THE FIRST THREE SECTIONS OF NEWTON'S PRINCIPIA,

WITH

AN APPENDIX;

AND THE

NINTH AND ELEVENTH SECTIONS.

EDITED BY

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The following pages with a few alterations were originally taken from the Manuscripts, which had been used in St John's College, and were printed with the view of saving to the Student the trouble of copying them. The few Propositions of the Seventh and Eighth Sections, now generally read in the University, are given in the Appendix, which also contains some examples of the applications of the principles of the first three Sections.

Sedbergh,
January, 1843.

SECTION I.

OF THE METHOD OF LIMITS AND LIMITING RATIOS.

DEF. 1. THE *Limit* of a continually increasing or decreasing quantity or ratio is that quantity or ratio, to which it continually approximates, but to which, though it may approach nearer than by any assignable difference, it never becomes actually equal.

OBS. The limit of a varying quantity or ratio is frequently called the *ultimate* value of that quantity or ratio; when we say that one quantity is *ultimately* equal to another, it is not to be inferred that the two quantities are ever equal, though their difference may be less than any assignable quantity.

DEF. 2. Quantities or the ratios of quantities tend continually to equality, when the ratio of the difference to either of them continually decreases.

LEMMA I.

Quantities and the ratios of quantities, which tend continually to equality, and whose difference may be made to bear to either of them a ratio less than any finite ratio, have their limits equal.

For if the limits be not equal, let L and $L + D$ represent them; then the ratio of the difference of the limits to one of them

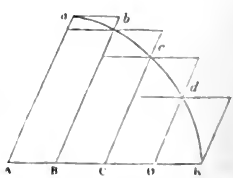
$$= D : L \quad \text{or} \quad D : L + D.$$

Now since the quantities or ratios tend continually to equality, the ratio of their difference to either of them must always be greater than that of the difference of their limits to either of the limits, that is than $D : L$ or $D : L + D$, either of which is a finite ratio. But by the hypothesis the ratio of their difference to either of them may be made less than any finite ratio, which is absurd; therefore the limits are not unequal, that is, they are equal.

Cor. Hence if the quantities or ratios be finite, the limit of their difference, as they tend continually to equality, must equal 0. If they be indefinitely great, the limit of their difference may be a finite quantity or ratio, for it would bear to either of them an indefinitely small ratio. Lastly, if they be indefinitely small, it must be a quantity or ratio, which vanishes compared with either of them.

Lemma II.

If in any figure AKa, bounded by the straight lines Aa, AK, and the curve line Ka, there be inscribed any number of parallelograms Ab, Bc, Cd....on equal bases AB, BC, CD,..., and the parallelograms Ba, Cb, Dc....be completed; then if the number of these parallelograms be increased and their breadths diminished indefinitely, the limit of the sum of each series will be the curvilinear area AKa.

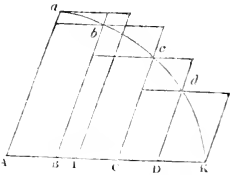


For as their bases are diminished, each series of parallelograms continually approximates to the area AKa. Also the difference between the two series is the sum of the parallelograms ab, bc, cd... which sum is equal to the parallelogram aB, for the base of each is equal to AB, and the sum of their altitudes to that of aB, and by diminishing the bases this difference, and therefore, *a fortiori*, the difference of either series and the area AKa may be made less than any assignable quantity, and therefore, by Lemma 1, the limit of either series is the curvilinear area AKa.

Lemma III.

If the two series of parallelograms be described in the same manner as in the last Lemma, except that their bases are not all equal, the limit of each series, when their bases are diminished indefinitely, is in this case also the curvilinear area AKa.

For take AF equal to the greatest base, and complete the parallelogram Fa; then this parallelogram, which is evidently greater than the difference between the two series of parallelograms, may, by diminishing the base, be made less than any assignable quantity. Hence the difference between the two series, and therefore *a fortiori*, the difference between each series and the area AKa may be made less than any assignable quantity; and they tend continually to equality, therefore, by Lemma 1, the limit of each series is the curvilinear area AKa.



Cor. 1. If the chords ab, bc, cd... be drawn, the limit of the area bounded by Aa, AK and the chords, when the bases AB, BC, CD.... are diminished indefinitely, is the curvilinear area AKa, for it always lies between this area, and the inner series of parallelograms.

Cor. 2. The limit of the figure bounded by Aa, AK and the tangents through a, b, c... is the same curvilinear area, since it lies always between the curvilinear area, and the outer series of parallelograms.

Cor. 3. The curve line aK is the limit of the boundary formed by the chords.

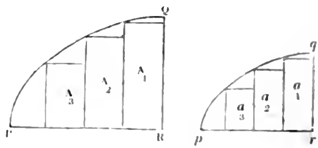
Lemma IV.

If in two curvilinear figures there can be inscribed the same number of parallelograms, which, when their number is increased, and their breadths diminished indefinitely, are ultimately to each other in a given ratio, the areas of the curvilinear figures will be in that ratio.

Let PQR, pqr be the figures, and let the parallelograms A_1, A_2, A_3, ... be inscribed in the one, and a_1, a_2, a_3, ... in the other,

and let $\frac{A_1}{a_1} = m + x_1$, $\frac{A_2}{a_2} = m + x_2$, $\frac{A_3}{a_3} = m + x_3$, &c. = &c.

$x_1 x_2 x_3 \ldots$ being quantities, which vanish, when the breadths of the parallelograms are diminished indefinitely, so that according to the hypothesis,



$$\text{limit } \frac{A_1}{a_1} = m = \text{limit } \frac{A_2}{a_2} = \text{limit } \frac{A_3}{a_3} = \text{\&c.}$$

Hence $A_1 = m a_1 + x_1 a_1$, $A_2 = m a_2 + x_2 a_2$, $A_3 = m a_3 + x_3 a_3$, &c. = &c.

$$\therefore\ A_1 + A_2 + A_3 + \ldots = m(a_1 + a_2 + a_3 + \ldots) + x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$$

$$\therefore\ \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

$$\therefore\ \text{limit } \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \text{limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

Now since x_1 vanishes in the limit, $x_1 a_1$ vanishes compared with a_1; similarly $x_2 a_2$ vanishes compared with a_2, $x_3 a_3$ compared with a_3, and so on; the number of terms also in the two series is the same, therefore ultimately $x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$ vanishes compared with $a_1 + a_2 + a_3 + \ldots$,

$$\text{or limit } \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = 0.$$

$$\text{Also limit } \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = \frac{\text{area } PQR}{\text{area } pqr},$$

$$\therefore \frac{\text{area } PQR}{\text{area } pqr} = m.$$

Cor. If there be two quantities of any kind, which are divided into the same number of parts, and if these parts, when their number is continually increased and the magnitude of each continually diminished, be to each other in a given ratio, the whole quantities will be in that ratio.

For if the parts be substituted for the parallelograms, and the whole quantities for the figures PQR, pqr, the reasoning will be the same in the two cases.

Def. 1. A *curve* is a line traced out by a moving point, which is continually changing the direction of its motion.

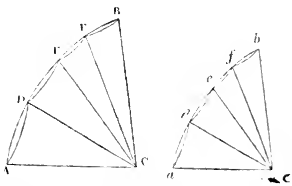
Def. 2. One curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other.

Obs. The curves and curvilinear figures, treated of in this Section, are always supposed to lie in one plane.

Lemma V.

The homologous sides of all similar curvilinear figures are proportionals, and their areas are in the duplicate ratio of the sides.

Let ACB, acb be two similar figures, of which the sides AB, AC, BC, are homologous to ab, ac, bc, respectively; then by definition, if $ADEBC$ be a polygon inscribed in ABC, a similar polygon $adebc$ may be inscribed in abc. Let such polygons be inscribed; and join CD, CE, &c. cd, ce, &c. dividing the polygons into the same number of similar triangles.



$$\therefore\ AD : AC = ad : ac,$$

$$\text{alt}^{\text{do}}\ AD : ad = AC : ac,$$

$$\text{Similarly}\ DE : de = DC : dc = AC : ac,$$

$$EF : ef = AC : ac,$$

....

therefore, componendo

$$AD + DE + EF + \&c. : ad + de + ef + \ldots = AC : ac.$$

Now this being always true, will be true when the number of sides is increased, and their magnitudes diminished, without limit ;

$$\therefore \text{limit}\ AD + DE + EF + \ldots : \text{limit}\ ad + de + ef + \ldots = AC : ac,$$

and therefore by Lem. III, Cor. 3,

$$ADB : adb = AC : ac$$

$$= BC : bc.$$

Again, polygon $ADEBC$: polygon $adebc = AC^2 : ac^2$,

and this being always true will be true in the limit;

$$\therefore \text{limit polygon}\ ADEBC : \text{limit}\ adebc = AC^2 : ac^2;$$

therefore by Lem. III, Cor. 1,

curvilinear figure ABC : curvilinear fig. $abc = AC^2 : ac^2$

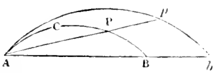
$$= \overline{ADB}|^2 : \overline{adb}|^2$$

$$= BC^2 : bc^2.$$

COR. If ACB, acb be two similar figures, and CE, ce be equally inclined to AC, ac, then $AC : CE = ac : ce$. Hence also this definition :

Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such, that if any two other distances be drawn equally inclined to the former, the four are proportional.

PROB. Let the chord *AB* of the curve *ACB* be produced to *b*, to describe on *Ab* a curve similar to *ACB*.



In *ACB* take any point *P*, join *AP*, and produce *AP* to *p*, so that $Ap : Ab = AP : AB$; then if the curve *Apb* be the locus of all points, whose position is determined in the same manner as that of *p*, it will be similar to the curve *APB*.

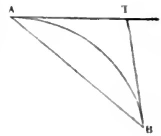
DEF. 1. The *tangent* to a curve *AB* at *A* is the straight line, in which the generating point would move, if instead of changing the direction of its motion it moved on in the direction which it had at *A*.

DEF. 2. The curvature of a curve is said to be *continued* through a point, when the curve is wholly convex to the tangent at that point, and on the same side of it, and when the change of direction is not abrupt, but gradual; that is, if *ATU*, *BT*, (Fig. Lem. VI.) be tangents at *A* and *B*, in a curve of continued curvature, the angle *BTU* as *B* moves up to *A*, diminishes through every change of magnitude from its original value and ultimately vanishes.

LEMMA VI.

If ACB be an arc of continued curvature, AB the chord, and ATU the tangent at A, the angle BAT between the chord and tangent, as B moves along the curve towards A and ultimately coincides with that point, continually diminishes and ultimately vanishes.

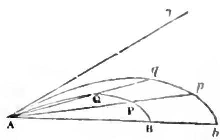
Let the tangents at *A* and *B* meet in the point *T*; then the angle *BTU* measures the change in the direction of the



motion of the generating point which takes place in passing from B to A, and since the curvature is continued, this angle, as B moves towards and ultimately coincides with A, continually diminishes and ultimately vanishes, therefore *a fortiori* the interior angle BAT continually diminishes and ultimately vanishes.

Cor. Similar conterminous arcs, which have their chords coincident, have a common tangent.

Let the similar conterminous arcs, APB, apb have their chords AB, Ab coincident, and let APp, AQq be any other coincident chords; then since the curves are similar $AP : Ap = AB : Ab = AQ : Aq$, therefore the arcs AP, Ap are similar, that is, the chords of the similar arcs AP, Ap coincide. Now let P and p move up to A, the arcs AP, Ap, since they are always similar, will vanish together, and APp in its ultimate position will be a tangent to each, that is, the arcs APB, Apb have a common tangent.

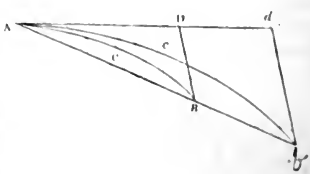


Def. The *subtense* of an arc is a straight line drawn from one extremity of the arc to meet at a finite angle the tangent to the arc at its other extremity.

Lemma VII.

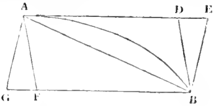
If BD be a subtense of the arc ACB of continued curvature, the chord AB, the arc ACB, and the tangent AD, when BD moves parallel to itself up to A, are ultimately equal to each other.

Produce AD to any fixed point d, and draw db parallel to DB to meet AB produced in b; on Ab describe the arc Acb similar to ACB, and as B moves up to A, let Acb so alter its form as to be always



similar to ACB; hence the two arcs have a common tangent, and the three lines AB, ACB, AD are always proportional to Ab, Acb, Ad. Now as B moves up to A, the angle bAd continually diminishes and ultimately vanishes, (Lemma VI.), the point b moves up to and coincides with d, and therefore Ab and Ad are ultimately equal; also, since db ultimately vanishes, $Ad + db$ is ultimately equal to Ab, therefore the arc Acb, which is greater than Ab and less than $Ad + db$, is ultimately equal to Ab. Hence AB, ACB, AD, which are always proportional to Ab, Acb, Ad, are ultimately equal to each other.

Cor. 1. Since the proof holds whatever be the inclination of BD to the tangent, provided it be finite, if BE be a subtense making any other finite angle with AD, the tangents AE, AD, and the chord and arc are ultimately equal.

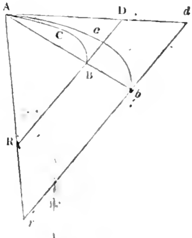


Cor. 2. Also if the parallelograms $ADBF$, $AEBG$ be completed, since AD, AE are always equal to BF, BG respectively, the lines AD, AE, BF, BG are ultimately equal to the chord and arc; and in all geometrical investigations the ultimate values of all these lines may be used indiscriminately for each other.

Lemma VIII.

If the straight lines AR, DBR, which meet in R, make with the chord AB, the arc ACB, and the tangent AD, the triangles ABR, $ACBR$, ADR; these three triangles, when B moves up to A, are ultimately similar and equal to each other.

Produce AD to a fixed point d, and draw dbr parallel to DBR, meeting AB, AR produced in b, r. On Ab describe the arc Acb similar to ACB, and let it so alter its form, as B moves up to A, as to be always similar to ACB. Then the two arcs will have a common tangent ADd, and the three triangles ABR, $ACBR$, ADR will be always similar to



the three Abr, $Acbr$, Adr respectively, and will bear each to each the same ratio, viz. that of $RA^2 : rA^2$; hence, alternando,

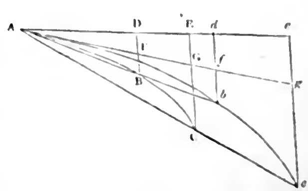
$$ABR : ACBR : ADR = Abr : Acbr : Adr.$$

Now let BD move parallel to itself up to A, then the angle bAd continually diminishes and ultimately vanishes; and Ab and therefore the intermediate arc Acb ultimately coincide with Ad; hence the triangles Abr, $Acbr$, are ultimately similar and equal to Adr; therefore the triangles ABR, $ACBR$, ADR, which are always proportional to them, are ultimately similar and equal to each other.

Obs. In the Lemma RBD is supposed to move parallel to itself towards A, that is, b moves along rd fixed, and the triangles Abr, $Acbr$, Adr are always finite; but the same thing will be true if RBD revolve round R fixed, in which case also, though r moves off to an infinite distance and the triangles Abr, $Acbr$, Adr increase indefinitely, they will be ultimately similar and equal to each other.

Lemma IX.

If the right line AE and the arc ABC, given in position, cut each other in a finite angle at A, and the ordinates BD, CE be drawn, making any other given angle with AE; when BD, CE move parallel to themselves up to A, the limiting ratio of area ABD : area ACE equals that of $AD^2 : AE^2$.



Produce AE to a fixed point e, and take Ad in Ae such, that $Ad : Ae = AD : AE$. Draw db, ec parallel to DB, or EC, meeting the chords AB, AC produced in b, c; and on Ac describe an arc similar to ABC: this arc shall pass through b, for by similar triangles and by construction,

$$AB : Ab = AD : Ad = AE : Ae = AC : Ac,$$

and therefore (Cor. Lemma v.) b is a point in the arc. As B and C move up to A, let the curve Abc so alter its form as to be always similar to ABC, then the area ABD will be always similar to Abd, and ACE to Ace. Hence

$$\text{area } ABD : \text{area } Abd = AD^2 : Ad^2 = AE^2 : Ae^2$$
$$= \text{area } ACE : \text{area } Ace,$$
$$\therefore \text{area } ABD : \text{area } ACE = \text{area } Abd : \text{area } Ace.$$

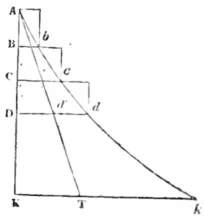
Also the two arcs being similar have a common tangent at A, let this be $AFGfg$; and let BD, CE move parallel to themselves up to A; then the angle cAg continually diminishes and ultimately vanishes, and therefore

$$\text{L.R.* area } Abd : \text{area } Ace = \text{L.R. } \triangle\, Afd : \triangle\, Age$$
$$= \text{L.R. } Ad^2 : Ae^2.$$
$$\text{Hence L.R. area } ABD : \text{area } ACE = \text{L.R. area } Abd : \text{area } Ace$$
$$= \text{L.R. } Ad^2 : Ae^2$$
$$= \text{L.R. } AD^2 : AE^2.$$

Lemma X.

The spaces, described from rest by a body acted on by any finite force, are in the beginning of the motion as the squares of the times, in which they are described.

Def. A finite accelerating or retarding force is such, that the ratio of the time to the velocity generated or destroyed in that time is finite.



