BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF BIOLOGICAL PROBLEMS

FOUNDED BY
W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY
KARL PEARSON

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CONTENTS OF VOLUME XIII.

Memoirs.

Miscellanea.

List of Plates.

PART I.

PARTS II AND III.

PART IV.

Vol. XIII. Part I October, 1920

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BIOMETRIKA

THE FUNDAMENTAL PROBLEM OF PRACTICAL STATISTICS.

BY KARL PEARSON, F.R.S.

(1) SOME excuse must be given for once more returning to a problem which is as ancient as Bayes, has been apparently treated from most aspects by Laplace, and has been criticised and re-criticised by Boole, De Morgan, Venn and Edgeworth. The problem I refer to is that of "inverse probabilities" and in practical statistics it takes the following form:

An "event" has occurred p times out of $p+q=n$ trials, where we have no *a priori* knowledge of the frequency of the event in the total population of occurrences. What is the probability of its occurring r times in a further $r+s=m$ trials?

In statistical language a first sample of n shows p marked individuals, and we require a measure of the accordance which future samples are likely to give with this result. For example, a medical treatment is found to be successful in p out of n cases, we require some measure of the probable stability of this ratio. It is on this stability and its limits according to the size of the first sample that the whole practical doctrine of statistics, which is the theory of sampling, actually depends. We usually state the "probable errors" of results without visualising the strength or weakness of the logic behind them, and without generally realising that if the views of some authors be correct our superstructure is built, if not on a quicksand, at least in uncomfortable nearness to an abyss.

As stated above, the problem had been considered in 1774 by Laplace* whose approximation by aid of Stirling's Theorem leads us directly to the normal curve. I shall later on repeat and to some extent modify on a broader basis Laplace's investigation. But Laplace was really only following Bayes†—and for our purposes

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* *Mémoires de mathématique et de physique présentés à l'Académie par divers savans*, T. VI. p. 6. Paris, 1774.

† Bayes' work was communicated after his death to the Royal Society by Price: see *Phil. Trans.* Vol. LIII. pp. 269—271, 370—375. Condorcet also gave the main result in 1783, *Histoire de l'Académie*, 1786, p. 539, Paris, 1788 and also in the "Essai sur l'application de l'analyse à la probabilité...," p. 188, Paris, 1785. Thus Condorcet wrote between the publication of Laplace's memoir of 1774 and of the *Théorie analytique des Probabilités*, Paris, 1812. Condorcet, however, supplied the combination factor omitted by Laplace. At the same time (*Essai*, p. lxxxiii) he admitted Bayes and Price's priority, while remarking that Laplace had been the first to treat it analytically. He recognised the existence of the hypothesis of "the stability of the statistical ratios," it is not equally clear that he recognised the need for "the equal distribution of ignorance." I do not think it is correct to say that Laplace was the first to treat the problem analytically. It all turns on the evaluation of the incomplete B-function. The methods of quadrature of Bayes and Price may be somewhat primitive, but I cannot see that they are much rougher than those used on this occasion by Laplace. There is no special merit in reducing any integral to terms in exponentials, unless these give an adequate approximation to the sought value. And Laplace does not really measure the closeness of his approximation nor indicate where it fails.

the fundamental principle on which Laplace's more complete solution is based may be extricated from Bayes' much more involved reasoning. None of the early writers on this topic—all approaching the subject from the mathematical theory of games of chance—seem to have had the least inkling of the enormous extension of their ideas, which would result in recent times from the application of the theory of random sampling to every phase of our knowledge and experience—economic, social, medical, and anthropological—and to all branches of observation whether astronomical, physical or psychical. Hence they must not be too severely handled if their "bag of balls" hypotheses seem too flimsy a structure for modern statistical theory.

Bayes in his original paper supposes balls rolled on a table and *equally likely to halt anywhere on the breadth of the table.* Condorcet and Laplace only generalise this in appearance by supposing all probabilities equally likely before the event. It is the same hypothesis expressed in more general words. Each repetition is supposed to be independent and the elementary algebra view of compound probability is accepted without hesitation. Thus we have two hypotheses to deal with:

(i) the hypothesis that *a priori* we ought to distribute our ignorance of the chance of a marked individual occurring *equally,*

(ii) the hypothesis that earlier occurrences do not modify the chance of later trials.

