UNIVERSITY OF CALIFORNIA PUBLICATIONS
IN
AGRICULTURAL SCIENCES

Vol. 4, No. 9, pp. 233-245, 3 figures in text December 20, 1921

THE ALINEMENT CHART METHOD OF PREPARING TREE VOLUME TABLES[1]

BY
DONALD BRUCE

The chief use of the alinement chart[2] is to express with the simplest possible system of lines a law the equation of which is known. The underlying principle is so flexible that almost any formula can be expressed thereby, although the most striking advantages over a system of rectangular coördinates do not appear unless three or more variables are involved. The axes may be parallel or converging, straight or curved, and graduated either uniformly or with intervals which vary in accordance with some given law, the form of the graph depending on the form of the equation. It follows from this very flexibility that such charts are, in general, unsuited for use with empirical data. The following pages, however, describe an exception to this general rule in which one type of alinement chart may be advantageously used in the preparation of tree volume tables, although the form of the equation of such a table is yet unknown.

The most suitable type of chart can be determined by working out an approximate algebraic expression for the volume of a tree in terms of its diameter and height. This expression is complicated by the fact that, for almost all American tables, volumes must be computed in board feet as scaled by some log rule, instead of in cubic feet. The starting point must therefore be the equation of the volume of a

[1] Acknowledgment is made to Professor Frank Irwin, of the Department of Mathematics of this University, for assistance in connection with the analytic features of this study.

[2] A complete discussion of the theory of alinement charts may be found in such works as: J. Lipka, *Graphical and Mechanical Computation;* J. B. Peddle, *The Construction of Graphic Charts;* and M. d'Ocagne, *Traité de Nomographie.* For a discussion of the application of certain simple types to some formulae of forest mensuration see D. Bruce, ''Alinement Charts in Forest Mensuration,'' *Journal of Forestry,* XVII, 7, 773 (November, 1919).

log in board feet first formulated by Professor Daniels,[3] i.e., $v = ad^2 + bd + c$ (I). For simplicity, let us apply this formula to a tree of uniform taper up to its merchantable top limit, that is, one which is a frustum of a cone.

Let $V =$ volume of a tree in feet b. m.
Let $v =$ volume of a log in feet b. m.
Let $D =$ d. i. b. stump (assumed equivalent to d. b. h.).
Let $d =$ top diameter of a log.
Let $H =$ height in logs of tree.
Let $t =$ top d. i. b. of tree.

It is evident that the taper of the tree $= D - t$, and that the taper per log $= \frac{D-t}{H}$. Therefore the top diameters of the several logs of the trees are the terms of the following series: t, $t + \frac{D-t}{H}$, $t + \frac{2(D-t)}{H}$, $t + \frac{3(D-t)}{H}$, ... to H terms, and the volumes of the same are the terms of the following series, each top diameter being successively substituted in I:

$$at^2+bt+c;$$

$$at^2 + \frac{2at}{H}(D-t) + \frac{a}{H^2}(D-t)^2 + bt + \frac{b}{H}(D-t) + c;$$

$$at^2 + \frac{4at}{H}(D-t) + \frac{4a}{H^2}(D-t)^2 + bt + \frac{2b}{H}(D-t) + c;$$

$$at^2 + \frac{6at}{H}(D-t) + \frac{9a}{H^2}(D-t)^2 + bt + \frac{3b}{H}(D-t) + c;$$

$$at^2 + \frac{8at}{H}(D-t) + \frac{16a}{H^2}(D-t)^2 + bt + \frac{4b}{H}(D-t) + c;$$

.... to H terms.

$V =$ the sum of this series to H terms. This may be obtained by the differential method in which a new series of first differences is derived by subtracting each term from that which follows it, and this process is repeated, successively obtaining a series of second differences, third differences, etc., until all terms become zero and the series vanishes.

[3] See A. L. Daniels, *Measurement of Sawlogs*, Vermont Agr. Exp. Sta., Bulletin 102, 1903.

The sum then equals $na + \frac{n(n-1)}{\lfloor 2} d_1 + n \frac{(n-1)(n-2)}{\lfloor 3} d_2 +$ .... where n = number of terms, a = the first term of the original series, d_1 = the first term of series of first differences, d_2 = the first term of series of second differences, etc.

