VELOCITY DEPENDENT BOSON EXCHANGE POTENTIALS AND NUCLEON-NUCLEON SCATTERING IN THE BORN APPROXIMATION

By

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THE LIGHTS OF BHAGAWAD GITA

यद्यदाचरति श्रेष्ठस्तत्तदेवेतरो जनः ।
स यत्प्रमाणं कुरुते लोकस्तदनुवर्तते ॥
(अध्याय ३) [२१]

That what is done by great men, common people do; the standard the wise man setteth up by that the multitudes go.
Ch. III.-21.

Dedicated to

My Physics Teachers

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CHAPTER I

INTRODUCTION

One of the most outstanding problems in Physics in this century has been the effort to achieve understanding of the nuclear forces. There exists no complete theory to date which can give a description of the nuclear forces and strong interactions. In particular, an unusually large number of physicists have worked on this problem because the two-body systems constitute the simplest systems from which the information about the nature of the interaction can be, hopefully, obtained. But even the greatest physicists have wondered about the immense complexities involved in these areas.

From time to time in the last few decades, our knowledge about the force field has increased. Yukawa's theory in 1935 which was basically a Lagrangian field theory was the first major theoretical breakthrough. Later on, various groups and individuals attempted to parameterize the nuclear-nucleon interaction using potential models and invariance arguments. A considerable amount of field theoretic work was done in this area. More recently, other methods of calculating the nucleon-nucleon interaction have emerged and have broadened our understanding of the subject, but as yet there has been no established superiority of these methods over and above the field theoretic ones.

In field theory, there has been a very interesting and important development in which the concepts of generalized quantum electrodynamics were developed around 1930. These concepts were furthur generalized and applied to the problem of nucleon-nucleon force from meson field theoretic viewpoint by Green (1, 62, 63, 64) in the period 1947-1950. This theoretical work constitutes an important description of the nucleon-nucleon force. This work is developed from the generalizations of Dirac, Fock, and Podolsky's (7) and Fock's (2) work in quantum electrodynamics. The later successes of field theory have come, from the electrodynamics point of view, from the formalisms of Schwinger, but the meson theory aspect of this problem had to wait for corresponding experimental developments. At least one such breakthrough has occurred due to the discovery of vector mesons. Thus the work that was dormant for this period in which symmetries and transformations associated with the field were used has to be re-examined. The work of Green, mentioned above, will constitute the theoretical basis of this study. Among others, a few distinctly distinguishable features of this work are as follows. It introduces the field theories with higher derivatives of field coordinates in the Lagrangian formalism, and as a consequence, introduces the modified Yukawa forms with subtractive mesons. This avoids the singularities in many ways. It also considers the applications of Fock's (2) methods to derive the nucleon-nucleon interaction which is quadratic in coupling constant even when multiple meson exchange processes are present. In addition, this work considers the questions regarding the correct field-theoretic description of the vector meson field. It also considers the tensorial characters of the field and its five dimensional aspects. Among other important aspects of this work are the fact that Green (6, 52) realized the scalar-vector cancellation of the static terms and the importance of the velocity dependence and of the other relativistic terms as early as 1948. The five-vector form, which is a synthesis of the scalar and vector meson fields, constitutes an important aspect of the work that will be presented in coming chapters and Green's (52) recognition of the significance of cancellation in 1948 was based upon physical intuition.

A definite influence of Podolsky's (43) and Kemmer's (4) earlier works can be seen in the work of Green (62, 63, 64). Equally important aspects of this work in relation to two particle Dirac equation come from the study and use of Breit's (51, 56, 65) work on reduction of Diracian forms to corresponding Pauli forms. This has helped our understanding of the components of nucleon-nucleon force immensely, and will constitute a very important aspect of this study.

A very large number of review articles and important publications exist which are almost impossible to compile, but for the sake of a general background and different viewpoints on the nucleon-nucleon force the following references might satisfy the reader. They are numbered 12, 15 through 30, 44 through 50, and are given in the Bibliography.

The author's involvement with parts of this problem occured when he started to look for the latest status on the nucleon-nucleon force in order to go for many-body calculations and came across the work of Bryan and Scott (68) who considered One Boson Exchange Potentials derived from Pole projections and the similarities of their work with Green's work were immediately recognized. A close look proved that these two agreed in every respect for the OBEP, except that they had neglected the velocity dependent parts of the interaction. Green and Sharma (54) then undertook the comparative study and the agreements obtained were exciting. As a result, a Born approximation study was undertaken by the author with a particular emphasis on the velocity dependence and a phase shift program was developed. The agreements within the Born approximation limits will be the topic of subsequent chapters. The results obtained will be discussed in the last chapter in a summarized form. In particular, Chapter II concerns itself with the presentation of old work by Green and others with a viewpoint of presenting derivation of OBEP. Chapter III is devoted to the study of reduction of most general Diracian forms and their application to specific interactions, while Chapter IV is concerned with a comparison of various velocity dependent potentials occuring phenomenologically in recent literature. In Chapter V a theoretical derivation of the phase shift formulae is presented both for uncoupled and the coupled states, while Chapter VI deals with the results on the phase shift calculations and comparisons with the experimental values. In Chapter VII, later aspects of the research regarding other methods of the reduction and 2π- Exchange Potential are presented while Chapter VIII concludes this study and points to various possible future directions. We will confine ourselves , in the following study, to the problem of accounting for the nucleon-nucleon elastic scattering data for non-relativistic energies (25 - 310 Mev) before production and other inelastic processes become important. This will be done only in the Born approximation. We hope that this study will illuminate many important aspects of the nucleon-nucleon interaction and at the same time will help clarify the validity of various theoretical works on which it relies.

CHAPTER II

DERIVATIONS OF THE ONE BOSON EXCHANGE POTENTIALS

Section - 1 Introduction

In this chapter an attempt is made to derive the relativistic nucleon-nucleon interactions due to the exchange of vector, pseudoscalar, and scalar mesons. The quantum field theoretical treatment of this problem is due to an early derivation by Green (1). Recent successes of Meson Exchange Potentials have lead to a re-examination of problems connected with vector mesons by Rochleder and Green (8, 11) while the multiple meson exchange processes have been again studied by Chern and Green (9). There are many different methods of deriving nucleon-nucleon interactions in One Boson Exchange approximation and they all give equivalent results in the static limit. But Two Boson Exchange Potentials and the relativistic terms due to various methods are not exactly the same. Some of the important methods for the derivation of meson theoretical nucleon-nucleon interactions are given in the Review article by Moravcsik and Noyes (12). Little has been published about the approach that will be sketched in the next sections, this being developed by Green (1) on the generalization of quantum electrodynamics using Fock-functional and multitime formalism, the reason for its dormant period being inadequate experimental knowledge about the properties of various mesons. Many other books and review articles [for example, Mandl (10), Hulthén (21), Wentzel (22), etc.] give details of some of the other methods. Perturbation theory, Tamm-Dancoff methods, S matrix theory (using Feynman Diagrams), Dispersion theory, and Fock formalism are the important methods quite recently used. Fock formalism, as treated by Green (1), was used for deriving a general interaction applicable to scalar, vector, and pseudoscalar mesons, while the vector-meson field with auxiliary condition (Kemmer-Proca interaction) was treated by perturbation theory as recently reported by Rochleder and Green (8, 11). Also it will be outlined that if the auxiliary condition is treated as a condition on the state vector of two-nucleon system, then vector meson interaction, as shown by Green (53), is of the form of Breit's operator in electrodynamics. The Two Boson Exchange term from the Fock methods were also calculated by Green.

Section - 2 A Sketch Of The Derivation Of One Boson Exchange Potentials For Scalar, Pseudoscalar, and Unconstrained Vector Meson Fields

The generalization of the mathematical apparatus and the second-quantization techniques of Fock in meson field theory were made by Green (1, 4, 62, 63, 64) which also embodied various other interesting and important aspects of field theory. He thus derived the nucleon-nucleon interactions due to the presence of scalar, pseudoscalar, and the vector meson fields. In the lowest order, when only the first term of such interactions were considered, they corresponded to One Boson Exchange Potentials. These agree in their static terms with the conventionally derived OBEP from perturbation theory (3, 4, 21) or from Feynman diagrams and the S matrix theory (68). However, their relativistic terms and fourth order terms like the Two Boson Exchange Potentials do not necessarily agree with each other.

General ideas about fields and the interaction are described in Schweber, Bethe, and de Hoffmann (13) where the concepts about the field and the Lagrangian interaction theory and field quantization are described starting with first principles and commutation rules. A field function $\varphi(\vec{r},t)$ can be decomposed into positive and negative frequency components in terms of a Fourier Transform as

$$\varphi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int \left[\phi(\vec{k})\, e^{i k_\alpha x_\alpha} + \right.$$
$$\left. + \phi^+(\vec{k})\, e^{-i k_\alpha x_\alpha} \right] d\vec{k}$$
$$(\text{II}-2.1)$$

where $\vec{k}$ is the three vector such that $k_\alpha = (\vec{k}, k)$, $x_\alpha = (\vec{x}, t)$, $k = \omega / c$. Thus $e^{-i\omega t}$ is the negative frequency component and so on. If the relativistic relation between energy and momentum of the particle is

$$k^2 = \vec{k}^2 + \mu^2 ,$$
$$(\text{II}-2.2)$$

then the Klein-Gordon equation for the field

$$[\Box^2 - \mu^2]\, \varphi(\vec{r},t) = [\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \mu^2]\, \varphi(\vec{r},t)$$
$$= 0 \qquad (\text{II}-2.3)$$

is satisfied.

To sketch Green's (1) meson field theoretically derived results with Fock techniques, we give very briefly the following summary. All of these results are either due to the work of Green (1, 62, 63, 64) or Rochleder and Green (8, 11) or based on Fock's (2) article on electrodynamics. For the case of Boson fields, we have the symmetric functions with regards to the exchange of the particle coordinates. The creation and destruction operators $b^+(\vec{k})$ and $b(\vec{k})$ are defined from (II-2.1) as

$$b(\vec{k}) = \phi(\vec{k}) \left(\frac{c\hbar}{2k}\right)^{-1/2} ,$$

and

$$b^{+}(\vec{k}) = \phi^{+}(\vec{k}) \left(\frac{c\hbar}{2k}\right)^{-1/2} \tag{II-2.4}$$

which satisfy

$$[\, b(\vec{k}), b^{+}(\vec{k}') \,] = \delta(\vec{k} - \vec{k}') \tag{II-2.5}$$

The Hamiltonian H and the number operator N can now be written in terms of these operators as

$$H = \frac{\hbar c}{2} \int k \, \delta(\vec{k} - \vec{k}') \, d\vec{k}' + \hbar c \int k \, b^{+}(\vec{k}) \cdot$$

$$\cdot \, b(\vec{k}) \, d\vec{k},$$

$$N = \int b^{+}(\vec{k}) \, b(\vec{k}) \, d\vec{k} \tag{II-2.6}$$

In the Fock formalism a special representation is chosen so that the adjoint operator $b^{+}(\vec{k})$ corresponds to a multiplication by a function $\mathfrak{b}(\vec{k})$ while the arbitrary state Ω on which these operators act can be expanded in terms of these functions and is a functional of the function $\mathfrak{b}(\vec{k})$ which can be written as

$$\Omega = \psi_0 + \frac{1}{\sqrt{1!}} \int \psi(\vec{k}) \, \mathfrak{b}(\vec{k}) \, d\vec{k} +$$

$$+ \frac{1}{\sqrt{2!}} \iint \psi(\vec{k}_1, \vec{k}_2) \, \mathfrak{b}(\vec{k}_1) \, \mathfrak{b}(\vec{k}_2) \, d\vec{k}_1 \, d\vec{k}_2 +$$

$$+ \cdots\cdots$$

$$= \Omega_0 + \Omega_1 + \Omega_2 + \cdots + \Omega_n + \cdots\cdots \tag{II-2.7}$$

Fock also defines a functional derivative with respect to the function $b(\vec{k})$ as

$$\frac{\delta'\Omega}{\delta b(\vec{k})} = A \ ,$$

where

$$\Omega = \int A \ b(\vec{k})\, d\vec{k} \ . \qquad (\text{II}-2.8)$$

If

$$\Omega = b(\vec{k}') = \int \delta(\vec{k}-\vec{k}')\, b(\vec{k}')\, d\vec{k}'$$

then

$$\frac{\delta'\Omega}{\delta b(\vec{k})} = \delta(\vec{k}-\vec{k}') \qquad (\text{II}-2.9)$$

For an arbitrary functional, therefore, we thus have

$$\frac{\delta'}{\delta b(\vec{k}')}\, b(\vec{k})\Omega - b(\vec{k})\frac{\delta'\Omega}{\delta b(\vec{k}')} = \delta(\vec{k}-\vec{k}')\,\Omega, \qquad (\text{II}-2.10)$$

which is a special representation of (II-2.5) and therefore the functional derivative is equivalent to $b(\vec{k})$ or

$$b^{+}(\vec{k}) \longrightarrow b(\vec{k})$$

$$b(\vec{k}) \longrightarrow \frac{\delta'}{\delta b(\vec{k})} \qquad (\text{II}-2.11)$$

We notice the similarity between $(\vec{r}, \vec{p})$ and (b^{+}, b) in quantum mechanical sense.

In meson theory, Green (1) derived the nucleon-nucleon interaction using this Fock functional formalism (2). These derivations were based on generalizations of Dirac, Fock, and Podolsky's (7) multitime formalism. The two particle

Dirac equation in the manifest covariant form can be written as

$$\left[c\,\vec{\alpha}^{(s)} \cdot \vec{p}^{(s)} + \beta^{(s)} M^{(s)} c^2 - i\hbar \frac{\partial}{\partial t_s} \right] \Psi(\vec{r}_1, \vec{r}_2, t_1, t_2) = 0 \qquad \text{(a)} \quad (\text{II}-2.12)$$

or

$$\left[H_p^{(s)} - i\hbar \frac{\partial}{\partial t_s} \right] \Psi = 0 \qquad \text{(b)} \, (\text{II}-2.12)$$

The interactions must be added to this in an invariant form. Thus with a scalar interaction $\varphi(\vec{r}, t)$ we get

$$\left[c\,\vec{\alpha}^{(1)} \cdot \vec{p}^{(1)} + \beta^{(1)} M^{(1)} c^2 - g\,\beta^{(1)} \varphi(\vec{r}_1, t_1) - i\hbar \frac{\partial}{\partial t_1} \right] \cdot$$

$$\cdot\, \Psi(\varphi, \vec{r}_1, \vec{r}_2, t_1, t_2) = 0$$

and with a similar equation for particle 2, we get

$$\left[H_p^{(1)} + H_p^{(2)} - g\,\beta^{(1)} \varphi^{(1)} - g\,\beta^{(2)} \varphi^{(2)} \right] \Psi = i\hbar \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \Psi \qquad (\text{II}-2.13)$$

We specialize to a special frame which in some ways destroys the totally covariant nature of 2-particle Dirac equation and set

$$\psi'(\varphi, \vec{r}_1, \vec{r}_2, t) = \Psi(\varphi, \vec{r}_1, \vec{r}_2, t_1 = t, t_2 = t)$$

and thus

$$\frac{\partial \psi'}{\partial t} = \frac{\partial \Psi}{\partial t_1} + \frac{\partial \Psi}{\partial t_2} \cdot \qquad (\text{II}-2.14)$$

Let us now denote

$$D = \sum_s H_P^{(s)} - i\hbar \frac{\partial}{\partial t}\ , \quad H_f = \int \hbar c k\, b^+ b\, d\vec{k} \qquad (\text{II}-2.15)$$

where H_f is the free field Hamiltonian. The interaction Hamiltonian $H_i^{(s)}$ was written by Green (1) in the following form

$$H_i = -g\beta\,\varphi(\vec{k}) = -g\beta\frac{1}{(2\pi)^{3/2}}\int\left[\phi(\vec{k})\,b^+(\vec{k}) + \right.$$

$$\left. + \phi^+(\vec{k})\,b(\vec{k})\right] d\vec{k}$$

$$= -\int g^+\, b(\vec{k})\,d\vec{k} - \int g\; b^+(\vec{k})\,d\vec{k} \qquad (\text{II}-2.16)$$

where for the scalar case we have

$$g(\vec{k}) = \sum_s g\,(2\pi)^{-3/2}\beta^{(s)} c\,(\hbar/2\omega)^{\frac{1}{2}}\, e^{-i\vec{k}\cdot\vec{x}^{(s)}}, \qquad (\text{II}-2.16)$$

(with the replacement of $\beta^{(s)}$ by $(\beta\gamma_5)^{(s)}$ for pseudoscalar and by $(\vec{\alpha}^{(s)}, i\text{I})$ for unconstrained vector interactions) and $\omega = c(\vec{k}^2 + K^2)^{1/2}$, where K is the inverse Compton wavelength associated with the meson field.

The wave function of the 2-particle system, the functional Ω , can be now expanded into the eigenfunctionals of 1, 2, . . . Boson wave functions, according to the equation (II-2.7), where Ω_n is defined by comparison with the previous step. If we substitute (II-2.7), (II-2.16) into

$$\left[D - \int g^+ b\, d\vec{k} + \int g\, b^+ d\vec{k} + \hbar c \int k\, b^+ b\, d\vec{k}\right]\Omega = 0\ , \qquad (\text{II}-2.17)$$

and collect the coefficients of $b(\vec{k})$, $b(\vec{k}_1)b(\vec{k}_2)$, $b(\vec{k}_1)b(\vec{k}_2)b(\vec{k}_3)$ etc. and equate it to zero, we obtain a set of coupled integro-differential equations because each of the b's are arbitrary functions. Thus we obtain in general

$$D\psi_q + \hbar c\,(k_1 + \cdots + k_q) =$$
$$= \sqrt{q+1} \int g^+(\vec{k})\, \psi_{q+1}(\vec{k}_1, \ldots, \vec{k}_{q+1}) \cdot d\vec{k}_1 \cdots d\vec{k}_{q+1} -$$
$$- \frac{1}{\sqrt{q}} \left[g(\vec{k}_1)\, \psi_{q-1}(k_2, \ldots, k_q) + g(\vec{k}_q) \cdot \psi_{q-1}(k_1, \ldots, k_{q-1}) \right], \qquad (\text{II}-2.18)$$

and explicitly this can be written as a set of coupled equations as follows.

$$D\psi_0 - \int g^+(\vec{k})\, \psi_1(\vec{k})\, d\vec{k} = 0, \qquad (a)$$

$$D\psi_1 + \hbar\omega_1 \psi_1 - \sqrt{2} \int g^+(\vec{k})\, \psi_2(\vec{k}, \vec{k}_1)\, d\vec{k} - \frac{1}{\sqrt{1}}\, g(\vec{k}_1)\, \psi_0 = 0, \qquad (b)$$

$$D\psi_2 + \hbar(\omega_1 + \omega_2)\psi_2 - \sqrt{3} \int g^+(\vec{k}) \cdot \psi_3(\vec{k}, \vec{k}_1, \vec{k}_2)\, d\vec{k} - \frac{1}{\sqrt{2}} \left[g(\vec{k}_1) \cdot \psi_1(\vec{k}_2) + g(\vec{k}_2)\, \psi_1(\vec{k}_1) \right] = 0, \qquad (c)$$

- etc.

(II-2.19)

But from (b) to first order

$$\psi_1 = \frac{g(\vec{k_1})\psi_0}{\hbar\omega_1} \qquad (\text{II}-2.20)$$

and substituting in (a) we get

$$D\psi_0 = \int \frac{g^+(\vec{k})\, g(\vec{k})\, d\vec{k}}{\hbar\omega}\psi_0$$

$$= -\,{}^2U\,\psi_0 \qquad (a)\ (\text{II}-2.21)$$

If the approximations made in (II-2.20) are now improved and ψ_2 and ψ_1 are determined in terms of ψ_0 and the neglected differences are picked up, then to the next orders the interaction takes the form

$$D\psi_0 = {}^2U\psi_0 + {}^4U\psi_0 + {}^6U\psi_0 + \cdots \qquad (b)\ (\text{II}-2.21)$$

where

$${}^2U = -\int \frac{g^+ g}{\hbar\omega}\, d\vec{k},$$

$${}^4U = +\int \left\{ g^+ [H_p, g]/(\hbar\omega)^2 \right\} d\vec{k},$$

$${}^6U = -\int \left\{ g^+ [H_p, [H_p, g]]/(\hbar\omega)^3 \right\} d\vec{k}$$

$$(\text{II}-2.22)$$

These coupled equations for meson fields and the above results were obtained by Green (1) and have the elegance that the ${}^2U$ represents only one meson processes and corresponds to the OBEP, and is quadratic in the coupling. Also the ${}^4U$ represents the Two Boson Exchange interaction. The direct reference to the field has been eliminated from these interactions and these are therefore our nucleon-nucleon interactions. Another beauty of these results is that the multiple meson exchanges have all quadratic terms in the coupling constants and thus this suggests that the perturbation approaches in coupling constants can be avoided by this method of deriving the interaction. Based on these results of Green, we will show that these interactions, (mostly the OBEP) represent a major component of the nucleon-nucleon force by experimental comparisons. We postpone the discussion of ${}^4U$ for Chapter VII and only make an important remark that in deriving the ${}^4U$, no considerations were made about the normalization of the Ω, which is a functional, depending on the meson fields, number and the nucleon coordinates and is really an infinite sum. Since the multiple meson processes are assumed to be successively less important, this series has to be chopped off at a particular point. But originally $(\Omega, \Omega) = 1$ and this requires a normalized Ω when the series is chopped off. These questions are being considered by Chern and Green (9) and any changes in this form of the interaction due to normalization, might influence our expressions for $2\pi-$ exchange interaction in Diracian and Pauli form which will be given in Chapter VII.

Now we proceed to get the Diracian form of the interaction in the more familiar form. We take the pseudo-scalar case $\mathcal{G}$ where $\beta^{(s)}$ is replaced by $(\beta\gamma_5)^{(s)}$ in (II-2.16). With some careful operator algebra, we get

$$^2U = -\frac{g^2}{(2\pi)^3}\left[\int \left\{ \frac{1}{(\vec{k}^2+K^2)} - \frac{(\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)}}{(\vec{k}^2+K^2)} \cdot \cos\left[\vec{k}\cdot(\vec{x}^{(1)}-\vec{x}^{(2)})\right]\right\} d\vec{k}\right] \qquad (\text{II}-2.23)$$

where we used the fact that $(\beta\gamma_5)^2 = 1$. Making a proper choice of axes and with

$$\vec{r} = \vec{x}^{(1)} - \vec{x}^{(2)}, \quad \vec{k}\cdot\vec{r} = kr\cos\theta,$$
$$z = kr, \quad a = Kr \qquad (\text{II}-2.24)$$

we obtain after carrying out the angular integrations

$$^2U = -\frac{g^2}{2\pi^2}\left[-\int_0^\infty \frac{k^2\,dk}{(\vec{k}^2+K^2)} - \frac{(\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)}}{r} \cdot \int_0^\infty \frac{z\sin z\,dz}{z^2+a^2}\right] \qquad (\text{II}-2.25)$$

The first integral, with the transformation $k = K\tan\eta$, can be translated into $\int_0^{\pi/2} \tan^2\eta\,d\eta = -\frac{K\pi}{2} + K(\tan\eta)_0^{\pi/2}$ which blows up and this denotes the self-energy term, which also has the classical interpretation of the interaction of the field upon itself. This generally diverges but disappears when Green's (62) generalized meson theory, with higher derivatives in the Lagrangian, is considered. The second term is a known integral which gives the Yukawa form or it can be evaluated in the complex plane by properly contouring around the poles ($\pm i\kappa$). The final result is therefore

$$^{2}U = (\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)} J(r) = V_P^D ,$$

$$J(r) = g^2 (\hbar c) \left(e^{-kr}/r \right) . \tag{II-2.26}$$

The analogous results are obtained when β, $(\vec{\alpha}, iI)$, are replaced for $(\beta\gamma_5)$. They are

$$V_S^D = - \beta^{(1)} \beta^{(2)} J(r)$$

and

$$V_{UV}^D = \left[1 - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} \right] J(r) . \tag{II-2.27}$$

These are our OBEP results for the scalar, pseudoscalar, and the undetermined vector meson interactions.

Section - 3 Vector Meson Interactions With The Auxiliary Condition

We mention briefly the vector meson interactions derived in meson field theory by Kemmer (3, 4) and by Green (63) from the considerations of the auxiliary conditions analogous to those occuring in the quantum electrodynamics. The result obtained by Green is more general in field theoretic sense because of Lagrangians with higher derivatives in the field coordinates but in the usual limit the interaction reduces to the analogous Breit vector interaction in electrodynamics. However, Breit (65) obtained a similar result with the opposite sign and without specification of the static Green function in 1938 from the considerations of relativistic invariance. If we consider the vector meson field without the auxiliary condition, the resulting Hamiltonian is not positive definite because of the time like component of the vector meson field. The free field Lagrangian for the nucleon field is

$$\mathcal{L}_0 = -\frac{1}{2}\sum_{\mu=1}^{4} \bar{\Psi}(\gamma_\mu \partial_\mu + M)\Psi + H.c. \qquad (\text{II}-3.1)$$

where the notation is due to Schweber (14). Field theories with indefinite metric reported by Arcms and Sudarshan (113) constitute interesting aspects but do not form a part of the present work.

The time like components of the vector field $\phi_\mu = (\phi_i, \phi_4), (\phi_4 = i\phi)$ satisfy the wrong sign commutation rules ($\phi_4^* = i\phi^*$)

$$[\phi^*(\vec{k}), \phi(\vec{k})] = \frac{c\hbar}{2k} \delta(\vec{k}-\vec{k}')$$

while

$$[\phi_i^*(\vec{k}), \phi_j(\vec{k}')] = \left(\frac{c\hbar}{2k}\right) \delta_{ij} \delta(\vec{k}-\vec{k}') \qquad (\text{II}-3.2)$$

The expansion of the fields into fourier components with the definition $\vec{A} \equiv \phi_\alpha \hat{i}_\alpha$, $\alpha = 1, 2, 3$, yield the equations

$$[i\{(\vec{k}\cdot\vec{A}) - k\phi\} + KB]\Psi = 0$$

and

$$[-i(\vec{k}\cdot\vec{A}^* - k\phi^*) + KB^*]\Psi = 0$$
$$(\text{II}-3.3)$$

where the auxiliary condition with the matter (mesons) is used as a constraint on the state vector

$$[\partial_\mu \phi_\mu + KB]\Psi = 0 .$$
$$(\text{II}-3.4)$$

In electrodynamics where $\mu = 0$, the two conditions are compatible because the transverse and longitudinal quanta (photons) cancel each other. But because of the mass term, it is not so in meson theory. Green (63) used the auxiliary condition as a constraint on the field equation and obtained the Breit vector interaction through the following scheme. If $X\Psi = 0$ and $Y\Psi = 0$, then $[X, Y]\Psi = 0$ must not imply new condition and for (II-3.3) it yields that

$$[\vec{k}^2 + k^2 + K^2]\Psi = 0 \tag{II-3.5}$$

which is not a new condition of the Klein-Gordon equation is satisfied by the field quanta. With q, q^+ analogous to longitudinal quanta in electrodynamics, the following commutation rules are satisfied.

$$q = \frac{\vec{k}\cdot\vec{A} - iKB}{k}$$

$$q^+ = \frac{\vec{k}\cdot A^+ + iKB}{k} \tag{II-3.6}$$

$$[q^+, q] = -\left(\frac{c\hbar}{2k}\right)\delta(\vec{k} - \vec{k}') \,. \tag{II-3.7}$$

Now the application of Fock functional formalism analogous to the last section yields

$$q^+ \to Q$$

$$q \longrightarrow \left(\frac{c\hbar}{2k}\right)\frac{\delta}{\delta Q} \,. \tag{II-3.8}$$

As a result the Dirac equation in multitime formalism, and the auxiliary conditions (II-3.3) take the form

$$\left[Q - \phi + \frac{f}{2k^2}\right]\Psi = 0 \quad ,$$

$$\left[Q^+ - \phi^+ + \frac{f^+}{2k^2}\right]\Psi = 0$$

$$\Big[c\,\vec{\alpha}^{(s)}\cdot\vec{p}^{(s)} + \beta^{(s)} M c^2 + g\,\phi(\vec{r}_s, t_s) -$$

$$- g\,\vec{\alpha}\cdot\vec{A}(\vec{r}_s, t_s) - i\hbar\frac{\partial}{\partial t_s}\Big]\Psi = 0 \qquad (\text{II} - 3.9)$$

where

$$\Psi = e^{\chi}\Omega \,,$$

$$\chi = \frac{2}{c\hbar}\int Q\,\phi\,k\,d\vec{k} + \frac{1}{c\hbar}\int\frac{Q f^+}{k}\,d\vec{k} -$$

$$-\frac{1}{c\hbar}\int\frac{Q^+ f}{k}\,d\vec{k} + \chi'$$

with

$$\chi' = \frac{1}{4\pi c}\int\frac{f^+ f}{k^2}\,d\vec{k} \qquad (\text{II} - 3.10)$$

and

$$f = \frac{g}{(2\pi)^{3/2}}\sum_s e^{-i k_\nu x_\nu^{(s)}} \,.$$

To get a Dirac equation in Ω we use the above canonical transformation and the transformed operators are given by the rule

$$F' = e^{-\chi} F e^{\chi} = F - [\chi, F] + \frac{[\chi,[\chi,F]]}{2!} + \cdots \qquad (\text{II} - 3.11)$$

and thus

$$Q' = Q + \phi - \frac{f}{2k^2} ,$$

$$\phi^{+\prime} = Q^+ + \phi^+ + \frac{f^+}{2k^2} , \qquad (\text{II}-3.12)$$

With some more simplification the generalized fields are used to calculate transformed momenta $\vec{P}'^{(s)}$ and the energy $T'^{(s)}$ in terms of various known quantities. The steps are quite mathematical and for details the reader is referred to Green (63) and Rochleder and Green (8, 11). The results are

$$\vec{P}^{(s)\prime} = \vec{P}^{(s)} - \frac{g}{c} \vec{D}(\vec{r}_s, t_s) ,$$

$$T^{(s)\prime} = i\hbar \frac{\partial}{\partial t_s} - \frac{g_s}{c} \frac{\partial u_s}{\partial t_s}$$

$$- \text{Self Energy Terms.} \qquad (\text{II}-3.13)$$

where

$$\vec{D}(\vec{k}) = \vec{A}(\vec{k}) - \frac{\vec{k}}{k} Q(\vec{k}) ,$$

$$u_s(\vec{k}) = \sum_{(u)}{}' \frac{g_u}{(2\pi)^3} \int \frac{\sin(\varphi_s - \varphi_u)}{k^2} d\vec{k} ,$$

and

$$\varphi_s = k_\nu x_\nu^{(s)} . \qquad (\text{II}-3.14)$$

Thus the new transformed Dirac equation can be written as

$$\Big[c\, \vec{\alpha}^{(s)} \cdot \vec{p}^{(s)} - g\, \vec{\alpha}^{(s)} \cdot \vec{D}(\vec{r}_s, t_s) + \beta^{(s)} M c^2 -$$

$$- i\hbar \frac{\partial}{\partial t_s} + \frac{g_s}{c} \frac{\partial u_s}{\partial t_s} + \text{Self Energy} \Big] \Omega = 0$$

(II-3.15)

So far every calculation is in relativistic notation. Now we set the particle and field times equal $t_s = t$ and obtain

$$V_s = \frac{g_s}{c} \frac{\partial u_s}{\partial t} = g_s \sum_u{}' \frac{g_u}{4\pi} \frac{e^{-kr}}{r}$$

where $\vec{r} = \vec{x}^{(s)} - \vec{x}^{(u)}$. Thus we see that by eliminating the auxiliary condition and transforming into new field quantities, we have obtained automatically a static Yukawa term. Since there is no auxiliary condition remaining, and there is only a vector field $\vec{D}$ remaining in the two particle Dirac equation, the remaining procedure is similar to that given in Section -2, where we expand again in terms of creation and destruction operators. The Dirac equation can be rewritten as (H_Y = Static Yukawa term)

$$\Big[H_P + H_Y - i\hbar \frac{\partial}{\partial t} \Big] \Omega = \Big[\sum g\, \vec{\alpha}^{(s)} \cdot \vec{D}(\vec{r}_s, t_s) \Big] \cdot$$

$$\Omega = \Big[\sum_s g_s \Big\{ \vec{\alpha}^{(s)} \cdot \vec{A}_T(\vec{r}_s, t_s) + \vec{\alpha}^{(s)} \cdot \vec{D}_3(\vec{r}_s, t_s) \Big\} \Big] \cdot$$

$$\cdot\, \Omega$$

(II-3.16)

where

$$\vec{A}_T(\vec{k}) = \left(\frac{c\hbar}{2k}\right)^{1/2} \sum_j \hat{e}_j \, b(\vec{k}, j),$$

$$\vec{D}_3(k) = \left(\frac{c\hbar}{2k}\right)^{1/2} \hat{e}_3 \, b(\vec{k}, 3).$$

$$(\text{II}-3.17)$$

and

$$g^+(\vec{k}, j) = \frac{1}{(2\pi)^{3/2}}\left(\frac{c\hbar}{2k}\right)^{1/2} \sum_s g \, \vec{\alpha}^{(s)} \cdot \hat{e}_j \cdot e^{-ik_\nu x_\nu^{(s)}}$$

$$g^+(\vec{k}, 3) = \frac{1}{(2\pi)^{3/2}}\left(\frac{c\hbar}{2k}\right)^{1/2} \sum_s g \, \vec{\alpha}^{(s)} \cdot \hat{e}_3 \, e^{ik_\nu x_\nu^{(s)}}.$$

$$(\text{II}-3.18)$$

where the $\hat{e}$'s are unit vectors and j takes on the values 1, 2. The perturbation from Fock techniques give, to the lowest order, the One Boson Exchange Potential as

$$\left(H_p + H_\gamma - i\hbar\frac{\partial}{\partial t}\right)\Omega = -{}^2V\Omega \qquad \text{(a)} \quad (\text{II}-3.19)$$

where

$${}^2V = -\int \frac{g^+(\vec{k}, j)\, g(\vec{k}, j)}{\hbar\omega}\, d\vec{k} \qquad \text{(b)} \quad (\text{II}-3.19)$$

with j = 1, 2, 3. A simplification analogous to Section - 2 yields

$${}^2V = \frac{g^2}{4\pi}\left\{-\frac{\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}}{2} + \frac{(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})}{2r}\frac{d}{dr}\right\}\bar{J}(r)$$

$$= \mathcal{B}\, J(r)$$

$$(\text{II}-3.20)$$

where

$$\bar{J}(r) = \frac{e^{-kr}}{r} \qquad (\text{II}-3.20)$$

This result is shown explicitly by Green (63) together with the subtractive meson in the modified Yukawa form. Thus the vector interaction so obtained is

$$V^{D}_{BV} = (1 + \tilde{B}) J(r) \qquad (\text{II}-3.21)$$

which is the same as derived by Breit (51, 65) from relativistic invariance in nuclear physics without specification of $J(r)$ and by Green (63) from meson field theory.

A third form of vector meson interaction can be derived if the auxiliary condition is treated as a second field equation. The result thus obtained is the Kemmer-vector interaction (4). Thus $\partial_\mu A_\mu = 0$ holds as other field equation. Various theoretical questions about the compatibility of the auxiliary condition with the generalized vector field and the scalar field have been discussed by Rochleder and Green (8, 11) and Stückelnberg (5). The reader is referred to the derivation by perturbation theory by Kemmer (4) or by Fock methods by Rochleder and Green (8, 11). The result in the Diracian form for OBEP is

$$V^{D}_{KV} = \left[1 - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} + \frac{1}{k^2} (\vec{\alpha}^{(1)} \cdot \vec{\nabla}) (\vec{\alpha}^{(2)} \cdot \vec{\nabla})\right] J(r) \qquad (\text{II}-3.22)$$

This differs from the unconstrained vector interaction by the last term. Thus we have completed our study on the One Boson Exchange Potentials from meson field theoretic point of view. We now proceed to solve the two particle Dirac equation in non-relativistic limits. All of these derivations presented above are due to Green (1, 62, 63, 64), Kemmer (4), and due to other standard results from the field theory for some of the interactions. Their inclusion here is for the sake of completeness of the work that will follow in the coming chapters and should be regarded as a background discussion for the convenience to the reader.

CHAPTER III

REDUCTION TO LARGE COMPONENTS - BREIT'S METHOD

Section - 1 Introduction

In the last chapter we sketched field theoretic methods of deriving nucleon-nucleon interactions due to exchange of scalar, vector, and pseudoscalar mesons. In all cases the final interactions contained no reference to the field quantities, except indirectly through meson masses and coupling constants, and were functions of distance of separation between the particles $(r = |\vec{r}| = |\vec{r}^{(1)} - \vec{r}^{(2)}|)$ and of the Dirac matrices $(\vec{\alpha}, \beta, \gamma_5 \text{ etc.})$. This type of interaction will be called Diracian Interaction and the correct quantum-mechanical description of 2-particle system will be assumed to be given by a 2-particle Dirac equation.

In this chapter we describe a method of reducing such 2-particle Dirac equation to a Schrödinger-Pauli form. The method is originally due to Breit (51) and was dealt with in the framework of approximately relativistic interactions between two nucleons, the forms of interactions being fixed by arguments of relativistic invariance. In the framework of field theory, most of the interactions were derived by Green (52) and the method of reduction to large components was used extensively by Green (53) in the period 1948-1950. The details of most of this work are still unpublished. The results of this method of reduction on five vector (scalar + vector) and pseudoscalar interactions were published by Green and Sharma (54), although the consequences of some of these reduced forms were reported earlier by Green (55). Some details of this method and its connections with pseudoscalar meson field have since been published by Breit (56) but not in enough detail so that the origin of various terms occuring in reduced forms could be seen. Also, for some interactions, the Schrödinger-Pauli form has not been derived before. It is the purpose of coming sections to provide this detail and then specialize the general Schrödinger-Pauli form to include specific interactions.

A procedure of reducing one particle Dirac equation to "large components" has been described by Bethe and Salpeter (57) as Pauli approximation. Although their discussion is with reference to one and two electron problems, various arguments are common to one and two nucleon problems also. It should also be kept in mind that the two particle Dirac equation, as it will be treated in this chapter, is not fully Lorentz invariant. It can be written in the general form as

$$\left\{ E - H_p^{(1)} - H_p^{(2)} - V^D \right\} \Omega = 0 \qquad (\text{III}-1.1)$$

where E is the total energy, $H_p^{(i)}$ is free particle Dirac Hamiltonian, V^D is the Diracian Interaction and Ω is the

16-component column wave function made from 4-component wave functions for nucleon 1 and 2. The nature of V^D is decided in our case by One Boson Exchange effects in lowest order of perturbation theory or of the second quantization procedures. It should also be understood that various effects are neglected in deriving V^D, like effect of motion of nucleons during the exchange of mesons, two or more meson exchanges, virtual processes like emission and reabsorption of mesons leading to self-energy divergences, and of pair processes, some of which were referred to in the last chapter. There are other methods of treating 2-nucleon problems which are fully covariant [reference (57), section 42] and the Bethe-Salpeter equation is one of such formalisms. It is, however, very tedious to solve and, therefore, we shall confine ourselves to the 2-particle Dirac equation only, which despite its complexity can be solved in a straightforward way.

Section-2 Reduction of Two Particle Dirac Equation To Schrödinger-Pauli Form

The two particle Dirac equation for stationary states of two nucleon system can thus be written as

$$\left\{E - H_P^{(1)} - H_P^{(2)} - V^D\right\}\Omega = 0 ,$$

$$H_P^{(i)} = -c\,(\vec{\alpha}^{(i)} \cdot \vec{p}^{(i)}) - \beta^{(i)} M c^2 ,$$

$$E = W + 2Mc^2 ,$$

$$M = M^{(1)} = M^{(2)} \qquad (\text{III}-2.1)$$

where symbols are expressed above and interaction between the two nucleons is represented by V^D. The wave function for such a system is a 16-component column vector which is obtained as a result of direct product of two 4-component column vectors representing wave function for each nucleon. This can be expressed as

$$\Omega = \Omega^{(1)} \otimes \Omega^{(2)} = \begin{pmatrix} A^{(1)} \\ C^{(1)} \end{pmatrix} \otimes \begin{pmatrix} A^{(2)} \\ C^{(2)} \end{pmatrix} =$$

$$= \begin{pmatrix} A^{(1)}A^{(2)} & A^{(1)}C^{(2)} \\ C^{(1)}A^{(2)} & C^{(1)}C^{(2)} \end{pmatrix} = \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} \qquad (\text{III}-2.2)$$

where $A^{(1)} = \begin{pmatrix} A_1^{(1)} \\ A_2^{(1)} \end{pmatrix}$, etc., are each a two component spinor and instead of writing a 16-component wave function in a column, Green (53) originally arranged them in above notation for compactness. We have not followed the order in which the components occur in the direct product but rather this grouping is done so as to collect those components which transform in the same way when operated by Dirac matrices. This grouping will, therefore, imply that for all the interactions concerned each 4-component part (such as φ, ψ, χ_1 or χ_2) will keep itself as an identity when acted upon by Dirac matrices. This grouping is, therefore, similar to the special cases (involving lower and upper components of column vectors for each nucleon) given by Chraplyvy (58), the details of which will be discussed in Chapter VI. We are thus summarizing, in a compact notation, what would correspond to 16-coupled equations and writing them as only four coupled equations involving φ, ψ, χ_1 and χ_2 each of which is a 4-component column vector. In the process of reduction we will eliminate all other components in terms of the largest. The purpose of doing so is to see the relativistic effects explicitly in a Schrödinger-like equation.

We now introduce the necessary matrix algebra developed by Green (53) for going from Dirac to Pauli matrices. Let us consider two independent sets of 2 x 2 matrices, one set acting on 4-component wave functions for the particle concerned, denoted by ρ, and another set affecting the spin parts in usual sense and called σ or Pauli spin matrices. We can now decompose any Dirac matrix as a direct product of these 2 x 2 matrices, the details of which are shown below. The same can be achieved alternatively by considering 4 x 4 matrices labeled as ρ and σ by Dirac (59) or by Rosenfeld (60), and then going to their Pauli limits (or nonrelativistic limits), but in that case Dirac matrices are the result of ordinary matrix multiplications between ρ's and σ's and are not direct products. The direct product notation here involves attaching the matrix on right to each element of matrix on left, i.e.,

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} . \tag{III-2.3}$$

We define

$$I_\rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$I_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{III-2.4}$$

With the discussed definition of direct product, the Dirac matrices can be decomposed as

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_1 \otimes \sigma_x = \alpha_x$$

$$\alpha_y = \rho_1 \otimes \sigma_y \quad , \quad \alpha_z = \rho_1 \otimes \sigma_z$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \rho_3 \otimes I_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\beta\gamma_5 = i\beta\alpha_x\alpha_y\alpha_z = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -i(\rho_2 \otimes I_\sigma)$$

Summarizing, we have

$$\vec{\alpha} = \rho_1 \otimes \vec{\sigma}$$

$$\beta = \rho_3 \otimes I_\sigma$$

$$\beta\gamma_5 = -i\rho_2 \otimes I_\sigma \qquad \text{(III-2.5)}$$

We retain the Pauli spin matrices as operators which will appear even in the reduced form and will occur in the Schrödinger-Pauli equation in familiar forms, while we show below the effect of ρ matrices on two particle wave function Ω , where superscripts refer to the particle. We have to remember that at least one operator should act for each particle wave function for the operation to be meaningful. For example

$$\rho_1^{(1)}\rho_1^{(2)}\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{(1)} \begin{pmatrix} A^{(1)} \\ C^{(1)} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^{(2)} \\ C^{(2)} \end{pmatrix} = \begin{pmatrix} C^{(1)} \\ A^{(1)} \end{pmatrix} \otimes \begin{pmatrix} C^{(2)} \\ A^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} C^{(1)}C^{(2)} & C^{(1)}A^{(2)} \\ A^{(1)}C^{(2)} & A^{(1)}A^{(2)} \end{pmatrix} = \begin{pmatrix} \psi & \chi_2 \\ \chi_1 & \varphi \end{pmatrix} .$$

In a similar way, it follows that

$$\rho_2^{(1)}\rho_2^{(2)}\Omega = \begin{pmatrix} -\psi & \chi_2 \\ \chi_1 & -\varphi \end{pmatrix}$$

$$\rho_3^{(1)}\rho_3^{(2)}\Omega = \begin{pmatrix} \varphi & -\chi_1 \\ -\chi_2 & \psi \end{pmatrix}$$

$$\rho_1^{(1)} I_\rho^{(2)}\Omega = \begin{pmatrix} \chi_2 & \psi \\ \varphi & \chi_1 \end{pmatrix}$$

$$I_\rho^{(1)}\rho_1^{(2)}\Omega = \begin{pmatrix} \chi_1 & \varphi \\ \psi & \chi_2 \end{pmatrix}$$

$$\rho_3^{(1)} I_\rho^{(2)}\Omega = \begin{pmatrix} \varphi & \chi_1 \\ -\chi_2 & -\psi \end{pmatrix}$$

$$I_\rho^{(1)}\rho_3^{(2)}\Omega = \begin{pmatrix} \varphi & -\chi_1 \\ \chi_2 & -\psi \end{pmatrix} \qquad (\text{III}-2.6)$$

Now we have the required algebra and notation for the reduction. We rewrite our Dirac equation for two particles as

$$\left[E + \left\{ c\,(\vec{\alpha}^{(1)}\cdot\vec{p}^{(1)}) + \beta^{(1)}Mc^2 + c\,(\vec{\alpha}^{(2)}\cdot\vec{p}^{(2)}) + \beta^{(2)}Mc^2 \right\} - V^D \right]\Omega = 0 \, . \qquad (\text{III}-2.7)$$

With the help of (III-2.5) we rewrite this as

$$\Big[E\,(I_\rho \otimes I_\sigma)^{(1)}(I_\rho \otimes I_\sigma)^{(2)} + c\,(\rho_1 \otimes \vec{\sigma}\cdot\vec{p})^{(1)}(I_\rho \otimes I_\sigma)^{(2)} +$$

$$+ c\,(I_\rho \otimes I_\sigma)^{(1)}(\rho_1 \otimes \vec{\sigma}\cdot\vec{p})^{(2)} + Mc^2 \{(\rho_3 \otimes I_\sigma)^{(1)}(I_\rho \otimes I_\sigma)^{(2)} +$$

$$+ (I_\rho \otimes I_\sigma)^{(1)}(\rho_3 \otimes I_\sigma)^{(2)} \} - V^D \Big]\,\Omega = 0 \,. \tag{III-2.8}$$

Using (III-2.6) we get

$$\Big[E \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} + c\,(\vec{\sigma}^{(1)}\cdot\vec{p}^{(1)}) \begin{pmatrix} \chi_2 & \psi \\ \varphi & \chi_1 \end{pmatrix} + c\,(\vec{\sigma}^{(2)}\cdot\vec{p}^{(2)}) \begin{pmatrix} \chi_1 & \varphi \\ \psi & \chi_2 \end{pmatrix} +$$

$$+ 2Mc^2 \begin{pmatrix} \varphi & 0 \\ 0 & -\psi \end{pmatrix} \Big] = V^D \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} . \tag{III-2.9}$$

All of the interactions that we will use can be placed in the form

$$V^D \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} = \begin{pmatrix} V_a\varphi + V_b\psi & V_c\chi_1 + V_d\chi_2 \\ V_d\chi_1 + V_c\chi_2 & V_b\varphi + V_a\psi \end{pmatrix} . \tag{III-2.10}$$

Thus the two particle Dirac equation can be written as four coupled equations.

$$\begin{aligned}
(E + 2Mc^2)\varphi + P_2\chi_1 + P_1\chi_2 &= V_a\varphi + V_b\psi \\
P_2\varphi + E\chi_1 + P_1\psi &= V_c\chi_1 + V_d\chi_2 \\
P_1\varphi + E\chi_2 + P_2\psi &= V_d\chi_1 + V_c\chi_2 \\
P_1\chi_1 + P_2\chi_2 + (E - 2Mc^2)\psi &= V_b\varphi + V_a\psi
\end{aligned} \tag{III-2.11}$$

where

$$P_i = c\,(\vec{\sigma}^{(i)}\cdot\vec{p}^{(i)}) .$$

These equations have also been reported by Breit (56) but only for the pseudoscalar interaction. Defining

$$E = W + 2Mc^2$$

$$\epsilon_a = \frac{W - V_a}{2Mc^2} \quad \text{e.t.c.}$$

we rewrite (III-2.11) as

$$4Mc^2\left(1 + \frac{\epsilon_a}{2}\right)\varphi - V_b\psi + P_2\chi_1 + P_1\chi_2 = 0 \qquad (a)$$

$$2Mc^2\left(1 + \epsilon_c\right)\chi_1 - V_d\chi_2 + P_2\varphi + P_1\psi = 0 \qquad (b)$$

$$2Mc^2\left(1 + \epsilon_c\right)\chi_2 - V_d\chi_1 + P_1\varphi + P_2\psi = 0 \qquad (c)$$

$$(W - V_a)\psi - V_b\varphi + P_1\chi_1 + P_2\chi_2 = 0 \,. \qquad (d)$$

(III-2.12)

We solve for χ_1 and χ_2 in terms of φ and ψ using (III-2.12 b and c) and then we use (III-2.12 a) to get φ in terms of ψ . This is then used to express χ_1 and χ_2 only in terms of ψ . Then we use (III-2.12 d) to get an equation in ψ. It is clear from (III-2.12 d) that ψ is the large component because every other component has Mc^2 multiplying it. In deriving χ_1 and χ_2 we assume that V_c and V_d commute as indeed they do. Then we find

$$\chi_1 = -\Theta_1\left\{\Theta_2\varphi + \Theta_3\psi\right\} \qquad (a)$$

and

$$\chi_2 = -\Theta_1\left\{\Theta_3\varphi + \Theta_2\psi\right\} \,, \qquad (b)$$

(III-2.13)

where

$$\theta_1 = \left[4M^2c^4(1+\epsilon_c)^2 - V_d^2 \right]^{-1} , \qquad (a)$$

$$\theta_2 = \left[2Mc^2(1+\epsilon_c) P_2 + V_d P_1 \right] , \qquad (b)$$

and

$$\theta_3 = \left[2Mc^2(1+\epsilon_c) P_1 + V_d P_2 \right] . \qquad (c)$$

(III-2.14)

From (III-2.12 a) we have

$$\varphi = \theta_4 \theta_5 \psi , \qquad (a)$$

$$\theta_4 = \left[4Mc^2(1+\tfrac{\epsilon_a}{2}) - (P_2\theta_1\theta_2 + P_1\theta_1\theta_3) \right]^{-1} ,$$

and (b)

$$\theta_5 = V_b + P_2\theta_1\theta_3 + P_1\theta_1\theta_2 . \qquad (c)$$

(III-2.15)

Using (III-2.15 a) in (III-2.13) we see that

$$\chi_1 = -\theta_1 \left\{ \theta_3 + \theta_2\theta_4\theta_5 \right\} \psi$$

and

(III-2.16)

$$\chi_2 = -\theta_1 \left\{ \theta_2 + \theta_3\theta_4\theta_5 \right\} \psi .$$

We use (III-2.15) and (III-2.16) in (III-2.12) and we find an equation in ψ, the large component, which is exact since no approximations have been made thus far. Therefore,

$$\Big[(W-V_a) - V_b\theta_4\theta_5 - P_1\theta_1(\theta_3 + \theta_2\theta_4\theta_5) - P_2\theta_1(\theta_2 + \theta_3\theta_4\theta_5) \Big] \psi = 0 .$$

(III-2.17)

This we know is only a formal way of writing the equation compact form and $\theta'\mathcal{S}$ involve interactions and spin and momentum operators and therefore must be treated carefully. Now we have to start approximations to reduce this equation to the more familiar Schrödinger-Pauli form.

Approximations

A correct relativistic treatment of the problem of nucleon-nucleon force would require the exact solution of the two particle Dirac equation. The preliminary results of such a recent study by Sawada and Green (61) have been reported. The detailed study is still in progress. The purpose of present work has been to consider nucleon-nucleon force problems for non-relativistic energies. Therefore it should be possible to develop (III-2.17) in a series, using as the parameter of largeness the rest mass of the nucleons. We assume the interactions to be small as compared with this quantity. In other words the expansion parameters are $(i = a, b, c, d)$

$$\frac{W}{Mc^2} \ll 1, \quad \frac{V_i}{Mc^2} \ll 1, \quad \frac{P_1}{Mc^2} \ll 1, \quad \frac{P_2}{Mc^2} \ll 1,$$

each of which is assumed to be less than unity. Before proceeding further, let us examine the nature of these approximations.

$\frac{W}{Mc^2}$ implies that the external kinetic energy imparted to the system should be small compared to its rest energy. Since for the elastic scattering problem we shall always be below 400 Mev in the laboratory framework, this condition will naturally be satisfied. The second condition however, is a severe restriction in that V_i/Mc^2 is not necessarily small if Vi is the Yukawa form whose radial dependence is $J(r) = g^2(\hbar c)r^{-1}\exp(-\kappa r)$. This function goes to infinity at the origin and may not be small even at other distances depending on the nature, mass, and coupling of the mesons. The quantitative behavior of these forms will be clear from the potentials reported in Chapter IV. Detailed studies of this problem both in quantum electrodynamics and in quantum field theory have been made by Green (62, 63, 64) on the basis of Lagrangians having higher derivatives of fields. As a consequence various new features and theoretical questions regarding subtractive particles have emerged which may prove helpful in solving the puzzles regarding the nature of nucleon-nucleon force at short distances. In our study we have only used the subtractive meson as a cut-off employing it in the same way as phenomenological cut-off's are employed in literature. Thus we use the form

$$J(r) = g^2(\hbar c)\left[\frac{e^{-\kappa r}}{r} - \frac{e^{-\Lambda r}}{r}\right]$$

which unlike the simple Yukawa expression does not blow up near the origin. The physical uncertainties of interactions due to heavier mass mesons and various other processes are thus nicely parameterized in terms of cut-off masses. At the same time the device helps to make our expansions valid. The third assumption P_1/Mc^2 implies, in classical terms, v/c to be small which restricts the velocity of relative motion of nucleons which we can consider.

