3.053 = .484727$,

then,

$\log 30.53 = 1.484727$	$\log .3053 = 9.484727 - 10$
$\log 305.3 = 2.484727$	$\log .03053 = 8.484727 - 10$
$\log 3053. = 3.484727$	$\log .003053 = 7.484727 - 10$
etc.	etc.

We may now see the reason why, as stated in Art. 450, only the mantissæ are given in the table; for if we wish to find the logarithm of any number, we have only to find the mantissa of the sequence of figures composing it from the table, and can prefix the proper characteristic, depending on the position of the decimal point, in accordance with the rules stated in Art. 451. This property of logarithms is only enjoyed by the common system, and constitutes its superiority over all others.

460. *Given the logarithm of a quantity to a certain base, to calculate the logarithm of the same quantity to any other base.*

Assume the equations,

$$\left.\begin{matrix} a^x = m \\ b^y = m \end{matrix}\right\} \text{ whence, } \left\{\begin{matrix} x = \log_a m \\ y = \log_b m \end{matrix}\right.$$

From the assumed equations, $a^x = b^y$

Hence, $$(a^x)^{\frac{1}{y}} = (b^y)^{\frac{1}{y}}, \text{ or } a^{\frac{x}{y}} = b$$

Whence, $$\frac{x}{y} = \log_a b$$

or, $$y = \frac{x}{\log_a b}$$

Substituting the values of x and y,

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

That is, if we know the logarithm of m to a certain base, a, its logarithm to any other base, b, is found by dividing by the logarithm of b to the base a.

461. *To show that* $\log_a b \times \log_b a = 1$, *for any values of* a *and* b.

Assume the equation,

$$a^x = b, \text{ whence } x = \log_a b$$

Taking the $\frac{1}{x}$ power of both members,

$$(a^x)^{\frac{1}{x}} = b^{\frac{1}{x}}, \text{ or } b^{\frac{1}{x}} = a$$

Whence,
$$\frac{1}{x} = \log_b a$$

Therefore,
$$\log_a b \times \log_b a = x \times \frac{1}{x} = 1.$$

462. We append a few examples to illustrate the applications of Arts. 455, 456, 457, and 458.

1. $\log \left(\frac{a}{b}\right)^{\frac{c}{d}} = \frac{c}{d} \log \frac{a}{b},$ (Art. 457)

$= \frac{c}{d} (\log a - \log b),$ (Art. 456).

2. $\log \frac{\sqrt[n]{a} \times \sqrt[m]{b}}{\sqrt[p]{c}} = \log (\sqrt[n]{a} \times \sqrt[m]{b}) - \log \sqrt[p]{c},$ (Art. 456)

$= \log \sqrt[n]{a} + \log \sqrt[m]{b} - \log \sqrt[p]{c},$ (Art. 455)

$= \frac{1}{n} \log a + \frac{1}{m} \log b - \frac{1}{p} \log c,$ (Art. 458).

The following are proposed as exercises:

3. $\log \frac{a\,b\,c}{d\,e} = \log a + \log b + \log c - \log d - \log e.$

4. $\log\left(\sqrt[n]{a} \times b^3 \times c^{\frac{d}{2}}\right) = \frac{1}{n}\log a + 3\log b + \frac{d}{2}\log c.$

5. $\log \frac{2^{\frac{2}{3}}}{3^{\frac{5}{6}}} = \frac{2}{3}\log 2 - \frac{5}{6}\log 3.$

6. $\log \sqrt[n]{\frac{a^2}{b\,c}} = \frac{1}{n}(2\log a - \log b - \log c).$

7. $\log \frac{\sqrt[n]{a\,b}}{\sqrt[m]{c}} = \frac{1}{n}(\log a + \log b) - \frac{1}{m}\log c.$

8. $\log \frac{\sqrt[4]{a}}{b\,c^{\frac{2}{3}}\,d^2} = \frac{1}{4}\log a - \log b - \frac{2}{3}\log c - 2\log d.$

9. $\log\left(\sqrt[5]{\frac{a}{b}} \div (c\,d)^{-\frac{m}{n}}\right) = \frac{1}{5}(\log a - \log b) + \frac{m}{n}(\log c + \log d).$

USE OF THE TABLE.

463. The table (Appendix) gives the mantissæ of the logarithms of all numbers from 1 to 10000, calculated to six decimal places. On the first page of the table are the logarithms of the numbers between 1 and 100. This table is added simply for convenience, as the same mantissæ are to be found in the rest of the table.

To find the logarithm of any number consisting of four figures.

Find, in the column headed N, the first three figures of the given number. Then the mantissa of the required logarithm will be found in the horizontal line corresponding, in the vertical column which has the fourth figure of the given number at the top. If only the last four figures of the mantissa are found, the first two figures may be obtained from the nearest mantissa above, in the same vertical column, which consists of six figures. Finally, prefix the proper characteristic (Art. 451).

For example, $\log 140.8 = 2.148603$
$\log .05837 = 8.766190 - 10$
$\log 8516. = 3.930236$

For a number consisting of one or two figures, use the first page of the table, which needs no explanation; for a number of three figures, look in the column headed N, and take the mantissa corresponding in the column headed 0. For example, $\log 94.6 = 1.975891$.

464. *To find the logarithm of a number of more than four figures.*

For example, let it be required to find log 3296.78.

From the table, we find $\log 3296 = 3.517987$
$\log 3297 = 3.518119$

That is, an increase of one unit in the number produces an increase of .000132 in the logarithm. Then evidently an increase of .78 unit in the number will produce an increase of $.78 \times .000132$ in the logarithm $= .000103$ to the nearest sixth decimal place.

Therefore, $\log 3296.78 = \log 3296 + .000103$
$= 3.517987 + .000103 = 3.518090$, *Ans.*

Note. The foregoing method is based upon the assumption that the differences of logarithms are proportional to the differences of their corresponding numbers, which is not strictly correct, but is sufficiently exact for practical purposes.

We derive the following rule from the above operation:

Find in the table the mantissa of the first four figures, without regard to the position of the decimal point.

Find the difference between this and the mantissa of the next higher number of four figures; (called the *tabular difference*, and to be found in the column headed D on each page; see Note on page 350.)

Multiply the tabular difference by the rest of the figures of the given number, with a decimal point before them.

Add the result to the mantissa of the first four figures.

Prefix the proper characteristic.

1. Find the logarithm of .02243076.

	Mantissa of 2243 = 350829
Tabular difference = 194	15
.076	350844
1.164	
13.58	
Correction = 14.744 = 15 nearly.	

Ans. 8.350844 − 10.

Note. To find the tabular difference mentally, subtract the last figure of the mantissa from the last figure of the next larger, and take the nearest whole number ending in that figure to the number in the column headed D in the same line. For instance, in finding log .02243076, the last figure of the mantissa of 2243 is 9, and of the next larger mantissa, 3; 9 from 13 leaves 4, and the nearest number ending in 4 to 193, the number in the column headed D, is 194, the proper tabular difference.

EXAMPLES.

Find the logarithms of the following numbers:

2. .053.	**6.** 33.6908.	**10.** 912.255.
3. 51.8.	**7.** .0602851.	**11.** .876092.
4. .2956.	**8.** 65000.63.	**12.** 7303.078.
5. 1.0274.	**9.** .001030741.	**13.** .0436927.

14. Given log 7.83 = .89376, log 7.84 = .89432; find log 78309.

15. Given log .05229 = 8.718419 − 10, log .05230 = 8.718502 − 10; find log 52.2938.

16 Given log 315.08 = 2.4984208, log 315.09 = 2.4984346; find log .003150823.

17. Given log 18.84 = 1.275081, log 18.87 = 1.275772; find log .188527.

18. Given log 9.5338 = .9792660, log 9.5342 = .9792843; find log 95 34071.

465. *To find the number corresponding to a logarithm.*

For example, let it be required to find the number whose logarithm is 3.693845.

Since the characteristic depends only on the position of the decimal point, and in no way affects the *sequence of figures* corresponding, we ought to obtain all of the number corresponding, except the decimal point, by considering the mantissa only. We find in the table the mantissa 693815, of which the corresponding number is 4941, and the mantissa 693903, of which the corresponding number is 4942.

That is, an increase of 88 in the mantissa produces an increase of one unit in the number corresponding. Hence, an increase of 30 in the mantissa will produce an increase of $\frac{30}{88}$ of a unit in the number, or .34 nearly. Therefore,

$$\text{Number corresponding} = 4941 + .34 = 4941.34, \textit{ Ans.}$$

We base the following rule on the above operation:

Find in the table the next less mantissa, the four figures corresponding, and the tabular difference.

Subtract the next less mantissa from the given mantissa.

Divide the remainder by the tabular difference; (the quotient in general cannot be depended upon to more than two decimal places.)

Annex all of the quotient except the decimal point to the first four figures of the number.

Point off.

Note. The rules for pointing off are the reverse of the rules for characteristic given in Art. 451:

I. *If* -10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of figures to the left of the decimal point.*

II. *If* -10 *is written after the mantissa, subtract the characteristic from* 9; *giving the number of zeros to be placed between the decimal point and first figure.*

1. Find the number whose logarithm is 7.950185 − 10.

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad\quad\quad 950185 \\ \text{Next less mantissa} = \underline{950170};\ \text{four figures corresponding} = 8916. \\ \text{Tabular difference} = 49)\ 15.00\ (.31\ \text{nearly.} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{147} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ 30 \end{array}$$

Therefore, number corresponding = .00891631, *Ans.*

EXAMPLES.

Find the numbers corresponding to the following:

2. 1.880814.

3. 9.470410 − 10.

4. 0.820204.

5. 4.745126.

6. 8.044891 − 10.

7. 2.270293.

8. 9.350064 − 10.

9. 3.000027.

10. 0.990191.

11. 7.115658 − 10.

12. 8.535003 − 10.

13. 1.670180.

14. Given log 113 = 2.05308, log 114 = 2.05690; find number corresponding to 1.05411.

15. Given log .08630 = 8.936011 − 10, log .08631 = 8.936061 − 10; find number corresponding to 0.936049.

16. Given log 2.0702 = .3160123, log 2.0703 = .3160333; find number corresponding to 9.3160138 − 10.

17. Given log 548.3 = 2.739018, log 548.9 = 2.739493; find number corresponding to 7.739416 − 10.

18. Given log 7.3488 = .8662164, log 7.3492 = .8662401; find number corresponding to 2.8662350.

466. In the application of Arts. 455, 456, 457, and 458, we have to perform the operations of Addition, Subtraction, Multiplication, and Division with logarithms. As some of the problems which may arise are peculiar, we give a few hints as to their solution, which will be found of service.

1. Addition. If, in the sum, − 10, − 20, − 30, etc., are written after the mantissa, and the characteristic standing be-

fore the mantissa is greater than 9, subtract from both parts of the logarithm such a multiple of 10 as will make the characteristic before the mantissa less than 10.

For example, 13.354802 — 10 should be changed to 3.354802; 28.964316 — 30 should be changed to 8.964316 — 10; etc.

2. SUBTRACTION. In subtracting a larger logarithm from a smaller, or in subtracting a negative logarithm from a positive, the characteristic of the minuend should be increased by 10, — 10 being written after the mantissa to compensate.

For example, to subtract 3.121468 from 2.503964, we write the minuend in the form 12.503964 — 10; subtracting from this 3.121468, we have as a result 9.382496 — 10.

To subtract 9.635321 — 10 from 9.583427 — 10, we write the minuend in the form 19.583427 — 20; subtracting from this 9.635321 — 10, we have as a result 9.948106 — 10.

3. MULTIPLICATION. The hint already given for reducing the result of Addition, applies with equal force to Multiplication.

To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

4. DIVISION. In dividing a negative logarithm, add to both parts of the logarithm such a multiple of 10 as will make the quantity after the mantissa exactly divisible by the divisor, with — 10 as the quotient.

For example, to divide 7.402938 — 10 by 6, we add 50 to both parts of the logarithm, giving 57.402938 — 60. Dividing this by 6, we have as a result 9.567156 — 10.

EXAMPLES.

1. Add 9.096004 — 10, 4.581726, and 8.447510 — 10.

2. Add 7.196070 — 10, 8.822209 — 10, and 2.205683.

3. Subtract 0.659321 from 0.511490.

4. Subtract 7.901338 — 10 from 1.009800.

5. Subtract 9.156243 — 10 from 8.750404 — 10.

6. Multiply $9.105107 - 10$ by 3.

7. Divide $8.452633 - 10$ by 4.

8. Divide $9.670392 - 10$ by 11.

9. Multiply $9.668311 - 10$ by $\frac{2}{7}$.

SOLUTIONS OF ARITHMETICAL PROBLEMS BY LOGARITHMS.

467. In finding the value of any arithmetical quantity by logarithms, we first find the logarithm of the quantity, as in Art. 462, by the aid of the table, and then find the number corresponding to the result.

1. Find the value of $.0631 \times 7.208 \times 512.72$.

By Art. 455, $\log (.0631 \times 7.208 \times 512.72) = \log .0631 + \log 7.208 + \log 512.72$

$$\begin{aligned} \log .0631 &= 8.800029 - 10 \\ \log 7.208 &= 0.857815 \\ \log 512.72 &= 2.709880 \end{aligned}$$

Adding, $\therefore \log \text{ of Ans.} = 12.367724 - 10$

$= 2.367724$ (Art. 466, 1)

Number corresponding to $2.367724 = 233.197$, *Ans.*

2. Find the value of $\frac{3368.52}{7980.04}$.

$$\log \frac{3368.52}{7980.04} = \log 3368.52 - \log 7980.04$$

$$\begin{aligned} \log 3368.52 &= 13.527439 - 10 \text{ (Art. 466, 2)} \\ \log 7980.04 &= 3.902005 \end{aligned}$$

Subtracting, $\therefore \log \text{ of Ans.} = 9.625434 - 10$

Number corresponding $= .422118$, *Ans.*

3. Find the value of $(.0980937)^5$.

$$\log (.0980937)^5 = 5 \times \log .0980937$$

$$\log .0980937 = 8.991641 - 10$$

$$5$$

Multiplying, $\therefore$ log of Ans. $= 44.958205 - 50$

$= 4.958205 - 10$

Number corresponding $= .0000090825$, *Ans.*

4. Find the value of $\sqrt[7]{2.36015}$.

$$\log \sqrt[7]{2.36015} = \frac{1}{7} \log 2.36015$$

$$\log 2.36015 = 0.372940$$

Dividing by 7, $\therefore$ log of Ans. $= 0.053277$

Number corresponding $= 1.13052$, *Ans.*

5. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \frac{1}{3} \log 5 - \frac{5}{6} \log 3$$

$\log 2 = 0.301030$

$\log 5 = 0.698970$; divide by $3 = 0.232990$

$\log 3 = 0.477121$ $\qquad$ 0.534020

Multiply by 5, $= 2.385605$; divide by $6 = 0.397601$

Subtracting, $\therefore$ log of Ans. $= 0.136419$

Number corresponding $= 1.36905$, *Ans.*

Note. The work of the next two examples will be exhibited in the customary form, the -10's being omitted after the mantissæ. See **Art. 451.**

6. Find the value of $\sqrt[7]{.00003591}$.

$$\log \sqrt[7]{.00003591} = \frac{1}{7} \log .00003591$$

$$\log .00003591 = 5.555215$$

$$7)\underline{5.555215}$$

$$\text{log of Ans.} = 9.365031 \quad \text{(Art. 466, 4)}$$

$$\textit{Ans.} = .231756.$$

7. Find the value of $\sqrt{\left(\frac{.032956}{7.96183}\right)}$.

$$\log \sqrt{\left(\frac{.032956}{7.96183}\right)} = \frac{1}{2} (\log .032956 - \log 7.96183)$$

$$\log .032956 = 8.517934$$

$$\log 7.96183 = \underline{0.901013}$$

$$2)\underline{7.616921}$$

$$\text{log of Ans.} = 8.808460$$

$$\textit{Ans.} = .0643369.$$

Note. In computations by logarithms, negative quantities are used as if they were positive; the *sign* of the result being determined irrespective of the logarithmic work.

EXAMPLES.

468. Calculate, by logarithms, the values of the following:

1. $9.23841 \times .00369822$.
2. $\dfrac{3.70963 \times 286.512}{1633.72}$.
3. $(23.8464)^6$.
4. $(-.000929687)^{\frac{2}{7}}$.
5. $\sqrt[8]{3}$.
6. $\sqrt{2}$.
7. $\sqrt[4]{5}$.
8. $\sqrt{.0042937}$.

9. $\sqrt[13]{-6829.586}.$

10. $(1.05624)^{112}.$

11. $(-.00200016)^{\frac{12}{13}}.$

12. $2^{\frac{3}{2}} \times (-3)^{\frac{2}{3}}.$

13. $\dfrac{5^{\frac{3}{7}}}{(-2)^{\frac{2}{9}}}.$

14. $\dfrac{3^{\frac{5}{8}}}{(-4)^{\frac{2}{3}}}.$

15. $\left(\dfrac{6}{7}\right)^{\frac{5}{2}}.$

16. $\sqrt[11]{7239.812}.$

17. $\sqrt[5]{.00230508}.$

18. $\sqrt[7]{-.000009506694}.$

19. $\left(\dfrac{35}{113}\right)^{\frac{3}{8}}.$

20. $\left(\dfrac{.0872635}{.132088}\right)^{\frac{5}{3}}.$

21. $\sqrt[9]{\dfrac{7}{3}}.$

22. $\sqrt[8]{\dfrac{21}{13}}.$

23. $\sqrt[5]{\dfrac{2}{3}} \div \sqrt[3]{\dfrac{3}{5}}.$

24. $\sqrt[8]{2} \times \sqrt[5]{3} \times \sqrt[7]{4}.$

25. $\sqrt[5]{\left(\dfrac{3258.826}{49309.8}\right)}.$

26. $\left(\dfrac{-31.6259}{429.0162}\right)^{\frac{3}{17}}.$

27. $\dfrac{(625.343)^{\frac{2}{3}}}{(.732465)^{\frac{3}{7}}}.$

28. $\dfrac{\sqrt[3]{.000128883}}{\sqrt[4]{.000827606}}.$

29. $\dfrac{(-.746892)^{\frac{5}{3}}}{-(.234521)^{\frac{7}{2}}}.$

30. $\dfrac{\sqrt[11]{.00730007}}{(.682913)^{\frac{5}{2}}}.$

31. $\dfrac{\sqrt{5.95463} \times \sqrt[8]{61.1998}}{\sqrt[5]{298.5434}}.$

32. $(538.217 \times .000596899)^{\frac{1}{8}}.$

33. $\dfrac{-304.698 \times .9026137}{-.00776129 \times -16923.24}.$

34. $(18.9503)^{11} \times (-.213675)^{14}.$

35. $\sqrt[6]{3734.89 \times .00001108184}$.

36. $(2.63172)^{\frac{3}{4}} \times (.712719)^{\frac{2}{5}}$.

37. $\dfrac{\sqrt[3]{-.00819323} \times (.0628513)^{\frac{3}{2}}}{-.9834171}$.

38. $\sqrt{.035} \times \sqrt[6]{.626671} \times \sqrt[3]{.00721033}$.

EXPONENTIAL EQUATIONS.

469. An **Exponential Equation** is one in which the unknown quantity occurs as an exponent.

To solve an equation of this form, take the logarithms of both members according to Art. 457; the result will be an equation which can be solved by ordinary algebraic methods.

1. Given $31^x = 23$; find the value of x.

Taking the logarithms of both members,

$$\log (31^x) = \log 23$$

or, by Art. 457, $$x \log 31 = \log 23$$

Whence, $$x = \frac{\log 23}{\log 31} = \frac{1.361728}{1.491362} = .913077, \textit{ Ans.}$$

The value of the fraction $\dfrac{1.361728}{1.491362}$ may be obtained by division, or better by logarithms, as in Art. 468.

2. Given $.2^x = 3$; find the value of x.

Taking the logarithms of both members,

$$x \log .2 = \log 3$$

Whence, $$x = \frac{\log 3}{\log .2} = \frac{.477121}{9.301030 - 10} = -\frac{.477121}{.698970}$$

We may find the value of the fraction by logarithms exactly as if it were positive, and prefix a — sign to the result. Thus,

$$\log .477121 = 9.678628 - 10$$
$$\log .698970 = 9.844458 - 10$$

Subtracting, $= 9.834170 - 10$

Number corresponding $= .682606$

Therefore, $x = -.682606$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $11^x = 3$. **5.** $13^x = .281$. **7.** $5^{x-3} = 8^{2x+1}$.

4. $.3^x = .8$. **6.** $.703^x = 1.09604$. **8.** $23^{3x+5} = 31^{2x-3}$.

APPLICATION OF LOGARITHMS TO PROBLEMS IN COMPOUND INTEREST.

470. Let $P =$ the principal, expressed in dollars.

Let $t =$ the interval of time during which simple interest is calculated, expressed in years and fractions of a year. For instance, if the interest is compounded annually, $t = 1$; if semi-annually, $t = \frac{1}{2}$; etc.

Let $R =$ the interest of one dollar for the time t.

Let $n =$ the number of years.

Let $A_1, A_2, A_3, \ldots\ldots$ be the amounts at the ends of the 1st, 2d, 3d, ... intervals.

Let A be the amount at the end of n years.

$$\text{Then } A_1 = P + PR = P(1 + R)$$
$$A_2 = A_1 + A_1 R = A_1(1 + R)$$
$$= P(1 + R)(1 + R) = P(1 + R)^2$$
$$A_3 = A_2 + A_2 R = A_2(1 + R)$$
$$= P(1 + R)^2(1 + R) = P(1 + R)^3$$

....

As there are $\frac{n}{t}$ intervals, the amount at the end of the last, according to the law observed above,

$$A = P(1+R)^{\frac{n}{t}}.$$

1. *Given P, t, R, and n, to find A.*

As $A = P(1+R)^{\frac{n}{t}}$, we have by logarithms,

$$\log A = \log P(1+R)^{\frac{n}{t}} = \log P + \log(1+R)^{\frac{n}{t}}$$

$$= \log P + \frac{n}{t}\log(1+R).$$

Example. What will be the amount of \$7,325.67 for 3 years 9 months at 7 per cent compound interest, the interest being compounded quarterly?

Here $P = 7325.67$, $t = \frac{1}{4}$, $R = .0175$, $n = 3\frac{3}{4}$, $\frac{n}{t} = 15$.

$$\log P = 3.864848$$

$\log(1+R) = 0.007534$; multiply by $15 = 0.113010$

Adding, $\therefore \log \text{ of } A = 3.977858$

Number corresponding, $A = \$9502.93$, *Ans.*

2. *Given t, R, n, and A, to find P.*

As $A = P(1+R)^{\frac{n}{t}}$, $\therefore P = \dfrac{A}{(1+R)^{\frac{n}{t}}}$; or, by logarithms,

$$\log P = \log A - \log(1+R)^{\frac{n}{t}} = \log A - \frac{n}{t}\log(1+R).$$

Example. What sum of money will amount to \$1,763.55 at 5 per cent compound interest in 3 years, the interest being compounded semi-annually?

$$\text{Here } t = \frac{1}{2},\ R = .025,\ n = 3,\ A = 1763.55,\ \frac{n}{t} = 6.$$

$$\log A = 3.246388$$

$\log(1 + R) = 0.010724$; multiply by $6 = \underline{0.064344}$

Subtracting, $\therefore \log P = 3.182044$

Number corresponding $=$ \$1520.70, *Ans.*

3. *Given P, t, R, and A, to find n.*

In Art. 470, 1, we showed that

$$\log A = \log P + \frac{n}{t} \log (1 + R)$$

$$\therefore \frac{n}{t} \log (1 + R) = \log A - \log P$$

$$\therefore n = \frac{t (\log A - \log P)}{\log (1 + R)}.$$

Example. In how many years will \$300.00 amount to \$400.00 at 6 per cent compound interest, the interest being compounded quarterly?

$$\text{Here } P = 300,\ t = \frac{1}{4},\ R = .015,\ A = 400.$$

$$\therefore n = \frac{\log 400 - \log 300}{4 \log 1.015} = \frac{2.602060 - 2.477121}{4 \times .006466} = \frac{.124939}{.025864}$$

$$= 4.83 \text{ years, } \textit{Ans.}$$

4. *Given P, t, n, and A, to find R.*

We showed, in Art. 470, 3, that $\frac{n}{t} \log(1 + R) = \log A - \log P$

$$\therefore \log (1 + R) = \frac{\log A - \log P}{\frac{n}{t}}.$$

Example. If \$500.00 at compound interest amounts to \$689.26 in 6 years and 6 months, the interest being compounded semi-annually, what is the rate per cent per annum?

Here $P = 500,\ t = \frac{1}{2},\ n = 6\frac{1}{2},\ A = 689.26,\ \frac{n}{t} = 13.$

$$\therefore \log(1+R) = \frac{\log 689.26 - \log 500}{13}$$

$$\log 689.26 = 2.838383$$

$$\log 500 \quad = \underline{2.698970}$$

Subtracting, $= 0.139413$

Dividing by 13, $\therefore \log(1+R) = 0.010724$

Number corresponding $= 1.025 = 1 + R$, or $R = .025$.

That is, one dollar gains \$.025 semi-annually; or the rate is 5 per cent per annum.

EXPONENTIAL AND LOGARITHMIC SERIES.

471. We know that for any values of n and x,

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}.$$

Expanding by the Binomial Theorem, we obtain

$$\left[1 + n\frac{1}{n} + \frac{n(n-1)}{\lfloor 2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{\lfloor 3}\frac{1}{n^3} + \ldots\ldots\right]^x$$

$$= 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{\lfloor 2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\lfloor 3}\frac{1}{n^3} + \ldots\ldots$$

or,
$$\left[1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \ldots\ldots\right]^x$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \ldots\ldots$$

This is true for all values of n; hence, it is true however large n may be. Suppose n to be indefinitely increased. Then the limiting values of the fractions $\frac{1}{n}$, $\frac{2}{n}$, etc., are 0 (Art. 210). Hence, at the limit, we have,

$$\left[1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\ldots\right]^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$$

The series in the bracket we denote by e; hence,

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots$$

472. *To expand a^x in powers of x.*

Let $a=e^m$; whence (Art. 444), $m=\log_e a$.

Then $a^x=e^{mx}=$ (Art. 471) $1+mx+\frac{m^2x^2}{\lfloor 2}+\frac{m^3x^3}{\lfloor 3}+\ldots\ldots$

Substituting the value of m,

$$a^x=1+(\log_e a)\,x+(\log_e a)^2\frac{x^2}{\lfloor 2}+(\log_e a)^3\frac{x^3}{\lfloor 3}+\ldots\ldots$$

This result is called the *Exponential Theorem.*

473. The system of logarithms which has e for its base, is called the *Napierian System*, from Napier, the inventor of logarithms. The value of e may be easily calculated from the series of Art. 471, and will be found to be 2.7182818......

474. *To expand* $\log_e(1+x)$ *in powers of x.*

$$a^x=\{1+(a-1)\}^x=1+x(a-1)+\frac{x(x-1)}{\lfloor 2}(a-1)^2$$

$$+\frac{x(x-1)(x-2)}{\lfloor 3}(a-1)^3+\ldots\ldots$$

$$=1+x\left\{(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}\ldots\ldots\right\}+\text{ terms con-}$$

taining x^2, x^3, etc.

But (Art. 472), $a^x = 1 + x(\log_e a) +$ terms containing x^2, x^3, etc.

As the two values of a^x are equal for all values of x, by the Theorem of Undetermined Coefficients the coefficients of x in the two expressions are equal; hence,

$$\log_e a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} \cdots\cdots$$

Putting $a = 1 + x$, and therefore $a - 1 = x$, we obtain

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots$$

Note. This formula might be used to calculate Napierian logarithms; but unless x is a very small fraction, the series in the second number is either divergent or converges very slowly, and hence is useless in most cases.