Series of first differences is:

$$\frac{2at}{H}(D-t) + \frac{a}{H^2}(D-t)^2 + \frac{b}{H}(D-t);$$

$$\frac{2at}{H}(D-t) + \frac{3a}{H^2}(D-t)^2 + \frac{b}{H}(D-t);$$

$$\frac{2at}{H}(D-t) + \frac{5a}{H^2}(D-t)^2 + \frac{b}{H}(D-t);$$

$$\frac{2at}{H}(D-t) + \frac{7a}{H^2}(D-t)^2 + \frac{b}{H}(D-t); \text{ etc.}$$

Series of second differences is:

$$\frac{2a(D-t)^2}{H^2}; \quad \frac{2a(D-t)^2}{H^2}; \quad \frac{2a(D-t)^2}{H^2}; \text{ etc.}$$

Series of the third differences is:

o; o; o; etc.

And V = sum of this series = $H(at^2 + bt + c) + \frac{H(H-1)}{2}$

$$\left\{ \frac{2at}{H}(D-t) + \frac{a}{H^2}(D-t)^2 + \frac{b}{H}(D-t) \right\} + \frac{H(H-1)(H-2)}{6}$$

$$\left\{ \frac{2a}{H^2}(D-t)^2. \right\}$$

Expanding and rearranging in terms of D, this becomes:

$$V = \left\{ \frac{a}{6}\frac{(H-1)(2H-1)}{H} \right\} D^2 + \frac{(H-1)}{6H} \left\{ (2at+3b)H + 2at \right\} D$$

$$+ \left\{ \frac{(2at^2 + 3bt + 6c)H}{6} + \frac{(3at^2 + 3bt)}{6} + \frac{at^2}{6H} \right\} \qquad \text{(II)}$$

Rearranging in terms of H, this may also be written:

$$V = \frac{H}{6}\left\{2aD^2 + (2at + 3b)\,D + 2at^2 + 3bt + 6c\right\} - \tfrac{1}{2}\left\{aD^2 + bD - t\,(at + b)\right\} + \frac{a}{6H}\,(D^2 - 2tD + t^2) \qquad \text{(III)}$$

Let us now apply this general formula to a specific case, for example, that of trees scaled by the Scribner log rule to a six-inch top cutting limit. A close approximation formula for this log rule (for 16-foot lengths) is:

$$V = .765d^2 - .55d - 21.$$

We therefore may assume

$$\begin{aligned} a &= .765 \\ b &= -.55 \\ c &= -21 \\ t &= 6. \end{aligned}$$

Substituting these values in III, we have:

$$V = H\,(.255D^2 + 1.255D - 13.47) - (.3825D^2 - .275D - 12.12) + \frac{1}{H}\,(.1275D^2 - 1.53D + 4.59). \qquad \text{(IV)}$$

Typical equations for the height class curves of a volume table in graphic form can now be found by substituting in IV given values of H; for example:

$$\text{For } H = 2,\ V = .19125D^2 + 2.02D - 12.52 \qquad \text{(V)}$$

$$H = 6,\ V = 1.16875D^2 + 7.55D - 67.93 \qquad \text{(VI)}$$

$$H = 10,\ V = 2.18025D^2 + 12.672D - 122.121 \qquad \text{(VII)}$$

Similarly, typical diameter class curves are:

$$\text{For } D = 10,\ V = 24.58H - 23.38 + \frac{2.04}{H} \qquad \text{(VIII)}$$

$$D = 20,\ V = 113.63H - 135.4 + \frac{25.99}{H} \qquad \text{(IX)}$$

$$D = 30,\ V = 253.68H - 323.88 + \frac{73.44}{H} \qquad \text{(X)}$$

$$D = 50,\ V = 686.78H - 930.38 + \frac{246.84}{H} \qquad \text{(XI)}$$

It will readibly be seen that V, VI, and VII are equations of parabolas, while VIII and IX and X and XI are hyperbolas. These deductions agree so well with the actual results obtained in volume tables constructed by the conventional method on a similar basis that it seems probable that the general form of the equation should apply at least approximately to actual trees as well as to the cone frusta on which it is based. Furthermore, it has been tentatively established, and without any conflicting evidence coming to light, that, in the case at least where a fixed top cutting limit is used, frustum form factors are functions of diameter and not of height. If this is true, such equations as VIII can be corrected to apply accurately to any given species by multiplying into them the proper form factors, which would merely change the values of the constants without affecting the form. Finally, the ease with which the alinement chart devised to apply to cone frusta works out for actual trees is the best proof of the adequacy of the equation.

