UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
BOOKSTACKS

Digitized by the Internet Archive
in 2012 with funding from
University of Illinois Urbana-Champaign

http://www.archive.org/details/utilityfunctions305roth

Faculty Working Papers

UTILITY FUNCTIONS FOR SIMPLE GAMES

Alvin E. Roth

#305

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

March 24, 1976

UTILITY FUNCTIONS FOR SIMPLE GAMES

Alvin E. Roth

#305

Utility Functions for Simple Games

By

Alvin E. Roth

The author wishes to acknowledge thoughtful comments given by D. Brown, following a colloquium at Yale University, and by P. Dubey after examining an outline of the results presented here.

1. Introduction

A simple game with transferable utility on a set N of players is one in which the characteristic function v takes on only the values 0 and 1, and in which v(S) = 1 implies v(T) = 1 for all $S \subset T \subset N$. Such games arise naturally as models of political or economic situations in which every coalition of players is either 'winning' or 'losing'.

In this paper we will investigate indices which reflect the relative power of each position (or player) in a simple game. We will show that both the Shapley-Shubik index and the Banzhaf index correspond to von Neumann-Morgenstern utility functions, which differ only in their posture towards risk. Chiefly, we will be applying the techniques developed in [9] to the results presented in [5] and [6].

2. Historical Background

A game on a finite universal set N of positions may be considered to be any function $v : 2^N \to R$ such that $v(\Phi) = 0$. In [10], the value of a game v is defined to be a vector valued function $\phi(v) = \{\phi_1(v), \ldots, \phi_n(v)\}$ which associates a real number $\phi_i(v)$ with each position $i \in N$, and which obeys the following conditions.

2.1) For each permutation /1 $\pi : N \to N$, $\phi_{\pi i}(\pi v) = \phi_i(v)$.

2.2) For each carrier /2 T of v, $\sum_{i \in T} \phi_i(v) = v(T)$.

2.3) For any games v and w, $\phi(v + w) = \phi(v) + \phi(w)$.

Shapley showed that the unique value defined on all games has the form

$$\phi_i(v) = \sum_{S \subset N} \left(\frac{(s-1)!(n-s)!}{n!}\right) \{v(S) - v(S - i)\}$$

where s and n denote the cardinality of the sets S and N.

In [11] this value is studied in the context of simple games. Observe that if v is a simple game, then the quantity $[v(S) - v(S - i)]$ equals 0 unless S is a winning coalition and (S - i) a losing coalition, in which case it equals 1. Consequently, if we suppose that players in a simple game v 'vote' in random order, then $\phi_i(v)$ is precisely the probability that player i will cast a 'pivotal' vote. As such, it can be viewed as an a priori index of power in simple games, and is referred to as the Shapley-Shubik index.

However if only simple games are to be considered, the conditions (2.1), (2.2), and (2.3) no longer specify a unique functional form. This is because condition (2.3) becomes non-binding, since the class of simple games is not closed under addition. (So if v and w are non-trivial simple games, $v(N) = w(N) = 1$, and the game $v + w$ is not simple, since $v(N) + w(N) = 2$.) Another value for simple games which has received attention in the literature is the Banzhaf index, first introduced in [1].

The Banzhaf index takes as a measure of power the relative ability of players to transform winning coalitions into losing coalitions, and vice versa. Define a swing for position $i \in N$ to be a pair (S, S - i) such that the coalition S is winning, and S - i is losing (i.e. $v(S) = 1$ and $v(S - i) = 0$). Let $\eta_i(v)$ denote the number of swings for position i in game v, and let $T(v) = \sum_{i \in N} \eta_i(v)$. Then the Banzhaf index of relative power for each position is

$$\beta_i(v) = \eta_i(v)/T(v) \qquad \text{for} \quad i = 1,\ldots,n.$$

We will refer to $\eta_i(v)$ as the non-normalized Banzhaf index.