With these justifications for expansion, we only retain the terms which will correct our potential to the order $1/M^2$ (or relativistic corrections of order up to v^2/c^2) and will also give first correction to the kinetic energy. Using the Binomial Expansion

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\cdots \qquad (\text{III}-2.18)$$

we obtain, using (III-2.14 a, b, c) and (III-2.15 b,c),

$$\theta_1 = \frac{1}{4M^2c^4(1+\epsilon_c)^2 - V_d^2}$$

$$= \frac{1}{4M^2c^4}\left(1 - 2\epsilon_c\right) , \qquad (a)$$

$$\theta_2 = 2Mc^2(1+\epsilon_c)P_2 + V_d P_1$$

$$= 2Mc^2\left[(1+\epsilon_c)P_2 + \frac{V_d}{2Mc^2}P_1\right] , \qquad (b)$$

$$\theta_3 = 2Mc^2\left[(1+\epsilon_c)P_1 + \frac{V_d}{2Mc^2}P_2\right] , \qquad (c)$$

$$\theta_4 = \left[4Mc^2(1+\epsilon_a/2) - (P_2\theta_1\theta_2 + P_1\theta_1\theta_3)\right]^{-1}$$

$$= \frac{1}{4Mc^2}\left[1 - \frac{\epsilon_a}{2} + \frac{(P_1^2+P_2^2)}{2Mc^2}\right] \qquad (d)$$

and

$$\theta_5 = V_b + P_2\theta_1\theta_3 + P_1\theta_1\theta_2 =$$

$$= V_b + \frac{2P_1P_2}{2Mc^2} + \frac{P_1V_dP_1}{4M^2c^4} + \frac{P_2V_dP_2}{4M^2c^4} - \frac{(P_2\epsilon_cP_1 + P_1\epsilon_cP_2)}{2Mc^2} . \qquad (e)\ (\text{III}-2.19)$$

Substituting (III-2.19) in (III-2.17) we get the equation in "large component" ψ correct to order $1/M^2$, as

$$\Big[\Big[(W-V_a) - \frac{V_b}{4Mc^2}\left(1-\frac{\epsilon_a}{2}+\frac{\{P_1^2+P_2^2\}}{2Mc^2}\right)\left(V_b+\frac{2P_1P_2-\{P_2\epsilon_cP_1+P_1\epsilon_cP_2\}}{2Mc^2}\right)$$

$$-P_1\left\{\frac{1}{2Mc^2}\left[P_2+\left(\frac{V_d}{2Mc^2}P_1-\epsilon_cP_2\right)\right]\frac{1}{4Mc^2}\left[1-\frac{\epsilon_a}{2}+\frac{(P_1^2+P_2^2)}{2Mc^2}\right]\cdot\right.$$

$$\left.\cdot\left[V_b+\frac{2P_1P_2}{2Mc^2}-\frac{(P_2\epsilon_cP_1+P_1\epsilon_cP_2)}{2Mc^2}\right]+\frac{1}{2Mc^2}\left[P_1+\left(\frac{V_d}{2Mc^2}P_2-\epsilon_cP_1\right)\right]\right\}$$

$$-P_2\left\{\frac{1}{2Mc^2}\left[P_1+\left(\frac{V_d}{2Mc^2}P_2-\epsilon_cP_1\right)\right]\frac{1}{4Mc^2}\left[1-\frac{\epsilon_a}{2}+\frac{(P_1^2+P_2^2)}{2Mc^2}\right]\cdot\right.$$

$$\cdot\left[V_b+\frac{2P_1P_2}{2Mc^2}-\frac{(P_2\epsilon_cP_1+P_1\epsilon_cP_2)}{2Mc^2}\right]+\frac{1}{2Mc^2}\left[P_2+\left(\frac{V_d}{2Mc^2}P_1-\right.\right.$$

$$\left.\left.\left.-\epsilon_cP_2\right)\right]\right\}\Big]\psi=0$$

Now also we reject terms of order higher than $1/M^2$ in the potential and retain p^4/M^3 type terms as correction to kinetic energy while taking the product of various factors in the last equation. The result is

$$\Big[W-V_a-\frac{V_b^2}{4Mc^2}-\frac{(P_1^2+P_2^2)}{2Mc^2}-\frac{2(V_bP_1P_2+P_1P_2V_b)}{8M^2c^4}+$$

$$+\frac{V_b\epsilon_aV_b}{8Mc^2}-\frac{V_b(P_1^2+P_2^2)V_b}{8M^2c^4}+\frac{1}{2Mc^2}(P_1\epsilon_cP_1+P_2\epsilon_cP_2)-$$

$$-\frac{(P_1V_dP_2+P_2V_dP_1)}{4M^2c^4}-\frac{2(P_1P_2)^2}{8M^3c^6}\Big]\psi=0. \qquad \text{(III-2.20)}$$

Using the proper definitions of ϵ's from (III-2.11) we rewrite

$$\Big[W-V_a-\frac{(P_1^2+P_2^2)}{2Mc^2}+\frac{W(P_1^2+P_2^2)}{4M^2c^4}-\frac{V_b^2}{4Mc^2}-\frac{(P_1P_2)^2}{4M^3c^6}-$$

$$-\frac{1}{4M^2c^4}\left\{P_1P_2V_b+V_bP_1P_2+P_1V_cP_1+P_2V_cP_2+P_1V_dP_2+P_2V_dP_1\right\}+$$

$$+\frac{V_b(W-V_a)V_b}{16M^2c^4}-\frac{V_b(P_1^2+P_2^2)V_b}{8M^2c^4}\Big]\psi=0. \qquad \text{(III-2.21)}$$

This equation as we see is already a Schrödinger type but we should remember that the wave function ψ is not normalized.

Normalization

We should clearly understand that ψ occuring in (III-2.21) is a 4-component wave function formed out of direct product of 2-component spinor wave functions for each particle and these two components are lower-lower components of original 4-component wave functions for particles one and two. Hence our equation (III-2.21) has reduced to the analogous form of what would be called 2-component equation for a single particle Dirac equation. We now consider the case of single particle Dirac equation, to see the effects of normalization as discussed by Breit (51). The Dirac equation for a free particle can be written in 2-component form as

$$\begin{aligned} (E+Mc^2)\,\Phi + c(\vec{\sigma}\cdot\vec{p})\Psi &= 0 \\ (E-Mc^2)\,\Psi + c(\vec{\sigma}\cdot\vec{p})\Phi &= 0 \,. \end{aligned} \qquad \text{(III-2.22)}$$

The symbols for 2-component Ψ, Φ are only for temporary use. To the lowest approximation

$$\Phi = \frac{(\vec{\sigma}\cdot\vec{p})}{2Mc}\Psi \qquad \text{(III-2.23)}$$

For a single particle case, the total probability of finding the particle is given by

$$\begin{aligned} \int \psi_\mu^+ \psi_\mu \, d\tau &= \int \left(\Phi^+\Phi + \Psi^+\Psi\right) d\tau \,, \quad (\mu=1,2,3,4) \\ &= \int \Psi^+ (1 + p^2/4M^2c^2)\, \Psi \, d\tau \,, \end{aligned} \qquad \text{(III-2.24)}$$

which means that if we define a 2-component wave function as

$$\Psi_0 = \left(1 + \frac{p^2}{8M^2c^2}\right)\Psi , \qquad (b)\ (\text{III-2.24})$$

then we have the wave function normalized. This transformation also leaves the resulting Hamiltonian in Hermitian form, as discussed by Breit (51).

This concept may be extended for 2-particle Dirac equations also. To do this we express φ, χ_1 and χ_2 in terms of ψ. By considering only interaction free terms of equation (III-2.11) and by using (III-2.15 a) and (III-2.16) we get to lowest order in $1/M$, the following expressions

$$\varphi \sim \frac{P_1 P_2}{4M^2c^4}\psi , \qquad (a)$$

$$\chi_1 \sim -\frac{P_1}{2Mc^2}\psi \qquad (b)$$

and

$$\chi_2 \sim -\frac{P_2}{2Mc^2}\psi . \qquad (c) \qquad (\text{III-2.25})$$

We only keep those terms in normalization which will be of order $1/M^2$ as they alone can influence the terms which we are looking for. Thus the normalization takes the form

$$1 = \iint \Omega^+\Omega \, d\tau_1 d\tau_2 = \iint (\psi^+\psi + \varphi^+\varphi + \chi_1^+\chi_1 + \chi_2^+\chi_2)\cdot d\tau_1 d\tau_2$$

$$= \iint \psi^+\left(1 + \frac{P_1^2 + P_2^2}{4M^2c^4}\right)\psi \, d\tau_1 d\tau_2 .$$

Therefore

$$\Psi = \left(1 + \frac{P_1^2 + P_2^2}{8M^2c^4}\right)\psi \equiv (1 + \nu/2)\psi \approx e^{\nu/2}\psi \qquad (\text{III-2.26})$$

It happens that even though $\iint \Psi^{+}\Psi \, d\tau_1 \, d\tau_2 = 1$, still we should not interpret $(\Psi^{+}\Psi)$ as the particle density. This has been discussed by Breit (65) in some detail.

Before we start normalization, we notice that equation (III-2.21) can be written as

$$W(1+\nu)\psi = [\ ]_{op}\,\psi\,, \qquad \text{(III-2.27)}$$

where $[\ ]_{op}$ involves remaining terms of (III-2.21). We can write this by multiplying with $(1-\nu)$ to get (to linear order in ν)

$$W\psi = (1-\nu)[\ ]_{op}\,\psi\,, \qquad \text{(III-2.28)}$$

which means explicitly

$$\Big[W - V_a - \frac{(P_1^2+P_2^2)}{2Mc^2} - \frac{V_b^2}{4Mc^2} - \frac{(P_1P_2)^2}{4M^3c^6} + \frac{V_b(W-V_a)V_b}{16M^2c^4} -$$

$$- \frac{V_b(P_1^2+P_2^2)V_b}{8M^2c^4} - \frac{1}{4M^2c^4}\Big\{ P_1P_2V_b + V_bP_1P_2 + P_1V_cP_1 +$$

$$+ P_2V_cP_2 + P_1V_dP_2 + P_2V_dP_1 \Big\} - \frac{(P_1^2+P_2^2)}{4M^2c^4}\,\cdot$$

$$\cdot\left(-V_a - \frac{(P_1^2+P_2^2)}{2Mc^2}\right)\Big]\psi = 0\,, \qquad \text{(III-2.29)}$$

where in $(\nu[\ ]_{op})$ we have neglected terms of order higher than $1/M^2$ in potential and $1/M^3$ in kinetic energy corrections. We can say by looking at the term $[(P_1^2+P_2^2)V_a]/4M^2c^4$, that the Hamiltonian is not Hermitian and this difficulty, as we shall see, will be remedied by normalization. Normalization here implies that we want an equation in normalized wave function and therefore the Hamiltonian acting on it will also be transformed. This canonical transformation can be seen mathematically as follows. We write (III-2.29) as

$$W\psi = [\;]_1 \psi \quad , \qquad \text{(III-2.30)}$$

where $[\;]_1$ contains remaining terms of (III-2.29). Then

$$W e^{-\nu/2}\Psi = [\;]_1 e^{-\nu/2}\Psi$$

or

$$W\Psi = \left\{ e^{\nu/2} [\;]_1 e^{-\nu/2} \right\} \Psi \equiv [\;]_1' \Psi \qquad \text{(III-2.31)}$$

which to the linear order in ν implies

$$[\;]_1' = (1 + \nu/2)[\;]_1 - [\;]_1(\nu/2) \,. \qquad \text{(III-2.32)}$$

Now we write our transformed equation (III-2.29) as

$$\Big[W - V_a - \frac{(P_1^2 + P_2^2)}{2Mc^2} - \frac{V_b^2}{4Mc^2} - \frac{(P_1 P_2)^2}{4M^3c^6} + \frac{V_b(W - V_a)V_b}{16M^2c^4} -$$

$$- \frac{V_b(P_1^2 + P_2^2)V_b}{8M^2c^4} - \frac{1}{4M^2c^4}\Big\{ P_1 P_2 V_b + V_b P_1 P_2 + P_1 V_c P_1 +$$

$$+ P_2 V_c P_2 + P_1 V_d P_2 + P_2 V_d P_1 \Big\} + \frac{(P_1^2 + P_2^2)V_a}{4M^2c^4} + \frac{(P_1^2 + P_2^2)^2}{8M^3c^6} +$$

$$+ \frac{(P_1^2 + P_2^2)}{8M^2c^2}\left(\frac{-(P_1^2 + P_2^2)}{2Mc^2} - V_a \right) - \left(-\frac{(P_1^2 + P_2^2)}{2Mc^2} - V_a \right)\frac{(P_1^2 + P_2^2)}{8M^2c^4} \Big] \Psi = 0 \,.$$

$$\text{(III-2.33)}$$

Here also we have retained terms consistent with previous arguments. We now get the final form of Schrödinger-Pauli equation in which the potential is Hermitian to order $1/M^2$, (or v^2/c^2) and we also get relativistic correction to the kinetic energy. We here make an assumption that the terms

$$\frac{V_b(W-V_a)V_b}{16M^2c^4} - \frac{V_b(P_1^2+P_2^2)V_b}{8M^2c^4}$$

(i.e. like V_i^2/M^2) are small as compared to V_i/M^2 terms and this is true only if V_i happens to be small. The other term $V_b^2/4Mc^2$ has only $1/M$ dependence and is a fairly important term which would be discussed in detail in Chapter VI. Hence for the moment we retain only linear terms in potential and obtain

$$\Big[W - \frac{(P_1^2+P_2^2)}{2M} + \frac{(P_1^2+P_2^2)^2}{8M^3c^6} - \frac{(P_1P_2)^2}{4M^3c^6} - V_a + $$
$$+ \frac{1}{8M^2c^4}\left\{(P_1^2+P_2^2)V_a + V_a(P_1^2+P_2^2)\right\} - \frac{1}{4M^2c^4}\Big\{P_1P_2V_b +$$
$$+ V_bP_1P_2 + P_1V_cP_1 + P_2V_cP_2 + P_1V_dP_2 + P_2V_dP_1\Big\}\Big]\Psi = 0 .$$
$$(\text{III}-2.34)$$

Now we specialize ourselves to the special reference system in which the total momentum of the 2-particles is zero but the relative momentum is not. We do not put any restrictions on V_i's (i = a, b, c, d) except that $[V_c, V_d] = 0$, but we assume the dependence of radial functions only upon relative distance between particles. Breit (65) took this approach to consider whether the transformation of coordinate systems would change the invariance arguments (to order v^2/c^2) discussed by him, but considered only stationary state (time independen) problem and commented that for that particular reason it was not necessary to decouple the center of mass and relative frame equations. The equations obtained in this approach would decouple if we require $[\,(\vec{p}^{(1)}+\vec{p}^{(2)})\equiv 0\,]$, a transformation to zero momentum in the center of mass frame. This particular physical frame is chosen by Breit (56) later and leads to similar equations as (III-2.34). Therefore if we set in our case that V's be radial functions of only r

$$r=|\vec{r}|=|\vec{r}^{(1)}-\vec{r}^{(2)}|,$$
$$\vec{R}=\tfrac{1}{2}(\vec{r}^{(1)}+\vec{r}^{(2)})$$
$$\vec{P}=\vec{p}^{(1)}+\vec{p}^{(2)}\equiv 0$$

and

$$\vec{p}=\tfrac{1}{2}(\vec{p}^{(1)}-\vec{p}^{(2)})$$
$$(\text{i.e. } \vec{p}^{(1)}=\vec{p},\quad \vec{p}^{(2)}=-\vec{p}) \qquad (\text{III-2.35})$$

then the wave equation for center of mass frame can be separated out and only the physically interesting part remains in the equation for relative motion between the particles (III-2.34), which in view of $P_1^2=P_2^2=c^2p^2$, $(P_1=c(\vec{\sigma}^{(1)}\cdot\vec{p}),\ P_2=c(\vec{\sigma}^{(2)}\cdot\vec{p}))$ becomes

$$\Big[\,W-\frac{p^2}{M}+\frac{p^4}{4M^3c^2}+\frac{1}{4M^2c^2}\{p^2V_a+V_ap^2\}-V_a+$$
$$+\frac{1}{4M^2c^2}\{(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})V_b+V_b(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\}-$$
$$-\frac{1}{4M^2c^2}\{(\vec{\sigma}^{(1)}\cdot\vec{p})V_c(\vec{\sigma}^{(1)}\cdot\vec{p})+(\vec{\sigma}^{(2)}\cdot\vec{p})V_c(\vec{\sigma}^{(2)}\cdot\vec{p})-$$
$$-(\vec{\sigma}^{(1)}\cdot\vec{p})V_d(\vec{\sigma}^{(2)}\cdot\vec{p})-(\vec{\sigma}^{(2)}\cdot\vec{p})V_d(\vec{\sigma}^{(1)}\cdot\vec{p})\}\,\Big]\Psi=0. \qquad (\text{III-2.36})$$

This result (III-2.36) in the most general form was obtained by Green (53) in 1948 and was applied then by him to many of the meson theoretic potentials that will be treated in the next section. This equation will form the basis of furthur work and other methods of checking this result, e.g. by mathematical manipulations of Breit's work, by methods of sums and differences, etc., will be discussed in Chapter VI. Green and Sharma (54), in 1965, published the results of this equation as applied to pseudoscalar and scalar + vector meson interactions in the light of a purely relativistic zero parameter model of nucleon-nucleon force which will be discussed in coming chapters. Now we proceed to apply this equation (III-2.36) to various forms of one Boson Exchange potentials discussed in Chapter II.

Section - 3 Application To Various Interactions

The interactions that we discussed in Chapter II can be written in the Diracian form as follows.

Scalar Meson Interaction

$$V_S^D = -\beta^{(1)}\beta^{(2)} J_S(r) . \qquad (\text{III}-3\ 1)$$

Unconstrained Vector Meson Interaction

$$V_{UV}^D = (\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J_{UV}(r) . \qquad (\text{III}-3\cdot 2)$$

(Derived in meson field theory by Green when no auxiliary condition is imposed.

Pseudoscalar Meson Interaction

$$V_P^D = (\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)} J_P(r) . \qquad (\text{III}-3\cdot 3)$$

Breit Vector Meson Interaction (Derived in meson field theory by Green, as a result of constraint due to auxiliary condition on state vector.)

$$V_{BV}^D = \left(\mathbb{1} - \frac{\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}}{2} + \frac{(\vec{\alpha}^{(1)} \cdot \vec{r})(\vec{\alpha}^{(2)} \cdot \vec{r})}{2r} \frac{d}{dr}\right) J_{BV}(r) . \qquad (\text{III}-3\cdot 4)$$

Kemmer Vector Meson Interaction (Derived by Kemmer using auxiliary condition to eliminate one component of vector field.)

$$V^{D}_{KV} = \left(\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} + \frac{1}{K^2} (\vec{\alpha}^{(1)} \cdot \vec{\nabla})(\vec{\alpha}^{(2)} \cdot \vec{\nabla}) \right) J_{KV}(r). \qquad (\text{III-3.5})$$

In (III-3.5) $K = m_V c/\hbar$ is the inverse compton wavelength of the meson under consideration.

In our notations we will quite often suppress the subscripts on J's and also the argument whenever it is needed for compactness of notation within each subsection but it should be understood that the nature, mass, and couplings of mesons are implicit in J and transformation properties in relativistic form are shown by the Dirac matrices ($\mathbb{1} \Rightarrow$ unit (4 x 4) matrix for each particle). We now show, by using matrix algebra developed in Section - 2, that all these interactions can be put in the same general form.

Using (III-2.5) and (III-2.6) we rewrite equations (III-3.1) through (III-3.5) as

$$V_S^D \Omega = -(\rho_3 \otimes I_\sigma)^{(1)} (\rho_3 \otimes I_\sigma)^{(2)} J_S(r) \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix}$$

or

$$-\beta^{(1)} \beta^{(2)} J_S \Omega = + J_S \begin{pmatrix} -\varphi & +\chi_1 \\ +\chi_2 & -\psi \end{pmatrix}, \qquad (\text{III-3.6})$$

$$\begin{aligned} V^{D}_{UV} \Omega &= (\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J_{UV} \Omega \\ &= J_{UV} \left[\begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} - (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \left(\rho_1^{(1)} \rho_1^{(2)} \right) \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} \right] \\ &= J_{UV} \left[\begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} - (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \begin{pmatrix} \psi & \chi_1 \\ \chi_2 & \varphi \end{pmatrix} \right], \end{aligned} \qquad (\text{III-3.7})$$

$$V^{D}_{P} \Omega = (\beta \gamma_5)^{(1)} (\beta \gamma_5)^{(2)} J_P \Omega = i^2 \rho_2^{(1)} \rho_2^{(2)} J_P \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} = J_P \begin{pmatrix} \psi & -\chi_2 \\ -\chi_1 & \varphi \end{pmatrix}; \qquad (\text{III-3.8})$$

$$V^D_{BV}\Omega = \left(1 - \frac{\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}}{2} + \frac{(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})}{2r}\frac{d}{dr}\right) J_{BV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix}$$

$$= \left[J_{BV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix} + \left(-\frac{(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})}{2} + \frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})}{2r}\frac{d}{dr}\right) J_{BV}\left(\rho_1^{(1)}\rho_1^{(2)}\right)\cdot\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix}\right]$$

$$= J_{BV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix} + \left(-\frac{(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})}{2} + \frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})}{2r}\frac{d}{dr}\right) J_{BV}\begin{pmatrix}\psi & \chi_2\\ \chi_1 & \varphi\end{pmatrix}, \qquad \text{(III-3.9)}$$

$$V^D_{KV}\Omega = \left(1 - \vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)} + \frac{1}{K^2}(\vec{\alpha}^{(1)}\cdot\vec{\nabla})(\vec{\alpha}^{(2)}\cdot\vec{\nabla})\right) J_{KV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix}$$

$$= J_{KV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix} + \left[-(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + \frac{1}{K^2}(\vec{\sigma}^{(1)}\cdot\vec{\nabla})(\vec{\sigma}^{(2)}\cdot\vec{\nabla})\right] J_{KV}\begin{pmatrix}\psi & \chi_2\\ \chi_1 & \varphi\end{pmatrix}, \qquad \text{(III-3.10)}$$

where the derivatives occuring in Diracian forms act, by definition from field theory, only on the J's and not on the wave function. Therefore, we write these in terms of the derivatives of the Yukawa form according to the definition

$$J_n(r) = \left(\frac{1}{r}\frac{d}{dr}\right) J_{n-1}(r), \quad J_1 = \left(\frac{1}{r}\frac{d}{dr}\right) J. \qquad \text{(III-3.11)}$$

Hence we can write (III-3.9) as

$$V^D_{BV}\Omega = J_{BV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix} + \left(-\frac{(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})}{2}\right) J_{BV}\begin{pmatrix}\psi & \chi_2\\ \chi_1 & \varphi\end{pmatrix} + \frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})}{2} J_1^{BV}\begin{pmatrix}\psi & \chi_2\\ \chi_1 & \varphi\end{pmatrix}$$

$$= J^{BV}\begin{pmatrix}\varphi & \chi_1\\ \chi_2 & \psi\end{pmatrix} + \left\{-\frac{(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})}{2} J^{BV} + \frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})}{2} J_1^{BV}\right\}\cdot\begin{pmatrix}\psi & \chi_2\\ \chi_1 & \varphi\end{pmatrix}, \qquad \text{(III-3.12)}$$

and from (III-3.10)

$$(\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \vec{\nabla}) J_{KV} = (\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \langle \vec{\nabla} J \rangle^{KV})$$

$$= (\vec{\sigma}^{(1)} \cdot \vec{\nabla}) \{ J_1{}^{KV} (\vec{\sigma}^{(2)} \cdot \vec{r}) \}$$

$$= (\vec{\sigma}^{(1)} \cdot \langle \vec{\nabla} J_1{}^{KV} \rangle)(\vec{\sigma}^{(2)} \cdot \vec{r}) + J_1{}^{KV} (\vec{\sigma}^{(1)} \cdot \vec{\nabla})(\vec{\sigma}^{(2)} \cdot \vec{r})$$

$$= J_2{}^{KV} (\vec{\sigma}^{(1)} \cdot \vec{r})(\vec{\sigma}^{(2)} \cdot \vec{r}) + J_1{}^{KV} (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \qquad \text{(III-3.13)}$$

which also implies

$$(\vec{\alpha}^{(1)} \cdot \vec{\nabla})(\vec{\alpha}^{(2)} \cdot \vec{\nabla}) J^{KV}$$

$$= (\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J_1^{KV} + (\vec{\alpha}^{(1)} \cdot \vec{r})(\vec{\alpha}^{(2)} \cdot \vec{r}) J_2^{KV} \qquad \text{(III-3.14)}$$

a relation that will prove very helpful later.

By looking at (III-3.6) through (III-3.11), we get the table 1 defining the V_i's (i = a, b, c, d) according to equation (III-2.10), $\sigma_{12} = (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$ and $\sigma_{12}^{r} = (\vec{\sigma}^{(1)} \cdot \vec{r})(\vec{\sigma}^{(2)} \cdot \vec{r})$.

Table 1

Meson Exchange Interaction

V_i	S	P	UV	BV	KV
V_a	$-J_S$	0	J^{UV}	J^{BV}	J^{KV}
V_b	0	J_P	$-\sigma_{12} J^{UV}$	$-\frac{\sigma_{12}}{2} J^{BV} + \sigma_{12}^{r} \cdot \frac{J_1{}^{BV}}{2}$	$-\sigma_{12} J^{KV} + \frac{1}{K^2}\{\sigma_{12} \cdot J_1{}^{KV} + \sigma_{12}^{r} J_2{}^{KV}\}$
V_c	J_S	0	J^{UV}	J^{BV}	J^{KV}
V_d	0	$-J_P$	$-\sigma_{12} J^{UV}$	$-\sigma_{12} J^{BV} + \sigma_{12}^{r} \cdot \frac{J_1{}^{BV}}{2}$	$-\sigma_{12} J^{KV} + \frac{1}{K^2}\{\sigma_{12} J_1{}^{KV} + \sigma_{12}^{r} J_2{}^{KV}\}$

The most general Schrödinger-Pauli form of interaction is given by (III-2.36) as

$$V^P = V_a - \frac{1}{4M^2c^2}\Big\{ p^2 V_a + V_a p^2 + (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})V_b + V_b(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(1)}\cdot\vec{p})V_c(\vec{\sigma}^{(1)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{p})\cdot V_c(\vec{\sigma}^{(2)}\cdot\vec{p}) + (\vec{\sigma}^{(1)}\cdot\vec{p})V_d(\vec{\sigma}^{(2)}\cdot\vec{p}) + (\vec{\sigma}^{(2)}\cdot\vec{p})V_d(\vec{\sigma}^{(1)}\cdot\vec{p})\Big\} \tag{III-3.15}$$

Now we go to special cases with the notation that an angled bracket $\langle\ \rangle$ implies that the operators in the bracket do not act outside the bracket, and therefore it should not be confused with usual notation for expectation value in quantum mechanics. Also we set temporarily $\hbar = 1$ so that $\vec{p} = -i\vec{\nabla}$ and also we suppress any superscripts or subscripts for J until we write the last step in such a process when we use them for differentiating various J's. We also list below the important operator-algebraic, vector and spinor identities which are used to reduce these interactions to the final form.

$$\vec{l} = \vec{r}\times\vec{p}$$

$$p^2(J\psi) = \{Jp^2 + 2\langle p\vec{J}\rangle\cdot\vec{p} + \langle p^2 J\rangle\}\psi \qquad (a)$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\vec{A}\cdot\vec{B}) + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \qquad (b)$$

$$\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times B \qquad (c)\ (III\text{-}3.16)$$

$$\vec{\nabla}\cdot(\phi\vec{A}) = \langle\vec{\nabla}\phi\rangle\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A}) \tag{d}$$

$$\vec{\nabla}\times(\phi\vec{A}) = \langle\vec{\nabla}\phi\rangle\times\vec{A} + \phi(\vec{\nabla}\times\vec{A}) \tag{e}$$

$$\vec{\nabla}\times\langle\vec{\nabla}\phi\rangle \equiv 0 \tag{f}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0 \tag{g}$$

$$\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C}) &= \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}, \quad \text{if } [\vec{A},\vec{B}]=0\\ &= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}, \quad \text{if } [\vec{B},\vec{C}]=0\end{aligned} \tag{h}$$

$$\begin{aligned}(\vec{A}\times\vec{B})\times\vec{C} &= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C}), \quad \text{if } [\vec{A},\vec{B}]=0\\ &= (\vec{A}\cdot\vec{C})\vec{B} - \vec{A}(\vec{B}\cdot\vec{C}), \quad \text{if } [\vec{B},\vec{C}]=0\end{aligned} \tag{i}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \langle\vec{\nabla}\times\vec{A}\rangle\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \tag{j}$$

$$\begin{aligned}\vec{\nabla}\times(\vec{A}\times\vec{B}) &= \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})A -\\ &\quad - (\vec{A}\cdot\vec{\nabla})\vec{B}, \quad \text{if } [\vec{A},\vec{B}]=0\end{aligned} \tag{k}$$

$$\vec{A}\times(\vec{\nabla}\times\vec{B}) = -\vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{A}\times(\vec{\nabla}\times\vec{B}) + (\vec{A}\cdot\vec{\nabla})\vec{B} \tag{l}$$

$$[p_x, x] = -i\hbar = -i, \quad [l_x, p_y] = i p_z$$

$$(\vec{p}\times\vec{r}) = -(\vec{r}\times\vec{p}) \tag{m}$$

$$[(\vec{r}\cdot\vec{p}), (\vec{\sigma}\cdot\vec{p})] = i\hbar(\vec{\sigma}\cdot\vec{p}) \tag{n}$$

$$[(\vec{\sigma}^{(1)}\cdot\vec{p}), (\vec{\sigma}^{(2)}\cdot\vec{l})] = i\hbar\{\vec{\sigma}^{(2)}\cdot(\vec{p}\times\vec{\sigma}^{(1)})\}$$

(p) (III-3.16)

$$[(\vec{\sigma}\cdot\vec{l}),(\vec{r}\cdot\vec{p})] = 0 \qquad (q)$$

$$[(\vec{\sigma}^{(1)}\cdot\vec{l}),(\vec{\sigma}^{(2)}\cdot\vec{l})] = i\hbar\{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})\} \qquad (r)$$

i.e.

$$(\vec{\sigma}^{(2)}\cdot\vec{l})(\vec{\sigma}^{(1)}\cdot\vec{l}) = \frac{1}{2}\{(\vec{\sigma}^{(1)}\cdot\vec{l})(\vec{\sigma}^{(2)}\cdot\vec{l}) + (\vec{\sigma}^{(2)}\cdot\vec{l})(\vec{\sigma}^{(1)}\cdot\vec{l})\} - \frac{i\hbar}{2}\{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})\} \qquad (s)$$

$$\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{r})\rangle = -3i\hbar \qquad (t)$$

$$(\vec{r}\cdot\vec{p})^2 = r^2p^2 + i\hbar(\vec{r}\cdot\vec{p}) - l^2 \,, \qquad (u)$$

$$(\text{III}-3.16)$$

(i) Scalar Meson Interaction

With V's defined by Table 1 we get Pauli form of interaction from (III-3.15)

$$V_S^P = -J - \frac{1}{4M^2c^2}\Big\{-p^2J - Jp^2 - (\vec{\sigma}^{(1)}\cdot\vec{p})J\cdot$$

$$\cdot(\vec{\sigma}^{(1)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{p})J(\vec{\sigma}^{(2)}\cdot\vec{p})\Big\}. \qquad \text{(III-3.17)}$$

With (III-3.16 a, b), $(\vec{\sigma}^{(1)}\cdot\vec{p})^2 = p^2$ etc., we get

$$V_S^P = -J + \frac{1}{4M^2c^2}\Big\{Jp^2 + 2\langle p\vec{J}\rangle\cdot\vec{p} + Jp^2 +$$

$$+\langle p^2J\rangle + 2Jp^2 + 2\langle p\vec{J}\rangle\cdot\vec{p} + i(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot(\langle p\vec{J}\rangle\times\vec{p})\Big\}.$$

Using

$$\langle p\vec{J}\rangle\cdot\vec{p} = -iJ_1(\vec{r}\cdot\vec{p}) \qquad \text{(III-3.18)}$$

and

$$\frac{\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}}{2} = \vec{S}, \qquad \text{(III-3.19)}$$

$$\vec{l} = (\vec{r}\times\vec{p}) \qquad \text{(III-3.20)}$$

we get

$$V_S^P = -J + \frac{1}{M^2c^2}\Big\{Jp^2 - iJ_1(\vec{r}\cdot\vec{p}) + \frac{\langle p^2J\rangle}{4} +$$

$$+\frac{J_1}{2}(\vec{l}\cdot\vec{S})\Big\}. \qquad \text{(III-3.21)}$$

(ii) Unconstrained Vector Meson Interaction

With the help of Table 1 and in analogy with (III-3.17), we write from (III-3.15)

$$V_{UV}^{P} = J - \frac{1}{4M^2c^2}\Big\{ p^2 J + J p^2 - (\vec{\sigma}^{(1)}\cdot\vec{p})\, J\, (\vec{\sigma}^{(1)}\cdot\vec{p}) -$$

$$- (\vec{\sigma}^{(2)}\cdot\vec{p})\, J\, (\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J -$$

$$- J\, (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J \,\cdot$$

$$\cdot\, (\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J\, (\vec{\sigma}^{(1)}\cdot\vec{p}) \Big\} \qquad (\text{III-3.22})$$

$$= J - \frac{1}{4M^2c^2}\Big[\langle p^2 J\rangle - 2J_1\, (\vec{l}\cdot\vec{S}) - (\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}) \Big] \qquad (\text{III-3.23})$$

where

$$\mathrm{I} \equiv (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J$$

$$\mathrm{II} \equiv J\, (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})$$

$$\mathrm{III} \equiv (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J\, (\vec{\sigma}^{(2)}\cdot\vec{p})$$

$$\mathrm{IV} \equiv (\vec{\sigma}^{(2)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, J\, (\vec{\sigma}^{(1)}\cdot\vec{p}) \qquad (\text{III-3.24})$$

It is best to consider these operations with the wave function ψ present so that various operations can be explicitly seen. With a considerable amount of reduction involving identities (III-3.16 b, c, d, e, g, j, k) we can write I as

$$(\mathrm{I})\,\psi \equiv (1 - \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\Big[2\,\langle \vec{p} J\rangle\cdot\vec{p} + \langle p^2 J\rangle +$$

$$+ J p^2 \Big]\psi + (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(J\psi) \,. \qquad (\text{III-3.25})$$

Similarly by using only (III-3.16 b, ℓ) we get

$$(\mathrm{II})\psi = (1 - \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(J p^2)\psi + + J\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\psi\rangle . \tag{III-3.26}$$

With the help of (III-3.16 b, c, f, g, etc.) identity we can write

$$(\mathrm{III})\psi = [\,J p^2 + \langle \vec{p}J\rangle\cdot\vec{p} + 2J_1(\vec{\ell}\cdot\vec{S}) + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\cdot \cdot J p^2 + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\langle\vec{p}J\rangle\cdot\vec{p}\,]\psi - (\vec{\sigma}^{(2)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(1)}\cdot\langle\vec{p}\psi\rangle) - - J\langle(\vec{\sigma}^{(1)}\cdot\vec{\nabla})(\vec{\sigma}^{(2)}\cdot\vec{\nabla})\psi\rangle . \tag{III-3.27}$$

The expression for (IV) ψ can be analogously obtained (and has also been checked) by interchanging particles 1 and 2 and we finally get

$$(\mathrm{III}+\mathrm{IV})\psi = 2J p^2 + 2\langle\vec{p}J\rangle\cdot\vec{p} + + 4J_1(\vec{\ell}\cdot\vec{S}) + 2(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(J p^2 + \langle\vec{p}J\rangle\cdot\vec{p}) - - (\vec{\sigma}^{(1)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(2)}\cdot\langle\vec{p}\psi\rangle) - (\vec{\sigma}^{(2)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(1)}\cdot\langle\vec{p}\psi\rangle) - - 2J\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\psi\rangle . \tag{III-3.28}$$

Also from (III-3.25) and (III-3.26)

$$(\mathrm{I}+\mathrm{II})\,\psi = \Big[2J p^2 + 2\langle \vec{p}J\rangle\cdot\vec{p} + \langle p^2 J\rangle -$$

$$-(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\left\{2Jp^2 + 2\langle\vec{p}J\rangle\cdot\vec{p} + \langle p^2 J\rangle\right\}\Big]\psi +$$

$$+(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(J\psi) + J\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\psi\rangle \;. \qquad (\mathrm{III}\text{-}3.29)$$

Now we write (III-3.23) in detail as

$$V^P_{UV}\,\psi = \Big[J - \frac{1}{4M^2c^2}\Big\{-4Jp^2 - 4\langle\vec{p}J\rangle\cdot\vec{p} +$$

$$+(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\langle p^2 J\rangle)\Big\}\psi - \frac{1}{4M^2c^2}\Big\{(\vec{\sigma}^{(1)}\cdot\langle\vec{p}J\rangle)\,\cdot$$

$$\cdot\,(\vec{\sigma}^{(2)}\cdot\langle\vec{p}\psi\rangle) + (\vec{\sigma}^{(2)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(1)}\cdot\langle\vec{p}\psi\rangle) -$$

$$-(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(J\psi) - J\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\psi\rangle\Big\} +$$

$$+\frac{6}{4M^2c^2}\,J_1(\vec{\ell}\cdot\vec{s})\,\psi\Big] . \qquad (\mathrm{III}\text{-}3.30)$$

But

$$(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})(J\psi) = \langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})J\rangle\psi +$$

$$+J\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\psi\rangle + (\vec{\sigma}^{(1)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(2)}\cdot\langle\vec{p}\psi\rangle) +$$

$$+(\vec{\sigma}^{(2)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(1)}\cdot\langle\vec{p}\psi\rangle) \quad . \qquad (\mathrm{III}\text{-}3.31)$$

And by writing in component form, we can show that

$$\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})J\rangle = -\left\{J_1\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + r^2 J_2\,(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})\,r^{-2}\right\} . \qquad (\mathrm{III}\text{-}3.32)$$

By substituting (III-3.31) and (III-3.32) into (III-3.30) we get the unconstrained vector interaction reduced to the following Pauli form

$$V^{P}_{UV} = \Big[J + \frac{1}{4M^2c^2} \Big\{ 4J p^2 + 4\langle \vec{p} J\rangle \cdot \vec{p} - $$
$$- \langle p^2 J\rangle \left(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right)\Big\} - \frac{1}{4Mc^2}\Big\{ J_1 \left(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right) +$$
$$+ r^2 J_2 \frac{\left(\vec{\sigma}^{(1)} \cdot \vec{r}\right)\left(\vec{\sigma}^{(2)} \cdot \vec{r}\right)}{r^2}\Big\} + \frac{3}{2}\frac{J_1}{M^2c^2}\left(\vec{l} \cdot \vec{S}\right)\Big] \cdot \qquad (\text{III}-3.33)$$

In order to bring this in conventional form involving tensor, spin-spin, spin orbit, and velocity dependent operators we define, as usual, the tensor operator

$$S_{12} \equiv \frac{3\left(\vec{\sigma}^{(1)} \cdot \vec{r}\right)\left(\vec{\sigma}^{(2)} \cdot \vec{r}\right)}{r^2} - \left(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right) \qquad (\text{III}-3.34)$$

and substitute in (III-3.33) and using

$$\langle p^2 J\rangle = -\left(r^2 J_2 + 3J_1\right), \qquad (\text{III}-3.35)$$

we get

$$V^{P}_{UV} = \Big[J + \frac{1}{M^2c^2}\Big\{ J p^2 - i J_1 \left(\vec{r} \cdot \vec{p}\right) - \frac{1}{6}\langle p^2 J\rangle \cdot$$
$$\cdot \left(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right) + \frac{3}{2} J_1 \left(\vec{l} \cdot \vec{S}\right) - \frac{1}{12} r^2 J_2 \left(S_{12}\right)\Big\}\Big], \qquad (\text{III}-3.36)$$

which becomes identical with original way of writing if we put $\vec{p} = -i\hbar\vec{\nabla}$ and insert $\hbar$ properly according to dimensional considerations, i.e.,

$$V^{P}_{UV} = J + \frac{\hbar^2}{M^2c^2}\Big\{ -J\nabla^2 - \langle \vec{\nabla} J\rangle \cdot \vec{\nabla} +$$
$$+ \frac{1}{6}\langle \nabla^2 J\rangle \left(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}\right) + \frac{3}{2r}\frac{dJ}{dr}\left(\vec{l} \cdot \vec{S}\right) - \frac{1}{12}\left(r^2 J_2\right) \cdot$$
$$\cdot \left(S_{12}\right)\Big\} . \qquad (\text{III}-3.37)$$

Similarly we can also write scalar interaction from (III-3.21) in this notation as

$$V_s^P = -J + \frac{\hbar^2}{M^2c^2}\Big\{-J\nabla^2 - \langle\vec{\nabla}J\rangle\cdot\vec{\nabla} - \Big\langle\frac{\nabla^2 J}{4}\Big\rangle +$$

$$+\frac{1}{2r}\frac{dJ}{dr}\,(\vec{l}\cdot\vec{s})\Big\}, \qquad \text{(III-3.38)}$$

which will be useful in the next chapter. These results were first derived by Green (53).

(iii) <u>Pseudoscalar Meson Interaction</u>

Again from Table 1 and Equation (III-3.15) we obtain

$$V_P^P = -\frac{1}{4M^2c^2}\Big\{(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})J + J(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) -$$

$$-(\vec{\sigma}^{(1)}\cdot\vec{p})J(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\vec{p})J(\vec{\sigma}^{(1)}\cdot\vec{p})\Big\} \qquad \text{(III-3.39)}$$

$$= -\frac{1}{4M^2c^2}\Big\{(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})J - J(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) -$$

$$-(\vec{\sigma}^{(1)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(2)}\cdot\vec{p}) - (\vec{\sigma}^{(2)}\cdot\langle\vec{p}J\rangle)(\vec{\sigma}^{(1)}\cdot\vec{p})\Big\}. \qquad \text{(III-3.40)}$$

With the help of (III-3.31) we get

$$V_P^P = \frac{-1}{4M^2c^2}\Big\{\langle(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})J\rangle\Big\}, \qquad \text{(III-3.41)}$$

using (III-3.32) we obtain

$$V_P^P = \frac{1}{4M^2c^2}\Big\{J_1\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + r^2 J_2\,\frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})}{r^2}\Big\}. \qquad \text{(III-3.42)}$$

With (III-3.34), (III-3.35) we get (inserting $\hbar^2$ also)

$$V_P^P = \frac{1}{12M^2c^2}\Big\{-\langle p^2 J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + r^2 J_2(S_{12})\Big\}, \qquad \text{(III-3.43)}$$

which can be written in final form published by Green (62) in field theoretic approach

$$V_P^P = \frac{\hbar^2}{12M^2c^2}\left\{\langle\nabla^2 J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + r^2 J_2\,(S_{12})\right\}. \tag{III-3.44}$$

(iv) Breit's Vector Meson Interaction

In this case we plan to use some of the results derived previously. In that process let us assume another Diracian interaction given by $V^D = \mathbb{1}\,J(r)$, where $\mathbb{1}$ is a (4 x 4) unit matrix and we plan to get its Pauli form and (III-2.6)

$$\begin{aligned} V^D\Omega &= \mathbb{1}\,J\,\Omega = (I_\rho \otimes I_\sigma)^{(1)}(I_\rho \otimes I_\sigma)^{(2)} J \cdot \\ &\qquad \cdot \begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix} \\ &= J\begin{pmatrix} \varphi & \chi_1 \\ \chi_2 & \psi \end{pmatrix}, \end{aligned} \tag{III-3.45}$$

which implies

$$V_a = V_c = J\,, \qquad V_b = V_d = 0\,, \tag{III-3.46}$$

and from (III-3.15)

$$\begin{aligned} V^P = J - \frac{1}{4M^2c^2}\{ & p^2 J + J p^2 - (\vec{\sigma}^{(1)}\cdot\vec{p})\,J\,(\vec{\sigma}^{(2)}\cdot\vec{p}) - \\ & - (\vec{\sigma}^{(2)}\cdot\vec{p})\,J\,(\vec{\sigma}^{(1)}\cdot\vec{p})\}\,. \end{aligned} \tag{III-3.47}$$

Therefore

$$[\mathbb{1}J]^P = J - \frac{1}{M^2c^2}\left\{\frac{\langle p^2 J\rangle}{4} - \frac{1}{2}J_1\,(\vec{\ell}\cdot\vec{S})\right\}, \tag{III-3.48}$$

where the last equation has been obtained by using Equations (III-3.16 a,b) and definitions of $\vec{\ell}$, $\vec{S}$ etc. From (III-3.36) we have the Pauli form of $[\mathbb{1} - \vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}]J$ and therefore we get Pauli form of $[\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}]J$ as follows (superscript P, as before, means Pauli form of)

$$\left\{[\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}]J\right\}^P = \frac{1}{M^2c^2}\left\{-\left\langle\frac{p^2J}{4}\right\rangle + J_1(\vec{l}\cdot\vec{s}) - \right.$$

$$- Jp^2 + iJ_1(\vec{r}\cdot\vec{p}) + \frac{1}{6}\langle p^2J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) -$$

$$\left. - \frac{3}{2}J_1(\vec{l}\cdot\vec{s}) + \frac{1}{12}r^2J_2\,(S_{12})\right\}. \qquad (\text{III-3.49})$$

Define

$$u = \frac{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})J_1}{2} \qquad (\text{III-3.50})$$

We can now write, using Table 1 and Equations (III-3.15), (III-3.48), and (III-3.49), the Pauli form of Breit interaction as given below.

$$V_{BV}^P = [\mathbb{1}J]^P - \left[\frac{(\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)})J}{2}\right]^P + \left[\frac{(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})J_1}{2r}\right]^P$$

$$= J - \frac{1}{4M^2c^2}\left\{p^2J + Jp^2 - (\vec{\sigma}^{(1)}\cdot\vec{p})J(\vec{\sigma}^{(1)}\cdot\vec{p}) - \right.$$

$$- (\vec{\sigma}^{(2)}\cdot\vec{p})J(\vec{\sigma}^{(2)}\cdot\vec{p}) + \left[(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\left(-\frac{\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}}{2}J +\right.\right.$$

$$\left. + u\right) + \left(-\frac{\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}}{2}J + u\right)(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) +$$

$$+ (\vec{\sigma}^{(1)}\cdot\vec{p})\left(-\frac{\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}}{2}J + u\right)(\vec{\sigma}^{(2)}\cdot\vec{p}) + (\vec{\sigma}^{(2)}\cdot\vec{p})\cdot$$

$$\left.\cdot\left(-\frac{\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}}{2}J + u\right)(\vec{\sigma}^{(1)}\cdot\vec{p})\right\} \qquad (\text{III-3.51})$$

$$= J + \frac{1}{M^2c^2}\Big\{ -\frac{\langle p^2 J\rangle}{4} + J_1(\vec{\ell}\cdot\vec{S}) + \frac{\langle p^2 J\rangle}{8} -$$

$$- \frac{J_1}{4}(\vec{\ell}\cdot\vec{S}) + \frac{Jp^2}{2} - \frac{i}{2}J_1(\vec{r}\cdot\vec{p}) - \frac{1}{12}\langle p^2 J\rangle \cdot$$

$$\cdot (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + \frac{3}{4}J_1(\vec{\ell}\cdot\vec{S}) - \frac{1}{24}(r^2 J_2)(S_{12})\Big\} -$$

$$- \frac{1}{4M^2c^2}\Big\{ (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\,u + u\,(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) +$$

$$+ (\vec{\sigma}^{(1)}\cdot\vec{p})\,u\,(\vec{\sigma}^{(2)}\cdot\vec{p}) + (\vec{\sigma}^{(2)}\cdot\vec{p})\,u\,(\vec{\sigma}^{(1)}\cdot\vec{p})\Big\}, \tag{III-3.52}$$

where u is given by (III-3.50). We now evaluate each term of last curly bracket of (III-3.52). Let

$$I \equiv 2u(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})$$

$$= J_1\,(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p}) \tag{III-3.53}$$

with the help of (III-3.16 n, p, b) we can write it as $(\hbar = 1)$

$$I = J_1\Big[(\vec{r}\cdot\vec{p})^2 + i(\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{r}\cdot\vec{p}) + i(\vec{r}\cdot\vec{p}) -$$

$$- 2(\vec{\ell}\cdot\vec{S}) - (\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{\sigma}^{(1)}\cdot\vec{\ell}) + i(\vec{r}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\ell}) -$$

$$- i(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p}) + i(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\vec{r}\cdot\vec{p})\Big] \tag{III-3.54}$$

With use of (III-3.16 q, s) we get

$$I = J_1\Big[(\vec{r}\cdot\vec{p})^2 + 2i(\vec{\ell}\cdot\vec{S})(\vec{r}\cdot\vec{p}) + i(\vec{r}\cdot\vec{p}) -$$

$$- 2(\vec{\ell}\cdot\vec{S}) + i(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\vec{r}\cdot\vec{p}) - \frac{1}{2}\Big\{(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell}) +$$

$$+ (\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{\sigma}^{(1)}\cdot\vec{\ell})\Big\} - \frac{i}{2}\Big\{(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p}) + (\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})\Big\}\Big]. \tag{III-3.55}$$

Now we define

$$\mathrm{II} \equiv \mathcal{Q}(\vec{\sigma}^{(1)}\cdot\vec{p})\,\mathcal{U}\,(\vec{\sigma}^{(2)}\cdot\vec{p})$$

$$= (\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})J_1\,(\vec{\sigma}^{(2)}\cdot\vec{p}) \quad . \tag{III-3.56}$$

But we first evaluate only $\mathcal{Q}(\vec{\sigma}^{(1)}\cdot\vec{p})\,\mathcal{U}$ which, with the help of (III-3.16 b, q) and some detail of careful operator algebraic operations, reduces to

$$\mathrm{II} = \left[(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})J_1\right](\vec{\sigma}^{(2)}\cdot\vec{p})$$

$$= \Big[-i(r^2J_2+4J_1)(\vec{\sigma}^{(2)}\cdot\vec{r}) - J_1\{\vec{\sigma}^{(1)}\cdot(\vec{r}\times\vec{\sigma}^{(2)})\} +$$

$$+ J_1(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{r}\cdot\vec{p}) - iJ_1(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{\ell})\Big](\vec{\sigma}^{(2)}\cdot\vec{p}) . \tag{III-3.57}$$

We evaluate each of these four major terms using various identities of (III-3.16) and sum up the result with careful ordering of the terms and the result is

$$\mathrm{II} = -i(r^2J_2+3J_1)(\vec{r}\cdot\vec{p}) + J_1(\vec{r}\cdot\vec{p})^2 + (r^2J_2+3J_1)\cdot$$

$$\cdot(\vec{\sigma}^{(2)}\cdot\vec{\ell}) + iJ_1(\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{r}\cdot\vec{p}) - iJ_1(\vec{r}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{\ell}) -$$

$$- iJ_1(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p}) + iJ_1(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p}) +$$

$$+ J_1(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell}) \quad . \tag{III-3.58}$$

We define below the similar term (III) which is obtained by interchanging particles 1 and 2 in (III-3.58) and has also been explicitly evaluated to see that this is really the case.

$$\mathrm{III} \equiv \mathcal{Q}(\vec{\sigma}^{(2)}\cdot\vec{p})\,\mathcal{U}\,(\vec{\sigma}^{(1)}\cdot\vec{p})$$

$$= \Big[-i\,(r^2J_2+3J_1)(\vec{r}\cdot\vec{p})+J_1\,(\vec{r}\cdot\vec{p})^2+(r^2J_2+3J_1)(\vec{\sigma}^{(1)}\cdot\vec{\ell})$$

$$+\,iJ_1\,(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{r}\cdot\vec{p})-iJ_1\,(\vec{r}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{\ell})-$$

$$-\,iJ_1\,(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{p})+iJ_1\,(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})+$$

$$+\,J_1\,(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell})\Big]\,. \qquad (\mathrm{III}-3.59)$$

Therefore

$$\mathrm{II}+\mathrm{III} = -2i\,(r^2J_2+3J_1)(\vec{r}\cdot\vec{p})+2J_1\,(\vec{r}\cdot\vec{p})^2+$$

$$+\,2(r^2J_2+3J_1)(\vec{\ell}\cdot\vec{S})+J_1\,\Big\{(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell})+$$

$$+\,(\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{\sigma}^{(1)}\cdot\vec{\ell})\Big\}\,. \qquad (\mathrm{III}-3.60)$$

The only remaining term of (III-3.52) is $\mathrm{IV} \equiv \mathcal{Q}(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(2)}\cdot\vec{p})\,\mathcal{U}$ which can also be written as below and evaluated with extensive reduction using many identities of (III-3.16). We first use (III-3.57) to give us $\mathcal{Q}(\vec{\sigma}^{(1)}\cdot\vec{p})\,\mathcal{U}$ and then get (IV).

$$\mathrm{IV} \equiv (\vec{\sigma}^{(2)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{p})(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})\,J_1$$

$$= -\big[r^4J_3+9r^2J_2+12J_1\big]-i(2r^2J_2+7J_1)\,\cdot$$

$$\cdot\,(\vec{r}\cdot\vec{p})+J_1\,(\vec{r}\cdot\vec{p})^2-(r^2J_2+2J_1)(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})+$$

$$+\,J_2(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r})-2(r^2J_2+4J_1)(\vec{\ell}\cdot\vec{S})-iJ_1\,\cdot$$

$$\cdot\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})(\vec{r}\cdot\vec{p})+iJ_1\,(\vec{\sigma}^{(2)}\cdot\vec{r})(\vec{\sigma}^{(1)}\cdot\vec{p})-2iJ_1\,(\vec{\ell}\cdot\vec{S})\,\cdot$$

$$\cdot\,(\vec{r}\cdot\vec{p})\;-\;J_1\,(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell})\,. \qquad (\mathrm{III}-3.61)$$

We use (III-3.16 s) for the last term above, add (I and IV) using (III-3.55), (III-3.61) and we get

$$\mathrm{I}+\mathrm{IV} = -\left[r^4 J_3 + 9r^2 J_2 + 12 J_1\right] - i\left(2r^2 J_2 + 6J_1\right)\cdot (\vec{r}\cdot\vec{p}) + 2J_1(\vec{r}\cdot\vec{p})^2 - (2r^2 J_2 + 5J_1)(\vec{\ell}\cdot\vec{s}) - (r^2 J_2 + 2J_1)(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + J_2(\vec{\sigma}^{(1)}\cdot\vec{r})(\vec{\sigma}^{(2)}\cdot\vec{r}) - J_1\left\{(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell}) + (\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{\sigma}^{(1)}\cdot\vec{\ell})\right\}. \qquad (\mathrm{III}\text{-}3.62)$$

But using (III-3.34) we can finally tabulate the coefficients of various operators coming from either (II + III) or (I + IV) to get a clearer picture of the resulting Pauli form. This will be shown in Table 2.