(2) The chief criticism of the theory of inverse probabilities has been based on the want of generality in the first hypothesis—what Boole has termed "the equal distribution of our ignorance" (*Laws of Thought,* p. 370, 1854) or the assigning to the appearance of a factor whose real probability is unknown to us all degrees of probability and then making these degrees all equally likely to occur. This is not the sole, but I think the chief feature of Boole's attack on the theory of inverse probabilities. Why should we, he asks, distribute our ignorance equally? Other sorts of distributions may occur, and we know do occur in chance problems. Why this assumption in the particular case? Dr Venn's strongly unfavourable criticism of inverse probabilities seems also based on objection to the principle of equal distribution of ignorance*. Strangely enough it seems to me, he is willing to admit the hypothesis when it is applied to a comparison of the probable contents of two bags of balls, while refusing to consider it valid in the case of a single bag. Thus he considers the problem: "of 10 cases treated by Lister's method seven did well and three suffered from blood-poisoning: of 14 treated with ordinary dressings nine did well and five had blood poisoning, what are the odds that the success of Lister's method was due to chance"? (p. 187) and remarks that the "bag of balls" method—i.e. the assumption of the equal prevalence of the different possible kinds of bag—seems "to be the only reasonable way of treating the problem, if it is to be considered capable of numerical solution at all." But I fail to see why the application of the hypothesis of the equal distribution of ignorance to Lister's method *and*

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* *The Logic of Chance*, p. 182 *et seq.*

to ordinary dressings is logically more legitimate than to either alone. The only argument must be that in some way a *wrong* distribution of the *a priori* chance of blood poisoning is more likely to correct itself if applied for comparative than for absolute purposes. Dr Venn does not show, however, that any distribution, right or wrong, would lead in the comparative case to the same result. I cannot help fancying that had Dr Venn come across the problem as a bag of balls problem he would have rejected the equal distribution of ignorance solution. But when he meets the same problem in a vital problem of conduct, he realises that some solution is essential, there is a wide experience in man of the stability of statistical ratios, and for practical purposes he needs to apply this even to small samples of new experiences. What he demands and rightly demands is a measure of the faith he is to put—as a guide to immediate conduct—in small experience, and Bayes' hypothesis gives him a rule, which if rough, corresponds not wholly badly with his sum total of small experiences. The mathematicians in their customary manner first stated an hypothesis which simplified their analysis without questioning whether it had any foundation in experience. But Maxwell's brilliant paradox—that the theory of probabilities is the only logic for the practical man—means more than it superficially conveys. The practical man is always working by comparative probabilities even if he has not reduced his appreciations to numbers. He would certainly scoff at the idea that a first sample should not influence his judgment of what a second sample or what the bulk would be like. For such an idea would render most actions in life impossible. If science cannot measure the degree of probability involved—so much the worse for science. The practical man will stick to his appreciative methods until it does, or will accept the results of inverse probability of the Bayes-Laplace brand till better are forthcoming.

Now let us see exactly where we stand. Notwithstanding the criticisms of Boole and Venn all branches of science have adopted the theory of "probable errors": they have applied past experience of limited samples to predict what deviations are likely to occur from this past experience in future experience, and mankind acts in accordance with a firm conviction in the relative stability of statistical ratios. But any numerical appreciation of the reasonableness of this conduct is apparently based on the "equal distribution of ignorance" or ultimately on such a quaint idea as that of Bayes that his balls might roll anywhere on the table with equal probability.

Edgeworth has done all that is feasible to transfer our faith in inverse probabilities from purely mathematical to more sound observational foundations. He wrote in 1884*, "The ridicule which has been heaped upon Bayes' theorem and the inverse method will be found only applicable to the pretence here deprecated of eliciting knowledge out of ignorance, something out of nothing. The most formidable objection is that which was made by Boole and is repeated by Mr Venn, Mr Pierce and others with approbation. Our procedure in treating one value as *a priori* not less likely than another is, it is said, of a quite arbitrary character, and

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* *Mind*, Vol. IX. p. 230.

apt to lead to different conclusions from the plausible one which we have reached by accident."

