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The Development of Arabic Mathematics: Between Arithmetic and Algebra

Roshdi Rashed

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BETWEEN ARITHMETIC AND ALGEBRA

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CONTENTS

EDITORIAL NOTE

Twenty years ago, Roshdi Rashed contributed a stimulating lecture at an international colloquium on philosophy, science and theology in the Middle Ages, held at the Boston University conference center nearby at Osgood Hill. Rashed's lecture initiated the colloquium and was entitled 'Recommencements de l'algèbre aux XIe et XIIe siècles'; it appeared in *The Cultural Context of Medieval Learning* (*Boston Studies*, Vol. 26, 1975), edited by the organizers John Murdoch and Edith Sylla. Cooperation and friendship developed between Rashed and our Center over the years, and personally as well. So it is now with pride and pleasure that I welcome this book.

I also note that Rashed's admirable fusion of interests in the external and the internal aspects of the history of science, indeed the varieties of developmental autonomy and developmental dependence, contributed again to international studies when he suggested that a colloquium be held on 'Sciences and empires' as a main research program for the REHSEIS group at CNRS-Paris (Research on Epistemology and History of Exact Sciences and Scientific Institutions group of the National Center for Scientific Research). The result was an international conference at UNESCO in Paris, with the opening lecture by Roshdi Rashed, entitled 'Science classique et science moderne à l'époque de l'expansion de la science européenne'; the proceedings appeared as *Science and Empires: Historical Studies about Scientific Development and European Expansion*, edited by Patrick Petitjean, Catherine Jami and Anne Marie Moulin (*Boston Studies*, Vol. 136, 1992).

And just last year, Rashed contributed his paper on 'Analysis and Synthesis According to Ibn al-Haytham' to the recent *Festschrift* for Marx Wartofsky, *Artifacts, Representations and Social Practice*, edited by Carol Gould and me (*Boston Studies*, Vol. 154, 1994).

So, much is required for Rashed's range of studies, and how arduous is his effort in coming to terms with the dialectic of the internal and the external, the dialectic within the internal, and within the complexity of the historical sociology of the external! The present book is a fine exploration of one crucial period in the history of mathematics, a stage in the coming of algebra.

April 1994

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PREFACE

These studies belong to epistemological history but they are not intended to revive an old debate on methodology in the history of science. Here as elsewhere, our work as historian is in fact subject to an epistemological principle; it is not a methodological *a priori*, but rather, on the contrary, the only viewpoint which enables us to describe and understand the facts at the same time. How can we set forth these facts, I would even say discover and restore them, without analyzing the conceptual configurations in which they are set, the connections they sustained with other configurations, the distortions to which they are subjected, and indeed the lack of understanding of which they are victims? There are so many considerations necessary for the restitution, at least partial, of this past and localized rational activity. To confine oneself to dating, to the search for influences, or just simply to recount the content of a text, is in fact of mediocre interest, even if made under the pretence of so-called analytical or bibliographical rigour. This epistemological conception is all the more necessary when a field of research is relatively unexplored, as is Arabic mathematics. It compels us to grasp the latent structures buried beneath the diversity of mathematical facts and masked by the dispersal of texts, many still in manuscript form, in various libraries throughout the world.

In *Between Arithmetic and Algebra*, the analysis of latent structures is conducted for algebra and algebraic calculus, numerical analysis and number theory, i.e. for the dialectic between algebra and arithmetic. In this analysis we evoke another dialectic between algebra and geometry and make some conjectures. The years following the French edition saw my investigations advance along two paths, presented mainly in three publications. The first is the mathematical work of one of the principal algebraists of the twelfth century discussed here, al-Ṭūsī; the establishment of the Arabic text, its translation into French with an accompanying mathematical and historical comment (Rashed, ed., 1986) enabled us to reconsider his theory of numerical equations in detail and also to describe another fundamental movement in the history of algebra, I mean the dialectic between algebra and geometry, and the foundation of traditional algebraic geometry. In a second publication (cf. *infra*, IV.5), we examined the presence of the reciprocal of Euclid's theorem on perfect numbers in Arabic mathematics. What was conjectured here is therefore now established, and the English translation has given us the opportunity to take these facts into account. A third publication (Rashed, 1991a) deals with methods of quadratic interpolation and attests to some results announced here earlier. These are the new facts of some significance which have enriched our knowledge of the areas studied here, all of which seem to confirm our earlier analysis.

I would like to thank Angela F. W. Armstrong who undertook the English translation and to express my gratitude to Professor R. S. Cohen for accepting it in his collection and for his care in revising the translation.

Bourg-la-Reine, 4 March 1993. ROSHDI RASHED

INTRODUCTION

Since the early nineteenth century, most histories of science have presented Arabic mathematics, like other scientific disciplines in that language, in a paradoxical light: though a fundamental chapter of the history of classical mathematics, they do not really belong. Historians of science cannot avoid coming across the works of Arab mathematicians in their research; they set foot on the stage of history either in person, or via Latin or Hebrew translations or, lastly, disguised in the works of scholars with direct access to the scientific language of Arabic, such as Leonardo of Pisa. But the rules for their appearance dictate the final scene, practically unchanged since the nineteenth century, by inviting them to slip away into the wings, to join all those other "extras" defined negatively by a term so loaded with ideology and myth that it speaks for itself: *non-Western* mathematics. Such a definition might be thought outdated, a vestige of nineteenth century traditional thought; on the contrary, it is still alive in the terminology of a large number of contemporary historians. So if the ideology that conjures up these pictures and myths of science is to be believed – analyzed in Appendix 1 – neither Arabic mathematics, nor Arabic science for that matter, have the right to claim a place in history.

As regards this ideology, it was undoubtedly sustained by an inherent weakness of historical research which is moreover far from innocent: a somewhat contradictory picture of Arabic mathematics is superimposed on its paradoxical aspect. That ideology, with few exceptions, such as the works of the eminent historian Woepcke in the last century, was able to channel the interest that stimulates historical research by narrowing its scope as well. Greek works translated into Arabic were given top priority; while original works written in Arabic were overlooked, as the dearth of critical editions and the inadequacy of empirical studies during the last two centuries bear witness. The result could only be unreliable and fragmentary knowledge. For example, in histories of Arabic science, a tenth-century mathematical genius often rubs shoulders with a dull, untalented fourteenth-century commentator, the only reason for this relationship being the availability of documents. As a result, Arabic mathematics quite often appears in numerous studies, not the worst, in a rather disconcerting light: a number of finds, theorems or propositions remarkable for their depth and potential, swamped in a mass of second-rate unrelated results. A contradictory picture that leaves the historian impassive nonetheless, he only pauses for results without really asking why they occurred.

The survival of this paradox, fanned by a certain ideology, in this drab and contradictory picture, the result of some practises, is due, partly at least, to historical procedures and methods. As in many other areas of the history of science, in most cases particular attention is usually directed towards establishing the succession of scientists. In this respect, the historian of Arabic mathematics closely resembles those working in other fields, for whom historical order amounts to the chronological succession of authors. This is not the place to engage in a methodological debate; let us just remark that, based on incomplete historical data, this order must be partial and unreliable. The mass of writings accumulated over seven centuries and more, recorded in hundreds of thousands of manuscripts scattered over the four corners of the earth, dooms empirical historical reconstruction procedures in advance to pure contingency. For instance, though centuries apart, two mathematicians are sometimes made to follow one another through current ignorance about their intermediaries. A general history is therefore understandably out of the question at the present time; but if we confine ourselves to one country or region, it then becomes illusory and out of all proportion to its real subject.

Our reason for recalling these features of the history of Arabic mathematics is not only to brush aside some ideas set forth and discussed later in this book, but also to inform the reader about a recent trend: the history of Arabic science has recently inspired an unprecedented interest and an evergrowing number of publications. However, this enthusiasm is not only characteristic of genuinely scrupulous historians concerned with understanding classical science; it also denotes a trend where, for good or bad reasons, apologists mingle with sensationalists. Only methodological rigour and strictness can guarantee us, as much as possible, against such endeavours.

Given these conditions, how can we set about isolating from among the host of facts, names and works, the hidden lines on which mathematical rationality or rationalities have developed? Essential for revealing the structure of mathematical activity at least seven centuries old, this theoretical question has a heuristic value as well, since it may indicate to the research worker where his priorities lie. By following this path I was able to reconstruct some hitherto unknown facts and, in particular, some theoretical trends embedded in empirical studies until now, and identify the main structure of Arabic mathematics. So let us go back to the basic rules for these procedures set forth in more detail in this volume.

To understand classical mathematics, and mathematics written in Arabic in particular, is to place ourselves first and foremost between algebra and arithmetic, on the one hand, and algebra and geometry on the other. Only this approach enables us to grasp the fundamental and radically new role of algebra in the formation of mathematical reasoning. And furthermore, it enables us to discern a movement for the reorganization and restructuring of these disciplines by each other; in other words, to see emerge a double dialectic between arithmetic and algebra, and algebra and geometry, a dialectic which, it should be emphasized, was in no way *a priori* as it came to light as our research progressed. It gradually emerged as an inductive movement for extending each discipline by overhauling its foundations, generalizing its concepts and methods, sometimes at the cost of rejecting and eliminating some of them. The essays collected here pay particular attention to establishing and describing the first movement between arithmetic and algebra. Analysis of the dialectic between algebra and geometry, simply mentioned in passing here, will form the subject of another book.

But to define the direction of this movement and grasp its scope, we have first of all to reconstitute one event in order to elucidate its significance: the appearance of al-Khwārizmī's work on algebra. For the first time in history, algebra is seen in this work as an autonomous discipline and in full possession of its name. We shall show that this early ninth-century work, though technically weak compared with the great Hellenistic compositions, remains nonetheless irreducible to those of Antiquity or late Antiquity. We shall therefore seek to isolate the very idea of this new discipline contained in it, the herald of an entire current of later research. And it is precisely this posterity that will give al-Khwārizmī's work its full historical dimension.

Although nothing or virtually nothing is known about al-Khwārizmī's predecessors and, consequently, the genesis of the first movement of algebra, we know at least that it was set in a non-Hellenistic arithmetical tradition, the very one from which two works on arithmetic by al-Khwārizmī arose, of which only one has survived in its Latin compilations. Be that as it may, from al-Khwārizmī's time, mathematicians took hold of this new discipline to develop algebraic calculation, theory of equations, indeterminate analysis, well before the translation of Diophantus' *Arithmetica*. We cite among others, the famous names of Ibn Turk, Abū Kāmil and Ibn al-Fatḥ.

However, about one and a half centuries after al-Khwārizmī, algebra, developed and enriched, was the subject of renewal, in fact a real new beginning. And arithmetic made this possible. If the term "arithmetization" has a non-figurative meaning, it is the most appropriate to describe the contributions of al-Karajī and his successors such as al-Shahrazūrī and al-Samawʾal. By "arithmetization" I mean the transposition and extension of elementary operations of arithmetic, algorithms like Euclidean division or the extraction of roots of algebraic expressions, in particular polynomials. Mathematicians succeeded in constituting polynomial algebra and reached a clearer understanding of the algebraic structure of real numbers through the arithmetization of algebra during the tenth and twelfth centuries. To use another language, this was how they carried out empirically finite algebraic extensions in the field of rational numbers.

From henceforth, the algebraic treatise will be constructed around these arithmetical operations and algorithms and includes, as an integral part, a chapter on indeterminate analysis. Attested in mathematical texts well before the translation of Diophantus' *Arithmetica*, indeterminate analysis came into its own with this Arabic translation, especially when it had received an algebraic interpretation at variance with its initial purpose, in our opinion.

Some of the studies included here concern the renewal of algebra, achieved by the arithmetization movement. Others examine the effects of renewed algebra on arithmetic and number theory. For instance, after defining the place and status of al-Karajī whose merit, since Woepcke, is appreciated, though his real objectives continue to be misunderstood, and after showing he was the creator of a school and tradition, not an isolated case, i.e. after defining this renewed algebra, we were in a position to establish that not only well-known but new results were organized into chapters which have never been reconstructed, let alone named. That these chapters are destined to be enriched is undisputed, nor is it excluded that additions will be made. Our only claim is to have reconstructed the main chapters according to which the arithmetic and algebraic acquisitions of Arabic mathematics were ordered and organized. But before setting them out, we want to show one result of renewed algebra related to proof techniques: finite mathematical induction as a demonstration method. It was during the tenth century that this method stood out from others used in arithmetical algebra. In our study entitled "Mathematical induction: al-Karajī, al-Samawʾal", we were led to dialecticize the regressive method in the history of science in order exactly to grasp the originality of the method of finite mathematical induction both as a concept and a technique.

If we now look at the chapters that comprise this field of mathematics, we find:

1. *Combinatorial Analysis*

Although considered until recently as an activity peculiar to Renaissance mathematicians and their successors, combinatorial analysis was in fact already part of Arabic mathematics (as we have shown) and its formation as calculus took place in two stages. First seen without unity, i.e. simply as calculus with the combinatorial features relegated to secondary importance, algebraists saw it as auxiliary "arithmetical method" for algebra; and on the other hand, lexicographers, and linguists in particular, saw it as a combinatorial practice, i.e. whose propositions were neither formulated in general terms or even less proved. In a second later stage, unity was essentially achieved by number mathematicians who concentrated on the study of the particular function, the number of divisors of a number. We establish the first stage in "Combinatorial analysis in Arabic mathematics" and reconstruct the second in "Amicable numbers, aliquot parts and figurate numbers in the thirteenth and fourteenth centuries".

2. *Numerical Analysis*

The application of traditional arithmetic to the new algebra provides the subject matter of this chapter where methods of numerical research are generalized, for example, the extraction of roots, with various methods for achieving it. We shall show how the invention of new fractions arose and present its theory at the occasion of the generalization of methods for extracting the *n*th root in particular. Cf. "The extraction of the *n*th root and the invention of decimal fractions".

3. *The Solution of Numerical Equations*

A result of the new algebra, this chapter, which took advantage of the preceding chapter as well as its development, is partly the result of the impossibility at that time of giving an algebraic solution of cubic equations. The mathematicians involved in the elaboration of this topic were, as we shall see, the same as those who belonged to another trend, i.e. geometer-algebraists. From below the surface and full of promise for the future, rich and profound ideas are seen to emerge, some of which were to prove functional and analytical. Cf. "The solution of numerical equations and algebra: al-Ṭūsī, Viète".

The application of algebra to number theory, inherited from Hellenistic mathematics, was to inaugurate classical number theory, which preserved its style until 1640 at least. The following chapters can be added to the above.

4. *New Diophantine Analysis*

Here we`are not concerned with traditional Diophantine analysis which as we said, was already part of algebra, but with Diophantine analysis integer. Emerging in the tenth century and based on, but also in spite of algebra, it began with numerical triangles, before extending to more complicated equations and systems of Diophantine equations. One of the main results is the statement of Fermat's conjecture for the case where $n = 3$ whose proof was unsuccessful. Cf. "Diophantine analysis in the tenth century: the example of al-Khāzin".

5. *Classical Theory of Numbers*

Lastly, in three successive studies, "Ibn al-Haytham and Wilson's theorem", "Amicable numbers, aliquot parts and figurate numbers in thirteenth and fourteenth centuries", and "Ibn al-Haytham and perfect numbers", we present new contributions to number theory, for example, a study of the characteristics of prime numbers, linear congruences, arithmetical functions. Particular stress will be laid on defining the style of that theory.

If we follow the dialectic between algebra and arithmetic, the main structure of both disciplines emerges. But diachrony also gives us access to the lines along which they developed. The body of results obtained shows that the void or quasi-void that most historians postulate between Alexandria and the Italian Republics, constituting an unsurmountable obstacle to their own understanding of the history of mathematics, is in fact quite full. Consequently, we had to consider afresh the problem of periodization in history. So it seemed appropriate to add a critical historical study of the notion of Western science itself and some reflections on the problem of periodization in the history of classical mathematics in an appendix. However, these results also show that science written in Arabic, called Arabic science for that reason only, whose only legitimate heirs were those who carried it on and developed it, is no more and no less than a moment, a stage in history, and should be considered as such if we want to avoid going astray or leading others astray as well.

1. AL-KHWĀRIZMĪ'S CONCEPT OF ALGEBRA

I

Muḥammad ibn Mūsā al-Khwārizmī[1] wrote his renowned *Kitāb al-jabr wa al-muqābala* (Musharrafa and Ahmad, eds., 1939) at Baghdad between A.D. 813 and 833, during the reign of al-Maᵓmūn. It was the first time in history that the term *algebra* appeared in a title to designate it as a discipline (al-Nadīm, Tajaddud, ed., 1971, pp. 338–341). Its recognition was not only insured by the title, but was confirmed by the formulation of a new technical vocabulary intended to specify its objects and procedures.

The event was crucial and was recognized as such, as much by ancient as by modern historians; its importance, furthermore, was not lost on the mathematical community of the time. Even in al-Khwārizmī's lifetime, or at least shortly afterwards, mathematicians were quick to comment on his book. Of his immediate successors alone, Ibn Turk, Thābit ibn Qurra, al-Ṣaydanānī, Sinān ibn al-Fatḥ, Abū Kāmil, and Abū al-Wafāᵓ al-Būzjānī should be cited. Some of these commentaries clearly constituted a fundamental contribution to the formulation of algebra. In writing the history of their disciplines, these mathematicians, like their successors, agreed unanimously to assign precedence to al-Khwārizmī[2]. There was one dissenting voice: that of Ibn Barza, who gave that honour to his grandfather, a claim rejected out of hand by Abū Kāmil Shujāᶜ.[3]

Although generally accepted, these few facts are difficult to interpret. In fact, as long as he remains unacquainted with the work of mathematicians preceding al-Khwārizmī, the historian is likely to find himself in a position which at first sight seems enigmatic. But this is a situation that, at least for the time being, remains irremediable.[4] The question is still open how this newly conceived discipline was developed, and how it is that this particular contribution, several aspects of which suggest the culmination of past activity, emerges as a radical departure. Unable to find a satisfactory answer, historians have for some time engaged in an endless debate around two complementary themes: the origins of algebra and the sources used by al-Khwārizmī, referring in turn to the Hellenistic mathematician (Euclid or Diophantus, depending upon the case) and to Indian or, more recently, Babylonian mathematics. The very fact that so many contradictory opinions exist indicates that no single one can prevail and that no historian has been able to establish effectively any kind of link between al-Khwārizmī and any of the supposed authors. The difficulty is the same when it concerns not his work as a whole but a chapter more restricted in scope, such as the measurement of surfaces and volumes, *al-Misāḥa*. One has only to remember the contradictory theories of the relationship between al-Khwārizmī's book and the *Mishnat ha-Middot*.[5]

It is not uncommon under these conditions for historians to have recourse to theories which raise more problems that they resolve, such as the well-known hypothesis of the "geometric algebra" of the Greeks. In addition to the problem of establishing the genesis of al-Khwārizmī's contribution to the history of algebra is another problem of a different order. Even if one accepts the breakdown of al-Khwārizmī's book to identify the traces of ancient mathematics, it must be allowed that these are merely vestiges which do not in the least clarify the theoretical form of the new knowledge. This section will be confined to an examination of this form in an attempt to grasp the concept of algebra as formulated by al-Khwārizmī. Then it will perhaps be possible to make a more rigorous enquiry into the genesis of al-Khwārizmī's algebra.

II

In the introduction to his book al-Khwārizmī announces his plan: to provide a manual which could be used for arithmetical problems, commercial transactions, in cases of inheritance, and land measurements (Musharrafa and Ahmad eds., 1939, p. 16). Sections of his book develop these themes.

The first section, a theoretical one, is concerned with this method – *ḥisāb* of algebra and *al-muqābala*, its early terminology and concepts. In the second, al-Khwārizmī determines the bases of the normal procedures which allow practical calculations to be reduced to fundamental algebraic types. The last sections, of purely practical intent, are concerned with the application of this method to commercial transactions, land surveys, geometric measurements, and, lastly, testaments. From reading al-Khwārizmī's book it is clear that algebra appears from the outset as a theoretical discipline, extended to be applicable to both the fields of numbers and of metric geometry.

Therefore, if algebra is an arithmetic – *ḥisāb*, as al-Khwārizmī writes – it is so for at least two reasons. On the one hand, the laws of calculation can be applied to different objects (arithmetic or geometric) once formulated in the primitive terms of algebra – number, unknown, and unknown squared (al-Khwārizmī had studied these laws from a book now preserved in its Latin translation).[6] On the other hand, algebra is found to have been envisaged from the outset with all the possibilities of application, and as such answers the practical needs of mathematics. It may be apodictic knowledge, but algebra is also an applied science. Its object is not specific since it is concerned with numbers as well as with geometric size.

It is impossible to overstress the originality of the conception and style of al-Khwārizmī's algebra, which did not rise from any "arithmetical" tradition, not even that of Diophantus. Analysis of al-Khwārizmī's book reveals two kinds of primitive terms: purely algebraic terms and those common to algebra and arithmetic. The algebraic terms, as already mentioned, are the unknown, designated without distinction – root, thing, and its square, *māl*. To these may be added rational positive numbers, the laws of arithmetic $\pm$, $\times/\div$, $\sqrt{\ }$, and the equals sign. All of these operations are designated by words of unequal occurrence. For example, speaking of multiplication, al-Khwārizmī uses *ḍrb* – to multiply – but uses *ḍʿf* just as often; and more rarely (two occurrences each) he uses *thny* and *thlt*. The relation *fī* functions equally as an operator of multiplication according to the model "*n fī* (in, by) *n*".

As regards algebraic terms proper, it would be astonishing if al-Khwārizmī knew only the two cited previously. In his book, it should be noted, he deals with a problem whose context suggests that he knew the third power, although he did not name this term. He writes in effect: "if one speaks of a square – *māl* – multiplied by its roots, it comes to three times the first square". By *māl* al-Khwārizmī almost invariably means the square of the unknown. He does express the same term *the thing*, at times. However, juxtaposed to the root, the term *māl* only expresses the first meaning, thus: $x^2 \cdot x = 3x^2$. This being established, even independently of the example, al-Khwārizmī must have known the cubic power, at least. It is known that the cube root was extracted at that time, as in the *Opuscula in the Measurement of the Plane and Spherical Figures* by Banū Mūsā,[7] and that the solid number can be found in al-Ḥajjāj's translation of Euclid's *Elements*. Banū Mūsā and al-Ḥajjāj were contemporaries and, so to speak, the colleagues of al-Khwārizmī, at the "House of Wisdom". The extension of the notion of algebraic power was effected, furthermore, after a single reading of al-Khwārizmī's book by at least two mathematicians independently of each other, Abū Kāmil Shujā^c and Sinān ibn al-Fatḥ.[8] The latter explicitly formulated the general concept of a positive integer power. It seems, then, that if al-Khwārizmī limited the usage of algebraic terms to two, it was not the result of a misunderstanding of the higher powers of the unknown, but reflects a whole concept of algebra, its domain, and its scope. It is equally necessary to return to the constituent concepts of algebraic theory in order to understand al-Khwārizmī's intention and at the same time grasp the sense and the import of this deliberate limitation to primitive terms.

The principal concepts used by al-Khwārizmī are first and second-degree equations, the related binomials and trinomials, the normal form, the algorithmic solution, and the demonstrability of the solution formulae. However, if one wishes to understand how these concepts are realized and coordinated in the original algebraic theory, the best method is a rapid perusal of al-Khwārizmī's explanation. Having introduced the terms of his theory, he writes: "Of these three types, some can be equal to others, as when you say: squares* are equal to roots, squares are equal to a number, roots are equal to a number." (Musharrafa and Ahmad, eds., 1939, p. 17) He continues: "I have found that these three types – *al-ḍurūb, modus* – which are the roots, the squares and the numbers combine, and that there are therefore three composite kinds – *ajnās muqtarina, genera composita* – which are squares, plus roots equal a number; squares plus a number equal roots; roots plus a number equal squares" (Musharrafa and Ahmad, eds., 1939, p. 18 and Latin translation, Libri, 1967, p. 255). It can then be seen that al-Khwārizmī retains three binomial equations and three trinomial equations:

$$ax^2 = bx,\ ax^2 = c,\ bx = c;$$
$$ax^2 + bx = c,\ ax^2 + c = bx,\ ax^2 = bx + c.$$

Even at this stage, al-Khwārizmī's text can be seen to be distinct not only from the Babylonian tablets, but also from Diophantus'

* square: *māl* (the square of the unknown).

Arithmetica. It no longer concerns a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study. On the other hand, the idea of an equation for its own sake appears from the beginning and, one could say, in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems. To gauge the extent of the breakthrough, it is sufficient to recall a survival of the ancient tradition in al-Khwārizmī's book. Having found the value of the unknown, he concludes by giving that of the square. This appears to derive from a tradition which was not concerned with the study of equations but with the solution of problems – with finding, for example, a square, such that its product by a given number is equal to its root by another given number.

In this situation it might be anticipated that al-Khwārizmī's exposition would evolve in an ever more general manner. Indeed, a second stage of generality is attained when the notion of normal form is introduced. Al-Khwārizmī requires the systematic reduction – *yarudd, reducere* – of each equation to the corresponding regular form. For the fourth equation, for example, he writes: "In the same way, if one postulates two *s*quares, or three, or more or less, reduce that to one *s*quare, and reduce the roots and numbers found at the same time to that to which you have reduced the *s*quares." (Musharrafa and Ahmad eds., 1939, p. 19) This is true, in particular, for trinomial equations:

$$x^2 + px = q \quad x^2 = px + q \quad x^2 + q = px.$$

Everything is therefore positioned to allow the establishment of algorithmic formulae to solutions. Al-Khwārizmī then deals with each of the three cases. The fact that the coefficients are given in no way diminishes the generality of the reasoning. Take the best-known case, the first of the three equations cited, with $p = 10$, $q = 39$. Al-Khwārizmī writes: "The rule in this – *fabābuhu, cujus regula* – is that you divide the roots into halves. In this problem five is the result, which you multiply by itself, you have twenty-five, you add it to thirty-nine, you have sixty-four; you take its root which is eight, you deduct from it half of the roots, which is five, and three is left, which is the root of the square you are looking for, and the *s*quare is nine (Musharrafa and Ahmad, eds., 1939, pp. 18–19: Libri, 1967, p. 255). In other words, he obtains the following expression:

$$x = \left[\left(\frac{p}{2}\right)^2 + q\right]^{\frac{1}{2}} - \frac{p}{2}. \tag{1}$$

For the other two cases, he obtains the following expressions:

$$x = \frac{p}{2} + \left[\left(\frac{p}{2}\right)^2 + q\right]^{\frac{1}{2}} \tag{2}$$

$$x = \frac{p}{2} \pm \left[\left(\frac{p}{2}\right)^2 - q\right]^{\frac{1}{2}}, \quad \text{if } \left(\frac{p}{2}\right)^2 > q. \tag{3}$$

In this last case, he condenses it: if $(p/2)^2 = q$, "then the root of the square is equal to half of the roots, exactly, without surplus or diminution"; if $(p/2)^2 < q$, "then the problem is impossible – *Fa-al-mas*ʾ*ala mustaḥīla, tunc quaestio est impossibilis*." (Musharrafa and Ahmad, eds., 1939, pp. 20–21; Libri, 1967, p. 257) To conclude this chapter, al-Khwārizmī writes:

These are the six types which I mentioned at the beginning of my book, I have finished the explanation and I have stated that there are three types, the roots of which are not divided into halves. I have shown the rules for these – *qiyāsahā, regular* – and their necessity – *iḍṭirārahā, necessitas*. Concerning those of the three remaining kinds, the roots of which must be divided into two, I have explained them with correct reasons (*bi-abwāb ṣaḥīḥa*), and I have invented for each a figure by which the reason for the division can be recognized. (Musharrafa and Ahmad eds., 1939, p. 21.)

Al-Khwārizmī demonstrates the different formulae not only algebraically with the aid of figures, as he writes – *ṣūra, forma* – that is, by means of the notion of the equality of surfaces. Each of these demonstrations is presented by al-Khwārizmī as the "cause" – ᶜ*illa* – of the solution. This treatment was very probably inspired by recent knowledge of the *Elements*. In addition, al-Khwārizmī not only requires each case to be demonstrated, but occasionally proposes two demonstrations for a single type of equation. Such requirements adequately illustrate the extent of progress and certainly distinguish al-Khwārizmī from the Babylonians, but also, in his systematic approach, from Diophantus. This brief summary shows how al-Khwārizmī's exposition evolves, and that he handles earlier concepts. All problems dealt with by means of algebra must be reduced to an equation, a single unknown, and rational positive coefficients of almost second degree. Such is the only equation admissible in al-Khwārizmī's book. Algebraic procedures, transposition and reduction, are then applied so that the equation may be written in a regular form. The procedures then make the idea of the solution as a simple question of decision possible: an algorithm for each class of problem. The formula for the solution is then justified mathematically with the help of a protogeometric demonstration. Al-Khwārizmī then finds himself in a position to write that everything stemming from algebra "must lead you to one of the six types which I have described in my book" (Musharrafa and Ahmad, eds., 1939, p. 27).

Al-Khwārizmī's exposition is followed by four brief chapters dedicated to the study of certain aspects of the application of elementary laws of arithmetic to the most simple expressions of algebra. He studies, in order, multiplication, addition and subtraction, and the division and extraction of the square root. For example, this is what he proposed to show in his short chapter on multiplication: "how to multiply objects [the unknown] which are roots, by each other, if they are single, or if they are added to a number, or if a number is subtracted from them" (Musharrafa and Ahmad, eds., 1939, p. 27), that is to say, products of the following type:

$$(a \pm bx)\,(c \pm dx) \qquad a, b, c, d \in \mathbb{Q}^{+}.$$

These chapters are far more important for the intention behind them than for the results that they enclose. If one considers al-Khwārizmī's declarations, the position that he gives these chapters (immediately after the theoretical study of the quadratic equation), and finally the autonomy conferred on each, it appears that the author wished to undertake the study of algebraic calculation for its own sake, that is, the properties of binomials and trinomials considered in the preceding section of the book. However rudimentary it may seem, this study nevertheless represents the first attempt at algebraic calculation *per se*, since the elements of this arithmetic do not simply emerge in the course of solving different problems, but provide the subject of relatively autonomous chapters.

Al-Khwārizmī's concept of algebra can now be grasped with greater precision: it concerns the theory of linear and quadratic equations with a single unknown, and the elementary arithmetic of relative binomials and trinomials. If al-Khwārizmī was confined to the second degree at best, it was simply through following the notion of solution and proof in the new discipline. The solution had to be general and calculable at the same time and in a mathematical fashion, that is, geometrically founded. In fact, only a solution by means of the root answered al-Khwārizmī's requirements. The restriction of degree, as well as that of the number of unsophisticated terms, is instantly explained.

From its true emergence, algebra can be seen as a theory of equations solved by means of radicals, and of algebraic calculations on related expressions, before the notion of the general polynomial had been formulated. This idea of algebra lasted long after al-Khwārizmī's death. His immediate successors were only concerned with the study of higher-degree equations, which could nevertheless be reduced to second-degree equations. Others would be tempted by the solution of cubic equations by radicals. Al-Khayyām's refusal to accept the solution of a cubic equation by the intersection of curves as algebraic (a definition he reserves for the solution by radicals) provides further evidence of al-Khwārizmī's influence.

After his theoretical chapters al-Khwārizmī returns to the different applications of the theory, arithmetic or geometric, most of which are now based on the generality of the theory. In each case he endeavoured to transpose the problem into algebraic terms in order to reduce it to demonstrated types of equation. It is not until the second section of the book that he meets certain problems of Diophantine analysis.[9]

It would be useless to look for a similar theory before al-Khwārizmī. Certainly it is possible to find one or another of his concepts in certain texts of Antiquity, or late Antiquity, but they are never found together and are never bound into a structure such as his. It is precisely this illustrated theoretical structure that explains the apparent technical weakness in al-Khwārizmī's algebra and the mathematician's intentionally innovative terminology. Compared with Diophantus' *Arithmetica*, for example, al-Khwārizmī's book appears to describe no more than an elementary algebraic technique; but this simplicity corresponds exactly to the limitations imposed by the constitution of the theory. Similarly, the terminology was designed to create a language capable of translating, without distinction, geometric and arithmetic terms. Thus, in expressing a requirement of the theory, the terms reflect the desire to distinguish the new knowledge.

No description of al-Khwārizmī's algebra is complete without indicating where its fertility lies. A science is defined not only by the procedures it develops, but also by its cumulative power, the obstacles generated and confronted – in short by all the channels of research that it is able to open. It is precisely in this area that al-Khwārizmī stands out from all possible predecessors: he alone determined the scope of a whole vein of algebraic research which has not yet been exhausted. This historical dimension of al-Khwārizmī's algebra must therefore be examined.

III

The different elements of classical algebra can be traced already in al-Khwārizmī's book. But to develop the traces effectively and to give body to al-Khwārizmī's concept of algebra, his successors had to diverge from his tracks. New inroads had to be made not only to surmount theoretical and technical problems which limited the execution of his programme (the resolution of the cubic equation by radicals, for example), but also to reorientate the project in a more arithmetical direction for the development of abstract algebraic calculation. One can therefore isolate two new departures for algebra and two areas of research undertaken following al-Khwārizmī's lead but in the opposite direction. The first was arithmetic, the second geometric; both modified the nature of the discipline profoundly. (Of course, I can only briefly outline the results of the arithmetic tradition of algebra.)

Shortly after al-Khwārizmī's death, perhaps even during his lifetime, attempts were made to follow his lead. While Ibn Turk adopted the theory of equations to give demonstrations (Sayili, 1962, in particular the Arabic text, pp. 144ff.), still protogeometric but firmer, al-Māhānī translated certain biquadratic problems from Book X of the *Elements* and cubic problems from Archimedes[10] into algebraic terms. Equally prompt was the extension of the notion of algebraic power. There are two testimonies confirming that this effort had been suggested by a reading of al-Khwārizmī's book: that of Abū Kāmil, whose work is known and has been analyzed, and that of Sinān ibn al-Fatḥ (Youschkevitch, 1976, pp. 52 ff.). The latter studied trinomial equations which, if divided by a power taken from the unknown, are brought into the realm of al-Khwārizmī's equations, in other words, equations which consist of the following terms:

$$ax^{2n+p}, \quad bx^{n+p}, \quad cx^p.$$

All these findings, especially Abū Kāmil's improved study of rational positive numbers, as well as other results obtained by algebraic mathematics in the study of algebraic irrationals, and finally the translation of Diophantus' *Arithmetica*, coincide with the development of the project of the "arithmetization of algebra" (as it has been called) by al-Karajī. It is concerned first, according to the words of a successor of al-Karajī, al-Samawʾal, "with operating on unknowns using all the arithmetical tools, in the same way as the arithmetician operates on the known", and with opting more and more for algebraic demonstrations at the expense of geometric demonstrations. Thus the movement is towards a systematic application of elementary arithmetic procedures to algebraic expressions. This application was made possible by the first elaboration in general terms of the idea of the polynomial. It is this application, evident in al-Karajī's work, which has permitted the extension of abstract algebra and the organization of the algebraic exposition around different arithmetic operations successively applied to algebraic expressions. Furthermore, it is in this way that the best treatises on classical algebra have been presented since then. It has been shown elsewhere in detail how such a project was constituted and what the principal results were (cf. *infra*, I.2 and I.3).

It would be too lengthy to enumerate the consequences of this arithmetization of algebra here; but it should be remembered that algebra itself is affected – the theory of numbers, numerical analysis, and the resolution of numerical equations, as well as rational Diophantine analysis and even the logic and philosophy of mathematics. I would like to conclude here on the theory of the equation itself in order to show, with the help of unpublished and unknown documents, that, contrary to the accepted idea, the successors of al-Karajī had in fact attempted an algebraic solution to the cubic equation.

First, with regard to the theory of equations, in *al-Fakhrī* as well as what has already been seen in Sinān ibn al-Fatḥ's work, the following equations can be found:

$$ax^{2n} + bx^n = c \quad ax^{2n} + c = bx^n \quad bx^n + c = ax^{2n}.$$

But al-Karajī himself says nothing about cubic equations (Rashed, 1986). It was one of his successors, al-Sulamī, who suggested that the question preoccupied algebraist mathematicians in the tradition of al-Karajī. He himself envisages two possible kinds:

$$x^3 + ax^2 + bx = c \quad \text{and} \quad x^3 + bx = ax^2 + c,$$

however, he imposes the condition $b = (a^2/3)$. He therefore gives a true positive root for each equation:

$$x = \left(\frac{a^3}{27} + c\right)^{\frac{1}{3}} - \frac{a}{3} \quad \text{and} \quad x = \left(c - \frac{a^3}{27}\right)^{\frac{1}{3}} + \frac{a}{3}.$$

We may, it seems, piece together the mathematician's progress in the following manner: by a related transformation, he restores the equation to its normal form. But instead of finding the discriminant, he annuls the coefficient of the first power of the unknown in order to bring the problem to that of the extraction of the cubic root. Thus, for example, for the first equation $x \longrightarrow y - a/3$ is taken for an affine transformation; the equation can be rewritten as follows:

$$y^3 + py - q = 0,$$

with $p = b - a^2/3$ and $q = c + a^3/27 + (b\ a/3 - a^3/9)$. For $b = a^2/3$, it comes to $y^3 = c + a^3/27$, whence the value of x. It should be remembered that the role of the discriminant was identified by Sharaf al-Dīn al-Ṭūsī in the particular case $x^3 - bx + c = 0$ (cf. *infra*, III).

As we have seen in the preceding pages, it was therefore al-Khwārizmī who established the unity of algebra, not merely because of the generality of mathematical entities which the discipline deals with, but above all because of the generality of his procedures. It is a question of successive operations designed to turn a numerical or geometrical problem into one of equations expressed in normal form, and of equations which lead to the canonical formulae of solutions, which, moreover, must be demonstrable and calculable.

The algebra worked out by al-Khwārizmī is a science of equations and algebraic calculations on related binomials and trinomials, an autonomous discipline which commanded already its proper historical perspective and carried with it the potential of a first reform, the arithmetization of algebra.

Al-Khwārizmī's contribution is clearly irreducible, reflecting the new type of mathematical rationality. If attempts to discover the origins of his algebra continue to fail, this failure may be as much the result of insufficient insight in analysis as of a lack of historical information, and of an uncontained regression in linguistic and ideological terms. Rather than ask what al-Khwārizmī may have read, it would be preferable to enquire how he conceived of what none of his predecessors had been able to conceive.

NOTES

1. This is the name of the author according to the testimonies of historians, bibliographers and mathematicians. Al-Ṭabarī in his *Tārīkh al-rusul wa al-mulūk* (Abū al-Faḍl, 1966) gave this name in relating the events of the year 210 of the Hegira: "Muḥammad ibn Mūsā al-Khwārizmī is reported to have said . . ." (vol. VIII, p. 609). However, speaking of the events of the year 232 of the Hegira, he gave a list of the names of astronomers present at the death of al-Wāthiq: "Amongst those present were: al-Ḥasan b. Sahl, the brother of al-Faḍl b. Sahl, and al-Faḍl b. Isḥāq, al-Hāshimī, Ismāʿīl b. Nawbakht and Muḥammad ibn Mūsā al-Khwārizmī al-Majūsī al-Quṭrubbullī and Sanad, the companion of Muḥammad b. al-Haytham and all those who were interested in the stars". If we compare al-Ṭabarī's testimonies, and taking into account the consensus of other authors, there is no need to be an expert on the period nor a philologist, to see that al-Ṭabarī's second citation should read: "Muḥammad ibn Mūsā al-Khwārizmī *and* al-Majūsī al-Quṭrubbullī . . .", and that there are two people (al-Khwārizmī and al-Majūsī al-Quṭrubbullī); the letter "wa" was omitted in the early copy.

   This would not be worth mentioning if a series of conclusions about al-Khwārizmī's personality, occasionally even the origins of his knowledge, had not been drawn. In his article on al-Khwārizmī in the *Dictionary of Scientific Biography*, G. J. Toomer, with naive confidence, constructed an entire fantasy on the error which cannot be denied the merit of making amusing reading.
2. Abū Kāmil described al-Khwārizmī as "The one who was the first to succeed in a book of algebra and *al-muqābala* and who pioneered and invented all the principles in it". Cf. Abū Kāmil, Istanbul, Beyazit, MS Kara Mustafa 379, f. 2^r.

   Cf. Sinān ibn al-Fatḥ, who only mentioned the name of al-Khwārizmī in the introduction to his small work and asserted that the science was his: "Muḥammad ibn Mūsā al-Khwārizmī wrote a book which he called "algebra and *al-muqābala*". Still later, al-Ḥasan b. Yūsuf on al-Khwārizmī wrote: "He was the first in the Islamic world to discover this science; consequently he was recognized as the Imām of arithmeticians and the master of that science". Lastly Ibn al-Malik al-Dimashqī may be cited: "You must understand that this science is the invention of that accomplished scholar Muḥammad ibn Mūsā al-Khwārizmī". Evidence to this effect is abundant and we could enumerate many other examples.
3. Ḥājjī Khalīfa cited a passage from a book by Abū Kāmil *al-Waṣāya bi-al-jabr*; Abū Kāmil mentioned another book by him and wrote: "I have established, in my second book, proof of the authority and precedent in algebra and *al-muqābala* of Muḥammad ibn Mūsā al-Khwārizmī, and I have answered that impetuous man Ibn Barza on his attribution to ʿAbd al-Ḥamīd, whom he said was his grandfather" (Khalīfa, 1943, vol. 2, pp. 1407–1408).
4. Nothing of great importance has emerged to clarify the history of mathematics in the first two centuries of the Hegira. In our study (cf. *infra*, IV.3) we showed that linguists, lexicographers, and in particular al-Khalīl b. Aḥmad (d. *circa* 786) possessed several combinatorial rules. It should not, however, be concluded that they were familiar with combinatorial analysis as such. The rules were applied but neither stated nor demonstrated. According to Ibn Khaldūn's later reference in his

Prolegomena, some elementary progressions can be found. Examination of other documents available in the fields of literature and philosophy etc. for lack of mathematical works, provide information too incomplete to lead to a decisive conclusion. The now lost arithmetic works of mathematicians like al-Khwārizmī himself in the third century of the Hegira are quite different. He drafted a book, *Kitāb al-jam*ᶜ *wa al-tafrīq* which is as yet untraced but is mentioned by Abū Kāmil in his *Algebra* (MS Kara Mustafa, 379, f. 110). If the contents of his work can be re-established, through patient effort, the direction of mathematics at that time could be established. But this is a privilege for the future.

5. S. Gandz saw the origins of al-Khwārizmī's section of "the measurement" of surfaces and volumes in this book (cf. Gandz, 1932). G. B. Sarfatti, on the other hand, located this text after al-Khwārizmī (Sarfatti, 1968).
6. Cf. Juschkewitsch (1964a). This work of al-Khwārizmī on arithmetic should not be confused with another by the same author, quoted by Abū Kāmil (see n. 4). In the latter al-Khwārizmī evidently deals with arithmetical problems.
7. Banū Mūsā, *Kitāb fī Ma*ᶜ*rifat Misāḥat al-Ashkāl*; cf. the (poor) transcription given in *Rasā*ᵓ*il al-Ṭūsī* (Hyderabad, ed., 1940), pp. 19ff.
8. Sinān ibn al-Fatḥ (Cairo, National Library, MS Riyāḍiyyāt 260, ff. 95ʳ–104ᵛ) introduced the power of the unknown in a general way. He wrote: ". . . let us call the first of these number, the second root, the third square (*māl*), the fourth cube (mukaᶜᶜab), the fifth square-square, the sixth *midād*, the seventh square cube, then you will have the eighth proportion, add the ninth, and as far as you wish. It is permissible to change the names, once you understand their intention. But as it is usual to give these names, we will conform.

   This is an example to show what has been described, following Indian mathematics

   one ten hundred thousand hundred thousand thousand thousand
   number root square cube square-square *midād* square-cube
   eight proportion ninth proportion

   (1) Sinān ibn al-Fatḥ claimed precedence for this generality; he wrote: "None of my predecessors in this science, whom I have heard of, has written a greater work in order to name 'the powers'. I have been pleased to compose this into a book wherein it is shown how to develop these names".
   (2) If the term *midād* is Arabic in origin, it must derived from the root *mdd*, which signifies the elongation of something, or the extension of one thing by another. It can also indicate the plural of *mudd*, a kind of measurement which originally signified the holding out of hand for food. The reason for such a choice to signify x^5, or the sixth position, is not clear. It is not impossible that it was borrowed from Persian to indicate the sixth position.
   (3) Ibn al-Fatḥ makes the power n^{th} correspond to the power $n + 1$.
   (4) Finally, the definition x^6 is multiplicative, contrary to all additive definitions known in Arabic.
9. This kind of problem is encountered in the chapter on wills. Cf. for example, Musharrafa and Ahmad, eds. (1939), pp. 76ff.

10. Cf. al-Māhānī's commentary on Book X of the *Elements*, Paris, Bibliothèque Nationale, *fonds arabe* MS 2457 (180^v–187^v), where $39x^2 = x^4 + 225/4$ can be found. Al-Khayyām reported that al-Māhānī: "was led to use algebra to analyse the lemma used by Archimedes, considering it as authorized in proposition 4 of the second book of his work on *The Sphere and the Cylinder*". Al-Khayyām continued, "Al-Māhānī arrived at cubes, squares and numbers forming an equation that he was unable to solve . . ." Cf. Rashed and Djebbar, eds. (1981), p. 11.

2. AL-KARAJĪ

Virtually nothing is known of al-Karajī's life; even his name is not certain. Since the translations by Woepcke and Hochheim he has been called al-Karkhī, a name adopted by historians of mathematics (Woepcke, 1853; Hochheim, 1878–80). In 1933, however, Giorgio Levi della Vida rejected this name for that of al-Karajī (1933, pp. 264ff). This debate would have been pointless if certain authors had not attempted to use the name of this mathematician to deduce his origins: Karkh, a suburb of Baghdad, or Karaj, an Iranian city. In the present state of our knowledge, della Vida's argument is plausible but not decisive. On the basis of the manuscripts consulted it is far from easy to decide in favour of either name.[1] Turning to the "commentators" does not take us any further.[2] For example, the *al-Bāhir fī al-jabr* of al-Samawʾal cites the name al-Karajī, as indicated in MS Aya Sofya 2718. On this basis some authors have sought to derive a definitive argument in favour of this name (Anbouba, ed., 1964, p. 11). On the other hand, another hitherto unknown manuscript of the same text (Esat Efendi 3155) gives the name al-Karkhī.[3] Because the use of the name al-Karajī is beginning to predominate – for no clear reasons – and because we do not wish to add to the already great confusion in the designation of Arab authors, we shall use the name al-Karajī – refraining from any speculation designed to infer our subject's origins from this name. It is sufficient to know that he lived and produced the bulk of his work in Baghdad at the end of the tenth century and the beginning of the eleventh and that he probably left that city for the "mountain countries"[4] where he appears to have ceased writing mathematical works in order to devote himself to composing works on engineering, as indicated by his book on the drilling of wells.

Al-Karajī's work holds an especially important place in the history of mathematics. Woepcke remarked that it "offers first the most complete or rather the only theory of algebraic calculus among the Arabs known to us up to the present time" (Woepcke, 1853, p. 4). It is true that al-Karajī employed an entirely new approach in the tradition of the Arab algebraists – al-Khwārizmī, Ibn al-Fatḥ, Abū Kāmil – commencing with an exposition of the theory of algebraic calculus (cf. *infra*, IV.3).

The more or less explicit aim of this exposition was to find means of realizing the autonomy and specificity of algebra, so as to be in a position to reject, in particular, the geometric representation of algebraic operations. What was actually at stake was a new beginning of algebra by means of the systematic application of the operations of arithmetic to the interval $[0, \infty]$. This arithmetization of algebra was based both on algebra, as conceived by al-Khwārizmī and developed by Abū Kāmil and many others, and on the translation of the *Arithmetica* of Diophantus, commented on and developed by such Arab mathematicians as Abū al Wafā᾽ al-Būzjānī (Medovoij, 1960). In brief, the discovery and reading of the arithmetical work of Diophantus, in the light of the algebraic conceptions and methods of al-Khwārizmī and other Arab algebraists, made possible a new departure in algebra by al-Karajī, the author of the first account of the algebra of polynomials.

In his treatise on algebra, *al-Fakhrī*, al-Karajī first presented a systematic study of algebraic exponents, then turned to the application of arithmetical operations to algebraic terms and expressions, and concluded with a first account of the algebra of polynomials. He studied (see Woepcke, 1853, p. 48) the two sequences $x, x^2, \ldots, x^9, \ldots; 1/x, 1/x^2, \ldots, 1/x^9, \ldots$ and, successively, formulated the following rules:

$$\frac{1}{x} : \frac{1}{x^2} = \frac{1}{x^2} : \frac{1}{x^3} = \ldots \tag{1}$$

$$\frac{1}{x} : \frac{1}{x^2} = \frac{x^2}{x} \ldots = \frac{1}{x^{n-1}} : \frac{1}{x^n} = \frac{x^n}{x^{n-1}} \tag{2}$$

$$\left.\begin{array}{l} \dfrac{1}{x} \cdot \dfrac{1}{x} = \dfrac{1}{x^2},\ \dfrac{1}{x^2} \cdot \dfrac{1}{x} = \dfrac{1}{x^3}, \ldots, \quad (3) \\[2ex] \quad \dfrac{1}{x^n} \cdot \dfrac{1}{x^m} = \dfrac{1}{x^{n+m}} \\[2ex] \dfrac{1}{x} \cdot x^2 = \dfrac{x^2}{x},\ \dfrac{1}{x} \cdot x^3 = \dfrac{x^3}{x}, \ldots, \quad (4) \\[2ex] \quad \dfrac{1}{x^n} \cdot x^m = \dfrac{x^m}{x^n} \end{array}\right\} \begin{array}{l} m = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \end{array}$$

In order to appreciate the importance of this study, it is necessary to see how al-Karajī's more or less immediate successors exploited it.

For example, al-Samaw'al (Rashed and Ahmad, eds., 1972, pp. 20ff of Arabic text) was able, on the basis of al-Karajī's work, to utilize the isomorphism of what would now be called the groups $(\mathbb{Z}, +)$ and $([x^n; n \in \mathbb{Z}], \times)$ in order to give for the first time, in all its generality, the rule equivalent to $x^m x^n = x^{m+n}$, where $m, n \in \mathbb{Z}$.

In applying arithmetical operations to algebraic terms and expressions, al-Karajī first considered the application of these rules to monomials before taking up "composed quantities", or polynomials. For multiplication he thus demonstrated the following rules: (1) $(a/b) \cdot c = ac/b$ and (2) $a/b \cdot c/d = ac/bd$, where a, b, c and d are monomials. He then treated the multiplication of polynomials, for which he gave the general rule. He proceeded in the same manner and with the same concern for the symmetry of the operations of addition and subtraction. Yet this algebra of polynomials was uneven. In division and the extraction of roots al-Karajī did not achieve the generality already attained for the other operations. Hence he considered only the division of one monomial by another and of a polynomial by a monomial. Nevertheless, these results permitted his successors – notably al-Samaw'al – to study, for the first time to our knowledge, divisibility in the ring $[Q(x) + Q(1/x)]$ and the approximation of whole fractions by elements of the same ring (Rashed and Ahmad, eds., 1972). As for the extraction of the square root of a polynomial, al-Karajī succeeded in giving a general method – the first in the history of mathematics – but it is valid only for positive coefficients. This method allowed al-Samaw'al to solve the problem for a polynomial with rational coefficients or, more precisely, to determine the root of a square element of the ring $[Q(x) + Q(1/x)]$ (Rashed and Ahmad, eds., 1972, p. 60 of the Arabic text). Al-Karajī's method consisted in giving first the development of $(x_1 + x_2 + x_3)^2$ – where x_1, x_2 and x_3 are monomials – for which he proposed the canonical form

$$x_1^2 + 2x_1x_2 + (x_2^2 + 2x_1x_3) + 2x_2x_3 + x_3^2.$$

This last expression is itself, in this case, a polynomial ordered according to decreasing powers. Al-Karajī then posed the inverse problem: finding the root of a five-term polynomial. He therefore considered this polynomial to be of the canonical form and proposed two methods. The first consisted of taking the sum of the roots of two extreme terms – if these exist – and the quotient of either the second term divided by twice the root of the first or of the fourth term divided by twice the root of the last.[5] The second method consisted in subtracting from the third term twice the product of the root of the first term times the root of the last term, then the root of the remainder from the subtraction is added to the roots of the extreme terms. Great care must be exercised here. This form is not restricted to a particular example, and al-Karajī's method, as can be seen in *al-Badī*ᶜ, is general (Rashed and Ahmad, eds. 1972).

Again with a view to extending algebraic computation, al-Karajī pursued the examination of the application of arithmetical operations to irrational terms and expressions.

"How multiplication, division, addition, subtraction, and the extraction of roots may be used [on irrational algebraic quantities]" (Anbouba, ed., 1964, p. 31 of the Arabic text). This was the problem posed by al-Karajī and used by al-Samawʾal as the title of the penultimate chapter of his work on the use of arithmetical tools on irrational quantities. The problem marked an important stage in al-Karajī's whole project and therefore also in the extension of the algebraic calculus. Just as he had explicitly and systematically applied the operations of elementary arithmetic to rational quantities, al-Karajī, in order to achieve his objectives, wished to extend this application to irrational quantities in order to show that they still retained their properties. This project, while conceived as purely theoretical, led to a greatly increased knowledge of the algebraic structure of real numbers. Clear progress indeed, but to make it possible it was necessary to risk a setback – a risk at which some today would be scandalized – in that it did not base the operation on the firm ground of the theory of real numbers. The arithmetician-algebraists were only interested in what we might call the algebra of $\mathbb{R}$ and did not attempt to construct the field of real numbers. Here progress was made in another algebraic field, that of geometrical algebra, later revived by al-Khayyām and Sharaf al-Dīn al-Ṭūsī (Rashed, ed., 1986). In the tradition of this algebra al-Karajī and al-Samawʾal could extend their algebraical operations to irrational quantities without questioning the reasons for their success or justifying the extension. Because an unfortunate lack of any such justification gave the impression of a setback, al-Karajī simultaneously adopted the definitions of Books VII and X of the *Elements*. While he borrowed from Book VII the definition of number as "a whole composed of unities" and of unity – not yet a number – as that which "qualifies by an existing whole", it is in conformity with Book X that he defined the concepts of incommensurability and irrationality. For Euclid, however, as for his commentators, these concepts only apply to geometrical objects or, in the expression of Pappus, they "are a property which is essentially geometrical". "Neither incommensurability nor irrationality", he continued, "can exist for numbers. Numbers are rational and commensurable" (Thomson, ed., 1930, p. 193).

Since al-Karajī explicitly used the Euclidean definitions as a point of departure, it would have been useful if he could have justified his use of them on incommensurable and irrational quantities. His works may be searched in vain for such an explanation. The only justification to be found is extrinsic and indirect and is based on his conception of algebra. Since algebra is concerned with both segments and numbers, the operations of algebra can be applied to any object, be it geometrical or arithmetical. Irrationals as well as rationals may be the solution of the unknown in algebraic operations precisely because they are concerned with both numbers and geometrical magnitudes. The absence of any intrinsic explanation seems to indicate that the extension of algebraic calculation – and therefore of algebra – needed for its development to forget the problems relative to the construction of $\mathbb{R}$ and to surmount any potential obstacle, in order to concentrate on the algebraic structure. An unjustified leap, indeed, but a fortunate one for the development of algebra. This is the exact meaning of al-Karajī when he writes, without transition immediately after referring to the definitions of Euclid, "I show you how these quantities [incommensurables, irrationals] are transposed into numbers" (Anbouba, ed., 1964, p. 29 of the Arabic text).

One of the consequences of this project, and not the least important, is the reinterpretation of Book X of the *Elements* (Van der Waerden, 1956; Vuillemin, 1962; Dedron and Itard, 1969). This had until then been considered by most mathematicians, even by one so important as Ibn al-Haytham, as merely a geometry book. For al-Karajī its concepts concerned magnitudes in general, both numerical and geometric, and by algebra he classified the theory in this book in what was later to be known as the theory of numbers. To extend the concepts of Book X of the *Elements* to all algebraic quantities al-Karajī began by increasing their number. "I say that the monomials are infinite: the first is absolutely rational, five for example, the second is potentially rational, as the root of ten, the third is defined by reference to its cube as the side of twenty, the fourth is the medial defined by reference to the square of its square, the fifth is the square of the quadrato-cube, then the side of the cubo-cube and so on to infinity" (Anbouba, ed., 1964, p. 29 of the Arabic text).

In the same way binomials can also be split infinitely. In this field, as in so many others, al-Samawʾal is continuing the work of al-Karajī. At the same time one contribution belongs to him alone and this is his generalization of the division of a polynomial with irrational terms (see introduction, Rashed and Ahmad, eds., 1972). He thus developed the calculus of radicals introduced by his predecessors. At the beginning of *al-Badīʿ* (Anbouba, ed., 1964, pp. 32ff. of the Arabic text and pp. 36ff. of French introduction) is a statement – for the monomials x_1, x_2 and the strictly positive natural integers, *m, n* – of the rules that make it possible to calculate the following:

$$x_1 \sqrt[n]{x_2}\,;\ \sqrt[n]{x_1}\cdot\sqrt[n]{x_2}\,;\ \sqrt[n]{x_1}\cdot\sqrt[m]{x_2}$$
$$\sqrt[n]{x_1}\,/\sqrt[n]{x_2}\,;\ \sqrt[n]{x_1}\,/\sqrt[m]{x_2}$$
$$\sqrt[n]{x_1} \pm \sqrt[n]{x_2}\,.$$

Al-Karajī next discussed the same operations carried out on polynomials and gave, among others, rules that allow calculation of expressions such as

$$\frac{\sqrt{x_1}}{\sqrt{x_2} - \sqrt{x_3}}\,;\ \frac{x_1}{\sqrt[4]{x_2} + \sqrt[4]{x_3}}\,;$$
$$\sqrt{x_1 + \sqrt{x_2}}\,;\ \sqrt{\sqrt{x_1} + \sqrt{x_2}}\,.$$

In addition he attempted, unsuccessfully, to calculate

$$\frac{x_1}{\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4}}.$$

In the same spirit al-Karajī took up binomial developments. In *al-Fakhrī* (Woepcke, 1853, p. 58) he gives the development of $(a + b)^3$, and in *al-Badīʿ* (Anbouba, ed., 1964, p. 33 of Arabic text) he presents those of $(a - b)^3$ and $(a + b)^4$. In a long text of al-Karajī reported by al-Samawʾal are the table of binomial coefficients, its formation law $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$, and the expansion $(a + b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^n$ for integer n (Rashed and Ahmad, eds., 1972).

To demonstrate the preceding proposition as well as the proposition $(ab)^n = a^n b^n$, where a and b are commutative and for all $n \in \mathbb{N}$, al-Samawʾal used a slightly old-fashioned form of mathematical induction. Before proceeding to demonstrate the two propositions he shows that multiplication is commutative and associative – $(ab)(cd) = (ac)(bd)$ – and recalls the distributivity of multiplication with respect to addition – $(a + b)\lambda = a\lambda + b\lambda$. He then uses the expansion of $(a + b)^{n-1}$ to prove the identity for $(a + b)^n$ and that of $(ab)^{n-1}$ to prove the identity for $(ab)^n$. It is the first time, as far as we know, that we find a proof that can be considered the beginning of mathematical induction.

Turning to the theory of numbers, al-Karajī pursued further the task of extending algebraic computation. He demonstrated the following theorems (Woepcke, 1853, pp. 59ff.):

$$\sum_{i=1}^{n} i = \frac{(n^2 + n)}{2} = n\left(\frac{1}{2} + \frac{n}{2}\right) \tag{1}$$

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i\left(\frac{2n}{3} + \frac{1}{3}\right). \tag{2}$$

Actually al-Karajī did not demonstrate this theorem; he only gave the equivalent form

$$\sum_{i=1}^{n} i^2 \Big/ \sum_{i=1}^{n} i = \left(\frac{2n}{3} + \frac{1}{3}\right).$$

The algebraic demonstration appeared for the first time in al-Samawʾal (Rashed and Ahmad, eds., 1972. pp. 56ff.):

$$\sum_{i=1}^{n-1} i(i + 1) = \left(\sum_{i=1}^{n} i\right)\left(\frac{2n}{3} - \frac{2}{3}\right) \tag{3}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 \tag{4}$$

$$\begin{aligned}\sum_{i=0}^{n-1} (2i + 1)(2i + 3) + \sum_{i=1}^{n} 2i(2i + 2) \\ = \left(\sum_{i=1}^{2n+2} i\right)\left(\frac{2}{3}[2n + 2] - \frac{5}{3}\right) + 1\end{aligned} \tag{5}$$

$$\begin{aligned}\sum_{i=1}^{n-2} i(i + 1)(i + 2) = \sum_{i=1}^{n-1} i^3 - \sum_{i=1}^{n-1} i \\ = \left(\sum_{i=1}^{n-1} i\right)^2 - \sum_{i=1}^{n-1} i.\end{aligned} \tag{6}$$

For al-Karajī, the "determination of unknowns starting from known premises" is the proper task of algebra.[6] The aim of algebra is to show how unknown quantities are determined by known quantities through the transformation of the given equations. This is obviously an analytic task, and algebra was already identified with the science of algebraic equations. One can thus understand the extension of algebraic computation and why al-Karajī's followers (Rashed and Ahmad, eds., 1972, pp. 73ff. of the Arabic text) did not hesitate to join algebra to analysis and, to a certain extent, oppose it to geometry, thus affirming its autonomy and its independence. Since al-Khwārizmī the unity of the algebraic object was no longer founded on the unity of mathematical entities but on that of operations. It was a question, on the one hand, of the operations necessary to reduce an arbitrary problem to one form of equation – or, more precisely, to one of the canonical types stated by al-Khwārizmī – and, on the other hand, of the operations necessary to give particular solutions, that is, the "canons". In the same fashion al-Karajī took up the six canonical equations (Woepcke, 1853, pp. 64ff.) – $ax = b$, $ax^2 = bx$, $ax^2 = b$, $ax^2 + bx = c$, $ax^2 + c = bx$, $bx + c = ax^2$ – in order to solve equations of higher degree: $ax^{2n} + bx^n = c$, $ax^{2n} + c = bx^n$, $bx^n + c = ax^{2n}$, $ax^{2n+m} = bx^{n+m} + cx^m$.

Next, following Abū Kāmil in particular, al-Karajī studied systems of linear equation (Woepcke, 1853, pp. 90–100) and solved, for example, the system $x/2 + w = s/2$, $2y/3 + w = s/3$, $5z/6 + w = s/6$, where $s = x + y + z$ and $w = 1/3(x/2 + y/3 + z/6)$.

The translation of the first seven books of Diophantus' *Arithmetica* revealed to al-Karajī the importance of at least two fields. Yet, unlike Diophantus, he wished to elaborate the theoretical aspect of the fields under consideration. Therefore al-Karajī benefited from both a conception of algebra renewed by al-Khwārizmī and a more developed theory of algebraic computation, and he was able, through his reading of Diophantus, to state in a general form propositions still implicit in Diophantus and to add to them others not initially foreseen. In *al-Fakhrī*, as in *al-Badī*ᶜ, by indeterminate analysis (*istiqrā*ʾ) (Woepcke, 1853, p. 72; Anbouba, ed., 1964, p. 62 of the Arabic text) al-Karajī meant "to put forward a composite quantity [that is, a polynomial or algebraic expression] formed from one, two, or three successive terms, understood as a square but the formulation of which is nonsquare and the root of which one wishes to extract."[7] By the solution in Q of a polynomial with rational coefficients al-Karajī proposed to find the values of x in Q such that $P(x)$ will be the square of a rational number. In order to solve in this sense, for example, $A(x) = ax^{2n} + bx^{2n-1}$, where $n = 1, 2, 3, \ldots$, divide by x^{2n-2} to arrive at the form $ax^2 + bx$, which should be set equal to a square polynomial of which the monomial of maximum degree is ax^2, such that the equation has a rational root.

Al-Karajī noted that problems of this type have an infinite number of solutions and proposed to solve many of them, some of which were borrowed from Diophantus while others were of his own devising. An exhaustive enumeration of these problems cannot be given here. We shall only present the principal types of algebraic expressions of polynomials that can be set equal to a square (Woepcke, 1853; Anbouba, ed., 1964).

1. Equations with one unknown:
   $ax^2 = u^2$
   $ax^2 + bx = u^2$ and in general $ax^{2n} + bx^{2n-1} = u^2$
   $ax^2 + b = u^2$ and in general $ax^{2n} + bx^{2n-2} = u^2$
   $ax^2 + bx + c = u^2$ and in general $ax^{2n} + bx^{2n-1} + cx^{2n-2} = u^2$
   $ax^3 + bx^2 = u^2$ and in general $ax^{2n+1} + bx^{2n} = u^2$
   for $n = 1, 2, 3 \ldots$

2. Equations with two unknowns:
   $x^2 + y^2 = u^2$, $x^3 \pm y^3 = u^2$, $(x^2)^{2m} \pm (y^3)^{2m+1} = u^2$
   $(x^{2m+1})^{2m+1} - (y^{2m})^{2m} = u^2$.

3. Equation with three unknowns:
   $x^2 + y^2 + z^2 \pm (x + y + z) = u^2$.

4. Two equations with one unknown:
   $\begin{cases} a_1x + b_1 = u_1^2 \\ a_2x + b_2 = u_2^2 \end{cases}$ and in general $\begin{cases} a_1x^{2n+1} + b_1x^{2n} = u_1^2 \\ a_2x^{2n+1} + b_2x^{2n} = u_2^2 \end{cases}$
   $\begin{cases} a_1x^2 + b_1x + c_1 = u_1^2 \\ a_2x^2 + b_2x + c_2 = u_2^2. \end{cases}$

5. Two equations with two unknowns:
   $\begin{cases} x^2 + y = u^2 \\ x + y^2 = v^2 \end{cases}$ $\begin{cases} x^2 - y = u^2 \\ y^2 - x = v^2 \end{cases}$ $\begin{cases} x^3 + y^2 = u^2 \\ x^3 - y^2 = v^2 \end{cases}$
   $\begin{cases} x^2 - y^3 = u^2 \\ x^2 + y^3 = v^2 \end{cases}$ $\begin{cases} x^2 + y^2 = u^2 \\ x^2 + y^2 \pm (x + y) = v^2 \end{cases}$

$$\begin{cases} x + y + x^2 = u^2 \\ x + y + y^2 = v^2. \end{cases}$$

6. Two equations with three unknowns:

$$\begin{cases} x^2 + z = u^2 \\ y^2 + z = v^2. \end{cases}$$

7. Three equations with two unknowns:

$$\begin{cases} x^2 + y^2 = u^2 \\ x^2 + y = v^2 \\ x + y^2 = w^2. \end{cases}$$

8. Three equations with three unknowns:

$$\begin{cases} x^2 + y = u^2 \\ y^2 + z = v^2 \\ z^2 + x = w^2 \end{cases} \qquad \begin{cases} x^2 - y = u^2 \\ y^2 - z = v^2 \\ z^2 - x = w^2. \end{cases}$$

$$\begin{cases} (x + y + z) - x^2 = u^2 \\ (x + y + z) - y^2 = v^2 \\ (x + y + z) - z^2 = w^2. \end{cases}$$

In al-Karajī's work there are other variations on the number of equations and unknowns, as well as a study of algebraic expressions and of polynomials that may be set equal to a cube. From a comparison of the problems solved by al-Karajī and those of Diophantus, it was found that "more than a third of the problems of the first book of Diophantus, the problems of the second book starting with the eighth, and virtually all the problems of the third book were included by al-Karajī in his collection." (Woepcke, 1853, p. 21.) It would be noted that al-Karajī added new problems.

Two sorts of preoccupations become evident in al-Karajī's solutions: to find methods of ever greater generality and to increase the number of cases in which the conditions of the solution should be examined. Hence, for the equation $ax^2 + bx + c = u^2$ – although he supposes that its solution requires that a and c be positive squares – he considers the various possibilities: a is a square, b is a square, neither a nor b is a square in $ax^2 - b = u^2$ but b/a is a square. In addition he shows that $\pm(bx - c) - x^2 = u^2$ has no rational solution unless $b^2/4 \pm c$ is the sum of two squares (Woepcke, 1853, p. 8).

The same preoccupation appears in his solution of the system $x^2 + y = u^2$ and $y^2 + x = v^2$, where he seeks first to transform $x = at$ and $y = bt$, $a > b$, in order to posit $(a - b)t = \lambda$; $a^2t^2 + bt = u^2$; $b^2t^2 + at = v^2$, and to solve the problem by means of the demonstrated identity.

$$\frac{1}{4}\left[\left(\frac{u-v}{\lambda}+\lambda\right)^2 - \left(\frac{u-v}{\lambda}-\lambda\right)^2\right] = u - v.$$

A great many other examples could be cited to illustrate al-Karajī's indisputable concern with generality and with the study of solutions, as well as a considerable number of other mathematical investigations and results. His most important work, however, remains the new start he gave to algebra, an arithmetization elicited by the discovery of Diophantus by a mathematician already familiar with the algebra of al-Khwārizmī. This new impetus was understood perfectly and extended by al-Karajī's direct successors, notably al-Samawʾal. It is this tradition, as all the evidence indicates, of which Leonardo Fibonacci had some knowledge, as perhaps did Levi ben Gerson.[8]

NOTES

1. No claim for completeness is made for this table, because of the dispersion of the Arabic MSS and their insufficient classification.

Title	al-Karkhī	al-Karajī
al-Fakhrī	BN Paris 2495 Esat Efendi Istanbul 3157 Cairo Nat. Lib., 21	Köprülü Istanbul 950
al-Kāfī	Gotha 1474 Alexandria 1030	Topkapi Sarayi, Istanbul A. 3135 Damat, Istanbul 855 Sbath Cairo 111
al-Badīʿ		Barberini Rome 36, 1
ʿIlal ḥisāb al-jabr	Hüsnü Pasha, Istanbul 257	Bodleian Library I, 968, 3

Inbaṭ al-miyāh al-khafiyyat	Publ. Hyderabad, 1945, on the basis of the MSS. of the library of Aya Sofya and of the library of Bankipore.

2. One encounters the same difficulties when one considers the MSS of the later Arab commentators and scholars. Thus in the commentaries of al-Shahrazūrī (Damat 855) and of Ibn al-Shaqqāq (Topkapi Sarayi A 3135), both of which refer to *al-Kāfī*, one finds the name al-Karajī, whereas in MS Alexandria 1030 one finds al-Karkhī.
3. This MS was classified as anonymous until the present author identified it as being the *al-Bāhir* of al-Samawʾal. See Rashed (1971); and Rashed and Ahmad, eds. (1972).
4. In Arabic dictionaries, the "mountain countries" include the cities located between "Azerbaijan, Arab Irak, Khurasan, Persia and the country of Deilem (a land bordering the Caspian Sea)."
5. For example, for the first method, to find the root of $x^6 + 4x^5 + (4x^4 + 6x^3) + 12x^2 + 9$, one takes the roots of x^3 and 9; one then divides $4x^5$ by x^3 or $12x^2$ by 3, in both cases one obtains $4x^2$. The root sought is thus $(x^3 + 2x^2 + 3)$.

   For the second method, take $x^8 + 2x^6 + 11x^4 + 10x^2 + 25$. One finds the roots of x^8 and 25; x^4 and 5, then subtracts as indicated to obtain x^4, the root of which is x^2. The root sought is thus $(x^4 + x^2 + 5)$. See Woepcke (1853, p. 55) and Anbouba, ed. (1964, p. 50 of the Arabic text).
6. Woepcke (1853, p. 63) with translation improved by comparison with MSS of the Bibliothèque Nationale, Paris.
7. Woepcke (1853) with translation improved by comparison with MSS of the Bibliothèque Nationale, Paris.
8. See the comparison by Woepcke (1853) and Sarton (1927, p. 596).

3. THE NEW BEGINNINGS OF ALGEBRA IN THE ELEVENTH AND TWELFTH CENTURIES

The history of classical algebra is still sometimes related as the succession of three events: the constitution of the theory of quadratic equations, the more or less general solution of the cubic equation, the introduction and development of algebraic symbolism. The name of al-Khwārizmī is frequently associated with the first event, mathematicians of the Italian school, notably Tartaglia and Cardano with the second, and lastly, the names of Viète and Descartes with the third.

By the nineteenth century, Woepcke's works on al-Karajī and al-Khayyām, and more recently Paul Luckey's work on al-Kāshī, had shown that the above outline is incomplete not to say inaccurate. Woepcke's translation of al-Khayyām's algebraic work revealed in particular that real progress in the theory of cubic equations had been accomplished long before the sixteenth century. Both authors, by their work on al-Karajī and al-Kāshī even went so far as to hint that the history of algebra could not be drawn independently of that of abstract algebraic calculus.

Despite such studies, some historians still conceive the history of classical algebra according to the same outline. However, historians are not entirely responsible for this situation: it stems partly from the fact that al-Karajī, al-Khayyām, and especially al-Kāshī, might themselves appear difficult to integrate into real mathematical traditions. Until quite recently incomplete and partial information about Arabic mathematics has presented and still presents such works as individual compositions through lack of knowledge about the tradition from which they originate.

Under these circumstances, it is understandable that the historian is naturally tempted to pose the controversial question of origins, which is automatically transformed into a question of originality.

In this section we want to review briefly these mathematical traditions in order to maintain that classical algebra was renewed in the later tenth century, and that this renewal was not confined to a reactivation of received algebra, but was a real beginning, or strictly speaking, a series of beginnings.

Two mathematical traditions to which algebra is connected may be recognized. The first is arithmetic – a "scientific art" – as Arab mathematicians and bibliographers called it, number theory and the art of calculus – or logistics – both closely related. This development, the work of Arab arithmeticians themselves, was also the result of the translation of Diophantus' *Arithmetica*. To renew this discipline, al-Karajī and his successors were to make the most of both the development and knowledge of algebra as practised since al-Khwārizmī. The second tradition is associated with the works of certain geometers, especially those concerned with infinitesimal determinations and those seeking to advance algebra by geometry. Representative of this tradition, al-Khayyām and Sharaf al-Dīn al-Ṭūsī were led, as we shall see, to the algebraic study of curves: they laid down the foundations of classical algebraic geometry.

To justify such claims, this brief account only aims to answer the following question: what did these beginnings consist of? What were their means and probable causes?

I

If we want briefly to characterize the task for algebraists, at least those belonging to the first tradition, one may say that their aim was the "arithmetization of algebra" as constituted earlier by al-Khwārizmī, and developed later by his successors such as Abū Kāmil (850–930). As al-Samawʾal was to write later, it consists of "operating on unknown quantities using arithmetical instruments as the arithmetician operates on known quantities". The task is clear and algebra acquires the meaning which will henceforth be its own: on the one hand, the systematic application of elementary arithmetical operations to algebraic expressions – algebraic unknowns – and, on the other, to consider algebraic expressions independently of what they may represent in order to apply the general operations applied to numbers.

In al-Karajī's work (d. early 11th c.), carried on and perfected by his successors, the accomplishment of this project led, as we can see, one century later with al-Samawʾal (d. 1174), to the extension of abstract algebraic calculus and the organisation of the algebraic exposition around the successive applications of different arithmetical operations. To be convinced we only need to skim through al-Karajī's *al-Fakhrī* or al-Samawʾal's *al-Bāhir*. The major result of their treatises on algebra was to provide greater knowledge about the algebraic structure of real numbers. But as this result and other less important ones they achieved were often attributed to later mathematicians such as Chuquet, Stifel, etc., and as those results express precisely a change in algebraic rationality, let us review here what we set forth elsewhere and give a rapid description of these authors' approach and prove the affirmation just advanced.

In *al-Fakhrī*, al-Karajī begins by studying the "successive powers of the unknown quantity". After stating verbally, i.e., not using symbols, that $x^m = x^{m-1}x$ for $m = 1, 2, \ldots, 9$, he writes, "this is so on to infinity", and "when any one of these powers is multiplied by a certain number of roots, the product is the order of the next power". It may therefore be said that al-Karajī defines $x^n = x^{n-1}x$ for any positive integer n.

Al-Karajī then tries to extend the notion of the algebraic power of a quantity, a power defined as it were by mathematical induction, to its inverse, and gives some important results such as $(1/x^n).\ (1/x^m) = 1/x^{n+m}$. This generalization was to be refined and completed by his successors who, with the definitions of the zero power. $x^0 = 1$ for $x \neq 0$, were finally able to state a rule equivalent to $x^n x^m = x^{n+m}$ for $m, n \in \mathbb{Z}$.

Only once the concept of algebraic power had been generalized could the application of arithmetical operations to algebraic expressions be attempted. The immediate consequence of this application was one of the first expositions of the algebra of polynomials.

In *al-Fakhrī*, al-Karajī is not satisfied with examining addition, subtraction, multiplication, division and the extraction of roots of monomials, but also that of polynomials. However, if he clearly states the general rules for +/–, ×, for polynomials, he does not do so for the division and extraction of roots. He only considers the division of a polynomial by a monomial; and if he extracts square roots, he confines himself to that of a polynomial with positive rational coefficients.

Al-Karajī's difficulties can moreover be explained by his own concept of the status of negative numbers. Although he had written in *al-Fakhrī* "that negative quantities must be counted as terms", customary practice apparently condemned the recognition of negative numbers to remain timid. Though he had no difficulty in accepting the subtraction of one positive number from another, he did not directly admit that $x - (-y) = x + y$. Under these circumstances, the difficulty of providing general rules for the division and extraction of the square root of polynomials with rational coefficients is understood. In the twelfth century however, al-Karajī's successors were to state the general rules for signs:

$$x \leq 0, y \geq 0 \Rightarrow xy \leq 0 \tag{1}$$

$$x \leq 0, y \leq 0 \Rightarrow xy \geq 0 \tag{2}$$

$$x \leq 0, y \geq 0 \Rightarrow x - y \leq 0 \tag{3}$$

$$x \leq 0, y \leq 0, |x| \geq |y| \Rightarrow x - y \leq 0 \tag{4}$$

$$x \leq 0, y \leq 0, |x| \leq |y| \Rightarrow x - y \geq 0 \tag{5}$$

$$x \geq 0 \Rightarrow 0 - x \leq 0 \tag{6}$$

$$x \leq 0 \Rightarrow 0 - x \geq 0 \tag{7}$$

or, as al-Samawʾal writes.

the product of a negative number – *al-nāqiṣ* – by a positive number – *al-zāʾid* – is negative, and by a negative number is positive. If we subtract a negative number from a higher negative number, the remainder is their negative difference. The difference remains positive if we subtract a negative number from a lower negative number. If we subtract a negative number from a positive number, the remainder is their positive sum. If we subtract a positive number from an empty power (*martaba khāliyya*), the remainder is the same negative, and if we subtract a negative number from an empty power, the remainder is the same positive number.

Equipped with these rules, al-Karajī's successors were able to accomplish the task and put forward a theory for dividing polynomials and extracting the square root from a polynomial with rational coefficients. The method al-Samawʾal proposed is none other than an extension of Euclid's algorithm for the division of integers to expressions of the form

$$f = \sum_{k=-m}^{n} a_k x^k \quad m, n \in \mathbb{Z}_+.$$

To be precise, it does not quite concern ordinary division in the ring of polynomials $K[x]$, K being a field, but in a ring $A[x] = [Q(x) + Q(1/x)]$. Moreover, al-Samawʾal is not explicitly interested in the degree of the remainder. However, the results of the division are correct, because dividing f by

$$g = \sum_{k=-m'}^{n'} b_k x^k \quad m', n' \in \mathbb{Z}_+$$

is the same as dividing $x^\alpha f$ by $x^\alpha g$, $\alpha = \sup(m, m')$; we are then led to the problem of division in $K[x]$.

It should be noted that the division in the ring $A[x]$ will be continued at least until the seventeenth century. Sometimes, moreover, instead of the elements of the ring $A[x]$, al-Samawʾal considers polynomials in the strict sense: in which case he defines the method for division with a remainder. In any case – further confirmation of the sufficiently elaborated concept of his approach – he indicates the elements of division by tables – elements of the ring $A[x]$ or $K[x]$ – by a sequence of positive or negative coefficients.

A no less important part of this algebra is the approximation of integer fractions by the elements of $A[x]$. We have for example

$$\frac{f(x)}{g(x)} = \frac{20x^2 + 30x}{6x^2 + 12} \approx \frac{10}{3} + \frac{5}{x} - \frac{20}{3x^2} - \frac{10}{x^3} + \frac{40}{3x^4} + \frac{20}{x^5} - \frac{80}{3x^6} - \frac{40}{x^7}$$

where al-Samawʾal obtains in a way a limited expansion of $\phi(x) = f(x)/g(x)$. This approximation fits only for sufficiently large values of x, a condition which was not mentioned by the author.

As they were able to extend ordinary division to polynomials, our algebraists follow an analogous approach in order to extract the square root of a polynomial. Al-Karajī had already proposed two methods to obtain the square root of a polynomial with positive rational coefficients. The two methods are based upon the development

$$(x + y + \ldots + w)^2 \\ = x^2 + (2x + y)y + \ldots + (2x + 2y + \ldots + w)w.$$

Al-Karajī's method had been generalized in the book *al-Bāhir* where the author proceeds to obtain the square root of a polynomial with rational coefficients, or more precisely to find the root of a quadratic element of a ring $A[x]$. Hence to obtain the square root

$$B = 25x^6 - 30x^5 + 9x^4 - 40x^3 + 84x^2 - 116x + 64 \\ - \frac{48}{x} + \frac{100}{x^2} - \frac{96}{x^3} + \frac{64}{x^4}$$

by the table method, he writes

$$\begin{aligned} B &= 25x^6 + (10x^3 - 3x^2)(-3x^2) + (10x^3 - 6x^2 - 4)(-4) \\ &\quad + \left(10x^3 - 6x^2 - 8 + \frac{6}{x}\right)\frac{6}{x} \\ &\quad + \left(10x^3 - 6x^2 - 8 + \frac{12}{x} - \frac{8}{x^2}\right)\left(-\frac{8}{x^2}\right) \\ &= \left(5x^3 - 3x^2 - 4 + \frac{6}{x} - \frac{8}{x^2}\right)^2 . \end{aligned}$$

Al-Samawʾal gives this example to illustrate a general method (*ṭarīq ᶜāmm*).

With the extension of algebraic calculus to rational expressions, al-Karajī and his predecessors continued to work towards the same goal and wants to show, as he writes: "how to operate on algebraic irrational quantities using multiplication, division, addition, subtraction and root extraction". Al-Samawʾal poses the question in almost identical terms: "how can arithmetical instruments be used for irrational quantities (*al-maqādīr al-ṣumm*)?"

Besides the mathematical results proper obtained by this extension, a study of special importance for the history of mathematics was undertaken: it mainly concerns so to speak the algebraic interpretation of the theory contained in Book X of the *Elements*, hitherto considered by mathematicians in the wake of Pappus, even as important as Ibn al-Haytham, as a work on geometry. With our algebraists these concepts refer henceforth to numerical and geometric magnitudes in general, and through the intermediary of algebra the theory takes its place in the field of number theory.

Fortunately, without posing the problem of the existence of the field of real numbers, al-Karajī and his successors starts with definitions from Book X and immediately place themselves on a general plane. To provide himself with conditions for recognizing that expressions obtained by the combination of several radicals are irrational, al-Karajī proceeds like Euclid, with one exception however, for he extends the concepts of Book X to any algebraic quantity.

In *al-Badīᶜ*, he writes:

Monomials are infinite: the first one is absolutely rational like five, the second is potentially rational, like the root of ten, the third is defined in relation to its cube, like the side of twenty, the fourth is the medial defined in relation to its square-square like the root of the root of ten, the fifth is the side of a quadrato-cube, then the side of the cubo-cube and is divided in the same way into infinity.

Binomials, like monomials, are divisible into infinity. Following this explanation, the mathematicians then give general rules for different operations, in particular:

$$x^{1/n}y^{1/m} = (x^m y^n)^{1/mn}$$
$$x^{1/n}y^{1/m} = (x^m/y^n)^{1/mn}$$
$$x^{1/m} \pm y^{1/m} = [y[(x/y)^{1/m} \pm 1]^m]^{1/m}$$

and, like al-Samawʾal, reconsider numerous problems in Book X in order to provide algebraic solutions equivalent to those of Euclid, or new solutions.

It was therefore with this tradition that polynomial algebra was constructed and a clearer knowledge of the algebraic structure of real numbers attained. Let us note, moreover, a new return to number theory, which furnished this algebraic discipline with the instruments it lacked. This return is oriented: from now on preference is given to algebraic proofs. This is precisely where a form of demonstration by finite mathematical induction appears.

In a chapter of *al-Fakhrī* entitled "Useful chapters and theorems for solving problems by algebra", and another text by the author preserved by his successor al-Samawʾal, al-Karajī reconsiders some of the problems of number theory, such as the sum of n natural prime integers, their squares and cubes, the binomial formula . . . Though al-Karajī does not provide a real proof for some theorems, and though these theorems continue to be presented without proof by some arithmeticians, such as al-Baghdādī (d. in 1037) in *al-Takmila*, for example, on the other hand, in the twelfth century they will be proved algebraically. They include the following properties:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \tag{2}$$

both proved by a clumsy form of mathematical induction called "regression", as I have shown elsewhere.

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad n \in \mathbb{N} \tag{3}$$

$$(ab)^n = a^n b^n, \quad a \text{ and } b \text{ commute and } n \in \mathbb{N} \tag{4}$$

both proved by a certain form of mathematical induction still in use to a certain extent in the seventeenth century.

But al-Karajī and his successors did not limit their investigations to the areas of algebra just examined. Their work covered many others: the theory of biquadratic equations, indeterminate analysis, systems of linear equations. In this last area, al-Samawʾal solved a system of 210 equations to 10 unknowns.

Apart from all these results and new methods related to the arithmetization of algebra, let us note the emergence of a reflection on mathematics, a philosophy not of the philosopher but of the mathematician. Even if this reflection or philosophy was thematic and unsystematic and even if, in comparison with the famous metaphysical systems of the Middle Ages, it may appear summary in structure and weak in argument, at least it had the advantage of being generated by mathematical activity itself. This probably explains why this reflection was not mentioned in the history of medieval thought, absorbed by traditional philosophy, "*kalām*", and the traditionalist reaction to tendencies represented by Ibn Taymiyya or Ibn Ḥazm. Whatever the case may be, even if this reflection borrowed its themes from Pappus or eventually Proclus, the intervention of the new algebra is obvious; it provided the topics with new subject matter.

Algebra is in fact the starting point for a reflection on the status of this discipline, its relation to geometry, its methods, the classification of problems and propositions. On this point it should be remembered that al-Samawʾal, after explicitly identifying algebra and analysis, was subsequently to modify the position of this topic which will remain essential in mathematical philosophy for many centuries: analysis and synthesis. Moreover, he referred to a work entirely devoted to this problem which unfortunately has not been rediscovered. The importance of this identification in the seventeenth century is well known. Moreover, al-Samawʾal, in the logical terminology of his age, but covering another range, provides a classification of mathematical propositions as important as it is difficult to interpret. He groups the propositions as follows:

I. necessary,
II. possible,
III. impossible.

I. *Necessary propositions*

(a) First sub-class

(a-1) "Propositions" or "problems where what is sought is to be found in all numbers", in other words, identities:

example: if $z = x + y$ then $z/x + z/y = (z/x)\cdot(z/y)$.

(a-2) "what is sought is found in an infinity of numbers", in other words, a proposition which has an infinite number of solutions without being an identity.

example: $x + 10 = a^2$
$x - 10 = b^2$

(a-3) "whose solutions are numerous but finite [in number]", many indeterminate problems are used as examples.

(a-4) "which has one solution":

example: $xa = u^2$, $xb = u \Rightarrow u = a/b$.

(b) Second sub-class
The author classifies "necessary propositions" a second time according to the number conditions they must verify, i.e., one or more conditions.

(b-1) only one condition: example, given two numbers a and b, determine x and y; $x^2 + y^2 = a$, $xy = b$; as a necessary condition we find $a \geq 2b$.

(b-2) several conditions, example: a system of equations of n equations to m unknowns, $m \leq n$.

II. *Possible propositions*

These are propositions for which the proof of either their truth or falsity is not known, as al-Samawʾal writes:

For any proposition the algebraist or geometer considers, he will inevitably obtain proof of its existence (the existence of its solutions), and he will then call it necessary or (of its impossibility), he will call it impossible, or again he will find neither proof of its existence, non-existence or impossibility, therefore he knows nothing about it and will then call it possible as he was unable to demonstrate either its existence or its non-existence. Since this (if we demonstrate existence and non-existence) would then refer to the negation of what is and to the impossibility of the necessary which is absurd.

We now know why the author unfortunately gives no examples. He only recalls that possible problems should not be mistaken for indeterminate problems: the latter are in fact necessary.

III. *Impossible propositions*

These are propositions such that "if one could assume the existence (of their solution), this existence would lead to an absurdity".

The least one can say is that this reflection on mathematical practice, in particular the new algebra, led the mathematician to deflect the Aristotelian notions of necessary, possible and impossible, toward those of calculability and semantic undecidability. They are moreover related to the notion of the solvability of an equation, and more generally, its calculability.

When al-Samawʾal speaks about a necessary proposition A, he means proving A or non-A, while by a possible proposition A, he means that A is undecidable or that there is hardly any method either to prove or refute A.

It is now clear how new mathematical practice stimulated philosophical mathematical reflection. In our opinion, the historian of medieval Arabic philosophy would be wrong to ignore it.