The Shapley-Shubik and the (normalized or non-normalized) Banzhaf indices yield different rankings of the positions in a given simple game.

Consequently it is desirable to find a common interpretation of the indices which will permit us to investigate their differences and similarities. This task is facilitated by the following two propositions, which are presented in [5] and [6].

Proposition 1 (Dubey): The Shapley-Shubik index is the unique function ϕ defined on simple games which satisfies conditions (2.1) and (2.2) and which has the property that

(2.4) For any simple games v, w $\quad \phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w)$

where the games $(v \vee w)$ and $(v \wedge w)$ are defined by

$$(v \vee w)(S) = \begin{cases} 1 & \text{if} \quad v(S) = 1 \quad \text{or} \quad w(S) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(v \wedge w)(S) = \begin{cases} 1 & \text{if} \quad v(S) = 1 \quad \text{and} \quad w(S) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2 (Dubey): The non-normalized Banzhaf index is the unique fun[illegible] defined on simple games which satisfies the following four conditions.

(2.5) If $i \in N$ is a dummy /3 in v, then $\eta_i(v) = 0$

(2.6) $\sum_{i \in N} \eta_i(v) = T(v)$

(2.7) For each permutation $\pi : N \to N$, $\eta_{\pi i}(\pi v) = \eta_i(v)$

(2.8) For any simple games, v, w $\quad \phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w)$.

In the next section, we shall use these propositions to show that ϕ, η, and β can be viewed as cardinal utility functions which differ only in their posture towards risk. It will be seen that conditions (2.2) or (2.6) express a posture towards one kind of risk, while the somewhat opaque conditions (2.4) or (2.8) express a posture towards another kind of risk.

3. Utility Functions for Simple Games

Let C be the class of simple games defined on a finite set N, and let M be the mixture space generated by $C \times N$. Then the elements of M are elements (w, i) of $C \times N$, and lotteries of the form $[p(w, i); (1 - p)(v, j)]$ where (w, i) and (v, j) are in $C \times N$, and p is a probability (i.e. $p \in [0, 1]$).[4]

Let P be a (strict) preference relation defined on M.[5] (Read $(w, i)P(v, j)$ as 'it is preferable to play position i in game w than to play position j in game v'). We will take P to be continuous on M; i.e. if a, b, $c \in M$ such that $aPbPc$, then there exists a (unique) $q \in (0, 1)$ such that $bI[qa; (1 - q)c]$.

Denote by v_R and v_0 the games defined by

$$v_R(S) = \begin{cases} 1 & \text{if} \quad R \subset S \\ 0 & \text{otherwise.} \end{cases} \quad ; \quad v_0(S) = 0 \quad \text{for all} \quad S \subset N$$

For each $i \in N$ denote by $D_i \subset C$ the set of simple games for which player i is a dummy. We will take P to have the following properties.

(3.1) For all $v \in C$, $i \in N$ and every permutation $\pi : N \to N$,

$(v, i)I(\pi v, \pi i)$

(3.2) For every $i \in N$, $v \in D_i$ implies $(v, i)I(v_0, i)$ and $(v_i, i)P(v_0, i)$.

for every $v \in C$, $(v_i, i)R(v, i)R(v_0, i)$.

It is well known (cf. [7]) that such a preference can be represented by a cardinal utility function θ; i.e. a function θ such that for all a, $b \in M$

$$\theta(a) > \theta(b) \quad \text{iff} \quad aPb,$$

and

$$\theta([pa; (1 - p)b]) = p\theta(a) + (1 - p)\theta(b).$$

Furthermore, θ is unique up to an affine transformation, so we can set $\theta(v_i, i) = 1$ and $\theta(v_0, i) = 0$. For an arbitrary element (v, i) of C × N we have

$$\theta_i(v) \equiv \theta(v, i) = q,$$

where q is the number such that

$$(v, i)I[q(v_i, i); (1 - q)(v_0, i)].$$

By the continuity of P and condition (3.2), $\theta_i(v)$ is well defined.