Edgeworth takes the bull by the horns and broadly speaking asserts that later experience has shown us that in innumerable cases where we were *a priori* ignorant the chances of given events really did not cluster. "We take our stand upon the fact that probability-constants occurring in nature present every variety of fractional value; and that natural constants in general are found to show no preference for one number rather than another. Acting on which supposition, while in particular cases we shall err, in the long run we shall find our account" (p. 231).

In other words Edgeworth returns to the appeal to experience from which Bayes and Laplace ought to have started. He asserts that in our ignorance we must appeal to our knowledge of kindred experience, and that statistical constants do not cluster in such experience. "I submit," he writes, "the assumption that any probability-constant about which we know nothing in particular is as likely to have one value as another is grounded upon the rough but solid experience that such constants do as a matter of fact, as often have one value as another" (p. 230). Unfortunately he does not give us any numerical data—except a few isolated illustrations—to prove that the universe of statistical ratios exhibits a rectangular frequency curve. Some captious critics might assert a conviction that *their* experience showed that chances ranged themselves like the horns of the bull, and that absolute non-occurrence and persistent occurrence were in the world at large the most emphasised probabilities. Our professor impaled thus upon the horns of the bull he has ventured to seize might leave the doctrine of inverse probabilities in a worse plight than he found it!—His practical safety depends on the extreme difficulty of testing by actual *numerical* experience the "fact" that natural constants in general show no preference for one number rather than another. At the best we may "feel" it is so, rather than hope to demonstrate it by forming a random sample which would cover the immensely wide fields of natural frequencies.

It has occurred to me, however, that possibly the bull itself is a chimera, and there may be no need whatever to master it. In short, is it not possible that any *continuous* distribution of *a priori* chances would lead us equally well to the Bayes-Laplace result? If this be so, then the main line of attack of its critics fails. If they then continue their assault on the ground that there are cases in which the repetitions of events are not independent, it is conceivable, that we who defend inverse probabilities as not only the basis of modern statistical theory, but also as the arithmological justification for the most ordinary actions and beliefs of the practical man can again counter our adversaries.

(3) Let us examine the problem on the lines of Bayes' original investigation into *a priori* chances. We may reduce that investigation to its simplest terms as follows: A line of length a is taken and a stroke made on it at random, and its position at distance x from one end is unknown. Afterwards n strokes are made at random on the line, and p fall in the segment 0 to x and $q = n - p$ in the segment x to a. The former are looked upon as "successes," the latter as "failures."

For a given position between x and $x+\delta x$, Bayes takes the total probability to be

$$\frac{\delta x}{a}\left(\frac{x}{a}\right)^p\left(1-\frac{x}{a}\right)^q.$$

It is against this first stage that the objection is raised that the chance of placing the first stroke is taken as $\delta x/a$, and of succeeding strokes x/a or $(a-x)/a$. All this is on the assumption that the first stroke *a priori* is equally likely to be anywhere in the line of length a, or freed from the analogy of strokes, the *a priori* chance is taken to have any value between 0 and 1 with equal likelihood. But there is really no necessity for this limitation. Suppose the frequency curve for strokes along the line to be given by any continuous function of x, say $y=\dfrac{\phi(x)}{a}$. Then the chance of a stroke occurring between x and $x+\delta x$ will be $\phi(x)\,\delta x/a$ instead of $\delta x/a$. Further, the chance P_x of a stroke afterwards occurring between 0 and x will be

$$P_x=\int_0^x \phi(x)\,dx/a,$$

and between x and a $\quad Q_x=1-P_x=\int_x^a \phi(x)\,dx/a.$

Clearly $P_0=0$ and $P_a=1$. Thus the probability of the combined event, since

$$\delta P_x=\phi(x)\,\delta x/a,$$

will be $\quad \delta P_x P_x{}^p\,Q_x{}^q\,\dfrac{(p+q)\,!}{p\,!\,q\,!}.$

Proceeding as in Bayes' or Laplace's manner, we have for the probability that the unknown original probability lies between P_b and P_c (i.e. x between b and c)

$$\frac{\int_{P_b}^{P_c} dP_x P_x{}^p\,(1-P_x)^q}{\int_0^1 dP_x P_x{}^p\,(1-P_x)^q}.$$

