## Faculty Working Papers

VALUES OF INFORMATION AND LIQUIDITY PREFERENCE: A COMMENTARY NOTE<br>Takeshi Murota

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# VALUES OF INFORMATION AND LIQUIDITY PREFERENCE: 

A COMMENTARY NOTE
by
Takeshi Murota

1. Introduction

In his recent article [2] Professor Arrow analyzed the Bayesian problems of decision making under uncertainty by casting them into an information thooretic framowork and proposed the concepts of the value of and demand for information. As a possible direction of extending his far-reaching ideas this note is intended to develop the following three aspects of importance in his article.

At first, we show that his definition of the value of information contains one logical slip, more precisely, a still remaining confusion of comparing the utility of income with the cost, the very same point that he keenly criticized in reference to other authors' preceding contributions in economic and statistical studios of information. In order to improve his result, we redefine the concept of the value of information in such a way that we can revive the essence of J. Marschak's proposal [10] of operationally referring the value of information as a demand price. We also obtain its precise formulation in the Arrow's special context of logarithmic utility function.

Secondly, we present a concept of the value of information in the supply sense to clarify the dual nature of the values of information viewed from its buyer's and seller's standpoints. In this regard the information

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is assigned the attribute of an object for interpersonal which flows in and out among individuais in a given economy under uncertainty.

Thirdly, we analyze the problem of a characterization of liquidity pref erence as behavior towards imperfect information, which seems to be intrinsi in his models of risk-bearing $[1,2]$. This problem was once quantified by Marschak [9] as early as 1949 and raken up again by Radner [12] and Hirshleifer [6] चather recently, while its thorough investigation is not availabl yet in the current economic literature. In the context of portfolio selection theory of Markowitz [8] and Tobin [14] one is supposed to reveal his preference about a given variety of risky assets in terms of his mean-variance utility, which is a derivative from his utility function of income and probability distributions of stochastic returns of assets. ${ }^{1}$ But in order fo our analysis of liquidity to be consistent with the conventional framework of finite-state general equilibrium models under uncertainty, we do not follow this mean-variable approach. Starting directly from an individual's utility function of income in Arrow's model, we atteropt to illustrate his be havior pattern tovards his imperfect knowledge on an uncertain nature from the angle of his optimai liquidity holding.

Though very primitive our results are in this note, they might serve one to initiate a construction of more general models which may capture important problems in the economics of uncertainty that have been outside of the scope of traditional literatures.

## 2. Basic Model

Let us sumarize Arrow's model [2] in the following manner. A decision maker's uncertain economic environment is assumed to be completely described

1 Mathematical structure of this iransformation from one utility to anotirer was elaborately investigated by Richter [13] and Chipman [4].


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by a finite probaoility space $(\Omega, P)$ and a random variabic $X: \Omega \rightarrow R$ on this space, where
(1) $\Omega=\{1, \ldots, S\}$ : an index set of finite $S$ possible states of nature.
(2) $P=\left\{\left(\mathrm{p}, \ldots, \mathrm{p}_{\mathrm{S}}\right) ; \Sigma \mathrm{p}_{\mathrm{i}}=1, \mathrm{P}: \geqq 0 ; \mathrm{i} \varepsilon \mathrm{S}_{2}\right\}$ : a decision maker's prior (objectively known or subjective) probability distribution on the occurrence of each state in $\Omega$, Where $p_{i}$ is the probability that state $i$ occurs.
(3) $X=\left\{X_{1}, \ldots, X_{S}\right\}$ : a given structure of monetary $r$, turns from each one dollar bet on the occurrence of each state in $\Omega$. This amounts to saying that the decision maker who bets one dollar on state $i$ receives $X_{i}$ dollars and nothing otherwise.

The decision maker is characterized by his initially held monetary resource which is normarized to the value 1 and his von Neuman-Morgenstern utility function of income:
(4) $U: R^{+} \rightarrow R$, where $U(y)$ is assumed to be monotone increasing, differentiable and of diminishing marginal utility in income $y$.

His action in this economy is confired to choosing an (S + 1) dimensional decision vector which is restricted to the feasible set $A$ of decisions defined as
(5) $A=\left\{\vec{a}=\left(a_{2}, \ldots, \vec{a}_{S}\right) ; a_{i}+b=1, a_{i} \geqslant 0\right.$ for $a l l$ i $\left.\varepsilon \Omega, b \geqq 0\right\}$, where $i_{i}$ is the amount bet by him on the occurrence of state $i$ and $b$ is the amount retained uninvested in a liquid form out of his initial response.

1 Each $X_{i}$ can be considered as a reciprocal of unit price of each i-th security. provided that Arrow regards this model as a further development of his now classic paper [1].

The decision maker will then face the problem:
(6) Given $(\Omega,[), X, \|$ and $A$, maximize, with respect'to a decision vector a $\varepsilon A$, the expected value of utility;

$$
\left\{E_{i p_{i}}\right\}=\sum_{i} \sum_{i} U\left(a_{i} x_{i}+b\right) .
$$

The situation which Arcon is concerned with is as follows. Suppose tnat the decision maker is presented an opportunity of acquiring a certain estimate on a true state of nature in the form of a message in a finite set given as
(7) $\Omega^{*}=\left\{1^{*}, \ldots, S^{*}\right\}:$ an index set of $S$ possible messages, where message
j* implies the estinate, "State $j$ will prevail." Since therc seems to be no danger of confusion, we shail use the rotation $j \equiv j *$ as long as $j \varepsilon \Omega^{*}$.