II

We have just seen that the project of algebraist-arithmeticians was directly presented under the theme of extension: that of the domain of application of arithmetical operations. The results these mathematicians obtained are important not only for themselves, but also because they made a new beginning of algebra possible. This new beginning is no longer connected with arithmetic but related to geometry. At the outset the dominant theme is not so much extension as systematization: the organization of the examination of cubic equations and the elaboration of its theory. In order to understand the scope of his task, we must go back to the theory of cubic equations, and first of all, to al-Khayyām's own exposition (1048–1131).

In his algebra al-Khayyām writes:

In this science (algebra) one encounters problems dependent on certain types of extremely difficult preliminary theorems, whose solution was unsuccessful for most of those who attempted it. As for the Ancients, no work from them dealing with the subject has come down to us; perhaps after having looked for solutions and having examined them, they were unable to fathom their difficulties; or perhaps their investigations did not require such an examination; or finally, their works on this subject, if they existed, have not been translated into our language. Al-Māhānī was one of the modern authors who conceived the idea of solving the auxiliary theorem used by Archimedes in the fourth proposition of the second book of his treatise on the sphere and cylinder algebraically. However, he was led to an equation involving cubes, squares and numbers which he failed to solve after giving it lengthy meditation. Therefore, this solution was declared impossible until the appearance of Jaʿfar al-Khāzin who solved the equation with the help of conic sections.

Al-Khayyām continues:

After him (al-Khāzin), all geometers would have needed a number of species of the aforesaid theorems, one solving one, the other another. But none of them wrote anything about the enumeration of its species, nor gave any exposition of each species, nor their demonstration, other than in relation to two species which I shall not fail to point out. I, on the contrary, have never ceased to genuinely hope to make known these species with precision as well as distinguishing for each species, the possible from the impossible, based on demonstrations.

So, in this important text for the history of algebra, al-Khayyām affirms that:

(1) Nothing came down from the Greeks concerning the *theory* of cubic equations. Though Archimedes states a geometrical problem capable of being reduced to a cubic equation, neither he nor his commentators were able to formulate this problem algebraically. This task devolved upon al-Māhānī, and the solution must be attributed to al-Khāzin. But neither they, nor their predecessors or contemporaries, attempted to elaborate a real theory of cubic equations.

(2) We must distinguish not only between a geometrical problem capable of being reduced to a cubic equation and its algebraic translation, but also between the solution to either one of these problems and the elaboration of a theory of cubic equations.

The problem of the status of this theory becomes clearer: does al-Khayyām's appreciation of his own contribution correspond to real history, at least as far as it is known?

It is common knowledge that Greek mathematicians encountered problems concerning the duplication of the cube, the trisection of an angle, both third-degree problems. Furthermore, Arab mathematicians knew and fully discussed the auxiliary propositions used by Archimedes, whose proof however is missing in the *Treatise on the Sphere and Cylinder*. It is also known that it can be reduced to a cubic equation of the form $x^3 - cx + a^2b = 0$ which was solved by Eutocius, and subsequently by Arab mathematicians such as Ibn al-Haytham. The means for achieving the solution is the intersection of the parabola $x^2 = ay$ and the hyperbola $y(c - x) = ab$. At no time, however, before al-Māhānī, did mathematicians think of reducing this or any other problem, such as the duplication of the cube ($x^3 = 2$) to their algebraic expression.

It is significant that the tendency to translate third-degree problems algebraically was reinforced in the tenth century for two reasons at least: the obvious progress of the theory of second-degree equations and the requirements of astronomy. The progress of the given theory provided algebraists with a model for algebraic solutions – using roots – to which they wanted to conform for higher degree equations, especially cubic equations. Astronomy directly posed numerous third-degree problems. Al-Māhānī (d. 874–884?) was himself an astronomer. But it was al-Bīrūnī (973–1048) in particular who, in order to determine the chord of certain angles for the construction of the sines table, explicitly formulated the following cubic equations

$x^3 - 3x - 1 = 0$ where x is the chord of an 80° angle.
$x^3 - 3x + 1 = 0$ where x is the chord of a 20° angle.

Both equations were solved by trial and error.

However, the algebraic translations of third-degree problems by al-Māhānī, al-Bīrūnī and other mathematicians contemporary to the latter, such as Abū al-Jūd ibn al-Layth, posed a previously inconceivable problem: can these problems be reduced to cubic equations? Furthermore, is it possible to group all third-degree problems under the same heading if not attempt a solution just as elegant – using roots – as that of second-degree equations, at least provide systematic solutions? Both questions would have been inconceivable before the development of the theory of biquadratic equations and abstract algebraic calculation, i.e. before the first renewal of algebra by al-Karajī. Neither Greek nor Arabic mathematicians, prior to this renewal, had posed this question. This problem and al-Khayyām's approach for providing a solution will constitute another beginning of algebra.

Before undertaking their solution, al-Khayyām therefore starts by classifying first, second and third-degree equations. This study has sometimes been assimilated with a geometrical theory of cubic equations. However, if by geometrical theory one means using geometrical figures to determine real positive roots of these equations, the assimilation is doubtless improper, since the geometrical figure only fulfilled an auxiliary function in al-Khayyām's algebra, and especially in that of his successor Sharaf al-Dīn al-Ṭūsī (fl. 1170). So, if solutions to these equations are obtained by means of the intersection of two conics, in both cases the intersection is still demonstrated algebraically, i.e. using curve equations. For instance, in the works of al-Khayyām and particularly those of al-Ṭūsī the following examples – among many without giving their proof in detail – may be found:

– The method for solving $x^3 + ax = b$ implies solving both equations simultaneously:

$$\left(x - \frac{1}{2}\frac{b}{a}\right)^2 + y^2 = \left(\frac{1}{2}\frac{b}{a}\right)^2 \quad \text{(equation of the circle)}$$

and

$$x^2 = \sqrt{a}y \quad \text{(equation of the parabola)}$$

where $\sqrt{a}$ is double the parameter of the parabola and b/a is the diameter of the circle.

– This gives the equation: $x(x^3 + ax - b) = 0$. By eliminating the trivial solution, we obtain the equation sought.
– The method of solving $x^3 = ax + b$ implies solving two equations simultaneously

$$x^2 = \sqrt{a}y \text{ (equation of the parabola)}$$

$$x\left(\frac{b}{a} + x\right) = y^2 \quad \text{(equation of the equilateral hyperbola)}$$

where $\sqrt{a}$ is double the parameter of the hyperbola, and b/a the transversel diameter of the hyperbola. Whence $x(x^3 - ax - b) = 0$. If we eliminate the trivial solution, we obtain our equation.

We could multiply such examples to show that a history of algebraic geometry, as yet unwritten, cannot succeed without examining the contribution of the algebraic trend to this discipline.

Just as important as this study is al-Ṭūsī's grasp and expression of the importance of the discriminant for the discussion on cubic equations. Thus when considering the existence of positive roots of the equation $x^3 + a = bx$ (a, $b \geq 0$), he first notes that any positive solution to this equation must be inferior or equal to $b^{1/2}$; for, if x_0 is a root, one obtains:

$$x_0^3 + a = bx_0$$

hence $\quad x_0^3 \leq bx_0$

hence $\quad x_0^2 \leq b$

on the other hand this root must satisfy the equality $bx - x^3 = a$.

Al-Ṭūsī seeks the value whereby $y = bx - x^3$ reaches its maximum, and finds $x = (b/3)^{1/2}$ by cancelling the first derivative. Therefore this maximum is

$$b(b/3)^{1/2} - (b/3)^{3/2} = 2(b/3)^{3/2}.$$

There exists therefore a positive root if and only if

$$a \leq 2(b/3)^{3/2} \Leftrightarrow \frac{b^2}{27} - \frac{a^2}{4} \geq 0.$$

The role of the discriminant $D = b^3/27 - a^2/4$ is thus established and elaborated algebraically for the study of the cubic equation.

Already located, the role of the discriminant is not generalized however: the discriminant does not yet intervene in canonical solutions, i.e. by radicals. In order to solve this difficulty these same mathematicians developed a method for solving numerical solutions to which the method named after Viète and notably after Ruffini-Horner, is usually related, as I shall show later on.

It was known that al-Khayyām had found such a method for solving the equations $x^n = q$. It was also known that before al-Khayyām, al-Bīrūnī had dealt with the same problem. But only the title of al-Bīrūnī's treatise has survived, and we only possess a summary of al-Khayyām's work, which would have helped to understand that this method was based on the development $(a + b + \ldots k)^n$, $n \in \mathbb{N}$. Thanks to al-Ṭūsī's *Treatise on Equations*, we now know of the existence of such a method, not only for equations of the type $x^n = q$, but for general cases. This method, applied by al-Ṭūsī to all equations studied, may be briefly described as follows:

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x = N$$

and put

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x.$$

The function is derivable several times. The interval to which the root belongs may be recognized, namely $x \in [10^r, 10^{r+1}]$, x is therefore of the form $\rho_0\ 10r + \rho_1\ 10^{r-1} + \ldots + \rho_r$ and such that $r = [m/n]$ where m is the decimal order of N, $[m/n]$ is the integer part of m/n.

- We determine $x_1 = \rho_0 10^r$ either by division or by seeking the highest integer of nth power contained in N.
- We set $N_1 = N - f(x_1)$

and $x = x_1 + x_2$, $N_1 = g(x_2)$ where g is a polynomial of x_2 of degree $n - 1$.

We obtain as the value approached by x_2, x'_2 defined by

$$N_1 = nx_1^{n-1}x'_2 + a_1(n - 1)\ x_1^{n-2}x'_2 + \ldots + 2a_{n-2}x_1x'_2 + a_{n-1}x'_2. \quad (1)$$

We recognize here the derivative of f at point x_1 and

$$x'^2 = \frac{N_1}{f'(x_1)}.$$

We then operate by successive iterations.

Assume $x_1, x'_2, \ldots, x'_{k-1}$ determined where

$$x = x_1 + x'_2 + \ldots + x'_{k-1} + x_k \quad k = 2, \ldots, n.$$

A value approached by x_k:x'_k is given by the formula

$$x'_k = \frac{N_k}{f'(x_{k-1})} \quad (2)$$

where

$$N_k = N - f(x_1 + x'_2 + \ldots x'_{k-1})$$
$$x_{k-1} = x_1 + x'_2 + \ldots + x'_{k-1}.$$

Thus the value of x will be $x_1 + x'_2 + \ldots + x'_n$ where the x'_i are given by (2).

However, although al-Ṭūsī only applies this method to third or lower degree equations, the subject of his treatise, everything implies that he understood it in general terms. Moreover, al-Khayyām's summary had already set out the general nature of the problem.

A method for solving numerical equations, the study of curves by means of equations, the localisation of the role of the discriminant in the solution of cubic equations: these are the chapters of the renewed algebra. The ground covered since al-Khwārizmī's work is therefore measured not only in terms of the extension of the discipline, but by a change in the meaning of algebraic knowledge as well. If algebra asserts itself as the science of algebraic equations, and the latter not only relates to numbers and segments but also refers to plane curves, then algebra incorporates techniques existing in a tradition that actively participated in its renewal: the work of infinitesimalists. Among these techniques we may cite the use of affine transformations by an infinitesimalist like Ibrāhīm ibn Sinān.

So this is how al-Ṭūsī, by means of the affine transformation $x \rightarrow x + a$ or $x \rightarrow a - x$ reduces the equations to be solved to other equations whose solution he knows.

To solve these equations, al-Ṭūsī examines the maximum of algebraic expressions. He systematically takes the first derivative of these expressions – without naming it however – which he sets equal to zero, and proves that the root of the equation obtained, substituted in the algebraic expression, gives the maximum.

Once he has found one of the roots of the cubic equation, to determine the other root, he occasionally examines a second-degree equation which is none other than the quotient of the division of the cubic equation by $(x - r)$ where r is the root found. In other terms, he knows that the polynomial $ax^3 + bx^2 + cx + d$ is divisible by $(x - r)$, if r is a root of the equation $ax^3 + bx + cx + d = 0$.

Lastly, after examining the equation, he tries to determine a higher and lower boundary for its real roots.

If we insist on recalling these results, it is not only to present still unknown historical facts, but above all to show the theoretical and technical level of this algebra and the complexity of the historical problems it raises as soon as one ceases to record results in order merely to understand their history. The use of the derivative by these algebraists arose with the solution of algebraic and numerical equations. It is common knowledge, however, that the use of "the first derivative", related to the search for maxima, was not new. However, employed for one or other example, its usage remained sporadic and it was only with these algebraists and al-Ṭūsī in particular, that the notion of derivative would become an integral part of the solution of algebraic and numerical equations. The generalization of this usage will be obtained in fact by that of the theory of equations, then in the process of being elaborated on the one hand, and investigations by mathematicians whose activities extended to other fields, on the other.

The works of Banū Mūsā, Ibn Qurra, his grandson Ibrāhīm ibn Sinān, al-Qūhī, Ibn al-Haytham and many other non-algebraists, on infinitesimal determinations had indirectly prepared the attempts of our algebraists. By their refusal to interpret algebraic operations geometrically, already observed in Banū Mūsā and reaffirmed by their successors, by their discovery of new arithmetical laws necessary for calculating surfaces and volumes, they provided those algebraists with techniques tested by research on maxima. But the basic enumeration and classification of third-degree problems, necessary for elaborating the theory of equations with which algebra has already merged, the search for a method of solving cubic equations, had extended the field of application of the infinitesimalists' techniques, and notably the search for the first derivative. Evidently thanks to infinitesimalists, enlarged by algebraists, the notion of the "derivative" was condemned to discretion as a result of the weakness of algebraic symbolism. In our opinion, this explains its systematic usage, though devoid of either name or title.

III

Barely half a century ago, Paul Tannery wrote that Arabic algebra "in no way supercedes the level attained by Diophantus". One is undoubtedly surprised to read such an appraisal especially after Woepcke's works. But it expresses more the ideological standpoint of the historian than conclusions to his own historical work. Moreover, if in the case of Tannery this ideology is obvious, it is often less so in other historians like Zeuthen, or more recently, Bourbaki.

If I insist on recalling Tannery's assessment, it is less to right an injustice to the history of algebra than to point out a major difficulty for the sociological study of the history of science. For someone like Tannery, for example, this study could only be the answer to the question: what were the cultural conditions that prevented any progress in algebra, for it to remain in the state reached by the Ancients. Failing questions about the conditions for algebraic production, priority was given to its absence. However, the above summary clearly indicates that we are necessarily led to ask why and how algebra was renewed, not only in relation to the Ancients – assuming moreover they possessed algebra – but also in relation to the earliest Arab algebraists: al-Khwārizmī and Abū Kāmil.

The way in which questions are asked, determined by the historian's ideology, inevitably leads to contradictory answers. Evidently for this question, ideology is found again in the formulation of the answer. Let us suppose for a moment that the second question is basically the right one, then nothing precludes looking for the answer in another directions. For instance, starting from the incomparable development of algebra in relation to Hellenistic and medieval Latin mathematics, Roger Arnaldez and the late Louis Massignon thought that Arabic as a Semitic language "resulted in inflecting the knowledge it expressed towards thinking which was an analytical, atomistic, occasionalist and apophtegmatic". A recent study of "the semantic involution of concept" explains how Semitic languages tend towards abbreviated and abstract formulation, "algebrisant", in contrast to "Aryan geometrisation". According to these authors, linguistic structure is therefore responsible for the development of "a science of algebraic constructions". It is therefore clear that even if the right question were asked, nothing safeguards the answer from another ideology which, in the preceding example, goes back to Ernest Renan.

Questions about the historical reasons for algebraic production must firstly imply therefore a rejection of ideology on more than one level: the question itself and elements for an answer. But a necessary, clearly inadequate condition for ideological neutrality would be in the first place knowledge about the state of the discipline examined. For the historian of Arabic science this knowledge remains fragmentary and imperfect. This simple observation shows that we are far from achieving the aim of this discussion and it is premature at the present time to pose the question of the social conditions of scientific production.

Two other details confirm our admittedly negative attitude. In fact, concerning algebra examined here, an algebraic problem can only be posed in intrinsic terms, and the autonomy of algebra has already been sought and confirmed for the production of theorems and the invention of propositions. The role of philosophies and ideologies is relegated to a second, far off stage. This epistemological enclosure is characteristic of all advanced disciplines and before being set, the problems of conditions for production must both mediated and fragmented. Its mediation requires taking all disciplines into account – in this case arithmetic, trigonometry, observational astronomy . . . – to which the discipline is linked. Its fragmentation requires knowing the respective weight of cultural factors that may influence scientific production in one way or another. For lack of details, we are inevitably victims of either one of these illusions – transcendental or empirical. The former mistakes the means of posing the problem for the problem itself and, consequently the components of Durkheimian, Weberian or even Marxist doctrines for the explanation itself. Too often these are general considerations that do not define the facts one is supposed to explain. Empirical illusion gives the impression that the enumeration of cultural components is a satisfactory answer. However, both illusions continue to dominate explanations of scientific production.

Moreover, for the case which interests us here, they can only be reinforced due to the scarcity of scientific studies on the Muslim empire and its economic system or systems.

But are we justified in taking refuge in this negative position by refusing to examine problems raised by this discussion? A rigorous attitude would incline us towards such an abstention, undoubtedly justified, but leaving room for the vaguest of considerations. I think it is important to take a risk here by exploiting the only possibility that remains: to formulate plausible conjectures that do not pretend to substitute a real answer, and to indicate one or more hypotheses for research. We must therefore commit ourselves to mediating the question of the social determination of new algebra, and instead of taking it as a starting-point, return to the disciplines that were most active at its birth.

Two disciplines among all others have contributed to the constitution of the new algebra: arithmetic and the various branches of observational astronomy. The former intervened in the transformation of the ancient algebra, as we have seen, by transposing its operations to [new] algebra, once these operations were identified and systematized and also the generalization of the algebraic expression of some of its techniques such as Euclid's algorithm for the division and extraction of the square root. Astronomy, by its own requirements, compelled the algebraist to re-examine the problem of numerical equations and study curves by means of equations.

The question of the social determination of the new algebra becomes clearer and is posed in the first place in connection with various branches of astronomy and arithmetic. Only the question of arithmetic concerns us here.

If we return to the work of arithmeticians before the rise of this algebra, mostly algebraists themselves, a twofold preoccupation is frequently observed: to develop their discipline and provide it with a "domain of application". By "domain of application" we mean an area of examples, not necessarily related, where mathematical instruments are applied in order to rationalize empirical practises, and therefore to solve practical problems theoretically. We may therefore appreciate the scope of the mathematical instrument, regardless of the significance of the example chosen or the effectiveness of the solution reached.

Theoretical development and the application of arithmetic to rationalizing empirical practice are the two tasks often assigned by mathematicians to their arithmetical treatises. They make it possible to identify some directions for research.

The composition and extent of the Abbasid empire brought together and confronted several arithmetics. Two of them, digital and Indian arithmetic, posed both theoretical and practical problems for mathematicians. Solicited in particular by state administrations, mathematicians attempted to develop both of them with other mathematical knowledge, justify their rules, compare them more or less implicitly, to base and facilitate their usage, and provide the civil servant with a sort of *vade-mecum*. Sometimes, moreover, the same mathematician composed a treatise for each branch of arithmetic: al-Karajī, for example.

That arithmetic treatises were at least partially a response to administrative needs, has been stated by the authors themselves. In *On the needs for scientific arithmetical science for "kuttāb" (writers, secretaries, civil servants in administrative offices), "ᶜummāl" (prefects, tax collectors) and others*, al-Būzjānī presents his treatise as a work

> comprising all that an experienced or novice, subordinate or chief in arithmetic needs to know, the art of civil servants (*ṣināᶜat al-kitāba*), the employment of land taxes and all kinds of business needed in *dīwāns*, proportions, multiplication, division, measurements, land taxes, distribution, exchange and all other practices used by various categories of men for doing business and which are useful to them in their daily life.

The same preoccupation is evident in al-Karajī's treatise, *al-Kāfī*.

The same concern is found in Indian treatises on arithmetic but more simply expressed. Thus Ibn Labbān (c. 1000) wrote in conclusion to his work: "these foundations are sufficient for all astronomical computations and current practises between men." His pupil, al-Nasawī (c. 1030), who had started by composing a treatise on arithmetic in Persian for the Rayy administration, later provided an Arabic version "so that men may use it for their various affairs and astronomers in their art".

We could multiply examples borrowed from that generation of mathematicians, i.e. at the end of the ninth century. In this period of the Abbasid empire, we witness:

(1) the consolidation and development of the administrative institutions of the empire;

(2) the multiplication of small-scale replicas of these institutions in the provinces following the declining power of the caliphs;

(3) the rise of a social class, "*kuttāb*", or civil servants, related to the increase in administrations (*dīwāns*) and their small-scale replicas.

The autonomous existence of this social class, and its sociological impact had already astonished contemporary historians: al-Ṭabarī, al-Ṣūlī, al-Masᶜūdī and especially al-Jahshayārī, in his work, *Al-Wuzarāʾ wa al-kuttāb*, gives a detailed description of them. Moreover, it is known that the arabisation of the *dīwāns* had begun fairly early, between 700 and 705, depending on the provinces, as al-Jahshayārī and al-Kindī the historian recall.

At the end of the Ummayad empire, one of these civil servants, Hārūn ibn ᶜAbd al-Ḥamīd had already drawn the ideal portrait of his colleagues. According to a text preserved by al-Jahshayārī and related by Ibn Khaldūn, we know he is an educated man with some knowledge of arithmetic. Besides moral and social qualities, he must have some knowledge of Arabic, history, arithmetic and religious science necessary for his work. This is what Adam Metz meant when he wrote that the civil servant "is the representative of literary culture and only deals with religious science according to the requirements of his work and intellectual level". He added: "It is this class of civil servants that most distinguishes the Muslim state from Europe in the early Middle Ages". Consequently, to educate of its members, this social class encouraged the production of treatises not only on arithmetic, but also on economics and geography, like Qudāma ibn Jaᶜfar's famous work on land tax, and dictionaries of the philosophical, economic, and scientific terminology of the age, such as al-Khwārizmī's *Mafātīḥ al-ᶜulūm*. To characterize this class, we could not improve on what Claude Cahen wrote:

Bureaucracy, i.e. a regime dominated by an army of specialized scribes who had become a sort of caste that survived the passing of caliphs and viziers; red tape, i.e. a regime where everything that could be recorded was inscribed in detail, according to technical and stylistic rules known only to them, to ensure a monopoly over the professions.

Financial, military, intelligence, correspondence (chancellery) and many other *dīwāns*, all shared a need for financial accountancy and reliable, accessible treatises on arithmetic for their daily activities.

However, what we agreed to describe as a "domain of application" for arithmetic is exactly constituted by the problems posed to *dīwān* civil servants. So chapters 4 and 5 of Abū al-Wafā᾿ al-Būzjānī are concerned with financial problems as such, while chapter 6 deals with matters such as the payment of soldiers, the granting or withholding of permits for commercial ships sailing on the river, merchants on the highways, the dispatch of correspondence and couriers, and all other business conducted by *dīwāns*.

The confrontation of both arithmetics makes it clear from the outset that easy, rapid handling have become preferential criteria. It was in fact to emphasize the importance of Indian arithmetic that al-Uqlīdisī advanced its practical value and wrote:

Most arithmeticians are obliged to use it in their work: since it is easy and immediate, requires little memorisation, provides quick answers, demands little thought . . . Therefore, we say it is a science and practice that requires a tool, such as a writer, an artisan, a knight need to conduct their affairs; since if the artisan has difficulty in finding what he needs for his trade, he will never succeed; to grasp it there is no difficulty, impossibility or preparation.

It was therefore apparently to fulfill two new requirements, and in conformity with new standards, that the mathematician went back to digital or Indian arithmetic, whose rules he set out to justify, and then to organise their exposition. This return, an implicit confrontation of arithmetics, brought out much more clearly than before the general and abstract nature of the concept of operation. Viewed in this way and partially systematized, operations would from now on be means of organizing arithmetic statements. The existence of several types of arithmetic resulted in relativizing systems of numeration in order to show that what is essential is the choice of a base and the operations to be applied. Al-Uqlīdisī had not hesitated to declare earlier: "It is possible to replace the nine figures (*ḥurūf*) by other figures, either *abjad*, Roman or Arabic figures", an idea so widespread that al-Khwārizmī the Writer could say in his cited work: "We shall write using these numbers (*abjad*) like Indian arithmeticians, which means using 9 figures from *alif* to *ṭa*ʾ, and this sign (٥) will be set in the empty squares in place of the zero in Indian computation".

In other words, once a base has been chosen, the figure used in Indian arithmetic can be replaced by any other numeration system. But given these conditions, operations no longer correspond to a particular notation of a system of numeration. Al-Karajī drew a sufficiently general distinction between two classes of data: rational and irrational quantities, the operations of multiplication, division, raising to powers, addition and subtraction.

It was precisely these operations that made it possible to systematically organize the exposition of the beginning of Indian arithmetic, and if they fulfilled a similar role for digital arithmetic, it was just as systematic though less complete. For instance, their exposition by al-Uqlīdisī, Ibn Labbān, and al-Nasawī is organized by the operations +/–, ×/÷ and the extraction of roots, while digital arithmetic is essentially organized by ×/÷, and only sometimes the extraction of roots, the composition law +/– being known.

Conceived in a more general and abstract way than in the past, with a direction for organization of treatises, the operations were available for other applications, and this is how they appeared to those who intended to extend algebraic calculation. They were also able to generalize results obtained by applying these operations to arithmetical operations to algebra. It was up to al-Karajī and his successors, al-Shahrazūrī and al-Samawʾal, accomplish this task.

DISCUSSION

R. Rashed: One of the topics of this symposium is the problem of the relation between science and society in the history of scientific thought. As I am particularly interested in algebra, I must first of all describe the state of this discipline as accurately as possible. Questions that will be asked about the relation between science and society are themselves determined by the knowledge the historian has of this science. The difficulty is even greater when it concerns mathematics in general and algebra in particular. I would like to say that algebra is both a privileged and a constraining field. Privileged in the sense that if a relation between science and society exists, it may be determined by what I call an "epistemological enclosure" for the production of mathematics. By "epistemological enclosure", I simply mean that on a certain level, a certain stage of scientific development, an algebraic theorem is produced, and only produced by a pre-existing series of other theorems; there are no external reasons. This enclosure makes it possible to bring out the relation science-society, and make it more apparent than in other disciplines which lack the same conceptual command. But this epistemological enclosure is justifiably quite constraining in the sense that if there exists a relation between science and society, or algebra and society, the number of intermediary disciplines must be multiplied in order to see on what level and how this relation occurs. I am going to show that it is quite impossible to examine the relation between algebra and society (or social conditions) without referring to arithmetic and astronomy, i.e. without enumerating the various branches of arithmetic and astronomy.

J. Gagné: You just said that the epistemological enclosure made the relation you study apparent. That is what I would like to clarify. You said it makes the relation more apparent. I wonder if, on the contrary, it doesn't make it less apparent.

R. Rashed: I prefer to use the words "privileged and constraining". For example, if algebra developed from the solution of practical problems, on this elementary level we can see immediately how practical reasons and problems intervene. For example, if algebra developed from the determining or sharing of inheritance, it must be integrated into an economic system which may be determined. One can see the relation directly. Inheritance distribution may use algebra, but algebra as it develops – and this is what I am trying to prove – does not need inheritance, which amounts to saying that theorems are not produced or invented for extrinsic reasons.

S. Victor: Once algebra becomes a science, it continues as an independent science, there I entirely agree with you. When these reasons are invoked to say there was *never* any relation between the inheritance distribution and the beginning of algebra, I do not agree, because a relation did exist at the beginning of algebra.

R. Rashed: That was perhaps valid for algebra in the early beginning with al-Khwārizmī, Abū Kāmil etc., but no longer so in the eleventh and twelfth centuries.

G. Beaujouan: It is a problem of getting under way; once a science has taken off, it continues to exist with its internal logic and is much less dependent on external stimulus than at the beginning.

R. Rashed: I didn't say there was an epistemological enclosure with al-Khwārizmī or Abū Kāmil. I was only talking about the eleventh and twelfth centuries.
J. Murdoch: But you have only excluded one kind of social relation, if you want to call it that. That is, you have excluded the influence of something exterior, or social on the invention, discovery, or production of a given theorem. But this ignores, it seems to me, a much more frequent kind of thing and that is, once a theorem is discovered, once it is established, what are the social factors acting upon its utilization, its application?
R. Rashed: I agree with the problem of applications. But if we go back to the constitution of algebra itself, we can see how social factors intervened, that is not in algebra as such, but by means of arithmetic, astronomy and disciplines which are not algebra.
J. Murdoch: But will we get a "third man" argument here? You say that for the application of algebra to external things you have to proceed by the intermediary of arithmetic. Well, let's take arithmetic. For the application of arithmetic – forget algebra for the moment – do you need another intermediary to apply arithmetic, and if so, why, and if not, why not?
R. Rashed: That depends on the state of arithmetic. Which is why I said we have to know what sort of arithmetic we mean. For the moment, I am simply trying to point out the difficulties concerning algebra. What is meant by "privileged" and "constraining" conditions for these disciplines that enables us to pose the problem of the relations between science and society? "Privileged" in the sense that where there is a relation, it will be more determined, more apparent than for example the relation between science and society for metaphysics, or physics, in the Middle Ages, where a body of ideologies may intervene. This is not the case for algebra. We have a discipline that is already neutral in relation to those ideologies. Therefore we can study the relation between science and society directly. But, on the other hand, we are bound by the level of this discipline; by the very fact that it is already scientific, our hands are rather tied for considering the intervention of social factors in its formation.
J. Murdoch: Yet you would claim that our hands are less tied when it comes to the consideration of the impact of social factors upon arithmetic.
E. Sylla: Isn't it just an historical fact that when you look in algebraic

works you don't see social connections, but when you look in arithmetic works you do, they tell you about applications?
R. Rashed: That is an historical fact, but there is something more than this fact. You have at least three systems of arithmetic – Indian, digital, and sexagesimal – and the question arises why, at a certain moment, did they try to unify arithmetic. What does it mean, unify arithmetic? And how did they do it? And what compelled them to do it? The conjecture I made in answer to these questions is very simple: it is the existence of a new social class, the class of scribes as a social organization. They requested this unified kind of arithmetic; they needed this kind of unification for their calculations. The development of arithmetic was produced by this kind of social need as can be proved by the books written by mathematicians like Abū al-Wafāʾ, al-Karajī, al-Shahrazūrī, al-Samawʾal, etc., especially for this new class, and by the kind of problems dealt with in these books.
A. Sabra: But you have to show this in a much more concrete way. That is to say, you have to show what kind of problems these people, these scribes, were working on and how and why they required this kind of unified arithmetic. Furthermore, I wonder whether the social factors proposed enable us to go beyond considerations that are at best vague.
J. Murdoch: Yes, but Roshdi has made it less vague in the sense that he has said that a socially catalyzed cause has conditioned developments in arithmetic which were in turn only a necessary condition for the development of algebra and that the development of algebra came out of this existing necessary condition in a totally internal way.
A. Sabra: Yes, all right, but what we have been discussing is algebra, not arithmetic. And what has come out of the discussion of algebra, and out of what Roshdi himself is saying, is that in the period he has been working on the development of algebra is an internal one. That makes sense. But one wonders about what happened in the period between al-Khwārizmī and Abū Kāmil. You don't talk about that. One doesn't know much about it and this makes treating it a bit difficult.
R. Rashed: Yes, it is difficult, just as all questions about origins are difficult. You cannot answer them. I even think it mistaken to ask them now. One gets at best anecdotal history.
A. Sabra: I don't think that looking for origins necessarily leads to a history which is just discoveries and anecdotes. In fact, I don't see how an historian of mathematics can get away from this question of origins.

I don't think you can ignore it. After all, it gives you a research program. One might see, for example, a certain similarity between a theorem that is found in one of those authors like al-Kāshī and something in China. Now of course it would be wrong to go on just from there and say: "See, similar, therefore this comes from that." No, this is not very interesting. But if this leads you to ask whether it is possible that such a transmission could have taken place, then it becomes fruitful. What you will then do with it as an historian is still a problem; I am not saying that history *ends* there.

R. Rashed: We must nevertheless foresee the danger of the question of origins, if it can be solved, being transformed into a question of originality.

G. Beaujouan: If it's about originality, we fall back on the issue of predecessors.

A. Sabra: That is why I was trying to protect myself by saying that what you do with the question of origins afterwards is something else. That is to say, the cluster of questions that are somehow involved in this concept of originality, which is not a clear concept, still remain. Take the work of someone like E. S. Kennedy. He is interested in questions of transmission. Yet Kennedy somewhere says in one of his articles that once you have a theorem, you are faced with something that has an intrinsic value. I was very much moved by that statement because it is true. The important thing for the historian is the intrinsic value that somehow resides in a new theorem, a new discovery. But I don't think that this releases the historian from further research. Because by bringing in questions of origins, the problem of originality and intrinsic value becomes more complicated. It also becomes more interesting and historically richer. Once you throw this away, it seems to me that eventually you will end up by doing, not history, but something like the philosophy of science. Thus, for the time being, I am saying that the questions of origins and originality still remain.

R. Rashed: But even the problem of origins includes at least two questions: firstly, a question of attitude, i.e. to pose the problem of origins without transforming it into one of originality. But there is also another issue, which is not to confuse or mistake historical genesis with the logical structure of the theory under consideration. Both issues are often mistaken for one another. And that is what makes it possible to find algebra in Euclid, information theory in Aristotle, and so on. So it's a question of decision and strategy. But that also depends on what history of science we mean and our knowledge of this history. For example, in the history of Arabic science, we don't even know who invented what. When Luckey and others after him wrote about al-Kāshī for example, they didn't know that al-Samawʾal and al-Ṭūsī possessed a large number of inventions attributed to al-Kāshī. Luckey and his successors interested in the question of origins went back to China to find them. That was as much a logical as an historical error. That's what the question of origins leads to, at least at the present time.

A. Sabra: What you are doing now is working in the research program of the historian filling in the gaps.

R. Rashed: I only said what the necessary conditions for useful work on origins are. These conditions may involve refusing easy solutions like continuity. It should be remembered that historical continuity is not necessarily logical continuity. To put a work in its historical context implies first of all analyzing it, grasping its logical structure. For example, to study al-Karajī's text as an algebraist, without understanding his essential contribution, and immediately to undertake work on origins, as is often the case, is to lose the essential, to lose al-Karajī's contribution. The search for the origins of his algebra, necessarily implies going back to al-Khwārizmī, Abū Kāmil. Even assuming we know all of al-Karajī's predecessors, we won't be able to understand if we stop his essential task there, i.e. a new departure for algebra by what I called the arithmetization of algebra. Perhaps we will succeed in finding origins when we separate historical genesis and logical structure; but then the question of origins will have been completely transformed.

A. Sabra: What you are saying is really not against this program; you are just saying that if you have to do it you should, of course, do it well.

J. Murdoch: One might say that your unhappiness with the history of mathematics has to do with the way it is usually done. That is, if one asks what the entities are between which we are trying to fill the gaps, in most histories of mathematics – Cantor, Tropfke, for example – what they're doing is concentrating on results or on theorems or on particular kinds of examples. That is what is traced. It is extremely difficult to find someone tracing within algebra, let us say, the use of false position; not just when it occurs, but why? Or who used the theory of proportion? Where? Why? That is, tracing methods, conceptions, apart from results. Now that kind of thing, it seems to me, is incredibly more productive.

4. MATHEMATICAL INDUCTION: AL-KARAJĪ AND AL-SAMAWʾAL

I

Since 1909, the history of mathematical induction has, on several occasions, been revised and rewritten. Perhaps, never to happen again in the history of science, it all began with a very brief paper; in three pages of the *Bulletin of the American Mathematical Society*, Georges Vacca[1] shattered an assertion almost unanimously accepted by historians: mathematical induction is the achievement of the seventeenth century and priority must be attributed to Pascal. But Maurolico, not Pascal, was the originator, "the first discoverer of the principle of mathematical induction". The first formulation of the principle of mathematical induction is not to be found in the *Traité du triangle arithmétique* either, nor independently, as has been said, in the works of Jacques Bernoulli (Cajori [1918], pp. 197ff.), but was expressed for the first time in the work of a sixteenth-century mathematician Maurolico. Fascinated by Vacca's discovery, some historians and not the least important, i.e. Cantor, Günther, and Bourbaki, integrated the newcomer, Maurolico, without further examination.

Apart from reservations that may be expressed concerning the intrinsic value of Vacca's article, it must at least be recognized that indirectly he challenged historians' certainty, and posed afresh both the problem of the history of the principle of mathematical induction and how to write it.

The most scholarly answer to this problem occurred forty-four years later in a criticism of Vacca. After detailed examination of Maurolico's work, Hans Freudenthal (1953) showed that there were at most three places where a clumsy form of mathematical induction could be observed, but Pascal was the first to give an abstract formulation of the principle of this induction. Though concerned with rehabilitating Pascal, Freudenthal nevertheless qualified his thesis: Maurolico had clearly recognized an archaic form of mathematical induction and Pascal, like so any others, had started from this form before going further to grasp the principle of mathematical induction in its abstract form.

Since Freudenthal's study, two other historians at least, who moreover referred to it, have taken up the problem afresh. Kokiti Hara (1962), a "Pascalian", disregarded Freudenthal's reservations, and made Pascal the absolute beginner, so to speak, in the history of mathematical induction. The other, Nahum L. Rabinovitch (1970) went explicitly back to Levi ben Gerson to show that he was "the earliest writer known to have used induction systematically in all generality and to have recognized it as a distinct mathematical procedure".

These investigations confirm that the problem posed in 1909 did indeed shift, only to be posed anew in identical terms.

We shall present unpublished documents that will complicate matters even further, and show that more elaborate attempts, not only before Maurolico but before Levi ben Gerson as well, are to be found in two mathematicians: one whose work is already known to historians – al-Karajī,[2] and the other whose importance has only recently come to light – al-Samawʾal.[3] But perhaps its very complexity makes the problem of the history of mathematical induction capable of receiving a precise answer. Here we may in fact ask a question forgotten with respect to Maurolico and Levi ben Gerson: why did al-Karajī and al-Samawʾal resort to new methods of proof? Only in so far as we can answer this question can we also hope to set forth the absence or presence of mathematical induction. Without this question, the history of the problem is identical to that of the rare text. After all, a historian as informed and experienced as Jean Itard[4] had already discovered mathematical induction in Euclid, while Freudenthal, no less informed and experienced, had traced these various attempts back to the prehistory of the concept. But as the history of science is not empirical archaeology, it must learn not only to identify a text, but also the language and style in which it is expressed. As a first step, let us start by establishing the text.

II

The first formulation of the binomial and the table of binomial coefficients, to our knowledge, is to be found in a text by al-Karajī, cited by al-Samawʾal in *al-Bāhir*. We note the existence of a type of proof we shall call R_1, whose successive stages we shall establish.

The author starts out by proving some propositions concerning the commutativity and associativity of multiplication and the distributivity of multiplication in relation to addition.

PROPOSITION 1. *Given any four numbers, the product of the multiplication of the first and the second by the multiplication of the third and the fourth, is equal to the multiplication of the first by the third and the multiplication of the second by the fourth.*[5]

$$\Leftrightarrow [(ab)(cd) = (ac)(bd)].$$

Proof. Given four numbers a, b, c, d. Multiply a by b to obtain e, a by c to obtain f, c by d to obtain g and b by d to obtain h. I say that the product eg is equal to fh. In fact, if we multiply b and c by a, we have two numbers e and f, and consequently, the ratio of e to f is the same as b to c. However, b to c is in the same ratio as h to g. Therefore e to f is in the same ratio as h to g. Consequently the product of e by g is equal to the product of f by h. Q.E.D.[6]

LEMMA. *Given three numbers a, b, c, we have $(ab)c = (ac)b$.*

The author recalls moreover the distributivity of multiplication in relation to addition.

PROPOSITION 2. *The product of the number $\overline{AB}$ ($\overline{AB} = \overline{AC} + \overline{CB}$), as Euclid showed,* wrote al-Samawʾal *(in Book 2 Figure 1) by any number is equal to the product of $\overline{AC}$ by this number, plus the product of $\overline{CB}$ by this same number.*[7]

$$\Leftrightarrow [(a + b)\lambda = (a)\lambda + (b)\lambda].$$

With the aid of these and other propositions relating to addition and multiplication, al-Samawʾal proposes to prove the two following expressions:

$$(a + b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m, \qquad n \in \mathbb{N} \tag{1}$$

$$(ab)^n = a^n b^n, \qquad n \in \mathbb{N} \tag{2}$$

To prove the first identity, al-Samawʾal assumes the reader to be familiar with the development $(a + b)^2$ given in al-Karajī's *al-Badīʿ*, and cited by the author in an earlier chapter. He proposes to prove the identity for $n = 3$. His proof consists of the following stages:

[1].

1.1. $(a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b) = (a + b)^3$
by the development of $(a + b)^2$

1.2. $(a + b)^3 = a^2(a + b) + (2ab)(a + b) + b^2(a + b)$
by Proposition 2

1.3. $= a^3 + a^2b + 2a^2b + 2ab^2 + b^2a + b^3$ by Propositions 1 and 2

1.4. $= a^3 + b^3 + 3a^2b + 3ab^2$ by grouping the terms.

[2]. He proves the identity in the same way for $n = 4$ using the development $(a + b)^3$. Here follows a full translation of the proof.

Any number divided into two parts, its square-square is equal to the square-square of each part, four times the product of each by the cube of the other, six times the product of the squares of each part.

EXAMPLE. Let the number $\overline{AB}$ be divided into two parts, $\overline{AC}$ and $\overline{CB}$. I say that the square-square of $\overline{AB}$ is equal to the sum of the square-square of $\overline{AC}$, the square-square of $\overline{CB}$, four times the product of $\overline{AC}$ by the cube of $\overline{CB}$, four times the product of $\overline{CB}$ by the cube of $\overline{AC}$, and six times the product of the square of $\overline{AC}$ by that of $\overline{CB}$.

Proof. The square-square of $\overline{AB}$ is the product of $\overline{AB}$ by its cube. But we showed in the preceding figures that the cube of $\overline{AB}$ is equal to the sum of the cube of $\overline{AC}$, of the cube of $\overline{CB}$, of three times the product of $\overline{AC}$ by the square of $\overline{CB}$, and three times the product of $\overline{CB}$ by the square of $\overline{AC}$.

But the product of $\overline{AB}$ by any number is equal to the sum of the product of this number by $\overline{AC}$ and its product by $\overline{CB}$.

Therefore we have, equal to the square-square of $\overline{AB}$, the sum of the product of the cube of $\overline{AC}$ by $\overline{AC}$ – the square-square of $\overline{AC}$ – and by $\overline{CB}$; the product of the cube of $\overline{CB}$ by $\overline{CB}$ – the square-square of $\overline{CB}$ – and by $\overline{AC}$; three times the product of the square of $\overline{CB}$ by $\overline{AC}$ multiplied by $\overline{AC}$ and by $\overline{CB}$, and three times the product of the square of $\overline{AC}$ by $\overline{CB}$ multiplied by $\overline{AC}$ and $\overline{CB}$. But three times the product of the square of $\overline{AC}$ by $\overline{CB}$ multiplied by $\overline{AC}$ is equal to three times the product of the cube of $\overline{AC}$ by $\overline{CB}$. Similarly, three times the product of the square of $\overline{AC}$ by $\overline{CB}$ multiplied by $\overline{CB}$ is equal to three times the product of the square of $\overline{BC}$ by the square of $\overline{CB}$. Similarly, three times the product of the square of $\overline{CB}$ by $\overline{AC}$ multiplied by $\overline{AC}$ is equal to three times the product of the square of $\overline{AC}$ by the square of $\overline{CB}$. Similarly, three times the product of the square of $\overline{CB}$ by $\overline{AC}$ multiplied by $\overline{CB}$, is equal to three times the product of the cube of $\overline{CB}$ by $\overline{AC}$. Therefore the square-square of $\overline{AB}$ is equal to the square-square of $\overline{AC}$, the square-square of $\overline{CB}$, four times the product of $\overline{AC}$ by the cube of $\overline{CB}$; four times the product of $\overline{CB}$ by the cube of $\overline{AC}$ and six times the product of the square of $\overline{AC}$ by the square of $\overline{CB}$. Q.E.D.[8]

[3]. For $n = 5$, he gives no proof but writes,

he who has understood what we have just said, can prove that for any number divided into two parts, its quadrato-cube is equal to the sum of the quadrato-cube of each of its parts, five times the product of each of its parts by the square-square of the other and ten times the product of the square of each of them by the cube of the other. *And so on in ascending order.*[9]

[4]. At this point, he gives the table of binomial coefficients from

al-Karajī's work[10] as a means of finding "the number in the development of squares and cubes up to the limit one wants". The table of coefficients is given according to the following model:

$$
\begin{array}{ccccc}
n=1 & n=2 & \dots & n-1=11 & n=12 \\
1 & 1 & & 1 & 1 \\
1 & 2 & & C^1_{n-1} & C^1_n \\
 & 1 & & C^2_{n-1} & C^2_n \\
 & & & \vdots & \vdots \\
 & & & \boxed{\begin{matrix} C^{m-1}_{n-1} \\ C^m_{n-1} \end{matrix}} & C^m_n \\
 & & & \vdots & \vdots \\
 & & & 1 & C^{n-1}_n \\
 & & & & 1
\end{array}
$$

on the other hand, the calculation of C_n^m assumes the binomial coefficient of index $(n - 1)$ known. In fact, the formation rule given by al-Karajī is equivalent to

$$C_n^m = C_{n-1}^{m-1} + C_{n-1}^m.$$

This method is, moreover, stated for integer n as large as desired.[11] To use al-Samawʾal's own words:

Let us now recall a principle for knowing the necessary number of multiplications of these degrees by each other, for any number divided into two parts. Al-Karajī said that in order to succeed, we must place 'one' on a table and 'one' below the first 'one', move the [first] 'one' into another column, add the [first] 'one' to the one below it. We thus obtain 'two', which we put below the [transferred] 'one', and we place the [second] 'one' below two. We have therefore 'one', 'two', and 'one'. This shows that for every number composed of two numbers, if we multiply each of them by itself once – since the two extremes are 'one' and 'one' – and if we multiply each one by the other twice – since the intermediary term is two – we obtain the square of this number. If we then transfer the 'one' in the second column into another column, then add 'one' [from the second column] to 'two' [below it]. We obtain 'three' to be written under the 'one' [in the third column], if we then add 'two' [from the second column] to 'one' below it, we have 'three' which is written under 'three', then we write 'one' under this 'three'; we thus obtain a third column whose numbers are 'one', 'three', 'three' and 'one'. This teaches us that the cube of any number composed of two numbers is given by the sum of the cube of each of them and three times the product of each of them by the square of the other. If we transfer again 'one' from the third column to another column, and if we add 'one' [from the third column] to 'three' below it, we have 'four' which is written under 'one'; if we then add 'three' to the 'three' below it, we obtain 6 which is written under 'four'; if we then add the second 'three' to the 'one' below it, we have 'four' which is written under 'six', then we place 'one' below 'four'; the result is another column whose numbers are 'one', 'four', $\overline{6}$, 'four' and 'one'. This teaches us that the formation of the square-square of a number consisting of two numbers is given by the square-square of each of them – since we have 'one' at each end – then by four times the product of each number by the cube of the other – since 'four' follows 'one' at either end – since the root multiplied by the cube is the square-square, and lastly by six times the product of the square of each by the square of the other – since the product of the square by the square is a square-square. Then if we transfer 'one' from the fourth column into the fifth column, and we add 'one' to 'four' below it, 'four' to 'six', 'six' to 'four' and 'four' to 'one', then we write down the results under the transferred 'one' in the foresaid manner, and lastly write down last the remaining 'one', we have a fifth column whose numbers are 'one', $\overline{5}$, 'ten', 'ten', $\overline{5}$ and 'one'. This teaches us that for any number divided into two parts, its quadrato-cube is equal to the quadrato-cube of each part – since both ends have 'one' and one – to five times the product of each one by the square-square of the other – since 'five' succeeds at both ends on either side, and six times the product of the square of each one by the cube of the other – since 'ten' succeeds each five. Each of these terms belongs to the kind of quadrato-cube since the product of the root by the square-square and that of the cube by the square, both give the quadrato-cube; we can thus find the numbers of squares and cubes to the power desired.[12]

x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12
	1	3	6	10	15	21	28	36	45	55	66
		1	4	10	20	35	56	84	120	165	220
			1	5	15	35	70	126	210	330	495
				1	6	21	56	126	252	462	792
					1	7	28	84	210	462	924
						1	8	36	120	330	792
							1	9	45	165	495
								1	10	55	220
									1	11	66
										1	12
											1

Fig. 1.

The second identity $(ab)^n = a^n b^n$ is proved by the same method. Al-Samaw᾿al supposes known the proof for $n = 2$, in Euclid's *Arithmetical Books*. In any case, proposition (1) makes the proof of (2) evident $(ab)(ab) = (ab)^2 = a^2b^2$. And as he states this identity immediately after Proposition 1 we might think the proof is made for a commutative group noted multiplicatively (a and b commute). Whatever the case, for $n = 3$ he states that the product of two cubic numbers is equal to the cube of the products of their sides.[13]

$$\Leftrightarrow [a^3b^3 = (ab)^3].$$

Proof. Let a and b be two cubic numbers, c and d their sides, e and f their squares, multiply c by d; $cd = g$; and a by b; $ab = h$; I say that h is equal to the cube of g. In Euclid's *Arithmetical Books*, it was shown that the product of the square number e by the square number f is equal to the square of g. If we multiply the last product by that of c by d, I mean by the number g, we obtain the cube of g. This results from the multiplication of the product of e by f by the product of c by d. But the product of c by f, multiplied by the product of e by d is equal to the product of c by e multiplied by the product of f by d as we showed in the preceding figure [Proposition 1]. Q.E.D.[14]

In other words, to prove that $a^3b^3 = (ab)^3$, he starts with $a^2b^2 = (ab)^2$, multiplies both members by (ab) and obtains $(ab)(a^2b^2) = (ab)(ab)^2 = (ab)^3$. But Proposition 1 gives $(ab)(a^2b^2) = (aa^2)(bb^2) = a^3b^3$.

He then proves the proposition for $n = 4$ and writes:

In the same manner, it can be shown that the quadrato-cube of the product of two numbers is equal to the product of the quadrato-cube of one by the quadrato-cube of the other and so on in ascending order.[15]

Not only the types of proofs we named R_1 are to be found in these authors, but also similar types of definition. Let us remember, for example, the definition of algebraic powers given by al-Karajī in *al-Fakhrī* and *al-Badī*ᶜ, and taken up by al-Samaw᾿al in *al-Bāhir*. The following table is set out verbally:[16]

$$\begin{aligned}
a &= a^1\\
a^2 &= a\cdot a\\
a^3 &= a^2\cdot a\\
a^4 &= a^3\cdot a = a^2\cdot a^2\\
a^5 &= a^4\cdot a = a^3\cdot a^2\\
a^6 &= a^5\cdot a = a^4\cdot a^2 = a^3\cdot a^3\\
a^7 &= a^6\cdot a = a^5\cdot a^2 = a^4\cdot a^3\\
a^8 &= a^7\cdot a = a^6\cdot a^2 = a^5\cdot a^3 = a^4\cdot a^4\\
a^9 &= a^8\cdot a = a^7\cdot a^2 = a^6\cdot a^3 = a^5\cdot a^4
\end{aligned}$$

"and these ascending powers follow the same ratios to infinity". In other words, x^n is defined by

$$x^n = x^{n-1}x \quad \text{for any } n \in \mathbb{N}.$$

III

To understand this kind of reasoning implies differentiating between others that resemble mathematical induction in order to prepare their comparison. An attempt in this direction had been made earlier. Freudenthal distinguished between two types of reasoning often mistaken for mathematical induction. One is "quasi-general", the other called "regression".

By "quasi-general", he designates a proof can be undertaken for any n (though, historically, it only operates on particular numbers). Even though the mathematician aimed at a true property for any n, he carried out his operations on particular numbers. While this reasoning may be considered an application of the principle of mathematical induction, it is not possible however, without stretching the interpretation, to attribute an explicit recognition of this principle to those who used it.

As an example of this proof Freudenthal takes Proposition 7 from Maurolico.[17] To show that $2 \sum_{k=1}^{n} k = n(n + 1)$, Maurolico writes for $n = 4$ only: $\sum_{k=1}^{n} k = 1 + 2 + \ldots + n$ and $\sum_{k=1}^{n} k = n + (n - 1) + \ldots + 1$; whence for addition $2 \sum_{k=1}^{n} k = n(n + 1)$. Then Freudenthal (1953, p. 22) writes:

Ohne Zweifel haben wir hier einen quasi-allgemeinen Beweis vor uns, wie man sich ihn exakter kaum wünschen kann. Man braucht nur n für 4, $n + 1$ für 5 einzusetzen, um einen echten allgemeinen Beweis zu erhalten.

In this type of reasoning the mathematician would sometimes have reasoned as follows: $P(1)$ is true by means of a quasi-general proof – $P(2)$ is also true, as well as $P(3)$ and $P(4)$; he concludes – more or less correctly – that this is so for all that follows.[18] Two factors seem to be inseparable for describing this type of proof: (1) the repetition of the quasi-general proof for each value taken for the variable; (2) the possession of a method independent of particular values taken by the variable, i.e. a method that makes it possible to prove any n in the same way as for a particular value, for example 4. It is understandable, therefore, that this proof, distinct from ordinary induction, cannot however be mistaken for mathematical induction.

The other type of proof – called "regression" – designates a primitive mathematical form of induction; though derived somewhat formally from mathematical induction, it cannot however be identified with it. It is mathematical induction repeated each time for the number immediately preceding. It is a repetition of mathematical induction taken from a value of the variable until the smallest value for the verified property is reached. Regression is often conducted in a quasi-general way, and avoids repeating the proof for values other than the one chosen initially. This form is closer to mathematical induction than any other, or as Freudenthal (1953, p. 27) wrote: "Bei grosser Liberalität darf man das Verfahren vielleicht als vollständige Induktion bezeichnen, obwohl der eigenartige formale Aufbau der vollständigen Induktion fehlt".

Before Pascal, real mathematical induction as such did not exist, but only two types of proof, and if Maurolico recognized mathematical induction, it was at the most as an archaic form of regression. This is Freudenthal's thesis in brief.

However, before proceeding further, we want to show that "quasi-general" and "regression" do not exhaust the description of all types of reasoning current in this field before Pascal; Freudenthal's generalization of Maurolico's example may limit understanding of the history of mathematical induction. To clarify these remarks, let us go back to some examples taken from al-Karajī and al-Samawʾal.

1. To prove $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i(i-1)$, or, as he writes: "The sum of the squares of the numbers that follow one another in natural order from one is equal to the sum of these numbers and the product of each of them by its predecessor".[19]

Schema of the proof:

$$
\begin{aligned}
&n^2 = n[(n-1) + (n-(n-1))] = n[(n-1)+1] = n(n-1) + n\\
&(n-1)^2 = (n-1)[(n-2) + ((n-1)-(n-2))] = (n-1)\,[(n-2)+1]\\
&\qquad\quad = (n-1)(n-2) + (n-1)\\
&\vdots\\
&1^2 = 1\cdot 1
\end{aligned}
$$

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= n^2 + (n-1)^2 + \ldots + 1^2 \\ &= [n(n-1) + (n-1)(n-2) + \ldots + 2\cdot 1] + [n + (n-1) \ldots + 1] \\ &= \sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n} i.\end{aligned}$$

However, this schema is expressed for $n = 4$. For instance, to prove the above proposition, after giving the statement in general terms, al-Samawʾal writes:

Example: let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ be numbers that follow each other in natural order from one. I say that the sum of the squares of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ is equal to the sum of $\overline{AE}$ and the products of $\overline{DE}\cdot\overline{CD}$, $\overline{DC}\cdot\overline{CB}$ and $\overline{CB}\cdot\overline{BA}$.

Proof.

$$\begin{aligned}\overline{DE}^2 &= \overline{DE}[\overline{CD} + (\overline{DE} - \overline{CD})] = \overline{DE}(\overline{CD} + 1) = \overline{DE}\cdot\overline{CD} + \overline{DE} \\ \overline{CD}^2 &= \overline{CD}[\overline{BC} + (\overline{CD} - \overline{BC})] = \overline{CD}(\overline{BC} + 1) = \overline{CD}\cdot\overline{BC} + \overline{CD} \\ \overline{BC}^2 &= \overline{BC}[\overline{AB} + (\overline{BC} - \overline{AB})] = \overline{BC}(\overline{AB} + 1) = \overline{BC}\cdot\overline{AB} + \overline{BC} \\ \overline{AB}^2 &= 1 = \overline{AB}\end{aligned}$$

where

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DE}^2 = (\overline{BC}\cdot\overline{AB} + \overline{CD}\cdot\overline{BC} + \overline{DE}\cdot\overline{CD}) + (\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE})$$

Q.E.D.[20]

2. To prove $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ "if we want to add the cubes of the number that follow one another from 1 [according to the natural order] we multiply their sum by itself. The product is the sum of their cubes".[21]

To prove this proposition al-Samawʾal first proves the following lemma:

LEMMA. *The cube of a number is equal to the sum of its square and the double product of this number by the sum of numbers that follow from one [according to the natural order] to the predecessor of this same number.*

$$\left[\Leftrightarrow n^3 = n^2 + 2n\sum_{i=1}^{n-1} i\right].$$ [22]

Schema of the proof.

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \Leftrightarrow 2\sum_{i=1}^{n-1} i = n(n-1)$$

$$\Leftrightarrow 2n\sum_{i=1}^{n-1} i = n^2(n-1) \Leftrightarrow n^3 = n^2 + 2n\sum_{i=1}^{n-1} i.$$

As expressed in *al-Bāhir*:

Example: let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ be these numbers. I say that the sum of the square of $\overline{DE}$ and the double product of $\overline{DE}$ by $\overline{AD}$ *is equal to the cube of* $\overline{DE}$.

Proof. $\overline{AD} = (\overline{CD}\cdot\overline{DE})/2$ where $2\overline{AD} = \overline{CD}\cdot\overline{DE}$, but by multiplying two numbers by $\overline{DE}$, we have $2\overline{AD}\cdot\overline{DE} = \overline{CD}\cdot\overline{DE}^2$, but $\overline{CD}\cdot\overline{DE}^2 = \overline{DE}^2(\overline{DE} - 1) = \overline{DE}^3 - \overline{DE}^2$, therefore $\overline{DE}^3 = \overline{DE}^2 + 2\overline{AD}\cdot\overline{DE}$. Q.E.D.[23]

Then al-Samawʾal can prove the proposition.[24]
Schema of the proof:

$$\left(\sum_{i=1}^{n}\right)^2 = \left(\sum_{i=1}^{n-1} i\right)^2 + n^2 + 2n\left(\sum_{i=1}^{n-1} i\right)$$

$$= n^3 + \left(\sum_{i=1}^{n-1} i\right)^2 \qquad \text{(Lemma)}$$

$$= n^3 + \left(\sum_{i=1}^{n-2} i\right)^2 + (n-1)^2 + 2(n-1)\left(\sum_{i=1}^{n-2} i\right)$$

$$= n^3 + (n-1)^3 + \left(\sum_{i=1}^{n-2} i\right) \qquad \text{(Lemma)}$$

$$= \ldots$$

$$= n^3 + (n-1)^2 + \ldots + 1^3 = \sum_{i=1}^{n} i^3.$$

As expressed in *al-Bāhir*:

Example: let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ be these numbers that follow one another in natural order from one. I say that the sum of the cubes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ is equal to the square of $\overline{AE}$.[25]

Proof.

$$\begin{aligned}
\overline{AE}^2 &= \overline{AD}^2 + \overline{DE}^2 + 2\overline{DE}\cdot\overline{AD} \\
&= \overline{DE}^3 + \overline{AD}^2 \quad \text{by applying the above figure} \\
&= \overline{DE}^3 + \overline{AC}^2 + \overline{CD}^2 + 2\overline{AC}\cdot\overline{CD} \\
&= \overline{DE}^3 + \overline{CD}^3 + \overline{AC}^2 \\
&= \overline{DE}^3 + \overline{CD}^3 + \overline{AB}^2 + \overline{BC}^2 + 2\overline{BC}\cdot\overline{AB} \\
&= \overline{DE}^3 + \overline{CD}^3 + \overline{BC}^3 + \overline{AB}^2
\end{aligned}$$

but $\overline{AB}^2 = \overline{AB}^3 = 1$, therefore $\overline{AE}^2 = \overline{DE}^3 + \overline{CD}^3 + \overline{BC}^3 + \overline{AB}^3$. Q.E.D.

In both these examples we observe two types of reasoning. The first one – R_2 – is illustrated by the proof of the lemma in the second example. The second – R_3 – was the one used to prove both propositions.

With R_2, al-Samawʾal's proof is only complete for $n = 4$. But, on the one hand, the statement of the proposition is general and, on the other, al-Samawʾal does not hesitate to use the same lemma without proof again for $n = 2$, $n = 3$. Everything indicates therefore that for the mathematician *the proof is the same* for any number as for 4; or yet again the proof is written in the same way for any n. R_2 may be considered a *quasi-general* reasoning and an application of mathematical induction, though without recognition of its principle. R_3 is different: it explicitly concerns establishing the procedure for the transition from n to $(n + 1)$, either by proving the lemma, or directly, in order to then proceed by successive reductions or *regression*. It is moreover true that R_2 and R_3 are used jointly as is easily observed: in the first example, R_2 intervenes for each equality; in the second example, R_2 intervenes for the binomial formula. In any case, R_3 is seen as an archaic form of mathematical induction.

Again it should be noted that here R_3 is a mastered technique: it was not simply used on rare occasions as was the case for Maurolico. To demonstrate the skill with which regression was practised, let us take al-Samawʾal's proof:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Moreover, it is often stated in histories of mathematics that this formula was proved by al-Karajī. However, this is not true. Al-Karajī only gave an equivalent formula $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i\left(\frac{2}{3}n + \frac{1}{3}\right)$ and there exists a proof using an archaic form of mathematical induction in Ibn al-Haytham's treatise, *On the Measurement of the Paraboloid*. Al-Samawʾal first wants to prove $(2n + 1)\sum_{i=1}^{n} i = 3\sum_{i=1}^{n} i^2$ from which the value of $\sum_{i=1}^{n} i^2$ is drawn. He first proves the following lemmas:

LEMMA 1. $(n+2)\sum_{i=1}^{n} i = n\sum_{i=1}^{n+1} i$. The proof of this lemma belongs to the quasi-general type outlined as follows:

$$(n+2)\sum_{i=1}^{n} i = \frac{n(n+1)}{2}(n+2) = n\left[\frac{n(n+1)}{2} + (n+1)\right] = n\sum_{i=1}^{n+1} i.$$

LEMMA 2. $n(n+1) + (n+1)(n+2) = 2(n+1)^2$ *for* $n \in \mathbb{N}$.

Proof of the schema.

$$\begin{aligned} n(n+1) &= (n+1)^2 - (n+1) \\ (n+1)(n+2) &= (n+1)^2 + (n+1) \end{aligned}$$

hence the lemma and hence we draw

$$(n+1)[n + (n+1) + (n+2)] = 3(n+1)^2$$

for $n \in \mathbb{N}$.

LEMMA 3. $n\sum_{i=1}^{n+1} i = n\sum_{i=1}^{n-2} i + 3n^2$. He uses the above lemma for his proof.

PROPOSITION. *Given the numbers $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, that follow one another in natural order from one, I say that the product of $\overline{AG}$* by *$\overline{FH}$ is equal to three times the sum of squares of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$.*[26]

Schema of the proof.

$$(2n+1)\sum_{i=1}^{n} i = n\sum_{i=1}^{n+1} i + (n+1)\sum_{i=1}^{n-1} i \quad \text{proved earlier.}$$

By Lemma 1 we have:

$$(n+1)\sum_{i=1}^{n-1} i = (n-1)\sum_{i=1}^{n} i = (n-1)\sum_{i=1}^{n-3} i + 3(n-1)^2$$

by Lemma 3.

Similarly, we have

$$n\sum_{i=1}^{n+1} i = n\sum_{i=1}^{n-2} i + 3n^2.$$

We have therefore

$$(2n+1)\sum_{i=1}^{n} i = 3n^2 + 3(n-1)^2 + n\sum_{i=1}^{n-2} i + (n-1)\sum_{i=1}^{n-3} i$$
$$= 3n^2 + 3(n-1)^2 + 3(n-2)^2 + 3(n-3)^2 + (n-2)\sum_{i=1}^{n-4} i + (n-3)\sum_{i=1}^{n-5} i$$

by applying the lemmas

$$= \ldots$$
$$= 3n^2 + 3(n-1)^2 + \ldots + 32^2 + 3 = 3\sum_{i=1}^{n} i^2.$$

As al-Samawʾal[27] expresses it:

$$\overline{AG}\cdot\overline{FH} = \overline{AH}\cdot\overline{FG} + \overline{AF}\cdot\overline{GH}$$

but

$$\overline{AF}\cdot\overline{GH} = \overline{AG}\cdot\overline{EF} = \overline{AD}\cdot\overline{EF} + 3\overline{EF}^2 \text{ and}$$
$$\overline{AH}\cdot\overline{FG} = \overline{AE}\cdot\overline{FG} + 3\overline{FG}^2,$$

therefore

$$\overline{AG}\cdot\overline{FH} = \overline{AE}\cdot\overline{FG} + 3\overline{FG}^2 + \overline{AD}\cdot\overline{EF} + 3\overline{EF}^2,$$

but

$$\overline{AE}\cdot\overline{FG} = \overline{AF}\cdot\overline{ED} = \overline{AC}\cdot\overline{DE} + 3\overline{DE}^2 \text{ and}$$
$$\overline{AD}\cdot\overline{EF} = \overline{AE}\cdot\overline{CD} = \overline{AB}\cdot\overline{CD} + 3\overline{CD}^2,$$

therefore

$$\overline{AG}\cdot\overline{FH} = \overline{AC}\cdot\overline{DE} + 3\overline{DE}^2 + 3\overline{FG}^2 + \overline{AB}\cdot\overline{CD} + 3\overline{CD}^2 + 3\overline{EF}^2,$$

but

$$\overline{AC}\cdot\overline{DE} = \overline{AD}\cdot\overline{BC} = 3\overline{BC}^2 \quad \text{since } \overline{AB} = 1$$

and

$$\overline{AB}\cdot\overline{CD} = \overline{AC}\cdot\overline{AB} = 3\overline{AB}^2 = 3 \quad \text{since } \overline{AB} = 1,$$

therefore

$$\overline{AG}\cdot\overline{FH} = 3\overline{FG}^2 + 3\overline{EF}^2 + 3\overline{DE}^2 + 3\overline{CD}^2 + 3\overline{BC}^2 + 3\overline{AB}^2.$$

Q.E.D.

In this chapter which deals mainly with the sum of n natural prime integers, the sum of their squares and the sum of their cubes, a fairly large number of examples are to be found, he often uses proof by regression, i.e. a form of primitive mathematical induction.

However, if we follow Freudenthal's analysis, we should find only two types of reasoning – R_2 and R_3 – before Pascal. His thesis is doubtless correct when it concerns Maurolico's work which he examined. However, the situation is different for the work of al-Karajī and al-Samawʾal. The first type of reasoning examined for the binomial development – R_1 – is in no way to be mistaken for R_2 and R_3. For R_1, we saw that an attempt was made to write the system of the transition from n to $n + 1$ in the same way, whatever the initial number. The idea is as follows: since the procedure of the transition from n to $n + 1$ is true, even if the transition is illustrated by a particular n, then it is true for any number. Or again the procedure of transition is the same whatever the number. Everything indicates that this reasoning, while not formalized or stated theoretically, differs not only from R_2 and R_3, but moreover apparently recognizes the axiom of mathematical induction.

IV

Can one speak about mathematical induction in connection with these attempts by al-Karajī and al-Samawʾal?

Two contradictory answers have been given to a similar question in order to characterize attempts where R_1 does not even appear. For instance, in spite of Freudenthal's study, Bourbaki (1960, p. 38) still wrote in 1960 that the principle of mathematical induction "had been clearly conceived and employed for the first time by the Italian F. Maurolico in the 16th century". Rabinovitch did not hesitate to certify Levi Ben Gerson's reasoning as mathematically inductive, though it was less elaborate in this respect than that of al-Karajī and al-Samawʾal.[28] Contrary to these positions, others with some restrictions like Freudenthal, sometimes without restriction like Hara,[29] grant Pascal the sole merit for applying mathematical induction.

These apparently exclusive positions do, however, have one point in common: they hinder understanding as to why new forms of mathematical reasoning arose. To refuse to describe the various attempts as mathematical induction in order to reserve it for Pascal bars the way to understanding the new forms of reasoning that arose with the renewal of algebra in the eleventh century.[30] On the contrary, to qualify everything as mathematically inductive, and as a result to find the principle of this reasoning everywhere, by suppressing differences prevents pinpointing the appearance of new forms. To evade such difficulties, it is not rare for a historian to choose an eclectic position. Bourbaki (1960), for example, after affirming the priority of Maurolico, then writes "that more or less conscious applications" of the principle of mathematical induction "are to be found in Antiquity". To avoid posing the problem, the origin of mathematical induction is frequently mentioned as well. The very ambiguity of the term may be convenient: origins are multiple, embedded in the memory of time, and make it possible to invert chronological and logical order without verification in the course of historical reconstitution. Is it not a matter of genesis where the two orders are indiscernable?

The difficulty of the problem as we can see, makes a unique answer impossible; an answer intimately linked to the point chosen from which to go back in time for our necessarily "recurrent history". To decide without further concern that one or another formulation of the principle of mathematical induction is sufficient, is to run the risk of integrating in the history of the principle what another choice might have assigned to a prehistoric museum. In fact, for the historian, the main problem lies in avoiding trivial historical "recurrence", which makes it impossible to reconstruct a mathematical activity whose history one is supposed to write.

If, by mathematical induction, we mean, like Peano, reasoning based on the following assertion, or another equivalent form:

> Let P a property defined on $\mathbb{N}$, if $P(1)$ and
> $[P(n) \Rightarrow P(n + 1)]$ then P is true for $n \in \mathbb{N}$,

it would be difficult to consider attempts before Pascal as mathematically inductive. Furthermore, Pascal's attempt could only be so described with much indulgence. If we confine ourselves to a rigorous formulation – which is essential – attempts that do not state the argument of induction – $P(n) \Rightarrow P(n + 1)$ – for any n in an explicit way will be rejected as outside mathematical induction. But as this rigour is related to a complete system of axioms – known as Peano's system – which includes precisely the exact formulation of the principle of mathematical induction, all earlier formulations are necessarily naive.

Therefore to ask about the status of attempts before Peano's formulation implies at least one question: are such naïvetés equally valid? Or do they differ significantly? After all, perhaps for a contemporary logician of our time, even Peano's formulation appears somewhat naive?

As far as we are concerned, let us go back briefly to Pascal. Passages in the *Traité du triangle arithmétique* (ed. Seuil, 1963, p. 57) on mathematical induction are known and often cited. We shall only cite the most significant passages:

> If an arithmetical triangle is found which has this quality . . . , I say that the following triangle will have the same property.
>
> Whence it follows that all arithmetical triangles have this quality. For the first has it by the first lemma, and it is even still evident in the second; therefore by the second lemma the next triangle will have it too, and consequently the next, and so on to infinity.

Pascal's formulation is undoubtedly much more abstract and elaborate than any previous attempt. Only thirty years before Pascal, Bachet was only capable of proving a less elaborate formulation of this reasoning. He wrote (1624, p. 5):

> Thus availing myself of what had been proved for three or four numbers, I completed the proof in the same way. And if six numbers were given, I would use what had been proved for five and so on, if more were given. Therefore the means of proof is universal and applicable to any multitude of numbers.

In spite of differences between Pascal's formulation and earlier attempts, they do have something in common: they are clearly shown by the use Pascal makes of his own principle. But it is at precisely this point that the scope and limits of Pascal's formulation may be understood.

1. Pascal, like his predecessors, applied the essential point of the principle of mathematical induction to its original field: combinatorial methods and related problems. But even if, here and there, this principle or similar types of reasoning are applied to other problems of number theory or algebra, combinatorial methods remain a privileged area for this application.

We saw that al-Karajī and al-Samawʾal use R_1 as a proof method in this field which will be used later as an exemplary field for illustrating the principle of mathematical induction. Well before Pascal, Levi Ben

Gerson, like Frénicle[31] (1605–1675) later on, applied a simpler but equivalent form of R_1 to the domain of permutations. Lastly, Bachet wants to prove that "if three or more numbers are multiplied together, the product will always be the same, in whatever order they are multiplied".[32] So the appearance of this type of reasoning may appear to be a technically adapted solution of a theoretical problem: to prove for the field of general questions of enumeration, or problems of the distribution of the elements of finite sets into various subsets ordered or not, according to different laws and later grouped together under the heading of combinatorial analysis.

2. Pascal, like his predecessors, stated the conclusion of the proof corresponding to his intuitive idea of $\mathbb{N}$ which limits the generality of the formulation. In fact $(\forall n)$ – with n integers – $P(n)$ is stated according to a concept of $\mathbb{N}$ which is none other than an intuitive description according to which the elements of this set are 0, 1, 2, 3 and "and so on to infinity".

3. Despite his general, clear formulation of the argument of mathematical induction as he applied it, Pascal reverted to his predecessor's usage. In fact, although it is recognized that $[P(n) \Rightarrow P(n+1)]$ is formulated verbally in a general way, i.e. for any integer and with the hypothesis: $P(n)$ is true, in practice, he only takes particular numbers 3 and 4. This is always the case in the two most important proofs where he applies the principle of mathematical induction. To prove the theorem equivalent to $C_n^p/C_n^{p+1} = (p+1)/(n-p)$, he verifies for $n = 1$, supposes it true for $n = 4$ and proves for $n = 5$. And he concludes in a way which in some respects is reminiscent of those who use R_1. He writes:

> It will be shown similarly for the following, since this proof is only founded on this proposition which is found in the preceding base and each cell is equal to its preceding base which is universally true.[33]

The other example is equivalent to

$$\phi(a, b) = \sum_{i=a}^{a+b-1} C_{a+b-1}^{i} \Big/ \sum_{k=0}^{a+b-1} C_{a+b-1}^{k}$$

where $\phi(a, b)$ is the sum attributed by the stake to player A in a game between two players A and B, and where a parts are missing in A and b parts in B. He also verifies the theorem for $n = 2$, supposes it true for $n = 4$ and proves it for $n = 5$. He concludes (ed. Seuil, 1963, pp. 60ff.) in a similar manner to the above conclusion.

Pascal, like his predecessors, never named the reasoning he used. The absence of a name apparently implies that this reasoning is only a specific procedure for a domain and not yet an autonomous proof, independent of a field of application, which would consequently require the attribution of a name. However, this name only appears – and this is significant – with the English school: George Peacock and Morgan (Cajori, 1918).

5. To judge the principle as a general method of proof at its true value would imply examining how Pascal's successors received it. If it had been understood as a general method, it would lead to at least two transformations: a clear distinction between complete induction and incomplete induction, and on the other hand, a rejection of proof by incomplete induction. However, neither this distinction nor this rejection were really accepted. Still in the eighteenth century, to take only a French example, we may read in the article on "Induction" in the *Encyclopédie méthodique*:

the signification of this term will be clearly understood for example. We have

$$(a + b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1\cdot 2}a^{m-2}b^2 +$$

$$\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a^{m-3}b^3 + \text{etc.}$$

He who without knowing the exact and general method of proving this formula, will conclude from having verified it in the case $m = 1$, $m = 2$, $m = 3$, etc. will judge by induction.

Therefore this method must only be used for lack of a more precise one, although in this case, it should be used with great circumspection, since sometimes wrong conclusions might be reached.

Even if after Pascal and independently of him, as in Jacques Bernoulli, we sometimes come across a distinction between complete mathematical induction and incomplete induction, it remains that this distinction is soon forgotten by its own author, which shows that at this period, at least, a real understanding of the necessity for mathematical induction was far from existing. In fact, J. Bernoulli not only established this distinction, but also denounced the use of incomplete induction as unscientific. Did he not write in his criticism of Wallis (1713, p. 95):

In fact, apart from the fact that the manner of proof by induction is scarcely scientific, moreover, it requires special work for each series.

And yet, not Wallis but also Montmort, de Moivre and Jacques Bernoulli[34] himself, continued to demonstrate and prove in one way or another using induction.

These various arguments show, I think, that Pascal, by his application of the principle of mathematical induction, and in some respects by his formulation of this principle, had not completely broken with an R_1 type reasoning. This explains why his principle could not impose itself as an independent method from a particular application, a principle which should have excluded all proof by simple induction for good. However, there is no question of denying the novelty of Pascal's formulation in relation to unformulated usages of R_1 or even earlier formulations such as that of Bachet. It is precisely this originality that enables a modern reader to recognize Pascal's principle, and despite a lack of precision in its outdated formulation, as a precursor of mathematical induction. In other words, if we start with formulations after Peano, we find strong traces that enable us to discover the principle of induction in Pascal. But if we start with Pascal, R_1 will be integrated as inductive mathematical reasoning, and regression as an archaic form of mathematical induction, an integration difficult to reach from Peano's formulation. Therefore, in a "recurrent history", Pascal's attempt appears as the culmination of that of al-Karajī and al-Samawʾal, while Peano's attempt appears to be the culmination of attempts which started with Pascal. To avoid making "recurrent" history commonplace, we must choose a starting point in the past that is a culmination, a culmination that is necessarily the culmination of a beginning. The double reference thus necessary for the historian enables us to conclude: al-Karajī's and al-Samawʾal's methods of proof – mainly R_1 and regression in some way – constitute the beginning of mathematical induction, if one starts with Pascal.

NOTES

1. G. Vacca (1909). Vacca was so convinced of the importance of his discovery that he published it in several other articles (1910) and (1911).
2. Al-Karajī (or al-Karkhī). Known since the translation of his book on Algebra by Woepcke and Hochheim on his book on Arithmetic (*Kāfī fī al-ḥisāb*). Little is known about his life other than he lived in Baghdad in the late 10th century and early 11th century. For a scientific bibliography on al-Karajī, see Anbouba's introduction (1964) and *infra*, I.2.
3. Al-Samawʾal ibn Yaḥyā ibn ʿAbbās al-Maghribī, died in 1174. Al-Samawʾal's autobiography is to be found in his polemical book, *Ifḥām al-Yahūd* (Perlmann,

ed., 1964). A scientific bibliography is given in Rashed and Ahmad, eds. (1972). This chapter is based on the MSS 2718 Aya Sofia and 3155 Esat ef.; page numbers refer to the first manuscript.

4. Itard (1961, p. 73) wrote: "However, some demonstrations by recurrence or complete induction may be found. We never come across the modern rather pedantic leitmotiv: 'we have checked the property for 2, we have shown that if is true for a number, it is true for the following, therefore it is general' and those who only see complete induction in this hackneyed expression have the right to say it does not appear in the *Elements*.

   We see it in prop. VII, 3, 27 and 36; VIII, 2, 4 and 13; IX, 8 and 9".
5. Al-Samawʾal, *al-Bāhir*, f. 43ᵛ, Rashed and Ahmad, eds. (1972), p. 104.
6. *Ibid.*, f. 43ᵛ, p. 104.
7. *Ibid.*, f. 44ᵛ, p. 106.
8. *Ibid.*, ff. 44ᵛ–45ʳ, pp. 107–108.
9. *Ibid.*, f. 45ᵛ, p. 109. Our italics.
10. Al-Samawʾal quoted *in extenso* the passage of this lost work quoted here.
11. To our knowledge, this is the first text, to state rules in such general terms. According to Needham (1959, p. 135), Yang Hui's book dates back to 1261, therefore one and a half centuries after al-Karajī's text. Al-Khayyām (1048–1131), in the wake of al-Karajī, and perhaps independently, was probably acquainted with these rules. Later in the 13th century, the same results are to be found in Naṣīr al-Dīn al-Ṭūsī (ed. A.S. Saïdan, 1967, p. 145), with one exception: the binomial formula is always written verbally

$$(a + b)^n - a^n = \sum C_n^m a^{n-m} b^m.$$

   They are also to be found in the 15th century in al-Kāshī's *Key to Arithmetic* (see Luckey, 1951, p. 24).

   For more details, see *infra*, pp. 261–274.
12. In the table reproduced here, we have replaced the terms, root, square, cube . . . with the symbols x, x^2, x^3, . . . Al-Samawʾal, *al-Bāhir*, ff. 45ᵛ–47ʳ, Rashed and Ahmad, eds. (1972), pp. 109–112.
13. *Ibid.*, f. 44ᵛ, p. 105.
14. *Ibid.*
15. *Ibid.*, ff. 45ʳ–45ᵛ, p. 108.
16. From *Fakhrī*, p. 48, reproduced in *al-Bāhir*, pp. 17–18.
17. Omnis radix multiplicata in radicem sequentem, producit duplum trianguli sibi collateralis. Demonstration: Exempli gratia, ducatur quaternarius in sequentem radicem, scilicet, quinarium: & producuntur 20. Aio, quod 20. duplus est ad triangulum ipsi quaternario collateralem. Sumantur enim ab unitate ad quaternarium radices; quibus applicetur totide & ordine praepostero ab unitate radices; singulae singulis: Sic enim fiet, ut crescentes cum decrescentibus singuli singulis conjuncti numeri faciant quatuor summas aequales: hoc est quatuor quinarios, quare earum aggregatum erit planus numerus, qui fit ex ductu quaternarij in quinarium: & idcirco 20, erit talis planus: Duplus autem est planus ipse ad triangulum quaternarij: quandoquidem, per diff. talis triangulus est aggregatum unius dictorum ordinum: quod

est dimidium plani: Igitur 20. duplus erit ad triangulum quaternarij. Et similiter in omni casu id quod proponitur demonstrabimus.

Cited by Freudenthal (1953, p. 21) from *Maurolico Arithmeticorum libri duo* in *Opuscula mathematica* (1575).

18. I.e. instead of reasoning as usual in an almost general way for a particular n, he repeats the same reasoning for some particular numbers.
19. Al-Samaw'al, *al-Bāhir*, f. 54^r, Rashed and Ahmad, eds. (1972), p. 127.
20. *Ibid.*, pp. 54^r, 54^v, pp. 127–128. Our translation, like other similar translations, follows the text closely, except for the substitution of signs for the terms: sum, difference, equality, etc.
21. *Ibid.*, ff. 61^v–62^r, p. 143.
22. *Ibid.*, f. 62^r, p. 143.
23. *Ibid.*, f. 62^v, p. 143.
24. *Ibid.*, ff. 62^r–62^v, pp. 143–144.
25. *Ibid.*, f. 62^v, p. 144.
26. *Ibid.*, f. 53^r, p. 125.
27. *Ibid.*, f. 53^v, p. 125.
28. This is basically the proof of a formula equivalent to $P_{n+1} = (n + 1)P_n$ where P_n is the set of permutations of n distinct elements. For Levi ben Gerson, see (1909, pp. 48ff.). See Carlebach (1910) and Rabinovitch (1970, p. 242).
29. Hara (1962, p. 287) wrote: "For the first time in Pascal, we encounter not only a systematic application but an almost completely abstract formulation of the method rigorously intended". Hara thought he conveyed not only his point of view but that of M. Freudenthal. However, the latter is apparently more reserved on this point as he wrote (1953, p. 33): "Nicht die Anwendung, auch nicht die systematische Anwendung ist das Auffallende, sondern die fast vollständig abstrakte Formulierung, die übrigens später nocheinmal, an anderen Objeckten, wiederholt wird".
30. This involves the application of arithmetic to algebra or the extension of algebra to elementary arithmetical operations, so these operations may be applied to $[0, \infty]$. The obvious result of this project was to define the frontiers between algebra and geometry and achieve autonomy and specificity for algebra. The principal means was to extend abstract algebraic calculation. As this renewal progressed, the following discoveries were made for the first time in history:
    (1) the multiplication and division of algebraic powers;
    (2) the theory of the division of polynomials;
    (3) the calculation of signs.

    Concurrently, we encounter the calculation of binomial coefficients and the binomial formula, including various enumeration problems, later to be known as combinatorial analysis. See Rashed (1974) and our introduction to al-Samaw'al (1972).
31. For Frénicle and for Levi ben Gerson earlier, this concerns permutation problems. In fact Frénicle proves a formula equivalent to $P_{n+1} = (n + 1)P_n$. For instance, having shown that $P_3 = 3P_2 = 6$; $P_4 = 4P_3 = 24$; $P_5 = 5P_4 = 120$, he writes (1729, p. 92), "and so on, we must multiply the preceding combination (read permutation) by the number of the given multitude; and that is clear proof which serves to demonstrate the construction of the table". For studies on Frénicle, see Coumet (1968, pp. 209ff.).

32. Bachet (1624, p. 2): Bachet's proof is based principally on type R_1. Here Bachet writes as follows (pp. 3ff.): "Euclid proved in VII-16, that for two numbers, whether the first is multiplied by the second, or the second by the first, the product is always the same. Here I want to prove that the same applies to three or more numbers. However, two or more numbers are said to be multiplied together when we multiply two together and the product of the one and the product of the other, and so on or as many numbers as there are.

    Firstly, let us take three numbers A, B, C, and by multiplying A by B we have D, which multiplied by C produces E. Let us change the order and multiply B by C which gives F which, multiplied by A gives G. Let us change the order once again and multiply A by C, which gives H which, multiplied by B gives K (these are all the different ways for multiplying three numbers together). I say that three products $E \cdot K \cdot G$ have the same number. Since B multiplied by $A \cdot C$. gives $D \cdot F$, there is a ratio of A to C and D to F therefore the same number is given by multiplying A by F and C by D in 19 in 7. Let us start with E and G the same number. Similarly, since C multiplied by A and B gives H and F, there is a ratio between A and B and between H and F. Therefore, the same number is given multiplying A by F and B by H. Therefore, the same number is given multiplying A by F and B by H, therefore KG is the same number. Consequently, all three $E \cdot K \cdot G$. are the same number which it what had to be proved.

    Now let us take four numbers $A \cdot B \cdot C \cdot D$ and multiply A by B and the product of C which gives E, which multiplied by D gives K. Then let us change the order and multiply D by C and the product by B which gives F which, multiplied by A gives H. I say that $K \cdot H$ is the same number and that the same number will always be produced in any way when we multiply the four numbers $A \cdot B \cdot C \cdot D$ together. Since by multiplying together on one side the three numbers $A \cdot B \cdot C$ and on the other side $D \cdot C \cdot B$, we find $B \cdot C$ on both sides, multiply $B \cdot C$ together, which gives G. However, by what was proved for three numbers the same E, I say, which is made by multiplying A by B and the product by C, the same E is reached as well by multiplying B by C and the product (i.e. G) by A. Similarly, we will prove that F is made by multiplying D by G, therefore as the same G multiplies the two $A \cdot D$ produces $E.F$. There is the same ratio in $A \cdot D$ between $E \cdot F$. Consequently, the same number is reached by multiplying A by F and D by E. Therefore, $K \cdot H$ are the same number. However, using the same means we always prove the same. Since of four numbers, by multiplying three together on one side and three on the other, there will always be two the same between three taken from one side and the other and consequently the same proof occurs".

33. Pascal (ed. Seuil, 1963), p. 53. This is the twelfth consequence which states: "in every arithmetical triangle, two contiguous cells having the same base, the higher is to the inferior like the multitude of the cells from the higher to the top of the base to the multitude of those from the inferior to the bottom inclusive".

34. For instance, after criticizing Wallis, J. Bernoulli proceeds by complete induction to find the general rule for the sum of squares, cubes, etc. and n first natural integers, i.e.

    $$\sum_{k=1}^{n} k^c$$

    with $c = 1, 2, 3, \ldots$ See Bernoulli (1713, pp. 96ff.).

THE EXTRACTION OF THE *n*TH ROOT AND THE INVENTION OF DECIMAL FRACTIONS (ELEVENTH TO TWELFTH CENTURIES)

INTRODUCTION

In the history of mathematics a discovery is sometimes seen to have no real effect for a more or less indefinite period of time. Buried in a "relative absence", it remains intact though outside the mathematical corpus in current use. One may speak of "absence" in so far as when the discovery occurred, it did not emerge as an active part of mathematical practice; but its "absence" is relative since the discovery did take place and was transmitted. Subsequently, even if its transmission was presented as the simple heritage of a succession of authors and not as the communication of a chapter of received mathematics, the discovery becomes an inalienable acquisition for the history of the discipline.

The invention of decimal fractions is a good illustration of the situation we have just described. Here as elsewhere, a rigorous examination will not fail to recognize the "absence" that shrouded it for some time as a component part of the history of the invention itself. However, not infrequently, when historians study inventions – in this case decimal fractions – they adopt towards it one of two attitudes, both of which, though exclusive, negate objective history.

In the most frequent case, they may record the discovery and its various dates in a completely empirical way without in the least explaining its relative eclipse, in which case a good explanation is no more than a clearly established succession, and the temptation is great to rush off to the archives in search of a possible forerunner. Of history, often all that remains is chronology, sometimes archaeology, and always a historical novel.

The more informed historian may isolate the conditions that made the invention possible and then interpret its interrupted advance in terms of theoretical or practical obstacles. He then runs the risk of reducing historical reality to these conditions, and producing as a result, instead of history, either a myth or at best a philosophy of history, since, just as important as the conditions of possibility of a conceptual invention are its own possibilities of application. The latter, by the modifications they enforce, the corrections they require, the extensions they sometimes necessitate, not only delimit the proper domain of existence for conceptual innovation, but even more confer on it true historical reality.

We shall see that the invention of decimal fractions is situated at the completion of two movements, already advanced in the twelfth century; the first aimed to renew algebra by arithmetic by means of the extension of abstract algebraic calculation; the other movement in progress at the same time, in which the theory of decimal fractions is integrated, effected a return to number theory and numerical analysis by renewed algebra. This return was also destined to advanced a topic previously confined to a simple collection of procedures and formulas: numerical methods of approximation.