Or, again, for the whole chance that there will be in a further m trials, r successes and s failures,

$$C_r=\frac{\int_0^1 dP_x P_x{}^{p+r}\,(1-P_x)^{q+s}}{\int_0^1 dP_x P_x{}^p\,(1-P_x)^q}\times\frac{(r+s)\,!}{r\,!\,s\,!}.$$

Now it is clear that the above integrals will take the same values

$$B(p+r+1,\ q+s+1)\quad\text{and}\quad B(p+1,\ q+1)$$

whether we replace P_x by x or not. But if we replace P_x by x, we have exactly the formula reached by Laplace on the basis of "the equal distribution of ignorance," to use Boole's phrase. Thus it would appear that the fundamental formula of Laplace*, i.e.

$$C_r=\frac{B(p+r+1,\ q+s+1)}{B(p+1,\ q+1)\,B(r+1,\ s+1)},$$

---

* Including the $B(r+1,\ s+1)$ term omitted by Laplace.

in no way depends on the equal distribution of ignorance. It is sufficient to assume any continuous distribution—which may vary from one type of *a priori* probability problem to a second—in order to reach the basis of the fundamental problem in practical statistics, i.e. the probability that in a second trial of m events following a p, q experience, r will be successful and s fail.

(4) I now propose to develop the above expression rather more completely than Laplace has done. We shall first replace the B-functions by Γ-functions. Then we shall use Stirling's theorem on the assumption that none of the quantities p, q, r or s are small. As we solely need to find the variation of C_r with r, we need not trouble about terms not involving r or s. We have

$$C_r = \frac{1}{B(p+1,\,q+1)} \frac{\Gamma(p+r+1)\,.\,\Gamma(q+s+1)\,.\,\Gamma(r+s+2)}{\Gamma(p+q+r+s+2)\,\Gamma(r+1)\,\Gamma(s+1)}$$

$$= \frac{e^{-n}}{B(p+1,\,q+1)} \frac{\Gamma(m+2)}{\Gamma(n+m+2)} \frac{(p+r)^{p+r+\frac{1}{2}}(q+s)^{q+s+\frac{1}{2}}}{r^{r+\frac{1}{2}}\,s^{s+\frac{1}{2}}} \times \chi,$$

where χ

$$= \frac{\left(1+\frac{1}{12(p+r)}+\frac{1}{288(p+r)^2}+\dots\right)\left(1+\frac{1}{12(q+s)}+\frac{1}{288(q+s)^2}+\dots\right)}{\left(1+\frac{1}{12r}+\frac{1}{288r^2}+\dots\right)\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\dots\right)}$$

We now take $$r = \frac{m}{n}p + h, \quad s = \frac{m}{n}q - h,$$

and put very approximately

$$\chi = e^{\frac{1}{12(p+r)}+\frac{1}{12(q+s)}-\frac{1}{12r}-\frac{1}{12s}}.$$

We may write

$$C_r = \left\{\frac{e^{-n}p^p q^q\,\Gamma(m+2)}{B(p+1,\,q+1)\,\Gamma(n+m+2)}\right\} \frac{\left(1+\frac{r}{p}\right)^{p+r+\frac{1}{2}}\left(1+\frac{s}{q}\right)^{q+s+\frac{1}{2}}}{\left(\frac{r}{p}\right)^{r+\frac{1}{2}}\left(\frac{s}{q}\right)^{s+\frac{1}{2}}}\chi.$$