More anslytically speaking, the acouisition of an estimation becomes possible Chrough a discrete commuication channel describable by means of $S \times S$ matrix (3 no.. ${ }^{-h}$ is defined as a chanmel metrix ${ }^{1}$
(8) $Q=\left\|q_{j \ldots}\right\|: \sum_{j} Q i=1$ and $q_{j i} \geq 0$ for $a l I$ i $\varepsilon \Omega$ and $j \varepsilon \Omega^{*}$,
where the $i$-th Low and $j-t h$ colum entry $q_{j i}$ of this matrix $Q$ signiffes the conditional probabinisy that message $j \varepsilon \Omega^{*}$ is sent from the channel while a true state is i.

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The unconditional and conditional probability distributions $\left\{p_{i}\right\}$ in (2) and $\left\{q_{j i}\right\}$ in (8) will then define the unconditional message probability distribution $\left\{q_{j}\right\}$ and the conditional pronability distribution $\left\{p_{i j}\right\}$ as

$$
\begin{equation*}
q_{j}=\sum_{i} p_{i} q_{j i} ; \quad \sum_{j} q_{j}=\lambda, q_{j} \geqq 0 \text { for all } j \varepsilon \Omega^{*} \tag{9}
\end{equation*}
$$

$$
\begin{equation*}
p_{i j}=p_{i} q_{j j} / q_{j} ; \sum_{i} p_{j j}=1, p_{i j} \geqslant 0 \text { for all i } \varepsilon \Omega \text { and } j \varepsilon \Omega^{*}, \tag{10}
\end{equation*}
$$

where $q_{j}$ is the probability that ressage $j$ is sent from the channel and $p_{i j}$ is the conditional probability that the true state is $i$ when message $j$ is sent. Given an information service in the form of a channel $Q=\left\|q_{j i}\right\|$ thus characterized, the decision maker previously ignorant on a true state up to his prior probability distribution $P$ can now take advantage of the con ditional probability distribution $\left\{p_{i j}\right\}$ to fom $(S+1)$ dimensional $S$ decision schedule vectors $\bar{a}(j)$ 's conditioned by each received message $j \varepsilon \Omega *$, within the restriction of his Eeasible sets $A_{j}{ }^{\text {'s }}$ of decision schedules defined as
(11) $A_{j}=\left\{\bar{a}_{j}=\left(a_{i}(j), \ldots, a_{S}(j), b(j)\right)\right.$;
$\sum_{i} a_{i}(j)+b(j)=1, a_{i}(j) \geqq 0, b(j) \geqq 0$ for $a l i$ i $\left.\varepsilon \Omega\right\}$, for all je $\Omega^{* *}$.

The components $\dot{a}_{i}(j)$ and $b(j)$ in a vector $\bar{a}_{j}$ signify the scheduled amount of money for bet on the occurrence of state it and amount of money retained in a. liquid form as functions of a transmitted message $j$. The decision maker's problem ( 6 ) then assumes a new form:
(12) Given $(\Omega, P), X, U, Q_{y}$ and $A_{j} ' S$, maximize, with respect to $S$ decision schedule vectoxs $\bar{a} E A_{j} ;$ all $j \in \Omega^{\star}$, the expected value of the conditional utility

$$
\left.\begin{array}{cl}
E & \{E U\} \\
\left\{q_{j}\right\} & \left\{p_{i j}\right\}
\end{array}\right\} \sum_{j} q_{j} \sum_{i} p_{i j} U\left[a_{i}(j) x_{i}+b(j)\right] .
$$

Our interest is then the evaluation of potential advantage of decision scheme (12) over the scheme (6), relative of a given deditional information through the channei (8). We are also concemed with the problem of how the level of optimal liquidity changes as the configuration of information changes. In this regard we need to introduce

DEFINITION I: With respect to the solution vectors $\bar{a}$ and $\bar{a}_{j}$ 's in the problems ( 6 ) and $(12)^{1}$, we define a component $b$ in $\bar{a}$ as the optimal liquidity and $\underset{\left\{q_{j}\right\}}{E}[b(j)] \equiv \underset{j}{i q_{j}} b(j)$ for $-b(j)^{\prime} s$ in $\bar{a}_{j}$ 's as the optimal average liquidity.

## 3. Demand Value of Information

Arrow has inclined to define the arithmetic difference between the maximands (5) and (11)--which has the dimension of utility of income--as the value of information, whicin presumbly has the dimension of money units. This approach is hardly justifiable excopt for the case where a utility function is dinear in income. Following the fully correct approach to this problen by La Valle [7], Hirshleifer [6] and Marschak and Radner [12] we introduce

DEFINITYON II: Given $(\Omega, P), X, U, A, Q, A_{i}$ 's and the payment schere of requiring to pay for information service fron the final outcome of the decision miker, a real number $V$ which satisf,es the equation:

$$
\begin{gather*}
\bar{a}_{j} \varepsilon A_{j} \sum_{j \in \Omega \Omega^{*} \sum_{j} q_{i} \sum_{i j} U\left[a_{i}(j) x_{i}+b(j)-V\right]}^{=\max _{\bar{a} \Leftrightarrow A} \sum_{i} p_{i} U\left(a_{i} x_{i}+b\right)} \tag{13}
\end{gather*}
$$

is defined as the demand value of informarion. with posterior payment.
The operational significance of the value of information thus posed resides in that $i t$ is the least upper bound of the buying price of information

1 It is plain fron an elementary result of convex analysis that the problems (6) and (12) have solutions because of the assumptions we imposed on the utility Eunction 11 in (d).
service, or more roughly, the maximum buying price of information in the sense that a presented information service is worth acquiring if its value $V$ exceeds the cost $C$ of acquiring it. This view revives the essence of Marschak's idea [10] on the reference of the value of information as a demand price, after freeing it from his dimensional problen which Arrow pointed out [2, p. 275]. This Definition II immediately leads us to PROPOSITION I: If a utility function $U$ in (4) is strictly increasing, continuous ${ }^{1}$ and of diminishing marginal utility in income, then the value $V$ of information in accordance with Definition II uniquely exists and it is nonnegative.