Table 2

Breit's Vector Interaction

OPERATORS	I + IV	II + III	TOTAL
1	$-[r^4J_3+9r^2J_2+12J_1]$	0	$-[r^4J_3+9r^2J_2+12J_1]$
$(\vec{r}\cdot\vec{p})$	$-i(2r^2J_2+6J_1)$	$-i(2r^2J_2+6J_1)$	$-i(4r^2J_2+12J_1)$
$(\vec{r}\cdot\vec{p})^2$	$2J_1$	$2J_1$	$4J_1$
$(\vec{\ell}\cdot\vec{s})$	$-2(r^2J_2+5J_1)$	$+2(r^2J_2+3J_1)$	$-4J_1$
(S_{12})	$\frac{1}{3}r^2J_2$	0	$\frac{1}{3}r^2J_2$
$(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})$	$-\frac{2}{3}(r^2J_2+3J_1)$	0	$-\frac{2}{3}(r^2J_2+3J_1)$
$(\vec{\sigma}^{(1)}\cdot\vec{\ell})(\vec{\sigma}^{(2)}\cdot\vec{\ell})+(\vec{\sigma}^{(2)}\cdot\vec{\ell})(\vec{\sigma}^{(1)}\cdot\vec{\ell})$	$-J_1$	$+J_1$	0

We substitute all these results from Table 2 into Equation (III-3.52) with use of (III-3.35) and get

$$V_{BV}^{P} = J + \frac{1}{M^2c^2}\Big\{ -\Big\langle\frac{p^2 J}{8}\Big\rangle + J_1(\vec{\ell}\cdot\vec{s}) + \frac{Jp^2}{2} -$$

$$- \frac{i}{2}J_1(\vec{r}\cdot\vec{p}) - \frac{1}{12}\langle p^2 J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) - \frac{1}{24}(r^2 J_2)(S_{12})\Big\}$$

$$+ \frac{1}{2M^2c^2}\Big\{\frac{1}{4}(r^4 J_3 + 9r^2 J_2 + 12 J_1) + i(r^2 J_2 +$$

$$+ 3J_1)(\vec{r}\cdot\vec{p}) - J_1(\vec{r}\cdot\vec{p})^2 + J_1(\vec{\ell}\cdot\vec{s}) - \frac{1}{12}r^2 J_2(S_{12})$$

$$- \frac{1}{6}\langle p^2 J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\Big\}. \qquad (III\text{-}3.63)$$

With (III-3.16 u) and

$$r\frac{d}{dr}\langle\nabla^2 J\rangle = r^4 J_3 + 5r^2 J_2 , \qquad (III\text{-}3.64)$$

we obtain

$$V_{BV}^{P} = J + \frac{1}{M^2c^2}\Big\{-\frac{1}{8}r\frac{d}{dr}\langle p^2 J\rangle - \frac{5}{8}\langle p^2 J\rangle +$$

$$+ \frac{i}{2}(J_1 + r^2 J_2)(\vec{r}\cdot\vec{p}) + \frac{1}{2}(J - r^2 J_1)p^2 -$$

$$- \frac{1}{6}\langle p^2 J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + \frac{3}{2}J_1(\vec{\ell}\cdot\vec{s}) +$$

$$+ \frac{1}{2}J_1(\vec{\ell}^{\,2}) - \frac{1}{12}(r^2 J_2)(S_{12})\Big\} \qquad (III\text{-}3.65)$$

This is the final form of Breit's (51) vector interaction and agrees with his Equation (17) of reference (51).

(v) Kemmer's Vector Meson Interaction

We can carefully use many of our previous results in order to derive the Pauli form of Kemmer's interaction. Using (III-3.14) we can write

$$\left[1 - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} + \frac{1}{K^2} (\vec{\alpha}^{(1)} \cdot \vec{\nabla}) (\vec{\alpha}^{(2)} \cdot \vec{\nabla}) \right] J$$

$$= \left[(1 - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J + \frac{1}{K^2} \left\{ (\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J_1 + \right.\right.$$

$$\left.\left. + (\vec{\alpha}^{(1)} \cdot \vec{r}) (\vec{\alpha}^{(2)} \cdot \vec{r}) J_2 \right\} \right] \tag{III-3.66}$$

The first term is known in Pauli form from the unconstrained vector meson interaction and the first term in the curly bracket can be deduced in Pauli form by using the result (III-3.49), except that J will be replaced by J_1. Similarly the last term in (III-3.66) in the curly bracket can be reduced to Pauli form by a careful use of second curly bracket of (III-3.63). We therefore start these mathematical manipulations with careful attention. From (III-3.49) and (III-3.11)

$$\left[(\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}) J_1 \right]^P = \frac{1}{M^2 \lambda^2} \left\{ - \left\langle \frac{p^2 J_1}{4} \right\rangle + \frac{J_2}{2} (\vec{\ell} \cdot \vec{s}) - \right.$$

$$- J_1 \, p^2 + i J_2 (\vec{r} \cdot \vec{p}) + \frac{1}{6} \langle p^2 J_1 \rangle (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) -$$

$$\left. - \frac{3}{2} J_2 (\vec{\ell} \cdot \vec{s}) + \frac{1}{12} (r^2 J_3) \, (S_{12}) \right\} . \tag{III-3.67}$$

Now let $v = J_1$, then from (III-3.63) we get

$$\left[(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})\,v_1\right]^P =$$

$$= \frac{1}{M^2c^2}\Big\{\frac{1}{4}(r^4v_3 + 9r^2v_2 + 12v_1) + i(r^2v_2 +$$

$$+ 3v_1\,(\vec{r}\cdot\vec{p}) - v_1(\vec{r}\cdot\vec{p})^2 + v_1(\vec{\ell}\cdot\vec{S}) -$$

$$- \frac{1}{12}(r^2v_2)\,(S_{12}) - \frac{1}{6}\langle p^2v\rangle\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\Big\},$$

$$\text{(III-3.68)}$$

where

$$v = J_1 = \frac{1}{r}\frac{d}{dr}J(r),\; v_1 = J_2$$

$$v_3 = J_4\,,\quad v_2 = J_3 \quad \text{e.t.c.}$$

$$\text{(III-3.69)}$$

and we use (III-3.16 u). Then

$$\left[(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})\,J_2\right]^P = \frac{1}{M^2c^2}\Big\{\frac{1}{4}(r^4J_4 + 9r^2J_3 +$$

$$+ 12J_2) + i(r^2J_3 + 2J_2)(\vec{r}\cdot\vec{p}) - J_2\,r^2p^2 +$$

$$+ J_2(\vec{\ell}^{\,2}) + J_2(\vec{\ell}\cdot\vec{S}) - \frac{1}{12}(r^2J_3)(S_{12}) +$$

$$+ \frac{1}{6}(r^2J_3 + 3J_2)(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\Big\} \quad \text{(III-3.70)}$$

Adding (III-3.36), (III-3.67), and (III-3.70) properly we obtain

$$V^{P}_{KV} = J + \frac{1}{M^2c^2}\Big\{ Jp^2 - iJ_1(\vec{r}\cdot\vec{p}) - \frac{1}{6}\langle p^2 J\rangle \bullet$$

$$\bullet (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + \frac{3}{2}J_1(\vec{\ell}\cdot\vec{S}) - \frac{1}{12}(r^2J_2)(S_{12})\Big\} +$$

$$+ \frac{1}{M^2c^2K^2}\Big\{\frac{1}{4}(r^2J_3 + 3J_2) - J_1p^2 + iJ_2(\vec{r}\cdot\vec{p}) +$$

$$+ \frac{1}{2}J_2(\vec{\ell}\cdot\vec{S}) - \frac{1}{6}(r^2J_3 + 3J_2)(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) -$$

$$- \frac{3}{2}J_2(\vec{\ell}\cdot\vec{S}) + \frac{r^2J_3}{12}(S_{12}) + \frac{1}{4}(r^4J_4 + 9r^2J_3 + 12J_2) +$$

$$+ i(r^2J_3 + 2J_2)(\vec{r}\cdot\vec{p}) - J_2r^2p^2 + J_2(\vec{\ell}^2) + J_2(\vec{\ell}\cdot\vec{S}) -$$

$$- \frac{1}{12}(r^2J_3)(S_{12}) + \frac{1}{6}(r^2J_3 + 3J_2)(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\Big\} \qquad (\text{III}-3\cdot 71)$$

$$= J + \frac{1}{M^2c^2}\Big[\Big\{ Jp^2 - iJ_1(\vec{r}\cdot\vec{p}) - \frac{1}{6}\langle p^2J\rangle(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) +$$

$$+ \frac{3}{2}J_1(\vec{\ell}\cdot\vec{S}) - \frac{1}{12}(r^2J_2)S_{12}\Big\} + \frac{1}{K^2}\Big\{\frac{1}{4}(r^4J_4 +$$

$$+ 10r^2J_3 + 15J_2) + i(r^2J_3 + 3J_2)(\vec{r}\cdot\vec{p}) - (r^2J_2 +$$

$$+ J_1)p^2 + J_2(\vec{\ell}^2)\Big\}\Big]. \qquad (\text{III}-3\cdot 72)$$

This is the final form of Kemmer's vector meson interaction and has been confirmed independently by Sawada (66) by working on each term explicitly.

Conculsions

We must emphasize that many pages of calculations are suppressed here in order to keep continuity of deductions among various forms and also to keep the size of this chapter limited, but essentially all the vector-operator and matrix algebraic identities given by (III-3.16) should enable a serious reader to derive these results with the help of well defined, necessary intermediate steps.

Another point of similarity among all these interactions is that we can always represent the general interaction in the following Pauli form:

$$V^P = V_C(r) + V_\nabla(r)\, i(\vec{r}\cdot\vec{p}) + V_\Delta(r)(p^2) + V_{SS}(r)\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + V_{LS}(r)\,(\vec{l}\cdot\vec{s}) + V_T(r)\,(S_{12}) + V_{LL}(r)\,(\vec{l}^{\,2}), \qquad \text{(III-3.73)}$$

in which V's represent the radial dependence of the potential and other operators show the various terms which have been used for a long time phenomenologically in nucleon-nucleon force problem, in nuclear physics and in atomic physics.

It should also be pointed out that we have neglected the quadratic terms in potential even to lowest order and, as promised before, they will be discussed in Chapter VII.

CHAPTER IV

VELOCITY DEPENDENT POTENTIALS

Section - 1 Comparison Of Currently Used Nucleon-Nucleon Potentials

As pointed out in Chapter I, the problem of finding a good nucleon-nucleon force has been a very challenging problem of this century in physics. A few recent developments in particle physics (discovery of vector mesons ω, ρ) have opened up new channels of approach which have greatly reduced the uncertainties in this field. Thus a new picture has emerged out in the last two to three years which tends to support the meson-exchange processes for accounting for the major components of nucleon-nucleon force. As we shall see, even the concept of potential becomes questionable and has to be generalized. This approach of One-Boson-Exchange-Potentials has been currently used by the following major groups (and many other individuals) and we narrate their conclusions. We do not claim, however, that this represents a complete review of the problem.

Within the framework of field theory and with definitions of potentials as non-relativistic limits of Diracian One-Boson-Exchange Interactions, Green (52) in 1949 derived various components of nucleon-nucleon potential due to exchange of scalar, vector, pseudoscalar, and pseudovector mesons (fields) but a quantitative analysis could not be presented due to lack of knowledge about meson masses and other properties. This work was re-examined by Green and Sharma (54, 67) in the light of known mesons and, as stated originally, this necessitated the presence of scalar meson. The interest in this problem arose when we were trying to find the best available nucleon-nucleon force to take as a starting point for many body calculations and came across the work of Bryan and Scott (68) who obtained expressions similar to those obtained by Green (52). A closer study revealed that although the two methods of deriving the nucleon-nucleon force were apparently different, that they gave exactly the same results as far as one-meson processes were concerned. In addition, we found that Bryan and Scott neglected the velocity dependent terms in their potentials. Later Bryan and Scott, in rechecking their work, found the velocity dependent terms.

In a later report Bryan and Arndt (69) showed enough details of their work which may be summarized by stating that they projected a pole term into angular momentum states, the poles corresponding to mesons. They relate Feynman T-matrix to Stapp's (70) M-matrix and then calculate the contribution to T-matrix as the Born Term in $|\ell s j\rangle$ representation which they indicate to be equivalent to choosing an interaction Lagrangian (e.g., pseudoscalar pole $\mathcal{L}_I = \frac{1}{\sqrt{4\pi}} g_{PS} \overline{\Psi}\gamma_5 \phi_{PS} \psi$, where ψ is nucleon wave function and ϕ_{PS} is the pseudoscalar meson field, etc.). In addition to direct coupling for scalar, pseudoscalar and vector mesons, they also calculate dipole type effects by considering the derivative coupling but in our work we consider the pole contributions (i.e., direct coupling only) to be major contributions and remaining discrepancies are considered to be accounted by Two-Boson-Exchanges and other effects which may contribute to the same order as the derivative coupling terms. We have thus been able to establish a very close contact with this group.

We confine our discussion to potentials alone for this chapter and come to the experimental phase shifts and scattering comparisons in the next chapter. Another approach of using multimeson resonances for NN Interaction has been due to Scotti and Wong (71) who use relativistic partial wave dispersion relations embodying single meson exchanges. They also have to postulate a scalar resonance like others in the field. They also introduce derivative coupling to the ρ — vector meson on the grounds of contributions (coming to anamalous magnetic moment of nucleon) from 2π -exchanges, but our work in the present phase will discuss these points later. It is also not clear whether the couplings and cut-offs (to be discussed shortly) correspond exactly to those of field theoretical definitions and a detailed comparison with their approach has still not been made.

Historically it is interesting to note that the presence of vector meson was pointed out by Breit (72) and Sakurai (73) in 1960. Breit's approach was a consideration of his own reduction (which has been discussed in Chapter III) and was also due to observations of experimental facts especially the spin orbit effects in the scattering data including electron-nucleon scattering experiments. But the mass of the particle could then be not fixed exactly. Sakurai (73) also discussed, almost at the same time in a general article, the whole approach to strong interaction physics, seemingly defending the meson-exchange processes and regarding them to be responsible for NN force. Later on ω, ρ mesons were confirmed supporting these observations. A similar phase exists today in the controversy about scalar mesons because experimental stage of affairs has not yet provided the final word. We feel that a scalar entity (may be even as an enhancement or S-wave $\pi-\pi$ resonance) is most likely to occur in nature and experimental data on nucleon-nucleon scattering can not be fitted without such an entity. We shall discuss shortly the idea of purely relativistic nucleon-nucleon force.

Another group which practically took the similar approach to that of Bryan and Scott and actually just before theirs, consisted of Japanese physicists, some of the important work being published by Hoshizaki, Otsuki, Watari, and Yonezawa (74) and by Sawada, Ueda, Watari, and Yonezawa (75). The latter eliminated the intermediate step of potentials and went directly to scattering data but their physical contents appear to be almost the same as those of Bryan and Scott.

Having given an idea about foresent nucleon-nucleon potentials we now proceed to discuss as to how our potentials compare with Bryan and Scott and also with best phenomenological models of Hamada and Johnston (76) and Lassila, Hull, Ruppel, McDonald, and Breit (77) which would be denoted (HJ) and (YALE) respectively. From last chapter, we can always write our potential in the general form

$$V = V_c + V_\nabla \, i(\vec{r}\cdot\vec{p}) + V_\Delta (p^2) + V_{SS}(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})$$
$$+ V_{LS}(\vec{\ell}\cdot\vec{S}) + V_T(S_{12}) + V_{LL}(\vec{\ell}^2) \; . \tag{IV-1.1}$$

At present we consider only the scalar, vector, and pseudoscalar meson exchange interactions given by (III-3.1), (III-3.2), and (III-3.3) and their Pauli forms given by (III-3.21), (III-3.36), and (III-3.44) respectively. The phenomenological groups have also cast their nucleon-nucleon potentials in the form of Equation (IV-1.1) but without velocity dependent terms and Bryan and Scott have reduced HJ quadratic $(\vec{\ell}\cdot\vec{S})$ term into these forms. Furthermore, since the scattering states of nucleon-nucleon system can be divided into isotopic spin singlets and triplets, arising from the requirements of charge independence [De Benedetti (41)] the potentials may only be unique for iso-singlet or iso-triplet states. The requirement of charge independence is that the nuclear forces between two nucleons be independent of the charge states. In other words the total isospin is conserved and forces are independent of T_z. Thus the forces $(nn) = (np) = (pp)$. Charge symmetry is a special case $(nn) = (pp)$. Conservation of isospin splits nucleon-nucleon states in to $T = 0$ and $T = 1$ states.

But the Hamiltonian should be independent of rotations in I-spin space. This implies that it can contain only scalar quantities formed out of 1, $\vec{\tau}^{(1)}$ and $\vec{\tau}^{(2)}$. Only two such linearly independent quantities exist. They are 1 and $[\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}]$.

There is an alternative set 1 and $\tau_T = \frac{1}{2}(1 + \vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})$ where the latter is denoted as Isotopic spin exchange operator. Exchange operators of similar type can be defined for spin and coordinate exchanges. The spatial exchange operator is also known as Majorana operator and for the case of two particles of equal mass in the C.M. frame it reduces to parity operator. Thus the exclusion principle can be rewritten as

$$\tau_r \, \tau_\sigma \, \tau_T = -1$$

or

$$(-1)^{L+S+T} = -1 \quad .$$

If the electromagnetic effects can be neglected, then T^2 becomes a good quantum number. The questions regarding the effects of charge independence and charge symmetry on the scattering cross sections have been summarized by de Benedetti (41). The phase shift approach is simple as $T=1$ phases are determined by p-p scattering. But $T=0$ phases in general are less certainly known because of difficulties in n-p scattering, etc.

Hence the most general Hamiltonian can be written as

$\left[V + \vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} V' \right]$. Also we make a hypothesis that nucleon-nucleon force is a purely relativistic force. This is a simple attractive model which will prove later to be reasonably successful. This requires that the scalar meson masses be the same as their corresponding vector particles. The Pauli form as given by Green and Sharma (54) is obtained by adding (III-3.21) and (III-3.36). Thus

$$V^D = \left[\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} - \beta^{(1)} \beta^{(2)} \right] J(r) \qquad \text{(IV-1.2)}$$

$$V^P = \frac{\hbar^2}{M^2 c^2} \left[- \frac{\langle \nabla^2 J \rangle}{4} - 2 J \nabla^2 - 2 J_1 (\vec{r} \cdot \vec{\nabla}) + \right.$$

$$\left. + \frac{\langle \nabla^2 J \rangle}{6} (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + 2 J_1 (\vec{l} \cdot \vec{S}) - \frac{1}{12} r^2 J_2 (S_{12}) \right]$$

$$\text{(IV-1.3)}$$

which is a purely relativistic scalar + vector (also called five vector model) meson exchange force. Together with this the pseudoscalar meson exchange force as given by (III-3.44)

$$V^P = \frac{\hbar^2}{12 M^2 c^2} \left[\langle \nabla^2 J \rangle (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + r^2 J_2 (S_{12}) \right] \qquad \text{(IV-1.4)}$$

is also purely relativistic in the sense that its static limit vanishes as is the case for (IV-1.3). This model thus has many essential features of nucleon-nucleon force that are experimentally established. We have the following mesons.

[(SPIN)$^{\text{PARITY}}$]	T = 0	T = 1
SCALAR (S) [0^+]	ω_s (782.8 Mev)	ρ_s (763 Mev)
VECTOR (V) [1^-]	ω (782.8 Mev)	ρ (763 Mev)
PSEUDOSCALAR (P) [0^-]	η (548.7 Mev)	π (137.5 Mev)

(IV-1.5)

The Yukawa form for each meson is taken to be

$$J(r) = g^2 (\hbar c) \frac{e^{-kr}}{r}$$

where

$$K = \frac{mc}{\hbar}, \qquad \text{(IV-1.6)}$$

$$K_N = \frac{Mc}{\hbar},$$

$$\hbar c = 197.32 \text{ Mev-Fermi}, \qquad \text{(IV-1.7)}$$

and

$$g^2 = 14.7 \quad (\text{coupling constant}).$$

The radial forms can now be expressed as

$$\langle \nabla^2 J \rangle = K^2 J - g^2 4\pi \delta(r)$$

$$r^2 J_2 = g^2 (\hbar c)(K^2 r^2 + 3Kr + 3) \frac{e^{-kr}}{r^3},$$

and

$$J_1 = -g^2 (\hbar c)(1 + Kr) \frac{e^{-kr}}{r^3}. \qquad \text{(IV-1.8)}$$

Since charge independence requirements [De Benedetti (41)] decouple the contributions of T = 0 and T = 1 mesons, we can now compute the contributions of various mesons to T = 0 or T = 1 potentials if the values of the coupling constants are known. Since this was a preliminary study for a comparison, we took the pion-nucleon coupling constant as determined by Hamilton and Woolcock (78) from π-N scattering analysis, (14.7) in above definition. This gives the potentials in Mev and range in fermis. The results for potentials associated with (Isoscalar and Isovector) Tensor, spin-spin, and spin orbit operators are presented in Figure 1. The mesons contributing to each potential are shown by (+++) and (...) symbols and by ω and ρ in Figure 1 we mean $(\omega + \omega_s)$ and $(\rho + \rho_s)$ contributions but care should be taken in noticing, for example, that (ω_s, ρ_s) do not contribute to V_{SS} and V_T at all, and that (π, η) do not contribute to V_{LS}. The superscripts on potentials denote the isospin value. Thus $V_T^{(0)}$ for (HJ) the dashed dot curves, and (YALE) dashed curves, are of opposite sign as compared to Bryan and Scott (continuous curves). This, as we see, can be interpreted as due to different couplings between ω and η mesons, which contribute with opposite signs to the tensor force. This shows how Figure 1(A) can be interpreted. For $V_{SS}^{(0)} \equiv V_{\sigma\sigma}^{(0)}$ the ω and η contribute with the same sign as Bryan and Scott's and meson theoretic potentials give opposite results to phenomenological results at inner distances as seen in Figure 1(B). The spin orbit effects for isoscalar states areeentirely due to $(\omega_s + \omega)$

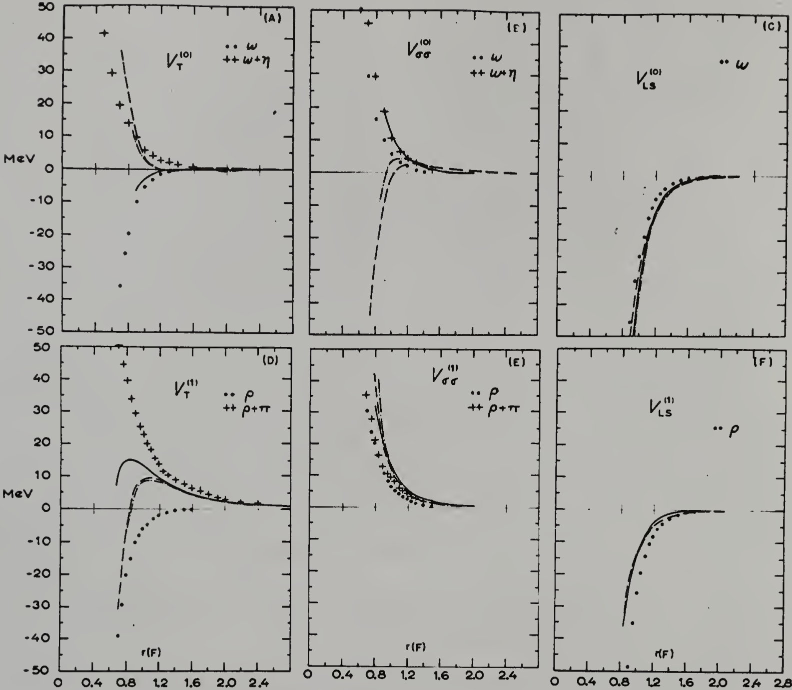

Figure 1. Comparison of the Tensor, Spin-Spin and Spin-Orbit Contributions of the Isoscalar and Isovector Mesons.

and all potentials beautifully agree in Figure 1C. Now we turn to isovector tensor potential $V_T^{(1)}$ and see that here the outer contribution beyond 1.4 fermi is only due to lightest mass π meson. This will guide our Born phases in the next chapter for higher angular momentum states. At the inner distances we have ρ meson playing an important role and thus the result is a sum of contributions of opposite signs and parameters can be adjusted to include the phenomenological or Bryan and Scott range (Figure 1D). Spin-spin potential $V_{\sigma\sigma}^{(1)}$ is also satisfactory (Figure 1E) but spin-orbit potential $V_{LS}^{(1)}$ needs some more adjustments (Figure 1F). Furthurmore we should notice that our present theory has to modify the behavior of potentials for inner distances and this will be based on theoretical and phenomenological cut-offs in the next sections but at the same time will give us more lattitude and parameterize the uncertainties about nucleon-nucleon force in the region where many difficult questions are yet to be resolved. It was only for these reasons that Bryan and Scott did not consider S-wave scattering in exact analysis and put all potentials to zero within the distance of about 0.6 fermi. We postpone our discussions for central and velocity dependent potentials for a later section and conclude that without any adjustments the agreements are very encouraging. In the next section we show two different ways of dealing with velocity dependent potentials and then give an account of phenomenological potentials with velocity dependence, which can be compared with our velocity dependent form factors.

Covariant solutions of Bethe-Salpeter equation in Schrödinger like reduced form have been worked by Biswas (89) and Green and Biswas (88) in instantaneous impulse approximation. They also get velocity dependent potentials and then go into effective potential approach.

Section - 2 Methods For Treating Velocity Dependent Potentials

In the last section we have compared various potentials (isoscalar and isovector) as contributions due to T = 0 and T = 1 pseudoscalar, scalar, and vector mesons. Hence if V_{Tot} denotes the total potential due to T = 0 and T = 1 mesons, we can write it as

$$V_{Tot} = V^{(0)} + (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) V^{(1)} \qquad (IV-2.1)$$

where $(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})$ is the state dependent isospin operator that is analogous to spin-spin operator and has value (+1) for iso-triplet states of the two nucleon system and (-3) for iso-singlet states, in analogy to $(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$.

If we denote

$$J^{s,p,v} = J^{(0)}_{s,p,v} + (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) J^{(1)}_{s,p,v} \qquad (IV-2.2)$$

then we can write the total potential using (IV-2.1), (III-3.21), (III-3.36), and (III-3.44) as

(over)

$$V_{Tot} = -J^s + \frac{\hbar^2}{M^2c^2}\left[-J^s\nabla^2 - J_1^s(\vec{r}\cdot\vec{\nabla}) - \frac{\langle\nabla^2 J^s\rangle}{4}\right] +$$

$$+ J^v + \frac{\hbar^2}{M^2c^2}\left[-J^v\nabla^2 - J_1^v(\vec{r}\cdot\vec{\nabla})\right] +$$

$$+ \left(\frac{\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}}{6}\right)\frac{\hbar^2}{M^2c^2}\left[\langle\nabla^2 J^v\rangle + \frac{1}{2}\langle\nabla^2 J^p\rangle\right] +$$

$$+ \left(\frac{\vec{\ell}\cdot\vec{s}}{2}\right)\frac{\hbar^2}{M^2c^2}\left[J_1^s + 3J_1^v\right] +$$

$$+ \frac{(S_{12})}{12}\frac{\hbar^2}{M^2c^2}\left[r^2 J_2^p - r^2 J_2^v\right]. \qquad \text{(IV-2.3)}$$

This can be rewritten as

$$V_{Tot} = (-J^s + J^v) - \frac{\hbar^2}{M}\left[\phi\nabla^2 + \frac{1}{r}\frac{d\phi}{dr}(\vec{r}\cdot\vec{\nabla})\right] -$$

$$-\frac{1}{4}\frac{\hbar^2}{M^2c^2}\langle\nabla^2 J^s\rangle + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})V'_{SS} + (\vec{\ell}\cdot\vec{s})V'_{LS} +$$

$$+ (S_{12})\,V'_T. \qquad \text{(IV-2.4)}$$

The meanings of V'_{SS}, V'_{LS} and V'_T are defined by comparison with (IV-2.3) and ϕ is defined as

$$\phi = \frac{J^s + J^v}{Mc^2}$$

$$= \frac{1}{Mc^2}\left[(J_s^{(0)} + J_v^{(0)}) + (\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})(J_s^{(1)} + J_v^{(1)})\right]. \qquad \text{(IV-2.5)}$$

Now we go to the effective mass or effective potential method of treating velocity dependent terms.

(i) <u>Effective Mass Method</u>

In this approach we make use of a transformation which allows our Schrödinger equation with velocity dependence to be placed in the usual radial form with an effective potential which turns out to be energy dependent. We write Schrödinger equation in the center of mass frame as

$$\left[-\frac{\hbar^2\nabla^2}{M} + V_{Tot}\right]\Psi = E_{CM}\Psi \tag{IV-2.6}$$

Defining

$$k = \sqrt{\frac{ME_{CM}}{\hbar^2}}$$

and

$$U = \frac{M}{\hbar^2}V'. \tag{IV-2.7}$$

we obtain

$$\Big[-(1+\phi)\nabla^2 - \frac{1}{r}\frac{d\phi}{dr}(\vec{r}\cdot\vec{\nabla}) - \frac{1}{4Mc^2}\langle\nabla^2 J^s\rangle + \frac{M}{\hbar^2}(-J^s + J^v) + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})U_{SS} + (\vec{\ell}\cdot\vec{S})U_{LS} + (S_{12})U_T\Big]\Psi = k^2\Psi \tag{IV-2.8}$$

Let us assume that we are considering such states of two particle system which are not coupled by tensor force and this is done only for the simplicity of notation because the tensor force does not involve any velocity dependence. We have thus only to be careful about ℓ-value dependence of wave functions that are involved with velocity dependent terms. Hence we can write our wave function in spherical polar coordinates as

$$\Psi(r,\theta,\varphi) \equiv \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} A_\ell R_\ell(r)\, Y_\ell^m(\theta,\varphi)$$

$$(\text{IV}-2.9)$$

Ordinarily tensor force mixes different ℓ values but let us confine only to uncoupled states. Then we can write

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R_\ell}{\partial r}\right) + \frac{r^2}{(1+\phi)}\Big(k^2 - \Big[\frac{M}{\hbar^2}(-J^s+J^v) -$$

$$-\frac{1}{4Mc^2}\langle\nabla^2 J^s\rangle + \frac{1}{r}\frac{d\phi}{dr}\frac{d}{dr} + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})U_{ss} +$$

$$+(\vec{\ell}\cdot\vec{S})\,U_{LS} + S_{12}U_T\Big]\Big)R_\ell = \ell(\ell+1)R_\ell$$

where we have used the following relations [see Schiff(79)]

$$Y_\ell^m(\theta,\varphi) = \Theta_\ell(\cos\theta)\,\Phi^m(\varphi)$$

$$\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2\Phi \quad \text{or} \quad \Phi = e^{\pm im\varphi}$$

and

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} \equiv -\ell(\ell+1)\,.$$

$$(\text{IV}-2.10)$$

These come from associated Legendre equations and the expansion of Laplacian into spherical polar coordinates. We reduce (IV-2.10) into more conventional form as

$$\frac{d^2R_\ell}{dr^2} + \frac{2}{r}\frac{dR_\ell}{dr} + \Big[k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{1}{(1+\phi)}\Big\{k^2\phi -$$

$$-\frac{\langle\nabla^2 J^s\rangle}{4Mc^2} - (-J^s+J^v)\frac{M}{\hbar^2} + U_{ss}(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + (\vec{\ell}\cdot\vec{S})U_{LS} +$$

$$+(S_{12})\,U_T\Big\}\Big]R_\ell + \frac{1}{(1+\phi)}\frac{1}{r}\frac{d\phi}{dr}\frac{dR_\ell}{dr} = 0\,.$$

$$(\text{IV}-2.11)$$

This looks like ordinary form of radial Schrödinger equation except for the last term arising from the gradient term in the potential. In an exact numerical method of solving second order differential equation, it would create no extra problem to deal with this term since the first derivative of the wave function is always evaluated. However, for the work here, it is convenient to eliminate it using the transformation

$$\chi_\ell(r) = r\,(1+\phi)^{1/2}\,R_\ell(r) \qquad \text{(IV-2.12)}$$

Letting primes denote differentiation with respect to r , we deduce the following relation by differentiating twice

$$\frac{(1+\phi)^{1/2}}{r}\,\chi_\ell'' = (1+\phi)\left[R_\ell'' + \frac{2}{r}R_\ell'\right] + \phi' R_\ell' + $$
$$+ \frac{1}{2}\left(\phi'' + \frac{2}{r}\phi'\right)R_\ell - \frac{1}{4}\,\frac{(\phi')^2}{(1+\phi)}\,R_\ell \,. \qquad \text{(IV-2.13)}$$

We substitute (IV-2.13) into (IV-2.11) with definition of χ_ℓ from (IV-2.12) to obtain

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} + \frac{1}{(1+\phi)}\left\{\frac{1}{2}\left(\phi'' + \frac{2}{r}\phi'\right) - \right.\right.$$
$$- \frac{1}{4}\,\frac{(\phi')^2}{(1+\phi)} + \frac{M}{\hbar^2}\left(-J^s + J^v\right) + k^2\phi - \frac{1}{4Mc^2}\langle\nabla^2 J^s\rangle +$$
$$\left.\left. + \left(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\right)U_{ss} + \left(\vec{\ell}\cdot\vec{s}\right)U_{Ls} + \left(S_{12}\right)U_T\right\}\right]\chi_\ell = 0 \qquad \text{(IV-2.14)}$$

Hence our effective potential is

$$U_{eff}(r,k) = \frac{1}{(1+\phi)}\left\{ \frac{1}{2}\langle\nabla^2\phi\rangle - \frac{1}{4}\frac{(\phi')^2}{(1+\phi)} + \right.$$

$$+ \frac{M}{\hbar^2}\left(-J^s + J^V\right) + k^2\phi - \frac{\langle\nabla^2 J^s\rangle}{4Mc^2} +$$

$$\left. + \left(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\right)U_{SS} + (\vec{\ell}\cdot\vec{S})\,U_{LS} + (S_{12})\,U_T \right\}. \qquad (\text{IV}-2.15)$$

We just want to conclude this approach with the comments that the $(1+\phi)$ factor occurs in the denominator and therefore it is not possible to see the explicit contributions to potential occuring from each meson or the separation and identification of various terms arising from velocity dependent terms. Since it is fairly important to understand these effects in the One-Boson-Exchange approach and in the study of velocity dependence, we decided to look at the problem in Born approximation where another method of treating velocity dependence is possible. This second approach has further advantages which will be discussed in the next chapter on phase shifts.

This effective potential method is, however, the only one known for exact solution of the velocity dependent problems and it should be born in mind that it imposes certain restrictions on potentials whose physical nature should be examined carefully.

(ii) Direct Method In Born Approximation

We can write the Schrödinger equation in this case also as in (IV-2.4) and (IV-2.5). Our intention in this approach is to make use of Bessel's equation and recurrence relations associated with them. Thus it is possible to do so only when the analytical properties of the wave functions are known. The "potential" in this approach is really the effect of various differentiation operators on known functions and is really $(V\Psi)$. This is reduced into an effective potential by dividing through Ψ, i.e., $V_{eff} = \Psi^{-1}(V\Psi)$, which is only a temporary process to look at the overall behavior of potential because, for physical quantities and experimentally related values, only the expectation values or overlap integrals among wave functions are involved. But nevertheless we decided to look at the effective potential in radial form to get some idea about the phase shift work where this ambiguity would disappear. The Schrödinger equation is written as (IV-2.6) and Ψ given by (IV-2.7) but we do not group the velocity dependent terms with kinetic energy and write

$$\nabla^2\Psi - \Big[\Big\{ \frac{M}{\hbar^2}(-J^s + J^v) - \frac{\langle\nabla^2 J^s\rangle}{4Mc^2} + (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)})\, V_{SS} + (\vec{L}\cdot\vec{S})\, V_{LS} + (S_{12})\, V_T \Big\}\Psi - \phi\,\langle\nabla^2\Psi\rangle - \frac{1}{r}\frac{d\phi}{dr}\big(\vec{r}\cdot\langle\vec{\nabla}\Psi\rangle\big)\Big] + k^2\Psi = 0 \qquad \text{(IV-2.16)}$$

By using (IV-2.9) and (IV-2.10) we have

$$\langle \nabla^2 R_\ell Y_\ell^m \rangle = \Big(R_\ell'' + \frac{2}{r}R_\ell' - \frac{\ell(\ell+1)}{r^2}R_\ell\Big) Y_\ell^m \qquad \text{(IV-2.17)}$$

and

$$\vec{r} \cdot \langle \vec{\nabla} \Psi \rangle = r \frac{dR_\ell}{dr} Y_\ell^m A_\ell \, .$$

$$(\text{IV}-2.18)$$

The radial parts only contribute in the scalar product in the latter case. Hence angular parts separate out and we get

$$R_\ell'' + \frac{2}{r} R_\ell' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] R_\ell - \left[\frac{M}{\hbar^2}(-J^s + J^v) - \right.$$

$$\left. - \frac{\langle \nabla^2 J^s \rangle}{4Mc^2} + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) U_{ss} + (\vec{\ell} \cdot \vec{s}) U_{Ls} + (S_{12}) U_T \right] R_\ell +$$

$$+ \phi \left(R_\ell'' + \frac{2}{r} R_\ell' - \frac{\ell(\ell+1)}{r^2} R_\ell \right) + \phi' R_\ell' = 0$$

$$(\text{IV}-2.19)$$

With the transformation

$$R_\ell(r) = r^{-1} \mathfrak{z}_\ell(r) \, , \qquad (\text{IV}-2.20)$$

we get

$$R_\ell'' + \frac{2}{r} R_\ell' = \mathfrak{z}_\ell'' / r \qquad (\text{IV}-2.21)$$

and on substituting in (IV-2.20) we deduce

$$\mathfrak{z}_\ell'' - \left[\frac{M}{\hbar^2}(-J^s + J^v) - \frac{\langle \nabla^2 J^s \rangle}{4Mc^2} + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) U_{ss} + \right.$$

$$\left. + (\vec{\ell} \cdot \vec{s}) U_{Ls} + (S_{12}) U_T \right] \mathfrak{z}_\ell + \left[k^2 - \frac{\ell(\ell+1)}{r^2} \right] \mathfrak{z}_\ell +$$

$$+ \phi \left[\mathfrak{z}_\ell'' - \frac{\ell(\ell+1)}{r^2} \mathfrak{z}_\ell \right] + r \phi' R_\ell' = 0 .$$

$$(\text{IV}-2.22)$$

Let every potential be set to zero, then the field-free solution $v_\ell(r)$ satisfies (Pipes, 80)

$$v_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} \right] v_\ell = 0 \,, \qquad \text{(IV-2.23)}$$

Now we use the radial form of the Born approximation and require that, to a fairly resonable extent, $R_\ell(r) \approx j_\ell(kr)$ For the velocity dependent part, the wave equation reduces to

$$(v_\ell(r) = r j_\ell(kr))\,, \qquad \text{(IV-2.24)}$$

$$z_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - U_{eff} \right] z_\ell = 0 \,, \qquad \text{(IV-2.25)}$$

where

$$U_{eff} = \frac{M}{\hbar^2} \frac{\langle V_{Tot}\, j_\ell(kr) \rangle}{j_\ell(kr)} =$$

$$= \left[\frac{M}{\hbar^2}(-J^\Delta + J^\nabla) - \frac{\langle \nabla^2 J^\Delta \rangle}{4Mc^2} + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) U_{SS} + (\vec{\ell} \cdot \vec{S}) U_{LS} + (S_{12}) U_T \right] - \frac{1}{j_\ell(kr)} \left[\phi\, k^2 j_\ell(kr) + \phi' j_\ell'(kr) \right] . \qquad \text{(IV-2.26)}$$

We have grouped the velocity dependent parts in the last bracket. Now we make use of a recurrance relation among Bessel functions

$$\frac{d j_\ell(kr)}{dr} = \frac{k}{(2\ell+1)} \left\{ \ell\, j_{\ell-1}(kr) - (\ell+1)\, j_{\ell+1}(kr) \right\} \qquad \text{(IV-2.27)}$$

which reduces our effective potential to

$$U_{eff} = \left\{ \frac{M}{\hbar^2}(-J^\Delta + J^\nabla) - \frac{1}{4Mc^2} \langle \nabla^2 J^\Delta \rangle + (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) U_{SS} + (\vec{\ell} \cdot \vec{S}) U_{LS} + (S_{12}) U_T \right\} - \left\{ k^2 \phi - \frac{k}{j_\ell(kr)(2\ell+1)} \left[\ell j_{\ell-1} - (\ell+1) j_{\ell+1} \right] \right\} , \qquad \text{(IV-2.28)}$$

which is the final form of effective potential in this approach. Now we see that in this case we do not have $(1+\phi)$ factor coming in the denominator so it is possible to identify contribution to potential from each meson and also we can identify the contribution from each velocity dependent term that we started with. This approach has been mainly used in establishing the contact with experimental data through phase shift calculations.

Section - 3 Phenomenological Velocity Dependent Potentials

We pointed out in Section - 1 of this chapter several groups working on the problem of nucleon-nucleon force through meson exchanges. Similarly the importance of velocity dependent potentials has also been recognized by many people in the field from phenomenological point of view. The concept of velocity dependent forces in classical physics, atomic physics, and nuclear physics (including nucleon-nucleon interactions) has been recently discussed in a historical way by Green (81) in a review article. This material also embodies the meson-theoretic nature but an attempt has been made to simplify the physical nature of the problem. Thus the velocity dependent, tensor, spin-spin, and spin-orbit force concepts have been described by Green (42) in analogy with simple classical electrodynamics.

In this brief survey we mention only the major similarities of such phenomenological attempts in this direction. It is interesting to note that Marshak and Okubo (82) commented in 1958 that nucleon-nucleon scattering data was not represented satisfactorily above 150 mev with just a linear term in momentum. Then they discussed a possible quadratic momentum dependence of the type of terms coming from Breit and Kemmer vector meson exchange interactions discussed in the last chapter. Simultaneous work was done by Moshinsky (83) which is a detailed work to understand two body velocity dependent potential in a nuclear structure calculation. This includes central and spin dependent forces in the discussion of shell model level-ordering utilizing Racah algebra and irreducible tensors. Another contemporary approach on the nuclear many body problem by Green in the same period will constitute the last section of this chapter.

An interesting and useful phenomenological approach of A. M. Green (84) on the nucleon-nucleon problem has been to use the form of potential as

$$V(r, \vec{p}) = V(r) + \frac{p^2}{m_0}\,\omega(r) + \omega(r)\,\frac{p^2}{m_0} \qquad (\text{IV}-3.1)$$

If we use the methods analogous to those used for effective mass method of the last section, we get analogous effective potential containing the factor $[1+2\omega(r)]$ in the denominator. He then parameterizes the form factor with the form

$$\omega(r) = c_1\, e^{-(0.6772\, c\,\mu r)^2}$$

and

$$\mu = 0.7082 \text{ fm}^{-1}. \qquad (\text{IV}-3.2)$$

where c_1 and c depend on nucleon-nucleon scattering states (e.g. $c_1 = 1.14$, $c = 3$).

Parallel developments in this area are due to Razavy, Field and Levinger (85) and also due to Rojo and Simmons (86). The former group takes the potential as

$$V(r, \vec{p}) = -V_0 \bar{J}_1(r) - \frac{\lambda}{M}\,\vec{p}\cdot\bar{J}_2(r)\,\vec{p}, \qquad (\text{IV}-3.3)$$

which for special choice of $\bar{J}_1 = \bar{J}_2 = J$ and $\lambda = 1$ in this notation becomes identical to our central and velocity dependent terms of purely relativistic $(\omega + \omega_s)$ case if J is Yukawa form and $k_N^2 V_0 \equiv K^2/4$. But instead they specialized to analytically solvable potential forms. They took in one case square well forms in which they could get analytic answers for S-wave phase shifts and simple forms for others. In another case they chose square well "Jost" potential in which they employ techniques in Born approximation that are similar to our indirect method but involve only one integral to be evaluated. This is so because of the simple form of their potential. A comparison of form factors is therefore not easy.

Rojo and Simmons use the same structure of potential as A. M. Green

$$v(r, \vec{p}) = -V_0 J(r) + \frac{\lambda}{M}\left[p^2 \omega(r) + \omega(r) p^2\right] \qquad (\text{IV}-3.4)$$

and we have carried out a detailed study of their work. Their form factor is

$$\lambda \omega(r) = 5\, e^{-3.6r}, \qquad (\text{IV}-3.5)$$

but the central part differs in their two sets of potentials. They also reduce them to effective potential formalism. Our detailed study revealed that a simple criterion for comparisons among A. M. Green, Rojo-Simmons, and Green-Sharma potentials could not be established if the central and velocity dependent terms were included together. The main reason for this being the different ways of parameterizing the potentials and also behavior of our potential at shorter distances. Especially until we reached the best set of couplings and cut-offs, this comparison would not be very meaningful in the radial form. It may well be the case that, when momentum dependences are involved, differently looking effective potentials may give the same phase shifts. Hence a true criterion was left for phase shift calculations. Instead of this we decided to compare the effective masses in various cases to get an idea of the range of velocity dependence when these factors occurred in the same place (i.e., in the denominator) in the effective potential (in the cases that are compared).

We might also mention a very recent preprint from Tabakin and Davies (87) describing the velocity dependence of the form

$$V(r, p^2) = -V_1(r) + p^2 e^{-s^2 p^2} V_2(r) + V_2(r)\, p^2 e^{-s^2 p^2},$$

which reduces to our form in the first term. But it is not clear whether the physical nature of the problem is less strongly velocity dependent, as they put it in the above potential.

Section - 4 Comparison Of Velocity Dependent Forms

Before we go to comparison with phenomenological velocity dependent form factors, we will sketch the results that are obtained if the velocity dependence is used with the direct method. This was shown in Equation (IV-2.28). In order to see the radial dependence, the last terms of (IV-2.28) with $\left[\frac{\langle \nabla^2 J^1 \rangle}{4Mc^2} \right]$ were used to group together central and velocity dependent terms. The trends were reported by Green and Sharma (54) and are shown in Figure 2. The potential is thus energy dependent and angular-momentum dependent and thus $\ell = 0$, $E_{CM} = 30$, 100 and 390 MeV are specified for the effective radial potential thus obtained which are shown in Figure 2a. Also we chose the wave function for deuteron as originally used by Green (52)

$$\psi_d = C \left(\frac{e^{-K_a r} - e^{-K_b r}}{r} \right) \qquad (IV-4.1)$$

with

$$4\pi C^2 = \frac{2(K_a + K_b) K_a K_b}{(K_a - K_b)^2}, \text{ and } K_a < K_b$$

(as required by normalization). We neglect the non-spherical contribution for an estimation of the effects. Thus the direct approach was applied to deuteron wave function and the results are shown by curve labeled (d) in Figure 2a. We see that the potential thus obtained (purely relativistic $5\omega \Rightarrow (\omega + \omega_3)$ being used) is attractive for deuteron, is attractive at lower energies, and becomes strongly repulsive at higher energies. This is what is precisely the requirement for the S-waves to become negative at higher energies and is normally met by using a hard core. The exact phase shift calculations will show no necessity for such a hard core. Figure 2B shows the same results for $\ell = 3$ and $E_{CM} = 30, 100$ and 320 MeV.

Now we turn to a comparison of effective masses. In every case we can put the central and velocity dependent terms in the form

$$V(r, \vec{p}) = V_{static} + \frac{1}{M(r)} p^2 + \langle \vec{p} \frac{1}{M(r)} \rangle \cdot \vec{p} \qquad (\text{IV}-4.2)$$

where

$$\left[\frac{M(r)}{M_0}\right]_{GS} = \frac{1}{1 + \frac{2J(r)}{M\mu^2}} ,$$

$$\left[\frac{M(r)}{M_0}\right]_{RS} = \frac{1}{1 + 2\lambda\, \omega_{RS}(r)}$$

and

$$\left[\frac{M(r)}{M_0}\right]_{AMG} = \frac{1}{1 + 2\,\omega_{AMG}(r)} , \qquad (\text{IV}-4.3)$$

where J is Yukawa's form for purely relativistic case and $\omega_{AMG}(r)$ and $\lambda\, \omega_{RS}(r)$ are given by (IV-3.2) and (IV-3.5) respectively. This radially dependent factor occurring in the effective potential can be interpreted as the effective mass of the system and Figure 2C shows a comparison among all three form factors. We observe that the agreement is encouraging. A detailed look at the radial dependence shows that we tend to drop to zero at 1.2 fermis where the others do the

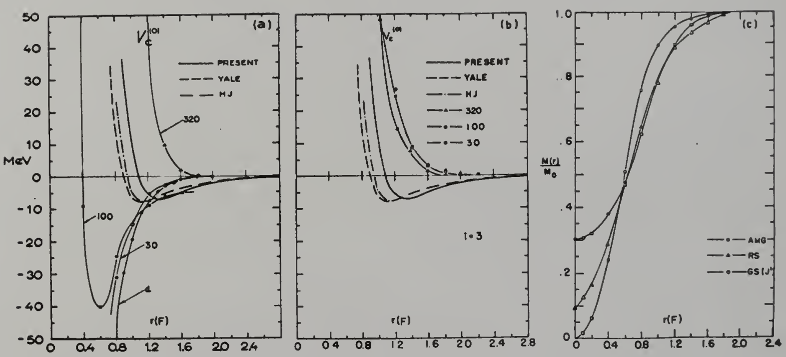

Figure 2. Velocity Dependent Potentials.

a) Effective Potentials by Direct Method for $\ell = 0$ and Deuteron.

b) Effective Potential by Direct Method for $\ell = 3$. The energies are in the C. M. frame.

c) Comparisons of Effectiveness.
AMG -- A. M. Green
RS -- Rojo Simmons
GS(J^1) -- Green - Sharma

same at 1.6 fermis. A larger coupling (to 5ω mesons) in our case could do the same. The behavior at short distances can be attributed to multiple meson exchanges and pair production processes.

Theoretically, Green's (62, 64) work on higher order field equations showed that infinite self energy terms were eliminated when certain algebraic relations were satisfied. These relations were mentioned by Green and Sharma (54) and the first correction gives us a subtractive meson which is equivalent to a phenomenological cut-off. In the Figure 3A we show how our effective mass will change if higher mass subtractive mesons with proper weights would occur in a realistic way. This approach not only removes the infinite self energy terms, but reduces the singularties in various terms of the potential and at the same time brings us closer to the phenomenological form factos. In Figure 3B we give the modification to the Yukawa form due to these subtractive conditions. The functions J^{I} through J^{IV} are discussed by Green (62) in detail [Equations 2.5 through 2.8 of (62)]. Physically interesting consequences and interpretations of these conditions in terms of indefinite metric in field theories arise but do not constitute a part of the present work.

Thus we have seen that the total picture on velocity dependent terms that emerges out of One-Boson-Exchange — Potentials is a very encouraging one and this leads us directly into a more meaningful area of direct contact with experimental data. In the next chapter we calculate the phase shifts in

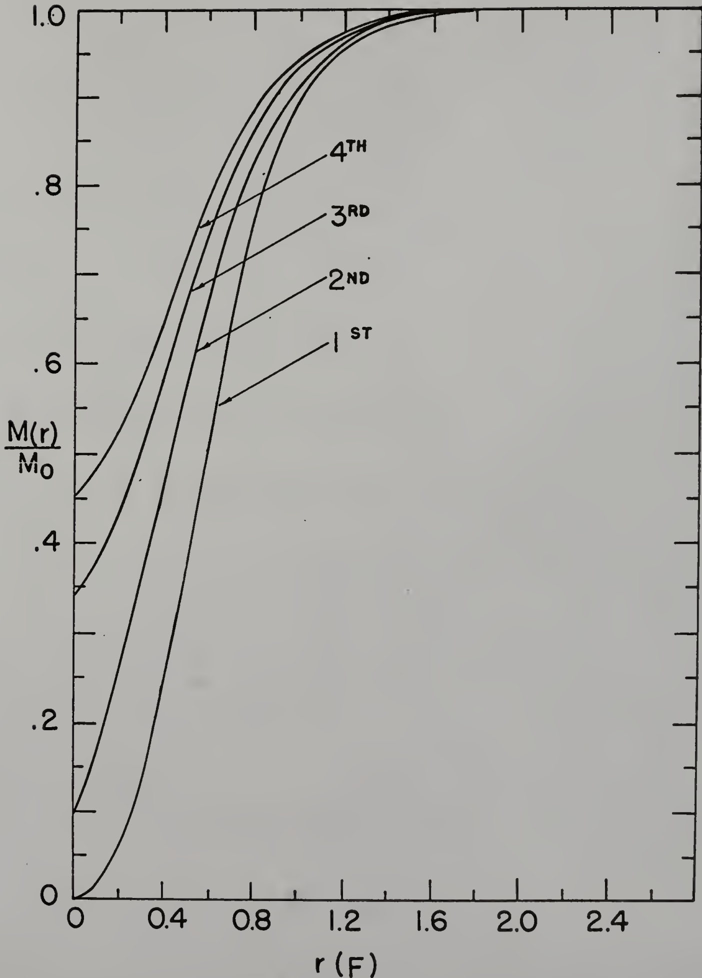

Figure 3A. Changes in the Effective Mass Due to Green's (62) Modified Yukawa Function.

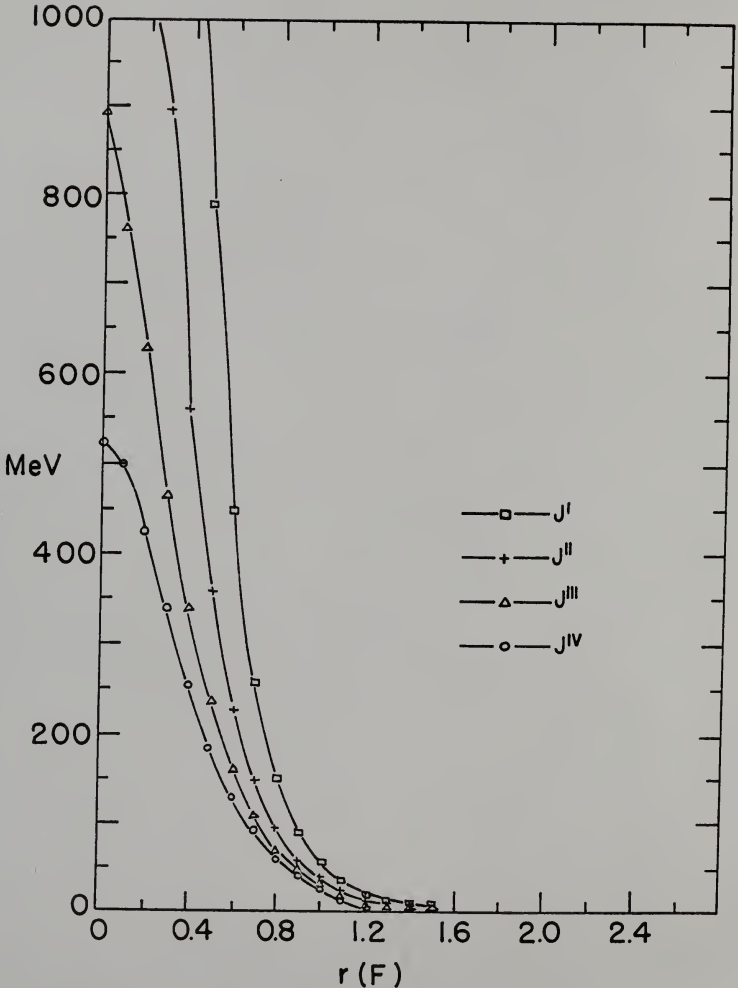

Figure 3B. Modifications of the Yukawa Function Due to Green's (62) Generalized Meson Theory.

Born approximation arising from this approach and compare them with the experimental facts.

Section - 5 Velocity Dependence In Nuclear Physics And Non-Locality

It is exciting to note that Green, et.al [References (31) through (39)] have worked extensively on the phenomenological velocity dependent potentials in nuclear physics. Thus the treatment of Schrödinger equation becomes quite difficult but the effective potential approach was used by them in this period (1955 - 1960). Thus the Schrödinger equation was mostly written as

$$\left\{ -\frac{\hbar^2}{8}\left[\nabla^2 \frac{1}{M(r)} + 2\,\vec{\nabla}\cdot\frac{1}{M(r)}\vec{\nabla} + \frac{1}{M(r)}\nabla^2\right] - V_0\,\mathcal{E}(r)\right\}\psi = E\psi \qquad (\text{IV}-5.1)$$

with

$$\frac{M(r)}{M_0} = \frac{1}{1+\beta\,\mathcal{E}(r)},$$

where β, V_0 are parameters and $\mathcal{E}(r)$ had been taken real or complex depending on the nature of the problem.

