MECHANICS OF MATERIALS

THEORETICAL MECHANICS

INTEGRAL CALCULUS

DIFFERENTIAL CALCULUS

ANALYTIC GEOMETRY

TRIGONOMETRY

ALGEBRA

WORKS BY

L. A. WATERBURY

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A VEST-POCKET

HANDBOOK

OF

MATHEMATICS

FOR

ENGINEERS

BY

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PREFACE.

This handbook is intended as a reference book, for the use of those who have studied or are studying the branches of mathematics usually taught in engineering courses. It is not intended for a text book, and does not, therefore, attempt to prove many of the formulæ which are given.

Most of the material in this book was obtained from the following sources: algebra from Hall & Knight's Algebra (Macmillan Co.); trigonometry from Bowser's Trigonometry; analytic geometry from Candy's Analytic Geometry; calculus from Taylor's Differential and Integral Calculus; theoretical mechanics from Church's Mechanics of Engineering; and mechanics of materials from Merriman's Mechanics of Materials; to all of which the writer is very much indebted and from all these Authors he has received permission to use the material. The reader is referred to these works for the proof and explanation of the various formulæ.

L. A. W.

TUCSON, ARIZ., March, 1908.

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CONTENTS.

ALGEBRA.

EXPONENTS AND LOGARITHMS.

If $a^m = b$, $m = \log_a b$. $a^m \cdot a^n = a^{m+n}$,

$\therefore$ $\log x \cdot y = \log x + \log y$. $a^m \div a^n = a^{m-n}$,

$\therefore$ $\log (x \div y) = \log x - \log y$.

$$(a^m)^2 = a^m \cdot a^m = a^{2m},$$

$\therefore$ $\log x^2 = 2 \cdot \log x$. $(a^m)^n = a^{m \cdot n}$,

$\therefore$ $\log x^n = n \cdot \log x$. $a^0 = 1$,

$\therefore$ $\log (1) = 0$.

The base of the common system of logarithms is 10.

The base of the natural system of logarithms is

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}}$$

$$+ \frac{1}{\underline{|5}} + \ldots = 2.7182818284.$$

The cologarithm of a number is the logarithm of its reciprocal. $\operatorname{Log}\left(\frac{1}{x}\right) = 0 - \log x$.

To transform a logarithm from base e to base 10, multiply by $\log_{10} e$.

$$\operatorname{Log}_{10} e = 0.43429448.$$

$$\operatorname{Log}_e 10 = 2.30258509.$$

$$\operatorname{Log}_{10} e = \frac{1}{\log_e 10}.$$

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QUADRATIC EQUATIONS.

$$ax^2+bx+c=0,$$

$$x=\frac{-b\pm\sqrt{b^2-4\,ac}}{2\,a}.$$

PROPORTION.

If $a:b::c:d$,

$$\frac{a}{b}=\frac{c}{d}, \quad \text{or} \quad \frac{b}{a}=\frac{d}{c},$$

$$ad=bc, \qquad \frac{a+b}{b}=\frac{c+d}{d},$$

$$\frac{a-b}{b}=\frac{c-d}{d}, \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

ARITHMETICAL PROGRESSION.

$$a,\ a+d,\ a+2\,d,\ \ldots$$

Last term, $L=a+(n-1)\,d$.

Sum of terms,

$$S=\frac{n}{2}(a+L)=\frac{n}{2}[2\,a+(n-1)\,d].$$

GEOMETRICAL PROGRESSION.

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

Last term, $L=ar^{n-1}$.

Geometric mean, $M=\sqrt{ab}$.

Sum, $$S=\frac{a\,(r^n-1)}{r-1}$$

$$=\frac{a\,(1-r^n)}{1-r}=\frac{rL-a}{r-1}.$$

For an infinite geometrical series, the sum to infinity is $S=\dfrac{a}{1-r}$.

HARMONIC PROGRESSION.

a, *b*, *c* are in harmonic progression if

$$\frac{a}{c}=\frac{a-b}{b-c},$$

or if $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in arithmetical progression.

PERMUTATIONS AND COMBINATIONS.

ab and *ba* are two permutations but only one combination.

The number of permutations possible of n things taken r at a time is

$$^nP_r=n\,(n-1)\,(n-2)\ldots(n-r+1).$$

$$^nP_n=\underline{\lfloor n}.$$

$$(\underline{\lfloor n}=1\times2\times3\times4\ldots\times n).$$

$$^nC_r=\frac{^nP_r}{\underline{\lfloor n}}=\frac{\underline{\lfloor n}}{\underline{\lfloor r}\,\underline{\lfloor n-r}}$$

$$={}^nC_{n-r}.$$

BINOMIAL THEOREM.

$$(a+b)^n=a^n+n\,.\,a^{n-1}\,.\,b$$

$$+\frac{n\,.\,(n-1)}{\underline{\lfloor 2}}\,.\,a^{n-2}\,.\,b^2$$

$$+\frac{n\,.\,(n-1)\,(n-2)}{\underline{\lfloor 3}}\,.\,a^{n-3}\,.\,b^3$$

$$+\ldots\qquad\ldots\qquad\ldots$$

SERIES.

1. An infinite series in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term.

2. An infinite series in which all of the terms are of the same sign is divergent if each term is greater than some finite quantity, however small.

3. An infinite series is convergent if from and after some fixed term the ratio of each term to the preceding term is numerically less than unity.

4. An infinite series in which all the terms are of the same sign is divergent if from and after some fixed term the ratio of each term to the preceding term is greater than unity, or is equal to unity.

5. If there are two infinite series in each of which all of the terms are positive, and if the ratio of the corresponding terms in the two series is always finite, the two series are both convergent, or both divergent.

DETERMINANTS.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 . b_2 . c_3 + a_2 . b_3 . c_1 + a_3 . b_1 . c_2 - a_1 . b_3 . c_2 - a_2 . b_1 . c_3 - a_3 . b_2 . c_1 .$$

If
$$\begin{cases} a_1 x + b_1 y + c_1 z + d_1 = 0, \\ a_2 x + b_2 y + c_2 z + d_2 = 0, \\ a_3 x + b_3 y + c_3 z + d_3 = 0, \end{cases}$$

then

$$\frac{x}{\begin{vmatrix} b_1 c_1 d_1 \\ b_2 c_2 d_2 \\ b_3 c_3 d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 c_1 d_1 \\ a_2 c_2 d_2 \\ a_3 c_3 d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 b_1 d_1 \\ a_2 b_2 d_2 \\ a_3 b_3 d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix}}$$

TRIGONOMETRY.

Radius = 1.

$AB = \sin\theta.$

$OA = \cos\theta.$

$CD = \tan\theta.$

$EF = \cot\theta.$

$OD = \sec\theta.$

$OF = \operatorname{cosec}\theta.$

$AC = \operatorname{vers}\theta = 1 - \cos\theta.$

$BG = \operatorname{covers}\theta = 1 - \sin\theta.$

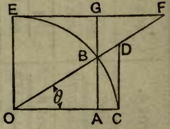

Fig. 1.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\sec^2\theta = 1 + \tan^2\theta.$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

$$\operatorname{exsec}\theta = \sec\theta - 1.$$

For θ in radians,

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor\underline{3}} + \frac{\theta^5}{\lfloor\underline{5}} - \frac{\theta^7}{\lfloor\underline{7}} + \dots$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{\lfloor\underline{4}} - \frac{\theta^6}{\lfloor\underline{6}} + \dots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2 \cdot \theta^5}{3 \cdot 5} + \frac{17\,\theta^7}{3 \cdot 3 \cdot 5 \cdot 7} + \dots$$

Function.	$90° - \theta$.	$\sin\theta$.	$\cos\theta$.	$\tan\theta$.	$\cot\theta$.	$\sec\theta$.	$\operatorname{cosec}\theta$.
$\sin\theta$	$\cos(90° - \theta)$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\pm\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sin(90° - \theta)$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\cot(90° - \theta)$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\tan(90° - \theta)$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\operatorname{cosec}(90° - \theta)$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\sec(90° - \theta)$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

$$\sin(A+B)=\sin A \,.\, \cos B+\cos A \,.\, \sin B.$$

$$\sin(A-B)=\sin A \,.\, \cos B-\cos A \,.\, \sin B.$$

$$\cos(A+B)=\cos A \,.\, \cos B-\sin A \,.\, \sin B.$$

$$\cos(A-B)=\cos A \,.\, \cos B+\sin A \,.\, \sin B.$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A \,.\, \tan B}.$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A \,.\, \tan B}.$$

$$\sin 2A=2 \,.\, \sin A \,.\, \cos A.$$

$$\cos 2A=\cos^2 A-\sin^2 A$$
$$=2\cos^2 A-1$$
$$=1-2 \,.\, \sin^2 A.$$

$$\tan 2A=\frac{2 \,.\, \tan A}{1-\tan^2 A}.$$

$$\sin\left(\frac{A}{2}\right)=\sqrt{\frac{1}{2}(1-\cos A)}.$$

$$\cos\left(\frac{A}{2}\right)=\sqrt{\frac{1}{2}(1+\cos A)}.$$

$$\tan\left(\frac{A}{2}\right)=\frac{1-\cos A}{\sin A}.$$

$$\sin 3A=3 \,.\, \sin A-4 \,.\, \sin^3 A.$$

$$\cos 3A=4\cos^3 A-3\cos A.$$

$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

$$\sin A+\sin B=2 \,.\, \sin\frac{A+B}{2} \,.\, \cos\frac{A-B}{2}.$$

$$\sin A-\sin B=2\cos\frac{A+B}{2} \,.\, \sin\frac{A-B}{2}.$$

$$\cos A+\cos B=2\cos\frac{A+B}{2} \,.\, \cos\frac{A-B}{2}.$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2} \,.\, \sin\frac{A-B}{2}.$$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A+B).$$

$$\frac{\sin A + \sin B}{\cos A - \cos B} = \cot \tfrac{1}{2}(A-B).$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A-B).$$

$$\frac{\sin A - \sin B}{\cos A - \cos B} = \cot \tfrac{1}{2}(A+B).$$

$$\frac{\cos A + \cos B}{\cos A - \cos B} = \cot\left(\frac{A+B}{2}\right).\cot\left(\frac{A-B}{2}\right).$$

PLANE TRIANGLES.

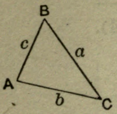

Fig. 2.

$$A + B + C = 180°.$$

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} . \cos \frac{B}{2} . \cos \frac{C}{2}.$$

$$\cos A + \cos B + \cos C = 1 + 4 . \sin \frac{A}{2} . \sin \frac{B}{2} . \sin \frac{C}{2}.$$

$$\tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a^2 = b^2 + c^2 - 2 . b . c . \cos A.$$

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$\text{Area} = \tfrac{1}{2}\, b \,.\, c \,.\, \sin A$$
$$= \frac{a^2 \sin B \,.\, \sin C}{2 \,.\, \sin A}$$
$$= \sqrt{s\,(s-a)\,(s-b)\,(s-c)},$$

where $s = \tfrac{1}{2}\,(a+b+c)$.

SPHERICAL TRIANGLES.

Center of sphere is at O.

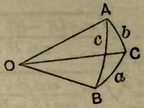

Fig. 3.

Right Spherical Triangles. Let C represent the right angle.

$$\cos c = \cos a \,.\, \cos b.$$
$$\sin b = \sin B \,.\, \sin c.$$
$$\tan a = \cos B \,.\, \tan c.$$
$$\tan a = \tan A \,.\, \sin b.$$
$$\tan A \,.\, \tan B = \frac{1}{\cos c}.$$
$$\cos A = \sin B \,.\, \cos a.$$

OBLIQUE SPHERICAL TRIANGLES.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \text{modulus}.$$

$$\cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A.$$
$$\cos A = -\cos B \,.\, \cos C + \sin B \,.\, \sin C \,.\, \cos a.$$
$$\cot a \,.\, \sin b = \cot A \,.\, \sin C + \cos C \,.\, \cos b.$$

Let $s = \tfrac{1}{2}\,(a+b+c),$

$S = \tfrac{1}{2}\,(A+B+C),$

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then $\sin\left(\frac{A}{2}\right)=\sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin b\,.\,\sin c}}.$

$$\cos\left(\frac{A}{2}\right)=\sqrt{\frac{\sin s\,.\,\sin(s-a)}{\sin b\,.\,\sin c}}.$$

$$\tan\left(\frac{A}{2}\right)=\sqrt{\frac{\sin(s-b)\,.\,\sin(s-c)}{\sin s\,.\,\sin(s-a)}}.$$

$$\sin\left(\frac{a}{2}\right)=\sqrt{-\frac{\cos S\,.\,\cos(S-A)}{\sin B\,.\,\sin C}}.$$

$$\cos\left(\frac{a}{2}\right)=\sqrt{\frac{\cos(S-B)\,.\,\cos(S-C)}{\sin B\,.\,\sin C}}.$$

$$\tan\left(\frac{a}{2}\right)=\sqrt{-\frac{\cos S\,.\,\cos(S-A)}{\cos(S-B)\,.\,\cos(S-C)}}.$$

ANALYTIC GEOMETRY.

TRANSFORMATION OF COÖRDINATES.

To transform an equation of a curve from one system of coördinates to another system, substitute for each variable its value in terms of variables of the new system.

Rectangular System. Old Axes Parallel to New Axes.

$x' = x - h.$

$y' = y - k.$

$x = x' + h.$

$y = y' + k.$

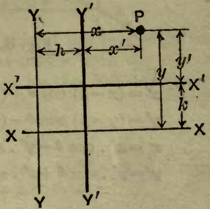

Fig. 4.

Rectangular System. Old Origin Coincident with New Origin.

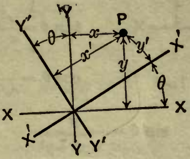

Fig. 5.

$x' = x \;.\cos\theta + y \;.\sin\theta.$

$y' = y \;.\cos\theta - x \;.\sin\theta.$

$x = x' .\cos\theta - y' .\sin\theta.$

$y = y' \;\; \cos\theta + x' .\sin\theta.$

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Rectangular System. Old Axes not Parallel to New Axes. Old Origin not Coincident with New Origin.