Proof: It is given in Mathematical Appendix at the end of this note.
While the investigation in the general properties of the demand value of information remains as an important problem, our immediate concern in this note is the special case of Arrow, i.e., the case where the sure system of bets exists, i.e., the random variable $X$ satisfies the inequality

$$
\begin{equation*}
\underset{i}{\sum\left(1 / X_{i}\right) \leqq 1, ~} \tag{14}
\end{equation*}
$$

and where
(15) $U(y)=\log y:$ the base of logarithm $=$ natural number e. Before proceeding our discussion, we have to note:

REMARK: (Arrow [2, p. 268]). Under the assumptions imposed on $U$ in (4), if $U^{\prime}(0)=+\infty_{s}$ then the decision maker will invest all his money if and only if there exists a sure system of bets expressed by (14).

It is obvious that the utility function (14) satisfies the condition in the above Remark. Confining ourselves to this Arrow's special case, we readily obtain

1 These assumptions are slightly different from the ones given by Arrow which are written out in (4) in Section 2.


PROPOSITION II: Under the assumptions (14) and (15), the value V of information in accordance with Definition II is an exponential function of the amount of information I conveyed by the channel (8) in the sense of Shannon, more precisely,
(16) $V=\left(1-e^{-I}\right) / \sum_{i}\left(1 / X_{i}\right)$,
where
(17)

$$
I \equiv I(Q / p) \equiv-\sum_{i} p_{i} \log p_{i}+\sum_{j} q_{j} \sum p_{i j} \log p_{i j} 0^{1}
$$

Proof: Using the customary Lagrangean method of maximization, we get the optimal solutions for (6) and (12) under assumptions (14) and (15) as

$$
\begin{aligned}
& a_{i}=p_{i} ; \quad b=0 \\
& a_{i}(j)=p_{i j}\left[1-V \sum_{k}\left(1 / X_{k}\right)\right]+V / X_{i} ; b(j)=0
\end{aligned}
$$

fo. ${ }^{\prime \prime}$ l $i \in \Omega$ and $j \in \Omega^{*}$. Evaluating the right and left hand sides of the definitional equation (13) in terms of these solutions, we obtain the equation
which amounts to the equation (16) in question.
q.e.d.

The formula (16) clarifies a rather misleading criticism of Arrow against Marschak. Having observed that the "value of information" in his own arbitrary difinition tums nut to be equal to the amount of information itself, Arrow concluded that if the cost of acquiring information is proportionate to the amount of information then there is no way for a decision maker to determine how much information or what channel he wants since both the value of

1
Readers who are not familiar with elementary concepts in information theory can refer to any textbook in this area, such as Ash [3].




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and cost of information are proportionate to each other [2, p. 275]. But this argument is based on his own dimensional confusion of comparing the utility of income with the cost, i.e., the same kind of comperison which Marschak made [10]. In fact, the value of information properly measured in monetary units in Proposition I is strictly concave in the amount of information so that his indeteminancy problem of the optimal amount of information does not occur at least within the conditions which he assumed.

It should be understood, however, that our criticism of Arrow using his own assumptions does not necessarily mean our full acceptance of all of his assumptions either. The proportional cost of information to its amount seems to be a very narrow assumption and there may be many economic situations in which the buyers of information face the price of information that is not quite proportional to its amount. One of the purposes of the next section is to show one such counterexample by investigating a case where the information service is a privately owned, perishable object for interpersonal exchange and which yields a supply price of information strictly convex in its amount rather than proportional to it.

## 4. Supply Value of Jnfo mation and Other Remarks

Our investigation in the value of information in the demand sense naturally leads us to characterize the similar problem from the supply side. Let us consider a decision maker in an environment similar to the one before but who initially own an information channel with its matrix (7) and who is ready to sell it out to somebody else. Symmetrically as in Definition II we introduce

DEFINITION III: FOr a similar decision maker as in Definition II who is characterized by $(\Omega, P), X, U$, and $A$ and privately owns a perishable informa-
tion service $Q_{\text {, a }}$ a real number $W$ which satisfies the equation

$$
\begin{align*}
\bar{a}_{j} \varepsilon A_{j} ; j & \varepsilon \Omega^{*} \sum_{j}^{\sum q_{j}} \sum_{i}^{\sum p_{i j}} U\left[a_{i}(j) X_{i}+b(j)\right]  \tag{18}\\
& =\max _{a \varepsilon A_{i}}^{\sum p_{i} U\left(a_{i} X_{i}+b+W\right)}
\end{align*}
$$

is defined as the supply value of information with posterior payment.
The value of $f$ of information thus defined ${ }^{2}$ is the greatest lower bound of the selliag price of information service, or roughly, a minimum selling price information in the sense that an information service is worse selling out if the value $W$ falls short of the revenue R. With respect to Arrow's snecial circumstance, we obrain

PROPOSITION III: Under the essumptions (14) and (15), the supply value $W$ of information in accordance with Definition III is given by

$$
\begin{equation*}
\left.W=\left(e^{I}-1\right) / \underset{i}{\sum\left(1 / X_{i}^{1}\right.}\right), \tag{18}
\end{equation*}
$$

where I is the amount of information given in (17).
The proff of this simple result may be omitted. The formula (19) iliustrates that there is no universal ground to support the cost of informa*ion proportional to its amount.