Let the straight line AK represent the time of the body's motion from rest, and Kk, drawn at right angles to AK, the last acquired velocity; suppose the time divided into equal intervals AB, BC, CD &c., and let Bb, Cc, Dd &c., drawn at right angles to AK, represent the velocities acquired in the times AB, AC, AD &c.; let $Abcdk$ be the curve passing through the extremities of all the ordinates thus drawn; and complete the parallelograms Ab, Bc, Cd &c.

* L.R. signifies "limit of the ratio" or "limiting ratio."

If now the force be supposed to act by impulses, which would cause the body to move uniformly during the times AB, BC, CD &c., with the velocities Bb, Cc, Dd &c. respectively, the spaces described in the 1st, 2d, 3d &c. intervals will be represented by the parallelograms Ab, Bc, Cd &c. On this supposition therefore, the space described in time AD : space in time AK = sum of the parallelograms in the former case : sum in the latter; and this being true always, will be true when the intervals are diminished and their number increased indefinitely, in which case the force, which was supposed to act by impulses, approximates to a continued force, and the sums of the parallelograms to the areas ADd, AKk, as their limits.

Hence

space in time AD : space in time AK = area ADd : area AKk.

Let the tangent at A cut Kk in T; now, the force being finite, the ratio AK : Kk is always finite; ∴ AK : KT, which equals L.R. AK : Kk is a finite ratio, and therefore,

$$\tan KAT \left(= \frac{KT}{KA}\right) \text{ is finite,}$$

or KA makes a finite angle with the curve at A;

Hence by Lemma IX.

$$\text{L.R. area } ADd : \text{area } AKk = \text{L.R. } AD^2 : AK^2,$$

and therefore in the beginning of the motion, space $\propto$ (time)^2.

Cor. 1. Force is measured by the velocity generated in any time divided by the time, the force being supposed to remain constant for that time. Hence if Dd' be the velocity generated by the force at A, continued constant, in time AD,

$$F \text{ at } A = \frac{Dd'}{AD},$$

and this being always true, will be true when AD is diminished indefinitely,

$$\therefore F = \text{limit}\,\frac{Dd'}{AD} = \text{limit}\,\frac{Dd}{AD}$$

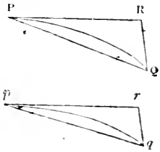
$$= \frac{KT}{AK} = \frac{KT.AK}{AK^2} = \frac{2\text{ triangle } AKT}{AK^2} = 2\text{ limit}\,\frac{\text{area } AKk}{AK^2}$$

$$= 2\text{ limit}\,\frac{\text{space}}{(\text{time})^2}.$$

Cor. 2. The effect produced by F upon the body is independent of any motion which it may have, when F begins to act upon it. Hence generally if S be the space, through which a force F, acting on a body moving in any orbit, draws the body in T'' from the place it would have occupied if the extraneous force had not acted, $F = 2\text{ limit}\,\frac{S}{T^2}$.

On the Curvature of Curve Lines.

Prop. I. If in PR, pr tangents at the points P, p in the curves PQ, pq, PR be taken equal to pr, and the subtenses QR, qr be drawn equally inclined to them, then when QR, qr move parallel to themselves to P, p,



$$\frac{\text{curvature of } PQ \text{ at } P}{\text{curvature of } pq \text{ at } p} = \text{limit}\,\frac{QR}{qr}.$$

Draw the chords PQ, pq,

$$\text{then } \frac{\text{curvature of } PQ \text{ at } P}{\text{curvature of } pq \text{ at } p} = \frac{\text{angle of contact at } P}{\text{angle of contact at } p}$$

$$= \text{limit}\,\frac{\text{angle } QPR}{\text{angle } qpr}$$

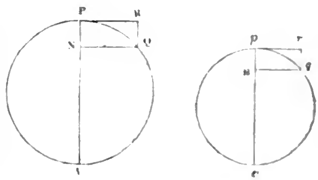
$$= \text{limit}\ \frac{\sin QPR}{\sin qpr}$$

$$= \text{limit}\ \frac{\frac{QR}{QP}\sin R}{\frac{qr}{qp}\sin r}$$

$$= \text{limit}\ \frac{QR}{qr}.$$

PROP. II. The curvatures in different circles vary inversely as the diameters.

Let PQV, pqr be two circles, draw the diameters PV, pr, and the tangents PR, pr. Take $PR = pr$, and draw the subtenses QR, qr parallel to the diameters, and QN, qn parallel to the tangents;



$$\text{then}\ \frac{QR}{qr} = \frac{PN}{pn} = \frac{QN^2}{NV} \div \frac{qn^2}{nv} = \frac{nv}{NV},$$

$$\therefore \frac{\text{curvature at } P}{\text{curvature at } p} = \text{limit}\ \frac{QR}{qr}$$

$$= \text{limit}\ \frac{nv}{NV}$$

$$= \frac{pv}{PV}$$

$$\text{or the curvature} \propto \frac{1}{\text{diameter}}.$$

Cor. Hence in the same circle the curvature is the same at every point.

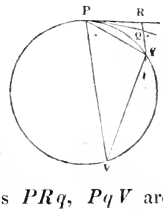
From this property of the circle, and also because by varying the diameter it may be made to have any curvature we please, the circle is made use of to measure the curvature at any proposed points of other curves.

Def. The *circle of curvature* at any point of a curve is that circle which has the same tangent and curvature as the curve has at that point, the curvatures being in the same direction.

Hence if QqR be a common subtense to the curve PQ and the circle Pq, and limit $\frac{QR}{qR} = 1$, Pq will be the circle of curvature at P.

The radius, diameter, and chord of the circle of curvature are generally called the radius, diameter, and chord of curvature.

Prop. III. If PqV be the circle of curvature at any point P, and PV a chord drawn in any given direction, then



$$PV = \text{limit} \frac{(\text{arc})^2}{\text{subtense parallel to the chord}}.$$

Take PQ an arc of the curve, through Q draw the subtense RQq parallel to PV, and join Pq, qV; then since the triangles PRq, PqV are evidently similar,

$$PV = \frac{Pq^2}{qR}.$$

Now this being true whatever be the magnitude of PQ, will be true when RQr moves parallel to itself up to P, in which case $Pq = PQ$ ultimately, and $qR = QR$ ultimately,

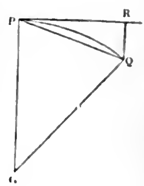
$$\therefore PV = \text{limit} \frac{Pq^2}{qR}$$

$$= \text{limit} \frac{(\text{arc } PQ)^2}{QR}.$$

Cor. Hence the diameter of curvature

$$= \text{limit} \frac{(\text{arc})^2}{\text{subtense perpendicular to the tangent}}.$$

Prop. IV. If in the curve PQ, PG and QG, drawn perpendicular to the tangent PR and the chord PQ respectively, intersect in G, then when Q moves up to P, the limit of PG is the diameter of curvature at P.



Draw the perpendicular subtense QR,

Then by similar triangles PGQ, PQR,

$$PG = \frac{PQ^2}{QR}:$$

$$\therefore \text{limit } PG = \text{limit} \frac{PQ^2}{QR}$$

$$= \text{limit} \frac{(\text{arc } PQ)^2}{QR}$$

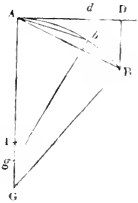
$$= \text{diameter of curvature at } P.$$

Def. The curvature of a curve at any point is said to be finite, when the diameter of curvature at that point is finite.

Lemma XI.

In curves of finite curvature the limiting ratio of the subtenses equals that of the squares of the conterminous arcs.

Let AbB be the curve having a finite curvature at A;



First, Let the subtenses bd, BD be perpendicular to the tangent at A. Draw bg BG at right angles to the chords Ab, AB, and let them meet AgG, which is drawn at right angles to the tangent AD, in the points g and G.

Then as b and B move up to A, g and G move up to I, the extremity of the diameter of curvature of A, as their limit. (Prop. IV.)

Now by similar triangles,

$$BD = \left(\frac{AB^2}{AG}\right), \quad bd = \frac{Ab^2}{Ag}$$

$$\therefore \; BD : bd = \frac{AB^2}{AG} : \frac{Ab^2}{Ag},$$

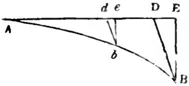
$$\therefore \; \text{L. R. } BD : bd = \text{L. R.} \frac{AB^2}{AG} : \frac{Ab^2}{Ag}$$

$$= \text{L. R. } AB^2 : Ab^2,$$

(since AG, Ag are ultimately equal to AI)

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$

Secondly, Let the subtenses be inclined at any equal angles to the tangent. Draw BE, be perpendicular to the tangent: then by similar triangles,



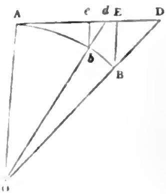
$$BD : BE = bd : be,$$

alternando $BD : bd = BE : be$;

$$\therefore \text{L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Thirdly, Let the subtenses, inclined at unequal angles to the tangent, converge to a point, and revolve round that point fixed, or approach to A according to any other given law.



Let O be the point in which DB, db meet when produced; draw BE, be always parallel to AO; then since the angles at D and d are always finite, AO must always be finite, and L. R. $DO : AO$ will be a ratio of equality, as also L. R. $dO : AO$.

But $BD : BE = DO : AO$ and $bd : be = dO : AO$ $\Big\}$, always and therefore ultimately;

$$\therefore \text{L. R. } BD : BE = \text{L. R. } bd : be;$$

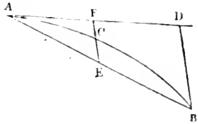
$$\therefore \text{L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. (arc } AB)^2 : (\text{arc } Ab)^2.$$

Cor. 1. Hence by Lemma vii. the limiting ratio of the subtenses will equal that of the squares of the arcs, chords, and tangents.

Theorem. *The subtense of an arc is ultimately equal to four times the parallel sagitta.*

Def. The sagitta of an arc is a line drawn at a finite angle to the chord from its middle point to meet the arc.



Let BD be a subtense of the arc AB, EC the sagitta parallel to it, bisecting the chord in E, and produced to meet the tangent in F.

Then by similar triangles,

$$AF = \tfrac{1}{2} AD, \quad \text{and} \quad EF = \tfrac{1}{2} BD.$$

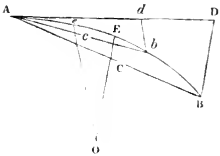
Also by the Lemma,

$$\text{L. R. } CF : BD = \text{L. R. } AF^2 : AD^2$$

$$= 1 : 4$$

$$\therefore \text{L. R. } CE : BD = 1 : 4.$$

Cor. 2. The limiting ratio of the sagittæ, which bisect the chords and converge to a given point, equals that of the squares of the arcs, chords, and tangents.



Let EC, ec be the sagittæ of the arcs AEB, Aeb, bisecting the chords AB, Ab in C, c; draw the subtenses BD, bd respectively parallel to them;

$$\text{then L. R. } EC : BD = 1 : 4$$

$$= \text{L. R. } ec : bd;$$

$$\therefore \text{L. R. } EC : ec = \text{L. R. } BD : bd;$$

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2$$

$$= \text{L. R. } (\text{chord } AB)^2 : (\text{chord } Ab)^2$$

$$= \text{L. R. } (\text{tangent } AD)^2 : (\text{tangent } Ad)^2.$$

Cor. 3. Hence if a body describe the arcs AB, Ab with any given velocity, the limiting ratio of the sagittæ will be that of the squares of the times, in which they are described.

Cor. 4. If the subtenses DB, db be perpendicular to the tangent, as in the first case of the Lemma,

$$\triangle ADB : \triangle Adb = AD \,.\, DB : Ad \,.\, db;$$

$$\begin{aligned} \therefore \text{L. R. } \triangle ADB : \triangle Adb &= \text{L. R. } AD \,.\, DB : Ad \,.\, db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}. \end{aligned}$$

Cor. 5. Since L. R. $DB : db =$ L. R. $AD^2 : ad^2$, the limiting form to which every curve of finite curuature approximates is that of the common parabola.

Hence also,

$$\begin{aligned} \text{L. R. area } ADB : \text{area } Adb &= \text{L. R. } \tfrac{1}{3} AD \,.\, DB : \tfrac{1}{3} Ad \,.\, db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}. \end{aligned}$$

Scholium to Lemma XI.

It was proved in the Lemma that if the curvature be finite, the subtense varies ultimately as the square of the conterminous arc; conversely,

If the subtense vary ultimately as the square of the arc, the curvature is finite, and if it vary according to any other power of the arc, the curvature is infinitely great or infinitely small.

Let PQ and Pq be arcs of a curve and circle, having a common tangent PR, and let RQq be a common subtense.

Since in the circle $qR \propto$ ult. PR^2, let $qR = a \,.\, PR^2$ ultimately,

and suppose that $QR \propto$ ult. PR^n and $\quad = b \,.\, PR^n$ ultimately,

$$\therefore \frac{\text{curvature of } PQ}{\text{curvature of } Pq} = \text{limit } \frac{QR}{qR} = \frac{b}{a} \,.\, \text{limit } PR^{n-2}.$$

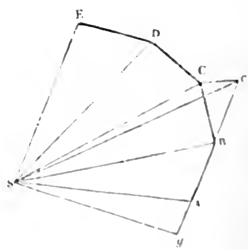
If $n = 2$, the curvature of the curve PQ bears a finite ratio to that of the circle, and is therefore finite. If n be greater than 2, limit $PR^{n-2} = 0$, and therefore the curvature of PQ is infinitely small compared with that of Pq, and the curve will lie between Pq and the tangent. If n be less than 2, limit $PR^{n-2} = \infty$, and therefore the curvature of PQ is infinitely great, and the curve will lie below Pq.

Cor. Since an infinite number of values may be given to n, to each of which there will be a corresponding curve, an infinite number of curves may be described between Pq and the tangent, corresponding to values of n greater than 2, and an infinite number below Pq, corresponding to values of n less than 2.

SECTION II.

ON THE MOTION OF A BODY, CONSIDERED AS A POINT, MOVING IN A NONRESISTING MEDIUM, AND ATTRACTED TO A SINGLE FIXED CENTER OF FORCE.

PROP. I. *If a body move in any orbit about a fixed center of force, the areas, described by lines drawn from the center to the body, lie in one plane, and are proportional to the times of describing them.*



Let S be the center of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in a given time (T). Then if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval (T); but at B suppose an instantaneous impulse communicated to it in direction BS, which causes it to move in direction BC; draw cC parallel to BS, then by the principles of Mechanics, the body at the end of the second interval will be found at C. Join SA, SB, Sc, SC. Since cC is parallel to BS, the triangle $SBC = SBc = SAB$, since $Bc = AB$; and these triangles are in the same plane, as no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of T'', in directions tending always to S, causing the body to describe CD, DE, &c. in the third, fourth, &c. intervals, the triangles SAB, SBC, SCD, &c. will be all equal, and will lie in the same plane; and their bases AB, BC, CD, &c. are described in equal times, therefore the area of any number of these triangles or the polygon $SABCDE$ varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, and the sum of the impulses to a continued force always tending to S, as their limits; and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

Obs. The area, described by the line joining S and the body, is frequently called the area described by the body round S.

Cor. 1. If V be the velocity of the body at A, and p the perpendicular from S upon the tangent at that point, the area described in $t'' = \frac{1}{2}p \,.\, t \,.\, v$.

Draw Sy perpendicular to AB; then since AB is ultimately the tangent at A, limit of $Sy = p$. Also if t be divided into n equal intervals, and AB be the space described in the first interval, the force in S being supposed, as in the Prop., not to act, $AB = \frac{t}{n} \,.\, v$.

Hence, polygonal area described in $t'' = n$. triangle SAB

$$= n \,.\, \tfrac{1}{2} Sy \,.\, \frac{t}{n} \,.\, v$$

$$= \tfrac{1}{2} Sy \,.\, t \,.\, v;$$

and the same is true in the limit.

$\therefore$ curvilinear area described in $t'' = \frac{1}{2} p \,.\, t \,.\, v$.

Cor. 2. Hence the time of describing any part of the orbit

$$= \frac{2}{p \,.\, v} \,.\, \text{area described.}$$

Cor. 3. If $t = 1$, area described in $1'' = \frac{1}{2} p \cdot v$.

Hence in different orbits, the velocity at any point

$$\propto \frac{\text{area described in } 1''}{\text{perpendicular from } S \text{ upon the tangent}},$$

and in the same orbit, the velocity

$$\propto \frac{1}{\text{perpendicular upon the tangent}}.$$

Prop. II. *If a body, moving in a curve, describe in one plane areas proportional to the times by lines drawn from the body to any point, the body is acted on by centripetal forces all tending to that point.* (Vide Fig. Prop. 1.)

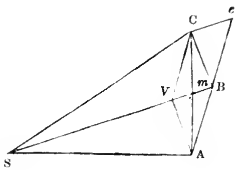
Let S be the point, about which areas proportional to the times are described; and suppose as in Prop. 1. that a body, unattracted by the force in S, describes the straight line AB in a given time T.

In AB produced take $Bc = AB$; then if suffered to proceed, the body would be at c at end of the second interval of T''. But at B suppose an impulse communicated, which causes it to describe BC in the second interval, such that the triangle SBC is equal to and in the same plane with the triangle SAB. Join cC, Sc.

Then the triangle $SBC = SAB = SBc$, therefore cC is parallel to BS, and therefore by the principles of Mechanics the impulse communicated at B tends to S. Similarly if D, E, &c. be the places of the body at the ends of the third, fourth, &c. intervals of T'', so that the triangles SAB, SBC, SCD, &c. are all equal, all the impulses communicated may be shewn to tend to S.

Now suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the polygon is the curvilinear area, and that of the sum of the impulses a continued force tending to S; and the above reasoning still holds in the limit, therefore the body is acted on by a continued force tending to S.