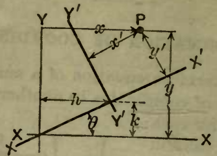

Fig. 6.

$$x' = (x-h)\cos\theta + (y-k)\sin\theta.$$
$$y' = (y-k)\cos\theta - (x-h)\sin\theta.$$
$$x = x'\,.\cos\theta - y'\,.\sin\theta + h.$$
$$y = y'\,.\cos\theta + x'\,.\sin\theta + k.$$

Polar and Rectangular Systems.

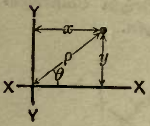

Fig. 7.

$$x = \rho\,.\cos\theta.$$
$$y = \rho\,.\sin\theta.$$
$$\rho = \sqrt{x^2+y^2}.$$
$$\tan\theta = \frac{y}{x}.$$
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}.$$
$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$
$$\cot\theta = \frac{x}{y}.$$
$$\sec\theta = \frac{\sqrt{x^2+y^2}}{x}.$$
$$\operatorname{cosec}\theta = \frac{\sqrt{x^2+y^2}}{y}.$$

THE STRAIGHT LINE.

Equations of Straight Line. An equation of the first degree containing but two variables can always be represented by a straight line.

The equation of the straight line may assume the following forms, for the rectangular system of coördinates.

$$Ax+By+C=0 \quad . \quad . \quad . \quad . \quad (1)$$

$$y=mx+k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which m is the value of the tangent of the angle which the line makes with the X-axis, and k is the intercept on the Y-axis between the line and the X-axis.

$$y-y'=A\ (x-x') \quad . \quad . \quad . \quad (3)$$

in which x', y' are the coördinates of a point of the line, and A is a constant.

$$y-y'=\frac{y'-y''}{x'-x''}(x-x') \quad . \quad . \quad (4)$$

in which x', y' and x'', y'' are the coördinates of two points of the line.

The *polar* equation of a straight line is

$$\rho \ . \cos\ (\theta-\alpha)=k \quad (5)$$

where k is the length of the normal ON.

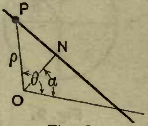

Fig. 8.

Distance between Two Points. The distance between two points, x', y' and x'', y'', is equal to

$$\sqrt{(x'-x'')^2+(y'-y'')^2}.$$

The distance between two points, ρ_1, θ_1, and ρ_2, θ_2, is equal to

$$\sqrt{\rho_1{}^2+\rho_2{}^2-2\ \rho_1 \ . \ \rho_2 \ . \cos\ (\theta_1-\theta_2)}.$$

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Angle between Two Lines. The angle between two lines, $y = m'x + k'$ and $y = m''x + k''$, is the difference between the two angles whose tangents are m' and m''.

Area of Triangle. The area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3), is equal to

$$\frac{1}{2} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

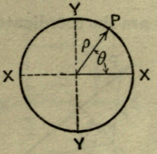
THE CIRCLE.

The most general equation of the circle, for rectangular coördinates, is

$$(x-a)^2 + (y-b)^2 = R^2,$$

in which a, b are the coordinates of the center of the circle, and R is the radius.

The following are special equations of the circle for rectangular and polar systems of coördinates.



$$x^2 + y^2 = R^2.$$
$$\rho = R.$$

Fig. 9.

$$y^2 = 2\,Rx - x^2.$$
$$\rho = 2\,R \,.\, \cos\theta.$$

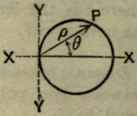


Fig. 10.

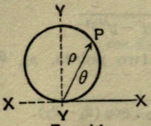


Fig. 11.

$$x^2 = 2\,Ry - y^2.$$
$$\rho = 2\,R \,.\, \sin\theta.$$

THE PARABOLA.

If the Y-axis coincides with the directrix, DM, then

$$y^2 = 4\,a\,(x-a).$$

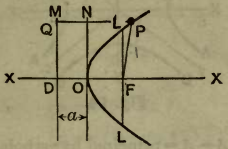


Fig. 12.

If the Y-axis coincides with ON, passing through the vertex, then

$$y^2 = 4\,ax.$$

In Fig. 12, F is the focus, $OF = OD = a$, and $L - L$ is the latus rectum $= 4\,a$.

Eccentricity, $e = \dfrac{FP}{PQ} = 1.$

THE ELLIPSE.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

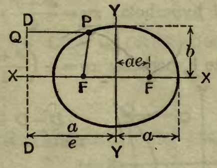


Fig. 13.

F, F are foci.
Eccentricity, $e < 1$.
The area of the ellipse is equal to πab.

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THE HYPERBOLA.

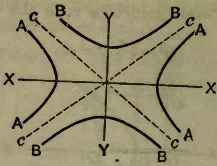

Fig. 14.

$A-A$ = principal hyperbola.
$B-B$ = conjugate hyperbola.
$c-c$ = asymptote.

Principal hyperbola:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Asymptotes: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$

Conjugate hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$

When referred to the asymptotes as axes, the equations become:

Principal hyperbola: $xy=\frac{a^2+b^2}{4}.$

Conjugate hyperbola: $xy=-\left(\frac{a^2+b^2}{4}\right).$

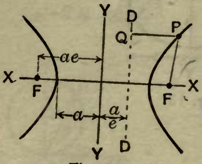

Fig. 15.

$D-D$ is the directrix.

F, F are foci.

$$\frac{FP}{PQ}=e>1.$$

THE CYCLOID.

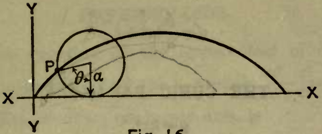

Fig. 16.

$$\begin{cases} x=a\,(\theta-\sin\theta), \\ y=a\,(1-\cos\theta), \end{cases}$$

or $$x=a\,.\,\text{vers}^{-1}\left(\frac{y}{a}\right)-\sqrt{2\,ay-y^2}\,.$$

THE SPIRAL OF ARCHIMEDES.

$$\rho=k\,.\,\theta.$$

THE RECIPROCAL OR HYPERBOLIC SPIRAL.

$$\rho=\frac{k}{\theta}.$$

THE PARABOLIC SPIRAL.

$$\rho^2=k\,.\,\theta.$$

THE LITUUS OR TRUMPET.

$$\rho^2=\frac{k}{\theta}.$$

THE LOGARITHMIC SPIRAL.

$$\log\rho=k\,.\,\theta.$$

If $k=1$, and logarithms to the base a are employed, then the equation may be written

$$\rho=a^\theta.$$

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THE CATENARY.

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

THE CUBIC PARABOLA.

$$y = kx^3.$$

THE SPHERE.

For the origin at the center,

$$x^2 + y^2 + z^2 = R^2,$$

where R is the radius.

CONES.

The equation of the cone generated by the line, $z = mx + c$, rotated about the Z-axis, is

$$x^2 + y^2 = \frac{(z-c)^2}{m^2}.$$

OBLATE SPHEROIDS.

The equation of the oblate spheroid generated by the ellipse, $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, rotated about its minor axis, is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

PROLATE SPHEROIDS.

The equation of the prolate spheroid generated by the ellipse, $\frac{x^2}{b^2} + \frac{z^2}{a^2} = 1$, rotated about its major axis, is

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1.$$

HYPERBOLOIDS.

The equation of the hyperboloid of one nappe, generated by the hyperbola, $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$, rotated about its conjugate axis, is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=1.$$

The equation of the hyperboloid of two nappes, generated by the hyperbola, $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$, rotated about its transverse axis, is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{b^2}=1.$$

THE PARABOLOID

The equation of the paraboloid of revolution generated by the parabola, $x^2=4az$ rotated about its axis, is

$$x^2+y^2=4\,az.$$

GENERAL EQUATION OF CONIC SECTION.

The general equation of any conic section, for which the Y-axis coincides with the directrix and the X-axis passes through the foci normal to the directrix, is

$$(x-k)^2+y^2=e^2x^2,$$

where k is the distance from the directrix to the focus, and e is the eccentricity.

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DIFFERENTIAL CALCULUS.

Variables will be represented by u, v, x, y, and z, and constants by a, b, m, and n.

D will be used as the sign for the derivative, and d as the sign for the differential.

$\mathrm{Sin}^{-1}\, x =$ angle whose sine is x.

$$D\,(fx) = \frac{d\,(fx)}{dx}$$

$$D_x y = \frac{dy}{dx}.$$

$\therefore$ To obtain the derivative of any function, drop the differential of the variable from the differential of the function.

$$D_x(fy) = D_y(fy)\,.\,D_x y.$$

$da = 0.$

$d\,(av) = a\,.\,dv.$

$d\,(u + v + x) = du + dv + dx.$

$d\,(x\,.\,y) = y\,.\,dx + x\,.\,dy.$

$d\,(u\,.\,v\,.\,x\,.\,y\,.\,.\,.) = (v\,.\,x\,.\,y\,.\,.\,.)\,du +$
$(u\,.\,x\,.\,y\,.\,.\,.)\,dv + (u\,.\,v\,.\,y\,.\,.\,.)\,dx +$
$(u\,.\,v\,.\,x\,.\,.\,.)\,dy + \,.\,.\,.$

$d\,(\log_e u) = \frac{du}{u}.$

$d\,(\log_a u) = \log_a e\,.\,\frac{du}{u}.$

$d\left(\frac{x}{y}\right) = \frac{y\,.\,dx - x\,.\,dy}{y^2}.$

$u = x^y$ $\qquad \frac{du}{dx} = y x^{y-1}$

$\frac{du}{}$ = $x^y \log x$

$$dx^y = y \cdot x^{y-1} \cdot dx + x^y \cdot \log_a x \cdot \frac{dx}{M},$$

where $M = \log_a e$.

$$d\,(b^y) = b^y \cdot \log_a b \cdot \frac{dy}{M}.$$

$$dx^a = a \cdot x^{a-1} \cdot dx.$$

$$d\sqrt{x} = \frac{dx}{2\sqrt{x}}.$$

$$d\,(\sin x) = \cos x \cdot dx.$$

$$d\,(\cos x) = -\sin x \cdot dx.$$

$$d\,(\tan x) = \sec^2 x \cdot dx.$$

$$d\,(\cot x) = -\operatorname{cosec}^2 x \cdot dx.$$

$$d\,(\sec x) = \sec x \cdot \tan x \cdot dx.$$

$$d\,(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x \cdot dx.$$

$$d\,(\operatorname{vers} x) = d\,(1 - \cos x) = +\sin x \cdot dx.$$

$$d\,(\operatorname{covers} x) = d\,(1 - \sin x) = -\cos x \cdot dx.$$

$$d\,(\sin^{-1} x) = dx/\sqrt{1 - x^2}.$$

$$d\,(\cos^{-1} x) = -dx/\sqrt{1 - x^2}.$$

$$d\,(\tan^{-1} x) = dx/(1 + x^2).$$

$$d\,(\cot^{-1} x) = -dx/(1 + x^2).$$

$$d\,(\sec^{-1} x) = dx/(x\sqrt{x^2 - 1}).$$

$$d\,(\operatorname{vers}^{-1} x) = dx/\sqrt{2\,x - x^2}.$$

$$d\,(\operatorname{covers}^{-1} x) = -dx/\sqrt{2\,x - x^2}.$$

To differentiate a function :

1. Find the value of the increment of the function in terms of the increments of its variables;

2. Consider the increments to be infinitesimals, and in all sums drop the infinitesimals of higher order than the first, and in the

remaining terms substitute differentials for increments.

For the *maximum* value of a function the first derivative is zero, and the second derivative is negative.

For the *minimum* value of a function the first derivative is zero, and the second derivative is positive.

If $\frac{Fx}{fx}$ assumes the form $\frac{0}{0}$, then

$$\frac{Fx}{fx}=\frac{D\,(Fx)}{D\,(fx)}.$$

Taylor's theorem is

$$f\,(x+h)=fx+h\,.\,D\,(fx)+\frac{h^2}{\lfloor 2}\,.\,D^2\,(fx)+\ldots$$

$$\ldots+\frac{h^n}{\lfloor n}\,.\,D^n\,(fx).$$

$$fx=f\,(0+x)=f\,(0)+$$

$$x\,.\,D\,(f\,0)+\frac{x^2}{\lfloor 2}\,.\,D^2\,(f\,0)+\ldots$$

The *radius of curvature* for a curve, $y=fx$, is

$$R=\frac{ds}{d\alpha}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{(dx)^2}}=\frac{(ds)^3}{dx\,.\,d^2y},$$

where s is length of curve.

INTEGRAL CALCULUS.