So far we have been assuming the posterior payment for an information unvice witheram frua orded on to the final outcomes of the decision. But if we cusider thase chere the motary paymen for information is taken out of or adeed on to the initial resource, then the sets of decision schedules given by (5) and (21) must be redefined accordingly. Wi.th respect


1 An information service may be said to be perishable if it does not matntain its service for owner's benefits once he sells it away.
${ }^{2}$ The proof of its unique existence and nonnegativity can be easily done similarly to the proof of Proposition I.


$$
\begin{align*}
A \tilde{v}^{(j)=} & \left\{\bar{a}_{j}=\left(a_{1}(j), \ldots, a_{S}(j), b(j)\right) ;\right.  \tag{20}\\
& \left.\sum_{i} a_{i}(j)+b(j)=1-\tilde{V}, a_{i}(j) \geqq 0 \text { for alli } \varepsilon \Omega, b(j) \geqq 0\right\}
\end{align*}
$$

$$
\begin{align*}
A \tilde{W}=\{\bar{a}= & \left(a_{1}, \ldots, a_{S}, b\right) ;  \tag{21}\\
& \left.\sum_{i} a_{i}+b+1+\tilde{W}, a_{i} \geqq 0 \text { for alli} \varepsilon \Omega, b \geqq 0\right\} .
\end{align*}
$$

NOTE: In Definition II, if we restrict decision schedule vectors $\bar{a}_{j}$ 's to the sets $A \tilde{V}(j)$ 's instead of $A_{j}$ 's, a real number $\tilde{V}$ satisfying the equation (13) can be defined as the demand value of information with prior payment. Similarly, in Definition III, with the restriction of a decision vector $\bar{a}$ to $A_{\tilde{W}}$ instead of $A$, real number $\tilde{W}$ satisfying the equation (21) is defined as the supply value of information with prior payment. Under the assumptions (14) and (15), the thus defined values $V$ and $W$ of information are formulated as

$$
\begin{align*}
& \tilde{V}=1-e^{-I}  \tag{22}\\
& \tilde{W}=e^{I}-1 . \tag{23}
\end{align*}
$$

Sumarizing the special results (16), (19), (22), and (23), we conclude this section with the following proposition whose proof may be unnecessary:

PROPOSTTION IV: Given $(\Omega, P), X, u, Q, A, A_{j} ' s, A \tilde{V}(j)$ 's, $A \tilde{W}$ and assumptions (14) and (15), the demand and supply values $V, \tilde{V}, W$, and $\tilde{W}$ of information with posterior and prior payments in accordance with their associated definitions are strictly increasing in the anount of information conveyed by a given channel Q relative to a given prior probability distribution $P$. They are nomnegative and become equal to zero if and only if the amount of information is zero, i.e., a given channel is useless. ${ }^{1}$ Moreover, the demand

1 A channel is said to be useless if $p_{i j}=p_{i}$ for all i $\varepsilon \Omega$ and $j \varepsilon \Omega^{*}$. For details of classification of channels, see, for example, Ash [3].

values of information are strictly concave and supply values strictly convex in the amount of information. The demand values of information with posterior and prior payments become identical functions of the amount of information if $\sum_{i}\left(1 / X_{i}\right)=1$, and the similar fact also holds for the supply values of information.

REMARK: The condition $\sum\left(1 / X_{i}\right)=1$ has a significant implication in the context of Arrow [I] if we regard $1 / X_{i}$ as a unit price of $i-t h$ security in an uncertain pure exchange conomy of $C$ comodities with $S$ possible states. Arron demonstrated that the optimal allocation of ( $S \times C$ ) contingent commodities, which appears to require to operate $(S \times C)$ markets can be achieved by operating only ( $S$ * $C$ ) markets, i.e., $S$ for securities and $C$ for commodities. The above condition exciudes the possibility of arbitrage between securities' and comodities' markets so that this economization of markets becomes meaningful enough.
5. Liquidity Preference as Behavior Towards Imperfect Information

Our discussion in the previous two sections was so dependent on Arrow's special case conditioned by the assumptions (14) and (15), especially by (14), that the problem of liquidity preferonce, which his article [2] rather implicirly points out, did not actually arise in our analysis. But it should be understood that Definition I and the optimization problems (6) and (13) in Section 2 of this note have already given us the necessary framework for the analysis of optimal liquidity. In contrast to the eraditional characterization of liquidity preference in terms of the mean-variance of probability distribution of risky assets, we are interested in the analysis which is directly based on a utility function of income from which the portfolio selection theorists supposedly deduce the associated mean-variance utility function.


Admittedly, the discrete description of uncertain returns from an investment may not be so practical and is quite foreign among their familiar continuous descriptions in the portfolio selection theory, except for a very few cases such 3 Shipman's analysis of the situation of two-point probability distribution $[4, p, 181]$. However, discrete models may be still interesting from a purely theoretical point of view because of their akinness to the general equilibrium models under uncertainty of Arrow-Debrue-Radner type as we mentioned in Section 1.

Generally speaking, we would like to know how a decision makex's optimal amount of liquidity holding changes as his knowledge on the uncertain nature changes due to additional information supplies to him under the condition

$$
\begin{equation*}
\sum_{i}\left(1 / X_{i}\right)>1 \tag{24}
\end{equation*}
$$

and without imposing too many assumptions on the properties of his utility function. But this general approach seems to be analytically very difficult. Therefore, we confine the scope of analysis to the case of logarithmic utilisy function (15) as before.

As an illustration of the nature of the problem which we are concermed with, let us consider the following simple numerical example:

$$
\Omega=\Omega^{*}=\{1,2,3\}
$$

$$
\begin{align*}
& \left(p_{1}, p_{2}, p_{3}\right)=(.05, .10, .85)  \tag{25}\\
& \left(x_{1}, X_{2}, x_{3}\right)=(5,2,2.5)^{1} .
\end{align*}
$$

Case 1: Given these datum and $U\left({ }^{\circ}\right)=\log \left({ }^{\prime}\right)$, the maximization problem (6) in section 2 yields the following corner optimum solution:

1 Note that $\sum_{i=1}\left(1 / X_{i}\right)=11 / 10>1$ so that the condition (24) is met.




 $\qquad$ 4) $1=$


$$
\left(a_{1}, a_{2}, a_{3}, b\right)=(0,0, .75, .25)
$$

where we obtain $b=.25$ as the optimal liquidity according to Definition I.