The factor in large curled brackets we will call C; it does not vary with r. The second factor, involving $\frac{r}{p}$ and $\frac{s}{q}$, we will call u, and the remainder is χ. We have

$$\log_e C_r = \log_e C + \log_e u + \log_e \chi,$$

and we need the development only of the last two logarithms. We shall write $(n+m)/n = 1+\rho$, and $p/n = P$, $q/n = Q$ will give the chances of success and failure as estimated on the first sample. We have

$$\log_e u = \{p(1+\rho)+h+\tfrac{1}{2}\}\log\left(1+\rho+\frac{h}{p}\right)+\{q(1+\rho)-h+\tfrac{1}{2}\}\left(\log 1+\rho-\frac{h}{q}\right)$$
$$-\{\rho p+h+\tfrac{1}{2}\}\log\left(\rho+\frac{h}{p}\right)-\{\rho q-h+\tfrac{1}{2}\}\log\left(\rho-\frac{h}{q}\right).$$

Expanding and collecting terms in powers of h, we have

$$\begin{aligned}\log_e u = {} & \{(p+q)(1+\rho)+1\}\log_e(1+\rho) - \{(p+q)\rho+1\}\log_e\rho - \tfrac{1}{2}h\frac{1}{\rho(1+\rho)}\frac{q-p}{pq} \\ & - \tfrac{1}{2}h^2\left(\frac{1}{\rho(1+\rho)}\frac{p+q}{pq} - \frac{1+2\rho}{2\rho^2(1+\rho)^2}\frac{p^2+q^2}{p^2q^2}\right) \\ & + \tfrac{1}{6}h^3\left\{\frac{1+2\rho}{\rho^2(1+\rho)^2}\left(\frac{1}{p^2}-\frac{1}{q^2}\right) - \frac{1+3\rho+3\rho^2}{\rho^3(1+\rho)^3}\left(\frac{1}{p^3}-\frac{1}{q^3}\right)\right\} + \text{etc.,} \\ = {} & \{n(1+\rho)+1\}\log_e(1+\rho) - \{n\rho+1\}\log_e\rho - \tfrac{1}{2}h\frac{1}{m(1+\rho)}\frac{Q-P}{PQ} \\ & + \tfrac{1}{2}h^2\left(\frac{1}{m(1+\rho)PQ} - \frac{1+2\rho}{2m^2(1+\rho)^2}\frac{1-2PQ}{P^2Q^2}\right) \\ & + \tfrac{1}{6}h^3\left(\frac{1+2\rho}{m^2(1+\rho)^2}\frac{Q-P}{P^2Q^2} - \frac{1+3\rho+3\rho^2}{m^3(1+\rho)^3}\frac{Q^3-P^3}{P^3Q^3}\right) + \text{etc.}\end{aligned}$$

Hence, putting $\sigma^2 = m(1+\rho)PQ$,

$$\begin{aligned}\log_e u = {} & \{n(1+\rho)+1\}\log_e(1+\rho) - (n\rho+1)\log_e\rho - \frac{1}{2}\frac{h}{\sigma}\frac{Q-P}{\sqrt{m}\sqrt{1+\rho}\sqrt{PQ}} \\ & - \frac{1}{2}\frac{h^2}{\sigma^2}\left(1 - \frac{1+2\rho}{2m(1+\rho)}\frac{1-2PQ}{PQ}\right) \\ & + \frac{1}{6}\frac{h^3}{\sigma^3}\left(\frac{1+2\rho}{\sqrt{m}\sqrt{1+\rho}}\frac{Q-P}{\sqrt{PQ}} - \frac{1+3\rho+3\rho^2}{m^{\frac{3}{2}}(1+\rho)^{\frac{3}{2}}}\frac{(Q-P)(1-QP)}{(PQ)^{\frac{3}{2}}}\right) + \text{etc.}\end{aligned}$$

Similarly

$$\begin{aligned}\log_e \chi = {} & -\frac{1}{12PQ}\frac{1}{m(1+\rho)} - \frac{1}{12}\frac{h}{\sigma}\frac{1+2\rho}{m^{\frac{3}{2}}(1+\rho)^{\frac{3}{2}}}\frac{P-Q}{(PQ)^{\frac{3}{2}}} \\ & - \frac{h^2}{12\sigma^2}\frac{1+3\rho+3\rho^2}{m^2(1+\rho^2)}\frac{1-3PQ}{PQ} - \text{.etc.}\end{aligned}$$