Wheeler (40) was first to recognize the connection of velocity dependence with non-local operators in his work on nucleon-nuclear potentials when one could write

$$V(\vec{r},\vec{p})\,\psi(\vec{r}) = \int K(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,')\,d\vec{r}\,',$$

where

$$K(\vec{r},\vec{r}') = \frac{1}{(2\pi\hbar)^3}\int V(\vec{r},\vec{p})\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, d\vec{p} \qquad (IV-5.2)$$

and symbols are self-explanatory. If the range of non-locality is assumed small then one can write the kernal as a smeared δ function.

$$K(\vec{r},\vec{r}') = -V_0\,\mathcal{E}\left(\left|\frac{\vec{r}+\vec{r}'}{2}\right|\right)\delta_b(\vec{r}-\vec{r}') . \qquad (IV-5.3)$$

For small non-locality $\vec{r}' = \vec{r} + b\vec{\rho}$ and with Taylor expansion around $b\vec{\rho} = 0$ and collection of terms in powers of b, we get

$$-V_0\,\mathcal{E}(\vec{r})\,\psi(\vec{r}) - \frac{b^2 V_0}{16}\left\{\nabla^2\mathcal{E}(r) + 2\vec{\nabla}\cdot\mathcal{E}(\vec{r})\vec{\nabla} + \mathcal{E}(r)\,\nabla^2\right\}\psi(\vec{r}) = E\,\psi(\vec{r}) \qquad (IV-5.4)$$

where properties of δ functions ($b^3\int\delta_b(\vec{\rho})\,d\vec{\rho} = 1$ and $b^3\int\rho^2\delta_b(\vec{\rho})\,d\vec{\rho} = 3/2$) make odd powers of b vanish. Hence we see that the non-locality to a certain degree can be brought into a velocity dependence. The physical nature of correlations in nuclear phenomena may be thus accounted by velocity dependence. These results have been shown by Green (81), Moravcsik (24), and Mott and Massey (98).

Thus we have concluded our discussion on velocity dependence and now we embark on the difficult task of confronting ourselves with the nucleon-nucleon scattering phase shifts in non-relativistic energy (elastic scattering) range (0 - 320 mev) with Born approximation. With this technique we hope to extract substantial information on nucleon-nucleon force in intermediate distances ranging from one to two fermis. This will be done in the light of the One Boson Exchange Potentials derived so far.

CHAPTER V

PHASE SHIFTS IN BORN APPROXIMATION FOR ONE BOSON EXCHANGE POTENTIALS

Section - 1 Classification of Nucleon-Nucleon Scattering States

In this chapter we will calculate the phase shifts corresponding to our One Boson Exchange Potentials developed in previous chapters. We confine ourselves to Born approximation because we want to treat velocity dependence in a direct way without going to the effective mass approach. We also want to estimate the importance of individual meson contributions and to examine the contribution from each term in the potential. At the same time, we would like to see the total picture of scattering states, judge the validity of Born approximation for lower angular momentum states, and compare these results with exact phase shift analysis. We also wish to study different forms of vector meson interactions. It may also be possible then to extract some information on the nature of nucleon-nucleon force and on possible corrections to the Born approximation. The exact phase shift analysis within the framework of the above potentials for $\mathcal{P}$—waves and higher has been reported by Bryan and Scott (68) and, subsequent to our publication, they have incorporated the effective potential approach to include velocity dependence in the exact phase shift calculation code. Thus we can not over emphasize their original results because of their neglect of velocity dependent terms. However, there are many features of their work that will be useful for discussion of the nature of the problem. All these reasons made us choose a Born approximation approach to nucleon-nucleon scattering.

Before we proceed we want to sketch the connection of phase shifts with experimental data. This itself is quite a detailed study but the voluminous work of Livermore's group (references 90 through 93) has enabled the field to have a unique set of phase shifts which utilizes scattering and polarization data to determine nucleon-nucleon elastic scattering matrix at energies of 25, 50, 95, 142 and 310 Mev in laboratory frame. This includes the analysis of (p, p) and (n, p) data. We shall keep coming to the discussion of these whenever it is necessary, and these will form the basis of our comparison. The most recent representations are due to Arndt and McGregor (93).

Now we proceed to classify the scattering states of two nucleon system each nucleon having spin magnitude 1/2. The total spin of the system S, can be 0 or 1 and the corresponding states are known as singlet or triplet spin states respectively. As is well known, the spin part of wave function is symmetric in triplet case and is anti-symmetric in the singlet case. Furthurmore, three types of usual symmetric spin wave functions are known. Isospin plays an analogous role with I-spin components for neutron and proton having magnitude 1/2. Thus we can get total I-spin T = 0 or 1 and similar arguments to those of spin hold. We know that the quantum mechanical problem and its complete degeneracy is removed if we know all different commuting operators which commute with the Hamiltonian. In alternative language, we can employ conservation principles derived from experiments and experience in strong interactions to decompose the $N-N$ scattering states into partial wave formalism. This can be done as follows.

Rotational invariance requires that the total angular momentum J be a constant of motion and therefore states with different J values will not mix together. Also we have seen that only singlet and triplet spin states are possible from vector addition theorem. If we adopt a general notation $^{(2S+1)}(\ell)_J$ to represent a state of the two body system in analogy with spectroscopic notation (as is usually done), then we get the following decoupled states.

$$\begin{array}{llll} {}^1S_0, & {}^3P_0 & & \\ {}^3S_1, & {}^1P_1, & {}^3P_1, & {}^3D_1 \\ {}^3P_2, & {}^1D_2, & {}^3D_2, & {}^3F_2 \\ {}^3D_3, & {}^1F_3, & {}^3F_3, & {}^3G_3 \\ {}^3F_4, & {}^1G_4, & {}^3G_4, & {}^3H_4 \end{array} \qquad (\text{V}-1.1)$$

where states in each line could, on the J conservation rule, get mixed together.

We know another tested and well known principle of Conservation of Parity in strong interactions. If the parity is conserved, then there should be no transition between states of even parity and odd parity. The concept of parity operator implies spacial reflection $(\vec{x} \to -\vec{x})$ through origin and it is idempotent. Its eigenvalues are given by $(-1)^{\ell}$. The states with even (odd) angular momenta have even (odd) parity. Thus the states with different parity can not mix and we have furthur classification

EVEN	ODD
$^1S_0$	$^3P_0$
$^3S_1$, $^3D_1$	$^3P_1$, $^1P_1$
$^1D_2$, $^3D_2$	$^3P_2$, $^3F_2$
$^3D_3$, $^3G_3$	$^3F_3$, $^1F_3$
$^1G_4$, $^3G_4$	$^3F_4$, $^3H_4$

(V - 1.2)

Next we have Isospin Conservation and Generalized Pauli Principle for identical particles. The neutron and proton are two different Isospin states of the same particle, the nucleon. If we have say (p,p) scattering then the states of NN scattering are confined to T = 1 and they have to furthur satisfy the Generalized Exclusion Principle. This states that the wave function of two nucleons must be antisymmetric relative to exchange of the space, spin, and isospin coordinates. This, therefore, restricts the states as follows

SINGLET EVEN		TRIPLET ODD
		$^3P_0$
$^1S_0$	$\underline{T=1}$	$^3P_1$
$^1D_2$		$^3P_2$, $^3F_2$
		$^3F_3$
$^1G_4$		$^3F_4$, $^3H_4$

(V-1.3)

But isospin conservation demands that the states of T = 0 do not mix with states of T = 1 and this therefore classifies the states of NN - system and also satisfies the exclusion principle which may be stated as

$$P_{\vec{r}}\, P_{\vec{\sigma}}\, P_{\vec{\tau}} = -1$$

or

$$l + S + T = \text{odd}$$

(V-1.4)

Thus we get for T = 0 states

TRIPLET EVEN		SINGLET ODD
$^3S_1$, $^3D_1$		$^1P_1$
$^3D_2$	$\underline{T=0}$	
$^3D_3$, $^3G_3$		$^1F_3$
$^3G_4$		$^1H_5$

(V-1.5)

Both (V-1.3) and (V-1.5) completely break the degeneracy of nucleon-nucleon scattering states. The only complication, that will be discussed later, is due to the triplet states with the same J but with $\ell = J \pm 1$, and they get mixed with each other. We will see that we need to define one more additional parameter for each J to completely represent them. The tensor force operator, as we shall see, mixes the states with different ℓ values. It is interesting to note that S has become a good quantum number as a by product of the exclusion principle.

Now we consider the effect of various operators acting on these states. The operators we have already discussed are the velocity dependent parts of potential and the remaining ones are $(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$, $(\vec{\ell} \cdot \vec{S})$, $(\vec{\ell}^{\,2})$ and (S_{12}). The effect of $(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$ can be seen by considering the effect on spin symmetric (triplet) functions

$$\alpha(1) \quad \alpha(2)$$

$$\frac{1}{\sqrt{2}}\left[\alpha(1)\ \beta(2) + \beta(1)\ \alpha(2) \right]$$

and

$$\beta(1)\ \beta(2)$$

where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{V-1.6}$$

which yield

$$\langle \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \rangle = +1 \ . \tag{a) (V-1.7}$$

For spin antisymmetric (singlet) wave function

$$\frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2) - \beta(1)\alpha(2) \right] \qquad (\text{V}-1.7\text{b})$$

which yields

$$\langle \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \rangle = -3 , \qquad (\text{V}-1.8)$$

where we used usual properties of Pauli matrices. Since $(\vec{\ell} \cdot \vec{S})$ operator commutes with $\vec{J}^2$, $\vec{\ell}^2$, $\vec{S}^2$ and J_z, we can write for $|J, S, \ell, J_z\rangle$ state the eigenvalues of $(\vec{\ell} \cdot \vec{S})$ operator as

$$\langle (\vec{\ell} \cdot \vec{S}) \rangle = \frac{1}{2}\left[J(J+1) - \ell(\ell+1) - S(S+1) \right] \qquad (\text{V}-1.9)$$

which could be also derived from

$$\vec{J}^2 = (\vec{\ell} + \vec{S})^2 = \vec{\ell}^2 + \vec{S}^2 + 2(\vec{\ell} \cdot S) \qquad (\text{V}-1.10)$$

and then substituting wave mechanical eigenvalues. We see that $\langle \vec{\ell} \cdot \vec{S} \rangle = 0$ for singlets. Depending on the value of ℓ we have

$$\langle \vec{\ell}^2 \rangle = \ell(\ell+1) \qquad (\text{V}-1.11)$$

because $[\vec{\ell}^2, J] = [\vec{\ell}^2, \vec{\ell}]$ which takes corresponding ℓ value for coupled states. The tensor force operator does not prove to be that simple because it has non-diagonal matrix elements in $|J, \ell, S\rangle$ representation. They connect states with $\ell = J \mp 1$ values. The states with $J = \ell$ are not connected with these because of parity conservation and for singlet states the tensor force averages to zero, identically. This can be seen by using (V-1.7b), definition of S_{12} in component form, and of spherical harmonics in terms of x, y, z. The values of tensor force operator for coupled triplet states were first derived by Bethe (94) and have been also discussed by Hulthe'n and Sugawara (21). Since $\vec{l}$ is not a good quantum number, the eigenfunctions of J, J_z and l are given by spin angular functions.

$$\Phi_{J, J_z, l}(\theta, \varphi, \text{spin}) = \sum_{l_z = J_z - 1}^{J_z + 1} C_{J J_z l l_z} Y_l^{l_z}(\theta, \varphi) \chi_{J_z - l_z} \qquad (\text{V}-1.12)$$

where Y_l^m are normalized spherical harmonics, χ's are triplet spin functions and C's are Clebsch-Gordon coefficients. The remaining task is a lengthy calculation which determines the matrix elements of tensor force by calculating

$$\langle J s l \,|S_{12}|\, J s l' \rangle \equiv \langle l \,|S_{12}|\, l' \rangle$$

$$= \int \left(\Phi_{J J_z l}, \; S_{12} \Phi_{J J_z l'} \right) d\Omega, \qquad (\text{V}-1.13)$$

where integral over θ, φ and summation for spin coordinates are implied. By using properties of C's, Y's and the $\vec{\sigma}$'s, the non-vanishing elements of the tensor force are obtained.

A parallel but different mathematical treatment is due to Corben and Schwinger (95) and Rarita and Schwinger (96) who utilize the fact that only three projections of spin are possible for triplets and calculate these matrix elements using vector spherical harmonics. It is an interesting alternative method of obtaining the same results.

The results for all these operators are

$$(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) = +1 \quad \text{triplets}$$
$$= -3 \quad \text{Singlets}$$

$$(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) = +1 \quad \text{(Isotriplets)}$$
$$= -3 \quad \text{(Isosinglets)}$$

$$\text{(V-1.14)}$$

$$(\vec{l} \cdot \vec{s}) = 0 \quad \text{singlets}$$
$$= -1 \quad (l = J)$$
$$= +l \quad (l = J+1)$$
$$= -(l+1) \quad (l = J-1) \qquad \text{(V-1.15)}$$

$$(\vec{l}^{\,2}) = l(l+1) \qquad \text{(V-1.16)}$$

$$\langle S_{12} \rangle = 0 \quad \text{(singlets)}$$

$$\langle l=J \,|\, S_{12} \,|\, l=J \rangle = 2$$

$$\langle J-1 \,|\, S_{12} \,|\, J-1 \rangle = -\frac{2(J-1)}{2J+1}$$

$$\langle J+1 \,|\, S_{12} \,|\, J+1 \rangle = \frac{-2(J+2)}{2J+1}$$

$$\langle J-1 \,|\, S_{12} \,|\, J+1 \rangle = \langle J+1 \,|\, S_{12} \,|\, J-1 \rangle \qquad \text{(V-1.17)}$$

These completely specify effects of all operators on all states of NN- system.

Section - 2 Explicit Forms Of One Boson Exchange Potentials

Now we proceed to reduce our potentials from functions to explicit forms which will be then used for the calculation of phase shifts. In this approach, the radial behavior of many terms will become clear. In effect we had done this when we compared various potentials for scalar, vector, and pseudoscalar cases. We also want to remind ourselves that both theoretical and phenomenological reasons support the modified Yukawa form with Green's subtractive meson as given by

$$J(r) = g^2 (\hbar c)\left[\frac{e^{-Kr}}{r} - \frac{e^{-\Lambda r}}{r}\right] \qquad (\text{V}-2.1)$$

where

$$K = \frac{mc}{\hbar}, \qquad \Lambda = \frac{m_T c}{\hbar} \qquad (\text{V}-2.2)$$

Thus Λ can be interpreted as a parameter which reduces uncertainties in the behavior of NN potentials for inner regions.

This form as well as more general forms of this type have been obtained by Green (62, 63, 64) in meson field theory on generalizations of Podolsky's (43) treatments in quantum electrodynamics. These were also discussed in Chapter IV when we compared our velocity dependent form factors with those of phenomenological potentials. Green showed with these modified Yukawa forms that infinite self energies did not appear and that the various potential terms were non-singular at origin. Later on this modified Yukawa form as well as other conditions have appeared with different names (e.g. Pauli-Villar of Faynman regularization). A full discussion and utilization of these conditions in terms of field theories with indefinite metric has not yet been fully exploited. From out standpoint here, this result of Green's gives a convenient cut-off parameter at a helpful place and enables us to allow for the uncertainty of nuclear forces at shorter distances. This parameterization, as mentioned before, compensates for higher mass mesons, pair processes, etc., and forms a potential without singularities. Let

$$k_N = \frac{Mc}{\hbar}$$

$$k = \sqrt{\frac{M E_{CM}}{\hbar^2}} = \sqrt{\frac{M E_{lab}}{2\hbar^2}}$$

and

$$z = kr \qquad (\text{V}-2.3)$$

The general potential is written as

$$U_{Tot} = U_c + U_\nabla\, i(\vec{r}\cdot\vec{p}) + U_\Delta\, p^2 + U_\gamma + $$
$$+ U_{SS}\,(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + U_{LS}\,(\vec{l}\cdot\vec{S}) + U_{LL}\,(\vec{l}^{\,2}) + $$
$$+ U_T\,(S_{12}) \qquad (\text{V}-2.4)$$

with

$$U_{Tot} = \frac{M}{\hbar^2}\, V_{Tot} \qquad (\text{V}-2.5)$$

For special cases, we have the following forms using (III-3.21), (III-3.36),

SCALAR

$$U^s_Y = -g^2 K_N \left(\frac{e^{-Kr}}{r} - \text{cutoff} \right)$$

$$U^s_c = -\frac{g^2}{4K_N} \left(K^2 \frac{e^{-Kr}}{r} - \text{cutoff} \right)$$

$$U^s_\nabla = \frac{g^2}{\hbar K_N} \left[(1+Kr)\, e^{-Kr} - \text{cutoff} \right] \frac{1}{r^3}$$

$$U^s_\Delta = \frac{g^2}{\hbar^2 K_N} \left[\frac{e^{-Kr}}{r} - \text{cutoff} \right]$$

$$U^s_{SS} = 0 = U^s_{LS} = U^s_{LL} = U^s_T \qquad (\text{V}-2.6)$$

VECTOR (Green's)

$$U^v_Y = -U^s_Y$$

$$U^v_c = U^v_{LL} = 0, \quad U^v_{LS} = 3U^s_{LS}$$

$$U^v_\nabla = U^s_\nabla, \quad U^v_T = -\frac{1}{12} \frac{g^2}{K_N r^3} \left[(K^2 r^2 + 3Kr + 3)\, e^{-Kr} - \text{cutoff} \right]$$

$$U^v_\Delta = U^s_\Delta$$

$$U^v_{SS} = \frac{g^2}{6K_N r} \left(K^2 e^{-Kr} - \text{cutoff} \right) \qquad (\text{V}-2.7)$$

PSEUDOSCALAR

$$U^{p}_{C} = U^{p}_{\nabla} = U^{p}_{\Delta} = U^{p}_{LL} \equiv 0$$

$$U^{p}_{SS} = \frac{1}{2} U^{v}_{SS}$$

$$U^{p}_{T} = - U^{v}_{T}$$

$$(\text{V} - 2.8)$$

BREIT VECTOR

$$U^{B}_{C} = \frac{g^2}{K_N} \left[\frac{5}{8} \left(k^2 e^{-kr} - \text{cutoff} \right) \frac{1}{r} - \frac{1}{8} \left\{ k^2 (1+kr) e^{-kr} - \text{cutoff} \right\} \frac{1}{r} \right]$$

$$U^{B}_{\nabla} = -\frac{1}{2} U^{v}_{\nabla} + \frac{g^2}{2K_N \hbar} \left[(k^2 r^2 + 3kr + 3) e^{-kr} - \text{cutoff} \right] \frac{1}{r^3}$$

$$U^{B}_{\Delta} = \frac{1}{2} U^{v}_{\Delta} + \frac{g^2}{2K_N \hbar^2} \left[(1+kr) e^{-kr} - \text{cutoff} \right] \frac{1}{r}$$

$$U^{B}_{SS} = U^{v}_{SS}, \quad U^{B}_{LS} = U^{v}_{LS}, \quad U^{B}_{T} = U^{v}_{T}$$

$$U^{B}_{LL} = -\frac{g^2}{2K_N} \left[(1+kr) e^{-kr} - \text{cutoff} \right] \frac{1}{r^3}$$

$$(\text{V} - 2.9)$$

KEMMER VECTOR

$$U_Y^K = U_Y^B$$

$$U_c^K = \frac{g^2}{4K^2K_N}\left[\frac{K^4e^{-Kr}}{r} - \text{cutoff}\right]$$

$$U_\nabla^K = U_\nabla^v + \frac{g^2}{K^2K_N\hbar}\left[-(K^3r^3+3K^2r^2+6Kr+6)e^{-Kr} - \text{cutoff}\right]\frac{1}{r^5}$$

$$U_\Delta^K = U_\Delta^v - \frac{g^2}{K^2K_N\hbar^2}\left[(K^2r^2+2Kr+2)e^{-Kr} - \text{cutoff}\right]\frac{1}{r^3}.$$

$$U_{SS}^K = U_{SS}^B = U_{SS}^v, \quad U_T^K = U_T^B = U_T^v$$

$$U_{LL}^K = \frac{g^2}{KK_N}\left[(K^2r^2+3Kr+3)e^{-Kr} - \text{cutoff}\right]\frac{1}{r^5}. \qquad (\text{V}-2.10)$$

We have carried our various differentiations and used notation that is obvious or is defined in this section. Again the relativistic nature of these interactions in apparent. The main static Yukawa type term in various cases cancels when the same mass m, coupling g^2, and cut-off mass m_c are used for the scalar and any of the three types of vector mesons. The purely relativistic forces in the three cases are, however, quite different and it is essential to carry out a careful study of them. This will form the subject of the next section where we present the phase shift formulation in Born approximation. Care should be taken in interpreting these formulae because both isoscalar and isovector meson contributions should be added properly $V^{(0)} + \vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} V^{(1)}$.

Section - 3 Born Approximation Phase Shifts For Velocity Dependent Potentials

Starting with first principles, we examine the meaning of Born approximation for velocity dependent potentials and the nature of restrictions arising out of it.

Assumptions

1. A Schrödinger-like equation can be written for such a potential with $V(r, \vec{p})$ replacing the usual $V(r)$.
2. The coulomb forces are small and can be neglected for (p, p) scattering.
3. The potential form factors are short ranged for reasonable masses and couplings in modified Yukawa forms.
4. The modified Yukawa form occurs with either the theoretical or phenomenological cut-off mesons in the form

$$J(r) = g^2 (\hbar c) \left(\frac{e^{-kr}}{r} - \frac{e^{-\Lambda r}}{r} \right), \qquad (\text{V}-3.1)$$

as given by Green (63, 64). Hence the Schrödinger equation can be written for scattering states of two nucleon system as

$$\left[-\frac{\hbar^2}{M} \nabla^2 + V(r, \vec{p}) \right] \Psi = E \Psi \qquad (\text{V}-3.2)$$

or

$$\left[-\nabla^2 + U(r, \vec{p}) \right] \Psi = k^2 \Psi \qquad (\text{V}-3.3)$$

where E is the energy in the center of mass frame. Let the solution of this equation, without interaction, be Φ. Then we have

$$[-\nabla^2]\Phi = k^2\Phi . \qquad (V-3.4)$$

Multiplying (V-3.3) by Φ and (V-3.4) by Ψ and subtracting we get

$$[\Phi\nabla^2\Psi - \Psi\nabla^2\Phi] = \Phi\, U(r, \vec{p})\, \Psi . \qquad (V-3.5)$$

If the solutions Φ, Ψ exist, then (V-3.5) holds and is an identity. Thus Φ, Ψ are supposed to be regular functions with meaningful second derivatives. This also implies that the operation $U(r, \vec{p})\Psi$ is meaningful because derivatives up to only second order occur in the potential operator as given by Equation (IV-2.3) (denoted by V_{Tot}).

We want to consider for a while only the left hand side (L.H.S.) of (V-3.5) with the Green theorem in mind. The conditions, which two scalar functions Φ and Ψ must satisfy for Green's theorem to be applicable, have been discussed in detail by Korn and Korn (97) and also by Mott and Massey (98). Thus a volume integral of the above expression can be changed by Green's theorem into a surface integral provided that the volume integrals are taken over an open singly connected bounded region V which is bounded by a two sided regular closed surface S . All functions are assumed to be single valued through out V and on S . Let us choose our volume to be bounded by two surfaces S_1 and S_2 with radii ϵ_1 and ϵ_2 . In the limit we shall make $\epsilon_1 \to 0$ and $\epsilon_2 \to \infty$. This will impose certain restrictions for the integrals to be defined in this limit and they will be now examined in the light of Velocity Dependent Potentials. There are no singularities in the region V and the J'^s vanish at reasonable distances (a few fermis). Thus with Green's theorem and L.H.S. of (V-3.5)

$$\int_{\epsilon_1}^{\epsilon_2}\iint_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV = \oiint_{S=S_1+S_2} (\Phi \vec{\nabla}\Psi - \Psi \vec{\nabla}\Phi) \cdot d\vec{s} \qquad (V-3.6)$$

where the contributions from the enclosing surface S are only due to S_1 and S_2. The surface S_1 shrinks to zero area when $\epsilon_1 \to 0$ and in order for the surface integral in (V-3.6) to exist we need two requirements. (1) <u>Ψ, Φ to be finite and continuous at origin</u>. (2) <u>$\vec{\nabla}\Psi$, $\vec{\nabla}\Phi$ (Gradients) be also finite and continuous at origin</u>. Those are the requirements of a quantum mechanical wave function as discussed by Schiff (7a) and the boundary conditions of this type are also discussed by Jackson (99) and Morse and Feshbach (100). Thus the requirements for this limit to exist are the requirements which are usual to the physical wave function and the velocity dependence of the potential does not affect these arguments until we consider the right hand side of (V-3.5).

Thus the only integral that survives in (V-3.6) is over surface S_2 which has to be considered when $\epsilon_2 \to \infty$ i.e. in the region where Ψ and Φ take on their asymptotic values. In order for the limit $\epsilon_1 \to 0$ to be meaningful, the integral

$$I = \underset{V}{\iiint_{\epsilon_1 \to 0}^{\epsilon_2 \to \infty}} \Phi \quad U(r, \vec{p}) \, \Psi \, dV \qquad (V-3.7)$$

should exist. Whether this imposes certain restrictions on velocity dependent parts or on other singular tensor and spin-orbit parts of U, will be seen in the following. We do not go to Born approximation and try to evaluate these conditions in the exact case. The integral (V-3.7) can be written with the potential with the velocity dependent part separated from the rest

$$\begin{aligned} I = \underset{V}{\iiint_{\epsilon_1 \to 0}^{\epsilon_2 \to \infty}} \Big\{ & \left[\Phi \, J \langle \nabla^2 \Psi \rangle + \Phi \langle \nabla \vec{J} \rangle . \langle \vec{\nabla} \Psi \rangle \right] + \\ & + \Phi \left[U_Y + U_C + U_{SS} (\vec{\sigma}^{(1)} . \vec{\sigma}^{(2)}) + U_{LS} (\vec{\ell} . \vec{s}) + \right. \\ & \left. + U_T (S_{12}) \right] \Psi \Big\} \, dV \\ \equiv \; & I_1 + I_2 \end{aligned} \qquad (V-3.8)$$

We want to change I_1, with velocity dependent part, into volume and surface integrals so that derivatives are transformed to Φ, the free field solution (plane wave) and the modified Yukawa form. Thus we shall be able to see the conditions at origin because we know the effect of these derivatives on Φ and J. In the upper limit $\epsilon_2 \to \infty$, the integrand identically vanishes because J's all vanish. Thus we have to consider only the limit $\epsilon_1 \to 0$.

We use the identities

$$\vec{\nabla} \cdot (\Phi J \langle \vec{\nabla} \Psi \rangle) = J \langle \vec{\nabla} \Phi \rangle \cdot \langle \vec{\nabla} \Psi \rangle +$$

$$+ \Phi \langle \vec{\nabla} J \rangle \cdot \langle \vec{\nabla} \Psi \rangle + \Phi J \langle \nabla^2 \Psi \rangle \tag{V-3.9}$$

and

$$\vec{\nabla} (J \Psi \langle \vec{\nabla} \Phi \rangle) = \langle \vec{\nabla} J \rangle \cdot \langle \nabla \Phi \rangle \Psi +$$

$$+ J \langle \vec{\nabla} \Phi \rangle \cdot \langle \vec{\nabla} \Psi \rangle + J \langle \nabla^2 \Phi \rangle \Psi \tag{V-3.10}$$

These, on application of Green's theroem (Gauss' divergence theorem) yield (with $\iint_{S_2} \to 0$)

$$I_1 = \iiint_{\substack{\epsilon_1 \to 0 \\ V}}^{\epsilon_2 \to \infty} \left[\Phi J \langle \nabla^2 \Psi \rangle + \Phi \langle \vec{\nabla} J \rangle \cdot \langle \vec{\nabla} \Psi \rangle \right] dv$$

$$= \iint_{S_1} \left[\Phi J \langle \vec{\nabla} \Psi \rangle - J \langle \vec{\nabla} \phi \rangle \psi \right] \cdot d\vec{S}_1 +$$

$$+ \iiint_{\substack{\epsilon_1 \to 0 \\ V}}^{\epsilon_2 \to \infty} \left[J \langle \nabla^2 \Phi \rangle \Psi + \langle \vec{\nabla} J \rangle \cdot \langle \vec{\nabla} \phi \rangle \Psi \right] dv \tag{V-3.11}$$

Now we use (V-3.4) and also note that

$$\langle \vec{\nabla} J \rangle \cdot \langle \vec{\nabla} \phi \rangle = J_1 \, r \frac{d}{dr} \Phi = \frac{dJ}{dr} \frac{d\Phi}{dr} \tag{V-3.12}$$

where the angular contributions are zero because $J \equiv J(r)$.

Using the already made assumptions (i.e. Ψ, Φ and

$\langle \nabla \vec{\Psi} \rangle$, $\langle \vec{\nabla} \Phi \rangle$, are finite and continuous at origin) we get the condition from surface integral that J can at most vary as $\frac{1}{r^2}$. And the volume integral yields that J and $\frac{dJ}{dr}$ may vary the most as $1/r^3$.

Now we consider I_2 which gives the contribution to the volume integral from the non-velocity dependent terms. Thus

$$I_2 = \iiint\limits_{\substack{\epsilon_1 \to 0 \\ V}}^{\epsilon_2 \to \infty} \Phi \left[U_Y + U_C + U_{SS}(\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}) + U_{LS}(\vec{\ell} \cdot \vec{s}) + U_T(S_{12}) \right] \Psi \, dV. \quad (\text{V}-3.13)$$

and this restricts the U's to vary at the most as $1/r^3$. By the term "at the most" we mean $1/(r^{3-\eta})$, $(\eta \to 0)$ such that the resulting integrand is integrable. The reader is reminded that this is so because the volume is proportional to r^3 and the surface is to r^2. If we carry out exact analysis with Yukawa form

$$J(r) = g^2 (\hbar c) \frac{e^{-kr}}{r}$$

only, then we see that

$$U_{LS} = U_T \propto \frac{1}{r^3} \text{ near the origin.}$$

Thus the Yukawa potential without the cut-off is not integrable. Also V_{SS} has a δ function because

$$\langle \nabla^2 J \rangle = g^2 (\hbar c) \left\langle \nabla^2 \frac{e^{-kr}}{r} \right\rangle = \left[k^2 e^{-kr}/r - 4\pi \delta(r) \right] g^2 (\hbar c), \quad (\text{V}-3.14)$$

as given by Wentzel (22). This would give a finite contribution from spin-spin term. These difficulties about tensor force and spin-orbit force disappear if Green's subtractive meson is present, i.e. if

$$J(r) = g^2 (\hbar c) \left[\frac{e^{-kr}}{r} - \frac{e^{-\Lambda r}}{r} \right] \quad (\text{V}-3.15)$$

So that near the origin the radial dependence of U_{LS} and U_T can be deduced by expanding the exponential and we obtain

$$U_{LS} \propto \frac{1}{r^3} \left[(1+kr)\left(1-kr+\frac{k^2r^2}{2}\right) - (1+\Lambda r)\left(1-\Lambda r+\frac{\Lambda^2 r^2}{2}\right) \right]$$

$$\propto \frac{(\Lambda^2-k^2)}{2r} + \cdots ,$$

$$U_T \propto \frac{1}{r^3} \left[(k^2r^2+3kr+3)\left(1-kr+\frac{k^2r^2}{2}\right) - (\Lambda^2r^2+3\Lambda r+3)\left(1-\Lambda r+\frac{\Lambda^2r^2}{2}\right) \right]$$

$$\propto \frac{\Lambda^2-k^2}{2r} + \cdots \quad (\text{V}-3.16)$$

where only singular term is proportional to $(1/r)$ which is quite acceptable in volume and surface integrals at origin.

Thus if we take modified Yukawa form with a cut-off, as originally derived by Green (63,64) based on generalizations of Podolsky's electrodynamic treatments, we can remove the difficulties with the tensor and the spin-orbit forces. Also the conditions from I_1 are also satisfied as

$$\frac{dJ}{dr} \propto \frac{1}{r^2}\left[(1+kr)\left(1-kr+\frac{k^2r^2}{2}\right) - \right.$$

$$\left. -(1+\Lambda r)\left(1-\Lambda r+\frac{\Lambda^2 r^2}{2}\right)\right]$$

$$\propto \frac{(\Lambda^2 - k^2)}{2} + \cdots\cdots ,$$

also

$$\langle \nabla^2 J \rangle \propto \left(k^2 \frac{e^{-kr}}{r} - \Lambda^2 \frac{e^{-\Lambda r}}{r} \right) \qquad (\text{V}-3.17)$$

thus the δ functions also cancel. In this way we conclude that neither the velocity dependent potentials nor the other potentials create any theoretical difficulties if we choose the modified Yukawa forms with Green's subtractive mesons. Thus we can meaningfully write (V-3.6), (V-3.7), and (V-3.5) as

$$\iint_{\substack{\epsilon_2 \to \infty \\ S_2}} (\Phi \vec{\nabla} \Psi - \Psi \vec{\nabla} \Phi) \cdot d\vec{S}_2 = \int_0^\infty \int_0^\pi \int_0^{2\pi} \Phi \, U(r, \vec{p}) \, \Psi \, dv \qquad (\text{V}-3.18)$$

where Φ, Ψ are functions of (r, θ, φ) and $d\vec{S}_2$ is drawn such that the outward normal $\hat{n}$ is in the direction of $\vec{r}$. Thus $d\vec{S}_2 = dS_2 \, \hat{n}$ and $(\vec{\nabla} \cdot \hat{n})$ imply that only the radial part will contribute.

We want to now decompose the incident plane wave and the outgoing spherical wave into partial waves in the usual manner. Thus we seek a solution of (V-3.3) which, in asymptotic form, is

$$\Psi(r,\theta,\varphi) \xrightarrow[r\to\infty]{} e^{ikz} + \frac{e^{ikr}}{r} f(\theta,\varphi). \qquad (\text{V}-3.19)$$

The symmetry along the z direction, along which the plane wave is incident, implies φ independence. These partial wave expansions are given in Mott and Massey (reference 98, page 21).

$$\Phi = e^{ikz} = e^{ikr\cos\theta}$$

or

$$\Phi = \sum_{l=0}^{\infty} (2l+1)\, i^l\, P_l(\cos\theta)\, j_l(kr)$$

$$j_l(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - l\pi/2)}{kr} \qquad (\text{V}-3.20)$$

Since axial symmetry about $\vec{k}$ exists, we can write

$$\Psi = \sum_{l=0}^{\infty} A_l\, P_l(\cos\theta)\, R_l(r) . \qquad (\text{V}-3.21)$$

Since the scattered part of the wave function must not have a e^{-ikr}/r (the converging wave) the A_l's have to be chosen for each l such that in the asymptotic region

$$A_l R_l(r) - (2l+1)\, i^l j_l(kr) \sim c_l \frac{e^{ikr}}{r} \qquad (\text{V}-3.22)$$

In the asymptotic region $R_l(r)$ can at the most differ from the force solution $j_l(kr)$ by a phase factor as the interaction vanishes. Thus we can write

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr}$$

(V-3.23)

The constant δ_ℓ (called the phase shift) therefore depends on k, ℓ and U and will contain the information about them which interests us. We expand $(\sin x)$ in terms of exponentials and collecting these terms we get L.H.S. of (V-3.22) as

$$\frac{e^{i(kr-\ell\pi/2)}}{2ikr}\left[A_\ell e^{i\delta_\ell} - (2\ell+1)\, i^\ell\right] -$$

$$- \frac{e^{-i(kr-\ell\pi/2)}}{2ikr}\left[A_\ell e^{-i\delta_\ell} - (2\ell+1)\, i^\ell\right].$$

From the second term, therefore,

$$A_\ell = (2\ell+1)\, i^\ell\, e^{i\delta_\ell}. \qquad \text{(V-3.24)}$$

Hence

$$\Psi = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell\, e^{i\delta_\ell}\, P_\ell(\cos\theta)\, R_\ell(r)$$

(V-3.25)

and from (V-3.19) for the asymptotic form of the scattered wave we obtain

$$f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)\left(e^{2i\delta_\ell} - 1\right) P_\ell(\cos\theta)$$

(V-3.26)

If we can obtain an alternative expression for $f(\theta)$ which depends on the potential, then by using orthonormality of Legendre polynomials we can project out a particular partial wave. For this purpose we introduce the integral form of Schrödinger equation

$$\left[\nabla^2 + k^2 - U(r, \vec{p})\right] \Psi(\vec{r}) = 0 \qquad (\text{V}-3.27)$$

where

$$k = \sqrt{\frac{ME}{\hbar^2}} \quad \text{and} \quad U(r, \vec{p}) = \frac{M}{\hbar^2} V_{Tot}(r, \vec{p}).$$

with

$$\mathcal{H}_0 = -\nabla^2, \quad \mathcal{E} = k^2 \qquad (\text{V}-3.28)$$

we write

$$(\mathcal{H}_0 - \mathcal{E})\, \Psi(r) = -U(r, \vec{p})\, \Psi(r) \qquad (\text{V}-3.29)$$

Green's function for this equation, with the outgoing part, is given by

$$G(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi} \int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\,' \qquad (\text{V}-3.30)$$

Hence a general solution is a sum of homogeneous solution and of the non-homogeneous part

$$\Psi(\vec{r}) = e^{ikz} - \frac{1}{4\pi} \int G(\vec{r}, \vec{r}\,')\, U(r, \vec{p})\, \Psi(\vec{r}\,')\, d\vec{r}\,' \qquad (\text{V}-3.31)$$

With the asymptotic expansions

$$|\vec{r} - \vec{r}\,'| = r - r' \cos\alpha = r - \hat{n}\cdot\vec{r}\,' \qquad (\text{V}-3.32)$$

we identify

$$f(\theta) = -\frac{1}{4\pi}\int \bar{e}^{i\vec{k}'\cdot\vec{r}'}\, U(r,\vec{p}')\,\Psi(\vec{r}')\,d\vec{r}' \qquad (V-3.33)$$

where α is the angle between $\vec{r}$ and $\vec{r}\,'$ and $\vec{k}'$ denotes the scattered wave propagation direction which is parallel to $\vec{r}$. It should be clarified here that $\vec{r}\,'$ is the integration variable which in spherical polar coordinates can be represented by $\vec{r}\,' = \vec{r}\,'(r', \zeta, \eta)$, while $\vec{k}'$ denotes the original direction of $\vec{r}$ occuring in (V-3.30) represented by $\vec{k}' = \vec{k}'(k, \theta, \varphi = 0)$, because the original incident wave was propagating along the z axis.

Now we make the Born approximation for velocity dependent potentials by replacing $\Psi(\vec{r}\,')$ by a plane wave. Thus we have

$$\begin{aligned}\Psi(\vec{r}\,') = \Phi(\vec{r}\,') &= e^{+i\vec{k}\cdot\vec{r}\,'} \\ &= e^{ikr'\cos\zeta} \qquad (V-3.34)\end{aligned}$$

where $\vec{k}$ is chosen along z - direction. For simplicity of notation, we put $\vec{r}\,' \equiv r$ for the integral in (V-3.33) and obtain in the Born approximation

$$f(\theta) = -\frac{1}{4\pi}\int \bar{e}^{i\vec{k}'\cdot\vec{r}}\, U(r,\vec{p})\, e^{ikr\cos\zeta}\,dr \qquad (V-3.35)$$

Now we use the expansions of plane and spherical waves in terms of spherical harmonics. The notation adapted here is due to De Benedetti (41) and Clebsch-Gordon coefficients used are also according to its phase conventions. Thus

$$e^{ikr\cos\zeta} = \sum_{\ell'=0}^{\infty} \sqrt{4\pi(2\ell'+1)}\,(i)^{\ell'} j_{\ell'}(kr)\, Y_{\ell'}^{0}(\zeta,\eta)$$

$$(\text{V}-3.36)$$

where $Y_\ell^0(\zeta,\eta)$ does not depend on η because $m' = 0$ for this case. The expansion for the spherical wave with proper identifications of angles is

$$e^{-i\vec{k}'\cdot\vec{r}} = \left[e^{i\vec{k}'\cdot\vec{r}} \right]^*$$

$$= 4\pi \sum_{\ell=0}^{\infty} (-i)^{\ell}\, j_\ell(kr)\, Y_\ell^0(\theta,0) \sum_{m=-\ell}^{+\ell} \cdot$$

$$\cdot\, Y_\ell^{m*}(\zeta,\eta)$$

$$(\text{V}-3.37)$$

where we used the fact that $\varphi = 0$ so only $m = 0$ term survives.

Now we write our potential as

$$U(r,\vec{p}) = U_0(r) + U_\nabla(r)\,(\vec{r}\cdot\vec{\nabla}) + U_\Delta(r)\,\nabla^2$$

$$(\text{V}-3.38)$$

where $U_0(r)$ contains the central, spin-spin, tensor (diagonal part for the uncoupled states), and spin-orbit parts for isoscalar and isovector mesons. Thus U_∇ and U_Δ are only functions of (r) and velocity dependence has been separated out for the most general form of the potential used in our cases. The effect of potential operators on plane wave is

$$\nabla^2 e^{ikz} = -k^2 e^{ikz},$$

$$(\vec{r}\cdot\vec{\nabla})\, e^{ikz} = r\frac{d}{dr}\left(e^{ikr\cos\zeta}\right)$$
$$= (ik\cos\zeta)\, r\, e^{ikz}. \quad (V-3.39)$$

We can also write

$$\cos\zeta = \sqrt{\frac{4\pi}{3}}\, Y_1^0(\zeta, \eta=0) \quad (V-3.40(a))$$

and thus

$$(\vec{r}\cdot\vec{\nabla})\, e^{ikz} = \sqrt{\frac{4\pi}{3}} \sum_{\ell'=0}^{\infty} i^{\ell'} j_{\ell'}(kr) \sqrt{4\pi(2\ell'+1)}\, Y_1^0(\zeta,0)\cdot$$
$$\cdot\, Y_{\ell'}^0(\zeta,0), \quad (V-3.40(b))$$

and with the use of additional theorem for spherical harmonics

$$Y_\ell^m(\theta,\varphi)\, Y_{\ell'}^{m'}(\theta,\varphi) = \sum_{L=|\ell-\ell'|}^{(\ell+\ell')} \left[\frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)}\right]^{1/2}\cdot$$
$$\cdot\, C(\ell\ell'L; m m')\, C(\ell\ell'L; 00)\, Y_L^{m+m'}(\theta,\varphi)$$
$$(V-3.40(c))$$

we obtain, using C's given by De Benedetti (41)

$$\cos\zeta\, Y_{\ell'}^0(\zeta,0) = \frac{\ell'}{\sqrt{(2\ell'-1)(2\ell'+1)}}\, Y_{\ell'-1}^0(\zeta,0) +$$
$$+ \frac{(\ell'+1)}{\sqrt{(2\ell'+3)(2\ell'+1)}}\, Y_{\ell'+1}^0(\zeta,0). \quad (V-3.40(d))$$

Now we substitute the results obtained in equations (V-3.36) through (V-3.40 d) into (V-3.35) and obtain

$$f(\theta) = -\frac{1}{4\pi}\int_0^\infty r^2 dr \int_0^\pi d(\cos\zeta)\int_0^{2\pi} d\eta \,\cdot$$

$$\cdot \left[\!\left[\left\{ 4\pi \sum_{l=0}^{\infty} (-i)^l j_l(kr)\, Y_l^0(\theta,0) \sum_{m=-l}^{+l} \cdot \right.\right.\right.$$

$$\left. \cdot\; Y_l^{m*}(\zeta,\eta) \right\} \left[U_0(r) - k^2 U_\Delta(r) + ikr\cos\zeta\, U_\nabla(r) \right] \cdot$$

$$\left.\left. \cdot \left\{ \sum_{l'=0}^{\infty} \sqrt{4\pi(2l'+1)}\; i^{l'} j_{l'}(kr)\, Y_{l'}^0(\zeta,\eta) \right\} \right]\!\right] \qquad \text{(V-3.40(e))}$$

Using (V-3.40 d) for the last term in the above equation and integrating over (ζ,η) and using the orthonormality condition

$$\int_0^\pi \int_0^{2\pi} Y_l^{m*}(\theta,\varphi)\, Y_l^m(\theta,\varphi)\, d(\cos\theta)\, d\varphi$$

$$= \delta_{ll'}\,\delta_{mm'} \qquad \text{(V-3.40(f))}$$

the sums over l' and m' are removed. Thus we obtain various orders of $j_{l'}(kr)$ contributing as

$$f(\theta) = -\sum_{l=0}^{\infty} (-i)^l Y_l^0(\theta,0) \int_0^\infty r^2 j_l(kr) \Big\{ \big[U_0(r) -$$

$$- k^2 U_\Delta(r) \big] \left(i^l j_l(kr) \sqrt{4\pi(2l+1)} \right. +$$

$$+ ik \left[\sqrt{4\pi}\; i^{l+1} \frac{(l+1)}{\sqrt{2l+1}}\, j_{l+1}(kr) + i^{l-1} \frac{\sqrt{4\pi}\, l}{\sqrt{2l+1}}\, j_{l-1}(kr) \right] \cdot$$

$$\cdot \left[r\, U_\nabla(r) \right] \Big\}\, dr$$

$$\text{(V-3.40(g))}$$

or

$$P_\ell^0(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^0(\theta, 0) \qquad (\text{V}-3.40\ h)$$

and we obtain the final expression as

$$f(\theta) = -\sum_{\ell=0}^{\infty} P_\ell^0(\cos\theta) \Big\{ (2\ell+1) \int_0^\infty r^2 \left[U_0(r) - k^2 U_\Delta(r) \right] \cdot$$

$$\cdot\, j_\ell^2(kr)\, dr + k \int_0^\infty \left[\ell\, j_{\ell-1}(kr) - (\ell+1)\, j_{\ell+1}(kr) \right] \cdot$$

$$\cdot\, r^2 \left[r\, U_\nabla(r) \right] j_\ell(kr)\, dr \Big\} \qquad (\text{V}-3.40(i))$$

Equating two expressions for $f(\theta)$ as given by (V-3.26) and (V-3.40 i), and then projecting a partial wave by multiplying through $P_{\ell'}(\cos\theta)$ and integrating with respect to $\cos\theta$, and using

$$\int_{-1}^{+1} P_\ell(x)\, P_{\ell'}(x) = \delta_{\ell\ell'} \left(\frac{2}{2\ell+1} \right) \qquad (\text{V}-3.40(j))$$

we obtain

$$\frac{2(e^{2i\delta_\ell} - 1)}{2ik} = -2 \Bigg[\!\!\Bigg[\int_0^\infty r^2 \left[U_0(r) - k^2 U_\Delta(r) \right] \cdot$$

$$\cdot\, j_\ell^2(kr)\, dr + \frac{k}{(2\ell+1)} \int_0^\infty r^2 \left[r\, U_\nabla(r) \right] j_\ell(kr) \cdot$$

$$\cdot \left[\ell\, j_{\ell-1}(kr) - (\ell+1)\, j_{\ell+1}(kr) \right] dr \Bigg]\!\!\Bigg] \qquad (\text{V}-3.40(k))$$

or

$$e^{i\delta_\ell} \sin\delta_\ell = -\Big[\int_0^\infty r^2 \left[U_0(r) - k^2 U_\Delta(r)\right] \cdot$$

$$\cdot\, j_\ell^2(kr)\, dr + \frac{k}{(2\ell+1)} \int_0^\infty r^2 \left(r U_\nabla(r)\right) j_\ell(kr) \cdot$$

$$\cdot \left\{ \ell\, j_{\ell-1}(kr) - (\ell+1)\, j_{\ell+1}(kr) \right\} dr$$

$$(\text{V}-3.40(\ell))$$

This result agrees with the result obtained by just carrying the partial wave expansion in Schrödinger's equation, and then letting the various derivatives operate on radial functions when the radial form of the Schrödinger equation is obtained by multiplying with $P_{\ell'}(\cos\theta)$ and using orthogonality condition. Then the phase shifts are obtained from the field free radial equation and radial Schrödinger equation by partial integration. This identification is made clear if we substitute $j_\ell(kr)$ for radial wave function in the Born approximation and identify

$$\frac{d j_\ell(kr)}{dr} = k \frac{d j_\ell(kr)}{d(kr)} = \frac{k}{(2\ell+1)} \left\{ \ell j_{\ell-1}(kr) - (\ell+1) j_{\ell+1}(kr) \right\}$$

$$(\text{V}-3.40(m))$$

Thus we can write the phase shift expression alternatively as

$$e^{i\delta_\ell} \sin\delta_\ell = -k \int_0^\infty j_\ell(kr) \left\langle U(\vec{r},\vec{p})\, j_\ell(kr) \right\rangle r^2 dr$$

$$(\text{V}-3.41\ (a))$$

which is the form of phase shift expressions used in the Born approximation for uncoupled states.

Let us for a moment concentrate on what we would get if we did not make Born approximation. A careful work on radial form of the equation gave the expression

$$\sin\delta_\ell = -k\int_0^\infty j_\ell(kr)\langle U(r,\vec{p})R_\ell(r)\rangle r^2 dr \qquad (\text{V-3.41 b})$$

The disappearance of $e^{i\delta_\ell}$ is due to substitution for Ψ and not Φ, the free wave function.

In the last case we chose our asymptotic form (which was regular at origin) to be

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr}. \qquad (\text{V-3.41c})$$

Alternatively, if we chose another form (regular at origin) as

$$\eta_\ell(r) \xrightarrow[r\to\infty]{} e^{i\delta_\ell}\,\frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} \qquad (\text{V-3.41(d)})$$

then our analogously derived expression is

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{e^{2i\delta_\ell}-1}{2i} = -k\int_0^\infty j_\ell(kr)\langle U(r,\vec{p})\cdot \cdot\,\eta_\ell(r)\rangle r^2 dr, \qquad (\text{V-3.41(e)})$$

in which L.H.S. is the same as we obtained in the Born approximation. If we chose still another form as

$$\zeta_\ell(r) \xrightarrow[r\to\infty]{} j_\ell(kr) + \tan\delta_\ell\, N_\ell(kr)$$

$$= \frac{\sin(kr - \ell\pi/2) + \tan\delta_\ell \cos(kr - \ell\pi/2)}{kr} \qquad (\text{V-3.41(f)})$$

where $N_\ell(kr)$ is spherical Neumann function, then the expression for phase shift is

$$\tan\delta_\ell = -k\int_0^\infty j_\ell(kr)\,\langle U(r,\vec{p})\, z_\ell(r)\rangle\, r^2 dr$$

(V-3.41(g))

These expressions have been discussed by Mott and Massey (reference 98, page 75) in the light of choice of asymptotic form, regular at origin, and the Green function method of most general solution of a second order differential equation. If we made the Born approxiamtion we would substitute $j_\ell(kr)$ for R_ℓ, η_ℓ or z_ℓ and thus the expressions for phases in the integrals would be the same in all three above cases. But the left hand sides of the phase shift expressions, although they look different, yield the same result for small phase shifts when

$$\mathrm{Re}\; e^{i\delta_\ell}\sin\delta_\ell = \sin\delta_\ell = \tan\delta_\ell$$

is satisfied. In our calculations, we have chosen the tangent representation for uncoupled phases. At this point we would like to remind the reader that there are other representations for getting phase shifts and in some of the iterative schemes, the Born approximation appears as successive approximation. Thus we can call them first, second, ... etc. Born approximations and the formalisms are presented in various books (Wu and Ohmura (23), etc.).

When we make Born approximation we have real quantities in the integral on R.H.S. provided that $U(r,\vec{p})$ is real. But if we relate the scattering amplitude to matrix and then require unitarity conditions, then we have complex phase shifts in general which are equated to a real integral only. Arndt, Bryan, and McGregor (101) make a discussion in this connection to make unitarity correction to their real Born amplitudes and fit NN scattering data (with four mesons) with O.B.E.P. The conformity of these corrections with dispersion relations and a geometric picture of such unitarization has been discussed by Moravcsik (102). One useful result is that the "true" phase δ' is related with Born phase as

$$\cos\delta' \sin\delta' = |e^{i\delta}\sin\delta| = \sin\delta$$

$$\delta = \delta'\left(1-\frac{\delta'^2}{2}\right), \quad \Delta\delta = \frac{1}{2}\delta^3$$

$$\frac{\Delta\delta}{\delta} = \frac{\delta^2}{2}$$

Therefore

$$\delta' = \delta\left(1+\frac{\delta^2}{2}\right) \qquad (\text{V}-3.41(h))$$

Thus the corrections are about 14% in the upper limit for uncoupled phases for a phase shift value of 30°. As Born approximation breaks down for very large phases, this point does not affect us seriously. Bryan and Arndt (69) have also indicated the differences due to K-matrix or S-matrix identification of Born amplitudes in their calculations and those of the Japanese group. These arguments are more complicated for coupled states and will also be neglected for simplicity.

We have sketched in the last chapter that non-locality to a certain degree of approximation is equivalent to quadratic velocity dependent terms. Mott and Massey (98) give a generalized phase shift formula for non-local potentials involving a kernel and double radial integration. This generalization is the same as replacing $U(r)\,G_\ell(r)$ of local problem by $\int K_\ell(r,r')\,G_\ell(r')\,dr'$.

Thus

$$\tan\delta_\ell = -k\int_0^\infty\int_0^\infty r r' j_\ell(kr)\,K_\ell(r,r')\,dr\,dr' \qquad (\text{V}-3.41(i))$$

apart from this, they give an extensive discussion and examples on non-local potentials and on velocity dependence in its connection with non-locality.

(i) Successive Born Approximation in Integral Form and Validity

In the following we give an integral equation, equivalent to Schrödinger equation from an academic point of view, to apply Born approximation as a successive approximation. The general nature of the nucleon-nucleon force problem in OBEP is so much numerically oriented that an extensive analysis of II-Born approximation is as complicated as the exact numerical integration. However, since the detailed numerical analysis even for first Born approximation will be found quite tedious, we leave the numerical test of this problem as an open question and present the formalism only. This will help us in understanding the Born approximation for coupled states also which will be treated in the next section. The approach is the Green function technique of solving differential equations in the integral forms, and we use its partial wave expansions. The Schrödinger equation is

$$\left[\nabla^2 + k^2 - U(r, \vec{p})\right] \psi(\vec{r}) = 0 \qquad (\text{V}-3.42)$$

where

$$k = \sqrt{\frac{M\,E_{cm}}{\hbar^2}} = \sqrt{\frac{M\,E_{lab}}{2\hbar^2}}$$

$$U(r, \vec{p}) = \frac{M}{\hbar^2} V_{Tot}(r, \vec{p}) . \qquad (\text{V}-3.43)$$

With

$$\mathcal{H}_0 = -\nabla^2 , \quad \mathcal{E} = k^2$$

we write (V-3.44)

$$(\mathcal{H}_0 - \mathcal{E})\, \Psi(\vec{r}\,') = \chi(\vec{r}) \qquad (\text{V}-3.45)$$

where

$$\chi(r) = -\, U(r, \vec{p})\, \Psi(\vec{r}) \qquad (\text{V}-3.46)$$

We also require that $\Psi(r)$ have the asymptotic form

$$\Psi(\vec{r}) \longrightarrow e^{ikz} + f(\theta)\, \frac{e^{ikr}}{r} . \qquad (\text{V}-3.47)$$

The solution Ψ can be written as a sum of homogeneous part-solution and an integral in terms of a kernel corresponding to an inhomogeneous part, i.e.

$$\Psi(\vec{r}) = \Phi(\vec{r}) + G(\vec{r}, \vec{r}\,')\, \chi(\vec{r}\,')\, d\vec{r}\,'$$

with the particular solution of homogeneous part

$$(\mathcal{H}_0 - \mathcal{E})\, \Phi_{\vec{k}}(\vec{r}) = 0, \; \Phi_{\vec{k}}(r) = e^{i\vec{k}\cdot\vec{r}} . \qquad (\text{V}-3.48)$$

The function $G(\vec{r},\vec{r}\,')$ is called Green's function for the operator $(\mathcal{H}_0-\mathcal{E})$ and satisfies

$$(\mathcal{H}_0-\mathcal{E})\; G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,'). \tag{V-3.49}$$

By expanding $\Psi(\vec{r})$ in a complete set of orthogonal functions $\Phi_{\vec{k}'}(\vec{r}\,')$, and using (V-3.45), it can be shown that the coefficients of expansion and Green's function are given by the following expressions and that $G(\vec{r},\vec{r}\,')$ satisfies (V-3.49).

$$\Psi(\vec{r}) = \int C_{\vec{k}'}\, \Phi_{\vec{k}'}(\vec{r}\,')\, d\vec{k}',$$

$$C_{\vec{k}'} = \frac{1}{(2\pi)^3(k^2-k'^2)} \int \Phi^*_{\vec{k}'}(\vec{r}\,')\, \chi(\vec{r}\,')\, d\vec{r}\,'$$

and

$$G_{\vec{k}}(\vec{r},\vec{r}\,') = -\frac{1}{(2\pi)^3}\int \frac{\Phi^*_{\vec{k}'}(\vec{r}\,')\,\Phi_{\vec{k}'}(\vec{r})}{(k^2-k'^2)}\, d\vec{k}' \tag{V-3.50}$$

Thus a general solution (V-3.47) is

$$\Psi(\vec{r}) = \Phi_{\vec{k}}(\vec{r}) - \int G(\vec{r},\vec{r}\,')\, U(\vec{r}\,';\vec{p}\,')\cdot \Psi(\vec{r}\,')\, d\vec{r}\,' \tag{V-3.51}$$

The $\vec{p}\,'$ dependence in potential will be changed to k' dependence if we make second Born approximation, i.e. $\Phi_{\vec{k}'}(\vec{r}\,')$ for $\Psi(\vec{r}\,')$ in the last term in (V-3.51).