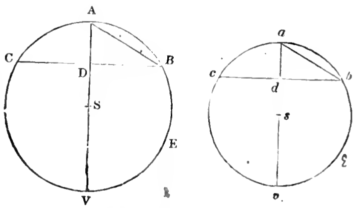
Cor. Draw CV parallel to AB meeting SB in V, and join AV. Then $CV = Bc = AB$, $\therefore$ AV is equal and parallel to CB, or $ABCV$ is a parallelogram. Draw the diagonal CA bisecting BV in m.



Now suppose $S'A'B'C'D'$ to be another orbit, in which the chords $A'B'$, $B'C'$ are described in the same time as either of the chords AB or BC; and let the same construction be made as in the former orbit, then impulse at B : impulse at $B' = cC : c'C' = Bm : B'm'$ and therefore force at B : force at $B' =$ L. R. $Bm : B'm'$; or the centripetal forces in different orbits are in the limiting ratio of the sagittæ of arcs described in equal times, which pass through the centers of force.

A prop: omitted here given in [illegible]

Prop. III. *The centripetal forces, by which bodies describe different circles with uniform velocities, tend to the centers of the circles, and are as the squares of the arcs, described in the same time, divided by the radii.* 1848



Since in each circle the motion is uniform, the arcs described are proportional to the times. But the sectors, i.e. the areas described about the centers of the circles, are as the arcs on which they stand, and are therefore proportional to the times; hence (Prop. II.) the forces tend to the centers of the circles.

Again let CAB, cab be arcs described in the same time in the circles, whose centers are S, s, and let A, a be their middle points; join AB, ab, and draw the diameters ASV, asv cutting the chords CB, cb in D, d; then (Prop. II. Cor.)

$$\text{Force at } A : \text{force at } a = \text{L. R. } AD : ad$$

$$= \text{L. R. } \frac{(\text{chord } AB)^2}{AV} : \frac{(\text{chord } ab)^2}{av}$$

$$= \text{L. R. } \frac{(\text{arc } AB)^2}{AS} : \frac{(\text{arc } ab)^2}{as}.$$

Take AE, ae any other arcs described in equal times;

$$\text{then } AE : ae = AB : ab,$$

and this being true whatever be the magnitudes of AB, ab will be true when they are diminished indefinitely,

$$\therefore AE : ae = \text{L. R. } AB : ab,$$

$$\text{and therefore force at } A : \text{force at } a = \frac{AE^2}{AS} : \frac{ae^2}{as}.$$

Cor. 1. Since AE = velocity × time, if V = velocity of the body, R = radius of the circle, and the time be given,

$$F \propto \frac{V^2}{R}.$$

Cor. 2. Let P equal the periodic time, then since $s = tv$,

$$2\pi R = P \, . \, V; \quad \therefore F \propto \frac{R^2}{P^2 \, . \, R} \propto \frac{R}{P^2}.$$

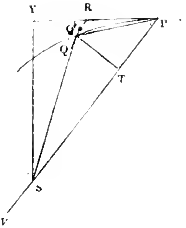
Cor. 3. If P be given, $F \propto R$. If $P \propto R^{\frac{3}{2}}$, $F \propto \frac{1}{R^2}$,

and generally if $P \propto R^n$, $F \propto \frac{1}{R^{2n-1}}$.

PROP. V. as in [illegible]

PROP. VI. *A body moving round a fixed center of force S, describes the arc PQ in T″; if F be the central force at P, and QR a subtense parallel to SP, when PQ and T are diminished indefinitely,*

$$F = 2 \text{ limit } \frac{QR}{T^2}.$$



The motion of the body on leaving P is compounded of two motions, one uniform in direction of the tangent PR, the other variable, arising from the action of S, by which the body is deflected from the tangent, and which tends continually, as Q approximates to P, to become parallel to PS. Let RQ' be the space, through which the force F, continued constant and always parallel to PS, would draw the body in T'',

$$\text{then } F = 2 \frac{Q'R}{T^2};$$

and this is true when T is diminished indefinitely, in which case $Q'R = QR$, and therefore

$$F = 2 \text{ limit } \frac{QR}{T^2}.*$$

COR. 1. Draw QT perpendicular to SP, and join SQ, QP; let h = 2 area described in 1″,

$$\text{then } \frac{T''}{1''} = \frac{2 \text{ area } PSQ}{h},$$

* The above expression for the force being obtained independently of the preceding propositions, it is not necessary that the areas described should be proportional to the times. It is therefore true in orbits described round several centers of force, in which case the expression represents the magnitude of the resultant of all the forces acting on the body at the point P. It is clear, however, that the equable description of areas is supposed to be preserved in the three succeeding corollaries. The result in Cor. 4, is general, and might easily be obtained from Cor. 3, in the particular case of the areas being described equably, by substituting for h its value $Sy \cdot V$, obtained in Cor. 2. Prop. 1.

$$\therefore \ F = 2 \text{ limit} \frac{QR}{T^2} = 2 \text{ limit} \frac{QR \ . \ h^2}{4 \, (\text{area } PSQ)^2}$$

$$= 2 \frac{h^2}{4} \text{ limit} \frac{QR}{(\Delta PSQ)^2} = 2 \frac{h^2}{4} \ . \text{ limit} \frac{QR}{\frac{1}{4} QT^2 \, . \, SP^2}$$

$$= \frac{2h^2}{SP^2} \ . \text{ limit} \frac{QR}{QT^2}.$$

Cor. 2. Draw SY perpendicular to the tangent PR, then since the angle QPR ultimately vanishes, the triangles QPT, SPY are ultimately similar;

$$\therefore \text{ limit} \frac{QT}{PQ} = \frac{SY}{SP}.$$

$$\therefore \text{ limit} \frac{QR}{QT^2} = \frac{SP^2}{SY^2} \text{ limit} \frac{QR}{PQ^2},$$

$$\therefore \ F = \frac{2h^2}{SY^2} \text{ limit} \frac{QR}{PQ^2}.$$

Cor. 3. If PV be the chord of curvature at P through S,

$$PV = \text{limit} \frac{PQ^2}{QR}, \quad \therefore \ F = \frac{2h^2}{SY^2 \, . \, PV}.$$

Obs. If A = the area described in P''. $h = \frac{2A}{P}$, which value may be substituted for h in the above expressions for the force.

Cor. 4. *The space, through which a body must descend from rest by the action of the force at P continued constant, in order to acquire the velocity at P, is $\frac{1}{4}$th of the chord of curvature PV.*

$$\text{Since limit} \frac{PR}{PQ} = 1, \quad F = 2 \text{ limit} \frac{QR}{T^2} = 2 \text{ limit} \frac{QR}{PQ^2} \, . \left(\frac{PR}{T}\right)^2.$$

Now limit $\frac{QR}{PQ^2} = \frac{1}{PV}$, and limit $\frac{PR}{T}$ = velocity at $P = V$;

$$\therefore F = \frac{2V^2}{PV}, \text{ and therefore } V^2 = F \cdot \frac{PV}{2}.$$

Let S = space due to V by the action of F continued constant,

$$\text{then, } V^2 = 2FS,$$

hence equating this to the above expression for V^2, we have

$$S = \tfrac{1}{4} PV.$$

Cor. 5. To find the velocity and periodic time of a body revolving in a circle and acted on by a centripetal force tending to the center of the circle.

$$\text{Here } PV = \text{the diameter} = 2R, \quad \therefore v = \sqrt{F \cdot R};$$

$$\text{Also } P = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi R}{\sqrt{F \cdot R}} = 2\pi \sqrt{\frac{R}{F}}.$$

Lemma. If P, p be points similarly situated in similar orbits described round S, s centers of force also similarly situated, and PV, pv be chords of curvature drawn through the centers of force,

$$SP : sp = PV : pv.$$

Take (Fig. Cor. 6.) PQ, pq similar arcs, and draw the subtenses QR, qr parallel to SP, sp: then by similar figures $SPRQ$, $sprq$,

$$SP^2 : sp^2 = PQ^2 : pq^2,$$

$$SP : sp = QR : qr,$$

$$\therefore SP : sp = \frac{PQ^2}{QR} : \frac{pq^2}{qr}$$

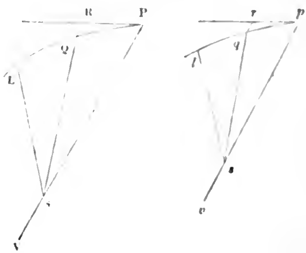
$$= \text{L. R. } \frac{PQ^2}{QR} : \frac{pq^2}{qr}$$

$$= PV : pv.$$

Cor. 6. If V, v be the velocities at P, p, points similarly situated in similar orbits, described round S, s centers of force, also similarly situated,

$$\text{Force at } P\,(F) : \text{force at } p\,(f) = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Let PQ, pq be arcs described in equal times, QR, qr subtenses parallel to SP, sp, and PV, pv chords of curvature at P, p through S, s.



Then since the times are equal,

$$F : f = \text{L.R. } QR : qr$$

$$= \text{L.R. } \frac{PQ^2}{PV} : \frac{pq^2}{pv},$$

$$\text{also } V : v = \text{L.R. } \frac{PQ}{T} : \frac{pq}{T}$$

$$= \text{L.R. } PQ : pq,$$

and since P and p are points similarly situated in similar orbits,

$$SP : sp = PV : pv,$$

by the Lemma in the preceding page,

$$\therefore F : f = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Cor. 7. If similar arcs of similar orbits be described in times T, t round S, s, centers of force similarly situated, (Fig. Cor. 6.)

$$F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Let PL, pl be similar arcs described in times T, t, and take PQ, pq other similar arcs described in times P, p; QR, qr subtenses parallel to SP, sp: then

$$F : f = \text{L. R. } \frac{QR}{P^2} : \frac{qr}{p^2},$$

join SQ, SL, sq, sl.

Then $T : P =$ area PSL : area PSQ (Prop. I.)

$=$ area psl : area psq, by similar figures,

$= t : p$, (Prop. I.)

$$\therefore\ T : t = P : p;$$

and this, being always true, will be true when P and p are diminished indefinitely,

$$\therefore\ T : t = \text{L. R. } P : p,$$

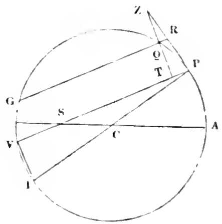
and by similar figures,

$SP : sp = QR : qr$ always and therefore in the limit;

$$\therefore\ F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Prop. VII. *A body revolves in the circumference of a circle, to find the law of force by which it is attracted to a given point.*

Let PAV be the circumference of the circle, and S the center of force; PQ an arc, QR a subtense parallel to SP, QT perpendicular to SP. Let RQ, and PS, produced if necessary, meet the circumference in G, V; draw the diameter PI, join IV, and produce TQ, PR to meet in Z. The triangles PTZ, PVI are evidently similar.



Hence $\dfrac{QR \cdot RG}{QT^2} = \dfrac{RP^2}{QT^2}$ (Euc. III. 36.) $= \dfrac{ZP^2}{ZT^2} = \dfrac{PI^2}{PV^2}$.

Now let Q move up to P,

$$\text{then limit } \frac{QR}{QT^2} = \text{limit } \frac{PI^2}{PV^2 \cdot RG}$$

$$= \frac{PI^2}{PV^3}, \text{ since limit } RG = PV.$$

$$\therefore \; F = \frac{2h^2}{SP^2} \cdot \text{limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{PI^2}{PV^3} = \frac{8h^2R^2}{SP^2 \cdot PV^3},$$

if R = radius of the circle.

Let μ represent that part of the expression for F, which in the same orbit is invariable; then in this case,

$$\mu = 8h^2R^2.$$

$$\text{Hence } F = \frac{\mu}{SP^2 \cdot PV^3},$$

and therefore in the same circle $\propto \dfrac{1}{SP^2 \cdot PV^3}$.

Cor. 1. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{4h^2 \cdot R^2}{SP^2 \cdot PV^2}, \text{ or } = \frac{\mu}{2 \cdot SP^2 \cdot PV^2};$$

$$\therefore \; V = \frac{2hR}{SP \cdot PV}, \text{ or } = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP \cdot PV}.$$

Obs. The quantity (μ) here introduced is that part of the general expression for the centripetal force in any orbit, which is invariable for all points in that orbit, and may always be determined, if the actual force at any given point be known. The force, by which a body is retained in a given curve, is in most cases undergoing a continual change in magnitude; but its magnitude at any given point is to be.

estimated by the effect it would produce, that is, by the velocity it would generate in a unit of time from rest, supposing it to remain constant for that time. Hence, if a second and a foot be the units of time and space, the magnitude of the centripetal force at any point is represented by twice the number of feet, which it would cause a body to describe from rest in 1″; if for instance, it draws a body from rest through 10 feet in 1″, its magnitude will be 20, and it will be to the force of gravity in the ratio of 20 : $32\frac{1}{5}$ or of 100 : 161. Suppose then in the preceding proposition, that the force at A, the extremity of the diameter through S, would if continued constant draw a body through (f) feet in 1″;

$$\text{then } \frac{\mu}{SA^2 \cdot (2R)^3} = 2f;$$

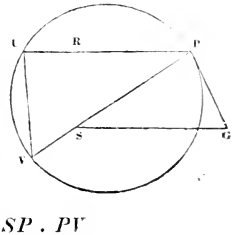
$$\therefore \mu = 2f \cdot SA^2, \ (2R)^3.$$

Cor. 2. Let S be in the circumference, then $PV = SP$.

$$\text{Hence } F = \frac{8h^2R^2}{SP^5}, \quad \text{or } = \frac{\mu}{SP^5}; \text{ and therefore, } \propto \frac{1}{SP^5},$$

$$V = \frac{2hR}{SP^2}, \quad \text{or } = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP^2}.$$

Cor. 3. To compare the forces, by which a body, attracted separately to two centers of force, may describe the same circle in the same periodic time.



Let R and S be the two centers of force; produce PR, PS if necessary to meet the circumference in U, V; draw SG parallel to RP to meet the tangent at P in G, and join UV; then the triangles SPG, PUV are evidently similar,

$$\therefore \frac{SG}{SP} = \frac{PV}{PU}, \quad \text{or } SG = \frac{SP \cdot PV}{PU}.$$

Also since the periodic time is the same, h, which

$$= \frac{2 \text{ area of circle}}{\text{periodic time}},$$

is the same for both centers, hence

$$F \text{ to } R : F \text{ to } S = \frac{1}{RP^2 \cdot PU^3} : \frac{1}{SP^2 \cdot PV^3}$$

$$= \frac{SP^3 \cdot PV^3}{PU^3} : RP^2 \cdot SP$$

$$= SG^3 : RP^2 \cdot SP.$$

Cor. 4. What has been proved in the last corollary in the case of the circle is true of any orbit described round two centers of force separately in the same periodic time. For if PUV be the circle of curvature at P, the expression for F, viz. 2 limit $\frac{QR}{QT^2}$, is the same in the curve and circle, and therefore what has been proved in the one case is true in the other. Hence generally in any orbit described in the same time round two centers of force,

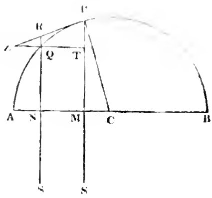
$$F \text{ to } R : F \text{ to } S = SG^3 : RP^2 \cdot SP.$$

If the periodic times are not the same,

$$F \text{ to } R : F \text{ to } S = \frac{SG^3}{P^2 \text{ round } R} : \frac{RP^2 \cdot SP}{P^2 \text{ round } S}.$$

Prop. VIII. *To find the law of force by which a body may describe a semicircle, the center of force being so distant, that all lines drawn from it to the body may be considered parallel.*

Let PQ be an arc of the semicircle, C the center; draw PS, QS parallel to each other towards the center of force, and CM perpendicular to PS; then CM produced both ways will determine the semicircle described. Draw QT perpendicular, and QR parallel to SP, and produce PR, TQ to meet in Z; join CP. The triangles PZT, CPM are evidently similar.



$$\therefore \frac{QR \cdot (RN + QN)}{QT^2} = \frac{RP^2}{QT^2} = \frac{ZP^2}{ZT^2} = \frac{CP^2}{PM^2};$$

$$\therefore \text{limit } \frac{QR}{QT^2} = \frac{CP^2}{2\,PM^3}, \quad \text{since limit } (RN + QN) = 2\,PM;$$

$$\therefore F = \frac{2h^2}{SP^2} \cdot \text{limit } \frac{QR}{QT^2}$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3}, \text{ or } = \frac{\mu}{PM^3}, \text{ and } \therefore \propto \frac{1}{PM^3}.$$

Cor. To find the velocity at any point.

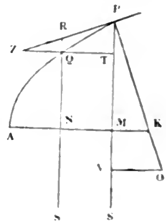
$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3} \cdot PM$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^2},$$

$$\therefore V = \frac{h \cdot CP}{SP \cdot PM}, \text{ or } = \frac{\sqrt{\mu}}{PM}.$$

Scholium to Prop. VIII.

If AQP be any conic section, it may be described by the action of a force tending to a point at an infinite distance, and varying inversely as the cube of the ordinate.



Let PO, the diameter of curvature at P, cut the axis of the conic section in K; draw OV perpendicular to PS, then PV is the chord of curvature at P in direction of the force; and complete the construction as in the proposition.