Next, it is necessary to determine this alinement form. Unfortunately a difficulty at once presents itself. The equation appears to be one of those rare instances which cannot be thus expressed.[4] It has been found by experiment, however, that if two parallel axes be assigned to V and H, the former graduated uniformly upward and the latter uniformly downward, all lines expressing a single value of D (taken from a table of values of volumes of cone frusta in board feet or calculated by the above formulae) will intersect nearly (but not quite) in a common point, and that these common points for a series of values of D lie almost (but not quite) in a straight line, which if produced will pass through the zero point on the V axis. Figure 1 illustrates this fact, although a larger scale is needed to bring out the failure of the lines to intersect perfectly.

The reason for this becomes evident upon analysis. Let the lower left-hand corner of figure 1 serve also as the origin of a system of rectangular coördinates with one unit equaling ten of the small squares. Also let b equal the width of the paper. Then any straight line used in solving values by the alinement chart can be expressed

[4] Only those equations can be expressed by an alinement chart that can be put in the determinant form

$$\begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} = 0$$

where f_i, g_i, and h_i are functions of x, $(i = 1, 2, 3)$.

as connecting the points $(0, 22 - 2H)$ and $\left\{ b, \frac{V}{100} \right\}$ or by the equation

$$Y = \left\{ \frac{V}{100} - 22 + 2H \right\} \frac{X}{b} + 22 - 2H. \qquad \text{(XI)}$$

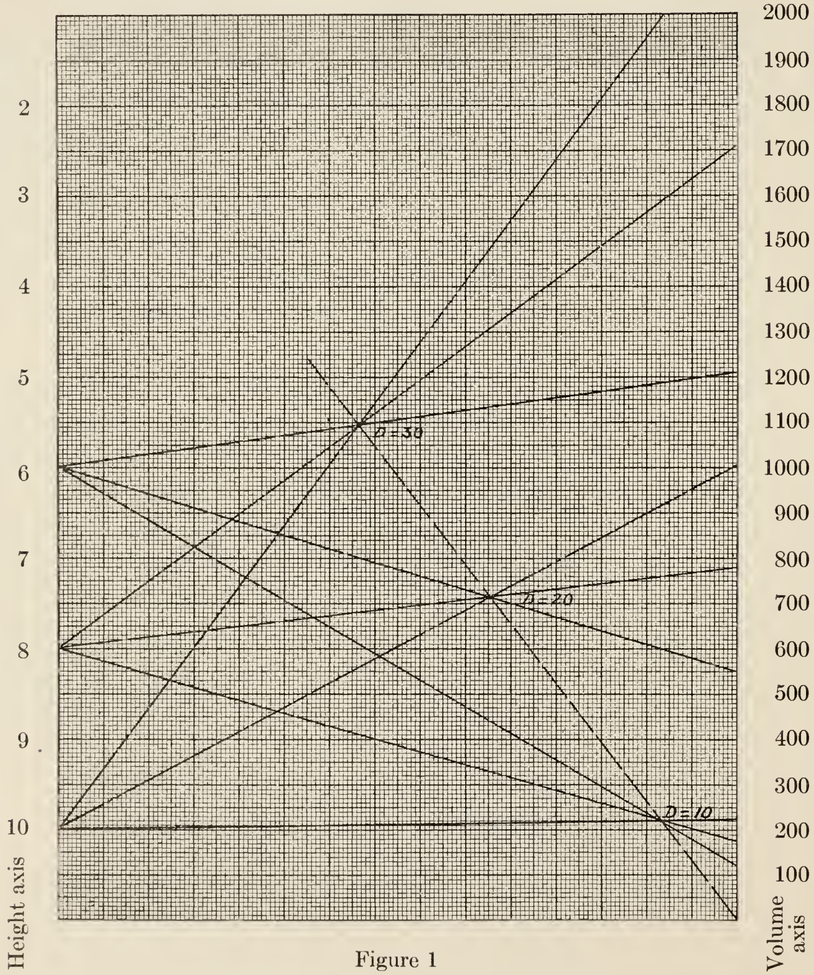

Figure 1

Alinement Form giving approximately correct results for volumes of cone frusta in feet b.m.