475. *To obtain a more convenient formula for calculating the Napierian logarithm of a number.*

By Art. 474, $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\cdots$

put $x = -x$,

$$\therefore \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\cdots$$

Subtracting,

$$\therefore \log_e(1+x) - \log_e(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\cdots$$

or, by Art. 456, $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\cdots\right)$

Let $$x = \frac{1}{2n+1}$$

$$\therefore \frac{1+x}{1-x} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} = \frac{2n+1+1}{2n+1-1} = \frac{2n+2}{2n} = \frac{n+1}{n}$$

Substituting, $\therefore \log_e \left(\frac{n+1}{n}\right) = \log_e (n+1) - \log_e n$

$$= 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\ldots\right)$$

$$\therefore \log_e (n+1) = \log_e n + 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\ldots\right).$$

476. To calculate $\log_e 2$, put $n = 1$ in the formula of Art. 475.

$$\therefore \log_e 2 = \log_e 1 + 2\left(\frac{1}{2+1} + \frac{1}{3(2+1)^3} + \frac{1}{5(2+1)^5} + \ldots\ldots\right)$$

or, since $\log_e 1 = 0$,

$$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15309} + \frac{1}{177147} + \frac{1}{1948617} + \ldots\ldots\right)$$

$$= 2\,(.3333333 + .0123457 + .0008230 + .0000653 + .0000056 + .0000005 + \ldots\ldots)$$

$= 2 \times .3465734 = .6931468 = .693147$, correct to the sixth decimal place.

From $\log_e 2$, we may calculate $\log_e 3$; and so on. We shall find $\log_e 10 = 2.302585$.

477. *To calculate the common logarithm of a number from its Napierian logarithm.*

By Art. 460, changing b to 10, and a to e, we obtain

$$\log_{10} m = \frac{\log_e m}{\log_e 10} = \frac{1}{2.302585} \log_e m = .4342945 \times \log_e m.$$

For instance, $\log_{10} 2 = .4342945 \times .693147 = .301030$.

The multiplier by which logarithms of any system are derived from the Napierian system, is called the *modulus* of that system. Hence, .4342945 is the modulus of the common system.

As tables of common logarithms are met with more frequently than tables of Napierian, a rule for changing common logarithms into Napierian may be found convenient.

RULE.

Divide the common logarithm by .4342945.

For example, to find the Napierian logarithm of 586.324,

$$\text{common log } 586.324 = 2.768138$$

Divide by .4342945, $\therefore$ Napierian log $586.324 = 6.373873$, *Ans.*

Another method would be to multiply the common logarithm by 2.302585, the reciprocal of .4342945.

Napierian logarithms are sometimes called *hyperbolic* logarithms, from having been originally derived from the hyperbola. They are also sometimes called *natural* logarithms, from being those which occur first in the investigation of a method of calculating logarithms. Napierian logarithms are seldom used in computation, but occur frequently in theoretical investigations.

ARITHMETICAL COMPLEMENT.

478. The **Arithmetical Complement** of the logarithm of any quantity is the logarithm of the reciprocal of that quantity.

For example, if $\log 4098 = 3.612572$, then

$$\text{ar. co. } \log 4098 = \log \frac{1}{4098} = \log 1 - \log 4098$$

$$= 0 - 3.612572 = 6.387428 - 10.$$

Again, if $\log .06689 = 8.825361 - 10$, then

$$\text{ar. co. } \log .06689 = \log \frac{1}{.06689} = 0 - (8.825361 - 10)$$

$$= 10 - 8.825361 = 1.174639.$$

The following rules will be evident from the preceding illustrations:

To find the arithmetical complement of a positive logarithm, subtract it from 10, *writing* -10 *after the mantissa.*

To find the arithmetical complement of a negative logarithm, subtract that portion of it besides the -10 *from* 10.

The only application of this is to exhibit the work of calculation by logarithms in a more compact form in certain cases. It depends on the principle that subtracting a logarithm or adding its arithmetical complement gives the same result.

For, suppose we are to calculate $\frac{a \times b}{c \times d}$ by logarithms.

$$\log \frac{a \times b}{c \times d} = \log \left(a \times b \times \frac{1}{c} \times \frac{1}{d}\right)$$

$$= \log a + \log b + \log \frac{1}{c} + \log \frac{1}{d}$$

$$= \log a + \log b + \text{ar. co. } \log c + \text{ar. co. } \log d.$$

That is, the work can be exhibited in the form of the addition of four logarithms, instead of the subtraction of the sum of two logarithms from the sum of two others. The principle is only applicable to the case of fractions; and the rule to be used is,

Add together the logarithms of the quantities in the numerator, and the arithmetical complements of the logarithms of the quantities in the denominator.

Example. Calculate the value of $\frac{79.23 \times 10.39}{613.8 \times .07723}$.

$$\log \frac{79.23 \times 10.39}{613.8 \times .07723} = \log 79.23 + \log 10.39 + \text{ar. co. } \log 613.8$$
$$+ \text{ar. co. } \log .07723$$

$$\begin{aligned} \log 79.23 &= 1.898890 \\ \log 10.39 &= 1.016616 \\ \text{ar. co. } \log 613.8 &= 7.211973 - 10 \\ \text{ar. co. } \log .07723 &= \underline{1.112214} \end{aligned}$$

Adding, $\therefore \log \text{ of Ans.} = 11.239693 - 10 = 1.239693$

Number corresponding $= 17.3657$, *Ans.*

Note. The arithmetical complement may be calculated mentally from the logarithm, by subtracting the last *significant* figure from 10, and all the others from 9.

MISCELLANEOUS EXAMPLES.

479. **1.** Find $\log_3 2187$. (See Art. 444.)

2. Find $\log_5 15625$.

3. Find the logarithm of $\frac{1}{64}$ to the base -2.

4. Find the logarithm of $\frac{1}{32}$ to the base 8.

5. Find the characteristic of $\log_2 183$.

6. Find the characteristic of $\log_5 4203$.

7. Given $\log 2 = .301030$, how many digits are there in 2^{17}?

8. Given $\log 3 = .477121$, how many digits are there in $3^{\frac{55}{4}}$?

9. Find $\log_{13} 56$. (See Art. 460.)

10. Find $\log_8 163$.

11. Find $\log_{20} 411$.

12. What sum of money will amount to \$8705.50, in 7 years, at 7 per cent compound interest, the interest being compounded annually?

13. In how many years will a sum of money double itself at 6 per cent compound interest, the interest being compounded semi-annually ?

14. What will be the amount of \$1000.00 for 38 years 3 months, at 6 per cent compound interest, the interest being compounded quarterly ?

15. At what rate per cent per annum will \$2500.00 amount to \$3187.29 in 3 years and 6 months, the interest being compounded quarterly ?

16. In how many years will \$9681.32 amount to \$15308.70 at 5 per cent compound interest, the interest being compounded annually ?

17. Using the table of common logarithms, find the Napierian logarithm of 52.9381 (Art. 477).

18. Find the Napierian logarithm of 1325.07.

19 Find the Napierian logarithm of .085623.

20. Find the Napierian logarithm of .342977.

XLII.—GENERAL THEORY OF EQUATIONS.

480. The *general form* of a complete equation of the *n*th degree is

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0$$

Where n is a positive integer, and the number of terms is $n+1$. The quantities $p, q, \ldots\ldots t, u, v$ are either positive or negative, integral or fractional; and the coefficient of x^n is unity.

481. In reducing an equation to the general form, all the terms must be transposed to the first member, and arranged according to the powers of x. If x^n has a coefficient, it may be removed by dividing the equation by that coefficient.

482. A **Root** of an equation is any real or imaginary expression, which, being substituted for its unknown quantity, satisfies the equation, or makes the first member equal to 0 (Art. 166).

We assume that every equation has at least one root.

483. An equation of the third degree containing only one unknown quantity, or one in which the cube is the highest power of the unknown quantity, is usually called a *cubic equation.*

484. An equation of the fourth degree containing only one unknown quantity is usually called a *biquadratic equation.*

DIVISIBILITY OF EQUATIONS.

485. *If a is a root of an equation in the form*

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0,$$

then the first member is divisible by $x - a$.

It is evident that the division of the first member by $x - a$ may be carried on until x disappears from the remainder. Let Q represent the quotient, and R the remainder, which is independent of x; then the given equation may be made to take the form

$$(x - a)\,Q + R = 0.$$

But if $x = a$, then $(x - a)\,Q = 0$, and, consequently,

$$R = 0;$$

that is, $x - a$ is a factor of the first member of the given equation, as it is contained in it without a remainder.

486. *Conversely, if the first member of an equation in the form*

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0$$

is divisible by $x - a$, then a is a root of the equation.

For, if the first member of the given equation is divisible by $x - a$, then the equation may be made to take the form

$$(x - a)\ Q = 0;$$

and it follows from Art. 330 that a is a root of this equation.

EXAMPLES.

By the method of Art. 486,

1. Prove that 3 is a root of the equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

2. Prove that -1 is a root of the equation $x^3 + 1 = 0$.

3. Prove that 1 is a root of the equation

$$x^3 + x^2 - 17x + 15 = 0.$$

4. Prove that -2 is a root of the equation

$$x^4 - 3x^2 + 4x + 4 = 0.$$

5. Prove that 4 is not a root of the equation

$$x^4 - 5x^3 + 5x^2 + 5x - 6 = 0.$$

NUMBER OF ROOTS.

487. *Every equation of the nth degree, containing but one unknown quantity, has n roots, and no more.*

Let a be a root of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0;$$

then, by Art. 485, the first member is divisible by $x - a$, and the equation may be made to take the form

$$(x - a)\ (x^{n-1} + p_1\,x^{n-2} + \ldots\ldots + u_1\,x + v_1) = 0.$$

The equation may be satisfied by making either factor of the first member equal to 0 (Art. 330); hence,

$$x - a = 0$$

and $$x^{n-1} + p_1\,x^{n-2} + \ldots\ldots + u_1\,x + v_1 = 0. \qquad (1)$$

But equation (1) must have some root, as b, and may be placed under the form

$$(x-b)\,(x^{n-2}+p_2\,x^{n-3}+\ldots\ldots+u_2\,x+v_2)=0;$$

which is satisfied by placing either factor of the first member equal to 0; and so on.

Since each of the factors $x-a$, $x-b$, etc., contains only the first power of x, it is evident that the original equation can be separated into as many such binomial factors as there are units in the exponent of the highest power of the unknown quantity, and no more; that is, into n factors, or

$$(x-a)\,(x-b)\,(x-c)\,\ldots\ldots\,(x-l)=0.$$

Hence, by Art. 330, the equation has the n roots a, b, c, ... l.

Moreover, if the equation had another root, as r, then it must contain another factor $x-r$, which is impossible.

488. It should be observed that the n binomial factors of which the general equation of the nth degree is composed, are not necessarily *unequal*; hence, two or more of the roots of an equation may be equal. Thus, the equation

$$x^3-6\,x^2+12\,x-8=0$$

may be factored so as to take the form

$$(x-2)\,(x-2)\,(x-2)=0, \text{ or } (x-2)^3=0;$$

and hence the three roots are 2, 2, and 2.

489. It will be readily seen that any equation, one of whose roots is known, may be depressed to another of the next lower degree, which shall contain the remaining roots. Hence, if all the roots of an equation are known except two, those may be obtained from the depressed equation, by the rules for quadratics.

1. One root of the equation $x^3+2\,x^2-23\,x-60=0$ is -3; what are the others?

Dividing $x^3 + 2x^2 - 23x - 60$ by $x + 3$, the given equation may be put in the form

$$(x + 3)(x^2 - x - 20) = 0.$$

Thus, the depressed equation is $x^2 - x - 20 = 0$.

Solving this by the rules for quadratics, we obtain $x = 5$ or -4; which are the remaining roots.

EXAMPLES.

2. One root of the equation $x^3 - 19x + 30 = 0$ is 2; what are the others?

3. Required the three roots of the equation $x^3 = a^3$, or $x^3 - a^3 = 0$.

4. One root of the equation $x^3 + x^2 - 16x + 20 = 0$ is -5; required the remaining roots.

5. Two roots of the equation $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0$ are 1 and 2; required the remaining roots.

6. One root of the equation $x^4 - 7x^3 + 3x + 3 = 0$ is 1; what equation contains the remaining roots?

7. One root of the equation $6x^3 - x^2 - 32x + 20 = 0$ is 2; what are the others?

8. Two roots of the equation $20x^4 - 169x^3 + 192x^2 + 97x - 140 = 0$ are 1 and 7; what are the others?

FORMATION OF EQUATIONS.

490. *An equation having any given roots may be formed by subtracting each root from the unknown quantity, and placing the product of these binomial factors equal to 0.*

For it is evident, from principles already established, that an equation having the n roots $a, b, c, \ldots\ldots l$ may be written in the form

$$(x - a)(x - b)(x - c) \ldots\ldots (x - l) = 0.$$

After performing the multiplication indicated, the equation will assume the form .

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0.$$

(Compare Art. 329.)

1. Form the equation whose roots are 1, 2, and -4.

Result, $$(x-1)\,(x-2)\,(x+4) = 0$$

or, $$x^3 + x^2 - 10\,x + 8 = 0.$$

EXAMPLES.

Form the equations whose roots are:

2. -1, -3, and -5.

3. 5, -2, and -3.

4. 1, $\frac{1}{2}$, and $\frac{1}{3}$.

5. ± 1 and ± 2.

6. 1, 2, 3, and 4.

7. 4, 4, and 5.

8. 0, -1, 3, and 4.

9. -5, $\frac{3}{4}$, -2, and $\frac{5}{3}$.

COMPOSITION OF COEFFICIENTS.

491. *The coefficient of the second term of an equation of the nth degree in its general form is the sum of all the roots with their signs changed; that of the third term is the sum of their products, taken two and two; that of the fourth term is the sum of their products, taken three and three, with their signs changed, etc.; and the last term is the product of all the roots with their signs changed.*

For, resuming the equation

$$(x-a)\,(x-b)\,(x-c)\,\ldots\ldots\,(x-k)\,(x-l) = 0,$$

if we perform the multiplication indicated, we obtain

$$(x-a)\,(x-b) = x^2 - (a+b)\,x + a\,b,$$

$$(x-a)\,(x-b)\,(x-c) = x^3 - (a+b+c)\,x + (a\,b + a\,c + b\,c)\,x - a\,b\,c,$$

and so on. When n factors have been multiplied, the coefficients of the general equation become

$$p = -a - b - c - \ldots\ldots - k - l$$
$$q = ab + ac + bc + \ldots\ldots + kl$$
$$r = -abc - abd - acd - \ldots\ldots - ikl$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$v = \pm abc \ldots\ldots kl$$

which corresponds with the enunciation of the proposition; the upper sign of the value of v being taken when n is even, and the lower sign when n is odd.

492. If $p = 0$, that is, if the second term of an equation be wanting, the sum of the roots will be 0.

If $v = 0$, that is, if the *absolute term* of an equation be wanting, at least one root must be 0.

493. Every rational root of an equation is a divisor of the last term.

494. When all the roots of an equation but two are known, the coefficient of the second term of the depressed equation (Art. 489) can be found by subtracting the sum of the known roots, with their signs changed, from the coefficient of the second term of the original equation. The absolute term of the depressed equation can be found by dividing the absolute term of the original equation by the product of the known roots with their signs changed.

EXAMPLES.

Find the sum and product of the roots in the following:

1. $x^3 - 7x + 6 = 0$. **2.** $2x^4 - 5x^3 - 17x^2 + 14x + 24 = 0$.

In the following example obtain the depressed equation by the method of Art. 494:

3. Two roots of the equation $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0$ are 2 and -1; what are the others?

FRACTIONAL ROOTS.

495. *An equation whose coefficients are all integral, the coefficient of the first term being unity, cannot have a rational fraction as a root.*

If possible, let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0,$$

where p, q, ..., t, u, v are integral. Then

$$\left(\frac{a}{b}\right)^n + p\left(\frac{a}{b}\right)^{n-1} + q\left(\frac{a}{b}\right)^{n-2} + \ldots\ldots + t\left(\frac{a}{b}\right)^2 + u\left(\frac{a}{b}\right) + v = 0.$$

Multiplying through by b^{n-1}, and transposing,

$$\frac{a^n}{b} = -(p\,a^{n-1} + q\,a^{n-2}\,b + \ldots\ldots + t\,a^2 b^{n-3} + u\,a\,b^{n-2} + v b^{n-1}).$$

Now, as $\frac{a}{b}$ is in its lowest terms, a and b can have no common divisor; therefore a^n and b can have no common divisor; hence $\frac{a^n}{b}$ is in its lowest terms. Thus, we have a fraction in its lowest terms equal to an entire quantity, which is impossible. Therefore no root of the equation can be a rational fraction.

Note. The equation may have an *irrational* fraction as a root, such as $\frac{2+\sqrt{3}}{4}$ for example. Such a root, whose value can only be expressed approximately by a decimal fraction, is called *incommensurable.*

IMAGINARY ROOTS.

496. *If the coefficients of an equation be real quantities, imaginary roots enter it by pairs, if at all.*

Suppose $a + b\sqrt{-1}$ to be a root of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0.$$

Substituting $a + b\sqrt{-1}$ for x, and developing each expression by the Binomial Theorem, all the *odd* terms of each series will contain either powers of a, or *even* powers of $b\sqrt{-1}$, and are therefore real; while all the *even* terms contain the *odd* powers of $b\sqrt{-1}$, and are therefore imaginary. Representing the sum of all the real quantities by P, and the sum of all the imaginary quantities by $Q\sqrt{-1}$, we have

$$P + Q\sqrt{-1} = 0.$$

This equation can be true only when both P and Q equal 0.

If we now substitute $a - b\sqrt{-1}$ for x, we find that the series differ from the former only in having their *even* or imaginary terms *negative*. Hence, we obtain as the first member

$$P - Q\sqrt{-1},$$

which must be equal to 0, for we have already shown that both P and Q equal 0. Thus, $a - b\sqrt{-1}$ satisfies the equation.

Similarly, we may show that if $b\sqrt{-1}$ is a root of the equation, then will $-b\sqrt{-1}$ also be a root of the equation.

497. The product of a pair of imaginary quantities is always positive. Thus,

$$(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2,$$

and

$$(b\sqrt{-1})(-b\sqrt{-1}) = b^2.$$

TRANSFORMATION OF EQUATIONS.

498. *To transform an equation into another which shall have the same roots with contrary signs.*

Let the given equation be

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0.$$

Put $x = -y$; then whatever value x may have, y will have the same value with its sign changed. The equation now becomes

$$(-y)^n + p(-y)^{n-1} + q(-y)^{n-2} + \ldots\ldots + t(-y)^2 + u(-y) + v = 0.$$

If n is even, the first term is positive, second term negative, and so on; and the equation may be written

$$y^n - p\,y^{n-1} + q\,y^{n-2} - \ldots\ldots + t\,y^2 - u\,y + v = 0. \qquad (1)$$

If n is odd, the first term is negative, second term positive, and so on; hence, changing all signs, we write the equation

$$y^n - p\,y^{n-1} + q\,y^{n-2} - \ldots\ldots - t\,y^2 + u\,y - v = 0. \qquad (2)$$

From (1) and (2) it is evident that to effect the desired transformation we have simply to *change the signs of the alternate terms, beginning with the second.*

Note. The preceding rule assumes that the given equation is *complete* (Art. 300); if it be incomplete, any missing term must be put in with zero as a coefficient.

1. Transform the equation $x^3 - 7x + 6 = 0$ into another which shall have the same roots with contrary signs.

We may write the equation $x^3 + 0 \,.\, x^2 - 7x + 6 = 0$.

Applying the rule,

$$x^3 - 0 \,.\, x^2 - 7x - 6 = 0, \text{ or } x^3 - 7x - 6 = 0, \textit{ Ans.}$$

EXAMPLES.

Transform the following equations into others which shall have the same roots with contrary signs:

2. $x^4 - 2x^3 + x - 132 = 0$. **3.** $x^5 - 3x^2 + 8 = 0$.

499. *To transform an equation into another whose roots shall be some multiple of those of the first.*

Let the given equation be

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0.$$

Put $x = \frac{y}{m}$; then whatever value x may have, y will have a value m times as great. The equation now becomes

$$\left(\frac{y}{m}\right)^n + p\left(\frac{y}{m}\right)^{n-1} + q\left(\frac{y}{m}\right)^{n-2} + \dots\dots + t\left(\frac{y}{m}\right)^2 + u\left(\frac{y}{m}\right) + v = 0.$$

Multiplying through by m^n, we have

$$y^n + pmy^{n-1} + qm^2y^{n-2} + \dots\dots + tm^{n-2}y^2 + um^{n-1}y + vm^n = 0.$$

Hence, to effect the desired transformation, *multiply the second term by the given factor, the third term by its square, and so on.*

Similarly, we may transform an equation into one whose roots shall be those of the first divided by some quantity.

1. Transform the equation $x^3 - 7x - 6 = 0$ into another whose roots shall be 4 times as great.

The equation may be written, $x^3 + 0 \cdot x^2 - 7x - 6 = 0$.

Then, by the rule,

$$x^3 - 4^2 \cdot 7x - 4^3 \cdot 6 = 0, \text{ or } x^3 - 112x - 384 = 0, \textit{ Ans.}$$

EXAMPLES.

2. Transform the equation $x^3 - 2x^2 + 5 = 0$ into another whose roots shall be 5 times as great.

3. Transform the equation $x^4 + \frac{3x^3}{4} - 27 = 0$ into another whose roots shall be one third as great.

500. *To transform an equation containing fractional coefficients into another whose coefficients are integral, that of the first term being unity.*

If in Art. 499 we assume m equal to the least common multiple of the denominators, it will always remove them; but often a smaller number can be found which will produce the same result.

1. Transform the equation $x^3 - \frac{x^2}{3} - \frac{x}{36} + \frac{1}{108} = 0$ into another whose coefficients shall be integral.

The least common multiple of the denominators is 108; so that one solution would be, by Art. 499,

$$x^3 - 108 \cdot \frac{x^2}{3} - 108^2 \cdot \frac{x}{36} + 108^3 \cdot \frac{1}{108} = 0.$$

An easier way, however, is as follows; the denominators may be written 3, $3^2 \times 2^2$, and $3^3 \times 2^2$, so that the multiplier 3×2 or 6 will remove them. Hence, by Art. 499, we have

$$x^3 - 6 \cdot \frac{x^2}{3} - 6^2 \cdot \frac{x}{36} + 6^3 \cdot \frac{1}{108} = 0, \quad \text{or} \quad x^3 - 2x^2 - x + 2 = 0,$$

whose roots are 6 times as great as those of the given equation.

EXAMPLES.

Transform the following equations into others whose coefficients shall be integral:

2. $x^3 + \frac{3x}{4} - \frac{7}{4} = 0.$

3. $x^2 - \frac{mx}{n} + \frac{a}{b} = 0.$

4. $x^3 + \frac{x^2}{30} - \frac{x}{5} - \frac{1}{30} = 0.$

5. $x^4 - 5x^3 - \frac{25x}{4} + \frac{3}{2} = 0.$

501. *To transform an equation into another whose roots shall be the reciprocals of those of the first.*

Let the given equation be

$$x^n + p x^{n-1} + q x^{n-2} + \ldots\ldots + t x^2 + u x + v = 0.$$

Put $x = \frac{1}{y}$; then whatever value x may have, y will be its reciprocal. The equation now becomes

$$\frac{1}{y^n} + \frac{p}{y^{n-1}} + \frac{q}{y^{n-2}} + \ldots\ldots + \frac{t}{y^2} + \frac{u}{y} + v = 0.$$

Multiplying through by y^n, and reversing the order,

$$v\,y^n + u\,y^{n-1} + t\,y^{n-2} + \ldots\ldots + q\,y^2 + p\,y + 1 = 0.$$

Dividing through by v,

$$y^n + \frac{u}{v}\,y^{n-1} + \frac{t}{v}\,y^{n-2} + \ldots\ldots + \frac{q}{v}\,y^2 + \frac{p}{v}\,y + \frac{1}{v} = 0.$$

Hence, to effect the transformation, *write the coefficients in reverse order, and then divide by the coefficient of the first term.*

EXAMPLES.

Transform the following equations into others whose roots shall be the reciprocals of those of the first:

1. $x^3 - 6\,x^2 + 11\,x - 6 = 0.$ **3.** $x^3 - 9\,x^2 + \dfrac{6\,x}{7} - \dfrac{1}{49} = 0.$

2. $x^4 - x^3 - 3\,x^2 + x + 2 = 0.$ **4.** $x^3 - 4\,x^2 + 9 = 0.$

502. *To transform an equation into another whose roots shall differ from those of the first by a given quantity.*

Let the given equation be

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0. \quad (1)$$

Put $x = y + r$, and we have

$$(y+r)^n + p\,(y+r)^{n-1} + \ldots\ldots + u\,(y+r) + v = 0. \quad (2)$$

Developing $(y+r)^n$, $(y+r)^{n-1}$, ..., by the Binomial Theorem, and collecting terms containing like powers of y, we have an equation of the form

$$y^n + p_1\,y^{n-1} + q_1\,y^{n-2} + \ldots\ldots + t_1\,y^2 + u_1\,y + v_1 = 0. \quad (3)$$

As $y = x - r$, the roots of (3) are evidently *less* by r than those of (1). By putting $x = y - r$, we shall obtain in the same way an equation whose roots are *greater* by r than those of (1).

503. If n is small, the operation indicated in Art. 502 may be effected with little trouble; but for equations of a higher degree a less tedious method is better.

If in (3) we put $y = x - r$, we shall have

$$(x-r)^n + p_1(x-r)^{n-1} + \dots\dots + u_1(x-r) + v_1 = 0, \qquad (4)$$

which is, of course, identical with (1), and must reduce to (1) when developed. If we divide (4) by $x - r$, we obtain

$$(x-r)^{n-1} + p_1(x-r)^{n-2} + q_1(x-r)^{n-3} + \dots\dots + u_1 \qquad (5)$$

as a quotient, with a remainder of v_1. Dividing (5) by $x - r$, we obtain a remainder of u_1; and so on, until we obtain all the coefficients of (3) as remainders.

Hence, to effect the desired transformation,

Divide the given equation by $x - r$ or $x + r$, according as the roots of the transformed equation are to be less or greater than those of the first by r, and the remainder will be the absolute term of the transformed equation. Divide the quotient just found by the same divisor, and the remainder will be the coefficient of the last term but one of the transformed equation; and so on.

504. 1. Transform the equation $x^3 + 3x^2 - 4x + 1 = 0$ into one whose roots shall be greater by 1.

Using the method of Art. 502, put $x = y - 1$.

Then, $\quad (y-1)^3 + 3(y-1)^2 - 4(y-1) + 1 = 0,$

or, $\quad y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 - 4y + 4 + 1 = 0,$

or, $\quad y^3 - 7y + 7 = 0,$ *Ans.*

EXAMPLES.

2. Transform the equation $x^3 - x - 6 = 0$ into one whose roots shall be less by 8.

3. Transform the equation $x^4 + 6x^3 - x^2 - 5x - 1 = 0$ into one whose roots shall be greater by 3.

505. *To transform a complete equation into one whose second term shall be wanting.*

The coefficient of y^{n-1} in (2), Art. 502, is $nr + p$. Hence, in (3), $p_1 = nr + p$. To make $p_1 = 0$, it is only necessary to make $nr + p = 0$, or $r = -\frac{p}{n}$; hence, to effect the desired transformation, put $x = y - \frac{p}{n}$; that is, *put x equal to y, minus the coefficient of the second term of the given equation divided by the degree of the equation.*

1. Transform the equation $x^3 - 6x^2 + 9x - 6 = 0$ into another whose second term shall be wanting.

Here $p = -6$, $n = 3$; then, put $x = y - \frac{-6}{3} = y + 2$.