By similar triangles ZPT, PMK,

$$\frac{QT^2}{QR} : \frac{RP^2}{QR} = ZT^2 : ZP^2 = PM^2 : PK^2,$$

and this being true always will be true when Q moves up to P,

$$\therefore \text{ L. R. } \frac{QT^2}{QR} : \frac{RP^2}{QR} = PM^2 : PK^2,$$

$$\text{and } PV : PO = PM : PK,$$

$$\therefore \text{ since limit } \frac{RP^2}{QR} = \text{limit } \frac{PQ^2}{QR} = PV,$$

$$\text{limit } \frac{QT^2}{QR} : PO = PM^3 : PK^3.$$

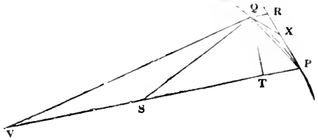
Now (Appendix, Arts. 4, 5.) in all conic sections, the diameter of curvature $= \frac{8}{L^2} \cdot PK^3$,

$$\therefore \text{ limit } \frac{QT^2}{QR} = \frac{8\,PM^3}{L^2},$$

$$\text{and } \therefore F = \frac{h^2 \cdot L^2}{4SP^2 \cdot PM^3} \propto \frac{1}{PM^3}.$$

Prop. IX. *To find the law of force tending to the pole, by which a body may describe an equiangular spiral.*

Def. An equiangular spiral is a spiral, which cuts every radius vector at the same given angle.



Let PQ be an arc of the spiral, S the center of force in the pole. QR a subtense parallel to SP, QT perpendicular to SP, and let the constant angle SPR, which the curve makes with the radius, $= a$. Let PV be the chord of curvature through S, and join PQ, QV;

$$\text{then } \frac{QT^2}{QR} = \frac{PR^2}{QR} \sin^2 a;$$

$$\therefore \text{ limit } \frac{QT^2}{QR} = \text{limit } \frac{PR^2}{QR} \sin^2 a = \text{limit } \frac{PQ^2}{QR} \sin^2 a = PV \sin^2 a.$$

Let the tangent at Q intersect PR in X. Then since SP, SQ make equal angles with the tangents at P, Q, the angles SPX, SQX are equal to two right angles, therefore the angle PSQ = angle QXR. Also since V is a point in the circumference of the circle of curvature, the angles XPQ, XQP are each ultimately equal to QVP. Hence the angle QXR, and therefore the angle QSP is ultimately double of the angle QVS, therefore $\angle\, SQV$ is ultimately equal to $\angle\, SVQ$, or $SV = SQ$ ultimately $= SP$. Hence $PV = 2SP$,

$$\text{and } \therefore\ F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{1}{2SP\sin^2 a}$$

$$= \frac{h^2}{\sin^2 a} \cdot \frac{1}{SP^3}, \text{ or } = \frac{\mu}{SP^3}, \text{ and } \therefore \propto \frac{1}{SP^3}.$$

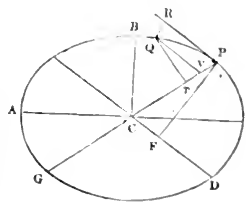
Cor. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2}{\sin^2 a} \cdot \frac{1}{SP^2}, \text{ or } \frac{\mu}{SP^2};$$

$$\therefore V = \frac{h}{\sin a} \cdot \frac{1}{SP}, \text{ or } \frac{\sqrt{\mu}}{SP}.$$

Prop. X. *A body describes an ellipse round a center of force in the center of the ellipse, to find the law of force.*

Let PQ be an arc of the ellipse, C the center, QR a subtense parallel to CP; AC, BC the semi-axes major and minor; QV parallel to PR; QT, PF perpendicular to CP and the semi-conjugate CD respectively, produce PC to meet the ellipse again in G; then the triangles QVT, PCF are evidently similar.



$$\text{Now } \frac{PV \cdot VG}{QV^2} = \frac{CP^2}{CD^2},$$

$$\text{and } \frac{QV^2}{QT^2} = \frac{CP^2}{PF^2};$$

$$\therefore \frac{PV \cdot VG}{QT^2} = \frac{CP^4}{PF^2 \cdot CD^2} = \frac{CP^4}{AC^2 \cdot BC^2};$$

$$\therefore \text{ limit } \frac{QR}{QT^2} = \text{limit } \frac{PV}{QT^2} = \frac{CP^4}{AC^2 \,.\, BC^2 \,.\, 2CP},$$

(since limit $VG = 2CP$)

$$= \frac{CP^3}{2AC^2 \,.\, BC^2};$$

$$\therefore F = \frac{2h^2}{CP^2} \text{ limit } \frac{QR}{QT^2} = \frac{h^2}{AC^2 \,.\, BC^2} \,.\, CP.$$

or $= \mu \,.\, CP$, and therefore $\propto CP$

Cor. 1. To find the velocity at any point.

$$V^2 = \tfrac{1}{2} F \,.\, PV = \tfrac{1}{2} \frac{h^2}{AC^2 \,.\, BC^2} CP \,.\, \frac{2CD^2}{CP}, \text{ since } PV = \frac{2CD^2}{CP}$$

$$= \frac{h^2}{AC^2 \,.\, BC^2} CD^2;$$

$$\therefore V = \frac{h}{AC \,.\, BC} \,.\, CD, \text{ or } \sqrt{\mu} \,.\, CD.$$

Cor. 2. To find the periodic time.

$$\text{Since } \mu = \frac{h^2}{AC^2 \,.\, BC^2}, \quad h = AC \,.\, BC \,.\, \sqrt{\mu};$$

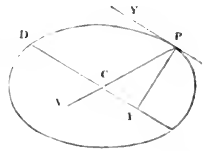
also the area of the ellipse $= \pi \, AC \,.\, BC$;

$$\therefore P = \frac{2 \text{ area of ellipse}}{h}$$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic times in all ellipses round the same center of force in the center are equal.

Cor. 3. *If a body be projected in a direction making any angle with its distance from a fixed point, and be attracted to that point by a force varying as the distance, it will describe an ellipse, whose center is the center of force.*



Let C be the center of force, P the point from which the body is projected in direction PY, V the velocity, and F the force at P.

Then space (s) due to the velocity at $P = \frac{V^2}{2F}$. In PC, produced if necessary, take $PV = 4s$, and draw CD parallel to PY and $= \sqrt{\frac{1}{2} CP . PV}$. With CP, CD as semi-conjugate diameters describe an ellipse, and suppose a body revolving in it to come to P; then it is moving in the direction of the tangent at P, that is, in a line parallel to CD or in direction PY. Also space due to velocity at $P = \frac{1}{4}$ chord of curvature at P,

$$= \frac{1}{4} . \frac{2\, CD^2}{CP} = \frac{1}{4} PV = s.$$

The force, distance, and law of force are the same also in both cases; hence the two bodies are under the same circumstances at P, and will therefore describe the same orbit; that is, the projected body will describe an ellipse, whose center is C.

If CPY be a right angle, and $s = \frac{1}{2} PC$, the orbit described will be a circle.

Cor. 4. To compare the velocity at P with the velocity in a circle, radius $= CP$, described round the same center of force.

$$V \text{. in ellipse} = \sqrt{\mu} . CD.$$

$$V \text{. in circle (radius} = CP) = \sqrt{F . CP}, \text{ (Prop. VI. Cor. 5.)}$$

$$= \sqrt{\mu} . CP;$$

$$\therefore V \text{. in ellipse} : V \text{. in circle (rad.} = CP) = CD : CP.$$

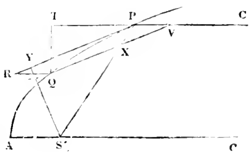
Scholium to Prop. X.

1. It was proved in the proposition, that, when a body moves in an ellipse round a center of force in the center, the force varies as the distance. The same is also true, when a body moves in an hyperbola, the construction and proof being exactly the same as for the ellipse.

2. If the orbit be a parabola, the center of force is removed to an infinite distance, and the force acts in lines parallel to the axis; in this case, since the difference of any two distances vanishes compared with the distances themselves, the force is invariable.

Or the following proof may be applied in the case of the parabola.

Let PQ be an arc of the parabola, A the vertex, S the focus; PC parallel to the axis, and therefore in the direction of the force; QR a subtense parallel to PC, and QV parallel to the tangent PR; QT, SY perpendicular to CP, PR.



Since $4SP \cdot PV = QV^2$, $\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP}$,

and by similar triangles, QTV, SPY,

$$\frac{QV^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA};$$

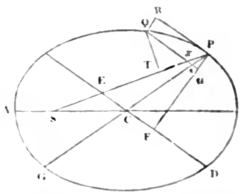
$$\therefore \text{ limit } \frac{QR}{QT^2} = \frac{1}{4SA};$$

$$\therefore F = \frac{2h^2}{CP^2} \cdot \frac{1}{4SA}.$$

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS, ABOUT A CENTER OF FORCE IN ONE OF THE FOCI.

PROP. XI. *A body revolves in an ellipse, to find the law of force tending to one of the foci.*



Let the focus S be the center of force, PQ an arc, QR a subtense parallel to SP; C the center of the ellipse, join PC and produce it to meet the ellipse in G; draw Qxv parallel to the tangent PR, cutting SP, CP in x, v; and QT, PF respectively perpendicular to SP, and the semi-conjugate diameter CD: and let E be the point, in which SP cuts CD, then $PE = AC$ the $\frac{1}{2}$ axis major.

By similar triangles, QxT, PEF,

$$\frac{Qx^2}{QT^2} = \frac{PE^2}{PF^2} = \frac{AC^2}{PF^2},$$

and by a property of the ellipse,

$$\frac{Pv}{Qv^2} = \frac{CP^2}{vG \cdot CD^2};$$

also by similar triangles, Pxv, PEC,

$$\frac{Px}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}.$$

Now $Px = QR$, Qx ultimately $= Qv$, and vG ultimately $= 2CP$; hence multiplying the above quantities together, and taking the limits of the products,

$$\text{limit } \frac{QR}{QT^2} = \frac{AC^3 . CP}{2CP . CD^2 . PF^2} = \frac{AC^3}{2AC^2 . BC^2},$$

$$= \frac{AC}{2BC^2} = \frac{1}{L};$$

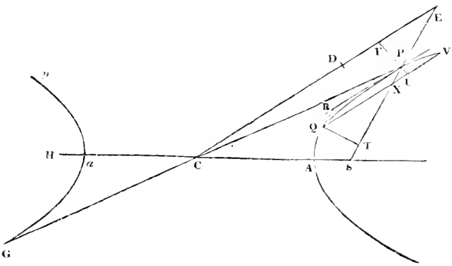
$$\therefore F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{L} \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2}$$

$$\propto \frac{1}{SP^2}.$$

Prop. XII. *A body moves in an hyperbola, to find the law of force tending to one of the foci.*

Let the center of force be in the focus S, and let the body move in the branch PA, which is nearer to S than

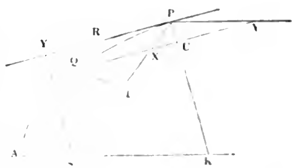


the other branch of the hyperbola. Then the same construction being made as in the ellipse, it may be shewn in precisely the same manner that the force

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2} \text{ and } \therefore \propto \frac{1}{SP^2}.$$

Cor. In the same manner it may be shewn, that if the body describes the opposite branch pa by a *repulsive* force proceeding from S, the force will vary as $\frac{1}{Sp^2}$.

Prop. XIII. *A body moves in a parabola, to find the law of force tending to the focus.*



Let A and S be the vertex and focus of the parabola, PQ an arc: QR a subtense parallel to SP, QXV parallel to the tangent PR, cutting SP in X, and the diameter through P in V; QT, SY perpendicular to SP, PR; L = latus rectum. Then by a property of the parabola $PV = PX$ and $\therefore = QR$; also $4SP \cdot PV = QV^2$.

Hence

$$\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP},$$

and by similar triangles,

$$\frac{QX^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA}.$$

Now QX ultimately $= QV$; hence multiplying these quantities together, and taking the limits,

$$\text{limit } \frac{QR}{QT^2} = \frac{1}{4SA} = \frac{1}{L},$$

$$\therefore\ F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2},$$

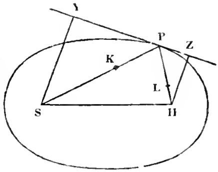
$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2},$$

$$\propto \frac{1}{SP^2}.$$

Cor. *If a body be projected at a given distance from a center of force, which* $\propto$ *(dist.)*$^{-2}$, *and in a direction making a finite angle with the distance, it will describe a conic section.*

Let S be the center of force, P the point and PY the direction of projection, F = the force at P, then if s be the space due to the velocity of projection, $s = \dfrac{(\text{velocity})^2}{2F}$, and is therefore known.

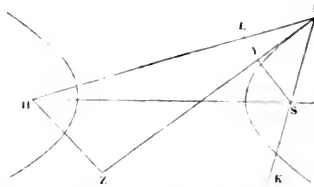
1. Let s be less than SP. In PS take $PK = s$, and draw PH, making with YP produced the $\angle HPZ = \angle SPY$; in PH take $PL = SK$, and let a circle described through the points S, K, L cut PL in H, then $PH \,.\, PL = PS \,.\, PK$. With foci S and H and axis major $= SP + HP$, describe an ellipse, and suppose a body revolving in this ellipse and acted on by the same force in S, to come to P; then space due to velocity at $P = \frac{1}{4}$ chord of curvature at P through S,



$$= \frac{(\frac{1}{2}\text{ conjugate diameter})^2}{\text{axis major}} = \frac{SP \,.\, HP}{SP + HP},$$

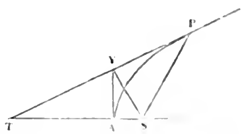
$$= \frac{SP}{1 + \dfrac{SP}{HP}} = \frac{SP}{1 + \dfrac{PL}{PK}} = PK.$$

Hence the velocity is the same in both cases; also the revolving body is moving in direction PY, since ZPy, making equal angles with SP, HP, is a tangent at P; and the force and the law of force are the same for both bodies; they will therefore describe the same curve, that is, the projected body will describe an ellipse.



2. Let s be greater than SP. In PS produced take $PK = s$; draw PH on the other side of PY, making the $\angle YPH = \angle YPS$, take $PL = SK$, and let a circle described through the points S, K, L cut PL produced in H: then if with foci S and H and axis major $= HP \sim SP$, an hyperbola be described, it may be shewn, as in the preceding case, that the body will move in the hyperbola thus constructed.

3. Let $s = SP$. Here $SK = 0$, and $\therefore PH = \dfrac{PK \cdot PS}{SK} = \infty$.



Let the circle described with center S and radius SP cut PY in T, join TS; draw SY, YA perpendicular to PT, TS respectively, and with focus S and vertex A describe a parabola; it will pass through the point P; for a parabola, whose focus is S, and which passing through P has PY for a tangent, will have its axis coincident with ST, and its latus rectum will $= 4\dfrac{SY^2}{SP}$, which $= 4SA$; hence, conversely, the parabola above described will pass through P, and it may be shewn as in the former cases, that the body will move in this parabola.

Prop. XIV. *If any number of bodies revolve about one common center of force, which varies as* $(dist.)^{-2}$, *and is the same at equal distances in all the orbits described, the latera recta of the orbits will be as the squares of the areas described in equal times.*

Let $\dfrac{\mu}{SP^2}$ be the force in any orbit at the distance SP, then since the forces at equal distances are equal, μ is the same for all the orbits:

Also by Props. XI, XII, XIII, $\mu = \frac{2h^2}{L}$,

$$\therefore L \propto h^2 \propto \left(\frac{\text{area described in a given time}}{\text{time}}\right)^2$$

$\propto$ (area)^2 described in a given time.

Prop. XV. *A body revolves in an ellipse round a center of force in the focus, to find the periodic time.*

Let AC, BC be the semi-axes major and minor, P the periodic time.

$$\text{Then } \frac{P''}{1''} = \frac{\text{area of the ellipse}}{\text{area described in } 1''},$$

$$\therefore P = \frac{\pi AC \,.\, BC}{\frac{1}{2}h},$$

and since $\frac{2h^2}{L} = \mu$, $h = \sqrt{\frac{\mu L}{2}} = \sqrt{\frac{\mu \,.\, BC^2}{AC}} = BC\sqrt{\frac{\mu}{AC}}$,

$$\therefore P = \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

Cor. Hence, the squares of the periodic times in all ellipses, described round the same center of force in the focus, are as the cubes of the major axes.

Prop. XVI. *To find the velocity at any point of a conic section, described about a center of force in the focus.*

Let V be the velocity at the point P,

$$V^2 = \tfrac{1}{2} F \,.\, PV = \frac{\mu}{SP^2} \,.\, \frac{PV}{2}.$$

Now in the ellipse and hyperbola,

$$\frac{PV}{2} = \frac{CD^2}{AC} = \frac{SP \,.\, HP}{AC} = SP.\left(2 \mp \frac{SP}{AC}\right),$$

and in the parabola,

$$\frac{PV}{2} = 2SP.$$

Hence in the ellipse $V = \sqrt{\frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right)}$, $v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$

in hyperbola $V = \sqrt{\frac{\mu}{SP}\left(2 + \frac{SP}{AC}\right)}$,

in parabola $V = \sqrt{\frac{2\mu}{SP}}$.

Cor. To compare the velocity at P with that of a body moving in a circle, radius $= SP$, and described round the same center of force.

Let U = velocity in the circle,

then (Prop. vi. Cor. 5),

$$U = \sqrt{F \cdot R} = \sqrt{\frac{\mu}{SP^2} \cdot SP} = \sqrt{\frac{\mu}{SP}},$$

$\therefore$ in ellipse $\frac{V}{U} = \sqrt{2 - \frac{SP}{AC}}$ which is less than $\sqrt{2}$,

in hyperbola $\frac{V}{U} = \sqrt{2 + \frac{SP}{AC}}$.........greater.........

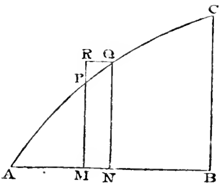
in parabola $\frac{V}{U} = \sqrt{2}$.

APPENDIX.