$\int dx = x + C$, where C is the constant of integration. The constant C must be added to all of the following forms.

$$\int (dx + dy + dz \ldots) = \int dx + \int dy + \int dz + \ldots$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1}.$$

$$\int \frac{dx}{x} = \log_e x.$$

$$\int a^x \cdot dx = \frac{a^x}{\log_e a}.$$

$$\int e^x \cdot dx = e^x.$$

$$\int a^x \cdot \log_e a \cdot dx = a^x.$$

$$\int \sin x \cdot dx = -\cos x \text{ or vers } x.$$

$$\int \cos x \cdot dx = \sin x \text{ or} - \text{covers } x.$$

$$\int \sec^2 x \cdot dx = \tan x.$$

$$\int \operatorname{cosec}^2 x \cdot dx = -\cot x.$$

$$\int \sec x \cdot \tan x \cdot dx = \sec x.$$

$$\int \operatorname{cosec} x \,.\, \cot x \,.\, dx = -\operatorname{cosec} x.$$

$$\int \tan x \,.\, dx = \log (\sec x).$$

$$\int \cot x \,.\, dx = \log (\sin x).$$

$$\int \operatorname{cosec} x \,.\, dx = \log \left(\tan \frac{x}{2}\right).$$

$$\int \sec x \,.\, dx = \log \left[\tan \left(\frac{x}{2} + \frac{\pi}{4}\right)\right].$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \,.\, \tan^{-1} \left(\frac{x}{a}\right), \text{ or}$$

$$= -\frac{1}{a} \,.\, \cot^{-1} \left(\frac{x}{a}\right).$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2\,a} \,.\, \log \left(\frac{x-a}{x+a}\right), \text{ or}$$

$$= \frac{1}{2\,a} \,.\, \log \left(\frac{a-x}{a+x}\right).$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a}\right) = -\cos^{-1} \left(\frac{x}{a}\right).$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}).$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \,.\, \sec^{-1} \left(\frac{x}{a}\right), \text{ or}$$

$$= -\frac{1}{a} \operatorname{cosec}^{-1} \left(\frac{x}{a}\right).$$

$$\int \frac{dx}{\sqrt{2\,ax - x^2}} = \operatorname{vers}^{-1} \left(\frac{x}{a}\right), \text{ or}$$

$$= -\operatorname{covers}^{-1} \left(\frac{x}{a}\right).$$

$$\int f(x)\, dx = Fx + C, \text{ if}$$

$$d(Fx) = fx \,.\, dx.$$

$$\int a \,.\, dx = a \int dx.$$

$$\int 0 = C.$$

$$\int x \,.\, dy = xy - \int y \,.\, dx.$$

$$\int \frac{x \,.\, dx}{a+bx} = \frac{1}{b^2}[a + bx - a \,.\, \log(a+bx)].$$

$$\int \frac{x \,.\, dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right].$$

$$\int \frac{x^2 \,.\, dx}{a+bx} = \frac{1}{b^3}\left[\frac{(a+bx)^2}{2} - 2a(a+bx) + a^2 \,.\, \log(a+bx)\right].$$

$$\int \frac{x^2 \,.\, dx}{(a+bx)^2} = \frac{1}{b^3}\left[a + bx - 2a \,.\, \log(a+bx) - \frac{a^2}{a+bx}\right].$$

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \,.\, \log\left(\frac{a+bx}{x}\right).$$

$$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \,.\, \log\left(\frac{a+bx}{x}\right).$$

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \,.\, \log\left(\frac{a+bx}{x}\right).$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \,.\, \tan^{-1}\left(x\sqrt{\frac{b}{a}}\right),$$

when $a > 0$ and $b > 0$.

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$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}} \cdot \log \frac{\sqrt{a}+x\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}},$$

when $a>0$ and $b<0$.

$$\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}.$$

$$\int \frac{dx}{(a+bx^2)^{n+1}} = \frac{1}{2na} \cdot \frac{x}{(a+bx^2)^n}$$

$$+\frac{2n-1}{2na} \int \frac{dx}{(a+bx^2)^n}.$$

$$\int \frac{x^2 \cdot dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}.$$

$$\int \frac{x^2 \cdot dx}{(a+bx^2)^{n+1}} = \frac{-x}{2nb(a+bx^2)^n}$$

$$+\frac{1}{2nb} \int \frac{dx}{(a+bx^2)^n}.$$

$$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \log\left(\frac{x^2}{a+bx^2}\right)$$

$$\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}.$$

$$\int \frac{dx}{x^2(a+bx^2)^{n+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^n}$$

$$-\frac{b}{a}\int \frac{dx}{(a+bx^2)^{n+1}}.$$

$$\int x^m \cdot (a+bx^n)^P \cdot dx =$$

$$\frac{x^{m-n+1} \cdot (a+bx^n)^{P+1}}{b(nP+m+1)}$$

$$-\frac{a(m-n+1)}{b(nP+m+1)} \cdot \int x^{m-n} \cdot (a+bx^n)^P \cdot dx,$$

or
$$=\frac{x^{m+1} \,.\, (a+bx^n)^P}{nP+m+1}$$
$$+\frac{anP}{nP+m+1}\int x^m \,.\, (a+bx^n)^{P-1} \,.\, dx,$$

or
$$=\frac{x^{m+1} \,.\, (a+bx^n)^{P+1}}{a\,(m+1)}$$
$$-\frac{b\,(nP+m+n+1)}{a\,(m+1)}\int x^{m+n} \,.\, (a+bx^n)^P \,.\, dx,$$

or
$$=-\frac{x^{m+1} \,.\, (a+bx^n)^{P+1}}{an\,(P+1)}$$
$$+\frac{nP+m+n+1}{an\,(P+1)}\int x^m \,.\, (a+bx^n)^{P+1} \,.\, dx.$$

$$\int\frac{dx}{ax^2+bx+c}=$$
$$\frac{2}{\sqrt{4\,ac-b^2}} \,.\, \tan^{-1}\left(\frac{2\,ax+b}{\sqrt{4\,ac-b^2}}\right),$$

or
$$=\frac{1}{\sqrt{b^2-4\,ac}} \,.\, \log\left(\frac{2\,ax+b-\sqrt{b^2-4\,ac}}{2\,ax+b+\sqrt{b^2-4\,ac}}\right).$$

$$\int\frac{x \,.\, dx}{ax^2+bx+c}=\frac{1}{2\,a} \,.\, \log\,(ax^2+bx+c)$$
$$-\frac{b}{2\,a}\int\frac{dx}{ax^2+bx+c}.$$

$$\int x\sqrt{a+bx} \,.\, dx=$$
$$-\frac{2\,(2\,a-3\,bx)\,(a+bx)^{\frac{3}{2}}}{15\,b^2}.$$

$$\int x^2 \,.\, \sqrt{a+bx} \,.\, dx=$$
$$\frac{2\,(8\,a^2-12\,abx+15\,b^2x^2)\,(a+bx)^3}{105\,b^3}.$$

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$$\int \frac{x^n \, . \, dx}{\sqrt{a+bx}} = \frac{2\,x^n\sqrt{a+bx}}{(2\,n+1)\,b} - \frac{2\,na}{(2\,n+1)\,b} \int \frac{x^{n-1} \, . \, dx}{\sqrt{a+bx}}.$$

$$\int \frac{x \, . \, dx}{\sqrt{a+bx}} = -\frac{2\,(2\,a-bx)\sqrt{a+bx}}{3\,b^2}.$$

$$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \, . \, \log \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}},$$

when $a > 0$,

or $$= \frac{2}{\sqrt{-a}} \, . \, \tan^{-1}\sqrt{\frac{a+bx}{-a}},$$

when $a < 0$.

$$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)\,ax^{n-1}} - \frac{(2\,n-3)\,b}{(2\,n-2)\,a} \int \frac{dx}{x^{n-1}\sqrt{a+bx}}.$$

$$\int \frac{\sqrt{a+bx}}{x} \, . \, dx = 2\sqrt{a+bx} + a\int \frac{dx}{x\sqrt{a+bx}}.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right).$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \, . \, \log\left(\frac{x}{a+\sqrt{a^2-x^2}}\right).$$

$$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = \frac{-\sqrt{a^2-x^2}}{a^2x}.$$

$$\int \sqrt{a^2-x^2} \, . \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \, . \, \sin^{-1}\left(\frac{x}{a}\right).$$

$$\int x^2\sqrt{a^2-x^2}\,.\,dx=$$

$$\frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{\sqrt{a^2-x^2}}{x}\,.\,dx=\sqrt{a^2-x^2}$$

$$-a\,.\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$$

$$\int\frac{\sqrt{a^2-x^2}}{x^2}\,.\,dx=\frac{-\sqrt{a^2-x^2}}{x}-\sin^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{x^2\,.\,dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{a^2\sqrt{a^2-x^2}}.$$

$$\int(a^2-x^2)^{\frac{3}{2}}.\,dx=$$

$$\frac{x}{8}(5\,a^2-2\,x^2)\sqrt{a^2-x^2}+\frac{3}{8}a^4\,.\sin^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{x^2\,.\,dx}{(a^2-x^2)^{\frac{3}{2}}}=\frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{dx}{\sqrt{x^2\pm a^2}}=\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\,.\sec^{-1}\left(\frac{x}{a}\right).$$

$$\int\frac{dx}{x\sqrt{x^2+a^2}}=\frac{1}{a}\,.\log\left(\frac{x}{a+\sqrt{x^2+a^2}}\right).$$

$$\int\frac{dx}{x^2\sqrt{x^2\pm a^2}}=\mp\frac{\sqrt{x^2\pm a^2}}{a^2x}.$$

$$\int\frac{dx}{x^3\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{2\,a^2x^2}+\frac{1}{2\,a^3}\sec^{-1}\frac{x}{a}.$$

$$\int \frac{dx}{x^3\sqrt{x^2+a^2}} = \frac{-\sqrt{x^2+a^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}.$$

$$\int \sqrt{x^2 \pm a^2}\cdot dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int x^2\sqrt{x^2\pm a^2}\cdot dx = \frac{x}{8}\left(2\,x^2\pm a^2\right)\sqrt{x^2\pm a^2} - \frac{a^4}{8}\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\cdot dx = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}.$$

$$\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} - a\cdot\log\frac{a+\sqrt{x^2+a^2}}{x}.$$

$$\int \frac{\sqrt{x^2\pm a^2}}{x^2}\cdot dx = \frac{-\sqrt{x^2\pm a^2}}{x} + \log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int \frac{x^2\,dx}{\sqrt{x^2\pm a^2}} = \frac{x}{2}\sqrt{x^2\pm a^2} \mp \frac{a^2}{2}\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int \frac{dx}{(x^2\pm a^2)^{\frac{3}{2}}} = \pm\frac{x}{a^2\sqrt{x^2\pm a^2}}.$$

$$\int \frac{x^2\,dx}{(x^2\pm a^2)^{\frac{3}{2}}} = \frac{-x}{\sqrt{x^2\pm a^2}} + \log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int (x^2\pm a^2)^{\frac{3}{2}}dx = \frac{x}{8}\left(2\,x^2\pm 5\,a^2\right)\sqrt{x^2\pm a^2} - \frac{3\,a^4}{8}\log\left(x+\sqrt{x^2\pm a^2}\right).$$

$$\int \frac{dx}{\sqrt{2\,ax - x^2}} = \text{vers}^{-1}\frac{x}{a}.$$

$$\int \frac{x^m dx}{\sqrt{2\,ax - x^2}} = -\frac{x^{m-1}\sqrt{2\,ax - x^2}}{m} + \frac{(2\,m-1)\,a}{m}\int \frac{x^{m-1}.\,dx}{\sqrt{2\,ax - x^2}}.$$

$$\int \frac{dx}{x^m\sqrt{2\,ax - x^2}} = -\frac{\sqrt{2\,ax - x^2}}{(2\,m-1)\,ax^m} + \frac{m-1}{(2\,m-1)\,a}\int \frac{dx}{x^{m-1}\sqrt{2\,ax - x^2}}.$$

$$\int \sqrt{2\,ax - x^2}.\,dx = \frac{x-a}{2}\sqrt{2\,ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$$

$$\int x^m\sqrt{2\,ax - x^2}.\,dx = -\frac{x^{m-1}\,(2\,ax - x^2)^{\frac{3}{2}}}{m+2} + \frac{(2\,m+1)\,a}{m+2}\int x^{m-1}.\,\sqrt{2\,ax - x^2}.\,dx.$$

$$\int \frac{\sqrt{2\,ax - x^2}}{x^m}.\,dx = \frac{-(2\,ax - x^2)^{\frac{3}{2}}}{(2\,m-3)\,ax^m} + \frac{m-3}{(2\,m-3)\,a}\int \frac{\sqrt{2\,ax - x^2}}{x^{m-1}}.\,dx.$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}}\log\left(2\,ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c}\right).$$

$$\int \sqrt{ax^2 + bx + c}.\,dx = \frac{2\,ax + b}{4\,a}\sqrt{ax^2 + bx + c} - \left(\frac{b^2 - 4\,ac}{8\,a}\right)\int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

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$$\int \frac{dx}{\sqrt{-ax^2+bx+c}} = \frac{1}{\sqrt{a}} \sin^{-1}\left(\frac{2\,ax-b}{\sqrt{b^2+4\,ac}}\right)$$

$$\int \sqrt{-ax^2+bx+c}\,.\,dx = \frac{2\,ax-b}{4\,a}\sqrt{-ax^2+bx+c} + \frac{b^2+4\,ac}{8\,a}\int \frac{dx}{\sqrt{-ax^2+bx+c}}.$$

$$\int \frac{x\,dx}{\sqrt{\pm ax^2+bx+c}} = \frac{\sqrt{\pm ax^2+bx+c}}{\pm a} \mp \frac{b}{2\,a}\int \frac{dx}{\sqrt{\pm ax^2+bx+c}}.$$

$$\int x\sqrt{\pm ax^2+bx+c}\,.\,dx = \frac{(\pm ax^2+bx+c)^{\frac{3}{2}}}{3\,a} \mp \frac{b}{2\,a}\int \sqrt{\pm ax^2+bx+c}\,.\,dx.$$

$$\int \sin^2 x\,.\,dx = \frac{x}{2} - \frac{1}{4}\sin(2\,x).$$

$$\int \cos^2 x\,.\,dx = \frac{x}{2} + \frac{1}{4}\sin(2\,x).$$

$$\int \sin^2 x\,.\,\cos^2 x\,.\,dx = \frac{1}{8}\left(x - \frac{1}{4}\sin 4\,x\right).$$

$$\int \sec x\,.\,\csc x\,.\,dx = \int \frac{dx}{\sin x\,.\,\cos x} = \log\tan x.$$

$$\int \sec^2 x\,.\,\csc^2 x\,.\,dx = \int \frac{dx}{\sin^2 x\,.\,\cos^2 x} = \tan x - \cot x.$$

$$\int \sin^m x\,.\,\cos^n x\,.\,dx = \frac{-\sin^{m-1} x\,.\,\cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x\,.\,\cos^n x\,.\,dx,$$

$$\text{or} \quad = \frac{\sin^{m+1} x \,.\, \cos^{n-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^m x \,.\, \cos^{n-2} x \,.\, dx.$$

$$\int \sin^m x \,.\, dx = -\frac{\sin^{m-1} x \,.\, \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x \,.\, dx.$$

$$\int \cos^n x \,.\, dx = \frac{\sin x \,.\, \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-1} x \,.\, dx.$$

$$\int \frac{\sin^m x}{\cos^n x} dx = \frac{\sin^{m+1} x}{(n-1)\cos^{n-1} x} + \frac{n-m-2}{n-1}\int \frac{\sin^m x \,.\, dx}{\cos^{n-2} x}.$$

$$\int \frac{\cos^n x}{\sin^m x} \,.\, dx = \frac{-\cos^{n+1} x}{(m-1)\sin^{m-1} x} + \frac{m-n-2}{m-1}\int \frac{\cos^n x \, dx}{\sin^{m-2} x}.$$

$$\int \frac{dx}{\sin^m x} = \frac{-\cos x}{(m-1)\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$$