An examination of the conditions of possibility which we shall continue later on, always possesses a hypothetical and sometimes a heuristic value. It has enabled us in fact to pinpoint a set of discoveries, and present for the first time unpublished and unknown documents which set forth decimal fractions for the first time, the so-called Ruffini–Horner method, as well as the general formula of approximation of the irrational root and other iterative methods for improving approximation. But it then seemed indispensable to try to understand why decimal fractions, already invented and endowed with theoretical means, and the object of effective transmission, had somehow remained outside history and, as a result, it was necessary to examine the conditions governing their application. This invention had to await the relatively late elaboration of the logarithmic function in order to encounter one of its first fields of application as well as its first plane of real existence.

But before developing this new history of decimal fractions, let us stop to consider the almost standard description historians usually provide.

It has in fact been customary to consider Simon Stevin's *La Disme*[1] as a work which sets forth decimal fractions for the first time.[2] For centuries nothing or practically nothing had arisen to challenge this conviction. Admittedly, when historians had reached a clearer understanding of Stevin's Western predecessors, they were somewhat disconcerted, but at no time did they question the priority of the Flemish mathematician.

Indeed, an intermittent usage of decimal fractions in mathematical works prior to Stevin had been observed; for instance, the names of

Christoff Rudolf, Peter Apian and many others have been cited, but it was soon admitted that their knowledge of decimal fractions remained fragmentary and partial. While Stevin gave an explicit exposition, they only gave them for specific problems. This opinion was only to be shattered in 1936 by the discovery of Bonfils' text of 1350 by Salomon Gandz and George Sarton,[3] and in particular Gandz's commentaries; for some time it was believed that the priority for the invention of decimal fractions must in fact be attributed *de facto* and *de jure* to Bonfils.

A balanced reading, freed from the artefact of a predecessor was soon to dispel the threat. In fact, at best in Bonfils' text one only encounters an unstructured programme of the theory of decimal fractions.[4] Nevertheless, whether it concerns a limited occurrence or an unstructured programme, these were the facts that historians wanted to integrate into the history of the invention of decimal fractions. Consequently, a doctrine was seen to emerge which may be summarized as follows: no attempt before Stevin had attained the level reached by him; at most his predecessors possessed a limited knowledge of decimal fractions.[5]

But fairly recent investigations have shown this viewpoint to be erroneous. In 1948, the German historian Paul Luckey (1951, pp. 102ff.), proved that in his *Key to Arithmetic* al-Kāshī (d. 1436/1437) gives an exposition of decimal fractions just as competent as that of Stevin. Furthermore, as his demonstration was irrefutable, historians rallied round Luckey in increasing numbers, and attributed the discovery, and the invention of the thing and the name, to al-Kāshī; this time Stevin's priority was seriously compromised. One would therefore expect the history of this chapter of mathematics to be rewritten, since discussion about priority involves the status of the theory itself. But instead of supplementary analysis, necessary for a clearer understanding of its status, only two attempts are to be found: an eclectic one purely and simply integrates the name of al-Kāshī into the old historical picture of decimal fractions. The other reiterates Gandz's mistake; it hastens to scrutinize the past in search of what might be called the "Bonfils" of al-Kāshī. So, just to give one example of the eclectic tendency, let us read what Struik wrote quite recently: "The introduction of decimal fractions as a common computational practice can be dated back to the Flemish pamphlet *De Thiende* published at Leyden in 1585, together with a French translation, *La Disme*, by the Flemish mathematician Simon Stevin (1548–1620), then settled in the northern Netherlands. It is true that decimal fractions were used by the Chinese many centuries before Stevin, and that the Persian astronomer al-Kāshī used both decimal and sexagesimal fractions with great ease in his *Key to Arithmetic* (Samarkand, early fifteenth century). It is also true that Renaissance mathematicians such as Christoff Rudolf (first half of the sixteenth century) occasionally used decimal fractions, in different types of notation".[6]

The traditional history of decimal fractions was thus expanded to incorporate al-Kāshī, after having implicitly limited the importance of his contribution. As a result, historians of the second tendency did their utmost to make the discovery of decimal fractions date back to the tenth century and attributed it to an Arab mathematician, al-Uqlīdisī.[7] Concerning his *Treatise of Arithmetic – al Fuṣūl* – A. Saidan (1966, p. 484) writes:

> The most remarkable idea in his work is that of decimal fractions. Al-Uqlīdisī uses decimal fractions as such, appreciates the importance of a decimal sign, and suggests a good one. Not al-Kāshī (d. 1436/7) who treated decimal fractions in his *Miftāḥ al-Ḥisāb*, but al-Uqlīdisī, who lived five centuries earlier, is the first Muslim mathematician so far known to write about decimal fractions.

This is a brief outline of our problem based, as can be seen, on the random discovery of texts. It will be understandable why the prudence shown by historians is only apparent, and leads them in many cases to overt eclecticism; moreover, peremptory affirmations such as those of Gandz on Bonfils and Saidan on al-Uqlīdisī, simply point to a hasty and strongly biased reading. Since, in the final count, must not the requisite for a history worthy of its name be to place an invention in its context, and by rigorous conceptual analysis first seek the conditions that made it possible?

In this precise case and in the final analysis, it concerns algebra. Our first task is to isolate these conditions.

I. NUMERICAL METHODS AND PROBLEMS OF APPROXIMATION

The simultaneous location of algebraic concepts and techniques which we made elsewhere (al-Samawʾal, ed., 1972), have enabled us to pinpoint a renewal of algebra in the eleventh century. Initiated by al-Karajī (late 10th–early 11th c.), pursued by his successors, in particular al-Samawʾal (d. 1174), it aimed to "operate on unknown quantities in the same way as the arithmetician operates on known quantities". In short, it means applying arithmetic to the algebra of al-Khwārizmī and his successors. The main instrument for the "arithmetization of algebra" (Rashed, 1974, pp. 63–69; *supra*, I.3), as we cited it, is the extension of abstract algebraic calculation. This instrument proved its effectiveness, not only for the development of algebra itself as "the arithmetic of unknowns", as it was then called, but also for the advancement of number theory and numerical methods as well.

The above interpretation just reviewed has, in our opinion, made it possible to reach a clearer understanding of one of the main tendencies of Arabic algebra. It might have remained, it is true, a simple, plausible, though unrestrictive interpretation. Its ability to account for the facts and concepts of al-Karajī's school, and its capacity to suggest paths which lead to the discovery of new data, suffice to confer upon it greater reliability.

Now an examination of the mathematical works of al-Karajī's school has enabled us to show that:

(1) Several inventions hitherto attributed to fifteenth and sixteenth century algebraists were in fact the work of this tradition. The results achieved by mathematicians from al-Karajī's school include complete theories such as polynomial algebra; key propositions – the binomial theorem and the table of coefficients; tested algorithms – e.g. the divisibility of polynomials; demonstration methods such as mathematical induction.

(2) Al-Kāshī's work is the culmination of a new start initiated by algebraists in the eleventh and twelfth centuries. It includes the essentials of their basic results.

The above description entitles us to advance the following hypothesis: decimal fractions, whose invention is still attributed to al-Kāshī, must have been the work of these eleventh and twelfth centuries algebraists. Did they not possess all the necessary theoretical means for its conception? However, out of all al-Karajī's successors, al-Samawʾal was the most instrumental in isolating an interpretation of the above hypothesis. His algebraic work, which we have already analyzed, is clearly presented as a theoretical and technical contribution for achieving al-Karajī's objective. Furthermore, his *Treatise on Algebra – al-Bāhir* – confirms that of all of al-Karajī's successors, he was undoubtedly one of the most committed to accomplishing his objective.

Now, in another treatise, a *Treatise on Arithmetic*, composed in 1172 (two years before his death), al-Samawʾal gives an exposition of decimal fractions.[8] This treatise is presented as a conclusion to his *Treatise on Algebra*, and consequently, as his ultimate mathematical achievement.

This biographical information enables us to draw a broad outline of the context of this invention.

The results achieved by renewed algebra apparently made possible an authoritative return to arithmetic, seen as its privileged field of application. The generalization of procedures and methods only used for particular cases by arithmeticians was then achieved, as well as providing them with others, unknown to them. This set of procedures and methods was henceforth to be part of what was later to be called "numerical analysis". The theory of decimal fractions saw the light of day as the completion of the first movement of the return as al-Samawʾal's *Treatise on Arithmetic* shows; it was not only a theory, but also a necessary technique for ensuring this return. In short, the first invention of decimal fractions is presented as the theoretical solution of a problem both theoretical and technical.

This description has enabled us to shift the dates of the various discoveries, including decimal fractions, back by two and a half centuries at least, and we are now in a position to pose questions which historians have neglected: why such inventions? why did they occur at that place and time?

To give a detailed description, we first need to understand the conceptual and technical configuration in which the theory of decimal fractions is integrated. However, in al-Samawʾal's *Treatise*, this theory follows several chapters dealing with problems of approximation and, in particular, the approximation of the nth positive root of a number. It involves the approximation of algebraic real numbers where each one is defined as the root of the equation $x^n = Q$, with $n = 2, 3, \ldots$, which cannot however be found with the aid of decimal numbers. By "to approach", al-Samawʾal intends to find a real number by means of a sequence of known numbers with an approximation which the mathematician can make as small as desired. It therefore implies measuring the difference between the nth irrational root and a sequence of rational numbers. For instance, he first writes in general terms:

What is extracted by approximation from irrational roots by means of calculation is by what we want to obtain a rational quantity close to the irrational root. There may exist a rational quantity closer to the irrational root than that. There may then exist a third rational quantity, closer to the irrational root than the second and the first quantity, since for any rational quantity supposed close to an irrational root, the difference between them is in truth a straight line, and a line is capable of being divided and partitioned indefinitely. Which is why it becomes possible to find continually a rational quantity close to the irrational root, and find another rational quantity closer than the first to the irrational, and so on indefinitely.[9]

This is the general problem raised in these chapters. Al-Samawʾal was moreover aware of the difficulty the above interpretation raises when it concerns powers higher than 3; a fascinating problem but beyond the scope of this study. Let us keep the general perspective in mind where the question of approximation is explicitly posed as a measurement of a difference, and see how this problem was introduced and solved.

1. *The Ruffini–Horner Method*

In an important paper published in 1948, Luckey (1948, pp. 217–274) established that al-Kāshī possessed a general method for the extraction of the *n*th root, which is none other than the application of the method invented by the nineteenth-century mathematicians Ruffini and Horner to a particular case. At that time the history of this method including al-Kāshī's other results was completely unknown; but as al-Kāshī, while not citing his predecessors, had not claimed the discovery for himself, historians abandoned their habitual caution and substituted history for myth: they alluded to a twelfth century Chinese source. This viewpoint has persisted since Luckey in spite of recent works[10] just as remarkable on the fifteenth century mathematicians.

We will show that al-Samawʾal's treatise of 1172 contains the Ruffini–Horner method, at least as it was formulated and applied by al-Kāshī about two and a half centuries later. Al-Samawʾal did not claim to be the author either; in his exposition he apparently assumed the reader familiar with the various operations involved and, in fact, it is correct that algebraic concepts and techniques necessary for its formulation go back to al-Karajī's school. We are now in a position to advance our hypothesis: as it appears in al-Samawʾal's *Treatise* of 1172, this method is the work of al-Karajī's school. But we must first of all define the method and clarify its formulation in the twelfth century. We shall avoid repetition by limiting ourselves to one example which fully describes it.

To extract the fifth root[11] of

$$Q = 0;\ 0,0,2,33,43,3,43,36,48,8,16,52,30.$$

which is equivalent to seeking the positive root of the equation

$$f(x) = x^5 - Q = 0. \tag{1}$$

Several stages in search of the solution can be distinguished.

Preliminaries

We first determine the positions of the form nk with $n = 5$ and $k \in \mathbb{Z}$. We obtain in particular the positions 0, –5, –10, –15. Call these positions perfect positions, i.e. positions where the different numbers of the positive root are to be found.

Each position will be marked twice – see Table I.[12]

We add the necessary number of zeros on the right-hand side to obtain the following segments

$$\begin{array}{rrrrr} & & 2 & 33 & 43 \\ 3 & 43 & 36 & 48 & 8 \\ 16 & 52 & 30 & 0 & 0 \end{array}$$

These operations are described by al-Samawʾal as follows:

You write this [Q] like the integers on a horizontal line, you start with the degrees which you write on the left, the far left, and the other positions extend from them to the right. You start with the degrees above which you write down zero or the giver sign – *al muʿṭiya* [the sign of the perfect position]. You pass over four positions and make a sign on the perfect position[13] where $\overline{43}$ is. You pass over four positions and make a sign above $\overline{8}$. In the last line you write down the same signs opposite the perfect positions. You leave the second, third, and fourth lines empty.[14]

TABLE I

first	0					0					0					0
fifth				2	33	43	3	43	36	48	8	16	52	30		
fourth																
third																
second																
first	0					0					0					0

Stage 1

(1) The interval of the root is clearly recognizable, that is $x_0 \in [60^{-1}, 60^0]$; so x_0 is of the form

$$x_1 60^{-1} + x_2 60^{-2} + \ldots + x_p 60^{-p} + r$$

where the x_i are not all nil.

The problem is therefore to determine $x_1, x_2, \ldots, x_p$ in succession. To determine x_1, al-Samawʾal writes

We start by carefully looking at the first perfect position[15] on the left-hand side which is the degree position, we find it empty. We move along to the next perfect position which contains ⟨53⟩ [43]. We seek the highest quantity from which we can subtract the square-cube of this position and the higher positions[16] that follow it, which is 3, ⟨11⟩ 43; we find 6. We write it down on the higher and lower lines. We look at the table of 6 in the sexagesimal tables. We multiply the higher by the lower, I mean 6 by 6, and we write down the product on the second line, and we multiply the higher by the second, and we write down the product on the third, and we multiply the higher by the third and we add the product to the fourth. We subtract the product of the higher by the fourth from the fifth, we obtain this figure [Table II]:[17]

TABLE II

first	0					6					0					0
fifth					24	7	3	43	36	48	8	16	52	30		
fourth					21	36										
third					3	36										
second						36										
first						6					0					

In the above passage just quoted and Table II, we observe that to determine x_1, al-Samawʾal does not seek a fraction but an integer whose fifth power may be subtracted from the first segment considered earlier as a segment of integers and no longer fractions. Similarly, in Table II, he writes down the successive powers of x_1 up to the order $n - 1 = 4$. He thus finds $x_1^2 = 36$, $x_1^3 = 3{,}36$, $x_1^4 = 21{,}36$.

What does this operation consist of exactly? It is in fact the first rule of the method. The mathematician dilates[18] the polynomial by a given positive number. In fact, after dilation of f or ratio $\alpha = 60$, we have

$$\begin{aligned} f_1(x) &= x^5 - 60^5\,Q = x^5 - Q_1 \\ &= x^5 - 2{,}33{,}43;3{,}43{,}36{,}48{,}8{,}16{,}52{,}30 = 0. \end{aligned} \tag{2}$$

To find the highest integer whose fifth power can be subtracted from the first segment simply means determining x_1 such that

$$x_1^5 \leqq Q_1 < (x_1 + 1)^5 \Leftrightarrow x_1^5 - Q_1 \leqq 0 < (x_1 + 1)^5 - Q_1. \tag{3}$$

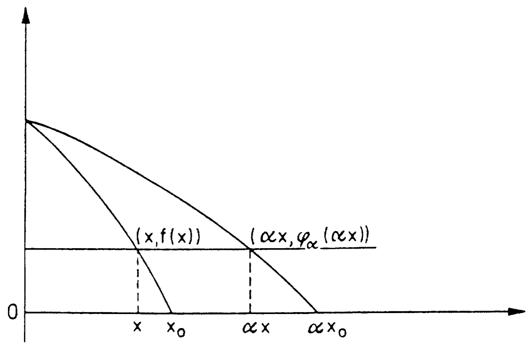


Fig. We remark that if $(x, f(x))$ is any point of the graph of f, then $(\alpha x, f(x))$ is the point of the correspondent graph of φ_α in affinity with ratio α and axis Ox.

(2) Al-Samawʾal then makes this laconic recommendation: "You finish calculating the four lines which must be noted down [in the table] once the *fourteen operations* have been completed".[19] This expression thus seems to suggest that it was an algorithm frequently used by contemporary mathematicians and not his own invention. If he had confined himself to this allusive formula, an essential part of our proof would have been missing; fortunately, several pages earlier, to solve the inverse problem under consideration here, al-Samawʾal had set forth these operations: to find the fifth power of a number.

After calculating the successive powers of x_1, he gives a table,[20] which we have left practically unaltered; we have just added the symbolical notation x_i^j and instead of writing $\frac{1}{48}$ for example, as he does, we use the notation 1, 48.

TABLE III

	first	second	third	fourth	fifth	straight line
5th	$x_1 = 6$	$x_1^2 = 36$	$x_1^3 = 3{,}36$	$x_1^4 = 21{,}36$	$x_1^5 = 2{,}9{,}36$	6
4th	$2x_1 = 12$	$3x_1^2 = 1{,}48$	$4x_1^3 = 14{,}24$	$5x_1^4 = 1{,}48{,}0$		
3rd	$3x_1 = 18$	$6x_1^2 = 3{,}36$	$10x_1^3 = 36{,}0$			
2nd	$4x_1 = 24$	$10x_1^2 = 6{,}0$				
1st	$5x_1 = 30$					

Before commenting on al-Samawʾal's explanation, let us read what he writes:

We add six on the right to six on the left, the left-hand term will be 12. We multiply 6 on the right by 12 on the left, we have 1,12 which we add to the second term [x_1^2]; the second term [$3x_1^2$] will be 1,48. We multiply 6 on the right by the second term 1,48, and we add the product to the third [x_1^3]; the third [$4x_1^3$] will be 14,24. We multiply 6 on the right by the third [$4x_1^3$] and we add the product to the fourth [x_1^4]; the fourth [$5x_1^4$] will be 1,48 [0]. We thus terminate calculating the fourth line.

We add 6 on the right to [12] on the left; we have 18. We multiply 6 on the right by 18 on the left, we have 1,48 which we add to the second [$3x_1^2$]; the second [$6x_1^2$] will be 3,36. We multiply 6 on the right by 3,36 which is in the second, and we add the product to the third, the third will be 36,0. We have thus terminated calculating the third line.

We add 6 on the right to 18 on the left, we have 24. We multiply 6 on the right by 24 on the left and we add the product to the second 3,36 and the second will be 6,0. We have thus terminated calculating the second line.

We add 6 on the right to 24 on the left, the left term will be 30.

Al-Samawʾal then turns to the elements of the diagonal and recalls that they are $30 = 5.6$, $6{,}0 = 10.6^2$, $36{,}0 = 10.6^3$, $1{,}48{,}0 = 5.6^4$, "as is necessary according to the numbers of the square root formula 5 10 10 5".[21]

Two questions arise here: what is this algorithm? Why is al-Samawʾal interested in the elements of the diagonal? We are going to show that this is the second rule of the method.

Let us start by answering the second question. It is clear that the algorithm had been formulated in order to obtain the elements of the diagonal. These elements are none other than the coefficients of the equation obtained by transformation of (2).

In fact, after dilating the function and thus obtaining (2), the mathematician reduces the roots of (2) to x_1. Let $x' = x - x_1$ be the reduced root of x_1, then

$$x = x' + x_1$$

and

$$f_1(x) = (x' + x_1)^5 - Q_1 = \sum_{p=1}^{5} C_5^p x'^p x_1^{5-p} - Q_2$$
$$\text{with } Q_2 = Q_1 - x_1^5.$$

The equation transformed by this reduction becomes

$$f_2(x) = \sum_{p=1}^{5} C_5^p x^p x_1^{5-p} - Q_2 = 0$$
$$(x \text{ being the reduced root}) \qquad (4)$$

hence

$$f_2(x) = x^5 + 30x^4 + 6{,}0x^3 + 36{,}0x^2 + 1{,}48{,}0x - 24{,}7;\ 3{,}43{,}36{,}48{,}8{,}16{,}52{,}30.$$

As for this algorithm it is none other than that of Horner applied to a particular case $x^n - Q = 0$. To determine it, it is sufficient to write Horner's algorithm for the preceding case and then compare it with the one given by al-Samawʾal. We have in fact:

$x_1 = 6$	1	0	0	0	0	$-Q_1$
	1	$x_1 = 6$	$x_1^2 = 36$	$x_1^3 = 3{,}36$	$x_1^4 = 21{,}36$	$-Q_2$
	1	$2x_1 = 12$	$3x_1^2 = 1{,}48$	$4x_1^3 = 14{,}24$	$5x_1^4 = 1{,}48{,}0$	
	1	$3x_1 = 18$	$6x_1^2 = 3{,}36$	$10x_1^3 = 36{,}0$		
	1	$4x_1 = 24$	$10x_1^2 = 6{,}0$			
	1	$5x_1 = 30$				
	1					

with

$$Q_1 = 2{,}33{,}43;\ 3{,}43{,}36{,}48{,}8{,}16{,}52{,}30$$
$$Q_2 = 24{,}7;\ 3{,}43{,}36{,}48{,}8{,}16{,}52{,}30.$$

Then, if we compare Horner's table with that of Samawʾal, we note that they are identical except that in his table al-Samawʾal omits:

1. the first column
2. the number $- Q_2$.

Nevertheless, the absolute value of this number is calculated in the line of the number whose fifth root we wish to extract. Lastly, these differences almost disappear when we remark that here and there, the rule for the formation of a triangle is basically the same. If we call $\alpha_{i,j}$ the elements of this triangle with $1 \leqq i \leqq n$, $1 \leqq j \leqq n - 1$, we have $\alpha_{i,j} = \alpha_{i-1,j} + x_1\alpha_{i,j-1}$.

(3) After dilating the function, finding the first figure of the root and transforming the equation by reducing its roots by this figure, he gives Table IV which expresses the transformed equation in another language.[22]

TABLE IV

first	0					6					0					0
fifth					24	7	3	43	36	48	8	16	52	30		
fourth				1	48											
third					36											
second					6											
first						30					0					

Stage 2

(1) Al-Samawʾal then recommends "transposing the four lines to obtain this table" [Table V].[23]

TABLE V

first						6					0					0
fifth					24	7	3	43	36	48	8	16	52	30		
fourth					1	48										
third							36									
second								6								
first										30	0					

When we examine this table closely, we noted that al-Samawʾal prepares the determination of the second figure of the root x_2 by repeating the same operations. In this way he reduces the search for x_2 to that of an integer and no longer a fraction. He therefore dilates f_2 by $\beta = 60$ to obtain

$$f_3(x) = \sum_{p=1}^{5} C_5^p 60^{5-p} x_1^{5-p} x^p - Q_3 = 0 \quad \text{with} \quad Q_3 = 60^5 Q_2 \tag{5}$$

hence $f_3(x) = x^5 + 30{,}0x^4 + 6{,}0{,}0x^3 + 36{,}0{,}0{,}0x^2 + 1{,}48{,}0{,}0{,}0{,}0x -$ $24{,}7{,}3{,}43{,}36{,}48{,}8;\ 16{,}52{,}30$.

This is the very expression that is to be found in Table V.

(2) We seek to determine x_2 such that

$$f_3(x_2) \leqq 0 < f_3(x_2 + 1) \Leftrightarrow f_3(x_2) + Q_3 \leqq Q_3 < f_3(x_2 + 1) + Q_3. \tag{6}$$

Let $x_2 = 12$ be the second figure of the root, we seek to reduce the roots of $f_3(x)$ of x_2. Let $x'' = x - x_2$ be the reduced root of x_2, then $x = x'' + x_2$ and

$$f_3(x) = \sum_{p=1}^{5} C_5^p 60^{5-p} x_1^{5-p} (x'' + x_2)^p - Q_3 = 0. \tag{7}$$

The equation transformed by this reduction with the aid of Horner's algorithm becomes $f_4(x) = \sum_{p=0}^{4} a_p x^{5-p} - Q_4 = 0$ with $a_0 = 1$; $a_1 =$ $31{,}0$; $a_2 = 6{,}24{,}24{,}0$; $a_3 = 39{,}43{,}16{,}48{,}0$; $a_4 = 2{,}3{,}8{,}10{,}4{,}48{,}0$; $Q_4 =$ $1{,}1{,}44{,}1{,}39{,}40{,}56;\ 16{,}52{,}30$.

This calculation was in fact completed by al-Samaw᾿al by means of two tables. The first (Table VI)[24] aims to calculate

$$Q_4 = Q_3 - [\{[(5x_1 60 + x_2)x_2 + 10x_1^2 60^2]x_2 + 10x_1^3 60^3\}x_2 + 5x_1^4 60^4]x_2.$$

TABLE VI

first	0					6					12					0
fifth					1	1	44	1	39	40	56	16	52	30		
fourth					1	55	26	38	29	45	36					
third							37	13	12	28	48					
second								6	6	2	24					
first										30	12					

The second table (Table VII) is used to calculate the remaining coefficients of the equation transformed by Horner's algorithm, with the reservations already expressed concerning Table III.

We can therefore write as before

TABLE VII

12	1	$5x_1 60 = 30{,}0$	$10x_1^2 60^2 = 6{,}0{,}0{,}0$	$10x_1^3 60^3 = 36{,}0{,}0{,}0{,}0$	$5x_1^4 60^4 = 1{,}48{,}0{,}0{,}0{,}0{,}0 - Q_3$
	1	30,12	6,6,2,24	37,13,12,28,48	1,55,26,38,29,45,36 $-Q_4$
	1	30,24	6,12,7,12	38,27,37,55,12	2,3,8,10,4,48,0
	1	30,36	6,18,14,24	39,43,16,48,0	
	1	30,48	6,24,24,0		
	1	31,0			
				Q_3 = 24,7,3,43,36,48,8; 16,52,30	
				Q_4 = 1,1,44,1,39,40,56; 16,52,30	

(3) When he has dilated the function, found the second figure of the root and transformed the equation by reducing its roots by this figure, he gives Table VIII[25] which expresses the transformed equation in another language.

TABLE VIII

first	0					6					12					0
fifth					1	1	44	1	39	40	56	16	52	30		
fourth					2	3	8	10	4	48						
third							39	43	16	48						
second								6	24	24						
first										31	0					0

One should remark though, that to find x_2 is difficult if, as is the case for x_1, one is satisfied to impose one condition: x_2 is the highest integer whose fifth power is contained in Q_3. Al-Samawʾal gives few details on this point; he simply recommends verifying the binomial development of exponent 5 by this figure. "We then seek [x_2]", he writes, "such that it verifies the conditions of the square cube, we find 12".[26]

If we want to clarify this expression somewhat, we can affirm that x_2 verifies (6), a condition equivalent to (3). To be even more precise, we shall write $f_3(y) = 0$ as follows:

$$[\{[(5x_1 60 + y)y + 10x_1^2 60^2]y + 10x_1^3 60^3\}y + 5x_1^4 60^4]y = Q_3.$$

We then arrive at an approximation y of x_2 if we divide Q_3 by $5x_1^4 60^4$. In this case, it is true, the term obtained may be greater than x_2, but we can then proceed by successive steps to determine x_2.

The above procedure can be interpreted in two other ways; the first one refers to a so to speak empirical observation: knowing that $5x_1^4 60^4 \leqq Q_3$, we operate by successive divisions and trial and error to determine x_2. The second introduces the notion of the derivative when the coefficients of y^k for $k > 1$ are omitted. Nothing justifies the presence of such a notion in the known work of al-Samawʾal. We shall see how the problem will be posed to a certain extent later.

Stage 3

Once the above calculation is completed, we start again to determine the third figure of the root, that is x_3. Unfortunately, at this point[27] the manuscript is mutilated and there is a real break in the text. In order to reconstruct this passage, therefore we shall turn to other examples developed by al-Samawʾal including his study of the inverse problem; a task all the easier since it concerns precisely the same operations. Similarly, al-Samawʾal seeks x_3 as an integer and not as a fraction. For instance, after the dilation of f_4 by $\gamma = 60$, we have

$$\begin{aligned} f_5(x) = x^5 &+ 31,0,0x^4 + 6,24,24,0,0x^3 + 39,43,16,48,0,0,0,0x^2 \\ &+ 2,3,8,10,4,48,0,0,0,0,0x - 1,1,44,1,39,40,56,16,52,30,0,0. \end{aligned} \tag{8}$$

Now let $x_3 = 30$, then

$$f_5(x_3) \leqq 0 < f_5(x_3 + 1) \Leftrightarrow f_5(x_3) + Q_5 \leqq Q_5 < f_5(x_3 + 1) + Q_5.$$

Let $x''' = x - x_3$ be the reduced root, which is equal in fact to 0 in the case under consideration, we have the transformed equation

$$\begin{aligned} f(x) &= x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x - Q_5 = g(x) - Q_5 = 0, \\ g(x) &= [\{[(a_1 60 + x)x + a_2 60^2]x + a_3 60^3\}x + a_4 60^4]x. \end{aligned} \tag{9}$$

the expression given by al-Samaw'al in the form of a table with the following lines

$$[(a_1 60 + x)x + a_2 60^2] = 6,24,39,30,15,0,$$
$$\{[(a_1 60 + x)x + a_2 60^2]x + a_3 60^3\} = 39,46,29,7,45,7,30,0,$$
$$\{[(a_1 60 + x)x + a_2 60^2]x + a_3 60^3\}x + a_4 60^4 = 2,3,28,3,19,21,52,33,45,0,0,$$

i.e. using Horners' algorithm.[28] Finally, we find the root

$$x_0 = ;x_1 x_2 x_3 = ;6,12,30.$$

Thus the only notable difference between al-Kāshī's method and that of eleventh and twelfth century mathematicians is neither the order of ideas nor the symbolism of the tables: it simply concerns presentation. In both expositions the mathematicians handle the same ideas that underlie the Ruffini–Horner method, at least for the particular case of $f(x) = x^n - Q = 0$. To solve this numerical equation the number Q is divided into segments in order to determine the interval of the positive root, then f is dilated or contracted depending on the case, and lastly, the roots of the transformed equation whose coefficients are obtained by means of Horner's algorithm are reduced. The method is repeated until the figures of the root have been used up. Such ideas had already been conceived and applied in a fully algebraic way.

As for the tables, their symbolic function for al-Kāshī like his predecessors was unquestionable; despite their clumsy symbolism, they made the expression of polynomials and operations on polynomials possible. Whether it concerns al-Kāshī or mathematicians from al-Karajī's school, they had recourse to the same symbolism, the only exception being that what al-Kāshī had combined in one table, in a more elegant and therefore less confused way, his predecessors had represented in a series of tables.

It remains to discover if al-Kāshī was acquainted with the arithmetical works of al-Karajī's school. He certainly made no mention of either al-Karajī or al-Samaw'al in his *Key to Arithmetic*, but this is not a decisive argument; it was not common practise then, nor is it today, to cite predecessors in mathematical treatises. However, the results we reached here and elsewhere, have enabled us to establish that the most important propositions in his *Key to Arithmetic*, the very ones that aroused the admiration of historians, are already to be found in al-Karajī's works and those of his successors. A philological study will confirm what the history of mathematics has established. Our certainty does not stop here, but we will only present it as a conjecture: might not al-Kāshī have had first-hand knowledge of al-Samawʾal's *Treatise* of 1172?

If we pursue our reasoning for the particular case of the Ruffini–Horner method, we can also establish a more or less direct filiation between al-Kāshī and his predecessors.

When we presented the previously unknown work of Sharaf al-Dīn al-Ṭūsī[29] for the first time, we drew the attention of historians to one of the most significant contributions of Arabic mathematics. On the subject we recalled that al-Kāshī apparently possessed an indirect knowledge at least of al-Ṭūsī's *Treatise on Equations*. We then gave a detailed exposition and explanation of al-Ṭūsī's method for solving numerical equations, "affected" equations, or polynomials. The tables in al-Ṭūsī's treatise omitted by the copyist, which we have laboriously reconstructed, are eloquent and even a superficial glance can identify a rather old-fashioned form of the Ruffini–Horner method, no longer for the particular case of the extraction of the *n*th root of a number, but for a general case. However, for want of undisputed historical proof, we have refrained from applying this name to al-Ṭūsī's method, estimating we could not take this step lightly. We have advanced the only thesis which then seemed founded: al-Ṭūsī's method, which is not necessarily his own invention, "is in a certain sense more 'modern' than that of Viète".[30]

At that point we lacked the means to affirm that it concerns the Ruffini–Horner method; we then lacked proof of the extraction of the *n*th root, and moreover, a real theory of decimal fractions necessary, as we shall see, for the application of this method in an author who used decimal notation himself, was absent. Today the situation is quite different. Thanks to the discovery of the Ruffini–Horner method in the writings of eleventh and twelfth century mathematicians, applied to the particular case of the *n*th root, and again the discovery of the theory of decimal fractions in those same mathematicians, we are at present in a position to state the problem of the generalization of this method in *historical* and no longer mathematical terms alone, and as a result, examine whether it is legitimate to apply the name of Ruffini–Horner to al-Ṭūsī's method. But to generalize a method is not to extend a set of procedures, all things being equal in other respects.

Now, taken as a the whole, al-Ṭūsī's work does not have its place in the line of the algebraist-arithmeticians of al-Karajī's school to which it is henceforth permitted to link al-Kāshī. Indeed, it represents an essential contribution to another algebra which aimed to study curves by means of equations, thus inaugurating the beginning of algebraic geometry. The importance of al-Ṭūsī's conception for our problem is therefore unquestionable. The generalization of the method undoubtedly requires that the mathematician possesses a surer conception of the phenomenon dealt with, and consequently justify the various operations that intervene in this method: in particular he must therefore justify dilation and solve the problem already encountered in al-Samawʾal's and al-Kāshī's expositions, and complicated by the transition to polynomial equations: the determination of the different figures of the roots starting with the second figure. Noncommittal as to how to find these figures, al-Samawʾal and al-Kāshī were still able to depend on successful trial and error. In this case, to arrive at a reasonably quick result, less empirical procedures are required. We shall confine ourself to one example from al-Ṭūsī[31] to illustrate the above affirmation, and show that the Ruffini–Horner method existed in a relatively general form before al-Kāshī. Let

$$f(x) = g(x) - N = 0$$

with

$$g(x) = x^3 + a_1x^2 + a_2x,$$
$$N = n_0 10^m + n_1 10^{m-1} + \ldots + n_m;$$

first determine the perfect positions of N, i.e. the positions of the form np with $n = 3$, $p \in \mathbb{Z}$. This means determining the segments of three figures that compose N. Let q_0 be the highest integer of the form np such that

$$0 \leqq q_0 \leqq m.$$

Let p_0 be such that

$$q_0 = np_0.$$

Let k_1 and k_2 be decimal orders of a_1 and a_2 respectively and $[k_2/2]$ the integer part of $k_2/2$.

Al-Ṭūsī distinguishes three cases:

$$(1)\quad p_0 > k_1 \quad \text{and} \quad p_0 > \left[\frac{k_2}{2}\right]$$

$$(2)\quad p_0 < \left[\frac{k_2}{2}\right] \quad \text{and} \quad k_1 < \left[\frac{k_2}{2}\right]$$

$$(3)\quad p_0 < k_1 \quad \text{and} \quad \left[\frac{k_2}{2}\right] < k_1.$$

We shall analyze the first case.

EXAMPLE. $f(x) = g(x) - N = x^3 + 12x^2 + 102x - 34345395 = 0$. Let x_0 be the positive root sought, we recognize that

$$x_0 \in [10^2, 10^3[$$

therefore

$$x_0 = \alpha_1 10^2 + \alpha_2 10 + \alpha_3.$$

(1) We first determine the perfect positions, i.e. from right to left, 5, 5, 4.
(2) We contract[32] f by $\beta_1 = 10^{-2}$; which is equivalent to setting

$$x = 10^2 x';$$

we obtain

$$f(10^2x') = (10^2x')^3 + 12(10^2x')^2 + 102(10^2x') - N = 0$$

which is equivalent to

$$f_1(x') = x'^3 + 0{,}12x'^2 + 0{,}0102x' - N_1 = g_1(x') - N_1 = 0$$

with $N_1 = 10^{-6}N = 34{,}345395$. We then have x'_1 the highest integer whose cube is contained in N_1. Such that

$$x'_1 = \alpha_1 = 3.$$

Therefore, if α_1 is the first figure of the root, we have

$$x_1 = 10^2x'_1 = 10^2\alpha_1 = 300.$$

(3) We reduce the roots of $f_1(x')$ of $x'_1 = 3$ by an old-fashioned form of Horner's algorithm. We thus obtain the coefficients of the transformed

equation

$$f_2(y) = f_1(y + x'_1)$$

with

$$y = x' - x'_1$$

and

$$f_2(y) = g_2(y) - N_2,$$

hence

$$\begin{aligned} N_2 &= N_1 - g_1(x'_1) f_2(y) \\ &= y^3 + (3x'_1 + 0{,}12)y^2 + (3x'^2_1 + 2 \times 0{,}12x'_1 + 0{,}0102)y \\ &\quad - [34{,}345395 - (x'^3_1 + 0{,}12x'^2_1 + 0{,}0102x'_1)] \\ &= y^3 + 9{,}12y^2 + 27{,}7302y - 6{,}234795. \end{aligned}$$

It will be noted that for the calculation of the coefficients of the transformed equation, al-Ṭūsī only effects the coefficient of y and N_2.
(4) He then expands f_2 by $\beta_2 = 10$, which is equivalent to posing $y = 10^{-1}y'$. He obtains

$$f_2(10^{-1}y') = 0;$$

which is equivalent to

$$f_3(y') = y'^3 + 91{,}2y'^2 + 2773{,}02y' - 6234{,}795 = g_3(y') - N_3 = 0.$$

One remarks that at the end of the preceding step al-Ṭūsī was preparing the search for the second figure of the root, or more precisely α_2. But if in the first step the real root sought was of the form $\alpha_1 10^2 + \alpha_2 10 + \alpha_3$, after contraction, extraction of the first figure, and reduction, the reduced root to be found is

$$f_2(y) = 0$$

it is therefore of the form $\alpha_2 10^{-1} + \alpha_3 10^{-2}$, which justifies dilation by $\beta_2 = 10$ to find α_2.

It is precisely at this point and without further explanation that he finds $\alpha_2 = 2$. If he does not explicitly indicate the method for determining α_2, the context nevertheless suggests a highly probable answer. In fact, al-Ṭūsī relates the determination of this figure in a direct way to a few operations and proceeds, moreover, in the same way throughout his

Treatise. Furthermore, everything implies that it concerns a method already known and applied.

Firstly, one remarks that to determine the second figure of the root, as well as the following figures, al-Ṭūsī no longer seeks the highest cube contained in N_3. Al-Ṭūsī was perfectly aware that this method was no longer valid since in this case y' determines the decimal order of the root. On the other hand, the determination of the second figure is directly linked to the calculation of N_3 and

$$(3x_1'^2 + 2 \times 0{,}12x_1' + 0{,}0102)10^2.$$

In fact al-Ṭūsī proceeds here as for the calculation of coefficients by means of Horner's triangle and selects N_2 and the coefficient of y, then N_3 and the coefficient of y'. At precisely this stage in his calculation he gives the value of α_2. Everything implies therefore that al-Ṭūsī determines an approached value of α_2 of the form

$$\frac{N_3}{10^2 g_1'(x_1')}$$

which is equivalent to

$$\alpha_2 10^{-1} \simeq \frac{N_2}{g_1'(x_1')}$$

which is the same as omitting the terms of an order higher than one in $g_3(y')$. The method followed to determine the third figure of the root confirms this interpretation.

Although al-Ṭūsī uses a "derivative" procedure in search of maxima in the same *Treatise*, in this instance the "derivative" only functions as an algebraic expression corresponding to a coefficient of y', and therefore necessarily a higher coefficient of the transformed equation. If the "derivative" in this case makes it possible to obtain an approached value of the second figure, it is by reason of its algebraic properties and in no way by virtue of its analytical meaning. This is no more than a procedure for the derivation of formal expressions. The same case occurs moreover in the "divisor" of the so-called Viète's method.[33]

(5) We reduce the roots of $f_3(y')$ of $\alpha_2 = x_2' = 2$ and by means of Horner's algorithm we obtain

$$f_4(z) = f_3(z + x_2') = g_4(z) - N_4 = 0$$

with

$$z = y' - x'_2 \quad \text{and} \quad N_4 = N_3 - g_3(x'_2)$$

hence

$$f_4(z) = z^3 + 97{,}2z^2 + 3149{,}82z - 315{,}955 = 0.$$

(6) We dilate f_4 by $\beta_3 = 10$.
(7) We recommence for the third figure of the root, which is found equal to 1.

In the case where

$$p_0 < \left[\frac{k_2}{2}\right] \quad \text{and} \quad k_1 < \left[\frac{k_2}{2}\right]$$

as for example

$$x^3 + 6x^2 + 3000000x = 996694407$$

and in the case where

$$p_0 < k_1 \quad \text{and} \quad \left[\frac{k_2}{2}\right] < k_1$$

as for example

$$x^3 + 30000x^2 + 20x = 3124315791.$$

Al-Ṭūsī first divides by the coefficient of x and x^2 respectively; this explains moreover the search for the highest cube contained in N.

It must then be noted that al-Ṭūsī justifies the operations of expansion, contraction and division in terms which Viète will use later for the same type of operation. In substance, it is a comparison between the various decimal orders that constitute $g(x)$ according to the various cases on the one hand, and the various segments of N on the other. From al-Ṭūsī to Viète the analogy between the language used and operations carried out is more than striking.

Lastly, note that al-Ṭūsī not only intends to determine the figure or the root, but also wants to give himself the means to verify the figure found at each stage. At each stage of the operation he must therefore compare the decimal order of the root sought with the decimal orders of the coefficients of the equation which leads to the ambiguity visible in the tabular notation.[34] Each term may in fact be read twice at least, according to the position chosen as that of the units: once before dilation or contraction, and once again after these operations have been carried out.

It is therefore established that if al-Kārajī's school was familiar with the Ruffini–Horner method for the particular case examined, this had already been generalized in the thirteenth century, i.e. two centuries before al-Kāshī by a mathematician whose work he knew of at least indirectly.

One final remark, if al-Ṭūsī only deals with third-degree equations – the subject of his *Treatise* – the application of his method to the case of polynomial equations of any degree does not require, as we have shown elsewhere,[35] any other notion unknown to the author. The functional language we have employed to present al-Ṭūsī's method, as well as those of al-Samawʾal and al-Kāshī should not of course mislead; the notion of function as such does not intervene; it is simply a conceptual short-cut which dispenses with algebraic expressions. Therefore, in our notation, $f(x)$ merely represents a polynomial.

2. *The Approximation of Irrational Roots of an Integer*

If we leave the history of the so-called Ruffini–Horner method to tackle the problem of the approximation of the irrational root of an integer, we encounter the same situation, identical names and similar commentaries, for example, the general formula attributed to al-Kāshī and ascribed by Luckey to a thirteenth-century Chinese origin. This myth was however somewhat shaken by the discovery of the same formula in the work of a mathematician about one and a half centuries before al-Kāshī: Naṣīr al-Dīn al-Ṭūsī. Once again we shall show that the rule and its formulation go back in actual fact to al-Karajī's school, therefore to the eleventh and twelfth centuries.

After his exposition of the Ruffini–Horner method, al-Samawʾal devotes a chapter to the problem of the approximation of the nth positive root of an integer, or more exactly, its fractional part.

If you extract the side of a square or a cube or any one of the other powers [*marātib*], if you know the integer side, I mean the side of the cube, or the square or one of the other powers, the closest to the one whose side is sought, and if a remainder is left that shows its side is irrational, if you want to extract fractions approaching the closest its integer from this remainder, you take the numbers [which] the law [namely the table of coefficients] gives you for this power, and you multiply each one by the number it defines; you add them all up, and you add one to the total in each case: what you obtain is the denominator of the remaining parts.[36]

One can rigorously affirm that here al-Samawʾal states a general rule for approaching the non-integer part of the rational root of an integer by fractions. Let us briefly retrace the method al-Samawʾal proposes for this rule. To solve the numerical equation

$$x^n = N; \qquad N \in \mathbb{N}.$$

He first finds the highest integer x_0 such that $x_0^n \leqq N$. Two cases exist:
(1) $x_0^n = N \Leftrightarrow x_0$ is the exact root found; we have seen that al-Samawʾal possesses a tested method for obtaining this result when possible.
(2) $x_0^n < N \Leftrightarrow N^{1/n}$ is irrational. In this case, he states the first approximation as

$$x' = x_0 + \frac{N - x^n{}_0}{\left[\sum_{k=1}^{n-1} \binom{n}{k} x_0^{n-k}\right] + 1} \tag{1}$$

i.e.

$$x' = x_0 + \frac{N - x_0^n}{(x_0 + 1)^n - x_0^n} \tag{2}$$

In the case of the cubic root, we obtain what Arab mathematicians called "conventional approximation".[37]

Al-Samawʾal then gives several examples to illustrate the application of this rule to different cases: square roots, cubic roots, roots of higher powers.[38] For example, he solves $x^5 = 250$ and writes:

Similarly, if we extract the side of the square-cube $\overline{250}$, we obtain $\overline{3}$ and $\overline{7}$ remains. And we find the numbers of the law of the square-cube $\overline{5}$ $\overline{10}$ $\overline{10}$ $\overline{5}$. We multiply $\overline{3}$, I mean the integer side by the first; and the square of three by the second; and the cube of three by the third; and the square-square of three by the fourth; and we add one to the total. We obtain $\overline{781}$, which is the denominator of the remaining parts. The side will be three units plus seven parts of $\overline{781}$ which is the side sought. And according to this rule [for the other powers].[39]

This approximation by default is similar in nature to the one expounded by al-Samawʾal's Arab predecessors, but much more general. While arithmeticians before al-Karajī's school (e.g. al-Nasawī) limited the application of this method to powers equal to or less than 3, here the rule is extended to a power of any degree as will be found later in al-Kāshī. Al-Samawʾal gives us no insight as to how he arrived at the above formula, but if we consider mathematical knowledge of the period, we are in a position to advance two hypotheses: we have here perhaps nothing more than a straightforward application of the binomial formula; or, a second hypothesis, we are perhaps confronted with the generalization of the rule of the false position – *regula falsi.*

In the first case suppose that $x_0 < N^{1/n} < x_0 + 1$ and set $N^{1/n} = x_0 + r$; we have

$$N = (x_0 + r)^n \Rightarrow N = \sum_{k=0}^{n} \binom{n}{k} x_0^{n-k} r^k$$

hence

$$r = \frac{N - x_0^n}{n x_0^{n-1} + \binom{n}{2} x_0^{n-2} r + \ldots + r^{n-1}}$$

where r is equivalent to the fractional part of (2) and consequently of (1).

In the second case, set

$$y = x^{1/n}, \quad x_1 = x_0^n \quad \text{and} \quad y_1 = x_0.$$

Set

$$x_2 = (x_0 + 1)^n \quad \text{and} \quad y_2 = x_0 + 1.$$

Finally, let $x = N = x_0^n + r$, and apply the formula of linear interpolation expressed in words by contemporary mathematicians. We have

$$\frac{y - y_1}{y_2 - y_1} \simeq \frac{x - x_1}{x_2 - x_1} \Rightarrow y \simeq y_1 + \frac{(y_2 - y_1)(x - x_1)}{x_2 - x_1}$$

hence

$$y \simeq x_0 + \frac{N - x_0^n}{(x_0 + 1)^n - x_0^n}$$

hence formula (2), and consequently formula (1).

In both cases, the approach followed supposes resorting to methods known and used by al-Karajī – the binomial formula, the table of coefficients, the rule of the false position – as we have seen. On the other hand, methods of linear interpolation were commonly applied by eleventh-century astronomers, if not earlier, as al-Bīrūnī (Kazim, 1951, pp. 161–170) shows. However, neither the presence of such mathematical methods, nor a reading of al-Samawʾal himself, authorizes us to credit him with the above rule of approximation. In his *Treatise on Algebra* – (*al-Bāhir*) – as in other works, al-Samawʾal explicitly claims his own inventions (al-Samawʾal, ed., 1972, p. 9; Rosenthal, 1950, pp. 560–564), but in the last folio of his *Treatise on Arithmetic*, where he writes that he introduced "inventions for which as far as we know, no one has preceded us", nowhere does he designate one. We shall adopt the same caution for the Ruffini–Horner method and attribute the formula and exposition of the method of approximation to al-Karajī's school as well.

3. *Methods and Procedures for Improving Approximation*

The above conclusion is also applicable to a set of procedures proposed by al-Samawʾal who aims to improve the approximation of the irrational root of an integer. The first one at least, about which we have very little historical information, is particularly significant: al-Samawʾal explicitly seeks to construct a sequence of rational numbers that converge towards a given algebraic real number. Moreover, as the procedure he seeks ought to allow him to derive the various approximations in a recursive way, he opts for a deliberately iterative method. But here, and this will still hold true in the seventeenth-century, the mathematician disregards theoretical problems of existence and even ignores all theoretical justification. Such considerations are recent; like seventeenth-century mathematicians studying numerical methods, al-Samawʾal only wants to obtain verifiable results. But before commenting, let us quote al-Samawʾal:

If you extract the irrational root of the number [. . .]* and if you want to equalize it, *taʿdīl* [improve the approximation] by this calculation, multiply the side [. . .]* by itself, and find the difference, *al-tafāwut*, between the product and the quantity whose root you wish to approach. Then divide this error by double the integer root; add the quotient of the division to the side if the error is negative, and subtract it from the side if the error is positive. You will thus find the side equalized [improved]. It is always closer to the truth than what precedes it.

Then multiply this equalized side by itself and know the quantity of the difference,

* Text mutilated.

which is the second error. It must be smaller than the preceding error. Divide this error by the double of the integer part of the side, the result is the third side. It must be closer to the truth than the second side. If you are satisfied with it, so much the better, if not you raise it to the square, and the [difference] that you manage to find between its square and the number whose root is sought must be smaller than the preceding error. You divide the difference by double the integer part of the side and you add the quotient to the side obtained earlier, I mean you add it* or subtract it according to whether the error is added or subtracted, you thus obtain the side [. . .]*.

Al-Samawʾal concludes: "We thus obtain rational quantities whose number is infinite, and each one is closer than the preceding one to the quantity we seek to approach".[40]

It must be noted that al-Samawʾal does not limit the usage of this method to the particular case $n = 2$, $n = 3$, but expounds it for general cases. He then writes, you must then divide the difference by double the $(n - 1)$th power of the integer part of the root, to which you add the sum of lower powers to $(n - (n - 1))$. In other words, al-Samawʾal finds the nth approached root for integer x.

Let a be an integer such that $x^{1/n} - 1 < a \leqq x^{1/n}$.

Let x_0 be a rational such that $x_0^{1/n} \leqq x^{1/n}$ and $a \leqq x_0^{1/n}$.

Set

$$\begin{aligned} x &= (a + \alpha)^n, \quad && \alpha \geqq 0, \\ x_0 &= (a + \beta)^n, \quad && 0 \leqq \beta \leqq \alpha. \end{aligned}$$

The approximation of order 1 is given by the formula

$$f(x) \simeq f(x_0) + \frac{x - x_0}{2a^{n-1} + \sum_{p=1}^{n-2} a^p} \qquad \text{with} \qquad f(u) = u^{\frac{1}{n}} .$$

By iteration, the approximation of order $k + 1$ $(k = 1, 2 \ldots)$ is written

$$f(x) \simeq f(x_k) + \frac{(x - x_k)}{2a^{n-1} + \sum_{p=1}^{n-2} a^p}$$

Al-Samawʾal gives two numerical examples;[41] we have chosen to present the easier of the two

$$n = 2, \qquad x = 5, \qquad x_0 = \frac{121}{25}, \qquad a = 2.$$

* Text mutilated.

For the first approximation, we have

$$\sqrt{x} \simeq \sqrt{x_0} + \frac{(x - x_0)}{2a} \Rightarrow \sqrt{5} \simeq \frac{11}{5} + \frac{1}{25}\,.$$

For the second approximation we have

$$\sqrt{x} \simeq \sqrt{x_1} + \frac{(x - x_1)}{2a}$$

with

$$x_1 = \left[f(x_0) + \frac{(x - x_0)}{2a}\right]^2 = \left[\frac{11}{5} + \frac{1}{25}\right]^2.$$

He next seeks the third approximation in the same way. Note that for $n = 2$, the expression

$$f(x) \simeq f(x_k) + \frac{(x - x_k)}{2a^{n-1} + \sum\limits_{p=1}^{n-2} a^p}$$

is close to

$$f(x) \simeq f(x_k) + \frac{(x - x_k)[f(x_k) - f(x_{k-1})]}{x_k - x_{k-1}}$$

which is the form of the *regula falsi*. For $n > 2$ the expression

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

which is equivalent to $1/(na^{n-1} + R(a))$ is replaced, i.e. corrected by a weight of greater magnitude which is

$$\frac{1}{2a^{n-1} + \sum\limits_{p=1}^{n-2} a^p}$$

That this method was deduced from the *regula falsi* is highly probable; al-Samawʾal had already applied this rule in *al-Bāhir* (ed., 1972, pp. 66ff., Fr. intr.), just like most other mathematicians of al-Karajī's school. The choice of the last "weight" was motivated by a theoretically unfounded generalization of the method. Comparatively simpler than the traditional method $[f(x) \simeq f(x_k) + (x - x_k)/2f(x_k)]$, though more longwinded in the case of the square root, it proves to be unsound for the *n*th root.[42]

Apart from the iterative method encountered here for the first time, al-Samawʾal's *Treatise* proposes other methods for improving approximation for the particular case of the square and cubic roots which were, on the other hand, already known to arithmeticians before al-Karajī's school, for example, al-Uqlīdisī (ed., 1973, pp. 416ff.), Abū Manṣūr al-Baghdādī[43] and many others. Their general formulation, attributed to al-Kāshī until now, dates back in fact to the twelfth-century. They include the following rules:[44]

$$\sqrt[n]{x} = \sqrt[n]{10^{nk}x}/10^k \qquad k = 1, 2, \ldots$$

$$\sqrt[n]{x} = \sqrt[n]{a^n x}/a \qquad a \text{ positive integer}$$

$$\sqrt[n]{x} = \sqrt[n]{(a^n \times b^n \times \ldots \times l^n)x}/a \times b \times \ldots \times l \qquad a, b, \ldots, l \text{ positive integers.}$$

II. THE INVENTION OF DECIMAL FRACTIONS

Before writing the history of decimal fractions, we do need to reiterate that it is one thing to use them on random occasions for the computation of common fractions and another to provide a conceptual and articulated exposition of the decimal representation of a fraction. Only in the case of decimal fractions can one distinguish a clear perception of the meaning of the notation for the mathematician, and affirm that he chose these fractions for themselves, deliberately opting for this representation. By failing to observe this elementary, even obvious rule, some historians working on the problem have been tempted to find an invention, though dated and localized, present everywhere; we shall simply quote Sarton's extensive classical study (1935, pp. 151–244) and Saidan's more recent article (1966, p. 489).

Whether we consider Arabic mathematics in the tenth and twelfth centuries or limit ourselves to al-Samawʾal's work, setting aside his *Treatise* of 1172 for the time being, in both cases one is surprised to find a usage of decimal fractions that implies no recognition of these fractions as such: one only needs to consider all the arithmetic operations carried out on common fractions whose denominator is a power of ten. It would be pointless here to accumulate such facts; a specific example, evidence in the strongest sense of the word, would be much more eloquent and allow us to raise a problem usually connected with the birth of decimal fractions. We shall see that several names, important ones at that, are associated with this problem; for instance, al-Samawʾal himself in the texts preceding his theoretical exposition of decimal fractions.

From the tenth century, if not before, in numerous Arabic treatises on arithmetic, one encounters a rule for approaching the irrational, square and cubic roots; a rule then called the "rule of zeros". The general formulation of this rule is to be found in al-Samawʾal's *Treatise*

$$(a)^{\frac{1}{n}} = \frac{(a \cdot 10^{nk})^{\frac{1}{n}}}{10^k} \quad \text{for} \quad k = 1, 2, \ldots$$

Now the approximation obtained according to this rule necessarily includes a decimal fraction. This observation led a historian such as Sarton (1935, pp. 168ff.) to attempt to integrate into the history of decimal fractions authors who proceeded by applying this rule. There is no basis for affirming that by following this procedure the mathematician has grasped and understood the decimal representation of the fraction; moreover, he would on occasion transform it directly into a sexagesimal fraction. For example, in a *Treatise on Arithmetic* composed in 952, to which we shall return later, al-Uqlīdisī (ed., 1973, pp. 133–134) states the "rule of zeros" for the particular cases of square and cubic roots, applies it to the approximation of the square root of 2, only to transform the result straight away into a sexagesimal fraction. The same approach for extracting the square root of 5 is to be found in another treatise on arithmetic composed by al-Baghdādī (d. 1037) and entitled *al-Takmila fī ʿilm al-ḥisāb*. Lastly, this method was also followed by an eleventh-century mathematician, al-Nasawī in his treatise *al-Muqniʿ*.[45] We could multiply examples, all of which would support this idea; whereas the mathematician immediately converts them into sexagesimal fractions and does not show sufficient interest in identifying decimal fractions. Of even greater significance are al-Samawʾal's writings before his *Treatise* of 1172. In a treatise entitled *al-Tabṣira fī ʿilm al-ḥisāb*, he recalls the "rule of zeros" and applies it to the extraction of the square root of 1020. He first obtains

> 31 plus nine hundred and thirty-seven parts of a thousand, you simplify it [. . .] and the answer will be 31 plus a half, plus two fifths, plus a fifth of a tenth, plus a tenth of a tenth, plus a half of a tenth of a tenth, plus a fifth of a tenth of a tenth, which is the square root of one thousand and twenty, whose difference with the truth is not perceptible.[46]

Our thesis is thus corroborated by these different examples: apparently none of these authors had effectively conceived the decimal representation of fractions. Nowhere is there yet a hint of the type of representation which will arise later and which was already present in al-Samawʾal's *Treatise* of 1172. Until then, one only encounters, at best, an intuition still submerged in empirical practice.

1. *Al-Karajī's School: Al-Samawʾal*

In his *Treatise* of 1172 one can note a sporadic usage[47] of decimal fractions; al-Samawʾal's theoretical exposition only comes at the end of his work; to be more precise, it follows an exposition of methods and problems of approximation described earlier. And as a result the final chapter is therefore, as we remarked, a direct development of the preceding chapters; one of al-Samawʾal's aims is to improve methods of approximation. This is therefore the context in which the exposition of decimal fractions intervenes and enables us to clarify its function in the overall plan of the work. Al-Samawʾal's real aim is to unify and generalize, as the title itself of the chapter on decimal fractions confirms: "On the place of a unique principle whereby one may determine all partition operations *(al-tafrīq)*, that is, division, the extraction of square roots, the extraction of a side for all powers and the correction of all fractions that appear in the operations, indefinitely".[48] By the expression the "indefinite correction of fractions", al-Samawʾal means to give to the latter a form so that they can be calculated like integers, and from then on, it is possible to correct the approximation of different operations indefinitely.

The title is a program in itself and its clarity makes commentary superfluous. Before giving its exposition, let us simply recall that the theory of decimal fractions was presented as a technical solution to the problem of approximation, theoretical and technical at the same time.

"Given that", writes al-Samawʾal,

proportional places, *(marātib)*, starting with the place of the units [10^0], follow one another indefinitely according to the tenth proportion, we therefore suppose that on the other side [of 10^0] the place of the parts [of ten follow one another] according to the same proportion, and the place of units [10^0] lies half-way between the place of the integers whose units are transferred in the same way indefinitely, and the place of indefinitely divisible parts.

We call the place that follows the place of units [10^0] the place of parts of tenths, and the one that follows this one the part of hundreds, and the one that which follows this one the part thousands and so on.

If, when calculating division, or the square root, or the side of a cube, or the square-square, or when, in another chapter on partition (*al-tafrīq*), we reach the place of the units [10^0], we do not stop calculating but transfer the lines [of the table] that should be transferred in the figure [table] below the row of the parts of the tenths for this place and what we obtain is a part of 10. If we continue our calculations, we transfer the [lines of the table] underneath the place of parts of tenths and under this place we obtain parts of hundreds. And here is the figure of the indicated places.[49]

parts of tens of thousands of thousands of thousands of thousands	0	
parts of thousands of thousands of thousands of thousands	0	4
parts of hundreds of thousands of thousands of thousands	0	
parts of tens of thousands of thousands of thousands	0	
parts of thousands of thousands of thousands	0	3
parts of hundreds of thousands of thousands	0	
parts of tens of thousands of thousands	0	
parts of thousands of thousands	0	2
parts of hundreds of thousands	0	
parts of tens of thousands	0	
parts of thousands	0	1
parts of hundreds	0	
parts of tens	0	
unit place	0	0
place of tens	0	
place of hundreds	0	
place of thousands	0	1
place of tens of thousands	0	
place of hundreds of thousands	0	
place of thousands of thousands	0	2
place of tens of thousands of thousands	0	
place of hundreds of thousands of thousands	0	

place of thousands of thousands of thousands	0	3
place of tens of thousands of thousands of thousands	0	
place of hundreds of thousands of thousands of thousands	0	
place of thousands of thousands of thousands of thousands	0	4
place of tens of thousands of thousands of thousands of thousands	0	

It may be observed that

(1) Al-Samawʾal starts by establishing the ratio: 1:10 = 10:100 = 100:1000, and so on to infinity.

(2) As his first sentence and table indicate, he sets $10^0 = 1$. The last line of the table shows that he explicitly writes the zero sign under the unit place.

(3) His objective is to extend the notion of the power of a quantity to its inverse. More precisely, having supposed $10^0 = 1$, he sets $1:\frac{1}{10} = \frac{1}{10}:\frac{1}{100}$, and so on to infinity.

(4) Lastly, he suggests that here the calculation is the same for algebraic quantities in general: the examples he gives later amply confirm this suggestion. To sum up, we shall say once $10^0 = 1$ is set, it involves placing two sequences $10, 10^2, \ldots; \frac{1}{10}, \frac{1}{10^2}, \ldots$, on either side of 10^0, and applying the general rules obtained by the algebraic calculation of powers. From now on any real number therefore admits a limited or unlimited decimal representation.

Through these results, al-Samawʾal achieves his project of generalization and formulates a unique principle enabling the indefinite correction of approximations. This theory can be explained, in this instance at least, by an extension of the concept of the power of an algebraic quantity to its inverse.

We have described elsewhere that the extension of this concept of algebraic power was the work of al-Karajī's school, and in particular, al-Samawʾal himself. These mathematicians found in it an instrument that enabled them to apply elementary arithmetic to polynomials and so achieve al-Karajī's project described above. But the greatest difficulty they had to overcome in order to achieve this extension was to give an explicit formulation of zero power: $x^0 = 1$, $x \neq 0$, for which al-Samawʾal himself gives a solution. Once this obstacle had been overcome, he was able to state the equivalent rule to

$$x^m x^n = x^{m+n} \qquad m, n \in \mathbb{Z}.$$

Thanks to tabular symbolism, al-Samaw'al places the two sequences of $x, x^2, \ldots; \frac{1}{x}, \frac{1}{x^2}, \ldots$, on either side of x^0; and his calculation of $x^n \frac{1}{x^{n'}}$, $n, n' \in \mathbb{Z}$, consists of counting from row n, n' rows in the direction of the unit. When it concerns the product $x^n x^{n'}$, he counts n' rows again, but in the opposite direction to the unit. The rule amounts to dealing with powers of the form $\frac{1}{x^{n'}}$ like $x^{-n'}$ and adding powers algebraically. Thus, after having drawn up the following table,[50] he writes:

If two powers lie on either side of the unit, from one of them we count in the direction of the unit, the number of elements of the table that separates the other power from the unit and the number is on the side of the unit. If two powers are on the same side of the unit, we count in the opposite direction to the unit.[51]

It is this conception that made possible the application of elementary arithmetical operations to algebraic expressions of the form

$$f(x) = \sum_{k=-m}^{n} a_k x^k \qquad m, n \in \mathbb{Z}+,$$

and to polynomials in particular.

In turn, all these results make it possible to elaborate the theory of decimal fractions. With al-Karajī's suggestion and the extensions he obtained, al-Samaw'al only needs to substitute x in the last table for 10 – which is what he did to achieve the table of decimal fractions; then adopt a notation previously used for the case of polynomials in general to obtain the decimal representation of any algebraic number; and lastly, apply the operations elaborated earlier for polynomials in the widest sense to these representations so as to obtain at the same time the rules for calculating fractions.

Everything concurs in testifying that the invention of this algebra was necessary for a truly general expression of decimal fractions. Once again it is seen that the road to scientific discovery is neither the shortest nor the most direct.

9	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	8	9
x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{x^3}$	$\frac{1}{x^4}$	$\frac{1}{x^5}$	$\frac{1}{x^6}$	$\frac{1}{x^7}$	$\frac{1}{x^8}$	$\frac{1}{x^9}$
512	256	128	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
19683	6561	2187	729	243	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{243}$	$\frac{1}{729}$	$\frac{1}{2187}$	$\frac{1}{6561}$	$\frac{1}{19683}$

Having reached this stage in his exposition of decimal fractions, al-Samawʾal was quite naturally confronted with the famous problem of notation, and consequently induced to deal with it indirectly at least. The solution of the problem, as we just remarked, was already given at the same time as decimal fractions. However, this notation had to fulfil two requirements whether expressed in words or symbolically. The first one, theoretical, was partially fulfilled by expressing polynomials by means of tables: the limited or unlimited representation of any real number then known had to be possible; the second, of a practical nature, concerns the possibility of *pronouncing* such a representation. Satisfying the latter condition is dependent on the integration of the body of decimal fractions into a practice other than purely bookish.

The significance of the problem of notation that faced al-Samawʾal is clear if we place it in its context, i.e. the algebra of the period as a whole. Everything seems to imply that, to be operative, this notation of decimal fractions had been chosen according to a notational system used in algebra. While not pretending here, of course, to study algebraic notation at the time of al-Samawʾal, let us simply recall that algebra was essentially verbal, but the absence of a symbolical notation was *partially* compensated by what we described earlier as the "tabular method". Its principle is simple: we write down verbally the various powers x^n, $n \in \mathbb{Z}$ on the first line. We write the coefficients on a second line below the first, and for each operation, we stipulate a set of rules for adding supplementary lines and transferring them.

If this method – tabular symbolism – was still rather cumbersome, it nevertheless enabled the execution of all algebraic operations on polynomials in general. It is undoubtedly due to its relative effectiveness that one must attribute the permanence of this method of notation by mathematicians several centuries later, such as Viète and Wallis.

In such a context the problem of the notation of decimal fractions had therefore to be posed in the tradition of the tabular method. Moreover, al-Samawʾal gives examples that tend to confirm our analysis; he applies the same operations he effected on integers written as decimals to decimal fractions without justification.

For instance, the first example given is the simple division of 210 by 13. Al-Samawʾal emphasizes in the first place that this division may be continued as far as one wants; to recall his own words: if we continue the operation "to the places we want, and if we restrict ourselves to five places",[52] we obtain the result written as follows:

16	1	5	3	8	4
integers	part of ten	parts of a hundred	parts of a thousand	parts of ten thousand	parts of one hundred thousand

As we can see this notation is based on the following principle: to separate the integer part and represent the fractional part according to al-Samawᵓal's technique also used in his *Algebra* for representing a polynomial. But if this notation was effective for the calculation of tables, it was difficult to pronounce and consequently restricted in practical scope.

Similarly, in other examples, al-Samawᵓal modified it in the sense just indicated: these corrections emphasize the succession of places rather than terms: parts of ten, parts of a hundred, parts of a thousand, etc. and make pronunciation possible. The improvement is clear in his second example, the extraction of the square root of 10, where he writes down the result as follows:[53]

tens	units	tenths	tenths of a tenth	tenths of a tenth of a tenth	tenths of a tenth of a tenth of a tenth	tenths of a tenth of a tenth of a tenth of a tenth	tenths of a tenth of a tenth of a tenth of a tenth of a tenth
	3	1	6	2	2	7	7

In fact, although the notational principle here is fundamentally identical to the previous one, al-Samawᵓal clearly wants to bring out the succession of places as well as their rank, by repeating the same term as often as necessary. Therefore, instead of using the clumsy expression "parts of ten, parts of a hundred, parts of a thousand, etc." The equivalent can be written:

$$\begin{array}{cccccccc} 10 & 10^{\circ} & \frac{1}{10} & \left(\frac{1}{10}\right)^2 & \left(\frac{1}{10}\right)^3 & \left(\frac{1}{10}\right)^4 & \left(\frac{1}{10}\right)^5 & \left(\frac{1}{10}\right)^6. \\ & 3 & 1 & 6 & 2 & 2 & 7 & 7 \end{array}$$

The *n*th place is thus indicated by the repetition of the term *n* times, the "tenth" – ᶜ*ushr*.

This unification assumes an importance that may elude a modern reader; it does in fact lie at the origin of a proper noun which was used later to name these fractions, ᶜ*ushria*, or *a*ᶜ*shāria*, i.e. tenths. That said, despite this improvement in notation, the difficulty remains if one wants to pronounce such a number. To get round this difficulty, al-Samawʾal was inspired by a notation then used for ordinary fractions and relates the fractional part to the same denominator; he thus arrives at the following notation,

$$\begin{array}{ccccccc} & & & & & & 3 \\ & 1 & 6 & 2 & 2 & 7 & 7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

which reads: 3 units plus 162277 of 1.000.000, or, as he writes:

> If we want all the fractions obtained to have the same denominator, we transfer the place of the units as well as [the places] that follow it – tenths, hundreds and the other [places] of integers to a higher line and underneath the other places [of the fraction]. We write down zeros, and after the zeros, one.[54]

Thanks to this notation and while respecting the same principle of separation between the integer and fractional parts, one manages to represent a pronounceable number.

To conclude his exposition, al-Samawʾal briefly recalls the initial purpose of the theory of decimal fractions: to enable the application of various operations – division, the extraction of *n*th root of fractions, in the same way as for integers ("as we operate on integers") and consequently make the indefinite correction of approximations more obvious and easier.

This reminder is followed by a second conclusion, where al-Samawʾal vigorously underlines the aim of the entire exposition: everything seems to imply that we are confronted with the grasp of an essential idea though, as will be understood later, as yet undemonstrated: the *n*th non-decimal root of any positive number is the limit of an ascending sequence $(a_n)_{n\geq 1}$ of decimal values, a_n being an approached value $\frac{1}{10^n}$ to the nearest decimal point.

"And thus", al-Samawʾal concludes, "we operate to determine the side of a cube, of a square-square, a square-cube and other [powers] this method enables us to understand fully the exact operations for partition and obtain an infinite number of answers, each one being more precise and closer to the truth than the preceding one".[55]

We have therefore seen with al-Samawʾal that the theory of decimal fractions was elaborated within the context of the problem of the extraction of the *n*th root of a number and problems of approximation. It remains for us to return to the forerunners of al-Karajī's school in order to show that it was effectively with mathematicians from this school that the first exposition of this theory is encountered.

2. *The Case of al-Uqlīdisī (952)*

The only known forerunner is al-Uqlīdisī, to whom historians quite recently thought it possible to grant a privileged status in the history of decimal fractions. Was the discovery of decimal fractions not attributed to him? Has it not been affirmed that he used them "as such" and "appreciated the importance of the decimal sign"?[56] Entrenched in this conviction, some historians have gone one step further and without further examination have declared they see al-Uqlīdisī's *Treatise* as "the explanation and application of decimal fractions".

It is appropriate to examine what induces these historians, though undoubtedly well informed, to arrive at such a reading, and in particular, to ask ourselves if this abusive historical recurrence should not be ascribed to the ambiguity of the text. On several occasions in his *Treatise*, al-Uqlīdisī undoubtedly poses specific problems which he solves by resorting to decimal fractions. We have already had the opportunity to set forth the "rule of zeros" that permits the solution of these problems, the extraction of the square and cubic root.

The other two problems are as follows:

(1) To increase – or decrease – a given number by its tenth, successively, as often as one wants.

(2) To halve an odd number a certain number of times, as well as the inverse operation.

Apart from these specific problems, nothing in al-Uqlīdisī's *Treatise* suggests a recourse to decimal fractions; it is moreover indisputable that he gave no exposition comparable to that of al-Samawʾal.

Given these conditions, one can ask in what way does al-Uqlīdisī's contribution differ from those we thought unable to integrate into the history of decimal fractions? In other words, was al-Uqlīdisī acquainted with decimal fractions other than in an intuitive or accidental way? To answer this well-defined question, let us return to one of his most important works. The first deals with raising a number of its tenth, five times. He writes (ed., 1973, p. 150):

> . . . we want to raise a number by its tenth five times. We write down this number as usual; we write it again below moved one place to the right; we therefore know its tenth, which we add to it. So we have added its tenth [to this number]. We put the resulting fraction in front of [this number] and we move it to the unit place after marking it [with the sign ′] thus. We add its tenth and so on five times.

He continues (ed., 1973, p. 150):

> For example, if we want to increase 135 by its tenth five times, we write it down again underneath moving one place to the right and we mark the place of the units, we thus obtain $\begin{smallmatrix}135\\ 135\end{smallmatrix}$ and the result is 1485. After finding a tenth, we add its tenth a second time, the result is $\begin{smallmatrix}1485\\ 1485\end{smallmatrix}$ which we add to it, the result is 16335, which is one hundred and three plus thirty-five of a hundred which is a quarter plus a tenth. After finding its tenth, we add its tenth, which we then add to it, the result is $\begin{smallmatrix}163'35\\ 16335'\end{smallmatrix}$ if we add it, the result is 179′685; what is in front of the place of the units which is 685, is raised to a thousand, since the place of the units is fourth for it. If we add its tenth a fourth time, the result is 1976535 and if we add its tenth, the result is 21741885 and we carry what is in the place of the units which is 41885 to a hundred thousand. We have added the tenth of 135 five times.

It was basically this passage that provided the basis for discerning the emergence of decimal fractions in al-Uqlīdisī's work. Such an interpretation apparently neglects serious difficulties which nonetheless confront a careful examination. A clear distinction should be drawn between what concerns ordinary division by one or another power [a positive integer of 10] and what refers to a deliberate usage of decimal fractions to understand the extension of the concept of place, and consequently, the exact meaning of the sign used. However, al-Uqlīdisī's silence on these various points, aggravated by the ambiguity of his expressions in general, do not facilitate attempts to unravel his real intentions. For the time being one negative certainty emerges: unlike al-Samawʾal, al-Uqlīdisī never formulates the idea of completing the sequence of powers of ten by that of their inverse after having defined zero power. That said, in the passage just quoted, three basic ideas emerge whose intuitive resonance may have misled historians: what they thought was a theoretical exposition was merely understood implicitly, and as a result, they have overestimated the author's contribution to decimal fractions. We note that al-Uqlīdisī

(1) repeats the same number by lowering its place;
(2) raises the fraction to the unit place;

(3) indicates the fraction by a sign.

However, such ideas raise more problems than they solve; for instance, (1) involves operations that govern all the others: the lowering of a place. But how does one proceed when by "place" – *manāzil* – one only means units, tenths, hundreds and their successive products? We may search al-Uqlīdisī's treatise in vain for another definition or usage of this fundamental concept.

Another text by al-Uqlīdisī, where the problem of division by ten is somewhat neutralized, may clarify the author's ideas. It concerns halving an odd integer as often as one wants.

The author states his rule as follows:

> Halving in any way is five in the place before it. And this necessitates that when we halve an odd number, we make the half of the unit five before it and we put over the unit place a mark [′][57] by which we distinguish the place.
>
> So the value of the unit place is a tenth to that before it. Now five can be halved just as integer numbers are halved and the value of the unit place in the second halving becomes hundreds and this may continue indefinitely.[58]

For reasons mentioned earlier, one should first of all be wary of translating this rule by the formula $\frac{1}{2}10^m = 5 \times 10^{m-1}$, $m \in \mathbb{Z}$, and secondly, keep in mind the two following ideas:

(1) By successive halving the unit place, though basically remaining the same, becomes that of tenths, hundreds, etc.

(2) Half of each position – units, tenths, thousands, etc. – is five before.

Whether one considers their formulation or applications, these ideas indicate that although al-Uqlīdisī has an intuitive grasp of the decimal representation of a fraction, he immediately moves away. This is where the real difficulty and limitation of al-Uqlīdisī's contribution concealed by "modern" interpretation lie. When al-Uqlīdisī assumes the unit place to be 10^k on kth division by 2, it is in order to retain positive integer powers. It is as if the calculation of fractions had to be transformed into the calculation of decimal integers and as if the sign [′] was intended to indicate the number of divisions: two in the case just mentioned, ten in the case before. Perhaps this explains why al-Uqlīdisī only resorted to these ideas for the particular cases of halving and dividing by ten. At no time does he consider applying these rules to fractions as simple as $\frac{19}{3}$, let alone the division of any two numbers.

Without minimising the importance of his intuition or the appropriateness of the sign chosen to indicate the unit place, we are nevertheless forced to conclude that this does not suffice to make al-Uqlīdisī the inventor of decimal fractions. He still lacks the means – polynomial algebra – to free himself from the immediate past, that is guess the form of what will arise much later, in short, invent. So his contribution remains a preliminary to its history, whereas al-Samawʾal's text already constitutes the first chapter.

3. *The Status of al-Kāshī*

It is difficult, even impossible, to describe how al-Samawʾal's work was received during the two and a half centuries that separate him from al-Kāshī (1436/7) and estimate the usage and scope of this theory of decimal fractions for mathematicians of that period. However, an examination of numerous treatises on arithmetic and algebra composed at that time, enables us to isolate a dominant trend: this exposition of decimal fractions remained outside active mathematical usage necessary for teaching, research and practice. But the fact that he was not mentioned by name in contemporary mathematical work does not permit us to infer that his work was neither transmitted nor commented on. On the contrary, it would not surprise us one day if one were to discover that al-Samawʾal's exposition had been pursued and improved by a thirteenth or fourteenth century mathematician. Such an eventuality would in no way modify the general trend indicated above which requires an explanation. So it is important to follow al-Samawʾal's exposition over two and a half centuries in order to study the changes it underwent. It is therefore quite natural to focus attention on the only known al-Samawʾal's successor to have taken up and made use of decimal fractions, namely al-Kāshī.

One observation should be made straight away: while the "thing" in the *Treatise* of 1172 is unquestionably present though unnamed, in al-Kāshī's *Key to Arithmetic*, it is designated: decimal fractions, "*al-Kusūr al-aʿshāriyya*". Whether this denomination is due to al-Kāshī or a predecessor, nothing enables us to reach a decision. We shall simply say that, like some new aspects we are about to examine, it is mentioned in al-Samawʾal's *Treatise*. Without overestimating the importance of the designation, we cannot remain unconcerned by the need and determination revealed by defining the "thing" by name. This need and determination may attest a knowledge of the mode of being of what is to be designated. In order to confirm this idea, we must now examine al-Kāshī's work where he mentions and uses decimal fractions; it concerns his two most important works, the *Treatise on the Circumference of a Circle* and a later one, the *Key to Arithmetic*.

In his *Treatise on the Circumference of a Circle* (*al-Risāla al-Muḥīṭiya*), edited, translated and analyzed in detail by Luckey (1953), al-Kāshī resorts to decimal fractions for the approximation of π. It is true that in this *Treatise* he had already reached an accurate approximation of π by computation based on traditional procedure (the calculation of the perimeter of inscribed and circumscribed polygons), though using a new and original method. He gives as an approximation of 2π, in sexagesimal notation first, the value

$$6;16,59,28,1,34,51,46,14,50.$$

In chapter eight of the same treatise (Luckey, 1953, Arabic text, p. 86) entitled, "On the conversion of the magnitude of a circumference in Indian figures knowing that half the diameter is one", al-Kāshī wants, as his title shows, to translate the above representation into decimal notation. He writes (Luckey, 1953, Arabic text, p. 86):

As the circumference is six times the half-diameter plus a fraction which we raised to the ninth [place 60^{-9}], we take this fraction [as the fraction] of denominator ten [times] a thousand repeated five times [$10 \times 1000^5 = 10^{16}$]; because only one part is higher than one ninth by the half of its tenth [$0 < 10^{-16} - 60^{-9} < \frac{1}{2}60^{-10}$].

It is precisely this last expression that ensures the concordance between the number of figures of both the sexagesimal and decimal systems. Al-Kāshī gives as follows

$$2\pi = 6{,}283\ 185\ 307\ 179\ 586\ 5.$$

The technical and mathematical competence of his exposition contrasts with the brevity and problematic nature of the explanation that follows. This was a general feature of the passage devoted to decimal fractions in the *Treatise on the Circumference of a Circle*. He explains as follows (Luckey, 1953, p. 87):

And know that two that is in the last place of fractions is like the minutes in relation to integer 6 by supposing that ten minutes make an integer. We can choose to call this place tenths. The figure eight on the right-hand side is like the seconds and we call it [this place] the second tenths; the 3 which follows [8] it is like the thirds and we call [this position] the third tenths and so on according to the calculations of astronomers.[59]

Which is why we start with a single denominator which is one. This method in Indian computation belongs to what we have revealed as well as displaying it in table form; and we have set out the figures from left to right . . .

Al-Kāshī's statement, we can see poses a problem for the historian: clarify what al-Kāshī means by "revealed". Since Luckey[60] it has been agreed that this revelation refers to decimal fractions themselves. However, knowledge of al-Samawʾal's *Treatise* makes a more objective reading possible: in fact it seems that al-Kāshī does not refer to decimal fractions here, but more exactly the decimal representation of 2π. Moreover, the entire passage is closely related to this representation, without in the least achieving, or hardly so, a more general formulation. Lastly, the elusive nature of the text confirms that for al-Kāshī, it does indeed concern an application of what had been acquired earlier.

But apart from the problem of attribution, definitively solved moreover as far as thirteenth and fourteenth century mathematicians are concerned, the above passage indicates two ideas still absent in the *Treatise* of 1172 and consequently, of prime importance for the history of decimal fractions.

(1) The analogy between both systems of fractions: the sexagesimal and the decimal systems.

(2) The usage of decimal fractions no longer for approaching algebraic real numbers, but real numbers such as π.

The *Treatise on the Circumference* is of little assistance if we want to examine the new ideas in depth and appreciate their extension; it is presented as a sort of research paper with no particular didactic aim. However, the *Key to Arithmetic*, a later work, is completely different in aim and style, a sum of arithmetic and algebra that is much more revealing; not only is the use of decimal fractions explained, but the explanation itself is given in general terms. "When we have", al-Kāshī writes (ed., 1967, p. 121) "proved the ratio of the circumference to the diameter, [. . .] and raised the fractions to the ninth place, we want to convert them into Indian figures so the calculator who is unfamiliar with astronomical computation is not reduced to powerlessness". His intention is therefore clear: to introduce an alternative, easier, more accessible system of fractions, but in which it is possible to effect the same operations already used in the sexagesimal system. At this point al-Kāshī draws an analogy between both systems for operations as well as denominations; an analogy ensured with mathematical status from the start. Since al-Samawʾal, it is known that the same notation for both systems is only a restriction to two given bases for a notation valid for any base. The reason for al-Kāshī's emphasis is therefore understandable when he writes (p. 79):

Astronomers use fractions related to one another *maʿṭūfa*, such that their successive denominators are sixty and the successive powers [of sixty] as far as one wants. They have left those that followed [60^{-k}, k fixed arbitrarily]. They called them respectively: minutes, seconds, thirds, fourths and so on. By analogy with astronomers, we have shown – *awradnā* – fractions whose successive denominators are ten and successive powers [of ten] as far as desired. We call them respectively: tenths, second tenths, third tenths and so on.

Once again al-Kāshī emphasizes the importance of this analogy in order finally to refer to what founds it: in the sexagesimal system, places are raised by 60, and the position of degrees lies between two sequences, one "ascending", the other "descending". In the decimal system the representation is identical provided 60 is replaced by 10 and the degrees by units. Al-Kāshī had, it is true, already set forth the same idea for any base a.

His expression is therefore identical to that of al-Samawʾal with one exception: in al-Samawʾal the analogy is merely latent, while al-Kāshī formulated it explicitly. That said, we only need to read al-Samawʾal's exposition of sexagesimal fractions and compare it with that of decimal fractions to conclude without exaggeration that this analogy was not only within his grasp but it also must undoubtedly have played a not inconsiderable historical role. However, the history of science is not the psychology of scholars, so this important mathematical achievement which makes explicit what was already present in his exposition though embedded, must be interpreted. In so far as the history of science is not a philosophy of scientists, one must interpret this important achievement of the mathematician who explicitly states what is already present in his exposition though hidden. In our opinion, this achievement is inseparable from the autonomy of the new system of fractions and refers to its independence as one of many equivalent systems and in particular the sexagesimal system. Thus it can be asserted that the statement of this analogy implies a grasp of its own possibilities of extension. In fact, as in the case of al-Samawʾal and that of al-Kāshī the degree of understanding of fractions is the same, but they are not exactly on the same plane of existence. What the analogy ensures for decimal fractions is an existence which, this time, crosses the limits of their initial field of exercise, the approximation of algebraic integers.

If the autonomy of the new system of fractions is understood in this way, we are in a position to clarify some otherwise unintelligible facts. Historians have observed the first one: it is explicitly acknowledged and for itself that the transition from one system to another is accounted for: the change of base.[61] The second refers to an evidently useless task undertaken by al-Kāshī which has always intrigued historians: why for the particular case of decimal fractions did he reformulate and justify what he had already formulated and justified for any base? As for the third, it has been overlooked by historians: the use of decimal fractions not only for approaching algebraic real numbers but real numbers as well. As we remarked earlier for π, in the *Key to Arithmetic* al-Kāshī makes similar calculations for areas: polygons, circles, circle sections, etc. In the course of his calculations he used a notation which is basically identical to that of al-Samawʾal.

Heir to al-Karajī's school, al-Kāshī can no longer be considered as the inventor of decimal fractions; it remains nonetheless, that in his exposition the mathematician, far from being a simple compiler, went one step beyond al-Samawʾal and represents an important dimension in the history of decimal fractions. Whether or not this progress was due to al-Kāshī, lack of knowledge about the intermediary period obliges us to leave this question unanswered for the moment; whatever the case, this tradition outlived al-Kāshī's work, and it is highly probable that it was transmitted to future generations through his intermediary.

There is no need for proof of his posterity as far as Arabic science is concerned: we know that al-Kāshī's work was read and cited by mathematicians. Suter (1900, p. 191), for example, pointed out that Taqī al-Dīn ibn Maʿrūf (d. 1585–1586) calculated decimal tables for sines and tangents. We note here that even in the seventeenth century, mathematicians like al-Yazdī (d. circa 1637), cited the *Key to Arithmetic* and decimal fractions as exposed by al-Kāshī. It is also of interest to note that al-Yazdī, though aware of these fractions, freely made use of ordinary and sexagesimal fractions, not only in his computations but also in his chapter on the theory of fractions.[62]

The situation is more complex, as will be understood, when one considers science in the West. It would be reasonable to presume that mathematicians knew about the results of Arab scientists in one way or another, but decisive proof that this knowledge included decimal fractions is still to be found. The discovery by H. Hunger and K. Vogel in 1963 of a Byzantine manuscript brought to Venice in 1562 provides an important element of this proof. The Byzantine author wrote: "The Turks multiply and divide fractions using a special means of calculation (δι'ἑνός λογαριασμοῦ). They introduced their fractions when they governed our land". The example[63] given by the Byzantine mathematician makes it possible to identify without hesitation the allusion to decimal fractions. Here we follow the conclusions of those are most familiar with this manuscript, Hunger and Vogel, who commented (1963, p. 104):

> Die von al-Kāšī gemachte geniale Erfindung der Einführung einer ⟨Kette des Aufsteigens and Absteigens⟩ auch im dekadischen Positionssystem wird in der untersuchten Handschrift wohl zum erstenmal im Abendland sichtbar. Wenn auch schon vor al-Kāšī Ansätze zu einer dezimalen Schreibung der Brüche, die den Indern nicht gelungen ist, vorliegen, so war dieser doch der erste, der wirklich auch mit den Dezimalbrüchen gerechnet hat, und diese persisch-türkische Kenntnis hat in Byzanz Eingang gefunden.

It suffices to say that the Byzantine author reproduces a part of fifteenth-century Arabic mathematical knowledge though in less elaborate terms. It is also probable that he was familiar with the works of one of al-Kāshī's successors. Nonetheless, it remains that the use of a vertical line (Tropfke, 1921, pp. 176ff.) to separate the fractional part – a procedure discovered by al-Kāshī – is attested in Western texts before 1562, i.e. before the Byzantine text was attested in Vienna. This notation was used by Rudolf, Apian and Cardano. It is also known that the mathematician Mīzrahī (b. in Constantinople in 1455) used the same sign in his *Sefer ha-Mispar* before Rudolf. All these indications oblige us to ask ourselves if the theory of decimal fractions had not been transmitted to the West even earlier than 1562, and if this transmission was not characterized by a relative loss of information.

Whatever the case, the fact remains that the various formulations of decimal fractions, not only those given here but others encountered later in Viète, Stevin and many others, remained relatively outside mathematical practice. We have to await the elaboration of the logarithmic function, notably with Napier, for decimal fractions to be integrated into a mathematical corpus effectively practised.

In the eleventh and twelfth centuries arose propositions, methods and theories which, until now, had been dated at least two and a half centuries later, and which were in fact already organized and structured at that period. We have shown, for instance, that propositions relating to algebraic real numbers, the Ruffini–Horner method, the method of approximation, in particular what D. T. Whiteside calls the "al-Kāshī–Newton" method, the theory of decimal fractions, are in fact the work of eleventh and twelfth century mathematicians. Integrated into this body of problems and methods studied at the same period, the theory of decimal fractions is seen in a new light by historians who have a better grasp of the reasons for its invention and is capable of explaining, at least partially, its isolation and relative absence before the development of the logarithmic function. Our analysis confirms that the search for predecessors is historically unfounded and theoretically unjustified as we have seen for al-Uqlīdisī, for example. We shall reserve the next two particularly important questions for a subsequent study:

(1) Did the algebraists give an algebraic expression of methods already used by astronomers?