Note to Lemma II.

1. To find the area of a plane curve.

Let the area ABC be bounded by the curve AC and the straight lines AB, BC. Let AB be divided into n equal parts, and let MN be the r^{th} part from A; draw MP, NQ parallel to BC, and complete the parallelogram $MNQR$.



Let $AB = h$, then $MN = \frac{h}{n}$,

$NQ = y_r$,

$\angle ABC = i$,

area of parallelogram $RN = \frac{h}{n} y_r \sin i$.

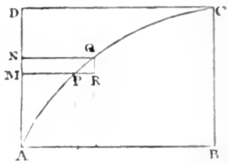
Therefore giving to r the values 1, 2, 3...n, the sum of the parallelograms described on all the parts

$$= \frac{h}{n} \sin i \, (y_1 + y_2 + y_3 + \ldots + y_n) = h \sin i \, . \, \Sigma \frac{y_r}{n} .$$

Therefore area of curvilinear figure $= h \sin i$. limit $\Sigma \frac{y_r}{n}$,

when n is infinite.

Ex. 1. To find the area of a portion of a parabola cut off by a diameter and one of its ordinates.



Let ABC be the parabolic area cut off by the diameter AB and a semi-ordinate BC. Complete the parallelogram $ABCD$; then AD is a tangent at A.

Let $AD = h$, $AB = k$, and let AN be the abscissa, and NQ, parallel to AB, the ordinate to the point Q; then by a property of the parabola,

$$\frac{QN}{AN^2} = \frac{AB}{AD^2}; \quad \therefore QN \text{ or } y_r = \frac{k}{h^2} \cdot \left(\frac{rh}{n}\right)^2 = k\,\frac{r^2}{n^2},$$

$$\therefore \text{ area } ADC = h \sin i \text{ . limit . } \Sigma \frac{y_r}{n} = kh \sin i \text{ limit } \Sigma \frac{r^2}{n^3},$$

$$= hk \sin i \text{ . limit . } \frac{1}{n^3} \cdot (1^2 + 2^2 + 3^2 + \dots + n^2),$$

$$= hk \sin i \text{ . limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3} hk \sin i$$

$$= \tfrac{1}{3} \text{ parallelogram } ABCD,$$

ag. P. 260.

and $\therefore$ parabolic area $ABC = \frac{2}{3}$ circumscribing parallelogram.

2. The volume of a solid of revolution may be deduced in a similar manner.

Let ABC (Fig. Art. 1.) be a plane curvilinear area by the revolution of which round AB the solid is generated, and let CB be perpendicular to AB. Then if AB $(= h)$ be divided into n equal parts, and the rectangular parallelogram RN be described on MN the r^{th} part, the cylinder generated by the revolution of RN round MN

$$= \frac{h}{n}\,\pi \,.\, QN^2 = \frac{h}{n}\,\pi \,.\, y_r^2,$$

and the volume of the solid

$$= \text{limit . sum of all such cylinders}$$

$$= \pi h \,.\, \text{limit } \Sigma \,.\, \frac{y_r^2}{n}, \text{ when } n \text{ is infinite.}$$

Ex. 2. To find the volume of a sphere.

Let ABC be a quadrant of the generating circle radius $= h$;

$$\text{then } y^2 = 2hx - x^2,$$

$$y_r^2 = 2h \frac{rh}{n} - \left(\frac{rh}{n}\right)^2 = h^2 \left(\frac{2r}{n} - \frac{r^2}{n^2}\right),$$

and therefore volume of hemisphere

$$= \pi h^3 \text{ limit} \left\{\frac{2}{n^2}(1 + 2 + \dots n) - \frac{1}{n^3}(1^2 + 2^2 + \dots + n^2)\right\}$$

$$= \pi h^3 \text{ limit} \left\{\frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right\}.$$

$$= \pi h^3 \left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi h^3,$$

and therefore volume of sphere

$$= \frac{4}{3}\pi h^3 = \frac{2}{3} \,.\, 2h \,.\, \pi h^2 = \frac{2}{3} \text{ circumscribing cylinder.}$$

Ex. 3. Similarly the volume of a cone and of a paraboloid may be shewn to be $\frac{1}{3}$ and $\frac{1}{2}$ of the circumscribing cylinder respectively.

3. To find the volume of a pyramid.

Let A be the area of the base of the pyramid, and let the perpendicular from the vertex upon the base $= h$. Divide h into n equal parts, and through the r^{th} point of division from the vertex draw a plane parallel to the base. Then the area of the section of the pyramid thus made

$$= A\frac{\left(\frac{rh}{n}\right)^2}{h^2} = A\frac{r^2}{n^2};$$

on this area as a base describe a right prism, whose altitude $= \frac{h}{n}$; then volume of prism

$$= A\frac{r^2}{n^2}\cdot\frac{h}{n} = Ah\frac{r^2}{n^3};$$

and therefore volume of pyramid = limit of sum of all such prisms

$$= Ah \text{ limit } \Sigma\frac{r^2}{n^3} = Ah \text{ limit } \frac{1}{n^3}(1^2 + 2^2 + \ldots\ldots + n^2)$$

$$= Ah \text{ limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3}Ah = \tfrac{1}{3}\text{ . base} \times \text{altitude.}$$

Note to Prop. III. on Curvature.

4. To find the chords of curvature through the center and focus, and the diameter of curvature, at any point of an ellipse and hyperbola. (Vide Figs. Prop. XI. and XII.)

Let Qv, a semi-ordinate to the diameter PCG, cut SP in x, CP in v, and PF, which is perpendicular to the semi-conjugate CD, in u.

Chord of curvature through C

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } CP}$$

$$= \text{limit} \frac{PQ^2}{Pv} = \text{limit} \frac{Qv^2}{Pv}$$

$$= \text{limit} \frac{CD^2}{CP^2} \cdot vG, \quad \text{since } \frac{Qv^2}{Pv \cdot vG} = \frac{CD^2}{CP^2}$$

$$= \frac{2\,CD^2}{CP}, \quad \text{since } vG \text{ ultimately} = 2\,CP.$$

Chord of curvature through S

$$= \text{limit} \frac{PQ^2}{\text{subtense parallel to } SP} = \text{limit} \frac{Qv^2}{Px}$$

$$= \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Px} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PE}$$

$$= \frac{2\,CD^2}{AC}, \quad \text{since } PE = AC.$$

Diameter of curvature

$$= \text{limit} \frac{PQ^2}{\text{subtense perpendicular to tangent}}$$

$$= \text{limit} \frac{Qv^2}{Pu} = \text{limit} \frac{Qv^2}{Pv} \cdot \frac{Pv}{Pu} = \text{limit} \frac{CD^2}{CP^2} \cdot vG \cdot \frac{PC}{PF}$$

$$= \frac{2\,CD^2}{PF}.$$

Cor. Let PF cut the axis major in K, then $PF \cdot PK = BC^2$;

also $CD \cdot PF = AC \cdot BC$,

$$\therefore \text{ diameter of curvature } = \frac{2AC^2 \cdot BC'^2}{PF^3} = \frac{2AC^2 \cdot BC^2 \cdot PK^3}{BC^6}$$

$$= \frac{8PK^3}{L^2}.$$

5. To find the chord of curvature through the focus, and the diameter of curvature at any point of a parabola. (Vide Fig. Prop. XIII.)

Let QV, a semi-ordinate to the diameter PV, cut SP in X, and the normal PK in U, draw SY perpendicular to the tangent at P; then $PX = PV$, hence

Chord of curvature through S

$$= \text{limit } \frac{PQ^2}{\text{subtense parallel to } SP}$$

$$= \text{limit } \frac{PQ^2}{PX} = \text{limit } \frac{QV^2}{PV}$$

$$= 4SP, \text{ since } QV^2 = 4SP \cdot PV.$$

Diameter of curvature

$$= \text{limit } \frac{PQ^2}{\text{subtense perpendicular to tangent}} = \text{limit } \frac{QV^2}{PU}$$

$$= \frac{4SP \cdot PV}{PU} = 4SP \cdot \frac{SP}{SY}, \text{ by sim}^{r}\text{. triangles } PVU,\ SPY,$$

$$= \frac{4SP^2}{SY}, \text{ or } = 4\sqrt{\frac{SP^3}{SA}}.$$

Cor. Let PY meet the axis in T, then

$$ST = SP = SK, \therefore SY = \tfrac{1}{2} PK;$$

hence diameter of curvature

$$= 4\frac{SP^2}{SY} = 4\frac{SY^4}{SA^2 . SY} = \frac{4}{SA^2} SY^3 = \frac{1}{2SA^2} . PK^3$$

$$= \frac{8PK^3}{L}.$$

Note to Prop. VI. Cor. 3.

6. If $SP = r$, and $SY = p$, $PV = \frac{2p}{dp}$.

$$\therefore F = \frac{2h^2}{SY^2 . PV} = \frac{2h^2}{p^2 . \frac{2p}{d_r p}}$$

$$= \frac{h^2}{p^3} . d_r p.$$

Again, if $r = \frac{1}{u}$, and $ASP = \theta$, AS being a fixed straight line drawn through S,

$$\frac{1}{p^2} = (d_\theta u)^2 + u^2;$$

$$\therefore \frac{1}{p^3} d_r p = -\frac{1}{p^3} . d_u p . u^2 = -\frac{1}{p^3} \frac{d_\theta p}{d_\theta u} . u^2$$

$$= (d_\theta u . d_\theta^2 u + u\, d_\theta u) \frac{u^2}{d_\theta u}$$

$$= u^2 (d_\theta^2 u + u);$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u)$$

Ex. 1. To find the law of force, by which a body may describe the curve, whose equation is $p = \dfrac{ar}{\sqrt{b^2 + r^2}}$, round a center of force in the pole.

$$\frac{1}{p^2} = \frac{b^2}{a^2 r^2} + \frac{1}{a^2},$$

$$\text{and } \therefore \frac{1}{p^3} d_r p = \frac{b^2}{a^2 r^3};$$

$$\therefore F = \frac{h^2}{p^3} d_r p = \frac{h^2 b^2}{a^2 r^3} \propto \frac{1}{r^3}.$$

Ex. 2. To find the law of force by which a body may describe a conic section, round a center of force in the focus.

Let P be any point in the curve, S the focus, A the extremity of the axis major;

$$SP = r = \frac{1}{u}, \quad ASP = \theta,$$

$$\text{then } r = \frac{m}{1 + e\cos\theta};$$

$$\therefore u = \frac{1}{m}(1 + e\cos\theta),$$

$$d_\theta u = -\frac{1}{m} e \sin\theta,$$

$$d_\theta^2 u = -\frac{1}{m} e \cos\theta;$$

$$\therefore F = h^2 u^2 (d_\theta^2 u + u)$$

$$= h^2 u^2 \frac{1}{m}$$

$$= \frac{h^2}{m r^2} \propto \frac{1}{r^2}$$

Note to Prop. X. Cor. 3.

7. To find the magnitude and position of the axes of the orbit described.

Let $CP = r$, $CPy = \alpha$, $s \left(= \frac{V^2}{2F}\right)$ = space due to velocity of projection, a and b the semi-axes of the orbit described.

$$CD = \sqrt{\tfrac{1}{2} CP \cdot PV} = \sqrt{2rs},$$

$$\left.\begin{aligned} a^2 + b^2 (= CP^2 + CD^2) &= r^2 + 2rs \\ ab \quad (= \quad CD \cdot PF) &= \sqrt{2rs} \cdot r \sin\alpha \end{aligned}\right\},$$

from which two equations a and b, and therefore e, the eccentricity, may be determined.

Also, if θ be the inclination of axis major to CP,

$$r^2 = \frac{b^2}{1 - e^2 \cos^2\theta},$$

$$\therefore \cos^2\theta = \frac{1}{e^2}\left(1 - \frac{b^2}{r^2}\right),$$

from which θ may be found.

Note to Prop. XIII. Cor.

8. To find the magnitude and position of the axis of the orbit described.

Let $SP = r$, $\angle SPY = \alpha$, draw SY, HZ perpendicular to YPZ; and let a, b, L be the semiaxes and latus rectum of the orbit;

$$\text{then } PL \cdot PH = PK \cdot PS,$$

$$\text{or } (r \sim s) \cdot (2a \mp r) = s \cdot r,$$

$$\therefore 2a(r \sim s) - r^2 = 0,$$

$$\therefore \ a = \frac{r^2}{2(r \sim s)},$$

hence the magnitude of the axis major is independent of the direction of projection.

$$\text{Again, } b = \sqrt{SY \,.\, HZ} = \sqrt{SP \sin a \,.\, HP \sin a}$$

$$= \frac{r \,.\, s^{\frac{1}{2}} \,.\, \sin a}{(r \sim s)^{\frac{1}{2}}};$$

$$\therefore \ L = \frac{2b^2}{a}$$

$$= 4s \,.\, \sin^2 a.$$

Again, $$YR = SH \,.\, \sin ASY,$$

$$\text{or } (SP + HP) \cos a = 2e \,.\, AC \sin ASY,$$

$$\therefore \ \sin ASY = \frac{1}{e} \cos a,$$

which equation, since $e = \sqrt{1 \mp \frac{b^2}{a^2}}$ is known, determines the position of the axis major.

ON ANGULAR VELOCITY.

2. When a body P moves in an orbit, its angular velocity round any point S, (fig. page 28.) is measured by the angle uniformly described by SP round S in 1″, in the same manner as linear velocity is measured by the line uniformly described in 1″. If the angular motion of SP be not uniform, the angular velocity at any point is measured by the angle, which would be described in 1″, if the angular motion of SP were to continue uniform for that time. Hence if the angular motion be not uniform, and PSQ be the angle described in T'' after leaving P, the angular velocity

$$= \text{limit } \frac{\text{angle } PSQ}{T},$$

for this is the angle which would be described in 1″, if the angular motion at P were to continue uniform for that time.

Prop. *If a body be moving in any orbit round a center of force S, the angular velocity at any point P*

$$=\frac{h}{SP^2}.$$

Let PSQ be the angle described in T''; with center S and radius SQ, describe a circular arc cutting SP in T, and draw SY perpendicular to the tangent at P; then the triangle PTQ may be considered as ultimately rectilinear, and similar to SYP, hence

$$\angle^r \text{ vel. at } P = \text{limit } \frac{\angle PSQ}{T} = \text{limit } \frac{QT}{SQ\,.\,T}$$

$$= \text{limit } \frac{PQ\,.\,SY}{SP^2\,.\,T}, \text{ since limit } \frac{QT}{PQ} = \frac{SY}{SP}$$

$$= \frac{SY\,.\,\text{vel. at } P}{SP^2}, \text{ since limit } \frac{PQ}{T} = \text{vel. at } P$$

$$= \frac{h}{SP^2}, \text{ (Prop. I. Cor. 3).}$$

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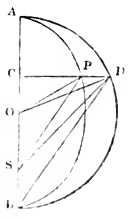
10. *Force varying as* (*distance*)$^{-2}$. *To find the time of motion and the velocity acquired by a body falling through a given space from rest.* (Props. XXXIII. and XXXVI.)



Let S be the center of force, A the point from which the body begins to fall;

$$\frac{\mu}{SP^2} = \text{force at distance } SP.$$

Let APB be a semi-ellipse, focus S and axis major ASB; ADB a semi-circle, whose diameter is ASB; and suppose a body revolving in the ellipse round the focus S to come to P; bisect AB in O, draw DPC perpendicular to AB, and join OP, OD.

Then the time through $AP \propto$ area $ASP \propto$ area ASD, and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point P with C, or the body is moving in the straight line AC; the point B also coincides with S, since $AS \cdot SB = (\frac{1}{2}\text{ axis minor})^2$; and since space due to velocity at $A = \frac{1}{4}$ chord of curvature at A through $S = \frac{1}{4}$ latus rectum $= \dfrac{(\text{axis minor})^2}{4AB} = 0$, the body begins to move from rest at A.

Hence time from rest through $AC \propto$ area ABD,

$$\therefore \frac{\text{time through } AC}{\text{time through } AB (= \frac{1}{2} \text{periodic time in ellipse})} = \frac{\text{area } ABD}{\text{semi-circle } ABD};$$

$$\therefore \text{ time through } AC = \frac{\pi \cdot AO^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \frac{\frac{1}{2} AO \cdot (AD + CD)}{\frac{1}{2}\pi \cdot AO^2}$$

$$= \sqrt{\frac{AS}{2\mu}} \cdot (AD + CD).$$

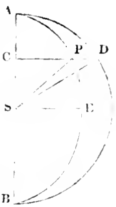
Again, velocity at $P = \sqrt{\dfrac{\mu}{AO} \cdot \dfrac{HP}{SP}}$ (Prop. XVI.), and when the ellipse coincides with the axis major,

$$\text{velocity at } C = \sqrt{\frac{2\mu}{AS} \cdot \frac{AB - BC}{BC}} = \sqrt{\frac{2\mu}{AS} \cdot \frac{AC}{SC}}.$$

Cor. Time through $AS = \sqrt{\dfrac{AS}{2\mu}}\, \pi \dfrac{AS}{2} = \dfrac{\pi \left(\dfrac{AS}{2}\right)^{\frac{3}{2}}}{\sqrt{\mu}}$

$= \frac{1}{2}$ per. time in an ellipse, of which AS is the axis major.

11. *Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest.* (Prop. xxxviii.)



Let S be the center of force, A the place from which the body begins to fall: on $AB = 2AS$ describe a semi-ellipse APB, and a semi-circle ADB, and let a body moving in the ellipse come to P. Draw DPC perpendicular to AB, and join SP, SD.