Now from equation IV, $V = AH + B + \frac{C}{H}$ (where A, B, C are functions of D) and the equations of two such lines corresponding to any values of H, such as H_1 and H_2, and having a common value of D, will then be (from equation XI):

$$Y = \left\{ \frac{AH_1 + B + \dfrac{C}{H_1}}{100} - 22 + 2H_1 \right\} \frac{X}{b} + 22 - 2H_1$$

and

$$Y = \left\{ \frac{AH_2 + B + \dfrac{C}{H_2}}{100} - 22 + 2H_2 \right\} \frac{X}{b} + 22 - 2H_2$$

The point of intersection of these two lines can now be found by solving these two equations simultaneously, and when this is done the following values of X and Y are obtained:

$$X = \frac{200b}{A - \dfrac{C}{H_1 H_2} + 200} \qquad \text{(XIII)}$$

$$Y = \frac{22\left\{ A - \dfrac{C}{H_1 H_2} \right\} + 2B + \dfrac{2C}{H_1} + \dfrac{2C}{H_2}}{A - \dfrac{C}{H_1 H_2} + 200} \qquad \text{(XIV)}$$

It will be seen that, for such a range of values for D and H as are actually encountered, these two equations approximate quite closely to

$$X = \frac{200b}{A + 200} \qquad \text{(XV)}$$

$$Y = \frac{22A + 2B}{A + 200} \qquad \text{(XVI)}$$

Since both X and Y in this last pair of equations are independent of H, this approximation explains the approximate common intersection of all lines having a given value of D. They can, moreover, be combined to give an equation of the curve upon which all these common intersections fall, but the result is in a form too complicated to be of much value, although it can be readily identified as an equation of a conic section (obviously a very straight portion of a hyperbola). The curve can be plotted more simply by means of equations XV and XVI, and will be found to be very nearly a straight line passing through the zero point on the V axis.

It seems as if a regraduation of the H axis might be made to result in perfect instead of approximate intersections. It will be found that this can be readily accomplished for any given value of D on the assumption that the H graduating distance from a fixed point $= KH + L + \frac{M}{H}$ where K, L, and M are constants. But unfortunately these three constants prove to be themselves functions of D. In other words, different sets of graduations would have to be used for each value of the diameter, which is obviously impracticable.

Each value of H, however, is in practice associated with a rather narrow range of D values. It is therefore possible to modify the positions of the H graduations empirically so as to result in a decided improvement in the intersections, and at the same time a slight readjustment of the D axis can be made with advantage. The best results appear to be obtained by the following plan:

1. Graduate the V axis as already described.

2. Graduate the H axis as already described but omit all values under that of 5 logs.

3. Select a few definite values of D well distributed over the desired range, and for each calculate the volumes for three or more values of H ($H = 5$ or over). If a table of cone frusta is available, these volumes may, of course, be taken therefrom.

4. Draw straight lines from each value of H to the corresponding value of V, and select points which appear to be the averages of the intersections of lines relating to each common value of D.

5. Draw a curve through the points thus selected. For most work a sufficiently close approximation will be found to be a straight line passing through the V origin.

6. Enter on this curve or straight line the D graduations indicated by the straight lines of step 4.

7. Obtain two or three values for V corresponding to the lower values of H (under 5 logs) and to the values of D already graduated. By drawing the appropriate straight lines their intersections with the H axis will indicate the best average positions of the smaller H graduations. These should then be entered on that axis.

8. Complete the graduation of the D axis by intersection, using for each value of D appropriate values of H.

The following table, worked out by the above process, may also be used directly to save time and labor.

TABLE I

GRADUATION OF *H* AXIS

Height in logs	Distance from fixed point on axis: Values to be used where the heights are in logs and tenths of logs	Distance from fixed point on axis: Values to be used where the heights are in feet
Intersection with the diagonal axis	1.26	20.16
2	2.08	33.28
3	3.03	48.48
4	4.01	64.16
5	5	80
6	6	96
7	7	112
8	8	128
9	9	144
10	10	160
11	11	176

In the application of this theory to the making of a volume table the successive steps may be as follows:

1. Prepare an incomplete alinement graph (fig. 2) such as has been illustrated in Figure 1, but with the *H* axis graduated in accordance with the table just given.

2. Draw a diagonal straight line representing the *D* axis between the point representing its intersection with the *H* axis and the zero point on the *V* axis.