Result, $\quad (y+2)^3 - 6(y+2)^2 + 9(y+2) - 6 = 0,$

or, $\quad y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 9y + 18 - 6 = 0,$

or, $\quad y^3 - 3y - 4 = 0$, *Ans.*

EXAMPLES.

Transform the following equations into others whose second terms shall be wanting:

2. $x^2 - px + q = 0$.

3. $x^3 + x^2 + 4 = 0$.

4. $x^3 + 6x^2 - 3x + 4 = 0$.

5. $x^4 - 4x^3 - 5x - 1 = 0$.

DESCARTES' RULE OF SIGNS.

506. A *Permanence* of sign occurs when two successive terms of a series have the *same* sign.

A *Variation* of sign occurs when two successive terms of a series have *contrary* signs.

DESCARTES' RULE.

507. *A complete equation cannot have a greater number of positive roots than it has variations of sign, nor a greater number of negative roots than it has permanences of sign.*

Let any complete equation have the following signs:

$$+ + + - + - + - -$$

in which there are three permanences and five variations.

If we introduce a new positive root a, we multiply this by $x - a$ (Art. 490). Writing only the *signs* which occur in the operation, we have

$$\begin{array}{cccccccccc}
+ & + & + & - & + & - & + & - & - & \\
+ & - & & & & & & & & \\
\hline
+ & + & + & - & + & - & + & - & - & \\
 & - & - & - & + & - & + & - & + & + \\
\hline
+ & \pm & \pm & - & + & - & + & - & \pm & + \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

a double sign being placed wherever the sign of a term is ambiguous.

However the double signs are taken, there must be at least one variation between 1 and 4, and one between 8 and 10, and there are evidently four between 4 and 8; or in all there are at least six variations in the result. As in the original equation there were five variations, the introduction of a positive root has caused at least one additional variation; and as this is true of any positive root, there must be at least as many variations of sign as there are positive roots.

Similarly, by introducing the factor $x + a$, we may show that there are at least as many permanences of sign as there are negative roots.

If the equation is incomplete, any missing term must be supplied with ± 0 as its coefficient before applying Descartes' Rule.

508. In any complete equation, the sum of the number of permanences and variations is equal to the number of terms less one, or is equal to the degree of the equation (Art. 480). Hence, when the roots are all real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences (Art. 487).

A complete equation whose terms are all positive can have no positive root; and one whose terms are alternately positive and negative can have no negative root.

509. In an incomplete equation, imaginary roots may sometimes be discovered by means of the double sign of 0 in the missing terms. Thus, in the equation

$$x^3 + x^2 \pm 0\,x + 4 = 0$$

if we take the upper sign, there is no variation, and consequently no positive root; if we take the lower sign, there is but one permanence, and hence but one negative root. Therefore, as the equation has three roots (Art. 487), two of them must be imaginary.

In general, whenever the term which precedes a missing term has the same sign as that which follows, the equation must have imaginary roots; where it has the opposite sign, the equation may or may not have imaginary roots, but Descartes' Rule does not detect them. If two or more successive terms of an equation be wanting, there must be imaginary roots.

Note. In all applications of Descartes' Rule, the equation must contain a term independent of x, that is, no root must be equal to zero (Art. 330); for a zero root cannot be considered as either positive or negative.

EXAMPLES.

510. The roots of the following equations being all real, determine their signs:

1. $x^3 - 3\,x - 2 = 0.$
2. $x^3 - 10\,x + 3 = 0.$
3. $x^3 - 7\,x^2 + 36 = 0.$
4. $x^4 - 2\,x^3 - 13\,x^2 + 38\,x - 24 = 0.$

5. What are the signs of the roots of the equation $x^3 + x^2 - 4 = 0$?

DERIVED POLYNOMIALS.

511. If we take the polynomial

$$a\,x^n + b\,x^{n-1} + c\,x^{n-2} + \ldots\ldots$$

and multiply each term by the exponent of x in that term, and then diminish the exponent by 1, the result

$$n\,a\,x^{n-1} + (n-1)\,b\,x^{n-2} + (n-2)\,c\,x^{n-3} + \dots\dots$$

is called the *first derived polynomial* or *first derivative* of the given polynomial.

The *second derived polynomial* or *second derivative* is the first derived polynomial of the first derivative; and so on. The given polynomial is sometimes called the *primitive polynomial.*

A *derived equation* is one whose first member is a derivative of the first member of another.

1. Find the successive derivatives of $x^3 + 5\,x^2 + 3\,x + 9$.

Result: First, $3\,x^2 + 10\,x + 3$.
Second, $6\,x + 10$.
Third, 6.
Fourth, 0.

EXAMPLES.

Find the successive derivatives of the following:

2. $x^3 - 5\,x^2 + 6\,x - 2$.

3. $2\,x^2 - x - 7$.

4. $a\,x^4 - b\,x^3 + c\,x - 3\,d$.

5. $7\,x^4 - 13\,x^2 + 8\,x - 1$.

EQUAL ROOTS.

512. Let the roots of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \dots\dots + t\,x^2 + u\,x + v = 0 \qquad (1)$$

be $a, b, c, \dots\dots$. Then (Art. 490), we have

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \dots\dots = (x-a)\,(x-b)\,(x-c)\,\dots\dots$$

Putting $x + y$ in place of x,

$$(x+y)^n + p\,(x+y)^{n-1} + \dots = (y + \overline{x-a})\,(y + \overline{x-b})\dots \qquad (2)$$

By Art. 399, the coefficient of y in the first member is

$$n\,x^{n-1} + p\,(n-1)\,x^{n-2} + q\,(n-2)\,x^{n-3} + \dots\dots \qquad (3)$$

which, we observe, is the first derivative of (1); and, as in Art. 491, regarding $x-a$, $x-b$, ... as single terms, the coefficient of y in the second member is

$$\left.\begin{array}{l}(x-b)(x-c)(x-d)\ \ldots\ldots \text{ to } n-1 \text{ factors} \\ +(x-a)(x-c)(x-d)\ \ldots\ldots \text{ to } n-1 \text{ factors} \\ +(x-a)(x-b)(x-d)\ \ldots\ldots \text{ to } n-1 \text{ factors} \\ +\ \ldots\ldots\end{array}\right\} \quad (4)$$

As (2) is identical, by Art. 413 these coefficients are equal.

Now if $b=a$, that is, if equation (1) has two roots equal to a, every term of (4) will be divisible by $x-a$, hence (3) will be divisible by the same factor; therefore (Art. 486) the first derived equation of (1) will have one root equal to a. Similarly, if $c=b=a$, that is, if (1) has three roots equal to a, (3) will have two roots equal to a; and so on. Or, in general,

If an equation has n roots equal to a, its first derived equation will have $n-1$ roots equal to a.

513. From the principle demonstrated in Art. 512, it is evident that to determine the existence of equal roots in an equation we must

Find the greatest common divisor of the first member and its first derivative. If there is no common divisor there can be no equal roots. If there is a greatest common divisor, by placing it equal to zero and solving the resulting equation we shall obtain the required roots.

The number of times that each root is found in the given equation is one more than the number of times it is found in the equation formed from the greatest common divisor.

If the first member of the given equation be divided by the greatest common divisor, the depressed equation will contain the remaining roots of the original equation.

1. Find the roots of the equation

$$x^4-14x^3+61x^2-84x+36=0.$$

Here the first derivative is $4x^3-42x^2+122x-84$; the greatest common divisor of this and the given first member is $x^2 - 7x + 6$. Placing $x^2 - 7x + 6 = 0$, we have, by the rules of quadratics, or by factoring, $x = 1$ or 6. Therefore the roots of the given equation are 1, 1, 6, and 6.

EXAMPLES.

Find all the roots of the following:

2. $x^3 - 8x^2 + 13x - 6 = 0$.
3. $x^3 - 7x^2 + 16x - 12 = 0$.
4. $x^4 - 6x^2 - 8x - 3 = 0$.
5. $x^4 - 24x^2 + 64x - 48 = 0$.

514. When the equation formed from the greatest common divisor is of too high a degree to be conveniently solved, we may in certain cases compare it with its own derived equation, and thus obtain a common divisor of a lower degree. Of course this can only be done when the equation formed from the greatest common divisor has equal roots.

For example, required all the roots of

$$x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0. \qquad (1)$$

Here the first derivative is $5x^4 - 52x^3 + 201x^2 - 342x + 216$; the greatest common divisor of this and the given first member is $x^3 - 8x^2 + 21x - 18$. We have then to solve the equation

$$x^3 - 8x^2 + 21x - 18 = 0. \qquad (2)$$

The first derivative of (2) is $3x^2 - 16x + 21$; the greatest common divisor of this and $x^3 - 8x^2 + 21x - 18$ is $x - 3$. Solving $x - 3 = 0$, we have $x = 3$; hence two of the roots of (2) are equal to 3. Dividing the first member of (2) by $(x-3)^2$ or by $x^2 - 6x + 9$, the depressed equation is

$$x - 2 = 0, \text{ whence } x = 2.$$

Thus the three roots of (2) are 3, 3, and 2. Hence, the five roots of (1) are 3, 3, 3, 2, and 2.

515. If an equation has two roots equal in magnitude, but opposite in sign, by changing the signs of the alternate terms beginning with the second we shall obtain an equation with these same two roots (Art. 498); then evidently the greatest common divisor of the two first members placed equal to zero will determine the roots.

For example, required all the roots of

$$x^4 + 3x^3 - 13x^2 - 27x + 36 = 0. \qquad (1)$$

Changing the signs of the alternate terms, we have

$$x^4 - 3x^3 - 13x^2 + 27x + 36,$$

the greatest common divisor of which and the given first member is $x^2 - 9$; solving $x^2 - 9 = 0$, we have $x = 3$ or -3, thus giving two of the roots of (1). Dividing the first member of (1) by $x^2 - 9$, we have for the depressed equation

$$x^2 + 3x - 4 = 0,$$

whence $x = 1$ or -4. Thus the roots of (1) are 3, -3, 1, or -4.

LIMITS OF THE ROOTS OF AN EQUATION.

516. A polynomial of the form

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + tx^2 + ux + v$$

which we shall represent by X, may also be expressed thus (Art. 490):

$$(x-a)(x-b)(x-c) \ldots\ldots (x-l)\,Y$$

in which $a, b, c, \ldots\ldots l$ are the real, unequal roots of the equation $X = 0$, in the order of their magnitude, a being *algebraically* the smallest; and Y the product of all the factors containing imaginary roots, which must always be positive, and cannot affect the sign of X, for each pair of imaginary roots (Art. 497) produces a positive factor.

Suppose x to commence at any value less than a, and to assume in succession all possible values up to some quantity greater than l. When x is less than a, each of the factors $x - a$, $x - b$, ... is negative, and therefore X is either positive or negative, according as the degree is even or odd.

When $x = a$, $X = 0$. When x is greater than a, and less than b, $x - a$ becomes positive, and the sign of X changes. Also, when the value of x is made equal to b, and then greater, X first becomes 0 and then changes sign; and so on, for each real root.

When x has any value greater than l, X must be positive; for all its factors are positive.

517. *If two numbers, when substituted for the unknown quantity in an equation, give results having a different sign, at least one root lies between those numbers.*

It is evident, from Art. 516, that if X has a different sign for two values of x, some *odd* number of roots lies between them.

When the numbers substituted differ by unity, it is evident that the integral part of the root is known.

EXAMPLES.

1. What is the first figure of a root of the equation $x^3 + 3x^2 - 7x - 8 = 0$?

Here, if $x = 2$, the first member becomes -2; and if $x = 3$, the first member becomes 25; therefore at least one root lies between 2 and 3. Hence 2 is the first figure of a root.

2. Find the integral parts of all the roots of the equation $x^3 - 6x^2 + 3x + 9 = 0$.

3. Find the first figure of a root of the equation $x^3 - 2x - 50 = 0$.

4. Find the first figure of a root of the equation $x^4 - 2x^3 + 3x^2 - x - 5 = 0$.

5. Find the integral part of a root of the equation $2x^4 + x^3 - 7x^2 - 11x - 4 = 0$.

518. *To find the superior limit of the positive roots of an equation.*

Let the equation be

$$X = x^n + px^{n-1} + qx^{n-2} + \dots\dots + tx^2 + ux + v = 0. \qquad (1)$$

Let r be the numerical value of the greatest negative coefficient, and x^{n-s} the highest power of x which has a negative coefficient. Then the first s terms have positive coefficients.

Now X will be positive when x is positive, provided

$$x^n - r\,x^{n-s} - r\,x^{n-s-1} - \ldots\ldots - r\,x^2 - r\,x - r \qquad (2)$$

is positive; for, since r is the numerically greatest negative coefficient, and all terms up to the $(s+1)$th are positive, X is equal to (2) plus a *positive* quantity.

We may write (2)

$$x^n - r\,(x^{n-s} + x^{n-s-1} + \ldots\ldots + x^2 + x + 1),$$

or (Art. 120), $$x^n - r\,\frac{x^{n-s+1}-1}{x-1}. \qquad (3)$$

Then X will be positive when (3) is positive. But if x is greater than unity, (3) is evidently greater than

$$x^n - r\,\frac{x^{n-s+1}}{x-1}.$$

Therefore X will be positive when this is positive; or, when $(x-1)\,x^n - r\,x^{n-s+1}$ is positive; or, when $(x-1)\,x^{s-1} - r$ is positive.

But $(x-1)\,x^{s-1} - r$ is greater than $(x-1)\,(x-1)^{s-1} - r$ or $(x-1)^s - r$; therefore X will be positive when $(x-1)^s - r$ is positive or equal to zero; or, when $(x-1)^s = r$ or $> r$; or, when $x - 1 = \sqrt[s]{r}$ or $> \sqrt[s]{r}$; or, when $x = 1 + \sqrt[s]{r}$ or $> 1 + \sqrt[s]{r}$.

That is, when $x = 1 + \sqrt[s]{r}$ or any greater value, X is positive, which is impossible, as it must equal zero. Hence x must be less than $1 + \sqrt[s]{r}$; or, $1 + \sqrt[s]{r}$ is the superior limit of the positive roots.

519. *To find the inferior limit of the negative roots of an equation.*

By changing the signs of the alternate terms beginning with the second, we shall obtain an equation having the same roots with contrary signs (Art. 498).

Then evidently the superior limit of the positive roots of the transformed equation, obtained as in Art. 518, will by a change of sign become the inferior limit of the negative roots of the given equation.

Note. In applying the principles of the preceding articles to determine the limits of the roots of an equation, the absolute term must be taken as the coefficient of x^0.

520. **1.** Find the superior limit of the positive roots of

$$x^4 + 4x^3 - 19x^2 - 46x + 120 = 0.$$

Here. $r = 46$, and $n - s = 2$; or, as $n = 4$, $s = 2$. Then by Art. 518, the required limit is $1 + \sqrt{46}$, or 8 in whole numbers.

2. Find the inferior limit of the negative roots of

$$x^3 - x^2 - 14x + 24 = 0. \quad (1)$$

Changing the signs of the alternate terms beginning with the second, we have

$$x^3 + x^2 - 14x - 24 = 0. \quad (2)$$

Here $r = 24$, and $n - s = 1$, or $s = 2$. Then the superior limit of the positive roots of (2) is $1 + \sqrt{24}$; therefore the inferior limit of the negative roots of (1) is $-(1 + \sqrt{24})$.

EXAMPLES.

Find the superior limits of the positive roots of the following:

3. $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0.$ **4.** $x^4 - 15x^2 + 10x + 24 = 0.$

Find the inferior limits of the negative roots of the following:

5. $x^3 - 2x^2 - 5x + 6 = 0.$ **6.** $x^4 - 5x^3 + 5x^2 + 5x + 6 = 0.$

STURM'S THEOREM.

521. *To determine the number and situation of the real roots of an equation.*

A perfect solution of this difficult problem was first obtained by Sturm, in 1829. As the theorem determines the number of real roots, the number of imaginary roots also becomes known (Art. 487).

522. Let X denote the first member of

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0,$$

from which the equal roots have been removed (Art. 512).

Let X_1 denote the first derivative of X (Art. 511).

Divide X by X_1, and we shall obtain a quotient Q_1, with a remainder of a lower degree than X_1. Denote this remainder, *with its signs changed*, by X_2, divide X_1 by X_2, and so on; the operation being the same as in finding the greatest common divisor, except that the signs of every remainder must be changed, while no other change of signs is admissible. As the equation $X = 0$ has been freed from equal roots, there can be no common divisor of X and X_1, and the last remainder, X_n, will be independent of x.

The successive operations may be represented by the following equations:

$$X = X_1\,Q_1 - X_2 \qquad (1)$$

$$X_1 = X_2\,Q_2 - X_3 \qquad (2)$$

$$X_2 = X_3\,Q_3 - X_4 \qquad (3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_{n-2} = X_{n-1}\,Q_{n-1} - X_n$$

The expressions X, X_1, $X_2, \ldots\ldots X_n$ are called *Sturm's Functions.*

STURM'S THEOREM.

523. *If any two numbers, a and b, be substituted for x in Sturm's Functions, and the signs noted, the difference between the number of variations in the first case and that in the second is equal to the number of real roots of the given equation lying between a and b.*

The demonstration of Sturm's Theorem depends upon the following principles:

(A). *Two consecutive functions cannot both become 0 for the same value of x.*

For, if $X_1 = 0$ and $X_2 = 0$, then by (2), Art. 522, $X_3 = 0$; and if $X_2 = 0$ and $X_3 = 0$, by (3), $X_4 = 0$; and so on, till $X_n = 0$. But as X_n is independent of x, it cannot become 0 for any value of x. Hence no two consecutive functions can become zero for the same value of x.

(B). *If any function, except X and X_n, becomes 0 for a particular value of x, the two adjacent functions must have opposite signs.*

For, if $X_2 = 0$, we have by (2), Art. 522, $X_1 = -X_3$; that is, X_1 and X_3 must have opposite signs, for by (A) neither can be equal to zero.

(C). *When any function, except X and X_n, changes its sign for different values of x, the number of variations is not affected.*

No change of sign can take place in any one of Sturm's Functions except when x passes through a value which reduces that function to zero.

Now, let c be a root of the equation $X_2 = 0$; d and e quantities respectively a little less and a little greater than c, so taken that no root of $X_1 = 0$ or of $X_3 = 0$ is comprised between them. Then, as x changes from d to e, no change of sign takes place in X_1 or X_3, while X_2 reduces to zero and may change sign. And as by (B), when $X_2 = 0$, X_1 and X_3 have opposite signs, the only effect of a change in the sign of X_2 is that what was originally a permanence and a variation is now a variation and a permanence; that is, the permanence and variation exchange places. Hence a change in the sign of X_2 does not affect the *number* of variations.

As X_n is independent of x, it can never change sign for any value of x. Therefore a change in the number of variations can be caused only by a change in the sign of the given function X.

(D). *When the function X changes its sign for successive increasing values of x, the number of variations is diminished by one.*

Let m be a root of the equation $X = 0$; $m - y$ and $m + y$ quantities respectively a little less and a little greater than m, so taken that no root of $X_1 = 0$ is comprised between them. Then, as x changes from $m - y$ to $m + y$, no change of sign takes place in X_1, while X reduces to zero and changes sign.

Putting $m + y$ in place of x in X, we have

$$(m + y)^n + p\,(m + y)^{n-1} + \ldots\ldots + u\,(m + y) + v.$$

Developing the terms by the Binomial Theorem, and collecting terms containing like powers of y, we have

$$\begin{aligned} & m^n + p\,m^{n-1} + \ldots\ldots + u\,m + v \\ & + y\left[n\,m^{n-1} + p\,(n-1)\,m^{n-2} + \ldots\ldots + u\right] \\ & + \text{terms containing } y^2, y^3, \ldots\ldots\, y^n. \end{aligned}$$

Representing the coefficient of y, which we observe is the value of X_1 when x is put equal to m, by A; the coefficient of y^2 by B; and so on, we have

$$m^n + p\,m^{n-1} + \ldots\ldots + u\,m + v + A\,y + B\,y^2 + \ldots\ldots + K\,y^n. \quad (1)$$

But as $x = m$ reduces X to 0, we have

$$m^n + p\,m^{n-1} + \ldots\ldots + u\,m + v = 0.$$

Hence (1) may be written

$$A\,y + B\,y^2 + \ldots\ldots + K\,y^n. \quad (2)$$

Now y may be taken so small that the sign of (2) will be the same as the sign of its first term. That is, when x is a little greater than m, the sign of X is the same as the sign of X_1.

Similarly, by substituting $m - y$ for x in X, we shall arrive at the expression

$$-A\,y + B\,y^2 - C\,y^3 + \ldots\ldots,$$

where as before y may be taken so small that the sign of the whole expression will be the same as that of its first term. That is, when x is a little less than m, the sign of X is the reverse of the sign of X_1.

Thus we see that as x changes from $m - y$ to $m + y$, the signs of X and X_1 are different before x equals m, and alike afterwards. Hence, when X changes its sign a variation is changed into a permanence, or the number of variations is diminished by one.

We may now prove Sturm's Theorem; for as x changes from a to b, supposing a less than b, a variation is changed to a permanence each time that X reduces to 0 and changes sign, and only then, for no change of sign in any of the other functions can affect the number of variations. And as X reduces to zero only when x is equal to some root of the equation $X = 0$, it follows that the number of variations lost in passing from a to b is equal to the number of real roots of the equation $X = 0$ comprised between a and b.

524. When $-\infty$ and $+\infty$ are substituted for x, or when the superior limit of the positive roots and the inferior limit of the negative roots are substituted for x, the whole number of real roots of the equation $X = 0$ becomes known.

The substitution of $-\infty$ and 0 will give the whole number of negative roots, and the substitution of $+\infty$ and 0 will give the whole number of positive roots. If the roots are all real, Descartes' Rule (Art. 507) will effect the same object.

The substitution of various numbers for x will show between what numbers the roots lie, or fix the *limits* of the roots.

525. X and X_1 must change signs alternately, as they are always unlike in sign just before X changes sign (Art. 523, (D)). Hence, when the roots of $X = 0$ and of $X_1 = 0$ are all real, each root of $X_1 = 0$ must be intermediate in value between two roots of $X = 0$. For this reason the first derived equation is often called the *limiting* or *separating* equation.

526. In the process of finding X_2, X_3, etc., any *positive* numerical factors may be omitted or introduced at pleasure, as the *sign* of the result is not affected thereby. In this way fractions may be avoided.

In substituting $-\infty$ and $+\infty$, the first term of each function determines the sign, for in any expression, as

$$a\,x^n + b\,x^{n-1} + \ldots\ldots + k,$$

where x may be made as great as we please, it may be taken so great that the sign of the whole expression will be the same as that of its first term.

527. 1. Determine the number and situation of the real roots of the equation

$$x^3 - 4\,x^2 - x + 4 = 0.$$

Here, the first derivative, $X_1 = 3\,x^2 - 8\,x - 1$. Multiplying $x^3 - 4\,x^2 - x + 4$ by 3 so as to make its first term divisible by $3\,x^2$,

$$\begin{array}{r|rrrl}
3\,x^2 - 8\,x - 1\,) & 3\,x^3 - 12\,x^2 & - \;\;3\,x & + 12 & (\,x \\
 & 3\,x^3 - \;\;8\,x^2 & - \;\;\;x & & \\ \hline
 & -\;\;4\,x^2 & - \;\;2\,x & + 12 & \\
 & & & \frac{3}{2} & \\ \hline
 & -\;\;6\,x^2 & - \;\;3\,x & + 18 & (-2 \\
 & -\;\;6\,x^2 & + 16\,x & + \;\;2 & \\ \hline
 & & -19\,x & + 16 & \quad \therefore X_2 = 19\,x - 16.
\end{array}$$

$$\begin{array}{r|rrrl}
 & 3\,x^2 & - \;\;8\,x & - \;\;1 & \\
 & & & 19 & \\ \hline
19\,x - 16\,) & 57\,x^2 & - 152\,x & - \;\;19 & (\,3\,x \\
 & 57\,x^2 & - \;\;48\,x & & \\ \hline
 & & -104\,x & - \;\;19 & \\
 & & & 19 & \\ \hline
 & & -1976\,x & - \;\;361 & (-104 \\
 & & -1976\,x & + 1664 & \\ \hline
 & & & -2025 & \quad \therefore X_3 = 2025.
\end{array}$$

Thus we have, $X = x^3 - 4x^2 - x + 4$; $X_2 = 19x - 16$.

$X_1 = 3x^2 - 8x - 1$; $X_3 = 2025$.

The last step of the division may be omitted, for we only wish the *sign* of X_3, and that may be seen by inspection when $-104x - 19$ is obtained.

We first substitute $-\infty$ for x in each function, and obtain three variations of sign; similarly $+\infty$ gives no variation; hence the three roots are all real. Substituting 0, we have two variations; comparing this with the former results, we see that one root is negative and the other two are positive. The same result could have been obtained by Descartes' Rule, as all the roots are real. We now substitute various numbers to determine the limits of the roots.

The table presents the results in a connected form:

	X	X_1	X_2	X_3	
When $x = -\infty$,	$-$	$+$	$-$	$+$	3 variations.
" $x = -2$,	$-$	$+$	$-$	$+$	3 variations.
" $x = -1$,	0	$+$	$-$	$+$	
" $x = 0$,	$+$	$-$	$-$	$+$	2 variations.
" $x = 1$,	0	$-$	$+$	$+$	
" $x = 2$,	$-$	$-$	$+$	$+$	1 variation.
" $x = 3$,	$-$	$+$	$+$	$+$	1 variation.
" $x = 4$,	0	$+$	$+$	$+$	
" $x = 5$,	$+$	$+$	$+$	$+$	no variation.
" $x = \infty$,	$+$	$+$	$+$	$+$	no variation.

Then by Sturm's Theorem we know that there is one root between -2 and 0, one between 0 and 2, and one between 3 and 5. In fact, as $X = 0$ when $x = -1$, 1, and 4, these are the three roots of the equation.

2. Determine the number and situation of the real roots of

$$X = x^4 - 3x^3 + 3x^2 - 3x + \frac{5}{2} = 0.$$

Note. In substituting the various numbers to determine the situation of the roots, it is best to work from 0 in either direction, stopping when the number of variations is the same as has been previously found for $+\infty$ or $-\infty$, as the case may be.

Here we find $X_1 = 4x^3 - 9x^2 + 6x - 3$; $X_3 = -92x + 129$;

$X_2 = 3x^2 + 18x - 31$; $X_4 = -1163$.

Substituting $+\infty$ for x, we obtain one variation; similarly, 0 gives three variations, and $-\infty$ gives three variations. Hence there are only two real roots, both of which are positive. We then substitute values of x from 0 upwards, giving the following results:

	X	X_1	X_2	X_3	X_4	
When $x = 0$,	+	−	−	+	−	3 variations.
" $x = 1$,	+	−	−	+	−	3 variations.
" $x = 2$,	+	+	+	−	−	1 variation.
" $x = \infty$,	+	+	+	−	−	1 variation.

Hence there are two roots between 1 and 2; and as the equation has four roots, there must be two imaginary roots.

EXAMPLES.

Determine the number and situation of the real roots of the following equations:

3. $x^3 - x^2 - 2x + 1 = 0$.
6. $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$.

4. $x^3 - 7x + 7 = 0$.
7. $2x^4 - 3x^3 + 17x^2 - 3x + 15 = 0$.

5. $x^3 - 2x - 5 = 0$.
8. $x^4 - 4x^3 - 3x + 27 = 0$.