Then time through $AP \propto$ area $ASP \propto$ area ASD, and this being true, whatever be the axis minor of the ellipse, will be true when it is diminished without limit, in which case the body will be at C, having fallen from rest at A,

$\therefore$ time through $AC \propto$ area ASD;

$$\therefore \frac{\text{time through } AC}{\text{time through } AS\,(=\frac{1}{4}\text{periodic time in a circle})}$$

$$= \frac{\text{sector } ASD}{\frac{1}{4}\text{area of a circle}};$$

$$\therefore \text{time through } AC = \frac{\pi}{2\sqrt{\mu}} \cdot \frac{\frac{1}{2}AS \,.\, AD}{\frac{1}{4}\pi AS^2}$$

$$= \frac{AD}{AS\sqrt{\mu}}.$$

Again, let SE be the semi-axis minor,

then vel. at P = semi-conjugate at $P \,.\, \sqrt{\mu}$ (Prop. x. Cor. 1.)

$$= \sqrt{AS^2 + SE^2 - SP^2} \,.\, \sqrt{\mu},$$

$$\therefore \text{vel. at } C = \sqrt{AS^2 - SC^2} \,.\, \sqrt{\mu}$$

$$= CD\sqrt{\mu}.$$

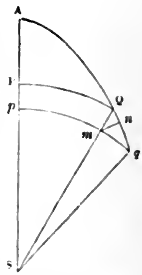
COR. Time to center of force $= \dfrac{\frac{1}{2}\pi AS}{AS\sqrt{\mu}} = \frac{1}{4}\,\dfrac{2\pi}{\sqrt{\mu}}$

$= \frac{1}{4}$ per. time in an ellipse, force in center.

Hence the times through all distances to the center of force are equal.

Vel. acquired in falling through $AS = AS\sqrt{\mu}$.

12. *If the velocities of two bodies, one of which is falling directly towards a center of force, and the other describing a curve about that center, be equal at any equal distances, they will always be equal at equal distances.* (Prop. XL.)



Let S be the center of force, and let one of the bodies be moving in the straight line APS and the other in the curve AQq; with radii SQ, Sq describe the circular arcs QP, qp: let SQ cut pq in m, and draw mn perpendicular to Qq; and suppose the velocities of the bodies at P and Q to be equal.

Since the centripetal forces at P and Q are equal, Pp, Qm may be taken to represent them: Pp is wholly effective in accelerating P, but the effective part of Qm is Qn, nm being wholly employed in retaining the body in the curve. Also since the velocities at P and Q are equal, the times of describing Pp and Qq, when the spaces are diminished indefinitely, are proportional to Pp and Qq; hence

force at P : force at $Q = Pp : Qn$

and time through Pp : time through $Qq = Pp : Qq$;

$\therefore$ velocity added in describing Pp : velocity added in describing Qq

$= Pp^2 : Qn\,.\,Qq = Qm^2 : Qn\,.\,Qq$

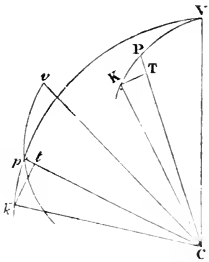
$= 1 : 1$,

and the same may be shewn at all corresponding points equally distant from S, therefore, *If the velocities, &c.*

SECTION IX.

ON THE POSITION OF THE APSIDES IN ORBITS VERY NEARLY CIRCULAR.

PROP. XLIII. *The orbit in which a body moves revolves round the center of force with an angular velocity, which always bears a fixed ratio to that of the body; to shew that the body may be made to move in the revolving orbit in the same manner as in the orbit at rest by the action of a force tending to the same center.*



Let C be the center of force, and when the body in the fixed orbit VCP has described the arc VP, let vCp be the position of the revolving orbit, and p that of the body moving in it; then $\angle\, vCp = \angle\, VCP$. Also let the angular velocity of the orbit be to that of P as $G - F : F$.

The angles VCv, VCP begin together at V, and their contemporary increments are as the angular velocities of Cv and CP, that is, as $G - F : F$, therefore the angles themselves are in that ratio, or

$$VCv : VCP \text{ or } vCp = G - F : F;$$

$$\therefore \text{ componendo } VCp : VCP = G : F;$$

hence, if the angle VCp be always taken $= \dfrac{G}{F} \times$ angle VCP, and $Cp = CP$, Vp the locus of p will be the curve traced out in fixed space by a body p moving in the revolving orbit in the same manner as P in the fixed orbit.

Also the body may describe the orbit Vp by the action of a force placed in C.

For let PCK, pCk be the areas described by CP, Cp in the same small increment of time; draw KT, kt perpendicular to CP, Cp; then the contemporary increments of the areas, described by p and P, are ultimately as

$$Cp \, . \, kt : CP \, . \, KT = Cp^2 \, . \sin pCk : CP^2 \, . \sin PCK$$
$$= \angle pCk : \angle PCK = \angle^r \text{ vel. of } Cp : \angle^r \text{ vel. of } CP = G : F,$$

and the whole areas begin together at V, therefore they are themselves in the same ratio; hence area $VCp \propto$ area $VCP \propto$ the time (Prop. 1); and therefore (Prop. 2) a body may be made to move in the orbit Vp by a proper centripetal force placed in C.

Def. An apse or apside is a point in an orbit at which the direction of the body's motion is perpendicular to the distance; and the angle between two consecutive apsidal distances is called the apsidal angle.

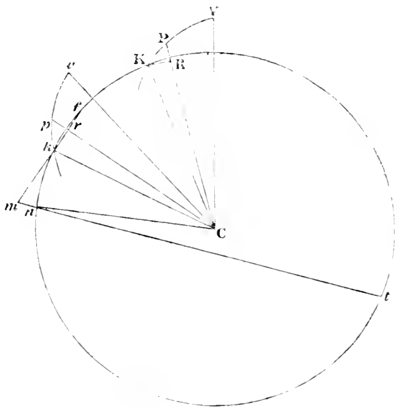
Cor. *If a be the apsidal angle in the orbit* VP, *the corresponding apsidal angle in the orbit* $\mathrm{Vp} = \frac{\mathrm{G}}{\mathrm{F}} a$.

For the motion of p is compounded of two motions, one arising from the angular motion of the orbit, and therefore perpendicular to the distance, and the other the same as the motion of P in the fixed orbit; hence when the latter body is at an apse, the whole motion of p will be perpendicular to the distance, or p will be at an apse; also the angles described in the same time in the orbits Vp and VP are always as $G : F$,

$$\therefore \text{ apsidal angle in orbit } Vp = \frac{G}{F} a.$$

Prop. XLIV. *To find the difference of the forces, by which the bodies are retained in the fixed and revolving orbits.*

Let P and p be contemporary positions of the two bodies, PK a small arc of the fixed orbit described in t''; take $pk = Pk$, and with radius CK or Ck describe the circle Kk;



draw KR, kr perpendicular to CP, Cp and in rk, produced if necessary, take $rm = \frac{G}{F}.kr$. Let the velocities of P and p be each resolved into two, one central or in the direction of the distance, and the other transverse or perpendicular to it; then since PK is very small, PR and RK may be taken to represent P's central and transverse velocities respectively; and since the angular motion of the orbit affects only the transverse motion of p, $pr = PR$ will represent p's central motion: also transverse vel. = angular vel. × dist.;

$$\therefore \text{transv. vel. of } p : \text{transv. vel. of } P = \angle^r \text{vel. of } p : \angle^r \text{vel. of } P$$
$$= G : F;$$

$$\therefore \text{ transverse vel. of } p = \frac{G}{F} . KR = rm.$$

Hence, in consequence of the two motions pr, rm, p will be at m, when P is at K. But if we take

$$\angle VCn = \frac{G}{F} \angle VCK, \quad \text{and} \quad Cn = CK,$$

p must be at n, when P is at K, in order that it may move in the manner required; join mn; then an additional force must have acted on p, sufficient to draw it through mn in t'', and therefore the difference of the forces on P and p

$$= 2 \text{ lim. } \frac{mn}{t^2} \quad \text{(Lem. x. Cor. 2).}$$

Let mn, mr produced cut the circle again in t and f,

then $mn = \dfrac{mk . mf}{mt}$. Now $mr = \dfrac{G}{F} . kr$, $\therefore mk = \dfrac{G - F}{F} kr$,

and $mf = \dfrac{G + F}{F} . kr$; $\therefore mk . mf = \dfrac{G^2 - F^2}{F^2} . kr^2$.

Let $h = 2$ area described by P in $1''$,

$$\therefore h = 2 \text{ lim. } \frac{\text{area } PCK}{t} = \text{lim. } \frac{CP . KR}{t}; \quad \therefore \text{lim. } \frac{KR}{t} = \frac{h}{CP};$$

also mt ultimately passes through C and equals $2CP$;

$$\therefore 2 . \text{lim. } \frac{mn}{t^2} = 2 . \text{lim. } \frac{G^2 - F^2}{F^2} . \frac{kr^2}{t^2 . 2CP};$$

$$\therefore \text{force on } p - \text{force on } P = \frac{G^2 - F^2}{F^2} . \frac{h^2}{CP^3}, \text{ and } \therefore \propto \frac{1}{CP^3}.$$

Prop. XLV. *The law of force in an orbit nearly circular being given, to find an approximate value of the apsidal angle.*

Let $\frac{1}{r^3} . fr$ be the force at any distance r, a the greatest value of r, and $a - x$ any other value; then

$$\frac{1}{r^3} fr = \frac{1}{r^3} f(a - x),$$

which being expanded in a series ascending by powers of x

$$= \frac{1}{r^3} (fa - f'a . x + \&c.) = \frac{1}{r^3} (fa - f'a . x) \text{ very nearly,}$$

since x is very small.

Let VP (Fig. Prop. XLIII.) be an ellipse of small eccentricity, C the focus, CV the greatest distance $= a$, L the latus rectum, and let $\frac{F^2}{a^2} =$ force at V; then (Prop. XI.) if $h = 2$ area described in $1''$ by a body revolving in the ellipse round a center of force in the focus,

$$F^2 = \frac{2h^2}{L} = \frac{h^2}{a}, \text{ since } L = 2a \text{ nearly; hence,}$$

$$\text{force on } p = \frac{F^2}{Cp^2} + \frac{G^2 - F^2}{F^2} . \frac{h^2}{Cp^3} \text{ (Prop. XLIV.)}$$

$$= \frac{1}{Cp^3} \{F^2 \, Cp + (G^2 - F^2) \, a\}$$

$$= \frac{1}{r^3} \{F^2 (a - x) + (G^2 - F^2) \, a\} \text{ since } Cp = r \text{ or } a - x,$$

$$= \frac{1}{r^3} (G^2 a - F^2 x).$$

Now the values of G and F being indeterminate, this expression may be made equal to the above value of the force in the orbit, of which the apsidal angle is required, that is,

$$G^2 a - F^2 x = fa - f'ax,$$

from which equation, since it must hold true for the different values of x, we obtain

$$G^2 a = fa, \text{ and } F^2 = f'a, \text{ and therefore, } \frac{G}{F} = \sqrt{\frac{fa}{af'a}}.$$

Now since the proposed orbit is nearly circular, (vel.)^2 at apsidal distance (a) = force × a nearly, $= \frac{1}{a^2} \cdot fa$, and since at an apse the velocity is wholly transverse, (vel.)^2 at V in orbit $Vp = \frac{G^2}{F^2} \cdot$ (vel.)^2 at V in orbit VP, $= \frac{G^2}{F^2} \cdot \frac{F^2}{a} = \frac{G^2}{a}$ = (vel.)^2 in proposed orbit, since $G^2 a = fa$. Since then in the orbit Vp, and in that of which the apsidal angle is required, the apsidal distances and the forces at equal distances, as well as the velocities at the apsidal distances are equal, the orbits will be similar, and the apsidal angles equal; but the apsidal angle in the orbit Vp

$$= \frac{G}{F} \cdot 180^\circ \text{ (Prop. XLIII. Cor.)} = \sqrt{\frac{fa}{af'a}}\, 180^\circ;$$

and therefore the apsidal angle required $= \sqrt{\frac{fa}{af'a}}\, 180^\circ$.

Ex. 1. Let the force $= \mu r^{n-3}$;

$$\therefore \text{ force } = \frac{\mu}{r^3} r^n = \frac{\mu}{r^3}(a - x)^n$$

$$= \frac{\mu}{r^3}(a^n - n a^{n-1} . x), \text{ nearly};$$

$$\therefore fa = \mu a^n, \quad f'a = \mu n a^{n-1};$$

$$\therefore \text{ apsidal angle } = \frac{180^\circ}{\sqrt{n}}.$$

Ex. 2. Let the force $= \frac{\mu r^m + \nu r^n}{r^3}$,

$$\therefore \text{ force } = \frac{1}{r^3}\{\mu (a - x)^m + \nu . (a - x)^n\}$$

$$= \frac{1}{r^3}\{\mu a^m + \nu a^n - (m \mu a^{m-1} + n \nu a^{n-1}) x + \&c.\};$$

$$\therefore fa = \mu a^m + \nu a^n,$$

$$f'a = m \mu a^{m-1} + n \nu a^{n-1};$$

$$\therefore \text{ apsidal angle } = \sqrt{\left\{\frac{\mu a^m + \nu a^n}{m \mu a^m + n \nu a^n}\right\}} . 180^\circ.$$

If $a = 1$, apsidal angle $= \sqrt{\left\{\frac{\mu + \nu}{m\mu + n\nu}\right\}} 180^\circ$.

In this manner, as will be shewn in the next section, the motion of the moon's apsides might be found approximately, if the direction of the disturbing force of the sun upon the moon tended wholly to the earth's center; but since this is not the case, their motion cannot be determined by the method here proposed.

SECTION XI.

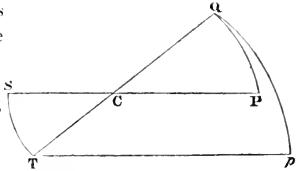
ON THE MOTION OF BODIES MUTUALLY ATTRACTING EACH OTHER.

THE motion of a physical point, attracted to an immovable center of force, has been explained in the preceding sections. We now proceed to consider the motions of mutually attracting bodies, of which the masses bear a finite ratio to each other. In this case the attracting body placed in the center of force is no longer immovable, for by the third law of motion the actions of the attracting and attracted bodies are mutual and equal; so that if M represent the mutual attraction of two bodies, whose masses are S and P, the bodies themselves will be acted on by accelerating forces $\frac{M}{S}$ and $\frac{M}{P}$ respectively, and a motion will consequently be generated in each, the nature of which it is now proposed to investigate.

PROP. LVII. *Two bodies attracting each other describe similar figures about their center of gravity, and about each other.*

Let S and P be the bodies, join SP and take $SC : SP = P : S + P$, then C is their center of gravity. If C be in motion, let a motion always equal and opposite to that of C be applied to the system, then C will continue at rest; and since the same motion applied to all the parts of a system produces no alteration in their relative motions, the relative orbits described by S and P about C and about each other will not be affected.

Let ST and PQ be arcs described in the same time round C; then



$$TC : CQ = P : S = SC : CP,$$

$$\therefore\ TC : SC = CQ : CP,$$

and angle SCT = angle PCQ; therefore ST and PQ are similar figures, and they are the figures described about the center of gravity.

Again, draw Tp parallel and equal to Sp. To a spectator at S, who is insensible of his own motion and refers the whole motion to P, P at first will be seen in the direction SCP or Tp, and afterwards in the direction TQ, and will therefore appear to have described the angle pTQ about S,

$$\text{and } SP \text{ or } Tp : CP = S + P : S = TQ : CQ;$$

$$\therefore\ Tp : TQ = CP : CQ,$$

and angle pTQ = angle PCQ, therefore the curves pQ and PQ are similar, that is, the figure described by P round S in motion is similar to the figures described by P and S round their center of gravity.

Prop. LVIII. *An orbit similar and equal to the apparent orbit of* P *round* S *in motion may be described round* S *fixed, by the action of the same central force.*

Let PQ and ST be the similar orbits described by P and S round C, their center of gravity. Take

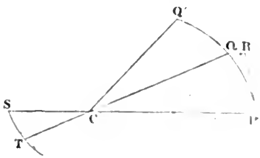
$$Sp = SP, \quad \angle\, pSq = \angle\, PCQ,$$

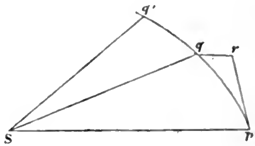
and take Sq such, that

$$Sq : Sp = CQ : CP = TQ : SP;$$

$$\therefore\ Sq = TQ,$$

and therefore q traces out the apparent orbit of P. Draw the subtenses QR, qr, parallel to CP, Sp, and meeting the tangents at P, p in R, r.





Let a body be projected from p with a velocity v, which is to V the velocity at P,

$$\text{as } \sqrt{S+P} : \sqrt{S}, \text{ as } \sqrt{Sp} : \sqrt{CP}, \text{ as } \sqrt{pr} : \sqrt{PR}$$

by similar figures, and let T, t be the times of describing PR, pr; then ultimately

$$\frac{T}{t} = \frac{PR}{V} \div \frac{pr}{v} = \frac{PR}{pr} \cdot \sqrt{\frac{pr}{PR}} = \sqrt{\frac{PR}{pr}} = \sqrt{\frac{QR}{qr}}.$$

Also the force being the same,

$$\frac{\text{space through which } P \text{ is drawn in } T''}{\text{space through which } p \text{ is drawn in } t''} = \frac{T^2}{t^2} = \frac{QR}{qr} \text{ ultimately},$$

but RQ = space through which P is drawn in T'',

$\therefore\ rq$ = ... p .... t'';

and therefore q is the place of the body at the end of t''; it will also continue in the curve, for the forces being equal and the orbits similar, the resolved parts of the forces in the directions of the tangents will be equal at all corresponding points in the arcs PQ, pq; hence the increments of the velocities continually generated, as the bodies describe the arcs, will be ultimately as the times of describing similar arcs, that is, as $T : t$, as $\sqrt{S} : \sqrt{S + P}$;

$\therefore$ componendo, vel. at q : vel. at $Q = \sqrt{S + P} : \sqrt{S}$,

hence the body is under the same circumstances as at p, and will therefore continue in the curve.