The Green function for outgoing scattered waves can be decomposed into partial waves as

$$G_{\vec{k}}(\vec{r},\vec{r}') = \sum_{\ell=0}^{\infty} G_\ell(\vec{r},\vec{r}') \frac{(2\ell+1)}{4\pi r r'} P_\ell(\cos\Theta) \qquad (V-3.52)$$

with

$$\Theta = \vec{r}\,\hat{}\,\vec{r}'$$

and

$$G_\ell(r,r') = -k r r' j_\ell(kr_<)\,\eta_\ell(kr_>) \qquad (V-3.53)$$

where j_ℓ, η_ℓ are spherical Bessel and Neumann functions respectively and $r_<, r_>$ are the smaller and larger functions of r, r' respectively. These are given by Rohrlich and Eisenstein (103). We also decompose the wave function into partial waves as

$$\Psi(\vec{r}) = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell R_\ell(r)\, P_\ell(\cos\Theta) \qquad (V-3.54)$$

Thus

$$R_\ell(r) = j_\ell(kr) - \int_0^\infty G_\ell(r,r')\, U(r',\vec{p}')\, R_\ell(r')\,dr'. \qquad (V-3.55)$$

We now obtain it explicitly with $G_\ell(r,r')$ substituted from (V-3.53) and make <u>II-Born Approximation</u> by substituting $j_\ell(kr')$ for $R_\ell(r')$ in (V-3.55).

$$R_\ell(r) = j_\ell(kr) + k\,\eta_\ell(kr)\int_0^r r'^2 j_\ell(kr')\, U(r',\vec{p}')\cdot$$
$$\cdot\; j_\ell(kr')\,dr' + k\, j_\ell(kr)\int_r^\infty r'^2 \eta_\ell(kr')\cdot$$
$$\cdot\; U(r',\vec{p}')\, j_\ell(kr')\,dr', \qquad (V-3.56)$$

or

$$R_\ell(r) = j_\ell(kr) + k\left[\eta_\ell(kr)\,A(r) + \right.$$
$$\left. + \; j_\ell(kr)\;B(r)\right] \qquad (V-3.57)$$

where

$$A(r) \equiv \int_0^r r'^2 j_\ell(kr')\,U(r',\vec{p}')\,j_\ell(kr')\,dr'$$

and

$$B(r) \equiv \int_r^\infty r'^2 \eta_\ell(kr')\;U(r',\vec{p}')\,j_\ell(kr')\,dr'. \qquad (V-3.58)$$

One way to decide the validity of the Born approximation is to see whether the correction terms due to II-Born approximation are small in the region of interest. These regions may be divided classically for partial waves according to the simple relation $kr \approx \ell$. For those regions it should be seen whether $\Delta_\ell \equiv k[\eta_\ell A + j_\ell B]$ is such that $|j_\ell(kr)|^2 > |\Delta_\ell(r)|^2$. Another approach would be to compute $R_\ell(r)$ using (V-3.57) and then carry out a phase shift analysis. Even the determination of $R_\ell(r)$ would require another integral to be evaluated. This would make it a complicated expensive numerical approach because of the nature of $U(r, \vec{p})$ and an exact treatment might even be simpler if numerical schemes are to be followed.

We should be careful about this validity criterion which assumes that a Born Expansion is valid. This severely restricts the problem when the expansion parameter associated with the potential (in this case g^2) is large. The convergance criteria of this type have been extensively discussed by many authors [Wu and Ohmura (23), Manning (104)]. Thus these methods of iteration are limited by the usual reasons of large couplings. Now we start deriving the phase shift expressions to be used for uncoupled states, using only the first Born approximation.

(ii) Direct Born Phases

Thus we are prepared to write the expressions for phase shifts in the Born approximation that will use the direct method of treating velocity dependence and will later be used for numerical analysis. We use the explicit formulae for potentials as given in the last section. The expression for uncoupled states is given by (V-3.34) in accordance with discussion on tangent representation for the phase

$$\tan \delta_\ell = - k \int_0^\infty r^2 j_\ell(kr) \langle U(r, \vec{p}) j_\ell(kr) \rangle dr \qquad (\text{V}-3.59)$$

where $U(r, \vec{p})$ for a sum of various interactions is given by (V-2.3) through (V-2.10). Only one of the vector interactions is to be chosen and the numerical program is so written that any of the vector interactions can be taken into the calculation along with the scalar and pseudoscalar interactions. This enables a comparison among various vector interactions. Actually the parts that are common are calculated only once and extra contributions are calculated by evaluating extra integrals, for the Breit and Kemmer cases, and are multiplied by appropriate numerical coefficients. We see that the potential is a sum of terms with separable contributions from each meson. We proceed symbolically to evaluate the contribution of each term to the phase shift (δ_ℓ) by the symbol δ , subscripted to denote a corresponding term in the potentials. In all cases (within a numerical coefficient depending on the nature of the meson and of the state) the functional form for each term is the same for all mesons. This is not true of the Breit and the Kemmer vector interactions where additional terms occur which are grouped and called δ^B_E or δ^K_E depending on the case. The description of codes will provide the detail later. Let us define

$$
\begin{aligned}
z &= kr \\
k &= \sqrt{\frac{M E_L}{2\hbar^2}} \\
\mu_N &= \frac{K_N}{k} = \frac{Mc}{\hbar k} \\
\mu &= \frac{K}{k} = \frac{mc}{\hbar k} \\
\mu_c &= \frac{\Lambda}{k} = \frac{m_c c}{\hbar k}.
\end{aligned}
\qquad (\text{V}-3.60)
$$

where K, Λ are chosen properly for the mesons. Then we have the modified Yukawa form with

$$J(r) = g^2 (\hbar c) \left[\frac{e^{-kr}}{r} - \frac{e^{-\Lambda r}}{r} \right] \tag{V-3.61}$$

Using (V-3.60) , (V-3.61), (IV-2.3) (IV-2.26) and (IV-2.28) we get

$$\delta_Y = -k \int_0^\infty (-J) \frac{M}{\hbar^2} r^2 j_\ell^2(kr)\,dr$$
$$= g^2 \mu_N \int_0^\infty (e^{-\mu z} - e^{-\mu_c z})\, z\, j_\ell^2(z)\,dz$$

where we chose the sign corresponding to J_Λ , for this case, but coefficients will be properly accounted for in the code.

Thus

$$\begin{aligned}
\delta_Y &= g^2 \mu_N A \\
\delta_C &= \frac{1}{4} \frac{g^2}{\mu_N} F \\
\delta_\nabla &= -\frac{g^2}{\mu_N} B \\
\delta_\Delta &= -\frac{g^2}{\mu_N} A \\
\delta_{SS} &= -\frac{1}{6} \langle \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \rangle \frac{g^2}{\mu_N} F \\
\delta_{LS} &= \frac{1}{2} \langle \vec{\ell} \cdot \vec{S} \rangle \frac{g^2}{\mu_N} C \\
\delta_T &= \frac{1}{12} \langle S_{12} \rangle_{\ell\ell} \frac{g^2}{\mu_N} D
\end{aligned} \tag{V-3.62}$$

where

$$A = \int_0^\infty (e^{-\mu z} - e^{-\mu_c z}) z\, j_\ell^2(z)\, dz$$

$$B = \int_0^\infty \left[(1+\mu z) e^{-\mu z} - (1+\mu_c z) e^{-\mu_c z}\right] \cdot$$

$$\cdot \left\{\frac{1}{(2\ell+1)} \left[\ell\, j_{\ell-1}(z) - (\ell+1)\, j_{\ell+1}(z)\right] j_\ell(z)\, dz\right.$$

$$C = \int_0^\infty \left[\left(\frac{1}{z} + \mu\right) e^{-\mu z} - \left(\frac{1}{z} + \mu_c\right) e^{-\mu_c z}\right] j_\ell^2(z)\, dz.$$

$$D = \int_0^\infty \left[(\mu^2 z^2 + 3\mu z + 3) e^{-\mu z} - (\mu_c^2 z^2 + 3\mu_c z + 3) \cdot\right.$$

$$\left. \cdot\, e^{-\mu_c z}\right] \left(\frac{1}{z}\right) j_\ell^2(z)\, dz.$$

$$F = \int_0^\infty (\mu^2 e^{-\mu z} - \mu_c^2 e^{-\mu_c z}) z\, j_\ell^2(z)\, dz.$$

$$(\mathrm{V} - 3.63)$$

These are the actual phase shift expressions used in the numerical analysis with the direct method for uncoupled states. The extra contributions from the Kemmer and the Breit vector meson interactions are given below.

$$\delta_E^B = \frac{g^2}{\mu_N}\left[-\frac{5}{8} F + \frac{1}{8} B_C - \frac{1}{2} B_\nabla - \frac{1}{2} B_\Delta + \right.$$

$$\left. + \frac{1}{2} \ell(\ell+1)\, C\right] \qquad (\mathrm{V} - 3.64)$$

where C, F are defined above and the other integrals are

$$B_C = \int_0^\infty \left[\mu^2(1+\mu z)\, e^{-\mu z} - \text{cutoff}\right] j_\ell^2(z)\, dz$$

$$B_\nabla = \int_0^\infty \left[(\mu^2 z^2 + 3\mu z + 3)\, e^{-\mu z} - \text{cutoff}\right] \cdot$$

$$\cdot \left\{ \frac{1}{(2\ell+1)} \left[\ell\, j_{\ell-1}(z) - (\ell+1)\, j_{\ell+1}(z)\right] j_\ell(z) \right\} dz$$

$$B_\Delta = \int_0^\infty \left[(1+\mu z)\, e^{-\mu z} - \text{cutoff}\right] z\, j_\ell^2(z)\, dz$$

$$\text{(a)} \quad (\text{V}-3.65)$$

Similarly for the Kemmer interaction

$$\delta_E^K = \frac{g^2}{\mu_N \mu^2} \left[-\frac{1}{4} K_C + K_\Delta + K_\nabla + K_{\ell\ell}(-\ell)(\ell+1) \right]$$

where

$$(\text{V}-3.65)(b)$$

$$K_C = \int_0^\infty \left(\mu^4 e^{-\mu z} - \text{cutoff}\right) z\, j_\ell^2(z)\, dz$$

$$K_\nabla = \int_0^\infty \left[(\mu^3 z^3 + 3\mu^2 z^2 + 6\mu z + 6)\, e^{-\mu z} - \right.$$

$$\left. - \text{cutoff} \right] \left\{ \frac{1}{2\ell+1} \left[\ell\, j_{\ell-1}(z) - (\ell+1)\, j_{\ell+1}(z) \right] \right\} dz$$

$$K_\Delta = \int_0^\infty \left[(\mu^2 z^2 + 2\mu z + 2)\, e^{-\mu z} - \text{cutoff} \right] \cdot$$

$$\cdot \left(\frac{1}{z}\right) j_\ell^2(z)\, dz$$

$$K_{\ell\ell} = \int_0^\infty \left[(\mu^2 z^2 + 3\mu z + 3)\, e^{-\mu z} - \text{cutoff} \right] \cdot$$

$$\cdot \left(\frac{1}{z^3}\right) j_\ell^2(z)\, dz \qquad (\text{V}-3.66)$$

Thus we see that for Kemmer's interaction our criterion for potential is not satisfied because of the highly singular nature of the potential. Hence for $S-$waves the phase shift expressions are not valid and even the cut-off does not help us. We must mention a very important fact about the cut-off for Kemmer's interaction. This interaction contained the square of inverse Compton wave length of the meson in the denominator. This part explicitly depends upon the nature of the meson. Thus a cut-off which comes from theoretical assumptions can only remove this difficulty. However, we have taken a phenomenological view point which means a substitution of the cut-off only in J and so the concerned interaction is

$$V_v^K = \left[1 - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} + \frac{1}{K^2}(\vec{\alpha}^{(1)} \cdot \vec{\nabla})(\vec{\alpha}^{(2)} \cdot \vec{\nabla})\right] J(r)$$

with the modified Yukawa form with Green's subtractive form,

$$J(r) = g^2(\hbar c)\left[\frac{e^{-Kr}}{r} - \frac{e^{-\Lambda r}}{r}\right] \qquad (\text{V}-3.67)$$

as before. Thus this may not be the same as obtained from theoretical considerations.

(iii) <u>Effective Born Phases</u>

In this case we use the effective mass method of treating the velocity dependence. The general arguments of the Born approximation for velocity dependence in this case are not very much complicated due to the fact that the potential is expressed already in a radial form. Thus we can directly start with the radial form of the Schrödinger type equation with velocity dependent potentials as given by (IV-2.14) and (IV-2.15) which define the effective potential and we can write

$$\chi_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]\chi_\ell - U_{eff}(r,k)\chi_\ell = 0 \qquad (\text{V}-3.68\,(a))$$

where $\chi_\ell(r) = r(1+\phi)^{1/2} R_\ell(r) \longrightarrow$ is given by (IV-2.12). Let the field free solution be called which satisfies

$$\eta_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]\eta_\ell = 0$$

$$(\text{V}-3.68\,(b))$$

hence $\eta_\ell = r\, j_\ell(kr)$. Multiplying (V-3.68 a) by η_ℓ and (V-3.68 b) by χ_ℓ and subtracting we get

$$(\eta_\ell \chi_\ell'' - \chi_\ell \eta_\ell'') = \eta_\ell\, U_{eff}(r,k)\chi_\ell \qquad (\text{V}-3.68\,(c))$$

Integrating with respect to r we obtain

$$(\eta_\ell \chi_\ell' - \chi_\ell \eta_\ell')_0^\infty = \int_0^\infty \eta_\ell\, U_{eff}(r,k)\chi_\ell\, dr$$

$$(\text{V}-3.68\,(d))$$

We have assumed that both $j_\ell(kr)$ and $R_\ell(r)$ are regular solutions at the origin and therefore the lower limit of L.H.S. vanishes. For the upper limit we substitute the asymptotic forms

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr}$$

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2)}{kr} .$$

Substituting proper differentiations into L.H.S. and using trigonometric relations we get

$$\text{L.H.S.} = -\frac{1}{k}\sin\delta_\ell \qquad (\text{V}-3.68(e))$$

We make the Born approximation for the R.H.S. of (V-3.68 d) in the partial wave form by replacing χ_ℓ by η_ℓ and obtain

$$\text{R.H.S.} = \int_0^\infty r^2\, U_{eff}(r,k)\, j_\ell^2(kr)\, dr \qquad (\text{V}-3.68(f))$$

and thus we obtain the expression for phase shifts as

$$\sin\delta_\ell = -k\int_0^\infty r^2\, U_{eff}(r,k)\, j_\ell^2(kr)\, dr \qquad (\text{V}-3.68\ g)$$

Actually we use tangent expression for the phase as discussed before. Thus by starting with partial wave expansion we obtain the above formula for phases. This approach would also apply to the Direct Method because only radial parts of the differential operators act on the $j_\ell(kr)$ and we have to replace $U_{eff}(r,k)\, j_\ell(kr)$ by $\langle U(r,\vec{p})\, j_\ell(kr)\rangle$ to obtain the analogous expressions.

Thus in this approach the zero of $(1+\phi)$ has to be avoided by further restriction of the parameters. We analogously define $\delta^{(i)}$ with the exception that δ_∇ be the contribution of $(\phi')^2/4(1+\phi)^2$ term and δ_Δ that of $\frac{1}{2}\langle\nabla^2\phi\rangle$, thus individual meson contributions can not be separated out from the phase shift. These subscripts, as well as others, therefore do not truly denote the analogous contributions from potentials but only approximately do so, (because of $(1+\phi)$ factors, transformation (IV-2.12) etc.). They are adopted only for convenience in approximate grouping of the terms and to conform to numerical analysis notation. This applies only to the effective mass approach. Let

$$D_n = (1+\phi)$$

$$D_n = 1 + \frac{g^2}{\mu_N}\left[\left(\frac{e^{-\mu z}-e^{-\mu_c z}}{z}\right)_S^{(0)} + \left(\frac{e^{-\mu z}-e^{-\mu_c z}}{z}\right)_V^{(0)} + \right.$$

$$\left. +(\vec{\tau}^{(1)}\cdot\vec{\tau}^{(2)})\,\frac{g^2}{\mu_N}\left[\left(\frac{e^{-\mu z}-e^{-\mu_c z}}{z}\right)_S^{(1)} + \left(\frac{e^{-\mu z}-e^{-\mu_c z}}{z}\right)_V\right]\right. \qquad \text{(V-3.69 a)}$$

We see that D_n is a dimensionless function. Using (IV-2.3), (IV-2.5) and (IV-2.15) and (V-3.41 a) together with the above equations we get

$$\delta_Y = -k\int_0^\infty \frac{1}{D_n}(-J)\,\frac{M}{\hbar^2}\,r^2 j_\ell^2(kr)\,dr$$

$$= g^2\,\frac{M_c k}{\hbar}\cdot\frac{1}{k^2}\int_0^\infty \frac{1}{D_n}\left(e^{-\mu z}-e^{-\mu_c z}\right) z\, j_\ell^2(z)\,dz$$

$$= g^2\mu_N\int_0^\infty \frac{1}{D_n}\left(e^{-\mu z}-e^{-\mu_c z}\right) z\, j_\ell^2(z)\,dz$$

or

$$\delta_Y = g^2\mu_N A \qquad \text{(V 3.69 b)}$$

where

$$A = \int_0^\infty \frac{1}{D_n}\left(e^{-\mu z} - \text{cutoff}\right) z\, j_\ell^2(z)\,dz. \qquad \text{(V-3.70)}$$

In this case we chose a sign corresponding to J^{s}. Similar reduction on other parts of potential yields

$$\delta_Y = g^2 \mu_N A$$

$$\delta_C = \frac{1}{4} \frac{g^2}{\mu_N} F$$

$$\delta_\nabla = - B(1)$$

$$\delta_\Delta = - \frac{g^2}{\mu_N} \left(A + \frac{1}{2} F \right)$$

$$\delta_{SS} = - \frac{\langle \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \rangle}{6} \frac{g^2}{\mu_N} F$$

$$\delta_{LS} = \frac{\langle \vec{l} \cdot \vec{S} \rangle}{2} \frac{g^2}{\mu_N} G$$

$$\delta_T = \frac{\langle S_{12} \rangle}{12} \frac{g^2}{\mu_N} D$$

$$(\mathrm{V}-3.71)$$

where

$$A = \int_0^\infty \frac{1}{D_n} \left(e^{-\mu z} - \text{cutoff} \right) z \, j_l^2(z) \, dz$$

$$B(1) = \frac{1}{4} \int_0^\infty \left[\left(\frac{d\phi}{dz} \right) \cdot \frac{1}{(1+\phi)} \right]^2 z^2 j_l^2(z) \, dz$$

$$G = \int_0^\infty \frac{1}{D_n} \left[\left(\mu + \frac{1}{z} \right) e^{-\mu z} - \text{cutoff} \right] j_l^2(z) \, dz$$

$$D = \int_0^\infty \left[\left(\mu^2 z + 3\mu + 3/z \right) e^{-\mu z} \text{ cutoff} \right] \frac{1}{D_n} \cdot j_l^2(z) \, dz$$

$$F = \int_0^\infty \frac{1}{D_n} \left(\mu^2 e^{-\mu z} \text{ cutoff} \right) z \, j_l^2(z) \, dz .$$

$$(\mathrm{V}-3.72)$$

These phase contributions have to be multiplied by proper coefficients and summed over all mesons appropriately. Then the sum total of all such sums is $\tan \delta_l$. These integrals are to be evaluated numerically and the details will be discussed in the next chapter on numerical analysis and computer programs. The only phase contribution δ_v (symbolic), is directly evaluated as the sum of all scalar and vector (T = 0 and 1) mesons due to the necessity of its functional form. There also has to be made a check on D_n . If is zero or negative, this procedure breaks down. Only unconstrained vector, scalar, and pseudoscalar interactions for uncoupled states will be treated in effective approach.

Section - 4 Born Phases For Coupled States

We have seen that velocity dependence complicates the conventional method of treating the potentials. Another complication is introduced when non-central forces play an appreciable role. In nucleon-nucleon scattering the tensor force has been long recognized phenomenologically and meson theoretically, and represents a major contribution from the meson. The methods of treating the non-central forces have been discussed extensively in literature [Wu and Ohmura (23), Goldberger and Watson (105), Blatt and Biedenharn (106)]. Indeed the contribution to the quadrupole moment of deuteron from tensor forces has been an historic topic in nuclear physics and constitutes an important quantity in the characterization of low energy nuclear phenomena. The complication due to the tensor force operator arises from the fact that the orbital angular momentum ℓ is no longer a conserved quantity as discussed in Section - 1. But there are only $\ell = J \pm 1$ states which can get coupled in our problem and the matrix elements of non-vanishing contributions were given there. The angular parts can thus be eliminated since their effects are embodied in the matrix elements of S_{12}.

Thus we obtain two coupled equations for each the total angular momentum-quantum number and these couple the $\ell = J-1$ to $\ell' = J+1$ states of orbital angular momentum. While the spectroscopic notation is not valid, nevertheless it is used conventionally. For example $^3S_1$ and $^3D_1$ get coupled and the statement " $^3S_1$ phase" has no precise meaning. It turns out that because of its singular nature, the tensor potential is a major part of the potentials. However, if we assume it to be not too large as compared to the other potentials, then we can assume the mixing to be small, at least at lower energies. The usual tensor part is non-velocity dependent and therefore poses no complication from this standpoint. We will therefore concern ourselves only with the coupled aspects of the problem. Thus we can talk about predominantly $^3S_1$ or $^3D_1$ phases in this interpretation. The two radially coupled equations are

$$\frac{d^2u_\ell}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u_\ell = -\left\{V_0 u_\ell + \langle S_{12}\rangle_{\ell\ell'} V_T w_{\ell'}\right\} \tag{V-4.1}$$

$$\frac{d^2w_{\ell'}}{dr^2} + \left(k^2 - \frac{\ell'(\ell'+1)}{r^2}\right)w_{\ell'} = -\left\{V_0' w_{\ell'} + \langle S_{12}\rangle_{\ell\ell'} \cdot V_T u_\ell\right\} \tag{V-4.2}$$

where $u_\ell(r)$, $w_{\ell'}(r)$ are radial wave functions with

$$\begin{aligned} \ell &= J-1 \\ \ell' &= J+1 \end{aligned} \tag{V-4.3}$$

and

$$V_0 = \frac{M}{\hbar^2}\Big(V_Y + V_C + i V_\nabla (\vec{r}\cdot\vec{p}) + V_\Delta(p^2) + $$

$$+ V_{SS}(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + V_{LS}(\vec{\ell}\cdot\vec{s}) + V_T \langle S_{12}\rangle_{\ell\ell},$$

with similar expression for V_0' corresponding to $\ell' = J+1$.

$$V_0' = V_0 \,(\text{for } \ell = \ell')$$

We should carefully interpret $V_0 u_\ell$, etc. for velocity dependent terms where it is actually $\gamma V_0 R_\ell$, etc. But this notation is used for compactness. Also the diagonal parts of tensor force are grouped with V_0. We intend to solve this problem in integral form using Green's functions and scattered and incident wave formalism. These two second order coupled differential equations have to posses two linearly independent solutions. These solutions will be specialized in the Born approximation. Two channel scattering of this type is specialized in our case by the fact that we choose incident waves only in $\ell = J-1$ state for first set of solutions and in $\ell' = J+1$ state for second set of solutions. The plane waves are decomposed into partial waves. These two initial state solutions are linearly independent. But the scattered waves in both cases will contain parts coming from both channels $\ell = J-1$ and $\ell' = J+1$. Thus a pure incident ℓ wave when scattered in a potential with tensor force, will contain both ℓ and ℓ' parts in the scattered wave. But predominant ℓ or ℓ' wave will be assumed to be present. This approach has been the mathematical content of the appendix by Stapp, Ypsilantis and Metropolis [called SYM reference (70)] who write the expressions for phase shifts in the Born approximation. We will use the matrix form of integral representation of the Schrödinger equation.

The general solution $\Psi(r)$ in terms of the partial waves would be for the scattered part

$$\Psi_{sc}(r) = \begin{pmatrix} A_\ell u_\ell(r) \\ A_{\ell'} w_{\ell'}(r) \end{pmatrix}, \quad \text{for } \Psi_{in} = \begin{pmatrix} a_\ell u_\ell(r) \\ 0 \end{pmatrix},$$

but we would hope that $A_{\ell'}$ would be small because initally it was zero. We know that the ratio $A_\ell / A_{\ell'}$ should be related to the mixing parameter which roughly designates the amount of mixing from the coupled state. Thus the parts, $w_{\ell'}(r)$, are perturbations on the wave function for the incident wave in $\ell = J-1$ channel and similarly for the incident wave in the $\ell' = J+1$ channel, $u_\ell(r)$ is a perturbation. Let us also examine (V-4.1) and (V-4.2). These should be written for each independent set. Thus $V w_{\ell'}$ will be small for the set corresponding to predominant ℓ channel and $V u_\ell$ will be small for the other set. In other words, the two sets of equations are after neglecting second order effects

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = -V_0 u_\ell$$

$$\frac{d^2 w_{\ell'}}{dr^2} + \left[k^2 - \frac{\ell'(\ell'+1)}{r^2}\right] w_{\ell'} = -\left\{ \langle S_{12} \rangle_{\ell\ell'} \cdot V_T w_{\ell'} \right\} \qquad (\text{V}-4.6)$$

[$(\ell = J-1)$ called α wave], and $(\ell' = J+1)$ for β wave

$$\frac{d^2 u_\ell}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = -\left\{ \langle S_{12} \rangle_{\ell\ell'} V_T w_{\ell'} \right\}$$

$$\frac{d^2 w_{\ell'}}{dr^2} + \left[k^2 - \frac{\ell'(\ell'+1)}{r^2}\right] w_{\ell'} = -\left\{ V_0' w_{\ell'} \right\}$$

$$(\text{V}-4.6)$$

This approximation removes the uncertainty about the ratio of coefficients of $u_\ell, w_{\ell'}$ which are related to ϵ_J, the mixing parameter (to be defined later) the presence of which would imply only a self consistent iterative solution because ϵ_J is to be determined. This would pose no additional problem in an exact case but we want to make a Born approximation for coupled states. Since potential is assumed small and the particular wave function u_ℓ or $w_{\ell'}$ depending on the set, is small so Vu_ℓ or $Vw_{\ell'}$ are neglected as second order effects as compared to $Vw_{\ell'}$ and Vu_ℓ respectively, in a corresponding predominant channel. Let us start with the α-wave where only $\ell = J-1$ is present in incident channel. The Green functions for (V-4.5) are given by Wu and Ohmura (23) and also by Rohrlich and Eisenstein (103) in the real form. Since they satisfy

$$\left\{\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right\} G_\ell(r, r') = -\delta(r-r')$$

$$(\text{V}-4.7)$$

and a similar equation for $G_{\ell'}(r, r')$, they can be written in the partial wave decomposition as

$$G_{\ell}(r,r') = -\frac{1}{k}\left\{kr_< j_\ell(kr_<)\, kr_> \left[\eta_\ell(kr_>) - - i j_\ell(kr_>)\right]\right\}$$

$$G_{\ell'}(r,r') = \text{Above eq}^{\underline{n}} \text{ with } \ell=\ell'$$

$$(\text{V}-4\cdot 8)$$

where $r_<, r_>$ are the smaller and the larger of r, r' respectively. They are chosen with spherical Hankel functions so that complex forms can be used to denote the incoming or outgoing waves as we will see below. These Green functions vanish at $r_< = 0$. Thus for a fixed J value (index suppressed) we have the solution $\Psi(r)$ given in matrix form

$$\Psi(r) = \begin{pmatrix} r j_\ell(kr) \\ 0 \end{pmatrix} + \int_0^\infty \mathbb{G}(r,r')\, \mathbb{V}(r')\, \Psi(r')\, dr' \qquad (\text{V}-4\cdot 9)$$

the general form of which in uncoupled case was discussed in the last section. The free wave solution has been chosen as discussed,

$$\Phi(r) = \begin{pmatrix} r j_\ell(kr) \\ 0 \end{pmatrix} \qquad (\text{V}-4\cdot 10)$$

where

$$\mathbb{V} = \begin{pmatrix} -V_0(r) & -\langle S_{12}\rangle_{\ell\ell'} V_T(r) \\ -\langle S_{12}\rangle_{\ell\ell'} V_T & -V_0'(r) \end{pmatrix} \qquad (\text{V}-4\cdot 11)$$

and

$$\mathbb{G}(r, r') = \begin{pmatrix} G_\ell(r, r') & 0 \\ 0 & G_{\ell'}(r, r') \end{pmatrix} \qquad (V-4.12)$$

G_ℓ , $G_{\ell'}$ being given by (V-4.8).

Now we make the Born approximation for the integral equation (V-4.9) where $\Psi(r')$ is not known and is replaced by $\Phi(r')$ i.e.

$$\Psi(r') = \Phi(r') = \begin{pmatrix} r' j_\ell(kr') \\ 0 \end{pmatrix} \qquad (V-4.13)$$

Thus we substitute (V-4.11) through (V-4.13) in (V-4.9) and obtain

$$\Psi(r) = \begin{pmatrix} r j_\ell(kr) \\ 0 \end{pmatrix} + (-1)\int_0^\infty r' dr' \begin{pmatrix} G_\ell V_0 j_\ell \\ G_{\ell'} \langle S_{12}\rangle_{\ell\ell'} V_T j_\ell \end{pmatrix} \qquad (V-4.14)$$

With

$$\begin{aligned} h_\ell^{(1)} &= j_\ell + i\eta_\ell \\ h_\ell^{(2)} &= j_\ell - i\eta_\ell \end{aligned} \qquad (V-4.15)$$

the upper integral in (V-4.14) becomes, using (V-4.8),

$$\int_0^\infty r' dr' \, G_\ell(r, r') \left\{-V_0(r') j_\ell(kr')\right\}_I$$

$$= kr \, i h_\ell^{(1)}(kr) \int_0^r r'^2 j_\ell(kr') \{ \ \}_I \, dr' +$$

$$+ kr \, j_\ell(kr) \int_r^\infty h_\ell^{(1)}(kr') r'^2 \{ \ \}_I \, dr'. \qquad (V-4.15)$$

We consider analogous expression for the lower integral also. This choice of breaking the integrals is made (Wu and Ohmura, reference 23, page 15) so as to prevent the expressions from blowing up at the origin or at infinity. Thus the first integral in (V-4.15) vanishes if $r \to 0$ and the second one vanishes if $r \to \infty$. We have to choose the asymptotic case when the second integral vanishes, because we are interested in extracting the phase shifts out of the scattered wave in the asymptotic region. With the realtions

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \frac{-i}{2kr}\left[e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}\right],$$

$$ih_\ell^{(1)}(kr) \xrightarrow[r\to\infty]{} \frac{e^{i(kr-\ell\pi/2)}}{kr}$$

and

$$-ih_\ell^{(2)}(kr) \xrightarrow[r\to\infty]{} \frac{e^{-i(kr-\ell\pi/2)}}{kr} \qquad (\text{V}-4.16)$$

we obtain the asymptotic form of $\Psi(r)$ as

$$\Psi(r) \xrightarrow[r\to\infty]{} -\frac{1}{2ik}\left\{\begin{pmatrix} e^{-i(kr-\ell\pi/2)} \\ 0 \end{pmatrix} - \begin{pmatrix} e^{i(kr-\ell\pi/2)} \\ 0 \end{pmatrix}\right\} +$$

$$+\begin{pmatrix} e^{i(kr-\ell\pi/2)} \int_0^\infty r'^2 j_\ell(kr')\left\{-V_0(r')j_\ell(kr')\right\}dr' \\ e^{i(kr-\ell'\pi/2)} \int_0^\infty r' j_{\ell'}(kr')\left\{-\langle S_{12}\rangle_{\ell\ell'} V_T(r') j_{\ell'}(kr')\right\} dr' \end{pmatrix} \qquad (\text{V}-4.17)$$

We multiply this throughout by a factor $(-2ik)$ and obtain the correct asymptotic form

$$\Psi(r) \xrightarrow[r\to\infty]{} \begin{pmatrix} e^{-i(kr-\ell\pi/2)} \\ 0 \end{pmatrix} - \begin{pmatrix} e^{i(kr-\ell\pi/2)}\{I_1(2ik)+1\} \\ e^{i(kr-\ell'\pi/2)}\{I_2(2ik)\} \end{pmatrix}$$

(V-4.18)

where I_1 and I_2 are integrals of (V-4.17). We now have acquired the correct asymptotic form for the wave function.

This two channel scattering in general is characterized by two amplitudes $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ for the initial wave and two $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ for the scattered wave. The scattering matrix is thus described by four elements. We will see in the following that only three parameters are needed to characterize the S- matrix (for a particular J value) completely. This 2 x 2 matrix has to be unitary from the arguments of conservation of flux and ahs to be symmetric from arguments of time reversal. These points have been discussed by Moravcsik (24) and also by McGregor, Moravcsik, and Stapp (25). The S- matrix, by definition, depends only on the asymptotic form of the wave function. The consequences of time reversal invariance are that S- be symmetric (discussed in reference 24). A general symmetric, unitary 2 x 2 matrix has three degrees of freedom and can be expressed in either of the following forms.

$$S = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_\ell} & 0 \\ 0 & e^{2i\delta_{\ell'}} \end{pmatrix} \begin{pmatrix} \cos\epsilon & +\sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$$

(V-4.19)

or

$$S = \begin{pmatrix} e^{i\bar{\delta}_\ell} & 0 \\ 0 & e^{i\bar{\delta}_{\ell'}} \end{pmatrix} \begin{pmatrix} \cos 2\bar{\epsilon} & i \sin 2\bar{\epsilon} \\ i \sin 2\bar{\epsilon} & \cos 2\bar{\epsilon} \end{pmatrix} \begin{pmatrix} e^{i\bar{\delta}_\ell} & 0 \\ 0 & e^{i\bar{\delta}_{\ell'}} \end{pmatrix} \qquad (\text{V}-4.20)$$

The individual matrices are unitary and S– is symmetric. The former is Blatt and Biedenharn's (106) way of characterizing the phases and the other is Stapp's (107) method of characterizing nuclear "bar" phases. The corrections and transformations between the two sets are discussed by SYM (70) and we will return to them later. At present we confine ourselves to the Blatt and Biedenharn phases.

In an alternative way we can start with the asymptotic form as given by Hulthe'n and Sugawara (21) in which $u_\ell(r)$ and $w_{\ell'}(r)$ are forced to

$$u_\ell(r) \underset{r\to\infty}{\longrightarrow} A_1 \, e^{-i(kr-\ell\pi/2)} - B_1 \, e^{+i(kr-\ell\pi/2)}$$

$$w_{\ell'}(r) \underset{r\to\infty}{\longrightarrow} 0 - B_2 \, e^{i(kr-\ell'\pi/2)} \qquad (\text{V}-4.21)$$

for the α wave. Now the scattering matrix S–is defined by

$$B = S \, A$$

or

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad (\text{V}-4.22)$$

where A'^s are incoming amplitudes and B'^s are outgoing ones. For the special case of α-wave, this reduces to

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} S_{11} A_1 \\ S_{21} A_1 \end{pmatrix} \qquad (\text{V}-4.23)$$

Substituting in (V-4.21) we get

$$\Psi(r) \xrightarrow[r\to\infty]{} A_1 \left[\begin{pmatrix} e^{i(kr-\ell\pi/2)} \\ 0 \end{pmatrix} - \begin{pmatrix} e^{i(kr-\ell\pi/2)} S_{11} \\ e^{i(kr-\ell'\pi/2)} S_{21} \end{pmatrix} \right]$$

$$(\text{V}-4.24)$$

And by comparison with (V-4.18), we get

$$S_{11} = 2ik(I_1) + 1$$

$$= -2ik \int_0^\infty r^2 [j_\ell(kr)]\, V_0(r, \vec{p})\, j_\ell(kr)\, dr + 1$$

$$= 2ik\, V_{11} + 1$$

$$S_{12} = S_{21} = -2ik \int_0^\infty r^2 j_\ell^2(kr)\, V_T \langle S_{12} \rangle_{\ell\ell'}\, dr$$

$$= V_{12} \qquad (\text{V}-4.25)$$

Similarly by going to β wave or $\ell' = J+1$ channel predominant scattering we get

$$S_{22} = -2ik \int_0^\infty r^2 j_{\ell'}(kr)\, V_0'(r, \vec{p})\, j_{\ell'}(kr)\, dr + 1$$

$$= V_{22}\,(2ik) + 1$$

$$S_{12} = S_{21} = V_{12}. \qquad (\text{V}-4.26)$$

Thus we have been able to express the scattering matrix elements in terms of the potential matrix elements. We would like to draw the attention of the reader to the fact that it expresses the familiar result in terms of $T-$ matrix formalism, i.e.

$$S = 2iT + 1$$

where $T^B = \langle V \rangle$ expresses the fact that T matrix elements to the lowest order are just the potential matrix elements. The integral formulae for this relation are expressed by Kerman, McManus, and Thaler (108), and also by Wu and Ohmura. Now we can choose any representation that characterizes the $S-$ matrix and these three quantities V_{11}, V_{22} and $V_{12} = V_{21}$, will enable us to determine three parameters δ_{J-1}, δ_{J+1}, ε_J (or those of the other set) completely.

The scattering matrix S is diagonal in the coupled case so in the case of vanishing coupling if the phases are defined by $\delta_{J\pm 1}$ then S can be expressed as

$$S = U^{-1} e^{2i\Delta} U \tag{V-4.27}$$

where

$$\Delta = \begin{pmatrix} \delta_{J-1} & 0 \\ 0 & \delta_{J+1} \end{pmatrix} \tag{V-4.28}$$

and U is a unitary matrix which is characterized by one parameter ε_J as

$$U = \begin{pmatrix} \cos\epsilon_J & \sin\epsilon_J \\ -\sin\epsilon_J & \cos\epsilon_J \end{pmatrix} \quad (\text{V}-4.29)$$

Thus S – can be explicitly written as

$$S = \begin{pmatrix} \cos^2\epsilon_J\, e^{2i\delta_{J-1}} + \sin^2\epsilon_J\, e^{2i\delta_{J+1}}, & \frac{1}{2}\sin 2\epsilon_J\left(e^{2i\delta_{J-1}} - e^{2i\delta_{J+1}}\right) \\ \frac{1}{2}\sin 2\epsilon_J\left(e^{2i\delta_{J-1}} - e^{2i\delta_{J+1}}\right), & \cos^2\epsilon_J\, e^{2i\delta_{J-1}} + \sin^2\epsilon_J\, e^{2i\delta_{J+1}} \end{pmatrix}$$

$$(\text{V}-4.30)$$

$$= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} .$$

Also we can write, using (V-4.25) and (V-4.26),

$$V = \left(\frac{S-1}{2i}\right)$$

$$= \begin{pmatrix} \cos^2\epsilon_J \tan\delta_{J-1} + \sin^2\epsilon_J \tan\delta_{J+1}, & \frac{1}{2}\sin 2\epsilon_J(\tan\delta_{J-1} - \tan\delta_{J+1}) \\ \frac{1}{2}\sin 2\epsilon_J(\tan\delta_{J-1} - \tan\delta_{J+1}), & \sin^2\epsilon_J \tan\delta_{J-1} + \cos^2\epsilon_J \tan\delta_{J+1} \end{pmatrix}$$

$$= \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \quad (\text{V}-4.31)$$

These formulae are identical with those of SYM (70) in their appendix if we identify $V_{11} = X_{J-1}$, $V_{22} = X_{J+1}$ and $V_{12} = y_J$. In the above relations we have again used the fact that

$$\frac{e^{2i\delta}-1}{2i} = e^{i\delta}\sin\delta = \tan\delta \qquad (\text{V}-4.32)$$

for small phase shifts, as discussed in Section - 3.

We want to express the nuclear Blatt-Biedenharm (BB) phases in terms of potential matrix elements. Thus using notation of (V-4.32) we obtain from (V-4.31)

$$X_{J\mp 1} = \frac{1+\cos 2\varepsilon}{2}\tan\delta_{J\mp 1} + \frac{1-\cos 2\varepsilon}{2}\tan\delta_{J\pm 1}$$

with some algebraic and trigonometric manipulations we obtain

$$X_{J-1} - X_{J+1} = \cos 2\varepsilon\left(\tan\delta_{J-1} - \tan\delta_{J+1}\right)$$

and

$$\tan 2\varepsilon = \frac{2y_J}{(X_{J-1} - X_{J+1})}, \qquad (\text{V}-4.33)$$

With

$$\tan\delta_{J-1} - \tan\delta_{J+1} = \frac{2y_J}{\sin 2\varepsilon} \qquad (\text{V}-4.34)\ (a)$$

[obtained from (V-4.31), and more algebraic and trigonometric deductions] we obtain(V- 4.34 b)

$$\sin 2\varepsilon = \pm \sqrt{\frac{4y_J^2}{(x_{J-1}-x_{J+1})^2+4y_J^2}}, \qquad (\text{V}-4.35)$$

From (V-4.33) above eqn is obtained.

$$2\tan\delta_{J\mp 1} = x_{J\mp 1} + \frac{2y_J}{\sin 2\varepsilon} - 2y_J \cos 2\varepsilon, \qquad (\text{V}-4.34)(b)$$

and thus by solving (V-4.34) we get

$$\tan\delta_{J\mp 1} = \frac{1}{2}(x_{J-1}-x_{J+1}) \pm \frac{1}{2}\left\{(x_{J-1}-x_{J+1})^2+(2y_J)^2\right\}^{1/2} \qquad (\text{V}-4.36)$$

Thus we have obtained expressions for $\tan 2\varepsilon$ and $\tan\delta_{J\mp 1}$ and the sign of the root is decided by the requirement that in the limit of $\varepsilon_J \to 0$ or vanishing coupling these expressions (V-4.36) reduce to corresponding expressions for phase shifts in the uncoupled cases.

There is an alternative viewpoint which we can take to derive the same results. We want to solve the eigenvalue problem for the S- matrix which can be reduced to the diagonalization of a 2 x 2 matrix, i.e. to solve

$$S\binom{a}{b} = \lambda\binom{a}{b}$$

or

$$\left[(2iV+1) - \lambda 1\right]\binom{a}{b} = 0 \qquad (\text{V}-4.37)$$

or

$$\begin{pmatrix} V_{11}' - \lambda & V_{12} \\ V_{12} & V_{22}' - \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$V_{11}' = 2iV_{11} + 1 , \quad V_{22}' = V_{22}(2i) + 1 . \qquad (\text{V}-4.38)$$

The determinant equated to zero gives

$$\frac{\lambda - 1}{2i} = \frac{1}{2}(V_{11} + V_{22}) \pm \frac{1}{2}\sqrt{(V_{11} - V_{22})^2 + (2V_{12})^2} \qquad (\text{V}-4.39)$$

since

$$S\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{2i\delta_{J-1}} & 0 \\ 0 & e^{2i\delta_{J+1}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} ,$$

therefore

$$\lambda_{J\mp 1} = e^{2i\delta_{J\mp 1}} \qquad (\text{V}-4.40)$$

and we get the same result (V-4.36). We have to find the eigenvector uniquely, up to a ratio, and from the δ_{J-1} expression we get

$$\left(\frac{a}{b}\right)_{J-1} = \left[\frac{(V_{22} - V_{11}) + \sqrt{(V_{11} + V_{22})^2 + (2V_{12})^2}}{2V_{12}}\right] \qquad (\text{V}-4.41)$$

Similarly from the δ_{J+1} expression we obtain

$$\left(\frac{a}{b}\right)_{J+1} = \left[\frac{2V_{12}}{(V_{11} - V_{22}) + \sqrt{(V_{11} + V_{22})^2 + (2V_{12})^2}}\right] \qquad (\text{V}-4.42)$$

If we identify $\tan \varepsilon_J = (a/b)$, we can derive (V-4.33) and also prove that

$$(\tan \epsilon_J)_{J-1} = (\tan \epsilon_J)_{J+1} = \tan \epsilon$$

provided

$$4V_{12}^2 = 4V_{12}^2$$

which is always true. Just as a check on the numerical procedure, we calculate both expressions and verify that they are the same. We thus use (V-4.41), (V-4.42), and (V-4.36) for our calculations of phase shifts $\delta_{J\pm 1}$ and the mixing parameter ϵ_J in the BB representation.

We could directly use the alternative defintion of S-matrix and calculate the recently quoted "bar" phases and mixing parameter in an analogous way. The reasons for using "bar" phases are given in SYM (70) where they discuss the coulomb phase shifts and the subtraction of coulomb effects from nuclear effects in the asymptotic region. The conversion from nuclear phases to the total phases is simple in "bar" phases as

$$\bar{\delta}^N_{J\pm 1} = \bar{\delta}_{J\pm 1, J} - \Phi_{J\pm 1}$$

$$\bar{\epsilon}^N_J = \bar{\epsilon}_J \qquad (\text{V}-4.44)$$

where Φ_ℓ denotes pure coulomb phases. In our cases we neglect the coulomb phases but compute bar phases for convenience of comparisons. Thus the phases and parameters obtained by these two methods may be completely different. There exist the relations which connect these two phases.

which are obtained by Stapp (107) and also given in SYM (70). These are

$$\delta_{J+1} + \delta_{J-1} = \bar{\delta}_{J+1} + \bar{\delta}_{J-1} ,$$

$$\sin\left(\bar{\delta}_{J-1} - \bar{\delta}_{J+1}\right) = \frac{\tan 2\bar{\varepsilon}_J}{\tan 2\varepsilon_J} ,$$

and

$$\sin(\delta_{J-1} - \delta_{J+1}) = \frac{\sin 2\bar{\varepsilon}_J}{\sin 2\varepsilon_J} , \qquad (\text{V}-4.45)$$

which are the relations we have also used in going from "BB" to "bar" phases. These transformations are numerically carried out because solving for one set in terms of the other set is not straight-forward, and the problem has to be dealt with numerically in any case. Thus we have presented an extensive account of dealing with the coupled phases in Born approximation. We would like to point out that for the coupled states the nature of the problem is such that individual mesonic contributions can not be studied and also individual terms in the potentials can not be identified. Thus we rely on uncoupled states for this aspect of the problem.

We have thus given all the required theoretical formalism for phase shift study and now we discuss the numerical analysis approach in the next chapter.

CHAPTER VI

NUMERICAL ANALYSIS AND COMPARISON WITH EXPERIMENTAL PHASES

Section - 1 Numerical Analysis And Computer Programs

In Section - 3 and Section - 4 of the last chapter we obtained analytic expressions for uncoupled and coupled phase shifts which were functions of meson masses, cut-off mases, coupling constants, scattering energy, and orbital angular momentum, etc. It is seen by a careful look at them that similar expressions for different mesons and for different couplings are to be evaluated. The nature of integrals involved is such that it is not possible to evaluate them analytically into a simple closed form. A close look at them revealed that they could be changed into an infinite confluent hypergeometric series but the convergence criterion were not simple and the series was oscillating with alternating signs. Even if there was a closed expression for these individual integrals, the variations of parameters, together with energy and the partial waves would have been very involved. Thus a computer oriented approach to this problem would have been essentially required. These were the reasons why Born approximation phases were developed in computer programs with these facilities in mind. Also the purpose of seeing the individual meson or individual term contributions could be mechanized relatively simply on the computer. These individual meson (or term) contributions are evaluated on the particular assumption that no other term except the one under consideration contributes. Thus the sum of these phases may not equal the total phase. In other words, we can say that the tangent of a sum of functions is not distributive (i.e., does not equal the sum of tangents of each function). These integrals can be easily evaluated numerically because they are in general smooth functions and in every case the factor $e^{-\mu z}$ makes them vanish at a reasonable distance from the origin (within 3-4 fermis). Since the potentials are short ranged only a few partial waves are scattered. The H waves are very small and, therefore, experimentally not known to a good accuracy. Hence, our phase shift calculations will be done only for the P, D, F and G waves, for both uncoupled and coupled states. S waves can be calculated in the same manner but will be excluded from our discussion due to the physical restrictions on the Born approximation. It might be mentioned that the final forms of these programs developed through many stages as the author's familiarity with programming developed. For example, these integrals were evaluated in a few different ways before bringing them into present form.

Returning to the problem, we summarize below the scheme of this work. Other than commonly available functions, we need spherical Bessel functions of desired argument and order. They are generated for a fixed set of energies, value and mesh widths, and are stored in an array for the main programs. Another important part is the Numerical Integration subroutine which uses Newton-Coles formula. We shall list all these programs in Appendix - A. A general criterion of using 60 points for integration between 0.001 and 5.0 fermis was established to be useful and economic for the accuracy desired in this problem. For uncoupled phases unconstrained Breit's or Kemmer's vector interactions can be used or the choice allows only the unconstrained vector interaction to be chosen. In either case, the scalar and pseudoscalar interactions are common as they are described in Chapter III. The phases are evaluated in degrees and energy in units of Mev. In this program, a subroutine is defined which contains the portions connected with numerical integration. Before this part the program reads in an array of multiplication constants for various mesons and various terms. With these coefficients and integrals we then calculate the appropriate contributions to integrals. Then their sum is also evaluated. The inverse tangent of the sum is the required phase in radians. To see the individual contribution to the phase, each meson or term value is changed to the radian by taking the inverse tangent if the phase is large or by using the expansion for the inverse tangent if it is small (i.e., less than 0.5 radian). Let

$$x = \tan \delta_i = \delta_i - \frac{\delta_i^3}{3}$$

or

$$\delta_i = x - x^3/3 . \qquad (\text{VI}-1.1)$$

This approximation is good up to 25°-30° or about half-radian. Thus individual contributions δ_i in degrees are also printed out. In case of the Kemmer and Breit interactions, the extra contribution to the phases are evaluated and added with original parts having appropriate numerical coefficients.

For the effective mass approach, the differences arise in coefficients, and the subroutine for integration is incorporated in the main program. The general layout is the same except for "gradient term" for which only a sum for proper mesons can be evaluated.

For the coupled states, a major portion of the uncoupled phase program is used but parts of it have to be looped twice to account for χ_{J-1} and χ_{J+1} which correspond to diagonal matrix elements of the potential. These parts are almost the same as the uncoupled program. In addition the non-diagonal matrix elements of pot tial are also evaluated. The last section of this program involves the calculation of coupled phases and mixing parameter in the Blatt-Biedenharn approach and then they are numerically converted to the "bar" phases. The mixing parameter in BB form is evaluated twice just for a check on the programs.

As a further study we used the exact phase shift code of Professor B. L. Scott (Long Beach State University) for comparisons. This code has been modified by Dr. T. Sawada

(University of Florida) to include different cut-off masses.

(i) <u>Spherical Bessel Functions</u>

These functions are well known to a physicist and we developed a code that used a series expansion for $j_\ell(z)$ for $z \leq 1$ and $\sin z$, $\cos z$ and recurrence relation forms for $z > 1$. Specifically

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{\ell+1/2} \left[\sum_{k=0}^{30} \frac{\{-z^2/4\}^k}{k!\,\Gamma(\ell+\frac{1}{2}+1+k)}\right] \qquad (\text{VI}-1.2)$$

for

$$0 < z \leq 1$$

where

$$\Gamma(n+3/2) = \frac{1.3.5.\cdots(2n+1)}{2^{n+1}} \Gamma(1/2) . \qquad (\text{VI}-1.3)$$

as given in the Handbook of Mathematical Functions (109).

For

$$z > 1$$

$$j_0(z) = \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

and

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z \qquad (\text{VI}-1.4)$$

were calculated and the higher order spherical Bessel functions were generated by the recurrence formula

$$j_{\ell+1}(z) = \frac{(2\ell+1)}{z} j_\ell(z) - j_{\ell-1}(z) \qquad (\text{VI}-1.5)$$

These formulae are given in Schiff (79).

(ii) Weddle's Rule For Numerical Integration

If a function $f(x)$ is continuous over intervals (a, b) then the numerical integration of such a function is done by method of approximate quadratures by replacing

$$I = \int_a^b f(x)\, dx$$

by

$$\int_a^b \phi(x)\, dx = A_0 y_0 + A_1 y_1 + \cdots + A_n y_n . \qquad (\text{VI}-1.6)$$

where $A_m's$ are independent of the $y's$. The error is minimized by a proper choice of $A_m's$. If we write the differences in terms of the $y_m's$ then the Newton-Coles formula follows for the $A_m's$ as given in Marganau and Murphy (110),

$$A_m = \frac{(-1)^{n-m}\, h}{m!\,(n-m)!} \int_0^n \frac{z(z-1)(z-2)\cdots(z-n)\, dz}{(z-m)} \qquad (\text{VI}-1.7)$$

where z is the variable of integration, h the interval and n the number of sub-divisions of each interval. The cases $n = 1$ and 2 correspond to trapezoidal and Simpson's rules respectively. We take a more accurate $n = 6$ case known as Weddle's rule which yields to a formula

$$I = h\left[\frac{41}{140}(y_0 + y_n) + \frac{54}{35}(y_1 + y_5) + \right.$$

$$\left. + \frac{27}{140}(y_2 + y_4) + \frac{68}{35} y_3 + \cdots \right] \qquad (\text{VI}-1.8)$$

(These agree with Table 4 of reference 110, page 460.) This method restricts the total number of integration intervals to be of the form $6m$ or the number of points to be $(6m+1)$, for which the integrand has to be evaluated and multiplied properly. The differences are of the order greater than the sixth and this is one of the most accurate formulae for numerical integration. Its accuracy for 25 points has been confirmed to second decimal place as compared to exactly known integrals.