XLIII. — SOLUTION OF HIGHER NUMERICAL EQUATIONS.

528. The real roots of the higher numerical equations in general can only be obtained by tentative methods, or by methods which involve approximation. Cubic and biquadratic equations may be considered as included in the class of higher equations; for their general solutions are complicated, and only of limited application. No general solution of an equation of a degree higher than the fourth can be obtained.

COMMENSURABLE ROOTS.

529. A *commensurable* root is one which can be exactly expressed as an integer or fraction without using irrational quantities.

An *incommensurable* root is one which can only be expressed approximately by means of a decimal fraction.

530. Any equation containing fractional coefficients may be transformed into another whose coefficients are entire, that of the first term being unity (Art. 500), and such an equation cannot have a root equal to a rational fraction (Art. 495); hence, to find all commensurable roots, we have only to find all integral roots.

531. As every rational root of an equation in its general form is a divisor of the last term (Art. 493), to find the commensurable roots we have only to *ascertain by trial what integral divisors of the absolute term are roots of the equation.*

The trial may be made by substituting each divisor, both with the positive and the negative sign, in the equation; or by dividing the first member of the equation by the unknown quantity minus the supposed root (Art. 486). In substituting very small numbers, such as ± 1, the former method may be most convenient; but when an actual root has once been used, the latter method will give at once the depressed equation, which may be used in obtaining the other roots.

532. When the number of divisors of the last term is large, this process of successive trials becomes tedious, and a better method, known as the *Method of Divisors,* may be adopted.

If a is a root of the equation

$$x^4 + p\,x^3 + q\,x^2 + t\,x + u = 0,$$

then $$a^4 + p\,a^3 + q\,a^2 + t\,a + u = 0.$$

Transposing and dividing by a,

$$\frac{u}{a} = -t - q\,a - p\,a^2 - a^3, \qquad (1)$$

whence we see that $\frac{u}{a}$ must be an integer.

Equation (1) may be written

$$\frac{u}{a} + t = -q\,a - p\,a^2 - a^3.$$

Denoting $\frac{u}{a} + t$ by t', and dividing by a,

$$\frac{t'}{a} = -q - p\,a - a^2,$$

whence $\frac{t'}{a}$ must be an integer.

Proceeding in this way, we see that if a is a root of the equation, $\frac{u}{a} + t$ or t', $\frac{t'}{a} + q$ or q', and $\frac{q'}{a} + p$ or p' must be integers, and $\frac{p'}{a} + 1$ must equal zero.

Hence the following

RULE.

Divide the absolute term of the equation by one of its integral divisors, and to the quotient add the coefficient of x.

Divide this sum by the same divisor, and, if the quotient is an integer, add to it the coefficient of x^2.

Proceed in the same manner with each coefficient in regular order, and, if the divisor is a root of the equation, each quotient will be entire, and the last quotient added to the coefficient of the highest power of x will equal 0.

Equal roots, if any, should be removed before applying the rule; and the labor may often be diminished by obtaining the superior limit to the positive and inferior limit to the negative roots of the equation, for no number need be tried which does not fall between these limits.

1. Find the roots of the equation

$$x^3 - 6x^2 + 27x - 38 = 0.$$

By Descartes' Rule, we see that the equation has no negative root; and the only positive divisors of 38 are 1, 2, 19, and 38. By substitution we see that 1 is not a root of the equation.

Dividing the first member by $x - 2$, we obtain $x^2 - 4x + 19$ as a quotient. Hence 2 is a root, and the depressed equation is $x^2 - 4x + 19 = 0$, from which we obtain

$$x = \frac{4 \pm \sqrt{16 - 76}}{2} = 2 \pm \sqrt{-15}$$

as the remaining roots. Hence,

$$x = 2, \text{ or } 2 \pm \sqrt{-15}, \textit{ Ans.}$$

2. Find the roots of the equation

$$8x^4 - 4x^3 - 14x^2 + x + 3 = 0.$$

We may write the equation

$$x^4 - \frac{x^3}{2} - \frac{7x^2}{4} + \frac{x}{8} + \frac{3}{8} = 0.$$

Proceeding as in Art. 500, we see that the multiplier 2 will remove the fractional coefficients. We then have the equation

$$x^4 - x^3 - 7x^2 + x + 6 = 0, \qquad (1)$$

whose roots are twice those of the given equation (Art. 499).

The divisors of 6 are ± 1, ± 2, ± 3, and ± 6.

By putting x equal to $+1$ and -1 in (1), it is readily seen that both are roots of the equation, and the other roots can be found from the depressed equation. But all of the rational roots may be obtained by the rule.

It is customary to abridge the work as follows:

Divisors,	6,	3,	2,	1,	-1,	-2,	-3,	-6
1st Quotients,	1,	2,	3,	6,	-6,	-3,	-2,	-1
Adding 1,	2,	3,	4,	7,	-5,	-2,	-1,	0
2d Quotients,		1,	2,	7,	5,	1,		0
Adding -7,		-6,	-5,	0,	-2,	-6,		-7
3d Quotients,		-2,		0,	2,	3		
Adding -1,		-3,		-1,	1,	2		
4th Quotients,		-1,		-1,	-1,	-1		
Adding 1,		0,		0,	0,	0		

As 6, 2, -3, and -6 give fractional quotients at different stages of the operation, they cannot be roots of the given equation, and are rejected. 3, 1, -1, and -2 give entire quotients, and in each case the last quotient added to the coefficient of x^4 gives zero; hence they are the four roots of equation (1), and $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$, and -1 are the four roots of the given equation.

EXAMPLES.

Find all the commensurable roots of the following equations, and the remaining roots when possible by methods already given:

3. $x^3 + 6x^2 + 11x + 6 = 0.$

4. $x^3 + 3x^2 - 4x - 12 = 0.$

5. $x^4 - 4x^3 - 8x + 32 = 0.$

6. $4x^3 - 16x^2 - 9x + 36 = 0.$

7. $x^3 - 3x^2 + x + 2 = 0.$

8. $x^3 - 7x^2 + 36 = 0.$

9. $x^3 - 6x^2 + 10x - 8 = 0.$

10. $x^3 - 6x^2 + 11x - 6 = 0.$

11. $2x^3 - 3x^2 + 16x - 24 = 0.$

12. $x^5 - 2x^3 - 16 = 0.$

13. $x^4 - 9x^3 + 23x^2 - 20x + 15 = 0.$

14. $x^4 + x^3 - 29x^2 - 9x + 180 = 0.$

RECURRING OR RECIPROCAL EQUATIONS.

533. A **Recurring Equation** is one in which the coefficients of any two terms equally distant from the extremes of the first member are equal.

The equal coefficients may have the same sign, or opposite signs; but a part cannot have the same sign, and a part opposite signs, in the same equation. Also, if the degree be even, and the equal coefficients have opposite signs, the middle term must be wanting. Thus,

$$x^4 - 5x^3 + 6x^2 - 5x + 1 = 0,$$

$$5x^5 - 51x^4 + 160x^3 - 160x^2 + 51x - 5 = 0,$$

$$x^6 - x^5 + x^4 - x^2 + x - 1 = 0,$$

are recurring equations.

534. *If any quantity is a root of a recurring equation, the reciprocal of that quantity is also a root of the same equation.*

Let $x^n + px^{n-1} + qx^{n-2} + \ldots \pm (\ldots + qx^2 + px + 1) = 0 \quad (1)$

be the equation. Substitute $\frac{1}{y}$ for x; then

$$\frac{1}{y^n} + \frac{p}{y^{n-1}} + \frac{q}{y^{n-2}} + \ldots\ldots \pm \left(\ldots\ldots + \frac{q}{y^2} + \frac{p}{y} + 1\right) = 0$$

Multiplying each term by y^n,

$$(1 + py + qy^2 + \ldots) \pm (\ldots + qy^{n-2} + py^{n-1} + y^n) = 0 \quad (2)$$

Now, (1) and (2) take precisely the same form on changing the $\pm$ sign to the first parenthesis in equation (2), and hence they must have the same roots. Now, if a is a root of (1), as $y = \frac{1}{x}$, $\frac{1}{a}$ must be a root of (2); but, as (1) and (2) have the same roots, $\frac{1}{a}$ must also be a root of (1). In like manner, if b is a root of (1), $\frac{1}{b}$ is also a root of (1).

On account of the property just demonstrated, *recurring* equations are also called *reciprocal* equations; the former term relating to their *coefficients*, and the latter to their *roots*.

535. *One root of a recurring equation of an odd degree is* -1 *when the equal coefficients have the same sign, and* $+1$ *when they have opposite signs.*

A recurring equation of an odd degree, as

$$x^{2m+1} + p x^{2m} + q x^{2m-1} + \dots \pm (\dots + q x^2 + p x + 1) = 0 \qquad (3)$$

has an even number of terms, and may be written in one of the following forms,

$$(x^{2m+1} + 1) + p\,(x^{2m} + x) + q\,(x^{2m-1} + x^2) + \dots\dots = 0,$$

$$(x^{2m+1} - 1) + p\,(x^{2m} - x) + q\,(x^{2m-1} - x^2) + \dots\dots = 0.$$

If -1 be substituted for x in the first form, or $+1$ in the second, the first member will become 0; hence, -1 is a root of the first and $+1$ a root of the second.

If equation (3) be divided by $x \pm 1$, both forms will reduce to the following form,

$$x^{2m} + p x^{2m-1} + q x^{2m-2} + \dots\dots + q x^2 + p x + 1 = 0, \qquad (4)$$

a recurring equation of an even degree in which the equal coefficients have the same sign. Hence, a recurring equation of an odd degree may always be depressed to one of an even degree.

536. *Two roots of a recurring equation of an even degree are* $+1$ *and* -1 *when the equal coefficients have opposite signs.*

Let

$$x^{2m} + p\,x^{2m-1} + q\,x^{2m-2} + \dots\dots - (\dots\dots + q\,x^2 + p\,x + 1) = 0$$

be such an equation. As the middle term must be wanting (Art. 533), the equation may be written in the form

$$(x^{2m} - 1) + p\,x\,(x^{2m-2} - 1) + q\,x^2\,(x^{2m-4} - 1) + \dots\dots = 0 \quad (5)$$

which is divisible by both $x - 1$ and $x + 1$, or by $x^2 - 1$ (Art. 120). Hence, both $+1$ and -1 are roots of the equation.

If equation (5) be divided by $x^2 - 1$, it will be depressed two degrees, and become a recurring equation of an even degree, in which the equal coefficients have the same sign (Art. 120). Hence, every recurring equation may be depressed to the form of equation (4), Art. 535.

537. *Every recurring equation of an even degree, whose equal coefficients have the same sign, may be reduced to an equation of half that degree.*

Let

$$x^{2m} + p\,x^{2m-1} + q\,x^{2m-2} + \ldots\ldots + q\,x^2 + p\,x + 1 = 0$$

be such an equation. Dividing it by x^m, we may write it

$$\left(x^m + \frac{1}{x^m}\right) + p\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + q\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots = 0 \quad (6)$$

the middle term if present becoming a known quantity.

Put $\quad x + \dfrac{1}{x} = y$

Then,

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

$$x^3 + \frac{1}{x^3} = y^3 - 3\left(x + \frac{1}{x}\right) = y^3 - 3\,y$$

$$x^4 + \frac{1}{x^4} = (y^2 - 2)^2 - 2 = y^4 - 4\,y^2 + 2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x^m + \frac{1}{x^m} = y^m - m\,y^{m-2} + \ldots\ldots$$

Substituting these values in (6), we have an equation of the form

$$y^m + p_1\,y^{m-1} + q_1\,y^{m-2} + \ldots\ldots = 0.$$

After this equation is solved, we can immediately find x from the equation $x + \dfrac{1}{x} = y$.

538. It thus appears that any recurring equation of the $(2\,m + 1)$th degree, one of the $(2\,m + 2)$th degree whose equal coefficients have opposite signs, and one of the $2m$th degree whose equal coefficients have the same sign, may each be reduced to an equation of the mth degree.

EXAMPLES.

1. Given $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$, to find x.

Dividing by x^2, $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0.$

Substituting y for $x + \frac{1}{x}$, and $y^2 - 2$ for $x^2 + \frac{1}{x^2}$, we have

$$y^2 - 2 - 5y + 6 = 0.$$

Whence, $y = 4$ or 1.

If $y = 4$, $x + \frac{1}{x} = 4$, or $x^2 - 4x = -1$;

Whence, $x = 2 \pm \sqrt{3}$.

If $y = 1$, $x + \frac{1}{x} = 1$, or $x^2 - x = -1$;

Whence, $x = \frac{1 \pm \sqrt{-3}}{2}$.

Note. That $2 - \sqrt{3}$ and $\frac{1 - \sqrt{-3}}{2}$ are reciprocals of $2 + \sqrt{3}$ and $\frac{1 + \sqrt{-3}}{2}$ may easily be shown by reducing $\frac{1}{2 + \sqrt{3}}$ and $\frac{2}{1 + \sqrt{-3}}$ to equivalent fractions with rational denominators (Art. 279).

Solve the following equations:

2. $x^5 - 11x^4 + 17x^3 + 17x^2 - 11x + 1 = 0.$

3. $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0.$

4. $x^6 - x^5 + x^4 - x^2 + x - 1 = 0.$

5. $x^3 + px^2 + px + 1 = 0.$

6. $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$

7. $5x^5 - 51x^4 + 160x^3 - 160x^2 + 51x - 5 = 0.$

8. $x^4 + 5x^3 + 5x + 1 = 0.$

9. $x^5 = -1$, or $x^5 + 1 = 0$. (See Art. 332.)

10. $x^5 - 32 = 0$. (Let $x = 2y$.)

CARDAN'S METHOD FOR THE SOLUTION OF CUBIC EQUATIONS.

539. In order to solve a cubic equation by Cardan's method, it must first be transformed, if necessary, into another cubic equation in which the square of the unknown quantity shall be wanting.

By Art. 505, this may be done by substituting for x, y minus the coefficient of x^2 divided by 3.

540. If the first power of the unknown quantity be wanting in the given equation, we may obtain the result by a simpler method, as follows:

Let $x^3 + ax^2 + c = 0$ be such an equation.

Substituting $\frac{1}{y}$ for x, we have

$$\frac{1}{y^3} + \frac{a}{y^2} + c = 0, \text{ or } cy^3 + ay + 1 = 0.$$

541. *To solve a cubic equation in the form* $x^3 + px + q = 0$.

Put $x = z - \frac{p}{3z}$, and the equation becomes

$$z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3} + pz - \frac{p^2}{3z} + q = 0,$$

or, $$z^3 - \frac{p^3}{27z^3} + q = 0; \text{ or, } 27z^6 + 27qz^3 - p^3 = 0.$$

This is an equation in the quadratic form, and may be solved by the method of Art. 313; and after z is known, x may be found directly from the equation $x = z - \frac{p}{3z}$.

We have then for solving cubic equations the following

RULE.

If necessary, transform the equation into another cubic equation in which the square of the unknown quantity shall be wanting (Arts. 539 and 540).

If y be the unknown quantity in the resulting equation, substitute for it z minus the coefficient of y divided by $3z$.

EXAMPLES.

1. Solve the equation $x^3 - 9x + 28 = 0$.

Substituting $z + \frac{3}{z}$ for x,

$$z^3 + 9z + \frac{27}{z} + \frac{27}{z^3} - 9z - \frac{27}{z} + 28 = 0,$$

or, $$z^3 + \frac{27}{z^3} + 28 = 0; \text{ or, } z^6 + 28z^3 = -27.$$

Solving by quadratics, $z^3 = -1$ or -27.

Whence, $z = -1$ or -3.

If $z = -1$, $x = z + \frac{3}{z} = -1 - 3 = -4$.

If $z = -3$, $x = -3 - 1 = -4$.

Hence, one root of the equation is -4. Dividing the first member of the given equation by $x + 4$, we obtain as the depressed equation,

$$x^2 - 4x + 7 = 0.$$

Whence, $x = 2 \pm \sqrt{-3}$, the remaining roots.

2. Solve the equation $x^3 - 24x^2 - 24x - 25 = 0$.

Putting $x = y + 8$ (Art. 539), we obtain

$$y^3 + 24y^2 + 192y + 512 - 24y^2 - 384y - 1536 - 24y - 192 - 25 = 0,$$

or, $$y^3 - 216y - 1241 = 0.$$

Putting $y = z + \frac{72}{z}$, we have

$$z^3 + 216z + \frac{15552}{z} + \frac{373248}{z^3} - 216z - \frac{15552}{z} - 1241 = 0,$$

or, $z^3 + \frac{373248}{z^3} - 1241 = 0$; or, $z^6 - 1241z^3 + 373248 = 0$.

Whence, $z^3 = 729$ or 512, and $z = 9$ or 8.

Therefore, $y = 9 + \frac{72}{9}$ or $8 + \frac{72}{8} = 17$, and $x = y + 8 = 25$.

Hence, one root of the equation is 25. Dividing the first member of the given equation by $x - 25$, we have as the depressed equation

$$x^2 + x + 1 = 0.$$

Whence, $x = \frac{-1 \pm \sqrt{-3}}{2}$, the remaining roots.

Solve the following equations:

3. $x^3 - 6x + 9 = 0$.

4. $x^3 - 6x^2 + 57x - 196 = 0$.

5. $x^3 - 4x^2 - 3x + 18 = 0$.

6. $x^3 + 9x^2 - 21x + 11 = 0$.

7. $x^3 - 2x^2 + 2x - 1 = 0$.

8. $x^3 - 4x^2 + 4x - 3 = 0$.

9. $x^3 - 3x^2 + 4 = 0$.

10. Obtain one root of the equation $x^3 + 6x - 2 = 0$.

542. In the cubic equation $x^3 + px + q = 0$, when p is negative, and $\frac{-p^3}{27} > \frac{q^2}{4}$, Cardan's method involves imaginary expressions; but it may be shown in that case that the three roots of the equation are then real and unequal.

Thus, in solving the equation $x^3 - 6x + 4 = 0$.

Substituting $z + \frac{2}{z}$ for x, we have

$$z^3 + 6z + \frac{12}{z} + \frac{8}{z^3} - 6z - \frac{12}{z} + 4 = 0,$$

or, $$z^3 + \frac{8}{z^3} + 4 = 0;\ \text{or,}\ z^6 + 4z^3 + 8 = 0.$$

Whence, $$z^3 = -2 \pm \sqrt{-4},\ \text{or}\ -2 \pm 2\sqrt{-1},$$

or, $$z = \sqrt[3]{-2 + 2\sqrt{-1}}\ \text{or}\ \sqrt[3]{-2 - 2\sqrt{-1}}.$$

It may be proved by trial that $1 + \sqrt{-1}$ is the cube root of $-2 + 2\sqrt{-1}$, and $1 - \sqrt{-1}$ of $-2 - 2\sqrt{-1}$. Hence,

$$z = 1 + \sqrt{-1}\ \text{or}\ 1 - \sqrt{-1}.$$

If $z = 1 + \sqrt{-1}$,

$$x = z + \frac{2}{z} = 1 + \sqrt{-1} + \frac{2}{1 + \sqrt{-1}} = \frac{2\sqrt{-1} + 2}{1 + \sqrt{-1}} = 2.$$

Hence, one root of the equation is 2. Dividing the first member of the given equation by $x - 2$, we have as the depressed equation

$$x^2 + 2x - 2 = 0.$$

Whence, $$x = -1 \pm \sqrt{3},\ \text{the remaining roots.}$$

543 We have no general rule for the extraction of the cube root of a binomial surd; so that in examples like that in the preceding article, unless the value of z can be obtained by inspection, it is impossible to find the real values of x by Cardan's method. . In this case, the real values of x can always be found by a method involving Trigonometry.

BIQUADRATIC EQUATIONS.

544. General solutions of biquadratic equations have been obtained by Descartes, Simpson, Euler, and others. Some of them require the second term of the equation to be removed, while others do not. All of them depend upon the solution of a cubic equation by Cardan's method, and will of course fail when that fails (Art. 542). They are practically of little value, especially as numerical equations of all degrees can be readily solved by methods of approximation.

INCOMMENSURABLE ROOTS.

545. If a higher numerical equation is found to contain no commensurable roots, or if, after removing the commensurable roots, the depressed equation is still of a higher degree, the irrational or incommensurable roots must next be sought. The integral parts of these roots may be found by Sturm's Theorem or by Art. 517, and the decimal parts by any one of the three following methods of approximation.

HORNER'S METHOD.

546. Suppose a root of the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} + \ldots\ldots + t\,x^2 + u\,x + v = 0 \qquad (1)$$

is found to lie between a and $a+1$. Transform the equation into another whose roots shall be less by a (Art. 502), and we shall have an equation in the form

$$y^n + p'\,y^{n-1} + q'\,y^{n-2} + \ldots\ldots + t'\,y^2 + u'\,y + v' = 0, \qquad (2)$$

one of whose roots is less than 1. If that root is found to lie between the decimal fractions a' tenths and $a'+1$ tenths, transform equation (2) into another whose roots shall be less by a' tenths, and we shall have an equation in the form

$$z^n + p''\,z^{n-1} + q''\,z^{n-2} + \ldots\ldots + t''\,z^2 + u''\,z + v'' = 0 \qquad (3)$$

one of whose roots is less than .1. If that root is found to lie between the decimal fractions a'' hundredths and $a''+1$ hundredths, transform equation (3) into another whose roots shall be less by a'' hundredths; and so on.

Thus we obtain

$$x = a + a' + a'' + \ldots\ldots$$

to any desired degree of accuracy.

As y and z in equations (2) and (3) are fractional, their higher powers are comparatively small; hence approximate values of y and z may be found by considering the last two terms only, from which we have

$$y = -\frac{v'}{u'} \text{ and } z = -\frac{v''}{u''}.$$

Thus approximate values of a', a'', ... may be found in this way, and with greater accuracy the smaller they become.

Hence a positive incommensurable root of the equation may be found by the following

RULE.

Find by Sturm's Theorem the initial part of the root, and transform the given equation into one whose roots are less by this initial part.

Divide the absolute term of the transformed equation by the coefficient of the first power of the unknown quantity for the next figure of the root.

Transform this last equation into another whose roots are less by the figure of the root last found, divide as before for the next figure of the root; and so on.

547. A negative root may be found by changing the signs of the alternate terms of the equation beginning with the second, and finding the corresponding positive root of the transformed equation (Art. 498). This by a change of sign becomes the required negative root.

548. In obtaining the approximate value of any one of the quantities a', a'', ... by the rule, we are liable to get too great a result; a similar case occurs in extracting the square or cube root of a number. We may discover such an error by observing the signs of the last two terms of the next transformed equation; for, as the figures of the root as obtained in succession are to be *added*, it follows that a', a'',...... must be positive quantities, so that the last two terms of the transformed equation must be of *opposite sign*. We then diminish the approximate value until a result is found which satisfies this condition.

549. If in any transformed equation the coefficient of the first power of the unknown quantity should be zero, the next figure of the root may be obtained by *dividing the absolute term by the coefficient of the square of the unknown quantity, and taking the square root of the result.*

For, if in equation (2), Art. 546, $u' = 0$, we have, approximately,

$$t' y^2 + v' = 0, \text{ whence } y = \sqrt{-\frac{v'}{t'}}.$$

We proceed in a similar manner if any number of the coefficients immediately preceding the absolute term reduce to zero.

550. **1.** Solve the equation $x^3 - 3x^2 - 2x + 5 = 0$.

By Sturm's Theorem, the equation has three real roots; one between 3 and 4, another between 1 and 2, the third between -1 and -2.

To find the first root, we transform the equation into another whose roots are less by 3, which by Art. 503 is effected as follows:

Dividing $x^3 - 3x^2 - 2x + 5$ by $x - 3$, we have $x^2 - 2$ as a quotient and -1 as a remainder. Dividing $x^2 - 2$ by $x - 3$, we have $x + 3$ as a quotient and 7 as a remainder. Dividing $x + 3$ by $x - 3$, we have 1 as a quotient and 6 as a remainder. Hence the transformed equation is

$$x^3 + 6x^2 + 7x - 1 = 0,$$

whose roots are less by 3 than those of the given equation.

Note. The operations of division in Horner's Method are usually performed by a method known as *Synthetic Division*. For example, let it be required to divide $x^3 - 19x + 30$ by $x - 2$.

$$\begin{array}{l|l}
x^3 \pm 0x^2 - 19x + 30 & x - 2 \\
\underline{x^3 - 2x^2} & \overline{x^2 + 2x - 15} \\
\quad 2x^2 & \\
\quad \underline{2x^2 - 4x} & \\
\qquad -15x & \\
\qquad \underline{-15x + 30} & \\
\qquad\qquad 0 &
\end{array}$$

The first term of each partial product may be omitted, as it is merely a repetition of the term immediately above. Also the remaining term of each partial product may be *added* to the corresponding term of the dividend, provided we change the sign of the second term of the divisor before multiplying. Also the powers of x may be omitted, as we need only consider the *coefficients* in order to obtain the remainder.

The work now stands

$$\begin{array}{llll|l} 1 \pm 0 & -19 & +30 & & 1+2 \\ \underline{+2} & & & & \overline{1+2-15} \\ +2 & & & & \\ & \underline{+4} & & & \\ & -15 & & & \\ & & \underline{-30} & & \\ & & 0 & & \end{array}$$

As the first term of the divisor is 1, it is usually omitted, and the first terms of the dividends constitute the quotient. Raising the oblique columns we have the following concise form:

Dividend,	$1 \pm 0 - 19 + 30$	$\lfloor +2$
Partial Products,	$+2 + \ \ 4 - 30$	
Quotient,	$1 + 2 - 15, + \ 0$	Remainder.

Here we use only the second term of the divisor *with its sign changed;* each term of the quotient is the sum of the terms in the vertical column under which it stands, and each term of the second line is obtained by multiplying the preceding term of the quotient by the divisor as written.

By the method of Synthetic Division, the work of transforming the given equation into one whose roots are less by 3 stands as follows:

$$\begin{array}{llll|l} 1 & -\ 3 & -\ 2 & +\ 5 & +3 \\ & \underline{+\ 3} & \underline{\ \ \ 0} & \underline{-\ 6} & \\ & \ \ \ 0 & -\ 2 & -\ 1, & \text{1st Remainder.} \\ & \underline{+\ 3} & \underline{+\ 9} & & \\ & +\ 3 & +\ 7, & \text{2d Remainder.} & \\ & \underline{+\ 3} & & & \\ & +\ 6, & \text{3d Remainder.} & & \end{array}$$

Thus the transformed equation is, as before,

$$x^3 + 6x^2 + 7x - 1 = 0. \qquad (1)$$

Dividing 1 by 7 we obtain .1 as the next figure of the root, and we proceed to transform equation (1) into another whose roots shall be less by .1.

$$
\begin{array}{rrrrl}
1 & 6 & 7 & -1 & \underline{|\;.1} \\
 & .1 & .61 & .761 & \\
\hline
 & 6.1 & 7.61 & -.239 & \\
 & .1 & .62 & & \\
\hline
 & 6.2 & 8.23 & & \\
 & .1 & & & \\
\hline
 & 6.3 & & &
\end{array}
$$

Thus the transformed equation is

$$x^3 + 6.3\,x^2 + 8.23\,x - .239 = 0,$$

whence by dividing .239 by 8.23 we obtain .02 for the next root figure; and so on. Thus the first root is, approximately, 3.12.

Similarly, the second root may be shown to be 1.201 approximately.