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$$

$$\int \tan^n x \,.\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \,.\, dx.$$

$$\int \cot^n x \,.\, dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \,.\, dx.$$

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \,.\, \tan\frac{x}{2}\right),$$

if $a^2 > b^2$,

$$= \frac{1}{\sqrt{b^2-a^2}} \cdot \log \frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}},$$

if $a^2 < b^2$.

$$\int x^m \cdot \sin x \cdot dx =$$

$$-x^m \cos x + m\int x^{m-1} \cos x\, dx.$$

$$\int x^m \cdot \cos x \cdot dx =$$

$$x^m \sin x - m\int x^{m-1} \overset{\text{sin}}{\cos} x\, dx.$$

$$\int \frac{\sin x}{x}\, dx = x - \frac{x^3}{3\,\lfloor\underline{3}} + \frac{x^5}{5\,\lfloor\underline{5}} - \frac{x^7}{7\,\lfloor\underline{7}} + \ldots$$

$$\int \frac{\sin x}{x^m}\, dx = \frac{-1}{m-1}\,\frac{\sin x}{x^{m-1}} + \frac{1}{m-1}\int \frac{\cos x\, dx}{x^{m-1}}.$$

$$\int \frac{\cos x}{x}\, dx = \log x - \frac{x^2}{2\,\lfloor\underline{2}} + \frac{x^4}{4\,\lfloor\underline{4}} - \frac{x^6}{6\,\lfloor\underline{6}} + \ldots,$$

$$\int \frac{\cos x}{x^m}\, dx = \frac{-1}{m-1} \cdot \frac{\cos x}{x^{m-1}} - \frac{1}{m-1}\int \frac{\sin x\, dx}{x^{m-1}}.$$

$$\int x \sin^{-1} x \cdot dx =$$

$$\frac{1}{4}[(2x^2-1)\sin^{-1} x + x\sqrt{1-x^2}].$$

$$\int x^n \sin^{-1} x \cdot dx =$$

$$\frac{x^{n+1}\sin^{-1} x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\, dx}{\sqrt{1-x^2}}.$$

$$\int x^n \cos^{-1} x \cdot dx =$$

$$\frac{x^{n+1}\cos^{-1} x}{n+1} + \frac{1}{n+1}\int \frac{x^{n+1}\, dx}{\sqrt{1-x^2}}.$$

$$\int x^n \tan^{-1} x \, . \, dx =$$

$$\frac{x^{n+1} \tan x}{n+1} - \frac{1}{n+1} \int \frac{x^{n+1}\, dx}{1+x^2} \, .$$

$$\int x^n \log x \, . \, dx = x^{n+1} \left[\frac{\log x}{n+1} - \frac{1}{(n+1)^2} \right] .$$

$$\int \frac{dx}{\sqrt{(x-a)\,(b-x)}} = 2 \cot^{-1} \sqrt{\frac{b-x}{x-a}}$$

$$= 2 \sin^{-1} \sqrt{\frac{x-a}{b-a}} \, .$$

$$\int \frac{dx}{x \sqrt{x^n + a^2}} = \frac{1}{an} \log \frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a} \, .$$

$$\int \frac{dx}{x \sqrt{x^n - a^2}} = \frac{2}{an} \sec^{-1} \frac{x^{\frac{n}{2}}}{a} \, .$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \cdot \log \frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}},$$

if $a^2 < b^2$.

ERRATA

The formula (p. 34) which reads

$$\int x^m \cdot \cos x \cdot dx =$$

$$x^m \sin x - m \int x^{m-1} \cos x \, dx$$

should read

$$\int x^m \cdot \cos x \cdot dx =$$

$$x^m \cdot \sin x - m \int x^{m-1} \cdot \sin x \cdot dx$$

$\frac{\;}{4}[(2x^2-1)\sin$ ...

$$\int x^n \sin^{-1} x \cdot dx =$$

$$\frac{x^{n+1}\sin^{-1} x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}.$$

$$\int x^n \cos^{-1} x \cdot dx =$$

$$\frac{x^{n+1}\cos^{-1} x}{n+1} + \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}},$$

$$\int x^n \tan^{-1} x \, . \, dx =$$

$$\frac{x^{n+1} \tan x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\, dx}{1+x^2}.$$

$$\int x^n \log x \, . \, dx = x^{n+1}\left[\frac{\log x}{n+1} - \frac{1}{(n+1)^2}\right].$$

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx.$$

$$\int \frac{e^{ax}}{x^n}\, dx = \frac{-1}{n-1} \cdot \frac{e^{ax}}{x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}}\, dx.$$

$$\int e^{ax} \log x \, . \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a}\int \frac{e^{ax}\, dx}{x}.$$

$$\int e^{ax} \sin (nx) \, . \, dx = e^{ax}\left(\frac{a \sin [nx] - n \cos [nx]}{a^2+n^2}\right).$$

$$\int e^{ax} \cos (nx)\, dx = e^{ax}\left[\frac{a(\cos (nx) + n \sin (nx))}{a^2+n^2}\right].$$

$$\int \sqrt{\frac{a+x}{b+x}} \, . \, dx = \sqrt{(a+x)\,(b+x)}$$

$$+ (a-b) \log (\sqrt{a+x} + \sqrt{b+x}).$$

$$\int \sqrt{\frac{a-x}{b+x}}\, dx = \sqrt{(a-x)\,(b+x)}$$

$$+ (a+b) \sin^{-1} \sqrt{\frac{b+x}{a+b}}.$$

$$\int \frac{dx}{\sqrt{(x-a)\,(b-x)}} = 2 \cot^{-1} \sqrt{\frac{b-x}{x-a}}$$

$$= 2 \sin^{-1} \sqrt{\frac{x-a}{b-a}}.$$

$$\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an} \log \frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$$

$$\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an} \sec^{-1} \frac{x^{\frac{n}{2}}}{a}.$$

THEORETICAL MECHANICS.

NOTATION.

A = area.

a = acceleration.

a_n = normal acceleration.

a_t = tangential acceleration.

b = breadth.

C_x = component of force parallel to the X-axis.

C_y = component of force parallel to the Y-axis.

C_z = component of force parallel to the Z-axis.

d = depth or distance. Also the sign of the differential.

F = force.

F_n = normal force or component of force.

F_t = tangential force or component of force.

f = coefficient of friction. Also the sign of a function of a variable.

g = acceleration due to gravity = 32.2. (The exact value is $32.1808 - 0.0821 \cos 2L$, where L is the latitude.)

h = distance from center of moments to line of force.

I = moment of inertia.

I_g = moment of inertia referred to center of gravity.

I_{gx} = moment of inertia about an axis through the center of gravity and parallel to the X-axis.

I_0 = polar moment of inertia about the pole 0.
I_x = moment of inertia about the X-axis.
I_y = moment of inertia about the Y-axis.
I_z = moment of inertia about the Z-axis.
J = product of inertia. (Subscripts are similar to those for I.)
K = a constant.
L = power.
M = moment of a force.
m = mass = $\frac{W}{g}$.
N = a normal force or component of a force.
P = point considered.
R = resultant of a system of forces.
r = radius of gyration.
s = space.
T = tangential force or component of a force.
t = time.
V = volume.
v = velocity.
v_0 = initial velocity.
v_t = tangential velocity.
v_x = velocity parallel to the X-axis.
v_y = velocity parallel to the Y-axis.
W = weight.
w = work.
x, y, z = rectangular coördinates of a point.
ρ, θ = polar coördinates of a point.
$\bar{\rho}$ = distance from pole to center of gravity.
a = angle.
ϕ = angle of friction.

THEORETICAL MECHANICS
MECHANICS OF MATERIALS

STATICS.

Equilibrium of Forces.

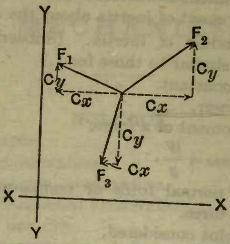


Fig. 17.

For a system of *concurrent* forces in equilibrium in one plane:

$$\Sigma C_x = 0.$$
$$\Sigma C_y = 0.$$

($C_x = F \cos \alpha$, $C_y = F \sin \alpha$, where α is the angle which F makes with $X - X$.)

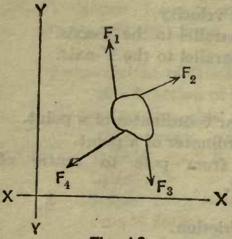


Fig. 18.

For a system of *non-concurrent* forces in equilibrium in one plane:

$$\Sigma C_x = 0.$$
$$\Sigma C_y = 0.$$
$$\Sigma M = 0.$$

Also, if $\Sigma M = 0$,

$$\Sigma M = \Sigma C_x y + \Sigma C_y x.$$

If three forces are in equilibrium they must be concurrent or parallel.

If a system of non-concurrent forces in space is in equilibrium, the plane systems formed by projecting the given system upon three coördinate planes must each be in equilibrium.

A couple consists of two equal and opposite parallel forces acting on a rigid body at a fixed distance apart.

The moment of a couple is equal to the product of one force by the distance between the two forces.

Center of Pressure.

F_1, F_2, F_3, etc., are parallel.

$$\bar{x} = \frac{\Sigma F x}{\Sigma F}.$$

$$R = \Sigma F.$$

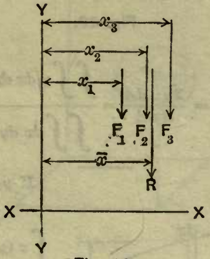


Fig. 19.

If F is the force exerted by a variable pressure, then

$$\bar{x} = \frac{\int xF\,dx}{\int F\,dx}.$$

Center of Gravity.

For an area,

$$\bar{x} = \frac{\Sigma x \,.\, dA}{\Sigma \, dA} = \frac{\iint x \, dx \, dy}{\iint dx \, dy}.$$

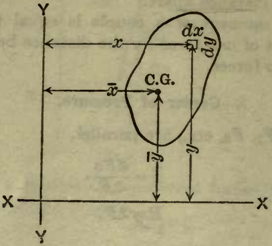


Fig. 20.

$$\bar{y} = \frac{\Sigma y \,.\, dA}{\Sigma \, dA} = \frac{\iint y \, dx \, dy}{\iint dx \, dy}.$$

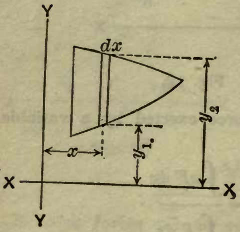


Fig. 21.

If $y_2 - y_1 = fx$,

$$\bar{x} = \frac{\int x \, (y_2 - y_1) \, dx}{\int (y_2 - y_1) \, dx} = \frac{\int x \,.\, fx \,.\, dx}{\int fx \,.\, dx}.$$

If $x_2 - x_1 = fx$

$$\bar{y} = \frac{\int y\,(x_2 - x_1)\,dy}{\int (x_2 - x_1)\,dy} = \frac{\int y \,.\, fy \,.\, dy}{\int fy \,.\, dy}.$$

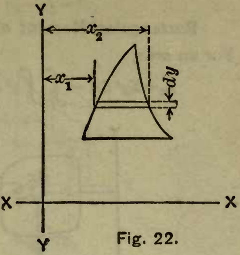


Fig. 22.

For a homogeneous mass,

$$\bar{x} = \frac{\Sigma x\,dm}{\Sigma\,dm} = \frac{\int\int\int x\,dx\,dy\,dz}{\int\int\int dx\,dy\,dz}.$$

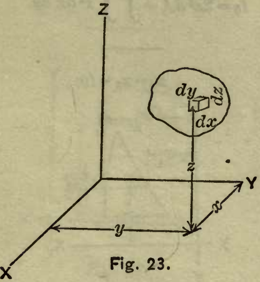


Fig. 23.

$$\bar{y} = \frac{\Sigma y\,dm}{\Sigma\,dm} = \frac{\int\int\int y\,dx\,dy\,dz}{\int\int\int dx\,dy\,dz}.$$

$$\bar{z} = \frac{\Sigma z\,dm}{\Sigma\,dm} = \frac{\int\int\int z\,dx\,dy\,dz}{\int\int\int dx\,dy\,dz}.$$

Rectangular Moment of Inertia.

For an area,

$$I_x = \Sigma y^2\, dA = \int\int y^2\, dx\, dy.$$

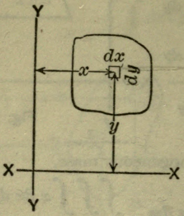


Fig. 24.

$$I_y = \Sigma x^2\, dA = \int\int x^2\, dx\, dy.$$

If $$x_2 - x_1 = fy,$$

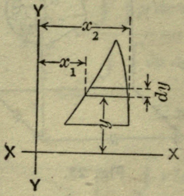


Fig. 25.

$$I_x = \Sigma y^2\, dA$$

$$= \int y^2 \,.\, (x_2 - x_1)\, dy$$

$$= \int y^2 \,.\, fy \,.\, dy.$$

If
$$y_2 - y_1 = fx,$$

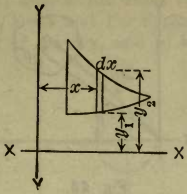


Fig. 26.

$$I_y = \Sigma x^2\, dA$$
$$= \int x^2\,(y_2 - y_1)\, dx$$
$$= \int x^2 \,.\, fx \,.\, dx.$$

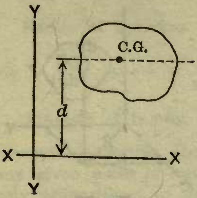


Fig. 27.

$$I_x = I_{gx} + A \,.\, d^2.$$

Polar Moment of Inertia.