(2) What did they contribute to the prehistory, if not the history, of analysis?

This implies that an active mathematical tradition was established in the eleventh and twelfth centuries. We have drawn up their identification sheet: al-Karajī's school. This name also designates a programme formulated by al-Karajī and pursued by his successors: the arithmetization of algebra as it was then called, the constitution of algebra as "the arithmetic of unknown quantities". Lastly, it also suggests that historical research should go back to the drawing board to fill in evident gaps and compensate for obvious ignorance. For example, we now know that the status attributed to al-Kāshī by traditional historiography is not his; until now enjoying the privilege of artificial independence and isolated from tradition by a difference creator of myths, al-Kāshī has quite naturally regained the place that has never ceased to be his as the undisputed successor of al-Karajī's school. As a result the picture of Arabic algebra drawn by traditional historiography must be corrected or rather drastically altered. The first task of this project will be to redraw the classical picture of the beginnings of Arabic algebra and its transmission to Western mathematicians in the Middle Ages and the Renaissance. The main task is not to discover lost manuscripts, exhibit forgotten works, in short, to establish facts alone, but above all to supply hypotheses necessary for this research. In this respect, documentary evidence is so abundant and dispersed, and studies so rare, that even purely descriptive historiography remains hazardous unless theoretical guidelines exist. It is therefore opportune to define the theoretical directions that inspired and governed the discoveries of mathematicians in the eleventh and twelfth centuries.

As we have seen, the work of al-Karajī's school on polynomial expressions paved the way for new investigations connected with the extension accomplished earlier of algebraic calculus, which was to find fruitful application in a field other than that of algebra. This new field of application for algebraic calculus had already been defined, though partially, by arithmeticians before al-Karajī's school. They extracted square and cubic roots, and possessed formulas for the approximation of the same powers. But for want of abstract algebraic calculus, these arithmeticians were unable to generalize either their methods or algorithms. The generalization of algebraic calculus required the renewal of algebra by al-Karajī's school in order to become a component part of the area of numerical analysis that includes methods for solving "pure powers", according to the sixteenth-century expression, and a variety of other procedures for approaching positive roots. Algebraist-arithmeticians, it is true, then introduced these methods without any concern for rigour or theoretical justification. The first formulation of theoretical questions, and in particular, problems of the existence of roots, had to await another algebraic tradition, that of algebraist-geometers like al-Ṭūsī. The practical orientation of the algebraist-mathematicians, which was still alive in the seventeenth century, was partly the result of their objective itself: to exploit the results obtained in algebra in order to pursue and extend a set of problems previously considered by arithmeticians. They therefore went back to arithmetic in order to discover the applied extension of an algebra in some areas of arithmetic which had just been renewed by arithmetic. It was during this dual, or, if you prefer, dialectic movement between algebra and arithmetic, that mathematicians looked for new methods they wished to be iterative, capable of approximation in a recursive way. And this clearly defined program is the conclusion to the theoretical and practical configuration wherein lies the invention of decimal fractions.

APPENDIX

Al-Samawʾal, *Treatise on Arithmetic* 110^{v}–114^{r}.

الباب الخامس عشر من المقالة الخامسة ١١٠- ب
في وجود مخرج الكسور البالغة
لصحاح المضلعات الصم*

إذا استخرجت ضلع مربع أو مكعب أو غير ذلك من المراتب وعلمت صحاح الضلع، أعني ضلع أقرب مكعب أو مال أو غير ذلك من المضلعات [أو] ‹من› المطلوب ضلعه، وبقيت منه بقية دالة على صمم ضلعه، وأردت أن تستخرج من تلك البقية الكسور البالغة لتلك الصحاح، أخذت أعداد القانون لذلك المضلع وضربت كل واحد منها في العدد الذي يرسمه وجمعت المبلغ وزدت جملته واحداً أبداً، فما اجتمع فهو مخرج الأجزاء الباقية.

مثال ذلك: أنّا أخذنا جذر ٦٠، فألقينا منه أقرب المجذورات إليه وهو
١١١-١ ٤٩ فبقي / ١١، ووجدنا قانون المال ٢، فضربناه في الجذر الحاصل وهو ٧
فخرج ١٤ زدنا عليه واحداً فبلغ ١٥ نسبنا منه ١١ الباقية فكان ١١ جزءاً
من ١٥، فصار الجذر الحاصل ٧ و١١ جزءاً من ١٥.

* النص غير منقوط في مواضع جمة وقد قمنا بتنقيطه دون الإشارة إلى ذلك، وعلامة الصفر في النص هي ٥، ولقد بدلناها بنقطة حتى لا تلتبس مع الخمسة واستثنينا من ذلك جدول القوى العشرية. واستعملنا الرموز التالية في التحقيق: < > ما بينهما كلامنا، [ ] نقترح حذف ما بينهما – 4 مربع: مربعات – 8 يرسمه: مطموسة في النص ولكن مكررة في الهامش – 9 جملته: حلبه – 12 ووجدنا: غير واضحة في الأصل.

وأيضاً، استخرجنا ضلع مكعب هو ١٠، فخرج ٢ وهو صحاح الضلع وبقي ٢، ووجدنا أعداد سطر قانون الكعب ٣ ٣، فضربنا أولها في صحاح الضلع والثاني في مربع صحاح الضلع وزدنا على المبلغ واحداً <فصار> [ضلع] ١٩ وهو مخرج الأجزاء الباقية نسبنا منه البقية التي بقيت وهي ٢، فصار الضلع الحاصل ٢ و٢ من ١٩.

وأيضاً، استخرجنا ضلع مال مال هو ٤٠، فخرج ٢ وبقي ٢٤، ووجدنا
أعداد قانون مال مال ٤ ٦ ٤، فضربنا الأول في صحاح الضلع، وذلك
اثنان، والثاني في مربعه، أعني مربع ضلع الاثنين، والثالث في مكعبه أعني
مكعب الاثنين الذي هو صحاح الضلع، وزدنا على المبلغ واحداً فبلغ ٦٥
وهو مخرج الأجزاء الباقية، فصار الضلع اثنين و٢٤ جزءاً من ٦٥. وأيضاً،
استخرجنا ضلع مال كعب مبلغه / ٢٥٠، فخرج ٣ وبقي ٧، ووجدنا ١١١-ب
أعداد قانون مال كعب ٥ ١٠ ١٠ ٥، فضربنا الثلاثة أعني صحاح الضلع
في الأول ومربع الثلاثة في الثاني ومكعب الثلاثة في الثالث ومال مال
الثلاثة في الرابع وزدنا على المبلغ واحداً فاجتمع ٧٨١ وهو مخرج الأجزاء
الباقية، فصار الضلع ثلاثة آحاد وسبعة أجزاء من ٧٨١ وهو الضلع
المطلوب. وعلى هذا القياس.

1 هو (الأولى): مكررة – 6 ٢٤: ٢٤ – 13 في (الثانية): و – 15 ثلاثة: مطموسة في الأصل.

الباب السادس عشر من المقالة الخامسة
في وضع أصل واحد تحدد به جميع أعمال التفريق
التي هي القسمة والتجذير والتضليع لجميع هذه المراتب
وتصحيح الكسور الواقعة في هذه الأعمال بغير نهاية

كما أن المراتب المتناسبة المبتدئة من مرتبة الآحاد تتوالى على نسبة العشر بغير نهاية، كذلك نتوهم في الجهة الأخرى مراتب الأجزاء على تلك النسبة ومرتبة الآحاد كالواسطة بين مراتب العدد الصحاح التي تتضاعف آحادها على نسبة العشر وأمثاله بغير نهاية وبين مراتب الأجزاء المتجزئة بغير نهاية./ ونسمي المرتبة التالية لمرتبة الآحاد مرتبة أجزاء العشرات ١١٢-١ والتالية لها أجزاء المئات والتالية لها أجزاء ألوف وعلى هذا القياس.

وإذا انتهينا في حساب القسمة أو التجذير أو تضليع الكعب أو مال المال أو غير ذلك من أبواب التفريق إلى مرتبة الآحاد لم نقطع الحساب عندها، لكنا ننقل السطور التي يجب نقلها على الرسم إلى تحت مرتبة أجزاء العشرات، وما حصل في هذه المرتبة فهو أجزاء من ١٠. وإذا أتينا على شروط الحساب نقلنا إلى تحت مرتبة أجزاء المئات، فما خرج في هذه المرتبة فهو أجزاء من مائة، وهذه صورة المراتب المشار إليها:

8 وأمثاله: وأمثال – 13 ننقل: مطموسة في الأصل.

أجزاء عشرات الوف الوف الوف الوف	أجزاء الوف الوف الوف الوف	أجزاء مئات الوف الوف الوف	أجزاء عشرات الوف الوف الوف	أجزاء الوف الوف الوف	أجزاء مئات الوف الوف	أجزاء عشرات الوف الوف	أجزاء الوف الوف	أجزاء مئات الوف	أجزاء عشرات الوف	أجزاء الوف	أجزاء المئات	أجزاء العشرات	مرتبة الآحاد	مرتبة العشرات	مرتبة المئات	مرتبة الالوف	مرتبة عشرات الالوف	مرتبة مئات الالوف	مرتبة الوف الالوف	مرتبة عشرات الوف الالوف	مرتبة مئات الوف الالوف	مرتبة الوف الوف الالوف	مرتبة عشرات الوف الوف الالوف	مرتبة مئات الوف الوف الالوف	مرتبة الوف الوف الوف الالوف
٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥	٥
	٤			٣			٢			١			٥			١			٢			٣			٤

فإذا قسمنا $\overline{\text{٢١٠}}$ على $\overline{\text{١٣}}$ وخرج من القسمة $\overline{\text{١٦}}$ وبقي $\overline{\text{٢}}$ كتبنا ذلك في هذه الصورة:

١٦
٢...............
١٣

ثم فلننقل المقسوم عليه مرتبة إلى اليمين ونتمم العمل كما بينا في القسمة إلى أن يخرج مهما شئنا من المراتب. فإذا اقتصرنا على خمس مراتب حصل ما هذه صورته:

أجزاء من مائة ألف	٣ ٨	٤	خمس خمس عشر عشر عشر
أجزاء من عشرة آلاف	١	٨	خمس خمس عشر عشر
أجزاء من ألف		٣	خمس عشر عشر وعشر عشر عشر
أجزاء من مائة		٥	نصف عشر
أجزاء من عشرة		١	عشر
		١٦	صحاح

8 العلامة التي تحت مرتبة الآحاد هي في الأصل هكذا – 13 فلننقل: فنقسم.

وذلك ستة عشر أحدا وجزء من عشر وخمسة أجزاء / من مائة ١١٣ - ١
وثلاثة أجزاء من ألف وثمانية أجزاء من عشرة آلاف وأربعة أجزاء من مائة ألف. وذلك ستة عشر وعشر ونصف عشر وخمس عشر عشر وعشر عشر عشر وخمس خمس خمس عشر وخمس خمس عشر عشر.

فإن أردنا جذر عشرة خرج ٣ ونضعف الجذر الأسفل وننقله مرتبة فيحصل ما هذه صورته:

أجزاء من عشرة آلاف ألف	أجزاء من ألف ألف	أجزاء من مائة ألف	أجزاء من عشرة آلاف	أجزاء من ألف	أجزاء المائة	مرتبة أجزاء العشرة	مرتبة الآحاد	مرتبة العشرات
٠	٠	٠	٠	٠	٠	٠	٣	٠
٠	٠	٠	٠	٠	٠	٠	‹٠›	‹١›
٠	٠	٠	٠	٠	٠	٠	١	

ثم نتمم العمل في التجذير والنقل كما نعمل في الصحاح، فيحصل ما هذه صورته:

أعشار عشر عشر عشر عشر عشر عشر	أعشار عشر عشر عشر عشر عشر	أعشار عشر عشر عشر عشر	أعشار عشر عشر عشر	أعشار عشر عشر	أعشار عشر	أعشار	آحاد	عشرات
	٧	٧	٢	٢	٦	١	٣	

2 آلاف: ألف – 25 ١٠٣

/ وذلك ثلاثة آحاد [وخمس] وعشر وثلاثة أخماس عشر [عشر] ١١٣- ب وخمس عشر عشر [عشر] وخمس عشر عشر عشر [عشر] ونصف عشر عشر عشر [عشر] وخمس عشر عشر عشر عشر [عشر] ونصف عشر عشر عشر عشر [عشر] وخمس عشر عشر عشر عشر عشر [عشر].

وينبغي أن نعلم طريق نسبة هذه الأعداد الحاصلة في هذه المراتب، فإنه من السهولة على غاية لا يحتاج معها إلى إعمال الفكر والقياس.

مثاله: أنّا أردنا أن ننسب هذه الستة لينطق بمقدارها، فنسبناها إلى العشرة التي بها تتناسب هذه المراتب، فكان ذلك ثلاثة أخماس، وأضفنا إلى ذلك لفظ العشر بعدد المراتب التي بين الستة وبين مرتبة الآحاد، فصار ثلاثة أخماس عشر عشر، وعلى هذا القياس تتناسب سائر المراتب.

وإن أردنا أن تكون جميع الكسور الحاصلة أجزاء من مخرج واحد نقلنا مرتبة الآحاد مع ما يتلوها من العشرات والمئات وغير ذلك من الصحاح إلى سطر / أعلى، وكتبنا تحت المراتب الباقية أصفاراً، وكتبنا ١١٤-١ بعد الأصفار واحداً، فيكون الحاصل هكذا، وذلك ثلاثة آحاد و١٦٢٢٧٧ [٣] جزءاً من [٠]٠٠٠٠٠٠١
٢ [١]<١>

7 الفكر: الفلره – 14 مع ما: معما – 15 وكتبنا: مطموسة / وكتبنا: مطموسة.

٣
[٣] ١ ٦ ٢ ٢ ٧ ٧
١ . . . .

وهكذا نعمل في تجذير ضلع الكعب ومال مال ومال كعب وغير ذلك.
ويمكننا بهذا الطريق استقصاء تدقيق أعمال التفريق وأن نستخرج به جوابات لا نهاية لعددها كل واحد منها أدق وأقرب إلى الحقيقة من الذي قبله.

1 الواحد تحت الصفر الرابع في الأصل – 3 في الأصل هناك صفر تحت الثلاثة وواحد بعدها – 4 تجذير، تحرير.

NOTES

1. See the reproduction of *De Thiende* and its English translation by Robert Norton (1608) in Struik (1958, pp. 386ff.). See also the French edition in Sarton (1935, pp. 230–244).
2. For instance, E. J. Dijksterhuis (1970, pp. 16 and 18), the best specialist on Stevin wrote: "Stevin's main contribution to the development of mathematics being his introduction of what are usually called decimal fractions", and further on: "Yet none of the steps taken by Regiomontanus and other writers is comparable in importance and scope with the progress achieved by Stevin in his *De Thiende*."

   See also Sarton (1935, p. 174): "There are many examples of decimal fractions before 1585 yet no formal and complete definition of them, not to speak of a formal introduction of them into the general system of numbers". We could multiply such references and quote numerous authors who consistently repeat this opinion. We shall simply give a recent example, that of J. F. Scott, who wrote in 1969 (p. 127): "Nevertheless, it was not until the close of the sixteenth century that we detect the first methodical approach to the system. In 1585 there appeared a short tract *La Disme* by Stevin . . . In this, the principles of the system, and the advantages which would follow from its use, are clearly set forth."
3. See Gandz's article with a preface by Sarton (1936, pp. 16–45).
4. Although Gandz (1936, p. 21) wrote: "The invention of Bonfils introduces two new elements; the decimal fractions and the exponential calculus", all that may be deduced from the translation of Bonfils' Hebrew text, reproduced by Gandz, is summarized in Juschkewitsch (1964b, p. 241) as follows: "Die kurze Skizze eines Systems von 'Primen', 'Sekunden', 'Terzen' usw. in einer Handschrift des jüdischen Mathematikers Immanuel ben Jacob Bonfils, der im 14. Jahrhundert in Tarascon gelebt hat, ist im Vergleich zur Dezimalbruchlehre al-Kāsīs völlig unbedeutend. – Dabei hat Bonfils keinerlei Berechnungen mit Hilfe von Dezimalbrüchen vorgenommen."
5. Tropfke (1930, p. 178) may be cited among many, who wrote: "Wenn noch andere Männer neben Stevin als Erfinder der Dezimalbrüche genannt werden, so ist das nicht zu verwundern. Die Erfindung der Dezimalbruchrechnung lag gleichsam in der Luft; Gelehrte aus allen Ländern beteiligten sich an ihr." This same idea is expressed by Sarton (1935, p. 173).

   Lastly, see Cajori (1928, I, p. 314): "The invention of decimal fractions is usually ascribed to the Belgian Simon Stevin, in his *La Disme* published in 1585. But at an earlier date several other writers came so close to this invention, and at a later date other writers advanced the same ideas, more or less independently, that rival candidates for the honor of invention were bound to be advanced. The *La Disme* of Stevin marked a full grasp of the nature and importance of decimal fractions, but labored under the burden of a clumsy notation".
6. Struik (1969, p. 7). A less eclectic but more embarrassed position is held by H. Gericke and K. Vogel, the German translators of Stevin's *La Disme*. They wrote: "Al-Kāschī bringt aber nicht nur die vollständige Theorie, sondern er führt auch die Rechnungen gelegentlich im einzelnen vor, einschließlich der Verwandlung von Sexagesimalzahlen und Brüchen in Dezimale und umgekehrt wobei er zur Trennung von Ganzen und Brüchen sich verschiedener Methoden bedient . . ."

In fact, according to these authors (1965, p. 45) the only difference with Stevin is described as follows: "Was aber bei ihm im Gegensatz zu Stevin auch nicht zu finden ist und was diesem ein Hauptanliegen war, ist die konsequente Anwendung auf alle Masse, deren dezimale Einteilung von grösster praktischer Bedeutung sein musste". On the one hand it is known that these applications had no real impact, and on the other, that al-Kāshī, as we shall see, proceeds with conversions other than certain measurements current at that time: it would therefore be incorrect to emphasize the difference. See Gericke and Vogel (1965, pp. 44, 45).

7. See A. Saidan's edition of *Kitāb al-Fuṣūl fī al-Ḥisāb al-Hindī*, 1973.
8. We know through ancient Arabic bibliographers that al-Samawʾal wrote a *Treatise on Arithmetic* entitled *al-Qiwāmī fī al-Ḥisāb al-Hindī*. The thirteenth-century bibliographer Ibn Abī Uṣaybiʿa wrote in his *Tabaqāt* (1965, p. 462) that al-Samawʾal had completed this treatise in 568 of the Hegira (1172–1173).

   See also Suter (1900) and Sezgin (1974, p. 197). The entire work is still lost. But a work by al-Samawʾal entitled *al-Maqāla al-thālitha fī ʿilm al-misāḥa al-hindiyya* ("The third volume of the work on Indian mensuration") is to be found in the Biblioteca Medicea Laurenziana, Orient. ms. 238. This manuscript comprises 115 in-folio pages and the copy, dated 751 of the Hegira, is in poor condition. The above title is obviously incorrect. I take this opportunity to thank Sezgin (1974, p. 197), who sent me a microfilm copy of this manuscript, whose existence he mentioned in the above work, observed quite correctly that the manuscript "hat trozt ihres Titels mit der indischen Ausmessung nicht direkt zu tun". We can effectively show that it is a change with 115 in-folio pages from *al-Qiwāmī* ("Treatise on Arithmetic"). The copist wrote at the end (ff. 114^r, 114^v): "We have completed the book al-Samawʾal composed at Baku and finished on 29th of the month of Ramadan the year five hundred and sixty-eight". He also mentioned possessing a copy written by al-Samawʾal himself. The subject, dates and attribution leave no doubt as to the identity of the manuscript. We set forth the results of this discovery for the first time at the Congress of the History of Arabic Sciences at Aleppo and we have undertaken a critical edition of this difficult text.
9. Al-Samawʾal, *al-Qiwāmī*, f. 32^r.
10. Juschkewitsch (1964b) where the author apparently repeats the conclusions of the analysis in the introduction of the Russian translation of al-Kāshī's work (Al-Kāshī 1956). A. S. al-Demerdash and M. H. al-Cheikh, the editors of al-Kāshī's work (1967) apparently shared Luckey's opinion. Lastly, in a detailed study by Dakhel (1960), Luckey's analysis and point of view is repeated.
11. *Al-Qiwāmī*, f. 108^r. Al-Samawʾal used the *Jummal* system to express numbers. It consists of the 28 letters of the Arabic alphabet arranged in an ancient Semitic order for expressing numbers. For typographical reasons we have used the corresponding numbers.
12. *Ibid.*, f. 108^r.
13. The literal translation of *al-muʿṭiya* is "indicator", which means "which gives one of the figures of the root". The text would have been incomprehensible if we had retained this translation.
14. *Al-Qiwāmī*, ff. 108^r–108^v.
15. *Al-muʿṭiya* (see note 13).

16. *Al-marāfī*^{c}.
17. *Al-Qiwāmī*, ff. 108^{r}–109^{r}. In the ms. the left-hand column headings are omitted both in this and the following tables.
18. Let f be a real function with a real continuous variable: α a real number strictly positive, we call the dilation of f of ratio α the application φ_∞; $\varphi_\infty(x) = f(\alpha^{-1}x)$ for any x.
19. *Al-Qiwāmī*, f. 109^{r}.
20. *Ibid.*, f. 104^{v}.
21. *Ibid.*, ff. 104^{r}–105^{r}.
22. *Ibid.*, f. 109^{r}.
23. *Ibid.*, f. 109^{v}.
24. *Ibid.*
25. *Ibid.*
26. *Ibid.*
27. *Ibid.* There is obviously an important break in the manuscript. We have been able to establish that f. 110^{r} follows f. 69^{r}.
28. *Ibid.*, f. 109^{r}.
29. *Infra*, Chap. III, pp. 153ff.
30. *Infra*, Chap. III, pp. 201–202, note 12.
31. *Infra*, p. 158. See also al-Ṭūsī (1986), I, pp. 32ff.
32. I.e. expansion of the relation $0 < \alpha < 1$.
33. *Infra*, p. 164.
34. *Ibid.*
35. *Infra*, pp. 162ff.
36. *Al-Qiwāmī*, f. 110r.
37. *Infra*, pp. 202–203, note 15.
38. *Al-Qiwāmī*, f. 111^{r}.
39. *Ibid.*
40. *Al-Qiwāmī*, ff. 68^{r} and 68^{v}.
41. *Ibid.*, ff. 68^{v}, 69^{r}, 69^{v}.
42. This point was brought to our attention after publication of this article in a correspondence with M. Bruins and also independently by Waterhouse (1978).
43. *Al-Takmila fī al-Ḥisāb*, MS 2708, Lâleli, Istanbul, 22^{r}ff. and 29^{r}ff.
44. *Al-Qiwāmī*, ff. 58^{r}ff.; 62^{r}ff.; 64^{r}ff.
45. Al-Nasawī, *al-Muqni*^{c} *fī al-ḥisāb al-hindī*, MS Leiden arabe, n°556, ff. 21, 22.
46. Al-Samaw$^{\supset}$al, *al-Tabṣira fī* ^{c}*ilm al-ḥisāb*, MS Oxford Bod. Hunt. 194, f. 18^{r}.
47. See *Al-Qiwāmī*, f. 27^{v}.
48. *Ibid.*, f. 111^{r}.
49. *Ibid.*, ff. 111^{r}, 111^{v}.
50. Al-Samaw$^{\supset}$al, *Al-Bāhir*, 1972, pp. 21 and 22 (Arabic text) and pp. 18 and 19 (French introduction).
51. *Ibid.*
52. *Al-Qiwāmī*, f. 112^{v}.
53. *Ibid.*, f. 113^{r}.
54. *Ibid.*, f. 114^{r}.
55. *Ibid.*

56. Saidan (1966). The same idea was expressed on several occasions in his edition of al-Uqlīdisī's book. He writes for example: "What inclines us to be proud of al-Uqlīdisī is that he was the first to deal with decimal fractions, suggest a sign for separating integers from fractions, and treat fractions as he treats integers. Before al-Uqlīdisī was known, common opinion held that the first to deal with decimal fractions was Ǧamšīd ben Masᶜūd al-Kāshī", p. 524.
57. In his English translation of this passage, A. Saidan (1966, p. 485) integrated the sign in the text. We may therefore read: ". . . and mark the unit place with the mark ′ over it . . .". However, if we consult the manuscript of *al-Fuṣūl*, this sign is missing. Nor is it to be found in Saidan's edition, only in his English translation. For this reason we prefer to use the manuscript although we always refer to Saidan's edition, easier to consult.
58. Al-Uqlīdisī gives the following example: "we want to separate $\overline{19}$ into two halves five times, we say: half of $\overline{9}$ is four and a half, we write half $\overline{5}$ in front of $\overline{4}$, we divide ten into two halves, and we mark the unit position, we obtain $\overline{95}$; we separate five and nine into two halves, we obtain $\overline{475}$, which we divide into two halves, we obtain $\overline{2375}$ and the unit position is ⟨the position⟩ of thousands for what precedes it. If we want to pronounce what we have obtained, we say that the separation into two halves makes two, plus $\overline{375}$ of a thousand. We separate this into two halves, we obtain $\overline{11875}$, we separate into two halves a fifth time, we obtain $\overline{059375}$, we find $\overline{59375}$ in Saidan's edition which is $\overline{59375}$ of a hundred thousand. Its ratio is said to be a half plus a half eighth plus a quarter of an eighth". *Al-Fuṣūl*, MS 802, Yeni Cami, Istanbul, 58ʳ. See al-Uqlīdisī, *Al-Fuṣūl*, pp. 145–146.
59. In the text one reads "the calculation of stars", this is probably a copyist's error. The accepted term is the calculation of astronomers.
60. Luckey (1951, p. 103) wrote: "Während also K. die ganzen wie die gebrochenen Sechzigerzahlen von Vorgängern übernahm, schreibt er sich wiederholt ausdrücklich die Einführung der Dezimalbrüche zu. Meines Wissens fand man bisher zwar in keinem älteren arabischen Texte, wohl aber in Schriften, die arabisches Gut wiedergeben oder auf solchem fussen, den Gedanken ausgesprochen, daß an die Stelle der Grundzahl 60 der Sexagesimalbrüche eine andere Grundzahl treten könne, als welche im ⟨Algorismus de minutiis⟩ von Seitenstetten aus dem 14. Jahrhundert neben 12 auch 10 genannt sein soll. Auf das, was Immanuel Bonfils aus Tarascon über Dezimalbrüche sagt, soll später eingegangen werden. Der Gedanke der Dezimalbrüche mag also im Mittelalter in der Luft gelegen haben. Wie andere vor und nach ihm, so kann auch K. sehr wohl selbaständig den Einfall gehabt haben, nach dem Vorbild der Sechzigerbrüche Dezimalbrüche einzuführen. Jedenfalls aber hat man bisher in keiner vor seine Zeit fallenden Schrift eine ausführliche praktische Durchführung der Methode der Dezimalbrüche im Positionsystem, wie er eine solche bringt, nachgewiesen."
61. Luckey (1951, pp. 115ff.). Let us remark moreover that the problem of the periodicity of the fraction may occur during the solution to this problem. We know that it is always possible to express a decimal fraction exactly by a sexagesimal fraction; but it is not always possible to express a sexagesimal number by a limited decimal fraction. In the French translation of the section on the history of Arabic mathematics, Youschkevitch (1976) wrote: "Note that al-Kāshī neither mentioned nor

remarked on the obvious periodicity of the fraction 0,141 (592) he obtained". In a note on the French translation (p. 168), he recalled a remark made by Carra de Vaux: the periodicity of a sexagesimal fraction is indicated by the thirteenth century mathematician al-Mārdīnī.

We are now able to show that the periodicity of a sexagesimal fraction was already known in the twelfth century. For instance, to convert a fraction – e.g. 4/11 – into a sexagesimal fraction, al-Samawʾal obtains ;21,49,5,27,16. He then writes (*al-Qiwāmī*, f. 89ʳ): "And these five figures repeat themselves indefinitely. If we are satisfied with limiting ourselves to a tenth of a tenth [60^{-20}] for example, the answer is ;21,49,5,27,16,21,49,5,27,16,21,49,5,27,16,21,49,5,27,16 and if we want a more correct [number] than this we repeat these five figures up to a higher [place] of these places". These computations demonstrate that the problem of periodicity at least was already known in the twelfth century.

62. MS Hazinesi, 1993, Istanbul. See in particular f. 49ʳ.
63. Hunger and Vogel (1963). The example given is the following: to calculate the price of $153^{1/2}$ measures of salt, the price of each being $16^{1/4}$ aspra, i.e. $153^{1/2}$ $16^{1/4}$. The author wrote that the Turks put 5 instead of half, and 25 instead of a quarter. So we obtain 2494375, from which we separate the last three figures which are in the three last positions (ψηφιὰ). The calculation was made as follows:

α ε γ | ε		1 5 3 | 5
α ς | β ε		1 6 | 2 5 3
ζ ς ζ ε	γ	7 6 7 5 8
γ · ζ	η	3 0 7 0
θ β α ·	τᾶυτα εἰσιυ	9 2 1 0
α ε γ ε	i.e.	1 5 3 5
β δ θ δ | γ ζ ε		2 4 9 4 | 3 7 5

We note that the zero (οὐδέυ) is indicated by a point, to what digits the Greek letters correspond in the positional number position and that the fractional part is separated by a vertical line.

THE SOLUTION OF NUMERICAL EQUATIONS AND ALGEBRA: SHARAF AL-DĪN AL-ṬŪSĪ AND VIÈTE

I

In the beginning was Viète. Thomas Harriot, William Oughtred, Claude-François Dechales, John Pell . . . , all improved the method in one way or another,[1] Newton[2] took it up later. Modified by Joseph Raphson, this method is known as Newton's method in textbooks on numerical calculation. Lagrange, J. R. Mouraille and Fourier[3] attempted to solve its difficulties. Independently of each other, Ruffini and Horner[4] developed investigations by Viète and Newton: they proposed a more practical algorithm for extracting the root of a numerical equation of any degree.

This is the accepted outline for recounting the history of this method. Historians of mathematics not the least important – i.e. Montucla, Hankel, Cantor, Wieleitner, Cajori, Tropfke – all have recognized Viète's priority and commented on Newton's improvement; some were subsequently to describe the refinements made by Ruffini and Horner. In the early nineteenth century, the same outline was accepted by Lagrange, in his *Traité de la résolution des équations numériques de tous les degrés* he wrote

> Viète was the first to be interested in the resolution of equations of any degree. He has shown in his treatise *De numerosa potestatum adfectarum resolutione* how one may solve several equations of this kind by analogous operations used for the extraction of roots of numbers. Harriot, Oughtred, Pell, etc. endeavoured to facilitate the application of this method by giving specific rules to reduce trial and error depending on the different cases that occur in equations in relation to the signs of their terms. But the multitude of operations required and the uncertainty of success in a large number of cases made them abandon it altogether.

Lagrange (1798, pp. 16–17) went on to write: "Viète's method was followed by that of Newton which is only a method of approximation".

This historical outline, expressed in similar terms, is frequently encountered in later histories of mathematics. Just a few years later, Montucla (1799, t. 1, pp. 603–604) was to write

Among Viète's purely analytical discoveries, one should also include his general method for the solution of equations of any degree. Reflecting on the nature of ordinary equations, Mr Viète remarked that they were only incomplete powers, and he conceived the idea that, as one extracts the roots of imperfect numerical powers by approximation, so one could extract the roots of equations, which would give the value of the unknown. Consequently, he proposed rules for that purpose in his work entitled *De numerosa potestatum adfectarum resolutione*; they are similar to those which are used to extract the root of a full power and may be conveniently employed for cubic equations. Harriot devoted half of his *Artis Analyticae praxis* to developing them; they are also explained in Oughtred, Wallis and de Lagni's Algebra. Wallis even used them to solve a fourth degree equation, and he pursued his approximation to the eleventh decimal place. But one has to be endowed with a mind as capable as this geometer to undertake such a laborious task. Today we possess more convenient methods of approximation. . . .

The reason why we insisted on citing such long extracts is because they give a precise description of the overall historical and analytical picture of our problem since Viète. Later on, Ruffini and Horner will be correctly placed in the finished picture, and the result will appear as the so-to-speak almost definitive history of the problem, not only in the writings of historians, but also in historical notes by mathematicians: Young, Burnside, Whittaker and Robinson,[5] among many others.

While this historical picture was endlessly repeated throughout the ninteenth century, around the 1850s investigations by Sédillot and Woepcke[6] had already implicitly started to undermine the confidence that might be granted this outline. Examining the prolegomens of Arab astronomers and mathematicians on Ulugh Beg's astronomical tables, they revealed the existence of methods of approximation for solving numerical equations: these methods were multiple, and moreover, highly developed. Confronted with their particularly fine appearance, Hankel (1874, p. 292) did not hesitate to write about one of them: "Diese schöne Methode der Auflösung numerischer Gleichungen steht allen seit Viète im Occident erfundenen Approximationsmethoden an Feinheit und Eleganz nicht nach". He affirmed, moreover, that this was the first instance of successive numerical approximations encountered in the history of mathematics.

The discovery by Sédillot and Woepcke certainly threw doubt on the traditional version of the history of our problem. But this doubt could only be implicit, insofar as the work of the mathematician Shalabī did not contain a systematic treatment of the so-called problem, but only considered a particular case, the calculation of the approached value of sin 1°. This may explain why the investigations of Sédillot and Woepcke went almost unnoticed. But Shalabī had cited al-Kāshī, his fifteenth century master of algebra: full attention then focused on the latter. In 1874, Hankel (1874, pp. 292–293) suggested, though still unable to justify it, the importance of al-Kāshī for the history of our problem. It is true that John Tytler (1820) had already underlined the fact fifty years earlier.

However, it was only in 1948 that Luckey presented the first extensive and detailed study of al-Kāshī's work; this time the traditional historical picture was explicitly shaken. In a fundamental article,[7] Luckey demonstrated that al-Kāshī was not only the inventor of decimal fractions, but, moreover, he already possessed the so-called Ruffini–Horner method.

The discovery was so important and knowledge of the history of mathematics before al-Kāshī so fragmentary and uncertain, that Luckey and historians of mathematics who followed him, experienced the greatest difficulty in placing al-Kāshī's work in a historical context. An understandable difficulty if one considers the scope of the *Key to Arithmetic*;[6] by its standard it stood in sharp contrast with most other works then known to historians.

To circumvent the problem without solving it, historians of science sometimes, and those of Arabic science often, implicitly changed their angle of approach. Instead of placing al-Kāshī's results, they were minimized, and instead of looking for conditions which made the algebra of the *Key to Arithmetic* possible, they only aimed at identifying an eventual forerunner. A clear definition of al-Kāshī's algebraic activity in the first place would undoubtedly have made it possible to place it historically afterwards. But this approach was usually reversed. Only the results, irrespective of an analysis of his activity, were considered, and as was common in this field, the search for a forerunner went back to Alexandria. But as the Alexandrians knew of no similar method, and al-Kāshī dated back to the fourteenth-fifteenth centuries, and as a method for extracting the root of a numerical equation similar to al-Kāshī's had been found in thirteenth century China, a Chinese origin was suggested and affirmed without sufficient detail.[9] As a result, Luckey's interpretation and analysis were accepted by some historians of mathematics without discrimination.

This historical approach is questionable not only for its conclusions, but even more for its suppositions: the history of mathematics is conceived of as mathematical results almost independently of the conceptual sequences they produced. Of history, there only remains an account o[f] a succession of facts, the transmission of propositions. While not wishing, in turn, to adopt a position which reduces individual mathematical results attained by mathematicians to a mere indication of the theory on which they are based, it may be admitted that a result is the same for two different writers if, on the one hand, the norms of the discipline that control them are the same, and if, on the other, the interests that guided these mathematicians are similar.

For the historian of a specific problem, for instance the solution of numerical equations, what is important is to situate it in relation to disciplines to which it is attached: algebra and arithmetic. Since 1948, however, our knowledge of the history of these disciplines among Arabs has made some progress. The name of al-Samawʾal permits a greater understanding of al-Kāshī's contribution to the knowledge of decimal fractions. That of al-Karajī and his successors, al-Shahrazūrī and al-Samawʾal,[10] as we have shown, are rigorous proof that the *Key to Arithmetic* was only the culmination of a longstanding and, for the age, intense activity in arithmetic and algebra. The names of al-Khayyām[11] and Sharaf al-Dīn al-Ṭūsī,[12] the importance of whose algebraic work we shall show for the first time, are of capital importance, not only for algebra but also for algebraic geometry.

Seen in this theoretical and historical perspective, the problem of numerical equations must clearly be stated in different terms. We shall therefore state and justify the two following propositions.

1. *Al-Kāshī's work – not only on numerical but also on decimal fractions – is the culmination of a renewal undertaken by eleventh and twelfth century algebraists. The hypothesis of a Chinese origin therefore appears historically superfluous and theoretically unfounded.*

Two sets of instruments, theoretical and technical, were then required to state the problem of the solution of numerical equations. On the one hand, a complete algebra of polynomials,[13] a knowledge of the binomial formula for any positive integer power,[14] a tested algorithm for extracting numerical roots, capable of being generalized;[15] on the other, an extension of the theory of equations for understanding second-degree equations or those that can be reduced to them; lastly, an introduction to the study of curves using algebra to deal with problems of approximation.

If these combined means were already at the disposal of mathematicians, it was due to the existence of two trends since the eleventh century that aimed at reactivating algebra and extending its field. Moreover, the history of this science after the eleventh century is incomprehensible if the existence of both trends is not sufficiently stressed.

One was closely related to the application of arithmetic to algebra and indirect attempts to extend the notion of number; al-Karajī's work, pursued by his successors, like al-Samawʾal, provided the first set of instruments noted earlier. The other, associated with attempts to advance algebra through geometry, quite naturally led to the algebraic study of curves, and thus enabled laying the foundations of algebraic geometry; marked by the names of al-Khayyām and Sharaf al-Dīn al-Ṭūsī, provided the second set of required instruments. These mathematicians, as we shall see, were instrumental in posing the problem of numerical equations.

It is also probable that, confronted with the problem of finding a brief and elegant algebraic solution to third-degree equations, the same mathematicians, in the process of constituting its theory, were induced to find alternative numerical methods for a solution, numerical methods. The theoretical obstacle not only possessed a stabilizing influence, but a heuristic function as well.

2. *Al-Ṭūsī already possessed a method to which the bulk of Viète's method is linked and the outline retained by historians needs to be modified once again.*

In other words, though al-Kāshī's method may be preferred to that of Ruffini–Horner, it was as if Viète's method necessarily had to precede that of the last two mathematicians. But whereas Ruffini and Horner rediscovered al-Kāshī's method using mathematics renovated by analysis, the method from which Viète drew most of his basic ideas was based on mathematics which remained, whatever is said, fundamentally the same. This poses a problem for the historian of Viète's thought.

But to avoid embarking on a historical approach we have just criticized, we are obliged to pursue, at least briefly within the limits of this study, the problem in its field and context, i.e. al-Ṭūsī's algebra. Here again we shall show the early stages of algebraic geometry. Let us start with an exposition of al-Ṭūsī's method and its relation to that of Viète.

II

In a well-known, often cited but soon forgotten passage, al-Khayyām (1048–1131) wrote

The Indians possess methods for finding the sides of squares and cubes based on such knowledge of the squares of nine figures, that is the square of 1, 2, 3, etc., and also the products formed by multiplying them by each other, i.e. the product of 2, 3, etc. I have composed a work to demonstrate the accuracy of these methods, and have proved that they do lead to the sought aim. *I have moreover increased the species, that is I have shown how to find the sides of the square-square, quadrato-cube, cubo-cube, etc. to any length, which has not been made before now.* The proofs I gave on this occasion are only arithmetic proofs based on the arithmetical parts of Euclid's *Elements*.[16]

Al-Khayyām's attempt was neither the only one nor the first. Al-Bīrūnī (973–1050), belonging to an earlier generation of mathematicians, had at least composed a work comprising one hundred in-folio pages whose exact title is: *The extraction of cubes and higher roots in the degrees of calculation* (*Fi Istikhrāj al-kiᶜāb wa-aḍlāᶜ mā warāʾahu min marātib al-ḥisāb*).[17]

Until now this information had rightly been taken at face value: an indication of past remains. It is true that both works remained undiscovered; of one, we only possess a summary or "abstract", and of the other, only the title. The summary, though concise, enables one to believe that al-Khayyām possessed a method for extracting roots to any degree, and that this method was based on the expansion of $(a + b + \dots + k)^n$, $n \in \mathbb{N}$, and therefore *a fortiori*, on knowledge of the formula of binomial expansion and the law for formulating the table of coefficients. In sixteenth-century terms, al-Khayyām possessed a method for extracting roots from "pure powers", in fact the same as that of Stifel and Viète for the same powers. Naturally, for want of other documents which develop al-Khayyām's ideas in similar or different terms, the latter conclusion remains hypothetical. With al-Ṭūsī's work, however, the hypothesis is confirmed both by the method he used and by his silence.

Al-Ṭūsī's method is partly based on knowledge of the expansion mentioned by al-Khayyām, and presented moreover as the generalization of the extraction of the root of "pure powers" from "affected powers", to use the same language. In fact, only the general case, that of affected equations, interested al-Ṭūsī and the treatment of this case apparently generalized work accomplished earlier by al-Khayyām. His silence is nonetheless significant; we mean that al-Ṭūsī made no mention of the particular problem of $x^n = N$, $n = 2, 3$, whose solution is knowingly left to the reader. It was as if this extraction was within the grasp of those who studied mathematics at that time: al-Ṭūsī then claimed for himself the general problem of affected equations.

Can one affirm, in spite of that, that al-Ṭūsī had himself generalized al-Khayyām's method? Lack of knowledge about the intermediaries between al-Khayyām and al-Ṭūsī makes attribution uncertain. Uncertainty does stem from another silence on the part of al-Ṭūsī: though he made no mention of an eventual inventor of the method, he did not claim its paternity for himself. No name is mentioned in the manuscript we possess. The use of tables[18] alone to expound his method indicates nothing in particular, insofar as, since the eleventh century at least, arithmeticians like Kūshyār ibn-Labbān used al-Ṭūsī's tables to extract square and cubic roots. So that all that can be said is that the method al-Ṭūsī used, called al-Ṭūsī's method, was formulated after al-Khayyām but before al-Ṭūsī, or by either of these mathematicians, and in any case in the algebraist trend.[19]

But what does this method consist of?

Al-Ṭūsī used the same approach throughout his work: he first discussed the existence of roots for each equation, then set forth how to solve the numerical equation corresponding to the equation just discussed. To recapitulate all the equations solved by al-Ṭūsī is beyond the scope of this book. We shall give enough examples for a complete description of his method, and in the first place, we shall limit ourselves to a paraphrase of al-Ṭūsī's text (ed., 1986, I, pp. 25–30; ff. 46^v–49^r) at the risk of appearing longwinded.

$$x^2 + a_1x = N$$

with $\quad N = n_0 10^m + n_1 10^{m-1} + \ldots + n_m.$

The ranks modified by roots determine $\left[\frac{m}{2}\right]$ intervals, $\left[\frac{m}{2}\right]$ is the integer part of $\frac{m}{2}$, and is compared with k, the decimal order of a_1. We have two cases $\left[\frac{m}{2}\right] > k$; $\left[\frac{m}{2}\right] \leqq k$.

1. First case $\left[\frac{m}{2}\right] > k$; example $x^2 + 31x = 112992$.

1-1. N is divided into 2-figure segments starting from the right-hand side. The zeros above the figures in Table I indicate this partition.

If the order of $N = m$, 5 here, then the number of digits is 6, and we therefore have three digits for the root x; and we have $r = \left[\frac{m}{2}\right] = 2$ and consequently a possible order for x.

The order of $a_1 = 31$ is 1 and $r - k = 1$. At the bottom of the table we write $a_1 10^2$, here 31.10^2.

TABLE I

$x^2 + 31x = 112992$
$a_1 = 31$
$f(x) = x^2 + 31x$

				3	2	1
			3	2		
		3				
		0		0		0
N	1	1	2	9	9	2
x_1^2		9				
$a_1 x_1$			9	3		
				0		0
$N_1 = N - x_1^2 - a_1 x_1$		1	3	6	9	2
x_2^2				4		
$(2x_1 + 31)\ x_2$		1	2	6	2	
						0
$N_2 = N_1 - x_2^2 - (2x_1 + 31)\ x_2$				6	7	2
x_3^2						1
$[2(x_1 + x_2) + 31]\ x_3$				6	7	1
$N_3 = N_2 - x_3^2 - [2(x_1 + x_2) + 31]\ x_3 = 0$						
$2(x_1 + x_2) + 31$				6	7	1
$(2x_1 + 2x_2 + 31)\ 10$			6	7	1	
$(2x_1 + 31)\ 10$			6	3	1	
$(2x_1 + 31)\ 10^2$		6	3	1		
$a_1 10^2$			3	1		

1-2. We find the last digit of the root by determining the highest square contained in the last segment of N. Let 9 be this square – we set $x_1 = 3.10^2$. We carry x_1^2, $31x_1$ to the top of the table and subtract the whole from N. We thus obtain $N - f(x_1) = N_1$; where $f(x_1) = x_1^2 + a_1x_1$.

1-3. We set $x = x_1 + y$. It will give

$$x_1^2 + y^2 + 2x_1y + 31(x_1 + y) = N$$

hence

$$y^2(2x_1 + 31)y = N_1.$$

1-4. We divide N_1 like N, and use the same reasoning. We thus determine $r_1 = \left[\frac{m_1}{2}\right]$; m_1 the order of N_1. We now turn to the term of the second member and place it at the bottom of the table $(2x_1 + 31)10^{\left[\frac{m_1}{2}\right]}$. We note that the last digit of this number is set below the last figure of N_1 and it is higher. Therefore, since the square of y must be added to $(2x_1 + 31)y$, their sum remains higher than N_1.

We have therefore established that digit 3 is found in the last digit of the root.

We therefore go forward one unit and seek y of the order $\left[\frac{m_1}{2}\right] - 1$. Here the order of y is equal to 1.

For the order

$$\alpha^2 \cdot 10^2 + 6 \cdot 10^2\alpha \cdot 10 = 10^4$$

where

$$\alpha^2 + 60\alpha = 10^2.$$

We therefore divide 130 by 60 or 13 by 6, and obtain an approached value of $y = x_2$ by omitting the terms of y order higher than 1 in N_1 and obtain $x_2 = 20$.

1-5. We carry x_2^2, $(2x_1 + 31)x_2$ to the top of the table and subtract it from N_1.

We thus obtain $N_1 - x_2^2 - (2x_1 + 31)x_2 = N_2$.

1-6. We start again and find x_3 such that $x = x_1 + x_2 + x_3$.

We have $(x_1 + x_2 + x_3)^2 + 31(x_1 + x_2 + x_3) = N$

hence $x_3^2 + x_3[(2x_1 + 2x_2) + 31)]x_3 = N_2$.

We separate N_2 into 2-figure segments and we determine an order $m_2 = 2$, $\left[\frac{m_2}{2}\right] = 1$. We see if order 1 suits x_3. We write at the bottom of the table $[2(x_1 + x_2) + 31]10$.

We repeat the comparison with the order obtained by m_2. But as the number obtained is greater than m_2, we find that two is the second figure of the root. We then determine x_3.

TABLE II

$$x^2 + 578\,442 = 2\,123\,x$$
$$a_1 = 2\,123$$
$$f(x) = x^2 - 2\,123\,x$$

				3	2	1
			3	2		
		3				
		0		0		0
N	5	7	8	4	4	2
$x_1(a_1 - x_1)$	5	4	6	9		
				0		0
$N_1 = N - f(x_1)$		3	1	5	4	2
$(a_1 - 2x_1 - x_2)\,x_2$		3	0	0	6	
						0
$N_2 = N - f(x_1 + x_2)$			1	4	8	2
$(a_1 - 2x_1 - 2x_2 - x_3)\,x_3$			1	4	8	2
$N_3 = 0 = N - f(x_1 + x_2 + x_3)$						
$a_1 - 2x_1 - 2x_2 - x_3$			1	4	8	2
$a_1 - 2x_1 - 2x_2$			1	4	8	3
$(a_1 - 2x_1 - 2x_2)\,10$		1	4	8	3	
$(a_1 - 2x_1 - x_2)\,10$		1	5	0	3	
$(a_1 - 2x_1)\,10$		1	5	2	3	
$(a_1 - 2x_1)\,10^2$	1	5	2	3		
$(a_1 - x_1)\,10^2$	1	8	2	3		
$a_1 10^2$	2	1	2	3		

1-7. We shift the last line of the bottom table and find x_3 of order 0. We see that $x_3 = 1$.

1-8. We verify that $N_3 = N - x_3^2 - [2(x_1 + x_2) + 31]x_3 = 0$.

Al-Ṭūsī gives a comprehensive table, eliminated by the copyist, our reconstruction is based on the author's verbal description; on the side of the table we have only replaced what al-Ṭūsī writes in words by symbols.

Second case. $\left[\frac{m}{2}\right] \leqq k$.

In the second case, $\left[\frac{m}{2}\right] \leqq k$, to determine the first figure of the root, al-Ṭūsī proceeds either by dividing N by a_1, or by finding the highest square. Sometimes division indicates this figure, sometimes not at all. Otherwise, the method is identical. It is also used by negative coefficients with slight changes in notation. For instance, for $x^2 + 578442 = 2123x$, we have the Table IV (al-Ṭūsī, ed., 1986, I, pp. 34–384; ff. 51^r–52^v):

It is therefore clear that al-Ṭūsī applies his method to the equation $x^2 + a_1x = N$ with $a_1 \in \mathbb{Z}$. The method includes cubic equations, the main subject of his *Treatise*, with no change of fundamental ideas or notable modification in representation. Here are some examples (al-Ṭūsī, ed., 1986, I, pp. 78–89; ff. 76^v–82^r):

$$x^3 + a_1x^2 + a_2x = N.$$

Al-Ṭūsī always distinguishes three cases.

First case. $\left[\frac{m}{3}\right] > k_1$ and $\left[\frac{m}{3}\right] > \left[\frac{k_2}{2}\right]$, k_1 and k_2 being the order of a_1 and a_2, respectively.

Example: $x^3 + 12x^2 + 102x = 34345395$. The discussion is the same as for the second-degree equation: transposing the earlier discussion to the case where $n = 3$. So from now on, we shall only give the tables.

Second case. $\left[\frac{m}{3}\right] < \left[\frac{k_2}{2}\right]$ and $k < \left[\frac{k_2}{2}\right]$ with k_1 and k_2 the orders of a_1 and a_2 respectively.

Example: $x^3 + 6x^2 + 3\,000\,000x = 996\,694\,407$.

Third case. $\left[\frac{m}{3}\right] < k_1$ and $\left[\frac{k_2}{2}\right] < k_1$ as for example $x^3 + 30\,000x^2 + 20x = 3\,124\,315\,791$.

The method is the same except for changes imposed by the above

TABLE III

$$x^3 + 12x^2 + 102x = 34\,345\,395$$
$$a_1 = 12$$
$$a_2 = 102$$
$$f(x) = x^3 + 12x^2 + 102x$$

						3	2	1
				3	2			
		3						
		0			0			0
N	3	4	3	4	5	3	9	5
x_1^3	2	7						
$3(x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2)\,x_1$		1	1	1	0	6		
					0			0
$N_1 = N - f(x_1) = N - x_1^3 - a_1x_1^2 - a_2x_1$		6	2	3	4	7	9	5
x_2^3					8			
$3[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2]\,x_2$		5	9	1	0	8	4	
$N_2 = N - f(x_1 + x_2)$			3	1	5	9	5	5
x_3^3								1
$3[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2 + (x_1 + \frac{1}{3}a_1 + x_2)\,x_2 + (x_1 + x_2 + \frac{1}{3}a_1)\,x_3]\,x_3$			3	1	5	9	5	4
$[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2 + (x_1 + \frac{1}{3}a_1 + x_2)\,x_2 + (x_1 + x_2 + \frac{1}{3}a_1)\,x_3]$			1	0	5	3	1	8
$[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2 + (x_1 + \frac{1}{3}a_1 + x_2)\,x_2]$			1	0	4	9	9	4
$[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2 + (x_1 + \frac{1}{3}a_1 + x_2)\,x_2]\,10$		1	0	4	9	9	4	
$[(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2) + (x_1 + \frac{1}{3}a_1)\,x_2]\,10$			9	8	5	1	4	
$(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2)\,10$			9	2	4	3	4	
$(x_1^2 + 2x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2)\,10^2$		9	2	4	3	4		
$(x_1\,\frac{1}{3}a_1 + \frac{1}{3}a_2)\,10^2$			1	2	3	4		
$\frac{1}{3}a_1\,10^4 + \frac{1}{3}a_2\,10^2$				4	3	4		
$\frac{1}{3}a_2$							3	4
$\frac{1}{3}a_2\,10^2$					3	4		
$\frac{1}{3}a_2\,10^4$			3	4				

conditions: here al-Ṭūsī proposes to divide by the "number of squares" (the coefficient of x^2) to obtain first the first figure of the root, or as he writes:

"We set [in the table] the number of squares as divisor, the number as dividend, we extract the coefficient and we know its order".

TABLE IV

$$x^3 + 6x + 3\,000\,000\,x = 996\,694\,407$$
$$a_1 = 6$$
$$a_2 = 3\,000\,000$$
$$f(x) = x^3 + 6x + 3\,000\,000\,x$$

			3		3	2	3	2	1
			0			0			0
N	9	9	6	6	9	4	4	0	7
x_1^3			2	7					
$3(x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2)\,x_1$	9				5	4			
						0			0
$N_1 = N - f(x_1)$		6	9	1	5	4	4	0	7
x_2^3						8			
$3[x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2]\,x_2$		6	5	8	3	4	4		
									0
$N_2 = N - f(x_1 + x_2)$			3	3	1	2	0	0	7
x_3^3									1
$3[(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2 + (x_1 + \tfrac{1}{3}a_1 + x_2)\,x_2 + (x_1 + x_2 + \tfrac{1}{3}a_1)\,x_3]\,x_3$			3	3	1	2	0	0	6
$[(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2 + (x_1 + \tfrac{1}{3}a_1 + x_2)\,x_2 + (x_1 + x_2 + \tfrac{1}{3}a_1)\,x_3]$			1	1	0	4	0	0	2
$[(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2 + (x_1 + \tfrac{1}{3}a_1 + x_2)\,x_2]$			1	1	0	3	6	8	
$[(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2 + (x_1 + \tfrac{1}{3}a_1 + x_2)\,x_2]\,10$		1	1	0	3	6	8		
$[(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2) + (x_1 + \tfrac{1}{3}a_1)\,x_2]\,10$		1		9	7	2	4		
$(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2)\,10$		1		9	1	2			
$(x_1^2 + 2x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2)\,10^2$	1		9	1	2				
$(x_1 \tfrac{1}{3}a_1 + \tfrac{1}{3}a_2)\,10^2$	1				6				
$\tfrac{1}{3}a_1 10^4 + \tfrac{1}{3}a_2 10^2$	1				2				

Lastly, to show that al-Ṭūsī applies his method to a polynomial function with coefficients in $\mathbb{Z}$, we shall take as a last equation:

$$x^3 - a_1x^2 - a_2x - c = 0,$$

only considering the first case $\left[\frac{m}{3}\right] > k_1$ and $\left[\frac{m}{3}\right] > \left[\frac{k_2}{2}\right]$.

Example:

$$\begin{array}{r} 0\ \ 0\ \ 0 \\ x^3 = 30x^2 + 600x + 29\,792\,331 \end{array}$$

dealt with in Table V.

These various examples show that al-Ṭūsī's method is well ordered and general. Though its generality is still implicit to a certain extent, probably for various reasons, we can still grasp its significance. Al-Ṭūsī's text is in fact highly condensed, as if it was originally intended for a certain style of teaching, and therefore necessarily accompanied by a spoken commentary. This is the state of the only manuscript so far identified and transcription errors do not facilitate its comprehension. Other intrinsic reasons complicate the task. Perhaps the implicit presence of notions as important as that of the "derivative" make the author's wording allusive. Lacking an autonomous status, unnamed, the notion itself and its presence, remain a problem to be solved rather than a means to a solution, as we shall soon see

$$x^3 + a_1x^2 + a_2x = N. \tag{1}$$

We know the root is written

$$x = \alpha 10^2 + \beta 10 + \gamma$$

and al-Ṭūsī will determine, in succession, α, β, γ.

We designate by f the function of a real variable $f(x) = x^3 + a_1x^2 + a_2x$. A comparison with the decimal order of the root found and the order of coefficients of (1) enables us, as we have seen, to control the choice of coefficients of the various figures of the root. The determination of these same figures in the strict sense and almost mechanical is as follows:

The determination of $x_1 = \alpha 10^2$ is carried out either by division, or by finding the highest cube contained in N, depending on the case.

We write $x = x_1 + x_2$ and seek to determine x_2. According to (1) we have

$$N = f(x_1 + x_2) = f(x_1) + N_1$$

where

$$N_1 = (3x_1^2 + 2a_1x_1 + a_2)x_2 + (3x_1 + a_1)\, x_2^2 + x_2^3.$$

TABLE V

$$x^3 - 30x^2 - 600x = 29\,792\,331$$
$$a_1 = -30$$
$$a_2 = -600$$
$$f(x) = x^3 - 30x^2 - 600x$$

						3	2	1
				3	2			
		3						
		0			0			0
N	2	9	7	9	2	3	3	1
x_1^3	2	7						
$+3(\frac{1}{3}x_1a_1 + \frac{1}{3}a_2)\,x_1$		2	8	8				
					0			0
$N_1 = N - f(x_1) = N - x_1^3 + a_1x_1^2 + a_2x_1$		5	6	7	2	3	3	1
x_2^3					8			
$3[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2]\,x_2$		5	3	7	6			
								0
$N_2 = N - f(x_1 + x_2)$			2	8	8	3	3	1
x_3^3								1
$3[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2 + (x_1 - \frac{1}{3}a_1 + x_2)\,x_2 + ((x_1 + x_2) - a_1)\,x_3]\,x_3$			2	8	8	3	3	
$N_3 = N - f(x_1 + x_2 + x_3)$								
$[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2 + (x_1 - \frac{1}{3}a_1 + x_2)\,x_2 + ((x_1 + x_2) - a_1)\,x_3]$				9	6	1	1	
$[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2 + (x_1 - \frac{1}{3}a_1 + x_2)\,x_2]$				9	5	8		
$[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2 + (x_1 - \frac{1}{3}a_1 + x_2)\,x_2]\,10$			9	5	8			
$[(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2) + (x_1 - \frac{1}{3}a_1)\,x_2]\,10$			8	9	6			
$(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2)\,10$			8	3	8			
$(x_1^2 - 2x_1\frac{1}{3}a_1 - \frac{1}{3}a_2)\,10^2$		8	3	2				
$\frac{1}{3}x_1a_1\,10^2 + \frac{1}{3}a_2\,10^2$			3	2				
$\frac{1}{3}a_2\,10^3$				2				
$\frac{1}{3}a_1\,10^4$			1					

N_1 is determined by the choice of x_1. Al-Ṭūsī obtains an approached value for x_2, that is x'_2 by omitting in N_1 terms of x_2 of a higher order than 1, and obtains

$$x'_2 = \frac{N_1}{3x_1^2 + 2a_1x_1 + a_2} = \frac{N_1}{f'(x_1)} \tag{2}$$

f' designates the derivative function of f'. We now write $x = ((x_1 + x'_2) + x_3)$ and seek to determine x_3. We set

$$N_2 = N - f(x_1 + x'_2) = 3(x_1 + x'_2)^2x_3 + 2a_1(x_1 + x'_2)x_3 + a_2x_3 + 3(x_1 + x'_2)x_3^2 + x_3^2 + x_3^3,$$

we use N_2 to determine x_3 in the same way we used N_1 to determine x'_2. In other words, the method is general, and if al-Ṭūsī only applies it to third-degree or lesser equations it is insofar as it constituted the theory of equations. The general case requires no other notions unknown to the author. Let the equation be

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x = N \tag{3}$$

and set

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x,$$

the function is derivable several times like all functions studied by al-Ṭūsī. The interval to which the root corresponds may be known, let $x \in [10^r, 10^{r+1}]$, x is therefore of the form $\rho_0 10^r + \rho_1 10^{r-1} + \ldots + \rho_r$ and such that $r = \left[\frac{m}{n}\right]$ where m is the decimal order of N.

We determine x_1 as above, that is, either by division or by finding the highest integer of nth power contained in N.

We set $N_1 = N - f(x_1)$ and $x = x_1 + x_2$, $N_1 = g(x_2)$, where g is a polynomial in x_2 of degree $n - 1$. We obtain the approached value of x_2, x'_2 defined by

$$N_1 = nx_1^{n-1}x'_2 + a_1(n - 1)x_1^{n-2}x'_2 + \ldots + 2a_{n-2}x_1x'_2 + a_{n-1}x'_2. \tag{4}$$

The derivative of f at point x_1 is recognized here and

$$x'_2 = \frac{N_1}{f'(x_1)}. \tag{5}$$

We operate by successive iterations. Let $x_1, x'_2, \ldots, x'_{k-1}$ such that

$$x = x_1 + x'_2 + \ldots + x'_{k-1} + x_k \qquad k = 2, \ldots, n$$

an approached value of x_k:x'_k is given by the formula

$$x'_k = \frac{N_k}{f'(X_{k-1})} \tag{6}$$

where

$$N_k = N - f(x_1 + x'_2 + \ldots + x'_{k-1})$$
$$X_{k-1} = x_1 + x'_2 + \ldots + x'_{k-1}.$$

So an approached value of x will be $x_1 + x'_2 + \ldots + x'_n$ where x'_i are given by the formula (6).

We therefore see that generalization does not require the introduction of new concepts not already used in the example al-Ṭūsī studied.

We should not be surprised by (4). In fact, if f is a polynomial of degree n,

$$f(x_1 + x_2) = f(x_1) + x_2 f'(x_1) + \frac{x_2^2}{2} f''(x_1) + \ldots + x_2^n \tag{7}$$

and similarly

$$\begin{aligned} &f(x_1 + x'_2 + \ldots + x'_{k-1} + x_k) &(8)\\ &\quad = f(x_1 + x'_2 + \ldots + x'_{k-1}) + x_k f'(x_1 + x'_2 + \ldots + x'_{k-1}) + \ldots + x_k^n \end{aligned}$$

which explains formula (6).

But when we argue in terms of the "derivative" are we not surreptiously introducing a meaning alien to al-Ṭūsī's theory? We shall return to this problem; for the time being we only need to remark that:

(1) In all these examples, and in a perfectly regulated way, al-Ṭūsī systematically uses for divisors expressions that correspond algebraically to the first derivative.

(2) In this field, even if functions are not yet explicitly mentioned, the idea is there, especially when it concerns determining the positive integer root of a numerical equation using a method of successive approximations. On the other hand, even if al-Ṭūsī only sought positive integer roots, his method is capable of obtaining negative roots of (1) as well. It is sufficient to apply it by changing $f(x)$ to $f(-x)$.

(3) As we shall see, the algebraic expression of the "derivative" is used throughout the discussion of the problem of the existence of roots of an algebraic equations. Al-Ṭūsī always treats numerical equations

as examples of algebraic equations for which the existence of roots had been demonstrated earlier.

Before tackling these questions, that is before pinpointing the solution of numerical equations in al-Ṭūsī's algebraic work, let us examine the relation between al-Ṭūsī's method and that of Viète.

III

Viète's work on numerical equations is just as difficult to grasp as that of al-Ṭūsī. As we said, al-Ṭūsī used the method as part of acquired mathematical knowledge. It would however be vain to seek in his work by whom and how this knowledge was transmitted. The substance of the method resides in the tables. But apart from some justification of the comparison of decimal orders, the expansion of the formula $(a + b + \ldots + k)^n$, $n = 2, 3$, division, and terms to be written in the table, the text remains silent about al-Ṭūsī's personal contribution and what he might have borrowed from his predecessors.

We might have expected a different situation with Viète, but this is not the case. Apart from similar justifications, barely more explicit than those of al-Ṭūsī, and though his work was printed and not a manuscript, we only find general considerations about the "analytical way".

In both cases, to know the inventor is of less interest from a biographical viewpoint than a real problem of interpretation: what mathematical concepts were elaborated to invent this method? As far as Viète is concerned, it was precisely on the question of these concepts that opinions diverge. The conflict between interpretations started in the last century; to be convinced we only need to quote the names of some eminent historians: Hankel, Ritter, Cantor, Eneström and Tropfke.

Viète's text is of little help and the essential is to be found once again in the tables. Nonetheless this text informs us that the solution of "affected powers" is based on the same method as that of "pure powers".[20]

The solution is "analytical", i.e. it follows the converse approach to that used for the formation of affected powers by respecting the place, order, increase and decrease of coefficients as well as those of the unknown quantities.[21]

If these considerations resemble those of al-Ṭūsī, though expressed in a language we know, one significant difference between both mathematicians must strike us first of all. Whereas al-Ṭūsī starts by proving the existence of one or more positive roots of equations, of which numerical equations are an example, Viète never poses this problem in his work and presents the numerical equations to be solved with preliminary justification. This difference was, moreover, to be a subject of reflection for those who, victims of the myth propagated by Renan, Tannery etc., contrast the practical computational aspect of Arabic mathematics with the theoretical character of Hellenic and Renaissance mathematics. To study Viète's method, let us first consider the following equation:

$$1Q + 7N \text{ equals } 6\ 0\ 7\ 5\ 0.$$

Like al-Ṭūsī, Viète starts by differentiating between 2-figure segments starting from the right-hand side; instead of placing zeros above the square orders, he places dots below these same orders.

He writes (ed., 1970, p. 174) in fact: "Ex adfecto igitur quadrato ut eruantur latera, sedes unitatum quadrata singularia metientium per binas alternas, ut in analysi puri quadrati, distinguuntur figuras punctis commode a dextra ad laevam subtus collocatis". He then gives the following tables.

We conclude that if $1Q + 7N$ equals 60750, $1N$ is 243 "in exactly the same way as that of the composition, Viète writes, but in the opposite direction".

The best way to compare Viète's method with that of al-Ṭūsī is to take Viète's example and apply al-Ṭūsī's method: see Table VII. We then note that part 1 of 6 and two parts of (1,1′) of 7; 2 of 6 (2,2′) of 7; 3 of 6 and (3,3′) of 7 respectively, are equivalent.

Furthermore, when we know that in his description al-Ṭūsī not only gives combined tables omitted by the copyist, but partial tables as well, we are not surprised by the similarity. The only difference is that instead of writing zeros above the figures, Viète writes them below, and instead of putting divisors right at the bottom of the table for multiplication accurate to one coefficient, he sets them out at the top of the table. The difference is insignificant and does not affect the identity of either method.

The similarity persists when we consider other cases of second-degree equations. For instance, in the case where $\left[\frac{m}{2}\right] < k$ we saw that al-Ṭūsī shifted the coefficient back in order to effect division. Viète expresses himself in equivalent terms when he writes (ed., 1970, p. 175): "Coëfficiens itaque ad succedentes sedes ordine revocanda est, donec sit locus divisioni, à qua tunc opus inchoare magis consentaneum est". Like his example, $954N + 1Q$ equals 18 487.

TABLE VI

I. *Extraction of the first partial side*

linear coefficient		7 ·	· ·	sublateral, as many lateral dots as there are quadratic dots	$\left\{ \begin{array}{lll} & 0 & 0\ 0 \\ \hline N. & 2 & 4\ \cdot \\ Q. & 4 & 16\ \cdot \end{array} \right.$ as many zeros as there are quadratic dots or partial sides
	6 · *Qj*	0 7 *N* · *Qij*	5 0 *N* : *Qiij*	quadratic dots	
planes to be subtracted	4	1 4		square of the first side plane of the first side by the coefficient	
sums of the planes to be extracted	4	1 4			
remainder of the affected square to be solved	1	9 3	5 0		

II. *Extraction of the second partial side*

upper partial part of divisors	linear coefficient				7		
remainder of the affected square to be solved		1	9	3	5	0	
lower partial part of divisors	twice the first side		4				
sum of divisors			4	0	7		
planes to be subtracted		1	6				second side by twice the first square of the second side
			1	6			
				2	8		
sum of planes to be subtracted		1	7	8	8		second side by the coefficient
remainder of the affected square to be solved			1	4	7	0	

III. *Extraction of the third partial side as if it were the second*

upper partial part of divisors	linear coefficient				7	0 0
					.	N 2 3
remainder of the affected square to be solved		1	4	7	0	Q 576.9
					.	
lower partial part of divisors	twice the extracted side		4	8		
			4	8	7	
sum of divisors		1	4	4		second side by twice the first
planes to be subtracted					9	square of the second side
				2	1	second side by the coefficient
sum of planes to be subtracted equal to the remainder of the affected square to be solved		1	4	7	0	
				— -		

But as the choice of divisors is important for interpreting the method, let us examine Viète's text for this example. In the third rule of his conclusion, on the composition of divisors, their order and place after extraction of the first partial side, Viète writes (ed., 1970, p. 226)

Tertia cura esto, ut post eductionem primi lateris singularis & emendatam congrua subductione expositam resolutioni magitudinem, dividentes scansoriae in suo collocentur situ & ordine, tam superius quam inferius. Ac inferius quidem collocentur multiplices laterum elicitorum gradus parodici, ipsimet qui dividerent in analysi purae potestatis, ut pote

In analysi quadrati, duplum lateris eliciti.

In analysi cubi, Prima, dividens scansoria magnitudo, triplum lateris eliciti.

Secunda, triplum quadratum ejusdem.

TABLE VII

			2	4	3
		2	4		
	2				
	0		0		0
	6	0	7	5	0
x_1^2	4				
$a_1 x_1$		1	4		
			0		0
$N_1 = N - f(x_1)$	1	9	3	5	0
x_2^2		1	6		
$(2x_1 + a_1)\, x_2$	1	6	2	8	
					0
$N_2 = N - f(x_1 + x_2)$		1	4	7	0
x_3^2					9
$(2x_1 + 2x_2 + a_1)\, x_3$		1	4	6	1
$N_3 = N - f(x_1 + x_2 + x_3)$					
$2x_1 + 2x_2 + a_1$			4	8	7
$(2x_1 + 2x_2 + a_1)\, 10$		4	8	7	
$(2x_1 + a_1)\, 10$		4		7	
$(2x_1 + a_1)\, 10^2$	4		7		
$a_1 10^2$			7		

TABLE VIII

I. *Extraction of the first partial side*

coefficient plane		3	0 . . .	sublateral, as many partial sides as there are cubic dots or cubic places	0 0 0 N. 2. 4 Q. 4 16 C. 8. 64	as many zeros as there are cubic dots or partial sides
affected cube to be solved	1 4 . Cj	3 5 6 Q N . Cij	1 9 7 Q N . Ciij			
solids to be subtracted first	8	6	0	cube of the first side first side by the plane coefficient		
sum of solids to be subtracted	8	0 0 6	0			
remainder of the affected cube to be solved	6	3 5 0	1 9 7			

II. *Extraction of the second partial side*

upper part of the divisors (linear coefficient)		•				3	0		
							•	•	
remainder of the affected cube to be subtracted		6	3	5	0	1	9	7	
					•			•	
lower part of the divisors	triple square of the first side	1	2						
	triple of the first side			6					
sum of the divisors		1	2	6	0	3	0		
solids to be subtracted		4	8						second side by the triple square of the first side
			9	6					square of the second side by the triple of the first side
				6	4				cube of the second side
					1		2	0	second side by the plane coefficient
sum of the solids to be subtracted		5	8	2	5	2	0		
remainder of the affected cube to be resolved			5	2	4	9	9	7	

III. *Extraction of the third partial side as if it were the second*

upper part of the divisors	{plane coefficient		3 0 .	0 0 0
remainder of the affected cube to be resolved		5 2 4	9 9 7	N 2 4 3.
				Q 576. 9.
lower part of the divisors	triple square of the first side	1 7 2	8	C 27
	triple of the first side		7 2	
sum of the divisors		1 7 3	5 5 0	
solids to be subtracted		5 1 8	4	second side by the triple square of the first side
		6	4 8	square of the second side by the triple of the first side
			2 7	cube of the second side
			9 0	second side by the plane coefficient
sum of solids to be subtracted equal to the remainder of the cube to be resolved		5 2 4	9 9 7	

If we apply what Viète wrote to the example dealt with, we take $2x_1$ for the part of divisors, not omitting of course to put them in their appropriate place and order. Then it remains to write down "magnitudes which are coefficients" with the same care among the higher divisors, in this case a_1, and lastly, for the sum of divisors we have $2x_1 + a_1$ to determine x_2.

It may therefore be affirmed that for second-degree equations, there is no notable difference between al-Ṭūsī's method and that of Viète. Is there any significant difference as far as higher degree equations are concerned?

To examine this question we shall proceed in the same way as for the above example of the equation $IC + 30N$ equals 14,356,197. We may expect to see the only important difference between the two methods emerge, and in fact, Viète's text leads us to assume that the sum of divisors for determining x_2 will be $3x_1^2 + 3x_1 + a_1$ in this case, and the nature of the method is somewhat altered.

Therefore, if $IC + 30N$ equals 14,356,197, IN and 243, following exactly the same path but in the opposite direction to that of the composition.

To compare the methods of both authors, let us take the same example used by al-Ṭūsī (Table IX).

We note therefore that "the sum of the divisors" ceases to be identical when we apply al-Ṭūsī's method to Viète's example. Whereas this sum is 1 260 300 in the second part of Viète's table, it is 1 200 300 according to al-Ṭūsī's method. Similarly, in Part Three of Viète's table, it is 173 550, while it should be 175 090 for al-Ṭūsī. What does this difference really imply?

To understand this difference, let us return to the equation $x^3 + a_1x^2 + a_2x = N$, discussed earlier. We have seen in fact that

$$x_2' = \frac{N_1}{3x_1^2 + 2a_1x_1 + a_2}$$

where

$$N_1 = (3x_1^2 + 2a_1x_1 + a_2)x_2 + (3x_1 + a_1)x_2^2 + x_2^3.$$

For Viète, we have

$$N_1 = (3x_1^2 + 2a_1x_1 + a_2)x_2 + (3x_1 + a_1)\{x_2\}x_2 + x_2^3$$

TABLE IX

		0			0			0
	1	4	3	5	6	1	9	7
x_1^3		8						
$a_1 x_1$					6			
					0			0
$N_1 = N - f(x_1)$		6	3	5	0	1	9	7
x_2^3				6	4			
$3(x_1^2 + \frac{1}{3}a_1 + x_1 x_2)\ x_2$		5	7	6	1	2		
								0
$N_2 = N - f(x_1 + x_2)$			5	2	4	9	9	7
x_3^3							2	7
$3(x_1^2 + \frac{1}{3}a_1 + x_1 x_2 + x_2^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)\ x_3$			5	2	4	9	7	
$N_3 = N - f(x_1 + x_2 + x_3)$								
$x_1^2 + \frac{1}{3}a_1 + x_1 x_2 + x_2^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)$				5	8	3	3	
$(x_1^2 + \frac{1}{3}a_1 + x_1 x_2 + x_2^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)\ 10$			5	8	3	3		
$(x_1^2 + \frac{1}{3}a_1 + x_1 x_2 + x_2^2 + x_1 x_2)\ 10$			5	7	6	1		
$(x_1^2 + \frac{1}{3}a_1 + x_1 x_2)\ 10$			4	8		1		
$(x_1^2 + \frac{1}{3}a_1)\ 10$			4			1		
$(x_1^2 + \frac{1}{3}a_1)\ 10^2$		4			1			
$\frac{1}{3}a_1 10^2$					1			

where $\{x_2\}$ will be replaced by 10 in the division and al-Ṭūsī preceding formula becomes, with Viète

$$x_2' = \frac{N_1}{(3x_1^2 + 2a_1x_1 + a_2) + 10(3x_1 + a_1)} .$$

So, more generally if we return to equation (3), for Viète (5) becomes

$$x_2' = \frac{N_1}{f'(x_1) + \frac{10^{m-1}}{2} f''(x_1) + \ldots + \frac{10^{(n-2)(m-1)}}{(n-1)!} f^{n-1}(x_1)}$$

from which we may deduce the formula corresponding to (6).

This is the only important difference between the two methods, a difference which we insisted on taking to extremes to emphasize our comparison. Nevertheless, in our opinion, Viète's method remains essentially close to that of al-Ṭūsī. The problem at stake is not so overdetermined that it may induce such a similarity by itself. Procedures, expositions, details of presentation resemble one another to such a degree that one is entitled to ask the question: Was Viète in contact with this trend of Arabic mathematics, of which al-Ṭūsī was one of its representatives?

IV

The current state of the history of algebra does not enable us to state clearly from what angle and by what means the work of these algebraists might have been known at the time of Viète. At the very least we may assume that if it was transmitted, it had entailed alterations; al-Ṭūsī's method is in one sense more "modern" than that of Viète.

Of the two, al-Ṭūsī's method is in fact closer to that of Newton and Raphson. But before drawing an overhasty conclusion, one particularly significant point needs to be clarified: the set of interpretations just stated may appear coherent from a mathematical point of view, it may even be discussed and verified; but it may appear excessive from a historical point of view. Do we in fact have the right to substitute the term "derivative" for algebraic expressions which, in another language, correspond, nonetheless, to the notion of the "derivative"? In short, are we entitled to use a language other than that of the theory whose history we are intending to write?

A satisfactory answer to these questions would force us to write another history: a history of the notion of the derivative. Not to identify a historical entity given once and for all in an ahistorical transcendency, but on the contrary to recognize a mathematical entity that is expressed in a necessarily obsolete language or style by which it is defined in the exposition and demonstration. An impossible task to achieve within the limits of this study, since it implies reviewing the mathematical thought of one of the greatest schools of Arabic mathematics which includes names as renowned as Thābit b. Qurra, Ibrāhīm b. Sinān, Alhazen, al-Qūhī *et al.*

It suffices here to show how al-Ṭūsī systematically applied the notion of the "derivative" elsewhere in his work. We shall therefore limit ourselves to showing that al-Ṭūsī thought "function", but systematically used another form of this notion which will later be named the derivative.

The significance and place of his method for solving numerical equations will then be understood. To illustrate this thesis, let us return to his algebra.

Al-Ṭūsī's treatise is, as we said, a treatise on equations whose subject is inscribed in its title. More precisely, it is the elaboration of the algebra of third or lower degree equations stated verbally. Furthermore, al-Ṭūsī (1986, I, pp. 16–17; ff. 42^v–43^r) gives al-Khayyām's classification:

(1) $x = a$	(2) $x^2 = a$
(3) $x^2 = ax$	(4) $x^3 = ax^2$
(5) $x^3 = ax$	(6) $x^3 = a$
(7) $x^2 + ax = b$	(8) $ax + b = x^2$
(9) $x^2 + a = bx$	(10) $x^3 + ax^2 = bx$
(11) $ax^2 + bx = x^3$	(12) $x^3 + ax = bx^2$
(13) $x^3 + bx^2 = a$	(14) $x^3 + a = bx$
(15) $a + bx = x^3$	(16) $x^3 + bx^2 = a$
(17) $x^3 + a = bx^2$	(18) $a + bx^2 = x^3$
(19) $x^3 + ax^2 + bx = c$	(20) $a + bx + cx^2 = x^3$
(21) $x^3 + a + bx = cx^2$	(22) $x^3 + ax^2 + b = cx$
(23) $a + x^3 = bx + cx^2$	(24) $x^3 + ax^2 = bx + c$
(25) $x^3 + ax = bx^2 + c$.	

The history of this theory is written elsewhere, here we only need to recall al-Ṭūsī's main approaches:

1. To solve equations, he divides them into two groups: one contains equations which always have solutions (al-Ṭūsī gives them); the other corresponds to equations with solutions only under certain conditions. He then undertakes the discussion.

2. He reduced the equations to be solved to other equations whose solution is known using the affine transformation $x \rightarrow x + a$ or $x \rightarrow a - x$.

3. To solve the equations, he examines the maximum of algebraic expressions. He takes the "first derivative" of these expressions which he cancels, and proved that the root of the equation obtained, substituted in the algebraic expression, gives the maximum.

4. He examines neither volume nor surface "maxima", but "limits".

5. When he has found one of the roots of a cubic equation, to determine another root he would sometimes study a second-degree equation which is only the quotient of the divisor of a cubic equation by $(x - r)$, where r is the root found. In other words, he knows that the

polynomial $ax^3 + bx^2 + cx + d$ is divisible by $(x - r)$, if r is a root of the equation $ax^3 + bx^2 + cx + d = 0$.

6. He might sometimes encounter a second-degree equation of a type not yet studied, for example $x^2 - bx = c$; he then reduces it by affine transformation to a type of known equation.

7. When he has examined the equation, he attempts to determine the higher and lower limit of its roots.

8. If similar equations are grouped together, for example, $ax^3 + bx = c$, $ax^3 + c = bx$, $ax^3 = bx + c$, which are equivalent to $ax^3 + bx + c = 0$, we can then find *a posteriori* the formula named after Cardano. In other words, this formula exists on specific occasions, but not globally in the case of real roots.

What we have just said about al-Ṭūsī's basic approaches will enable us to explain how the method of numerical solution was invented, and better still, how this method employed the notion of derivative. But some of the above affirmations may have been surprising; we are fully aware of their import, and first we need to produce evidence. And as the only convincing proof is to let al-Ṭūsī's text speak for itself, we shall take three examples from his work. The first, to show the algebraic study of curves, the second to develop a discussion which announces Cardano by the presence of the formula named after Cardano, the third to reveal how affine transformation, divisibility and the derivative are combined in the solution of the equation.

1. *To solve the equation* $x^3 = ax + b$ [22]

In his introduction al-Ṭūsī has already given:

– The equation of the parabola in relation to two perpendicular axes, where one is the axis of the parabola and the other the tangent at the summit of the parabola.

– The equation of the hyperbola in relation to two perpendicular axes where one is the axis of the hyperbola and the other the tangent of the summit of the hyperbola.

– The equation of an equilateral hyperbola in relation to its asymptotes.

To solve the proposed equation, he proceeds as follows. Let $AB = \sqrt{a}$, $AC = \frac{b}{AB^2} = \frac{b}{a}$. Construct parabola P with summit A and the *latus rectum* – double the parameter – $\sqrt{a}$; hyperbola E with summit A and a transversal diameter AC (see figure).

Al-Ṭūsī first proves that the two conics intersect one another at a point other than A. His proof is as follows: the equation of P (see below) gives $\overline{AS}^2 = \overline{BM}^2$, hence

$$AS = BM = AN = NM; \tag{1}$$

equation E gives

$$NC \times AN = \overline{QN}^2, \text{ but } NC \times AN > \overline{AN}^2 = \overline{NM}^2$$

hence

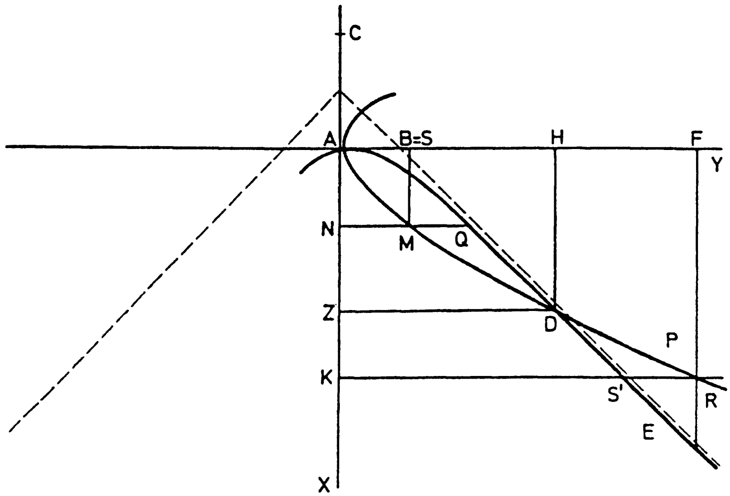
$$QN > NM. \tag{2}$$

which shows that M is inside E. Now take AF such that

$$AF > 4AB \tag{3}$$

and

$$AF \times AB > \overline{AC}^2 \tag{4}$$



by the equation of the parabola, we have

$$AF \times AB = \overline{RF}^2 > \overline{AC}^2 \quad \text{hence} \quad RF > AC \text{ or } AK > AC$$

hence

$$2AK > KC > 2AC. \tag{5}$$

With the equation P and (3), we obtain

$$\frac{\overline{RK}^2}{\overline{RF}^2} = \frac{\overline{AF}^2}{AF \cdot AB} = \frac{AF}{AB} > 4$$

which gives with (5)

$$RK > 2RF = 2AK > KC; \tag{6}$$

but as $AK \times KC = \overline{KS'}^2$ (the equation of E), we deduce that $RK > KS'$, which proves that R is outside E, where P and E intersect each other at point D. I say that AZ is the solution sought. In fact, the equation of P and E give

$$\frac{AB}{AZ} = \frac{AB}{DH} = \frac{DH}{AH} = \frac{AZ}{DZ} = \frac{DZ}{CZ}$$

hence

$$\frac{AB}{AZ} = \frac{AZ}{DZ} = \frac{DZ}{CZ}$$

hence

$$\frac{\overline{AB}^2}{\overline{AZ}^2} = \frac{AZ}{CZ}$$

hence

$$\overline{AB}^2 \times CZ = \overline{AZ}^3$$

an equality that may be expressed as

$$\overline{AB}^2 \times AC + \overline{AB}^2 \times AZ = \overline{AZ}^3,$$

which proves that AZ is a solution.

Using a different symbolism from that of al-Ṭūsī, followed closely above, equations of P and E in relation to axes AX, AY are

$$x^2 = \sqrt{a}\,y\,,$$

$$x\left(\frac{b}{a} + x\right) = y^2$$

where $x(x^3 - ax - b) = 0$. If we eliminate the trivial solution $x = 0$, we obtain our equation.

Al-Ṭūsī's exposition leads us to conclude: (1) the intersection of the parabola and hyperbola is proved algebraically, i.e. with curve equations. (2) At this stage it is possible to believe that al-Ṭūsī was trying to solve the cubic equation geometrically. The use of terms such as "inner" and "outer" in his proof may reinforce this conviction. If we take a closer look, we are led to another conclusion. In fact al-Ṭūsī was not in the least obliged to consider the figure, and he used curve equations. This usage is clear in the above and following examples. Moreover, as he works in the positive field and as the overall configuration of curves is absent, the terms "inner" and "outer" correspond, in al-Ṭūsī's usage, to the greatest and smallest. More precisely, what we have here is not geometry but a geometrical intuition of the argument of continuity. Expressed in terms different from those of al-Ṭūsī, the author wants to show, given $f(x) = \frac{1}{\sqrt{a}}\, x^2$; $g(x) = \sqrt{x^2 + \frac{b}{a}\, x}$, $a > 0$, that if there exists α and β such that $(f - g)(\alpha) > 0$ and $(f - g)(\beta) < 0$, then there exists a point $\gamma \in \,]\alpha, \beta[$, such that $(f - g)(\gamma) = 0$. The "geometrical" intuition used by al-Ṭūsī is therefore based on a notion of continuity of $(f - g)$, f and g continuous.

2. *To solve the equation* $x^3 + a = bx$ [23]

Al-Ṭūsī first states that x^2 must be smaller than b, and then writes the equation of the form $x(b - x^2) = a$. More precisely, he assumes that b = area of the square AC (see Figure 1), x^2 = area of the square AY, then the equation becomes $AT[CY] = a$, with $[CY]$ = area AC − area $AY = (AB + AT) \times BT$. He is then led to study the maximum of $x(b - x^2) = AT[CY]$ with $0 < AT < AB$. He then gives the following lemma:

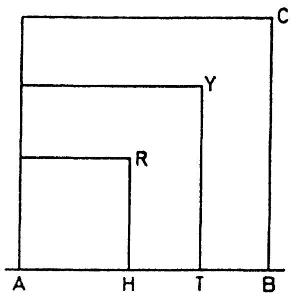


Fig. 1.

LEMMA. *The area of the square* $AR = {}^1\!/_3$ *of the area of the square* AC, *then* $AT[CY]$ *reaches its maximum for* $AT = AH$.

(1) He assumes $AT > AH$ and proves that $AH[CR] > AT\,[CY]$ as follows, we have

$$[CR] \times AH = [CY] \times AH + [YR]AH, \tag{1}$$

$$[CY] \times AT = [CY] \times AH + [CY] \times HT. \tag{2}$$

Since area $AR = {}^1\!/_3$ area $AC = \overline{AH}^2$, we obtain

$$[CR] = (AB + AH)BH = 2\overline{AH}^2. \tag{3}$$

On the other hand, we have

$$(AT + AH) \times AH = (TH + 2AH) \times AH > 2\overline{AH}^2. \tag{4}$$

taking account of (3)

$$(AB + AT) \times BT = [CY] < [CR] = 2\overline{AH}^2. \tag{5}$$

(4) and (5) give

$$(AB + AT)BT < (AT + AH) \times AH$$

hence

$$\frac{BT}{TH} \times \frac{AB + AT}{AT + AH} < \frac{BT}{TH} \times \frac{AH}{BT}$$

which gives $[CY] \times HT < [YR] \times AH$. From (1) and (2), we finally obtain $[CR] \times AH > [CY] \times AT$.