The general formula is given by

$$I_i = \int_{r_0}^{r_{max}} e^{-\mu z} z^{p} j_m(z)\, j_n(z)\, dz \qquad (\text{VI}-1.9)$$

The sums of such forms are evaluated for phase shift integrals.

(iii) Notation

It will be an essential duplication to list the names of variables and their corresponding physical quantities. Any reader reasonably familiar with Fortran - 2, and the facets of this problem can follow a parallel nomenclature of the text symbols to those used in the program. For example, δ_∇ will be denoted as DGRAD, for phases. The mesons are also designated by their names although in the sequence they occur as $\omega_s, \omega, \eta, \rho_s, \rho, \pi$ and ϕ but the last one is left out for most cases because it is heavy (1020 Mev) and influences mostly S-waves. Its effects will be discussed in the next section. The W's are the numerical coefficients and alternate indices in their array are reminiscent of older approaches. The three phase shift programs which are listed in Appendix - A could be, with a considerable effort, changed to one single code but the effort spent on it may imply an equivalent spending of time as compared to the advantage. Hence they are not given in most compact form. The results of these analyses will be discussed in the next two sections.

Section - 2 Variations Of Potential Parameters

Any theoretical description of physical phenomena has to face, at one stage or another, the task of a successful connection with experimental facts. A proper time exists for any reasonable hypothesis to be tested experimentally and vice-versa. It is often not possible to do so simultaneously. Thus this study had begun almost two decades ago (Green, reference 53) as pointed out in the first chapter but had to await a variety of experimental results. The task is yet unfinished but a great deal of uncertainty in the field is reduced today. It has been shown by various workers in the field that one-particle-exchange models can explain to a reasonable degree almost all the awaitable data on nucleon-nucleon force. Although in almost all the cases a strict meaning of potentials breaks down, it is possible to reasonably generalize such a concept for non-relativistic problems. This has been the topic of the past few chapters. Now we start within the limits of our model, in the Born approximation, to establish this contact through comparison with sets of phases (obtained from a variety of experimental data) by Arndt and McGregor (93).

We briefly recall that the Born approximation implies replacement of scattered wave by a plane wave and thus depends heavily on the smallness of the phases or of the potentials. Hence for small potentials we expect to get good results in the Born approximation. Sometimes it so happens that even lower angular momentum partial waves have an effectively small interaction and thus the Born approximation will be found quite accurate for them. For higher ℓ values and higher energies (for ordinary potentials) the Born approximation is generally good. But for velocity (energy) dependent potentials, we have to be careful about this statement because the interaction might become larger at higher energies and plane wave approximation might start loosing its accuracy. But since higher partial waves sample only the outermost regions of meson exchange potentials (short ranged) we expect them to be quite good in the Born approximation. Let us start with classical relation for angular momentum associated with a particle scattered by a center of force. The impact parameter r is defined as the perpendicular distance between the center of force and the incident velocity. The angular momentum ℓ associated with velocity v of the particle is given by

$$\ell = m v r = p r \qquad \text{(VI-2.1)}$$

or

$$\ell = r \sqrt{2ME} \qquad \text{(VI-2.2)}$$

But quantum mechanically

$$\vec{\ell}^{\,2} = \ell(\ell+1)\hbar^2$$

and also $\vec{p} = \hbar \vec{k}$ and we obtain

$$\hbar k r = \sqrt{\ell(\ell+1)}\,\hbar \quad . \qquad \text{(VI-2.3)}$$

For convenience we use a good approximation for the root

$$\ell(\ell+1) = \ell^2+\ell \approx \ell^2+\ell+\tfrac{1}{4} = (\ell+1/2)^2 \qquad (\text{VI}-2.4)$$

along with the definition

$$\begin{aligned} k &= \sqrt{ME/\hbar^2} \\ &= \sqrt{\frac{2ME_L}{\hbar^2}} \\ &= \frac{1}{\sqrt{82.94}}\sqrt{E_L} \simeq \frac{1}{9}\sqrt{E_L} \end{aligned} \qquad (\text{VI}-2.6)$$

Thus we finally have approximately

$$r \approx 9(\ell+1/2)/\sqrt{E_L} \qquad (\text{VI}-2.5)$$

Hence we can assign a classical impact parameter or a quantum mechanical region of importance for each of the partial waves for a given energy. Taking the example of 320 Mev we have $k \sim 2 \; fm^{-1}$. Thus P- waves according to (VI-2.5) would be affected up to a distance of about 0.75 fermis, D- waves up to about 1.25 fermis, F- waves up to 1.75 fermis, G- waves up to 2.25 fermis, and so on. These estimates will change slightly with energy. The approximation which is not very good for S- waves denotes a short range of about 0.25 fermi but these are only approximate characterizations because of quantum mechanical effects, and also due to mixing in various phases. Thus we can say that heavy vector mesons will not play a great role in phenomena involving partial waves higher than D- waves and π- meson being the lighest would affect the higher partial waves. The cut-offs associated with the already discussed modified Yukawa form with Green's subtractive meson will also affect the phases according to this rule, thus higher mass cut-offs will affect $S-$ waves only.

With these ideas in mind, we now report the results on the phase shifts. The experimentally determined quantities are the masses of ω, η, ρ and π mesons and the coupling constant of π meson as determined by Hamilton and Woolcock (78) from $\pi-N$ scattering experiments. Other coupling constants and cut-off masses are the parameters of our theory. However, in the following we will try to minimize our parameters, see the effects of slight and considerable deviations from the purely relativistic model and will, in the end, confine ourselves again to the exactly determined phases within the purely relativistic model.

We list in Table - 3, various parameters that have been used for the calculations of phase shifts in the Born approximation and plot the results of these calculations in Figure 4. Starting with our original study, we try to establish parameters that will fit the uncoupled states reasonably well in the Born approximation. Then we try to make use of the exact phase shift analysis to be referred to in the coming results. These more or less represent the results in which accumulated experience is used due to various other variations that are not reported here. We only give those variations which led us to some definite and interesting physical conclusions. We have not mechanized our searching procedures, because we wanted to see the sensitivity of various parameters with regards to our physical intuitions and also according to our accumulated experience and therefore no least square criterion has been used in the present study. Our thinking is in favor of purely relativisitc models of nucleon-nucleon force together with the contributions from various other terms from multiple meson exchanges, and from other processes within the OBEP and outside it and we want to assign the descrepencies of agreement to these rather than adjust the parameters in this respect.

We adopt the following notation for this discussion. The parameters of Table - 3 will be designated by numbers in parentheses and the Figure - 4's corresponding to them will be referred to by lower case letters in parentheses.

We represent the phase shifts corresponding to the simple purely relativistic model of Green and Sharma (54) in which the strengths of nuclear forces are assumed to be the same for all the mesons exchanged. The parameters (1) are reduced to minimum and this is an overly simplified model (the Zero Parameter Model). The results are presented in (a), for uncoupled phases, by continuous curves. For a completely unadjusted model, the results are quite encouraging. The signs of phases and their behaviors are all in the right direction in relation to the phases of Arndt and MacGregor (93) (extracted from experimental data). The phases $^1P_1$ and $^1D_2$ are bad & are the phases which sense the innermost part of the potential, of the phases reported in (a). Higher phases mostly sense the pion contribution denoted by OPEP (or π in other figures)

TABLE - 3

Parameter Variations

m = Meson Mass in Mev λ = Cut-off Mass in Mev

Constants

$m_\omega = 782.8$; $m_\eta = 548.7$; $m_\pi = 137.5$; $m_\rho = 763.$; $g_\pi^2 = 14.7$

Parameter Variations

	1	2	3	4	5
m_{ω_s}	m_ω	m_ω	m_ω	m_ω	m_ω
m_{ρ_s}	m_ρ	m_ρ	m_ρ	m_ρ	m_ρ
λ_ω	1500	1500	1000	3000	1500
λ_{ω_s}	1500	1500	1000	3000	1500
λ_ρ	1500	1500	1000	3000	1500
λ_{ρ_s}	1500	1500	1000	3000	1500
λ_η	1500	1500	1000	3000	1500
λ_π	1500	1500	1000	3000	1500
g_ω^2	g_π^2	20.0	g_π^2	g_π^2	10.0
$g_{\omega_s}^2$	g_π^2	20.0	g_π^2	g_π^2	g_π^2
g_ρ^2	g_π^2	$g_\pi^2/9$	g_π^2	g_π^2	10.0
$g_{\rho_s}^2$	g_π^2	$g_\pi^2/9$	g_π^2	g_π^2	g_π^2
g_η^2	g_π^2	g_π^2	g_π^2	g_π^2	g_π^2

TABLE - 3 (cont.)

Parameter Variations

	6	7	8	9	10
m_{ω_s}	m_ω	750	700	650	650
m_{ρ_s}	m_ρ	750	700	650	650
λ_ω	1500	1500	1500	1500	1500
λ_{ω_s}	1500	1500	1500	1500	1500
λ_ρ	1500	1500	1500	1500	1500
λ_{ρ_s}	1500	1500	1500	1500	1500
λ_η	1500	1500	1500	1500	1500
λ_π	1500	1500	1500	1500	1500
g_ω^2	5.0	g_π^2	g_π^2	g_π^2	6.7
$g_{\omega_s}^2$	g_π^2	g_π^2	g_π^2	g_π^2	10.0
g_ρ^2	5.0	g_π^2	g_π^2	g_π^2	5.0
$g_{\rho_s}^2$	g_π^2	g_π^2	g_π^2	g_π^2	7.5
g_η^2	g_π^2	g_π^2	g_π^2	g_π^2	5.0

TABLE - 3 (cont.)

	Parameter Variations				
	11	12	13	14	15
m_{ω_s}	m_ω	m_ω	m_ω	m_ω	m_ω
m_{ρ_s}	m_ρ	--	--	--	--
λ_ω	2000	1000	1000	1000	1600
λ_{ω_s}	2000	1000	1000	1000	1600
λ_ρ	1000	--	--	--	--
λ_{ρ_s}	1000	--	--	--	--
λ_η	1000	--	1500	1500	--
λ_π	550	550	275	1500	750
g^2_ω	10.0	20.0	20.0	20.0	$3g^2_\pi$
$g^2_{\omega_s}$	10.0	20.0	20.0	20.0	$3g^2_\pi$
g^2_ρ	2.5	--	--	--	--
$g^2_{\rho_s}$	2.5	--	--	--	--
g^2_η	g^2_π	--	g^2_π	g^2_π	--

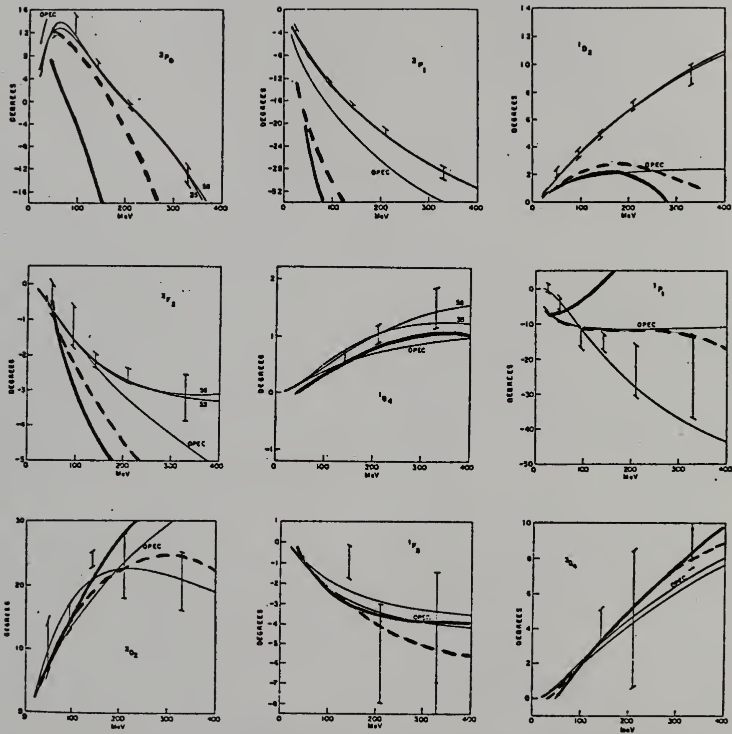

Figure 4A. Phase Shifts Corresponding to the Parameter of Table 3. (1) solid curves, (2) dashed curves.

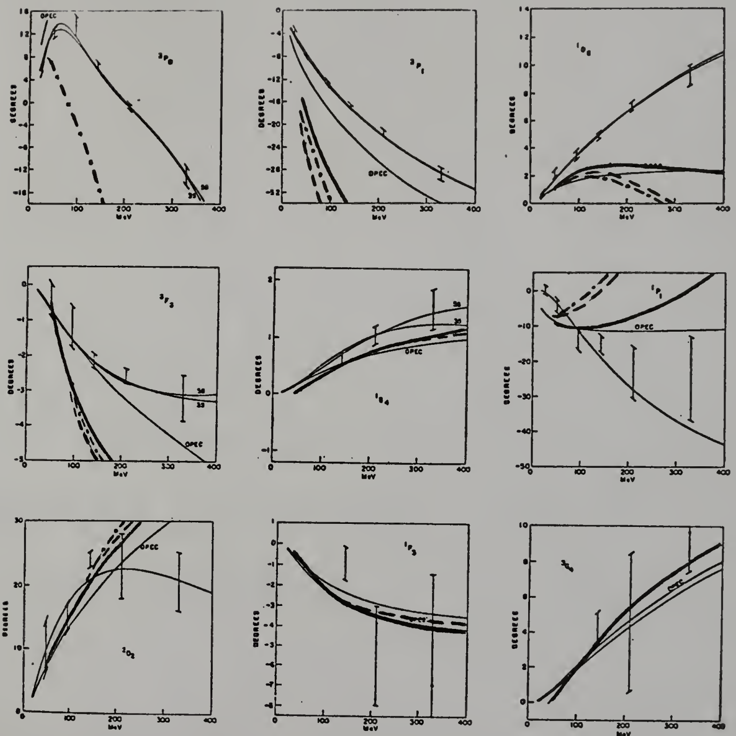

Figure 4B. Phase Shifts Corresponding to the Parameter of Table 3. (3) solid curves, (4) dot-dash curves, (1) dashed curves.

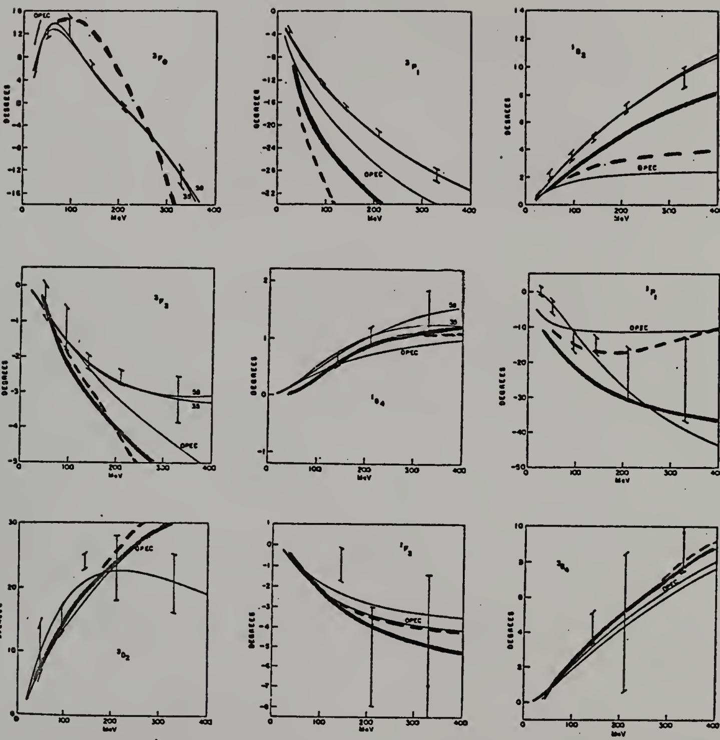

Figure 4C. Phase Shifts Corresponding to the Parameter of Table 3. (5) dashed curves, (6) solid curves.

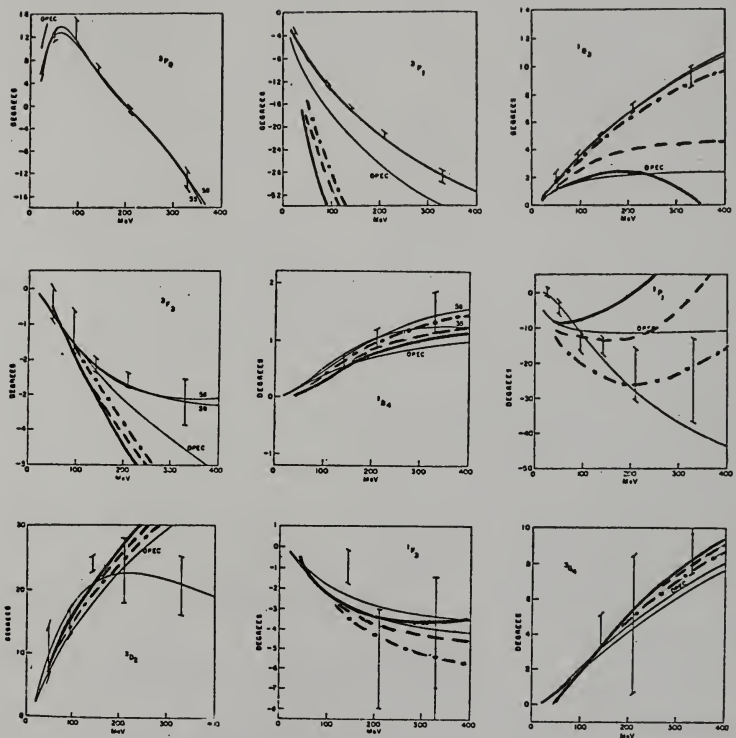

Figure 4D. Phase Shifts Corresponding to the Parameter of Table 3. (7) solid curves, (8) dashed curves, (9) dot-dashed curves.

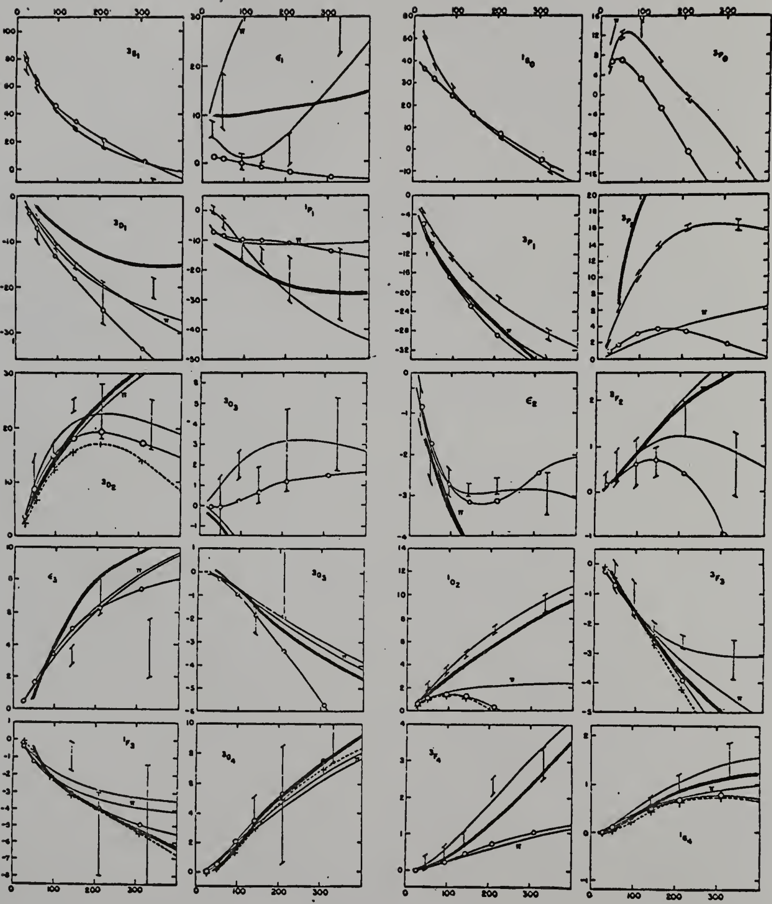

Figure 4E. Phase Shifts Corresponding to the Parameter of Table 3. (10) solid curves.

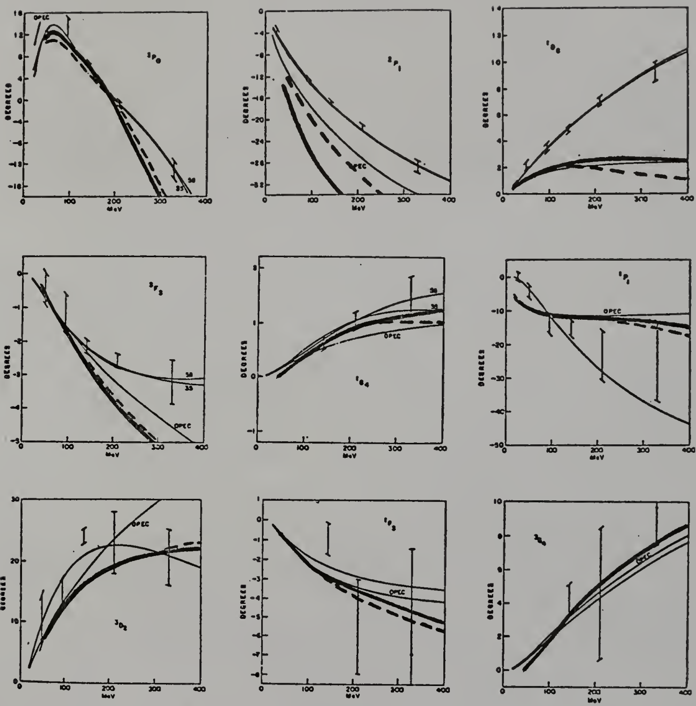

Figure 4F. Phase Shifts Corresponding to the Parameter of Table 3. (11) solid curves, (12) dashed curves.

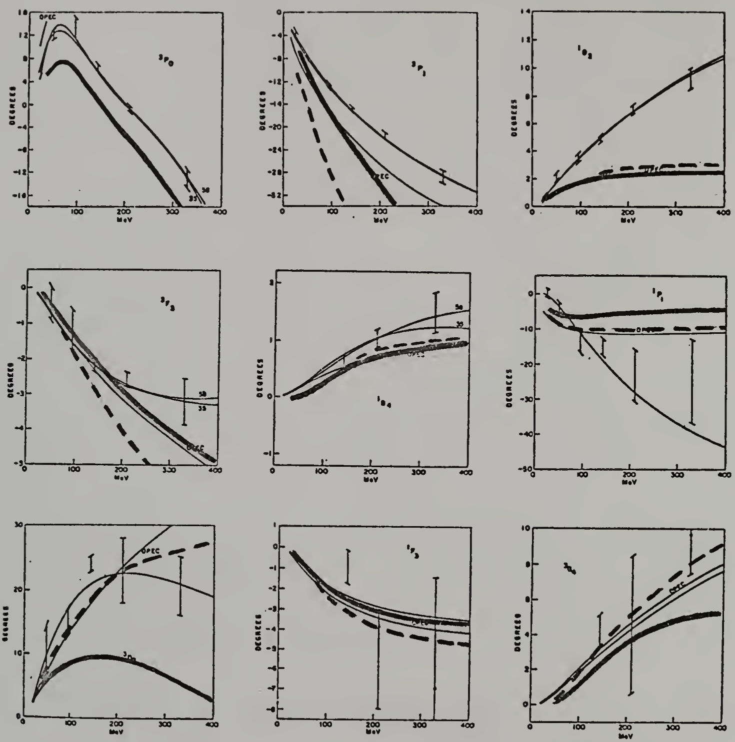

Figure 4G. Phase Shifts Corresponding to the Parameter of Table 3. (13) solid curves, (14) dashed curves.

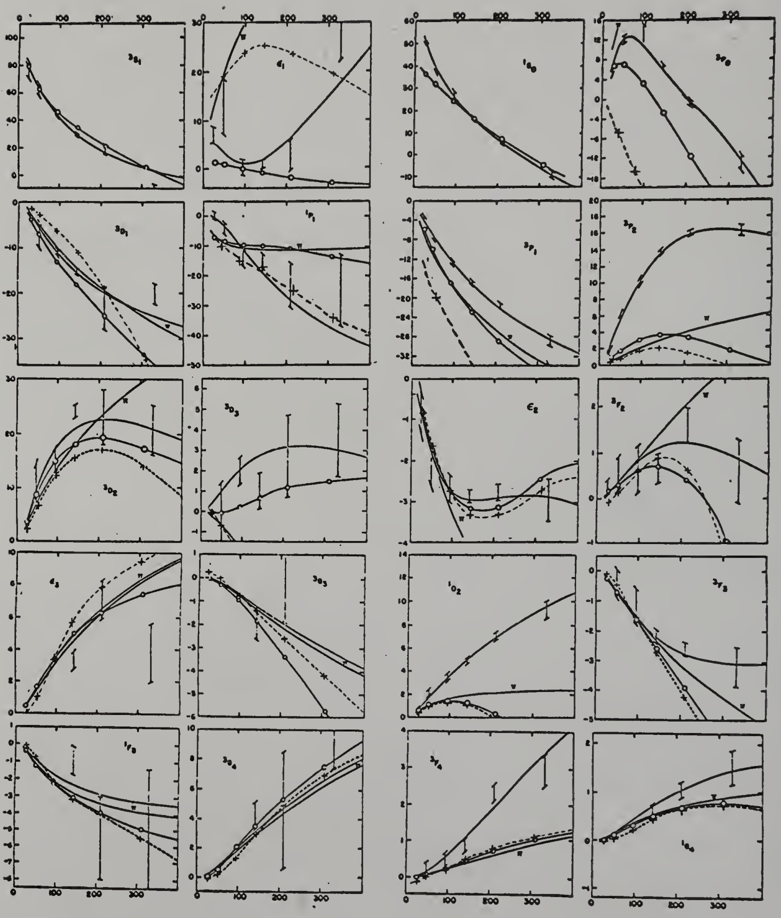

Figure 4H. Phase Shifts Corresponding to the Parameter of Table 3. (15) dashed curves with (+) symbols at proper energies in Mev in laboratory frame, circled curves denote exact phase shifts.

and they are generally good. Encouraged by this, we carried out a few different combinations of the coupling constants. Out of some such sets the ones in which the vector mesons occured with smaller couplings proved a little better at the first sight. We should be careful in considering the couplings and cut-offs in the Born approximation. The higher phases decide and give information only about the pion coupling constant which is fixed and is known ($g_{\pi}^2 = 14.7$) in our case. Thus the lower phases will decide the other Born parameters. The results for lower phases are generally larger in the Born approximation than in the exact calculations. Thus an attempt to calculate the exact phases after the parameters are determined by the Born approximation should carefully allow for the uncertainties of the Born results for the lower phases. Alternatively, the Born parameters (granting the exact parameters to denote the physical values for nuclear forces) also contain the corrections for the inaccuracy of the Born approximation for lower partial waves. This tends to yield smaller couplings in the Born approximation.

The parameters (2) are given by the dashed curves in (a). Here we have reduced the coupling of ρ– meson by nearly a factor of 10 while we have enhanced that of the slightly. The results become better in every case. We should thus expect the results to be less strongly dependent on Isovector mesons (except the π). This aspect of the study has been made by Green, Sawada, and Sharma (111) and will be discussed later. We should also mention that the model stays purely relativistic only as long as $J_s = J_v$ which requires that $m_\omega = m_{\omega_s}$, $g_\omega^2 = g_{\omega_s}^2$, and $\lambda_\omega = \lambda_{\omega_s}$ and similar relations for the isovector, scalar and vector mesons (ρ_s and ρ).

Staying within the purely relativistic model, we decided to see the sensitivity of our results as a function of the cut-off mass which was kept the same for all the mesons. The results for (3), (4), and (1) are presented in Figure (b). The solid curves present the cut-off mass λ = 1000 Mev for all mesons. The dashed curves represent λ = 1500 Mev and dot dashed curves represent λ = 3000 Mev. Hence a change in cut-off mass from 1500 to 3000 Mev produces very small changes in these phases, while a change from 1000 to 1500 Mev produces a comparatively big change. Thus our results for 1500 Mev cut-off or higher are not sensitive to it. A smaller mass for the cut-off implies that we have modified the Yukawa form starting with larger distances and vice-versa. With different cut-offs we can thus properly weigh the important regions of the OBEP. For example the π meson is very light and its potential has the longest range. We also think that 2π— exchange processes start taking place in a correlated as well as uncorrelated way as we move towards the origin. Thus the cut-off for the π meson is a very helpful device to truncate or reduce the effects of OPEP in the regions where other effects like these may be important.

Having seen the general trend of phase shifts we decided to break our relativistic model just to see the influence of other parameters. We believe that the nuclear forces are mainly relativistic but we have to be open minded about possible outcomes of future experiments regarding the scalar meson. There is a good reason to believe that they will be experimentally more difficult to discover if their masses are at the same place as those of the ω and the ρ (and the ϕ) mesons. Recently many groups (112) have reported such a meson. It may also be an enhancement in meson-meson scattering or simply an S wave $\pi-\pi$ resonance. Thus the question of the existence of scalar mesons and the positions of their peaks are quite important from the standpoint of our results and for theories characterizing the nucleon-nucleon force. This requirement of additional attraction in central potentials is the common ingredient of almost all current theories which account for nucleon-nucleon force (69, 71, 74, and 54), and more or less balances the repulsive static term coming from the vector mesons. Bryan and Scott (69) however make the scalar meson mass quite light as compared to the vector meson.

We decided to see the effects of approximately relativistic models in the light of scalar-vector couplings and the masses. The parameters (5) and (6) are denoted in (c) by dashed and continuous curves respectively and the changes from previous sets are that we make vector meson couplings smaller as compared to scalar meson couplings and this has the effect of introducing extra attraction. As a result, the phases $^1P_1$ and $^1D_2$ become better. Thus we need basically a short range attraction. All other phases improve. We exclude $^3P_0$ for this case. It is sensitive to cut-offs and will be discussed in context with other parameters.

Approximately the same and somewhat better results can be obtained if we make the scalar meson mass lighter as compared with the vector mass. This is done for the parameters (7), (8), and (9) which are given in (d) by continuous, dashed and dot dashed curves respectively. Thus we have seen that as far as parametric analysis is concerned, we can adjust various degrees of freedom at our disposal and account for the uncoupled states.

Such an attempt was made in the Born approximation for uncoupled states and a reasonable fit to the uncoupled states was obtained for (10) shown by continuous lines in (e). An attempt to fit $^1P_1$, $^3P_1$, and $^1D_2$ along with higher partial waves for uncoupled phases were considered and the Born parameters for this broken relativistic model were determined. The model is quite broken in this case because scalar mesons are lighter. But we still maintain a major cancellation of the static term. However the gains are not too much and so we will return to purely relativistic models again. Therefore we started out to test the conclusions derived from the uncoupled phases. Since only a few constraints are contained in the uncoupled phases, the other constraints are determined by the coupled phases. Thus all the phases taken together can characterize the two body scattering data. We wanted to determine how much information could be obtained purely from the considerations of uncoupled phases. For this test we evaluated these parameters (10) for uncoupled Born phases and then calculated the coupled phases (with mixing parameters) for the set (10) by the methods described in the last chapter. The results for these are also included in (e). We do not present these for S-waves for which the Born results are not very meaningful. We see that ϵ_1 and $^3D_1$ are quite good in the Born approximation while $^3P_2$ is far too positive and $^3F_2$ is reasonably good. Another exception is $^3D_3$ which always follows the OPEP trend while the experimental values are quite different from it. This is true of all the variations made on the coupled states. The quantity ϵ_3 and the phases $^3G_3$ and $^3F_4$, however, are quite good. Thus we have predicted the coupled phase shifts within the limits of our model quite well.

We should be careful about the high energy behavior of coupled phases in general. The amount of mixture of the two coupled states in question becomes larger with increasing energy and the coupled non-predominant wave function starts becoming important. But we made approximations which neglected the second order perturbations in the coupled states which might become important at this stage. This reason, together with the strongly velocity dependent forces, may be sufficient reasons for deviations from the Born approximation at higher energies. But in general our previous conclusions still hold. The lower partial waves are thus more uncertain in the Born approximation. Thus (10) represents the approximately broken relativistic model (e).

But we have to confine ourselves to a physically reasonable picture of the nucleon-nucleon force because as yet there are many open questions connected with the meaning, derivation, and validity of OBEP. These may very well account for the discrepencies in the exceptional phases. The sources of these additional terms have been mentioned before and are mainly confined towards the origin, except probably for 2π-exchanges which may be important for the intermediate (1.5 to 2.0 fermis) range. Unless the questions are resolved regarding multiple meson exchanges, pair processes, derivative couplings, other mesons, and other field theoretic symmetries together with the mass of the scalar entity, it is not possible to attach much meaning to the parameters as determined. In other words, we do not want to search the parameters for the sake of experimental agreements without considering the limits of the physically feasible model. And this line of thinking brought us back to explore the limits of the purely relativistic model of nucleon-nucleon force.

Thus using the previous experience but still staying within the purely relativistic model, the effect of the ρ meson was reduced by a lighter cut-off and small coupling and the pion cut-off mass was kept at $4m_\pi$. The results of these variations are given by (11) and are plotted in (f) by the continuous lines. We see that this model is almost as good as that given by (10) except for $^3P_1$ and $^1D_2$ phases and this is because of the purely relativistic model which lacks the attraction needed for the $^1D_2$ phase. Another step towards further simplification was taken by excluding the ρ_s, ρ, and η mesons and just calling it $(\omega_s + \omega + \pi)$ model in which the coupling of the ω_s and ω are increased slightly. The results for these parameters (12) are given by dashed curves in (f). Our results are almost always better even after removing ρ_s, ρ, and η mesons from the scene. Thus the purely relativistic ($\omega_s + \omega + \pi$) model is quite encouraging. A number of variations were made on this set of parameters and also the ones including the η meson. We denote by (13) and (14) the parameters that differ only in the π meson cut-off and the results are shown in (g) by continuous and dashed curves respectively. The phases $^3P_0$ (dashed curves are out of the scale), $^3D_2$ and $^3P_1$ are most affected because the tensor force from the pion becomes quite large for these phases in the inner regions and so the π cut-off is quite important for adjustments on these phases. However we note that an intermediate value of 700 to 800 Mev for π cut-off should give a reasonable fit for these phases.

Now we proceed to discuss our final set and one of the latest results for which a considerable search has been made by Green, Sawada, and Sharma (111) within the relativistic ($\omega_s + \omega + \pi$) model. The exact phase shift code was used in this study to fit the $S-$ waves and higher as close as they could be allowed within the limits of a simple model like this which is given by (15) in (h). Here the results of phase shift calculations with exact code are given by solid lines with open circles, the experimental and OPEP curves are given by thin continuous lines. In other figures, by "continuous curves" we meant the <u>dominant curves</u> for the Born calculations. The Born phases are given by dashed lines with symbol <u>+'s</u>. The symbols (+) and (0) in the curves denote the theoretical values of phases carried out at 25, 50, 95, 142, 210, and 310 Mev lab energies. The coupling constants thus determined are quite large $3 g_{\pi}^2$ and they are required by the constraint to fit the $S-$ waves. The cut-offs of 1600 Mev for ω and ω_s are forced to be the same in purely relativistic model and the pion cut-off is searched at 750 Mev. Thus the number of parameters is <u>three</u> for this model and the exact fits to the $S-$ waves are excellent. The other phases are quite close in exact analysis. The Born phases follow the trend of the exact phases. Thus the Born approximation always gives larger phases for waves or more precisely for ϵ_1 , $^3P_0$, $^3P_1$, and $^1P_1$ phases. The agreement for the Born approximations for higher phases is quite good and the only exception is the coupled $^3D_3$ phase which follows the OPEP trend for every calculation done on coupled phases. In general, thus we see that for both coupled and uncoupled phases the Born approximation can be relied upon for $D-$ waves and higher. Also for $P-$ waves it always gives the right trend and simple criteria like reducing the Born phase by 50% to get the corresponding exact $P-$ wave phases may be established. In general it is surprisingly good at low energies.

Section - 3 Vector Mesons And The Nucleon-Nucleon Interaction

We have reported in Chapter II three different methods of deriving the OBEP due to the vector meson field. Their Diracian forms are given by (III-3.2), (III-3.5), and (III-3.4) or explicitly by

$$V^D = \left[\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} \right] J \tag{VI-3.1}$$

$$V^D = \left[\mathbb{1} - \vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)} + \frac{1}{K^2} (\vec{\alpha}^{(1)} \cdot \vec{\nabla})(\vec{\alpha}^{(2)} \cdot \vec{\nabla}) \right] J \tag{VI-3.2}$$

and

$$V^D = \left[\mathbb{1} - \frac{\vec{\alpha}^{(1)} \cdot \vec{\alpha}^{(2)}}{2} + \frac{(\vec{\alpha}^{(1)} \cdot \vec{r})(\vec{\alpha}^{(2)} \cdot \vec{r})}{2r} \frac{d}{dr} \right] J \tag{VI-3.3}$$

Their Pauli forms are given by (III-3.36), (III-3.72), and (III-3.65) respectively. All of the three forms of vector meson interactions have a common Yukawa term while they differ considerably in their relativistic terms as can be seen by looking at their Pauli forms or at their explicit forms as given in Chapter V, Section - 2. Since the vector meson interactions play a very important role in nucleon-nucleon force for lower partial waves, as is evident from the successes of OBEP models, and since the relativistic terms are very important for all these theories, it is necessary to decide upon the nature of the vector interaction. These three different forms as mentioned before, imply different field theoretic derivations which involve the discussions of auxillary conditions and the constraints on the wave function. Because of the importance of vector mesons, we decided to study all these three interactions in detail in the Born approximation and calculated uncoupled phases for them. To get realistic values within the limits of the Born approximation for these vector interactions, we decided to confine ourselves to the purely relativistic model and included the scalar meson interaction given by (III-3.1) and (III-3.21) and also the pseudoscalar interaction as given by (III-3.3) and (III-3.44). This brings out the relativistic effects to the utmost importance and the differences, if any, will become larger. The results of these phase for the parameters (15) of Table 3 at 50 and 310 Mev laboratory energy are given in Table 4. Equations (1), (2), and (3) in the table denote (VI-3.1), (VI-3.2) and (VI-3.3) or the unconstrained, the Kemmer, and the Breit vector interactions respectively. The experimental values (93) are shown to give an idea of the deviations of the Born approximation, but only in a crude way. The exact phases for these parameters (15) of Table 3 are plotted in Figure - 4(h) which gives a more precise idea of the validity of the Born approximation.

We notice that the phases are almost identical for P— waves and higher. The only noticeable exception is

Table 4. Born Phase Shifts For Vector Interactions

	E_{Lab} = 50 Mev				E_{Lab} = 310 Mev			
Phases	Eq. (1)	Eq. (2)	Eq. (3)	Expt.	Eq. (1)	Eq. (2)	Eq. (3)	Expt.
$^3P_0$	-7.60	-7.61	-7.29	12.08	-75.90	-75.91	-74.86	-9.92
$^3P_1$	-20.17	-20.18	-19.90	-7.98	-68.40	-68.41	-65.95	-27.50
$^1P_1$	-9.64	-9.64	-9.33	-4.28	-33.32	-33.39	-20.45	-38.00
$^3D_2$	6.54	6.54	10.31	13.23	13.23	13.23	15.41	21.80
$^1D_2$	0.870	0.870	0.880	2.16	-5.37	-5.27	-2.95	9.20
$^1F_3$	-0.654	-0.654	-0.654	-0.90	-5.12	-5.12	-4.78	-3.19
$^3F_3$	-.457	-.457	-.456	-0.45	-6.91	-6.91	-6.58	-3.12
$^1G_4$	-0.034	-0.034	-0.034	0.09	0.74	0.74	0.79	1.35
$^3G_4$	0.198	0.198	0.198	0.20	6.95	6.95	7.00	6.30

$^1P_1$ phase which differs by about 10° at 310 Mev lab energy. The only other phase is $^1D_2$ which shows a difference of about 2°. For these phases the tensor and the spin orbit potentials vanish and they depend strongly on the central, velocity dependent and spin-spin forces and thus sense the inner regions of potential as compared to some of the other phases given here. The meaning of this comparison is lost for F- and G- waves where the only contribution to the phases is from the pion and vector mesons have little affect after a distance of about 1.8 fermis for the range of parameters used in all of our discussions. This aspect led to the question of whether these identical results were due to the fact that P- waves and higher were not affected by vector mesons (masses over 750 Mev) or due to the fact that these three interactions were the same in terms of their relativistic terms for the distances that generally affect the P- and D- waves. Therefore we carried out the calculations for a number of different sets of parameters for the uncoupled states with all three vector interactions and realized that our conclusions were quite general and that in all of the cases tried, we reached the same result, i.e., the relativistic terms of all three interactions are the same for P waves and higher with the exception of $^1P_1$ and there are consistently differing results for $^1D_2$ also, although by only a small amount. This suggests that the three vector interactions are likely to give different results for S-waves. There are other reasons besides $^1P_1$ and $^1D_2$ which also support this observation. One of the important ones among them is the fact that the Kemmer vector interaction has quite singular terms in its Pauli form. Even with the introduction of the modified Yukawa form with Green's subtractive meson, it remains singular and the S- waves, even in the Born approximation, can not be calculated. This difficulty could probably be removed if one studies this problem in the light of Green's higher derivative Lagrangians (62, 63, 64). Another difficulty that is connected with this interaction is that the meson compton wavelength occurs in the interaction in the last term. Thus, the way in which a cut-off should be applied to this interaction is not clear. A more conclusive statement about these problems lies in furthur theoretical studies and for our purposes we took the phenomenological view point for this aspect and used the modified Yukawa form for J occuring in (VI-3.2), without worrying about the different form of the Diracian interaction that may arise in this way. These problems will affect the S- waves which are not presented here because both the Born approximation and OBEP become very uncertain for these phases.

To remove the doubt about the question of whether vector mesons affect the P- and D- waves, we consider $^1P_1$, $^1D_2$, $^3P_1$, and $^3D_2$ phases for the parameters (1) of Table 3 at 310 Mev lab energy. If we also look at a typical computer output given in Appendix B, we can notice that by adding the phases for each meson in a vertical column, we can get a rough estimate of the contribution by that meson to a particular phase at the given energy. This was the advantage of our Born approximation where individual potential terms or meson contributions could be estimated approximately and relatively easily. But for large phases, we should really account for the fact that $\tan\delta \neq \tan(\sum_i \delta_i)$ But this certainly gives a rough estimate of the contribution from each meson. For the four cases discussed, the contributions of $(\omega + \rho)$ vector mesons are more than -25° for $^3P_1$, 7° for $^3D_2$, 20° for $^1P_1$, and -2.5° for $^1D_2$ for the interaction (VI-3.1). The phases as calculated from all three vector interactions at the same energy do not differ from each other by more than 3° for $^3P_1$, 2° for $^3D_2$, 10° for $^1P_1$ and 2° for $^1D_2$. This result therefore decisively proves that the vector mesons do affect the P– and D–waves and that the relativistic terms of the three vector interactions under consideration give almost the same results for the P– waves and higher. Thus any furthur study on vector mesons for S– waves should also resolve the questions connected with the cut-off, the OBEP, and also as to which one of the three field theoretically derived forms should be used.

Section - 4 Comparison Of Direct And Effective Mass Methods With Born Phases

We discussed in Chapter IV two different methods of treating the velocity dependent potentials in the Born approximation. The direct method employed the use of Bessel's equation and the recurrence relations among the Bessel functions for treating the first and second derivatives occuring in the potential. The effective mass method, however, made use of a transformation that eliminated the first derivative while the potential terms with the second derivative were grouped together with the kinetic energy terms. This gave rise to a factor $(1+\phi)$ in the denominator of the effective potential where

$$\phi = \phi^{(0)} + (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\, \phi^{(1)} \qquad (\text{VI}-4.1)$$

and

$$\phi^{(i)} = \frac{J_s^{(i)} + J_v^{(i)}}{M r^2}, \quad i = 0, 1. \qquad (\text{VI}-4.2)$$

For the iso-triplets, $\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} = 1$ and therefore the denominator term $(1+\phi)$ can not blow up for T = 1 state potentials. However, if the isovector, scalar, and vector mesons (ρ_s and ρ) are important then, since we have for iso-singlets $\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)} = -3$, the quantity $(1+\phi)$ can be zero or even negative. Thus the effective potential can blow up for

T = 1 states if ($\omega + \omega_s$) and ($\rho + \rho_s$) play the same part in nucleon-nucleon forces. Since there is no a priori reason to believe that they do not, the effective mass method of treating velocity dependence in the Born approximation has the inherent weakness that for certain feasible values of parameters, the method breaks down. The exact treatment of the Schrödinger equation also employs the same method, hence it also runs into the same problem. We have also shown that the non-locality and the velocity dependence to the quadratic order in momentum can be shown to be equivalent. Thus we can change our velocity dependent form of Schrödinger type equation into an integro-differential equation which can then be solved by standard methods for integral equations. This difficulty arises from a mathematical transformation and grouping of terms and if this can be avoided in the integral equation methods, then velocity dependence might be treated by determinantal methods of the type used for the solutions of Fredholm or Volterra type of integral equations. This type of integro-differential equation occurs in the cascade theory of cosmic ray showers also. No attempt is made here to solve it by those methods.

However, it seemed of interest to compare the two methods and we carried out a calculation for the uncoupled states using both methods for the constants of Table 3 and parameters $g_\omega^2 = g_{\omega_s}^2 = 3g_\pi^2$, $\lambda_\omega = \lambda_{\omega_s} = 1600$ Mev.,

$g_\rho^2 = g_{\rho_s}^2 = g_\pi^2/2$, $\lambda_\rho = \lambda_{\rho_s} = 1600$ Mev.,

without the η meson and $\lambda_\pi = 750$ Mev. in the notation of the Table 3. This set does not make the effective mass negative and so both the methods are valid for these parameters. The results are shown in Table 5 for E_{lab} = 50 and 310 Mev.

TABLE 5

Comparison of Direct and Effective Mass Methods

Phases	50 Mev Direct	50 Mev Effective	310 Mev Direct	310 Mev Effective
$^3P_0$	-11.1	-0.10	-78.3	-61.5
$^3P_1$	-22.6	-19.1	-71.3	-58.3
$^1P_1$	-8.91	-10.3	-11.7	-18.7
$^3D_2$	7.20	7.16	23.5	22.1
$^1D_2$	0.92	0.82	-6.49	-7.94
$^3F_3$	-0.49	-0.49	-7.68	-7.95

We see that for P— waves the phases are different and for D- waves the differences become smaller. The ($1+\phi$) factor affects only up to D-waves and quickly reaches the value unity in a purely relativistic model and so we expect the results to be the same for higher waves and they indeed are. Since the explicit effect of velocity dependence and other terms and mesons can not be seen in this method [because ($1+\phi$) occurs with every term and some terms come from the transformation] it is possible that these differences are due to different effective potentials in the two cases. However the differences are not extremely large for $\mathcal{P}-$ and $\mathcal{D}-$waves. If we take the parameters (1) of Table 3, we see that the effective potential blows up and if the potential is put to zero up to this point where it blows up (in this case, 0.67 fermi), the phases take on unphysical values. For example $^3D_2$ phase normally +25 to +30° now becomes -85° at 310 Mev and this is due to a chopping off of the potential. In general the T = 0 phases start becoming bad as the parameters reach the limit such that $\phi = -1/2$. But T = 1 phases are still reliable. Thus we have seen the limitations of the effective mass method.

<u>Effect of the Heavier ϕ Meson</u>

We decided to look at the total effect of heavy isoscalar vector ϕ meson (1020 Mev) which in SU_3 theory is generally mixed with the ω and various accounts of their mixing are important from that point of view. We do not expect to see a considerable influence on our phases because it will affect $S-$waves mostly but its effects on $\mathcal{P}-$waves may be noticeable. However we did not confine ourselves to a purely relativistic model for this case since we wanted to see its total effect including its static term. We thus do not have a scalar meson corresponding to ϕ in seeing its influence. The parameters (1) and (15) of Table 3 together with $g_\phi^2 = g_\pi^2$ and $\lambda_\phi = 2000$ Mev. in the previous notation are shown with and without ϕ in Table 6.

TABLE 6

Effects of ϕ — Vector Meson

PARAMETERS	(1) of Table 3		(15) of Table 3	
Phases	Without ϕ	With ϕ	Without ϕ	With ϕ
$^3P_0$	-58.0	-70.0	-75.9	-78.9
$^3P_1$	-66.9	-71.1	-68.4	-72.1
$^1P_1$	30.1	14.9	-33.3	-44.2
$^3D_2$	38.0	36.1	13.2	11.8
$^1D_2$	-0.83	-2.53	-5.27	-6.95

$E_{lab} = 310$ Mev.

Hence we see that ϕ meson creates a small total influence and its relativistic terms are not too crucial for our discussion.

CHAPTER VII

ALTERNATIVE METHODS OF THE REDUCTION AND EXPLORATORY DIRECTIONS

Section 1 - Alternative Methods of the Reduction to Large Components.

In this chapter we report the work which was either done by the author during the last stages (Section 1) or for which no definite conclusions have been reached (Section 2) as yet. We now give an alternative derivation of the method of reduction to large components which has the mathematical elegance that the momentum does not appear in the binomial expansions in the denominator. Also, the four component form of the Dirac equation is simple to deal with. In this method we go to the sums and differences of φ, ψ, χ_1 and χ_2 (notation of Chapter III), and wish to derive the same result under the similar approximations as done in the Chapter III. We start with the equations (III-2.11) and by defining (Green unpublished notes 1949)

$$\begin{aligned} \psi_s &= \varphi + \psi \\ \psi_d &= \psi - \varphi \\ \chi_s &= \chi_1 + \chi_2 \\ \chi_d &= \chi_1 - \chi_2 \end{aligned} \qquad (\text{VII}-1.1)$$

and

$$V_1 = V_a + V_b, \quad V_2 = V_a - V_b, \quad V_3 = V_c + V_d$$
$$V_4 = V_c - V_d, \qquad (\text{VII}-1.2)$$

together with

$$P_s = P_1 + P_2 = c\,(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) \cdot \vec{p},$$
$$P_d = P_1 - P_2 = c\,(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \cdot \vec{p}, \qquad (\text{VII}-1.3)$$

We obtain from (III-2.11) the Dirac equation in component form

$$\begin{aligned} (E-V_1)\psi_s - 2Mc^2\psi_d + P_s\chi_s &= 0 \\ -2Mc^2\psi_s - (E-V_2)\psi_d + P_d\chi_d &= 0 \\ P_s\psi_s + (E-V_3)\chi_s &= 0 \\ P_d\psi_d + (E-V_4)\chi_d &= 0 \end{aligned} \qquad (\text{VII}-1.4)$$

therefore

$$\begin{aligned} \chi_s &= -(E-V_3)^{-1} P_s\psi_s \\ \chi_d &= -(E-V_4)^{-1} P_d\psi_d \end{aligned} \qquad (\text{VII}-1.5)$$

and substitution in first two equations of (VII-1.4) yields two equations in ψ_s and ψ_d. Thus we conclude that by going to sums and differences we have obtained the same result in equivalent approximations.

$$\begin{aligned} \left[(E-V_1) - P_s(E-V_3)^{-1}P_s\right]\psi_s - 2Mc^2\psi_d &= 0 \\ -2Mc^2\psi_s + \left[(E-V_2) - P_d(E-V_4)^{-1}P_d\right]\psi_d &= 0 \end{aligned} \qquad (\text{VII}-1.6)$$

Hence $(c=1)$

$$\psi_s = \frac{1}{2M}\left[(E-V_2) - P_d(E-V_4)^{-1}P_d\right]\psi_d, \quad (\text{VII}-1.7)$$

substituting in (VII-1.6)

$$\left\{\left[(E-V_1) - P_s(E-V_3)^{-1}P_s\right]\left(\frac{1}{2M}\right)\left[(E-V_2) - P_d(E-V_4)^{-1}P_d\right] - 2M\right\}\psi_d = 0, \quad (\text{VII}-1.8)$$

or

$$\left\{\left[(E-V_1) - P_s(E-V_3)^{-1}P_s\right]\left(\frac{1}{2M}\right)\left[(E-V_2) - P_d(E-V_4)^{-1}P_d\right] - 2M\right\}\psi_d = 0. \quad (\text{VII-1.8})$$

Also from (VII-1.6)

$$\psi_d = \frac{1}{2M}\left[(E-V_1) - P_s(E-V_3)^{-1}P_s\right]\psi_s\,,$$

therefore

$$\left\{\frac{1}{2M}\left[(E-V_2) - P_d(E-V_4)^{-1}P_d\right]\left[(E-V_1) - P_s(E-V_3)^{-1}P_s\right] - 2M\right\}\psi_s = 0. \quad \begin{matrix}(a)\end{matrix}\ (\text{VII-1.9})$$

But as noted in 1958 by Breit and recently by Sawada

$$P_s\, P_d \equiv 0 \qquad (\text{VII-1.10})$$

Therefore

$$\left[-2M + \frac{1}{2M}A + \frac{1}{2M}B\right]\psi_d = 0, \quad (\text{VII-1.11}) \quad (a)$$

where

$$A = (E-V_1)(E-V_2) - E\left[P_s\frac{1}{E-V_3}P_s + P_d\frac{1}{E-V_4}P_d\right] \quad (b)$$

$$B = \left[P_s\frac{1}{E-V_3}P_s V_2 + V_1 P_d\frac{1}{E-V_4}P_d\right] \quad (c).$$

Similarly from (VII-1.9a)

$$\left\{\frac{1}{2M}\left[(E-V_1)(E-V_2) - E\left\{P_s\frac{1}{E-V_3}P_s + P_d\frac{1}{E-V_4}P_d\right\} + P_d\frac{1}{E-V_4}P_d V_1 + V_2 P_s\frac{1}{E-V_3}P_s\right] - 2M\right\}\psi_s = 0. \quad (a)$$

Let

$$G = \left\{V_2 P_s\frac{1}{E-V_3}P_s + P_d\frac{1}{E-V_4}P_d V_1\right\}, \quad (b)$$

$$D = \left\{-2M + \frac{1}{2M}A\right\}, \quad (c)\ (\text{VII-1.12})$$

Thus both the equations can be written as

$$\left[D + \frac{1}{2M}G\right]\psi_s = 0, \quad (a)$$

$$\left[D + \frac{1}{2M}B\right]\psi_d = 0. \quad (b) \quad (\text{VII-1.13})$$

These two equations in ψ_s and ψ_d were obtained by Sawada and Green and are the basis of their exact relativistic study of the two particle Dirac equation(61). We wish to use their starting point as a basis for our approximate treatment. Rearranging the terms with

$$\frac{1}{E-V_3} = \frac{1}{E}\left(1+\frac{V_3}{E-V_3}\right), \quad \frac{1}{E-V_4} = \frac{1}{E}\left(1+\frac{V_4}{E-V_4}\right) \qquad (\text{VII}-1.14)$$

$$A = (E-V_1)(E-V_2) - \left[P_s^2 + P_d^2 + P_s\frac{V_3}{E-V_3}P_s + P_d\frac{V_4}{E-V_4}P_d\right] \quad (a)$$

$$B = \frac{1}{E}\left(P_s^2V_2 + V_1P_d^2\right) + \frac{1}{E}\left(P_s\frac{V_3}{E-V_3}P_sV_2 + V_1P_d\frac{V_4}{E-V_4}P_d\right) \quad (b) \qquad (\text{VII}-1.15)$$

But we are neglecting V^2 terms, hence

$$B = \frac{1}{E}\left(P_s^2V_2 + V_1P_d^2\right) \qquad (a)$$

$$C = \frac{1}{E}\left(V_2P_s^2 + P_d^2V_1\right) \qquad (b)$$

$$D = 2M\left(\frac{A}{4M^2} - 1\right) \qquad (c)$$

Let

$$F = \left(\frac{A}{4M^2} - 1\right) \qquad (d) \qquad (\text{VII}-1.16)$$

Then

$$\left[F + \frac{1}{4M^2}C\right]\psi_s = 0$$

or

$$\left[F + \frac{1}{4M^2}B\right]\psi_d = 0 \qquad (\text{VII}-1.17)$$

$$\left[F + \frac{C}{4M^2}\right](\varphi + \psi) = 0$$

$$\left[F + \frac{B}{4M^2}\right](\psi - \varphi) = 0 \qquad (\text{VII}-1.18)$$

Adding and subtracting

$$\left[2F + \frac{B+C}{4M^2}\right]\psi + \left(\frac{C-B}{4M^2}\right)\varphi = 0 \quad (a)$$

$$\left[2F + \frac{B+C}{4M^2}\right]\varphi + \left(\frac{C-B}{4M^2}\right)\psi = 0 \quad (b) \qquad (\text{VII}-1.19)$$

Writing explicitly

$$\Big[\frac{1}{4M^2}\Big((E-V_1)(E-V_2)-\Big\{P_s^2+P_d^2+P_s\frac{V_3}{E-V_3}P_s+P_d\frac{V_4}{E-V_4}P_d\Big\}\Big)$$

$$-1+\frac{1}{8M^2E}\left(V_2P_s^2+P_s^2V_2+V_1P_d^2+P_d^2V_1\right)\Big]\varphi+\frac{1}{8M^2}\Big[V_2P_s^2-$$

$$-P_s^2V_2+P_d^2V_1-V_1P_d^2\Big]\psi=0 \qquad (\text{VII}-1.20)$$

We know that the term with (-1) is large as compared to the terms with ($1/M^2$) hence equation (VII-1.20) is an equation in which φ is the small component and should be expressed in terms of ψ from (VII-1.19b). Also, we see that equation (VII-1.19a) has the φ term correct up to $(1/M^2)$ order already and so even approximate expression (first order) for φ in terms of ψ will suffice to make (VII-1.19a) an equation in large component ψ.