For an area,

$$I_0 = \Sigma \rho^2\, dA = \int\int \rho^2 \,.\, d\rho \,.\, d\theta.$$

Since $\rho^2 = x^2 + y^2$,

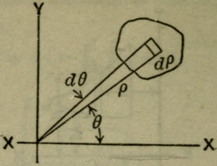


Fig. 28.

$$I_0 = I_x + I_y.$$

For a mass,

$$I_z = \Sigma \rho^2 \, dm$$
$$= k \int\int\int \rho^2 \, dx \, dy \, dz$$
$$= k \int\int\int (x^2 + y^2) \, dx \, dy \, dz,$$

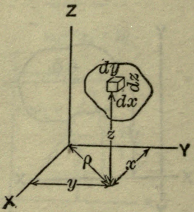


Fig. 29.

where k is the weight per cubic unit divided by g.

Product of Inertia.

$$J = \Sigma xy \, dA$$
$$= \int\int x \, . \, y \, . \, dx \, . \, dy.$$
$$J_1 = J_{c.g.} + Akh,$$

where J_1 is the value of J referred to $X-X$ and $Y-Y$, $J_{c.g.}$ is the value of J for axes parallel to $X-X$ and $Y-Y$ passing through the center of gravity, and h, k are the co-

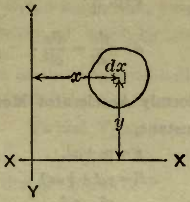


Fig. 30.

ordinates of the center of gravity referred to $X-X$ and $Y-Y$.

(See "A Complete Analysis of General Flexure in a Straight Bar of Uniform Cross-Section," by L. J. Johnson, *Trans. Am. Soc. C. E.*, Vol. LVI, 1906.)

Radius of Gyration.

$$r=\sqrt{\frac{I}{A}}, \quad \text{or } r=\sqrt{\frac{I}{m}}.$$

Ellipsoid of Inertia.

The moments of inertia about all axes through any given point of any rigid body are inversely proportional to the squares of the diameters which they intercept in an imaginary ellipsoid, whose center is the given point, and whose position depends upon the distribution of the mass and the location of the given point. This ellipsoid is the ellipsoid of inertia for the body. The axes which contain the principal diameters of the ellipsoid are called the principal axes of the body for the given point.

DYNAMICS.

Velocity and Acceleration.

$$v = \frac{ds}{dt}.$$

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Uniformly Accelerated Motion.

If a is constant,

$$v = v_0 + at.$$

$$S = v_0t + \tfrac{1}{2}\,at^2$$

$$= \frac{v^2 - v_0^2}{2\,a}$$

$$= \tfrac{1}{2}\,(v_0 + v)\,t.$$

$$v\,dv = a\,ds.$$

Falling Bodies.

For a body falling in a vacuum, $a = g$, hence

$$v = v_0 + gt.$$

$$S = v_0t + \tfrac{1}{2}\,gt^2$$

$$= \frac{v^2 - v_0^2}{2\,g}$$

$$= \tfrac{1}{2}\,(v_0 + v)\,t.$$

Force and Acceleration.

$$F = m\,.\,a = \frac{W}{g}\,.\,a.$$

Direct Central Impact.

For two inelastic bodies, let

m_1 = mass of first body.
m_2 = mass of second body.
v_1 = original velocity of first body.
v_2 = original velocity of second body.
v = common velocity after impact.

Then $$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

For two elastic bodies having velocities k_1 and k_2 after impact,

$$m_1 v_1 + m_2 v_2 = m_1 k_1 + m_2 k_2.$$

The product of mass by its velocity is momentum.

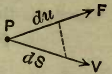
The sum of the momenta before and after impact is constant.

Virtual Velocities.

F = force.

v = direction of motion of P.

du = virtual velocity of force.



$$\frac{du}{dt} = \text{velocity of force.}$$

$$\frac{ds}{dt} = \text{velocity of } P.$$

$$F \cdot du = \text{virtual moment of force.}$$

The virtual moment of a force is equal to the algebraic sum of the virtual moments of its components.

For a system of concurrent forces in equilibrium, $\Sigma F \cdot du = 0$.

For any small displacement or motion of a rigid body in equilibrium under non-concurrent forces in a plane, with all points of the body moving parallel to this plane,

$$\Sigma F \cdot du = 0.$$

Curvilinear Motion of a Point.

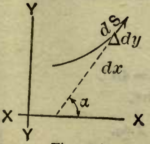


Fig. 31.

$$v_t = \frac{ds}{dt}.$$

$$v_t^2 = \left(\frac{ds}{dt}\right)^2$$

$$= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

$$v_t^2 = v_x^2 + v_y^2.$$

$$a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$= a_x \cos a + a_y \sin a.$$

$$a_n = a_y \cos a - a_x \sin a.$$

$$a_n = \frac{v_t^2}{r},$$

where r is the radius of curvature.

$$F = m \cdot a, \therefore$$

$$F_n = \frac{m \cdot v_t^2}{r},$$

where r is the radius of curvature.

$$F_t = m \cdot a_x \cos a + m \cdot a_y \sin a$$

$$= m \cdot a_t.$$

$$\frac{v^2 - v_0^2}{2} = \int a_t \, ds.$$

Projectiles.

Neglecting resistance of air,

$$x = v_0 \cos a_0 \cdot t.$$

$$y = v_0 \sin a_0 \cdot t - \tfrac{1}{2} gt^2,$$

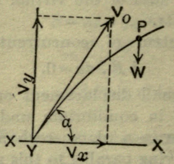


Fig. 32.

or $$y = x \tan a_0 - \frac{gx^2}{2 v_0^2 \cos^2 a_0}.$$

Horizontal range,

$$x_r = \frac{v_0}{g} \sin 2 a_0,$$

which is a maximum for $a_0 = 45°$.

The greatest height of ascent,

$$y_m = \frac{v_0}{2g} \sin 2 a_0.$$

Translation of Rigid Body.

$$dF_x = a_x \, . \, dm.$$

$$R_x = \int a_x \, . \, dm.$$

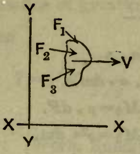


Fig. 33.

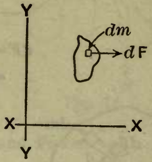


Fig. 34.

The resultant force must act in a line through the center of gravity and parallel to the direction of motion.

Rotation of a Rigid Body.

Let O be the axis of rotation.

θ = angular space passed over by any line from O.

α = angular acceleration.

ω = angular velocity.

Then

$$\omega = \frac{d\theta}{dt}.$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

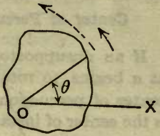


Fig. 35.

$$\omega \, d\omega = \alpha \, d\theta.$$

For uniform acceleration, $\alpha = k$, $\therefore$

$$\omega = \omega_0 + kt.$$

$$\theta = \omega_0 t + \tfrac{1}{2} kt^2$$

$$= \frac{\omega^2 - \omega_0^2}{2\,\alpha}$$

$$= \frac{\omega_0 + \omega}{2} \, . \, t.$$

For a point ρ distant from O,

$$v_t = \rho \cdot \omega.$$

$$a = \rho \cdot \alpha.$$

$$s = \rho \cdot \theta.$$

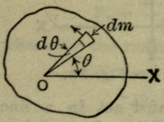


Fig. 36.

$$\begin{aligned} dF &= dm \cdot \alpha \\ &= \rho \cdot \alpha \cdot dm. \end{aligned}$$

$$dM_0 = \rho \cdot dF.$$

$$dM_0 = \rho^2 \alpha \; dm.$$

$$\begin{aligned} M_0 &= \int \rho^2 \cdot \alpha \cdot dm \\ &= \alpha \int \rho^2 dm \\ &= \alpha \cdot I. \end{aligned}$$

For a mass m concentrated ρ distant from O,

$$M_0 = \alpha \rho^2 m.$$

Center of Percussion or Oscillation.

If an unsupported bar upon being struck at a begins to rotate about b, then a is the center of percussion for b as a center, and b is the center of instantaneous rotation.

$$\begin{aligned} Fh &= \int \rho^2 \cdot \alpha \cdot dm \\ &= \alpha I_b. \end{aligned}$$

$$dF = \alpha \cdot \rho \cdot dm.$$

$$\begin{aligned} F &= \alpha \int \rho \cdot dm \\ &= \alpha \cdot \bar{\rho} \cdot m. \end{aligned}$$

$$h = \frac{I_b}{\bar{\rho} m} = \frac{r^2}{\bar{\rho}}.$$

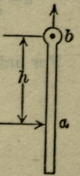


Fig. 37.

Pendulum.

t = time of oscillation from one extreme position to the other.

r = radius of gyration.

Then

$$T=\pi\sqrt{\frac{r^2}{p \cdot g}}.$$

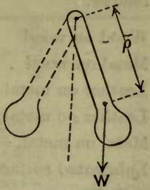


Fig. 38.

Work, Energy, and Power.

Work is equal to the product of the force by the distance through which it acts.

$$w=F \cdot S.$$

Power is the rate of doing work.

$$L=\frac{w}{t}.$$

1 H.P. = 33,000 ft.-lb. per min. = 550 ft.-lb. per sec.

Energy is the capacity or ability to do work.

K.E. = Energy of a moving body.

$$\text{K.E.}=\tfrac{1}{2}\,mv^2.$$

For rotation,

$$\text{K.E.}=\tfrac{1}{2}\,I \cdot \omega^2.$$

Friction.

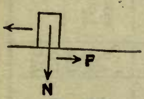


Fig. 39.

F = friction.

N = normal force.

f = coefficient of friction.

$$F=f \cdot N.$$

Angle of friction,

$$\phi=\tan^{-1}\frac{F}{N}.$$

Average values of f for motion are as follows:

Wood on wood	.25–.50
Metal on wood	.50–.60
Leather on metal	0.56
Leather on metal, lubricated . . .	0.15
Metal on metal, — dry	0.15–.24
Lubricated surfaces:	
Ordinary	0.08
Best	0.03–0.36

For values of f for rest add 40 per cent to above values.

Friction of Belt.

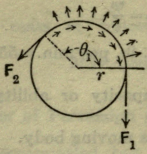


Fig. 40.

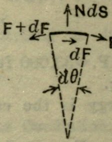


Fig. 41.

$$dF = f \cdot N\,ds$$

$$= f\,\frac{F}{r}\,ds.$$

$$ds = r\,d\theta$$

$$f \cdot d\theta = \frac{dF}{F}.$$

$$\therefore\ f \cdot \theta_1 = \log_e \left[\frac{F_2}{F_1}\right],$$

or $F_1 \cdot e^{f \cdot \theta_1} = F_2$, where θ_1 is in radians.

MECHANICS OF MATERIALS.

Material.	Wt. Lb. per Cu. Ft.	Ultimate Strength.				Elas. Limit.		Fac. of Saf.			Modulus of Elas.	
		Tension.	Flexure.	Comp.	Shear.	Tension.	Comp.	Steady.	Var.	Shock.	Tension and Comp.	Shear.
Cast Iron . .	450	20,000	35,000	90,000	18,000	6,000	20,000	6	10	20	15,000,000	6,000,000
Wrought Iron	480	50,000	50,000	50,000	40,000	25,000	25,000	4	6	10	25,000,000	10,000,000
Struct. Steel.	490	60,000	60,000	60,000	50,000	35,000	35,000	4	6	10	30,000,000	12,000,000
Strong Steel .	490	100,000	110,000	120,000	80,000	60,000	60,000	5	8	15	30,000,000	...
Brick . . .	125	...	800	3,000	1,000	...	1,000	15	25	40	2,000,000	...
Stone. . . .	160	...	2,000	6,000	1,500	...	2,000	15	25	40	6,000,000	...
Concrete . .	150	300	...	3,000	1,000	...	1,000	6	10	25	2,000,000	...
Timber . . .	40	10,000	9,000	8,000	500 3,000	3,000	3,000	8	10	15	1,500,000	400,000
Hemlock .	25	8,000	6,000	5,000	350 2,500	...	...	..	..	..	...	...
White Pine	27	8,000	6,000	5,500	500 2,500	...	...	..	..	..	...	...
Chestnut .	40	12,000	7,000	5,000	600 1,500	...	...	..	..	..	...	...
Red Oak .	42	9,000	7,000	6,000	600 4,000	...	...	..	..	..	...	...
Yellow Pine	45	15,000	11,000	9,000	600 4,000	...	...	..	..	..	...	...
White Oak	48	12,000	10,000	8,000	600 4,000	...	...	..	..	..	...	...

NOTATION.

A = area.
b = breadth.
d = depth.
E = modulus of elasticity.
e = total deformation.
F = force of load.
I = moment of inertia.
I_0 = polar moment of inertia.
J = product of inertia.
l = length.
M = moment.
R = resultant of forces.
r = radius of gyration.
S = unit stress.
s = section modulus.
V = vertical shear.
W = total weight.
w = weight per lineal unit.
Δ = maximum deflection.
ϵ = unit deformation.

Direct Stress.

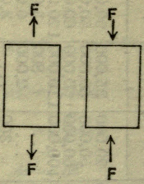


Fig. 42.

$$S = \frac{F}{A}.$$

$$\epsilon = \frac{e}{l}.$$

$$E = \frac{S}{\epsilon} = \frac{Fl}{eA}.$$

Eccentric Loads.*

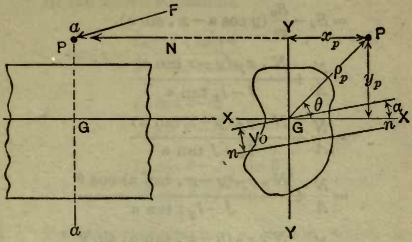


Fig. 43.

Consider a section $a-a$ perpendicular to axis of a bar, and take axes of coördinates through center of gravity.