By Art. 547, the third root is the positive root of the equation $x^3 + 3\,x^2 - 2\,x - 5 = 0$ with its sign changed. The successive transformations are usually written in connection as in the following form, where the coefficients of the different transformed equations are indicated by (1), (2), (3), ... The work may also be contracted by dropping such decimal figures from the right of each column as are not needed for the required degree of accuracy.

$$
\begin{array}{rrrrl}
1 & 3 & -2 & -5 & \underline{|\;1.33\ldots\ldots} \\
 & 1 & 4 & 2 & \\
 & \overline{4} & \overline{2} & (1)\;\overline{-3} & \\
 & 1 & 5 & 2.667 & \\
 & \overline{5} & (1)\;\overline{7} & (2)\;\overline{-\;\;.333} & \\
 & 1 & 1.89 & & \\
 & (1)\;\overline{6} & \overline{8.89} & & \\
 & .3 & 1.98 & & \\
 & \overline{6.3} & (2)\;\overline{10.87} & & \\
 & .3 & & & \\
 & \overline{6.6} & & & \\
 & .3 & & & \\
 & (2)\;\overline{6.9} & & &
\end{array}
$$

Hence, the third root is -1.33 approximately.

EXAMPLES.

Find the real roots of the following equations:

2. $x^3 - 2x - 5 = 0$.

3. $x^3 + x^2 - 500 = 0$.

4. $x^3 - 7x + 7 = 0$.

5. $x^3 - 17x^2 + 54x - 350 = 0$.

6. $x^4 - 4x^3 - 3x + 27 = 0$.

7. $x^4 - 12x^2 + 12x - 3 = 0$.

APPROXIMATION BY DOUBLE POSITION.

551. Find two numbers, a and b, the one greater and the other less than a root of the equation (Arts. 517 or 521), and suppose a to be nearer the root than b. Substitute them separately for x in the given equation, and let A and B represent the values of the first member thus obtained. If a and b were the true roots, A and B would each be 0; hence the latter may be considered as the errors which result from substituting a and b for x. Although not strictly correct, yet, for the purpose of approximation, we may assume that

$$A : B = x - a : x - b$$

Whence (Art. 348), $A - B : A = b - a : x - a$

or (Art. 345), $A - B : b - a = A : x - a$ (1)

and,
$$x - a = \frac{A\,(b-a)}{A - B}$$

or,
$$x = a + \frac{A\,(b-a)}{A - B}.$$

From (1), we see that, approximately,

As the difference of the errors is to the difference of the two assumed numbers, so is either error to the correction of its assumed number.

Adding this correction when its assumed number is too small, or subtracting when too large, we obtain a nearer approximation to the true root. This result and another assumed number may now be used as new values of a and b, for obtaining a still nearer approximation; and so on.

It is best to employ two assumed quantities that shall differ from each other only by unity in the last figure on the right. It is also best to use the smaller error.

This method of approximation has the advantage of being applicable to equations in any form. It may, therefore, be applied to radical and exponential equations, and others not reduced to the general form (Art. 480).

EXAMPLES.

1. Find a root of the equation $x^3 + x^2 + x - 100 = 0$.

When 4 and 5 are substituted for x in the equation, the results are -16 and $+55$, respectively; hence $a = 4$, $b = 5$, $A = -16$, and $B = 55$. According to the formula, the first approximation gives

$$x = 4 + \frac{-16\,(5-4)}{-16-55} = 4 + \frac{16}{71} = 4.2 +.$$

As the true root is greater than 4.2, we now assume 4.2 and 4.3 as a and b. Substituting these values for x in the given equation, we obtain -4.072 and $+2.297$; therefore 4.3 is nearer the true root than 4.2.

Hence, $$x = 4.3 - \frac{2.297\,(4.3-4.2)}{2.297+4.072} = 4.3 - \frac{.2297}{6.369}$$
$$= 4.3 - .036 = 4.264.$$

Substituting 4.264 and 4.265 for x, and stating the result in the form of a proportion, we have

$$.0276 + .0365 : .001 = .0276 : \text{correction of } 4.264.$$

Whence the correction $= .00043+$.

Hence, $$x = 4.264 + .00043 = 4.26443+,\ \textit{Ans.}$$

Find one root of each of the following equations:

2. $x^3 - 2x - 50 = 0$.

3. $x^3 + 10x^2 + 5x - 260 = 0$.

4. $x^3 + 8x^2 + 6x - 75.9 = 0$.

5. $x^3 + \frac{11x}{16} - \frac{3}{4} = 0$.

6. $x^4 - 3x^2 - 75x - 10000 = 0$.

7. $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$.

NEWTON'S METHOD OF APPROXIMATION.

552. Find two numbers, one greater and the other less than a root of the equation (Arts. 517 or 521). Let a be one of those numbers, the nearest to the root, if it can be ascertained. Substitute $a + y$ for x in the given equation; then y is small, and by omitting y^2, y^3, ..., a value of y is obtained, which, added to a, gives b, a closer approximation to the value of x. Now substitute $b + z$ for x in the given equation, and a second approximation may be obtained by the same process as before. By proceeding in this way, the value of the root may be obtained to any required degree of accuracy.

The assumed value of x should be nearer to one root than to any other, in order to secure accuracy in the approximation.

EXAMPLES.

1. Find the real root of the equation $x^3 - 2x - 5 = 0$.

When 2 and 3 are substituted for x in the equation, the results are -1 and $+16$ respectively; hence a root lies between 2 and 3, and near to 2. Substitute $2 + y$ for x, and there results

$$y^3 + 6y^2 + 10y - 1 = 0.$$

Whence, approximately, $y = .1$.

Now substitute $2.1 + z$ for x, and there results

$$.061 + 11.23z + \ldots\ldots = 0.$$

Whence, approximately, $z = -\frac{.061}{11.23} = -.0054$, and

$$x = 2.1 - .0054 = 2.0946, \text{ nearly.}$$

Find one root of each of the following equations:

2. $x^3 - 3x + 1 = 0$. **3.** $x^3 - 15x^2 + 63x - 50 = 0$.

ANSWERS TO EXAMPLES.

In the following collection of the answers to the examples and problems given in the preceding portion of the text-book, those answers are omitted which, if given, would destroy the utility of the problem.

Art. 47; pages 10 and 11.

1. 93.	**5.** 408.	**9.** $5\frac{2}{3}$.	**13.** 36.	**17.** $11\frac{2}{5}$.
2. 136.	**6.** 254.	**10.** $13\frac{4}{9}$.	**14.** 48.	**18.** 9.
3. 127.	**7.** 24.	**11.** 4.	**15.** 3.	**19.** 10.
4. 156.	**8.** 310.	**12.** $1\frac{4}{5}$.	**16.** 4.	**20.** 76.

Art. 60; page 18.

6. $14a - 9mp^2$. **7.** x. **8.** $8ab - 4cd$. **10.** $3mn^2 - 2x^2y$.
11. $39a^2 - 24ab + 5b^2$. **12.** $-a + 3c + 2$.
13. $x - y + 3m + 3n$. **14.** $3a + 3b + 3c + 3d$. **15.** x.
16. $n + r$. **17.** $6mn - ab - 4c + 3x + 3m^2 - 4p$.
18. $4a - 2b - 12 - 3c - d + 4x^2 - 18m$. **19.** $6a^3$.
20. $14\sqrt{x}$. **21.** $7ab + 7(a + b)$. **22.** $16\sqrt{y} - 4(a - b)$.

Art. 66; page 21.

6. $-3ab + 4cd - 5ax$. **7.** $6x + 12y - 8a + 4$.
8. $-4abc - 14x - 2y - 148$. **9.** $2\sqrt{a} - 4y^2 + 12a + 1$.
11. $14x^2 - 8y^2 + 5ab - 7$. **12.** $2b - 2c$. **13.** $6b + 1$.
14. $4m - 8n - r + 3s$. **15.** $6d - 2b - 3a - 3c$.
16. $5m^2 + 9n^3 - 71x$. **17.** $2b$. **18.** $a - b - 3c$.

Art. 74; page 24.

4. $a-b+c+d-e$. **5.** $2a+2$. **6.** $x-y$. **7.** $a-3b+c$.
8. $5m^2-6n-4a$. **9.** $6m-3n$. **10.** $4x+2y$.
11. $-3b-7c$. **12.** $a-c$. **13.** $9a+1$. **14.** $6m+2$.

Art. 86; pages 29 and 30.

3. $6x^3-16x^2y+6xy^2+4y^3$. **4.** x^4+4x+3. **5.** $a^2-b^2+2bc-c^2$. **6.** $-6a^2+16ab-8b^2$. **7.** b^3-a^3. **8.** ax^4-a. **9.** $30a^3-43a^2b+39ab^2-20b^3$. **10.** $6x^4+13x^3-70x^2+71x-20$. **11.** $-x^5-37x^2+70x-50$. **12.** $-6x^5-25x^4+7x^3+81x^2+3x-28$. **13.** $2a^5b^2-3a^4b^3-7a^3b^4+4a^2b^5$. **14.** $4x^{2m+7}y^3-16x^{m+6}y^{n+1}+12x^5y^{2n-1}$. **15.** $12x^6+7x^4+5x^3+10x-4$. **16.** m^5+n^5. **17.** $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

Art. 87; page 30.

2. $6a^2+11ab+4b^2$. **3.** a^5+x^5. **4.** $a^8-2a^4x^4+x^8$. **5.** $2a^{m+1}-2a^{n+1}-a^{m+n}+a^{2n}$. **6.** $1-x^8$. **7.** $a^3+3a^2x-10ax^2-24x^3$. **8.** $a^5-5a^4+10a^3-10a^2+5a-1$.

Art. 101; pages 37 and 38.

3. $ax-2$. **4.** $3b^2-4a^2$. **5.** $4a^2-3b^2$. **6.** $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$. **7.** $a^2-ax+x^2+\dfrac{x^3}{a+x}$. **8.** $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$. **9.** $2x^2-7x-8$. **10.** $5x^2-4x+3$.
11. x^2-2x-3. **12.** $x^4+x^3y+x^2y^2+xy^3+y^4$.
13. $3x^3-2x^2+x-5$. **14.** $2x^3-x+1$. **15.** $a-b+c$.
16. x^2-3x-y. **17.** $x+y$. **18.** $a^n-b^m+c^r$.
19. $1+2a+2a^2+2a^3+\dots$ **20.** $a-ax+ax^2-ax^3+\dots$

21. $a^4 - a^3 b + a^2 b^2 - a b^3 + b^4$. **22.** $2 a^3 - 2 a^2 - 3 a - 2$.
23. $- x^2 - 2 x - 4$. **24.** $x^3 - x + 2$. **25.** $2 a^2 - a b + 2 b^2$.

Art. 107; page 40.

23. $1 - a^2 + 2 a b - b^2$. **24.** $a^2 - b^2 - 2 b c - c^2$.
25. $a^2 - 2 a b + b^2 - c^2$. **26.** $c^2 - a^2 + 2 a b - b^2$.
27. $a^2 + 2 a b + b^2 - c^2 + 2 c d - d^2$. **28.** $a^2 - 2 a b + b^2 - c^2 + 2 c d - d^2$. **29.** $a^2 + 2 a b + b^2 - c^2 - 2 c d - d^2$.

Art. 115; page 42.

3. $(a + x)(b + y)$. **5.** $(x + 2)(x - y)$. **7.** $(x^2 - y^2)(m - n)$.
4. $(a - m)(c + d)$. **6.** $(a - b)(a^2 + b^2)$. **8.** $(x + 1)(x^2 + 1)$.
9. $(3 x + 2)(2 x^2 - 3)$. **12.** $(a b - c d)(a c + b d)$.
10. $(2 x - 3 y)(4 c + d)$. **13.** $(m^2 x - n y)(n^2 x - m y)$.
11. $(2 - 7 m^2)(3 n - 4 m)$. **14.** $(4 m n - 7 x y)(3 a b + 5 c d)$.

Art. 117; page 45.

9. $(a + b + c + d)(a + b - c - d)$.
10. $(a - c + b)(a - c - b)$. **11.** $(m + x - y)(m - x + y)$.
12. $(x - m + y - n)(x - m - y + n)$.
16. $(x + y + 2)(x + y - 2)$. **18.** $(3 c + d + 1)(3 c + d - 1)$.
17. $(a + b - c)(a - b + c)$. **19.** $(3 + x^2 - 2 y)(3 - x^2 + 2 y)$.
20. $(2 a - b + 3 d)(2 a - b - 3 d)$.
21. $(2 m^2 + 2 b - 1)(2 m^2 - 2 b + 1)$.
22. $(a - m + b + n)(a - m - b - n)$.
23. $(a + m + b - n)(a + m - b + n)$.
24. $(x - c + y - d)(x - c - y + d)$.

Art. 118; page 49.

25. $(x^2 - 24)(x^2 - 5)$. **27.** $(x y^3 + 12)(x y^3 - 10)$.
26. $(c^3 + 11)(c^3 + 1)$. **28.** $(a b^2 - 16)(a b^2 + 9)$.

29. $(x+20n)(x+5n)$.
30. $(m^2+11n^2)(m^2-6n^2)$.
31. $(a-b-4)(a-b+1)$.
32. $(x+y-5)(x+y-2)$.
33. $(x-8y^2z)(x+6y^2z)$.
34. $(m+n+2)(m+n-1)$.

Art. 121; page 53.

3. $3ab(a+2)^2$.
4. $5xy^2(3x-4y^2)^2$.
5. $2xy(3x+y)(3x-y)$.
6. $x(x+7)(x+1)$.
7. $3a^2(a-5)(a-2)$.
8. $2cm(c+7)(c-3)$.
9. $xy(m-6)(m+2)$.
10. $4ab(2a+b)(4a^2-2ab+b^2)$.
11. $(n-1)(n^2+n+1)(n^6+n^3+1)$.
12. $(x^2+y^2)(x+y)(x-y)$.
13. $(x^4+m^4)(x^2+m^2)(x+m)(x-m)$.
14. $(m+n)(m-n)(m^2+mn+n^2)(m^2-mn+n^2)$.
15. $(a+c)(a^2-ac+c^2)(a^6-a^3c^3+c^6)$.
16. $(2a+1)(2a-1)(4a^2+2a+1)(4a^2-2a+1)$.

Art. 125; pages 55 and 56.

3. ax.
4. $m+n$.
5. x^2+1.
6. $x+7$.
7. $2x-3$.
8. $3x-4$.
9. $x(x-1)$.
10. $a-2b$.
11. $x+6$.
12. $2x+5$.
13. $ax(x-1)$.
14. $4x-1$.

Art. 126; page 61.

6. $2x+3$.
7. $8x-7$.
8. $x-1$.
9. $3x+4$.
10. $2x-5$.
11. $5x+3$.
12. $x+2$.
13. $2x-1$.
14. x^2+x+1.
15. $a-x$.
16. x^2-2.
17. $2(x+y)$.
18. $2a-3x$.
19. $3x+2$.

Art. 130; page 63.

2. $120a^4b^2c$.
3. $30x^3y^2z^3$.
4. $36a^5b^4$.
5. $480m^3n^2x^2y^2$.
6. $840a^2c^2d^3$.
7. $252x^3y^3z^3$.
8. $1080a^2b^2c^3d^4$.
9. $168mn^2x^3y^3$.

Art. 131; pages 63 and 64.

2. $ax(x+a)(x-a)(x^2+ax+a^2)$. 7. $a\,x\,(x-3)\,(x-7)\,(x+8)$.
3. $12\,a\,b\,c\,(a+b)\,(a-b)$. 8. $(2x+1)(2x-1)^2(4x^2+2x+1)$.
4. $x\,(x+1)\,(x-1)\,(x^2-x+1)$. 9. $3\,a\,b\,(x-y)^2\,(a-b)$.
5. $24(1+x)\,(1-x)\,(1+x^2)$. 10. $2\,a\,x^2(3x+2)^2(9x^2-6x+4)$.
6. $(x+1)(x-2)(x+3)(x+4)$. 11. $(x-1)(x-3)\,(x+4)(x-5)$.
12. $(x+y+z)\,(x+y-z)\,(x-y+z)$.

Art. 132; page 65.

2. $(3x-4)\,(4x-5)\,(2x+7)$. 4. $(a^2-2\,a-2)\,(a+3)\,(2\,a-1)$.
3. $(4x+1)\,(2x+7)\,(3x-8)$. 5. $(2x+3)(x^2-x+1)\,(x^2+x-2)$.
6. $(a-b)\,(a^2-a\,b+b^2)\,(a^2+2\,a\,b+b^2)$.
7. $a\,x\,(x+1)\,(x^2-x-1)\,(x^2+x+1)$.
8. $x\,(x-5)\,(2\,x^2-x-2)\,(3\,x^2+x-1)$.

If the above expressions are expanded, the answers take the following forms:

2. $24\,x^3+22\,x^2-177\,x+140$. 4. $2\,a^4+a^3-17\,a^2-4\,a+6$.
3. $24\,x^3+26\,x^2-219\,x-56$. 5. $2\,x^5+3\,x^4-4\,x^3+5\,x-6$.
6. $a^5-a^3b^2+a^2\,b^3-b^5$.
7. $a\,x^6+a\,x^5-a\,x^4-3\,a\,x^3-3\,a\,x^2-a\,x$.
8. $6\,x^6-31\,x^5-4\,x^4+44\,x^3+7\,x^2-10\,x$.

Art. 148; page 71.

10. $\frac{c\,d}{3\,x\,y}$. 14. $\frac{5-2\,c}{5+2\,c}$. 18. $\frac{2\,x^2\,y}{5-x}$.
11. $\frac{x^3}{2\,y^2}$. 15. $\frac{a\,(2+3\,n)}{b\,(2-3\,n)}$. 19. $\frac{2+x}{x\,(7-x)}$.
12. $\frac{x-5}{x+7}$. 16. $\frac{4\,x^2-2\,x\,y+y^2}{2\,x-y}$. 20. $\frac{c-d}{c+d}$.
13. $\frac{m-2}{m+9}$. 17. $\frac{9\,y^2+15\,y+25}{3\,y-5}$. 21. $\frac{m-n^2}{m^2-n}$.

Art. 149; page 72.

2. $\frac{3x-7}{4x+1}$. 4. $\frac{m-1}{6m-5}$. 6. $\frac{3x-2}{x+3}$. 8. $\frac{2x+5}{2x-7}$.

3. $\frac{5a+7}{a-2}$. 5. $\frac{x+2}{x-3}$. 7. $\frac{2x-3}{2x-5}$. 9. $\frac{6a-1}{5a-7}$.

10. $\frac{x^2-3x+1}{x^2-x+3}$. 11. $\frac{2x^2-x-2}{2x^2+3x+1}$.

Art. 150; page 73.

3. $a-\frac{a^2}{b}$. 4. x^2-xy+y^2. 5. $\frac{2x}{5}-\frac{3}{5}-\frac{4}{5x}$.

6. $\frac{x^2}{3}-\frac{x}{3}+\frac{7}{3}-\frac{2}{x}$. 7. $\frac{a}{2b}-\frac{3}{2}+\frac{2b}{a}$. 8. $2x+6+\frac{23}{x-3}$.

9. x^2+x+1. 10. $2+\frac{3}{2x^2-x+1}$. 11. $x-2+\frac{2x-4}{x^2+x-1}$.

12. $x+\frac{x-2}{2x^2-3x+3}$.

Art. 151; page 74.

2. $\frac{(x-1)^2}{x-3}$. 3. $\frac{an+b^2-cd}{n}$. 4. $\frac{56x-4n^2-5a}{8}$.

5. $\frac{(x+1)^2}{x}$. 6. $\frac{2ab}{a+b}$. 7. $\frac{a^2+2b^2}{2a}$. 8. $\frac{a^3+b^3}{a-b}$.

9. $\frac{6x^2-7x-1}{2x-1}$. 10. $-\frac{2b^2}{a+b}$. 11. $\frac{x^3-2x^2-3x}{x-2}$.

Art. 152; pages 76 and 77.

3. $\frac{27ab}{72}, \frac{16ac}{72}, \frac{30bc}{72}$. 4. $\frac{3x^2y}{30}, \frac{2xyz}{30}, \frac{7yz^2}{30}$.

5. $\frac{18y^2z^2}{12xyz}, \frac{16x^2z^2}{12xyz}, \frac{15x^2y^2}{12xyz}$.

6. $\frac{40c^2-10c}{30abc}, \frac{18b^2-12b}{30abc}, \frac{25a^2}{30abc}$. 7. $\frac{2x}{a^3x^3}, \frac{3a^2}{a^3x^3}, \frac{4ax^2}{a^3x^3}$.

8. $\dfrac{100\,a\,y\,z^3}{120\,x^2\,y^2\,z^2}$, $\dfrac{45\,b\,x^3\,z}{120\,x^2\,y^2\,z^2}$, $\dfrac{84\,c\,x\,y^3-12\,m\,x\,y^2}{120\,x^2\,y^2\,z^2}$.

9. $\dfrac{(a+b)\,(a^2+b^2)}{a^4-b^4}$, $\dfrac{(a-b)\,(a^2+b^2)}{a^4-b^4}$, $\dfrac{a^2-b^2}{a^4-b^4}$.

10. $\dfrac{x^2-9}{(x-1)(x-2)(x-3)}$, $\dfrac{x^2-1}{(x-1)(x-2)(x-3)}$, $\dfrac{x^2-4}{(x-1)(x-2)(x-3)}$.

11. $\dfrac{2\,a\,(a+2)}{(a-2)(a+2)(a+3)}$, $\dfrac{3\,b\,(a-2)}{(a-2)(a+2)(a+3)}$, $\dfrac{4\,c\,(a+3)}{(a-2)(a+2)(a+3)}$.

12. $\dfrac{x^3+2\,x^2+2\,x+1}{(x+1)\,(x^3-1)}$, $\dfrac{x^2+x+1}{(x+1)\,(x^3-1)}$, $\dfrac{x+1}{(x+1)\,(x^3-1)}$.

13. $\dfrac{6\,a^2\,b^2}{6\,a\,b\,(a-b)\,(m+n)}$, $\dfrac{3\,b\,(m^2-n^2)}{6\,a\,b\,(a-b)\,(m+n)}$, $\dfrac{2\,a\,(a^2-b^2)}{6\,a\,b\,(a-b)\,(m+n)}$.

15. $\dfrac{3(a+1)}{a^2-1}$, $\dfrac{2\,(a-1)}{a^2-1}$, $\dfrac{2-a}{a^2-1}$. 16. $\dfrac{1-x}{1-x^2}$, $\dfrac{x^2-x-2}{1-x^2}$, $\dfrac{3}{1-x^2}$.

17. $\dfrac{c^2-d^2}{(a^2-b^2)\,(c-d)}$, $\dfrac{(x-1)\,(a+b)}{(a^2-b^2)\,(c-d)}$, $\dfrac{(a-b)^2}{(a^2-b^2)\,(c-d)}$.

Art. 153; page 78.

2. $\dfrac{(a-b)^2}{a^2-b^2}$. 3. $\dfrac{x^2+9\,x+8}{x^2+5\,x-24}$. 4. $\dfrac{9\,m^2-4}{6\,m^2-19\,m+10}$.

5. $\dfrac{4\,(a^2+a\,b+b^2)}{a^3-b^3}$. 6. $\dfrac{1-x^2}{1-x}$.

Art. 154; pages 80 to 82.

4. $\dfrac{12\,x+7}{36}$. 5. $\dfrac{6\,a+5\,b}{10\,a^2\,b^2}$. 6. $-\dfrac{a+3}{24}$. 7. $\dfrac{3\,m^2\,n^2-4}{6\,m^2\,n^3}$.

8. $\dfrac{5\,b^2+4\,a^2}{120\,a\,b}$. 9. $\dfrac{5\,a+b}{24}$. 10. $\dfrac{4\,a\,b-b-4\,a^3}{12\,a^3\,b}$. 11. $\dfrac{1}{15}$.

12. $\dfrac{m}{42}$. 13. $\dfrac{3\,x-2}{18\,x^2}$. 14. $-\dfrac{1}{60}$. 15. $\dfrac{4bcd+6acd-3abd-2abc}{48\,a\,b\,c\,d}$.

17. $\dfrac{5}{6+x-x^2}$. 18. $\dfrac{1}{x^2+15\,x+56}$. 19. $\dfrac{2\,(a^2+b^2)}{a^2-b^2}$.

20. $\frac{4x}{1-x^2}$. 21. $\frac{a+b}{a-b}$. 22. $\frac{4xy^2}{x^4-y^4}$. 23. $\frac{(x+2)^2}{(x+1)(x^3-1)}$.

24. $\frac{13-18x}{(x+1)(x+2)(x-3)}$. 26. $\frac{1}{b-a}$. 27. $\frac{a^2-14a+1}{6(a^2-1)}$.

28. $\frac{3}{9x-x^3}$. 29. $\frac{3x^2}{x^2-1}$. 30. 0. 31. $-\frac{5}{(x-2)(x-3)(x-4)}$.

Art. 155; pages 83 to 85.

2. $\frac{a^5b^3c}{m^4n^3d}$. 3. $\frac{12a^4bx}{35h^5m}$. 4. $\frac{1}{a}$. 5. $\frac{1}{4}$. 6. $\frac{1}{2}$.

7. $\frac{z}{4xy}$. 8. a^3. 9. $\frac{11m^2n^2}{4}$. 11. $\frac{3x-1}{x-2}$. 12. $\frac{10x}{3}$.

13. $\frac{b(a-b)}{x(a+b)}$. 14. $\frac{a-b}{a^2}$. 15. $\frac{1-y}{x}$. 16. $\frac{x^2-x-20}{x^2}$.

17. $\frac{a(x-2)}{a+1}$. 18. $\frac{x}{x-2}$. 19. $\frac{x+5}{x^2}$. 20. $\frac{x^2+5x+6}{x^2}$.

21. $\frac{x^2}{x+2}$. 22. x^2+xy. 23. 1. 24. 2. 25. $\frac{y^2}{x^2+y^2}$.

Art. 156; page 86.

3. $\frac{91m^2}{6n^2}$. 5. $\frac{3ny^3}{5mx}$. 7. $\frac{1}{x}$. 9. $\frac{a+1}{a+5}$.

4. $\frac{a^2}{8b^3mn^2}$. 6. $\frac{3(x-4)}{x^3}$. 8. $\frac{3x-2y}{x+y}$. 10. x.

Art. 157; pages 88 and 89.

4. $\frac{a}{bm+bn}$. 5. $\frac{acn+bn}{cnx-cm}$. 6. $\frac{3m-n}{3x}$. 7. $\frac{4y-4x+2a}{31}$.

8. $x-1$. 9. x^2-x+1. 10. $a+b$. 11. $\frac{x^2y^3+1}{xy^3-2y^2}$.

12. $\frac{a-b}{a^2b}$. 13. $\frac{x-4}{x+6}$. 14. x. 15. $\frac{4}{3x+3}$. 16. $\frac{ab}{a^2+b^2}$.

17. 1. **18.** $-\frac{mn(m-n)^2}{m^4+m^2n^2+n^4}$. **19.** $\frac{x-a}{x+2a}$.

Art. 175; pages 94 and 95.

2. $aenx-bcen=bdnx-bem$. **3.** $6bx-8a^2=3-2abx$.
4. $bdex-adex+bcex-abd=0$. **5.** $12x+5x=6x-1320$.
6. $9x-12a=10x+24-4b$. **7.** $28x-4x+560=14x+7x+728$.
8. $4ax-6c-5a^3x+2a^3bd=0$. **9.** $10x-32x-312=21-52x$.
11. $3x-2a-2x=45$. **12.** $abx+b^2-cx-d=ac$.
13. $3-3x-2-2x=0$. **14.** $6x^2+3x-6x^2+18-4x-2=0$.
15. $3x-3-2x-2-5x=0$. **16.** $6x+6-15x+45-20x-10=0$.

Art. 177; pages 97 to 102.