We have yet to completely specify the preference P. We do so by expressing the preferences involving two kinds of risk.

(3.3) Ordinary risk neutrality: for all simple games v, w

$$[\tfrac{1}{2} (v, i); \tfrac{1}{2} (w, i)]I[\tfrac{1}{2} (v \vee w); \tfrac{1}{2} (v \wedge w)]$$

(3.4) Strategic risk neutrality: for all $R \subset N$ and $i \in R$,

$$(v_R, i)I[\tfrac{1}{r} (v_i, i); (1 - \tfrac{1}{r})(v_0, i)].$$

Condition (3.3) specifies indifference between two lotteries: note that the condition is plausible, since $(v \vee w) \geq v$ and $(v \wedge w) \leq w$. Condition (3.4) specifies indifference between playing the game v_R as one of r players in the unique minimal winning coalition, or participating in a lottery which gives probability $\frac{1}{r}$ of being a dictater and probability $(1 - \frac{1}{r})$ of being a dummy. Note that the risk involved in playing the game v_R is strategic rather than probabilistic—no gamble is involved.

We can now state the following theorem.

Theorem 1: If P is a preference obeying conditions (3.1) through (3.4), then

the unique utility θ such that $\theta_i(v_i) = 1$ and $\theta_i(v_0) = 0$ is equal to the Shapley-Shubik index.

Lemma 1: If P obeys condition (3.1) then, for every $v \in C$, $i \in N$ and permutation $\pi : N \to N$,

$$\theta_i(v) = \theta_{\pi i}(\pi v).$$

Proof: Immediate from (3.1) and the definition of utility.

Lemma 2: If P obeys conditions (3.1), (3.2) and (3.4) then for each $R \subseteq N$,

$$\theta_i(v_R) = \begin{cases} \frac{1}{r} & \text{if } i \in R \\ 0 & \text{if } i \notin R. \end{cases}$$

Proof: If $i \notin R$ then $(v_R, i) I (v_0, i)$ by (3.2), and so $\theta_i(v_R) = \theta_i(v_0) = 0$. If $i \in R$, then $\theta_i(v_R) = \frac{1}{r}$ by (3.4) and the definition of the utility θ.

Lemma 3: If P obeys condition (3.3), then

$$\theta_i(v \vee w) + \theta_i(v \wedge w) = \theta_i(v) + \theta_i(w).$$

Proof: From the definition of utility,

$$\theta[\tfrac{1}{2}\left((v \vee w), i\right); \tfrac{1}{2}\left((v \wedge w), i\right)]$$

$$= \frac{1}{2}\,\theta\left((v \vee w), i\right) + \frac{1}{2}\,\theta\left((v \wedge w), i\right)$$

and

$$\theta[\tfrac{1}{2}(v, i); \tfrac{1}{2}(w, i)] = \frac{1}{2}\,\theta(v, i) + \frac{1}{2}\,\theta(w, i).$$

Consequently, by condition (3.3) we have

$$\frac{1}{2}\,\theta(v, i) + \frac{1}{2}\,\theta(w, i) = \frac{1}{2}\,\theta\left((v \vee w), i\right) + \frac{1}{2}\,\theta\left((v \wedge w), i\right)$$

So far we have demonstrated that θ obeys conditions (2.1) and (2.4), and that for every $R \subseteq N$, $\theta(v_R) = \phi(v_R)$; i.e. θ coincides with the Shapley-Shubik index on the games v_R. (Note that conditions (2.1) and (2.2) determine the value of $\phi(v_R)$.) To complete the proof of the theorem, we show that θ coincides with ϕ on every game $v \in C$.