Really we have

$$\varphi \approx \frac{1}{8M^2}\left[V_2P_s^2-P_s^2V_2+P_d^2V_1-V_1P_d^2\right]\psi \qquad (\text{VII}-1.21)$$

This term in second part of (VII-1.19a) would give square of potential terms only but we neglect that part, and our equation in large component is

$$\Big[\!\Big[\frac{1}{4M^2}\left[4M^2+4WM+W^2-(W+2M)(V_1+V_2)+V_1V_2-4p^2\right]-1-$$

$$-\frac{1}{4M^2}\Big\{P_s\frac{V_3}{2M+(W-V_3)}P_s+P_d\frac{V_4}{2M+(W-V_4)}P_d\Big\}+$$

$$+\frac{1}{8M^2(2M+W)}\left(V_2P_s^2+P_s^2V_2+V_1P_d^2+P_d^2V_1\right)\Big]\!\Big] \qquad (\text{VII}-1.22)$$

or

$$\Big[W\Big(1-\frac{V_a}{2M}\Big)-V_a-\frac{p^2}{M}-\frac{1}{8M^2}\Big\{P_sV_3P_s+P_dV_4P_d\Big\}+$$

$$+\frac{1}{16M^2}\left(V_2P_s^2+P_s^2V_2+V_1P_d^2+P_d^2V_1\right)\Big]\psi=0 \qquad (\text{VII}-1.23)$$

Since $W = p^2/M + V_a$,

to lowest order from above, therefore

$$W^2/4M = p^4/4M^3+(p^2V_a+V_ap^2)/4M^2+V_a^2/4M.$$

NORMALIZATION - Neglecting potential (for consistency of two methods)

$$\chi_s = -\frac{1}{2M} P_s \psi, \qquad \chi_d = -\frac{1}{2M} P_d \psi$$

$$\chi_1 = -\frac{1}{2M} (\vec\sigma^{(1)}\cdot\vec p)\psi, \quad \chi_2 = +\frac{1}{2M}(\vec\sigma^{(2)}\cdot\vec p)\psi$$

$$\therefore \quad \iint (\psi^+\psi + \varphi^+\varphi + \chi_1^+\chi_1 + \chi_2^+\chi_2)\,d\tau_1 d\tau_2 = \iint \psi^+\left(1+\frac{p^2}{2M^2}\right)\psi\,d\tau_1 d\tau_2$$

or

$$\Psi = (1 + p^2/4M^2)\psi = (1+\nu/2)\psi \approx e^{\nu/2}\psi. \quad \text{(VII-1.24)}$$

Then

$$W(1+\nu')\psi = [\;]\,\psi$$

$$\Rightarrow \quad W e^{\nu'}\psi = [\;]\, e^{-\nu/2}\Psi$$

or

$$W e^{-\nu'} e^{\nu'}\psi = e^{-\nu'}[\;]\,e^{-\nu/2}\Psi$$

or

$$W\Psi = \left\{(1+\nu/2-\nu')[\;] - [\;]\,\nu/2\right\}\Psi$$

(VII-1.25)

we get on simplification

$$\Big[W - \frac{p^2}{M} - V_a - \frac{1}{4M^2}\{p^2 V_a + V_a p^2\} - \frac{1}{8M^2}\{P_s V_3 P_s + P_d V_4 P_d\} + \frac{1}{16M^2}\{V_a(P_s^2+P_d^2) + (P_s^2+P_d^2)V_a - V_b(P_s^2-P_d^2) - (P_s^2-P_d^2)V_b\} + \frac{p^4}{4M^3} + \frac{1}{4M^2}(p^2 V_a + V_a p^2) + \frac{V_a^2}{4M}\Big]\Psi = 0,$$

(VII-1.26)

with

$$P_s^2 + P_d^2 = 4p^2, \qquad P_s^2 - P_d^2 = -4(\vec\sigma^{(1)}\cdot\vec p)(\vec\sigma^{(2)}\cdot\vec p),$$

and retaining only linear terms we get

$$\Big[W - V_a - \frac{p^2}{M} - \frac{1}{4M^2}\{V_a p^2 + p^2 V_a\} - \frac{1}{8M^2}\{P_s V_3 P_s + P_d V_4 P_d\} + \frac{1}{4M^2}\{V_a p^2 + p^2 V_a + (\vec\sigma^{(1)}\cdot\vec p)(\vec\sigma^{(2)}\cdot\vec p)V_b + V_b(\vec\sigma^{(1)}\cdot\vec p)(\vec\sigma^{(2)}\cdot\vec p) + \frac{p^4}{4M^3} + \frac{1}{4M^2}(p^2 V_a + V_a p^2)\Big]\Psi = 0$$

(VII-1.27)

$$\therefore \quad V^{Pauli} = \Big[V_a - \frac{1}{4M^2} \Big\{ (\vec{\sigma}^{(1)} \cdot \vec{p})(\vec{\sigma}^{(2)} \cdot \vec{p}) V_b + V_b (\vec{\sigma}^{(1)} \cdot \vec{p}) \cdot$$

$$\cdot (\vec{\sigma}^{(2)} \cdot \vec{p}) + p^2 V_a + V_a p^2 \Big\} + \frac{1}{8M^2} \Big\{ P_s (V_c + V_d) P_s +$$

$$+ P_d (V_c - V_d) P_d \Big\} \Big] \qquad (VII-1.28)$$

Thus (VII-1.28) is the same as (III-3.15). Thus we conclude that by going to sums and differences we have obtained the same result in equivalent approximations.

Section 2 - Exploratory Directions-Two Boson Exchange Contributions

As mentioned in Chapter II, the two meson processes in the Fock formalism could be accounted for by carrying the series to one higher order and with various improvements and iterations Green (1) derived the Two-Boson-Exchange-Potential in which various relations used resulted in the quadratic dependence on the coupling constant. The details about various physical processes and normalization are currently being studied by Chern and Green (9) and final conclusions are yet to be reached. However, for our purposes we took the form of Two-Dion-Exchange-Potential as given by Green (1) (Eq. II - 2.22) and proceeded to reduce it to the Diracian form. Then an attempt was made to use the method of reduction and the interaction was brought to the Schrodinger-Pauli form. There are quite a few calculational steps involved which will not be provided here, but if the reader is familiar with Chapter III, there will be little difficulty in deriving them. From (II - 2.22) and (II - 2.16) with $(\beta\gamma_5)^{(\lambda)}$ for $\beta^{(\lambda)}$, we obtain

$$^4U = \int \mathcal{G}^+ \left[H_p , \mathcal{G} \right] / (\hbar\omega)^2 \, dk^{\prime} \qquad (\text{VII}-2.1)$$

$$= \frac{g^2}{4\pi^2} \Bigg[\text{Self Energy} + i \Big[\frac{(I_1 - I_2)}{r} \Big\{ \Gamma_5^{(1)} \Big(\alpha\Gamma_5 \cdot \frac{\vec{r}}{r} \Big)^{(2)} + \Gamma_5^{(2)} \Big(\vec{\alpha}\Gamma_5 \cdot \frac{\vec{r}}{r} \Big)^{(1)} \Big\} + I_1 \Big\{ \Gamma_5^{(1)} \big(\Gamma_5 \vec{\alpha} \cdot \vec{\nabla} \big)^{(2)} + \Gamma_5^{(2)} \big(\Gamma_5 \vec{\alpha} \cdot \vec{\nabla} \big)^{(1)} \Big\} \Big] + \frac{2M\kappa}{\hbar} I_1 \Big\{ \Gamma_5^{(1)} \gamma_5^{(2)} + \Gamma_5^{(2)} \gamma_5^{(1)} \Big\} \Bigg] \qquad (\text{VII}-2.2)$$

where

$$\Gamma_5 = \beta\gamma_5$$

$$I_1 = \int_0^\infty \frac{(\sin z)\, z\, dz}{(z^2+a^2)^{3/2}}\ , \quad I_2 = \int_0^\infty \frac{z^2 \cos z\, dz}{(z^2+a^2)^{3/2}}$$

and

$$z = kr, \qquad a = kr \qquad \text{(VII-2.3)}$$

To reduce this Diracian form to the Pauli form we take our wave function Ω from Chapter III and develop the algebra analogous to the one given there and obtain

$$\begin{aligned} {}^4U\Omega = \frac{g^2}{4\pi^2}\Big[(\text{self energy})\Omega - i\left(\frac{I_1-I_2}{r}\right)\Big\{\left(\frac{\vec\sigma^{(1)}\cdot\vec r}{r}\right)\begin{pmatrix}\chi_1 & -\varphi\\ -\psi & \chi_2\end{pmatrix} - \\ -\left(\frac{\vec\sigma^{(2)}\cdot\vec r}{r}\right)\begin{pmatrix}\chi_2 & -\psi\\ -\varphi & \chi_1\end{pmatrix}\Big\} + iI_1\Big\{(\vec\sigma\cdot\vec\nabla)\begin{pmatrix}\chi_1 & -\varphi\\ -\psi & \chi_2\end{pmatrix} + \\ + (\vec\sigma\cdot\vec\nabla)^{(2)}\begin{pmatrix}\chi_2 & -\psi\\ -\varphi & \chi_1\end{pmatrix}\Big\} + \frac{4Mc}{\hbar} I_1 \begin{pmatrix}-\psi & 0\\ 0 & \varphi\end{pmatrix}\Big] \end{aligned} \qquad \text{(VII-2.4)}$$

We substitute this result together with the one pion contribution and obtain after neglecting the self energy terms, various higher order terms in $(1/M)$ and J, while the normalization is being carried only to the previous order

$$\begin{aligned} V_{PS}^{(2+4)}\,\Psi = \frac{\hbar^2}{12M^2c^2}\Big[r\,\frac{d}{dr}\Big(\frac{1}{r}\frac{d}{dr}(J+J_I)\Big) S_{12} + \\ + \langle\nabla^2(J+J_I)\rangle\,(\vec\sigma^{(1)}\cdot\vec\sigma^{(2)})\Big] + \frac{\hbar^2}{4M^2c^2}\Big[(\vec\sigma^{(1)}\cdot\langle\vec\nabla J\rangle)\cdot \\ \cdot(\vec\sigma^{(2)}\cdot\langle\vec\nabla\Psi\rangle) + (\vec\sigma^{(2)}\cdot\langle\vec\nabla J\rangle)(\vec\sigma^{(1)}\cdot\langle\vec\nabla\Psi\rangle)\Big] + O\left(\frac{1}{M^3}\right) \end{aligned}$$
$$\text{(VII-2.5)}$$

where

$$J = \frac{g^2}{4\pi}\, e^{-kr}/r\ , \qquad J_I = 4\left(\frac{Mc}{\hbar}\right)\frac{g^2}{4\pi^2}\,K_0(kr). \qquad \text{(VII-2.6)}$$

The function K_0 is the modified Bessel function of zero order. In order to compare these effects as compared to OPEP we tried to determine (J_I/J) and if their coupling constants were taken to be the same (which may not be true as the 2π -exchange would be partially accounted for by resonances also) then it was found that J_I was ten times larger as compared to J at a distance of about 1.5 fermis and five times as large at 0.5 fermis. This indicates therefore that 2π -processes might be very important in the intermediate range where we are looking for a correct analysis for ${}^1D_2$ and ${}^1P_1$ phases, etc. Thus, the 2π -exchanges should be also studied and the questions about their couplings should also be decided.

Section 3 - Discussion on the Methods of Reduction

In the derivation of the Schrodinger-Pauli forms of various interactions in Chapter III we met quite a few terms which were quadratic in the potentials. These terms partly come from the method of reduction to large components and partly from the structure of two particle Dirac equation whenever some of the components are expressed in terms of the others. Breit (56), Green (53) and recently Sawada (66) have noticed these terms in their reductions of the Dirac equation, even in exactly relativistic treatments. An extensive discussion about these can be found in Breit's (56) paper where he relates this term to the soft core. But a very clear conclusion about the results from such a core is yet to come. We see that only the pseudoscalar and vector mesons contribute to this term. We call this the non-linear term.

$$V_{NL} = J^2 / 4Mc^2 \qquad (\text{VII}-3.1)$$

The explicit effects of this term can also be calculated in our computer programs where we have an input variable ANL which decides whether the non-linear term in the interactions is to be considered or not. With the coupling constant 14.7 taken for the vector and pseudoscalar mesons, we found the contributions from this term to be about ten times as large as due to various other terms. However, we do not yet understand the origin and the physical interpretations of this term and further work will shed more light as to whether this term would survive after the contributions from the Two-Boson-Exchange-Potentials.

We have seen in the last chapter that the contributions from 2π-exchange are also large. Thus these two effects (i.e. 2π - exchange and $J^2/4M\lambda^2$ terms) are quite important if the coupling associated with the former is comparable with the latter, especially for the intermediate range where we want to be more certain about our knowledge of the nucleon-nucleon interaction in order to be able to fit the P and the D waves. We assume for most purposes that the already established OPEP and the OBEP are the most important contributions for non-relativistic energies and this indeed is so because the coupling of the and its contribution for the higher waves are reasonably well established. However, for the relativistic problems the ideas about the OBEP alone being the principal contribution may not hold.

Another area of interest is to explore various methods of reduction to the large component. In the literature the only other reference found is that of Chraplyvy (58) and the previous paper by him referred in (58). There is also a complicated reduction of the two particle Dirac equation carried out. The Hamiltonian is reduced by using various matrix product properties in to even-even, even-odd, odd-even and odd-odd parts as defined in details in the reference (58). It is not clear to us whether the two methods of reduction are equivalent or not since most of the terms referred by him are the same as those obtained by Breit's method of reduction to the large components. However, a detailed comparison to orders higher than $1/M^2$ might give significant results. We have also obtained an agreement for our purely relativistic model for its reduced terms using Breit's (54) results on Pauli forms of the vector interactions treated there. If from (51) we take

$$H = -c(\vec{\alpha}^{(1)}\cdot\vec{p}^{(1)}) - c(\vec{\alpha}^{(2)}\cdot\vec{p}^{(2)}) - (\beta^{(1)}+\beta^{(2)})Mc^2 - J + Q \qquad (\text{VII}-3.2)$$

and

$$H = -c(\vec{\alpha}^{(1)}\cdot\vec{p}^{(1)}) - c(\vec{\alpha}^{(2)}\cdot\vec{p}^{(2)}) - (\beta^{(1)}+\beta^{(2)})Mc^2 - \beta^{(1)}\beta^{(2)}J + Q \qquad (\text{VII}-3.3)$$

as given by Eqs. (16.1) and (16.2) of (51) with Q defined by (16.3)

$$Q = \frac{1}{2}(\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)})J - \frac{1}{2}\frac{(\vec{\alpha}^{(1)}\cdot\vec{r})(\vec{\alpha}^{(2)}\cdot\vec{r})}{r}\frac{dJ}{dr} \qquad (\text{VII}-3.4)$$

then proper substitutions of $-2J$ and $+2J$ in the Dirac equations for these Hamiltonians and simplifications yield to

$$\left\{E + c(\vec{\alpha}^{(1)}\cdot\vec{p}^{(1)}) + c(\vec{\alpha}^{(2)}\cdot\vec{p}^{(2)}) + (\beta^{(1)}+\beta^{(2)})Mc^2 - (1 - \vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)} - \beta^{(1)}\beta^{(2)})J\right\}\psi = 0, \qquad (\text{VII}-3.5)$$

and therefore the corresponding Pauli forms can be obtained by using the results on the Pauli forms given in (51). The results agree with those found in Chapter III or in the Section 1 of this chapter, thus providing a check on the method by which the Dirac equation is reduced to Schrödinger-Pauli form at least to the order v^2/c^2. The higher orders and normalization connected with them should be carefully examined but a relativistically correct treatment should serve the purpose better.

It would also be appropriate here to understand the structure of the equations obtained in these methods of reduction to the large components. The conclusions obtained are not unexpected but clear our understanding of the expansions involved in approximations that are made in obtaining the Schrodinger-Pauli form.

We obtained equation (VII-1.8) in ψ_s and equation (VII-1.9) in ψ_d which can be written as (Sawada, 66)

$$(\theta_1 - 2M)\,\psi + [\theta_1 + 2M]\,\varphi = 0 \qquad \text{(VII-3.6)}$$

and

$$(\theta_2 - 2M)\,\psi - [\theta_2 + 2M]\,\varphi = 0, \qquad \text{(VII-3.7)}$$

where θ_1 and θ_2 are defined from (VII-1.8) and (VII-1.9). The expressions for φ obtained from these two equations are

$$\varphi = \frac{1}{(\theta_2 + 2M)} (\theta_2 - 2M) \psi \qquad (VII-3.8)$$

and

$$\varphi = -\frac{1}{(\theta_1 + 2M)} (\theta_1 - 2M) \psi \qquad (VII-3.9)$$

If we require that these two equations in φ are the same then

$$\left[\frac{1}{(\theta_2+2M)} (\theta_2 - 2M) + \frac{1}{(\theta_1+2M)} (\theta_1 - 2M) \right] \psi = 0 \qquad (VII-3.10)$$

should be satisfied. But if (VII-1.8) and (VII-1.9) are reduced to a single equation in either ψ_s or ψ_d from substitution of one in terms of the other then the two forms obtained are

$$\left[(\theta_1 - 2M) + (\theta_1 + 2M) \frac{1}{(\theta_2 - 2M)} (\theta_2 - 2M) \right] \psi = 0 \qquad (VII-3.11)$$

and

$$\left[(\theta_2 - 2M) + (\theta_2 + 2M) \frac{1}{(\theta_1 + 2M)} (\theta_1 - 2M) \right] \psi = 0 \qquad (VII-3.12)$$

If we multiply (VII-3.11) by $(\theta_1 + 2M)^{-1}$ from the left and (VII-3.12) by $(\theta_2 + 2M)^{-1}$ from the left then the equation (VII-3.10) is obtained.

This observation makes differently looking forms consistent before the reduction is made. But if different looking forms of these operator equations are used, then the orders of expansion should be kept in proper sequence to obtain the same results.

In chapter III we gave another method of the reduction to large components. There also we used an equation expressing

φ in terms of ψ . If we had used another equation for this substitution the we would have reached the similar conclusions as are obtained from the equations used in this chapter.

Thus in these chapters we have given the furthur work of what was reported by Green and Sharma (115) and the work includes some more recent aspects and the discussions on the vector mesons as discussed by Sharma , Rochleder and Green(114).

CHAPTER VIII

DISCUSSIONS

Thus an attempt has been made to understand the nucleon-nucleon (NN) interaction in terms of One-Boson-Exchange-Potential (OBEP) model and to account for NN elastic scattering phase-shifts in the Born approximation for non-relativistic energies (25 - 310 Mev) in the laboratory frame. Theoretically, the OBEP were derived by Green (1), the results being given in the Diracian form (Chapter II). The two particle Dirac equation was reduced to a Schrodinger-Pauli form using the method similar to Breit's method of reduction to large components (Chapter III). A general form of interaction was then obtained in the Pauli form which was applied to scalar, pseudoscalar and three different forms of vector meson exchange potentials. They all were put in to the general form

$$V^P = V_C + V_\nabla (\vec{r}\cdot\vec{\nabla}) + V_\Delta(\nabla^2) + V_{SS}\ (\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}) + \\ + V_{LS}\ (\vec{\ell}\cdot\vec{S}) + V_T\ (S_{12}) + V_{NL}\ ,$$

where the V's denoted the radial forms and velocity dependent features of the nucleon-nucleon force together with more established tensor, spin-spin and spin-orbit effects. The velocity dependence was studied in great detail to see its effects and the general formalism for phase-shift expressions with velocity dependent potentials were also presented (Chapter V). Special methods were required to deal with the Schrodinger-type equation with velocity dependence. The method of effective mass generally used in exact treatments was discussed. Another method in the Born-approximation was the direct-method which treated the velocity dependence directly in phase-shift integrals without going to an effective potential. The experimentally known mesons π, ω, ρ and η were used together with two postulated scalar mesons in Green's purely relativistic model (cancellation of static scalar-vector terms) as well as in broken-relativistic models and use was made of modified Yukawa form with a higher mean subtractive meson coming from Green's work on Lagrangian with higher derivatives in the field coordinator. This modified form successfully removed the singularities from the integrands and at the same time nicely parameterized the OBEP in the inner regions where various other processes could be important. The scalar meson masses are very important for the results presented in this study on phase shifts. Only phases higher than S-waves are considered. The Born phases are found to be quite reliable for D-waves and higher and could be corrected empirically for P-waves. However, some general information was obtained from P-wave Born phases. It was shown that couplings determined from the Born-approximation are usually smaller and therefore Born-approximation generally gives larger phases for P-waves. Comparisons are made for certain sets of parameters between exact and Born phases and the differences are large only for the P-waves. Coupled states are treated according to Stapp's (70) formalism and are also found to conform to these conclusions.

Three different vector meson interactions have been treated and the Born phases resulting from them indicate that their relativistic terms agree for P-waves and higher. Various theoretical questions should be resolved before treating the S-waves, for which these interactions are likely to differ. The effective mass method has been shown to fail if ρ meson parameters play an important role. The direct Born approximation method can help in that case and also the effects of various mesons and various terms in the interactions can be seen relatively explicitly in the Born approximation. The Born approximation thus seems to have given us an overall picture of various important and emerging phenomena in accounting for the nucleon-nucleon interaction. However, the definition of velocity dependent potentials and the validity of the OBEP become questionable at shorter distances where S and P waves might be affected. Multiple meson exchanges and the pair productions become quite important for distances less than 0.6 fermis or so and various definitions and the reduction procedures have to be examined. The quantity $J/M c^2$ may thus be no longer a small quantity for these distances. Therefore a relativistically correct approach to this problem is currently being taken by Sawada and Green (61). The results of this approach should clarify more the understanding of nucleon-nucleon force for still inner distances.

Another important area of approach should be to account for the inelastic processes like the Bion production. It is also worth noting that recently the attempts have begun to obtain the mesons as bound states of the nucleon-antinucleon system and then use these parameters for accounting the nucleon-nucleon scattering data also. Recent studies (61) are proceeding in the same direction.

A very crucial requirement at this stage is to come to a conclusion about the scalar mesons and to decide their "masses" of the positions of their resonances. It is likely that a scalar peak might be hidden under the ω or the ρ peaks and the scalar mass can radically change the phase shift results.

A few words must be said about the possibility that other interactions might also bridge the gap that seems to occur at the intermediate distances from 1.2 to 2.0 fermis. The sources of this might be in the derivative couplings of the ρ meson as used by Bryan and Scott (68) or in the Two-Boson-Exchange-Potentials. Another difficulty is regarding the $\frac{1}{4}(J^2/Mr^2)$ term as discussed by Breit (56) which should be properly accounted for, for exact treatments.

Thus we finally conclude that our understanding of the nucleon-nucleon interaction has increased by an order of magnitude due to the discovery of various vector and the pseudoscalar mesons. However, the need to establish a scalar entity (that gives proper attraction) still remains an open question. Our understanding breaks down for distances around 0.5 fermi or more and further work is needed to explore this region. The Born-approximation has been pushed far enough and should be corrected for S and P waves, however the advantages obtained from it are quite significant from the point of view of individual meson contributions or the terms. The treatment of the velocity dependent potentials can also be carried out in the Born-approximation where the effective mass method of exact treatment breaks down.

Appendix A

COMPUTER PROGRAMS

1. SPHERICAL BESSEL FUNCTIONS

```
      FUNCTION SFBESS (M,Z)
C     FUNCTION SFBESS(M,Z) USING ASCENDING SERIES
C     THIS IS A MORE REFINED CODE WHICH CALCULATES BESSEL FU
                                         NCTIONS MORE
C      CORRECTLY BY POLYNOMIAL APPROXIMATION IN THE RANGE O
                                         TO 1 FOR ITS
C      ARGUMENT.REF.HANDBOOK MATH FUNCTIONS NATL BUREAU OF S
                                         TANDARDS.
      DIMENSION VALUE(15)
      LIMIT=M+1
      IF (Z-1.0 ) 231,231,232
  231 CONST = 1.253314137
      CONSTZ = ( CONST/(SQRTF(Z)))
      ZJ = (Z/3.0)**2
    7 FM  =M
      U=FM+0.5
      ZU=(Z/2.0)**U
      FACTOR  =  ((CONST*ZU)/SQRTF(Z))
      ZZ= (Z/2.0)**2
      MULT =1
      MULT1 =1
      AMULT2  =1.0
      AMULT3  =1.0
      MA=M+1
      MM=  2*MA-1
      DO 237 I=1,MM,2
      MULT = MULT*I
  237 CONTINUE
      AMULT =  MULT
      GAMA = (((1.772453851)*AMULT)/(2.0**MA))
      TERM = 1.0/GAMA
      DO 238 KK= 1,30
      DO 239 I=1,KK
      AI  =  I
      AMULT2  =AMULT2*AI
  239 CONTINUE
      FACTL  =AMULT2
      MMM = M+1+KK
      MK1  =2*MMM-1
      DO 247 I =1,MK1,2
      AI  = I
      AMULT3  =AMULT3 *AI
  247 CONTINUE
      GAMA = (((1.772453851)*AMULT3)/(2.0**MMM))
      TERM  = TERM +((-ZZ)**KK/(FACTL*GAMA))
  238 CONTINUE
```

Appendix A continued

```
      VALUE (LIMIT)  = FACTOR * TERM
      GO  TO  4
232   VALUE(1)=SINF(Z)/Z
      VALUE(2)=SINF(Z)/Z**2.0-COSF(Z)/Z
      VALUE(3)=(3.0/Z**3.0-1.0/Z)*SINF(Z)-3.0/Z**2.0*COSF(Z)
      IF (LIMIT-3) 4, 4, 6
    6 K=3
    5 FK  =K-1
      VALUE(K+1) =  ((2.0*FK+1.0)/Z)* VALUE(K) - VALUE(K-1)
      K=K+1
      IF (LIMIT-K) 20, 20, 5
   20 SFBESS=VALUE(K)
   21 RETURN
    4 SFBESS=VALUE(LIMIT)
      GO TO 21
      END
```

2. WEDDLE'S RULE FOR NUMERICAL INTEGRATION

```
C      ---------- WEDDLE'S RULE FOR NUMERICAL INTEGRATION ---
                                            -------
C      NOTATION--
C         DELTA = WIDTH OF EACH INTERVAL.
C         N = THE NUMBER OF POINTS AT WHICH THE INTEGRAND IS
                                            TO BE
C       EVALUATED. N IS AN INTEGER AND MUST BE OF THE FORM 6*
                                            M+1, WHERE
C       M IS AN INTEGER.
C         Y = A 1-DIMENSIONAL ARRAY OF THE N VALUES OF THE IN
                                            TEGRAND.
C         VALUE = THE VALUE OF THE INTEGRAL.
C      NOTE OF CAUTION---
C              THE NUMBER OF INTERVALS MUST BE A MULTIPLE OF
                                            6 AND
C              HENCE N = 6*M+1.
       SUBROUTINE WEDDLE (DELTA, Y, N, VALUE)
                                                        W 01
       DIMENSION  Y(1201 )
       VALUE = 0.
                                                        W 03
       M = N - 6
                                                        W 04
         SUM1=0.0
        SUM2=0.0
        SUM3=0.0
        SUM4=0.0
       DO 15 I=1,M,6
                                                        W 005
        SUM1=SUM1+Y(I)
        SUM2=SUM2+Y(I+1)+Y(I+5)
          SUM3=SUM3+ Y(I+2)+Y(I+4)
        SUM4=SUM4+Y(I+3)
    15 CONTINUE
                                                        W 08
          VALUE=VALUE+82.0*SUM1+216.0*SUM2+27.0*SUM3+272.0*SU
                                            M4
        VALUE=DELTA*(VALUE+41.0*(Y(N)-Y(1)))/140.0
       RETURN
                                                        W 10
       END
                                                        W
```

3. UNCOUPLED BORN PHASES

```
C        UNCOUPLED BORN PHASES(DIRECT METHOD)
C       IMPROVED DIRECT UNMIXED BORN PHASES WITH CUT OFF
C       THIS CODE WILL CALCULATE PHASE SHIFTS FOR SINGLET AND
                                      TRIPLET
C       (UNMIXED) STATES OF NUCLEON-NUCLEON SYSTEM IN BORN APP
                                     ROXIMATION
C       THIS CODE WILL CALCULATE BORN PHASES BY DIRECT METHOD
                                     IN WHICH THE
C       OPERATORS OF VELOCITY DEPENDENT POTENTIAL ACT ON THE B
                                     ESSEL FUNCT'
C       IONS AND THEN THE RECURRENCE RELATIONS ARE USEDTO REMO
                                     VE FIRST AND
C       SECONDDERIVATIVES. THEN THE PHASE SHIFT EXPRESSIONS IN
                                     VOLVE ONLY
C       SIX  INTEGRALS A B C D E AND F.  INTEGRALS ARE EVALUAT
                                     ED USING
C       WEDDLE,S RULE. FUNCTION SFBESS IS NEEDE D  AND NO OF P
                                     OINTS OF
C       INTEGRATION MUST BE 6*M AND SO INPOINTSARE6M+1.
C       AMASS IS THE MASS OF MESON IN MEV   . GSQ IS THE COUPL
                                     ING CONSTANT
C       SS IS THE EXPECTATION VALUE OF (SIGMA1.SIGMA2) FOR THE
                                      STATE .
C        ALS IS THE SPIN ORBIT OPERATOR EXPECTATION VALUE FOR
                                     THE STATE
C       THAT IS (L.S).S12 IS THE TENSOR OPERATOR EXPECTATION V
                                     ALUE FOR THE
C       STATE. L.S AND S12 AVERAGE TO ZERO FOR SINGLET STATES.
C        ANL IS THE COEFFICIENT THAT MAKES NON LINEAR TERM ZER
                                     O OR TAKES
C       IT INTO ACCOUNT
C       J IS THE NO OF MESONS 12  OR 14 (FOR INCLUDING PHI) IN
                                     CLUDING CUT OFF
C       NBK CHARACTERISES  CHOICE OF CALCULATING BREIT,S AND K
                                     EMMER,S
C       VECTOR MESON INTERACTIONS   . IT CAN BE ZERO OR ONE.
C       IDENTIFICATIONS OF THREE VECTOR INTERACTIONS ARE MADE
                                     ACCORDING
C       TO THE NAMES OF THE PEOPLE WHO USED THEM HISTORICALLY.
        COMMON RO,RMAX,XN,UF,AK,FL,AF,BF,CF,DF,EF,SS,ALS,S12,A
                                     NL ,UFC    ,
     1  ELAB,BES ,FF ,HBF,HKF,NBK
        DIMENSIONW(20,20),GSQ(15),AMASS(15),U(15),A(15),B(15),
                                      C(15),D(15)
     1,E(15),G(15),DY(15),DC(15),DGRAD(15),DDELSQ(15),DSS(15
                                     ),DLS(15),DT
```

```
     2(15),  DNL(15)     ,CAPPA(15)    ,RY(15),RC(15),RGRAD(15
                                              ),RDELSQ(15)
     3 ,RSS(15),RLS(15),RT(15),RNL(15) ,BES(6,8,100)  ,F(15)
                                                  ,HB(15),
     4 RB(15),DB(15),HK(15),RK(15),DK(15),RGRADB(15),RDLSQB(
                                              15),
     5  DGRADB(15),DDLSQB(15)
      CALL BESTO
  18  READ INPUT TAPE 5,9,J    ,NBK
C     IF BREIT KEMMER NOT NEEDED NBK =0 OTHERWISE =1
 9    FORMAT(I2,I1)
 973  IF ( J-12) 967,967,966
966  READ INPUT TAPE 5,43,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 43   FORMAT(        12F6.0/2F6.0/12F6.0/2F6.0)
      GO TO 19
 967 READINPUT TAPE 5,968,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 968  FORMAT(        12F6.0/       12F6.0)
 19   WRITE OUT PUT TAPE 6,1150
1150 FORMAT( 1H1, 25H  DIRECT    BORN PHASES         )
 971 READ INPUT TAPE 5,972,ELAB,FL,T,SS,ALS,S12,ANL,R0,RMAX
                                              ,XN
 972 FORMAT(10F6.0)
      IF (ELAB- 1000.0) 969,973,973
969  IF (SS-1.0 )31,32,31
  32 WRITE OUTPUT T  APE 6,115
 115 FORMAT(     5X,30H THIS IS A TRIPLET SPIN STATE
                                              )
      GO TO 199
  31 IF (SS+ 3.0) 41,42,41
41      WRITE OUT PUT TAPE 6,125
 125 FORMAT (5X    ,20H WRONG INPUT DATA           )
      CALL EXIT
  42 WRITE OUTPUT  TAPE 6,135
 135 FORMAT(     5X, 30H THIS IS A SINGLET SPIN STATE
                                              )
 199  ENERGY = ELAB
      WRITE OUTPUT TAPE 6,55,ENERGY,FL,T,    SS,ALS,S12  ,ANL
                                              ,R0,RMAX,XN
  55 FORMAT (     ////1X,  9H ENERGY = ,F6.1,4H L = ,F4.1,9H
                                               T1.T2 =  ,
     1 F5.1 ,7H S.S = ,F5.1,7H L.S =  ,F5.1, 7H S12 = ,F5.1,
                                              5H ANL=,F3.1
     2 , 4H R0=,F8.4,6H RMAX=,F4.1,4H XN=,F5.1    )
      AK = SQRTF(ELAB/(2.0*41.469))
      UN= 4.7583/AK
      DO 600 I=1,J
      CAPPA(I)=AMASS(I)/197.32
      U(I) = CAPPA(I)/AK
 600 CONTINUE
      W(1,1) =1.0
      W(1,3)= -1.0
      W(1,5)=0.0
```

```
W(1,7)=T
W(1,9)=-T
W(1,11)=0.0
W(2,1)=1.0
W(2,3)=0.0
W(2,5)=0.0
W(2,7)= T
W(2,9)= 0.0
W(2,11)=0.0
W(3,1)=1.0
W(3,3)=1.0
W(3,5)=0.0
W(3,7)= T
W(3,9)= T
W(3,11)=0.0
W(4,1)=1.0
W(4,3)= 1.0
 W(4,5)=0.0
W(4,7)= T
W(4,9)= T
W(4,11)=0.0
W(5,1)=0.0
W(5,3)= 1.0
W(5,5)=1.0/2.0
W(5,7)=0.0
W(5,9)= T
W(5,11)= (1.0/2.0)*T
W(6,1)=1.0
W(6,3)=3.0
W(6,5)= 0.0
W(6,7)= T
W(6,9)= 3.0*T
W(6,11)=0.0
W(7,1)=0.0
W(7,3)=1.0
W(7,5)=-1.0
W(7,7)= 0.0
W(7,9)= T
W(7,11)= -T
W(8,1)=0.0
W(8,3)=1.0
W(8,5)=1.0
W(8,7)=0.0
W(8,9)= T
W(8,11)=  T

 W(1,13)=W(1,3)
W(2,13)=W(2,3)
 W(3,13) =W(3,3)
 W(4,13) =W(4,3)
  W(5,13)=W(5,3)
W(6,13)=W(6,3)
W(7,13)=W(7,3)
W(8,13)=W(8,3)
W(10,1)=0.0
```

```
      W(10,3)=1.0
      W(10,5)=0.0
      W(10,7)=0.0
      W( 10,9)=T
      W(10,11)=0.0
      W(10,13)=W(10,3)
      W(11,1)=0.0
      W(11,3)= 1.0/2.0
      W(11,5)=0.0
      W(11,7)=0.0
      W(11,9)= T/2.0
      W(11,11)=0.0
      W(11,13)=W(11,3)
      W(12,1)=0.0
      W(12,3)= 1.0/2.0
      W(12,5)=0.0
      W(12, 7)=0.0
      W(12,9)= T/2.0
      W(12,11)=0.0
      W(12,13)=W(12,3)
      SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      SUM6=0.0
      SUM7=0.0
      SUM8=0.0
      SUM10=0.0
      SUM11=0.0
      SUM12=0.0
      SUM13=0.0
      RADIAN =57.29578
      JM=J-1
      JJ=J+1
      I= -1
    3 I=I+2
      IF(I-JJ)5,6,41
    5 IF(I-1)41,1,12
   12 M=I
   10 M=M-2
      IF(M) 1,41,2
    2 IF (AMASS(I)-AMASS(M)) 10,7,10
1     UF =U(I)
      UFC = U(I+1)
      CALL BGKINT
      A(I) =AF
      B(I) =BF
      C(I) =CF
      D(I) = DF
      E(I) =EF
      F(I)=FF
      IF (NBK)471,472,471
  471 HB(I)=HBF
```

```
        HK(I)=HKF
 472  GO TO 3
 7     A(I) = A(M)
      B(I) = B(M)
      C(I) = C(M)
       D(I) = D(M)
       E(I) = E(M)
       F(I)=F(M)
        HB(I)=HB(M)
        HK(I)=HK(M)
      GO TO 3
  6      GO TO 1321
1321   WRITEOUTPUTTAPE 6,33 ,   (A(I),B(I),C(I),D(I),E(I),F(
                                        I),HB(I),
    1    HK(I),I=1,JM,2)
 33    FORMAT ( 10X,8E13.6)
       DO 100  I=1,JM,2
      RY(I) =  GSQ(I)*UN*A(I)*W(1,I)
      SUM1=SUM1+RY(I)
      RC(I) =  (0.25*GSQ(I)*          F(I)*W(2,I)/UN)
      SUM2=SUM2+RC(I)
      RGRAD(I)=-GSQ(I)/UN *B(I)*W(3,I)
      SUM3=SUM3+RGRAD(I)
      RDELSQ(I)=  -(GSQ(I)/UN)*A(I)*W(4,I)
      SUM4=SUM4+RDELSQ(I)
      RSS(I)= -(1.0/6.0)*(GSQ(I)/UN)*SS          *W(5,I)*F(
                                        I)
      SUM5=SUM5+RSS(I)
      RLS(I)= (ALS/2.0)*(GSQ(I)/UN)*C(I)*W(6,I)
      SUM6=SUM6+RLS(I)
      RT(I) =(S12/12.0)*(GSQ(I)/UN)*D(I)*W(7,I)
      SUM7=SUM7+RT(I)
      RNL(I) =  -((GSQ(I)**2)/4.0)*E(I)*W(8,I)*ANL
      SUM8=SUM8+RNL(I)
       Y1=RY(I)
       Y2=RC(I)
       Y3=RGRAD(I)
       Y4=RDELSQ(I)
       Y5=RSS(I)
       Y6=RLS(I)
       Y7=RT(I)
       Y8=RNL(I)
       YP=ABSF(Y1)
       YQ=ABSF(Y2)
       YR=ABSF(Y3)
       YS= ABSF(Y4)
       YT= ABSF(Y5)
       YU= ABSF(Y6)
       YV=  ABSF(Y7)
       YW= ABSF(Y8)
        IF(NBK) 411,412,411
411     RGRADB(I)=  (-GSQ(I)/UN)*B(I)*W(11,I)
        SUM12=SUM12+RGRADB(I)
     RDLSQB(I)= (-GSQ(I)/UN)*A(I)*W(12,I)
```

```
      SUM13=SUM13+RDLSQB(I)
      RB(I)=(GSQ(I)/UN)*HB(I)*W(10,I)
      SUM10=SUM10+RB(I)
      RK(I)=(GSQ(I)/(UN*(U(I)**2)))*W(10,I)*HK(I)
      SUM11=SUM11+RK(I)
      Y10 =RB(I)
      Y11=RK(I)
      Y12  =RGRADB(I)
      Y13=RDLSQB(I)
      DB(I)=   ATANF(Y10)*RADIAN
      DK(I)=ATANF(Y11)*RADIAN
      DGRADB(I)=ATANF(Y12)*RADIAN
      DDLSQB(I)=ATANF(Y13)*RADIAN
 412  IF(YP-0.5)201,202,202
 201  DY(I)=(Y1-(Y1**3)/3.0)*RADIAN
      GO TO 203
 202  DY(I)=ATANF(Y1)*RADIAN
 203  IF(YQ-0.5) 204,205,205
 204  DC(I)=(Y2-(Y2**3)/3.0)*RADIAN
      GO TO 206
 205  DC(I)=ATANF(Y2)*RADIAN
 206  IF(YR-0.5)207,208,208
 207  DGRAD(I)=(Y3-(Y3**3)/3.0)*RADIAN
      GO TO 209
 208  DGRAD(I)=ATANF(Y3) *RADIAN
 209  IF(YS-0.5)210,211,211
 210  DDELSQ(I)=(Y4-(Y4**3)/3.0)*RADIAN
      GO TO 212
 211  DDELSQ(I)=ATANF(Y4)*RADIAN
 212  IF (YT-0.5)213,214,214
 213  DSS(I)=(Y5-(Y5**3)/3.0)*RADIAN
      GO TO 215
 214  DSS(I)= ATANF(Y5)*RADIAN
 215  IF(YU-0.5) 216,217,217
 216  DLS(I)=(Y6-(Y6**3)/3.0)*RADIAN
      GO TO 218
 217  DLS(I)= ATANF(Y6)*RADIAN
 218  IF(YV-0.5)219,220,220
 219  DT(I)=(Y7-(Y7**3)/3.0)*RADIAN
      GO TO 221
 220  DT(I)=ATANF(Y7)*RADIAN
 221  IF(YW-0.5)222,223,223
 222  DNL(I)=(Y8-(Y8**3)/3.0)*RADIAN
      GO TO 100
 223  DNL(I)=ATANF(Y8)*RADIAN
 100  CONTINUE
      PHASEL= SUM1+SUM2+SUM3+SUM4+SUM5+SUM6+SUM7
      PHASE =PHASEL+SUM8
      SHIFTL=ATANF(PHASEL)*RADIAN
      SHIFT=ATANF(PHASE)*RADIAN
      IF(NBK)415,416,415
 415  PHASEK=PHASEL+SUM11
      PHASEB=SUM1+SUM2+SUM12 + SUM13+SUM5+SUM6+SUM7+SUM10
      SHIFTK= ATANF(PHASEK)*RADIAN
```

```
          SHIFTB= ATANF(PHASEB)*RADIAN
           DIFK = SHIFTL-SHIFTK
          DIFB = SHIFTL-SHIFTB
          DIFBK=SHIFTB-SHIFTK
             SUM10= ATANF(SUM10)*RADIAN
          SUM11= ATANF(SUM11)*RADIAN
          SUM12=ATANF(SUM12)*RADIAN
           SUM13= ATANF(SUM13) *RADIAN
 416    SUMP= ABSF(SUM1)
        SUMQ= ABSF(SUM2)
          SUMR= ABSF(SUM3)
        SUMS= ABSF( SUM4)
        SUMT= ABSF(SUM5)
        SUMU= ABSF(SUM6)
        SUMV= ABSF(SUM7)
        SUMW= ABSF(SUM8)
        IF(SUMP-0.5)224,225,225
 224    SUM1 =(SUM1-(SUM1**3)/3.0)*RADIAN
        GO TO 226
 225    SUM1=ATANF(SUM1)*RADIAN
 226    IF(SUMQ-0.5)227,228,228
227       SUM2=(SUM2-(SUM2**3)/3.0)      *RADIAN
        GO TO  229
 228    SUM2=ATANF(SUM2)*RADIAN
 229    IF(SUMR-0.5)230,231,231
 230    SUM3 = (SUM3-(SUM3**3)/3.0)*RADIAN
        GO TO 232
 231    SUM3=ATANF(SUM3)*RADIAN
 232    IF(SUMS-0.5)233,234,234
 233    SUM4=(SUM4-(SUM4**3)/3.0)*RADIAN
        GO TO 235
 234    SUM4=ATANF(SUM4)*RADIAN
 235    IF(SUMT-0.5)236,237,237
 236    SUM5=(SUM5-(SUM5**3)/3.0)*RADIAN
        GO TO  238
 237    SUM5=ATANF(SUM5)*RADIAN
 '238     IF(SUMU-0.5)239,240,240
 239     SUM6=(SUM6-(SUM6**3)/3.0)*RADIAN
         GO TO  241
 240    SUM6=ATANF(SUM6)*RADIAN
 241    IF(SUMV-0.5)242,243,243
 242     SUM7  =(SUM7-(SUM7**3)/3.0)*RADIAN
        GO TO 244
 243     SUM7=ATANF(SUM7)*RADIAN
 244    IF(SUMW-0.5)245,245,246
 245    SUM8=(SUM8-(SUM8**3)/3.0)*RADIAN
          GO TO 247
 246    SUM8=ATANF(SUM8)*RADIAN
 247  IF (J-12) 302,302,303
302   WRITE OUTPUT TAPE 6,15
   15 FORMAT(8X , 7H OMEGAS ,                         4X, 6H OMEGA ,
     1  4X,      4H ETA ,                  6X, 7H RHOS    ,     6X
                                            ,
```

```
    2 4H RHO ,                6X, 3H PI ,                6X, 5H S
                                               UM      //)
 304 WRITE OUTPUT TAPE 6,75,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                               ,JM,2)
  75 FORMAT(2X,(6H.MASS   , 6F10.2)//(2X, 6H  GSQ   , 6F10.2)
                                               )
     WRITE OUTPUT TAPE 6,76,(AMASS(I),I=2,J ,2),(GSQ(I),I=2
                                               ,J ,2)
  76 FORMAT(2X,(6H MASSC , 6F10.2)//(2X, 6H  GSQC , 6F10.2)
                                               )
402   WRITE OUTPUT TAPE 6,25,(DY(I),I=1,JM,2),SUM1,(DC(I),I=
                                               1,JM,2),SUM2
    1 , (DGRAD(I),I=1,JM,2),SUM3,(DDELSQ(I),I=1,JM,2),SUM4,
                                               (DSS(I),I=1,
    2 JM,2),SUM5,(DLS(I),I=1,JM,2),SUM6,(DT(I),I=1,JM,2),SU
                                               M7,(DNL(I),
    3 I=I,JM,2),SUM8
  25 FORMAT((       3X, 5H  DY ,7F10.5)//(3X, 5H  DC ,7F10.5
                                               )//(2X,
    1 6H DGRAD ,7F10.5)// (1X, 7H DDELSQ ,7F10.5)//(3X, 5H
                                               DSS ,7F10.5)
    2 //(3X, 5H DLS ,7F10.5)//(3X,5H  DT ,7F10.5)//(3X,5H
                                               DNL,7F10.5))
       GO TO 409
 303   WRITE OUTPUT TAPE 6, 16
  16 FORMAT(8X , 7H UMEGAS ,                    4X, 6H OMEGA ,
    1  4X,       4H ETA ,                6X, 7H RHOS    ,      6X
                                               ,
    2 4H RHO ,                6X, 3H PI ,6X,4H PHI ,  6X, 5H S
                                               UM      //)
     WRITE OUTPUT TAPE 6,78,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                               ,JM,2)
 78   FORMAT(2X,(6H MASS   , 7F10.2)//(2X, 6H  GSQ   , 7F10.2)
                                               )
     WRITE OUTPUT TAPE 6,77,(AMASS(I),I=2,J ,2),(GSQ(I),I=2
                                               ,J ,2)
 '77  FORMAT(2X,(6H MASSC , 7F10.2)//(2X, 6H  GSQC , 7F10.2)
                                               )
 403  WRITE OUTPUT TAPE 6,26,(DY(I),I=1,JM,2),SUM1,(DC(I),I=
                                               1,JM,2),SUM2
    1 , (DGRAD(I),I=1,JM,2),SUM3,(DDELSQ(I),I=1,JM,2),SUM4,
                                               (DSS(I),I=1,
    2 JM,2),SUM5,(DLS(I),I=1,JM,2),SUM6,(DT(I),I=1,JM,2),SU
                                               M7,(DNL(I),
    3 I=1,JM,2),SUM8
  26 FORMAT((       3X, 5H  DY ,8F10.5)//(3X, 5H  DC ,8F10.5
                                               )//(2X,
    1 6H DGRAD ,8F10.5)// (1X, 7H DDELSQ ,8F10.5)//(3X, 5H
                                               DSS ,8F10.5)
    2 //(3X, 5H DLS ,8F10.5)//(3X,5H  DT ,8F10.5)//(3X,5H
                                               DNL,8F10.5))
 409  WRITE OUTPUT TAPE 6,36,PHASE,PHASEL
  36   FORMAT  (5X,9H TANDG=  , F12.5,10H TANDLG=  , F12.5)
       WRITE UUT PUT TAPE 6,35,SHIFT ,SHIFTL
```

```
  35 FORMAT (1X,25H PHASE GREEN IN DEGREES = ,F12.5,20H PHA
                                                SE LINEAR GR
    1EEN=          ,F12.5    )
      IF(NBK) 421,422,421
421   IF(J-12) 452,451,452
451   WRITE OUTPUT TAPE 6,423,
    1                (DGRADB(I),I=1,JM,2),SUM12,(DDLSQB(I),I
                                                =1,JM,2),SUM
    1 13,(DK(I),I=1,JM,2),SUM11,(DB(I),I=1,JM,2),SUM10
 423 FORMAT((1X,7H DGRADB ,7F10.5)//(1X,7H DDLSQB ,7F10.5)/
                                                /(1X,7H DKMR
    1EX  ,7F10.5)//     (1X,7H DBRTEX  ,7F10.5     )    )
        GO TO  457
452   WRITE OUTPUT TAPE 6,456,
    1                (DGRADB(I),I=1,JM,2),SUM12,(DDLSQB(I),I
                                                =1,JM,2),SUM
    1 13,(DK(I),I=1,JM,2),SUM11,(DB(I),I=1,JM,2),SUM10
456  FORMAT((1X,7H DGRADB ,8F10.5)//(1X,7H DDLSQB ,8F10.5)/
                                                /(1X,7H DKMR
    1EX  ,8F10.5)//     (1X,7H DBRTEX  ,8F10.5     )   )
457   WRITE OUT PUT TAPE 6,461,
    1                PHASEK,PHASEB,SHIFTK,SHIFTB,DIFK,DIFB
                                                   ,DIFBK
461    FORMAT(5X,20H TAND(KEMMER)=          ,F12.5, 20H TAND(
                                                BREIT)=
    1          ,F12.5,//      15H PHASE KEMMER=  ,F12.5
                                                 ,15H PHASE
    2BREIT=   ,F12.5,//20H GREEN-KEMMER=          ,F12.5, 20
                                                H  GREEN-BRE
    3IT =              ,F12.5 , 20H BREIT-KEMMER=
                                                    ,F12.5)
422  GO TO 19
     END
```

SUBROUTINE "BGKINT"
Used for Calculating Integrals
for Uncoupled Phases

```
      SUBROUTINE BGKINT
C     BGKINT IS THE SUBROUTINE FOR BREIT GREEN KEMMER INTEGR
                                                   ATIONS
      COMMON RO,RMAX,XN,UF,AK,FL,AF,BF,CF,DF,EF,SS,ALS,S12,A
                                                   NL ,UFC    ,
     1  ELAB,BES ,FF      ,HBF,HKF,NBK
      DIMENSION Z(1201),GA(1201),GB(1201),GC(1201),GD(1201),
                                                   GE(1201)  ,
     1 BES(6,8,100)    , TF(1201)     ,THB(1201),THK(1201)
      ZO=RO*AK
      ZMAX =RMAX*AK
      DELZ =(ZMAX-ZO)/XN
      M=FL
      K=M-1
       N = M+1
      NN=XN
      L=NN+1
       EX1=EXPF(-UF*ZO)
       EX2=EXPF(-UFC*ZO)
        EXD=EXPF(-UF*DELZ)
       EXDC= EXPF(-UFC*DELZ)
      M1=M+1
      K1=K+1
      N1=N+1
        IF(ELAB-50.0    )      21,21,22
   21 I1=1
      GO TO 23
   22 IF (ELAB-142.0) 99,99,25
   99 I1=2
        GO TO 23
   25  IF ( ELAB-310.0    ) 26,26,27
   27 WRITE OUTPUT TAPE 6,28
   28    FORMAT(20H WRONG ENERGY                          )
         CALL EXIT
   26  I1=3
   23  DO 100 I=1,L
      AI=I
      Z(I) =ZO+(AI-1.0)*DELZ
      ZZ=Z(I)
       UZ=UF*ZZ
       UZC=UFC*ZZ
       UZ2=(UZ**2+2.0*UZ+2.0)
       UZ2C=(UZC**2+2.0*UZC+2.0)
         UZ3=(UZ**2+3.0*UZ+3.0)
       UZ3C=(UZC**2+3.0*UZC+3.0)
      BESN =BES(I1,N1,I)
```

```
      BESM=BES(I1,M1,I)
      IF (K1) 27,401,402
  402 BESK=BES(I1,K1,1)
      GO TO 404
  401  BESK = (COSF(ZZ)/ZZ)
  404 BESQ=(BESM)**2
        DBES=(FL*BESK-(FL+1.0)*BESN)*(1.0/(2.0*FL+1.0)   )
       EX = ( EX1 - EX2)
      GA(I) =EX*ZZ*BESQ
      GB(I) = ( EX + ( UF*EX1 -UFC*EX2 ) *ZZ)*( DBES*BESM)
       GC(I) = EX * ( BESQ/ZZ) + ( UF*EX1 -UFC*EX2 )*BESQ
       GD(I) = (((UF**2)*EX1-(UFC**2)*EX2)*ZZ+3.0*(UF*EX1-UF
                                                C*EX2))*BESQ
     1 +3.0*EX*(BESQ/ZZ)
      GE(I) =(EX**2)*BESQ
        TF(I)=((UF**2)*EX1-(UFC**2)*EX2)*BESQ*ZZ
           IF(NBK)405,406,405
 405     CK=((UF**4)*EX1-(UFC**4)*EX2)*ZZ*BESQ
      DLSQK=(UZ2*EX1-UZ2C*EX2)*(BESQ/ZZ)
      GRADK  =((UZ**3+3.0*UZ2)*EX1 -(UZC**3+3.0*UZ2C)*EX2 )*
                                                DBES*(BESM/
     1(ZZ**2))
        ELLK= (UZ3*EX1-UZ3C*EX2)*(BESQ/(ZZ**3))
       THK(I)=-0.25*CK+DLSQK+GRADK -FL*(FL+1.0)*ELLK
        CB=( (UF**2)*(1.0+UZ)*EX1-(UFC**2)*(1.0+UZC)*EX2)*ZZ
                                                *BESQ
        GRADB = (UZ3*EX1- UZ3C*EX2)*DBES*BESM
        DLSQB = ( (1.0+UZ)*EX1-(1.0+UZC)*EX2)*ZZ*BESQ
      THB(I)= 0.5*(-1.25*TF(I)+0.25*CB-GRADB-DLSQB +FL*(FL+1
                                                .0)*GC(I))
 406   EX1=EX1*EXD
       EX2=EX2*EXDC
 100   CONTINUE
      CALL WEDDLE(DELZ,GA,L,RESULT)
      AF = RESULT
      CALL WEDDLE(DELZ,GB,L,RESULT)
      BF = RESULT
       CALL WEDDLE(DELZ,TF,L,RESULT)
       FF=RESULT
       IF (NBK)407,408,407
  407   CALL WEDDLE(  DELZ,THB,L,RESULT          )
         HBF=RESULT
         CALL WEDDLE (DELZ,THK,L,RESULT)
         HKF = RESULT
 408   IF (ALS) 20,30,20
 20   CALL WEDDLE(DELZ,GC,L,RESULT)
      CF = RESULT
      GO TO 40
  30    CF = 0.0
  40  IF(S12) 50,60,50
  50  CALL WEDDLE(DELZ,GD,L,RESULT)
      DF = RESULT
       GO TO 70
  60   DF=0.0
```

```
 70    IF (ANL) 80,90,80
 80   CALL WEDDLE(DELZ,GE,L,RESULT)
       EF=RESULT
      GO TO 130
90      EF=0.0
 130 RETURN
        END
```