4. 3. **13.** 1. **22.** 72. **31.** 5. **41.** $-1\frac{1}{14}$.
5. 7. **14.** 2. **23.** 60. **32.** -5. **42.** $4\frac{1}{4}$.
6. -1. **15.** 2. **24.** 10. **33.** 4. **43.** $1\frac{5}{18}$.
7. 2. **16.** -4. **25.** $-2\frac{4}{9}$. **34.** -5. **44.** 0.
8. 1. **17.** 2. **26.** 56. **35.** -2. **45.** $1\frac{1}{2}$.
9. $\frac{3}{7}$. **18.** 1. **27.** $\frac{2}{7}$. **36.** $\frac{2}{3}$. **46.** $-\frac{12}{5}$.
11. $-\frac{3}{2}$. **20.** 3. **29.** -2. **37.** $-\frac{1}{2}$. **47.** 1.
12. 0. **21.** 5. **30.** $\frac{1}{3}$. **40.** -7. **50.** $\frac{3c-d}{2a+b}$.
51. $\frac{a^2+4a}{a^2-3a+2}$. **52.** $2b$. **53.** $\frac{5a}{2b}$. **54.** $\frac{4bc+a^4}{4a^2-b+16c}$.
55. $\frac{2a^2}{3b}$. **57.** $\frac{1}{b}$. **59.** $-\frac{1}{a+2}$. **61.** $-\frac{a}{3b}$. **64.** -3.
56. $\frac{a}{7}$. **58.** $12a^3$. **60.** ab. **63.** 2. **65.** 50.
66. $\frac{7}{10}$. **67.** 5. **68.** 0.

Art. 182; pages 108 to 113.

10. Horse, \$224; chaise, \$112. **11.** 37. **12.** 10 and 7.
13. 18 and 2. **14.** $58\frac{1}{2}$ and $41\frac{1}{2}$. **15.** A, 40; B, 20.
16. A, 60; B, 15. **17.** $1\frac{7}{8}$. **18.** $\frac{11}{12}$. **19.** $23\frac{1}{3}$.
20. 84. **21.** 36. **22.** Oxen, 12; cows, 24.
23. Wife, \$864; daughter, \$288; son, \$144.
24. Worked, 20; absent, 16. **25.** Horse, \$126; saddle, \$12.
26. Infantry, 2450; cavalry, 196; artillery, 98.
27. 144 sq. yds. **28.** Water, 1540; foot, 880; horse, 616.
29. \$1728. **30.** \$2000 at 6 p.c.; \$1200 at 5 p.c. **31.** 7.
32. 31. **33.** \$24. **34.** \$100. **35.** 142857.
36. A, \$$466\frac{2}{3}$; B, \$$533\frac{1}{3}$. **37.** 2 dollars, 20 dimes, 4 cents.
38. \$2.75. **39.** Men, \$25; women, \$21. **40.** 23 and 18.
41. 48 minutes. **42.** 12121 men; 110 on a side at first.
43. $5\frac{5}{11}$ minutes after 7. **44.** $43\frac{7}{11}$ minutes after 2.
45. $27\frac{3}{11}$ minutes after 5. **46.** 29 and 14.
47. 3377 ounces of gold; 783 ounces of silver. **48.** \$2000.
49. 30 bushels at 9 shillings; 10 at 13 shillings. **50.** 10 A.M.
51. \$1280. **52.** $21\frac{9}{11}$ minutes, or $54\frac{6}{11}$ minutes after 7.
53. $27\frac{3}{11}$ minutes after 4. **54.** $23\frac{3}{34}$ miles.
55. Greyhound, 72; fox, 108. **56.** 1 minute, $1\frac{683}{697}$ seconds.

Art. 192; pages 120 to 123.

3. $x=4,\ y=3$. **9.** $x=-2,\ y=10$. **15.** $x=12,\ y=18$.
4. $x=5,\ y=-2$. **10.** $x=12,\ y=8$. **16.** $x=35,\ y=-10$.
5. $x=7,\ y=5$. **11.** $x=-2,\ y=-10$. **17.** $x=-28,\ y=21$.
6. $x=-8,\ y=2$. **12.** $x=10,\ y=5$. **18.** $x=.4,\ y=.1$.
7. $x=5,\ y=7$. **13.** $x=7,\ y=11$. **19.** $x=1\frac{1}{3},\ y=3\frac{3}{4}$.
8. $x=-8,\ y=-12$. **14.** $x=11,\ y=-9$. **20.** $x=3,\ y=-2$.

21. $x=\dfrac{dm-bn}{ad-bc},\ y=\dfrac{an-cm}{ad-bc}$. **22.** $x=\dfrac{nr'+n'r}{mn'+m'n},$

$y = \frac{m' r - m r'}{m n' + m' n}$. **23.** $x = \frac{a c (b m + d n)}{a d + b c}$, $y = \frac{b d (c n - a m)}{a d + b c}$.

24. $x = \frac{1}{2 a}$, $y = \frac{1}{2 a}$. **25.** $x = 60$, $y = 40$. **26.** $x = \frac{25}{6}$, $y = \frac{5}{6}$.

27. $x = 1\frac{2}{13}$, $y = 4\frac{6}{13}$. **28.** $x = -6$, $y = -5$. **30.** $x = 4$, $y = 2$.

31. $x = -5$, $y = 3$. **32.** $x = -2$, $y = -1$. **33.** $x = \frac{b c - a d}{b n - d m}$,

$y = \frac{b c - a d}{c m - a n}$. **34.** $x = \frac{3}{a^2 b}$, $y = \frac{2}{a b^2}$. **35.** $x = \frac{1}{n}$, $y = \frac{1}{m}$.

Art. 194; pages 126 and 127.

3. $x = 23$, $y = 6$, $z = 24$.
6. $x = -5$, $y = -5$, $z = -5$.
4. $x = -2$, $y = 3$, $z = 7$.
7. $u = 4$, $x = 5$, $y = 6$, $z = 7$.
5. $x = 8$, $y = -3$, $z = -4$.
8. $x = 3$, $y = -1$, $z = 0$.

9. $x = \frac{1}{2}(b + c - a)$, $y = \frac{1}{2}(a + c - b)$, $y = \frac{1}{2}(a + b - c)$.

10. $x = \frac{13 a}{8}$, $y = \frac{7 a}{8}$, $z = \frac{a}{2}$. **11.** $x = -24$, $y = -48$, $z = 60$.

12. $u = -7$, $x = 3$, $y = -5$, $z = 1$.

13. $x = \frac{b^2 + c^2 - a^2}{2 b c}$, $y = \frac{a^2 + c^2 - b^2}{2 a c}$, $z = \frac{a^2 + b^2 - c^2}{2 a b}$.

14. $x = \frac{2}{9}$, $y = -\frac{3}{4}$, $z = -\frac{4}{7}$. **15.** $x = 1\frac{1}{2}$, $y = -1\frac{1}{3}$, $z = 1$.

16. $x = a b c$, $y = a b + a c + b c$, $z = a + b + c$.

17. $x = 7$, $y = -3$, $z = -5$. **18.** $x = \frac{a + 1}{c}$, $y = a - c$, $z = \frac{c - 1}{a}$.

Art. 195; pages 129 to 133.

4. A, 30; B, 20. **5.** $\frac{4}{15}$. **6.** Cows, 49; oxen, 40.
7. A, \$140; B, \$70. **8.** A, 98; B, 15. **9.** 32 and 18.
10. Man, 24; wife, 18. **11.** Worked, 6; absent, 4.
12. Horse, \$96; chaise, \$112. **13.** A, \$96; B, \$48.

14. 16 days. **15.** $13\frac{1}{3}$ bushels at 60 cts.; $26\frac{2}{3}$ at 90 cts. **16.** Wheat, 9; rye, 15. **17.** Income tax, \$20; assessed tax, \$30. **18.** A, \$500; B, \$700. **19.** 30 cents; 15 oranges. **20.** 1st, 8 cts.; 2d, 7 cts.; 3d, 4 cts. **21.** Better horse, \$40; poorer, \$30; harness, \$50. **22.** 10, 22, and 26. **23.** 246.

24. A, \$2000; B, \$3000; C, \$4000; D, \$5000.

25. A, 45; B, 55. **26.** A, \$20; B, \$30; C, \$40.

27. Whole sum, \$120; eldest, \$40; 2d, \$30; 3d, \$24; 4th, \$26.

28. Length, 30 rods; width, 20 rods; area, 600 sq. rods.

29. Going, 4 hours; returning, 6 hours.

30. A, $9\frac{3}{5}$ days; B, 16; C, 48. **31.** 1st rate, 6 p.c.; 2d, 5 p.c. **32.** 15 miles; $5\frac{1}{2}$ miles an hour. **33.** 30 miles an hour. **34.** A, 5; B, 6. **35.** First, 22; second, 10. **36.** A, 8; B, 6.

Art. 197; pages 136 and 137.

4. $\frac{abc}{ab+ac+bc}$. **5.** $1\frac{1}{4}$ hours. **6.** $\frac{ma}{m+n}$ and $\frac{na}{m+n}$.

7. 12 and 8. **8.** $\frac{an}{b-a}$. **9.** 12. **10.** $\frac{100a}{rt+100}$. **11.** \$2100.

12. $\frac{100(a-p)}{pr}$. **13.** $11\frac{1}{2}$. **14.** 1st, $\frac{a(c-b)}{a-b}$; 2d, $\frac{b(a-c)}{a-b}$.

15. 1st kind, 5; 2d, 10. **16.** $\frac{b+d}{a-c}$. **17.** $\frac{am+bn+cp}{a+b+c}$.

18. A, $\frac{amt}{mt+nt'+pt''}$; B, $\frac{ant'}{mt+nt'+pt''}$; C, $\frac{apt''}{mt+nt'+pt''}$.

Art. 205; page 141.

3. -2 rods. **4.** $\frac{-5}{-9}$. **5.** 105 and -15. **6.** In -30 years.

7. A, $-$\$1500; B, $-$\$500; that is, A was in debt \$1500, and B \$500. **8.** Man, \$3; son, $-$\$0.50; that is, the man was at an expense of 50 cents a day for his son's subsistence.

Art. 225; page 152.

4. $x > 5$. 5. $x > 15$, $x < 20$. 6. 4. 7. $x > 6\frac{8}{9}$, $y > 2\frac{8}{9}$.
8. $x > c$, $x < d$. 9. $x > 9\frac{3}{4}$, $y < 12\frac{1}{2}$. 10. 19 or 20.
11. Any no., integral or fractional, between 8 and 15. 12. 60.

Art. 229; page 155.

1. $a^3 - 3a^2b + 3ab^2 - b^3$. 2. $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$.

3. $1 + 3a^2 + 3b^2 + 3a^4 + 6a^2b^2 + 3b^4 + a^6 + 3a^4b^2 + 3a^2b^4 + b^6$.

4. $a^2 + 2am - 2an + m^2 - 2mn + n^2$.

5. $a^{4m} - 4a^{3m+n} + 6a^{2m+2n} - 4a^{m+3n} + a^{4n}$.

6. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Art. 230; page 156.

3. $4x^4 + 12x^3 + 25x^2 + 24x + 16$. 4. $4x^4 - 12x^3 + 11x^2 - 3x + \frac{1}{4}$.
6. $x^6 + 4x^5 + 6x^4 + 8x^3 + 9x^2 + 4x + 4$. 7. $1 - 4x + 10x^2 - 12x^3 + 9x^4$.

8. $1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$.

9. $x^6 - 8x^5 + 12x^4 + 10x^3 + 28x^2 + 12x + 9$.

10. $4x^6 + 4x^5 + 29x^4 + 10x^3 + 47x^2 - 14x + 1$.

11. $x^6 + 10x^5 + 23x^4 - 6x^3 + 21x^2 - 4x + 4$.

12. $9x^6 - 12x^5 - 2x^4 + 28x^3 - 15x^2 - 8x + 16$.

Art. 231; page 157.

2. $a^6 + 6a^4b + 12a^2b^2 + 8b^3$. 3. $8m^3 + 60m^2n + 150mn^2 + 125n^3$.
4. $27x^3 - 108x^2 + 144x - 64$. 5. $8x^9 - 36x^6 + 54x^3 - 27$.

6. $64x^6 - 48x^5y + 12x^4y^2 - x^3y^3$.

7. $27x^3y^3 + 135ab^2x^2y^2 + 225a^2b^4xy + 125a^3b^6$.

Art. 232; page 158.

3. $x^6 - 3x^5 + 5x^3 - 3x - 1$. 5. $8 - 24x + 36x^2 - 32x^3$
$+ 18x^4 - 6x^5 + x^6$. 6. $1 + 3x + 6x^2 + 10x^3 + 12x^4$

$+ 12x^5 + 10x^6 + 6x^7 + 3x^8 + x^9$. **7.** $8x^9 - 12x^8 + 30x^7 - 61x^6 + 66x^5 - 93x^4 + 98x^3 - 63x^2 + 54x - 27$.

Art. 239; pages 162 and 163.

2. $2x^2 - x - 1$. **5.** $3 - 2x + x^2$. **8.** $3x^2 - 4x - 5$.
3. $2a^2 - 4a + 2$. **6.** $5 + 3x + x^2$. **9.** $2x^2 - 5x + 8$.
4. $m + 1 - \frac{1}{m}$. **7.** $1 - 7x - 2x^2$. **10.** $a - b - c$.

11. $x - 2y + 3z$. **12.** $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\ldots$

13. $a + \frac{b}{2a} - \frac{b^2}{8a^3} + \frac{b^3}{16a^5} - \ldots\ldots$ **14.** $1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \ldots\ldots$

15. $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \ldots\ldots$

Art. 241; page 166.

2. 523.	**7.** $\frac{81}{95}$.	**12.** 900.8.	**17.** 13.15295.
3. 214.	**8.** 1.082.	**13.** .4125.	**18.** .88192.
4. 327.	**9.** 21.12.	**14.** 1.41421.	**19.** .43301.
5. 5.76.	**10.** .083.	**15.** 2.23607.	**20.** .57735.
6. .97.	**11.** .00328.	**16.** 5.56776.	**21.** .53452.

Art. 242; page 168.

1. 3.3166.	**3.** 7.81024968.	**5.** 27.94638.
2. 1.732051.	**4.** 11.446.	**6.** 113.7234.

Art. 243; pages 170 and 171.

2. $1 - 2y$. **4.** $4x - 3ab$. **6.** $y^2 - y - 1$. **8.** $x^2 - 2x + 1$.
3. $2x^2 + 3$. **5.** $x^2 + 2x - 4$. **7.** $x + \frac{1}{x}$. **9.** $a + b + c$.

10. $2x^2 - 3x - 1$. **11.** $x + \frac{1}{3x^2} - \frac{1}{9x^5} + \frac{5}{81x^8} - \ldots\ldots$

12. $x-\frac{a^3}{3x^2}-\frac{a^6}{9x^5}-\frac{5a^9}{81x^8}-\cdots$ 13. $2x^2-\frac{1}{4x^4}-\frac{1}{32x^{10}}-\frac{5}{768x^{16}}-\cdots$

Art. 245; page 173.

2. 123. 5. $\frac{31}{88}$. 8. 1.442. 11. .855.
3. .898. 6. 3.72. 9. 1.913. 12. .420.
4. 11.4. 7. .0803. 10. 5.963. 13. .561.

Art. 247; pages 175 and 176.

2. m^2-2m-4. 3. a^2-ax+x^2. 4. $2x-1$. 5. x^2-x+1.

Art. 248; page 176.

1. $2x-3y$. 2. a^2-1. 3. m^2-2m-3.

Art. 257; pages 180 and 181.

4. $c^{\frac{11}{3}}$. 5. $x^{-\frac{7}{4}}$. 6. $m^{\frac{9}{5}}$. 9. $-6ac^{\frac{11}{35}}$.

11. $a^4b^{-4}-2+a^{-4}b^4$. 12. $a-b$.

13. $a^{-5}-3a^{-3}b^2+a^{-2}b^3-2a^{-1}b^4$. 14. $18a^2b^2+10+2a^{-2}b^{-2}$.

15. $2x^{-1}y-10xy^{-1}+8x^3y^{-3}$. 16. $2-4x^{-\frac{4}{3}}y^{\frac{3}{2}}+2x^{-\frac{8}{3}}y^3$.

17. $6x^2-7x^{\frac{5}{3}}-19x^{\frac{4}{3}}+5x+9x^{\frac{2}{3}}-2x^{\frac{1}{3}}$. 18. $32ab^{-2}-50+18a^{-1}b^2$

Art. 258; pages 182 and 183.

5. $c^{-\frac{9}{4}}$. 6. $m^{\frac{12}{5}}$. 7. $x^{\frac{13}{12}}$. 8. $\frac{5ab^{-\frac{1}{3}}n}{2}$.

11. $a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}}$. 12. $a^{-2}-a^{-1}b^{-1}+b^{-2}$.

13. $x^{-3}y^2-x^{-2}y+2x^{-1}$. 14. $x^{\frac{2}{3}}y^{-1}-3+4x^{-\frac{2}{3}}y$.

15. $x^{-1}y^{-2}-x^{-2}y^{-3}-x^{-3}y^{-4}$. 16. $2x^{\frac{1}{2}}y^{-\frac{2}{3}}-x^{-\frac{1}{2}}y-x^{-\frac{3}{2}}y^{\frac{8}{3}}$.

Art. 260; page 184.

6. $x^{\frac{1}{6}}$. 7. $c^{-\frac{1}{5}}$. 8. m^{-1}. 9. y^{-3}. 10. a^{-3}. 11. n^{-1}.

Art. 262; page 186.

3. $9.$ 5. $\frac{1}{10000}$. 7. $4.$ 9. $\pm\frac{12500}{9}$.

4. $\pm 216.$ 6. $\pm\frac{1}{2187}$. 8. $-243.$ 10. $\pm\frac{9}{16}$.

Art. 263; pages 186 and 187.

5. $3x^{-2}y - 2x^{-1} - y^{-1}$. 6. $2x^{\frac{2}{3}} + xy^{-\frac{1}{4}} - 4x^{\frac{4}{3}}y^{-\frac{1}{2}}$.

7. $x^{\frac{3}{2}}y^{-\frac{1}{3}} - 2 + x^{-\frac{3}{2}}y^{\frac{1}{3}}$. 11. $2y^{\frac{2}{3}} - y^{\frac{1}{2}}x^{-1}$. 12. x^{r-m}.

13. x^{-2ab}. 14. a^{xy}. 15. a^{3x}. 16. $a^{-\frac{1}{2}}$. 17. x.

21 $\frac{a^{\frac{19}{6}} + a^{\frac{3}{2}}}{1 - 3a^3}$. 22. $\frac{b^2c^3d^4 - ac^3d^4}{ab^2d^4 + ab^2c^3}$. 23. $\frac{5x^3(x^2-1)}{3a}$.

Art. 267; page 189.

2. $\sqrt[6]{27}, \sqrt[6]{16}, \sqrt[6]{25}$. 3. $\sqrt[12]{625}, \sqrt[12]{216}, \sqrt[12]{49}$.

4. $\sqrt[12]{x^6y^6}, \sqrt[12]{x^4z^4}, \sqrt[12]{y^3z^3}$. 5. $\sqrt[15]{32a^5}, \sqrt[15]{27b^3}, \sqrt[15]{64c^3}$.

6. $\sqrt[12]{a^2+2ab+b^2}, \sqrt[12]{a^3-3a^2b+3ab^2-b^3}$. 7. $\sqrt[6]{a^6-3a^4x^2+3a^2x^4-x^6}$, $\sqrt[6]{a^6 - 2a^3x^3 + x^6}$. 8. $\sqrt{3}$. 9. $\sqrt[3]{2}$. 10. $\sqrt[4]{4}$.

Art. 269; page 190.

5. $\sqrt{5ab^3}$. 6. $\sqrt[m]{ab^2}$. 7. $\sqrt{\left(\frac{5a}{6b^3}\right)}$.

Art. 270; page 191.

11. $3xy\sqrt{2xy^2 - 3x^2y}$. 12. $(x-3)\sqrt{a}$. 13. $(x+y)\sqrt{x-y}$.

14. $(2x+3a)\sqrt[3]{5a}$. 15. $4ab\sqrt[3]{3ab^2+5b}$. 18. $\frac{1}{2}\sqrt{6}$.

19. $\frac{1}{6}\sqrt{30}$. 20. $\frac{1}{6}\sqrt{21}$. 21. $\frac{2a}{9}\sqrt{3}$. 22. $\frac{1}{2}\sqrt[3]{6x}$.

23. $\frac{1}{3}\sqrt[3]{15}$. **24.** $\frac{3ab}{10cd}\sqrt{10bcd}$. **25.** $\frac{y}{4a^2}\sqrt{14ax}$. **26.** $\frac{6}{77}\sqrt{7}$.

27. $\frac{b}{2(a+x)}\sqrt{a^2+ax}$. **28.** $\frac{a\sqrt{abc}}{b^2(a+b)}$.

Art. 272; page 192.

7. $\sqrt{x^2-1}$. **8.** $\sqrt[3]{(a-b)^4}$. **9.** $\sqrt{\left(\frac{1+a}{1-a}\right)}$.

Art. 273; pages 193 and 194.

3. $10\sqrt{2}$. **4.** $12\sqrt{3}$. **5.** $9\sqrt[3]{2}$. **6.** $\frac{38}{15}\sqrt{5}$.

7. $\frac{19}{36}\sqrt{6}$. **8.** $\frac{3}{4}\sqrt[3]{2}+\frac{1}{3}\sqrt[3]{18}$. **9.** $4\sqrt{5}$. **10.** $\frac{1}{15}\sqrt{15}$.

11. $6a\sqrt{3a}$. **12.** $\frac{313}{18}\sqrt{3}$. **13.** $\frac{35}{3}\sqrt[3]{2}+\frac{1}{2}\sqrt[3]{3}$.

14. $2\sqrt{x^2-y^2}$. **15.** $(2a-5b)\sqrt{7x}$.

Art. 274; pages 195 and 196.

5. $\sqrt[6]{a^3b^2x^5}$. **6.** $a^2\sqrt[6]{4500a}$. **7.** $\sqrt[12]{\left(\frac{27}{256x^{11}}\right)}$.

8. $\sqrt[10]{\frac{3125}{8}}$. **10.** $x+\sqrt{x}-6$. **11.** $21x-38\sqrt{x}+5$.

12. 2. **13.** -1. **14.** $x-y-z+2\sqrt{yz}$.

15. $4+2\sqrt{10}$. **16.** $56+12\sqrt{35}$. **17.** $36-32\sqrt{15}$.

18. $ax-x^2$. **19.** $m+n$. **20.** $14-4\sqrt{6}$.

21. $147+30\sqrt{24}$. **22.** $1+2a\sqrt{1-a^2}$. **23.** $2a-2\sqrt{a^2-b^2}$.

Art. 275; page 196.

6. $\sqrt[6]{\frac{8}{9}}$. **7.** $\sqrt[20]{\frac{16}{243}}$. **8.** $\sqrt[6]{18}$. **9.** $\sqrt[12]{32a}$.

Art. 276; page 197.

3. $\sqrt[5]{125}$. 4. $\sqrt{7}$. 5. $2304\,x^2$. 6. $a^4\,x^2$. 7. $\sqrt{a-b}$.
8. $81\,a^4\,b\,x\,\sqrt[8]{b\,x}$. 9. $x^2+2\,x+1$. 10. $16\,x^2-48$.

Art. 277; pages 198 and 199.

3. $\sqrt[4]{2}$. 4. $\sqrt{2}$. 5. $\sqrt[12]{a+b}$. 6. $\sqrt[8]{x-1}$. 7. $\sqrt{2}$
8. $\sqrt[5]{3}$. 9. $\sqrt{3}$. 10. $\sqrt[5]{x^2\,y^3}$. 11. $\sqrt{2}$.

Art. 278; page 200.

3. $\dfrac{3\sqrt{2}}{2}$. 4. $\dfrac{\sqrt[8]{4\,a^2}}{2\,a}$. 5. $\dfrac{5\sqrt[8]{2}}{2}$. 6. $\dfrac{2\,c\sqrt[4]{3\,a^2}}{3\,a}$.

Art. 279; page 201.

3. $\dfrac{12-4\sqrt{2}}{7}$. 4. $5+2\sqrt{3}$. 5. $2\sqrt{6}-5$.

6. $\dfrac{a+2\sqrt{a\,b}+b}{a-b}$. 7. $-\dfrac{16+7\sqrt{10}}{13}$.

8. $\dfrac{a-2\sqrt{a\,x}+x}{a-x}$. 9. $-\dfrac{a+3+3\sqrt{a+1}}{a}$.

10. $2a^2-1-2a\sqrt{a^2-1}$. 11. $\dfrac{a+\sqrt{a^2-x^2}}{x}$. 12. $\sqrt{a^4-1}-a^2$.

13. $\dfrac{x^2-2+x\sqrt{x^2-4}}{2}$. 14. $\dfrac{14\,x-24-11\sqrt{x^2-2\,x}}{18-5\,x}$.

Art. 281; page 202.

2. .894. 3. 7.243. 4. 3.365. 5. .101.

Art. 286; page 204.

4. $-8\sqrt{6}$. 5. $12\sqrt{a\,b}$. 6. 46. 7. 2.
8. $-a\,b\,c\sqrt{-1}$. 9. a^2+b. 10. 12. 12. $\sqrt{3}$.

13. $\sqrt{2}$. **14.** $\sqrt{5}$. **15.** $\sqrt{3}$. **16.** $\sqrt{-1}$. **17.** $1+\sqrt{-2}$.

18. $\frac{2(a^2-b)}{a^2+b}$. **19.** $1-4\sqrt{-3}$. **20.** $-100-18\sqrt{-2}$.

Art. 293; pages 207 and 208.

5. $\sqrt{7}+\sqrt{5}$. **8.** $5+\sqrt{10}$. **11.** $\sqrt{15}-\sqrt{5}$. **14.** $3-2\sqrt{-2}$.

6. $\sqrt{21}-\sqrt{3}$. **9.** $3-\sqrt{3}$. **12.** $3+\sqrt{5}$. **15.** $5+3\sqrt{-2}$.

7. $3+\sqrt{7}$. **10.** $\sqrt{5}-\sqrt{3}$. **13.** $7-3\sqrt{2}$. **16.** $6-\sqrt{-1}$.

17. $\sqrt{m+n}-\sqrt{m-n}$. **18.** $x-\sqrt{ax}$. **19.** $3+\sqrt{2}$.

20. $\sqrt{2}-1$. **21.** $2-\sqrt{3}$.

Art. 297; pages 209 and 210.

4. 17. **9.** 4. **14.** 4. **19.** -1. **24.** 4.

5. 19. **10.** 5. **15.** 81. **20.** -3. **25.** 5.

6. $7\frac{2}{3}$. **11.** -2. **16.** 4. **21.** 4. **26.** 3.

7. 2. **12.** $\frac{2}{3}$. **17.** 8. **22.** 12. **27.** 6.

8. 4. **13.** 4. **18.** -3. **23.** 25. **28.** 39.

29. $3\frac{1}{4}$. **30.** 3. **31.** 6. **32.** $3a-1$.

Art. 303; pages 212 and 213.

2. ± 3. **4.** $\pm\sqrt{\left(-\frac{3}{2}\right)}$. **6.** ± 7. **8.** ± 1. **10.** ± 3.

3. ± 5. **5.** ± 1. **7.** $\pm\sqrt{11}$. **9.** $\pm\frac{1}{2}$. **11.** $\pm\sqrt{19}$.

12. $\pm\sqrt{\left(\frac{c-b}{a}\right)}$. **13.** $\pm\sqrt{a+b}$.

Art. 310; pages 220 to 222.

10. 5 or -7. **11.** 11 or -2. **12.** 5 or 3.