<u>Proof of Theorem 1:</u> Let $v \in C$, and let $R_1, R_2, \ldots, R_k \subset N$ be all the distinct minimal winning coalitions[6] of v. Then we say the game v is in <u>class k</u>, and note that

$$v = v_{R_1} \vee v_{R_2} \vee \ldots \vee v_{R_k}.$$

If v is in class k = 0, then $v = v_0$ and $\theta(v) = \phi(v) = 0$. If v is in class k = 1, then $v = v_R$, and $\theta(v)$ is defined by Lemma 2, and is equal to $\phi(v)$.

Suppose that for games v in classes $k = 1,2,\ldots,m$ it has been shown that θ is well defined and coincides with the Shapley-Shubik index. Consider a game v in class m + 1. Then

$$v = v_{R_1} \vee v_{R_2} \vee \ldots \vee v_{R_m} \vee v_R = w \vee v_R,$$

where w is a game in class m.

So, by Lemma 3,

$$\theta_i(v) = \theta_i(w \vee v_R) = \theta_i(w) + \theta_i(v_R) - \theta_i(w \wedge v_R).$$

But we will show that the game $(w \wedge v_R)$ cannot be in a higher class than w, so by the inductive hypothesis the terms on the right hand side of the above expression are uniquely determined and equal to the Shapley-Shubik index. Consequently (from property (2.4) of ϕ) we will have shown that $\theta(v) = \phi(v)$

for all simple games v.

To see that the game $w' = (v \wedge v_R)$ cannot be in a higher class than the game w, consider a minimal winning coalition S of the game w'. By the definition of w' we know that $S \supset R$ and $w(S) = 1$. If $S = R$, then $w' = v_R$ and we are done (since except for the game v_0, every game has at least one minimal winning coalition). Otherwise $S = S' \cup R$ where S' is non-empty and disjoint from R. Then there exists an $S_1 \subset N$ such that $S_1 \supset S'$, and S_1 is minimal winning in the game w.

Consider now a coalition $T \neq S$ such that T is also minimal winning in w' and $T = T' \cup R$ where T' is non-empty and disjoint from R. Then there is a coalition $T_1 \supset T'$ which is minimal winning in w.

But any coalition which contains both T_1 and S_1 cannot be minimal winning in w, since it is not contained in any minimal winning coalition of w'. Consequently every minimal winning coalition in w' can be identified with a distinct minimal winning coalition in w, so w' cannot be in a higher class than w. This completes the proof.

So t.e Shapley-Shubik index s the utility func cion representing preferences described by conditions (3.1) through (3.4). Naturally, different preferences will give rise to different utility functions. Suppose, for instance, that the posture towards strategic risk is represented not by condition (3.4) but by the following condition for every $R \subset N$ and $i \in R$.

(3.5) $$\left[\frac{1}{T(v_R)} (v_R, i); \left(1 - \frac{1}{T(v_R)}\right)(v_0, i)\right] I \left[\frac{1}{r} (v_i, i); \left(1 - \frac{1}{r}\right)(v_0, i)\right]$$

Then the following theorem says that the non-normalized Banzhaf index is a cardinal utility for the preference relation P.

Theorem 2: If P is a preference obeying conditions (3.1), (3.2), (3.3) and

(3.5), then the unique utility θ such that $\theta_i(v_i) = 2^{n-1}$ and $\theta_i(v_0) = 0$ is equal to the non-normalized Banzhaf index.

The proof is precisely like the proof of Theorem 1, once it has been observed that condition (3.5) implies that

$$\theta_i(v_R) = \eta_i(v_R) = \begin{cases} \frac{T(v_R)}{r} & \text{for} \quad i \in R \\ 0 & \text{for} \quad i \notin R. \end{cases}$$

So the non-normalized Banzhaf index and the Shapley-Shubik index reflect preferences which differ only in their posture towards strategic risk. Similarly it is not difficult to show that the ordinary (normalized) Banzhaf index corresponds to preferences which obey condition (3.4) but not condition (3.3). That is, the Banzhaf index reflects preferences which are neutral to strategic risk, but not to ordinary risk. The normalization has the effect of changing the risk posture, since each game is normalized independently (i.e. each game v is normalized by $T(v)$.)