Case 2: Let us next consider the case where the costless information service is acquired in the following form of tertiary symmetric channel ${ }^{1}$ with error probability $\varepsilon=.25$ :

$$
Q_{.25}=\left(\begin{array}{lll}
.75 & .125 & .125 \\
.125 & .75 & .125 \\
.125 & .125 & .75
\end{array}\right)
$$

accompanied by the message probability distribution:

$$
\left(q_{1}, a_{2}, q_{3}\right)=(5 / 32,3 / 16,21 / 32)
$$

The problem (12) yields the set of optimal solutions:

$$
\left(\begin{array}{lll}
a_{1}(1) & a_{2}(1) & a_{3}(1) \\
a_{1}(2) & a_{2}(2) & a_{3}(2) \\
a_{1}(3) & a_{2}(3) & a_{3}(3) \\
b(3)
\end{array}\right)=\left(\begin{array}{cccc}
.2 & 0 & .6 & .2 \\
0 & 7 / 30 & 13 / 30 & 1 / 3 \\
0 & 0 & 20 / 21 & 1 / 21
\end{array}\right)
$$

The average optimal liquidity $\tilde{b}$ in accordance with Definition $I$ is then calculated as

$$
\tilde{b}=\sum_{j=1}^{3} q_{j} b(j)=31 / 320 \cong .097<.25=b
$$

We notice here that the liquidity holding conditioned by the transmitted message 2 is $1 / 3$ and is larger than the liquidity under no information, i.e. $b=.25$ but the liquidity averaged over message probability distribution is smaller than that value .25.

Case 3: Let us observe what the average liquidity is under the more accurate tertiary symetric chamel with error probability $\varepsilon=.04$ :

1 A channel characterizeri by a $S \times S$ matrix $Q=\left\|q_{j 1}\right\|$ is called a S-ary symmetric chanmel with errox probability $\varepsilon$ if $q_{j i}=1-\varepsilon$ for $j=i$ and aji $=\varepsilon /(S-1)$ for $j \neq i$; for ail $i, j=1, \ldots$..

$$
Q .04=\left(\begin{array}{lll}
.96 & .02 & .02 \\
.02 & .96 & .02 \\
.02 & .02 & .96
\end{array}\right)
$$

accompanied with the message probabjlity distribution:

$$
\left(q_{1}, q_{2}, q_{3}\right)=(.067, .114, .819)
$$

Under this channel the set of optimal solutions can be calculated as

$$
\left(\begin{array}{lll}
a_{2}(1) & a_{2}(1) & a_{3}(1) \\
a_{1}(2) & a_{2}(2) & a_{3}(2) \\
a_{1}(3) & b(2) \\
a_{2}(3) & a_{3}(3) & b(3)
\end{array}\right)=\left(\begin{array}{cccc}
173 / 268 & 0 & 0 & 95 / 268 \\
0 & 39 / 57 & 0 & 18 / 57 \\
0 & 0 & 814 / 819 & 5 / 819
\end{array}\right)
$$

The average optimal liquidity $\tilde{\sigma}$ is then

$$
\tilde{b}=259 / 4000=.065<.097 \approx \tilde{\square} .
$$

Observation of the above results in Cases 1, 2, and 3 given the initial datum (25) tells us the following facts and conjectures:

Note 1: As was ciearly stated and proved in Arrow [2], the optimal liquidicy holding turns out to be positive when the problem (6) or more generally the problem (12) yields comer optimun, which needs a full application of KuhnTucker Theorem in a differential form for it to be solved. From a qechnical point of view this difficulty may be one of the reasons why finite state approach to the liquidity preference theory based on a utility function of incone of von Neuman-Morgenstern type has not developed until roday.

Note 2: Even though a decision makex is assured that he will absolutely gain from investing all his money (in the above numerical example, $X_{i}>1$ for $i=1,2,3$, he may still prefer to hold some positive liquidity unless perfect information is given to him.

To qualify chis second Note, lei us first establish

LEMMA: Under the assumptions

$$
\begin{equation*}
U(0)=\log \left(0^{0}\right) \tag{15}
\end{equation*}
$$

$$
\begin{align*}
& \sum\left(1 / x_{\hat{i}}\right)>1  \tag{24}\\
& i  \tag{26}\\
& X_{i}>1 \text { for all } i \varepsilon \Omega,
\end{align*}
$$

the general solution to the problem (12) in Section 2 is written out as: For all j E $\Omega^{*}$ and with respect to index sets $\mathrm{H}_{j}$ and $K_{j}$ which are subsets of $\Omega$

$$
b(j)=\left(1-\sum_{h \varepsilon K_{j}} p_{k j}\right) /\left(1-\sum_{h \varepsilon H_{j}}\left(1 / X_{h}\right)\right)
$$

$$
a_{h j}=p_{h j}-\left(b(j) / x_{h}\right) ; \text { for } h \in H_{j}
$$

$$
a_{k j}=0 ; \text { for } k \in k_{j}
$$

where

$$
\begin{aligned}
& H_{j}=\left\{j \in \Omega ; a_{i j}>0\right\} \\
& K_{j}=\left\{i \in \Omega ; a_{i j}=0\right\}=\Omega H_{j} .
\end{aligned}
$$