13. -5 or -13.

14. $\frac{1}{3}$ or $-\frac{3}{2}$.

15. 2 or $\frac{7}{3}$.

16. $-\frac{1}{3}$ or $-\frac{7}{5}$.

17. $\frac{1 \pm \sqrt{-959}}{12}$.

18. $\frac{17 \pm \sqrt{337}}{4}$.

19. $-\frac{1}{6}$ or $-\frac{1}{2}$.

20. 1 or $-\frac{7}{4}$.

21. $\frac{11}{3}$ or -2.

22. $\frac{1 \pm \sqrt{409}}{8}$.

23. $\frac{15}{4}$ or $\frac{5}{2}$.

24. $3\frac{1}{2}$ or -1.

25. 13 or -2.

26. $\frac{1}{2}$ or $\frac{1}{14}$.

27. 1 or $3\frac{1}{4}$.

28. -4 or $-\frac{11}{6}$.

29. -4 or -1.

30. 2 or $\frac{1}{3}$.

31. 4 or $-1\frac{2}{3}$.

32. $4 \pm 2\sqrt{3}$.

33. 3 or -1.

34. 2 or $-\frac{4}{7}$.

35. 7 or $\frac{5}{6}$.

36. 4 or $-\frac{7}{4}$.

37. $-10 \pm \sqrt{78}$.

38. $-3\frac{1}{2}$ or $-2\frac{1}{2}$.

39. 1 or $\frac{7}{36}$.

40. 1 or $\frac{2}{9}$.

41. 5 or $\frac{16}{5}$.

42. 18 or 3.

43. -2 or $\frac{16}{23}$.

44. -3 or $2\frac{7}{22}$.

45. 2.

46. 4 or 0.

47. 3 or -2.

48. -2 or $\frac{12}{65}$.

49. $\pm \frac{2}{\sqrt{3}}$.

50. 25 or 3.

51. 6 or -2.

52. $\frac{b}{a}$ or $-\frac{d}{c}$.

53. $a \pm b$.

54. $\frac{3a}{4}$ or $\frac{a}{2}$.

55. $-a$ or $-b$.

56. 11 or $18\frac{1}{2}$.

57. 5 or -3.

58. $\frac{12 \pm \sqrt{-1}}{5}$.

59. $a-b$ or $-a-c$.

60. $\frac{2a-b}{ac}$ or $-\frac{3a+2b}{bc}$.

61. $\frac{a+b}{a-b}$ or $\frac{a-b}{a+b}$.

Art. 311; pages 224 to 227.

4. 12 rds. **5.** 40000 sq. rds., and 14400 sq. rds. **6.** 9 and 6. **7.** 16 and 10. **8.** 16. **9.** 3 inches. **10.** \$30. **11.** 14 and 5. **12.** \$2000. **13.** 18 bbls., at \$4 each. **14.** 256 sq. yds. **15.** 5. **16.** 7 and 8. **17.** 7, 8, and 9. **18.** Length, 125; breadth, 50. **19.** 9. **20.** 3712. **21.** 80. **22.** 20. **23.** Area of court, 529 square yards; width of walk, 4 yards. **24.** 36 bu. at \$1.40. **25.** Larger, \$77.17½; smaller, \$56.70. **26.** 1st, 14400; 2d, 625; or, 1st, 8464; 2d, 6561. **27.** 84. **28.** 6. **29.** Larger pipe, 5 hours; smaller, 7 hours. **30.** 38 or 266 miles. **31.** 70 miles.

Art. 314; pages 230 to 232.

5. ± 3 or $\pm\sqrt{-13}$. **6.** $\pm\frac{1}{2}$ or $\pm\frac{1}{\sqrt{5}}$. **7.** 1 or -2.

8. ± 1 or $\pm\frac{1}{9}$. **9.** ± 7 or ± 5. **10.** $\sqrt[3]{3}$ or $-\sqrt[3]{23}$.

11. ± 8 or $\pm\sqrt[4]{\left(-\frac{74^3}{125}\right)}$. **12.** 4 or $\sqrt[3]{49}$. **13.** 4 or 1.

14. 243 or $-\sqrt[3]{(28^5)}$. **15.** 4 or $7\frac{1}{9}$. **16.** 49 or 25.

18. 2, -2, 3, or 7. **19.** 3 or -1. **20.** ± 1 or ± 2.

21. 2 or -3. **23.** 1, -1, 5, or 7. **24.** 2, -3, 4, or -5.

25. 1, 2, -5, or 8. **26.** 1, -1, -6, or -8.

28. $3, -\frac{9}{2}$, or $\frac{-3\pm\sqrt{-55}}{4}$. **29.** 8, -2, or $3\pm\sqrt{110}$.

30. $\frac{3}{2}, -\frac{9}{2}$, or $\frac{-3\pm 2\sqrt{3}}{2}$. **31.** 1, 9, or $5\pm 2\sqrt{2}$.

32. $0, -5, \frac{1}{3}$, or $-\frac{16}{3}$.

Art. 317; page 234.

2. $x=2,\ y=\pm 1$; or, $x=-2,\ y=\pm 1$. **3.** $x=4,\ y=\pm 5$;

or, $x=-4,\ y=\pm 5$. **4.** $x=\frac{1}{3},\ y=\pm\frac{1}{2}$; or, $x=-\frac{1}{3},\ y=\pm\frac{1}{2}$.

5. $x=3,\ y=\pm\frac{1}{4}$; or, $x=-3,\ y=\pm\frac{1}{4}$.

Art. 318; page 235.

2. $x=7,\ y=-8$; or, $x=-8,\ y=7$.

3. $x=5,\ y=-2$; or, $x=-2,\ y=5$.

4. $x=3,\ y=4$; or, $x=-4,\ y=-3$.

5. $x=8,\ y=\frac{5}{2}$; or, $x=-\frac{5}{2},\ y=-8$.

6. $x=2,\ y=4$; or, $x=-\frac{1}{3},\ y=\frac{5}{3}$.

7. $x=2,\ y=-3$; or, $x=3,\ y=-2$.

8. $x=1,\ y=2$; or, $x=2,\ y=1$.

9. $x=3,\ y=2$; or, $x=-\frac{15}{13},\ y=\frac{62}{13}$.

10. $x=9,\ y=6$; or, $x=-6,\ y=-9$.

11. $x=2,\ y=9$; or, $x=9,\ y=2$.

12. $x=9,\ y=3$; or, $x=-3,\ y=-9$.

13. $x=6,\ y=-4$; or, $x=-4,\ y=6$.

14. $x=3,\ y=2$; or, $x=\frac{47}{3},\ y=-\frac{13}{3}$.

15. $x=5,\ y=3$; or, $x=-3,\ y=-5$.

16. $x=3,\ y=-7$; or, $x=-7,\ y=3$.

Art. 319; page 238.

4. $x=3,\ y=4$; $x=4,\ y=3$; $x=-3,\ y=-4$; or, $x=-4$, $y=-3$.

5. $x=6,\ y=7$; $x=7,\ y=6$; $x=-6,\ y=-7$; or, $x=-7$, $y=-6$.

6. $x=2,\ y=-3$; or, $x=-3,\ y=2$.

7. $x=-1,\ y=4$; or, $x=-4,\ y=1$.

8. $x=3,\ y=-2$; or, $x=-2,\ y=3$.

9. $x=4,\ y=-7$; or, $x=7,\ y=-4$.

10. $x=5,\ y=6$; or, $x=6,\ y=5$.

11. $x=5,\ y=2$; or, $x=-2,\ y=-5$.

Art. 320; pages 239 and 240.

2. $x=2,\ y=\frac{1}{2}$; $x=-2,\ y=-\frac{1}{2}$; $x=\sqrt{\frac{2}{5}},\ y=-2\sqrt{\frac{2}{5}}$;
or, $x=-\sqrt{\frac{2}{5}},\ y=2\sqrt{\frac{2}{5}}$.

3. $x=2,\ y=3$; $x=-2,\ y=-3$; $x=\frac{5}{\sqrt{31}},\ y=-\frac{6}{\sqrt{31}}$;
or, $x=-\frac{5}{\sqrt{31}},\ y=\frac{6}{\sqrt{31}}$.

4. $x=3,\ y=1$; $x=-3,\ y=-1$; $x=2\sqrt{2},\ y=\sqrt{2}$;
or, $x=-2\sqrt{2},\ y=-\sqrt{2}$.

5. $x=3,\ y=5$; $x=-3,\ y=-5$; $x=\frac{5}{3},\ y=\frac{13}{3}$; or, $x=-\frac{5}{3}$,
$y=-\frac{13}{3}$.

6. $x=2,\ y=-1$; $x=-2,\ y=1$; $x=\frac{5}{\sqrt{11}},\ y=\frac{7}{\sqrt{11}}$;
or, $x=-\frac{5}{\sqrt{11}},\ y=-\frac{7}{\sqrt{11}}$.

7. $x=2,\ y=1$; $x=-2,\ y=-1$; $x=7,\ y=-19$; or,
$x=-7,\ y=19$.

Art. 321; pages 243 and 244.

5. $x=1,\ y=8$; or, $x=8,\ y=1$.

6. $x=4,\ y=9$; or, $x=9,\ y=4$.

7. $x=2,\ y=3$; or, $x=3,\ y=2$.

8. $x=3, y=4$; $x=4, y=3$; $x=-4+\sqrt{-11}, y=-4-\sqrt{-11}$; or, $x=-4-\sqrt{-11}, y=-4+\sqrt{-11}$.

9. $x=4, y=5$; $x=16, y=-7$; $x=-12+\sqrt{58}, y=-1-\sqrt{58}$; or, $x=-12-\sqrt{58}, y=-1+\sqrt{58}$.

10. $x=4, y=2$; $x=-2, y=-4$; or, $x=0, y=0$.

11. $x=9, y=4$; or, $x=\frac{605}{117}, y=\frac{20}{117}$. **12.** $x=1, y=\frac{3}{2}$.

13. $x=3, y=2$; or, $x=2, y=3$. **14.** $x=9, y=4$.

15. $x=1, y=-3$; $x=-3, y=1$; $x=1+\sqrt{-2}, y=1-\sqrt{-2}$; or, $x=1-\sqrt{-2}, y=1+\sqrt{-2}$.

16. $x=1, y=-2$; $x=2, y=-1$; $x=\frac{3+\sqrt{-55}}{2}, y=\frac{-3+\sqrt{-55}}{2}$;

or, $x=\frac{3-\sqrt{-55}}{2}, y=\frac{-3-\sqrt{-55}}{2}$.

17. $x=2, y=3$; $x=-3, y=-2$; $x=\frac{-1+3\sqrt{-3}}{2}, y=\frac{1+3\sqrt{-3}}{2}$;

or, $x=\frac{-1-3\sqrt{-3}}{2}, y=\frac{1-3\sqrt{-3}}{2}$.

18. $x=3, y=2$; $x=2, y=3$; $x=\frac{-9+\sqrt{309}}{12}, y=\frac{-9-\sqrt{309}}{12}$;

or, $x=\frac{-9-\sqrt{309}}{12}, y=\frac{-9+\sqrt{309}}{12}$.

19. $x=1, y=-3$; $x=-1, y=3$; $x=14\frac{3}{4}, y=3\frac{7}{8}$; or, $x=-14\frac{3}{4}$, $y=-3\frac{7}{8}$.

20. $x=2, y=3$; or, $x=2\frac{4}{5}, y=1\frac{4}{5}$.

21. $x=4, y=2, z=3$; or, $x=\frac{4}{9}, y=\frac{22}{3}, z=\frac{59}{9}$.

22. $x=1, y=2, z=4$; $x=-1, y=-2, z=-4$; $x=9$, $y=-6, z=4$; or, $x=-9, y=6, z=-4$.

Art. 322; pages 246 to 248.

4. 12 and 7, or -12 and -7. **5.** 11 and 7, or -11 and -7.

6. A, \$2025; B, \$900; or, A, \$900; B, \$2025.

7. A, 25; B, 30. **8.** Length, 150 yds.; breadth, 100 yds. **9.** 13 and 6. **10.** A, $15; B, $80. **11.** 10 lbs., at 8 cts. **12.** A, $5; B, $120. **13.** Duck, $0.75; turkey, $1.25. **14.** Price, $1600; length, 160 rods; breadth, 40 rods. **15.** Larger, 864 sq. in.; smaller, 384. **16.** A, $275; B, $225. **17.** 1st rate, 7 p.c.; 2d, 6. **18.** A, 40 acres at $8; B, 64, at $5. **19.** Distance of towns, 450 miles; A, 30 miles a day; B, 25. **20.** 3 and 1; or, $2+\sqrt{7}$ and $2-\sqrt{7}$. **21.** Larger, 12 ft.; smaller, 9. **22.** Width of street, 63 ft.; length of ladder, 45. **23.** B, 15 days; C, 18 days. **24.** Length, 16 yds.; width, 2 yds.

Art. 328; page 253.

3. $(x+60)(x+13)$. **6.** $(x+13)(x-3)$. **9.** $(4x-1)(2x+5)$. **4.** $(x-9)(x-2)$. **7.** $(x-5)(2x+3)$. **10.** $(x-3)(4x-3)$. **5.** $(x-10)(x+6)$. **8.** $(7x+3)(3x+7)$. **11.** $(x+2)(2x-3)$. **12.** $(3x-2+\sqrt{3})(3x-2-\sqrt{3})$. **13.** $(\sqrt{17}+4+x)(\sqrt{17}-4-x)$. **14.** $(7x+1+2\sqrt{5})(7x+1-2\sqrt{5})$.

Art. 329; page 254.

2. $x^2+x=2$. **5.** $3x^2-2x=133$. **8.** $3x^2+17x=0$. **3.** $x^2-9x=-20$. **6.** $21x^2+44x=32$. **9.** $x^2-2x=4$. **4.** $5x^2-12x=9$. **7.** $6x^2+35x=-49$. **10.** $x^2-2mx=n-m^2$.

Art. 330; page 255.

7. 0 or $\frac{13}{2}$. **8.** 0 or -4. **9.** 0 or ± 3. **10.** $-\frac{5}{2}$ or $\frac{1}{3}$. **11.** $-\frac{b}{a}$ or $\frac{d}{c}$. **12.** ± 2 or ± 3. **13.** $-\frac{1}{3}$ or $\pm\frac{5}{2}$. **14.** $\pm\sqrt{a}$ or $\frac{a\pm\sqrt{a^2+4b}}{2}$. **15.** $0, -\frac{5}{2}, \frac{7}{3}$, or $-\frac{1}{4}$. **16.** $2, 3, -3, -4, \frac{1}{2}$, or -5.

Art. 331; page 256.

3. $(x + \sqrt{2x} + 1)(x - \sqrt{2x} + 1)$.
4. $(x + \sqrt{x} + 1)(x - \sqrt{x} + 1)$.
5. $(a + \sqrt{5ab} + b)(a - \sqrt{5ab} + b)$.
6. $(x^2 + 3xy + y^2)(x^2 - 3xy + y^2)$.
7. $(x + 1 + \sqrt{3x + 2})(x + 1 - \sqrt{3x + 2})$.
8. $(m^2 + mn + n^2)(m^2 - mn + n^2)$.

Art. 332; page 256.

2. $\frac{\sqrt{2} \pm \sqrt{-2}}{2}$ or $\frac{-\sqrt{2} \pm \sqrt{-2}}{2}$. 3. -1 or $\frac{1 \pm \sqrt{-3}}{2}$.

4. $a\left\{\frac{\sqrt{2} \pm \sqrt{-2}}{2}\right\}$ or $a\left\{\frac{-\sqrt{2} \pm \sqrt{-2}}{2}\right\}$. 5. $\frac{\sqrt{3} \pm \sqrt{-1}}{2}$

or $\frac{-\sqrt{3} \pm \sqrt{-1}}{2}$. 6. $\pm 1, \frac{1 \pm \sqrt{-3}}{2}$, or $\frac{-1 \pm \sqrt{-3}}{2}$.

7. $\frac{\sqrt{7} \pm \sqrt{-1}}{2\sqrt{2}}$ or $\frac{-\sqrt{7} \pm \sqrt{-1}}{2\sqrt{2}}$.

Art. 357; pages 269 and 270.

1. 4. 2. 11. 3. $\frac{1}{6}$. 4. $1\frac{4}{5}$. 5. ± 4. 6. ± 12.
7. ± 14. 8. 25 and 20. 9. 23 and 27. 10. 12 and 15.
11. 8 and 18. 12. 26 and 14. 13. 17 and 12. 14. 12 and 8.
15. First, 1 : 2; second, 2 : 1. 16. Females : males = 4 : 5.
17. 8 : 7.

Art. 365; page 273.

2. 4. 3. $y = 8z$. 4. $\frac{4}{3}$. 5. 4. 6. $y = \frac{14}{4 - 5x}$.
7. 10 inches. 8. $3(\sqrt{2} - 1)$ inches. 9. 143.

Art. 370; page 276.

3. $l=71, S=540$. **4.** $l=-69, S=-620$. **5.** $l=57, S=552$.

6. $l=-145, S=-2175$. **7.** $l=\frac{23}{12}, S=\frac{62}{3}$. **8.** $l=-\frac{3}{5}, S=0$.

9. $l=-\frac{5}{11}, S=\frac{1}{2}$. **10.** $l=\frac{137}{15}, S=\frac{917}{15}$. **11.** $l=5, S=17$.

12. $l=\frac{35}{4}, S=\frac{315}{2}$.

Art. 371; pages 278 and 279.

4. $a=3, S=741$. **5.** $a=\frac{3}{2}, l=-\frac{25}{2}$. **6.** $d=\frac{1}{3}, S=39$.

7. $d=-\frac{1}{12}, l=-\frac{5}{4}$. **8.** $a=5, d=-3$. **9.** $n=18, S=411$.

10. $d=-8, n=11$. **11.** $n=30, l=80$. **12.** $n=52, a=4$; or, $n=43, a=-5$. **13.** $n=16, l=-43$.

Art. 372; page 279.

2. $\frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}$. **3.** $\frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}$.

4. $-2, -3, -4, -5$. **5.** $-\frac{52}{7}, -\frac{48}{7}, -\frac{44}{7}, -\frac{40}{7}, -\frac{36}{7}, -\frac{32}{7}$.

6. $-\frac{2}{5}, \frac{6}{5}, \frac{14}{5}, \frac{22}{5}$. **7.** $\frac{am+b}{m+1}, \frac{a(m-1)+2b}{m+1}, \ldots\ldots$

Art. 373; page 281.

3. 2500. **4.** Last payment, \$103; amount, \$2704.

5. 4. **6.** After 9 days, at a distance of 90 leagues.

7. 4, 11, 18, and 25. **8.** 3. **9.** 0. **10.** 20 miles.

11. 2, 6, 10, and 14; or, $-2, -6, -10$, and -14. **12.** 8.

Art. 378; page 284.

4. $l = 2048$, $S = 4095$.

5. $l = \frac{64}{243}$, $S = \frac{2059}{243}$.

6. $l = 2048$, $S = 1638$.

7. $l = -\frac{1}{256}$, $S = \frac{341}{256}$.

8. $l = \frac{1}{2048}$, $S = \frac{2047}{2048}$.

9. $l = -\frac{729}{64}$, $S = -\frac{1261}{192}$.

10. $l = \frac{1}{32}$, $S = \frac{511}{32}$.

11. $l = -\frac{1}{324}$, $S = \frac{91}{162}$.

12. $l = -\frac{1}{768}$, $S = -\frac{341}{256}$.

13. $l = 192$, $S = 129$.

Art. 379; page 286.

4. $a = \frac{1}{2}$, $S = -\frac{341}{2}$.

5. $a = \frac{2}{3}$, $l = \frac{2}{6561}$.

6. $r = 3$, $S = 2186$; or, $r = -3$, $S = 1094$.

7. $r = -\frac{1}{4}$, $S = \frac{2457}{1024}$.

8. $n = 5$, $S = 121$.

9. $n = 7$, $r = \frac{1}{2}$.

10. $n = 6$, $l = -\frac{243}{2}$.

11. $n = 8$, $a = -1$.

Art. 380; pages 287 and 288.

3. 4.

4. $\frac{8}{3}$.

5. $-\frac{3}{4}$.

6. $-\frac{15}{4}$.

7. $\frac{9}{4}$.

8. $\frac{7}{25}$.

9. $\frac{160}{19}$.

10. $-\frac{10}{3}$.

Art. 381; page 288.

3. $\frac{2}{27}$.

4. $\frac{13}{27}$.

5. $\frac{11}{15}$.

6. $\frac{86}{165}$.

7. $\frac{17}{150}$.

8. $\frac{237}{1100}$.

Art. 382; pages 289 and 290.

3. $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}$. **4.** $\frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}, \frac{243}{2}$; or, $-\frac{3}{2}, \frac{9}{2}, -\frac{27}{2}, \frac{81}{2}, -\frac{243}{2}$. **5.** $-6, -18, -54, -162, -486, -1458$.

6. $-\frac{9}{4}, \frac{27}{16}, -\frac{81}{64}, \frac{243}{256}$. **7.** $\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \frac{3}{64}, \frac{3}{128}, \frac{3}{256}$;

or, $-\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}, -\frac{3}{64}, \frac{3}{128}, -\frac{3}{256}$.

Art. 383; page 291.

3. \$64. **4.** \$295.23. **5.** 3100 ft. **6.** 5, 10, 20, and 40; or, -15, 30, -60, and 120. **7.** -4. **8.** $\frac{1}{18}$.

Art. 386; page 292.

2. $\frac{15}{31}$. **3.** $-\frac{1}{78}$. **4.** $-\frac{3}{4}$. **5.** $\frac{ab}{an - bn + 2b - a}$.

Art. 387; page 293.

2. $\frac{48}{125}, \frac{24}{65}, \frac{16}{45}, \frac{12}{35}, \frac{48}{145}, \frac{8}{25}, \frac{48}{155}$.

3. $-\frac{5}{4}, -\frac{5}{3}, -\frac{5}{2}$. **4.** $7, -21, -\frac{21}{5}, -\frac{7}{3}, -\frac{21}{13}, -\frac{21}{17}$.

5. $\frac{(m+1)ab}{mb+a}, \frac{(m+1)ab}{mb+2a-b}, \frac{(m+1)ab}{mb+3a-2b}, \ldots\ldots$

Art. 397; pages 297 and 298.

4. Of 4 letters, 360; of 3, 120; of 6, 720; in all, 1956.

5. 1680. **6.** 3838380. **7.** 358800. **8.** 15120. **9.** 120.

10. 35. **11.** 15504. **12.** 31824. **13.** 77520. **14.** 648.

Art. 403; page 302.

5. $1 + 5c + 10c^2 + 10c^3 + 5c^4 + c^5$.

6. $a^6 + 6a^5x^3 + 15a^4x^6 + 20a^3x^9 + 15a^2x^{12} + 6ax^{15} + x^{18}$.

7. $x^8 - 8x^6y + 24x^4y^2 - 32x^2y^3 + 16y^4$.

8. $a^7b^7 - 7a^6b^6cd + 21a^5b^5c^2d^2 - 35a^4b^4c^3d^3 + 35a^3b^3c^4d^4 - 21a^2b^2c^5d^5 + 7abc^6d^6 - c^7d^7$.

9. $m^{12} + 18m^{10}n^2 + 135m^8n^4 + 540m^6n^6 + 1215m^4n^8 + 1458m^2n^{10} + 729n^{12}$.

10. $a^{-10} - 20a^{-8}x^{\frac{1}{2}} + 160a^{-6}x - 640a^{-4}x^{\frac{3}{2}} + 1280a^{-2}x^2 - 1024x^{\frac{5}{2}}$.

11. $c^{\frac{16}{3}} + 8c^{\frac{14}{3}}d^{\frac{3}{4}} + 28c^4d^{\frac{3}{2}} + 56c^{\frac{10}{3}}d^{\frac{9}{4}} + 70c^{\frac{8}{3}}d^3 + 56c^2d^{\frac{15}{4}} + 28c^{\frac{4}{3}}d^{\frac{9}{2}} + 8c^{\frac{2}{3}}d^{\frac{21}{4}} + d^6$.

12. $m^{-\frac{21}{5}} + 14m^{-\frac{18}{5}}n^3 + 84m^{-3}n^6 + 280m^{-\frac{12}{5}}n^9 + 560m^{-\frac{9}{5}}n^{12} + 672m^{-\frac{6}{5}}n^{15} + 448m^{-\frac{3}{5}}n^{18} + 128n^{21}$.

13. $a^{-4} - 4a^{-3}b^2x^{\frac{1}{3}} + 6a^{-2}b^4x^{\frac{2}{3}} - 4a^{-1}b^6x + b^8x^{\frac{4}{3}}$.

Art. 404; page 303.

2. $5005a^6x^9$.

3. $2002m^5$.

4. $-19448c^{10}d^7$.

5. $495a^8$.

6. $42240x^{-3}y^4$.

7. $262440a^2x^{-7}$.

Art. 405; page 304.

2. $1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$.

3. $x^6 + 9x^5 + 30x^4 + 45x^3 + 30x^2 + 9x + 1$.

4. $1 - 6x + 6x^2 + 16x^3 - 12x^4 - 24x^5 - 8x^6$.

5. $1 + 5x + 5x^2 - 10x^3 - 15x^4 + 11x^5 + 15x^6 - 10x^7 - 5x^8 + 5x^9 - x^{10}$.

Art. 414; page 309.

3. $1 - 2x + 2x^2 - 2x^3 + 2x^4 \ldots\ldots$

4. $3 + 19x + 95x^2 + 475x^3 + 2375x^4 \ldots\ldots$

5. $2 - x + 3x^2 - x^3 + 3x^4 \ldots\ldots$

6. $1 - 2x + 2x^3 - 2x^4 + 2x^6 \ldots\ldots$

7. $1 - 2x + 5x^2 - 16x^3 + 47x^4 \ldots\ldots$

8. $\frac{1}{2} + \frac{5x}{4} + \frac{7x^2}{8} + \frac{17x^3}{16} + \frac{31x^4}{32} \ldots\ldots$

9. $2 - 7x + 28x^2 - 91x^3 + 322x^4 \ldots\ldots$

10. $1 + \frac{2x}{3} - \frac{7x^2}{9} - \frac{13x^3}{27} + \frac{8x^4}{81} \ldots\ldots$

11. $\frac{1}{2} + \frac{3x}{4} + \frac{x^2}{8} + \frac{15x^3}{16} + \frac{49x^4}{32} \ldots\ldots$

Art. 415; page 310.

2. $\frac{2x^{-2}}{3} + \frac{4x^{-1}}{9} + \frac{8}{27} + \frac{16x}{81} + \frac{32x^2}{243} \ldots\ldots$

3. $x^{-1} + 3 + 2x - 5x^2 - 16x^3 \ldots\ldots$

4. $x^{-2} - x^{-1} - 2x + 2x^2 - 4x^3 \ldots\ldots$

Art. 416; page 311.

2. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} \cdots$

3. $1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \frac{5x^4}{8} \cdots$

4. $1 - x + x^2 + x^3 + \frac{x^4}{2} \cdots$

5. $1 + \frac{x}{2} + \frac{3x^2}{8} - \frac{3x^3}{16} + \frac{3x^4}{128} \cdots$

6. $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} \cdots$

7. $1 + \frac{x}{3} + \frac{2x^2}{9} - \frac{13x^3}{81} + \frac{8x^4}{243} \cdots$

Art. 418; page 314.

2. $\frac{3}{x+2} + \frac{2}{x-2}$.

3. $\frac{3}{x} - \frac{2}{x+3}$.

4. $\frac{4}{x-2} - \frac{1}{x}$.

5. $\frac{1}{x-4} + \frac{1}{x+1}$.