If we stop at terms in $\frac{1}{m}$ we see that $\log_e\chi$ contributes only to the constant term and this may be included in C. Into the same factor C may be put the logarithmic terms in $\log_e u$. Thus finally we have*:

$$\begin{aligned}C_r = {} & Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}\left(1-\frac{1+2\rho}{2m(1+\rho)}\frac{1-2PQ}{PQ}\right)} \\ & \times e^{-\frac{1}{2}\frac{h}{\sigma}\frac{Q-P}{\sqrt{m}\sqrt{1+\rho}\sqrt{PQ}} + \frac{1}{6}\frac{h^3}{\sigma^3}\frac{1+2\rho}{\sqrt{m}\sqrt{1+\rho}}\frac{Q-P}{\sqrt{PQ}}}.\end{aligned}$$

We note the following facts with regard to this result:

(i) The series converges with a factor $1/\sqrt{m}$. Unless this be small we cannot neglect the terms in h and h^3. In other words deviations in excess and defect of mP are *not* equally probable. Thus skew frequency rather than the Gaussian hypothesis is indicated from the very start of the investigation if our second sample be not considerable.

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* The existence of the term in $(h/\sigma)^3$ of the same order as that in h/σ prevents us treating the result as a normal curve with shifted centre.

(ii) Examining the odd terms we see that within the range

$$h = \pm \frac{3}{1+2\rho}\sigma = \pm \frac{3n}{n+2m}\sigma$$

positive deviations are less frequent than negative deviations, but beyond this range the positive are more frequent than the negative. In other words *large* excesses are more probable than large defects, but small excesses are less probable than small defects from the result of our first sample.

(iii) It is not legitimate to keep the term in h^2 in $\frac{1}{m}$, for this is really the order of the lowest term in h^4/σ^4. Without including that term our degree of approximation will only be to $1/\sqrt{m}$.

(iv) For the very narrow case of $Q = P$, or equal chance of success and failure, of course the odd terms vanish.

If we include in this case the lowest term in h^4/σ^4 we have

$$C_r = Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}\left(1-\frac{1+2\rho}{m(1+\rho)}\right)-\frac{1}{12}\frac{h^4}{\sigma^4}\frac{1+3\rho+3\rho^2}{m(1+\rho)}}.$$

The case of $P = Q$ is, however, of such secondary importance that this second approximation on Laplace's lines is of little significance*.

(v) If our second sample is so large that terms in $\frac{1}{\sqrt{m}}$ are negligible, then our frequency becomes

$$C_r = Ce^{-\frac{1}{2}\frac{h^2}{\sigma^2}}.$$

This result was reached by Laplace before Gauss. Neither stated the very narrow limitations of the formula.

Besides the condition that m is to be very large, we must also note that neither Q nor P can be very small. We obtain quite different results, if we suppose m large and P small so that mP remains finite.

(vi) Thus the Gauss-Laplacian distribution fails:

(*a*) for small samples. Its whole method of deduction is then wrong for Stirling's Theorem is invalid;

(*b*) when the sample is large, but the probability of occurrence is small, so that mP is finite and small.

The whole of this subject has now-a-days been reconsidered under the topics of "Small Samples" and the "Law of Small Numbers."

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* A method of investigating the frequency constants of such a distribution has recently been given by Forsyth, *Messenger of Mathematics*, Vol. XLVIII. pp. 131–44.

Laplace's method gives the "normal curve" as an approximation to the original series, when we make very limiting assumptions as to n, m, P. It led to his suggestion that $\int e^{-\frac{1}{2}x^2}dx$ should be tabulated, and eventually to the computation of the so-called "probability integral."

(5) It will be seen that Laplace's method is only a rough method of approximation to our original series for C_r. That series is only a special case of the general hypergeometrical series, and we have at once thrust upon us the problem: Can no better curve be found than the Gauss-Laplacian $y = y_0 e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ to represent the sum of any number of terms of the hypergeometrical series? The answer is that curves infinitely better which demand no restriction on the values of m, n, P can be determined, and these curves can with just as much validity be called probability curves as the above normal curve; and the integrals of their areas up to any given value have equal claim to be "probability integrals."