Proof: It is given in Mathematical Appendix.
From this result we inmediately obtain
THEOREM: If in the above Lemat the given chanzel is S-ary symmetric with exror probability $\varepsilon$, and if $\varepsilon$ is sufficiently small, then the optimal average liquidity $\tilde{b}$ in accordance with nefinition $r$ becomes proportional to the error probability independently of the variation of prior probability distribution $\left\{p_{i}\right\}$ and of system of bets $\left\{X_{i}\right\}$, more precisely, it is given by

[^1]
$$
\tilde{b}=\sum_{j=1}^{S} q_{j} b(j)=\frac{\varepsilon}{S-1} \sum_{i=1}^{S} \frac{1-p_{i}}{1-\frac{1}{x_{i}}} .
$$

Proof: It is also given in Mathematical Appendix.
This result captures an interesting behavior pattern of a risk averse decision maker who is characteristic in Arrow's model. Although he is absolutely sure to gain $\left(X_{i}>1\right.$ for all $\left.i \varepsilon \Omega\right)$ by investing ail his money $(=1)$ on bets, he keeps a certain amount of liquidity and his liquidity preference ceases only when perfect infomation ( $\varepsilon=0$ ) is given to him under the systen of bets (24). In contrast to this, his liquidity holding is always equal to zero regardless of his state of knowledge and it is so even under no information if he is presented a sure system of bets $\left(E\left(1 / X_{i}\right) \leqq 1\right)$. This rather drastic contrast of his behavior in two differi ent situations may be rephrased in such a way that in the Eormex a decision maker's knowledge on his uncertain environment does not matter at all for him to choose no ilquidity as optimal while in the latter it significantly matters, and in fect, zero liquidity is chosen only accompanied by perfect knowledge on the environment.

## 6. Summary

For the purpose of enriching the hypothetical themes proposed in Arrow's article on the value of and demand for information and of making them operationally workable in economic models of uncertainty, we established the concept of the demand values of information as its maximm buying prices. To ciarify his seeming attempt to regard information as an object of interpersonal exchange, we also defined the supply values of information as its minimum seiling prices so that one can analyze the rolos of information flow among individuals in an urcertain economy both from its buyer's and seller's viewpoincs.

To make sure that these new value concepts are not arbitrary trivia, we proved their existence and nonnegativity under loose assumptions. With respect to Arrow's special exariple based on a Bernoullian logarithmic utility function of income we obtained functional forms of those values of information which exponentially increase in the amount of information in the sense of Shannon.

We also noted that his original model intrinsically contains an analytical characterization of a rational individual's liquidity preference as behavior cowards inperfect information with somewhat different implications from the one in the traditional portfolio selection theory. By means of simple numerical illustrations and a limit theorem based on Kuhn-Tucker Theoren, we aralyzed an interesting behavior pattern of a risk averter in his optimal liquioity holding in a sensitive or nonsensitive response to his state of knowledge on the uncertain enviroment.

Adnittedyy, most of the propositions obtained in this note have meanings only for illustrative purposes because of the assumption of logarithmic utility function, and not for a general theory. By confining our analysis within Arrow's special case, we attempted to capture a few essential problems arising in an uncertain economy, which distinguish themselves from the economic problems in actutn world and which we may easily fail to notice if we entarge the scope of analysis too broadly.


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Ciend

A. Proof of Proposition $I$ in Section 3 .

Let $\bar{a}_{j}^{*}=\left(a_{1}(j) *, \ldots, a_{S}(j)^{*}, b(j) *\right) ; j \varepsilon \Omega^{*}$ be optimal solution vec. tors of the problem (12) so that
$(A-1) \quad \sum_{j} q_{j} \sum p_{i} U\left[a_{i}(j) * x_{i}+b(j) *\right] \geqq \sum q_{j} \sum_{i} p_{i j} U\left[a_{i}(j) X_{i}+b(j)\right]$

$$
\text { for all } \bar{a}_{j}=\left(a_{1}(j), \ldots, a_{S}(j), b(j)\right) ; j \varepsilon \Omega^{*} \text {. }
$$

And let $\bar{a}^{*}=\left(a_{2}^{*}, \ldots, a_{S}^{*}, b^{*}\right)$ be an optimal solution vector of the problem (6). At first we shall prove the quantity (the expected marginal utility due to additional information)
$(A-2) \quad \Delta E U=\sum_{j} q_{j} \sum_{i} p_{i j} U\left[a_{i}(j) * X_{i}+b(j) *\right]-\sum_{i} p_{i} U\left(a_{i}^{*} X_{i}+b^{*}\right)$
is nonnegative. Noting the fact that $\sum_{i} q_{i j}=1$ and $q_{j} p_{i j}=p_{i} q_{j i}$ for all i and $j$, we can rewrite $(A-2)$ into

$$
\begin{aligned}
\Delta E U & =\sum_{j} q_{j} \sum_{i} p_{i j} U\left[a_{i}(j) * x_{i}+b(j)^{*}\right]-\sum_{i} p_{i} \sum_{j} q_{j i} U\left(a_{i}^{*} X_{i}+b * j\right. \\
& =\sum_{j} q_{j} \sum_{i} p_{i j} U\left[a_{i}(j) * x_{i}+b(j)^{*}\right]-\sum_{j} q_{j} \sum_{i} p_{i j} U\left[a_{i}(j)^{\#} x_{i}+b(j)^{\#}\right]
\end{aligned}
$$

where we artificially introduced vectors $\bar{a}_{j}^{-\#}$ 's defined as

$$
\begin{aligned}
a_{j}^{-\#} & =\left(a_{i}(j)^{\frac{\pi}{n}}, \ldots, a_{s}(j)^{\#}, b(j)^{\#}\right) \\
& =\left(a_{i}^{*}, \ldots, a_{S}^{*}, b^{*}\right) .
\end{aligned}
$$

Hence from the inequality $(A-1)$, the right nand side of $(A-2)$ must be nonnegative, i.e., $\triangle E U \geqq 0$.