4. BORN PHASES FOR THE UNCOUPLED STATES WITH "EFFECTIVE MASS" METHOD

```
C        EFFECTIVE BORN PHASES
C      CORRECT EFFECTIVE POTENTIAL APPROACH TO BORN PHASES
C      THIS CODE WILL CALCULATE PHASE SHIFTS FOR SINGLET AND
                                              TRIPLET
C      (UNMIXED) STATES OF NUCLEON-NUCLEON SYSTEM IN BORN APP
                                              ROXIMATION
C      WILL BE DONE IN EFFECTIVE MASS APPROXIMATION WHERE 2ND
                                              DERIVATIVES
C      ARE REMOVED BY EFFECTIVE MASS AND FIRST BY TRANSFORMIN
                                              G THE WAVE FUNCTION
C      X=R*(1+PHE)**.5*U .THEN THE PHASE SHIFT EXPRESSIONS IN
                                              VOLVE ONLY
C      SIX INTEGRALS A B C D EFAND THE INTEGRALS ARE PERFORM
                                              ED USING
C      WEDDLE,S RULE. FUNCTION SFBESS IS NEEDE D  AND NO OF P
                                              OINTS OF
C      INTEGRATION MUST BE 6*M AND SO INPOINTSARE6M+1.
C        DELSQ MEANS LINEAR VEL DEP TERMS. SUM3 IS VEL DEP NO
                                              N LINEAR TERM
C      AMASS IS THE MASS OF MESON IN MEV   . GSQ IS THE COUPL
                                              ING CONSTANT
C      SS IS THE EXPECTATION VALUE OF (SIGMA1.SIGMA2) FOR THE
                                              STATE .
C       ALS IS THE SPIN ORBIT OPERATOR EXPECTATION VALUE FOR
                                              THE STATE
C      THAT IS (L.S).S12 IS THE TENSOR OPERATOR EXPECTATION V
                                              ALUE FOR THE
C      STATE. L.S AND S12 AVERAGE TO ZERO FOR SINGLET STATES.
C       ANL IS THE COEFFICIENT THAT MAKES NON LINEAR TERM ZER
                                              O OR TAKES
C      IT INTO ACCOUNT
C      J IS THE NO OF MESONS 12  OR 14 (FOR INCLUDING PHI) IN
                                              CLUDING CUT OFF
       COMMON RO,RMAX,XN,UF,AK,FL,AF,BF,CF,DF,EF,SS,ALS,S12,A
                                              NL ,UFC    ,
      1  ELAB,BES ,FF
       DIMENSIONW(10,20),GSQ(15),AMASS(15),U(15),A(15),B(15),
                                              C(15),D(15)
      1,E(15),G(15),DY(15),DC(15),DGRAD(15),DDELSQ(15),DSS(15
                                              ),DLS(15),DT
      2(15),  DNL(15)     ,CAPPA(15)    ,RY(15),RC(15),RGRAD(15
                                              ),RDELSQ(15)
      3 ,RSS(15),RLS(15),RT(15),RNL(15) ,BES(6,8,100)   ,F(15)
       DIMENSION Z(1201),GA(1201),GB(1201),GC(1201),GD(1201),
                                              GE(1201)
       DIMENSION DENOM(1201),PHEPR(1201   )     ,TF(1201)
```

```
      CALL BESTO
  18   READ INPUT TAPE 5,9,J
 9     FORMAT(I2)
 973   IF ( J-12) 967,967,966
966   READ INPUT TAPE 5,43,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 43    FORMAT(        12F6.0/2F6.0/12F6.0/2F6.0)
      GO TO 19
 967 READINPUT TAPE 5,968,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 968   FORMAT(        12F6.0/       12F6.0)
 19    WRITE OUT PUT TAPE 6,1150
1150 FORMAT( 1H1, 25H EFFECTIVE BORN PHASES       )
 971 READ INPUT TAPE 5,972,ELAB,FL,T,SS,ALS,S12,ANL,RO,RMAX
                                               ,XN
 972 FORMAT(10F6.0)
       IF (ELAB- 1000.0) 969,973,973
969   IF (SS-1.0 )31,32,31
  32 WRITE OUTPUT T  APE 6,115
 115 FORMAT(     5X,30H THIS IS A TRIPLET SPIN STATE
                                                )
      GO TO 199
  31 IF (SS+ 3.0) 41,42,41
  41 WRITE OUT PUT TAPE 6,125
 125 FORMAT (5X   ,20H WRONG INPUT DATA          )
     CALL EXIT
  42 WRITE OUTPUT  TAPE 6,135
 135 FORMAT(     5X, 30H THIS IS A SINGLET SPIN STATE
                                                  )
 199  ENERGY = ELAB
     WRITE OUTPUT TAPE 6,55,ENERGY,FL,T,    SS,ALS,S12  ,ANL
                                              ,RO,RMAX,XN
  55 FORMAT (     ////1X,  9H ENERGY = ,F6.1,4H L = ,F4.1,9H
                                                T1.T2 =   ,
   1 F5.1 ,7H S.S = ,F5.1,7H L.S =  ,F5.1, 7H S12 = ,F5.1,
                                              5H ANL=,F3.1
      , 4H RO=,F8.4,6H RMAX=,F4.1,4H XN=,F5.1    )
      AK = SQRTF(ELAB/(2.0*41.469))
     UN= 4.7583/AK
      DO 600 I=1,J
     CAPPA(I)=AMASS(I)/197.32
     U(I) = CAPPA(I)/AK
 600 CONTINUE
     W(1,1) =1.0
     W(1,3)= -1.0
     W(1,5)=0.0
     W(1,7)=T
     W(1,9)=-T
     W(1,11)=0.0
      W(2,1)=-1.0
     W(2,3)=0.0
     W(2,5)=0.0
      W(2,7)=-T
     W(2,9)=0.0
```

```
W(2,11)=0.0
W(3,1)=1.0
W(3,3)=1.0
W(3,5)=0.0
W(3,7)= T
W(3,9)= T
W(3,11)=0.0
W(4,1)=1.0
W(4,3)= 1.0
 W(4,5)=0.0
W(4,7)= T
W(4,9)= T
W(4,11)=0.0
W(5,1)=0.0
W(5,3)= 1.0
W(5,5)=1.0/2.0
W(5,7)=0.0
W(5,9)= T
W(5,11)= (1.0/2.0)*T
W(6,1)=1.0
W(6,3)=3.0
W(6,5)= 0.0
W(6,7)= T
W(6,9)= 3.0*T
W(6,11)=0.0
W(7,1)=0.0
W(7,3)=1.0
W(7,5)=-1.0
W(7,7)= 0.0
W(7,9)= T
W(7,11)= -T
W(8,1)=0.0
W(8,3)=1.0
W(8,5)=1.0
W(8,7)=0.0
W(8,9)= T
W(8,11)=  T
  W(1,13)=W(1,3)
 W(2,13)=W(2,3)
  W(3,13) =W(3,3)
  W(4,13) =W(4,3)
   W(5,13)=W(5,3)
 W(6,13)=W(6,3)
 W(7,13)=W(7,3)
 W(8,13)=W(8,3)
SUM1=0.0
SUM2=0.0
SUM3=0.0
SUM4=0.0
SUM5=0.0
SUM6=0.0
SUM7=0.0
SUM8=0.0
RADIAN =57.29578
```

```
       JM=J-1
      ZO=RO*AK
      ZMAX =RMAX*AK
      DELZ =(ZMAX-ZO)/XN
      M=FL
      NN=XN
      L=NN+1
      M1=M+1
        IF(ELAB-50.0    ) 521,521,522
521   I1=1
      GO TO523
 522 IF (ELAB-142.0)599,599,525
 599 I1=2
        GO TO523
525    IF ( ELAB-310.00 ) 526,526,527
 527 WRITE OUTPUT TAPE 6,528
528      FORMAT(20H WRONG ENERGY                        )
         CALL EXIT
526    I1=3
 523  DO 2000 K=1,JM,2
      UF  = U(K)
      UFC  = U(K+1)
       EX1=EXPF(-UF*ZO)
       EX2=EXPF(-UFC*ZO)
        EXD=EXPF(-UF*DELZ)
       EXDC= EXPF(-UFC*DELZ)
       DO 1000I=1,L
      AI=I
      Z(I) =ZO+(AI-1.0)*DELZ
      ZZ=Z(I)
      BESM=BES(I1,M1,I)
 404 BESQ=(BESM)**2
       EX = ( EX1 - EX2)
      IF(K -1) 527,572,5005
572   PSUM =0.0
       DPSUM =0.0
      DO  550 JJ=1,JM,2
        JI= (JJ  -1)/2+1
      GO TO (551,551,552,553,553,552,551),JI
 552 PHE =0.0
      DPHE =0.0
       GO TO  554
 551    PHE= GSQ(JJ)*(EXPF(-U(JJ)*ZZ)-EXPF(-U(JJ+1)*ZZ))/(UN
                                                        *ZZ)
       DPHE =-(PHE/ZZ)-(GSQ(JJ))*(U(JJ)*EXPF(-U(JJ)*ZZ)-U(JJ
                                                        +1)*EXPF(-
     1 U(JJ+1)*ZZ))/(UN*ZZ)
       GO TO 554
 553    PHE= GSQ(JJ)*(EXPF(-U(JJ)*ZZ)-EXPF(-U(JJ+1)*ZZ))/(UN
                                                        *ZZ)
       DPHE =-(PHE/ZZ)-(GSQ(JJ))*(U(JJ)*EXPF(-U(JJ)*ZZ)-U(JJ
                                                        +1)*EXPF(-
     1 U(JJ+1)*ZZ))/(UN*ZZ)
      PHE = T*PHE
```

```
      DPHE = T* DPHE
 554  PSUM= PSUM + PHE
      DPSUM = DPSUM + DPHE
 550  CONTINUE
 5004 DENOM(I)= 1.0+PSUM
      PHEPR(I)=DPSUM
 5005 IF (DENOM(I))555,555,571 | IF(K-1)5555,5555,5006
                                  555
 5555 RR= ZZ/AK
      WRITE OUTPUT TAPE 6,557,DENOM(I),RR
 557  FORMAT( 25H EFFECTIVE MASS NEGATIVE      ,E12.6 ,15H
                                          AT RADIUS=
    1      , F 10.6     )
 5006 KK=I
      GA(KK)=0.0
      GB(KK)=0.0
      GC(KK)=0.0
      GD(KK)=0.0
      GE(KK)=0.0
      TF(KK)=0.0
      GO TO 1000
 571  GA(I) = ( EX*ZZ*BESQ )/DENOM(I)
 5001 GC(I) =(EX * ( BESQ/ZZ) + ( UF*EX1 -UFC*EX2 )*BESQ )/
                                          DENOM(I)
 5002 GD(I) =((((UF**2)*EX1-(UFC**2)*EX2)*ZZ+3.0*(UF*EX1-UF
                                          C*EX2))*BESQ
    1 +3.0*EX*(BESQ/ZZ) )/DENOM(I)
 5003 GE(I)=((EX**2)*BESQ    )/DENOM (I)
      TF(I)=((UF**2)*EX1-(UFC**2)*EX2)*BESQ*ZZ      /DENOM(
                                          I)
      IF(K-1)527,574,1001
 574  GB(I) = 0.25*((PHEPR (I)/DENOM(I))**2)*(BESQ*(ZZ**2))
 1001 EX1=EX1*EXD
      EX2=EX2*EXDC
 1000 CONTINUE
      CALL WEDDLE(DELZ,GA,L,RESULT)
      A(K) =RESULT
      CALL WEDDLE(DELZ,TF,L,RESULT)
      F(K)=RESULT
      IF (K-1)861,861,862
 861  CALL WEDDLE(DELZ,GB,L,RESULT)
      B(K)=RESULT
 862  IF (ALS) 20,30,20
 20   CALL WEDDLE(DELZ,GC,L,RESULT)
      C(K)=RESULT
      GO TO 40
 30   C(K) = 0.0
 40   IF(S12) 50,60,50
 50   CALL WEDDLE(DELZ,GD,L,RESULT)
      D(K)= RESULT
      GO TO 70
 60   D(K)=0.0
 70   IF (ANL) 80,90,80
 80   CALL WEDDLE(DELZ,GE,L,RESULT)
      E(K)= RESULT
```

```
      GO TO 2000
 90   E(K)=0.0
 2000 CONTINUE
      DO 575 KK=3,JM,2
 575  B(KK)=B(1)
      WRITE OUTPUT TAPE 6,33,(A(I),B(I),C(I),D(I),E(I),I=1,J
                                               M,2)
   33 FORMAT (10X,5E15.6)
      DO 100  I=1,JM,2
      RY(I) =  GSQ(I)*UN*A(I)*W(1,I)
      SUM1=SUM1+RY(I)
      RC(I) =  (0.25*GSQ(I)*                 F(I)*W(2,I)/UN)
      SUM2=SUM2+RC(I)
       RGRAD(I)=0.0
      SUM3=SUM3+RGRAD(I)
      RDELSQ(I)=-(GSQ(I)/UN)*(A(I)+0.5*F(I))*W(4,I)
      SUM4=SUM4+RDELSQ(I)
      RSS(I)= -(1.0/6.0)*(GSQ(I)/UN)*SS              *W(5,I)*F(
                                             I)
      SUM5=SUM5+RSS(I)
      RLS(I)= (ALS/2.0)*(GSQ(I)/UN)*C(I)*W(6,I)
      SUM6=SUM6+RLS(I)
      RT(I) =(S12/12.0)*(GSQ(I)/UN)*D(I)*W(7,I)
      SUM7=SUM7+RT(I)
      RNL(I) =  -((GSQ(I)**2)/4.0)*E(I)*W(8,I)*ANL
      SUM8=SUM8+RNL(I)
       Y1=RY(I)
       Y2=RC(I)
       Y3=RGRAD(I)
       Y4=RDELSQ(I)
       Y5=RSS(I)
       Y6=RLS(I)
       Y7=RT(I)
       Y8=RNL(I)
       YP=ABSF(Y1)
       YQ=ABSF(Y2)
       YR=ABSF(Y3)
       YS= ABSF(Y4)
       YT= ABSF(Y5)
       YU= ABSF(Y6)
       YV=  ABSF(Y7)
       YW= ABSF(Y8)
       IF(YP-0.5)201,202,202
 201  DY(I)=(Y1-(Y1**3)/3.0)*RADIAN
      GO TO 203
 202  DY(I)=ATANF(Y1)*RADIAN
 203   IF(YQ-0.5) 204,205,205
 204    DC(I)=(Y2-(Y2**3)/3.0)*RADIAN
      GO TO 206
 205  DC(I)=ATANF(Y2)*RADIAN
 206  IF(YR-0.5)207,208,208
 207  DGRAD(I)=(Y3-(Y3**3)/3.0)*RADIAN
      GO TO 209
 208   DGRAD(I)=ATANF(Y3) *RADIAN
```

```
209   IF(YS-0.5)210,211,211
210   DDELSQ(I)=(Y4-(Y4**3)/3.0)*RADIAN
      GO TO 212
211   DDELSQ(I)=ATANF(Y4)*RADIAN
212   IF (YT-0.5)213,214,214
213   DSS(I)=(Y5-(Y5**3)/3.0)*RADIAN
      GO TO 215
214   DSS(I)= ATANF(Y5)*RADIAN
215   IF(YU-0.5) 216,217,217
216   DLS(I)=(Y6-(Y6**3)/3.0)*RADIAN
      GO TO 218
217   DLS(I)= ATANF(Y6)*RADIAN
218   IF(YV-0.5)219,220,220
219   DT(I)=(Y7-(Y7**3)/3.0)*RADIAN
      GO TO 221
220   DT(I)=ATANF(Y7)*RADIAN
221   IF(YW-0.5)222,223,223
222   DNL(I)=(Y8-(Y8**3)/3.0)*RADIAN
      GO TO 100
223   DNL(I)=ATANF(Y8)*RADIAN
100   CONTINUE
      PHASEL= SUM1+SUM2+SUM3+SUM4+SUM5+SUM6+SUM7
      PHASE =PHASEL+SUM8
      SUMP= ABSF(SUM1)
      SUMQ= ABSF(SUM2)
      SUM3=-8(1)
      SUMR=ABSF(SUM3)
      SUMS= ABSF( SUM4)
      SUMT= ABSF(SUM5)
      SUMU= ABSF(SUM6)
      SUMV= ABSF(SUM7)
      SUMW= ABSF(SUM8)
      IF(SUMP-0.5)224,225,225
224   SUM1 =(SUM1-(SUM1**3)/3.0)*RADIAN
      GO TO 226
225   SUM1=ATANF(SUM1)*RADIAN
226   IF(SUMQ-0.5)227,228,228
227   SUM2=(SUM2-(SUM2**3)/3.0)      *RADIAN
      GO TO  229
228   SUM2=ATANF(SUM2)*RADIAN
229   IF(SUMR-0.5)230,231,231
230   SUM3 = (SUM3-(SUM3**3)/3.0)*RADIAN
      GO TO 232
231   SUM3=ATANF(SUM3)*RADIAN
232   IF(SUMS-0.5)233,234,234
233   SUM4=(SUM4-(SUM4**3)/3.0)*RADIAN
      GO TO 235
234   SUM4=ATANF(SUM4)*RADIAN
235   IF(SUMT-0.5)236,237,237
236   SUM5=(SUM5-(SUM5**3)/3.0)*RADIAN
      GO TO  238
237   SUM5=ATANF(SUM5)*RADIAN
238   IF(SUMU-0.5)239,240,240
239   SUM6=(SUM6-(SUM6**3)/3.0)*RADIAN
```

```
      GO TO  241
240   SUM6=ATANF(SUM6)*RADIAN
241   IF(SUMV-0.5)242,243,243
242    SUM7  =(SUM7-(SUM7**3)/3.0)*RADIAN
      GO TO 244   .
243    SUM7=ATANF(SUM7)*RADIAN
244   IF(SUMW-0.5)245,245,246
245   SUM8=(SUM8-(SUM8**3)/3.0)*RADIAN
        GO TO 247
246   SUM8=ATANF(SUM8)*RADIAN
247    SHIFTL=ATANF(PHASEL)*RADIAN
      SHIFT=ATANF(PHASE)*RADIAN
     IF (J-12) 302,302,303
302   WRITE OUTPUT TAPE 6,15
   15 FORMAT(8X , 7H OMEGAS ,                    4X, 6H OMEGA ,
     1  4X,       4H ETA ,                  6X, 7H RHOS    ,     6X
                                            ,
     2 4H RHO ,                 6X, 3H PI ,                6X, 5H S
                                            UM      //)
  304 WRITE OUTPUT TAPE 6,75,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                            ,JM,2)
   75 FORMAT(2X,(6H MASS   , 6F10.2)//(2X, 6H  GSQ   , 6F10.2)
                                            )
      WRITE OUTPUT TAPE 6,76,(AMASS(I),I=2,J ,2),(GSQ(I),I=2
                                            ,J ,2)
   76 FORMAT(2X,(6H MASSC , 6F10.2)//(2X, 6H  GSQC , 6F10.2)
                                            )
402   WRITE OUTPUT TAPE 6,25,(DY(I),I=1,JM,2),SUM1,(DC(I),I=
                                            1,JM,2),SUM2
     1 , (DGRAD(I),I=1,JM,2),SUM3,(DDELSQ(I),I=1,JM,2),SUM4,
                                            (DSS(I),I=1,
     2 JM,2),SUM5,(DLS(I),I=1,JM,2),SUM6,(DT(I),I=1,JM,2),SU
                                            M7,(DNL(I),
     3 I=1,JM,2),SUM8
   25 FORMAT((      3X, 5H  DY ,7F10.5)//(3X, 5H  DC ,7F10.5
                                            )//(2X,
     1 6H DGRAD ,7F10.5)// (1X, 7H DDELSQ ,7F10.5)//(3X, 5H
                                            DSS ,7F10.5)
     2 //(3X, 5H DLS ,7F10.5)//(3X,5H  DT ,7F10.5)//(3X,5H
                                            DNL,7F10.5))
      GO TO 409
303   WRITE OUTPUT TAPE 6, 16
  16  FORMAT(8X , 7H OMEGAS ,                    4X, 6H OMEGA ,
     1 4X,       4H ETA ,                  6X, 7H RHOS    ,     6X
                                            ,
     2 4H RHO ,                 6X, 3H PI ,6X,4H PHI ,  6X, 5H S
                                            UM      //)
      WRITE OUTPUT TAPE 6,78,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                            ,JM,2)
 78   FORMAT(2X,(6H MASS   , 7F10.2)//(2X, 6H  GSQ   , 7F10.2)
                                            )
      WRITE OUTPUT TAPE 6,77,(AMASS(I),I=2,J ,2),(GSQ(I),I=2
                                            ,J ,2)
```

```
 77  FORMAT(2X,(6H MASSC , 7F10.2)//(2X, 6H  GSQC , 7F10.2)
                                                  )
403  WRITE OUTPUT TAPE 6,26,(DY(I),I=1,JM,2),SUM1,(DC(I),I=
                                                   1,JM,2),SUM2
    1 , (DGRAD(I),I=1,JM,2),SUM3,(DDELSQ(I),I=1,JM,2),SUM4,
                                                   (DSS(I),I=1,
    2 JM,2),SUM5,(DLS(I),I=1,JM,2),SUM6,(DT(I),I=1,JM,2),SU
                                                   M7,(DNL(I),
    3 I=1,JM,2),SUM8
  26 FORMAT((       3X, 5H  DY ,8F10.5)//(3X, 5H  DC ,8F10.5
                                                   )//(2X,
    1 6H DGRAD ,8F10.5)// (1X, 7H DDELSQ ,8F10.5)//(3X, 5H
                                                   DSS ,8F10.5)
    2 //(3X, 5H DLS ,8F10.5)//(3X,5H  DT ,8F10.5)//(3X,5H
                                                   DNL,8F10.5))
409  WRITE OUTPUT TAPE 6,36,PHASE,PHASEL
 36   FORMAT  (5X,9H TAND =   , F12.5,10H TANDL =   , F12.5)
      WRITE OUT PUT TAPE 6,35,SHIFT ,SHIFTL
  35 FORMAT (1X,25H PHASE SHIFT IN DEGREES = ,F12.5,20H PHA
                                                   SE LINEAR
    1=              ,F12.5     )
     GO TO 19
     END
```

5. BORN PHASES FOR THE COUPLED STATE

```
C     THIS IS A DIRECT COUPLED STATE CODE IN BORN APPROXIMAT
                                              ION AND
C     THE THREE MTRIX ELEMENTS OF TENSOR FORCE ARE THE INPUT
                                               WHILE THE
C     L.S SECOND ELEMENT  IS CALCULATED IN THE CODE USING TH
                                              E FIRST ONE
C     THERE ARE TWO SETS OF SAME INTEGRALS FOR L=J-1 ,L=J+1
                                              AND ONE FOR
C     THE INTEGRAL FOR NON DIAGONAL CASE. BOTH BLATT  BIEDEN
                                              HARN AND
C     BAR PHASES ARE CALCULATED.PHYS REV STAPP
C     THIS CODE WILL CALCULATE PHASE SHIFTS FOR       THE
                                               TRIPLET
C     (MIXED)   STATES OF NUCLEON-NUCLEON SYSTEM IN BORN APP
                                              ROXIMATION
C     THIS CODE WILL CALCULATE BORN PHASES BY DIRECT METHOD
                                              IN WHICH THE
C     OPERATORS OF VELOCITY DEPENDENT POTENTIAL ACT ON THE B
                                              ESSEL FUNCT'
C     IONS AND THEN THE RECURRENCE RELATIONS ARE USEDTO REMO
                                              VE FIRST AND
C     SECONDDERIVATIVES. THEN THE PHASE SHIFT EXPRESSIONS IN
                                              VOLVE ONLY
C     FIVE INTEGRALS A B C D E AND THE INTEGRALS ARE PERFORM
                                              ED USING
C     WEDDLE,S RULE. FUNCTION SFBESS IS NEEDE D  AND NO OF P
                                              OINTS OF
C     INTEGRATION MUST BE 6*M AND SO INPOINTSARE6M+1.
C     AMASS IS THE MASS OF MESON IN MEV    . GSQ IS THE COUPL
                                              ING CONSTANT
C     SS IS THE EXPECTATION VALUE OF (SIGMA1.SIGMA2) FOR THE
                                               STATE .
C      ALS IS THE SPIN ORBIT OPERATOR EXPECTATION VALUE FOR
                                              THE STATE
C     THAT IS (L.S).S12 IS THE TENSOR OPERATOR EXPECTATION V
                                              ALUE FOR THE
C     STATE. L.S AND S12 AVERAGE TO ZERO FOR SINGLET STATES.
C      ANL IS THE COEFFICIENT THAT MAKES NON LINEAR TERM ZER
                                              O OR TAKES
C     IT INTO ACCOUNT
C     J IS THE NO OF MESONS 12  OR 14 (FOR INCLUDING PHI) IN
                                              CLUDING CUT OFF
      COMMON RO,RMAX,XN,UF,AK,PQ,AF,BF,CF,DF,EF,SS,ALS,S12,A
                                              NL ,UFC    ,
     1  ELAB,BES ,FF,FJ,GF,KKK
```

```
      DIMENSIONW(10,20),GSQ(15),AMASS(15),U(15),A(15),B(15),
                                                 C(15),D(15)
     1,E(15),G(15),DY(15),DC(15),DGRAD(15),DDELSQ(15),DSS(15
                                                 ),DLS(15),DT
     2(15),  DNL(15)     ,CAPPA(15)    ,RY(15),RC(15),RGRAD(15
                                                 ),RDELSQ(15)
     3 ,RSS(15),RLS(15),RT(15),RNL(15) ,BES(6,8,100)  ,F(15)
      CALL BESTO
  18   READ INPUT TAPE 5,9,J
 9     FORMAT(I2)
 973   IF ( J-12) 967,967,966
966   READ INPUT TAPE 5,43,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 43    FORMAT(        12F6.0/2F6.0/12F6.0/2F6.0)
      GO TO 19
 967  READINPUT TAPE 5,968,
     1(AMASS(I),I=1,J ),(GSQ(I),I=1,J )
 968   FORMAT(        12F6.0/       12F6.0)
 19    WRITE OUT PUT TAPE 6,1150
1150  FORMAT( 1H1, 35H COUPLED DIRECT BORN PHASES
                                                   )
 971  READ INPUT TAPE 5,972,ELAB,FJ,T,SS,ALS,S1 ,S2,SN,ANL,R
                                                 0,RMAX,XN
 972  FORMAT(12F6.0)
       IF (ELAB- 1000.0) 969,973,973
969   IF (SS-1.0 )31,32,31
  32  WRITE OUTPUT T  APE 6,115
 115  FORMAT(     5X,30H THIS IS A TRIPLET SPIN STATE
                                                 )
      GO TO 199
  31  IF (SS+ 3.0) 41,42,41
  41  WRITE OUT PUT TAPE 6,125
 125  FORMAT (5X    ,20H WRONG INPUT DATA           )
      CALL EXIT
  42  WRITE OUTPUT  TAPE 6,135
 135  FORMAT(     5X, 30H THIS IS A SINGLET SPIN STATE
                                                   )
 199   ENERGY = ELAB
      WRITE OUTPUT TAPE 6,55,ENERGY,FJ,T,SS,ALS,S1,S2,SN,ANL
                                                 ,RO,RMAX,XN
  55  FORMAT (     ////1X,  9H ENERGY = ,F6.1,4H J = ,F4.1,9H
                                                 T1.T2 =  ,
     1 F5.1 ,7H S.S = ,F5.1,7H L.S =  ,F5.1, 10H S12(J-1)=
                                                 ,F7.3
     2 ,/10H S12(J+1)=  ,F7.3  ,10H S12(ND)=  ,F7.3,5H ANL=
                                                 ,F3.1
     3 , 4H RO=,F8.4,6H RMAX=,F4.1,4H XN=,F5.1   )
       AK = SQRTF(ELAB/(2.0*41.469))
      UN= 4.7583/AK
       DO 600 I=1,J
      CAPPA(I)=AMASS(I)/197.32
      U(I) = CAPPA(I)/AK
 600  CONTINUE
      W(1,1) =1.0
```

```
W(1,3)= -1.0
W(1,5)=0.0
W(1,7)=T
W(1,9)=-T
W(1,11)=0.0
W(2,1)=1.0
W(2,3)=0.0
W(2,5)=0.0
W(2,7)= T
W(2,9)= 0.0
W(2,11)=0.0
W(3,1)=1.0
W(3,3)=1.0
W(3,5)=0.0
W(3,7)= T
W(3,9)= T
W(3,11)=0.0
W(4,1)=1.0
W(4,3)= 1.0
 W(4,5)=0.0
W(4,7)= T
W(4,9)= T
W(4,11)=0.0
W(5,1)=0.0
W(5,3)= 1.0
W(5,5)=1.0/2.0
W(5,7)=0.0
W(5,9)= T
W(5,11)= (1.0/2.0)*T
W(6,1)=1.0
W(6,3)=3.0
W(6,5)= 0.0
W(6,7)= T
W(6,9)= 3.0*T
W(6,11)=0.0
W(7,1)=0.0
W(7,3)=1.0
W(7,5)=-1.0
W(7,7)= 0.0
W(7,9)= T
W(7,11)= -T
W(8,1)=0.0
W(8,3)=1.0
W(8,5)=1.0
W(8,7)=0.0
W(8,9)= T
W(8,11)=  T
  W(1,13)=W(1,3)
 W(2,13)=W(2,3)
  W(3,13) =W(3,3)
  W(4,13) =W(4,3)
   W(5,13)=W(5,3)
 W(6,13)=W(6,3)
 W(7,13)=W(7,3)
```

```
      W(8,13)=W(8,3)
      W(9,1)=0.0
      W(9,3)=1.0
      W(9,5)=-1.0
      W(9,7)=0.0
      W(9,9)=T
      W(9,11)=-T
      W(9,13)=W(9,3)
      RADIAN =57.29578
      JM=J-1
      JJ=J+1
      KKK=1
      S12=S1
      GO TO 1234
 1232 S12=S2
      ALS=-(ALS+3.0)
      GO TO 1234
 1233 S12=SN
 1234 I= -1
    3 I=I+2
      IF(I-JJ)5,6,41
    5 IF(I-1)41,1,12
   12 M=I
   10 M=M-2
      IF(M) 1,41,2
    2 IF (AMASS(I)-AMASS(M)) 10,7,10
 1    UF =U(I)
      UFC = U(I+1)
      CALL COPINT
      IF (KKK-3) 1253,1254,1254
 1254 G(I)=GF
      GO TO 1259
 1253 A(I) =AF
      B(I) =BF
      C(I) =CF
      D(I) = DF
      E(I) =EF
      F(I)=FF
      GO TO 3
    7 IF(KKK-3) 1258,1257,1257
 1257 G(I)=G(M)
      GO TO 1259
 1259 GO TO 3
 1258 A(I) = A(M)
      B(I) = B(M)
      C(I) = C(M)
      D(I) = D(M)
      E(I) = E(M)
      F(I)=F(M)
      GO TO 3
 6    WRITE OUTPUT TAPE 6,1255,KKK
 1255 FORMAT(5X, I3)
      SUM1=0.0
      SUM2=0.0
```

```
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      SUM6=0.0
      SUM7=0.0
      SUM8=0.0
      SUM9=0.0
      IF(KKK-3) 1251,1252,1252
 1252 WRITE OUTPUT TAPE 6,1256,(G(I),I=1,JM,2)
 1256 FORMAT(10X,5E15.6)
      DU 1300 I=1,JM,2
      RT(I) =(S12/12.0)*(GSQ(I)/UN)*G(I)*W(9,I)
 1300 SUM9=SUM9+RT(I)
      GO TO 1276
 1251 WRITE OUTPUT TAPE 6,33,(A(I),B(I),C(I),D(I),E(I),F(I),
                                                   I=1,JM,2)
   33 FORMAT (10X,6E15.6)
      DO 100  I=1,JM,2
      RY(I) =  GSQ(I)*UN*A(I)*W(1,I)
      SUM1=SUM1+RY(I)
      RC(I) =  (0.25*GSQ(I)*                  F(I)*W(2,I)/UN)
      SUM2=SUM2+RC(I)
      RGRAD(I)=-GSQ(I)/UN *B(I)*W(3,I)
      SUM3=SUM3+RGRAD(I)
      RDELSQ(I)=  -(GSQ(I)/UN)*A(I)*W(4,I)
      SUM4=SUM4+RDELSQ(I)
      RSS(I)= -(1.0/6.0)*(GSQ(I)/UN)*SS                *W(5,I)*F(
                                                   I)
      SUM5=SUM5+RSS(I)
      RLS(I)= (ALS/2.0)*(GSQ(I)/UN)*C(I)*W(6,I)
      SUM6=SUM6+RLS(I)
      RT(I) =(S12/12.0)*(GSQ(I)/UN)*D(I)*W(7,I)
      SUM7=SUM7+RT(I)
      RNL(I) =  -((GSQ(I)**2)/4.0)*E(I)*W(8,I)*ANL
      SUM8=SUM8+RNL(I)
  100 CONTINUE
      PHASEL= SUM1+SUM2+SUM3+SUM4+SUM5+SUM6+SUM7
      PHASE =PHASEL+SUM8
 247  SHIFTL=ATANF(PHASEL)*RADIAN
      SHIFT=ATANF(PHASE)*RADIAN
      IF(KKK-2) 1271,1272,41
 1271 IF (J-12) 302,302,303
 302  WRITE OUTPUT TAPE 6,15
   15 FORMAT(8X , 7H OMEGAS ,                           4X, 6H OMEGA ,
     1 4X,       4H ETA ,                     6X, 7H RHOS    ,      6X
                                                   ,
     2 4H RHO ,                  6X, 3H PI ,                    6X, 5H S
                                              UM        //)
  304 WRITE OUTPUT TAPE 6,75,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                              ,JM,2)
   75 FORMAT(2X,(6H MASS   , 6F10.2)//(2X, 6H  GSQ   , 6F10.2)
                                              )
      WRITE OUTPUT TAPE 6,76,(AMASS(I),I=2,J  ,2),(GSQ(I),I=2
                                              ,J  ,2)
```

```
  76 FORMAT(2X,(6H MASSC , 6F10.2)//(2X, 6H  GSQC , 6F10.2)
                                             )
      GO TO 409
 303  WRITE OUTPUT TAPE 6, 16
  16 FORMAT(8X , 7H OMEGAS ,                     4X, 6H OMEGA ,
    1  4X,      4H EIA ,              6X, 7H RHOS    ,     6X
                                             ,
    2 4H RHO ,              6X, 3H PI ,6X,4H PHI ,  6X, 5H S
                                           UM      //)
     WRITE OUTPUT TAPE 6,78,(AMASS(I),I=1,JM,2),(GSQ(I),I=1
                                            ,JM,2)
 78   FORMAT(2X,(6H MASS  , 7F10.2)//(2X, 6H  GSQ   , 7F10.2)
                                             )
     WRITE OUTPUT TAPE 6,77,(AMASS(I),I=2,J ,2),(GSQ(I),I=2
                                            ,J ,2)
  77  FORMAT(2X,(6H MASSC , 7F10.2)//(2X, 6H  GSQC , 7F10.2)
                                             )
 409  WRITE OUTPUT TAPE 6,36,PHASE,PHASEL
  36   FORMAT(5X, 25H X1 WITH NL TERM =           ,F12.6,5H
                                           X1 =,F12.6)
       X1=PHASE
      KKK=KKK+1
C     MAKE SURE YOU HAVE TAKEN OBVIOUSLY CARE OF BIG NL TERM
                                            TO THIS POINT
      GO TO 1232
 1272  WRITE OUT PUT TAPE 6,1275,PHASE ,PHASEL
 1275  FORMAT ( 5X, 25H X2 WITH NL TERM                 ,F12.6,7H
                                            X2 =    ,
    1  F12.6)
       X2=PHASE
      KKK=KKK+1
      GO TO 1233
 1276  YJ=SUM9
       WRITE OUTPUT TAPE 6,1277,YJ
 1277  FORMAT(5X, 6H YJ =   , F12.6)
      DIFR=X1-X2
       SUMM=X1+X2
     ROOT= SQRTF((X1-X2)**2+(2.0*YJ)**2)
      TD1=0.5*(SUMM+ROOT)
      TD2= 0.5*(SUMM-ROOT)
      D1=ATANF(TD1)
      D2=ATANF(TD2)
       D1P=D1*RADIAN
       D2P = D2*RADIAN
       TNE1 = (2.0*YJ)/(DIFR+ROOT)
      TNE2 = (-DIFR+ROOT)/(2.0*YJ)
      EJ  = ATANF(TNE1)
            EJ1P = ATANF(TNE2)*RADIAN
      EJP= EJ*RADIAN
      WRITE OUTPUT TAPE 6,721,D1P,D2P,EJP,FJ       ,EJ1P
 721  FORMAT(5X,45H BLATT BIEDENHARN PHASES IN DEGREES   D(J
                                          -1)=   , F12.
    16, 8H D(J+1)=   , F12.6, 8H E(J) =    ,F12.6 ,6H J=     ,
                                            F3.1  ,  7H
```

```
    2E(J)=    ,F12.6    )
      DA=D1
        DB=D2
      EPN=EJ
     SIN2EB=SINF(2.*EPN)*SINF(DA-DB)
     CSK2EB=1./SIN2EB**2
     TAN2EB=1./SQRTF(CSK2EB-1.)
     IF(SIN2EB)1001,2001,2001
1001 TAN2EB=-TAN2EB
2001 SINDLB=TAN2EB*COSF(2.*EPN)/SINF(2.*EPN)
       EJBP          =.5*ATANF(TAN2EB)*57.29578
     CSKDLB=1./SINDLB**2
     TANDLB=1./SQRTF(CSKDLB-1.)
     IF(SINDLB)3001,4001,4001
3001 TANDLB=-TANDLB
4001 DLBM=ATANF(TANDLB)
     DLBP=DA+DB
      DB1          =.5*(DLBM+DLBP)*57.29578
      DB2    =  DL1                    -DLBM*57.29578
       WRITE OUTPUT TAPE 6,722,DB1,DB2,EJBP
 722  FORMAT(5X, 45H STAPP BAR PHASES IN DEGREES   DB(J-1)=
                                                              ,
    1F12.6,10H DB(J+1)=    ,F12.6,7H EB(J)=   , F12.6)
     GO TO 19
     END
```

SUBROUTINE "COPINT"
for the Coupled State Integrals

```
      SUBROUTINE COPINT
C      SUBROUTINE COPINT FOR COUPLED STATES
      COMMON RO,RMAX,XN,UF,AK,PQ,AF,BF,CF,DF,EF,SS,ALS,S12,A
                                                NL ,UFC    ,
     1  ELAB,BES ,FF,FJ,GF,KKK
      DIMENSION Z(1201),GA(1201),GB(1201),GC(1201),GD(1201),
                                                GE(1201)   ,
     1 BES(6,8,100)      ,GG(1201)    ,TF(1201)
      ZO=RO*AK
      ZMAX =RMAX*AK
      DELZ =(ZMAX-ZO)/XN
         GO TO (1241,1242,1243),KKK
 1241  FL=FJ-1.0
C      FL HERE IS NOT TO BE CONFUSED WITH FL IN THE COMMON OF
                                                 BESTO INDEX
        GO TO 1244
 1242   FL=FJ+1.0
        GO TO 1244
 1243  FL = FJ
 1244 M=FL
      K=M-1
       N = M+1
      NN=XN
      L=NN+1
       EX1=EXPF(-UF*ZO)
       EX2=EXPF(-UFC*ZO)
        EXD=EXPF(-UF*DELZ)
       EXDC= EXPF(-UFC*DELZ)
      M1=M+1
      K1=K+1
      N1=N+1
 1245   IF(ELAB-50.0    )      21,21,22
  21  I1=1
      GO TO 23
   22 IF (ELAB-142.0) 99,99,25
   99 I1=2
        GO TO 23
  25   IF ( ELAB-310.0    ) 26,26,27
  27  WRITE OUTPUT TAPE 6,28
  28     FORMAT(20H WRONG ENERGY                              )
         CALL EXIT
  26   I1=3
  23   DO 100 I=1,L
      AI=I
      Z(I) =ZO+(AI-1.0)*DELZ
      ZZ=Z(I)
```

```
      BESN =BES(I1,N1,I)
      BESM=BES(I1,M1,I)
      IF (K1) 27,401,402
 402  BESK=BES(I1,K1,I)
      GO TO 404   .
 401   BESK = (COSF(ZZ)/ZZ)
 404  BESQ=(BESM)**2
       EX = ( EX1 - EX2)
       IF(KKK-3)1246,1247,1247
1246  GA(I) =EX*ZZ*BESQ
      GB(I) = ( EX + ( UF*EX1 -UFC*EX2 ) *ZZ)*(FL*BESK-(FL+1
                                                    .0)*BESN)*BE
     1 SM*(1.0/(2.0*FL +1.0))
       GC(I) = EX * ( BESQ/ZZ) + ( UF*EX1 -UFC*EX2 )*BESQ
       GD(I) = (((UF**2)*EX1-(UFC**2)*EX2)*ZZ+3.0*(UF*EX1-UF
                                                    C*EX2))*BESQ
     1 +3.0*EX*(BESQ/ZZ)
      GE(I) =(EX**2)*BESQ
        TF(I)=((UF**2)*EX1-(UFC**2)*EX2)*BESQ*ZZ
       GO TO 1249
1247    GG(I)=(((UF**2)*EX1-(UFC**2)*EX2)*ZZ+3.0*(UF*EX1-UFC
                                                    *EX2))*(BESK
     1 * BESN)+3.0*EX*(BESN/ZZ)*BESK
1249   EX1=EX1*EXD
       EX2=EX2*EXDC
100    CONTINUE
       IF(KKK-3  )1248,132,132
1248  CALL WEDDLE(DELZ,GA,L,RESULT)
      AF = RESULT
      CALL WEDDLE(DELZ,GB,L,RESULT)
      BF = RESULT
       CALL WEDDLE(DELZ,TF,L,RESULT)
       FF=RESULT
       IF (ALS) 20,30,20
20    CALL WEDDLE(DELZ,GC,L,RESULT)
      CF = RESULT
      GO TO 40
 30     CF = 0.0
 40   IF(S12) 50,60,50
 50   CALL WEDDLE(DELZ,GD,L,RESULT)
      DF = RESULT
       GO TO 70
 60    DF=0.0
 70    IF (ANL) 80,90,80
 80   CALL WEDDLE(DELZ,GE,L,RESULT)
       EF=RESULT
      GO TO 130
90      EF=0.0
       GO TO 130
 132   CALL WEDDLE (DELZ,GG,L,RESULT)
        GF= RESULT
 130  RETURN
        END
```

```
DIRECT   BORN PHASES
   THIS IS A TRIPLET SPIN STATE


ENERGY = 310.0 L = 2.0 T1.T2 =  -3.0 S.S =   1.0 L.S =  -1.0 S12 =   2.0 ANL=0.  RO=  0.0010 RMAX= 5.0 XN= 60.0
          0.234138E-02 0.435806E-02 0.317928E-02 0.182573E-01 0.           0.871942E-02-0.770888E-02-0.131421E-05
          0.234138E-02 0.435806E-02 0.317928E-02 0.182573E-01 0.           0.871942E-02-0.770888E-02-0.131421E-05
          0.937147E-02 0.953017E-02 0.857958E-02 0.448046E-01 0.           0.190659E-01-0.192332E-01 0.160703E-05
          0.262624E-02 0.467089E-02 0.345074E-02 0.196975E-01 0.           0.934525E-02-0.832066E-02-0.113546E-05
          0.262624E-02 0.467089E-02 0.345074E-02 0.196975E-01 0.           0.934525E-02-0.832066E-02-0.113546E-05
          0.235697E-00 0.147381E-01 0.544910E-01 0.192541E-00 0.           0.290679E-01-0.131475E-00-0.834727E-03
          0.706792E-03 0.231068E-02 0.147475E-02 0.904740E-02 0.           0.462317E-02-0.381915E-02-0.272797E-05
       OMEGAS     OMEGA      ETA       RHOS        RHO        PI        PHI       SLM

 MASS     782.80    782.80    548.70    763.00    763.00    137.50   1019.50

  GSL      14.70     14.70     14.70     14.70     14.70     14.70     14.70
 MASSC   1500.00   1500.00   1500.00   1500.00   1500.00   1500.00   2000.00

  GSQC     14.70     14.70     17.70     14.70     14.70     14.70     14.70
   DY    4.84194  -4.84194  0.       -15.88981  15.88981  0.        -1.46482  -1.46482

   DC    0.74593  0.        0.        -2.39713  0.        0.        0.        -1.65210

 DGRAD  -1.49103  -1.49103  -0.        4.78407   4.78407  -0.       -0.79069   5.79708

DDELSQ  -0.80119  -0.80119  -0.        2.69419   2.69419  -0.       -0.24187   3.54346

  DSS   -0.       -0.49730  -0.54369  -0.        1.59860   2.48527  -0.26368   2.77896

  DLS   -0.54397  -1.63153  -0.        1.77075   5.29871  -0.       -0.75696   4.14503

   DT    0.        1.04119  -2.55373  0.        -3.36645  29.89861  0.51600  26.20679

  DNL   -0.       -0.       -0.       -0.       0.        0.        -0.       0.
    TANDG=       0.72880 TANDLG=       0.72880
PHASE GREEN IN DEGREES =   36.08449 PHASE LINEAR GREEN=    36.08449
DGRADB  -0.       -0.74564  -0.       -0.       2.39624  -0.       -0.39536   1.25638

DDLSQB  -0.       -0.40061  -0.       -0.       1.34784  -0.       -0.12094   0.82647

DKMREX  -0.       -0.00011   0.       -0.       0.00029  -0.       -0.00013   0.00005

DBRTEX  -0.       -2.63619  -0.       -0.       8.47976  -0.       -1.30672   4.58739
    TAND(KEMMER)=          0.72880 TAND(BREIT)=         0.68194

PHASE KEMMER=     36.08453 PHASE BREIT=      34.29167

GREEN-KEMMER=          -0.00003  GREEN-BREIT =          1.79282 BREIT-KEMMER=          -1.79286
```

Appendix B Cont.


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COUPLED DIRECT BORN PHASES
    THIS IS A TRIPLET SPIN STATE


ENERGY = 142.0 J = 3.0 T1.T2 =  -3.0 S.S =   1.0 L.S =   2.0 S12(J-1)= -0.556
S12(J+1)= -1.429 S12(ND)=   2.969 ANL=0.  RO=  0.0010 RMAX= 5.0 XN= 60.0
    1
          0.955736E-03  0.286648E-02  0.186122E-02  0.113207E-01  0.              0.573701E-02
          0.379222E-03  0.159819E-02  0.974046E-03  0.612102E-02  0.              0.319888E-02
          0.208414E-02  0.453279E-02  0.316792E-02  0.185749E-01  0.              0.907114E-02
          0.955736E-03  0.286648E-02  0.186122E-02  0.113207E-01  0.              0.573701E-02
          0.432667E-03  0.174172E-02  0.106975E-02  0.669538E-02  0.              0.348614E-02
          0.126749E-00  0.185165E-01  0.410445E-01  0.158690E-00  0.              0.355566E-01
      OMEGAS      OMEGA      ETA        RHOS        RHO        PI        SLM

 MASS     650.00    782.80    548.70    650.00    763.00    137.50

  GSW      10.00      6.70      5.00      7.50      5.00     14.70
 MASSC   1500.00   1500.00   1500.00   1500.00   1500.00   1500.00

  GSWC     10.00      6.70      5.00      7.50      5.00     14.70
   X1 WITH NL TERM =            -0.080655 X1 =   -0.080655
    2
          0.109156E-04  0.345129E-04  0.110529E-04  0.102150E-03  0.              0.689910E-04
          0.241088E-05  0.109955E-04  0.330206E-05  0.318962E-04  0.              0.219900E-04
          0.394605E-04  0.892058E-04  0.307306E-04  0.270208E-03  0.              0.178016E-03
          0.109156E-04  0.345129E-04  0.110529E-04  0.102150E-03  0.              0.689910E-04
          0.298583E-05  0.129514E-04  0.392030E-05  0.376619E-04  0.              0.259010E-04
          0.311311E-01  0.641785E-02  0.496043E-02  0.237103E-01  0.              0.882900E-02
   X2 WITH NL TERM              -0.025355 X2 =     -0.025355
    3
          0.765520E-03  0.321934E-03  0.155954E-02  0.765520E-03  0.364722E-03
          0.347963E-01
   YJ =      0.103655
   BLATT BIEDENHARN PHASES IN DEGREES   D(J-1)=    3.106636 D(J+1)=   -9.106161 E(J) =    52.467885 J=   3.0
   STAPP BAR PHASES IN DEGREES   DB(J-1)=          -4.596223 DB(J+1)=    -1.403301 EB(J)=    5.897077
```

Appendix B Cont.


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EFFECTIVE BORN PHASES
    THIS IS A SINGLET SPIN STATE


ENERGY = 310.0 L = 3.0 T1.T2 =  -3.0 S.S =  -3.0 L.S =   0.  S12 =   0.  ANL=0.  RO=  0.0010 RMAX= 5.0 XN= 60.0
EFFECTIVE MASS NEGATIVE 0.455091E 02 AT RADIUS=      0.001000
EFFECTIVE MASS NEGATIVE 0.280237E 02 AT RADIUS=      0.084317
EFFECTIVE MASS NEGATIVE 0.172614E 02 AT RADIUS=      0.167633
EFFECTIVE MASS NEGATIVE 0.105832E 02 AT RADIUS=      0.250950
EFFECTIVE MASS NEGATIVE 0.640536E 01 AT RADIUS=      0.334267
EFFECTIVE MASS NEGATIVE 0.377062E 01 AT RADIUS=      0.417583
EFFECTIVE MASS NEGATIVE 0.209582E 01 AT RADIUS=      0.500900
EFFECTIVE MASS NEGATIVE 0.102297E 01 AT RADIUS=      0.584217
EFFECTIVE MASS NEGATIVE 0.330611E-00 AT RADIUS=      0.667533
           0.589393E-02  0.411129E-01  0.           0.           0.
           0.589393E-02  0.411129E-01  0.           0.           0.
           0.175756E-01  0.411129E-01  0.           0.           0.
           0.642278E-02  0.411129E-01  0.           0.           0.
           0.642278E-02  0.411129E-01  0.           0.           0.
           0.241563E-00  0.411129E-01  0.           0.           0.
        OMEGAS     OMEGA      ETA       RHOS        RHO        PI         SUM

  MASS     782.80     782.80     548.70     763.00     763.00     137.50

  MSQ       14.70      14.70      14.70      14.70      14.70      14.70
  MASSC   1500.00    1500.00    1500.00    1500.00    1500.00    1500.00

  GSQL      14.70      14.70      17.70      14.70      14.70      14.70
  DY     12.03261  -12.03261   0.        -34.88125   34.88125   0.         -0.

  DC      1.45379  -0.        -0.         -4.32980  -0.        -0.         -2.88155

  DGRAD   0.        0.         0.          0.        0.         0.         -2.35427

  DDELSQ  0.89117   0.89117   -0.         -2.08145  -2.08145   -0.         -2.38088

  DSS    -0.       -2.90570   -1.14747    -0.        8.60986    6.54132   11.04818

  DLS     0.        0.         0.         -0.       -0.         0.          0.

  DT      0.        0.        -0.          0.       -0.         0.          0.

  DNL    -0.       -0.        -0.         -0.        0.         0.          0.
   TAND =       0.10340 TANEL =       0.10340
PHASE SHIFT IN DEGREES =      5.90325 PHASE LINEAR   =         5.90325
```

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BIOGRAPHICAL SKETCH

Ravi Dutta Sharma was born on September 4,1941, in Udaipur, India. He was brought up by his parents in a Sanskrit loving atmosphere in Jaipur, India. He obtained a medal for standing first in school at the high school examination in 1957, obtained distinctions and stood ninth in the order of merit in Rajasthan State, India. He passed the Intermediate in Science with biology (1959) and additionally with mathematics (1960), in order to be able to study physics, obtaining distinctions. He obtained a Bachelor of Science degree from the Rajasthan University, Jaipur in 1961. He obtained a Master of Science degree specializing in"Advanced mathematical physics, advanced quantum mechanics and nuclear physics" from the Institute of Science, Bombay University, India in 1963, standing first in this option. He was also a demonstrator in physics at Bhavan's College, Bombay, during 1961-62. In Sept,1963 he started working for the Doctor of Philosophy degree with a major in physics and a minor in mathematics and completed all the requirements in July, 1966. He also attended a Winter Institute in Quantum Chemistry in 1963-64. He has been elected a member of Sigma Pi Sigma and an associate member of Sigma Xi honorary scientific societies, has three publications and has been a member of various societies in India including the Experiment in International Living. He is also a member of the American Physical Society and has presented three papers in their meetings.

He has been married to Sneh Prabha since June, 1963 and has a son Amit born in Gainesville in August, 1964.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

(August, 1966)

Dean, College of
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