4. Discussion

We have seen that the difference between the Shapley-Shubik index and the non-normalized Banzhaf index results from different postures towards strategic risk. That is, the two indices reflect different attitudes towards the relative benefits of engaging in strategic interaction with other players in games of the form v_R.

The difference between the Shapley-Shubik index and the ordinary Banzhaf index, on the other hand, reflects different postures towards ordinary risk—the kind which results from lotteries, rather than from strategic interactions. Thus the difference between these two indices seems to be

essentially non-game-theoretic (cf. [3] pp 195-196)

It is natural to consider a whole spectrum of risk postures, and to examine the resulting utility functions. It seems likely that this point of view will serve to illuminate some of the recent work (cf. [2], [4], [8]) on alternative formulations of the value concept.

Footnotes

1) A permutation $\pi : N \to N$ is one-to-one and onto. The game πv is defined by $\pi v(\pi S) = v(S)$.

2) A carrier of a game v is any coalition $T \subseteq N$ such that for all $S \subseteq N$, $v(S) = v(S \cap T)$. The smallest carrier in a game may be viewed as the set of active players in the game.

3) A dummy in a game v is an $i \in N$ such that $\eta_i(v) = 0$.

4) We assume the usual properties of mixture spaces: i.e. for all elements $a, b \in M$ and all probabilities p and q, we have

$$[1a;\ (1 - 1)b] = a;$$

$$[pa;\ (1 - p)b] = [(1 - p)b;\ pa];$$

and
$$\left[q[pa;\ (1 - p)b];\ (1 - q)b\right] = [qpa;\ (1 - qp)b].$$

5) For $a, b \in M$ we write aIb (a is indifferent to b) if neither aPb nor bPa. We write aRb if either aPb or aIb, and assume that R is a transitive, complete order on M. Furthermore, we assume that if aIb then for every $c \in M$ and $p \in [0, 1]$

$$[pa;\ (1 - p)c]I[pb;\ (1 - p)c].$$

6) A coalition $R \subset N$ is minimal winning in v if $v(R) = 1$ and if $S \subset R$, $S \neq R$ implies $v(S) = 0$.

Bibliography

1. Banzhaf, J. F. III, "Weighted Voting Doesn't Work: A Mathematical Analysis," Rutgers Law Review, Vol. 19 (1965), 317-343.

2. Blair, D. H., "Power Valuations of Finite Voting Games," presented at Econometric Society Winter Meeting, Dallas, Texas, 1975.

3. Brams, S., Game Theory and Politics (New York: Free Press, 1975).

4. Brown, D. and Dubey, P., private communication (Cowles Foundation technical report in progress, 1976).

5. Dubey, P., "Some Results on Values of Finite and Infinite Games," (Ph.D. thesis, Cornell University, 1975).

6. Dubey, P., "On the Uniqueness of the Shapley Value," International Journal of Game Theory, Vol. 4, No. 3 (1975), 131-139.

7. Herstein, I. N. and Milnor, J., "An Axiomatic Approach to Measurable Utility," Econometrica, Vol. 21 (1953), 291-297.

8. Meyerson, R., "A Theory of Cooperative Games," (Ph.D. dissertation in progress, Harvard University).

9. Roth, A. E., "The Shapley Value as a von Neumann-Morgenstern Utility," Faculty Working Paper #297, College of Commerce and Business Administration, University of Illinois, Urbana.

10. Shapley, L. S., "A Value for n-Person Games," Annals of Mathematics Studies, Vol. 28 (1953), Princeton, New Jersey, 307-318.

11. Shapley, L. S. and Shubik, M., "A Method for Evaluating the Distribution of Power in a Committee System," American Political Review, Vol. 48 (1954), 787-792.

ND BY

UNIVERSITY OF ILLINOIS-URBANA
3 0112 060296636