Let us return to Laplace's starting-point on p. 6. We have at once for the ratio:

$$\frac{C_{r+1}}{C_r} = \frac{(p+r+1)s}{(r+1)(q+s)} = \frac{(p+r+1)(m-r)}{(r+1)(q+m-r)}.$$

Hence $$\frac{C_{r+1}-C_r}{C_{r+1}+C_r} = \frac{p(m+1)-n-rn}{m(p+2)+q+r\{2(m-1)-p+q\}-2r^2}.$$

Divide by n and put $p/n = P$, $q/n = Q$ as before, and we have

$$\frac{C_{r+1}-C_r}{C_{r+1}+C_r} = \frac{P(m+1)-1-r}{m\left(P+\frac{2}{n}\right)+Q+r\left(Q-P+2\frac{m-1}{n}\right)-\frac{2}{n}r^2}.$$

Now suppose $C_0, C_1 \dots C_r, C_{r+1} \dots$ to be plotted as a histogram, i.e. as a series of rectangles of base c and heights $C_0/c, C_1/c \dots C_r/c, C_{r+1}/c \dots$. Let the tops of the midordinates of these rectangles be joined, so as to form a polygonal figure $P_0, P_1, P_2 \dots P_r, P_{r+1} \dots$. If we take the origin at the ordinate C_0/c we shall have for abscissae of the angles $X_r = rc$, $X_{r+1} = r\pi c \dots$ and for ordinates $y_r = C_r/c$, $y_{r+1} = C_{r+1}/c$. Thus the abscissa and the ordinate of the midpoint of a side of the polygon will be $x_{r+\frac{1}{2}} = (r+\frac{1}{2})c$ and $y_{r+\frac{1}{2}} = \frac{1}{2}(C_{r+1}+C_r)/c$. Accordingly we have, if $\Delta x = c$:

$$\frac{1}{y_{r+\frac{1}{2}}}\frac{\Delta y_{r+\frac{1}{2}}}{\Delta x}$$
$$= \frac{2}{c}\,\frac{c\{P(m+1)-\frac{1}{2}\}-X_{r+\frac{1}{2}}}{c\left\{m\left(P+\frac{2}{n}\right)+Q\right\}+c\left\{(Q-P)+\frac{2(m-1)}{n}\right\}(X_{r+\frac{1}{2}}-\frac{1}{2}c)-\frac{2}{n}(X_{r+\frac{1}{2}}-\frac{1}{2}c)^2}.$$

Now let us transfer the origin to the mode, i.e. take

$$-x_{r+\frac{1}{2}} = c\{P(m+1)-\tfrac{1}{2}\}-X_{r+\frac{1}{2}}.$$

Hence after some reductions we find

$$\frac{1}{y_{r+\frac{1}{2}}}\frac{\Delta y_{r+\frac{1}{2}}}{\Delta x} = \frac{-x_{r+\frac{1}{2}}}{PQ(m+1)\left(1+\frac{m+1}{n}\right)c^2 + c\left\{(Q-P)\left(\frac{1}{2}+\frac{m+1}{n}\right)\right\}x_{r+\frac{1}{2}} - \frac{1}{n}x^2_{r+\frac{1}{2}}}$$

Take $$\sigma_0^2 = PQ(m+1)\left(1+\frac{m+1}{n}\right)c^2,$$

and we may write

$$\frac{\text{Slope of side of polygon } P_rP_{r+1}}{\text{Ordinate of midpoint of side}}$$

$$= -\frac{x_{r+\frac{1}{2}}}{\sigma_0^2}\,\frac{1}{1+\dfrac{(Q-P)\left(\frac{1}{2}+\frac{m+1}{n}\right)}{\sqrt{QP(m+1)\left(1+\frac{m+1}{n}\right)}}\dfrac{x_{r+\frac{1}{2}}}{\sigma_0} - \dfrac{1}{n}\dfrac{x^2_{r+\frac{1}{2}}}{\sigma_0^2}}.$$

Now Laplace's hypotheses amount to taking both n and m very large and n relatively large as compared with m. Thus we have, if we neglect the second and third term in the denominator,

$$\frac{\text{Slope of } P_rP_{r+1}}{\text{Ordinate}} = -\frac{x_{r+\frac{1}{2}}}{\sigma_0^2}.$$

The same result would be reached, if we took $Q = P$, i.e. took equal chances for each contributory event—which is really Gauss' equality of chance for errors in excess and defect—together with n the number of cause groups in the first sample very large.