Let x, y = coördinates of any point of section.

$n-n$ = neutral axis.

v = distance of any point from line through center of gravity and parallel to neutral axis, positive toward P.

v_0 = value of v for neutral axis.

F = force or resultant of forces acting at P.

N = component of F normal to section considered.

S_0 = unit stress at center of gravity.

$$S_0 = \frac{N}{A}.$$

---

* The method here presented is taken from a paper by L. J. Johnson, M. Am. Soc. C. E., "An Analysis of General Flexure in a Straight Bar of Uniform Cross Section," Trans., Am. Soc. C. E., volume LVI, p. 169, 1906.

$$S = S_0 - \frac{S_0}{v_0} \cdot v$$

$$= S_0 - \frac{S_0}{v_0} (y \cos \alpha - x \cdot \sin \alpha)$$

$$= \frac{N}{A} + \frac{N \cdot x_P (y - x \tan \alpha)}{J - I_y \tan \alpha}$$

$$= \frac{N}{A} + \frac{N \cdot y_P (y - x \tan \alpha)}{I_x - J \tan \alpha}$$

$$= \frac{N}{A} + \frac{N \cdot \rho_P (y - x \cdot \tan \alpha) \cos \theta}{J - I_y \cdot \tan \alpha}$$

$$= \frac{N}{A} + \frac{N \cdot \rho_P (y - x \cdot \tan \alpha) \sin \theta}{I_x - J \cdot \tan \alpha}$$

$$= \frac{N}{A} + \frac{N(y_P I_y - x_P J)\, y + N(x_P I_x - y_P J)\, x}{I_x I_y - J^2}$$

$$= \frac{N}{A} + N \cdot \rho_P \times$$

$$\left[\frac{(I_y \sin \theta - J \cdot \cos \theta)\, y + (I_x \cos \theta - J \sin \theta)\, x}{I_x I_y - J^2}\right].$$

In the above equations $\frac{N}{A}$ is the portion of S which is direct stress, and the other term is the portion due to the bending moment, $M = N \cdot \rho_P$. If s represent the section modulus

$$\left(\frac{I_x I_y - J^2}{(I_y \sin \theta - J \cdot \cos \theta)\, y + (I_x \cos \theta - J \cdot \sin \theta)\, x}\right),$$

then

$$S = \frac{N}{A} + \frac{M}{s}.$$

NOTE. — The values of the section modulus given in the handbooks are computed from the formula $s = \frac{I}{y}$, which is the value of

s for $J=0$ and for P located on $Y-Y$. For angles and Z-bars J does not equal zero.

In the above equations,

$$\tan a = \frac{I_x - J \cdot \tan\theta}{J - I_y \cdot \tan\theta}$$

$$= \frac{I_x \cot\theta - J}{J\cot\theta - I_y}$$

$$= \frac{I_x \cos\theta - J \cdot \sin\theta}{J\cos\theta - I_y \sin\theta}.$$

For any bar having a section which is symmetrical about either axis, $J=0$, and the values of S become

$$S = \frac{N}{A} + N \cdot \rho_P \left(\frac{I_y \sin\theta \cdot y + I_x \cos\theta \cdot x}{I_x I_y}\right).$$

If for a symmetrical section, P is on $Y-Y$, then $\sin\theta = 1$ and $\cos\theta = 0$, or

$$S = \frac{N}{A} + \frac{N \cdot \rho_P \cdot y}{I_x}$$

$$= \frac{N}{A} + \frac{M \cdot y}{I_x}.$$

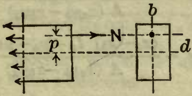


Fig. 44.

For a rectangular section, for which N is applied on $Y-Y$ and p distant from the axis of the bar, the extreme fiber stresses are

$$S = \frac{N}{A}\left(1 \pm 6\frac{p}{d}\right).$$

Equation of Neutral Axis.

The equation of the neutral axis for an eccentric load is

$$y = \left(\frac{x_P \cdot I_x - y_P \cdot J}{x_P \cdot J - y_P \cdot I_y}\right) x + \frac{I_x I_y - J^2}{A\ (x_P \cdot J - y_P \cdot I_y)} .$$

Kernel or Core-Section.

The kernel of a section (sometimes called the core-section) is the area within which P, the point of application of the resultant of the forces, must fall in order that the stress shall be of the same sign throughout the section. It is the area bounded by the locus of the P's corresponding to a series of neutral axes touching the periphery of the section but never crossing the section. For every side of the section there will be an apex of the kernel. If x_a, y_a and x_b, y_b are the coördinates of a and b, which are two consecutive vertices of the section, then the coördinates, x_{ab}, y_{ab}, of the vertex of the kernel corresponding to the side, ab, of the section will be

$$x_{ab} = -\frac{(x_a - x_b)\ J - (y_a - y_b)\ I_y}{A\ (x_a y_b - x_b y_a)},$$

$$y_{ab} = -\frac{(x_a - x_b)\ I_x - (y_a - y_b)\ J}{A\ (x_a y_b - x_b y_a)} .$$

If ab is parallel to $X-X$, then

$$x_{ab} = -\frac{J}{A \cdot y_a}, \quad y_{ab} = -\frac{I}{A \cdot y_a} .$$

If ab is parallel to $Y-Y$, then,

$$x_{ab} = -\frac{I_y}{A \cdot x_a}, \quad y_{ab} = -\frac{J}{A \cdot x_a}$$

The radii vectores of the kernel are lengths which for any θ need only be multiplied by the area of the section (A) to give the section modulus

$$\left(\frac{I_x I_y - J^2}{(I_y \sin\theta - J \cdot \cos\theta)\, y + (I_y \cdot \cos\theta - J \cdot \sin\theta) x}\right),$$

but these lengths must be considered positive if measured on the opposite side of G from P.

Section Modulus Polygons.

In the equation $S = \frac{N}{A} + \frac{M}{s}$ (see Eccentric Loads), s is the section modulus. The section modulus polygon is the polygon the lengths of whose radii vectores are the graphical representations of the values of s for extreme fibers for successive values of θ from 0 to 360 degrees. The section modulus polygon is a figure whose sides are parallel to the sides of the kernel of the given section but which lie on opposite sides of the center of gravity from the sides of the kernel.

The most general value of s is

$$\frac{I_x I_y - J^2}{(I_y \sin\theta - J \cos\theta)\, y + (I_y \cos\theta - J \cdot \sin\theta)\, x}.$$

For any section which is symmetrical about either axis, s becomes

$$s = \frac{I_x I_y}{I_y \sin\theta \cdot y + I_x \cos\theta \cdot x}.$$

For any symmetrical section for which P lies on $Y-Y$, $\theta = 90°$, hence

$$s = \frac{I_x}{y}.$$

If for any symmetrical section P lies on $X-X$, $\theta=0°$, hence

$$s=\frac{I_y}{x}.$$

There will be one vertex of the s-polygon for each side of the polygon bounding the section. If x_a, y_a and x_b, y_b, are the coördinates of a and b, two consecutive vertices of the bounding polygon of the section, then the coördinates of the vertex of the s-polygon corresponding to the side ab of the bounding polygon will be

$$_ax_{ab}=\frac{(x_a-x_b)\,J-(y_a-y_b)\,I_y}{x_ay_b-x_by_a},$$

$$y_{ab}=\frac{(x_a-x_b)\,I_x-(y_a-y_b)\,J}{x_ay_b-x_by_a}.$$

If ab is parallel to $X-X$,

$$x_{ab}=\frac{J}{y_a}, \qquad y_{ab}=\frac{I_x}{y_a}.$$

If ab is parallel to $Y-Y$,

$$x_{ab}=\frac{I_y}{x_a}, \qquad y_{ab}=\frac{J}{x_a}.$$

For sections symmetrical about either $X-X$, or $Y-Y$, $J=0$, and the values of $\frac{I_x}{y_a}$ and $\frac{I_y}{x_a}$ can be found in the handbooks issued by the steel companies, under the column marked "Section Modulus." The vertices can then be plotted and connected by straight lines to form the s-polygon. From this s-polygon the values of s for any value of θ can be obtained graphically.

The most advantageous plane of loading for any section will be that having the greatest value of s.

DIAGONAL STRESSES

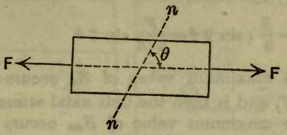


Fig. 45.

F = axial load.

A = area of section normal to axis of bar.

$n-n$ = any diagonal section.

θ = angle which $n-n$ makes with axis.

S = unit axial stress.

S_s = unit shear along plane normal to axis.

S_n = unit tension or compression normal to section $n-n$.

S_{sn} = unit shear along section $n-n$.

For *combined direct stress and vertical shear*,

$$S_n = \frac{S}{2}\,(1-\cos 2\,\theta) + S_s \,.\, \sin 2\,\theta.$$

$$S_{sn} = \frac{S}{2}\,.\, \sin 2\,\theta + S_s \,.\, \cos 2\,\theta.$$

The maximum or minimum value of S_n occurs when $\cot 2\,\theta = -\dfrac{S}{2\,S_s}$, and is

$$\text{max. } S_n = \frac{1}{2}\,S \pm \left(S_s^2 + \frac{S^2}{4}\right)^{\frac{1}{2}}.$$

The maximum value of S_{sn} occurs when $\tan 2\,\theta = +\dfrac{S}{2\,S_s}$, and is

$$\text{max. } S_{sn} = \left(S_s^2 + \frac{S^2}{4}\right)^{\frac{1}{2}}.$$

For *axial load only*, $S_s=0$, hence

$$S_n=\frac{S}{2}(1-\cos 2\theta)=S\,.\,\sin^2\theta=\frac{F}{A}\,.\,\sin^2\theta.$$

$$S_{sn}=\frac{S}{2}\,.\,\sin 2\theta=\frac{F}{2A}\sin 2\theta.$$

The maximum value of S_n occurs when $\theta=90°$, and is then the unit axial stress.

The maximum value of S_{sn} occurs when $\theta=45°$, and is $\frac{S}{2}$ or $\frac{F}{2A}$.

THIN PIPES, CYLINDERS, AND SPHERES.

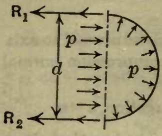


Fig. 46.

S = unit stress in metal.
t = thickness of metal.
d = diameter.
p = unit pressure of liquid or gas.
θ = angle which the direction of P makes with $X-X$.

For the transverse stress across a longitudinal section of a pipe or cylinder,

$$R_1=R_2=\tfrac{1}{2}\,\Sigma\, p\,.\,\cos\theta=\tfrac{1}{2}\,p\,.\,d.$$

$$S=\frac{R_1}{t}=\frac{p\,.\,d}{2\,t}.$$

For the longitudinal stress across a transverse section of a pipe, or for the stress in a thin hollow sphere,

$$S=\frac{p\,.\,\frac{1}{4}\,\pi d^2}{\pi d\,.\,t}=\frac{p\,.\,d}{4\,t},$$

which is one-half of the unit transverse stress in a pipe having the same diameter and thickness.

RIVETED JOINTS.

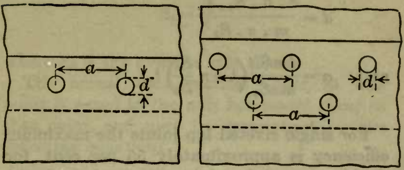


Fig. 47.

a = distance center to center of two consecutive rivets in one row.

d = diameter of rivet or rivet hole.

F = stress in unriveted plate in length a.

t = thickness of plate.

S_t = unit tensile stress.

S_c = unit compressive or bearing stress.

S_s = unit shearing stress.

e_t = efficiency of joint for tension.

e_c = efficiency of joint for compression.

e_s = efficiency of joint for shear.

m = number of shearing sections of rivets in distance a. (Notice that for butt joints each rivet has two shearing areas.)

n = number of bearing areas of rivets in distance a.

$$F = t\,(a-d)\,S_t = m \,.\, \frac{1}{4}\,\pi d^2 \,.\, S_s = n \,.\, t \,.\, d \,.\, S_c.$$

$$e_t = \frac{a-d}{a}.$$

$$e_s = \frac{m \,.\, \pi \,.\, d^2 S_s}{4 \,.\, a t S_t}.$$

$$e_c = \frac{n \,.\, d S_c}{a S_t}.$$

For maximum, efficiency, make $e_t = e_s = e_c$, for which

$$d = \frac{4 \cdot n \cdot S_c \cdot t}{m \cdot \pi \cdot S_s},$$

and $$a = \frac{4\, nS_c t}{m\pi S_s}\left(1 + n\frac{S_c}{S_t}\right) t.$$

For single riveted lap joints the maximum efficiency is approximately 55 per cent, for double riveted lap joints approximately 70 per cent, for triple riveted lap joints approximately 75 per cent, and for triple and double riveted butt joints approximately 80 per cent.

BEAMS.

Vertical Shear. The vertical shear at any given section of a horizontal beam is the sum of the vertical components of all of the stresses at that section. The vertical shear is equal to the sum of all the reactions of the supports upon the left of the given section minus the sum of all of the vertical loads on the left of the section.

For any beam the vertical shear upon the right side of the left support of any span is

$$V_1 = \frac{M_2 - M_1}{l} + \frac{1}{2}\, wl + \Sigma F\left(1 - \frac{a}{l}\right),$$

where

M_1 = the moment at the left support,

M_2 = the moment at the right support,

w = the uniform load per lineal unit,

F = any concentrated load,

a = the distance from the left support to F,

l = the length of span.

Shearing Stresses. If V = vertical shear at any section,

$$S_s = \frac{V}{A}$$

where S_s is the average unit shear.

The actual unit vertical shear at any point is equal to the unit horizontal shear at that point, and may be determined by the following equation:

$$S_s = \frac{V}{I \cdot b} \cdot \sum_y^c (y \cdot dA),*$$

where b is the breadth of the section at the given point, y is the distance of the point considered from the neutral axis, and c is the distance from the neutral axis to the extreme fiber on the same side as the point considered.