Cor. 1. *Two bodies, which attract each other with forces varying as the distance, describe similar ellipses about their center of gravity and about each other as centers.*

For the orbits described about C and about each other are similar to that described about S fixed, which in this case is an ellipse, whose center is S.

Cor. 2. *Two bodies, which attract each other with forces varying inversely as the square of the distance, describe similar ellipses about their center of gravity and about each other as foci.*

Cor. 3. *Two bodies revolving round their center of gravity describe round it areas proportional to the times.*

Let PQ, PQ' be arcs respectively similar to pq, pq', and let T, T', t, t', be the times of describing the four arcs respectively;

$$\text{now } \frac{t}{T} = \frac{\sqrt{S + P}}{\sqrt{S}} = \frac{t'}{T'};$$

$$\therefore \frac{t}{t'} = \frac{T}{T'};$$

also by similar figures,

$$\frac{\text{area } PCQ}{\text{area } PCQ'} = \frac{\text{area } pSq}{\text{area } pSq'} = \frac{t}{t'} = \frac{T}{T'};$$

$\therefore$ area $PCQ \propto$ time of describing it.

Prop. LIX. *The periodic time of* P *round* S *at rest : that of* P *or* S *round* C $= \sqrt{S+P} : \sqrt{S}$.

For the orbits, being similar, may be divided into the same number of similar parts, as pq, PQ in Prop. LVIII;

and time of describing pq

: time of describing $PQ = \sqrt{S+P} : \sqrt{S}$,

and the same being true for the times of describing all the similar arcs, we have componendo

periodic time of P round S at rest

: that of P or S round $C = \sqrt{S+P} : \sqrt{S}$.

Prop. LX. *Force* $\propto (dist)^{-2}$. *If* (a) *be the axis major of the apparent orbit described by* P *round* S *in motion,* (a') *that of an orbit described by* P *round* S *at rest in the same periodic time, then* $a : a' = \sqrt[3]{S+P} : \sqrt[3]{S}$.

Let $p'q'$ be the ellipse, of which a' is the axis major, that is, let $p'q'$ be an ellipse described by P round S at rest, in the same periodic time as that in which P describes an ellipse round S in motion, or as that in which PQ is described; and let pq be the apparent orbit described by P round S in motion; then, since the force in the two orbits is the same,

periodic time in $p'q'$: periodic time in $pq = a'^{\frac{3}{2}} : a^{\frac{3}{2}}$ (Prop. xv.)

also by Prop. LIX,

period. time in pq : period. time in $PQ = \sqrt{S+P} : \sqrt{S}$.

$\therefore$ period. time in $p'q'$: period. time in $PQ = \sqrt{(S+P)a'^3} : \sqrt{Sa^3}$,

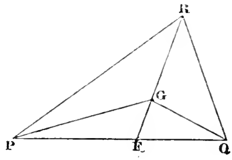
the first term of which proportion is equal to the second by the hypothesis,

$$\therefore (S+P)\,a'^3 = Sa^3;$$

$$\therefore a : a' = \sqrt[3]{S+P} : \sqrt[3]{S}.$$

Prop. LXIV. *To determine the motion of a system of bodies attracting each other with forces varying as the distance between their centers.*

Let P and Q be two bodies collected in their respective centers of gravity. Join PQ and take $PF : PQ = Q : P + Q$, then F is the center of gravity of P and Q; and $(P + Q) \cdot PF = Q \cdot PQ =$ force of Q on P; but $(P + Q)\ PF =$ the force, which two bodies equal to P and Q placed at F would exert on P, therefore P is attracted in the same manner as if a body equal to the sum of the bodies were placed at F, and will therefore describe an ellipse round F at rest as its center. Similarly Q will describe an ellipse round the same point as a center, and in the same periodic time, since the absolute force $P + Q$ is the same in both cases.



Let R be a third body, join RP, RQ, RF; the forces $R \cdot PR$ and $R \cdot QR$, which R exerts on P and Q, may be resolved respectively into $R \cdot PF$, $R \cdot FR$, and $R \cdot QF$, $R \cdot FR$; the force $R \cdot FR$, being the same for either body, produces no disturbance in their relative motions, and therefore the bodies will move in the same manner with respect to each other, as if that force did not act. The other forces $R \cdot PF$, $R \cdot QF$, varying as the distance of P and Q from F, will not cause any perturbations in the orbits described by P and Q round F, and therefore these bodies will still describe ellipses round F, but since the absolute force is increased in the ratio of $P + Q + R : P + Q$, the periodic time will be diminished in the ratio of $\sqrt{P + Q} : \sqrt{P + Q + R}$.

Again, in FR take $FG : FR = R : P + Q + R$, and join PG, QG; then G is the center of gravity of P, Q, R; and $R \cdot FR = (P + Q + R) \cdot FG$; hence the force which R exerts on P is equivalent to the forces $R \cdot PF$ and $(P + Q + R) \cdot FG$, and the force which Q exerts on P is equal to $(P + Q) \cdot PF$; hence the whole force on P is equal to $(P + Q + R)\ PF$ and $(P + Q + R)\ FG$, that is, to $(P + Q + R)$

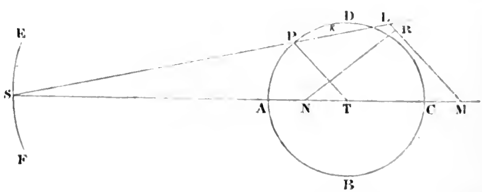
PG, and therefore P will describe an ellipse round G as a center. Similarly Q will describe an ellipse round the same point as a center, and therefore P and Q describe ellipses round their common center of gravity and round the center of gravity of the system.

In the same manner it may be shewn that P and R, and Q and R will describe ellipses round their common centers of gravity respectively, and round the center of gravity of the system; and the same may be proved of any number of bodies.

Prop. LXVI. *Force* $\propto$ *(dist.)*$^{-2}$. *Two bodies* S *and* P *revolve round a third* T *in such a manner, that* P *describes the interior orbit: to shew that* P *will describe round* T *areas more nearly proportional to the times, and a figure more nearly resembling an ellipse, if* T *be acted on by the attractions of the other two, than if it were either not attracted by them at all, or attracted much more or much less.*

Let PAB, ESF be the orbits of P and S respectively.

1. Let the orbits be in the same plane. Join SP, PT, TS, and in SP, produced if necessary, take KS equal to the mean distance of P from S, and let it represent the accele-



rating force of attraction of P to S at that distance; take also $LS = \frac{KS^2}{PS^2} \cdot KS$, then LS will represent the attraction of P to S at the distance PS. Draw LM parallel to PT meeting ST, produced if necessary, in M, and resolve LS into the forces LM, MS.

P is acted on by three forces, *LM*, *MS* and its original gravitation to *T*, the last of which would cause it to describe areas proportional to the times and an ellipse, focus *T*: the force *LM*, acting in the direction *PT*, does not affect the equable description of areas, but since by composition with the attraction of *T* on *P* it forms a force not varying as $(\text{dist.})^{-2}$, it will disturb the elliptic form of *P*'s orbit; and the force *MS*, neither acting in the direction *PT*, nor varying as $(\text{dist.})^{-2}$, will disturb both the equable description of areas and the elliptic form of the orbit.

Let *NS* represent the attraction of *S* on *T*; then if *MS* and *NS* are equal, these equal forces, acting in parallel directions on *P* and *T*, will not disturb the relative motions of the two bodies; but if they are unequal, the disturbing force on *P* will be represented by their difference *MN*; hence the less *MN* is, the smaller will be the disturbances produced: now since the distance of *P* from *S* is sometimes greater and sometimes less than that of *T* from *S*, the mean attraction *KS* of *P* to *S* differs less from *NS*, than if *T* were attracted by a *much* greater or *much* less force; that is, the disturbing force *MN* will be less, and therefore the equable description of areas and the elliptic form of *P*'s orbit will be less disturbed, if *T* be attracted by *S*, than if it were not attracted by *S* at all, or attracted much more or much less.

Def. The *Line of Nodes* is the straight line, in which the planes of the orbits of *P* and *S* intersect each other.

2. Let the orbits lie in different planes. The same construction being made, the force *LM* acting in direction *PT*, which is in the plane of *P*'s orbit, produces the same effect as in the first case, and has no tendency to draw *P* from the plane of its orbit. But *MN*, acting in a direction inclined to that plane, except when the line of nodes passes through *S*, not only produces the effects mentioned in the first case, but also tends to draw *P* from the plane of its orbit; and this and the other perturbations depending on the magnitude of *MN* will be least, when *MN* is least, that is, when *NS* is equal or nearly equal to *KS*, as before

Obs. In the proposition P is supposed to describe an orbit round T fixed; this cannot in reality be the case, as long as its magnitude bears a finite ratio to that of T; for, leaving out the consideration of the forces which S exerts, the two bodies P and T describe orbits about their center of gravity. The orbit here meant is the *apparent* orbit of P to a spectator at T, that is, the orbit $p\,Q$ in Prop. 57. If, however, we suppose a force applied every instant to P and T equal and opposite to that which P exerts on T, T will remain at rest, and the gravitation of P to T will be the sum of the attractions of T on P, and of P on T, acting in the direction PT; so that the whole gravitation of P to $T = \frac{P+T}{PT^2}$.

Prob. I. *To investigate expressions for the disturbing forces of* S *on* P, *on the supposition that* P's *orbit is circular, and coincident with the plane of* S's *orbit.*

Force of S on P represented by $LS = \frac{S}{SP^2}$,

∴ force of S on P in direction PT

$$= \frac{S}{SP^2} \cdot \frac{LM}{LS} = \frac{S}{SP^2} \cdot \frac{PT}{SP} = \frac{S \cdot PT}{SP^3} \quad \text{.........(1)},$$

this is called the *addititious* force, and is represented by LM.

Again, force of S on P in direction TS

$$= \frac{S}{SP^2} \cdot \frac{MS}{LS} = \frac{S}{SP^2} \cdot \frac{SP}{ST} = \frac{S \cdot ST}{SP^3},$$

and force of S on T in direction $TS = \frac{S}{ST^2}$;

∴ disturbing force of S on P in direction TS

$$= S\left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \quad \text{..................(2)},$$

this is called the *ablatitious* force, and is represented by MN.

Draw NR perpendicular to LM; then MN is equivalent to MR, RN,

$$RN = MN \sin NMR = S \cdot \left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \sin PTS, \ldots (3),$$

this force acts in the direction of the tangent at P, and is called the *tangential* force.

$$\text{Similarly, } MR = S \cdot \left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \cos PTS,$$

$$\text{hence } LR = LM - MR = \frac{S.PT}{SP^3} - S.\left\{\frac{ST}{SP^3} - \frac{1}{ST^2}\right\} \cos PTS, \ldots (4),$$

this force, which is the resultant of the disturbing forces of S on P in direction PT, is called the *central disturbing* force.

Hence the gravitation of P to T

$$= \frac{P+T}{PT^2} + S \cdot \left\{\frac{PT}{SP^3} - \left(\frac{ST}{SP^3} - \frac{1}{ST^2}\right) \cos PTS\right\}.$$

Prob. II. *To find approximate expressions for the above disturbing forces, when* ST *is very great compared with* PT.

$$SP = \{ST^2 - 2ST \cdot PT \cos PTS + PT^2\}^{\frac{1}{2}}$$

$$= ST \left\{1 - \frac{2PT}{ST} \cos PTS\right\}^{\frac{1}{2}} \text{ nearly};$$

$$\therefore \frac{1}{SP^3} = \frac{1}{ST^3}\left\{1 + \frac{3PT}{ST} \cos PTS\right\} \text{ nearly},$$

$$\therefore \frac{ST}{SP^3} - \frac{1}{ST^2} = \frac{3PT}{ST^3} \cos PTS,$$

hence the ablatitious force

$$= \frac{3S \cdot PT}{ST^3} \cos PTS,$$

the tangential force

$$= \frac{3S \cdot PT}{ST^3} \cos PTS \cdot \sin PTS = \frac{3S \cdot PT}{2ST^3} \sin 2PTS,$$

the central disturbing force

$$= \frac{S \,.\, PT}{ST^3}\left\{1 + \frac{3\,PT}{ST}\cos PTS\right\} - \frac{3S \,.\, PT}{ST^3}\cos^2 PTS$$

$$= \frac{S \,.\, PT}{ST^3}\,.\,\left\{1 - 3\cos^2 PTS\right\} \text{ nearly}$$

$$= -\frac{S \,.\, PT}{2ST^3}\left\{1 + 3\cos 2PTS\right\}.$$

Cor. 1. Let F = the mean central disturbing force, or the force, which, acting uniformly for a whole revolution of P round T, would produce the same effect as the variable central disturbing force; and let the four right angles through which TP moves in one revolution be divided into n equal angles; then

$$F = -\frac{S \,.\, PT}{2ST^3}\,.\,\frac{1}{n}\left\{n + 3\left(\cos\frac{4\pi}{n} + \cos\frac{8\pi}{n} + \cos\frac{12\pi}{n} + \ldots + \cos\frac{4n\pi}{n}\right)\right\} \text{ when } n \text{ is infinite,}$$

$$= -\frac{S \,.\, PT}{2ST^3}\left\{1 + \frac{3}{n}\,.\,\frac{\cos\left(\frac{n+1}{n}\,.\,2\pi\right)\sin 2\pi}{\sin\frac{2\pi}{n}}\right\} \text{ when } n \text{ is infinite,}$$

$$= -\frac{S \,.\, PT}{2ST^3},$$

and therefore the mean central disturbing force is ablatitious, and diminishes the gravitation of P to T.

Def. 1. P is said to be in *syzygy*, when its orthogonal projection on the plane of S's orbit lies either in ST or in ST produced, and in *quadrature* when the projections lie in a line drawn through T in the plane of S's orbit perpendicular to ST.

In the first nine corollaries to the Proposition the planes of the two orbits are supposed to coincide, and therefore P will be in syzygies at A and C, when crossing the line ST or ST produced, and in quadratures at B and D, 90° distant from A or C. S and P move in the directions ESF, *DAB*. The distance *PS* is supposed invariable, and so great as to be always considered parallel to *TS*. In the eighth and ninth corollaries the eccentricity of *P*'s orbit is taken into account, but the expressions above obtained for the disturbing forces on the supposition that *P*'s orbit is circular, may, on account of the smallness of the eccentricity, be applied without affecting the *general* correctness of the results deduced.

Cor. 2. If the planes of the two orbits coincide, the central disturbing force $= -\frac{2S \cdot PT}{ST^3}$ when *P* is in syzygies, and $= \frac{S \cdot PT}{ST^3}$ when *P* is in quadratures; and is therefore ablatitious in the former case, and addititious in the latter.

Def. 2. If the Earth, Moon and Sun be supposed to be represented by *T*, *P*, and *S*, the Moon is said to be in *perigee* when at the nearer, and in *apogee* when at the farther apse.

Corollaries to the Proposition.

Cor. 1. What has been proved as to the disturbances caused by *S* may be proved as to those produced by any other body revolving round *T*: hence if several bodies *P*, *S*, *R*, &c. revolve about another *T*, the motion of the innermost body *P* will be least disturbed by the attractions of *S*, *R*, &c. when *T* is attracted by the others in the same manner as they mutually attract each other.

Cor. 2. *The areas, described by* P *round* T *in the same given times, continually increase as* P *moves from quadrature to syzygy, and continually decrease from syzygy to quadrature.*

For the only part of the disturbing force, which affects the equable description of areas is the tangential force, and it acts in consequentiâ from upper quadrature to syzygy, and in antecedentiâ from syzygy to lower quadrature.

Similarly the areas described in the same given times increase continually from lower quadrature to syzygy, and decrease from syzygy to upper quadrature.

Cor. 3. *The velocity of P is greatest in syzygies, and least in quadratures.*

Cor. 4. *If P's orbit be originally circular, the curvature of the disturbed orbit will be greatest in quadratures, and least in syzygies.*

For the radius of curvature in an orbit nearly circular $\propto \dfrac{(\text{vel})^2}{\text{central force}}$, and therefore the curvature, which varies inversely as the radius varies as $\dfrac{\text{force}}{(\text{vel})^2}$. Now the force of P to T is greatest in quadratures, and least in syzygies, and the velocity of P is least in the former case, and greatest in the latter; hence on both accounts the curvature is greatest in quadratures and least in syzygies.

Cor. 5. *Hence P's orbit, if it be originally circular, will assume the form of an oval, whose axis major passes through quadratures, and axis minor through syzygies.*

Cor. 6. *To consider the effect produced by the disturbing forces on the periodic time of P round T.*

The tangential force accelerates and retards P's motion equally in a whole revolution, and therefore does not affect the periodic time. But the central disturbing force in a whole revolution diminishes the gravitation of P to T, and therefore increases the distance PT; hence the periodic time, which $\propto \dfrac{(\text{rad})^{\frac{3}{2}}}{\sqrt{\text{absolute force}}}$, will from both these causes be increased by the action of the central disturbing force.

Obs. If S approach towards the system T and P, the central disturbing force, which varies inversely as ST^3, will be increased, and consequently the gravitation of P to T will be still more diminished, and the distance PT increased; hence the periodic time will be still farther increased.