3. Graduate the *D* axis as follows: Each tree measurement (a sufficient number of which are supposed to be at hand) is used to draw a straight line between the point on the *H* axis corresponding to the height of the tree and the point on the *V* axis corresponding to the volume of the tree. The intersection of this with the *D* axis is an indication of the position of the *D* graduation corresponding to its diameter. The indications of a number of trees will naturally be more or less conflicting and the results must therefore be evened off by a graduating curve such as indicated in figure 3. In this curve the distance of each intersection above the base of the graph of figure 2 is plotted over its corresponding diameter. When all the points are thus plotted a smooth curve is drawn through them, and from this curve the *D* axis is finally graduated.

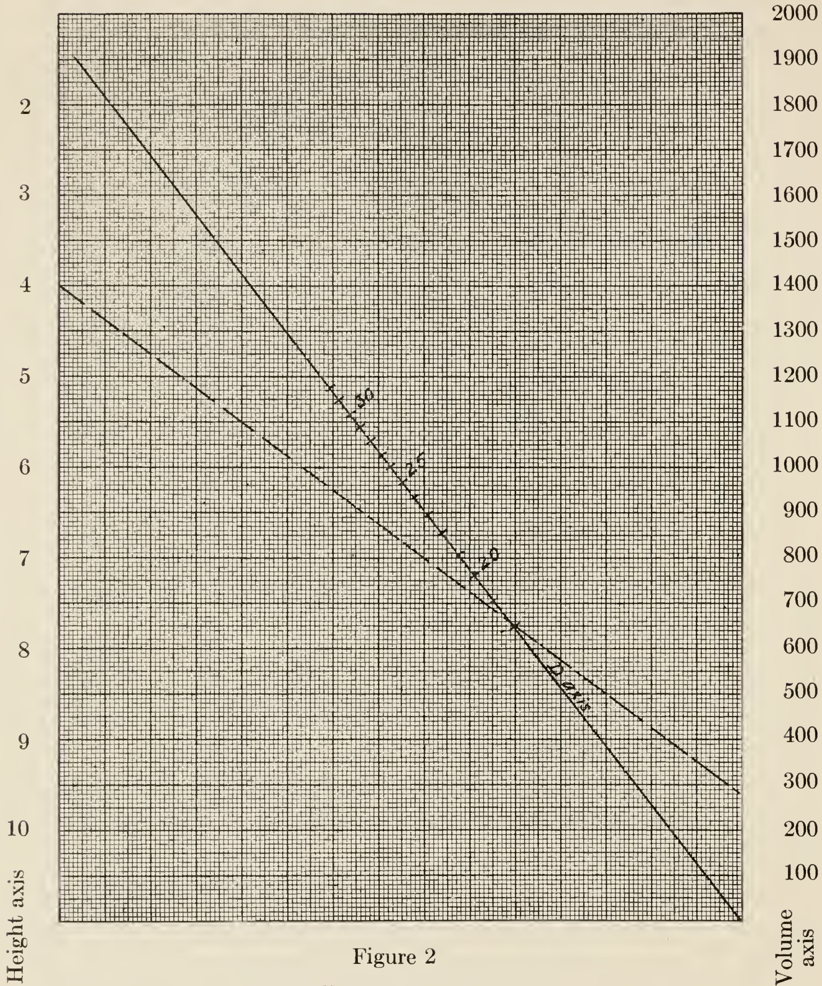

Figure 2

An alinement volume table.

Table II gives the basic tree data used in this illustrative instance. The broken line in figure 2 shows how the first values of this table are plotted. The left-hand point on figure 3 is plotted from the resulting intersection with the *D* axis.

It is immaterial whether each tree is thus used to determine a point on the graduating curve or whether the average volume, height, and diameter of the height-diameter classes are used. In the latter case, each point should, of course, be weighted in accordance with the number of trees which are thus averaged together. Figure 3 was drawn in accordance with the latter method.