(2) He assumes $AT < AH$ and undertakes a similar proof to the above. He then returns to the equation and distinguishes three cases:

$$[[CR] \times AH < a] \Leftrightarrow \left[\, 2\left(\frac{b}{3}\right)^{\frac{3}{2}} < a \,\right]; \tag{1}$$

the problem is then impossible

$$[CY] \times AT < [CR] \times AH < a$$

whatever T belongs to AB.

$$[[CR] \times AH = a] \Leftrightarrow \left[\, 2\left(\frac{b}{3}\right)^{\frac{3}{2}} = a \,\right]; \tag{2}$$

the problem has a unique solution, i.e. $\left(\frac{b}{3}\right)^{\frac{1}{2}}$. The proof given is a check. The unicity results from the lemma.

$$[[CR] \times AH > a] \Leftrightarrow \left[2\left(\frac{b}{3}\right)^{\frac{3}{2}} > a\right]; \tag{3}$$

the problem has two solutions, one smaller than AH, the other greater. Since $[CR] \times AH$ is greater than a, there exists a positive number K such that

$$[CR] \times AH = a + K. \tag{6}$$

Al-Ṭūsī then examines the equation

$$x^3 + K = HQ \times x^2$$

with QA equal to $2AH$ (see Figure 2). Al-Ṭūsī had studied this equation earlier. Let HL be its root, i.e.

$$\overline{HL}^3 + K = HQ \times \overline{HL}^2$$

taking $HT = HL$, it may be written as

$$\overline{HL}^2 \times QT = K. \tag{7}$$

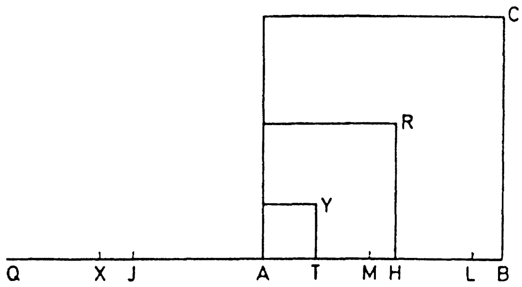


Fig. 2.

Al-Ṭūsī justifies this notation by the affine transformation $x \rightarrow \left(\frac{b}{3}\right)^{\frac{1}{2}} - x$. He wants to prove the existence of the smallest root, He first proves:

HL, i.e. HT is smaller than AH.

In fact, we have, successively

$$[CR] \times AH = 2\overline{AH}^3$$
$$\overline{AH}^2 \times AQ = 2\overline{AH}^3$$
$$[CR] \times AH = \overline{AH}^2 \times AQ = 2\overline{AH}^3 \tag{8}$$

and on the other hand

$$2BH \times AH + \overline{BH}^2 = [CR] = 2\overline{AH}^2$$

hence

$$BH < AH. \tag{9}$$

We then take $AM = BH$; we have

$$\overline{AM}^2 + 2AM \times AH = 2\overline{AH}^2$$

but

$$2HM \times AH + 2AM \times AH = 2\overline{AH}^2$$

hence

$$\overline{AM}^2 + 2AM \times AH = 2HM \times AH + 2AM \times AH$$

which gives

$$\overline{AM}^2 = 2HM \times AH,$$

hence

$$\frac{HM}{AM} = \frac{AM}{2AH} = \frac{AM}{AQ} \tag{10}$$

Let $XQ = AM$ and $XJ = MH$, then (10) is written

$$\frac{XJ}{XQ} = \frac{XQ}{JH}$$

hence by multiplying both members by $\dfrac{JQ + XQ}{XQ}$:

$$\frac{JQ + XQ}{XQ} \times \frac{XJ}{XQ} = \frac{JQ + XQ}{XQ} \times \frac{XQ}{JH}$$

which gives after simplification

$$\overline{JQ}^2 \times JH = \overline{BH}^2 \times BQ = 2\overline{AH}^3 > K$$

which gives, according to (9) and (7)

$$HL < HB < \overline{AH}.$$ [24]

Justification made, al-Ṭūsī continues as follows

$$\begin{aligned}
&[CR] \times AH = [CR] \times AT + [CR] \times HT,\\
&[CR] \times HT = 2\overline{AH}^2 \times HT,\\
&2\overline{AH}^2 \times HT = 2[RY] \times HT + 2\overline{AT}^2 \times HT,\\
&2[RY] \times HT = 2(HT \times AH + HT \times AT)HT,
\end{aligned}$$

hence

$$\begin{aligned}
[CR] \times AH &= [CR] \times AT + 2\overline{AT}^2 \times HT + 2\overline{HT}^2 \times AH + 2\overline{HT}^2 \times AT\\
&= [CR] \times AT + [RY]AT + \overline{HT}^2 \times QT\\
&= [CY] \times AT + \overline{HT}^2 \times QT
\end{aligned}$$

but

$$[CR] \times AH = a + K \quad \text{and} \quad \overline{HT}^2 \times QT = K \quad \text{(see (6), (7))}$$

hence

$$[CY] \times AT = a;$$

i.e. $(b - \overline{AT}^2)AT = a$, which proves that AT is a root of the equation, smaller than AH.

– When we examine al-Ṭūsī's proof, we observe that the auxiliary equation $x^3 + K = HQ \times x^2$ is obtained by the first equation using the affine transformation $x \to \left(\frac{b}{3}\right)^{\frac{1}{2}} - x$.

To obtain the root of the first equation we must of course subtract the root of the auxiliary equation from $\left(\frac{b}{3}\right)^{\frac{1}{2}}$. Which is what al-Ṭūsī did.

– To find the other root of the equation greater than AH, al-Ṭūsī solves the equation $x^3 + HQx^2 = K$ obtained from the initial equation by the affine transformation $x \to \left(\frac{b}{3}\right)^{\frac{1}{2}} + x$.

When he has examined the equation and found the root, al-Ṭūsī adds $\left(\frac{b}{3}\right)^{\frac{1}{2}}$ to it to obtain the root sought.

– Al-Ṭūsī's discussion may be compared with what that of Cardano for the same equation.

$x^3 + a = bx$ *Ars magna*, ch. XIII.

Al-Ṭūsī first discusses the (positive) roots of the equation $x^3 + a = bx$ ($a \geqq 0$, $b \geqq 0$). He first notes that any positive solution of this equation must be lower or equal to $b^{\frac{1}{2}}$, since if x_0 is a root, we obtain

$$x_0^3 + a = bx_0$$

hence

$$x_0^3 \leqq bx_0$$

hence

$$x_0^2 \leqq b;$$

on the other hand, this root must verify the equality

$$bx - x^3 = a.$$

Al-Ṭūsī seeks the value where $y = bx - x^3$ reaches its maximum and finds $x = \left(\frac{b}{3}\right)^{\frac{1}{2}}$ by cancelling the first derivative. This maximum is therefore

$$b \times \left(\frac{b}{3}\right)^{\frac{1}{2}} - \left(\frac{b}{3}\right)^{\frac{3}{2}} = 2\left(\frac{b}{3}\right)^{\frac{3}{2}}$$

There exists a positive root if and only if

$$a \leqq 2\left(\frac{b}{3}\right)^{\frac{3}{2}} \Leftrightarrow \frac{b^3}{27} - \frac{a^2}{4} \geqq 0.$$

The role of the discriminant $D = \frac{b^3}{27} - \frac{a^3}{4}$ is thus established and elaborated algebraically for the study of the cubic equation.

To confirm the set of propositions advanced earlier, let us only examine *two cases* from al-Ṭūsī's discussion of the following problem.

3. *To solve the equation* $x^3 + a = bx^2 + cx$ [25] (1)

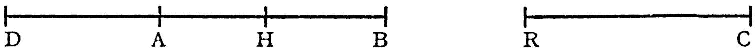
We shall follow his discussion by keeping close to al-Ṭūsī's text. He distinguishes three cases:

$$\text{I} - b = \sqrt{c}.$$

He first proves that the problem is impossible if $a > b^3$. Let

$$AB = \sqrt{c}$$
$$RC = b = AB.$$

Assume the problem possible, and distinguish two cases



1. BD is a root $> AB$. Substituting in (1) we obtain

$$\overline{BD}^3 - AB \times \overline{BD}^2 = \overline{AB}^2 \times BD - a$$

hence

$$\overline{AB}^2 \times BD - \overline{BD}^2 \times AD = a;$$

on the other hand, we have

$$\overline{AB}^2 \times BD - \overline{AB}^2 \times AD = \overline{AB}^3$$

hence

$$\overline{AB}^3 - a = (\overline{BD}^2 - \overline{AB}^2) \times AD = (AB + BD) \times \overline{AD}^2 \geqq 0$$

hence

$$a \leqq \overline{AB}^3.$$

2. BH is root $< AB$. Similarly as in the previous case, we find that

$$a - \overline{AB}^2 \times BH = \overline{BH}^2 \times AH,$$
$$\overline{AB}^3 - \overline{AB}^2 \times BH = \overline{AB}^2 \times AH$$

hence

$$a - \overline{AB}^2 \times BH \leqq \overline{AB}^3 - \overline{AB}^2 \times BH$$

hence

$$a \leqq \overline{AB}^3.$$

In both cases where the problem is possible, we should have $a \leq \overline{AB}^3 = b^3$.

Al-Ṭūsī then examines the following three cases:

(1) $a > \overline{AB}^3$; we have seen that the problem is impossible.

(2) $a = \overline{AB}^3$; there is one solution AB. Al-Ṭūsī's demonstration is a simple verification.

(3) $a < \overline{AB}^3$; there are two solutions.

In fact, let $BK = AB$ (see Figure 3) and the equation

$$x^3 + AK\,x^2 = \overline{AB}^3 - a \tag{2}$$

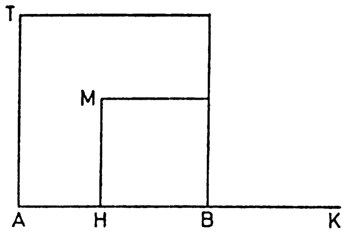


Fig. 3.

an equation studied earlier in al-Ṭūsī's treatise. Let AD be the solution of (2), then BD is the solution of equation (1). In fact as AD is the root of (2), we obtain

$$\overline{AD}^2 \times DK + a = \overline{AB}^3, \tag{3}$$

but

$$\overline{AD}^2 \times DK = AD \times AD(DB + AB) = [TM] \times AD$$

then (3) is written

$$[TM] \times AD + a = \overline{AB}^3;$$

we then have, successively,

$$[TM] \times AD + \overline{AB}^2 \times AD + a = \overline{AB}^3 + \overline{AB}^2 \times AD,$$
$$\overline{BD}^2 \times AD + a = \overline{AB}^2 \times BD,$$
$$\overline{BD}^2 \times AD + \overline{DB}^2 \times AB + a = \overline{AB}^2 \times BD + \overline{DB}^2 \times AB,$$
$$\overline{BD}^2 \times BD + a = \overline{AB}^2 \times BD + \overline{BD}^2 \times AB,$$

which proves that BD is a root of the equation sought. Al-Ṭūsī then gives additional information about the root BD: BD has a *higher limit.*[26] In fact, we have seen (3) that

$$\overline{AB}^3 - a = \overline{AD}^2 \times DK$$

hence

$$\overline{AD}^2 \times DK < \overline{AB}^3;$$

as $DK > AB$ we obtain $AD < AB$, hence

$$DB = AD + AB < 2AB.$$

To find the other solution, al-Ṭūsī takes the equation

$$x^3 + \overline{AB}^3 - a = AK \times x^2. \tag{4}$$

If AH is a root of (4) (AH is smaller than AB according to al-Ṭūsī's earlier study on this type of equation (see Figure 4)), then BH is a root of equation (1).

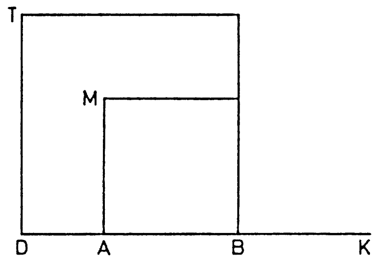


Fig. 4.

In fact AH being a root of (4) we have

$$\overline{AH}^2 \times HK = \overline{AB}^3 - a, \tag{5}$$

but

$$\begin{aligned}\overline{AB}^3 &= \overline{AB}^2 \times BH + \overline{AB}^2 \times AH \\ &= \overline{AB}^2 \times BH + \overline{BH}^2 \times AH + [TM] \times AH \\ &= \overline{AB}^2 \times BH + \overline{BH}^2 \times AH + (AB + BH)\overline{AH}^2 \\ &= \overline{AB}^2 \times BH + \overline{BH}^2 \times AH + \overline{AH}^2 \times HK;\end{aligned}$$

on the other hand, from (5) we have

$$\overline{AB}^3 = a + \overline{AH}^2 \times HK$$

hence

$$\overline{AB}^2 \times BH + \overline{BH}^2 \times AH = a$$

hence

$$\overline{BH}^3 + a = \overline{AB}^2 \times BH + AB \times \overline{BH}^2$$

which proves that BH is a solution of (1).

Al-Ṭūsī proves in a similar way to the one used in the case of the first root that BH has a lower limit.

REMARK. In the case where there are two solutions, from equation (1) and with these two affine transformations

$$x \rightarrow x + AB$$
$$x \rightarrow AB - x;$$

al-Ṭūsī obtains, respectively,

$$x^3 + AK\, x^2 = \overline{AB}^3 - a$$
$$x^3 + \overline{AB}^3 - a = AK\, x^2.$$

He adds the root of the first transformation to AB, and subtracts the root of the second to obtain the roots sought.

$$\text{II} - b > \sqrt{c}$$

Let $AB = \sqrt{c}$, $BC = b$ and $x = BH$; equation (1) is written

$$\overline{BH}^2 \times BC - \overline{BH}^3 + \overline{AB}^2 \times BH = a.$$

Al-Ṭūsī is induced to study the *maximum of the expression*

$$bx^2 - x^3 + cx = \overline{BH}^2 \times BC - \overline{BH}^3 + \overline{AB}^2 \times BH.$$

Al-Ṭūsī's result is given by the following lemma.

LEMMA. *Let the second-degree equation be*

$$\frac{2}{3} BC \cdot x + \frac{1}{3} \overline{AB}^2 = x^2; \tag{6}$$

let BD be the root of this equation, then whatever H different from D, we have:

$$\overline{BH}^2 \times BC - \overline{BH}^3 + \overline{AB}^2 \times BH < \overline{BD}^2 \times BC - \overline{BD}^3 + \overline{AB}^2 \times BD. \tag{7}$$

Al-Ṭūsī first proves that $AB < BD < BC$.

We have seen according to (6) that BD is the sum of

$$x_1 = \frac{2}{3} BC \tag{8}$$

$$x_2 = \frac{1}{3} \frac{\overline{AB}^2}{BD} \tag{9}$$

Then if $BD = AB$, we obtain from (6)

$$\overline{AB}^2 = \frac{2}{3} BC \times AB + \frac{1}{3} \overline{AB}^2 > \overline{AB}^2$$

which is absurd. If $BD < AB$, we have from (9) $x_2 > \frac{1}{3} AB$, and on the other hand, from (8): $x_1 > \frac{2}{3} AB$ where

$$BD = x_1 + x_2 > \frac{1}{3} AB + \frac{2}{3} AB = AB$$

which is also absurd. On the other hand, as $BD > AB$ we obtain from (9)

$$x_2 < \frac{1}{3} AB < \frac{1}{3} BC;$$

then from (8), we obtain

$$BD = x_1 + x_2 > \frac{1}{3} BC + \frac{2}{3} BC = BC.$$

Al-Ṭūsī goes back to the proof of the lemma, and distinguishes several cases

(1) $BC > BH > BD$ (see Figure 5).

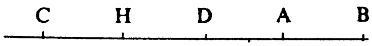


Fig. 5.

(7) is then written

$$\begin{aligned}
&\overline{BH}^2 \times CH + \overline{AB}^2 \times BH < \overline{BD}^2 \times CD + \overline{AB}^2 \times BD,\\
&\overline{BD}^2 \times CD + \overline{AB}^2 \times BD\\
&\quad = \overline{BD}^2 \times CH + \overline{BD}^2 \times HD + \overline{AB}^2 \times BD,\\
&\overline{BH}^2 \times CH + \overline{AB}^2 \times BH = \overline{BD}^2 \times CH\\
&\quad + (BD + BH)HD \times CH + \overline{AB}^2 \times BD + \overline{AB}^2 \times DH.
\end{aligned}$$

The difference between both members of (7) is therefore

$$\begin{aligned}
&\overline{BD}^2 \times HD - \overline{AB}^2 \times DH - (BD + BH)HD \times CH\\
&\quad = (BD + AB)AD \times HD - (BD + BH)HD \times CH.
\end{aligned}$$

To prove the lemma therefore implies proving that

$$(BD + AB)AD > (BD + BH) \times CH.$$

however

$$\begin{aligned}
&2BD \times CD = 2BD \times DH + 2DB \times CH,\\
&(BH + DB)CH = 2DB \times CH + DH \times CH,
\end{aligned}$$

but

$$2BD \times DH > DH \times CH$$

since according to (8), we have

$$2BD > 2 \cdot \frac{2}{3}CB > CB > CH$$

hence

$$2BD \times CD > (BH + DB)CH,$$

but from (6), we obtain by substituting BD for x, all simplifications made

$$2BD \times CD = (BD + BA)AD \tag{9'}$$

hence

$$(BD + BA)AD > (BH + DB)CH.$$

The lemma is therefore proved in this case.

(2) $BH = BC.$

In this case (7) yields

$$\overline{AB}^2 \times BC < \overline{BD}^2 \times CD + \overline{AB}^2 \times BD;$$

while the difference between both members yields

$$\overline{BD}^2 \times CD + \overline{AB}^2 \times BD - \overline{AB}^2 \times BC =$$
$$\overline{BD}^2 \times CD - \overline{AB}^2 \times CD > 0(BD > AB).$$

Which proves the lemma in this case as well.

(3) $BH > BC$ (see Figure 6).

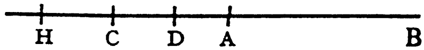


Fig. 6.

In this case (7) is written

$$\overline{AB}^2 \times BH - \overline{BH}^2 \times CH < \overline{BD}^2 \times CD + \overline{AB}^2 \times BD$$

but as $BH > AB$, we have

$$\overline{AB}^2 \times BH - \overline{BH}^2 \times CH < \overline{AB}^2 \times BH - \overline{AB}^2 \times CH = \overline{AB}^2 \times CB$$

but we have seen in case (2) that

$$\overline{AB}^2 \times BC < \overline{BD}^2 \times CD + \overline{AB}^2 \times BD$$

which proves the lemma in this case.

(4) $AB < BH < BD$.
(5) $AB = BH$.
(6) $AB > BH$.

These cases are treated in the same way.

Call the maximum found S; i.e.

$$S = \overline{BD}^2 \times CD + \overline{AB}^2 \times BD$$

and return to equation (1). Al-Ṭūsī distinguishes three cases

(1) $S < a$ – there is no solution;
(2) $S = a$ – it has one solution i.e. BD itself;
(3) $S > a$ – it has two solutions: one root higher than BD and the other smaller than BD.

To find the highest root

(a) $a > \overline{AB}^2 \times BC.$

In this case there is a solution lying strictly between BD and BC. In fact let

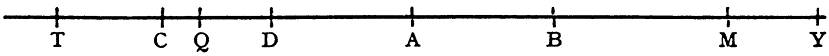


Fig. 7.

$$\left.\begin{array}{l} BY = BD \\ MY = CD \end{array}\right\} \tag{10}$$

and let the equation be

$$x^3 + DM\,x^2 = S - a. \tag{11}$$

Al-Ṭūsī proves that if we add DB to the root of this equation, we obtain the root sought for equation (1). But before proving this proposition, he proves that the root of (11), *i.e.* DQ, is smaller than DC.

In fact

$$\begin{aligned} S - a &< S - \overline{BA}^2 \times CB \\ &= \overline{BD}^2 \times CD + \overline{AB}^2 \times BD - \overline{AB}^2 \times CB \\ &= \overline{BD}^2 \times CD - \overline{AB}^2 \times CD = (BD + AB) \times AD \times CD \end{aligned}$$

but according to (9′), we have

$$(BD + AB)AD = 2DB \times DC,$$

hence

$$\begin{aligned} S - a < 2DB \times \overline{DC}^2 = DY \times \overline{CD}^2 &= \overline{CD}^2 \times CM \\ &= \overline{CD}^3 + \overline{CD}^2 \times DM, \end{aligned}$$

but

$$S - a = \overline{DQ}^3 + \overline{DQ}^2 \times DM$$

hence

$$DQ < CD.$$

Al-Ṭūsī then proves that BQ is the root sought. In fact according to

(9′) we have

$$\begin{aligned}(DB + BA)DA \times DQ &= YD \times CD \times DQ\\ &= YD \times DQ \times CQ + CM \times \overline{DQ}^2\\ &= YD \times DQ \times CQ + \overline{DQ}^2 \times QM + \overline{DQ}^2 \times CQ\\ &= DQ \times CQ \times YQ + \overline{DQ}^2 \times QM\\ &= (QB + BD)DQ \times CQ + \overline{DQ}^2 \times QM.\end{aligned}$$

Add $\overline{BA}^2 \times BQ$ to the extreme members, we have

$$\begin{aligned}&\overline{BD}^2 \times DQ + \overline{BA}^2 \times BD\\ &\quad = (QB + BD)DQ \times CQ + \overline{DQ}^2 \times QM + \overline{BA}^2 \times BQ;\end{aligned}$$

add $\overline{BD}^2 \times CQ$ to both members, we obtain

$$\begin{aligned}S &= \overline{BD}^2 \times CD + \overline{BA}^2 \times BD\\ &= \overline{BQ}^2 \times CQ + \overline{BA}^2 \times BQ + \overline{DQ}^2 \times QM. \qquad (12)\end{aligned}$$

But DQ being the root of (11) we obtain

$$\overline{DQ}^3 + DM \times \overline{DQ}^2 = S - a$$

hence

$$\overline{DQ}^2(DQ + DM) = S - a$$

hence

$$\overline{DQ}^2 \times QM = S - a;$$

substitute in (12) we finally obtain

$$\overline{BQ}^2 \times CQ + \overline{BA}^2 \times BQ = a$$

hence

$$\overline{BQ}^2(BC - BQ) + \overline{BA}^2 \times BQ = a$$

hence

$$\overline{BQ}^3 + a = BC \times \overline{BQ}^2 + \overline{BA}^2 \times BQ$$

which proves that BQ is a root of equation (1).

(b) $\quad a = \overline{BA}^2 \times BC.$

Al-Ṭūsī proves by simple verification that BC is the root sought.

(c) $\quad a < \overline{BA}^2 \times BC.$

Al-Ṭūsī finds the root of (11) and proves that if BD is added to it, we obtain the solution sought. To establish this proof, he first verifies that the root of (11) namely DT, is greater than CD.

In fact as $a < \overline{BA}^2 \times BC$ assumed, we have

$$S - a > S - \overline{AB}^2 \times CB$$

but

$$\begin{aligned} S - a &= \overline{DT}^3 + DM \times \overline{DT}^2 \\ S - \overline{AB}^2 \times CB &= \overline{BD}^2 \times CD + \overline{AB}^2 \times BD - \overline{AB}^2 \times CB \\ &= \overline{CD}^3 + DM \times \overline{CD}^2 \end{aligned}$$

hence

$$\overline{CD}^3 + DM \times \overline{CD}^2 < \overline{DT}^3 + DM \times \overline{DT}^2$$

hence

$$CD < DT.$$

Al-Ṭūsī then turns to equation (1). We know that S is, by definition, the value of the expression

$$BC \times x^2 + \overline{AB}^2 x - x^3 \text{ for } x = BD.$$

We have therefore

$$BC \times \overline{BD}^2 + \overline{AB}^2 \times BD = S + \overline{BD}^3.$$

If we add $\overline{BA}^2 \times DT$ to both members, we obtain

$$\overline{AB}^2 \times BT + BC \times \overline{BD}^2 = S + \overline{BD}^3 + \overline{BA}^2 \times DT.$$

If we add $(\overline{TB}^2 - \overline{DB}^2)BC$, we obtain

$$\begin{aligned} &\overline{AB}^2 \times BT + BC \times \overline{BT}^2 \\ &\quad = S + \overline{BD}^3 + \overline{BA}^2 \times DT + (\overline{TB}^2 - \overline{BD}^2)BC. \end{aligned}$$

Lastly, adding

$$\mathcal{H} = (\overline{TB}^2 - \overline{BD}^2)TC + (\overline{BD}^2 - \overline{BA}^2)TD$$

to both members, we obtain

$$\overline{AB}^2 \times BT + BC \times \overline{BT}^2 + \mathcal{H} = S + \overline{BT}^3. \tag{13}$$

But

$$(\overline{BD}^2 - \overline{BA}^2)TD = (BD + BA) \times AD \times TD = 2BD \times CD \times TD$$

according to (9′)

$$(\overline{TB}^2 - \overline{BD}^2)TC$$
$$= (TB + BD) \times BT \times TC = 2TC \times BD \times DT + \overline{DT}^2 \times TC$$

hence

$$\mathcal{H} = 2BD \times TD \times DT + \overline{DT}^2 \times TC$$
$$= MC \times \overline{TD}^2 + \overline{DT}^2 \times TC = \overline{TD}^2 \times MT.$$

But as TD is the root of (11), we have

$$S - a = \overline{TD}^3 + DM \times \overline{TD}^2 = \overline{TD}^2 \times MT$$

hence

$$\mathcal{H} = S - a.$$

Substituting in (13), we obtain

$$\overline{BT}^3 + a = \overline{AB}^2 \times BT + BC \times \overline{BT}^2$$

which proves that BT is the solution sought and $BT > BD$; $BT > BC$.

Al-Ṭūsī also studies the following problem: Given that AB, BC, then the family of roots of equations of the family $(x^3 + a = \overline{AB}^2 + BC\, x^2;$ $0 < a < \overline{AB}^2 \times BC)$, greater than BD, is the interval $]BD, BT_1[$, where BT_1 is the root of the equation $x^2 = \overline{AB}^2 + BC\, x$.

In fact, BT_1 is the root of no equation of the family, since we have

$$\overline{AB}^2 \times BT_1 + BC \times \overline{BT}_1^2 = \overline{BT}_1^3 < \overline{BT}_1^3 + a.$$

On the other hand, for whatever BQ belonging to $]BD, BT[$, there exists a such that BQ is the root of the equation $x^3 + a = \overline{AB}^2\, x + BC\, x^2$.

In fact

$$\overline{BT}^3 - \overline{BQ}^3 = \overline{BT}^3 - \overline{BQ}^2(BT_1 - T_1Q)$$
$$= (\overline{BT}^2 - \overline{BQ}^2)BT_1 + \overline{BQ}^2 \times T_1Q,$$
$$\overline{BT}^3 - (\overline{AB}^2 \times BQ + BC \times \overline{BQ}^2)$$
$$= \overline{AB}^2 \times BT_1 + BC \times \overline{BT}_1^2 - \overline{AB}^2 \times BQ - BC \times \overline{BQ}^2$$
$$= (\overline{BT}_1^2 - \overline{BQ}^2)BC + \overline{AB}^2 \times T_1Q.$$

By comparison, we find that

$$\overline{BQ}^3 < \overline{AB}^2 \times BQ + BC \times \overline{BQ}^2.$$

Therefore, there exists a such that

$$\overline{BQ}^3 + a = \overline{AB}^2 \times BQ + BC \times \overline{BQ}^2.$$

We therefore conclude that in this case al-Ṭūsī was induced first to find the maximum of the expression $bx^2 + cx - x^3$. To determine the maximum, he cancels $2/3bx + 1/3c = x^2$, i.e. in another language, the first derivative of this equation.

When he has shown that the root DB corresponds to the maximum, once found, namely S, he distinguishes three cases:

(1) $S < a$, no solution,
(2) $S = a$, one solution,
(3) $S > a$, two solutions.

The last case is divided into three cases

(a) $a > \overline{BA}^2 \times CB$.

He transforms the equation by $x \to DB + x$ and finds $x^3 + DMx^2 = S - a$ which he had examined earlier.

To find the other root, he transforms the equation by $x \to BD - x$

(b) $a = \overline{BA}^2 \times BC$;

the solution sought is BC. The proof is a verification.

(c) $a < \overline{BA}^2 \times BC$.

He solves the equation transformed by $x \to DB + x$ to find one of the two roots.

To find the other root he solves the one transformed by $x \to DB - x$.

V

If by the geometric theory of cubic equations we mean using geometrical figures to determine the real roots of these equations, al-Ṭūsī goes far beyond this framework. The examples given in his own style show that his concern was an entirely different attempt. The geometrical figure only played an auxiliary role and al-Ṭūsī, far from limiting himself to it, thought "function" and studied curves by their equations.

We have discussed elsewhere for itself this fundamental stage in the history of algebraic geometry. It has been established that a history of algebraic geometry is inconceivable without a study of this trend of Arabic algebra, initiated by al-Khayyām and developed by al-Ṭūsī, and still to be undertaken.

It was with these algebraists that the use of the "derivative" in the discussion of algebraic equations was seen to emerge. It is common knowledge, however, that related to the search for maxima, the use of the "first derivative" was not new. Called on, however, for a particular example, its usage remained sporadic and only became an integral part of the solution of cubic equations with al-Ṭūsī. The generalization of its usage was in fact determined by that of the theory of equations which was being elaborated at that time. One important difference: generalization was the result of progress accomplished in the field of algebra, and investigations by mathematicians whose activities extended to other fields.

The works of Banū Mūsā, Ibn Qurra, his grand-son Ibrāhīm ibn Sinān, al-Qūhī, Ibn al-Haytham, and many other non-algebraists on infinitesimal determinations, indirectly prepared the way for attempts such as that of al-Ṭūsī. An informed and meticulous history of differential concepts before the beginnings by Newton and Leibniz would show to what extent we may affirm that the mathematicians cited earlier completed the study of these determinations.

Their refusal to interpret algebraic operations geometrically, manifest in Banū Mūsā's work and reaffirmed by their successors, their discovery of new arithmetic laws for calculating surfaces and volumes, enabled them to generalize the concept of number.

However, despite the obvious significance of their results, the calculus of infinitesimal determinations could not be transformed into differential and integral calculus as seen in Newton and Leibniz, since the lack of a comprehensive and efficient algebraic symbolism was a serious obstacle to this transformation. However, it was precisely this symbolism that made it possible to *name* the notion, present in mathematical research: the derivative.

The question is still open however: how did al-Ṭūsī systematically use an unnamed notion? The fact can only be explained outside the "infinitesimalist" tradition, and was only made possible by the extension of algebra itself. The simple enumeration and classification of equations, necessary for elaborating a theory of equations with which algebra had begun to merge, the search for a method for solving cubic equations led to an extension of the field of application of the notion of "derivative" and the generalization of this application.

The notion of the "derivative", the work of infinitesimalists extended by algebraists, was therefore doomed to oblivion because of the weakness of algebraic symbolism. We know moreover that this weakness persisted; even in the seventeenth century, algebraic symbolism was more adapted to differential concepts than algebra itself. The least we may therefore affirm is that mathematicians who had systematized the usage of a notion present, though unnamed, were capable of extending it to a neighbouring field of algebraic equations, i.e. the solution of numerical equations. This was the case of al-Ṭūsī.

So replaced in context – algebra – the solution of numerical equations brought out a meaning it had always possessed: to compensate for the absence of an algebraic solution, explicitly by radicals, for third-degree or higher equations. Even the specific presence if not global, of the so-called "Cardano" formula could not replace this solution. However, algebra contained conceptual means for posing the problem of numerical equations of any degree.

This algebra itself, al-Ṭūsī's method for solving numerical equations, shows that the history of Arabic algebra and the Renaissance is still to be written. To stimulate this work we shall conclude with a conjecture for historians: this algebraic tradition – that of al-Khayyām and al-Ṭūsī – survived and was known to sixteenth-century algebraists, of whom Viète was the most important.

NOTES

1. Harriot (1631), pp. 117–180. Herigone (1634), vol. 2, pp. 266ff. Oughtred (1652), pp. 121–196. Dechales (1647), 1, pp. 646–652. Prestet (1689), vol. 2, pp. 432–440. Wallis (1685), pp. 113–117.
2. In his famous letter dated 26th June 1676, Newton wrote: "Extractiones in numeris, sed methodus Vietae et Oughtredi nostri huic negotio minus idonea est, quapropter aliam excogitare adactus sum . . .".

   Newton expounded his method in a letter to Oldenbourg and Leibniz dated 26th July 1677; See Gerhardt (1962), pp. 179–192. See also his letter to Collins dated 20th June 1674: Turnbull (1959), pp. 309–310. Turnbull gives the references to other letters on the same problem. We know the method is to be found in *De analysi per aequationes numero terminorum infinitas* (1669), reproduced in *Methodus fluxionum et serierum infinitarum* (1671) and only published in 1736. The first edited summary is in Wallis (1685), pp. 381–383. See also Buffon's "Introduction" (1740) and Cajori (1911), pp. 29–30. Lastly, see Whiteside (1964), I, pp. 928ff.

3. Lagrange (1878), pp. 159ff. Mouraille (1768), Part 1. Fourier (1830). See also Cajori (1910–11), pp. 132–137.
4. Horner (1819), Part 1, pp. 308–335 and also Smith (1959), 1, pp. 232–252.
5. Burnside and Panton (1912), 1, note B: "The first attempt at a general solution by approximation of numerical equations was published in the year 1600 by Vieta. Cardan had previously applied the rule of 'false position' (called by him 'regula aurea') to the cubic; but the results obtained by this method were of little value". Whittaker and Robinson (1926), see ch. 6, par. 41. Young (1843), pp. 248ff.
6. Sédillot (1847), pp. 69–83. Woepcke (1854a), p. xix. To calculate the value of sine 1° the equation $X = (X^3 + A)B$ must be solved; B is of a higher order than X. The method exposed by Shalabī is based on an idea common to a family of approximation methods: replace the initial equation by a linear or an approached equation as often as one wants.

   He sets it out as follows

$$X = \sum_{k=0}^{\infty} \frac{1}{m^k} x_k, \quad A = am + \sum_{k=0}^{\infty} \frac{1}{m^k} a_k, \quad B = bm$$

   hence

$$X^3 = \left(\sum_{k=0}^{\infty} \frac{1}{m^k} x_k \right)^3 = (bx_0 - a)m + \sum_{k=1}^{\infty} (bx_k - a_{k-1}) \frac{1}{m^{k-1}}$$

   hence, using the method of indeterminate coefficients, we have x_k for $k = 0, 1, 2, \ldots$ but then a_k, b are integers, x_k are not usually integers. We then take the integer part and obtain

$$X = \sum_{k=0}^{\infty} \frac{1}{m^k} [x_k].$$

   This type of solution consists of "substituting the third-degree equation proposed by an infinity of linear equations". A detailed description of this method is to be found in Woepcke (1854a) and Woepcke (1854b) pp. 153ff. See also Hankel (1874), pp. 290–292.
7. Luckey (1948), pp. 217–274. To give a general summary of al-Kāshī's method it should be recalled that the author solves the equation $x^n - Q = 0$. He takes as a first approximation the highest integer lower than $Q^{1/n}$, that is $x_0 = [Q^{1/n}]$, and obtains

$$Q = x^n = (x_0 + x_1)^n$$

   hence

$$(x_0^n - Q) + nx_0^{n-1}x_1 \approx 0$$

$$x_1 \approx \frac{Q - x_0^n}{nx_0^{n-1}}$$

   hence

$$x = Q^{1/n} \approx x_0 + \frac{Q - x_0^n}{nx_0^{n-1}} .$$

hence Luckey showed that al-Kāshī used Horner's table to calculate the coefficients for each transformed function.

8. Luckey (1951) analyzed the work and translated some passages. The only edition of this work is by A. S. al-Demerdash and M. H. al-Cheikh (1967). A Russian translation of al-Kāshī's work with a photographic reproduction exists, without a critical edition of the text. The translation is by Rosenfeld: al-Kāshī (1956).
9. Luckey (1948), p. 248.
10. See our edition: al-Samawʾal (1972) and *supra*, pp. 22–33.
11. See Woepcke's translation: Woepcke (1851).
12. The case of Sharaf al-Dīn al-Ṭūsī is not uncommon in the history of mathematics. The importance of his work, frequently reaffirmed by algebraists, contrasts with a total lack of historical studies on his work. And if the same situation is almost true for al-Samawʾal, the case of al-Ṭūsī is more surprising, it reveals the inadequacy of historical research in this field and renders our knowledge of Arabic algebra and the Renaissance precarious. Al-Ṭūsī was not only cited by Arab algebraists but by historians as well. For instance, al-Kāshī cited him in *The Key to Arithmetic* on the solution of cubic equations, p. 198. He was also cited by the famous Kamāl al-Dīn al-Fārisī in MS Shehid Ali Pasha 1972, *Asās al-Qawāʿid fī uṣūl al-fawāʾid* ("The foundation of rules . . ."). Several references show that some algebraists recognized the importance of al-Ṭūsī's contribution. For instance, the early thirteenth-century mathematician Shams al-Dīn Ismāʿīl ibn Ibrāhīm al-Mārdīnī attributed the invention of the "table method" to him, *i.e.* the numerical solution of cubic equations. See *Niṣāb al-ḥabr*, MS Istanbul, Feyzullah, 1366. Al-Ṭūsī was no less unknown to ancient or modern biographers. Sarton wrote his biography and recalled that he composed "a treatise on algebra . . . in 1209–10 [which] is only known through a commentary [*talkhīs*] by an unknown author". See Sarton (1950), II, pp. 622–623. This fact was mentioned in Suter's bibliography earlier and in the MS in the India Office. See Suter (1900, p. 134). See also Brockelmann (1898), 1, p. 472. Al-Ṭūsī is also known as the inventor of the *Linear Astrolabe* since the translations by Suter and Carra de Vaux. See Carra de Vaux (1895), pp. 464–516. Also Suter (1895), pp. 13–18 and (1896), pp. 13–15. Ancient biographers, at least those we were able to consult, cites al-Ṭūsī but without giving any important biographical details. See al-Qifṭī (ed., 1903), p. 426. Ibn Khallikān (ed., 1978), p. 314 where we may read: "Shaykh Sharaf al-Dīn al-Muẓaffar ibn Muḥammad ibn al-Muẓaffar al-Ṭūsī is the inventor of the linear astrolabe called the stick". Tashkupri-Zadeh (1968), p. 392: "Sharaf al-Dīn, Muḥammad Masʿūd ibn Masʿūd al-Masʿūdī [composed] a 'development' [the opposite of a summary] in this discipline [Algebra]". As yet virtually nothing is known about his life. He lived in the twelfth century, taught in Damascus where the famous Muhadhdhab al-Dīn ibn al-Dīn al-Ḥājib was his pupil, and in Mosul where his pupils were Kamāl al-Dīn ibn Yūnus and Muḥammad ibn ʿAbd al-Karīm al-Ḥarithī, and lastly, in Baghdad. From Ṭus in Khurasan, he probably died in the last quarter of the 12th century.

    Of his works, the following are known: *Risāla fī ṣanʿ al-asṭurlāb al-musaṭṭaḥ*, MS Leiden 591 ("A Discourse on the linear astrolabe"); "A Reply to a geometrical question asked by his friend Shams al-Dīn", cited by Suter; *Risāla fī al-ḫaṭṭayn alladhayni yaqrubāni wa lā yataqābalāni*, cited by Brockelmann (1898), I, supp. Bd. p. 850, probably a treatise on asymptotes; lastly, the *Algebra*.

The only surviving copy of his treatise on algebra entitled *On equations* (al-Ṭūsī, ed., 1986) is in the India office, London. For a translation and complete study of this work including a new translation of al-Khayyām's work see al-Ṭūsī (ed., 1986). The text is not a "commentary" as Sarton wrote, but really a summary as the unknown author explained: "In this work I wanted to summarize the art of algebra and *al-muqābala*, adapt what has survived from the great philosopher Sharaf al-Dīn al-Muẓaffar ibn Muḥammad al-Ṭūsī, and reduce his overlengthy exposition to a moderate size; I eliminated the tables he drew up to make his computations and solve his problems". The manuscript is therefore an adaptation of al-Ṭūsī's algebraic work minus the tables and some figures; if we include the copyist's errors, it is understandable why it is difficult to read. This probably explains why it has never aroused the curiosity of historians. The manuscript comprises 154 in-folio pages.

13. We now know that this work on algebra was completed by Karajī and his successors such as al-Samawʾal. See al-Samawʾal (ed., 1972), and Rashed (1971).
14. *Infra*, I.4, pp. 64–68.
15. In this respect, al-Ṭūsī's work may only be understood if sufficient emphasis is laid on the development of method invented to extract roots from a number. Two movements have marked the history of this problem: the first was dominated by the first algebraist, al-Khwārizmī, while the second was almost achieved by the renovator of algebra: al-Karajī. If the Arabic text of al-Khwārizmī's arithmetic remains lost, its Latin translation has, on the other hand, survived, see Vogel (1963). This text teaches us that al-Khwārizmī saw the problem of the extraction of the square root as a step towards the systematic study of arithmetical operations. He gave the rule of approximation for the square root of a number N; $N = a^2 + r$, $\sqrt{N} = a + r/2a$. This fact was confirmed by al-Khwārizmī's Arab successors. For instance, al-Baghdādī (d. 1037), in his work *al-Takmila* (MS Laleli 2708/1 Istanbul), attributed this approximation to al-Khwārizmī, and recalled that later Arab mathematicians deliberately abandoned it as unsatisfactory for values such as $\sqrt{2}$, $\sqrt{3}$. Much more important than the rule of approximation are al-Khwārizmī's fundamental ideas on the subject and which may be Hindu in origin. He uses both the development $(a + b)^2$ and the notation for $N = n_0 10^{m-1} + \ldots + n_m$. His method consists of:

(1) differentiating between two-digit groups, i.e. the positions 10^{2k}, $k = 0, 1, 2, \ldots$ the set of digits of the number from which one wants to extract the root, working from right to left.

(2) then finding a number whose square is the highest square included in the last group (on the left) of two digits. This number will be the first digit of the root, that is a, written in its decimal order.

(3) subtracting a^2 to obtain the first remainder, and determining the second digit of the root with its decimal order, that is b, subtract $2ab$, b^2 from the first remainder and so on. Al-Khwārizmī's computation was not direct and his exposition remained imperfect. Later on, Arab mathematicians attempted to improve the approximation, and perfected the representation of the method, and finally extended it to the extraction of higher order roots: these were the three goals al-Khwārizmī's successors attempted to reach.

For instance, al-Uqlīdisī (ed., 1973) first gives the approximation $\sqrt{N} = a + r/(2a + 1)$, but recalls that if al-Khwārizmī's approximation was in excess, it was by default and $\sqrt{N} = a + r/2a + 1/2$. Kūshyār ibn Labbān (1000 c.) wanted to improve the result and representation. For instance, in the example $N = 65342$, he proposes the following figures:

$$\text{(1)}\quad \begin{array}{lllll} 2 & & & & \\ 6 & 5 & 3 & 4 & 2 \\ 4 & & & & \end{array} \left\{ \begin{array}{l} \quad a = 2.10^2 \\ \Leftrightarrow N \\ \quad 2a \end{array} \right.$$

$$\text{(2)}\quad \begin{array}{lllll} & 2 & & & \\ 2 & 5 & 3 & 4 & 2 \\ & 4 & & & \end{array} \left\{ \begin{array}{l} \quad a \\ \Leftrightarrow N - a^2 \\ \quad 2a \end{array} \right.$$

$$\text{(3)}\quad \begin{array}{lllll} & 2 & 5 & & \\ 2 & 5 & 3 & 4 & 2 \\ & 4 & 5 & & \end{array} \left\{ \begin{array}{l} \quad (a + b);\ b = 5 \times 10 \\ \Leftrightarrow N - a^2 \\ \quad (2a + b);\ (2a + b)b \leqq N - a^2 \end{array} \right.$$

$$\text{(4)}\quad \begin{array}{llll} & 2 & 5 & \\ 2 & 8 & 4 & 2 \\ & 4 & 5 & \end{array} \left\{ \begin{array}{l} \quad a + b \\ \Leftrightarrow N - a^2 - (2a + b)b \\ \quad (2a + b) \end{array} \right.$$

$$\text{(5)}\quad \begin{array}{lll} 2 & 5 & 5 \\ 3 & 1 & 7 \\ 5 & 1 & 1 \end{array} \left\{ \begin{array}{l} \quad (a + b + c);\ c = 5 \times 10^0 \\ \Leftrightarrow N - a^2 - (2a + b)b - (2a + 2b + c)c \\ \quad 2(a + b + c) + r \end{array} \right.$$

hence $\sqrt{N} = (a + b + c) + r/[2(a + b + c) + 1]$ or $r = N - a^2 - (2a + b)b - (2a + 2b + c)c$. In this representation, he endeavours, as we see, to give the figure of the roots higher than N written out in full, and explicitly marks the positions and decimal order of each number in order to make the procedure uniform or "standard". See Saidan (1967), pp. 65–66 and the English translation with an historical introduction: Levey and Petruck (1965). Ibn Labbān's pupil, al-Nasawī went further, at least as far as the roots of fractional numbers are concerned. Arab mathematicians were later to improve on the representation and indicate the group of two digits by small circles, like Sharaf al-Dīn al-Ṭūsī. Kūshyār ibn Labbān and his pupil, al-Nasawī, did not stop there, they extended the same method to the extraction of cubic roots. They therefore used the development $(a + b + \ldots + k)^3$ and always the decimal decomposition. They gave $N = a^3 + r$, $\sqrt[3]{N} = a + \dfrac{r}{3a^2 + 1}$ their own formula, while other Arabic mathematicians used what Naṣīr al-Dīn al-Ṭūsī was to call "conventional approximation": $\sqrt[3]{N} = a + \dfrac{r}{3a^2 + 3a + 1}$, i.e. an approximation that is to be found later in Leonardo of Pisa. See al-Ṭūsī (ed., 1967), pp. 141ff. This method of extracting the roots of "pure powers" according to sixteenth-century terminology, was to be found together with some inessential variations by mathematicians before al-Ṭūsī. So we have attempted to show the result rather than a real history of the problem. See also Suter (1966), pp. 113–119; Luckey (1948).

16. Woepcke (1851), p. 13.

17. See Schau (1923), 8, p. xxxii; Wiedemann and Suter (1920–21); Boilot (1955), 2, p. 187.
18. There is evidence of a generalized usage of tables in al-Samawʾal (ed., 1972).
19. Only al-Mārdīnī expressly attributed the invention of this method to al-Ṭūsī (ed., 1986), I. But in absence of further confirmation, this evidence is not decisive.
20. Viète (ed., 1970): "Numerosam resolutionem potestatum purarum imitatur proxime resolutio adfectarum potestatum . . .", p. 173; see also p. 224.
21. Viète (ed., 1970), p. 173: "Intelliguntur videlicet componi adfectae potestates à duobus quoque lateribus, immiscentibus se subgradualibus magnitudinibus, una vel pluribus, & in eadem resolvuntur contraria compositionis via, observato coefficientium subgradualium, sicut potestatis, & parodicorum graduum, congruente situ, ordine, lege, & progressu."
22. Al-Ṭūsī (1986), I, pp. 47–48; ff. 58^r–59^v.
23. Al-Ṭūsī (1986), II, pp. 19–32; ff. 113^r–120^r.
24. The conclusion $HL < HB$ is not valid for any value of a; it is not indispensable here as it may be shown directly that $HL < HA$ is what al-Ṭūsī seeks. HL is root of the equation $K = HQx^2 - x^3$ where $HQ = 3AH$. Put $f(x) = 3AHx^2 - x^3$.

    We have

    $$f'(x) = 3x(2AH - x),$$

    hence $f'(x) = 0$ for $x = 0$ and $x = 2AH$ and $f'(x) > 0$ for $x \in]0, 2AH[$.

    On $]0, AH[$ f is ascending monotonous with $f(0) = 0$ and $f(AH) = 2AH^3$, where for $K < 2AH^3$ there exists $x_0 \in]0, AH[$ such that $f(x_0) = K$. Therefore if we set $x_0 = HL$, we have $HL < AH$.
25. Al-Ṭūsī (1986), II, pp. 71–72; f. 143^r.
26. Al-Ṭūsī (1986), II, p. 73; f. 144^r; the expression for the higher limit is: *nihāya fī al-ʿiẓam*.

1. DIOPHANTINE ANALYSIS IN THE TENTH CENTURY: AL-KHĀZIN

SUMMARY. First introduced in the 10th century, Diophantus' *Arithmetica* contributed much to the development of mathematics in the Middle Ages. Most notably it permitted the extension of classical Diophantine analysis, which existed already among the Arabic algebraists, independently of the Arabic translation of Diophantus.

Less well-known but more original, is the contribution of the *Arithmetica* to the development of new research on modern Diophantine analysis, as that term was understood by Bachet de Méziriac and Fermat. The examination of two unpublished documents in this article demonstrates this fact more clearly than before. The author shows that this research, inspired by a reading of Diophantus, was the work of mathematicians who deliberately placed themselves outside the algebraic tradition and chose an intentionally different style from that of the *Arithmetica*.

The contribution of Diophantus' *Arithmetica* to Arabic mathematics is both of greater and lesser importance than is generally granted. Many historians, after interpreting Diophantus' works algebraically, project their interpretation on history and overestimate the contribution of this mathematician to the constitution and development of this science. Despite diverging opinions, they all agree that the *Arithmetica* is a succession of numerical problems, mostly equivalent to indeterminate equations (or systems of equations) of degree ≤ 9, with one or more unknowns and only contains rational quantities. The solutions of these equations must be positive rational numbers, integers if possible, but no requirement is formulated on this point. The *Arithmetica* only deals with positive rational numbers, algebraic irrational numbers are not considered for themselves, nor are there general criteria of rationality. If Diophantus sometimes examines conditions for rationality, it is only to seek a positive rational solution. So Diophantus' work is in the final count interpreted in terms of variables, powers, parameters and general solutions. For instance, when Diophantus wants "to divide a given square into two other squares", the statement is immediately translated as a second-degree indeterminate problem with two variables, equivalent to the equation $x^2 + y^2 = a^2$. And as the mathematician, in the course of his solution, assigns a particular value to a given a, this is seen as a representation of any parameter for similar cases.

This interpretation may no doubt enlighten the historian embarking on an examination of the internal cohesion and organization of the *Arithmetica*. When attributed to the author himself, however, it raises at least two problems in the field of historiography; it may substantiate the idea that the introduction of Diophantus was a source of algebra. Moreover, it precludes understanding a second trend whereby mathematicians took Diophantus' work for what it was in fact: a book on arithmetic.

It is a common fact that algebra received its name, was constituted as an autonomous discipline and developed on the conceptual and the technical level (including the study of indeterminate equations) before Qusṭā b. Lūqā's translation of the *Arithmetica* (Rashed, 1974, 1975). For the history of Arabic mathematics, it may therefore be affirmed that, though Diophantus preceded al-Khwārizmī by several centuries, he was really his successor. The algebraic interpretation therefore introduces an error of perspective into the history of mathematics. And yet the paternity of such an interpretation must be imputed to Arab algebraists themselves. It suffices to remember that the title itself, *Arithmetica*, was simply translated as *The Art of Algebra* (Rashed, 1975). We shall show, on the other hand, that Diophantine analysis in the ring of relative integers, i.e. as Bachet de Méziriac understood it, arose in the tenth century under the direct impetus of the Arabic version of the *Arithmetica*. Now an algebraic interpretation contributes little to understanding the novel contribution of Diophantus' work, which forms the main topic of this chapter.

In any case, clearly the historical question of the influence of the *Arithmetica* cannot be correctly answered unless, subsequent to theoretical analysis in the first place, Diophantine *mathesis* itself has been grasped. We carried out this analysis elsewhere (Diophantus, 1984, cf. Introduction), and advanced a thesis that may appear paradoxical: in the tenth century the *Arithmetica* contributed more to the constitution of a field that will bear Diophantus' name than to algebra.

In the tenth century, we come across several works relating to Diophantine analysis as understood in the sixteenth and seventeenth centuries. These works that might appear no more than individual and unrelated, acquire a more defined status when related to the introduction of Diophantus; they then appear as elements of a trend in research largely stimulated by an arithmetical reading of Diophantus, and prepared, negatively at least, by the integration of Diophantine equations with rational solutions into algebra. It remains for us to show briefly how this arithmetical interpretation came about.

However, in the *Arithmetica*, the mathematicians' aim is clear: to construct a theory of arithmetic ἀριθμητικὴ θεωρία, whose elements are numbers seen as pluralities of units μονάδων πλῆθος, and fractional parts as fractions of magnitudes. The elements of the theory are not only present themselves, but also as species of numbers. The meaning of the term εἶδος, translated by Qusṭā b. Lūqā as *naw*ᶜ and later on by Bachet, as *species*, is in no way limited to "the power of the unknown". It can be shown that this term is applied without distinction to both the power of a determined plurality and the power of a number of any plurality, i.e. temporarily undetermined, though always determined once the solution is found. The last number is the unnamed number, ἄλογος ἀριθμός, *shay*ᵓ. For a clearer understanding of the notion of species, it should be recalled that Diophantus discusses three different species: the linear number, *naw*ᶜ*u al-*ᶜ*adadi al-khaṭṭī*, the plane number, *naw*ᶜ*u al-*ᶜ*adadi al-saṭḥī* and the solid number, *naw*ᶜ*u al-*ᶜ*adadi al-jismī*. These three basic species correspond to the three magnitudes set out in Book Δ of the *Metaphysics*, obtainable by infinite divisibility according to quantity. Diophantus only discusses the nature of numbers, φύσις, *ṭab*ᶜ, in relation to the three species. Three classes of numbers exist: firstly, the number commensurable with the unit, divisible in one way only; secondly, the number commensurable with power, divisible in two ways, i.e. by two numbers equal to its sides; and thirdly, the class of numbers commensurable with the cube and divisible in three ways. These species engender all the others which must ultimately be named after them. For instance, the square-square, the square-square-square, the square-cubo-cube are squares; the cubo-cube-cube is a cube. In other words, the species are only engendered by *composition*, and the power of each is necessarily a multiple of 2 or 3. So we now understand why the Arabic translation of Book IV entitled *On Squares and Cubes*, deals with square-squares, square-cube-cubes and cubo-cube-cubes as well. This also explains why the square-cube, though defined by Diophantus, never appears in statements of problems in the *Arithmetica* (in either the Greek or Arabic versions) and, furthermore, why the square-square-cube is completely absent from Diophantus' text. But this notion of species permits recognition of the same number as belonging to several species: the importance of this aspect was known not only for the formulation of problems, but their solution as well. At the same time the composition of the *Arithmetica* is clarified: the combination of species with each other under certain conditions using the operations of elementary arithmetic. The solution of these problems is to try to proceed in each case "until there remains only one species on either side".

However, a systematic examination of the text reveals, that by "solution" Diophantus means determinate numbers, i.e. positive rational numbers. On occasion, before embarking on the discussion, he imposes additional conditions on given numbers and parameters, such as the problem admits only one rational solution, in which case Diophantus qualifies the problem πλασματιχός, an expression perfectly expressed by *muhayaʾa*, suitably determined. This concept of the solution explains why Diophantus never differentiates between determinate and indeterminate problems and why impossible problems as such were never studied. We know in fact that in his classification of problems, groups of determinate problems were inserted between indeterminate problems. We also know that problems that should have been included in the *Arithmetica*, such as the one equivalent to $x^3 + y^3 = z^3$, are missing.

If, in the course of his solution, Diophantus proceeded with the substitution, elimination and displacement of species, in short using algebraic techniques, the *Arithmetica* is not a treatise on algebra. According to our terminology, it is definitely a book on arithmetic, not in the ring of relative integers, but the half-field of positive rational numbers; the main responsibility for the development of algebraic techniques, invaluable for Arabic algebraists, must apparently be imputed to the relatively narrow framework of the half-field.

Seen in the light of the new algebra constituted by al-Khwārizmī and his successors, the *Arithmetica* found its place among works on indeterminate analysis. It even communicated considerable impetus to the development of this area designated by its own title: *fī al-istiqrāʾ*[1] as al-Karajī's works show, for example. It is clearly Diophantine analysis in the half-field of positive rational numbers.

So Diophantus' influence on Arab algebraists is more apparent in the area of extension than that of innovation; but it is also noted that rational Diophantine analysis was fully integrated into algebra by means of indeterminate analysis.

This was the situation encountered by some other tenth-century mathematicians, seldom algebraists; these mathematicians belonged to the Euclidean tradition in a certain sense. They were not only well acquainted with contemporary algebra but with Diophantus' work as well.

For Euclideans, arithmetic remained the arithmetic of integers represented by line segments. Unlike Diophantus' *Arithmetica*, this representation made it possible to conform to proof requirements as defined and practised in the arithmetical books of the *Elements*.

Moreover, their familiarity with the algebra and work of Diophantus made them avoid indeterminate equations with rational solutions as such in order to concentrate on problems common to the *Elements* and Diophantus' *Arithmetica*: the theory of Pythagorean triplets for example. It was this combination of both arithmetics – i.e. a reading of Diophantus in the light of Euclid – that led them quite naturally to Diophantine analysis as understood in the sixteenth and seventeenth centuries, and to encounter other problems it contained, for instance, the representation of integers as the sum of squares, quadratic congruence, etc.; which explains why proposition III-19 of the *Arithmetica* occupied such a privileged position in their work.

As yet little is known about this trend. In the nineteenth century Woepcke translated and analyzed two papers by mathematicians dealing with some themes of Diophantine analysis. The first was anonymous (Woepcke, 1861), the second by al-Khāzin (Woepcke, 1861); both dealt with Pythagorean numerical triangles. With his customary perspicacity, Woepcke drew the attention of historians to the existence of these investigations before the sixteenth century. We have noted the importance of the problem for a larger group of tenth-century mathematicians. We remarked that, in *al-Bāhir*, al-Samawʾal[2] not only quotes Diophantus when discussing numerical right triangles, but also al-Sijzī[3] and Ibn al-Haytham.[4] More recently, Adel Anbouba[5] quite rightly emphasized the importance of this trend among tenth-century Arabic mathematicians, in particular al-Khāzin. However, we know that two other tracts on numerical right triangles have survived. The first by Abū al-Jūd b. al-Layth;[6] the second, much more important, by al-Khāzin. There is no question of writing about the history of this theory here. We shall simply isolate some of its features before examining two contemporary tracts, examples of the state and style of Diophantine analysis in the tenth century, one by al-Khāzin, the other anonymous.

(1) These mathematicians clearly underlined the originality of their investigations, unknown not only to the Ancients but also to their contemporaries. For instance, after giving the principle for generating numerical right-angle triangles, the anonymous author writes:

This is the principle for the knowledge of the hypotenuses of triangles which are the principles of species.[7] I have found no reference to it in ancient books,[8] and none of the Modern authors who composed works on arithmetic mention it either. And I think that it was revealed to no one before myself.

(2) They clearly differentiates between indeterminate analysis and new analysis. So al-Khāzin assigns all problems lacking an integer solution to algebra.

(3) These mathematicians occasionally cited Diophantus directly: al-Khāzin referred to Book III-19. This confirms what we demonstrated earlier: Book III of the Greek text and the Arabic translation are identical and follow the same order.

(4) Both tracts introduce the basic concepts of the new analysis: the primitive triangle, the generator, and in particular, the representation of the solution in relation to a particular modulus. For instance, the anonymous author states that any element of the sequence of primitive Pythagorean triplets is such that the hypotenuse is one of either form, 5 (mod. 12) or 1 (mod. 12).

(5) The study of impossible problems such as $x^3 + y^3 = z^3$.

(6) The study of congruent numbers.

(7) The use of the Euclidean term of segments in order to prove the various propositions.

In conclusion, and to illustrate this field of investigation, let us now consider: I. the examination of al-Khāzin's text and II. Fermat's theory for $n = 3$.

I. AL-KHĀZIN'S EPISTLE ON NUMERICAL RIGHT-ANGLE TRIANGLES[9]

In this epistle, which we shall closely follow and analyze here, al-Khāzin states and proves the following three lemmas.

LEMMA 1. *There exists no couple of square odd integers whose sum is a square.*[10]

Proof. Let (a, b) be a couple of square odd integers such that

$$a + b = c, \ c \text{ is a square.} \tag{1}$$

Let $a = x^2$, $b = y^2$, $c = z^2$, (1) is written

$$x^2 + y^2 = z^2.$$

As a and b are odd, c is even; therefore x and y are odd and z is even.

From (1), we deduce

$$x^2 = z^2 - y^2 = (z + y)(z - y) = (z - y)^2 + 2y(z - y). \tag{2}$$

However $(z - y)$ is odd, therefore $z - y = 2p + 1$.

On the other hand,

$$x^2 = [x + (z - y)][x - (z - y)] + (z - y)^2. \tag{3}$$

From (2) and (3) we deduce

$$2y(z - y) = [x + (z - y)][x - (z - y)].$$

Assume

$$z - y = 2p + 1,$$

then $x + (z - y)$ is even, and $x - (z - y)$ is even; the second member is divisible by 4; but in the first member, $y(z - y)$ is odd. The equality is therefore impossible. Hence the conclusion.

REMARK. The demonstration is made using segments and proposition IX-22 of the *Elements*. Proposition VIII-22 of the *Elements* is indicated in the text, but IX-22 is written in the margin in the same hand.

It is therefore clear that

if $\quad a \equiv 1 \pmod 4$

$\qquad b \equiv 1 \pmod 4$

then $\quad c \equiv 2 \pmod 4$

and there is no square of the form 2 (mod. 4).

LEMMA 2. *It is impossible that the sides of two squares whose sum is a square be evenways even.*[11]

Proof. Assume $x = 2^m$, $y = 2^n$ with $m < n$.

If $p = n - m$, then $\dfrac{x}{y} = \dfrac{1}{2^p}$, whence we deduce

$$\left(\frac{x}{y}\right)^2 = \frac{1}{2^{2p}} \quad \text{and} \quad \frac{x^2}{x^2 + y^2} = \frac{1}{1 + 2^{2p}}.$$

But $1 + 2^{2p}$ is not a square [since two squares are never consecutive],* therefore $x^2 + y^2$ is not a square either.

REMARK. He proves therefore that

$$x^2 + y^2 = 2^{2m}(1 + 2^{2p})$$

is never a square.

Al-Khāzin's proof is made using segments and to conclude, implicitly uses proposition VIII-24 of the *Elements*.

LEMMA 3. $(a + b)^2 = b^2 + 4\,\frac{a}{2}\left(b + \frac{a}{2}\right).$

The identity[12] is verified for a even and b odd, for a and b even, using proposition II-8 of the *Elements*.

PROPOSITION 1. *Find two square numbers, one even, the other odd co-prime, whose sum is a square.*[13] *Which means: find primitive Pythagorean triplets.*[14]

Analysis

Suppose these numbers exist. Let x, y be two numbers such that x is even and y odd, and that:

$$x^2 + y^2 = z^2. \tag{1}$$

Set $t = z - y$, t is even since y and z are odd, and it gives[15]

$$z = \left(y + \frac{t}{2}\right) + \frac{t}{2}. \tag{2}$$

According to Lemma 3, we have

$$z^2 = y^2 + 4\left(y + \frac{t}{2}\right) \cdot \frac{t}{2},$$

hence

$$x^2 = 4\left(y + \frac{t}{2}\right) \cdot \frac{t}{2},$$

* Passages inserted between square brackets are not in the text.

therefore $\left(y + \frac{t}{2}\right) \cdot \frac{t}{2}$ is a square, and so is $\left(y + \frac{t}{2}\right) / \frac{t}{2}$.

Write

$$\left(y + \frac{t}{2}\right) / \frac{t}{2} = \frac{p^2}{q^2} \quad \text{with} \quad \begin{matrix} (p, q) = 1 \\ (p > q) \end{matrix}$$

p and q have different parities according to (2).

Hence

$$y = p^2 - q^2, \quad x = 2pq, \quad z = p^2 + q^2.$$

REMARKS.

(1) It is clear that al-Khāzin implicitly uses several propositions from the *Elements* – VIII-9, 24, 26; IX-2 – in his analysis. That he did not quote them explicitly is proof the *Elements* already constituted a common source for mathematicians.

(2) Al-Khāzin does not give the synthesis of this proposition. It was given, it is true, in X-29, Lemma 1 of the *Elements*. If we combine al-Khāzin's analysis with Euclid's synthesis, we obtain the following theorem (Hardy and Wright, 1965, th. 225).

Let x, y, z be three numbers such that $x > 0$, $y > 0$, $z > 0$, $(x, y) = 1$, x even.

The following conditions are equivalent:

(a) We have $x^2 + y^2 = z^2$.

(b) There exists a couple of integers (p, q) such that $p > q > 0$, $(p, q) = 1$ and p and q have opposite parities, such that

$$x = 2pq, \quad y = p^2 - q^2, \quad z = p^2 + q^2. \tag{*}$$

It results from X-29, Lemma 1 of Euclid, that $(b) \Rightarrow (a)$; and al-Khāzin's proposition that $(a) \Rightarrow (b)$. He calls this last implication analysis.

It remains to prove that *Euclid's application*

$$\varepsilon: (p, q) \rightarrow (x, y, z)$$

defined by the relation [*] is surjective; which is easily seen (Hardy and Wright, 1965) and strongly emphasized by al-Khāzin, though not demonstrated. Furthermore, he points out that if x and y are both even, they result from a couple (p, q) with $(p, q) = 1$. In other words, al-Khāzin indicates that the triplets (x, y, z) where x and y are even, are also in the image of Euclid's application: we have the same formulae as above.

Al-Khāzin next affirms for numerical examples that Euclid's application is homogenous for degree 2. If $p' = \lambda p$ and $q' = \lambda q$, by setting

$$(x, y, z) = \varepsilon(p, q)$$
$$(x', y', z') = \varepsilon(p', q')$$

we have

$$\varepsilon(\lambda p, \lambda q) = \lambda^2 \varepsilon(p, q),$$

i.e.

$$(x', y', z') = (\lambda^2 x, \lambda^2 y, \lambda^2 z).$$

Next he deals with the two following problems:

PROBLEM 1. There exists a family of square numbers such that if we add 1 to each of them, each sum is a multiple of 5.

It therefore consists of numbers that verify the relation

$$(x^2 + 1) \equiv 0 \pmod 5.$$

As a solution, al-Khāzin gives numbers such that:

$$x \equiv 2 \pmod 5$$
$$x \equiv 3 \pmod 5.$$

It should be noted that in this instance we are confronted with a very ancient, if not one of the earliest examples of the solution of polynomial equations modulus a given integer. In modern terms, -1 is a quadratic residue modulus 5. We are already within the field of congruence theory.

PROBLEM 2. To find square numbers multiples of 9 and 16 such that their sum is a multiple of 5.

Al-Khāzin's expression of this problem is somewhat confused. He proceeds as follows

We know that:

If $p = 2$, $q = 1$, we have $x = 4$, $y = 3$, $z = 5$, a primitive triplet that answers the problem.

If $p = 3$, $q = 1$, we have $x = 6$, $y = 8$, $z = 10$, a non-primitive triplet, a multiple of triplet (3, 4, 5) whose squares are respectively multiples of 9, 16 and 5.

If $p = 3$, $q = 2$, we have $x = 12$, $y = 5$, $z = 13$. This triplet is unsuitable.

If $p = 4$, $q = 1$, we have $x = 8$, $y = 15$, $z = 17$, a triplet to be excluded since z is not a multiple of 5.

If $p = 11$, $q = 2$, we have $x = 44$, $y = 117$, $z = 125$; therefore x^2 is a multiple of 16, y^2 is a multiple of 9, and z^2 a multiple of 5. But we note that this triplet is not a multiple of the first, and that x^2, y^2 are not multiples – "equimultiples" – of 16 and 9. According to al-Khāzin this is because 125 is decomposed into the sum of two squares in two different ways

$$125 = 100 + 25 = 4 + 121.$$

Therefore, for $p = 10$, $q = 5$, we have $x = 4.25$, $y = 3.25$, $z = 5.25$, a multiple triplet of the first triplet.

From the triplet (3, 4, 5) which corresponds to $p = 2$, $q = 1$, we therefore obtain $p = 2\lambda$, $q = \lambda$, $(4\lambda^2, 3\lambda^2, 5\lambda^2)$, a triplet that answers the problem for any λ. But other primitive solutions exist, for example (44, 117, 125) and each of them is associated with a family of solutions.

REMARK. If we examine the above closely, we note that al-Khāzin states the problem

$$x^2 + y^2 = x^2$$
$$x = 4u, \quad y = 3v, \quad z = 5w.$$

Problem 1 implies that if $z = p^2 + 1$, then $p \equiv 2$ or $3 \pmod 5$.

As if to relate it with problem 1, al-Khāzin cites the couples $(p, q) = (2, 1)$ and $(p, q) = (3, 1)$ as solutions to problem 2 without further explanation.

A family of solutions to problem 2 obviously consists of the multiple triplets of (3, 4, 5)

$$u = v = w,$$

but he notes that there are other solutions. He finds for example

$$u = 11, \quad v = 39, \quad w = 25.$$

However, all the above considerations show that it consists of solving the equation

$$16u^2 + 9v^2 = 25w^2$$

whose set of solutions is in bijection with that of the solutions of the equation

$$x^2 + y^2 = z^2$$

by the relation (Mordell, 1969, p. 43)

$$\begin{array}{llll} dx = 4u, & dy = 3v, & dz = 5w & (u, v, w) = 1 \\ \delta u = 15x, & \delta v = 20y, & \delta w = 12z & (x, y, z) = 1 \end{array}$$

PROPOSITION 2. *We can find n square integers whose sum is a square*[16] [$\Leftrightarrow$ *to resolve into integers* $x_1^2 + x_2^2 + \ldots + x_n^2 = x^2$].

Proof: $n = 2$.

Al-Khāzin first proves the identity:

$$p^2q^2 + \left(\frac{p^2 - q^2}{2}\right)^2 = \left(\frac{p^2 + q^2}{2}\right)^2.$$

The identity yields the solution:

$$(x_1, x_2, x) = \left(pq, \frac{p^2 - q^2}{2}, \frac{p^2 + q^2}{2}\right)$$

for any couple (p, q) such that $p > q$ and with p and q of the same parity.

We note that the identity

$$4p^2q^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2$$

yields the solution

$$(x_1, x_2, x) = (2pq, p^2 - q^2, p^2 + q^2)$$

for any couple (p, q) such that $p > q$. This solution was examined earlier, and we saw that the triplet obtained was primitive if $(p, q) = 1$ and p and q of different parities

$$n = 3.$$

Al-Khāzin proves the identity

$$p^2q^2 + p^2r^2 + \left(\frac{p^2 - q^2 - r^2}{2}\right)^2 = \left(\frac{p^2 + q^2 + r^2}{2}\right)^2.$$

This identity yields the solution

$$(x_1, x_2, x_3, x) = \left(pq, pr, \frac{p^2 - q^2 - r^2}{2}, \frac{p^2 + q^2 + r^2}{2}\right)$$

an integer solution if $p^2 - q^2 - r^2$ and $p^2 + q^2 + r^2$ are even, which requires p^2 and $q^2 + r^2$ of the same parity.

$$p^2 \text{ even and } (q^2 + r^2) \text{ even} \Leftrightarrow \begin{cases} p \text{ even, } q \text{ even, } r \text{ even} \\ p \text{ even, } q \text{ odd, } r \text{ odd} \end{cases}$$

$$p^2 \text{ odd and } (q^2 + r^2) \text{ odd} \quad \Leftrightarrow \begin{cases} p \text{ odd, } q \text{ even, } r \text{ odd} \\ p \text{ odd, } q \text{ odd, } r \text{ even.} \end{cases}$$

The triplet (p, q, r) must therefore consist of three even numbers, or one even and two odd numbers.

The identity

$$4p^2q^2 + 4\, p^2r^2 + (p^2 - q^2 - r^2)^2 = (p^2 + q^2 + r^2)^2$$

yields the solution

$$(x_1, x_2, x_3, x) = (2pq, 2pr, p^2 - q^2 - r^2, p^2 + q^2 + r^2).$$

For any triplet (p^2, q^2, r^2) such that $p^2 > q^2 + r^2$, if $(p, q, r) = 1$ then $(x_1, x_2, x_3, x) = 1$.

REMARKS.

(1) Al-Khāzin's reasoning is general, even if he stops at the case $n = 3$.

In fact, let $p_1, p_2, \ldots, p_n$ be integers such that

$$p_n^2 > \sum_{i=1}^{n-1} p_i^2.$$

We have in fact

$$p_n^2 \sum_{i=1}^{n-1} p_i^2 + \frac{1}{4}\left[p_n^2 - \sum_{i=1}^{n-1} p_i^2\right]^2 = \frac{1}{4}\left(\sum_{i=1}^{n} p_i^2\right)^2;$$

hence

$$x_r^2 = p_r^2 p_n^2 \quad (r = 1, 2, \ldots, n-1),$$

$$x_n^2 = \frac{1}{4}\left[p_n^2 - \sum_{i=1}^{n-1} p_i^2\right]^2,$$

$$x^2 = \frac{1}{4}\left(\sum_{i=1}^{n} p_i^2\right)^2.$$

For an integer solution, p_n^2 and $\sum_{i=1}^{n-1} p_i^2$ must be of the same parity.

Now let the identity be

$$4p_n^2 \sum_{i=1}^{n-1} p_i^2 + \left[p_n^2 - \sum_{i=1}^{n-1} p_i^2\right]^2 = \left(\sum_{i=1}^{n} p_i^2\right)^2;$$

we draw the solution

$$x_r^2 = 4p_r^2 p_n^2 \quad (r = 1, 2, \ldots, n-1);$$

$$x_n^2 = \left[p_n^2 - \sum_{i=1}^{n-1} p_i^2\right]^2,$$

$$x^2 = \left(\sum_{i=1}^{n} p_i^2\right)^2.$$

If $(p, \ldots, p_n) = 1$, we then have a primitive solution which is easy to prove.

(2) Al-Khāzin uses segments to prove the identities.

(3) If the solution is not an integer, i.e. if p_n^2 and $\sum_{i=1}^{n-1} p_i^2$ are not of the same parity, then according to al-Khāzin, the problem is algebraic, i.e. it concerns "the indeterminate analysis" of algebraists, since the solution is fractional.

PROPOSITION 3. *To resolve into integers*[17]

$$x^2 + y^2 = z^4. \tag{1}$$

Let (p, q, r) be a Pythagorean triplet, and set: $x = 2pq$, $y = p^2 - q^2$; we then have $x^2 + y^2 = r^4$, according to the identity already seen

$$4p^2q^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2. \tag{2}$$

It is sufficient therefore to use Euclid's application again, by setting

$$p = 2uv, \quad q = u^2 - v^2, \quad r = u^2 + v^2.$$

EXAMPLE. $u = 2$, $v = 1$, whence $p = 4$, $q = 3$ and $x = 24$, $y = 7$ and $z = r = 5$.

PROPOSITION 4. *To resolve into integers*[18]

$$x^4 + y^2 = z^2. \tag{1}$$

First method. We have the following identity

$$4\left(u^4 \cdot \frac{1}{4}v^4\right) = u^4v^4.$$

Set $p = u^2$, $q = \frac{1}{2}v^2$, we have

$$x^4 = 4p^2q^2 = u^4v^4, \quad y^2 = (p^2 - q^2)^2 = \left(u^4 - \frac{1}{4}v^4\right)^2,$$

$$z^2 = (p^2 + q^2)^2 = \left(u^4 + \frac{1}{4}v^4\right),$$

EXAMPLE. $u = 1$, $v = 2$, $x = 2$, $y = 3$, $z = 5$.

REMARK. Equation (1) is equivalent to

$$\begin{cases} x^2 = \xi \\ \xi^2 + y^2 = z^2 \end{cases}$$

$$\xi = 2pq, \quad y = p^2 - q^2, \quad z = p^2 + q^2.$$

This yields the relation $2pq = x^2$ which is satisfied if $2pq$ is a square. Al-Khāzin suggests taking $p = u^2$, $q = \frac{v^2}{2}$; whence $x = uv$.

Second method. Having found a particular solution with $z^2 = (p^2 + q^2)^2 = 25$ – therefore $p^2 + q^2 = 5$ – he seeks a solution with $z^2 = 25\lambda^2$.

We have

$$4\lambda^2p^2q^2 + (\lambda p^2 - \lambda q^2)^2 = \lambda^2(p^2 + q^2)^2$$

and want to find $(\lambda p^2 - \lambda q^2)^2$ biquadratic, therefore $\lambda(p^2 - q^2)$ square.

He therefore seeks two integers u, v such that: $u = \lambda p^2$, $v = \lambda q^2$ with $(u - v)$ square and

$$\frac{u}{v} = \frac{p^2}{q^2}, \quad p^2 + q^2 = 5.$$

EXAMPLE. $u = 12, \; v = 3, \; u - v = 9, \; \dfrac{u}{v} = \dfrac{1}{4},$

REMARK. The equation $x^2 + y^4 = z^2$ is equivalent to

$$\begin{cases} y^2 = \eta \\ x^2 + \eta^2 = z^2 \end{cases}$$

$$x = 2pq, \quad \eta = p^2 - q^2, \quad z = p^2 + q^2.$$

It is therefore sufficient that $p^2 - q^2 = y^2$, in other words, $p^2 = y^2 + q^2$ (Pythagoras). Al-Khāzin applies this method in the particular case (3, 4, 5).

Al-Khāzin sets forth other investigations which are merely variations on the above.

PROPOSITION 5. *For any number decomposable into two squares, its double is decomposable into two squares, and also the double of the latter and so on into infinity.*[19]

Proof. Let

$$k = x^2 + y^2, \quad \text{with} \quad x \neq y. \tag{1}$$

According to the *Elements* (II-10, VII-5 and VII-9) we have, for any couple of numbers (a, b)

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2), \tag{2}$$

therefore

$$2k = (x + y)^2 + (x - y)^2$$
$$2k = x_1^2 + y_1^2 \quad \text{with} \quad x_1 = x + y, \quad y_1 = x - y; \quad x > y.$$

Similarly

$$2^2k = (x_1 + y_1)^2 + (x_1 - y_1)^2 = x_2^2 + y_2^2$$

and [by mathematical induction] we have

$$2^nk = x_n^2 + y_n^2.$$

REMARK. Here the proof is algebraic, and al-Khāzin only mentions simplifications given by VII-5 and VII-9, clearly based on an algebraic interpretation of II-10 which he does not quote explicitly.

PROPOSITION 6. *For any even number decomposable into two squares, its half is decomposable into two squares and so on as far as one wants.*[20]

Proof. The identity (2) allows us to write

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = \frac{x^2+y^2}{2} \quad \text{with} \quad x > y.$$

if $k = x^2 + y^2$ with k even, then

$$\frac{1}{2}k = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2.$$

Therefore x and y must be of the same parity so that $x + y$ and $x - y$ are both even, and $\frac{x+y}{2} = x_1$ and $\frac{x-y}{2} = y_1$ are integers.

We have therefore

$$\frac{1}{2}k = x_1^2 + y_1^2.$$

Similarly, we have

$$\left(\frac{1}{2}\right)^2 k = x_2^2 + y_2^2$$

with $x_2 = \frac{x_1 + y_1}{2}$ and $y_2 = \frac{x_1 - y_1}{2}$ [by mathematical induction], with conditions of parity, we have

$$\left(\frac{1}{2}\right)^n k = \left(\frac{x_{n-1} + y_{n-1}}{2}\right)^2 + \left(\frac{x_{n-1} - y_{n-1}}{2}\right)^2.$$

REMARK. As al-Khāzin conceives numbers as "the plurality of units", he restricts this proposition to even numbers. He writes as follows: "Which is why, if the number that is decomposed into two squares is odd, its half will be a fraction and it is not decomposed into two square numbers, since the number, as we said, is composed of integer units".

And so al-Khāzin arrives at the central problem of his paper. He writes: "After introducing the above, we reach the goal pursued, which is to show: if we choose one number from the numbers, how to find a square number such that if we add the given number to it and if we take away the same number from it, the sum and the difference will be two squares". Dickson (1919, II, pp. 459ff.) has given a summary of the history of this problem. Let us note here that this problem was dealt with in an anonymous manuscript (Dickson, 1919, II), and the author gave the table of numbers that solves the problems. Al-Khāzin takes another path: he looks for the conditions necessary for solving this system; which is why he starts with "analysis". His theorem may be presented as follows:

THEOREM.[21] *Let a be a given natural integer, the following conditions are equivalent*:

(a) *the system*

$$\begin{cases} x^2 + a = y_1^2 \\ x^2 - a = y_2^2 \end{cases} \quad (y_1 > x > y_2) \tag{1}$$

admits a solution;

(b) *there exists a couple of integers* (u, v) *such that*

$$\begin{cases} u^2 + v^2 = x^2 \\ 2uv = a. \end{cases} \tag{2}$$

Under these conditions, a is of the form 4k, and k is not a power of 2.

(a) $\Rightarrow$ (b). Assume that (1) admits a solution, we have therefore

$$2x^2 = y_1^2 + y_2^2. \tag{3}$$

According to Lemma 1, it is easily deduced that the integers y_1, y_2 have the same parity, and this enables us to define the integers u, v by

$$u = \frac{y_1 + y_2}{2}, \quad v = \frac{y_1 - y_2}{2}. \tag{4}$$

We have therefore

$$u^2 + v^2 = \left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2$$

$$= \frac{1}{2}(y_1^2 + y_2^2) = x^2 \tag{5}$$

and

$$2uv = 2\left(\frac{y_1 + y_2}{2}\right)\left(\frac{y_1 - y_2}{2}\right) = \frac{1}{2}(y_1^2 - y_2^2) = a.$$

(b) $\Rightarrow$ (a). Let (u, v) satisfy (2); put $y_1 = u + v$, $y_2 = u - v$. We then have

$$y_1^2 = u^2 + 2uv + v^2 = x^2 + a$$
$$y_2^2 = u^2 - 2uv + v^2 = x^2 - a.$$

If $u^2 + v^2 = x^2$, u and v cannot both be odd (Lemma 1); therefore one of them is even and $a = 2uv$ is clearly of the form sought $4k$. Al-Khāzin notes that 24 is the smallest integer a that verifies these conditions.

EXAMPLE. $u = 4$, $v = 3$, $u^2 + v^2 = 5^2$, $x^2 = 5^2$, $y_1 = u + v = 7$, $y_2 = u - v = 1$, integers that verify

$$\begin{cases} 5^2 + 24 = 7^2 \\ 5^2 - 24 = 1^2. \end{cases}$$

REMARKS.

(1) Here al-Khāzin proceeds using the Diophantine method of "the double equation".[22] In fact, in (3) he changes the linear variable (4).

(2) Al-Khāzin calls u and v two associated integers – *qarīnayn*.

(3) He writes: "There exists no multiple of 24 such that its half is divisible by two associated numbers before 240." However, a few sentences later, he gives $a = 96 = 4.24$, whose half $48 = 6.8$ verifies the conditions, and we have:

$$\begin{cases} 10^2 + 96 = 14^2 \\ 10^2 - 96 = 2^2 \end{cases}.$$

But in this case we have no primitive solution to (5). Was this why he excluded this case? There is no basis for an answer.

(4) For $a = 240$, we have

$$\begin{cases} 17^2 + 240 = 23^2 \\ 17^2 - 240 = 7^2 \end{cases}.$$

(5) Al-Khāzin deals with problems that have a rational solution; in this case he writes, "we express (*lafaẓa*) by using the term indeterminate analysis in the algebraists' sense of *māl*, square". The distinction is important for understanding al-Khāzin's method. For instance, in the case where a is divisible by two squares, (1) is rewritten

$$\begin{cases} x_1^2 + a_1 = z_1^2 \\ x_1^2 - a_1 = z_2^2 \end{cases} \quad \text{with } x_1 \text{ fractional.}$$

For $a = 240$, therefore divisible by 16 and 4, the system is rewritten in two ways

$$\begin{cases} \left(\frac{17}{2}\right)^2 + 60 = \left(\frac{23}{2}\right)^2 \\ \left(\frac{17}{2}\right)^2 - 60 = \left(\frac{7}{4}\right)^2 \end{cases} \quad \text{and} \quad \begin{cases} \left(\frac{17}{4}\right)^2 + 15 = \left(\frac{23}{4}\right)^2 \\ \left(\frac{17}{4}\right)^2 - 15 = \left(\frac{7}{4}\right)^2 \end{cases}.$$

(6) He sets out several particular methods for solving (1), either as integers or as rational numbers like the algebraists; the following are the most important:

(a) Assume the given number is written $a = \left(1 + \frac{1}{2}\right)t^2$. In this case $x = \left(1 + \frac{1}{4}\right)t$, and (1) is rewritten

$$\begin{cases} \frac{25}{16}t^2 + \frac{3}{2}t^2 = y_1^2 = \left(\frac{7}{4}\right)^2 t^2 \\ \frac{25}{16}t^2 - \frac{3}{2}t^2 = y_2^2 = \left(\frac{1}{4}\right)^2 t^2 \end{cases}.$$

Set $t = 8$, we have $x = 10$, $a = 96$.

(b) A method according to "the rules of the art", *al-ṭarīq al-ṣināᶜī* of algebra, or the canonical method of algebraists.

To solve (1), let us first find x_1, such that

$$x_1^4 + \left(\frac{a}{2}\right)^2 = z^2.$$

We have

$$x_1^2 + \left(\frac{a}{2x_1}\right)^2 = \left(\frac{z}{x_1}\right)^2 = z_1^2.$$

Put

$$x = z_1 = \frac{z}{x_1}.$$

(1) is rewritten

$$\begin{cases} z_1^2 + a = \left(x_1 + \dfrac{a}{2x_1}\right)^2 \\ z_1^2 - a = \left(x_1 - \dfrac{a}{2x_1}\right)^2 . \end{cases}$$

Al-Khāzin, after having noted that the system

$$\begin{cases} x^2 + 20 = y_1^2 \\ x^2 - 20 = y_2^2 \end{cases}$$

is impossible for integers, applies the above method for solving into rational numbers, i.e. like algebraists.

He takes $x_1 = \dfrac{3}{2}$, we have

$$\left(\frac{3}{2}\right)^4 + 10^2 = \left(\frac{41}{4}\right)^2$$

hence

$$x = z_1 = \frac{41}{6}, \quad y_1 = \frac{49}{6}, \quad y_2 = \frac{31}{6}.$$

(7) To ensure that $u^2 + v^2 = z^2$ $(u > v)$, al-Khāzin gives the obvious criteria

$$v^2 = 2uq + r^2 \quad \text{with} \quad r = q$$

in fact, in this case, we have $z^2 = (u + q)^2$.

(8) Note again that whenever al-Khāzin discusses the rational solution, he explicitly mentions "the art of algebra" in one way or another.[23]

Properties of numbers "where each one is decomposed into the sum of two squares.[24]

Al-Khāzin writes: "This clarifies the preliminary – *al-muqaddima* – introduced by Diophantus in problem 19 in Book Three of his work on algebra". This remark of major historical significance requires comment. Let us first recall Diophantus' text. In Book III – in Greek – of the *Arithmetica*, Diophantus wants to solve the problem equivalent to

$$(x_1 + x_2 + x_3 + x_4)^2 \pm x_i = y_i^2 \quad \text{with} \quad i = 1, 2, 3, 4.$$

To solve this problem, he starts with a preliminary on rational right-angle triangles, i.e. by discussing a problem equivalent to

$$x^2 + y^2 = z^2.$$

On this occasion, he remarks that if b and c are the two sides of a right-angle triangle with hypotenuse a, then

$$a^2 \pm 2bc = b^2 + c^2 \pm 2bc = (b \pm c)^2.$$

He recalls, moreover, the result of problem II-9: "To divide a given square into two squares in an infinite number of ways", and seeks to represent an integer n as the sum of two squares in four different ways. To solve the last problem, he considers two right-angle triangles "in the smallest ratio", i.e. whose sides are co-prime integers. He finds (3, 4, 5) and (5, 12, 13), and concludes that the product of their hypotenuses may be represented as the sum of two squares in two different ways

$$65 = 16 + 49 = 1 + 64.$$

The least we can say is that in this instance Diophantus poses the problem of the decomposition of a integer as the sum of squares of integers.

Strictly speaking, this preliminary may be considered as a lemma to III-19; which is why al-Khāzin uses "*al-muqaddima allatī qaddamahā*", "the introduced preliminary" here, clearly distinguishing between this preliminary or lemma and the proposition itself. Both the extant Greek text and the Arabic translation summarized by al-Karajī confirm that this preliminary was always an integral part of the proposition itself. In this summary, both the preliminary or lemma and the problem itself were given by the title of III-19.

On the other hand, it is known that it was the same problem that led Bachet, and later Fermat, to examine the representation of an integer, notably prime numbers as sums of squares. See Fermat's note VII (Fermat, 1896, vol. 3, pp. 243ff.). Therefore the origin of this investigation apparently lies in the tenth century, as al-Khāzin's text proves.

PROPOSITION 7. *If an integer is decomposed into the sum of two squares, then its square is also decomposed into the sum of two squares.*[25]

Let $n = p^2 + q^2$ (n, p, q integers).

We have: $n^2 = (p^2 + q^2)^2 = 4p^2q^2 + (p^2 - q^2)^2$.

PROPOSITION 8. *If a number is decomposed into two plane numbers whose factors are proportional, then its square is decomposed into two squares.*

Let $n = pq + rs$ (n, p, q, r, s integers) with $\frac{p}{r} = \frac{q}{s}$.

We have

$$n^2 = 4pqrs + (pq - rs)^2.$$

Now $pqrs$ is a square; since if $\frac{p}{q} = \frac{r}{s} = k$, then $\frac{pq}{q^2} = \frac{rs}{s^2} = k$ and $\frac{pqrs}{q^2s^2} = k^2$.

PROPOSITION 9. *If a square number is decomposed into the sum of two squares, then its square is decomposed into two squares in two different ways.*

Let $n^2 = p^2 + q^2$ (n, p, q integers).