Cor. 7. *The orbit of P being supposed nearly circular, to consider the effect of the central disturbing force on the motion of its apsides during a whole revolution.*

Let $PT = r$, and let $\frac{\mu}{r^2}$ represent the force of T on P; then if νr represent the addititious force, when P is in quadrature, $-2\nu r$ will represent the ablatitious force when P is in syzygy; and therefore the whole attractions of P to T in quadrature and syzygy respectively will be $\frac{\mu}{r^2} + \nu r$, and $\frac{\mu}{r^2} - 2\nu r$. Hence if the force in quadratures prevailed for a whole revolution, the apsidal angle would $= \sqrt{\frac{\mu + \nu}{\mu + 4\nu}} . 360^0$. which is less than 360^0, or the apside would regrede; and if the force in syzygies prevailed for the same time, it would $= \sqrt{\frac{\mu - 2\nu}{\mu - 8\nu}} . 360^0$, which is greater than 360^0, or the apside would progrede. At any other point the apside will regrede or progrede, according as the disturbing force at that point increases or diminishes the gravitation of P to T; but the gravitation is on the whole diminished by the central disturbing force, and therefore its tendency is to make the apsides progrede.

Obs. In investigating in this and the following Corollaries the effects produced on P's orbit by the different disturbing forces, it is to be observed that only general results are obtained: the disturbing force may be supposed to act by impulses, its effects are then examined at the points where its action is most effective, and from these a general conclusion is drawn as to its effect in a whole revolution of P.

Cor. 8. *The orbit to P being supposed eccentric, to consider the effect of the central disturbing force on the motion of its apsides.*

1. Let the apsidal line be in a syzygy; draw the tangent Py in the direction of P's motion. As P approaches perigee,

the central disturbing force being ablatitious*, tends to draw P from T; hence the *acute* angle TPy is increased by it, or P arrives at an apse (π) sooner than it would have done in the undisturbed orbit; therefore the apsidal line regredes. For a short time after passing perigee, the disturbing force, being still ablatitious, tends to increase the *obtuse* angle TPy, so that P appears to have proceeded from an apse (π') still more distant than π: hence if the disturbing force now ceased acting, so that P described an undisturbed ellipse, the apogee, found by producing $\pi' T$, will have regreded more than that found by producing πT, and therefore both before and after perigee, the tendency of the central disturbing force is to make the apsidal line regrede. As P approaches near to apogee, the disturbing force being still ablatitious increases the obtuse angle TPy, and $\therefore$ P arrives at the apse later than it would otherwise have done, or the line of apses progredes; and in like manner as before it may be shewn to progrede still farther after P leaves apogee; hence when P is near apogee the line of apses is progressive. Now the disturbing force, varying as PT, is greater in the latter case than in the former, hence the progression of the apsidal line, when P is near apogee, is greater than the regression, when P is near perigee. When P is near the extremities of the latus rectum, it may be easily seen by reasoning similar to the above, that the effect of the disturbing force is to make the apsidal line regrede at one extremity, and progrede at the other, and PT being in this case the same for both, the regression will equal the progression. Similarly, at other intermediate points between syzygies and quadratures, the disturbing forces in a whole revolution will in a great measure counteract each other, and their effects need not be considered; hence when the apsidal line is in syzygy, the effect of the central disturbing force is to make it progrede.

2. Let the apsidal line be in quadrature: then at the apsides the disturbing force is addititious; and it may be shewn as above, that when P is near perigee, the apsidal line

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* In this and the remaining Corollaries, the central disturbing force is called ablatitious, when it acts in the direction TP, and therefore tends to diminish the gravitation of P to T.

progredes, and regredes when P is near apogee; and the regression in this case is greater than the progression; therefore since the whole motion of the apsides for other positions of P is inconsiderable, in this position the apsidal line is regressive.

The apsidal line then progredes when in syzygy, and regredes in quadrature; but the progression exceeds the regression; for the former is due to the difference of the ablatitious forces at apogee and perigee, when the apsidal line is in syzygy, and the latter to the difference of the addititious forces at the same point, when that line is in quadrature, and the former difference equals twice the latter. As the line of apses by the actual motion of S appears to revolve from syzygy to quadrature, the progression for the same reason exceeds the regression; hence during a whole revolution of S the effect of the central disturbing force is to make the line of apses progrede.

Moreover, when the apsidal line is in syzygy and therefore progressive, it is moving in the same direction as S, and thus continues longer in syzygy than if S were quiescent, and hence the progression is increased. When the apsidal line is in quadrature, the contrary takes place, and the regression is not so great as if S were stationary. (Vid. Prof. Airy's "Gravitation.") moreover when apse is in syz: the L's motion " slower at apogee. also effect are still more import

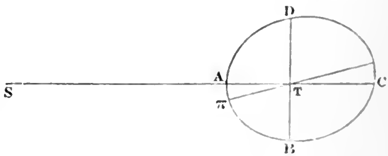
Cor. 9. *To consider the effect of the central disturbing force on the eccentricity of* P's *orbit.*

As P moves from perigee to apogee, the ablatitious force tends to increase, and the addititious force tends to diminish the *obtuse* angle TPy, which the tangent Py makes with PT; also the velocity at any point, and therefore the axis major remains nearly unaltered; therefore in the former case the form of the orbit departs farther from, and in the latter approaches nearer to that of a circle; that is, the tendency of the ablatitious force is to increase, and that of the addititious to diminish the eccentricity. As P moves from apogee to perigee, the *acute* angle TPy is increased by the former force, and diminished by the latter, that is, the eccentricity is diminished by the ablatitious and increased by the addititious force.

1. When the line of apsides is in either syzygy or quadrature, the effects in either case of these disturbing forces separately, as P moves from perigee to apogee, are equal and opposite to those produced by them during P's motion from apogee to perigee; and therefore the eccentricity of P's orbit in either of these positions of the apsidal line is unaltered by the central disturbing force.

2. Let the perigee π lie between lower quadrature and nearer syzygy.

At A and C the disturbing force is ablatitious, and at the former point P is moving towards, and at the latter from perigee; hence at A the force tends to diminish, and at C to increase the eccentricity; but TC is greater than TA, and $2 \times$ distance is a measure of the ablatitious force at these points, therefore the combined effects of the forces at A and C will increase the eccentricity. At B and D the force is addititious, and at B, P is moving from, and at D towards perigee, hence the tendency of the force at B is to diminish, and at D to increase the eccentricity; but TD is greater than TB, and the distance is a measure of the addititious force at these points, therefore on the whole the forces at B and D increase the eccentricity. Hence in this position of the apsidal line the eccentricity is increased in a whole revolution.



Now as S moves in a direction parallel to AB, π moves from quadrature towards syzygy, and therefore $2(TC - TA)$ continually increases, and $TD - TB$ decreases, but the former difference increases faster than the latter decreases; hence, as the perigee moves from quadrature to syzygy, the eccentricity is continually increasing.

3. By reasoning similar to the above it may be shewn, that as the perigee moves from syzygy to upper quadrature,

the eccentricity is continually decreasing; that it increases as it moves through DC, and decreases through CB; so that generally, *the eccentricity continually increases as the apsidal line revolves from quadrature to syzygy, and decreases as that line revolves from syzygy to quadrature.*

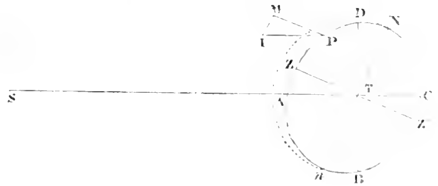
Cor. 10. *To consider the effects produced on the inclination of* P's *orbit to that of* S *by the ablatitious force.*

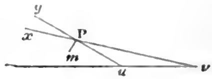
Let Nn be the line of nodes: through P draw PI parallel to TS to represent the ablatitious force at P, IM perpendicular to the plane of P's orbit, and join PM: the force PI may be resolved into the two PM, MI, of which the latter only affects the inclination of the orbit; and since during P's motion from upper to lower quadrature the ablatitious force acts in direction TS or PI, and through the remaining part of the orbit in direction ST or IP, the perpendicular force in the former case acts in direction MI, and in the latter in direction IM; hence the perpendicular force tends towards the plane of S's orbit through Dn and BN, and from it through nB and ND; and similarly, whatever be the position of the nodal line, the perpendicular force tends towards the plane of S's orbit, except when P is between quadrature and the nearer node.

In the plane of P's orbit, draw ZTZ' perpendicular to Nn.

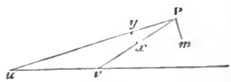
1. When the nodes are in syzygy, since no part of the disturbing force acts out of the plane of P's orbit, the inclination will not be affected by it.

2. Let the node N lie between upper quadrature and farther syzygy, and let the portion NPn of its orbit be above the plane of that of S. From upper quadrature to Z, P is

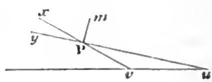




moving from the plane of S's orbit; let Py the tangent at P (fig. 2.) be produced backward to meet that plane in u, draw Pm parallel to MI; then Px the new direction of P's motion will fall between Py and Pm, and when produced backwards will cut the plane in v at a less angle than that in which yP cuts it, and therefore the inclination of P's orbit, the position of which is determined by the point T and the direction of P's motion, is diminished. From Z to n, P is moving *towards* the plane of S's orbit, and therefore, as appears from fig. 3, Px will cut the plane at a greater angle than that in which Py cuts it, or the inclination is increased.



From n to lower quadrature P is moving from the plane, and the perpendicular force now tends from the plane, and therefore, as in fig. 4, the inclination is increased.



In a similar manner it may be shewn, that as P moves from B to Z' the inclination is diminished, that it increases from Z' to N, and also from N to D; hence if $NTD = a$, the inclination, in this position of the line of nodes, is increased, while P describes $180^0 + 2a^0$ and diminished through $180^0 - 2a^0$.

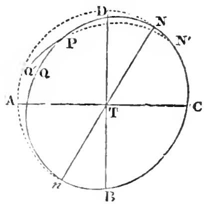
3. When the nodes are in quadrature the inclination is as much increased as it is diminished, and therefore at the end of one revolution it is unaffected by the ablatitious force.

4. Let N lie between C and B at an angular distance (a) from B; then it may be shewn by reasoning similar to the above, that in this position the inclination is increased, while P moves through $180^0 - 2a^0$, and diminished through $180^0 + 2a^0$.

As the node recedes (see Cor. 11.) from quadrature to syzygy, the inclination is increased, and from syzygy to quadrature it is as much diminished, so that in a whole revolution of the nodes the inclination is neither increased nor diminished. The inclination is a maximum when the nodes are in syzygy, and a minimum when they are in quadrature; and least of all when the nodes are in quadrature and P in syzygy.

Cor. 11. *To consider the effects produced on the motion of the Nodes by the ablatitious force.*

Let P be the place of the body; resolve the ablatitious force at P into two, one perpendicular to and the other in the plane of P's orbit; and let PQ be a small arc of the orbit which P would describe, were there no perpendicular force; PQ' a small arc of the disturbed orbit.



Then it is manifest that when P is ascending from the node, N' the node of PQ' will lie behind or before N, that is, the node will be retrograde or progressive, according as Q' is at a less or greater distance from the plane of S's orbit than Q, that is, according as the perpendicular force tends towards or from that plane; and the same is true of the node n, when P is approaching that node. Now by what has been shewn in the first part of Cor. 10, the force tends always towards the plane, except between quadrature and the nearer node; hence the motion of the node is always retrograde, except when P is moving between quadrature and the nearer node.

If a be the angular distance of the node from quadrature, the node will be progressive while P moves through $2a^0$, and retrograde through $360^0 - 2a^0$.

Since a is less than 90 except at syzygy, the nodes in a whole revolution of P regrede more than they progrede.

If the nodes be in quadratures, they will regrede during the whole revolution: when they are in syzygies, the disturbing force acting in the plane of P's orbit, produces no effect upon the node, which therefore remains stationary; it will however pass out of syzygy by the motion of S, and become retrograde.

Cor. 12. *The effects produced by the disturbing forces are greater, when P is in conjunction than when in opposition.*

For when P is at nearer syzygy or in conjunction, the addititious force $= \frac{S \cdot PT}{SA^3}$, and when at farther syzygy or in opposition, it $= \frac{S \cdot PT}{SC^3}$; and SA being less than SC, the former value is greater than the latter. Also in the former case the ablatitious force $= \frac{3S \cdot PT}{SA^3}$, and in the latter it $= \frac{3S \cdot PT}{SC^3}$, and therefore is greater in conjunction than in opposition. Hence, the effects produced by these forces will be greater in conjunction than in opposition.

Cor. 13. The reasoning employed in this proposition is wholly independent of the magnitude of S; if therefore S be so great, that the system of P and T revolves round S fixed, the disturbing forces will be of the same *kind* as when S moved round T fixed; but since each varies as S, they will all be increased in the same ratio as that in which we suppose S to be increased.

Cor. 14. *If S and the distance ST vary, whilst the system of P and T remains the same, the angular error of P as seen from T, produced in a given time by the disturbing force of S, will vary inversely as the square of the periodic time of T round S, or directly as the cube of the apparent diameter of S as seen from T.*

For let S' and $S'T'$ be other values of S and ST; then in any given position of P, since PT is the same, the disturbing forces of S on P are to those of S' as $\frac{S}{ST^3} : \frac{S'}{S'T'^3}$, and therefore the linear errors produced by them in the same unit of time are in the same ratio, and PT being given, the angular errors as seen from T will be proportional to the linear errors; and the same being true of all corresponding angular errors, componendo, the angular errors generated in a given time will be as $\frac{S}{ST^3} : \frac{S'}{S'T^3}$, that is, by Prop. xv

$$\text{as } \frac{1}{(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S} : \frac{1}{(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S'},$$

and therefore the angular error varies inversely as

$$(\text{per}^{\text{c}}.\ \text{time})^2 \text{ of } T \text{ round } S.$$

Also if D = diameter of S, $S \propto D^3$, and therefore angular error $\propto \frac{D^3}{ST^3} \propto$ the cube of the apparent diameter of S, as seen from T.

Cor. 15. *If there be two systems* P, T, S *and* P′, T′, S′, *such that* S : S′ = T : T′, *and* PT : ST = P′T′ : S′T′; *and if the orbits of* P *and* P′ *be similar and similarly situated, their periodic angular errors round* T *and* T′ *arising from the disturbing forces of* S *and* S′ *will be equal.*

The bodies P and P' at any two similarly situated points in each orbit, are similarly acted on by proportional forces, and therefore the linear errors, generated while they move through small similar parts of their orbits, will be similar and proportional, and will therefore be respectively as the diameters of the orbits; hence, the angular errors through those small parts will be equal; and this being true of the errors through all corresponding parts, the periodic angular errors will be equal.

Cor. 16. *In any two systems* P, T, S *and* P′, T′, S′, *in which the orbits of* P *and* P′ *are similar and similarly situated, to compare the periodic angular errors round* T *and* T′.

Let P and p be the periodic times of T round S and of P round T,
P' and p' .... T' ...... S' and of P' ...... T'.

In TS, produced if necessary, place a body s such that $s : S' = T : T'$, and at a distance sT from T, such that

$$sT : PT = S'T' : P'T'; \ \therefore s = \frac{T}{T'} S', \text{ and } sT = \frac{PT}{P'T'} \cdot S'T'.$$

Then by Cor. 15, the periodic angular errors in the system P', T', S' equal the errors in the system P, T, s. Again, by Cor. 14, in the systems P, T, S and P, T, s, the angular errors in a given time, and therefore the periodic angular errors are

$$\text{as } \frac{S}{ST^3} : \frac{s}{sT^3} \text{ as } \frac{S}{ST^3} : \frac{S'}{S'T'^3} \cdot \frac{T}{T'} \cdot \frac{P'T'^3}{PT^3},$$

$$\text{as } \frac{S}{ST^3} \cdot \frac{PT^3}{T} : \frac{S'}{S'T'^3} \cdot \frac{P'T'^3}{T'} \text{ as } \frac{p^2}{P^2} : \frac{p'^2}{P'^2};$$

therefore the periodic angular errors in the systems P, T, S and P', T', S' are as $\frac{p^2}{P^2} : \frac{p'^2}{P'^2}$.

Hence, if the orbits of two satellites be similar, and equally inclined to the orbits of their primaries, the periodic angular errors in their orbits will vary directly as the squares of the periodic times of the satellites, and inversely as the squares of those of the primaries.

The errors here spoken of are the angular motions of the nodal line, apsidal line, &c.

Cor. 17. *To compare the mean addititious force with the force of* T *on* P.

Let P be the periodic time of T round S.

p that of P and T round their center of gravity;

therefore $\sqrt{\frac{P+T}{T}} \cdot p =$ time in which P would revolve round T at rest at the same distance TP, by Prop. LIX.

$$\text{Now, mean addititious force} = \frac{S \cdot PT}{ST^3},$$

$$\text{and force of } S \text{ on } T = \frac{S}{ST^2};$$

$\therefore$ mean addititious force : force of S on $T = PT : ST$,

and by Prop. IV,

$$\text{force of } S \text{ on } T : \text{force of } T \text{ on } P = \frac{ST}{P^2} : \frac{PT}{p^2} \cdot \frac{T}{P+T};$$

$$\therefore \text{mean addititious force} : \text{force of } T \text{ on } P = \frac{1}{P^2} : \frac{1}{p^2} \cdot \frac{T}{P+T}.$$

The force of T on P here spoken of is that with which T alone draws P, and this force is to that with which P and T are drawn towards each other as $T : P + T$; hence compounding this with the above proportion, we have

mean addit[s]. force : force of P and T towards each other $= \frac{1}{P^2} : \frac{1}{p^2}$.

THE END.

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