The maximum value of S_s occurs at the neutral axis, and is

$$\max. S_s = \frac{V}{I \cdot b} \int_0^c y \cdot dA = \frac{V}{I \cdot b} \cdot A_1 y_1,$$

where A_1 is the area of the portion of the section on one side of the neutral axis, and y_1 is the distance from the neutral axis to the center of gravity of the portion of the section on one side of the neutral axis.

For a rectangular section, the maximum unit shear is $\frac{3}{2}$ of the mean unit shear.

For *Diagonal Shear*, see Diagonal Stresses, page 61.

Bending Moment. The bending moment at any point for any beam is

$$M = M_1 + V_1 x - \tfrac{1}{2} wx^2 - \Sigma F (x - a),$$

---

* See "Merriman's Mechanics of Materials," page 269.

where

M = bending moment at section considered,
M_1 = bending moment at the left support,
V_1 = vertical shear upon the right side of the left support,
w = uniform load including weight of beam, per lineal unit,
F = any concentrated load upon the left of the section considered,
x = distance from the left support to the section considered,
a = distance from left support to F.

For any beam of one span V_1 is equal to the reaction at the left support.

The maximum values of M occur at those sections for which $\frac{dM}{dx} = 0$, that is, where the shear passes through zero.

The values of M for special cases are given in Table of Beams, page 68.

Theorem of Three Moments. For any two consecutive spans of a continuous beam, let

M_1 = moment at the left support,
M_2 = moment at the middle support,
M_3 = moment at the right support,
l_1 = length of the first span,
l_2 = length of the second span,
l = length of span for equal spans,
w_1 = uniform load per lineal unit on first span,
w_2 = uniform load per lineal unit on second span,
F_1 = any concentrated load on the first span,
F_2 = any concentrated load on the second span,
a_1 = distance from first support to F_1,
a_2 = distance from middle support to F_2.

Then, for *uniform loads only*,

$$M_1 l_1 + 2\,M_2\,(l_1 + l_2) + M_3 l_2 = -\tfrac{1}{4}\,w_1 l_1^3 - \tfrac{1}{4}\,w_2 l_2^3.$$

For *equal spans* with *equal uniform loads*,

$$M_1 + 4\,M_2 + M_3 = -\tfrac{1}{2}\,wl^2.$$

For *concentrated loads only*,

$$M_1 l_1 + 2\,M_2\,(l_1 + l_2) + M_3 l_2$$
$$= -F_1\left(a_1 l_1 - \frac{a_1^3}{l_1}\right) - F_2\left(2\,a_2 l_2 - 3\,a_2^2 + \frac{a_2^3}{l_2}\right).$$

Flexural Stresses. The *tensile* and *compressive stresses* in a beam, produced by bending, are the same as the stresses upon a section having an eccentric load, due to the moment of that load. Therefore, for *pure flexure* the tensile and compressive stresses for the extreme fibers of any section can be determined by placing $\frac{N}{A} = 0$ in the formula for S given under Eccentric Loads, which gives

$$S = \frac{M}{s},$$

where s is the section modulus, the values for which are given under Section Modulus Polygons.

For *combined flexure and direct stress*, the tensile and compressive stresses are given by the formulæ for Eccentric Loads.

Elastic Curves. The curve which is assumed by the neutral surface of a beam under load is called the elastic curve.

The *radius of curvature* of the elastic curve is

$$R = \frac{EI}{M} = \frac{dl^3}{dx \cdot d^2y} = \frac{dx^2}{d^2y},$$

from which the equation of the elastic curve can be obtained, for any particular case, by placing M equal to $EI\frac{d^2y}{dx^2}$, and by making two integrations to obtain an equation in terms of x and y.

The *deflection* of a beam at any given point is obtained by substituting the particular value of x in the equation of the elastic curve and solving for y. The maximum deflection occurs at the section for which

$$\frac{dy}{dx}=0.$$

(For particular cases, see Table of Beams.)

TABLE OF BEAMS.

NOTE. — The equations for elastic curves and the values of Δ apply only to beams of uniform section.

Beams Supported at Both Ends and Uniformly Loaded.

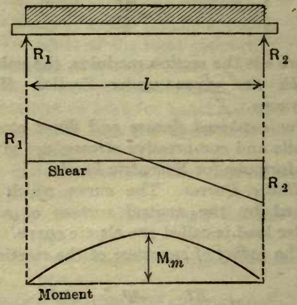


Fig. 48.

$$R_1 = R_2 = \frac{1}{2}\, wl = \frac{W}{2}.$$

$$V = R_1 - wx.$$

$$M = R_1 x - \frac{1}{2}\, wx^2$$

$$= \frac{1}{2}\, wlx - \frac{1}{2}\, wx^2$$

$$= \frac{1}{2}\, Wx - \frac{1}{2}\, wx^2.$$

$$M_m = \frac{1}{8}\, wl^2 = \frac{1}{8}\, Wl.$$

$$EI\,\frac{d^2y}{dx^2} = \frac{1}{2}\, wlx - \frac{1}{2}\, wx^2.$$

$$24\, EIy = w\,(-x^4 + 2\, lx^3 - l^3x).$$

$$y = \Delta \text{ when } x = \frac{l}{2}\text{, or}$$

$$\Delta = \frac{5}{384}\,\frac{wl^4}{EI} = \frac{5}{384}\,\frac{Wl^3}{EI}.$$

Beam Supported at Both Ends and Loaded with a Concentrated Load at Center of Span.

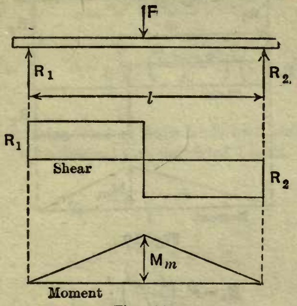


Fig. 49.

$$R_1 = R_2 = \frac{1}{2}\, F.$$

$V = R_1$, or $V = R_2$.

$M = R_1 x$, on the left of F,

$= R_1 x - F\left(x - \frac{l}{2}\right)$, on the right of F.

$$M_m = \frac{1}{4} Fl.$$

$EI \frac{d^2y}{dx^2} = \frac{1}{2} Fx$, on the left of F.

$48\, EIy = F\,(4\, x^3 - 3\, l^2 x)$, on the left of F.

$$\Delta = \frac{1}{48} \frac{Fl^3}{EI}.$$

(For both uniform and concentrated loads, combine the results for each.)

Beam Supported at Both Ends and Loaded with a Concentrated Load Distant *a* from the Left Support.

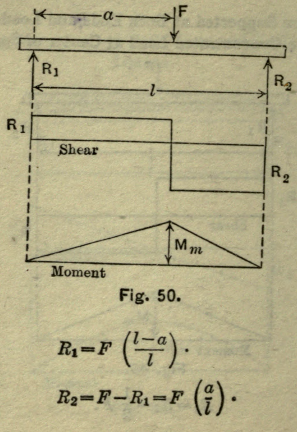


Fig. 50.

$$R_1 = F\left(\frac{l-a}{l}\right).$$

$$R_2 = F - R_1 = F\left(\frac{a}{l}\right).$$

$V = R_1$, on the left of F,
$= R_2$, on the right of F.

$M = R_1 x$, on the left of F,
$= R_1 x - F\ (x-a)$, on the right of F.

$$M_m = Fa\left(1 - \frac{a}{l}\right).$$

$EI\dfrac{d^2y}{dx^2} = R_1 x$, on the left of F,

$= R_1 x - F\ (x-a)$, on the right of F.

$EIy = \dfrac{1}{6}\ R_1 x^3 + c_1 x + c_2$, on the left of F,

$= \dfrac{1}{6} R_1 x^3 - \dfrac{1}{6} F x^3 + \dfrac{1}{2} F a x^2 + c_3 x + c_4$, on the right of F.

$$6\ EIy = F\left(1 - \frac{a}{l}\right)x^3 - F\left(2al - 3\ a^2 + \frac{a^3}{l}\right)x.$$

The maximum deflection (Δ) occurs at the section for which

$$x = \sqrt{\frac{2\ al - a^2}{3}},$$

and is
$$\Delta = \frac{F}{3\ EI}\left(\frac{2\ al - a^2}{3}\right)^{\frac{3}{2}}\left(1 - \frac{a}{l}\right).$$

Beam Supported at Both Ends and Loaded with Several Concentrated Loads.

$$R_1 = \frac{\Sigma\ F\ (l-a)}{l}.$$

$$R_2 = \frac{\Sigma\ Fa}{l} = \Sigma\ F - R_1.$$

$$V = R_1 - \sum_0^x F.$$

$$M = R_1 x - \sum_0^x F\ \ (x-a).$$

The maximum moment (Mm) occurs at the section for which $R_1 - \sum_0^x F = 0$, that is, where the vertical shear is zero.

For a system of movable loads the maximum moment will occur under one of the loads, the loads being in such a position

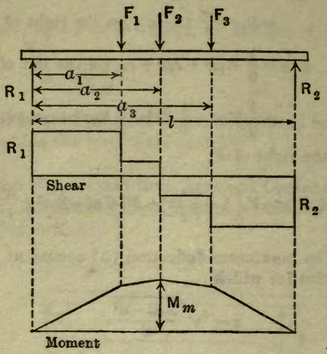


Fig. 51.

that the center of the span is midway between the center of gravity of all the loads and the section at which the maximum moment occurs.

The maximum deflection of a beam loaded with several loads is the sum of the deflections produced by each load at the section at which the maximum deflection for the entire system of loads occurs. The deflections produced by each load can be obtained by means of the equation of the elastic curve for a single load.

Cantilever Beam with Uniform Load.

$$R_1 = wl = W$$

$$R_2 = 0.$$

$$V = R_1 - wx.$$

$$M = \frac{1}{2} w (l-x)^2,$$

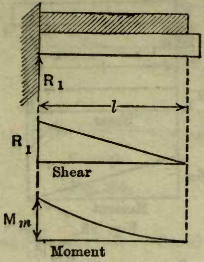


Fig. 52.

or if x is taken from the free end,

$$M = \frac{1}{2} wx^2.$$

$$M_m = \frac{1}{2} wl^2 = \frac{1}{2} Wl.$$

$$EI \frac{d^2y}{dx^2} = \frac{1}{2} wl^2 - wlx + \frac{1}{2} wx^2.$$

$$24\, EIy = wx^4 - 4\, wlx^3 + 6\, wl^2x^2.$$

$$\Delta = \frac{1}{8} \frac{wl^4}{EI} = \frac{1}{8} \frac{Wl^3}{EI}.$$

Cantilever Beam with Concentrated Load at the Free End.

$$R_1 = F.$$

$$R_2 = 0.$$

$$V = R_1.$$

$$M = F\,(l - x).$$

$$M_m = Fl.$$

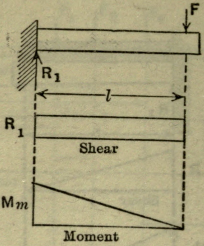


Fig. 53.

$$EI\frac{d^2y}{dx^2} = F\,(l - x).$$

$$6\,EIy = 3\,Flx^2 - Fx^3.$$

$$\Delta = \frac{1}{3}\frac{Fl^3}{EI}.$$

Beam Fixed at Both Ends and Uniformly Loaded.

$$R_1 = R_2 = \frac{1}{2}\,wl = \frac{1}{2}\,W.$$

$$V = R_1 - wx.$$

$$M = -\frac{1}{12}\,wl^2 + \frac{1}{2}\,wlx - \frac{1}{2}\,wx^2.$$

$$M_o = \frac{1}{24}\,wl^2 = \frac{1}{24}\,Wl.$$

$$M_m = \frac{1}{12} wl^2 = \frac{1}{12} Wl.$$

$$EI\frac{d^2y}{dx^2} = M_1 + \frac{1}{2} wlx - \frac{1}{2} wx^2.$$

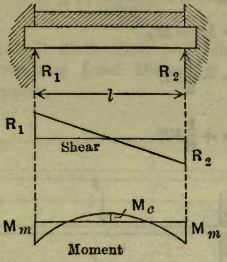


Fig. 54.

By placing $\frac{dy}{dx} = 0$ when $x = 0$ and when $x = l$,

$$M_1 = -\frac{1}{12} wl^2.$$

$$24\ EIy = w\,(-l^2x^2 + 2\,lx^3 - x^4).$$

$$\Delta = \frac{1}{384}\frac{Wl^4}{EI} = \frac{1}{384} Wl^3.$$

Beam Fixed at Both Ends and Loaded at the Center of the Span with a Concentrated Load.

$$R_1 = R_2 = \frac{1}{2} F.$$

$V = R_1$, on the left of F,

$= R_2$, on the right of F.

$$M = -\frac{1}{8}Fl + \frac{1}{2}Fx, \text{ on the left of } F,$$

$$= -\frac{1}{8}Fl + \frac{1}{2}Fx - F\left(x - \frac{l}{2}\right),$$

on the right of F.

$$M_m = M_1 = M_c.$$

$$M_1 = -\frac{1}{8}Fl.$$

$$M_c = +\frac{1}{8}Fl.$$

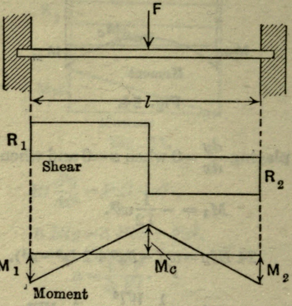


Fig. 55.

$$EI\frac{d^2y}{dx^2} = M_1 + \frac{1}{2}Fx, \text{ on the left of } F,$$

$$= M_1 + \frac{1}{2}Fx - F\left(x - \frac{l}{2}\right),$$

on the right of F.

By placing $\frac{dy}{dx} = 0$ when $x = 0$ and when $x = \frac{l}{2}$,

$$M_1 = -\frac{1}{8}Fl.$$

$48\ EIy = 4\ Fx^3 - 3\ Flx^2$, on the left of F.

$$\Delta = \frac{1}{192}\frac{Fl^3}{EI}.$$

Beam Fixed at Both Ends and Loaded with a Concentrated Load Distant a from the Left Support.