Let us define the real-valued functions $f$ and $g$ in the following manner:

$$
\begin{aligned}
& f\left(Z ; \bar{a}_{j} \varepsilon A_{j} ; j \varepsilon \Omega^{*}\right)=\sum_{j} a_{j} \sum_{i} p_{i j} U\left[a_{i}(j) X_{i}+b(j)-Z\right] \\
& g(Z)=\bar{a}_{j} \varepsilon A_{j} ; j \varepsilon \Omega_{\Omega^{*}} f\left(Z ; \tilde{a}_{j} \varepsilon A_{j} ; j \varepsilon \Omega^{*}\right)
\end{aligned}
$$

Since $U$ is strictly increasing by assumption, $f\left(Z, \bar{a}_{j} \equiv A_{j} ; j \varepsilon \Omega_{i}\right)^{\prime}$ is strictly decreasing in $Z$ for each $\left(\bar{a}_{j} \varepsilon A_{j} ; j \varepsilon \Omega_{Z^{*}}\right)$. Hence $g(Z)$ is strictly decreasing
in 2. On the other hand, from the already proved fact $E U \geq 0$ we obtain $(A-3) \quad g(0)=\max _{a \in A} \sum_{i} p_{i} H\left(n_{i} X_{i}+b\right)$.

If we let $Z$ be sufficiently large, $g(2)$ can be made not to exceed the right hand side of the equation $(A-3)$. If we note that the continuity of $U$ implies the continuity of $g$, then from the well-cstabisher property of continuous functions there exists $V$ which satisfies
$(A-4) \quad g(V)=\sum_{a \in A \sum_{i}}^{\max _{i} \sum_{i} U\left(a_{i} x_{i}+b\right)}$
and from the strict monotonicity of $g$ we can conclude that this $V$ is unique. If it is negative, then from the strict decreasingness of $g$ and from ( $A-3$ ) we get

$$
g(V)>g(0)=\max _{a \in A} \sum_{i} U\left(a_{i} x_{i}+b\right)
$$

Since this contradicts with $(A-4)$ itself, $V$ must be nonnegative.
q.e.d.
E. Proof of Lemme in Section 5.

We are going to consider the problein:
(B-1) Maximize, with respect to $\bar{a}_{1} \in A_{1}, \ldots, \bar{a}_{S} \in A_{S}$,

$$
\sum_{j} q_{j} \sum p_{i}{ }_{i j} \log \left[a_{i}(j) x_{i}+b(j)\right]
$$

By introducing $S$ Lagrangean multipliters $\lambda_{1}, \ldots, \lambda_{S}$, we rewrite the problem (B-1) inco the problem of maximizing

$$
\begin{align*}
& L\left(\bar{a}_{1}, \ldots, \bar{a}_{S} i_{1}, \ldots \lambda_{s}\right)  \tag{B-2}\\
& \quad=\sum_{j} q_{j} \sum_{j} p_{i j} \log \left[a_{i}(j) x_{i}+b(j)\right]+\sum_{j} \lambda_{j}\left[1-\left(\sum_{i} a_{i}(j)+b(j)\right)\right] .
\end{align*}
$$

Zonditions for maximization are:
For each $j \in \Omega *$

$(B-3)$

$$
\frac{\hat{o}_{L}}{\partial a_{i}(j)}=\frac{q_{j} p_{i j} X_{i}}{a_{j}(j) X_{i}+b(j)}-\lambda_{j} \leq 0 \text {; Equality holds if } a_{i}(j)>0
$$

(B-4) $\quad \frac{\partial L}{\partial b(j)}=\sum_{i} \frac{q_{j} p_{i j}}{a_{i}(j) X_{i}+b(j)}-\lambda_{j} \leqq 0$; Equality holas if $b(j)>0$
(B-5) $\frac{\partial L}{\partial \lambda_{j}}=1-\left(\sum_{i} a_{i}(j)+b(j)\right) \leqq 0 ;$ Equality holds if $\lambda_{j}>0$.
If we assume that $\lambda_{j}$ 's are nonpositive, then this assumption violates the conditions ( $B-3$ ) and ( $B-4$ ) under supposedly positive values of $p_{i}$ 's and $X_{i}$ 's and nonnegative values of $a_{i}(j)^{\prime} s$. Hence $\lambda_{j}>0$ for all $j \varepsilon \Omega^{*}$ and then the conditions ( $B-5$ ) must be read as

$$
\left(B-5^{\prime}\right) \quad 1-\left(\sum_{i} a_{i}(j)+b(j)\right)=0 \text { for all } j \varepsilon \Omega^{*} \text {. }
$$

On the other hand, from one of the results of Arrow (see Remark in Section 3, page 7 of this notej we know that $b(j)$ 's are all positive. Hence the conditions ( $B-4$ ) must be read as
$\left(B-4^{1}\right)$

$$
\sum_{i} \frac{p_{i j}}{a_{i}(j) X_{i}+b(j)}=A_{j},
$$

Where we started to use the notation $\Lambda_{j}=i_{j} / q_{j}$.
For conventence we define the index sets $H_{j}$ and $K_{j}$ as

$$
\begin{aligned}
& H_{j}=\left\{i \in \Omega ; a_{i}(j)>u\right\} \\
& K_{j}=\left\{i \in \Omega ; a_{i}(j)=0\right\}
\end{aligned}
$$

Then the conditions (B-3) yield

$$
\frac{p_{h j} X_{h}}{a_{h}(j) x_{h}+b(j)}=A_{j} ; \text { for al2 } h^{H} H_{j}
$$

$$
\begin{equation*}
\frac{p_{k j} X_{k}}{a_{k}(j) X_{k}+b(j)}=\frac{P_{k j} X_{k}}{b(j)} \leqq A_{j} ; \text { for } a I 1 k \varepsilon K_{j} . \tag{B-7}
\end{equation*}
$$