4. The volume table can now be read from the completed alinement chart, the result being given in Table III.

TABLE II

BASIC TREE DATA, WHITE FIR, STANISLAUS NATIONAL FOREST

(Measurements taken by U. S. Forest Service)

No. of trees in class	Average D. B. H. inches	Average merchantable height in 16-ft. logs	Average Vol. ft. B. M.
7	18.1	4.0	280
3	19.5	3.9	330
8	19.9	4.0	340
5	19.8	5.0	500
6	21.9	4.4	540
7	22.2	5.1	640
6	21.8	6.0	670
1	22.8	6.5	810
2	23.9	4.1	510
1	23.4	4.2	490
9	23.7	5.2	640
6	24.1	6.2	970
2	24.6	7.3	1230
4	25.5	5.0	660
9	25.9	5.6	950
12	26.3	6.0	970
9	26.2	7.0	1270
1	25.8	7.6	1510
4	27.7	5.6	1050
10	28.3	6.3	1250
7	28.1	7.5	1450
1	27.6	8.3	1760
1	29.2	4.6	1090
7	29.9	6.0	1300
7	29.8	6.4	1280
12	29.7	7.3	1640
3	30.2	7.9	1720
16	31.6	6.6	1600

TABLE III

VOLUME TABLE READ FROM FIGURE 2

	Height in 16-ft. logs				
D. B. H.	4	5	6	7	8
18	280	380			
20	350	480			
22	430	590	740	900	
24	510	690	880	1060	
26		790	1000	1220	1430
28		850	1130	1370	1600
30		1010	1280	1550	1820
32		1140	1450	1750	2060

In most cases it will be found that the graduating curve of figure 3 can be entered on the same sheet as the alinement chart without confusion and with a saving in time and convenience. In step 3, moreover, it is not necessary actually to draw the various straight lines, which are apt to become confusing if many tree measurements are available. Instead, a straight edge may be laid across the proper values and its intersections with the intermediate axis noted.

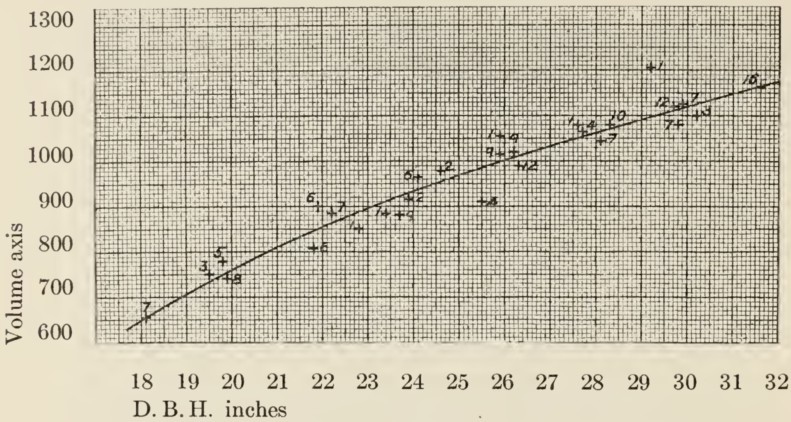

Figure 3

The graduating curve.

There are several advantages in this method of preparing a volume table. In the first place, the curve drawing is simplified. In place of the system of curves which have to be harmonized in the usual plan only a single graduating curve need be drawn, and since this is based on all of the tree measurements available it is much better defined and more easily and accurately located. As compared with the frustum form factor method, which also uses a single curve, the use of the alinement method saves considerable time, since the calculation of the form factors is avoided; the result is, however, practically identical, for if the frustum form factors of such a table as Table III be calculated they will be found to vary with diameter but not materially with height.

Secondly, exterpolations are easily (perhaps almost too easily) made, especially in height, and with far more certainty than is possible by the normal system of curve extension.

Lastly, the resulting alinement chart can be read with great accuracy for fractions of inches in diameter and fractions of logs in height.

This is not true of the conventional method, where graphic interpolations between the harmonized curves are both slow and inaccurate and where arithmetical interpolations in the final table are exceedingly laborious. For certain problems of forest mensuration this advantage is highly important, although it is of little weight in connection with ordinary timber cruising. The method appears superior in accuracy to the ordinary plan, especially where the amount of data available is limited. Volume tables have been made by both methods from tree data for three species, including the species used in illustrating this paper. The results appear as follows:

	Aggregate difference between all trees as actually scaled and as read by table		Average deviation between individual tree volumes as scaled and as read by table	
Basic data	Conventional	Alinement	Conventional	Alinement
145 trees, western larch	1.5%	0.2%	5.8%	5.1%
166 trees, western white pine	2.1%	0.3%	3.8%	3.9%
166 trees, white fir	0.5%	0.2%	7.1%	6.3%

It will be observed that the result by the alinement method is much superior as a whole for each species, and is better, on the average, in detail as well.