We have

$$n^4 = n^2 \cdot n^2 = n^2p^2 + n^2q^2.$$

On the other hand, we have

$$n^4 = 4p^2q^2 + (p^2 - q^2)^2.$$

PROPOSITION 10. *The product of two numbers where each is decomposed into two squares is decomposed in the sum of two squares in two different ways.*

Let $m = p^2 + q^2$, $n = r^2 + s^2$ (m, n, p, q, r, s integers).

We have

$$\begin{aligned}(p^2 + q^2)(r^2 + s^2) &= p^2r^2 + p^2s^2 + q^2r^2 + q^2s^2 \\ &= p^2r^2 + q^2s^2 + 2pqrs + p^2s^2 + q^2r^2 - 2pqrs\end{aligned}$$

hence

$$\begin{aligned}mn = (p^2 + q^2)(r^2 + s^2) &= (pr + qs)^2 + (ps - qr)^2 \\ &= (pr - qs)^2 + (ps + qr)^2.\end{aligned}$$

EXAMPLE.

$$\begin{array}{lll} 5 = 4 + 1 & p = 2, & q = 1 \\ 13 = 4 + 9 & r = 2, & s = 3. \end{array}$$

We have

$$5 \times 13 = 65 = (4 - 3)^2 + (2 + 6)^2 = 1^2 + 8^2$$
$$= (4 + 3)^3 + (6 - 2)^2 = 7^2 + 4^2.$$

REMARK. It is therefore clear that this proposition directly refers to Diophantus III-19; but al-Khāzin explicitly shows that it is a consequence of the identity

$$(p^2 + q^2)(r^2 + s^2) = (pr + qs)^2 + (ps - qr)^2$$
$$= (ps + qr)^2 + (pr - qs)^2$$

which is one of the first known decompositions of quadratic forms. Note that this identity was not mentioned explicitly by Diophantus.

PROPOSITION 11. *The product of two numbers where one is decomposed into the sum of two squares in two different ways and the other is decomposed into the sum of two squares in one way, is decomposed into the sum of two squares in four different ways.*[26]

Let $m = p^2 + q^2 = p_1^2 + q_1^2$, $n = r^2 + s^2$.

We have as above

$$mn = (pr + qs)^2 + (ps - qr)^2 = (ps + qr)^2 + (pr - qs)^2$$
$$= (p_1r + q_1s)^2 + (p_1s - q_1r)^2 = (p_1s + q_1r)^2 + (p_1r - q_1s)^2.$$

PROPOSITION 12. *The product of two numbers where one is decomposed into the sum of two squares in two different ways and the other is a square which is decomposed in one way into the sum of two squares, is decomposed into the sum of squares in six different ways.*[27]

Let $m = p^2 + q^2 = p_1^2 + q_1^2$, $n^2 = r^2 + s^2$.

We have

$$mn^2 = (pr + qs)^2 + (ps - qr)^2 = (ps + qr)^2 + (pr - qs)^2$$
$$= (p_1r + q_1s)^2 + (p_1s - q_1r)^2 = (p_1s + q_1r)^2 + (p_1r - q_1s)^2$$
$$= p^2(r^2 + s^2) + q^2(r^2 + s^2)$$
$$= p_1^2(r^2 + s^2) + q_1^2(r^2 + s^2).$$

PROPOSITION 13. *The square of a number decomposed into the sum of two squares in two different ways is decomposed into the sum of two squares in four different ways.*

Let $m = p^2 + q^2 = p_1^2 + q_1^2$.

We have

$$m^2 = 4p^2q^2 + (p^2 - q^2)^2$$
$$= 4p_1^2q_1^2 + (p_1^2 - q_1^2)^2.$$

On the other hand, we have

$$m^2 = (pp_1 + qq_1)^2 + (pq_1 - qp_1)^2$$
$$= (pq_1 + qp_1)^2 + (pp_1 - qq_1)^2.$$

Let[28]

$$m = 65 = 64 + 1 = 16 + 49.$$

We have

$$\begin{aligned} m^2 &= 16^2 + 63^2 = 256 + 3\,969 \\ &= 56^2 + 33^2 = 3\,136 + 1\,089 \\ &= 60^2 + 25^2 = 3\,600 + 625 \\ &= 39^2 + 52^2 = 1\,521 + 2\,704. \end{aligned}$$

Al-Khāzin recapitulates these values as follows in the table below

3 969	63	16	256
3 600	60	25	625
3 136	56	33	1 089
2 704	52	39	1 521

He concludes as follows:

The square of sixty-five, with the squares into which it is decomposed, is what Diophantus introduced beforehand in the problem we mentioned [III-19] which is to find four numbers such that, if we add each one to the square of their sum, what is obtained has a root, and if we subtract each one from it, what remains has a root.

The above is an almost literal translation of III-19 of the *Arithmetica*. Al-Khāzin terminates the propositions by affirming that this preliminary to III-19 may lead to the following proposition, which he did not prove. The proof can effectively be made using the procedures employed. We shall give it in another language.

PROPOSITION 14. *To find four different numbers such that their sum is a square and any sum of two of them is a square* (Diophantus, ed., 1984, III-6).

This problem is written

$$(\Sigma_0)\begin{cases} x_1 + x_2 + x_3 + x_4 = z^2 \\ x_1 + x_2 = u_1^2 \\ x_3 + x_4 = u_2^2 \\ x_1 + x_3 = v_1^2 \\ x_2 + x_4 = v_2^2 \\ x_1 + x_4 = w_1^2 \\ x_2 + x_3 = w_2^2 \end{cases}$$

Consider the system

$$(\Sigma)\begin{cases} u_1^2 + u_2^2 = z^2 \\ v_1^2 + v_2^2 = z^2 \\ w_1^2 + w_2^2 = z^2 \end{cases}$$

Then, if we put

$$\begin{cases} 2x_1 = u_1^2 + v_1^2 - w_2^2 = u_1^2 + w_1^2 - v_2^2 = u_1^2 + v_1^2 + w_1^2 - z^2 \\ 2x_2 = u_1^2 - v_1^2 + w_2^2 = u_1^2 - w_1^2 + v_2^2 = u_1^2 - v_1^2 - w_1^2 + z^2 \\ 2x_3 = u_2^2 + v_1^2 - w_1^2 = u_2^2 - v_2^2 + w_2^2 = -u_1^2 + v_1^2 - w_1^2 + z^2 \\ 2x_4 = u_2^2 - v_1^2 + w_1^2 = u_2^2 + v_2^2 - w_2^2 = -u_1^2 - v_1^2 + w_1^2 + z^2 \end{cases},$$

we obtain a solution of system (Σ_0). Reciprocally, all solutions of (Σ_0) give a solution of (Σ).

The solutions of system (Σ) are given by the relations (C).

$$(C)\begin{cases} z = d_1(p_1^2 + q_1^2), & u_1 = d_1(p_1^2 - q_1^2), & u_2 = 2d_1p_1q_1 \\ z = d_2(p_2^2 + q_2^2), & v_1 = d_2(p_2^2 - q_2^2), & v_2 = 2d_2p_2q_2 \\ z = d_3(p_3^2 + q_3^2), & w_1 = d_3(p_3^2 - q_3^2), & w_2 = 2d_3p_3q_3 \end{cases}$$

with $d_i \in \mathbb{N}$, $(p_i, q_i) = 1$, $(i = 1, 2, 3)$.

So that the system (Σ) admits integer solutions, it is necessary and sufficient that u_1, v_1, w_1 be integers, and that $u_1 + v_1 + w_1$ (therefore z) are odd. We then choose the integers p_i and q_i such that $(p_i, q_i) = 1$ and if

$$N = \text{ppcm}(p_1^2 + q_1^2, p_2^2 + q_2^2, p_3^2 + q_3^2),$$

we take z the multiple of N and

$$d_i = \frac{z}{p_i^2 + q_i^2}$$

The method is the same for the problem

$$\begin{cases} \sum_{i=1}^{n} x_i = z^2 \\ x_i + x_j = u_{ij}^2 \quad (i < j) \qquad (\binom{n}{2} \text{ equations}) \end{cases}$$

when n is even ≥ 4.

II. ABŪ JAᶜFAR: ON THE DIOPHANTINE EQUATION $x^3 + y^3 = z^3$

We know from al-Khāzin that the tenth-century mathematician al-Khujandī stated Fermat's theorem for $n = 3$. Al-Khāzin laconically affirms[29] that the latter's proof was false

I demonstrated earlier, he wrote, in fact, that what Abū Muḥammad al-Khujandī advanced – may God have mercy on him – in his demonstration, that the sum of two cubic numbers is not a cube, is defective and incorrect.

Until now there was no support for this significant piece of evidence.

However, we have discovered a manuscript[30] attributed to Abū Jaᶜfar, which states this theorem including an attempt to prove it. Not only does this manuscript enable us to grasp why the tenth-century mathematician was unable to consider the case where $n > 3$, it also coincides on all points with al-Khāzin's affirmations, except that it was attributed to Abū Jaᶜfar and not to al-Khujandī. Apart from this attribution, the author only indicates his interest in Diophantine analysis and number theory. All we know is that Abū Jaᶜfar al-Khāzin fits this description. It would however be surprising if he were the author of a text whose proof is so obviously defective. How, after his denunciation of al-Khujandī's proof, could he in turn have followed such an erroneous approach? Unless we are to suppose al-Khāzin himself was completely off the track.

It may refer to a tract where al-Khāzin quotes al-Khujandī's text, a hypothesis justified by the title itself: "This is the proof using segments according to – ᶜ*an* – the master Abū Jaᶜfar . . ."? But such a conjecture does not stand up to one simple observation: al-Khāzin's criticism of al-Khujandī's proof is, as we have seen, not to be found in this text.

Though not wishing to draw a conclusion, we may advance that this tract dates back to the time of al-Khujandī and al-Khāzin, in other words the tenth century. It is therefore the work of one of those mathematicians interested in Diophantine analysis. From this period on Fermat's theorem for $n = 3$ was studied by mathematicians and became famous enough to attract the attention of philosophers themselves. For instance, in the early eleventh century, Avicenna recalled in his *Compendium al-Shifāʾ* (ed., 1956, V, pp. 194–195) that this theorem had not yet been demonstrated:

Is the sum of two cubic numbers, he wrote, a cube in the same way as the sum of two square numbers is a square?

In the rediscovered text the problem is stated in almost identical terms.

After stating the theorem in a perfectly clear way, the author sets out to prove it, starting with the identity

$$z^3 - y^3 = y^2(z - y) + (z + y)(z - y)z \qquad (z > y).$$

To prove the theorem, he starts by interpreting this identity geometrically; he observes that the second member corresponds to a volume, but is not a cube. He therefore deduces that the first member is not a cube. The confusion between the geometrical figure and its volume – elementary – even for the age – is no basis for assessing his mathematical competence. It may in fact result from a determination to avoid difficulties by justifying a proposition that the author "intuitively" knew to be true. It may even be assumed that this conviction itself was based on numerous numerical trails. The geometrical approach, which moreover permits the introduction of proof methods in Diophantine analysis – a decisive step in its constitution – functions here as a real obstacle. The proof is unsuccessful, the geometrical approach intrinsically hindering a more general formulation of the problem. The case where $n = 4$ no longer depends on a geometrical interpretation. To alleviate the difficulties of the problem of proof and the generalization of the formulation would have required adopting a more arithmetical approach.

Even if we had to wait for Fermat and Euler to accomplish this task, in spite of everything this problem did not cease to preoccupy Arab mathematicians. For instance, algebraist-mathematicians such as Ibn al-Khawwām in the twelfth century and his famous fourteenth-century commentator, Kamāl al-Dīn al-Fārisī, noted the impossibility of $x^4 + y^4 = z^4$ without proving it.

Numerous other examples of the existence of Fermat's theorem in Arabic mathematics for the case cited above could be given. For the time being we shall be content with giving the translation of Abū Ja^{c}far's text whose historical significance is evident.

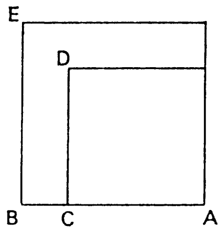
This is the proof using segments according to the master Abū Ja^{c}far, may God have mercy on him.

It is impossible that the sum of two cubic numbers be a cubic number, while it is possible that the sum of two square numbers is a square number, and it is impossible that a cubic number is divisible into two cubic numbers, while it is possible that a square number is divisible into two square numbers.

We shall show this as follows: for two cubic numbers, their difference is the sum of the product of the square of the smallest side by the difference of two sides, plus the product of the sum of two sides by their difference and then by the greatest side.

Suppose *AB* any number, and divide it into two different parts at point *C*. Construct two squares *AD* and *AE* on them. The product of the sum of *AC* and *AB* by *BC*, which is the difference between two sides, is the difference between two squares *AD* and *AE*, which is the area of the border.* The product of the square *AD* by *AC* is a cube with side *AC* and the product of the square *AE* by *AB* is a cube with side *AB*. But the sum of the products of the square *AD* by *AC* and ⟨the square *AD*⟩** by *CB* – which are both perpendicular in depth at point *C* – and the product of the border area by *AB* – which is perpendicular in depth to point *E* – is the cube of *AB*.

Subtract the product of the square *AD* by *AC*, which is the cube of *AC*. Therefore the remainder is the product of the square *AD* by *CB*, plus the product of the sum of two sides by their difference, and then by *AB* – ⟨this last product⟩*** is the product of the area of the border by *AB* – which is the difference between two cubes. But this difference is not a cubic number, because it is not the product of a square number by its side. We have therefore shown that if we subtract a cubic number from a cubic number, the remainder is not a cubic number.



And similarly, a cubic number is not divisible into two cubic numbers.

Now suppose two different cubic numbers such that their sides are *AB* and *BC*. Let *BC* be the largest side. The side of the sum of two cubes is therefore greater than *BC*.

* It consists of the area comprised by two squares – Ed.

** Our angle brackets – Ed.

*** Our angle brackets – Ed.

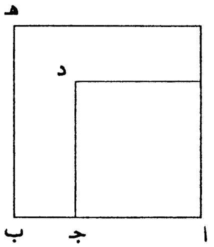
Set this side *BD*. If *BD* is the side of a cube, then if we subtract its cube from the cube of *BC*, the remainder is equal to cube of *AB*. But we have shown that if we subtract a cubic number from a cubic number, the remainder is not a cubic number. Therefore *BD* is not the side of a cube, and the sum of two cubes *AB* and *BC* is not a cubic number. Which is what we looked for.

هذا هو البرهان المخطوطي عن الشيخ أبي جعفر
رحمه الله

لا يمكن أن يجتمع من عددين مكعبين عدد مكعب كما قد يمكن أن يجتمع من عددين مربعين عدد مربع ولا أن ينقسم عدد مكعب إلى عددين مكعبين كما قد ينقسم عدد مربع إلى عددين مربعين.

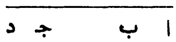
ونبين ذلك هكذا: كل عددين مكعبين فإن فضل ما بينهما هو الذي يجتمع من ضرب مربع الضلع الأقلّ في فضل ما بين الضلعين ومن ضرب مجموع الضلعين في فضل ما بينهما ثم في الضلع الأكبر.

فنفرض ا ب أي عدد اتفق ونقسمه على جـ بقسمين مختلفين، ونعمل عليهما مربعي ا د ا هـ، فيكون المجتمع من ضرب مجموع ا جـ ا ب في جـ ب – الذي هو فضل ما بين الضلعين – هو فضل ما بين مربعي ا د ا هـ الذي هو العلم. وضرب مربع ا د في ا جـ هو مكعب ضلع ا جـ، وضرب مربع ا هـ في ا ب هو مكعب ضلع ا ب. ولكن الذي يجتمع من ضرب مربع ا د في ا جـ وفي جـ ب – اللذين جميعهما قائم في السمك على نقطة جـ – ومن ضرب العلم في ا ب – الذي هو قائم في السمك على هـ – هو مكعب ا ب.



ونلقي ضرب مربع ا د في ا جـ – الذي هو كعب ا جـ – فيبقى ضرب مربع ا د في جـ ب مع ضرب مجموع الضلعين في فضل ما بينهما ثم في

ا ب – وهو ضـرب العلم في ا ب – فضل المكعـبين. وهذا الفـضل ليس بعدد مكعب لأنه غير مجتمع من ضرب عدد مربع في ضلعه. فإذن قد تبيّن أنه: إذا نُقص عدد مكعب من عـدد مكعب فـإن الباقي عـدد غيـر مكعب.



وكذلك لا ينقسم عدد مكعب إلى عددين مكعبين.

ثم لنفـرض عـددين مكعـبين مختلفين يكون ضلعاهمـا ا ب ب ج، ونجعل أكبر الضلعين ب ج، فيكون ضلع مجمـوع المكعبين أكبر من ب ج، فنجعله ب د. فإن كان ب د ضلع مكعب فإنه إذا نُقص من مكعبه مكعبُ ب ج بقي الباقي مثل مكعب ا ب. وقـد بينا أنه إذا نُقص عدد مكعب من عـدد مكعب فـإن البـاقي عـدد غيـر مكعب، فـإذن ب د ليس بـضلع مكعب ولا مجموع مكعبي ا ب ب ج بعدد مكعب، وهو المطلوب.

3 عدد مكعب (الثانية)؛ عددين مكعبين.

NOTES

1. Although not in the Koran, this term is used for ἐπαγωγὴ in several translations of Aristotle. However, the meaning of "to induct" is contained in the Arabic verb root. For instance, in *Lisān* – written in the thirteenth century based on much earlier evidence – *qarawā*, *iqtarā* and *istaqrā* (countries, people and things), it implies considering and examining them in turn. This meaning of the verb was retained later by all dictionaries and glossaries without exception (see, for example, Al-Tahānawī, 1862, p. 1229). From the 10th century on, this term designed indeterminate analysis. For details of the discussion, see Diophantus (ed., 1984).
2. Al-Samawʾal, *Al-Bāhir* (ed., 1972); see the Arabic text pp. 146–151 and the French introduction, pp. 64–66.
3. *Ibid.*
4. *Ibid.*
5. See Anbouba (1978). In particular, his remarks on al-Khāzin's work (pp. 91–92) analyzed in I. See also the appendix (pp. 98–100), where Anbouba corrects an error made by Woepcke and commonly accepted since then, which consists of inventing a second author – Abū Jaʿfar Muḥammad b. al-Ḥusayn – to whom some of al-Khāzin's works were attributed. A supplementary argument in favour of Anbouba's correction is the following: the work *Iṣlāḥ al-Makhrūṭāt* ("The Reform of Conics"), Alger MS 1446(10) is attributed to the second Abū Jaʿfar. However, examination shows that this manuscript is identical to the one explicitly attributed to al-Khāzin: Bodleian, Huntingdon MS 237, ff. 78^v–123^v.
6. Leiden, Or. MS 168(14), ff. 116^r–134^r.
7. Here, as in al-Khāzin's text, two terms are used to designate primitive triangles: "principle of kinds" – *aṣlu al-ajnās* – or "primitive" – *awwalī.*
8. "Ancient" means "Hellenistic".
9. Bibl. Nat., Paris, Fonds arabe MS 2457/49, ff. 204^r–215^r. This manuscript was transcribed in 359 of the Hegira (969) by the mathematician al-Sijzī.
10. *Epistle*, f. 204.
11. *Ibid.*, f. 204^v. Note that "evenways even" designates numbers like 2^n. See Nicomachus of Gerasa (1866, p. 15. l. 4–10). See also Thābit b. Qurra's translation (1958, p. 20, l. 23–25 and 21, l. 1–2). See also Euclid's definition in the *Elements*, book VII, def. 8.
12. *Epistle*, f. 205^r.
13. *Ibid.*
14. A triplet (x, y, z) is called primitive if three numbers are co-prime.
15. Al-Khāzin calls $\left(y + \frac{t}{2}\right)$ "a composite number" and $\frac{t}{2}$ "the difference".
16. *Epistle*, ff. 206^v–207^v.
17. *Ibid.*, f. 207^v.
18. *Ibid.*, ff. 207^v–208^r.
19. *Ibid.*, ff. 208^v–209^r.
20. *Ibid.*, f. 209^r.
21. *Ibid.*, ff. 209^v–211^r.
22. See *Arithmetica*, II–11; and also Itard's commentary (1967, pp. 46ff.).

23. *Epistle*, f. 212^{v}, l. 2.
24. *Ibid.*, f. 213^{r}.
25. *Ibid.*
26. *Ibid.*, ff. 213^{v}–214^{r}.
27. *Ibid.*, f. 214^{r}.
28. *Ibid*, ff. 214^{v}–215^{r}.
29. See al-Khāzin's *Epistle* to al-Ḥāsib, Bibl. Nat., Paris, Fonds arabe MS 2457, f. 86^{v}. See also Woepcke's translation (1861, p. 39).
30. Bodleian, Thurston MS 3, f. 140^{r}.

2. IBN AL-HAYTHAM AND WILSON'S THEOREM

In 1770, Edward Waring (1770, p. 218) recorded the birth certificate of Wilson's theorem in two sentences:

Sit n numerus primus, & $\dfrac{1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) + 1}{n}$ erit integer numerus, e.g. $\dfrac{1 \times 2 + 1}{3} = 1$, $\dfrac{1 \cdot 2 \cdot 3 \cdot 4 + 1}{5} = 5$, $\dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1}{7} = 103$ etc. Hanc maxime elegantem primorum numerorum proprietatem invenit vir clarissimus, rerumque mathematicorum peritissimus Joannes Wilson Armiger.

Though this theorem had always been attributed to Wilson, at no time did E. Waring intimate that the latter had proved it; moreover, all the evidence goes to show that Wilson did not possess the proof of the theorem that bears his name. Therefore, after quoting this and other related theorems, Waring (1770, p. 218) writes:

Demonstrationes vero hujusmodi propositionum eo magis difficiles erunt, quod nulla fingi potest notatio, quae primum numerum exprimit.

Wilson's priority, unanimously accepted by historians until then, was only to be shaken by greater knowledge of Leibniz's manuscripts. At the end of the last century Vacca was able to discover an equivalent formulation of this theorem in Leibniz's work, and consequently, much earlier than that of Wilson. And actually Leibniz's text leaves no room for doubt:

Productus continuorum usque ad numerum qui antepraecedit datum divisus per datum relinquit 1, si datus sit primitivus. Si datus sit derivativus, relinquet numerum qui cum dato habeat communem mensuram unitate majorem.[1]

Leibniz's proposition may be translated as follows:

If p is a prime number, then $(p-2)! \equiv 1 \pmod{p}$.

This theorem was not to be proved until 1771. Lagrange himself gave the proof in two ways: the first was direct; the second consisted of deducing Wilson's theorem from Fermat's small theorem. Moreover, Lagrange (1869, 3, pp. 425–434) proved the reciprocal of Wilson's statement, so we finally arrive at the following theorem:

If $n > 1$, the two following conditions are equivalent:
(a) n is prime
(b) $(n - 1)! \equiv -1 (\mathrm{mod}\ n)$.

This is the picture of the known history of Wilson's theorem. However, well before Leibniz, a tenth-century mathematician stated the same theorem in terms just as precise as those of Waring. We are going to show that in his *Opuscula* (whose edition and translation are presented here),[2] the famous mathematician and physicist Ibn al-Haytham (d. after 1040), in the course of his solution of the problem of linear congruences, introduced Wilson's theorem as a proposition that clearly states "a necessary property" of prime numbers, in other words a property belonging to them "exclusively".

A good method is to start by following Ibn al-Haytham's own order of exposition to see how he set the so-called theorem in his own study and grasp the function he assigned to it.

In his *Opuscula* Ibn al-Haytham proposes to solve the system

$$\begin{cases} x \equiv 1 & (\mathrm{mod}\ m_i) \\ x \equiv 0 & (\mathrm{mod}\ p) \end{cases} \tag{1}$$

with p a prime number and $1 < m_i \leq p-1$. We are therefore confronted with a particular case of the famous Chinese theorem.[3]

After affirming that it concerns a problem admitting an infinity of integer solutions, Ibn al-Haytham suggests two methods for solving it. The first was designated by the author as "canonical" or normal; in fact it only gives one solution. As for the second, it enables him to find all solutions. However, it is precisely the first, "normal" method that is based on Wilson's theorem, stated verbally and equivalent to the following:

> if p is any prime number, then the sum
>
> $$2.3 \ldots (p - 1) + 1$$
>
> is divisible by p; and if we divide it by any one of the numbers $2, 3, \ldots, (p - 1)$, the remainder will always be the unit.

This theorem clearly enables to obtain the solution of (1):

$$x = (p - 1)! + 1. \tag{2}$$

This preceding value does in fact immediately satisfy the first equation of (1) and also verifies, according to the theorem, the second equation of (1).

Ibn al-Haytham then introduces his second method capable of providing all solutions. This method was explicitly based on three ideas, two of which may be considered technical lemmas. Let us present them as they appear in the *Opuscula*:

(a) if m is the ppcm of m_i; then $(p, m) = 1$.

(b) if x_0 is a solution of the first equation of (1),
then the general solution of this equation is of the form

$$x = x_0 + \lambda m \quad (\lambda \text{ an arbitrary integer})$$

(c) if r is such that

$$m \equiv r (\text{mod } p) \quad (0 < r < p),$$

then $(r, p) = 1$.

Now write (1)

$$\begin{cases} x \equiv 1 & (\text{mod } m) \\ x \equiv 0 & (\text{mod } p) \end{cases} \tag{3}$$

and find a number s such that

$$\begin{cases} s - 1 \equiv 0 & (\text{mod } r) \\ x \quad\;\;\, \equiv 0 & (\text{mod } p) \end{cases} \tag{4}$$

Let $s = p + kp$. The number $(p + kp)$ satisfies the second equation of (4) whatever k. Therefore find the smallest integer k such that $(p + kp)$ satisfies the first equation of the system. In this case, we must have

$$(p - 1) + kp \equiv 0 \ (\text{mod } r). \tag{5}$$

As it appears in the *Opuscula*, Ibn al-Haytham's approach appears strictly inductive: to satisfy (5), he adds p to the number $(p - 1)$ as often as necessary.

But as close examination shows, Ibn al-Haytham does not fail to observe that this approach was only possible if $(p, r) = 1$. What does this requirement mean? It can be supposed here that Ibn al-Haytham was somehow acquainted with Bezout's theorem. In fact, as $(p, r) = 1$, there exists k and h natural integers such that

$$(k + 1)\, p - hr = 1 \tag{6}$$

Let k_0 and h_0 be the smallest integers verifying (6). We have finally

$$s = p + k_0 p$$

or

$$s = 1 + h_0 r$$

hence

$$h_0 = \frac{s-1}{r}.$$

Now consider the number $\frac{m(s-1)}{r} + 1$; it verifies the first equation of (3).

But this number is rewritten

$$mh_0 + 1 \quad \text{with} \quad m = pq + r$$

hence

$$mh_0 + 1 = h_0 pq + h_0 r + 1 = (h_0 q + 1 + k_0)p,$$

which verifies the second equation of (3). The smallest solution sought is therefore

$$x = m\,\frac{(s-1)}{r} + 1 = mh_0 + 1,$$

and the general solution is written:

$$x = \frac{m}{r}\,[(s-1) + nrp] + 1 = \frac{m}{r}\,[(p-1) + (k_0 + nr)p] + 1$$

or

$$x = m(h_0 + np) + 1;$$

therefore

$$x \equiv (mh_0 + 1)(\text{mod } p).$$

If, like Ibn al-Haytham, we note $k = k_0 + nr$ in the general solution, this number corresponds to the general solution of equation (6) which also gives $h = k_0 + np$. Once again we may ask ourselves if Ibn al-Haytham's inductive method was not intended as an attempt to solve Bezout's equation.

The above exposition leads to the heart of the problem of Wilson's equation. Of the two methods Ibn al-Haytham proposed for solving the system of congruences and thus achieve the aim of his *Opuscula*, the second is sufficient since it enables to obtain the general solution. And we shall see both Ibn al-Haytham's Arab and Latin successors had perfectly grasped this point since they only retained the second method.

Therefore, if Ibn al-Haytham insisted on presenting the first method, distinguished moreover by the attribution of a title – canonical or normal –, it was because he aimed at Wilson's theorem itself.

So Wilson's theorem therefore appears as a result based on a reflection on the properties of prime numbers for solving the famous "Chinese problem", but in a sense independent of this very solution.

It should be noted again that both methods reveal what the above analysis implies: Ibn al-Haytham was somehow acquainted with Bezout's theorem. If this was the case, he should have been in a position to prove Wilson's theorem. But for lack of contemporary documents that give Bezout's theorem "as such" and not just implicitly, any conclusion on this issue remains doomed to conjecture. This is nevertheless the direction that two series of arguments oblige us to follow.

Firstly, several recent discoveries in the history of mathematics of this period have shown that it is often risky to accept as a historical fact what is simply the result of our current ignorance, itself ascribable to texts which are temporarily or definitely lost.

Nevertheless, our knowledge of works on number theory at that period still remains, it is true, fragmentary and even more so since many of them, including those of Ibn al-Haytham are lost for the time being. For lack of sources, the historian is therefore obliged to rely on conjecture.

However, an examination of the standard already attained in number theory at this period, combined with a reflection on Ibn al-Haytham's approach set in the terms of Bezout's theorem, renders the assumption that the latter must have been unknown to tenth-century mathematicians highly unlikely. To which it must be added that this theorem was not only known to Indian mathematicians[4] but appeared, for particular cases at least, in a text directly dependent on Arabic mathematics.[5]

Secondly, the very way in which Ibn al-Haytham set forth Wilson's theorem is further confirmation of this approach. Any reader, somewhat familiar with Ibn al-Haytham's writings on mathematics and optics, cannot ignore either his scrupulous requirements for proof, or the prolixity of his supplementary commentaries addressed to the reader. Both characteristics are sometimes pushed to the extreme, and thus excessively lengthen his exposition, without affecting its clarity.

However, in the *Opuscula* analyzed here, more particularly when it concerns the statement of Wilson's theorem, for once the exposition surprises by its conciseness, all the more so since the author implies that he has formulated an essential property of prime numbers (and we are in fact in the presence of one of the first criteria capable of identifying prime numbers).

It may therefore be concluded that Wilson's theorem was not introduced for the first time in this *Opuscula*, but was included as a proposition familiar to the reader. We might have found in Ibn al-Haytham's other works on number theory explanations of how he proved this theorem, and as a result, what he knew about Bezout's theorem, but those works, as we said, still remain lost.

So our question becomes more precise: how was Ibn al-Haytham able to prove Wilson's theorem? If we accept the conjecture that he knew Bezout's theorem as plausible, we may reconstitute his method as follows: set

$$E = \{1, 2, \ldots, p-1\}$$

and show: if $a \in E$, then there exists $b \in E$, unique, such that

$$ab \equiv 1 (\mathrm{mod}\ p). \tag{7}$$

Since $(a, p) = 1$, there exists at least one couple of integers (x, y), such that

$$ax - py \equiv 1 (\mathrm{mod}\ p)$$

hence

$$ax \equiv 1 (\mathrm{mod}\ p).$$

Let b be the remainder of the division of x by p. We know b is unique, $b \in E$; b verifies (7). But a and b can be equal. In this case we have

$$a \in E \quad \text{and} \quad a^2 \equiv 1\ (\mathrm{mod}\ p) \Leftrightarrow \begin{cases} a \in E \\ a \equiv 1 \quad (\mathrm{mod}\ p), & \text{or} \\ a \equiv -1 \quad (\mathrm{mod}\ p) \end{cases}$$

therefore $a = 1$ or $a = p - 1$. For instance, for any $a \in E$, such that $a \neq 1$ and $a \neq p - 1$, there exists $b \in E$, $b \neq a$, such that we have (7);

hence

$$(p-1)! \equiv -1 (\mathrm{mod}\ p).$$

This was apparently the method adopted by Ibn al-Haytham. There exists, it is true, another proof given by Carmichael (1929, pp. 44, 45) using inscribed polygons, requiring nothing unknown to Ibn al-Haytham, and only including means he often used in other works. Nonetheless, it remains that if we establish a relation between the ideas of both methods of solving the problem of linear congruences, the above reconstitution is the most plausible.

Again we should ask ourselves why Ibn al-Haytham stated this question whose solution led him to apply Wilson's theorem. To clarify the context in which Ibn al-Haytham's problem is stated, let us turn for a moment to his successors. Let us consider, for example, a thirteenth-century treatise on algebra,[6] still unedited, where everything indicates that it was designed for teaching and was in fact more a compilation than a piece of original research. Ibn al-Haytham's problem is to be found in a chapter dealing primarily with Diophantine analysis, where the author, after an exposition of some of the basic principles of the Pythagorean theory of triplets, first deals with some second-degree Diophantine problems,[7] and finally gives the solution of Ibn al-Haytham's problem. He then writes: "To find a number such that if divided by 2, 3, 4, 5 or 6, one remains, and if divided by 7 nothing remains". He presents his solutions as follows:

> Let us then find the smallest common multiple number by which we divide, we find sixty, to which we add the unit that the enquirer suggested as a remainder. We have sixty-one. If we divide it by seven, five remains which we subtract from 7; 2 remains. By induction we find a number such that, if we multiply it by sixty and if we divide the result by seven, two remains. We find four. We multiply four by sixty, and obtain 240. If we divide it by seven, two remains, as we said. We add 240 to 61, we have 301, which is the number sought. This problem can include impossible cases.[8]

The summary by the thirteenth-century mathematician is quite inferior to Ibn al-Haytham's text. Whether the problem was borrowed directly from Ibn al-Haytham or a compiler is unimportant here; it should only be observed that the method relating to Wilson's theorem disappears to the advantage of the second[9] which alone remains and may be formulated as follows:

The system (1) is rewritten in the form:

$$\begin{cases} (1)\ x \equiv a & (\text{mod } m), \\ (2)\ x \equiv b & (\text{mod } p) \end{cases} \quad (m = 60,\ \ p = 7,\ \ a = 1,\ \ b = 0);$$

we have $x' = a + m$ the solution of (1). Let z be the residue (mod p) of x', then

$$x' = a + m \equiv z(\text{mod } p) \qquad (z = 5).$$

Given y such that

$$y + z \equiv b(\text{mod } p) \qquad (y = 2).$$

Suppose t found such that $mt \equiv y(\text{mod } p)$ and put $x = x' + mt$; then

$$x \equiv x'(\text{mod } m) = (a + m)(\text{mod } m) \equiv a(\text{mod } m),$$
$$x = x' + mt \equiv x' + y(\text{mod } p) \equiv (z + y)(\text{mod } p) \equiv b(\text{mod } p).$$

Therefore it remains to solve

$$mt \equiv y(\text{mod } p).$$

In other words, it remains to solve the equation

$$mt - pk = y$$

i.e.

$$60t - 7k = 2.$$

By continuous fractions we obtain $t = 4$ and $k = 34$.

To be understood mathematically, this reformulation like that of Ibn al-Haytham, supposes a knowledge of Bezout's theorem; it has however the advantage over the former of illustrating the context wherein the entire problem is situated. As understood by Ibn al-Haytham's successors and this thirteenth-century mathematician in particular, it concerns a study belonging to new Diophantine analysis, a discipline that arose out of the encounter between two traditions in the tenth century: one was number theory in the sense of Euclid's *Arithmetical Books*, the other found its full extension after the translation of Diophantus' *Arithmetica*.

Various commentaries on Euclid, stemming from the first tradition, including those of Ibn al-Haytham himself, are known to us. Let us also mention the new results obtained by Thābit ibn Qurra in his study of perfect and amicable numbers. Whatever these results, they all refer to an identical concept of arithmetic: the arithmetic of integers represented by line segments where one can only proceed using proofs similar to those of Euclid in the *Elements*, themselves made possible by this representation of numbers. The norm for proof was not only restrictive; rather for Ibn al-Haytham it made explicit the distinction between the two kinds of arithmetic; the one found in the work of Nicomachus of Gerasa, and that of the *Elements*. While the former only proceeded by induction, the latter was based on proofs. To designate the first kind, Arabic mathematicians retained the Greek term, ἡ ἀριθμητική, with its phonetic Arabic transcription, *al-arithmāṭīqī*, whereas the term *ᶜilm al-ᶜadād*, "the science of numbers", was applied to the second.

This was how Ibn al-Haytham himself differentiated between them:

Properties of numbers are shown in two ways: the first is by induction, since if we take the numbers one by one and if we distinguish between them, we find by distinguishing and considering all their properties, and to find the number in this way is called *al-arithmāṭīqī*. This is shown in a book on *al-arithmāṭīqī* [Nicomachus of Gerasa]. The other way of showing properties of numbers proceeds by proofs and deductions. All properties of numbers grasped by proofs are contained in these three books [of Euclid] or in those which refer to them.[10]

As for the second tradition, we showed elsewhere[11] that the introduction of Diophantus' *Arithmetica* in the tenth century lay at the beginning of the new Diophantine analysis; it concerns Diophantine integer analysis, no longer based on an algebraic, but a Euclidean reading of Diophantus' work. Even if the authors of the new Diophantine analysis such as al-Khujandī or al-Khāzin, for example, were able to borrow a number of methods of proof from algebra, notwithstanding this they clearly distinguish between their work and that of the algebraists. They thus deal with several themes, the most important of which are: the theory of Pythagorean triplets, the problem of congruent numbers, the representation of integers as the sum of squares, the impossibility of integers of the Diophantine equation $x^3 + y^3 = z^3$, etc. It was investigations such as these that led mathematicians, so to speak, to take an interest in problems which later became part of the theory of congruences.

Let us leave this general outline of Diophantine analysis in the tenth century to return to the case of Ibn al-Haytham. Firstly, it should be recalled that though he belonged to the Euclidean tradition for number theory, he also wrote a commentary on the five books of Diophantus' *Arithmetica*[12]. It is also known that he composed several works on the theory of number and arithmetic,[13] of which, unfortunately, only the titles remain; but at least we know that he dealt with Diophantine analysis in those works. A passage quoted by the twelfth-century algebraist al-Samawʾal[14] is proof that he was also interested in numerical right-angle triangles, a theme privileged, so to speak, by the new Diophantine analysis.

So it was therefore within the framework of new Diophantine analysis that the problem of linear congruences quite naturally arose, and within this same framework that the theorem incorrectly named after Wilson was stated.

In the name of God, the Clement and Merciful.
God the Almighty

A TREATISE BY AL-ḤASAN B. AL-ḤASAN B. AL-HAYTHAM ON THE SOLUTION OF A NUMERICAL PROBLEM*

Problem

To find a number such that if we divide by two, one remains; if we divide by three, one remains; if we divide by four, one remains; if we divide by five, one remains; if we divide by six, one remains; if we divide by seven, there is no remainder.

Solution

The problem is indeterminate, that is it admits of many solutions. There are two methods to find them. One of them is the canonical method: we multiply the numbers mentioned that divide the number sought by each other; we add one to the product; this is the number sought. I mean to say we multiply two by three, the product by four, then the product by five, and finally the product by six; we then add one to the product; this is the number sought.

The product of these numbers by each other according to the order mentioned is 720; we add one to 720, we have 721 which will be the

* Translated from the Arabic: MSS Loth 734 (f. 121), India Office Library and Tehran, Milli Malik 3086, f. 62^{v}–66^{r}.

number sought, because 720 is divisible by two since it has a half, by three since is has a third, by four since it has a quarter, by five since it has a fifth, by six since it has a sixth. Therefore, if 720 is divided by each of these numbers, then if we divide 721 by each of these numbers, one always remains, and 721 is divisible by seven because it is a seventh. The number sought corresponding to the property mentioned above is 721.

We can find the number sought by another method, which is the method by which we show that this problem has several solutions and even an infinity of solutions. It consists of finding the smallest number that has a half, a third, a quarter, a fifth and a sixth, i.e. the smallest multiple number of the numbers under seven, which is sixty. We divide sixty by seven, four remains. We then seek a number that has a seventh such that if we subtract one, the remainder has a quarter. There are many numbers that possess this property: the method to find these numbers is to take seven, subtract one, six remains; we add seven, seven, to six until we reach a number that has a quarter. If the [repeated] addition then gives a number with a quarter, we add one to this number. The sum will have a seventh.

Example. We add seven to six, we have 13 which has no quarter; we then add seven to 13, we have twenty which has a quarter; we then add one to 20, we have 21 which has a seventh. We take a quarter of 20 which is 5, we multiply it by 60, we therefore have 300, to which we add one; we therefore have 301, which is the number sought and thus, since 300 has a half, a third, a quarter, a fifth, a sixth, therefore three hundred is divisible by 2, 3, 4, 5 and 6.

And if three hundred is divisible by these numbers with no remainder, then if we divide three hundred and one by each of these numbers, one remains and 301 has a seventh, it is therefore divisible by 7 and nothing remains. Therefore 301 is the number sought.

Similarly, if we take six to which we add seven, seven until we obtain the sum 20, to which we then add seven, seven, four times, the sum will have a quarter, and if we add one the sum will have a seventh. If we add seven, seven, four times to 20, that gives 48 which has a quarter, and if we add one to 48, we have 49 which has a seventh. We then take the quarter of 48 which is 12 which we multiply by 60; we then have 720 to which we add one, we have at last 721 which is the number sought, and which is the number obtained by the first method.

Similarly, if we add seven, seven, four times to 48, that will give 76 which has a quarter; if we add one to 76, that will give 77, which has a seventh. We take a quarter of 76, which is 19, which we multiply by 60, we will then have 1140, to which we add one; we therefore have 1141, which is the number sought, since 1140 has a half, a third, quarter, a fifth and a sixth, and 1141 has a seventh. Similarly, if we add seven, seven, four times to 76, we have 104; if we take its quarter which is 26, if we multiply it by 60, and if we add one to the product, we have the number sought. And so on indefinitely,* each time we add seven to the number obtained, seven, four times, we take the quarter of the sum, we multiply it by 60 and we add one, we have the number sought.

In this way, there may be found an infinity of numbers where each one is divisible by 2, 3, 4, 5, 6 and has one as remainder, and where each one is divisible by seven. If this is the case, instead of adding seven, seven four times, to 20 and taking a quarter of the sum, we add to five, which is a quarter of 20, seven once, and we obtain 12. And similarly, for 48, instead of adding seven, seven, four times and taking the quarter of this sum, we add seven once to 12. The method for finding the numbers sought is to take a quarter of twenty which is five, to which we add seven, seven and so on, indefinitely. If we then multiply each of these numbers by sixty, and if we add one to the product, each of the numbers obtained according to this order is the number sought. This is the solution to the problem.

Having shown this, we say that this property** is necessary for any prime number, that is for any prime number – which is the number that is only a multiple of the unit – if we multiply the numbers that precede it by each other in the way shown, and if we add one to the product, then if we divide the sum by each of the numbers that precede the prime number, one remains and if we divide by the prime number, there is no remainder.

And, similarly, according to the second way, if we find the smallest multiple number of the numbers that precede the prime number, that is the smallest number whose homonymous parts are the numbers that precede the prime number; if then we divide this number by the prime number, we keep the remainder and keep the homonymous part of this remainder to use it as a measure; for example, if we divide the number 60 by 7, four remains whose homonymous part is a quarter which is

* Literal meaning: perpetually.

** Literal meaning: notion, concept.

the measure. The homonymous part of a number is the number that divides the number of which it is part as many times as there are units in the number which we said is its homonym. We therefore keep the homonymous part of the remainder; we take the prime number from which we subtract one, as we did for seven, we add the prime number to the difference* as many times as necessary until we arrive at the number which has the remainder as homonymous part, that is the part kept.

From the number obtained, we take the homonymous part of the remainder and we multiply it by the number which is the smallest number whose homonymous parts are the numbers that precede the prime number, we add one to the result, what we obtain is the number sought. If we add to the number that is the homonymous part of the remainder the prime number as many times as one wants, if we multiply each of these numbers by the number that is the smallest number that has the parts mentioned, successively, and if we add to each of them the number one, each of the numbers obtained according to this property is the number sought. For instance, if we multiply each of the numbers 12 and 19 by 60 and if we add one to each of their products, we have the number sought [each time]. Therefore, if we divide ⟨one of the numbers obtained according to this property⟩ by the smallest number whose homonymous parts are the numbers that precede the prime number, the remainder is only one. If we subtract one from the prime number and multiply the difference [after dividing the smallest number whose homonymous parts are the numbers that precede the prime number and after adding the number seven to the result as often as one wants] by the smallest number, whose homonymous parts are the numbers that precede the prime number, we add one to the result and we have the number sought.

If we follow this method for any prime number, then for any number found in this way, if we divide it by each of the numbers that precede the prime number, one remains and if we divide by the prime number, there is no remainder.

What we have just mentioned includes the answers to all problems of this type and may God assist us.

The answer to the numerical problem is found. Praise be to God Lord of the World; Blessed be his Prophet Muhammad, the Elected one and all his own.

* Literal meaning: remainder.

بسم الله الرحمن الرحيم ا- ١٢١- و

العزة لله ط-٦٢- ظ

قول للحسن بن الحسن بن الهيثم
في استخراج مسألة عددية*

المسألة

نريد أن نجد عدداً إذا قسم على اثنين بقي منه واحد وإن قسم على ثلاثة بقي منه واحد وإن قسم على أربعة بقي منه واحد وإن قسم على خمسة بقي منه واحد وإن قسم على ستة بقي منه واحد وإن قسم على سبعة لم يبق منه شيء .

الجواب

هذه المسألة سيالة، أعني لها أجوبة كثيرة، ولوجودها طريقان . أحد الطريقين وهو القانون أن نضرب الأعداد المذكورة التي يقسم عليها العدد بعضها في بعض فما اجتمع منها يزاد عليه واحد ، وهو العدد المطلوب. أعني أن نضرب اثنين في ثلاثة ثم ما اجتمع منه في أربعة ثم ما اجتمع منه في خمسة ثم ما اجتمع منه في ستة ثم يزاد على ما اجتمع / من ذلك ط-٦٣-ظ

* حققنا هذا النص على مخطوطتين، الأولى هي India Office Library, Loth 734, fol. 121، ورمزنا لها بـ [ا]، والثانية هي Tchran, Milli Malik 3086, fol. 62v-66r، ورمزنا لها بـ [ط]، وبمقارنة المخطوطتين يتضح أن الثانية نقلت عن الأولى أو عن أصلها على أكثر تقدير - 2 العزة لله: ناقصة [ط] – 3 للحسن: الشيخ أبي الحسن [ط] – 4 مسألة: مسئلة، وردت هكذا في النص ولن نشير إليها مرة أخرى [ا، ط] – 5 المسألة: كتبها أيضاً في الهامش [ط] – 6 قسم (الأولى): اقسم [ط] – 7 ثلاثة ... على (الأولى): ناقصة [ط] – 10 الجواب: كتبها أيضاً في الهامش [ط] – 12 يقسم: نقسمه [ط] – 13 عليه: عليها [ط] – 13-15 واحد ... في (الأولى): ناقصة [ط] – 15 منه: ناقصة [ط].

واحد، وهو العدد المطلوب. والذي يجتمع من ضرب هذه الأعداد بعضها في بعض على الترتيب الذي ذكرناه هو ٧٢٠، فيزاد على ٧٢٠ واحد فيكون ٧٢١ فهو العدد المطلوب. وذلك أن ٧٢٠ تنقسم على اثنين لأن لها نصف وتنقسم على ثلاثة لأن لها ثلث وتنقسم على أربعة لأن لها ربع وتنقسم على خمسة لأن لها خمس وتنقسم على ستة لأن لها سدس، وإذا كانت ٧٢٠ تنقسم على كل واحد من هذه الأعداد، فإن ٧٢١ إذا قسمت على كل واحد من هذه الأعداد بقي منها أبداً واحد، و٧٢١ تنقسم على ٧ لأن لها سبع. فالعدد المطلوب الذي على هذه الصفة المتقدم ذكرها هو ٧٢١. وقد يوجد العدد المطلوب بطريق آخر وهو الطريق الذي به نبين أن لهذه المسألة عدّة أجوبة، بل أجوبة بلا نهاية. وهو أن يوجد أقل عدد له نصف وثلث وربع وخمس وسدس، أعني أقل عدد يعدّه الأعداد التي قبل السبعة، وهو ستون./ ونقسم الستين على سبعة فيبقى ا-٦٣-ظ أربعة، فنطلب عدداً له سبع وإذا نقص منه واحد كان للباقي ربع. وقد يوجد أعداد كثيرة على هذه الصفة، وطريق وجود هذه الأعداد هو أن يؤخذ السبعة فينقص منها واحد فيبقى ستة فيضاف إلى ستة سبعة سبعة إلى أن ينتهي إلى عدد له ربع. فإذا انتهى التزيّد إلى عدد له ربع أضيف إلى ذلك العدد واحد فيكون للجميع سُبع.

ومثال ذلك: يضاف إلى الستة سبعة فيكون ١٣ وليس لها ربع، فيضاف إلى ١٣ سبعة فيكون ٢٠ ولها ربع، فيضاف إلى ٢٠ واحد فيكون ٢١ ولها سبع، فيؤخذ ربع ٢٠ وهو ٥ فيضرب في ٦٠ فيكون ثلاثمائة فيضاف إليها

2 ٧٢٠: كتب ناسخ [ط] الأعداد منطوقة أحيانا وأرقاما أحيانا، ولن نثبت هذا – 3 المطلوب: ناقصة [ا] / وذلك: الواو ناقصة، و كتبها دائما "ذالك" [ط] / تنقسم: ينقسم، وهي جائزة على اعتبار العدد ولكننا آثرنا التصحيح، ولن نشير إليها فيما بعد – 5 ستة: ناقصة [ط] – 6 وإذا: فإذا [ط] / ٧٢١: أعاد الناسخ ٧٢ تحت ٧٢ من ٧٢١ [ا] – 8 هذه: ناقصة [ا] – 10 نبين: تبين [ا] / أجوبة بلا: ناقصة [ط] – 11 يعدّه: بعيده [ط] – 12 ونقسم: ويقسم [ا، ط] – 13 فنطلب: فيطلب [ا] / للباقي: الباقي [ا، ط] – 14 أعداد: اعدادا [ط] – 15 فينقص: فنقص [ط] / واحد: واحداً [ط] / فيضاف: ويضاف [ط] / سبعة (الثانية): ناقصة، وكتب "فيكون" ثم ضرب عليها بالقلم [ط] – 16 إلى عدد (الثانية): على العدد الذي [ط] – 19 ٢٠ (الأولى): ٢ [ط]

واحد فيكون ٣٠١ وهو العدد المطلوب. وذلك أن ٣٠٠ لها نصف وثلث وربع وخمس وسدس، فالثلاثمائة تنقسم على ٢ وعلى ٣ وعلى ٤ وعلى ٥ وعلى ٦، وإذا كانت ٣٠٠ تنقسم على هذه الأعداد ولا يبقى منها شيء فالثلاثمائة وواحد إذا قسمت على كل واحد من هذه الأعداد بقي منها واحد، و٣٠١ لها سبع فهي تنقسم على ٧ ولا يبقى منها شيء، فالثلاثمائة والواحد هو العدد المطلوب. وأيضًا فإنا إذا أخذنا الستة وأضفنا إليها سبعة سبعة حتى يصير ٢٠ ثم أضفنا إليها أيضًا بعد ذلك سبعة سبعة أربع مرات كان لما يجتمع ربع وكان إذا / زيد عليه واحد كان لما يجتمع سبع. ط-٦٤-و وإذا أضيف إلى ٢٠ سبعة سبعة أربع مرات كان من ذلك ٤٨ ولها ربع، وإذا أضيف إلى ٤٨ واحد كان ٤٩ ولها سبع، فيؤخذ ربع ٤٨ وهو ١٢ فيضرب في ٦٠ فيكون ٧٢٠ فيضاف إليها واحد فيكون ٧٢١ وهو العدد المطلوب، وهو العدد الذي خرج بالوجه الأول. وكذلك إن أضيف إلى ٤٨ سبعة سبعة أربع مرات صارت ٧٦ ولها ربع وإذا أضيف إلى ٧٦ واحد صارت ٧٧ ولها سبع، فيؤخذ ربع ٧٦ وهو ١٩ فيضرب في ٦٠ فيكون ١١٤٠ فيضاف إليه واحد فيكون ١١٤١ وهو العدد المطلوب. وذلك أن ١١٤٠ لها نصف وثلث وربع وخمس / و سدس، و١١٤١ لها سبع. وأيضًا فإنه إذا أضيف إلى ٧٦ سبعة سبعة أربع مرات كان من ذلك ١٠٤، فإذا أخذ ربعها وهو ٢٦ وضرب في ٦٠ وأضيف إلى ما يخرج من الضرب واحد / كان ذلك هو العدد المطلوب. وكذلك دائمًا كلما أضيف إلى ط-٦٤-ظ العدد الذي يُنتهى إليه سبعة سبعة أربع مرات وأخذ ربع ما اجتمع وضرب في ٦٠ وزيد عليه واحد كان منه العدد المطلوب.

2 وعلى ٥: ناقصة [ا] – 3 منها: منه [ط] – 5-4 وواحد ... فالثلاثمائة: ناقصة [ط] – 6 والواحد: فواحد [ط] – 7-6 وأضفنا ... ٢٠: ناقصة [ط] – 7 أيضًا: ناقصة [ا] / سبعة (الثانية): كرر ناسخ [ط] بعدها "حتى يصير عشرين" – 8-9 لما يجتمع ربع ... كان: مكررة [ط] – 10 ولها: وله [ط] / ٤٨: ثمنية [ط] – 12 المطلوب، وهو العدد الذي خرج بالوجه الأول: الذي خرج بالوجه الأول المطلوب وهو العدد [ط] – 13 ولها ربع: ناقصة [ط] – 14 سبع: مكررة [ط] – 15 ١١٤٠: ١١٦٠ [ط] / ١١٤١: ١١٦١ [ط] / أن: ناقصة [ا] – 16 وربع:

فعلى هذا الوجه يمكن أن يوجد أعداد بلا نهاية كل واحد منها ينقسم على ٢ و٣ و٤ و٥ و ٦ ويبقى من كل واحد منها واحد وكل واحد منها ينقسم على سبعة. وإذا كان ذلك كذلك فإنه بدل ما يزاد على ٢٠ سبعة سبعة أربع مرات ويؤخذ ربع ما يجتمع يزاد على ٥ التي هي ربع ٢٠ سبعة واحدة فيكون ١٢. وكذلك ٤٨ بدل ما يزاد عليها سبعة سبعة أربع مرات ويؤخذ ربع ما يجتمع يزاد على ١٢ سبعة واحدة. وطريق وجود الأعداد المطلوبة هو أن يؤخذ ربع ٢٠ وهو ٥ ويزاد عليها سبعة سبعة أبداً بلا نهاية، ثم كل واحد من هذه الأعداد إذا ضرب في ٦٠ وزيد على ما اجتمع واحد كان كل واحد من الأعداد التي تجتمع على هذا الترتيب هو العدد المطلوب. وهذا هو الجواب عن المسألة.

وإذ قد تبيّن ذلك فإنّا نقول إن/ هذا المعنى يلزم في كل عدد أوّل، ط-٦٥-و
أعني أن كل عدد أوّل – وهو الذي لا يعدّه إلا الواحد فقط – فإنه إذا ضربت الأعداد التي قبله بعضها في بعض على الوجه الذي قدمناه وزيد على ما يجتمع واحد كان الذي يجتمع إذا قسم على كل واحد من الأعداد التي قبل العدد الأوّل بقي منه واحد وإذا قسم على العدد الأول لم يبق منه شيء.

وعلى الوجه الآخر أيضاً: إذا وجد أقل عدد يعدّه الأعداد التي قبل العدد الأول، أعني أقلّ عدد له الأجزاء السمّية الأعداد التي قبل العدد الأول، ثم قسم هذا العدد على العدد الأول فما بقي حُفظ، ونحفظ الجزء السمّي لهذه البقية لنجعل القياس إليه. كما إذا قسم عدد ٦٠ على ٧ بقي منه ٤ والجزء السمّي لها – الذي هو الربع – كان القياس. والجزء السمّي

2 كل (الثانية): ناقصة [ا] – 3 كذلك: ناقصة [ا] / بدل: يدل [ا، ط] / يزاد: زاد [ط] – 5 بدل: يدل [ا] /سبعة سبعة: سبعة سبعة سبعة [ط] – 6 واحدة: واحد [ط] – 9 تجتمع: يجتمع [ا] – 11 المعنى: المبنى [ط] – 12 وهو: ناقصة [ط] – 13 قدمناه: قدمنا [ا] – 14 على (الثانية): ناقصة [ط] – 20 كما: كيما [ا] – 21 والجزء (الأولى): وبالجز [ا، ط].

للعدد هو الذي يعدّ العدد الذي هو جزء له مرات بقدر آحاد العدد الذي يقال له إنه سمّيه. فإذا حفظ الجزء السمّي للبقية يؤخذ العدد الأول فينقص منه واحد كما فعل بالسبعة فما بقي يضاف إليه العدد الأول مرة بعد مرة إلى أن ينتهي إلى / عدد له الجزء السمّي للبقية، أعني الجزء ط-٦٥-ظ الذي حفظ، ثم يؤخذ من هذا العدد الذي ينتهي إليه الجزء السمّي للبقية ويضرب في العدد الذي هو أقلّ عدد له الأجزاء السمّية الأعداد التي قبل العدد الأول، فما خرج يضاف إليه واحد، وهو العدد المطلوب. ثم إذا أضيف إلى العدد الذي هو الجزء السمّي للبقية العدد الأول مرة بعد مرة ثم ضرب كل واحد من هذه الأعداد في العدد الذي هو أقل عدد له الأجزاء المذكورة واحداً بعد واحد وزيد على كل واحد منها واحد كان كل واحد من الأعداد التي تجتمع على هذه الصفة هو العدد المطلوب. كما إذا ضرب كل واحد من $\overline{\text{١٢}}$ و$\overline{\text{١٩}}$ في $\overline{\text{٦٠}}$ وزيد على <كل> واحد مما يخرج من الضرب واحد كان منه العدد المطلوب. فإن قسم <عدد من الأعداد التي تجتمع على هذه الصفة على> العدد الذي هو أقلّ عدد له الأجزاء السمّية الأعداد التي قبل العدد الأول [و]كان الذي يبقى واحد فقط، <ثم> نقص من العدد الأول واحد وضرب الباقي <بعد أن قسم على الذي هو أقل عدد له الأجزاء السمّية الأعداد التي قبل العدد الأول وأضيف إلى ما اجتمع سبعة مرة بعد مرة كم شئنا> في العدد الذي هو أقلّ عدد له الأجزاء / السمّية الأعداد التي قبل العدد الأول، فما خرج ط-٦٦-و يزاد عليه واحد وهو العدد المطلوب.

1 العدد (الثانية): ناقصة [ط] – 9 ضرب: صرف [ا] – 10 الأجزاء: الآحر [ا] / واحداً: واحد، وفوقها علامة قد تكون الألف الذي نسيها الناسخ ثم عاد فأضافها [ا] / كل واحد: مكررة [ا] / منها واحد: مكررة [ط] – 12 $\overline{\text{١٢}}$ و$\overline{\text{١٩}}$: اثنى عشر من ١٩ [ط] / $\overline{\text{٦٠}}$: كتبها أولاً $\overline{\text{٣٠}}$ ثم صححها عليها ثم أعاد الستة تحتها [ا] / مما: كتب في الهامش "فما" [ط] – 13 فإن: فإذا [ط] – 15 [و]كان: فكان [ط] – 16 من: ناقصة [ط].

فإذا سلكت هذه الطريقة في كل عدد أول كـان كل عدد يوجد على هذا الوجه إذا قسم على كل واحد من الأعداد التي قبل العدد الأول بقي منه واحد وإذا قسم على العدد الأول لم يبق منه شيء . فهذا الذي ذكرناه يستوعب أجوبة جميع المسائل التي من هذا الجنس وبالله التوفيق.

تم جواب المسألة العددية والحمد لله ربّ العالمين والصلوة على رسوله محمّد المصطفى وآله أجمعين.

2 هذا: هذه [ط] – 4 يستوعب: بسوغب [ا] / المسائل: المنايل [ط] – 5 رسوله: ناقصة [ط] – 6 المصطفى: ناقصة [ط] / أجمعين: كتب بعدها "وسلم تسليماً" [ط].

NOTES

1. Vacca (1899, p. 114, n. 2). Mahnke (1912, p. 42 note) was justified in writing: "Leibniz hat nun seinen induktiv gefundenen Satz noch bei der nächsten Primzahl, $p = 17$, nachgeprüft, sich dabei aber verrechnet. Er gibt nämlich an: $11! \equiv 16 \ldots 15! \equiv 16$, $16! \equiv 1 (\text{mod } 17)$, während in Wirklichkeit $11! \equiv 1 \ldots 15! \equiv 1$, $16! \equiv 16$ ist. Durch diesen Rechnenfehler ist er veranlasst worden, seinen richtigen Satz abzuändern und noch den falschen Zusatz zu machen: . . . relinquit {1 vel complementum ad 1}, d.h. $p - 1$. In der Tat ist ja bei seiner Rechnung $15! \equiv 17 - 1$, während in Wirklichkeit $15! \equiv 1$ ist. So erklärt sich dieser falsche Zusatz, der Vacca unverständlich war".
2. Wiedemann (reprint 1970, see 1, pp. 529–531) made a paraphrase not a rigorous translation into German of this text, edited here for the first time.

   Wiedemann's summary was published a second time (1926–7, vol. 2, p. 756). This eminent historian did not remark that Ibn al-Haytham stated and used Wilson's theorem.
3. Shanks (1978), pp. 204–205.
4. Colebrooke (1817). See the introd., pp. xvii, xviii and Bhāskara's *Arithmetic*, ch. XII, notably pp. 115 and 116 where he solved the equation $100x + 90 = 63y$. See also Brahmagupta's *Algebra*, ch. 1.
5. Curtze (1902).
6. *Nūr al-dalāla fī ᶜilm al-jabr wa al-muqābala* by ᶜAbd al-ᶜAzīz b. ᶜAbd al-Jabbār, University of Tehran, MS.4409, 64ff. The author mentioned the "master of his master Sharaf al-Dīn al-Ṭūsī". As the latter died in the early thirteenth century, we may assume that our author lived in the first half of the same century. The entire manuscript is devoted to arithmetical algebra and is apparently a comprehensive summary of al-Samawʾal's treatise *al-Bāhir* (see below). It is therefore basically a study of elementary arithmetical operations on polynomials. This is the precise context where the author repeated the formula for binomial development and the table of coefficients as given by al-Karajī and related by al-Samawʾal in *al-Bāhir*.
7. From the 9th chapter onwards (ff. 40ff.) the author studies problems of Diophantine analysis. He starts with the theory of Pythagorean triplets before proceeding to examine other Diophantine problems. As it is exactly within this framework that he takes up Ibn al-Haytham's problem, let us give some examples of the problems dealt with so as to reconstruct the context. Therefore, let (x, y, z) be a Pythagorean triplet; the author gives the following propositions and identities:

$$
\begin{aligned}
&z^2 \pm 2xy = \alpha^2, \\
&2(z - x)(z - y) = [z - (x + y)]^2, \\
&2xy + (x + y + z)^2 = 2(x + y + z)(x + y).
\end{aligned}
$$

   Next he solves the problem of congruent numbers:

$$
\begin{cases} x^2 + a = y_1^2, \\ x^2 - a = y_2^2. \end{cases}
$$

   Then he deals with problems borrowed from Arabic predecessors or the translation of Diophantus' *Arithmetica*; as, e.g.:

$$\begin{cases} x + y + z = a, \\ \frac{1}{2}xy = a. \end{cases}$$

$$\begin{cases} x + y = a, \\ x + b = y_1^2, \\ y + c = y_2^2. \end{cases}$$

He then deals with the problem of the representation of an integer by the sum of two squares, a problem raised in Diophantus' *Arithmetica* II-19 and taken up by Arabic mathematicians such as al-Khāzin who worked on Diophantine integer analysis. For instance he states:

> Let $N = a^2 + b^2$, then N can be decomposed into the sum of two squares in a infinite number of ways.

To prove this proposition, he recalls the following two identities, both explicitly stated by al-Khāzin:

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1y_2 \pm y_1x_2)^2 + (x_1x_2 \mp y_1y_2)^2,$$
$$(u^2 + v^2)^2 = (u^2 - v^2)^2 + 4u^2v^2.$$

Now, let

$$A = (a^2 + b^2)\,(u^2 + v^2)^2.$$

A is rewritten

$$A = (a^2 + b^2)\,[(u^2 - v^2)^2 + 4u^2\,v^2]$$

hence

$$a^2 + b^2 = \left[\frac{2uva + b(u^2 - v^2)}{u^2 + v^2}\right]^2 + \left[\frac{a(u^2 - v^2) - 2uvb}{u^2 + v^2}\right]^2.$$

It is clear that if a, b, u and v are integers such that $(u, v) = 1$ and $(u^2 + v^2)$ is a common divisor of a and b, then the numbers found are integers. He moreover deals with several Diophantine problems, as for example:

$$\begin{cases} ax^2 + x = y_1^2, \\ bx^2 - x = y_2^2. \end{cases}$$

8. Al-Jabbār, *Nūr al-dalāla . . .* , *op. cit*, ff. 59–60.
9. The problem of linear congruences and its solution is one of many borrowings Fibonacci made from Arabic mathematics. A comparison between Fibonacci's text, quoted below and that of Ibn al-Haytham, clearly illustrates that the first summarized the second. In comparison with Ibn al-Haytham's exposition, this much less rigorous summary is not without ambiguity. Similarly, in a text by the thirteenth-century mathematician Ibn ᶜAbd al-Jabbār, the method connected with Wilson's theorem is occulted in Fibonacci; so that we may ask if the Fibonacci's summary is not really based on a compilation of Ibn al-Haytham's *Opuscula*. Fibonacci writes

as follows (1857, pp. 281–282): "Est numerus qui, cum diuiditur per 2, uel per 3, uel per 4, aut per 5, seu per 6, semper superat ex eo 1 indiuisibile; per 7 uero integraliter diuiditur. Queritur, qui sit numerus ille: quia preponitur, quod semper superat 1, cum diuiditur per 2, uel per 3, uel per 4, uel per 5, uel per 6; ergo extracto ipso 1 de numero, diuidetur residuum per unumquemque suprascriptorum integraliter: quare reperias numerum, in quo reperiantur $\frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{1}{5}\frac{1}{6}$; eritque numerus ille 60; quem diuide per 7, superant 4, qui uellent esse 6. Ideo quia totus numerus per 7 diuiditur; ergo numerus, qui fuerit unum minus eo, cum per 7 diuidatur, 6 inde superare necesse est, hoc est 1, minus septenario numero: quare duplicetur 60, uel triplicetur, uel multiplicetur per alium quemlibet numerum, donec multiplicatio ascendat in talem numerum, qui cum dividatur per 7, remaneant inde 6; eritque numerus ille 5, in quo 60 multiplicanda sunt; ex qua multiplicatione ueniunt 300: quibus superaddatur 1, erunt 301; et talis est numerus ille. Similiter si 420, que integraliter diuiduntur per omnes predictos numeros, addideris cum 301 semel, uel quotiens uolueris, procreabitur numerus quesitus semper, uidelicet qui diuidetur integraliter per 7, et per omnes reliquos, cum diuisus fuerit, remanebit 1."

10. Ibn al-Haytham, *Sharḥ muṣādarāt Kitāb Uqlīdis*, "Commentary on the Premises of Euclid's *Elements*". Feyzullah, Istanbul, MS 1359, f. 213^{v}.
11. *Supra*, pp, 205–237.
12. We know from the thirteenth-century bibliographer, Ibn Abī Uṣaybiᶜa (ed., 1965) that Ibn al-Haytham dictated five books of commentaries "on Diophantus' work on algebraic problems" to Isḥāq b. Yūnus, an Egyptian doctor.
13. Among the works on number theory quoted by Ibn Abī Uṣaybiᶜa (ed., 1965, p. 554) based on Ibn al-Haytham's handwritten list is "The sum of principles of arithmetic" and described by Ibn al-Haytham as follows: "It is a book where I have determined the principles of all kinds of arithmetic starting with what Euclid set down in the Elements of Geometry and Arithmetic. I have seen to it that the solution of arithmetical problems is conducted according to geometrical analysis and numerical determination. In this book I have avoided what algebraists have laid down as well as their expressions".
14. It concerns a problem of right-angle number triangles: to show how to construct a right-angle triangle such that one of its two sides is equal to the given number. It therefore means solving the problem in integers

$$a^2 + y^2 = z^2 \quad (a \text{ given});$$

Ibn al-Haytham sets

$$y = \frac{a^2 - 1}{2} .$$

hence

$$z = \frac{a^2 + 1}{2} .$$

This method, equivalent to the one related by Proclus in his *Commentary on the*

First Book of Euclid's Elements, only gives one solution. Al-Samawʾal justifiably criticized this method and formulated another which gives an infinity of solutions, by setting

$$y = \frac{a^2 - b^2}{2b} \quad (b < a).$$

Al-Samawʾal's generalization was quite clearly suggested by the first method. See Al-Samawʾal (ed., 1972, p. 148 of the Arabic text, p. 65 commentary in French).

3. ALGEBRA AND LINGUISTICS: COMBINATORIAL ANALYSIS IN ARABIC SCIENCE

If we set aside probability theory, combinatorial analysis was most often exercised within the scope of two disciplines: algebra and linguistic studies whether of language in general or the language of philosophy.[1] Everyone knows that, especially since the early eighteenth century, notably with Bernoulli and Montmort,[2] combinatorial analysis expanded rapidly in answer to the needs of probability theory, insofar as it concerned problems of partition of sets of events and not exclusively numbers. It is common knowledge, moreover, that before the favourable encounter for the unprecedented development of combinatorial analysis, algebraists and linguists had already produced and used some of its methods. At least this is how Arabic mathematicians and linguists discovered combinatorial analysis.

A closer look reveals that, as was still the case to a certain extent in the sixteenth if not the seventeenth century, Arab scholars separate what we combine, though only recently, in the concept of combinatorial analysis. While the algebraist had difficulty in recognizing the instrument the linguist employed as his own, the linguist endeavoured to reinvent an instrument of which the algebraist already possessed some elements. This fragmentary theoretical awareness was, moreover, discreet in Arabic science and it was not felt necessary, as in the seventeenth century, to designate combinatorial analysis by any particular name. It seems the linguist discovered combinatorial methods quite naturally when he undertook a logical exploration of certain linguistic phenomena. As for the algebraist, he designated procedures, but not yet an activity which, as an organized activity, would require the attribution of a title. However, questions about the fragmentation and discreteness of theoretical awareness – the unity of combinatorial analysis – imply differentiating between the specific goals of the linguist and the algebraist. We shall see that, if for the linguist combinatorial analysis was a means of rationalizing an old practice, for the algebraist it was, all things considered, just a technical instrument to base a theoretical problem: a different concept of algebra or rather the project of autonomous algebra. An instrument in both cases, it should undoubtedly be emphasized that it was, firstly, a means of solving a practical problem theoretically, and secondly, a means forged in the course of the solution of a theoretical problem. The difference in projects is, in our opinion, responsible for the misunderstanding in which the unity of combinatorial analysis was held, but if this difference is overlooked, the gates are flung wide open to misinterpretations about combinatorial activity, by unifying what scholars see as unrelated and complex.

Again it should be noted that both meanings of combinatorial analysis, whatever their difference, share at least one condition of possibility which may be briefly summarized as a change in the relationship between two terms: science and art. To found the autonomy of algebra is to aim towards its constitution as a science, but this amounts to admitting that a science could also be an art, therefore capable of being without the assurance of an object because it has several – arithmetic and geometry – in short, to conceive science without any predication of being. The linguist by his conception of the technical treatment of art, that of the lexicographer, also abolished an old distinction between science and art insofar as he intended to attribute scientific status to knowledge conceived in its possibilities of practical realization and whose aim is external to it. Now, if a clearer understanding of this change refers at least partially to the sociology of knowledge,[3] it remains that, sensed though never grasped, it has been the pretext for judgements about the pragmatic spirit of Arabic science in opposition to the theoretical spirit of Hellenic science, judgements often reiterated since Renan and later on by Duhem and Tannery.

Within the limits of this paper, it is clearly impossible to write a history, however condensed, of combinatorial analysis. But the importance of the problem is not the only reason why we consider it worth discussing.

In this particular, relatively unknown area of Arabic science we also want to make a stand against a certain history of science whose erudition does not conceal a bias towards continuity, which often hinders the reconstruction of a historically dated and geographically located rational activity. Combinatorial analysis is an exemplary case in more than one respect since, outside the Hellenic tradition, champions of continuity have often considered it a deviant case to be disregarded or reduced to so-called "analytical, atomistic, occasionalistic and apophtegmatic thinking" peculiar to Arab scholars. But if by reconstruction we basically mean understanding, the references must be multiplied, i.e. this activity must be reconsidered in the light of two series of questions: one which Arab scholars asked themselves – scientific and extra-scientific questions – and others, which mature science will answer later. We shall examine in succession the emergence of combinatorial analysis in algebra and in linguistics.

The first recourse to combinatorial analysis in algebra is often dated back to the eleventh century and, more precisely, ascribed to a still undiscovered work by al-Khayyām: this is the prevailing opinion among historians of mathematics. In fact, in the second half of the tenth century emerged a certain interest in combinatorial analysis designed to improve and extend algebraic calculus in the particular field of algebraic equations and problems of the extraction of roots, as the titles of essays by Abū al-Wafā᾿ (940–998), and those of the famous astronomer-mathematician al-Bīrūnī (973–1048), bear witness. However, this historical fact has not received the explanation it deserves. Why did combinatorial analysis develop in Arabic science in the eleventh century? The question remains unanswered; either it is disregarded altogether or allusions are made to the results of a fortuitous influence – still unproved! – of Chinese or Hindu science or the result of fate and chance.

A closer look reveals that it was precisely during the same period that the notion of the autonomy and specificity of algebra was developed, an autonomy which implied not only a separation from geometry but furthermore, and above all, the arithmetization of algebra.[4] Let us summarize this programme briefly: arithmetic was applied to algebra so that the latter kept for the variables $x \in [0, \infty[$, ($x \in]-\infty, 0]$ is introduced by the definition $x = -y$; $y \in [0, \infty[$), the basic operations of arithmetic: $+/-$, $\times/\div$. The most surprising fact here is that the algebraists who strove harder than others to achieve the autonomy of algebra were the very same that developed combinatorial methods. This development is seen as the outcome of a *deliberate* return by algebraists to arithmetic following the requirements of the new project in order to find necessary means. To clarify these affirmations, we must briefly recall how algebra developed in the ninth century after al-Khwārizmī (ed., 1939, pp. 16–17) and in spite of him.

Uncertainty about whether to attribute the paternity of algebra to Diophantus in order to reserve it for al-Khwārizmī is justified in so far as, unlike Diophantus, al-Khwārizmī considers algebra for its own sake and no longer as a means of solving problems of number theory. From now on the main object of algebra, as will be repeated later, is "the absolute number and measurable magnitudes (being) unknown but related to something known in order to be determined". The main object of algebraic knowledge is therefore to determine operations "by means of which one is in a position to make the above mentioned kind of determination of unknowns, whether numerical or geometric". Confronted with the diversity of "mathematical beings" – geometrical, arithmetical – the unity of the algebraic object is based solely on the generality of operations required for reducing any problem to a form or equation or, preferably, to one of the six canonical types given by al-Khwārizmī.

$$\begin{array}{llll} (1)\ ax^2 = bx & (4)\ ax^2 + bx = c & \\ (2)\ ax^2 = c & (5)\ ax^2 + c = bx & a,\ b,\ c > 0 \\ (3)\ bx = c & (6)\ bx + c = ax^2 & \end{array}$$

on the one hand, and by the generality of operations for deducing particular solutions, i.e. a "canon",[5] on the other. Insofar as al-Khwārizmī abolished, as we said, the opposition between science and art, this object – the operations – may be considered an object of science.

An operation is an object of theoretical knowledge, though without reference to a theory of algebraic being. It is also the object of knowledge of an activity whose finality is external to it since conceived in its possibilities, either to reduce a problem to a certain form, or to derive particular solutions in a perfectly regulated way. A twelfth-century algebraist, al-Samawʾal, seems to have grasped this situation; for him, in algebra unlike geometry, "the beginning of knowledge is the end of action and the end of knowledge is the beginning of action". But if the abolition of the opposition between science and art – whether dialectical or exclusive for each kind – lies at the origin of the science of algebra, to find the specificity of this science implies defining its autonomy. Now al-Khwārizmī's algebra was still to confront the obstacle of geometrical proof: when he sought to determine the condition for the existence of roots to solve second-degree equations, his proof was geometrical and his rules for solution only yielded the positive root.[6]

While pursuing his investigations, al-Khwārizmī's successors reacted against the insufficiency of the geometrical demonstration in algebra as we said earlier. However, the sensed need for a numerical proof was itself only possible with the extension of algebraic calculus and its domain, then its subsequent systematization. Al-Khwārizmī's immediate successors set to work on this task without delay: Abū Kāmil (850–930) (Suter, 1910, pp. 100–120; Levey, 1966) integrated irrational numbers as an object of calculation in their own right, considered as roots and coefficients. Operations required for the solution of systems of linear equations with several unknowns and the extraction of roots of algebraic polynomials, etc. were developed. The systematization, in particular of the theory of equations, was to take place precisely in the tenth century: a complete classification of canonical types of cubic equations (10) was then attempted, for example, al-Khayyām (Woepcke, 1851).

The extension and systematization of algebraic calculus permitted the formulation of the idea of algebraic proof insofar as they provided the elements for a possible realization. In the early tenth century, i.e. shortly before al-Khayyām, al-Karajī (late 10th century) one of the most active scholars in this field, undertook to provide not only a geometrical proof, but another algebraic proof of the problems under consideration. Al-Khayyām was not satisfied with achieving the co-existence of both proofs but made the reason for it clear in a text-programme. After giving the solution of the third-degree equation with the aid of conic sections, he writes: ". . . and know that the geometrical proof of these procedures is not a substitute for their numerical proof, when the object of a problem is a number and not a measurable magnitude".[7]

In the same vein, al-Samawʾal in the twelfth century apparently required algebraic proof inasfar as algebra, unlike geometry, is an analytical approach to mathematical problems or as he wrote: "algebra is a part of the art of analysis, whereas in geometry we can determine the unknown quantity without analysis."[8]

It was precisely during the extension and systematization underlined above of an algebra, one of whose central items is the theory of equations, that algebraists returned to arithmetic to develop combinatorial analysis. It is then understandable why the search for techniques to extract any higher degree roots took on such special importance for them. In the course of elaborating these techniques they turned towards combinatorial analysis to discover, on the one hand, the table of binomial coefficients and its rule of formation, on the other, the binomial formula for integer powers expressed in words. Lastly, it is known through al-

Samawʾal that al-Karajī constructed Pascal's triangle for this calculation. This text included the table of binomial coefficients, its law of formation

$$C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$$

and the development

$$(a + b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

for integer n (*supra*, pp. 66–67). To our knowledge, this is the first occurrence of these rules stated in such terms. It had probably been formulated by al-Khayyām as well in a work as yet undiscovered. He writes in fact in his algebra (Woepcke, 1851):

> The Indians possess methods for finding the sides of squares and cubes, based on such knowledge of a small sequence of numbers, i.e. the knowledge of the squares of nine figures, that is the squares of one, two, three, etc. as well as the products formed by multiplying them by each other, that is the product of two by three etc. I composed a work to prove the correctness of such methods, and I proved that they do in fact lead to the object sought. I have, in addition, increased the species, i.e. I have taught how to find the sides of a square-square, quadrato-cube, cubo-cube etc. to any extension which has not been done before. The proofs I gave on this occasion were only arithmetical proofs based on the arithmetical sections of Euclid's *Elements*.

Later on in the thirteenth century the same results will be found, except that the binomial formula is still expressed verbally:

$$(a + b)^n - a^n = \sum_{m=1}^{n} C_n^m a^{n-m} b^m. \quad [9]$$

And again, in the fifteenth century, in al-Kāshī's *Key to Arithmetic*.[10]

While combinatorial analysis followed this path in algebra, a parallel development was taking place in linguistics. Less important in this case for its mathematical results, combinatorial analysis indicated a field outside mathematics where it could be exercised. It is this attempt neglected by historians of science which we shall now consider.

The enduring interest Arabs showed in their own language struck not only modern Western orientalists but also earlier Arab historians. Inseparable from progress in modern linguistics, the surprise of the former is prompted not only by the multiplicity and diversity of research in linguistics by Arab scholars, but also by a structuralist slant ahead of its time to be found in their work, perhaps shared by all who endeavour to analyze their own language as the Hindu example, among many, shows. Classical Arab historians saw in it an event as important and original as the constitution of logic.[11] In fact, apart from linguists themselves, jurists (e.g. al-Baṣrī, 1964, 1, pp. 15ff), theologian-philosophers,[12] and classifiers (Meyerhof and Ṣubhī, 1932; Ibn al-Bayṭār, ed., 1877–83), have, for various reasons, always paid particular attention to a reflection on language; some for a rational exploration of linguistic phenomena, others to solve the arduous question of eternity or the creation of divine speech, and still others to present a logical mode for the classification of empirical matter: plants, curative substances, etc. For linguists, this interest, probably religious in origin, was soon to be secularized. Stimulated by a dual necessity to create a conservatory of words and meanings, and elaborate the syntactical rules of divine speech in order to present the original means of revelation expressed in the language of "pagans", this task was dictated by the rapid extension of the new religion in the absence of a specific institution for ensuring a standardized interpretation of the sacred word, the prime source for the doctrinal unification of peoples of varying languages, cultures and traditions. With these motivations relegated to the background, secularization was soon to enable early linguistics to treat the sacred word and pre-Islamic pagan poetry in the same way. However, for grammarians turned lexicographers, what they meant at first by lexicon was a specific glossary on a subject or region explaining out-of-date words or difficult meanings. For the Arabs, as for other cultures, it concerned lexicons whose scope was limited and arrangement uncertain. In such glossaries the underlying principle behind the composition or the arrangement of words was basically semantic.

The idea of replacing the work of lexicographic monography by a lexicon of all the words of a language arose for the first time with al-Khalīl Ibn Aḥmad: it was in answer to this practical problem that language was proposed as an object of combinatorial analysis.

Al-Khalīl aimed at rationalizing the empirical work of the lexicographer, or better still, seeking a theoretical solution to the practical problem of composing a lexicon of the Arabic language. The task was not self-evident since the semantic principle of classification common to ancient lexicons was difficult to generalize and as a result inefficient. Such a generalization would have required a founded and exact system of concepts. The state of research in semantics in the ninth century, not to mention the present situation, did not permit the elaboration of such a system. The composition of a lexicon of a language could therefore only be a recomposition, and language was therefore subject to analysis for an exhaustive enumeration of all words.[13] Only under this condition would any word appear once and once only in the lexicon. Al-Khalīl's objective becomes clearer: on the one hand, an exhaustive enumeration, and on the other, a bijective mapping of the set of words to a lexicon entry; these are its conditions insofar as they are the restrictions to which any principle for the composition of a lexicon must submit. To these two internal restrictions must be added a third external one: the need to devise a lexicon accessible and easy to handle for a potential user.[14] However, as these restrictions are clearly formal, so must the principle of composition be of the same nature. Its constitution therefore requires a preliminary elaboration, if not of a theory of the ideal functioning of language, at least a doctrine of all linguistic phenomena based on the reconstitution of vocabulary alone, i.e. linguistic elements that remain identical when the meaning of words varies. But to elaborate this doctrine the lexicographer must be seconded by phonologist. Only their collaboration will effectively prepare the rise of combinatorial analysis.

Al-Khalīl's doctrine may be reduced to one basic proposition: a language is a phonetically realized part of possible language.[15] If, in fact, the r by r arrangement of the letters of the alphabet – with $1 < r \leq 5$ according to the number of letters of the root, as we shall see, gives us the set of roots – and, consequently the set of words of possible language, wrote al-Khalīl, only one part, restricted by the rules of incompatibility between root phonemes, will constitute a language. The compilation of a dictionary therefore implies constituting possible language before extracting, in accordance with the so-called rules, all the words that comply with them, an important thesis whose formulation nonetheless required a phonological study which al-Khalīl undertook at the outset. To achieve this he exploited his knowledge of music as he had done earlier in his research on metrics. By distinguishing between two levels of analysis – sign and meaning – he was able to reconstitute language from signs alone. This differentiation also suggested another: between a regular, musical sound and an irregular, aperiodic sound, i.e. between vowels and consonants. Consonants were then classified according to their point of articulation. Starting with laryngals and ending with labials, he drew up the following classes (ed., 1967, pp. 52–53, 65):

1. ᶜ h ḥ kh ġ
2. k q
3. j sh ḍ
4. ṣ s z
5. ṭ d t
6. ẓ dh th
7. r l n
8. f b m
9. ū ā ī ᵓ

In some classes he distinguishes between voiced and voiceless letters. For instance, in the first class ᶜ is voiced while ḥ is voiceless, in the fifth class d is voiced and t voiceless (ed., 1967, p. 64). A study of al-Khalīl's classification and his explanations in *Kitāb al-ᶜAyn*, clearly reveal, in the light of modern phonetics, that the distribution of sounds into groups according to points of articulation, and the opposition voiced/voiceless, are generally correctly conceived. However, the order of consonants within each class remains rather approximate and al-Khalīl's pupils, for example, Sībawayh, were to resume his analysis in order to perfect it.

Before applying his knowledge of lexicography to the task in hand, the phonologist will first of all exploit it with a view to the morphological study of Arabic, thus considerably facilitating the lexicographer's approach.[16] He thus discovered the morphological characteristic of Arabic and Semitic languages in general: the importance of roots for deriving vocabularies and the relatively small number of roots. The root as a group of consonants and only consonants, signified, usually attached to a generic signifier, could not be viewed as a unit of theoretical analysis until the above distinctions between signified and signifier, on the one hand, and vowel and consonant, on the other, had been made. Moreover, the roots are limited in form, five-letter roots at the most, usually three; so to find the number of all possible roots he only needed to calculate the permutation of a group of numbers at the most equal to 5. After this calculation, al-Khalīl then turns to the composition of his dictionary. The method is simple: first calculate the number of unrepeatable combinations of the letters of the alphabet taken r by r with $r = 2, \ldots, 5$, then calculate the number of permutations of each group of r letters.[17] In other words, he calculated $A_n^r = r!.\ C_n^r$, n being the number of the letters of the alphabet $1 < r \leq 5$.[18] For example, for $r = 3$, this method gives him all the three-letter roots of possible language. These phonological and morphological considerations led him to consider a problem which Arab linguists after him, like Abū ᶜAlī ibn Fāris, Ibn Jinnī, al-Suyūṭī were to develop: the incompatibility between the phonemes of a root. Rules of incompatibility[19] enable extraction of a certain number of roots from possible language, and thus identify those that should be included in the dictionary. We cannot go into details about the rules of incompatibility here. Let us summarize them as follows: the first two consonants of a root do not belong to the same class of localisation, nor rarely to neighbouring classes of localisation. The last two consonants of the root are subject to the same rule but may be similar. Words are derived from roots in finite patterns, themselves the object of a combinatorial. Neither these patterns nor their combination were as yet explicitly recognized by al-Khalīl. This will only occur when Arabic phonology like morphology is considered for its own sake and not from a specifically lexicographic point of view. Al-Khalīl's pupils and successors were to carry on this work, whereas in the *Kitāb al-ᶜAyn* the derivation of words apparently lacked any obvious rule.

By way of conclusion, let us recall the following points:

(1) Combinatorial analysis, developed in different ways by linguists and algebraists, was a response to two different goals: one was to solve a practical problem theoretically, the other, on the contrary, to provide the basis of a theoretical concept.

(2) The fragmentary and discrete knowledge of the unity of combinatorial analysis is seen in the absence of a particular term for designating it, and refers to the difference in goals.

(3) In both cases, however, combinatorial analysis is the result of an essential transformation of the concept of the relationship between art and science in a tradition outside Arabic Hellenism. In fact this transformation sheds some light on the emergence of two scientific disciplines that set themselves as a field for the development and the exercise of combinatorial analysis.

In the area of historical reconstruction, these hypotheses of an epistemological nature enable us on the one hand, to incorporate into the history of science a branch whose exclusion from science was unimaginable to ancient Arab scholars; and, on the other, to return to the eleventh century for reasons given above to seek texts on combinatorial analysis, and consequently, place the appearance of the first known work in this field two centuries earlier. As a result epistemological history not only permits therefore the comprehension of a rational activity situated in space and time but also ensures a better reconstruction of its history.

NOTES

1. As, for example, the philosophical alphabet proposed in Avicenna's "Nayrūzia epistle", and especially Raymond Lull's attempts, notably in *Ars Magna*, see Platzeck (1964, 1, pp. 298ff.) On the history of Lull's school, see Risse (1964, 1, pp. 582ff.)

   One had wanted to see in combinatorial analysis the origin of a tendency which, from Lull to Leibniz, would have led to the foundation of logical calculus. But it is now known that, as Risse has clearly shown, there is no continuity between the questions and solutions of Lull and his school and Leibnizian thought. Lull's attempt was the starting point for a metaphysics, rather than a logic.
2. It concerns Bernoulli's *Ars conjectandi* (1713, pp. 72–137), the second part of which concerns "The doctrine of permutations and combinations" and the *Traité des combinaisons* composed by Montmort for the second edition of his *Essai d'analyse sur les jeux du hasard* (1713, pp. 1–72).
3. The field of the sociology of knowledge can for the most part be divided into three tendencies. The first wants to explain the configuration of scientific knowledge by its connections with the structure of the means and relations of production: this is the Marxist thesis. The second finds this explanation in the constitution of collective representations themselves, the manifestations and constituent parts of collective consciousness, whether transcendental as in Durkheim or immanent as in Gurvitch, of social totality. The third does not acknowledge that either the sociology of knowledge or any other sociology is entitled to "explain" but only to interpret meaning by means of eidetic reduction. For Weber as for many others after him, it concerns subtracting knowledge from its history in order to understand it as a rather pallid projection of an ideal type.