6. $\frac{7}{x-7} - \frac{6}{x-6}$.

7. $\frac{2}{2x-5} - \frac{3}{3x+1}$.

8. $\frac{1}{3+4x} + \frac{2}{3-x}$.

9. $\frac{1}{6(x+1)} - \frac{1}{2(x-1)} + \frac{4}{3(x-2)}$.

Art. 419; page 316.

2. $\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{(x+1)^3}$.

3. $\frac{2}{x-5}-\frac{3}{(x-5)^2}$.

4. $\frac{1}{x-2}+\frac{4}{(x-2)^2}+\frac{4}{(x-2)^3}$.

5. $\frac{3}{x+1}-\frac{6}{(x+1)^2}-\frac{1}{(x+1)^3}$.

6. $\frac{3}{2(2x-5)}-\frac{5}{2(2x-5)^2}$.

7. $\frac{2}{3x+2}-\frac{4}{(3x+2)^2}-\frac{3}{(3x+2)^3}$.

Art. 420; page 317.

2. $\frac{2}{x}-\frac{3}{x+2}-\frac{5}{(x+2)^2}$.

3. $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{(x-2)^2}$.

4. $\frac{5}{x}-\frac{1}{x^2}-\frac{5}{x+1}-\frac{4}{(x+1)^2}$.

5. $\frac{1}{x}-\frac{2}{x^2}+\frac{3}{x^3}-\frac{4}{x+5}$.

6. $\frac{2}{x-2}-\frac{1}{2x-3}+\frac{3}{(2x-3)^2}$.

7. $\frac{5}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{5}{x+1}-\frac{4}{(x+1)^2}$.

Art. 422; page 320.

3. $x=y-y^2+y^3-y^4\ldots$

4. $x=\frac{y}{2}-\frac{3y^3}{16}+\frac{19y^5}{128}-\frac{19y^7}{128}\ldots$

5. $x=y+y^3+2y^5+5y^7\ldots$

6. $x=(y-1)-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}-\frac{(y-1)^4}{4}\ldots$

7. $x=y+\frac{y^3}{3}+\frac{2y^5}{15}+\frac{17y^7}{315}\ldots$

8. $x=\frac{y}{3}+\frac{2y^2}{27}-\frac{y^3}{243}-\frac{14y^4}{2187}\ldots$

Art. 425; page 325.

4. $a^{\frac{5}{2}}+\frac{5}{2}a^{\frac{3}{2}}x+\frac{15}{8}a^{\frac{1}{2}}x^2+\frac{5}{16}a^{-\frac{1}{2}}x^3-\frac{5}{128}a^{-\frac{3}{2}}x^4\ldots\ldots$

5. $1-6x+21x^2-56x^3+126x^4 \ldots\ldots$

6. $1+\frac{3}{5}x+\frac{12}{25}x^2+\frac{52}{125}x^3+\frac{234}{625}x^4 \ldots\ldots$

7. $a^{\frac{1}{2}}-\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{8}a^{-\frac{3}{2}}x^2-\frac{1}{16}a^{-\frac{5}{2}}x^3-\frac{5}{128}a^{-\frac{7}{2}}x^4 \ldots\ldots$

8. $1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\frac{35}{243}x^4 \ldots\ldots$

9. $a^{-3}+3a^{-4}x+6a^{-5}x^2+10a^{-6}x^3+15a^{-7}x^4 \ldots\ldots$

10. $c^{-\frac{3}{2}}-c^{-3}d+c^{-\frac{9}{2}}d^2-c^{-6}d^3+c^{-\frac{15}{2}}d^4 \ldots\ldots$

11. $x^{-\frac{1}{3}}-2x^{\frac{1}{6}}y-x^{\frac{2}{3}}y^2-\frac{4}{3}x^{\frac{7}{6}}y^3-\frac{7}{3}x^{\frac{5}{3}}y^4 \ldots\ldots$

12. $m+3m^{\frac{5}{3}}n^{\frac{3}{2}}+\frac{15}{2}m^{\frac{7}{3}}n^3+\frac{35}{2}m^3n^{\frac{9}{2}}+\frac{315}{8}m^{\frac{11}{3}}n^6 \ldots\ldots$

13. $1-10xy^{-1}+80x^2y^{-2}-\frac{1760}{3}x^3y^{-3}+\frac{12320}{3}x^4y^{-4} \ldots\ldots$

14. $x^3+3x^{-1}ab-\frac{3}{2}x^{-5}a^2b^2+\frac{5}{2}x^{-9}a^3b^3-\frac{45}{8}x^{-13}a^4b^4 \ldots\ldots$

15. $a^4+12a^5y^{-2}+90a^6y^{-4}+540a^7y^{-6}+2835a^8y^{-8} \ldots\ldots$

Art. 426; page 326.

3. $\frac{33a^{-\frac{13}{2}}x^7}{2048}$.

5. $\frac{315a^8}{128}$.

7. $\frac{44x^{\frac{14}{3}}y^8}{6561}$.

4. $84m^6$.

6. $-\frac{663x^{-\frac{21}{2}}y^{15}}{8192}$.

8. $210n^{\frac{22}{3}}c^{-8}$.

9. $-\frac{308}{3}a^{-\frac{34}{9}}x^{-5}$.

10. $36x^{-30}y^{-10}z^{-\frac{14}{3}}$.

Art. 427; page 327.

3. 3.14138.

5. 9.94987.

7. 2.03054.

4. 2.08008.

6. 1.96101.

8. 2.97183.

Art. 435; pages 331 and 332.

2. $\frac{1+x}{1-x-x^2}$. 4. $\frac{4-11x}{1-5x+6x^2}$. 6. $\frac{2+5a+5a^2}{(1+a)^3}$.

3. $\frac{a}{b+cx}$. 5. $\frac{1+x}{1-2x+x^2}$. 7. $\frac{3-x-6x^2}{1-2x-x^2+2x^3}$.

8. $\frac{1+2x}{1-x-x^2}$ 9. $\frac{2+2x-3x^2}{1-x+x^2-x^3}$.

Art. 440; page 336.

3. 3. 4. -14. 5. 30. 6. 1365. 7. 5050.

8. 225. 9. $\frac{n^4+2n^3+n^2}{4}$. 10. $\frac{6n^5+15n^4+10n^3-n}{30}$.

11. 165. 12. 5525.

Art. 443; pages 338 and 339.

3. 4.0514. 4. 3.634241. 5. 2.23830. 6. 44.24.

7. $1.356.

Art. 455; page 344.

1. 1.681241. 4. 1.991226. 7. 2.225309. 10. 3.489536.
2. 2.644438. 5. 1.924279. 8. 3.848558. 11. 4.191785.
3. 1.748188. 6. 2.753582. 9. 2.702430. 12. 4.158543.

Art. 456; page 345.

1. 1.176091. 4. 2.243038. 7. 0.853872.
2. 2.096910. 5. 0.522879. 8. 1.066947.
3. 0.154902. 6. 1.045758. 9. 0.735954.

Art. 464; page 350.

2. 8.724276 − 10. 4. 9.470704 − 10. 6. 1.527511.
3. 1.714330. 5. 0.011739. 7. 8.780210 − 10.

8. 4.812917. **11.** 9.942550 — 10. **14.** 4.89381.
9. 7.013150 — 10. **12.** 3.863506. **15.** 1.718451.
10. 2.960116. **13.** 8.640409 — 10. **16.** 7.4984240 — 10.
17. 9.275374 — 10. **18.** 1.9792784.

Art. 465; page 352.

2. 76. **7.** 186.334. **12.** .034277.
3. .2954. **8.** .223905. **13.** 46.7929.
4. 6.61005. **9.** 1000.06. **14.** 11.327.
5. 55606.5. **10.** 9.77667. **15.** 8.63076.
6. .011089. **11.** .00130514. **16.** .2070207.
17. .00548803. **18.** 734.9114.

Art. 466; pages 353 and 354.

1. 2.125240. **4.** 3.108462. **7.** 9.613158 — 10.
2. 8.223962 — 10. **5.** 9.594161 — 10. **8.** 9.970036 — 10.
3. 9.852169 — 10. **6.** 7.315321 — 10. **9.** 9.905232 — 10.

Art. 468; pages 356 to 358.

1. .0341657. **13.** 1.70869. **25.** .580799.
2. .650573. **14.** .788547. **26.** — .631188.
3. 13560.2. **15.** .680192. **27.** 83.5656.
4. .136085. **16.** 2.24328. **28.** .297812.
5. 1.14720. **17.** .296850. **29.** 98.4295.
6. 1.41421. **18.** — .191680. **30.** 1.65900.
7. 1.49535. **19.** .644349. **31.** 3.07616.
8. .0655264. **20.** .501126. **32.** .867674.
9. — 1.97221. **21.** 1.09872. **33.** — 2.09389.
10. 458.623. **22.** 1.06178. **34.** 46809.2.
11. — .000113607. **23.** 1.09328. **35.** .588142.
12. 5.88336. **24.** 1.65601. **36.** 1.80446.
37. .00323011. **38.** .0334343.

The following are the values of the expressions in Art. 468, when calculated by seven-figure logarithms:

1. .03416568.	**13.** 1.708689.	**25.** .5807987.
2. .6505727.	**14.** .7885469.	**26.** — .6311888.
3. 1.356027.	**15.** .6801947.	**27.** 83.56558.
4. .1360851.	**16.** 2.243284.	**28.** .2978123.
5. 1.147203.	**17.** .2968501.	**29.** 98.42991.
6. 1.414214.	**18.** — .1916795.	**30.** 1.658989.
7. 1.495349.	**19.** .6443490.	**31.** 3.076162.
8. .06552632.	**20.** .5011282.	**32.** .8676754.
9. — 1.972211.	**21.** 1.098718.	**33.** — 2.093891.
10. 458.5759.	**22.** 1.061780.	**34.** 46808.95.
11. — .0001136063.	**23.** 1.093280.	**35.** .5881412.
12. 5.883366.	**24.** 1.656005.	**36.** 1.804459.
37. .003230121.	**38.** .03343431.	

Art. 469; page 359.

3. .458156.	**5.** — .494903.	**7.** — 2.70951.
4. .185339.	**6.** — .260231.	**8.** — 10.2341.

The results with seven-figure logarithms are as follows:

3. .4581568.	**5.** — .4949028.	**7.** — 2.709513.
4. .1853394.	**6.** — .2602272.	**8.** — 10.23414.

Art. 479; pages 368 and 369.

1. 7.	**3.** — 6.	**5.** 7.	**7.** 6.
2. 6.	**4.** $-\frac{5}{3}$.	**6.** 5.	**8.** 7.

9. 1.56937.	**13.** 11.725 yrs.	**17.** 3.96913.
10. 2.44958.	**14.** $ 9756.59.	**18.** 7.18923.
11. 2.00906.	**15.** 7 per cent.	**19.** — 2.4578.
12. $ 5421.33.	**16.** 9.392 yrs.	**20.** — 1.07009.

The results of the last 12 examples, using seven-figure logarithms, are as follows:

9. 1.569369.	**13.** 11.725 yrs.	**17.** 3.969124.
10. 2.449576.	**14.** \$9756.59.	**18.** 7.18922.
11. 2.009056.	**15.** 7 per cent.	**19.** -2.457802.
12. \$5421.35.	**16.** 9.392 yrs.	**20.** -1.070092.

Art. 489; page 373.

2. 3 and -5. **3.** a and $\frac{a}{2}(-1 \pm \sqrt{-3})$. **4.** 2 and 2. **5.** ± 4.

6. $x^3 - 6x^2 - 6x - 3 = 0$. **7.** $\frac{2}{3}$ and $-\frac{5}{2}$. **8.** $\frac{5}{4}$ and $-\frac{4}{5}$.

Art. 490; page 374.

2. $x^3 + 9x^2 + 23x + 15 = 0$. **4.** $6x^3 - 11x^2 + 6x - 1 = 0$.

3. $x^3 - 19x - 30 = 0$. **5.** $x^4 - 5x^2 + 4 = 0$.

6. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

7. $x^3 - 13x^2 + 56x - 80 = 0$.

8. $x^4 - 6x^3 + 5x^2 + 12x = 0$.

9. $12x^4 + 55x^3 - 68x^2 - 185x + 150 = 0$.

Art. 494; page 375.

1. Sum, 0; product, -6. **2.** Sum, $\frac{5}{2}$; product, 12.

3. $2 \pm 2\sqrt{2}$.

Art. 504; page 382.

2. $y^3 + 24y^2 + 191y + 498 = 0$. **3.** $y^4 - 6y^3 - y^2 + 55y - 76 = 0$.

Art. 505; page 383.

2. $y^2 - \frac{p^2}{4} + q = 0$. **4.** $y^3 - 15y + 26 = 0$.

3. $y^3 - \frac{y}{3} + \frac{110}{27} = 0$. **5.** $y^4 - 6y^2 - 13y - 9 = 0$.

Art. 513; page 388.

2. 1, 1, and 6.
3. 2, 2, and 3.
4. $-1, -1, -1$, and 3.
5. 2, 2, 2, and -6.

Art. 517; page 390.

2. -1, 1, and 5.
3. 3.
4. 1.
5. 2.

Art. 520; page 392.

3. $1+\sqrt{14}$.
4. $1+\sqrt{15}$.
5. $-(1+\sqrt{6})$.
6. $-(1+\sqrt[3]{5})$.

Art. 527; page 399.

3. Three; respectively between 0 and 1, 1 and 2, and -1 and -2.
4. Three; two between 1 and 2, and one between -3 and -4.
5. One; between 2 and 3.
6. Four; respectively between 0 and 1, 1 and 2, 2 and 3, and -2 and -3.
7. None.
8. Two; respectively between 2 and 3, and 3 and 4.

Art. 532; page 403.

3. $-1, -2$, and -3.
4. $2, -2$, and -3.
5. 2, 4, and $-1 \pm \sqrt{-3}$.
6. $\frac{3}{2}$, 4, and $-\frac{3}{2}$.
7. 2, and $\frac{1 \pm \sqrt{5}}{2}$.
8. 3, 6, and -2.
9. 4, and $1 \pm \sqrt{-1}$.
10. 1, 2, and 3.
11. $\frac{3}{2}$, and $\pm 2\sqrt{-2}$.
12. 2.
13. 3.
14. 3, 4, -3, and -5.

Art. 538; pages 407 and 408.

2. $-1, \frac{9 \pm \sqrt{77}}{2}$, or $\frac{3 \pm \sqrt{5}}{2}$.

3. $-1, 1, 1$, or $\frac{-3 \pm \sqrt{5}}{2}$.

4. $\pm 1, \pm \sqrt{-1}$, or $\frac{1 \pm \sqrt{-3}}{2}$.

5. -1 or $\frac{1-p \pm \sqrt{p^2-2p-3}}{2}$

6. $2, \frac{1}{2}, -3$, or $-\frac{1}{3}$.

7. $1, 5, \frac{1}{5}$, or $2 \pm \sqrt{3}$.

8. $\frac{\sqrt{33}-5 \pm \sqrt{42-10\sqrt{33}}}{4}$, or $\frac{-\sqrt{33}-5 \pm \sqrt{42+10\sqrt{33}}}{4}$.

9. $-1, \frac{1+\sqrt{5} \pm \sqrt{2\sqrt{5}-10}}{4}$, or $\frac{1-\sqrt{5} \pm \sqrt{-2\sqrt{5}-10}}{4}$.

10. $2, \frac{-1-\sqrt{5} \pm \sqrt{2\sqrt{5}-10}}{2}$, or $\frac{-1+\sqrt{5} \pm \sqrt{-2\sqrt{5}-10}}{2}$.

Art. 541; page 410.

3. -3 or $\frac{3 \pm \sqrt{-3}}{2}$.

4. 4 or $1 \pm 4\sqrt{-3}$.

5. $3, 3$, or -2.

6. $1, 1$, or -11.

7. 1 or $\frac{1 \pm \sqrt{-3}}{2}$.

8. 3 or $\frac{1 \pm \sqrt{-3}}{2}$.

9. $2, 2$, or -1.

10. $\sqrt[3]{4} - \sqrt[3]{2}$.

Art. 550; page 417.

2. 2.09455. 3. 7.61728. 4. 1.3569, 1.6920, and -3.0489.

5. 14.95407. 6. 2.2674 and 3.6796.

7. 2.85808, .60602, .44328, and -3.90738.

Art. 551; page 419.

2. 3.864854.
3. 4.11799.
4. 2.4257.
5. .66437.
6. 10.2609.
7. 8.414455.

Art. 552; page 420.

2. 1.53209. 3. 1.02804.

A

TABLE,

CONTAINING THE

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8706	9116	9532	9947	020361	020775	416
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702	314
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911	309
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982	307
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	5852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068	301
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689	289
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558	287
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051	281
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	310056	310268	310481	310693	310906	311118	311330	311542	212
5	311754	311966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	320146	320354	320562	320769	320977	321184	321391	321598	321805	322012	207
210	322219	322426	322633	322839	323046	323252	323458	323665	323871	324077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	330008	330211	203
4	330414	330617	330819	331022	331225	331427	331630	331832	2034	2236	202
5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	340047	340246	199
9	340444	340642	340841	341039	341237	341435	341632	341830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	344196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	350054	194
4	350248	350442	350636	350829	351023	351216	351410	351603	351796	1989	193
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835	360025	360215	360404	360593	360783	360972	361161	361350	361539	189
230	361728	361917	362105	362294	362482	362671	362859	363048	363236	363424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	380030	181
240	380211	380392	380573	380754	380934	381115	381296	381476	381656	381837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
5	9166	9343	9520	9698	9875	390051	390228	390405	390582	390759	177
6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	398114	398287	398461	398634	398808	398981	399154	399328	399501	173
1	9674	9847	400020	400192	400365	400538	400711	400883	401056	401228	173
2	401401	401573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	410102	410271	410440	410609	410777	410946	411114	411283	411451	169
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	415140	415307	415474	415641	415808	415974	416141	416308	416474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	420121	420286	420451	420616	420781	420945	421110	421275	421439	165
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9	9752	9914	430075	430236	430398	430559	430720	430881	431042	431203	161
270	431364	431525	431685	431846	432007	432167	432328	432488	432649	432809	161
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
5	9333	9491	9648	9806	9964	440122	440279	440437	440594	440752	158
6	440909	441066	441224	441381	441538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
N	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
280	447158	447313	447468	447623	447778	447933	448088	448242	448397	448552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	450095	154
2	450249	450403	450557	450711	450865	451018	451172	451326	451479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	460146	460296	460447	460597	460748	151
9	460898	461048	461198	461348	461499	1649	1799	1948	2098	2248	150
290	462398	462548	462697	462847	462997	463146	463296	463445	463594	463744	150
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
5	9822	9969	470116	470263	470410	470557	470704	470851	470998	471145	147
6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	477121	477266	477411	477555	477700	477844	477989	478133	478278	478422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
2	480007	480151	480294	480438	480582	480725	480869	481012	481156	481299	144
3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	490099	490239	490380	490520	490661	490801	490941	491081	491222	140
310	491362	491502	491642	491782	491922	492062	492201	492341	492481	492621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
5	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	500099	500236	500374	500511	500648	500785	500922	137
7	501059	501196	501333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	505150	505286	505421	505557	505693	505828	505964	506099	506234	506370	136
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
3	9203	9337	9471	9606	9740	9874	510009	510143	510277	510411	134
4	510545	510679	510813	510947	511081	511215	1349	1482	1616	1750	134
5	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	518514	518646	518777	518909	519040	519171	519303	519434	519566	519697	131
1	9828	9959	520090	520221	520353	520484	520615	520745	520876	521007	131
2	521138	521269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
5	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	530072	128
9	530200	530328	530456	530584	530712	530840	530968	531096	531223	1351	128
N.	0	1	2	3	4	5	6	7	8	9	D.

N	0	1	2	3	4	5	6	7	8	9	D.
340	531479	531607	531734	531862	531990	532117	532245	532372	532500	532627	128
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
5	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
6	9076	9202	9327	9452	9578	9703	9829	9954	540079	540204	125
7	540329	540455	540580	540705	540830	540955	541080	541205	1330	1454	125
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	544068	544192	544316	544440	544564	544688	544812	544936	545060	545183	124
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	550106	123
5	550228	550351	550473	550595	550717	550840	550962	551084	551206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	556303	556423	556544	556664	556785	556905	557026	557146	557267	557387	120
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
3	9907	560026	560146	560265	560385	560504	560624	560743	560863	560982	119
4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	568319	568436	568554	568671	568788	568905	569023	569140	569257	117
1	9374	9491	9608	9725	9842	9959	570076	570193	570309	570426	117
2	570543	570660	570776	570893	571010	571126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	579784	579898	580012	580126	580241	580355	580469	580583	580697	580811	114
1	580925	581039	1153	1267	1381	1495	1608	1722	1836	1950	114
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
5	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
9	9950	590061	590173	590284	590396	590507	590619	590730	590842	590953	112
390	591065	591176	591287	591399	591510	591621	591732	591843	591955	592066	111
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
5	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
8	9883	9992	600101	600210	600319	600428	600537	600646	600755	600864	109
9	600973	601082	1191	1299	1408	1517	1625	1734	1843	1951	109
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
400	602060	602169	602277	602386	602494	602603	602711	602819	602928	603036	108
1	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
7	9594	9701	9808	9914	610021	610128	610234	610341	610447	610554	107
8	610660	610767	610873	610979	1086	1192	1298	1405	1511	1617	106
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	612784	612890	612996	613102	613207	613313	613419	613525	613630	613736	106
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
5	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
7	620136	620240	620344	620448	620552	620656	620760	620864	620968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	623249	623353	623456	623559	623663	623766	623869	623973	624076	624179	103
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326	102
7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	633468	633569	633670	633771	633872	633973	634074	634175	634276	634376	101
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	101
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
5	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	100
6	9486	9586	9686	9785	9885	9984	640084	640183	640283	640382	99
7	640481	640581	640680	640779	640879	640978	1077	1177	1276	1375	99
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	643453	643551	643650	643749	643847	643946	644044	644143	644242	644340	98
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
6	9335	9432	9530	9627	9724	9821	9919	650016	650113	650210	97
7	650308	650405	650502	650599	650696	650793	650890	0987	1084	1181	97
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	653213	653309	653405	653502	653598	653695	653791	653888	653984	654080	96
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
7	9916	660011	660106	660201	660296	660391	660486	660581	660676	660771	95
8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
460	662758	662852	662947	663041	663135	663230	663324	663418	663512	663607	94
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
7	9317	9410	9503	9596	9689	9782	9875	9967	670060	670153	93
8	670246	670339	670431	670524	670617	670710	670802	670895	0988	1080	93
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	672098	672190	672283	672375	672467	672560	672652	672744	672836	672929	92
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245	91
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151	91
480	681241	681332	681422	681513	681603	681693	681784	681874	681964	682055	90
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107	89
490	690196	690285	690373	690462	690550	690639	690728	690816	690905	690993	89
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
500	698970	699057	699144	699231	699317	699404	699491	699578	699664	699751	87
1	9838	9924	700011	700098	700184	700271	700358	700444	700531	700617	87
2	700704	700790	0877	0963	1050	1136	1222	1309	1395	1482	86
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	707570	707655	707740	707826	707911	707996	708081	708166	708251	708336	85
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	710033	85
3	710117	710202	710287	710371	710456	710540	710625	710710	710794	0879	85
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
520	716003	716087	716170	716254	716337	716421	716504	716588	716671	716754	83
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	724276	724358	724440	724522	724604	724685	724767	724849	724931	725013	82
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	732394	732474	732555	732635	732715	732796	732876	732956	733037	733117	80
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
550	740363	740442	740521	740600	740678	740757	740836	740915	740994	741073	79
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	748266	748343	748421	748498	748576	748653	748731	748808	748885	77
1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	755875	755951	756027	756103	756180	756256	756332	756408	756484	756560	76
1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
5	9668	9743	9819	9894	9970	760045	760121	760196	760272	760347	75
6	760422	760498	760573	760649	760724	0799	0875	0950	1025	1101	75
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
580	763428	763503	763578	763653	763727	763802	763877	763952	764027	764101	75
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
590	770852	770926	770999	771073	771146	771220	771293	771367	771440	771514	74
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970	71
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	792392	792462	792532	792602	792672	792742	792812	792882	792952	793022	70
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	799409	799478	799547	799616	799685	799754	799823	799892	799961	69
1	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
640	806180	806248	806316	806384	806451	806519	806587	806655	806723	806790	68
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	0425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	819544	819610	819676	819741	819807	819873	819939	820004	820070	820136	66
1	820201	820267	820333	820399	820464	820530	820595	0661	0727	0792	66
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	826140	826204	826269	826334	826399	826464	826528	826593	826658	65
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
6	9947	830011	830075	830139	830204	830268	830332	830396	830460	830525	64
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	832509	832573	832637	832700	832764	832828	832892	832956	833020	833083	64
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	838849	838912	838975	839038	839101	839164	839227	839289	839352	839415	63
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	840043	63
2	840106	840169	840232	840294	840357	840420	840482	840545	840608	0671	63
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
N	0	1	2	3	4	5	6	7	8	9	D

N.	0	1	2	3	4	5	6	7	8	9	D.
700	845098	845160	845222	845284	845346	845408	845470	845532	845594	845656	62
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	851258	851320	851381	851442	851503	851564	851625	851686	851747	851809	61
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	857332	857393	857453	857513	857574	857634	857694	857755	857815	857875	60
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
4	9739	9799	9859	9918	9978	860038	860098	860158	860218	860278	60
5	860338	860398	860458	860518	860578	0637	0697	0757	0817	0877	60
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	863323	863382	863442	863501	863561	863620	863680	863739	863799	863858	59
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	869232	869290	869349	869408	869466	869525	869584	869642	869701	869760	59
1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930	58
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	875061	875119	875177	875235	875293	875351	875409	875466	875524	875582	58
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
9	880242	880299	880356	880413	880471	880528	0585	0642	0699	0756	57
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
760	880814	880871	880928	880985	881042	881099	881156	881213	881271	881328	57
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	886491	886547	886604	886660	886716	886773	886829	886885	886942	886998	56
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	892150	892206	892262	892317	892373	892429	892484	892540	892595	56
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	897627	897682	897737	897792	897847	897902	897959	898012	898067	898122	55
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
4	9821	9875	9930	9985	900039	900094	900149	900203	900258	900312	55
5	900367	900422	900476	900531	0586	0640	0695	0749	0804	0859	55
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	903090	903144	903199	903253	903307	903361	903416	903470	903524	903578	54
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	908539	908592	908646	908699	908753	908807	908860	908914	908967	54
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037	53
3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
820	913814	913867	913920	913973	914026	914079	914132	914184	914237	914290	53
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	919130	919183	919235	919287	919340	919392	919444	919496	919549	52
1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	924331	924383	924434	924486	924538	924589	924641	924693	924744	52
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	929470	929521	929572	929623	929674	929725	929776	929827	929879	51
1	9930	9981	930032	930083	930134	930185	930236	930287	930338	930389	51
2	930440	930491	0542	0592	0643	0694	0745	0796	0847	0898	51
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	934549	934599	934650	934700	934751	934801	934852	934902	934953	50
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	969569	939619	939669	939719	939769	939819	939869	939918	939968	50
1	940018	940068	940118	940168	940218	940267	940317	940367	940417	940467	50
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
8	3445	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927	49
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	949439	949488	949536	949585	949634	949683	949731	949780	949829	49
1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316	49
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	954291	954339	954387	954435	954484	954532	954580	954628	954677	48
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	959089	959137	959185	959232	959280	959328	959375	959423	959471	48
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	963835	963882	963929	963977	964024	964071	964118	964165	964212	47
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903	47
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300	47
4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765	46
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543	46
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.