The above equation (5-6) is rewritten as
(B-8)

$$
\frac{p_{h j}}{a_{h}(j) X_{n}+b(j)}=\Lambda_{j} / X_{h} ; \text { for all } h \varepsilon h_{j}
$$

Summing up both sides of ( $B-8$ ) over the set $H_{j}$, we obtain

$$
\begin{equation*}
\sum_{h \in H} \frac{p_{h j}}{a_{h}(j) X_{h}+b(j)}=\Lambda_{j} \sum_{h \in H}\left(1 / X_{h}\right) . \tag{B-9}
\end{equation*}
$$

Dividing the first two terms of $(B-7)$ by $X_{k}(\neq 0)$ and summing up the results over the set K , we obtain
(B-10)

$$
\sum_{k} \frac{p_{k j}}{a_{k}(j) X_{k}+b(j)}=\frac{1}{b(j)} k \sum_{k} p_{k j} .
$$

Since $H V K=\Omega$, the equations $(B-4),(B-9)$ and $(B-10)$ amount to:

$$
\begin{equation*}
A_{j} \sum_{h \in H}\left(1 / X_{h}\right)+\frac{1}{b(j)} \sum_{k \in K} P_{k}=A_{j} \tag{B-11}
\end{equation*}
$$

On the other hand, from the equations ( $B-6$ ) we have

$$
\begin{equation*}
a_{h 2}(j)=\frac{p_{h j}}{\Lambda_{j}}-\frac{b(j)}{x_{h}} \text { for all } h \varepsilon H \tag{B-12}
\end{equation*}
$$

Hence, by noting ( $B-5^{\prime}$ )

$$
\begin{align*}
\sum_{i} a_{i}(i)+b(j) & =\sum_{h \in H} a_{h}(j)+\sum_{k} \sum_{k} a_{k}(j)+b(j)  \tag{B-13}\\
& =\left(1 / A_{j}\right) \sum_{h \in H} p_{h j}-b(j) \sum_{h \varepsilon H}\left(1 / X_{h}\right)+b(j) \\
& =1 .
\end{align*}
$$

From the equation (B-11) we know

$$
\Delta(j) \sum_{k \in H}\left(1 / K_{h}\right)=b(j)-\left(1 / \Lambda_{j}\right) \sum_{k \in k}^{\sum} p_{k j}
$$

From ( $B-13$ ) and ( $B-14$ ) we obtain

$$
\left(1 / \Lambda_{j}\right) \quad \sum_{i} p_{i j}=13
$$

which amount to $\Lambda_{j}=1$ (i.e., $\lambda_{j}=q_{j}$ ) for all $j \varepsilon \Omega^{*}$ since $\sum_{i} p_{i j}=1$ for ail $j \in \Omega^{*}$. Therefore, (B-I4) yields
( $B-15$ )

$$
b(j)=\frac{\sum_{\sum_{k} p_{k j}}^{1-\sum_{h \in H}\left(I / X_{h}\right)}}{1-\sum_{h \in H}\left(1 / X_{h}\right)} ; j \varepsilon \Omega^{*} .
$$

In terms of these solutions for $b(j)$ 's we obtain

$$
a_{i}(j)=\left\{\begin{array}{cc}
p_{i j}-b(j) / X_{i} & \text { if } i \varepsilon H_{j} \\
0 & \text { if } i \varepsilon K_{j}
\end{array}\right.
$$

q.e.d:
C. Proof of Theorem in Section 5.

If a given chanel $Q=\| q_{j i}!\mid$ is $S$-amy symmetric with error probability $\varepsilon$, i.e., $q_{j i}=1-E$ for $j=I$ and $q_{j i}=\varepsilon /(S-1)$ for $j \neq i$, then $p_{i j} \rightarrow 1$ if $i=j$ and $p_{i j} \rightarrow 0$ if $i \neq j$ as $\varepsilon \rightarrow 0$ and the sets $H_{j}$ 's in the above proof of Lemma shrink to singleton sets $\{j\}$ 's. Fence from the result (B-15) we get $(C,-1)$ $b(j)=\left(1-p_{j j}\right) /\left(1-\left(i / X_{j}\right)\right)$; for all $j$ E $\Omega^{*}$.
We further obtain

$$
\begin{aligned}
q_{j}\left(1-p_{j j}\right) & =q_{j}-q_{j} p_{j j}=\sum_{i} p_{i} q_{j i}-p_{j} q_{j j} \\
& =\sum_{i \neq j} E_{i} q_{j i}=\frac{\varepsilon}{S-1} \sum_{i \neq j} p_{i} \\
& =\frac{\varepsilon}{S-I}\left(1-p_{j}\right)
\end{aligned}
$$

Combining these results with ( $C-1$ ), we obtain the optimal average liquidity b $a s$

$$
\tilde{b} \equiv \sum_{j \varepsilon \Omega_{i *}^{*}} q_{j} b(j)=\frac{\varepsilon}{S-1} \sum_{j \varepsilon \Omega^{*}} \frac{1-p_{j}}{1-\frac{1}{X_{j}}} .
$$

q.e.d.


[^0]:    1 The word, "chunnel," doos not have to be understood literally in the marrok sense of mathematical communcation theory. Ne may regard it as an coerational tool of quantitatively descriting degrees of accuracy in any kind of predictive activities guch as research, sampling, human verbal conversation and the like which yield certain estimation on a true state in the stochastic neture.

[^1]:    ${ }^{1}$ As for the definition of symmetric channel, see the footnote of page 14 of this note.