   If, for the first tendency, the achievements have unfortunately not yet exceeded the level of affirmations too general to be really demonstrative – such as the rise of the commercial bourgeoisie and the beginnings of classical science or "technological" development in the Renaissance and the beginning of mechanics, as some non-Marxists influenced by Marxist thought affirm – both the other tendencies are dangerous. The Durkheimian tendency proposes as an instrument for analysis a still vague concept given as defined: the collective consciousness. The Weberian trend satisfies neither the scientist nor the philosopher: the former will refuse the status of science for a sociology that mistakes the task of the scientist – the construction of theoretical models – with that of the philosopher – the interpretation of meanings; the second will claim guarantees that neither the Weberians nor the phenomenologists are in a position to give: how in fact can they affirm that eidetic connections are not especially in this field, the products of contingencies?

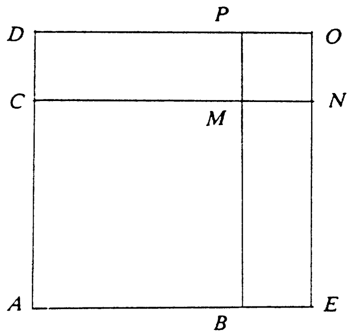
   If, therefore, the internal uncertainties of the sociology of knowledge itself must be overcome before attempting to exploit it in the history and philosophy of science, it is indispensable to start by reconstructing the history of the scientific activity one wants to explain. This history was often deficient and particularly in the domain of Arabic science. However, only at this price will sociological discourse cease to oscillate between programmatic affirmations without scope of realisation and general statements devoid of any explanatory value.

4. This fact has not been sufficiently stressed and distinguished from another tendency that sought to improve and extend the application of algebra to arithmetic. In fact, both tendencies coexisted in Arab mathematicians. As for the arithmetization of algebra, it arose in particular with al-Karajī and after him. In his introduction to al-Karajī's *al-Fakhrī*, Woepcke (1853, p. 7) remarked: "The Author frequently observed that we should be prepared for the intelligence of the rules of algebraic calculus – *Ḥisāb al-Majhūlāt* – by the rules of common arithmetic – *Ḥisāb al-Maʿlūmāt*". However, this remark is only a very superficial expression of al-Karajī's task. This book concerns a systematic and deliberate introduction of arithmetical operations to algebra, so that the following operations ×, ÷, +, –, exist not only for numbers but also for algebraic structures as well. This application of arithmetic is revealed by algebraists – e.g. al-Karajī – as a necessary means for organizing and extending the algebraic exposition. The specificity of algebraic demonstration is then discovered.

   For instance, after examining the powers of the unknown by examining simultaneously x, x^2, x^3, . . . and $1/x$, $1/x^2$, $1/x^3$, . . . al-Karajī went on to introduce from the start arithmetical operations for rational algebraic expressions. See Woepcke, 1853, ch. 3–8.

5. The idea of canon was one of the central ideas of al-Khwārizmī's work. It systematically follows the solution of each type of equation in approximately the same terms. It should be remarked here that for lack of symbolism, a brief and limited vocabulary expresses the idea of canon by the repetition of almost identical terms.

6. Al-Khwārizmī's demonstration is geometric. For the above equation $x^2 + 10x = 39$, he takes two perpendicular segments $AB = AC = x$. He then takes $CD = BE = 5 = (10/2)$. The sum of the surfaces $ABMC$, $BENM$ and $DCMP$ is equal to 39, that of the square $AEOD$ is $25 + 39 = 64$ therefore $x + 5 = 8 = \sqrt{64}$, hence $x = 3$.



7. Woepcke (1951), p. 9. In Woepcke's translation, we replaced "ne rend pas superflu" ("is not made superfluous") by "ne remplace pas" ("does not replace") which is a more rigourous translation of al-Khayyām's sentence.

8. For al-Samawʾal's *al-Bāhir*, we consulted MS 2718 Aya Sofia, 113 f°, see f° 27ᵛ.

9. This text was translated into Russian by Ahmadov and Rosenfeld (1963, pp. 431–444).
10. Luckey, 1951, pp. 24ff. and Luckey, 1948, pp. 217–274.

    Luckey's basic thesis is that al-Kāshī at least rediscovered the table of binomial coefficients by using the procedure later known as the Ruffini–Horner procedure. The idea deserves detailed discussion carried out elsewhere (cf. above).
11. Cf. al-Rāzī (ed., n.d., p. 99): "The relationship of al-Shāfiʿī to the science of *uṣūl al-Fiqh* – canonical jurisprudence – is the same one which links Aristotle with logic and al-Khalīl ibn Aḥmad with metrics. Before Aristotle, men used to discuss and confront each other following their natural inclination for lack of a universal law for organizing definitions and proofs. It goes without saying that their thoughts were confused and vague because nature alone without reference to universal law can do no better. Having noted that, Aristotle retired for some time and then presented his science of logic whereby he proposed to mankind a universal law to which to refer for the knowledge of definitions and concepts. In the same way, before al-Khalīl ibn Aḥmad, poets used to compose poems based on nature alone, while al-Khalīl presented his science of metrics which consisted of a universal law of the merits and defects of poetry. And in the same way again, before Imām al-Shāfiʿī, men used to deal with questions relating to *uṣūl al-Fiqh*, argue and confront each other without however possessing a universal law to which to refer for a knowledge of judicial proofs, the mode of controversy and demonstration. Al-Shāfiʿī deduced from the *uṣūl al-Fiqh* and proposed to mankind a universal law to which to refer for the knowledge of degrees of judicial proof". Classical Arabic tradition differentiated on the whole between two classes of science: ancient sciences – *ʿulūm al-awāʾil* – and Arabic sciences – *ʿulūm al-ʿarab*, i.e. linguistic sciences where the authors recognized no priority to Hellenic science.
12. Cf. by way of example in the "great theologico-philosophical encyclopaedia", al-Jabbār (ed., 1961, t.1), on the creation of divine speech.

    In works by Muʿtazilites as in those of linguists, theoretical discussions developed at length the question of the origin and nature of language Cf. e.g. the work by the linguist al-Suyūṭī (ed., n.d.), t.1, pp. 7ff.
13. On the history of lexicography and to locate the importance of lexicographic study, see Keith (1924); Müller (1913, II); Collant (1932); Krumbacher (1896).
14. In the beginning of *Kitāb al-ʿAyn* (ed., 1967, p. 52) we read: "This is what al-Khalīl ibn Aḥmad al-Baṣrī composed . . . the consonants *alif, bāʾ, tāʾ, thāʾ* . . . by which Arabs speak, the pivot of their words and expressions, so that none are omitted. In this way he wanted to understand Arabic poetry and speech so that nothing would be irregular for him".
15. This theory presented in the text attributed to al-Khalīl, was taken up and developed later by Abū ʿAlī ibn Fāris, Ibn Jinnī and al-Suyūṭī . . . The latter also related in his work cited above the opinions of Ibn Fāris and Ibn Jinnī. See al-Khalīl (ed., 1967, pp. 240ff.).
16. Al-Khalīl (1967, p. 55) wrote: "In Arabic no construction of noun or verb is composed of more than five consonants. For any adjunctions exceeding five consonants to be found in a noun or verb, know that they are adjunctions to a construction which does not belong to the root of the word".
17. "Know that a two-letter word permutes in two ways, like *qd, dq, shd, dsh*, a three-

letter word permutes in six ways, like *ḍrb, ḍbr, brḍ, bḍr, rḍb, rbḍ*, and is called sextuple, a four-letter word permutes in twenty-four ways since its consonants which number is four are multiplied by the modes of the normal three-letter word which are six and therefore give twenty-four where we write down those language used whereas we eliminate those it does not . . . and a five-letter word permutes in one hundred and twenty ways, since its consonants which number is five are multiplied by the modes of the four-letter word which are 24 and this gives one hundred and twenty ways where the smallest part is retained and the largest is not".

In other words, to seek the number of permutations of r consonants, we find the number of $r - 1$ consonants multiplied by r, ou $r! = r(r - 1)!$

18. The calculation attributed by Abū Ḥamza to al-Khalīl and reported by al-Suyūṭī (ed., n.d., p. 74) is correct, that is A_n^r for $n = 28$, $r = 2, \ldots, 5$. As well as this calculation, the composition of the lexicon itself makes it possible to propose the corresponding formula. The method was often cited and much later is still to be found in Ibn Khaldūn's *Prolegomena*, taken as an element of cultural background. To find C_n^r for $r = 2$ for example, he proceeded empirically, taking the first consonant which he combined with the others and thus obtained 27 words. He then took the second consonant which he combined with 26 to give 26 words, and so on. He added the combinations and multiplied the sum by two to give the arrays. For $r = 3$, he proceeded in the same way, but considered two consonant words as one consonant which he combined with the remaining 26 consonants. With 27 roots composed of 2 consonants he formed 26 three-consonant roots, and so on. He added them and multiplied by 6 to form the arrays, and so on for $r = 4$ and $r = 5$.
19. See al-Khalīl (ed., 1967, p. 63) for the consonant ᶜ.

Al-Khalīl (ed., 1967, p. 68) writes: "The ᶜ is not combined with the ḥ as one word because of the proximity of the point of articulation".

Again, "But the ᶜ is negligible with the following consonants: j, h, ḥ, kh" (ed., 1967, p. 69).

However, it remains that after al-Khalīl these problems were to be the subject of systematic investigations.

4. AMICABLE NUMBERS, ALIQUOT PARTS AND FIGURATE NUMBERS IN THE THIRTEENTH AND FOURTEENTH CENTURIES

INTRODUCTION

It is often difficult to comprehend the genesis of concepts and techniques in number theory in the sixteenth and seventeenth centuries and, as a result, follow their filiation. Not infrequently, however, when one sets out to write the history of this theory, instead of being recognized and assessed, the difficulty is evaded by making one big jump over the centuries. So, nothing prevents placing Bachet de Méziriac or Fermat immediately after Euclid and Diophantus. Such an viewpoint is doubly misleading: it not only truncates history, but distorts an appreciation of the novel significance of more than one arithmetician in the sixteenth and seventeenth centuries as well. How to determine real changes in style that may have occurred at this time, and identify their manifestation in a rigorous way, if Bachet and Fermat simply follow Euclid and Diophantus? Under these circumstances, how to avoid a global judgement of classical arithmetic, which expresses more often than not an inability to discern differences?

Since the nineteenth century, nevertheless, one figure – Leonardo of Pisa, alias Fibonacci – has consistently disturbed this picture. His work, including significant results and methods in number theory, was known to many mathematicians who, like Luca Pacioli,[1] transmitted and developed his work. No one is unaware, however, that Fibonacci was directly in touch with Arabic mathematics. Therefore greater knowledge about its history would at least enable us to tackle the arduous question of the genesis of concepts and techniques, if not pose the more epistemological problem of the scientific style and the novel contribution of the seventeenth century.

But, if we are to believe most historians the return to Arabic mathematics is not in the least compulsory. In fact, well-informed specialists, whose sincerity is unquestioned, are unanimous in insisting that in number theory, unlike algebra and trigonometry, for example, Arab mathematicians were outstanding, neither for the originality of their discoveries, nor the significance of their results. In comparison with other mathematical disciplines, therefore, number theory apparently fails to come up to expectations: neither by its results nor even its mistakes can it lay claim to historical fame. So the history of number theory can be written at the expense of the contribution of Arab mathematicians. In contradiction with these theses, however, two facts emerge from Woepcke's work in the nineteenth century which should have put historians on their guard: the first occurrence of Fermat's theorem, and Thābit b. Qurra's theorem on amicable numbers.[2]

Over the past few years we have pointed out the inaccuracy of this view of the history of number theory, at least in one important area: Diophantine integer analysis. This area arose in the tenth century and was established through algebra developed since al-Khwārizmī and also against it, and based on a Euclidean, not a Diophantine interpretation of Diophantus' *Arithmetica*, recently translated by Qusṭā b. Lūqā. We set forth below (*infra*, pp. 205–237 and pp. 238–261) the contributions of al-Khujandī, al-Khāzin, Ibn al-Haytham among many others, towards the elaboration of Diophantine integer analysis in the tenth century.

Here we shall pursue our investigations once again in another area of number theory, an area closely related to the tradition of Euclid's *Elements*: the study of aliquot parts required by the study of perfect and amicable numbers in particular. Significant for the history of the elementary theory of numbers, this study is apparently exemplary on two counts. Firstly, the history of aliquot parts, and notably amicable numbers,[3] has been written many times over in terms considered definitive. Secondly, according to its history, it apparently evolved independently of other mathematical disciplines, and as a result, lacked real significance for number theory as a whole. Based on a series of unedited documents, some hitherto unknown, we shall demonstrate that this was not the case. Furthermore, we shall show that the application of algebraic concepts and means to the traditional, Euclidean, domain of number theory by thirteenth-century mathematicians (at the latest) achieved results attributed until now to seventeenth century mathematicians: for instance, the study of two elementary arithmetical functions, figurate numbers, combinatorial analysis, and amicable numbers themselves. Let us start by examining their history.

1. THĀBIT B. QURRA'S THEOREM AND THE CALCULATION OF AMICABLE NUMBERS

1.1. It all really started with Thābit b. Qurra. Before his work, amicable numbers, unlike perfect numbers, lacked the comprehensive theory they deserved. We know that the "perfect number" in the Euclidean sense was the subject of a theory that appears at the end of Book IX of the *Elements*.[4] The famous proposition IX-36 on perfect numbers was presented in a purely speculative, *a priori* way; furthermore, we might ask ourselves why the Greeks paid particular attention to such questions. Among all the various hypotheses on the subject, the most attractive is that of Friedrich Hultsch in the late nineteenth century: it shows that it was in fact a theoretical translation of logistic procedures known since the Egyptians.[5] But the situation is entirely different for amicable numbers. They were only mentioned in later evidence in reference to mystical or esthetic traditions. Iamblichus of Chalcis (1894, p. 35), the most renowned author for such evidence, like Thābit b. Qurra later on, ascribed a knowledge of these numbers to the Pythagoreans. Though they are of course legendary accounts, they have the advantage, in the case of Ibn Qurra at least, of revealing his scientific intentions. For instance, in the preamble to his tract on amicable numbers, Thābit ibn Qurra recalls that "Pythagoras and the ancient philosophers of his school" used two species of numbers: perfect and amicable numbers. Nicomachus of Gerasa, Ibn Qurra continues, gave the rule for determining perfect numbers though without proof. On the other hand, Euclid gave both the rule and its proof. As for amicable numbers, Ibn Qurra observes that "neither of these authors either mentioned or showed any interest in them".[6]

The dissymmetry between perfect and amicable numbers, the disproportion between the mystical importance of amicable numbers and mathematical knowledge about them are so many historical facts on the eve of Ibn Qurra's proof. If we confine ourselves to mathematical knowledge alone, it is in fact limited to a definition and the calculation of the couple [220, 284]. So Ibn Qurra draws up a new programme whose accomplishment he outlines in these terms: "Since the matter of them (the amicable numbers) has occurred to my mind, and since I have derived a proof for them, I did not wish to write it (i.e. the rule) without proving it perfectly, because they has been mentioned in this way (i.e. they have been neglected by Euclid and Nicomachus). I shall therefore prove it with complete clarity after introducing the necessary lemmas".[7] And, in fact, after establishing the necessary lemmas, he proved the theorem that bears his name.

Before presenting the theorem, let us recall some definitions: the aliquot parts of a natural integer n or its proper divisors, are its divisors excluding n. Call $\sigma(n)$ their sum, and $\sigma(n) = \sigma_0(n) + n$, the sum of divisors. Integer n is called

abundant	if	$\sigma_0(n) > n$,
deficient	if	$\sigma_0(n) < n$,
perfect	if	$\sigma_0(n) = n$.

Two natural integers m, n are called amicable[8] if

$$\sigma_0(m) = n \qquad \sigma_0(n) = m.$$

In the tenth century[9] the notion of equivalent integers $m, n, \ldots, r$ (*muta*ᶜ*ādila*) was also introduced,

$$\sigma_0(m) = \sigma_0(n) = \ldots = \sigma_0(r);$$

and that of the sub-double,[10] though unnamed.

$$\sigma_0(n) = 2n.$$

IBN QURRA'S THEOREM. *For* $n > 1$, *let* $p_n = 3.2^n - 1$, $q_n = 9.2^{2n-1} - 1$; *if* p_{n-1}, p_n, q_n *are prime, then* $a = 2^n p_{n-1} p_n$ *and* $b = 2^n q_n$ *are amicable, a is abundant and b is deficient.*[11]

To establish the theorem, Ibn Qurra proves nine lemmas divided into two groups. The first three lemmas deal with the determination of the aliquot parts of an natural integer. Here Ibn Qurra touches on two themes which were to be systematically developed by his successors. He then turns to the decomposition of an integer into prime factors and considers procedures for combinatorial analysis *avant la lettre*. He thus proves in succession:

– "Any plane number whose two sides are prime is measured by no number other than itself".[12] The lemma is clearly a particular case of proposition IX-14 of the *Elements* (voir *infra*, p. 289).

– "Any plane number, one of whose sides is prime and the other a composite number, is measured by its two sides and by any number which measures its composite side, and by any product of its first side by one of the numbers that measures its composite side, and is measured by no other number".[13]

– "Any plane number whose two sides are composite, is measured by its two sides, by any number which measures each of its sides, by any product of each of these two sides by the number which measures the other, and by any product of any number which measures one of the two sides by any number which measures the other; and is measured by no number other than these".[14]

Ibn Qurra proceeds in the same way to prove the three lemmas; he starts by establishing that each element of the set of proper divisors of a number divides this number; he then shows by *reductio ad absurdum* that there exists no other element that divides this number not a member of this set. Moreover, the three lemmas correspond to the same situation, made more complicated each time. We see therefore that after completion of the first attempt to elaborate a theory of amicable numbers, Ibn Qurra has already caught sight of the basic problems for the history of arithmetic: factorization into prime elements and resort to a possible combinatorial for enumerating these elements.

The second group of lemmas deals more specifically with the formation of perfect, abundant and deficient numbers. This was in fact a return to traditional research on the characteristic of the proper divisors of a number. Thābit b. Qurra proves two propositions – the first of which is important for the history of perfect numbers. It is rewritten:[15]

Let $s = \sum_{k=0}^{n} 2^k$, p a prime odd number; then

$$\begin{aligned} \sigma_0(2^n s) &= 2^n s \quad \text{if} \quad s \text{ a prime number.} \\ \sigma_0(2^n p) &> 2^n p \quad \text{if} \quad p < s \\ \sigma_0(2^n p) &< 2^n p \quad \text{if} \quad p > s \end{aligned}$$

and

$$|\sigma_0(2^n p) - 2^n p| = |s - p|.$$

If the first proposition gives a procedure for generating perfect Euclidean numbers, abundant and deficient numbers, the second gives another for the last two cases. It is rewritten:[16]

Let $p_1, p_2 > 2$ be two distinct prime numbers, then

$$\sigma_0(2^n p_1 p_2) > 2^n p_1 p_2 \quad \text{if} \quad p_1 p_2 < (2^{n+1} - 1)(1 + p_1 + p_2)$$

and

$$\sigma_0(2^n p_1 p_2) < 2^n p_1 p_2 \quad \text{if} \quad p_1 p_2 > (2^{n+1} - 1)(1 + p_1 + p_2).$$

An examination of Ibn Qurra's tract reveals that the study of amicable numbers is not just a part of a wider ensemble that also includes the formation of abundant, deficient and perfect numbers but, furthermore, it requires closer investigation into the properties of proper divisors as well. Then the lines of this investigation still embedded in traditional arithmetic are implicitly drawn: factorization and combinatorial methods. If they came to dominate, it was due to a repeated call for the notion of prime number. More than before, it had become necessary to ensure whether given numbers were prime or not. This orientation, as we shall see with Ibn Qurra, will receive further confirmation with his successors; the notion of prime numbers must be assigned a much more central position than had been the case in the past.

1.2. If we set aside the mystical aspect of amicable numbers in order to concentrate on their mathematical aspect alone, we are obliged to note that the history of these numbers merges with that of the knowledge and transmission of Ibn Qurra's work.[17] The paucity of this already meagre history is greater if, for a wide variety of reasons, Ibn Qurra is required to step down. However, this is what some historians meant when they asserted that Ibn Qurra's theorem, buried in oblivion like the mathematician himself, was rediscovered complete and independently by Fermat and Descartes; moreover, not until Woepcke's translation in the nineteenth century would the theorem relinquish the names of Fermat and Descartes. Yet again, according to this viewpoint, the study initiated by Ibn Qurra could not have been mathematically active, since, forgotten, it had not given rise to research.

Such a position would be shattered by recent studies of certain works by Arabic-speaking mathematicians after Ibn Qurra: for example, al-Kāshī's *Key to Arithmetic* (d. 1436/7), and quite recently, the opuscula by Ibn al-Bannāʾ)'s commentator (probably 14th c. as we shall see). Both works attest in fact that Ibn Qurra's theorem was known to mathematicians in the fourteenth and fifteenth centuries. But if we succeed in showing that neither al-Kāshī nor the fourteenth century commentator are isolated cases, that since its formulation this theorem was transmitted without interruption, and lastly that since its diffusion was not confined to mathematicians alone, but also won over philosophers, the opinion which affirms the eclipse of Ibn Qurra's theorem must collapse. While not pretending to be exhaustive, a fanciful claim in view of the present state of research in Arabic mathematics, it is sufficient to select a few significant landmarks, and stress the importance of this research on aliquot parts in each case.

In the second half of the tenth century, Abū al-Ṣaqr al-Qabīṣī[18] in a small treatise on arithmetic, examines perfect numbers, recalls the rule for the formation of Euclidean perfect numbers, then examines amicable numbers about which he cited Ibn Qurra's theorem.[19] Some decades later, two other studies much more significant than that of al-Qabīṣī are to be found. The first is by al-Karajī, the famous tenth century algebraist, and appeared in his work on arithmetic, *al-Badī*ᶜ (*Ḥisāb*), the second is by the arithmetician Abū Ṭāhir al-Baghdādī (d. 1037). The first study is the most useful among known texts for identifying the state of research on this subject a century or so after Ibn Qurra. Before examination and by its very presence in *al-Badī*ᶜ, we know that the theory of amicable numbers not only attracted mathematicians of the calibre of al-Karajī, but was also considered worthy enough to appear in a work for experienced mathematicians. Moreover, it was for them that al-Karajī on his own admission,[20] wrote *al-Badī*ᶜ which included a chapter on the theory of amicable numbers. This chapter consists essentially of Ibn Qurra's two preceding propositions and his theorem; however, al-Karajī proposes to prove these propositions again in a really general way or, in his own terms, to give "a universal proof" (al-Karajī, ed., 1964, p. 28), while his predecessors only gave a quasi-general proof, i.e. immediately generalized once $n = 2, 3, 4, 5$ etc. is established. Any requirement for the representation of numbers by segments, even if aimed at helping the imagination, is excluded from the proof. So, even if for simple technical reasons[21] his attempt was a failure, nonetheless al-Karajī's work ensured the circulation of Ibn Qurra's theorem. As for al-Baghdādī, he apparently synthesized contemporary research on aliquot parts in his important treatise on arithmetic *al-Takmila.*[22] His exposition is arranged according to the following plan: he starts by repeating the definitions of the various species of numbers, notably abundant and deficient numbers including perfect numbers with some of their properties; he then introduces equivalent numbers, and lastly, concludes with amicable numbers. This still unpublished treatise is proof that contemporary mathematicians were acquainted with several propositions unanimously attributed to later mathematicians. For instance, al-Baghdādī states a result commonly attributed to Bachet de Méziriac: the smallest odd abundant number is 945.[23] On the other hand, in his study of perfect numbers, he disputes the validity of an affirmation which already figured in Nicomachus of Gerasa, and was reiterated in the sixteenth century. He writes as follows: "He who affirms that there is only one perfect number in each ᶜ*aqd* ⟨power of ten⟩ is wrong; there is no perfect number between ten thousand and one hundred thousand. He who affirms that all perfect numbers start with the figure six or eight are right".[24] Next he recalls the rule Euclid stated for forming perfect numbers before proposing another equivalent rule which he claims as his own discovery. This rule is rewritten:

$$\textit{if } \sigma_0(2^n) = 2^n - 1 \textit{ is prime}$$
$$\textit{then } 1 + 2 + \ldots + (2^n - 1) \textit{ is a perfect number}$$

or, in his own words: "We discovered another method ⟨other than that of Euclid⟩ for the formation of perfect numbers: if ⟨the sum⟩ of the aliquot parts of an evenways even number is prime, then the sum of integers starting with the unit, is a perfect number".[25] So instead of resorting to geometrical progressions, arithmetic progressions suffice to form Euclidean perfect numbers. Besides this result, commonly attributed to the seventeenth-century mathematician Jan Brożek (Dickson, 1966, pp. 13–14), al-Baghdādī states other less important ones,[26] and completes his exposition with amicable numbers to which he applied a slight variation of Ibn Qurra's theorem. Moreover, this theorem was at that time apparently part of basic arithmetical knowledge, since it is to be found in the work of an author who died the same year as al-Baghdādī: the arithmetical book of Avicenna's famous philosophical synthesis, *al-Shifā*ᵓ.[27]

Most of the above results are to be found in treatises devoted to teaching two centuries later. In an early thirteenth-century treatise, al-Zanjānī (still alive in 1257) took up al-Baghdādī's results in almost identical terms; he touched on the problem of sub-double numbers, and, once again, gave Ibn Qurra's theorem on amicable numbers.[28] Yet the most significant attempt to give a new proof of this theorem was achieved in the late thirteenth–early fourteenth century by Kamāl al-Dīn Fārisī (d. 1320), to whom we shall return at length later. Other mathematicians may be added: al-Tanūkhī[29] in the thirteenth century, Ibn al-Bannāᵓs fourteenth-century commentator,[30] as well as al-Umawī.[31] From the fifteenth century onwards, al-Kāshī,[32] Sharaf al-Dīn al-Yazdī,[33] and Muḥammad Bāqir al-Yazdī[34] may be cited. Many other names could be added to the above, but their diversity in space and time are sufficient proof of the uninterrupted circulation and continuous transmission of this theorem. Did this tradition – and this is the question here – remain unknown to European mathematicians? Though highly unlikely, such an eventuality nevertheless remains probable until clear documentary proof is discovered. On the other hand, if we consider that the author in question was already known in Europe for several of his works on astronomy and statics, the probability is even less.

Independently of this tradition or not, Fermat and Descartes stated the same theorem in turn between 1636 and 1638. Here, as for Arab mathematicians, research on amicable numbers is set in the configuration of perfect numbers, which includes sub-multiple numbers and, more generally, aliquot parts. Consequently, it is important to know the advance of seventeenth-century mathematicians in this field over their Arab predecessors in order to give a correct assessment of the role of these investigations for the history of number theory from Fermat onwards. As we know, Fermat's early works were devoted primarily to a study of sub-double numbers, the number of aliquot parts of a positive integer and amicable numbers, a conclusion drawn from a systematic examination of his correspondence between 1636 and August 1638, and some passages in Mersenne bearing the stamp of Fermat. In his general preface to *Harmonie Universelle* (1636), Mersenne gives the famous couple of amicable numbers named after Fermat. In the second part of the same work (1637), in a passage just as famous, Mersenne sets out Ibn Qurra's theorem, probably based on Fermat. Fermat (1894, t. II, p. 20) wrote to Mersenne on 24 June 1636: "Some time ago now, I sent the proposition on aliquot parts to Mr de Beaugrand with a construction for finding infinite numbers of the same nature. If he has not lost it, he will let you know about it". On 22 September, in the same year, he wrote to Roberval (1894, t. II, p. 72): "this is also how I found infinite numbers that make the same thing as 220 and 284, that is the parts of the first equals the second, and those of the second the first".

On 31 March 1638 in a letter adressed to Mersenne (ed., 1962, t. VII, p. 131), Descartes gives, in turn, the same theorem with this comment: "I do not need to add its demonstration, as I am saving time and, concerning problems, it is sufficient to give the *facit* since those who proposed it can examine whether or not it was solved correctly". Moreover, Fermat and Descartes were apparently just as convinced as their Arab predecessors that Ibn Qurra's theorem gives an infinity of solutions, or as Descartes wrote, this rule "contains an infinity of solutions".[35] However, during this period as we said, the works of

Descartes and Fermat on arithmetic focused on sub-double numbers, the numbers of aliquot parts and amicable numbers. If we set aside the study of aliquot parts for the time being in order to concentrate on amicable numbers, we note that both mathematicians simply rediscovered Ibn Qurra's theorem without proving it. And yet for some historians the seventeenth-century contributions are seen as a reactivation of number theory; and it is true the study of two prime elementary arithmetic functions are seen to take shape. On the other hand, the methods employed by Descartes and Fermat in their works are algebraic rather than strictly arithmetical. If we reconsider the advance of these mathematicians over their Arab predecessors, our question becomes more precise: when and why did this reactivation occur? More exactly, when and why were algebraic methods applied to Euclidean number theory? Before reconsidering this question, let us start by examining how Thābit b. Qurra's theorem was applied to the calculation of couples of amicable numbers.

1.3. One would have expected Ibn Qurra's theorem to have encouraged mathematicians to multiply calculations of amicable numbers, all the more so because the latter were invested with social and psychological virtues. Yet this was not the case. In fact, only three couples are known before Euler. The first couple (220, 284) is of uncertain origin, though mentioned earlier by Iamblichus (1894), who attributed it to Pythagoras; Thābit b. Qurra was not tempted to pursue his calculations further. The two other couples (17296, 18416) and (9363584, 9437056) are named after Fermat and Descartes respectively, to whom the first calculation is usually attributed. But it has recently been shown that Fermat's couple had been calculated by Ibn Bannāʾ's commentator (Souissi, 1976, p. 202) in the late fourteenth century. Here we want to show that it had already been calculated at least a century earlier, before 1320, and it was subsequently known to many mathematicians; as for Descartes' couple, it was known well before the works of the philosopher, as we shall see. But beyond the question of historical priority lies a much more important one, that of the techniques on which it was based. So we shall concentrate on the means implemented for the calculation of amicable numbers.

In his tract, edited elsewhere (Rashed, 1982b), al-Fārisī is not satisfied to give the calculation of "Fermat's couple" but states a complete justification for it as well. He starts with $n = 4$; then $p_3 = 23$, $p_4 = 47$,

$q_4 = 1151$. Then using several propositions including Eratosthenes' sieve, he shows that 1151 is prime; the first two numbers are obviously prime. He applies the theorem and then obtained Fermat's couple. He goes on to write:

Determine the sum of aliquot parts of the prime number ⟨17296⟩, itself a product of 16 by 1081. The sum of aliquot parts of the first is therefore 15, and that of the second 71. Multiply the sum of aliquot parts of the first – which is 15 – by the second plus the sum of its aliquot parts, i.e. 1152, we have 17280. Then multiply the first – which is 16 – by the sum of the aliquot parts of the second, i.e. 71, we obtain 1136, which added to what we have obtained gives 18416, the higher of two amicable numbers.

He continues (Rashed, 1982b, proposition 27):

We determine the aliquot parts of the second number ⟨18416⟩ in the same way by finding the aliquot parts of its two sides, which are 16 and 1151. The sum of the aliquot parts of the first ⟨16⟩ is 15 and that of the ⟨parts⟩ of second ⟨1151⟩ is one. Multiply the sum of the aliquot parts of the first by the second ⟨plus the aliquot parts of the second⟩ – ⟨the sum⟩ of the aliquot parts of its two sides ⟨of 18416⟩, one of them is the unit, whose unit is one, being known – that is, 1152, we obtain 17280. Then multiply the first, that is 16, by the sum of the aliquot parts of the second, we have 16, which we add to the first ⟨result⟩, which gives 17 296.

Therefore, to prove that Fermat's couple is really a couple of amicable numbers, al-Fārisī, as we saw, proceeds as follows

$$\begin{aligned}\sigma_0(17296) = \sigma_0(2^4\cdot 23\cdot 47) &= \sigma_0(2^4)\sigma(23\cdot 47) + 2^4\sigma_0(23\cdot 47)\\ &= 15(71 + 1081) + 16\cdot 71 = 18416.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_0(18416) = \sigma_0(2^4\cdot 1151) &= \sigma_0(2^4)\sigma(1151) + 16\sigma_0(1151)\\ &= 15\cdot 1152 + 16 = 17296.\end{aligned}$$

Al-Fārisī therefore proceeds with the aid of the properties – established beforehand – of the sum function of the aliquot parts of a number. This is a rapid survey of the ground covered since Ibn Qurra's method. Moreover, the calculation of the same couple was apparently known to mathematicians as on at least two occasions it is to be found in texts clearly devoted to teaching. The first text was composed by Ibn al-Bannāʾs commentator (Souissi, 1976) mentioned earlier; the second, recently discovered, by al-Tanūkhī.[36] In both cases we have an example of a direct application of Ibn Qurra's theorem without verification with the aid of the sum function. Whatever the case, the existence of this calculation in texts less advanced mathematically than al-Fārisī's tract, enables us to affirm without risk that this couple was part of received mathematical knowledge in the fourteenth century.

The situation is different for the calculation of Descartes' couple. The early seventeenth-century mathematician al-Yazdī had in fact claimed this result for himself. In his *Treatise on Arithmetic*, widely circulated as the number of surviving manuscripts bear witness,[37] after stating a theorem equivalent to that of Ibn Qurra, he examines the case $n = 7$; $p_6 = 191$, $p_7 = 383$, $q_7 = 73727$, and obtains Descartes' couple. In his own words:

Example: we found that 192 and 384 of this sequence $\langle (6.2^k)_{k\geq 1}\rangle$ are suitable for that. If we subtract the unit from each of them, the remainder is 191 and 383, two numbers whose product is 73153: the third odd number. The sum of these three odd numbers is 73727 which is an odd prime number. The third of the highest is 128 which, multiplied by the third odd number $\langle p_6 p_7\rangle$, gives the smallest of two amicable numbers, that is, 9363584. We then multiply by the sum of two odd numbers which is 574, which gives 73472; which we add to the first number obtained, which gives 9437056, the higher of the two.[38]

Al-Yazdī then gives Table 1 (see below) which recapitulates the calculation of aliquot parts.

TABLE 1

Aliquot parts of the highest		Aliquot parts of the smallest			
sum of odd numbers $[p_6 + p_7 + p_6 p_7 = q_7]$	unit $[2^n]$	third odd number $[p_6 \cdot p_7]$	second odd number $[p_7]$	first odd number $[p_6]$	unit $[2^n]$
73727	1	73153	383	191	1
147454	2	146306	766	382	2
294908	4	292612	1532	764	4
589816	8	585224	3064	1528	8
1179632	16	1170448	6128	3056	16
2359264	32	2340896	12256	6112	32
4718528	64	4681792	24512	12224	64
9437056	128	9363584	49024	24448	128

As may be seen, far from being forgotten, Ibn Qurra's theorem was still alive at the end of the fifteenth century; moreover, pairs of amicable numbers commonly attributed to seventeenth-century mathematicians had been known for a long time. Still more generally, several results relating to these numbers, perfect numbers including the study of aliquot parts, whose discovery was ascribed to later mathematicians, had already been proved by their Arab predecessors. Whatever the significance of these results, the essential, as we said, lies elsewhere: the study of elementary arithmetical functions in the thirteenth century, and the much earlier introduction to algebraic methods in number theory. The intervention of algebraic methods had been remarked on, moreover, by later Arab mathematicians, one of whom recalls, in an outline of the history of amicable numbers, that there were several methods for determining them: "the one mentioned by Thābit ibn Qurra of Ḥarrān, which is the geometrical method which he proved; the one mentioned by Abū al-Wafāʾ Muḥammad b. Muḥammad al-Būzjānī; the one mentioned by Abū al-Ḥasan ʿAlī b. Yūnis al-Miṣrī; and lastly, the one of algebra and *al-Muqābala*".[39] Though we have not succeeded in discovering the manuscripts of the last two authors, al-Fārisī's tract, however, gives us ample means to review the problem of the usage of algebraic procedures in Euclidean number theory.

2. AL-FĀRISĪ: A NEW STUDY ON ALIQUOT PARTS. THE FUNDAMENTAL THEOREM OF ARITHMETIC, ELEMENTARY ARITHMETICAL FUNCTIONS, FIGURATE NUMBERS

2.1. Kamāl al-Dīn al-Fārisī's avowed purpose in his tract on amicable numbers[40] could not be clearer: to prove Ibn Qurra's theorem afresh but in a different way. He intended to base the new proof on a systematic knowledge of the divisors of an integer and the operations that can be applied to them. Such a project must in fact have led him to a radical reorganization of this area of number theory; and al-Fārisī's approach induced him not only to modify Euclidean arithmetic, locally at least, but also gave rise to new topics in number theory. To achieve such a study he was obliged to develop what Ibn Qurra had only hinted at in passing, in particular factorization and combinatorial methods; it was therefore necessary to ensure the existence and uniqueness of the factorization of an integer in order to introduce combinatorial methods and reach an exact knowledge of the number of divisors or proper divisors: consequently, this implied engaging in a new study of elementary arithmetic functions. No wonder therefore that al-Fārisī's tract opens with three propositions clearly intended to state and prove what will much later be called the fundamental theorem of arithmetic.

PROPOSITION 1. *Any composite number is necessarily decomposed into a finite number of prime factors of which it is the product* (Rashed, 1982b, proposition 1).

Al-Fārisī's proof can be summarized as follows: let $a > 1$ be an integer; a has a prime divisor b according to VII-13 of the *Elements*. Then a is written

$$a = bc,$$

with c integer such that $1 \leq c < a$.

If c is prime, then the proposition is proved; otherwise c has a prime divisor d such that

$$c = de,$$

with $1 \leq e < c$.

If e is prime, it gives

$$a = bde,$$

and the proposition is proved; otherwise, the procedure is repeated, and after a finite number of times, a prime number k reached, such that

$$a = bde \ldots k;$$

otherwise, al-Fārisī writes (Rashed, 1982b, proposition 1), "it would necessarily follow that the finite is composed of a product of an infinity of numbers, which is absurd".

Consequently, after proving the existence of the resolution into a finite number of prime factors, al-Fārisī makes a clumsy attempt to prove its uniqueness. He first proves the following two propositions:

PROPOSITION 2. *Let a and b be two integers, each one decomposed into distinct prime factors,* $p_1, p_2, \ldots, p_n$, *then* a and b *are identical* (Rashed, 1982b, proposition 4).

By identical – *mutamāthil* – al-Fārisī, like his contemporaries, implies that both numbers are considered as two magnitudes whose ratio is equal to the unit. Such a notion of identity apparently refers therefore to a ratio between two geometrical representations subjacent to two integers. This concept, directly related to Euclidean arithmetic where integers represent line segments, outlived al-Fārisī since it will be found later in Euler. In any case, a and b are identical if a measures

b as often as b measures a. The primacy of the geometrical representation is of little help in formulating the proof of uniqueness for al-Fārisī. Next al-Fārisī justified the reciprocal of the above proposition without really proving it:

PROPOSITION 3. *Let a and b be two different (non-identical) integers, then their respective factorization into prime factors differs either by the number of factors or by the multiplicity of any one of them* (Rashed, 1982b, proposition 5).

The last two propositions were obviously intended to establish the uniqueness of the factorization into prime factors. However, they were clearly insufficient to enable al-Fārisī to achieve his aim. He should have stated and proved the reciprocal of Proposition 3, and it is surprising he did not adopt this approach. It is also surprising he did not follow the approach indicated in proposition IX-14 of Euclid's *Elements* either, which he knew perfectly, and what is more, had already been used by Ibn Qurra in his tract on amicable numbers.

So we have seen how al-Fārisī formulated the fundamental theorem of arithmetic and his attempt to prove it. Nevertheless, despite its deficiencies, al-Fārisī's contribution is still the first known version of the famous theorem. Whether he was the inventor or not is unimportant; on the other hand, what is capital is the intimate relation between a systematic study of the divisors of an integer – their sum and number – and the elaboration of this theorem which quite naturally appears to base this study. If this is the case, it is easily understood why the fundamental theorem of arithmetic was absent from Euclid's *Elements*, though all the necessary means for its formulation and proof existed. This is an important point for the history of mathematics, and the subject of much controversy.

Among the great theorems, few indeed have such a meagre history as that of the fundamental theorem of arithmetic. In fact, apart from al-Fārisī's theorem just introduced, its history amounts to no more than a passing reference in Gauss' *Recherches arithmétiques* (1807, theorem 16). As to whether this theorem was known earlier the only answers are conflicting interpretations. The controversy originated in a commentary by Heath[41] on proposition IX-14 of the *Elements*, which reads: "If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it". In other words, the smallest common multiple of prime numbers has no prime divisors other than these numbers. Heath thought he had recognized the famous fundamental theorem in this proposition. This interpretation by the eminent historian was not only disputed by his successors, but even before it was formulated so to speak, by his predecessors. We have just seen that neither al-Fārisī, nor commentators on Euclid as distinguished as Ibn al-Haytham,[42] nor mathematicians of the class of al-Karajī (ed., 1964, p. 22), recognized IX-14 as what would become the fundamental theorem; this shows that Heath's interpretation was not really historical. It is understandable then that some historians, undaunted by a recurrent interpretation, had hesitated before adopting Heath's point of view. They were unanimous in recognizing that the famous theorem was completely absent in the *Elements*, though not unknown to Euclid; a position undistinguished by its clarity. For some, for example Jean Itard,[43] this absence is ascribable to Euclid's pedagogic preoccupation in the *Elements* which dissuaded him from exhausting his subject. For others such as Bourbaki,[44] Euclid did not formulate this theorem because he lacked an adequate terminology and notation for any powers. Still more recently,[45] while not abandoning Heath's interpretation, an attempt was made to reduce its scope by affirming that IX-14 is equivalent to a particular case of the fundamental theorem: that of natural integers without square factors, *i.e.* if $p_1 p_2 \dots p_n$ is a product of distinct prime numbers taken in pairs, then the only prime factors are $p_1, p_2, \dots, P_n$.

However IX-14 is interpreted, we can simply note the absence of any formulation or proof of the existence of the factorization of an integer into prime factors. At best therefore, IX-14 is still only a proof of the uniqueness of factorization, whose existence was merely postulated for the restrictive case described above. How not to be surprised then, by an approach which aims at proving uniqueness without proving existence, when all the necessary means for this proof exist? In fact, proposition VII-31 had been used to that purpose by Euclid's successors. Since it is inconceivable here to invoke circumstantial reasons to justify this absence, it is more advisable to yield to facts and admit, like some historians (Hardy and Wright, 1965; Bourbaki, 1960, I; Itard, 1961; Knorr, 1976), that in the *Elements* Euclid did not deal with the question of the factorization into prime factors, or at least it did not appear fundamental enough to him to devote a theory to it. If this hypothesis is correct, then the above debate briefly mentioned in these pages would appear historically superfluous. Heath thought he perceived a theorem where there was none; his detractors raised a controversy which should never have taken place, and a project was attributed to Euclid which was not his, only to be accused of faulty execution.

A study of Euclid's *Arithmetical Books* which deliberately brushes aside problems raised both by their genealogy and their destiny, i.e. their application to Book X, shows that their overall connection rules out any fundamental role for a theorem on decomposition. We first encounter Euclid's algorithm – sometimes called ἀνθυφαίρεσις, – as it was based on two propositions in Book VII. But, given the Euclidean concept of the unit (as a measurement of any number), and the number (as a *finite* plurality of units), the algorithm therefore establishes the existence of the highest common divisor. The importance of the notion of co-prime numbers followed by absolute prime numbers, whose existence and infinity are established in Book IX, is immediately apparent.

Nothing in the evolution of Euclid's exposition compels us to look for a theorem that is not fundamental, at least for the organization of Book IX, and will not provide the basis for other essential applications. This is precisely the case of the fundamental theorem of arithmetic.

If we confront this context of the *Elements* with that of the search for all the divisors of a natural integer for examining their sum and number, we will have a better grasp of the reasons that led a mathematician such as al-Fārisī to conceive this theorem. Indeed, if the theorem was formulated, it was with a view to prepare this study of divisors and introduce the necessary combinatorial methods for it, while all the required conditions for its proof had been consigned in the *Elements* long before. Destined therefore to making algebraic procedures applicable to Euclidean arithmetic, this theorem emerged naturally, and neither al-Fārisī nor his successors grasped either its fundamental role or its central place. To be truly identified as such, it was necessary to wait until it was established that it was not as "natural" as it appeared, in other words, it is not verified in the arithmetic of any ring of integers. But that is another topic.

2.2. With the above theorem and combinatorial procedures it is now possible to study two elementary arithmetical functions, though effective means for resolving into prime factors are still wanting. Since al-Baghdādī at least, mathematians use several lemmas to facilitate the use of Eratosthenes' sieve. The following is the most important:

LEMMA 4. *If a natural integer n has no prime divisor p such that $p^2 < n$, then n is prime.*

A lemma incorrectly attributed to Fibonacci.[46]

The study of arithmetical functions proper starts with Propositions 5 and 6, which are a good illustration of the general style of al-Fārisī's study as a whole.

PROPOSITION 5. *If we resolve a composite number into prime factors, then any number composed of two, or three of these factors and so on to any number composed of all its factors minus one, is an aliquot part of the given number* (Rashed, 1982b, proposition 6).

PROPOSITION 6. *Any number resolved into prime factors only has as its aliquot parts, the unit, its prime factors, the numbers composed of its factors, two by two, if their number is greater than two, three by three if their number is greater than three and so on until we reach the number composed of all the factors minus one* (Rashed, 1982b, proposition 9).

At the outset the problem was studied in a deliberately combinatorial style. Then followed two groups of propositions in succession. The first deals with the sum function of aliquot parts; as we have seen, Ibn Qurra and al-Baghdādī had obtained some partial results, but at no time had the arithmetical function been studied for itself; it is in al-Fārisī's tract that research wholly devoted to the topic appears for the first time. We shall give the most important propositions he stated and proved there.

PROPOSITION 7. *If $n = p_1 p_2$ with p_2 prime and $(p_1, p_2) = 1$, then*

$$\sigma_0(n) = p_2\sigma_0(p_1) + \sigma(p_1),$$

or, in his own words: "If we multiply a composite number by a prime number different from the prime factors of the composite number, then the sum of the aliquot parts of the product is equal to the product of the sum of the aliquot parts of the composite number by this prime number, plus the sum of the aliquot parts of the composite number, plus the composite number" (Rashed, 1982b, proposition 18).

Al-Fārisī's proof can be summarized as follows. Note $\mathcal{D}_0(n)$ the set of aliquot parts of an integer n, $\mathcal{I}$ the set of elements of the second side of the preceding relation. Al-Fārisī first shows that any element of $\mathcal{J}$ is a proper divisor of n, and therefore an element of $\mathfrak{D}_0(n)$, whence $\mathcal{J} \subset \mathfrak{D}_0(n)$. He then proves by *reductio ad absurdum* that $\mathfrak{D}_0(n)$ contains no other element not belonging to $\mathcal{J}$; whence the result. In fact, the idea subtending Proposition 7 is, as we shall see,

$$\sigma(n) = \sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2) = \sigma(p_1)(1 + p_2),$$

hence

$$\sigma_0(n) = \sigma(p_1)(1 + p_2) - p_1 p_2,$$

hence the result.

COROLLARY 8. *If $n = p^r$, p prime, then*

$$\sigma_0(n) = \sum_{k=0}^{r-1} p^k = \frac{p^r - 1}{p - 1}\ ;$$

this corollary had already been applied by al-Baghdādī, for example (Rashed, 1982b, proposition 27).

Next al-Fārisī considers a slightly more complicated situation, touched on by Ibn Qurra, and shows (Rashed, 1982b, proposition 21):

PROPOSITION 9. *If $n = p_1 p_2$, with $(p_1, p_2) = 1$, then*

$$\sigma_0(n) = p_1\sigma_0(p_2) + p_2\sigma_0(p_1) + \sigma_0(p_1)\sigma_0(p_2);$$

which is proof that he knows the expression

$$\sigma(p_1 p_2) = \sigma(p_1)\sigma(p_2),$$

and knows that the σ function is multiplicative. Let us read the statement of this proposition, where al-Fārisī uses a similar proof to that of the above proposition. He writes (Rashed, 1982b, proposition 21):

> If we multiply a composite number by a composite number, the sum of the aliquot parts of the product is equal to the product of the sum of the aliquot parts of the multiplier by the multiplicand, plus the product of the sum of the aliquot parts of the multiplicand by the multiplier plus the sum of its aliquot parts, which is the sum of aliquot parts of the product, from which we subtract the products of any proportion of four terms.

The last part of the sentence means, as his following exposition will show, that no couple of divisors of p_1 is proportional to any pair of divisors of p_2, and therefore $(p_1, p_2) = 1$. Otherwise, as al-Fārisī indicates further on, it is necessary that an aliquot part, at least of p_1 other than 1, also be an aliquot part of p_2. In the case where $(p_1, p_2) \neq 1$, the repeated terms must be subtracted. Lastly, al-Fārisī makes an unsuccessful attempt, as will be easily understood, at establishing an effective formula for the last case, i.e. when $n = p_1p_2$, with $(p_1, p_2) \neq 1$.

All these propositions on the sum function are to be found three centuries later. They appear in Descartes,[47] to whom historians ascribed the formulation. However, we now know they had been stated at the end of the thirteenth century by mathematicians who, unlike Descartes, also gave most of their proofs as well.

Still more important than the propositions themselves, is the path followed for their discovery, the method chosen by mathematicians for establishing them, which consequently determines mathematical rationality itself. In a letter addressed to Mersenne, dated 13 July 1638, Descartes mentioned (Mersenne, 1963, p. 345) this method in passing: "As for the method I used to find aliquot parts, I say that it is none other than my analysis, which I apply to this kind of problem and others as well; and I need time to explain it as a rule comprehensible for those using another method". Barely one month after Descartes, and independently of him, Fermat described his own method for finding aliquot parts expressed in the same terms. He wrote to Mersenne on 10 August 1638 (Mersenne, 1963, VII, p. 27): "Concerning the numbers of aliquot parts, if I have time I shall put my analytical method on this subject down in writing and will let you know". The terminology – analysis, analytical method – common to both, is not the result of chance, but merely the expression of unity of thought. In such a context, these terms, it is true, apparently designate mainly algebra in the sense of Viète, i.e. in our opinion it only differs very slightly from the discipline inherited from al-Karajī and his school (*supra*, pp. 22–33). It could be shown generally how, for both traditions, "analysis" was identified with or substituted for algebra; but if we restrict ourselves to the area of aliquot parts, to be convinced it suffices to read what Fermat wrote when began to work on the subject. On 16 December 1636, he wrote to Roberval (1896, II, p. 93): "Concerning numbers and their aliquot parts, I have found a general method for solving all these questions by algebra, on which I intend to write a small treatise". Fermat never wrote the treatise announced, but an identical role of algebra emerges from a reading of Descartes' *Excerpta Mathematica*; in this way he justified the interpretation of the term "analysis" and, in so doing, characterized the body of research on aliquot parts in the first half of the seventeenth century.

But we have just seen that the use of algebraic methods was by no means the prerogative of seventeenth-century mathematicians, and that it was in fact an acquisition of the thirteenth century at least. Now, it was precisely this application of algebra to the traditional field of Euclidean arithmetic, this use of algebraic methods in arithmetic, that distinguishes al-Fārisī's tract from not only the Alexandrian school, but Ibn Qurra and his successors as well. The study of the sum function of aliquot parts is sufficient proof, but its algebraic character is even more apparent on two occasions. Firstly, when al-Fārisī achieved his goal, and in so doing used algebraic methods: this was the new proof of Ibn Qurra's theorem. And the algebraic character emerges again when we estimate the extension of this field -- the study of aliquot parts – through the impetus of the intervention of algebraic procedures, and when we note its independence in relation to the initial goal which was to prove the theorem on amicable numbers. It particularly concerns the study of the function number of aliquot parts of an integer and the relationship between figurate numbers and combinations required by this new study.

To establish his new demonstration of Ibn Qurra's theorem, al-Fārisī starts by proving the following theorem (Rashed, 1982b, propositions 25 and 26).

LEMMA 10. *For any natural integer n, we have*

$$2^n q_n - 2^n p_{n-1} p_n + (2^{n+1} - 1) = q_n.$$

Al-Fārisī sets $2n = x$, which he calls a "thing" according to contemporary algebraic language, and infers:

$$p_{n-1} = \frac{3}{2}x - 1, \quad p_n = 3x - 1, \quad p_{n-1}p_n = \frac{9}{2}x(x-1) + 1,$$

$$q_n = \frac{9}{2}x^2 - 1;$$

it is then sufficient to substitute and identify to obtain the result.

Now let us examine al-Fārisī's proof (Rashed, 1982b, proposition 27) of Ibn Qurra's theorem. Since $(2^n, q_n) = 1$ and q_n is assumed prime, using Proposition 7, we may write:

$$\sigma_0(2^n q_n) = \sigma_0(2^n) q_n + \sigma(2^n); \qquad (1)$$

but with Corollary 8 we obtain

$$\sigma_0(2^n) = 2^n - 1 \qquad (2)$$

and

$$\sigma(2^n) = 2^{n+1} - 1; \tag{3}$$

substitute (2) and (3) in (1); it yields

$$\sigma_0(2^n q_n) = 2^n q_n - q_n + (2^{n+1} - 1)$$

and with Lemma 10 we obtain

$$\sigma_0(2^n q_n) = 2^n p_{n-1} p_n. \tag{4}$$

On the other hand, since $(2^n, p_{n-1}p_n) = 1$ according to Proposition 7 we have

$$\sigma_0(2^n p_{n-1} p_n) = \sigma_0(2^n) p_{n-1} p_n + \sigma(2^n)(1 + p_{n-1} + p_n); \tag{5}$$

by (2) and (3), it yields

$$\sigma_0(2^n p_{n-1} p_n) = (2^n - 1) p_{n-1} p_n + (2^{n+1} - 1)(1 + p_{n-1} + p_n);$$

but

$$(2^{n+1} - 1)(1 + p_{n-1} + p_n) = q_n + p_{n-1} p_n - (2^{n+1} - 1);$$

therefore, by substituting in (5), it gives

$$\sigma_0(2^n p_{n-1} p_n) = 2^n p_{n-1} p_n + q_n - (2^{n+1} - 1),$$

hence, according to Lemma 10

$$\sigma_0(2^n p_{n-1} p_n) = 2^n q_n. \tag{6}$$

From (4) and (6), we obtain the result, and Ibn Qurra's theorem is thus proved. This is al-Fārisī's exact method, except for the notation of course.

Though the sum function had been necessary for the proof, the number function of aliquot parts of an integer was not. It was therefore through his interest in the study of aliquot parts themselves, independently of Ibn Qurra's theorem, that al-Fārisī set out to study this last function. Denote $\tau_0(n)$, the number of aliquot parts of n, and $\tau(n) = \tau_0(n) + 1$ the divisors of n; al-Fārisī shows:

PROPOSITION 11. *If $n = p_1 p_2 \ldots p_r$ with $p_1, \ldots, p_r$ distinct prime factors, then*

$$\tau_0(n) = 1 + \binom{r}{1} + \ldots + \binom{r}{r-1}.$$

Al-Fārisī states this proposition attributed to Deidier,[48] as follows (Rashed, 1982b, proposition 9):

Let a be the number, it is resolved into prime factors, some or all of them may be equal or distinct. If all of them are equal, then one of the species of its side is in the homonymous position of the number of sides. The first power is the side, and the aliquot parts of the number are the side and species smaller than the number, and there are no other aliquot parts according to proposition 13 of Book IX ⟨of the *Elements*⟩ [$\Leftrightarrow$ if $a = p^n$, p prime, then the aliquot parts of a are p^k, $0 \leq k < n$]. If they are all distinct, let them be b, c, d, e. Multiply b by c, by d and by e; c by d and by e; d by e. We have six two-term combinations. Then, in turn set aside one factor and combine the three remaining factors; we have four three-term combinations. There are no aliquot parts left. So we have all the aliquot parts without exception: the unit, the prime factors and their combinations.

Before returning to the method to find these combinations, note that al-Fārisī uses the following proposition, but without establishing it completely:[49]

PROPOSITION 12. *If* $n = p_1^{e1} p_2^{e2} \ldots p_r^{er}$, with $p_1, p_2, \ldots, p_r$ *distinct prime factors, then*

$$\tau(n) = \prod_{i=1}^{r} (e_i + 1).$$

2.3. Is there a simple method for enumerating all the combinations necessary for calculating the number of aliquot parts of an integer? It was in response to this very practical question that an ancient chapter of arithmetic was taken up afresh: figurate numbers. As we shall see, the effectiveness of this resumption lies in the extension of the notion of a figurate number to any order, and the new, resolutely combinatorial interpretation that was made. On the one hand, we are no longer limited either to polygonal or pyramidal numbers and on the other, figurate numbers are identified with binomial coefficients, which will be the subject of combinatorial interpretation from henceforth. In this way, each term occurs as often as it is possible to transpose the letters that compose it. For instance, the coefficient of a^2b is given by the possible number of arrays of *aab*, i.e. *aab, aba, baa*; namely three. These two interdependent acts – extension and interpretation – assume a vital importance for the history of combinatorial analysis and have been attributed to Frénicle, René-François de Sluse and Pascal.[50] We propose to show that they had already been achieved by al-Fārisī's time at least.

Let us start by recalling what we showed earlier (*supra*, pp. 261–274), i.e. the existence of two combinatorial activities from the late tenth century. The first was the work of algebraists who, since al-Karajī, were acquainted with the arithmetical triangle and its rule of formation, and used combinatorial practice when dealing with systems of linear equations (Al-Samawʾal, ed., 1972, French introduction, pp. 77ff.). The second was the work of lexicographers in particular, who correctly combined, arranged and applied quite obviously general rules, but without bothering to formulate them explicitly. In the present state of our knowledge it appears that the focal point for both activities lies in the field of figurate numbers. The essential components of algebraic and lexicographic investigations had therefore been established, in all likelihood, before the end of the thirteenth century, and this unification is contained in al-Fārisī's tract. However, to appreciate the progress made before al-Fārisī, let us briefly recall the introduction of figurate numbers in Arabic mathematics.

Since Ibn Qurra's translation of Nicomachus of Gerasa's *Introductio arithmetica*, Arab arithmeticians were familiar with the table of polygonal numbers as given by Ibn Qurra in his translation:[51]

triangle number	1	3	6	10	15	21	28	36	45	$\frac{1}{2}n(n+1)$
square number	1	4	9	16	25	36	49	64	81	n^2
pentagonal number	1	5	12	22	35	51	70	92	117	$\frac{1}{2}n(3n-1)$
hexagonal number	1	6	15	28	45	66	91	120	153	$n(2n-1)$
heptagonal number	1	7	18	34	55	81	112	148	189	$\frac{1}{2}n(5n-3)$.

A reading of Nicomachus' text suffices to show that he knew the rule for forming this table, which may be rewritten

$$p_n^r = p_{n-1}^r + p_1^{r-1},$$

with p_n^r the unit of the nth row and the rth column.

From the tenth century onwards, this table was reproduced *ad lib* by increasing the number of rows and columns; for instance, in treatises on arithmetic such as those by al-Baghdādī, Avicenna, Ibn al-Bannāʾ and, later al-Umawī. Clear progress was accomplished in calculating the powers of n natural prime integers at the same time. This movement reached its climax in Ibn al-Haytham's proof[52] of an expression known to his predecessors, such as al-Qabīṣī,[53] and contemporaries such as al-Baghdādī:[54]

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

If we confine ourselves to figurate numbers, let us first study their sums. For example, this was how al-Baghdādī calculated pyramidal numbers, and proved that a pyramid of root n is of the form

$$\pi_n = \frac{1}{2}\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{6}.$$

He calculates solid numbers in the same way,[55] starting with n first squares, and next n first pentagonals:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\frac{1}{2}\sum_{k=1}^{n} k(3k-1) = \frac{n^2(n+1)}{2}.$$

Until then, contrary to what had been observed in the study of the powers of n first natural integers, nothing radically new nor particularly significant had been added to the achievements of Hellenic works on figurate numbers, apart from some results relating to the sum of these progressions and, more generally, greater knowledge about the properties of rows and columns;[56] to which should be added the rejection by most mathematicians of any geometrical representation of figurate numbers. Apart from these few features nothing else emerges from an examination of known documents.

However, at the end of the thirteenth century, two contributions apparently challenged this limitation of knowledge about figurate numbers, which is merely the result of a lack of sources. The first, partial it is true, was due to Ibn al-Bannāʾ; the second, much more general, was the work of al-Fārisī. Since the first died in 1321, and the second in 1320, and since Ibn al-Bannāʾ was Moroccan, while al-Fārisī was Persian, and lastly, since neither of them claimed the discovery, but both apparently expounded known results, for all these reasons, there are grounds for thinking that both mathematicians are inscribed in the filiation of a common heritage.

In the commentary on his own book on arithmetic,[57] Ibn al-Bannāʾ, after studying polygonal numbers, considers triangular numbers and numbers generated by their sums, i.e. figurate numbers of the 4th order. He then relates combinations used in lexicography and these figurate numbers. Such an act is important and must be analyzed in detail. Ibn al-Bannāʾ recalls that combinations of p elements in threes are given "by the sum of the triangular numbers whose side is always equal to the given number minus two; and the sum of triangular numbers is obtained by multiplying this last side by the product of two numbers that follow it, and by taking the sixth of the product".[58]

For instance, if we call p the number of items, F_k^3 the triangular numbers, Ibn al-Bannāʾ's assertion is written

$$\binom{P}{3} = \sum_{k=1}^{p-2} F_k = \frac{p(p-1)(p-2)}{6}.$$

To justify this result Ibn al-Bannāʾ returns to the general case of combination of p terms k by k. But this is precisely where he omits figurate numbers. Ibn al-Bannāʾ gives an equivalent expression to

$$\binom{p}{2} = \frac{p(p-1)}{2}$$

and

$$\binom{P}{3} = \frac{(p-2)}{3}\binom{P}{2},$$

before generalizing, or, as he wrote,

the ternary ⟨combination⟩ is thus obtained by multiplying the binary ⟨combination⟩ by the third of the third term preceding the given number; and so we always multiply the combination – *ʿadad al-tarkīb* – that precedes the combination sought by the number that precedes the given number, and whose distance to it is equal to the number of combinations sought. From the product, we take the part that names ⟨the number⟩ of combinations.[59]

In other words, Ibn al-Bannāʾ stated

$$\binom{p}{k} = \frac{p-(k-1)}{k}\binom{p}{k-1}. \quad (1)$$

Ibn al-Bannāʾ proved this relation using a kind of archaic recurrence, defined earlier (*supra*, pp. 62–84). Let us reproduce the proof in Ibn al-Bannāʾ's own words so as to appreciate his style which interests us particularly. Let p be the given number of elements to be combined.

To obtain two-term combinations, Ibn al-Bannāʾ gives no proof, but takes up "the sum of successive numbers from the unit to the number preceding the given number."[60] For three-element combinations, he writes:

each binary term is combined with one of the remaining ⟨elements⟩ of the given number. Ternary combinations are therefore the product of binary combinations by the given number minus 2, which is the third number preceding the given number $\left[(p-2)\binom{p}{2}\right]$. But since there are three binary combinations for each ternary combination, it follows that the ternary combination will be repeated three times: itself and two of its permutations. For example, if we combine a and b with c, we will have a and c with b, and b and c with a. But these three ternary combinations give the same ternary combination.[61]

Ibn al-Bannāʾ therefore concludes: "It is necessary to take the third of the binary combinations which is multiplied by the remainder of given the number $\left[\left(\frac{1}{3}\binom{p}{2}\right)(p-2)\right]$, or binary combinations are multiplied by the third of the remainder of the given number $\left[\binom{p}{2}\frac{(p-2)}{3}\right]$."[62] He used a similar reasoning for $k = 4$, concluded for $k = 5$ and so on for any k. From the above, Ibn al-Bannāʾ[63] derived the relation

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}, \tag{2}$$

which is to be found later in Cardano, and then Fermat.[64]

If we only consider the results, nothing is really surprising. Not only the multiplicative expression (2) but the relation (1) which made its determination possible, are easily deduced from the additive law of the formation of the table of binomial coefficients. This law, as we know, announced and proved three centuries earlier by al-Karajī, and reproduced by al-Samawʾal in the twelfth century, had not ceased to be transmitted.[65] It is true that Ibn al-Bannāʾ did not prove the case $\binom{n}{1}$; one might think he had wanted to avoid $\binom{n}{0}$ in this way, though given in the arithmetic triangle as it had been represented by al-Samawʾal, for example.[66] But he did not prove the case $\binom{n}{2}$ either, and was satisfied to write: "Binary combinations consist of an assembly of successive numbers from the unit to the number preceding the given number ⟨of elements⟩".[67] Although we are not in a position to confirm it, it seems unlikely that Ibn al-Bannāʾ or his sources were ignorant of this triangle. In fact the relative lack of an effective practice of combinatorial clashes with a formulation whose generality is such that it can only be justified outside this practice, in other words, in the mathematician's formulation of the rule of the formation of this triangle.

But, in our opinion, there is something more fundamental than these results: it is precisely the combinatorial appearance of Ibn al-Bannāʾ's exposition, together with the relation he partially established between figurate numbers and combinations. It concerns, in the first place, triangular numbers and combinations of p objects in twos, and then figurate numbers of order 4 and combinations of p objects in threes. Let us cite Ibn al-Bannāʾ:

> It follows from that, if one of two successive numbers is multiplied by half of the other, the product is equal ⟨to the number of⟩ binary combinations of the greater of the two and is equal to the triangular number ⟨whose side is⟩ the smaller of them following the above. If we multiply three successive numbers, the first by the half of the second, and the product by a third of the third, the product is equal ⟨to the number of⟩ ternary combinations of the greatest and is equal to the sum of successive triangular numbers up to the triangular number ⟨whose side is⟩ the smallest of these numbers.[68]

Such results are far from insignificant for the age. Let us just remember that, even in the seventeenth century, Bachet de Méziriac did not suggest anything more important on the subject.[69] But it is surprising that Ibn al-Bannāʾ stopped at two orders of figurate numbers and the relation between figurate numbers and combinations was so soon exhausted.

Our question is more precise: why, when he disposed of all the necessary arithmetical and combinatorial means for establishing the relation between figurate numbers and combinations in all generality, did Ibn al-Bannāʾ turn away so quickly? To answer these questions, we are apparently reduced to considering aliquot parts and arithmetic functions. In the chapter devoted to combinations of two types of figurate numbers, Ibn al-Bannāʾ apparently only aimed at showing the usefulness of figurate numbers for calculating "the combinations of ternary words" in the domain of lexicography and completely disregarded aliquot parts. Furthermore, in the same chapter amicable numbers are neglected as "useless".[70] The situation is entirely different when the mathematician is involved with the study of aliquot parts and has to find all the necessary combinations for the calculation of their number. He is then obliged to proceed to an entirely different order of generality and can no longer stop at what Pascal will call "the use of the arithmetic triangle for numerical orders". However, all the above has been found in al-Fārisī's tract.

And, indeed, the status of figurate numbers is radically different when one wishes to answer the question of the number of aliquot parts: the mathematician is no longer interested in a specific polygonal or pyramidal number but figurate numbers of any order. Such a level of abstraction calls for a general formulation. To form the latter, al-Fārisī states in an equivalent way the relation

$$F_p^q = \sum_{k=1}^{p} F_k^{q-1}, \tag{3}$$

with F_p^q the p^{th} the figurate number of order q, $F_1^q = 1$.

With the aid of (3) he constructs Table 2 (Rashed, 1982b, proposition 16) as an example.

TABLE 2

their numbers sums		first 2	second 3	third 4	fourth 5	fifth 6	sixth 7	seventh 8	eighth 9	ninth 10	tenth 11
first	1	3	6	10	15	21	28	36	45	55	66
second	1	4	10	20	35	56	84	120	165	220	286
third	1	5	15	35	70	126	210	330	495	715	1001
fourth	1	6	21	56	126	252	462	792	1287	2002	3003
fifth	1	7	28	84	210	462	924	1716	3003	5005	8008
sixth	1	8	36	120	330	792	1716	3432	6435	11440	19448
seventh	1	9	45	165	495	1287	3003	6435	12870	24310	43758
eighth	1	10	55	220	715	2002	5005	11440	24310	48620	92378
ninth	1	11	66	286	1001	3003	8008	19448	43758	92378	184756
tenth	1	12	78	364	1365	4368	12376	31824	75582	167960	352716

Al-Fārisī then shows an expression equivalent to

$$F_p^q = \binom{p+q-1}{q} \tag{4}$$

and thus establishes a relation between figurate numbers of any order and combinations. As a result, from henceforth, it is possible to refer to the table of figurate numbers to know the number of aliquot parts. He writes:

The method for recognizing binary aliquot parts ⟨two-term combinations⟩ or ternary or others with any number of sides, on condition that they all be prime and distinct, consists of seeking in the sequence of the homonymous sums of the number of time by which one combines [*ʿadad al-taʾlīf*] minus one, the number whose rank – that is the first numbers ⟨in the sequences⟩ of sums ⟨column indices⟩ – is homonymous of the number of sides minus the number of times by which we combine. It gives the number of combinations (Rashed, 1982b, proposition 17).

To clarify the idea, assume the given integer is decomposed into n prime distinct factors, and seek the number of aliquot parts of m elements, with $0 < m < n$. We therefore seek in the table the item to the $(m-1)$th row and the $(n-m)$th column. If we complete al-Fārisī's table by adding the row and column whose items are all equal to 1, i.e. F_k^0 and F_1^k, and then by adding the row whose items are natural integers, that is F_k^1, we then obtain F_{n-m+1}^m which, according to (4), is equal to $\binom{n}{m}$.[71] Note, moreover, G_n^m the elements of the above incomplete table. Then G_n^m is equal to F_{n+1}^{m+1}.

To prove the above proposition al-Fārisī proceeded in a fully combinatorial way by applying in succession the arithmetic triangle whose various "cells" are interpreted as combinations of p objects k by k. The combinatorial style clearly more mastered than that of Ibn al-Bannāʾ, dominated his entire work. This interpretation and style are as important for the history of combinatorial analysis as for number theory; so it is advisable to cite al-Fārisī himself. He starts by examining the case of an integer decomposed into five prime distinct factors and seeks the number of two-term aliquot parts to show that it is equal to $G_3^1 = F_4^2 = 10$. He repeats the same reasoning for an integer decomposed into six distinct prime factors, and still using combinatorial formulas, shows that the number of three-item aliquot parts is equal to $G_3^2 = F_4^3 = 20$. As the proposition is proved for five elements taken in twos and six elements taken in threes, al-Fārisī then assumes it true for n items taken k by k, with $(1 \leqq k \leqq n)$.

Let, a, b, c, d, e, be the sides, the binary combinations do not fail to include e among their sides or not. In the last case they include d or not. In this last case a or b or c must be cancelled. The binary combination of these three ⟨elements⟩ are therefore three, the homonymous rank of the number of sides minus the number of times by which we combine – i.e. two; which is the first rank of sums, homonymous of the number of times by which we combine, minus one, i.e. the first $\left[\binom{3}{2} = 3 = G_1^1 = F_2^2\right]$. The binary combinations including d but not e, have as the other side one of the three remaining, *a, b, c*. There are three of them. Binary combinations of *a, b, c, d* are thereore six according to the rule mentioned. Those including *e*, include as the other side one of the four remaining – *a, b, c, d*. There are also four of them. Binary combinations of these five ⟨sides⟩ are therefore ten, which is in the first sums at rank three, homonymous to the number of sides minus the number of times by which we combine (Rashed, 1982b).

Hampered by the incompleteness of his table, al-Fārisī omits some stages in his reasoning which may be summarized as follows. He first establishes that

$$\binom{4}{2} = \binom{3}{2} + \binom{3}{1} = F_2^2 + F_3^1 = F_1^1 + F_2^1 + F_3^1 = F_3^2,$$

and then

$$\binom{5}{2} = \binom{4}{2} + \binom{4}{1}$$

he then substitutes, and obtains

$$\binom{5}{2} = \binom{3}{2} + \binom{3}{1} + \binom{4}{1} = F_1^1 + F_2^1 + F_3^1 + F_4^1 = F_4^2.$$

This expansion does not force the meaning of al-Fārisī's thought and is amply confirmed by his proof of the following case: six factors, the number of combinations three by three. He writes (Rashed, 1982b):

Let *a, b, c, d, e, f* be these sides. Ternary combinations will not fail to have *f* as one of their sides or not. In the last case one of their sides is *e*, or not. In the last case, i.e. when they are combined with *a, b, c, d* – only four – it is necessary that ⟨each time⟩ one of them does not exist. We therefore have four $\left[\binom{4}{3}\right]$, i.e. the first term of the second sum $[G_1^2 = F_2^3]$. Each combination containing *e* but not *f*, contains as two other sides the two sides of one of the binary combinations of the remaining sides, i.e. *a b, c, d*, that is six $\left[\binom{4}{2}\right]$; these ternary combinations are also six. Ternary combinations from *a, b, c, d, e* – five – number ten $\left[\binom{5}{3} = \binom{4}{3} + \binom{4}{2}\right]$. Each combination containing *f* has as the two remaining sides, one of the binary combinations of *a, b, c, d, e* – the five remaining – which number ten $\left[\binom{5}{2}\right]$; they are also ternary and number ten. Ternary combinations from *a, b, c, d, e, f*, theretore number twenty $\left[\binom{6}{3} = \binom{5}{3} + \binom{5}{2}\right]$, which is the number of the second sum, homonymous of the number of times by which we combine minus one. of rank three, homonymous of the number of sides by which we combine $[G_3^2 = F_4^3]$.

We see therefore that al-Fārisī proceeds as above using the results just obtained; he establishes in succession

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2} = F_2^3 + F_3^2 = F_1^2 + F_2^2 + F_3^2 = F_3^3,$$

and

$$\binom{6}{3} = \binom{5}{3} + \binom{5}{2} = \binom{4}{3} + \binom{4}{2} + \binom{5}{2}$$

$$= F_1^2 + F_2^2 + F_3^2 + F_4^2 = F_4^3.$$

So al-Fārisī proved his proposition by resorting to an archaic recurrence mentioned earlier; but his stumbling block lies in a reasoning dependent on a double index without any form of symbolism at his disposal.

In his computation of combinations for determining the number of aliquot parts of an integer, al-Fārisī therefore takes up binomial coefficients again, but in order to provide a resolutely combinatorial interpretation. Such an act, the foundation of combinatorial analysis itself, also enabled him to conceive figurate numbers in an incomparably more general sense that anything encountered in the works of al-Fārisī's known predecessors and contemporaries. The above table – still used by Bernoulli[72] – only represents a model, a handy instrument for al-Fārisī; in his own words (Rashed, 1982b, proposition 16): it "facilitates obtaining [figurate numbers] for those that seek them" and "constitutes a model – *mithāl* – for he who wants to determine others from them". The first consequence of this generality was to reorientate mathematical language towards greater abstraction. It is true that well before al-Fārisī the geometrical representation of polygonal numbers had fallen into disuse, and if it survived on odd occasions, it was only as a relic to bolster the imagination of those for whom elementary courses in arithmetic were intended. Such a representation was naturally banished from a work of research such as al-Fārisī's tract; furthermore, nothing remains of the geometric qualification of numbers. Even terms such as triangular, pyramidal etc. for designating figurate numbers have disappeared. From henceforth al-Fārisī only uses the language of the sequences of sums – *mujtamiᶜāt* – of such an order. However, scholars like Frénicle, Pascal and Bernoulli will express themselves in identical terms later on.[73]

Compared with the study of his contemporary Ibn al-Bannāᵓ, al-

Fārisī's exposition is distinguished, not only by its generality, but by an entirely different subject matter. But, when compared with that of Pascal, i.e. the first "usage of the arithmetic triangle", it stands up to comparison by accounting for the problem examined, the generality achieved and the concern for proof that animates it, though it undoubtedly remains more opaque and elaborate. In both cases, it is true, as for Frénicle, generality is assured by a more or less direct but always effective knowledge of the arithmetic triangle. Again, in all three cases, a distinction must be drawn between the combinatorial approach and another, no less general, arithmetical approach. To elucidate the difference, in this case fundamental, let us examine a plausible interpretation of Fermat's proposition in the famous Letter XII: He wrote (1896, III, pp. 291–292): "In natural progression, we have double the triangle whose side is the last number, by multiplying the latter by the number immediately above; we have the triplet of the pyramid whose side is the last number by multiplying this by the triangle of the number immediatly above; we have the quadruple of the triangulo-triangle of the last number by multiplying this by the pyramid of the number immediatly higher and so on to infinity by a uniform method". This proposition is rewritten:

$$F_p^q = \frac{p}{q} F_{p+1}^{q-1}.$$

That this result was established before Fermat is relatively unimportant here.[74] Greater emphasis is laid on the method that enabled him to achieve it and about which we nothing certain as yet in the absence of proof.[75] Nonetheless, the expression "uniform method" apparently designates here a call for induction – not necessarily mathematical, as authorized by practice long after Fermat – and probably non-combinatorial.[76] Al-Fārisī's contribution, like that of Frénicle and Pascal, is undoubtedly less natural insofar as it employed means other than purely arithmetical ones. His contribution borrowed its generality from combinatorial interpretation and the study of the function: "the number of aliquot parts"; and it was precisely the last study that provoked the interpretation itself. So al-Fārisī's research could not have been anticipated by mathematicians who, according to an ancient tradition, were only interested in figurate numbers; but only by those, like Naṣīr al-Dīn al-Ṭūsī, who were interested in the combinatorial interpretation.[77]

CONCLUSION: CLASSICAL NUMBER THEORY

With Thābit ibn Qurra we are firmly placed in the framework of Hellenistic arithmetic. Himself a translator of Euclid and Nicomachus of Gerasa, it was in the wake of these mathematicians that he conceived and achieved a theory of amicable numbers. All his investigations on perfect numbers, his discovery of amicable numbers, like the works of his successors (e.g. al-Baghdādī) are inscribed in the tradition of Hellenistic arithmetic. But whereas this arithmetic, like others, was the subject of intense reactivation, algebraists were involved in expanding and even renewing their discipline; and this task was, as we showed elsewhere, accomplished with the aid of arithmetic. But what should be emphasized here is the central position of this algebra for the development of number theory.

This was how, with mathematicians such as al-Khāzin, in relation to and, in some respects against, this algebra, an important chapter of this theory was elaborated in the tenth century: Diophantine integer analysis. In the same century and early in the next, it was in connection with this analysis that one of the central issues of number theory was posed: the search for a necessary and sufficient condition for characterizing prime numbers. We have shown elsewhere how Ibn al-Haytham formulated the question and how he managed to answer it by stating Wilson's theorem. The dialectic between arithmetic and algebra encompasses an entire chapter of numerical analysis, including another devoted to the solution of numerical equations. But if one only considers number theory, it has been shown this time that such a dialectic did not dispense with the Euclidean legacy. It was in fact thanks to algebraic methods that al-Fārisī, notably, constructed a new chapter which may be entitled: aliquot parts and the combinatorial interpretation of figurate numbers. The introduction of algebraic methods themselves did not in fact obey any preconceived, careful and theoretically elaborated purpose: it quite naturally forced itself upon a mathematician trained in the algebraic tradition and grappling with the problem of Euclidean arithmetic, by offering him the means to abandon, at least locally, the framework of this arithmetic in order to join the vast domain of classical number theory. Within this discipline, areas and chapters that were no longer fully Euclidean will coexist with Euclidean arithmetic. We have already named two chapters, to which may now be added aliquot parts and figurate numbers.

On reading historians of mathematics, one notes that the discoveries and results recalled are generally considered the work of mid-seventeenth, if not eighteenth-century mathematicians; furthermore, it was precisely these results that provided the basis for a definition of a new arithmetical rationality. Has it not often been maintained that the study of aliquot parts by Descartes and Fermat inaugurated a new era, whereas that of figurate numbers by Frénicle and Pascal represented a new rationality? But beyond mere historical error, such a misunderstanding might distort the understanding of the history of number theory itself in the seventeenth century. By dint of seeing novelty where there was none, it was overlooked where it did in fact exist. Ignorant of the place of number theory in Arabic mathematics, one might have considered this chapter on aliquot parts and figurate numbers, like Diophantine integer analysis, to be the work of early seventeenth-century mathematicians. In our language this would amount to defining the originality of the latter by the intervention of algebraic methods in number theory. There is then a high risk of misunderstanding Fermat and limiting the scope of his work which developed, not as a continuation of but rather as a break with this so-called algebraic number theory. As a result, the beginning of purely arithmetical number theory around 1640 cannot be pinpointed with sufficient clarity, a problem amply discussed elsewhere (Rashed, 1984, introd.).

To conclude with Arabic mathematics, the thesis that number theory is their weak point does not stand up to the facts reconstructed throughout our various studies. We have shown that Diophantine integer analysis, the search for a criterion for recognizing prime numbers, the study of aliquot parts and figurate numbers, research on perfect numbers, are just so many chapters of new number theory elaborated from the ninth century onwards. These chapters alone suffice to impose a revision of the history of elementary number theory, and above all, a correction of commonly accepted periodization. Nothing validates the separation of works composed in the ninth century into different eras from those accomplished in later centuries, before 1640, insofar as until that date, results and methods did in fact refer to the same arithmetic. Classical number theory followed the Hellenistic period; it preceded the new theory that is seen to begin with Fermat's works.

NOTES

1. See, in particular, Pacioli (1494). Fibonacci's Italian successors who lived before Luca Pacioli were just as important. See, in particular, Picutti (1979).
2. Woepcke (1852), pp. 420–429 where he summarizes Ibn Qurra's *Opuscula*. We have consulted several manuscripts of this work, notably: Aya Sofya MS 4830, ff. 110^v–121^r; Paris, Bibliothèque Nationale, MS 2457, ff. 170^r–180^r.

   Other manuscripts of the same work exist and should be consulted for a truly scientific edition which is still to be made.
3. See, in particular, Dickson (1919, vol. I, p. 38); Borho (1979, 2. Auflage); Itard (1967, pp. 37–38).
4. Remember that in proposition IX-39, Euclid gives the sum of the geometric progression of ratio 2; and proposition IX-36 is rewritten:

   If $1 + 2 + 2^2 + \ldots + 2^p = 2^{p+1} - 1$ is a prime number, then $2^p(2^{p+1} - 1)$ is a perfect number.

   The definition of the perfect number as a number such that the sum of its aliquot parts is equal to it, or, as Euclid (*Elements*, VII-22) wrote: "τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν", appears for the first time here. It has prevailed since then as in Theon of Smyrna and Nicomachus. For instance, Theon (1892), p. 74), wrote: "χαὶ τέλειοι μέυ εἰσιυ οἱ τοῖς αὑτῶυ μέρεσιυ ἴσοι, ὡς ὁ τῶν ς'". Similarly, on perfect numbers, Nicomachus (1866, p. 39) wrote, "that it is always equal to its own parts", "[ἀιεὶ ἶσος τοῖς ἑαυτοῦ μέρεσιυ]" *cf. Introduction*, XVI, ed. R. Hoche, Leipzig, 1866, p. 39. This expression was perfectly rendered in Thābit b. Qurra's translation by "*Wa-lakinnahu abadan musāwin li-ajzaʾihi*". See Ibn Qurra (ed., 1958, p. 38).
5. Exploited by many later scholars, Hultsch's hypothesis was well summarized by Tannery whom Itard cited (1961, pp. 69–70).
6. See the introduction to his tract in the Bibliothèque Nationale, Paris, MS 2457, *op. cit.*
7. Ibn Qurra, *op. cit.*
8. The Arabic terminology in this field leaves no doubt about the Greek origin of the term. Moreover, Thābit b. Qurra's translation of Nicomachus' *Introduction* apparently established this terminology. For instance, the "perfect" number, τέλειος, was translated by *tāmm*, a term whose root like the Greek root, contains the idea of completion and achievement. The terms "abundant" and "deficient" ὑπερτελής and ἐλλιπής, were first given by Ibn Qurra as "more than perfect", *al-zāʾid ʿalā al-tamām*, and "less than perfect", *al-nāqis ʿan al-tamām*, respectively. This translation was forgotten later, and only *al-zāʾid*, abundant, and *al-nāqis*, deficient, were retained. The expression friendly (or amicable) number, φίλοι ἀριυμοί, was rendered by *al-ʿadād al-mutaḥāba.*
9. Abū Manṣūr ʿAbd al-Qāhir al-Baghdādī (d. 1037), *Al-Takmila fī al-Ḥisāb* (Istanbul, Laleli, Süleymaniye, MS 2708/1. f. 59).

   This definition of equivalent numbers is to be found in this text. The author then states the following problem; "Given a number, to find two numbers whose ⟨sum⟩ of the parts of each of them is equal to this given number". It therefore implies seeking the reciprocal image of the given number a by σ_0. Al-Baghdādī proceeds as follows:

"Subtract one from the given number and divide the remainder by the ⟨sum⟩ of two prime numbers, and next by the ⟨sum⟩ of the two other prime numbers and so on as far as two prime numbers may be divided. We then multiply both parts of the first division by each other, and we multiply both parts of the second division by each other, and next we proceed as before for the third division, the fourth and following divisions; ⟨the sum⟩ of the parts of each of the products is equal to the given number". For instance, let a be the given number. The problem is to find all the equivalent numbers related to a, i.e. σ_0^{-1} (a). Al-Baghdādī proceeds as follows: Find p_i, q_i prime ($i = 1, 2, \ldots$), such that

$$a = 1 + p_i + q_i,$$

it yields

$$\sigma_0^{-1}(a) = \{p_i q_i\} = \{b_i\} \quad (i = 1, 2, \ldots),$$

and b_i are equivalent numbers.

It is clear that $\sigma_0(b_i) = \sigma_0(p_i q_i) = a$ ($i = 1, 2, \ldots$).

Al-Baghdādī gives the example $a = 57$, $p_1 = 3$, $q_1 = 53$; $p_2 = 13$; $p_2 = 43$, where $b_1 = 159$, $b_2 = 559$; and thus only gives two elements of the reciprocal image.

In *ᶜUmdat al-Ḥisāb* (MS. Ahmad III, Topkapi Saray 3145) al-Zanjānī repeats the same definitions, takes the same example and finally gives

$$\sigma_0^{-1}(57) = \{159, 559, 703\}.$$

The study of these equivalent numbers is found in several later treatises on arithmetic.

10. In al-Zanjānī's treatise, *op. cit.*, we find these numbers again and the incorrect proposition which affirms that $n = 120$ is the only number that verifies $\sigma_0(n) = 2n$.
11. See Ibn Qurra, *op. cit.*, Prop. 10.
12. *Ibid.*, Prop. 1.
13. *Ibid.*, Prop. 2.
14. *Ibid.*, Prop. 3.
15. *Ibid.*, Prop. 5.
16. *Ibid.*, Prop. 6.
17. Euler obtained a generalization of Ibn Qurra's proposition, that sets.

$$a = 2^n - 1 + 2^{n-\alpha}, \quad b = 2^n - 1 + 2^{n+\alpha}, \quad c = (2^\alpha + 1)^2 2^{2n-\alpha} - 1,$$

three prime numbers; α is necessarily odd if c is a prime number. Ibn Qurra's theorem clearly corresponds to $\alpha = 1$. Cf. Lucas (1958), pp. 380–381.

Note also that Paganini found the couple (1184, 1210), unobtainable by Ibn Qurra's method. Cf. Dickson (1919), p. 47.

18. See *fī jamᶜi ᵓanwāᶜi min al-aᶜdād*, MS Aya Sofya 4832, ff. 85^v–88^v. However, some sentences are missing from the text, probably omitted by the copyist. Al-Qabīṣī forms successively

$$p_n = (2^{n+1} - 1) + 2^n, \quad p_{n-1} = (2^{n+1} - 1) - 2^{n-1}, \quad q_n = 2^{n+1}(2^{n+1} + 2^{n-2}) - 1.$$

19. Other recently discovered examples are proof of contemporary interest in this theorem, cf. *infra*, p. 330, note 1.
20. Al-Karajī (ed., 1964), Arabic text, p. 7. The chapter is intitled *Bāb fī zikr ṭalab al-ᶜadād al-mutaḥāba*, pp. 26ff.

21. Al-Karajī starts by deducing the definition of amicable numbers from the following proposition:

    Let (m, n) be a couple of amicable numbers, then it is necessary that one of these numbers be deficient (that is m, the other abundant (that is n) and that

    $$m - \sigma_0(m) = \sigma_0(n) - n.$$

    He then proves the two propositions Ibn Qurra gave above, before stating the proposition which may be rewritten as follows:

    Let p_1, p_2, q be three odd prime numbers such that $q > s = \sum_{k=0}^{n} 2^k$, $q - s = (1 + p_1 + p_2)s - p_1p_2$, then 2^nq is deficient, $2^np_1p_2$ is abundant, and 2^nq and $2^np_1p_2$ are amicable.

    To prove that 2^nq and $2^np_1p_2$ are amicable, al-Karajī uses the necessary condition given by the above proposition as a sufficient condition.
22. Al-Baghdādī, *al-Takmila.*
23. Al-Baghdādī (*al-Takmila*), writes: "the prime odd abundant number is twelve and any odd number smaller than nine hundred and forty-five is deficient, and the prime odd abundant number is nine hundred and forty-five".
24. Al-Baghdādī, *al-Takmila.*
25. *Ibid.*
26. For example, the numbers of the form 2^n are not perfect, since $\sigma_0(2^n) = 2^n - 1$.
27. Ibn Sīnā (ed., 1975, p. 28). If we correct the reading of the edition, Avicenna's text becomes crystal clear and is translated as follows:

    If $(2^{n+1} - 1)$, p_{n-1}, p_n are prime, then $2^np_{n-1}p_n$ and $2^n(p_{n-1} + p_n + p_{n-1}p_n) = 2^nq_n$ and are amicable.