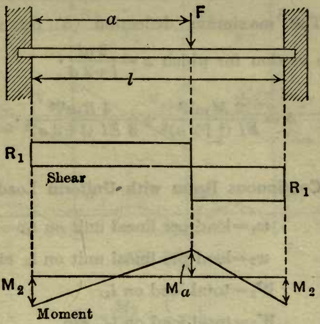


Fig. 56.

$$R_1 = F\left(1 - 3\frac{a^2}{l^2} + 2\frac{a^3}{l^3}\right).$$

$$R_2 = F\frac{a^2}{l^2}\left(3 - 2\frac{a}{l}\right).$$

$V = R_1$, on the left of F,
$\quad = R_2$, on the right of F.

$M = M_1 + R_1x$, on the left of F,
$\quad = M_1 + R_1x - F\ (x - a)$, on the right of F.

$$M_1 = -Fa\left(1 - 2\frac{a}{l} + \frac{a^2}{l^2}\right).$$

$$M_2 = -\frac{Fa^2}{l}\left(1 - \frac{a}{l}\right).$$

$$M_a = +F\frac{a^2}{l}\left(2 - 4\frac{a}{l} + 2\frac{a^2}{l^2}\right).$$

$$EI\frac{d^2y}{dx^2} = M_1 + R_1x, \text{ on the left of } F.$$

$$6\,EIy = 3\,M_1x^2 + R_1x^3, \text{ on the left of } F.$$

The maximum deflection (Δ) occurs at the section for which $x = \dfrac{2\,al}{l+2\,a}$.

$$\Delta = \frac{2\,M_1a^2l^2}{EI\,(l+2\,a)^2} + \frac{4\,R_1a^3l^3}{3\,EI\,(l+2\,a)^3}.$$

Continuous Beam with Uniform Loads.

w_1 = load per lineal unit on l_1.

w_2 = load per lineal unit on l_2, etc.

W_1 = total load on l_1.

W_2 = total load on l_2., etc.

$R_1 = V_1$.

$R_2 = V_{2a} + V_{2b}$.

$R_3 = V_{3a} + V_{3b}$.

$R_4 = V_{4a} + V_{4b}$.

$R_n = V_n$.

For a continuous beam *supported at the ends*,

$$O + 2\,M_2\,(l_1 + l_2) + M_3l_2 = -\frac{1}{4}\,w_1l_1^2 - \frac{1}{4}w_2l_2^2.$$

$$M_2l_2+2\,M_3\,(l_2+l_3)+M_4l_3$$
$$=-\frac{1}{4}w_2l_2^2-\frac{1}{4}w_3l_3^2, \text{ etc.}$$

$$M_{n-2}\,l_{n-2}+2\,M_{n-1}(l_{n-2}+l_{n-1})+O$$
$$=-\frac{1}{4}w_{n-2}l_{n-2}^2-\frac{1}{4}w_{n-1}l_{n-1}^2.$$

From the above simultaneous equations M_2, M_3, M_4, . . M_{n-1} can be determined.

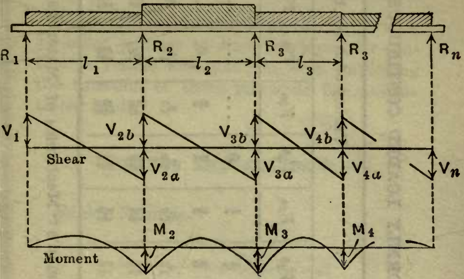


Fig. 57.

$$V_1=\frac{M_2}{l_1}+\frac{1}{2}w_1l_1.$$

$$V_{2a}=W_1-V_1.$$

$$V_{2b}=\frac{M_3-M_2}{l_2}+\frac{1}{2}w_2l_2.$$

$$V_{3a}=W_2-V_{2b}, \text{ etc.}$$

For equal spans with equal uniform load over the entire beam, the ends of the beam resting upon supports, the moment at any support is Kwl^2 or KWl, and the vertical shear is Nwl or NW, where K and N have the values given in the following table:

COEFFICIENTS FOR UNIFORMLY LOADED CONTINUOUS BEAMS.*

No. of Spans	Values of K for Moment						Values of N for Shear											
	M_1	M_2	M_3	M_4	M_5	M_6	V_{1a}	V_{1b}	V_{2a}	V_{2b}	V_{3a}	V_{3b}	V_{4a}	V_{4b}	V_{5a}	V_{5b}	V_{6a}	V_{6b}
1	0	0	...	...	...	...	0	$\frac{1}{2}$	$\frac{1}{2}$	0	...	...	...	...	...	...	...	...
2	0	$\frac{1}{8}$	0	...	...	...	0	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	0	...	...	...	...	...	...
3	0	$\frac{1}{10}$	$\frac{1}{10}$	0	...	...	0	$\frac{4}{10}$	$\frac{6}{10}$	$\frac{5}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{4}{10}$	0	...	...	...	...
4	0	$\frac{3}{28}$	$\frac{2}{28}$	$\frac{3}{28}$	0	...	0	$\frac{11}{28}$	$\frac{17}{28}$	$\frac{15}{28}$	$\frac{13}{28}$	$\frac{13}{28}$	$\frac{15}{28}$	$\frac{17}{28}$	$\frac{11}{28}$	0	...	...
5	0	$\frac{4}{38}$	$\frac{3}{38}$	$\frac{3}{38}$	$\frac{4}{38}$	0	0	$\frac{15}{38}$	$\frac{23}{38}$	$\frac{20}{38}$	$\frac{18}{38}$	$\frac{19}{38}$	$\frac{19}{38}$	$\frac{18}{38}$	$\frac{20}{38}$	$\frac{23}{38}$	$\frac{15}{38}$	0

* Taken from Merriman's "Mechanics of Materials."

For a *continuous beam with fixed ends* consider an imaginary span to be added at each end of the beam, with the free ends resting upon supports. Then write the equation of three moments for each two consecutive spans, making $l=0$ for the first and last spans, and compute the moments at the supports as shown above.

Continuous Beams with Concentrated Loads.

Determine the moments at the supports in a similar manner to that employed for continuous beams with uniform load, employing the equation of three moments for concentrated loads.

STRUTS AND COLUMNS.

Euler's Formula.

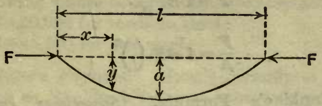


Fig. 58.

$$EI\ \frac{d^2y}{dx^2} = -Fy.$$

$$dx = \left(\sqrt{\frac{EI}{F}}\right)\left(\frac{dy}{\sqrt{a^2-y^2}}\right).$$

$$x = \sqrt{\frac{EI}{F}}\ .\ \sin^{-1}\left(\frac{y}{a}\right),\ \text{or}$$

$$y = a\ .\ \sin\left(x\sqrt{\frac{F}{EI}}\right).$$

Since $y=a$ when $x=\frac{l}{2}$, $\frac{l}{2}\sqrt{\frac{F}{EI}}$ must equal $\frac{\pi}{2}$, or

$$F=EI\frac{\pi^2}{l^2}.$$

$$\frac{F}{A}=\pi^2E\left(\frac{r}{l}\right)^2, \textit{ for round ends.}$$

For *one end round and the other end fixed,* replace l by $\frac{4}{3}l$ and π by 2π, which gives

$$F=\frac{9}{4}EI\frac{\pi^2}{l^2}.$$

$$\frac{F}{A}=\frac{9}{4}\pi^2E\left(\frac{r}{l}\right)^2.$$

For *both ends fixed,* replace l by $\frac{3}{2}l$ and π by 3π, in the formula for round ends, which gives

$$F=4EI\frac{\pi^2}{l^2}.$$

$$\frac{F}{A}=4\pi^2E\left(\frac{r}{l}\right)^2.$$

Rankine's Formula. (*Sometimes called Gordon's Formula.*)

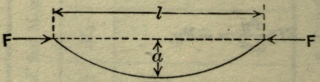


Fig. 59.

From the formula for eccentric loads for a symmetrical section (page 57), the maximum stress will be

$$S=\frac{F}{A}+\frac{My}{I},$$

where y is the distance from the neutral axis to the extreme fiber.

But, $I=Ar^2$, $M=Fa$ and $a=K\frac{l^2}{y}$, where K is some constant depending upon character and condition of the column. Hence

$$S=\frac{F}{A}\left[1+K\left(\frac{l}{r}\right)^2\right], \text{ or}$$

$$\frac{F}{A}=\frac{S}{1+K\left(\frac{l}{r}\right)^2}.$$

Cambria handbook gives the following values of K for steel columns:

$\frac{1}{36{,}000}$ for both ends fixed,

$\frac{1}{24{,}000}$ for one end fixed,

$\frac{1}{18{,}000}$ for pin connected ends.

The above values are to be used with following values of S for ultimate strength:

$S=50{,}000$ for medium steel.

$S=45{,}000$ for soft steel.

Ritter's Formula. Ritter's formula is the same as Rankine's formula except that the expression $\frac{S_e}{nE}$ is used for K, in which S_e is the elastic limit of the material, and n is equal to π^2 for round ends, $\frac{9}{4}\pi^2$ for one end round and one end fixed, and $4\pi^2$ for both ends fixed.

The Straight Line Formula. The straight line formula is

$$\frac{F}{A}=S-C\frac{l}{r},$$

where C is a constant depending upon the character and condition of the column.

Merriman gives the value of C in the above equation to be

$$C=\frac{2}{3}S\left(\frac{S}{3nE}\right)^{\frac{1}{2}},$$

which is obtained by making the straight line a tangent to the curve for Euler's formula passing through the point S for $\frac{l}{r}=0$.

The following values of S and C for allowable stresses are given in Cooper's Specifications for Railroad Bridges, 1906.

$S=10{,}000$, $C=45$,
for live load on chords,
$S=20{,}000$, $C=90$,
for dead load on chords,
$S=8{,}500$, $C=45$,
for live load on posts of through bridges,
$S=17{,}000$, $C=90$,
for dead load on posts of through bridges,
$S=9{,}000$, $C=40$,
for live load on posts of deck bridges,
$S=18{,}000$, $C=80$,
for dead load on posts of deck bridges,
$S=13{,}000$, $C=60$,
for wind load on lateral struts.

Engineering News Formula. The *Engineering News*, Vol. 57, No. 1, Jan. 3, 1907, gives the following formula:

$$S=\frac{F}{A}\left(1+\frac{ay}{r^2}\right),$$

which is the same as Rankine's formula given on page 87, allowing the eccentricity a to remain in the formula instead of substituting $K\frac{l^2}{y}$. The value of a to be used may be considered to represent the eccentricity due to imperfection in manufacture (since it is impossible to obtain the ideal straight column), plus the additional eccentricity due to the failure to obtain an axial load. The proper value of a to obtain correctly proportioned columns might be determined empirically by experiment, or it may be determined by comparison with column formulæ in use which have been found to give correct results.

For any formula of the Rankine type,

$$a = K\frac{l^2}{y}.$$

In the article above mentioned the values of a for a number of formulæ in use are thus computed, the mean values being as follows:

$$a = 0.000051\frac{l^2}{y}, \text{ for steel,}$$

$$a = 0.000177\frac{l^2}{y}, \text{ for cast iron,}$$

$$a = 0.000164\frac{l^2}{y}, \text{ for timber.}$$

For any formula of the straight line type

$$a = \frac{CAlr}{Fy}.$$

In the article above mentioned the values of a for a number of formulæ of the straight line type have been computed, using $\frac{r}{y} = 0.8$, the mean values being as follows:

$a = 0.0053\ l$, for steel,
$a = 0.0015\ l$, for cast iron.
$a = 0.0044\ l$, for timber.

Eccentrically Loaded Columns. To the quantity a in the Engineering News formula add the eccentricity of the load at the end of the column, that is

$$S=\frac{F}{A}\left[1+\frac{(e+a)\,y}{r^2}\right],$$

where e = eccentricity of load at the end of the column.

To determine the maximum stress of an eccentrically loaded column by Rankine's formula replace a in the above formula by its equivalent $K\frac{l^2}{y}$, which gives

$$S=\frac{F}{A}\left[1+K\frac{l^2}{r^2}+\frac{ey}{r^2}\right].$$

TORSION.

Circular Sections.

Twisting moment, $M=Fa$.

Circular Sections

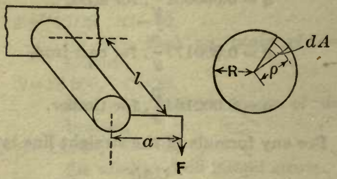


Fig. 60. Fig. 61.

Resisting moment, $M_r=\int\frac{\rho^2}{R}SdA$, where S is the shearing stress at the extreme fiber.

$$M=M_r, \text{ or}$$

$$M=\frac{SI_0}{R},$$

where I_0 is the polar moment of inertia.

For a *solid round shaft* $\frac{I_0}{R}=\frac{1}{2}\pi R^3$, hence

$$M=\frac{1}{2}\pi R^3 S, \quad \text{or} \quad S=\frac{2M}{\pi R^3}.$$

Non-Circular Sections. (Taken from Merriman's "Mechanics of Materials.") For non-circular sections the above formulæ are only approximate.

For an *elliptical section* whose major axis is m and whose minor axis is n the maximum stress is

$$S=\frac{16\,Fa}{\pi mn^2}, \text{ or}$$

$$M=\frac{\pi mn^2 S}{16}.$$

For a *rectangular section* whose long side is m and whose short side is n, the maximum stress is

$$S=\frac{9}{2}\frac{Fa}{mn^2}, \text{ or}$$

$$M=\frac{2}{9}mn^2 S.$$

Transmission of Power. The horse-power which is transmitted by a shaft is

$$\text{H.P.}=\frac{2\,\pi a\,.\,F\,.\,\omega}{550\times 12},$$

where a = moment arm in inches,

ω = number of revolutions per sec.

But, $Fa=\frac{SI_0}{R}$, hence

$$\text{H.P.}=\frac{2\,\pi\omega SI_0}{550\times 12\,R}=0.000952\frac{\omega SI_0}{R}.$$

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