    Therefore if we add the condition q_n prime, we find Ibn Qurra's theorem again, with the superfluous condition $(2^{n+1} - 1)$ prime.
28. Al-Zanjānī, ᶜ*Umdat al-Ḥisāb*, f. 69ʳ. Al-Zanjānī was a compiler, his treatise, as yet unstudied, is a good example of what a man well versed in arithmetic would know in the first half of the twelfth century. Note, however, that he wrote (f. 68ᵛ): "any perfect number greater than six is evenways even odd". Is this a clumsy way of affirming that all even perfect numbers are Euclidean? It seems highly probable, in any case, that contemporary mathematicians dealt with the characterization of perfect numbers as the above affirmation goes to show at least. It is also true that they were interested in the computation of perfect numbers. Several indications in later treatises show that they calculated perfect numbers other than those given by Nicomachus of Gerasa, e.g. the fifth perfect number.
29. Zayn al-Dīn al-Tanūkhī. See his *Treatise on Arithmetic*, MS Vatican 317,2 f. 78. Cf. our edition Rashed (1982b). According to ᶜUmar Kaḥāla (1957, vol. 6, p. 286), Tanūkhī was a thirteenth-century linguist.
30. This is in fact an important text edited by Souissi (1976), pp. 193–209. Souissi (1975) gives a complete French translation of this manuscript.

    There is however no evidence for categorically attributing this manuscript to Ibn al-Bannāᵓ. The comparison between Ibn al-Bannāᵓ's two treatises, *i.e. Talkhiṣ A*ᶜ*māl al-Ḥisāb* and *Raf*ᶜ *al-Ḥijāb* and a study of the manuscript itself, seems to indicate that it is probably an opuscula by Ibn al-Bannāᵓ's commentator, e.g.

Ibn Ḥaydūr (d. 1413), whose name was cited in another opuscula in the same collection which includes this study of amicable numbers. A hypothesis that Souissi did not reject when submitted to him in a recent correspondence. So it is in all likelihood a later manuscript, dating back to the fourteenth century, and its author apparently gives a known rather than a recently discovered result.

31. Al-Umawī (ed., 1981), p. 34. Yaᶜīsh ibn Ibrāhīm al-Umawī was a fourteenth-century mathematician. In his formulation of Ibn Qurra's theorem, he added, like Avicenna, the condition $(2^{n+1} - 1)$ prime.
32. Al-Kāshī (ed., 1977), pp. 484ff. Here al-Kāshī gives Ibn Qurra's theorem again, but without indicating that that q_n must be a prime number. This error was noted by al-Kāshī's successor, Muḥammad Bāqir al-Yazdī, who noted that it involved another: al-Kāshī considered 20234 and 2296 to be amicable; he was not aware of his error as he was also wrong when he stated the proper divisor of 2296.

    According to M. b. al-Yazdī, following al-Kāshī, Sharaf al-Dīn al-Yazdī, also made a mistake in *Kunh al-murād fī ᶜilm wifq wa al-aᶜdād.*
33. Sharaf al-Dīn al-Yazdī, see note above.
34. Muḥammad Bāqir al-Yazdī, *ᶜUyūn al-ḥisāb*. See Rashed (1982b).
35. Mersenne (1962, VII, p. 131).

    It is still not known whether there exists an infinite number of couples (m, n) of amicable numbers. The present situation (1981) may be summarized as follows: Note $A(x)$ the number of couples, with $m < n < x$. Erdös conjectures that

    $$A(x) = o(x/(\ln x)^k) \quad \text{for any } k.$$

    Bomerance confirms that

    $$A(x) \leqq x \exp\{-(\ln x)^{1/3}\}.$$

    See Guy (1980, 1, pp. 31–33).
36. According to the dates of al-Tanūkhī (13th c.), the calculation of Fermat's couple was apparently known before al-Fārisī; not in the same way of course, but the result at least was known before 1307.
37. Al-Yazdī died around 1637. We have identified a large number of copies of his manuscript scattered in various libraries all over the world, proof of its circulation.
38. For the text and table see Rashed (1982b).
39. This is an elementary treatise by Muḥammad b. al-Ḥasan b. Ibrāhīm al-ᶜAṭṭār al-Asfardī, *al-Lubāb fī al-ḥisāb*, Bodleian, MS Marsh 663 (10), f. 238ᵛ.
40. Al-Fārisī's tract entitled, *Tadhkirat al-aḥbāb fī bayān al-taḥāb*, may be translated as *A Tract for friends to show amicability*. This manuscript, whose importance was emphasized by ancient bibliographers, was only recently lost. Just to give one example, this is what Tāshkupri-Zadeh (ed., 1968, I, p. 396) wrote: "Concerning the method for determining amicable numbers, it was completely shown by numerical demonstrations in the book *A Tract for friends to show amicability*. This is a rare book, proof of the preeminence of its author and his distinction in mathematics".

    We have edited al-Fārisī's text and we shall refer in future to the page numbers of our edition, Rashed (1982b).
41. Heath (1956, 2, p. 403) writes: "In other words, a number can be resolved into prime factors in only one way".

He took up the same idea in (1921), 1, "From Thales to Euclid", p. 401 and (1931), p. 241.

42. Ibn al-Haytham, *Fi ḥall shukūk Uqlīdis fī al-Uṣūl*, Istanbul, MS 800. He wrote (f. 139^v) on IX-14 and XI-15: "There is no doubt about either of them nor a difference of demonstration. Their reason is the figures whose correctness we have shown".
43. Itard (1961, p. 68) wrote: "In the *Elements* we must look neither for commutativity or associativity of a product of several factors, nor composition of a number into a product of prime factors, nor a search for all divisors". He then wonders whether one can assumes that they were unknown to Euclid and writes: "This would be to misunderstand the character of a treatise like the *Elements* where the essential mathematical facts for all future research are generally established in a logical but rather tortuous way, but where no attempts are made to exhaust the subject and where its applications are carefully avoided".

    The position adopted since 1938 by Hardy and Wright (1965, p. 182) should be noted: "It might seem strange at first that Euclid, having gone so far, could not prove the fundamental theorem itself; but this view would rest on a misconception. Euclid had no formal calculus of multiplication and exponentiation, and it would have been most difficult for him even to state the theorem. He had not even a *term* for the product of more than three factors. The omission of the fundamental theorem is in no way casual or accidental; Euclid knew very well that the theory of numbers turned upon his algorithm, and drew from it all the return he could".
44. Bourbaki (1960, p. 110 and p. 111 note). Bourbaki added: "To support this hypothesis, we may also remark that the proof of the theorem of perfect numbers is, in fact, basically no more than a particular case of a theorem of unique decomposition into prime factors. Moreover, all the evidence concurs to prove that, from his time, the decomposition of a number into prime factors was known and commonly used, but no complete proof of the theorem of decomposition has been found before that given by Gauss at the beginning of his *Disquisitiones*".
45. Mullin (1965, pp. 218–222). Hendy (1975, pp. 189–191 and lastly, the thoughtful reconsideration of this problem by Knorr (1976, pp. 353–368).
46. See how he applies this rule, par. 15. Note moreover when examining the decomposition of any number and applying Eratosthenes' sieve, al-Fārisī gives other propositions. For instance, after recalling the decimal expression of integer N

    $$N = a_n\, 10^n + \ldots + a_1 10 + a_0,$$

    he affirms, e.g.

    - as $N \equiv a_0$ (mod. 10) then N is odd when a_0 is odd.
    - similarly, N is divisible by 5 if $a_0 = 0$ or $a_0 = 5$.

    Like so many others these propositions aim at determining whether the number is prime or not by examining its last figure (or figures).
47. Descartes (ed., 1966, X, pp. 300–302). In his *Excerpta Mathematica*, IV, *De partibus Aliquotis Numerorum*, Descartes gives several of the above propositions without proof. For instance, he states Corollary 8 as follows:

    "Numerus autem primus, saepius per seipsum multiplicatus, sicuti a^n, partes aliquotas habet $\frac{a^n - 1}{a - 1}$. Hoc est: seipsum minus 1, divisum sua radice minus 1" (p. 301).

He proceeds and states Proposition 7 in these terms

> "Si reperire velimus partes aliquotas numeri cujusdam primi, per alium numerum multiplicati, cujus jam habemus partes aliquotas, veluti si partes aliquotae numeri a sint b, & x sit numerus primus, partes aliquotae numeri $\langle ax \rangle$ sunt $bx + a + b$", *op. cit.*

He states Proposition 9 in the same way:

> "Si habemus duos numeros primos inter se, eorumque partes aliquotas, habemus etiam partes aliquotas producti ipsorum: veluti, si unus sit a ejusque partes aliquotae sint b, alter vero sit c, cujus partes aliquotae sint d, partes aliquotae ac erunt $ad + bc + bd$", *op. cit.*

48. To "find how many divisors a number has" Deidier (1739, p. 311) wrote: "If the simple divisors of the number are all unequal, we must disregard the unit and then examine how other simple divisors may produce different products taken in pairs, in triplets, in quaduplets, etc., after which add to the number found that of simple divisors including the unit; and the total sum will be the number of different divisors which the proposed number may have".

    Note that Deidier (1739, p. 330) states al-Fārisī's proposition for divisors, i.e. $\tau(n)$. He does not prove it, but verifies it with a numerical example. The calculation of the sum ($\tau(n) = 2^n$), in this case is not stated in a general way but is nevertheless verified by the example $n = 30030$.
49. Rashed (1986, par. 28). Several centuries later, Montmort (1713, p. 55) stated the same rule as follows: "Let it be assumed that we want to find the number of divisors of the literal quantity a^5b^3ccde, by counting the unit as divisor, we will find according to the ordinary rule, which is to multiply the exponents by each other where each one increased by the unit, that this number is 288 . . .".
50. Pascal (ed., 1963, pp. 54–55). Remember that the *Traité du Triangle arithmétique* dates back to 1645. However, some of Frénicle's investigations in the *Abrégé* were known to Mersenne, therefore before 1648 when the latter died. One of his investigations deals with figurative numbers relating to combinatorial analysis; see Coumet (1968, pp. 328–330).

    On René-François de Sluse, see Dickson (1919, II, p. 9). These results are to be found in other 17th century mathematicians; e.g. Wallis (1685, pp. 485–486).
51. Nicomachus' *Introduction* translated by Thābit ibn Qurra (ed., 1958, p. 77). Note that the table in the Greek edition has a supplementary column whose elements are successively (55, 100, 145, 190, 235). See Nicomachus (1866), p. 97.

    Note also, that this table or one of its variations, is to be found in most elementary treatises on arithmetic.
52. See Suter's translation of Ibn al-Haytham's treatise (1912), pp. 296ff.).

    See also our edition and translation of the same text Rashed (1981, pp. 199ff.).
53. Al-Qabīṣī, *op. cit.*, ff. 86^v–87^r.

54. Al-Baghdādī, *op. cit.*, f. 65^r.
55. *Ibid.*, ff. 64^r–65^v.
56. The presentation and study of the progressions these lines and columns may represent.
57. *Raf*ᶜ *al-ḥijāb* ᶜ*an wujūh* ᶜ*amāl al-ḥisāb*, Bibliothèque Nationale, Tunis, MS 9722, ff. 1–45. We are indebted to M. Souissi for a copy of this manuscript. On the life and work of Ibn al-Bannā᾿, see Souissi's introduction to his edition of *Talkhiṣ* ᶜ*amāl al-ḥisāb* (1969, Arabic text, pp. 15ff. and French text, pp. 17ff.).
58. Ibn al-Bannā᾿, *op. cit.*, f. 15^v.
59. *Ibid.*
60. *Ibid.*
61. *Ibid.*, f. 16^r.
62. *Ibid.*
63. *Ibid.*, f. 16^v. "Set the numbers which we multiply with each other, where each one exceeds the other by a unit, where the highest is the number ⟨of terms⟩ of the expression [*p*] and whose number is the number of combinations [*k*]. Next set and numbers by which we are going to divide them, and where each one exceeds the other by a unit, and such that the highest among them is the number of given combinations [*k*] and begins with the unit, two ⟨. . .⟩. Let us then eliminate the numbers common to the prime numbers and the second numbers; when we do that, all the numbers by which we have divided are always eliminated; we multiply the remaining prime numbers by each other; we obtain the number of combinations ⟨of elements⟩ of this expression".
64. Boyer (1950, pp. 387–390). Cf. Letter 4 November, 1636 to Roberval, where Fermat presented this proposition not as a combinatorial but as arithmetic. He wrote (Mersenne, VI, pp. 146–147): "Here is nevertheless a beautiful proposition which might be of some use to you. At least, if was by this means that I succeeded. It is a rule I found to give the sum, not only of triangles, which is what Bachet and others did, but also of pyramids, triangulo-triangulorum, etc. to infinity. Here is the proposition:

    Ultimum latus in latus proxime majus facit duplum trianguli.

    Ultimum latus in triangulum lateris proxime majoris facit triplum pyramidis.

    Ultimum latus in pyramidem lateris proxime majoris facit quadruplum triangulotrianguli.

    Et eo in infinitum progressu."

65. We can in fact show that the diffusion of knowledge of the arithmetical triangle in Arabic mathematics was uninterrupted from the late 10th century until the 17th century.
66. Al-Samaw᾿al (ed., 1972), see the photographic reproduction of the triangle.
67 Ibn al-Bannā᾿, *op. cit.*, f. 16^r.
68. *Ibid.*, f. 16^v.
69. Bachet de Méziriac gave the following proposition in "Appendicis ad Librum de Numeris polygonis". Liber Secundus, prop. 17 (1621, p. 49):

    "Si numerus secetur in duas partes, tum in tres, tum in quatuor, tum in quinque, & sic deinceps, & quaelibet pars unius sectionis comparetur, cuilibet ex aliis partibus eiusdem sectionis, continget hanc comparationem in prima sectione fieri semel, in secunda ter, in tertia sexies, in quarta decies, & sic continue per numeros triangulos ascendendo." see *Diophanti Alexandrini Arithmeticorum*, p. 49.

For instance, after comparing triangular numbers and the combinations of n objects 2 by 2, Bachet did not establish the general interpretation.

70. Ibn al-Bannā᾿, *op. cit.*, f. 17ʳ.
71. Strictly, al-Fārisī did not forget F_k^0. which he wrote down in a table beside the prime, second, etc. sums.
72. Bernoulli (1713, p. 114). Bernoulli completed the table by adding F_k^0 and F_1^k. It is interesting to compare the following commentary by Bernoulli (p. 113) with al-Fārisī: "Hinc vero haud difficult colligimus, uniones omnium serierum rursus efficere seriem monadum, bibiones seriem lateralium, terniones trigonalium, caeterasque combinationes majorum exponentium itidem constituere series aliorum figuratorum altioris generis, prorsus ut combinationes praecedd. Capitum, hoc solo cum discrimine, quod ibi series a cyphris, hic ab ipsis statim unitatibus incipiant".

    He them introduced his table. It had been given earlier but in a condensed form by Pascal (1963, p. 55) Also probably by Frénicle before him, see Coumet (1968, p. 331). It also appears in several treatises on arithmetic later on; cf., e.g. Deidier (1739, p. 322).
73. Frénicle (1729, p. 54) speaks of "triangular powers", Pascal (1963) speaks of "numerical orders" and Bernoulli (1713) of "Series of other figures of a high order", "series aliorum figuratorum altioris generis".
74. Fermat (1896, I, p. 341) affirmed that it was his own discovery; he wrote: "Propositionem pulcherrimam et mirabilem, quam nos invenimus, hoc is loco sine demonstratione apponemus . . .". But, according to Charles Henry *op. cit.*, IV, p. 234, the result had been given earlier by Briggs.
75. Fermat also wrote here: "Cujus demonstrationem margini inserere nec vacat, nec licet"; *op. cit.*, t. I, p. 341.
76. According to Fermat, it concerns more than a "standard method" edition, which requires more space than the Bachet edition of the *Arithmetic* of Diophantus. These extremely brief indications, however, appear to be more suited to the arithmetical demonstration analogous to the one given here by combinatorial techniques. The latter makes it possible quickly to identify the binomial coefficients and the figurative numbers, thus.

$$F_{p-q+1}^q = \binom{p}{q};$$

from which follows

$$F_p^q = \binom{p+q-1}{q},$$

and finally

$$pF_{p+1}^{q-1} = p\,\frac{(p+q-1)\ldots(p+1)}{(q-1)!} = q\,\frac{(p+q-1)\ldots p}{q!} = pF_p^q.$$

Otherwise at greater length, and moreover with greater difficulty, would be an arithmetical demonstration similar to that which we have shown here. We first prove the following lemma:

LEMMA. $F^q_{n+p} = \sum\limits_{\substack{i,j=0\\ i+j=q}}^{q} F^i_n F^j_n.$

For $q = 1$, the previous relation is verified immediately. For $q = 2$, it follows

$$F^2_{n+p} = 1 + 2 + \ldots + n + (n+1) + \ldots + (n+p)$$
$$= F^2_n + F^1_n F^1_p + F^2_p = \sum_{\substack{i,j=0\\ i+j=2}}^{2} F^i_n F^j_p.$$

Let us suppose that the previous relation is true for q, and consider

$$F^{q+1}_{n+p} = F^q_1 + \ldots + F^q_n + F^q_{n+1} + \ldots + F^q_{n+p},$$

according to the definition of figurative numbers. It follows, by the hypotheses of recurrence

$$F^{q+1}_{n+p} = F^{q+1}_n + \sum_{\substack{i,j=0\\ i+j=q}}^{q} F^i_n F^j_1 + \sum_{\substack{i,j=0\\ i+j=q}}^{q} F^i_n F^j_2 + \ldots + \sum_{\substack{i,j=0\\ i+j=q}}^{q} F^i_n F^j_p.$$

From

$$F^{q+1}_{n+p} = F^{q+1}_n + F^0_n \sum_{k=1}^{p} F^q_k + \ldots + F^q_n \sum_{k=1}^{n} F^0_k;$$

and, after definition of figurative numbers

$$F^{q+1}_{n+p} = F^{q+1}_n + F^0_n F^{q+1}_p + \ldots + F^q_n F^1_p;$$

but

$$F^{q+1}_n = F^{q+1}_n F^0_p;$$

therefore

$$F^{q+1}_{n+p} = \sum_{\substack{i,j=0\\ i+j=q+1}}^{q+1} F^i_n F^j_p,$$

PROPOSITION. $qF^q_p = pF^{q-1}_{p+1}.$

With the aid of the lemma, we can write

$$F_{p+1}^{q-1} = F_p^{q-1} + F_p^{q-2}F_1^1 + \ldots + F_p^{q-p-1}F_1^p + \ldots + F_1^{q-1},$$
$$F_{p+1}^{q-1} = F_{p-1}^{q-1} + F_{p-1}^{q-2}F_2^1 + \ldots + F_{p-1}^{q-p-1}F_2^p + \ldots + F_2^{q-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_{p+1}^{q-1} = F_{p-k}^{q-1} + F_{p-k}^{q-2}F_{k+1}^1 + \ldots + F_{p-k}^{q-p-1}F_{k+1}^p + \ldots + F_{k+1}^{q-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_{p+1}^{q-1} = F_1^{q-1} + F_1^{q-2}F_n^1 + \ldots + F_1^{q-p-1}F_n^p + \ldots + F_n^{q-1};$$

in the addition column by column each sum is equal to F_p^q.

In effect

$$F_{p-q}^{q-p-1}F_{k+1}^p = F_{p-q}^{p-k-1}F_{p+1}^k,$$

according to the definition of figurate numbers F_p^q. We have therefore

$$\sum_{k=0}^{p-1} F_{q-p}^{p-k-1}F_{p+1}^k = F_{q+1}^{p-1} = F_p^q.$$

Finally we obtain

$$pF_{p+1}^{q-1} = qF_p^q.$$

Directly based on the definition of figurate numbers and the preceding lemma, the style of this proof may resemble the one Fermat was thinking of when he made this remark, i.e. around 1638, according to the latest date given for letter XII.

77. A reading of al-Fārisī's treatise on algebra, i.e. his commentary on Ibn al-Khawwām's *al-Bahāʾia*, denotes a certain familiarity with combinatorial methods. For instance, when for algebraic needs he raises the square of a polynomial, he proves that the number of arrangements with repetition of the n terms taken in pairs is n^2. Cf. the chapter on the extraction of roots.

5. IBN AL-HAYTHAM AND PERFECT NUMBERS

Two earlier studies (Rashed, 1982b, pp. 209–278; *supra*, pp. 275–319) on the history of number theory in Arabic mathematics showed how, starting in the ninth century with Thābit ibn Qurra, a vast body of research centered on abundant, deficient, perfect, amicable and equivalent numbers was constructed. At first glance, within this body perfect numbers appear to have been neglected; compared with amicable numbers, writings on perfect numbers that have survived are fewer and less significant (Rashed, 1982b, pp. 209–278; *supra*, pp. 275–319).

Contrary to this impression, however, we conjectured that research on perfect numbers must have been more substantial than documents at our disposal lead us to believe and, furthermore, that before the twelfth century attempts were made to prove the reciprocal of Euclid's theorem. Two groups of facts corroborate our conjecture: the results obtained by the mathematician al-Baghdādī (d. 1037) (Rashed, 1984, pp. 263, 267–268) and the echo transmitted by a twelfth-century mathematician, al-Zanjānī, whose arithmetical work remains relatively unstudied despite his significance.

Al-Baghdādī's results attest that contemporary mathematicians had overtaken Euclid and Nicomachus in their understanding of perfect numbers. As for al-Zanjānī's testimony, it suggests that attempts had been made to *characterize* these numbers. But which mathematician capable of conceiving such a characterization pioneered research in this field? Several candidates from the tenth-eleventh centuries stand out about whom we possess little information. We only need to take the example of al-Anṭakī who pursued his research in arithmetic, some fragments of which we have just identified,[1] indications of the path that lies ahead in order to perfect our knowledge of the history of number theory at that time. But despite our incomplete knowledge of the subject, it is thanks to Ibn al-Haytham that we shall validate our conjecture by showing that he not only characterized prime numbers by stating Wilson's theorem (*supra*, pp. 238–261), but also perfect numbers. Seven centuries before Euler, he stated and attempted to prove the reciprocal of Euclid's theorem in IX-36 on perfect numbers. Still more precisely, he states and proves the theorem that may be rewritten as follows:

THEOREM. *Let n be an even number, $s(n)$ the sum of the proper divisors of n. The following conditions are equivalent:*
(a) *if $n = 2^p(2^{p+1} - 1)$ with $(2^{p+1} - 1)$ prime, then $s(n) = n$*
(b) *if $s(n) = n$, then $n = 2^p(2^{p+1} - 1)$, with $(2^{p+1} - 1)$ prime.*

We know that (a) is IX-36 of the *Elements*. Ibn al-Haytham comments on Euclid's proposition on two occasions, and takes up its proof. In his *Commentary on Euclid's Postulates* he mentions the proposition in these terms:

Euclid said next that the perfect number is equal to the sum of the parts that divide it. This is the definition of the perfect number. There exists a number that verifies this property. Euclid has in fact shown at the end of Book Nine how to find the numbers that verify this property.[2]

In a second book, *On the solution of doubts concerning Euclid*,[3] Ibn al-Haytham goes into more detail about this proposition which he proves again. This time he asks the "cause" (*al-ᶜilla*) why only n has proper divisors.

$$\{1, 2, \ldots, 2^p, (2^{p+1} - 1), 2(2^{p+1} - 1), \ldots, 2^{p-1}(2^{p+1} - 1)\}.$$

He retains as the "first cause" the fact "that the sum of parts that are evenways even is an odd number".[4] It was in fact this property that Ibn al-Haytham developed to show that there exists no number that divides n other than those noted above.

But Ibn al-Haytham's real contribution is expounded elsewhere. To my knowledge, no one before him had stated the theorem, nor attempted to propose a proof. One of the examples Ibn al-Haytham chose in his treatise on *Analysis and Synthesis* was that of perfect numbers. He starts by recalling that Euclid only proceeded by synthesis and proposes to undertake analysis himself. Before commenting, let us present this paragraph of Ibn al-Haytham's treatise, established and translated elsewhere:[5]

"... *To find a perfect number*: the perfect number is that which is equal to the sum of the parts that measure it. This problem was set forth by Euclid at the end of the arithmetical chapters of his work. He did not propose his analysis and nothing in what he says shows how he found the perfect number by analysis. He only proposed its synthesis like the other problems included in his work. We shall show here how to find a perfect number by analysis before composing this analysis afterwards.

⟨1⟩ The analytical path to this problem. We set that we have found the perfect number, for example, let it be the number *AB* and let the parts that measure it be the numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N*. Let *AB* be equal to the sum of the numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N*. We next examine the properties of numbers with parts. If we examine the properties of numbers with parts, we find that it was shown in proposition 36 of Book 9[6] that if any number of numbers follow one another in the same ratio, and if we separate a number equal to the first from the second and the last, then the ratio of the remainder of the second to the first is equal to the ratio of the remainder of the last to the sum of all preceding numbers. It follows that, given successive proportional numbers that are in the double ratio, if we subtract from their second and last a number equal to the first, then the remainder of the second will be equal to the first and the remainder of the last equal to the sum of all preceding numbers. But for successive numbers which are in the double ratio, each one measures the greatest number and each one of them is part of the greatest number. It then follows that if the numbers *AB*, *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N* are in double ratio and are successive, then each of the numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N* is a part of *AB*, and if we subtract from *AB* a number equal to *N*, the remainder of *AB* is equal to the sum of remaining numbers that are parts of *AB*. But all *AB* is equal to the sum of parts, therefore the number *AB* is not in double ratio with all the successive remaining numbers.

⟨2⟩ Similarly, among the properties of successive proportional numbers that are in double ratio starting with one, if we subtract one from each of them, each remainder will be equal to the sum of numbers that precedes it, since if we subtract from the second a number equal to the first, which is one; the remainder will be one. It follows that for the numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N*, if some are successive in the double ratio starting with *AB*, and if the last of ⟨those⟩ which is the smallest of them is the double of that preceding it minus one, then all numbers following *AB* are parts of *AB*, and *AB* will be their sum.

* Let the numbers *AB*, *C*, *D*, *E*, *G* be successive in the double ratio, and let number *G* be smaller by one then the double of number *H*. Separate *KB* equal to *G* ⟨from *AB*⟩, then *AK* will be equal to the sum

of the numbers *C*, *D*, *E*, *G*, but *G* is smaller by one than the double of *H*, it is therefore equal to the sum of *H*, *I*, *L*, *M*, *N*, now *KB* is equal to *G*, and *KB* is equal to the sum of *H*, *I*, *L*, *M*, *N*. Therefore the whole of *AB* is equal to the sum of numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, *N*.* But the numbers *H*, *I*, *L*, *M*, *N* are successive numbers in the double ratio starting with the unit and *N* is the unit. But since *C* is half of *AB*, then *C* measures *AB* by the units of *M*, and *D* measures *AB* by the units of *L*, and *E* measures *AB* by the units of *I*, and so on for the remaining numbers. Therefore if the numbers *C*, *D*, *E*, *G* have the same number as the numbers *H*, *I*, *L*, *M*, then each of the numbers that follow *AB* measures *AB* by the units of one of the numbers that follows *N*, and each of the numbers that follow *N* measures *AB* by the units of one of the numbers that follows *AB*. Thus all the numbers are parts of *AB*, and no other number measures *AB*. If there are more numbers following *AB* than these following *N*, and if some numbers following *AB* measure *AB* by the units of the numbers that follow *N*, and if the remaining numbers following *AB* measure *AB* by the units of other numbers, then those other numbers are parts of *AB*. But *AB* has no other parts than the numbers *C*, *D*, *E*, *G*, *H*, *I*, *M*, *N*. Therefore the numbers following *AB* are not more numerous than the numbers following *N*. If the numbers that follow *AB* are less numerous than the numbers that follow *N*, then certain numbers following *N* measure *AB* by the units of the numbers that follow *AB*, and the remaining numbers that follow *N* measure *AB* by the unit of other numbers; these other numbers are therefore parts of *AB*. But *AB* has no parts other than the given numbers. The numbers that follow *AB* according to the double ratio are of the same number as those that follow *N*. Thus the numbers *C*, *D*, *E*, *G* are equal in number to the numbers *H*, *I*, *L*, *M*.

Similarly, if the number *G* has a part or parts, this or these parts measure *AB* since *G* measures *AB*. This or these parts are a part or parts of *AB*, and none of them is one of the numbers *C*, *D*, *E*, *G*, *H*, *I*, *L*, *M*, because none of the numbers *C*, *D*, *E*, is a part of *G* since each one of them is greater than *G*, and none of the numbers *H*, *I*, *L*, *M* measures *G*, because if we add one to *G*, then the numbers *H*, *I*, *L*, *M* measure it, and none of the numbers *H*, *I*, *L*, *M* measures the number one added, because each of them is greater than one, therefore none of the numbers *H*, *I*, *L*, *M* measure the number *G*, therefore none of them is a part of the number *G*; therefore if the number *G* has a part or parts different from the unit, then this part or these parts are parts of *AB* and each of them is other than the numbers C, D, E, G, H, I, L, M. But AB has no other parts than these numbers and the unit, therefore G is a prime number.

Analysis has shown that between the number AB and the number G there exists successive numbers whose ratio is the double, that the number G among them is prime, and that the number G is smaller by one unit than the double of one of the successive proportional numbers starting with one whose ratio is double. This notion is possible, that is the existence of a number among successive numbers whose ratio is double and starting with one, and such that if we take away one we have a prime number.

⟨3⟩ The synthesis of this problem will be made according to the following description. We consider evenways even numbers by induction, that is those numbers that are in double ratio starting with one. We subtract one from each of them, the one that will be prime, we double it as many times until the number of numbers doubled in this way is equal to the number of successive proportional numbers that precede this number counting the unit which is their first number. The greatest number obtained by doubling is a perfect number.

EXAMPLE: The numbers A, B, C, D, E, GH are successive numbers whose ratio is double. Among them A is equal to one, and if we subtract one from GH, the remainder will be a prime number. Subtract from GH the number on which is SH, there remains GS a prime number; we double GS until the number of doublings is equal to the number of numbers A, B, C, D, E. Given the number GS, I, K, L, NO. I say that NO is a perfect number.

Proof: Separate OP equal to GS; NP is therefore equal to the sum of numbers L, K, I, GS; but the number PO will therefore be equal to the sum of numbers E, D, C, B, A; the number NO will therefore be equal to the sum of the numbers A, B, C, D, E, GS, I, K, L. But the numbers L, K, I, GS measure NO by the units of one of the numbers E, D, C, B, and each of the numbers B, C, D, E, measures NO by the units of one of the numbers GS, I, K, L. All the numbers B, C, D, E, GS, I, K, L are parts of NO, but we showed that NO is equal to the sum of these numbers ⟨plus A which is equal to one⟩. It remains to be proved that no number other than these numbers measures NO. Let the number M measure NO; I say that M is one of the numbers B, C, D, E, GS, I, K, L. Let the number M measure NO by the units of number Q. If we then multiply it by Q, we have NO; but the number GS measures NO by the units of the number E; therefore if we multiply GS by E, we have NO, the product of E by GS is therefore equal to the product of M by Q. The ratio of GS to G is therefore equal to the ratio of M to E. Either GS measures Q, or not. If GS measures Q, then M measures E; but the numbers A, B, C, D, E are successive from the unit and proportional; the number that follows the units is a prime number, since it is two; therefore no number measures the greatest of one of them, as was shown in proposition 13 of Book 9; the number M is therefore one of the numbers B, C, D, E. If the number GS does not measure Q, then they are co-prime as was shown in proposition 31 of Book 7; if they are co-prime, then they are the two smallest numbers according to their ratio, as it was shown in proposition 22 of Book 7 and if the two numbers GS and Q are the two smallest numbers according to their ratio, then they measure the numbers which are in their ratio, as was shown in proposition 20 of Book 7. Therefore if the number GS does not measure Q, they are the two smallest numbers according to their ratio, and they measure the numbers which are in their ratio. But the ratio of GS to Q is equal to the ratio of M to E. Therefore the number Q measures E; therefore the number Q is one of the numbers [A], B, C, D. Therefore the number Q measures NO by the number of the units of one of the numbers GS, I, K, L. But Q measures NO by the number of the units of M, the number M is therefore one of the numbers GS, I, K, L.

Any number that measures NO is one of the numbers B, C, D, E, GS, I, K, L. And no number other than the numbers B, C, D, E, GS, I, K, L and A which is the unit measures NO. But the number NO is equal to the sum of these numbers; therefore the number NO is a perfect number".

An examination of the above text leaves no room for ambiguity as to Ibn al-Haytham's intentions: he wants to prove that any perfect number is of the Euclidean form. As can be seen, his exposition is not exempt from obscurities, ascribable not only to the author but also to the manuscript tradition of his treatise; for example, in order to find the coherence of his approach, one paragraph has to be transferred.[8] Now let us consider the various stages of Ibn al-Haytham's proof as carried out in this text.

Firstly, Ibn al-Haytham shows that, for n any even perfect number

$$s(2^p) = 1 + 2 + \ldots + 2^{p-1} = 2^p - 1, \tag{1}$$

whence

$$n = s(n) \neq 2^p. \tag{2}$$

He proves (2) by *reductio ad absurdum*: if $n = 2^p$, then $n - 1 = 1 + 2 + \ldots + 2^{p-1}$; and, according to (1), we obtain $n = n - 1$. Next he shows that if $n = 2^p g$ with $g = 2^{q+1} - 1$, then

$$n - g = g + 2g + \ldots + 2^{p-1}g \tag{3}$$

and

$$g = 1 + 2 + \ldots + 2q;$$

for n perfect we obtain

$$s(n) = n = (2^{p-1}g + \ldots + 2g + g) + (2q + \ldots + 2 + 1). \tag{4}$$

Name the two sets of divisors D_1 and D_2 respectively

$$\{2^{p-1}g, \ldots, 2g, g\} \quad \text{and} \quad \{2^q, \ldots, 2, 1\};$$

Ibn al-Ḥaytham then shows that

$$q = p. \tag{5}$$

He also reasons by *reductio ad absurdum*: to any $d_1 \in D_1$ must correspond $d_2 \in D_2$ and inversely. Therefore if one supposes $q <$ p or $q > p$ one arrives at a contradiction with one of the above conditions; hence the results.

Finally, he shows

$$g \text{ is prime.} \tag{6}$$

Suppose that g is not prime, then there exists $d|g$, $d \neq 1$. But $d|n$, then $d \in D_1 \cup D_2$. Thus $d < g$, therefore $d \notin D_1$; on the other hand, $d \neq 2^k$, therefore $d \notin D_2$; since the terms of D_2 are the divisors of $2^{q+1} = g + 1$; if follows that $d = 1$. So it is obvious that Ibn al-Haytham only gives in fact a partial reciprocal of Euclid's theorem. He does not show that of *all* the even numbers only the Euclidean even numbers are perfect; but only that among the even numbers of the form $2^p(2^{q+1} - 1)$ only those of Euclid are perfect.

Ibn al-Haytham then proceeds to the synthesis. He takes a number

$$n = 2^p g$$

with g a prime number such that

$$g = 2^{p+1} - 1 = \sum_{k=0}^{p} 2^k;$$

we have

$$n = g \sum_{k=0}^{p-1} 2^k + \sum_{k=0}^{p} 2^k.$$

Every number of D_1 or D_2 (for $q = p$) is truly a divisor of n. Suppose d is a divisor of n, then there exists e a divisor of n such that

$$d \cdot e = n = 2^p g;$$

we have

$$e/g = 2^p/d.$$

If g divides e, then d divides 2^p and $d \in D_2$. If g does not divide e, $(g, e) = 1$ because g is prime; then e divides 2^p, and $e = 2^k$ $(1 \leq k \leq p)$; therefore $d = g2^{p-k}$, $d \in D_1$. Any divisor of n appears in D_1 or D_2. We conclude that n is equal to the sum of its divisors; therefore n is perfect.

But this partial failure should not eclipse the essential: a deliberate attempt to characterize the set of perfect numbers. Now this approach is not only dated but also "localized": it is the offspring of the Arabic Euclidean tradition of number theory. A successor to Thābit ibn Qurra, like him Ibn al-Haytham took the arithmetical books of the *Elements* as a model. They were much more intent on establishing and proving certain properties of integers or classes of integers than on calculating particular numbers. Neither of them, let us emphasize again, wanted to extend his investigations to the calculation of numbers unknown to the Ancients, whether it concerns perfect or amicable numbers. Their lack of interest is even more obvious in Ibn al-Haytham's study of perfect numbers, in so far as it was given as an example in a treatise on analysis and synthesis for the entire body of mathematical disciplines with a propaedeutic and methodological bent.

One therefore expects to encounter this calculation in research by mathematicians who opted, either completely or partially, for the neo-Pythagorean style, in other words in the tradition of Nicomachus of

Gerasa's *Introductio Arithmetica.* The manuscripts at our disposal confirm this interpretation and authorize us, moreover, to surmise that this calculation was carried out quite early; we would not be surprised to find the calculation of new perfect numbers in writings dating back to the tenth-eleventh centuries, even though at the present time they are only to be found from the thirteenth century onwards in the writings of Arab Neo-Pythagoreans.

We shall now briefly consider the work of Ibn Fallūs (d. ca 1240), recently examined for the same reason (Brentjes, 1987, pp. 21–30) and also another much later scientist Ibn al-Malik al-Dimashqī. By this choice, we want to circumscribe the period before covering it by systematic analysis.

Ibn Fallūs' work is only a summarized commentary on Nicomachus' book and he did in fact state that it was "based on Nicomachus' book".[9] He starts by giving the Euclidean rule for the formation of perfect numbers, before writing down the following numbers in a table[10] – notwithstanding some mistakes made by the copyist:

6, 28, 496, 8128, 130816, 2096128, 33550336, 8589869056, 137438691328, 35184367894528.

But Ibn Fallūs' texts call for several remarks. Like the Neo-Pythagorean tradition, he does not prove but proceeds by incomplete induction and states rules. Furthermore, one observes that the above list of terms given as perfect numbers also includes numbers that are not perfect. Lastly, one notes the presence of three new perfect numbers; the fifth, sixth and seventh whose discovery is usually dated back to the mid-fifteenth century for the first and to the late sixteenth for the last two. The introduction of numbers in the list of perfect numbers that are not perfect cannot be neglected either since Ibn Fallūs never explicitly exposed his calculation, and in particular his calculation for checking that $(2^p - 1)$ is prime. The numbers are laid out in two rows in a table without comment. But before we clarify this point, let us draw a comparison with the explicit presentation by Ibn al-Mālik, himself heir to this chapter of number theory of his predecessors. In Book 7 of his treatise,[11] after giving the rule of the formation of perfect numbers. Ibn al-Mālik presents a table incompletely reproduced here:

TABLE

Divisible into two halves $[2^p]$	Non successive odd numbers $[2^{p+1} - 1]$	Successive perfect numbers $[2^p(2^{p+1} - 1)]$
2	3	6
4	7	28
8	15	0
16	31	496
32	63	0
64	127	8,128
128	255	0
256	511	130,816
512	1023	0
1024	2047	2,096,128
2048	4095	0
4096	8191	33,550,336

Now, though Ibn al-Malik stops at the fifth perfect number, he too combines numbers that are not perfect in his list. So this mistake, which is not peculiar to Ibn Fallūs, is no longer a simple error of calculation. Moreover, in the eyes of someone, though slightly familiar with mathematical writings of the age, even if second-rate, it is inconceivable that those mathematicians were incapable of recognizing whether 511 or 2047, for example, are prime or not. In our opinion, the reason for the error must be explained otherwise; it refers to Nicomachus' affirmation according to which there exists a perfect number in each decimal position. Refuted earlier by al-Baghdādī in the early eleventh century (*supra*, pp. 281–282), nonetheless this opinion outlived criticism as it may be found expressed by Ibn Fallūs: "All these numbers (i.e. perfect numbers) have two necessary properties [*lāzimān*]; the first one is that there exists one in each of the ranks of calculation: one in the units, i.e. six, one in the tenths, i.e. 28, and so on for the other ranks: the second is that their smallest extremity is two successive numbers that are six and eight".[12] It was apparently from such affirmations that mathematicians were overhasty in drawing their conviction that $(2^{p+1} - 1)$ is prime or not for $p = 8, 10, 22$, without undertaking the necessary proofs and without perceiving that they are in contradiction with their own calculations: there is no perfect number between 10 000 and 100 000.

In this way the history of number theory in Arabic mathematics, which we are attempting to constitute, is enriched with a chapter on perfect numbers. Theoretical research undertaken by Ibn al-Haytham, and by his contemporary al-Baghdādī, including the calculation of new perfect numbers, is evidence that, in spite of errors these numbers were the object of active research. Historical studies of the future will not fail to enrich this chapter whose existence we have just established. But as of now, Ibn al-Haytham by his discovery of Wilson's theorem and his statement of Euler's theorem, emerges as an important figure in the history of number theory.

NOTES

1. According to ancient bibliographers, in particular al-Nadīm, we know that al-Anṭakī (d. 376 H, i.e. 987) was a mathematician whose works deal mainly with arithmetic and number theory. Al-Nadīm attributed a commentary on Nicomachus of Gerasa's *Introductio Arithmetica* to him and a commentary on Euclid's *Elements*. If only for his dates the importance of al-Anṭakī's works for the history of numbers is understable. But his works remain undiscovered. We were fortunate enough to come across a later work that reproduces long passages from his commentary on the *Elements*. An analysis of the quoted fragments that are the only available texts of this author, show that it was an important work, comparable to works by al-Nayrīzī or his successor Ibn al-Haytham. In a later paper we shall re-examine these passages including the entire manuscript from which they are extracted (MS 992). For the present let us note that al-Anṭakī's interest in the study of prime numbers. He solved the following problem: to find two prime numbers a and b such that

$$a - s(a) = k,$$

$$s(b) - b = k',$$

and solves the case where

$$k = k'.$$

On the calculation of amicable numbers, if the author who consulted his text is to be believed, al-Anṭakī only gives the couple 220, 284; which indicates that, just like Thābit ibn Qurra, he was much more interested in theoretical research than in calculating new couples.

Apart from the passages by al-Anṭakī, the same manuscript contains fragments by al-Nayrīzī, al-Dimashqī, Ibn al-Haytham and also Ibn Hūd's book, *Al-Istikmāl*. Lastly, it gives us the calculation of the couple of amicable numbers 17296, 18416, named after Fermat which we found earlier in four different texts; which is proof that this couple was part of common knowledge of Arabic mathematicans from the late thirteenth century.

2. Ibn al-Haytham, *Sharḥ Muṣādarāt Uqlīdis*, Istanbul, MS Feyzullah 1359, f. 223^r.

3. Ibn al-Haytham, *Kitāb fī ḥall shukūk kitāb Uqlīdis fī al-Uṣūl*, Istanbul, University Library, MS 800, ff. 140^v – 142^r.
4. *Ibid.*, f. 142^r.
5. The translation of the Arabic text is based on all the available manuscripts and is published in Rashed (1991b).
6. 35 in the Heiberg edition.
7. 21 in the Heiberg edition.
8. This history is related in Rashed (1991b). Just note here that the paragraph marked by asterisks, should be inserted several lines before, i.e. before the sentence starting with "it follows that . . .". This point corresponds, moreover, to a break in the text.
9. Ibn Fallūs, *Kitāb Iʿdād al-Isrār fī Asrār al-Aʿdād*, MS Dār al-Kutub, 23317/B, ff. 62^r – 72^r; cf. f. 62^v.
10. *Ibid.*, f. 70^v.
11. Ibn al-Malik al-Dimashqī, *Al-Isʿāf al-Atamm bi-Ḥisāb al-Qalam*, Cairo, Riyāḍiyyāt, MS Dār al-Kutub, 182, f. 279.
12. Ibn Fallūs, *op. cit.*, f. 65^v.

APPENDIX I

THE NOTION OF WESTERN SCIENCE: "SCIENCE AS A WESTERN PHENOMENON"

Classical science is European and its origins are directly traceable to Greek philosophy and science. This tenet – for once an exception in the history of philosophy and science – has nevertheless survived intact despite numerous conflicting interpretations over the last two centuries. Almost without exception, philosophers accepted it as a postulate to characterize Classical Rationalism. Not only Kant but Comte, neo-Kantians as well as Neo-Positivists, Hegel as well as Husserl, Hegelians and phenomenologists as well as Marxists, acknowledged this postulate as the basis for their interpretations of Classical Modernity. The names of Bacon, Descartes and Galileo (the first name sometimes omitted and, depending on the circumstances, a number of others added) are still cited today as so many stages for the resumption of an advance interrupted by centuries of decadence, and as so many milestones along the path of a revolutionary return to Greek science and philosophy. The Platonic and Archimedian metaphors used by a Brunschvicg or a Koyré to characterize the modes of the existence of classical science bear witness to the fact that this return was understood by all to be, at one and the same time, the search for a model and the rediscovery of an ideal. One might impute this unanimity of philosophers to their approach which transcends immediate historical data, their concern for radicality, and their efforts at grasping what Husserl described as "the original phenomenon (*Urphänomen*) which characterizes Europe from the spiritual point of view"; and consequently, one might expect the stand taken by those directly in contact with the facts of the history of science to be entirely different. However, this is not the case: the same postulate is adopted by historians of science as a starting point for their work and especially their interpretations. In this respect, the difference between Poggendorff, Rosenberger, Dühring and Gerland on the one hand, and Duhem on the other in the history of physics, and between Tannery, Cantor and Bourbaki in the history of mathematics are infinitesimal. Whether the advent of classical science is interpreted as the product of a break with the Middle Ages, or, on the contrary, the thesis of continuity is defended, or, as is more frequently the case, an eclectic position is adopted, most historians agree in their almost implicit acceptance of this postulate.

Nowadays, despite works by Woepcke, Suter, Wiedemann and Luckey etc. on the history of Arabic science, the recent *Dictionary of Scientific Biography*, and Needham's research on the history of Chinese science, current historical works are based on a fundamentally identical concept. Moreover, though the very concept of the history of science and its methods have recently become the object of controversy and criticism, internalists and externalists, continuists and discontinuists, sociologists of science and conceptual analysts tacitly agree to leave the above doctrine outside discussion and consequently unquestioned. The same representation is encountered time and again: classical science, both in its modernity and historicity, appears in the final count as the work of European humanity alone; furthermore, it is essentially the means by which this branch of mankind is defined. In fact, only the scientific activities of European humanity are the objects of history. It is true that the existence of some scientific activity in other cultures is occasionally acknowledged. Nevertheless, it remains outside history or only integrated in so far as it contributed to science, which is essentially European; such contributions are merely additional techniques which in no way modify the intellectual configuration or the spirit of the latter.

The picture drawn of Arabic science is an excellent illustration of this approach: a museum of the Greek heritage, enriched by a few technical innovations or transmitted intact to the legitimate heirs of classical science. Without exception, scientific activity outside Europe, poorly integrated into the history of science, is the object of an anthropology of science whose academic translation is nothing more than Orientalism.

The consequences of this doctrine are not confined to the domain of science and its history and philosophy; its application in the nineteenth century is well-known. Similarly it is known to lie at the heart of a debate which bears the same name today as in the past: modernism *vs* tradition. In some Mediterranean and Asiatic countries in search of their identity today as was the case in eighteenth century Europe, science qualified as European is identified with modernism in the conflict that opposes the Ancients and the Moderns. When the historian of science questions the notion of Western science, he not only poses a problem for his discipline, but may also contribute towards answering a question of his own time. Let it be said straight away: our objective here is not to redress wrongs, nor contrast an alleged Oriental science with a science qualified as European. We just want to understand what the European determination of the concept of classical science implies, grasp its reasons and assess the importance of the specification, geographical at the least and doubtless anthropological, of a phenomenon whose definition necessarily requires universality.

To achieve this we shall therefore begin by sketching the present history of the notion of European science itself which, by all indications, stems from diverse and heterogeneous origins. Its subsequent confrontation including the doctrine it embodies with the facts of the history of science, will enable us to estimate its scope. For obvious reasons, we cannot claim to make an exhaustive study here, and even less, a definitive one. We shall limit ourselves to posing the problem and advancing several hypotheses with two restrictions however: the only non-European science under consideration here is the product of many cultures and of scholars of different beliefs and religions, all of whom wrote their science principally, if not exclusively, in Arabic; concerning the tenets of the history of science, those most cited are French historians.

The concept of European science is to be found in the works of eighteenth-century historians and philosophers. It then fulfilled two distinct but not unrelated functions: a means to define modernity in a dogmatic debate that persisted throughout the eighteenth century and also a component part of a naive diachrony whose aims remained polemic and critical. In the debate between the Ancients and Moderns, engaged early, to define modernity scholars and philosophers referred to a science, which combines reasoning and experiment: the preface to *Traité du vide* (Pascal, ed., 1963, p. 231) and, to some extent, *De la recherche de la vérité* (Malebranche, 1910, I, p. 139), are thus early seventeenth attempts at proving the superiority of the Moderns. Historical induction, or so-called historical induction, intended to provide this dogmatic debate with its concrete determination, thus rendering the superiority of the Moderns indisputable. This is also one of the reasons, and certainly not the least, why the history of science was introduced on the scene during the eighteenth century. But the West had already been identified with

Europe, and "Oriental wisdom" already contrasted with the natural philosophy of the post-Newtonian West, as in Montesquieu's *Persian Letters* (1721).[1]

Besides its critical and polemic role in a continuous and rebounding debate, the notion of Western science then assumed a function in the elaboration of history as the diachrony of the human spirit. It also intervened as a milestone in its own progressive movement, a movement regulated both by a cumulative order and a continuous elimination of its acquired errors. This is a brief outline of the representation given by a Fontenelle, a d'Alembert or a Condorcet. While the latter for example (like so many others subsequently), designated modernity by advancing the names of Bacon, Galileo and Descartes, he did so to name the transition from the "Eighth to the Ninth Epoch of the Historical Table" of humanity (Condorcet, 1966, p. 201) whose future merges with the indefinite becoming of the Age of Reason. Classical science is European and Western only insofar as it represents a stage in the continuous and normalized development of one and the same individuality: Humanity. For a Fontenelle, a d'Alembert or a Condorcet it would therefore be absurd to discover the origins of classical science in Greek science and philosophy alone: its qualification as European does not refer to any anthropology, but simply to a coincidence of empirical history and ideal history, the truth of the former. Of this conception Bossut's *Discours préliminaire* (1784, p. III) to the *Encyclopédie méthodique* offers an illustration, admittedly limited, for the history of science. The initial postulate of the historical table of progress for the exact sciences divided into three periods which confuses conjectures, alleged facts and real facts, is that ". . . all the eminent peoples of the ancient world appreciated and cultivated mathematics. The most renowned among them were the Chaldeans, the Egyptians, the Chinese, the Indians, the Greeks, the Romans, the Arabs, etc. in modern times, the Western nations of Europe." ". . . The progress accomplished by the Western nations of Europe in the sciences from the sixteenth century to our times utterly effaces those of other peoples".

While formulated in these terms in the eighteenth century, the notion of Western science changed in nature and scope at the turn of the nineteenth century. In short, with what Edgar Quinet called the "Oriental Renaissance"[2] (i.e. Orientalism) in the last century, the anthropological dimension completed its conceptualization which had been lacking up to that point. This Oriental Renaissance led to the discredit of science in the East, to which history-through-language added its alleged scientific support.

It is true that while the eighteenth-century concept still survived intermittently, notably among historians of astronomy, from the early nineteenth-century, the resources and ideas of Orientalism contributed most to the constitution of the historical themes of the various philosophies. In Germany as in France, philosophers of various tendencies staked themselves on Orientalism for diverse reasons to be sure, but in accordance with an identical representation: the East and West oppose each other not as geographical but as historical positivities. This opposition is not confined to a period of history but refers so to speak to the essence of each term. In this respect, Hegel's *Lectures on the Philosophy of History* (1963, pp. 82ff.; 1954, pp. 19–21) and other works like Joseph de Maistre's *Of the Pope* (1884, pp. 487ff.) may be invoked. At the same time, with the French Restoration philosophers and later the partisans of Saint-Simon, emerged the themes of the "Call of the Orient", the "Return to the Orient", which translated a reaction against science, and more generally, against Rationalism. But it was with the advent and growth of the German school of philology that the notion of science as a Western phenomenon was endowed with a "scientific" and no longer a purely philosophical basis, lacking until then.

The importance of this school for historical disciplines in general is well known, though its exact influence on the history of science is less known. However, everything indicates that its influence was not only direct but also indirect with the extension of this school into the study of mythology and religion. In any case, from the outset the works of Friedrich von Schlegel and Bopp in particular, placed the historian in a novel situation. From henceforth, his object constituted an irreducible totality in relation to its mode of existence and the nature of its elements. His method now compelled him to compare analogous totalities through the structures and functions they fulfilled. For Schlegel in 1808, as for Max Müller later, the model of historicity is natural history, and in linguistics, comparative grammar played the role of comparative anatomy. This method then led Schlegel to distinguish between two classes of language: flexional Indo-European languages and the others. The former are "noble", the latter less perfect. Sanskrit, and consequently German, considered the closest to it, is ". . . a systematic language and perfect from its conception"; it is ". . . the language of a people composed not of brutes, but of limpid intelligence".[3] There is nothing surprising in this statement; with the advent of the German school, we are already within the realm of the classification of mentalities. Neither von Schlegel, nor Bopp, nor Jacob Grimm later, would have disagreed with William von Humboldt when he saw language as the soul of a nation, its particular genius, its "*Weltanschauung*".

From now on the scene is set for the transition from the history of language to history-through-language.

Thanks to comparative philology and closely related to it, let us first note that the comparative study of religion and mythology was developed around the middle of the century by Alfred Kühn and Max Müller in particular. The classification of mentalities was perfected. On the basis of these tenets and dating from this period one of the most important attempts to establish the notion of science as Western and European in a so-called scientific manner was elaborated. Already apparent in Christian Lassen's synthetic work, this objective came into its own, this time in France, in the work of Ernest Renan.

Renan's avowed aim was to accomplish ". . . for the Semitic languages what Bopp had accomplished for Indo-European languages".[4] His task consisted of taking advantage of all the contemporary literature on comparative philology and mythology, in order to arrive at a so to speak eidetic description of the Semitic spirit and its manifestations in history. Now, for Renan like Lassen,[5] civilization is only shared by Aryans and Semites; the historian then only needed to evaluate their respective contributions in a differential and comparative way. From henceforth the notion of race constituted the foundation of historiography. But by "race" is merely understood the set of ". . . aptitudes and instincts recognizable solely through linguistics and the history of religion" (Renan, 1863, pp. 490–491). In the final analysis then, for reasons inherent in Semitic languages, the Semites unlike Indo-Europeans, did not and could not possess either philosophy or science. "The Semitic race", wrote Renan (1863, p. 16) "is distinguished almost exclusively by its negative features: it possesses neither mythology, nor epic poetry, nor science, nor philosophy, nor fiction, nor plastic arts, nor civil life". The Aryans, whatever their origin, define the West and Europe at one and the same time. In such a context Renan, who otherwise fought against miracles as a whole, nevertheless retained one: the "Greek Miracle."[6] As for Arabic science, it is, wrote Renan (1859, p. 89) ". . . a reflection of Greece, combined with Persian and Indian influences": in short, Arabic Science is an Aryan reflection.

Historians of science not only borrowed their representation of the Western essence of science from this tradition, but also some of their methods for describing and commenting on the evolution of science. For instance, they applied themselves to discovering the concepts and methods of science, followed their genesis and transmission through philological analysis of the terms and on the basis of the documents at their disposal. Like the historian of mythology or religion, the historian of science must be a philologist as well. Representations and methods that provide an anthropological basis of this concept of science as a Western phenomenon are from henceforth available. This was, for example, the situation of a Tannery, a Duhem or a Milhaud in France, all of whom borrowed Renan's representation and often, his own terminology as well![7] Even though a good number of historians had already abandoned this brand of anthropology, they nevertheless preserved and propagated a series of consequences it had engendered. These may be enumerated as follows:

1. Just as science in the Orient left no consequential traces in Greek science, Arabic science left none of consequence in classical science. It both cases, the discontinuity was such that the present could no longer recognize itself in its abandoned past.

2. Science after the Greeks was strictly dependent on them. According to Duhem (1965, p. 125), ". . . Arabic science only reproduced the teachings received from Greek science". In general terms, Tannery (1887, p. 6) recalled that the more one examines the Hindu and Arabic scholars, ". . . the more they appear dependent upon the Greeks . . . (and) . . . quite inferior to their predecessors in all respects".

3. Whereas Western science, in its origins as well as in the era of classical modernity, paid particular attention to theoretical foundations, Oriental science, even in its Arabic period, is defined essentially by its practical aims: it contrasts with the former, just as the science of an artisan attempting to dominate the rules of his art is contrasted with the knowledge of a philosopher-turned-scholar.

4. The hallmark of Western science in its Greek origins as well as in its modern renaissance, is its conformity to rigorous standards; in contrast, Oriental science in general, and Arabic science in particular, allows itself be carried away by empirical rules, calculatory methods, without checking the validity of each step as it proceeds. The case of Diophantus illustrates this idea perfectly: as a mathematician, wrote Tannery (1887, p. 5), ". . . Diophantus is hardly Greek". But when he

compared Diophantus' *Arithmetica* with Arabic algebra, he (1887) wrote that the latter ". . . in no way superseded the level attained by Diophantus".

5. The introduction of experimental norms which, according to historians, totally distinguishes Hellenistic science from classical science, is the achievement of Western science alone (Milhaud, 1893, p. 301).

So it is to Western science alone that we owe both the concept of science and the method of experimentation.

This concept of Western science, elaborated in the eighteenth century as an element of simple diachrony, was based on anthropology. If their origins were sometimes overlooked, these consequences still remained in force in the works of philosophers and historians, and particularly in works relating to classical science. We shall not oppose this ideology with another. We simply propose to confront some of these elements with the facts of the history of science, starting with algebra and concluding with the crucial problem of the relationship between mathematics and experimentation.

Neither algebra, nor other Arabic sciences, escapes the above characterization: practical aims, a calculatory appearance, an absence of rigour. It was precisely this that enabled Tannery to write that Arabic algebra had not attained the level of Diophantus. And, even more recently, this representation apparently authorized Bourbaki to exclude the Arabic period when he retraced the evolution of algebra. It goes without saying that we shall not engage in the discussion of contentious and – in our opinion – erroneous theses here, such as the existence of an algebraic theory in Diophantus' *Arithmetica*, or the existence of a geometric algebra, recognized as such, among the Greeks. We shall therefore confine our study to the problem of the Western nature of classical algebra. Has it not frequently been affirmed since Condorcet and Montucla up to Bourbaki, by way of Nesselman, Zeuthen, Tannery and Klein (to cite only a few names), that classical algebra is the work of the Italian School, perfected by Viète and Descartes? Did not Milhaud (1921) in the past, and Dieudonné (1974, I) more recently, trace the early history of algebraic geometry back to Descartes? The modern mathematician's text is, in this respect, significant: between the Greek prehistory of algebraic geometry and Descartes, Dieudonné finds only a void, which far from being frightening, is ideologically reassuring.

Despite the exemplary cases of Bourbaki and Dieudonné, some historians do occasionally cite al-Khwārizmī, his definition of algebra and solution of the quadratic equation, but usually to reduce Arabic algebra to its initiator. However, this restriction is serious and wrongs the history of algebra which is not a simple extension of al-Khwārizmī's algebra, but principally an attempt to surpass his achievements both theoretically and technically. Moreover, this attempt is not the result of the sum of individual works, but so to speak the outcome of genuine traditions active at that time. The first tradition conceived the precise objective of arithmetizing algebra inherited from al-Khwārizmī and his immediate successors. The second which overcame the obstacle of the solution by radicals of third and fourth-degree equations had, in its initial stage, formulated a geometric theory of equations before changing its viewpoint and studying known curves by means of their equations. In other words, this tradition engaged itself explicitly in the first research on algebraic geometry. Given these conditions, the traditional pattern of the history of algebra could only be a historical myth; its proof is furnished by recalling certain facts.

As we said, the first tradition set out to arithmetize inherited algebra. This theoretical programme was inaugurated at the end of the tenth century by al-Karajī, and was summarized by one of his successors, al-Samawʾal (d. 1174) as follows: "To operate on unknowns as the arithmeticians work on known quantities".

The orientation is clear and its execution was organized into two complementary stages: on the one hand, a systematic application of elementary arithmetic operations to algebraic expressions; on the other hand, algebraic expressions are considered independently from what they may represent so as to be able to apply them to operations hitherto restricted to numbers. Nevertheless, as we know, a programme is not defined by its theoretical aims alone, but also by the technical difficulties it must confront and resolve; one of the most important was the extension of abstract algebraic calculus. At this stage mathematicians in the eleventh and twelfth centuries had achieved some results which are unjustly and persistently attributed to mathematicians in the fifteenth and sixteenth centuries. Among those results may be cited: the extension of the idea of an algebraic power to its inverse after clearly defining the power of zero, the rule of signs in all generality, the binomial formula and the tables of coefficients, the algebra of polynomials, and above all, the algorithm of divisibility, and the approximation of whole fractions by elements of the algebra of polynomials.[8]

Subsequently, algebraists intended to apply this same extension of algebraic calculation to irrational algebraic expressions. The question al-Karajī asked was: "How to operate on irrational quantities using multiplication, division, addition, subtraction and extraction of roots?" In response to his question the mathematician gave, for the first time, apart from mathematical results, an algebraic interpretation of the theory contained in Book X of the *Elements*; now Pappus considered this book, as did a much later mathematician as important as Ibn al-Haytham, as a work on geometry owing to the basic traditional separation – in Aristotle as well as Euclid – between continuous and discontinuous magnitudes. With al-Karajī's school then a greater understanding of the structure of real algebraic numbers was achieved.

In addition, the works of this algebraic tradition paved the way for new research on number theory and numerical analysis (*supra*, pp. 85–146). An examination of numerical analysis, for example, reveals that after renewing algebra through arithmetic, mathematicians in the eleventh and twelfth centuries also effected a return movement to arithmetic to discover the applied extension of new algebra in some of its chapters. It is true that arithmeticians before the eleventh century and algebraists in the twelfth century extracted square and cubic roots and possessed formulas of approximation for the same powers. But for lack of abstract algebraic calculus, they were unable to generalize either their results, methods or algorithms. With the new algebra, the generality of algebraic calculus became a constituent part of numerical analysis which, until then, had only been a sum of procedures, if not prescriptions. It was in the course of this double movement between algebra and arithmetic that mathematicians of the eleventh and twelfth centuries achieved results still incorrectly attributed to mathematicians of the fifteenth and sixteenth centuries. For instance, the method attributed to Viète for the resolution of numerical equations, the method ascribed to Ruffini–Horner, general methods of approximation, in particular the one Whiteside designated as the al-Kāshī–Newton method, and lastly, the theory of decimal fractions. In addition to methods which were to be reiterative and capable of leading in a recursive way to approximations, eleventh and twelfth-century mathematicians also formulated new procedures of demonstration such as mathematical induction as it will still be found in the seventeenth century. Similarly, they engaged in new logicophilosophical debates: for example, the classification of algebraic propositions and the status of algebra in relation to geometry. It was their successors who were to tackle the problem of symbolism.

All of this is to say that certain concepts, methods and results attributed to Chuquet, Stifel, Faulhaber, Scheubel, Viète, Stevin, etc. were actually the work of this tradition of al-Karajī's school, known furthermore to Latin and Hebrew mathematicians.

We have just seen that one of the concepts elaborated by algebraist-arithmeticians at the end of the tenth century was that of polynomials. This algebraic tradition – the "arithmetic of unknowns" according to contemporary terminology –, paved the way for another algebraic tradition initiated by al-Khayyām (eleventh century) and renewed at the end of the twelfth century by Sharaf al-Dīn al-Ṭūsī. While the former formulated a geometric theory of equations for the first time, the latter left his decisive mark on the beginnings of algebraic geometry.

It is true that, unlike Alexandrian mathematicians, al-Khayyām's immediate predecessors such as al-Bīrūnī, al-Māhānī, and Abū al-Jūd, had already reduced problems of solids to third-degree equations, precisely through the concept of the polynomial. But it was with al-Khayyām (Woepcke, 1851) that these hitherto inconceivable questions were posed for the first time: can problems of straight lines, planes and solids be reduced to equations of corresponding degree on the one hand, and on the other, can the set of third-degree equations be re-aranged to seek, in the absence of a solution by radicals, solutions governed by means of the intersection of auxiliary curves? In answer to these perfectly delineated questions, al-Khayyām was led to formulate the geometric theory of third and lower degree equations. His successor, al-Ṭūsī, was soon to change his point of view: far from adhering to geometric figures, he thought henceforth in terms of functional relations and studied curves by means of equations. Even if al-Ṭūsī (*supra*, pp. 147–204) still solved equations by auxiliary curves for each case, nevertheless, the intersection of curves was proved algebraically by means of their equations. This is of capital importance, since the systematic usage of these proofs introduced the practice of instruments already available to tenth century mathematical analysts, so to speak: affine transformations, the study of the maxima of algebraic expressions, and with the aid of what was latter to be recognized as the derivative, the study of the upper and lower boundary of roots. It was in the course of these studies and the application of these methods that al-Ṭūsī grasped the importance of the discriminant of the cubic equation and gave locally the so-called Cardano formula just as it was found in the *Ars Magna*. Finally, without further developing the results obtained, we can say that both by the level of results as well as their style, we find al-Khayyām and al-Ṭūsī fully set in the field allegedly pioneered by Descartes.

Now, the exclusion of both traditions just examined, that of arithmeticians and geometers including analysts (before its time), from the history of algebra and the justification of this ban by invoking the practical and computational aims of Arabic mathematicians and an absence of rigorous standards of proof in their work is what permits the history of classical algebra to be written as the work of the Renaissance, which (to borrow Tannery's expression) culminated in the "Cartesian Revolution". The Western character of algebra then appears as the result of an oblique interpretation or truncated history, and sometimes both at once.

Among mathematical disciplines, algebra is neither a unique case, nor a privileged example. In varying degrees trigonometry, geometry, infinitesimal determinations and number theory are similarly illustrative of the above analysis. In a more general sense, optics, statics, mathematical geography, and astronomy are also no exception. For instance, recent works on the history of astronomy (some of which are in progress), render Tannery's understanding and interpretation of Arabic astronomers manifestly outdated, if not erroneous.[9] But since we assigned ourselves the task of examining the doctrine of the Western nature of classical science, we shall restrict our discussion to one essential component of this doctrine: experimentation.

Was the gulf between two periods of Western science – the Greek and the Renaissance – not often characterized by the introduction of experimental norms? General agreement between philosophers, historians and sociologists of science undoubtedly comes to a standstill at this point; the differences become apparent when they attempt to define the meaning, scope, and origins of experimental norms. For instance, in one case its origins are linked with the Augustinian-Platonic current, in a second with the Christian tradition and particularly the dogma of Incarnation,[10] in a third case with the engineers of the Renaissance, in a fourth with Bacon's *Novum Organum*, and lastly, in a fifth, with Gilbert, Harvey, Kepler and Galileo. Some of these attitudes were superimposed on each other, become entangled and contradictory, but all converge at one point: the Western nature of the new norms. Nevertheless, as early as the nineteenth century, some historians and philosophers such as Alexander von Humboldt in Germany and Cournot (1973, pp. 27–29) in France, diverge from this predominant position by attributing the origins of experimentation to the Arabic period. A correct analysis of the origins or beginnings of experimentation is not easy since no study of the interrelationship between various traditions and themes connected with the concept of experimentation exists. Perhaps such a history, before a history of the term itself, would enable us to account for the multiplicity of usages and the ambiguity of the concept. Two histories (as yet unwritten may I add) particularly are necessary for this analysis: a history of the relationship between art and science, and that of the links between mathematics and physics. At least until such studies are made, the problem of the origins of experimental norms will remain unsolved and subject to controversy. At most then, we can advance some hypotheses and invoke some facts, which will nevertheless suffice to show that the doctrine of the Western essence of classical science does not take objective history into account.

The history of the relationship between science and art places us in a position to understand when, why, and how it became accepted that knowledge may emanate from apodictic proofs and rules of practice, that a body of knowledge possesses the status of a science even though conceived in its possibilities of practical realization whose purpose is external. Now a weakening of the traditional opposition between science and art was apparently the work of all intellectual currents during the Arabic period. One fact stands out: whether Muslim traditionalists, rationalist theologians, scholars of different fields or even philosophers of the Hellenistic tradition, such as al-Kindī or al-Fārābī, they all somehow contributed toward a weakening of the traditional differentiation between science and art. In other respects this general trait undoubtedly stems from the no less general opinion of some historians regarding the practical spirit and realistic imagination of Arabic scholars. It is a fact that this new connection between science and art removed all obstacles from the introduction of the rules of Art, including its instruments, as objects of science and even more so of deductive reasoning. From now on knowledge may be accepted as scientific without conforming to either the Aristotelian or Euclidean pattern. The new concept of the status of science raised disciplines traditionally confined to the domain of art to the dignity of scientific understanding: for example, alchemy, particularly in the sense of Rhazes, medicine, pharmacology, music and lexicography. Nevertheless, whatever the importance of the new concept of the relationship between science and art, at most it could only lead to an extension of empirical research and a diffuse notion of experimentation. And in fact, one does witness at that time the multiplication and systematic application of empirical procedures: for example, classification in botany and linguistics, control experiments in medicine and alchemy, as well as clinical observations and comparative diagnoses by physicians. But until new links were established between mathematics and physics such a diffuse notion of experimentation could not acquire the dimension that determines it: a regulated and systematic component of proof. Ibn al-Haytham's work in the field of optics is where this new dimension primarily is seen to emerge.

It is generally accepted that the break with optics as the geometry of vision or light was definitely established with Ibn al-Haytham. Similarly, it is common knowledge that experimentation had indeed become a category of proof. Lastly, it is acknowleged that Ibn al-Haytham's successors, for example al-Fārisī, adopted experimental norms in their optical research (such as that performed on the rainbow). Now we must ask ourselves what Ibn al-Haytham understood by experimentation. In his work are to be found as many meanings of this term as many functions ensured by experimentation, as there are links between mathematics and physics. A close look at his writings will indicate that the term and its derivatives (to experiment, experimentation, experimenter) belong to several superimposed systems unlikely to be discerned by philological analysis alone. But if attention is fixed primarily on content rather than lexical form, one can distinguish several types of relationships between mathematics and physics, enabling us to pinpoint the corresponding functions of the idea of experimentation. In fact the links between mathematics and physics are established according to several modes, even if not specifically treated by Ibn al-Haytham; they underlie his work and lend themselves to analysis.[11]

For geometrical optics, reformed by Ibn al-Haytham himself, the only link between mathematics and physics is an isomorphism of structures. With his definition of a ray of light in particular, Ibn al-Haytham was able to formulate his theory of the phenomena of propagation, including the important phenomenon of diffusion, so that they perfectly related to geometry. Various experimental set-ups were then devised to ensure a technical check of the propositions controlled earlier on the linguistic plane by geometry. For instance, we have experiments designed to prove the laws and rules of geometrical optics. A further look at Ibn al-Haytham's work attests, moreover, to two important and often insufficiently stressed facts: firstly, some of Ibn al-Haytham's experiments were not simply designed to verify qualitative assertions but also to obtain quantitative results; secondly, the varied and for the age complex apparata devised by Ibn al-Haytham, were not limited to that of the astronomers.

Another type of relation between mathematics and physics is encountered in physical optics and, consequently, a second meaning of experimentation. While not opting for an atomistic theory, Ibn al-Haytham, for the sake of his own reform of geometrical optics, stated that light, or as he wrote: "the smallest of the lights", is a material being external to vision, which moves in time, changes its velocity according to its medium, follows the easiest path, and diminishes in intensity according to its distance from its source. At this stage mathematics were introduced into physical optics by means of analogies established between the movement patterns of a heavy body and those of reflection and refraction. In other words, mathematics were introduced into physical optics through the intermediary of the dynamic patterns of the movement of heavy bodies, themselves supposed already mathematized. This earlier mathematical treatment of physical notions was what enabled them to be transferred to the experimental plane. Although this situation might be somewhat approximate in nature and only indicative in function, it nevertheless furnished a level of existence for syntactically structured ideas: for instance, Ibn al-Haytham's pattern of the movements of a projectile, which was later taken up again by Kepler and Descartes.

A third type of experimentation, not practised by Ibn al-Haytham himself, but made possible by this reform and discoveries in optics, arose in the early fourteenth century in the work of his successor al-Fārisī. In this case, the links between mathematics and physics aimed at reconstructing a model and, consequently, by geometric means systematically reduce the propagation of light from a natural to an artificial object. The problem is therefore to define, for propagation, an analogic correspondence between natural and artificial objects which is truly ensured of mathematical status. For instance, the model of a glass sphere filled with water to explain the rainbow. Here the function of experimentation is to reproduce the physical conditions of a phenomena that cannot otherwise be studied directly or in detail. Two other types of experimentation can be added to the three cited above whose examination would involve a detailed exposition which we shall pass over in silence here. Despite their various functions, let us simply note that, unlike traditional astronomical observations, all three types of experimentation reveal not only a means of control, but also furnish a plane of existence for syntactically structured notions. In all three cases, it concerns situations where the scientist intends to realize his object physically in order to conceive it briefly as a means of realizing physically an object of thought unrealizable before. For instance, in the most elementary example of rectilinear propagation, Ibn al-Haytham did not consider any arbitrarily chosen aperture in a black box, but rather specific ones in accordance with specific geometric relations in order to realize as precisely as possible his concept of a ray.

Ibn al-Haytham's reform including the requirement of experimental norms as an integral part of proof outlived him. The genealogical succession from Ibn al-Haytham to Kepler and into the seventeenth century is established. Once again, the tenet of Western classical science is seen to lead, as clearly as in algebra, to a truncation of objective history through choices that must indeed be qualified as ideological.

In conclusion, let us recall three points:

1. Launched in the eighteenth century, the tenet of the Western nature of classical science for establishing a diachrony of Universal reason, owes to nineteenth-century Orientalism the picture we know today. It was then believed that one could deduce from cultural anthropology that classical science is European and that its origins are directly traceable to Greek science and philosophy.

2. On the one hand, the opposition between the East and the West underlies the criticism of science and rationalism in general; on the other, it excludes *de facto* and *de jure* the scientific production of the Orient from the history of science. To justify the exclusion of science written in Arabic from the history of science, one invokes its absence of rigour, its calculatory appearance and its practical aims. Furthermore, strictly dependent on Greek science and, lastly, incapable of introducing experimental norms, scientists of that time were relegated to the role of conscientious guardians of the Hellenistic museum. Though attenuated in the course of this century, particularly during the last twenty

years, this picture of Arabic science persists nonetheless in the ideology of historians.

3. Confronted with the facts, this tenet reveals a disdain for historical data and a capacity for ideological interpretation; notions that raise more issues than they solve are accepted as evidence. We thus have the notion of a Scientific Renaissance, though in many disciplines everything indicates that there was at most only reactivation. As recent endeavours show, such elements of pseudo-evidence were soon to provide the conceptual basis of a philosophy or sociology of science, as well as a starting point for theoretical elaboration in the history of science.

We must now ask ourselves, though rather unoptimistically, if the time is not ripe to abandon an anthropological characterization of classical science and its still lively traces in historical writings, to give back to the profession of historian of science the objectivity it requires, to discontinue the clandestine import and circulation of uncontrolled ideologies, to refrain from reductionist tendencies whatever their origin that favour similarities at the expense of differences, to be wary of writing history that relies on miracles (Greek for most historians, Arabic for Sarton). In short, if the moment has not come to write history without recourse to false evidence whose nationalistic motivations are barely concealed. The neutrality of the historian, a condition for a theoretical elaboration in the history of science, is not an *a priori* ethical value. It can only be the product of painstaking work, undeceived by myths engendered by the East-West couple. Above all, commonly accepted periodization in the history of science must be drastically changed. A new, differential periodization will break with the general history of science and refuse an unfounded identification between logical and historical time. It will place works written between the tenth and seventeenth centuries under the same heading of classical algebra or classical optics, for example. Consequently, not only the notion of classical science, a notion whose elements are heterogenous and situated on varying planes, but also that of medieval science, will be re-aligned. Classical science will reveal itself to be what it has never ceased to be: a product of the Mediterranean, not as such, but as the hub of exchanges between civilizations at the centre and on the outskirts of the ancient world. Only then can the historian of science clarify a debate currently involving several countries of the ancient world and lying at the heart of their cultures today, the debate between modernism and tradition.

NOTES

1. Montesquieu (1964). See Letters 104 and 135, in particular, Letter 97.
2. The title of a chapter of Quinet's *Génie des religions* (1841).
3. Schlegel (1837). Remember that, for Schlegel (1837, p. 51), language is divided into two types, flexional and inflexional languages, which "completely exhaust the entire range of language". According to him (pp. 54–61), Semitic languages are not flexional since the flexional structure based on roots was a borrowing.

   As for Indo-Germanic languages (Schlegel, 1937, p. 79), "they require the clearest and most penetrating intelligence as they express the highest notions of pure and universal thought as well as the entire range of consciousness".
4. Renan (1863, p. IX). Renan adhered to the vision and expression of German linguists. He wrote, for instance (1863, p. 18): "The unity and simplicity which characterize the Semitic race, are found in Semitic languages themselves. Abstraction is unknown to them and metaphysics impossible. As language is a necessary mould for the intellectual activities of a people, an idiom almost bereft of syntax, without variety of construction, deprived of conjunctions that establish such delicate relations between members of thought, that depict objects by their external qualities, must be eminently suited to the eloquent inspiration of visual thinkers and the image of fugitive impressions, but must reject any philosophy, any intellectual inspiration".

   Further on, we read (p. 22): "We may say that Aryan languages compared with Semitic languages are the languages of abstraction and metaphysics compared with those of realism and sensitivity".
5. Lassen (1847, I, pp. 414ff.). See, for example, p. 415: "Auch die Philosophie gehört den Semiten nicht, sie haben sich, und zwar nur die Araber, bei den Philosophen der Indogermanen eingemiethet. Ihre Anschauungen und Vorstellungen beherrschen ihrer Geist zu sehr, als dass er sich zum Festhalten des reinen Gedankens ruhig erheben und das allgemeine und nothwendige von seiner eigenen Individualität und deren Zufähligkeiten trennen könnte".
6. Milhaud (1893, p. 306) cited Renan as follows: "As for miracles, as Mr Renan said recently at the banquet of the Association of Greek Studies, there exists one in history: Ancient Greece. There is no doubt that around 500 B.C. a type of civilisation had achieved such perfection and accomplishment that all its predecessors faded into obscurity. This was truly the birth of reason and liberty". See also Renan (1883, p. 59).
7. See, e.g. Duhem (1965, II, p. 126) where he mentions "the realistic tendencies of the Arabic imagination".
8. See Woepcke's works, Anbouba's edition of al-Karajī's *al-Badī'* and our various studies on the history of this school of algebraists.
9. See. in particular, Carra de Vaux's translation of "Les sphères célestes selon Nasīrū Eddīn Attūsī", published in Tannery (1883, App. VI, pp. 337–361).
10. This point of view is illustrated by the Hegelian Alexander Kojève (1964, II, pp. 295–306).
11. See works by Wiedemann, M. Nazīf, Schramm, Sabra and ourselves on Ibn al-Haytham and al-Fārisī.

APPENDIX 2

PERIODIZATION IN CLASSICAL MATHEMATICS

The dichotomy between "medieval" and "modern" has dominated and continues to dominate the history of mathematics. At least until the mid-seventeenth century this dichotomy was considered an indispensable tool for periodization and, consequently, one of the pivots around which the diachrony of mathematics revolves. It is therefore understandable that for all sides involved in the famous debate on concepts and methods in the historiography of science still alive today, it is considered a postulate, or rather an assumption prior to any historical periodization of mathematics. To be convinced of the fact one only needs to consult historical compendia such as those of Montucla or Cantor, or more recent synthetical studies such as, for example, that of Bourbaki, or specialized monographs by Zeuthen in the past or Youschkevitch today. They all accept this postulate in order to apprehend, as they readily say, the rise of modern mathematics and agree to introduce an additional notion, that of Renaissance mathematics, which raises more problems than it solves. Therefore nothing could be more natural than a study of the dichotomy between "medieval" and "modern" in the light of progress accomplished during the last few decades in our knowledge of Arabic mathematics between the ninth and sixteenth centuries and the Latin world after the twelfth century.

Let us stop to consider this dichotomy itself. In works on the history of mathematics this well-known opposition between "medieval" and "modern" is not only used to mark stages in chronology, but is also invoked, deliberately or not, to designate two distinct mathematical positivities. It is relatively unimportant then that "modern mathematics" is considered as a radical departure from medieval mathematics, or its natural development, or even as a direct continuation of Hellenistic mathematics, thus completely bypassing medieval mathematics. Though at first glance contradictory and exclusive, such opinions do however converge when medieval mathematics is seen as constituting not only a historical entity, but standing in opposition to another entity created at the Renaissance.

But, whatever position is chosen, one is soon confronted with major difficulties. The first arises from the meaning of the term "medieval". Is one entitled to designate under this same heading Latin, Byzantine and Arabic mathematics, to mention only these? Is one justified in grouping them together by associating them with Chinese and Indian mathematics as some eminent historians do, on the pretext that they are contemporary to one another? Moreover, it is not uncommon that the historian, though without clearly formulating the problem, quite often avoids it altogether by writing separate histories of each of these three – or five – mathematics in turn. But as he is not entitled, either *de jure* or *de facto*, to presuppose a generically common feature for medieval mathematics as a whole, the dichotomy collapses, and with it the suggested periodization.

But the question raised by the use of the term "medieval" is more formidable when it concerns the same mathematics. One will look in vain for reasons to justify applying it, for example, to Leonardo of Pisa in the twelfth century, but not Luca Pacioli at the end of the fifteenth; or to al-Karajī at the end of the tenth, but not al-Yazdī at the end of the seventeenth.

Such questions, of an apparently methodological nature, turn out to be both historical and epistemological. And to clarify this dichotomy requires a better knowledge of the components of medieval mathematics and their basic features. These are precisely some of the aspects I wish to outline briefly here for the single case of mathematics written in Arabic between the ninth and seventeenth centuries.

Let us start with Baghdad in the early ninth century. The work of translating the great compositions of Hellenistic mathematics was at its peak. Two important, though underemphasized, aspects of this work are of prime importance: the translations were conducted by scientists and mathematicians, often of outstanding ability, such as Thābit ibn Qurra, for example, and they were stimulated by the most advanced research of their age. For instance, everything indicates that Qusṭā ibn Lūqā's translation of Diophantus' *Arithmetica* around 870, was stimulated by earlier research on indeterminate analysis or rather rational Diophantine analysis. Even the translation of *Burning Mirrors* by Diocles or Anthemius of Tralles had to satisfy the requirements of research in this field. We could multiply such examples, all of which reveal the close ties between translation on the one hand, and research and innovation on the other. To overlook this aspect of the question is to refrain from understanding the properties of these translations, as well as the circulation of translated knowledge and the ways and means for innovation.

In short, the work of translation represented a great moment in the expansion of Hellenistic mathematics in Arabic. Now, it was precisely at this time and in the same milieu – the Academy of Baghdad (the House of Wisdom) – that al-Khwārizmī composed a work whose subject matter and style were new. For the first time algebra emerged as a distinct and independent branch of mathematics. This was a crucial event and recognized as such by his contemporaries, not only by the style of this mathematics but also the ontology of its object, and above all, by the possibilities it opened up for the future. The style is in fact both algorithmic and demonstrative. Moreover, it was necessary to conceive a mathematical being general enough to receive determinations of different kinds, yet whose existence is independent of its own determinations. In al-Khwārizmī's earlier work (around 830), the algebraic object referred not only to rational numbers but also to irrational quantities or geometrical magnitudes as well. Algebra as a science had to be both apodictic and applied. This new ontology, including the combination of demonstrative and applied methods, impressed contemporary philosophers. Without further pursuing the point, let us just note that with this algebra one catches a glimpse of the incredible potential available to mathematics after the tenth century: the application of mathematical disciplines to each other. In other words, if algebra, due to the general nature of its object and the ontology thus introduced, made such applications possible, the number and differing nature of these applications will increasingly modify the configuration of mathematics after the tenth century.

Al-Khwārizmī's successors undertook a systematic application of arithmetic to algebra, algebra to arithmetic, both of them to trigonometry, algebra to the Euclidean theory of numbers, algebra to geometry, and geometry to algebra. These applications were always the starting point for new disciplines, or at least new topics. This was how the creation of polynomial algebra, combinatorial analysis, numerical analysis, the numerical solution of equations, the new elementary theory of numbers, and the geometric construction of equations arose. Other results should be included, such as the separation of Diophantine integer analysis from rational Diophantine analysis, now an independent chapter of algebra entitled "indeterminate analysis".

So from the ninth century onwards the mathematical scene was different; it had changed and its horizon had expanded. In the first place, we witness the extension of Hellenistic arithmetic and geometry. The theory of conics, the theory of parallels, Euclidean number theory, Archimedian methods for measuring surfaces and volumes, and isoperimetric problems: all these fields were to become subjects of study for the greatest names in mathematics -- Thābit ibn Qurra, al-Qūhī, Ibn al-Haytham, just to name a few – who by thorough research succeeded in developing them in the same style as that of their predecessors. Secondly, non-Hellenistic areas were carved out within the field of Hellenistic mathematics. As we have seen, algebraic methods provided the means to study not only first arithmetical functions but also sequences of figurative numbers, by promoting the creation of a new area in Euclidean number theory. Lastly, if we add the recently created disciplines mentioned above to the extension of the same field and the creation of new areas within it, a new picture of mathematics is seen to emerge. More detailed analysis will reveal a change in the relationship between ancient disciplines and the appearance of many other combinations. This change in relationship is of prime importance if the history of mathematics is to be understood. Let us take Book X of Euclid's *Elements* as an example. A work on geometry for Euclid, Pappus and even Ibn al-Haytham from the tenth century onwards, it came to be seen as a work on algebra concerned with, to use another language, finite algebraic extensions in the field of rational numbers.

The introduction of a new kind of proof – algebraic proof – hitherto unknown, was just as important as changes in the mathematical scene. In fact, even if the Euclidean or Archimedian type of geometrical proof continued to prevail, algebraic proofs began to dominate areas such as polynomial algebra, combinatorial analysis and the new number theory. Moreover, this proof alone was used to justify algorithms for solving algebraic or numerical equations.

Other important techniques came to light at the same time; for example, the study of local analysis will help us define this mathematics more clearly. To understand the rise of the local point of view, let us return to the dialectic between algebra and geometry, mentioned earlier.

Without any theoretical justification, tenth-century mathematicians initiated an entirely novel, dual translation: they translated solid problems, non-constructible with ruler and compass, into algebraic form, in particular the trisection of an angle, the two means and the regular heptagon.

At the same time some mathematicians and astronomers proceeded with this algebraic translation to determine the chords of angles in order to construct the table of sines. Al-Māhānī, al-Khāzin and al-Bīrūnī, among others, are landmarks in this trend of algebraic translation. These are therefore the main aspects of the dialectic between algebra and geometry. To be comprehensive we should also mention two obstacles which slowed down the advancement of the new mathematics by hampering, in particular, the expansion of local analysis: a lack of audacity in the use of negative numbers as such, even though they remained undefined, and a weakness of algebraic symbolism. All these questions will be subjects of concern for later mathematicians.

If we endorse the above analysis, then nothing justifies the creation of separate classifications for work accomplished during the ninth century on the one hand, and work produced later on in the early seventeenth century. In fact, everything suggests that it was basically the same mathematics. To be convinced of this we only need, for example, to draw a comparison between al-Samawʾal and Simon Stevin for algebra and numerical analysis; al-Fārisī and Descartes for number theory; al-Ṭūsī and Viète on methods for the numerical solution of equations; al-Ṭūsī and Fermat on research on *maxima*; al-Khāzin and Bachet de Méziriac on Diophantine integer analysis, etc. If, on the other hand, we disregard the work of al-Khwārizmī, Abū Kāmil, al-Karajī, to name only them, how is it possible to understand not only the work of Leonardo of Pisa and Italian mathematics, but also later seventeenth-century mathematicians?

The rift was neither necessarily sudden, nor did it occur simultaneously in all branches of mathematics. On the other hand, the lines of cleavage seldom encompass authors, but often run right across their work. Therefore, the new number theory did not commence with Descartes' and Fermat's theory of algebraic method as has been claimed; by proceeding in this way they merely rediscovered al-Fārisī's results. On the contrary, the origins of the new theory are to be found in the application of purely arithmetic methods, namely when Fermat, around 1640, invented the "infinite descent" and embarked on the study of some quadratic forms. Therefore the rift occurs in the heart of Fermat's work, somewhere around 1640. The situation is quite different concerning the geometric construction of equations: it was initiated by al-Khayyām, pursued by al-Ṭūsī, developed by Descartes and taken up by many other mathematicians at the end of the same century and even right up to the middle of the next century.

It is obvious that the frontiers between these different periods are hazy and interwoven and, of course, neatly defined periods exist only "in textbooks", as Alexandre Koyré once wrote. The first half of the seventeenth century is when this rift occurs and the dividing lines intertwine.

Historical and epistemological analysis that takes Arabic mathematics into account as well as its usage in Latin leads to the elaboration of a coherent picture of the configuration of mathematics between the ninth and seventeenth centuries. But the division of this configuration, and consequently, the resulting periodization cannot be adapted to the framework of the dichotomy between "medieval" and "modern". In the final analysis, periodization is seen as an inadequate transfer from political history which is doubly out of step in relation to the history of mathematics. The only periodization truly based on historical facts must surely be a differential one.

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See also Juschkewitsch.

INDEX

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