To get the Gaussian or normal curve we must then replace differences by differentials and we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{\sigma_0^2}.$$

Laplace, however, fails to introduce the condition that neither P nor Q is to be small which is essential.

Now there is a point here of considerable importance. We do *not* assume that differences may be replaced by differentials. What we have reached is a fundamental geometrical property of the hypergeometrical histogram and what we do is to seek for a continuous curve, which possesses this same geometrical property.

If our first sample be very large, i.e. we have exact knowledge of the bag or population contents, the last term in $\frac{1}{n}$ will vanish and we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{\sigma_0^2 + \frac{1}{2}c(Q-P)x},$$

where $$\sigma_0^2 = PQ(m+1)c^2.$$

This is the curve which corresponds as closely to the skew binomial histogram as the normal curve has been shown by Laplace to correspond to the symmetrical binomial histogram, or to the skew binomial even if n be very large and the bases of the histogram very small.

In general however when none of these conditions hold we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{a_0 + a_1 x + a_2 x^2},$$

where $\quad a_0 = PQ(m+1)\left(1 + \frac{m+1}{n}\right) c^2$ and is *positive*,

$$a_1 = (Q-P)\left(\frac{1}{2} + \frac{m+1}{n}\right) c \text{ and may be of either sign,}$$

$$a_2 = -\frac{1}{n} \text{ and is } \textit{negative}.$$

Thus for this particular case we may throw the result into the form

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{b_0(b_1 - x)(b_2 + x)} = \frac{1}{b_0(b_1+b_2)}\left\{\frac{b_2}{b_2+x} - \frac{b_1}{b_1-x}\right\}.$$

Hence

$$\log_e y = \text{constant} + \frac{1}{b_0(b_1+b_2)}\{b_2 \log_e(b_2+x) + b_1 \log_e(b_1 - x)\}.$$

Let

$$s_1 = \frac{b_1}{b_0(b_1+b_2)}, \quad s_2 = \frac{b_2}{b_0(b_1+b_2)}.$$

Thus

$$y = y_0\left(1 + \frac{x}{b_2}\right)^{s_2}\left(1 - \frac{x}{b_1}\right)^{s_1},$$

where y_0 is the modal ordinate.

This is a limited range curve, of which the partial area is expressible in terms of the incomplete B-function. In other words, the so-called "probability integral" is only a very special case of the incomplete B-function, which is in this sense the general probability integral.

We easily find

$$b = b_1 + b_2 = c\left\{\tfrac{1}{2}n + m + 1 - n\sqrt{PQ}\right\}^{\frac{1}{2}}\left\{\tfrac{1}{2}n + m + 1 + n\sqrt{PQ}\right\}^{\frac{1}{2}},$$

and

$$b_1 - b_2 = c(Q-P)\left(\tfrac{1}{2}n + m + 1\right).$$

Thus

$$b_1 = \tfrac{1}{2}b + \tfrac{1}{2}c(Q-P)\left(\tfrac{1}{2}n + m + 1\right),$$

$$b_2 = \tfrac{1}{2}b - \tfrac{1}{2}c(Q-P)\left(\tfrac{1}{2}n + m + 1\right).$$

Hence

$$s_1 = nb_1/b$$

$$= \tfrac{1}{2}n\left(1 + \frac{(Q-P)\left(\tfrac{1}{2}n + m + 1\right)}{\left\{\tfrac{1}{2}n + m + 1 - n\sqrt{PQ}\right\}^{\frac{1}{2}}\left\{\tfrac{1}{2}n + m + 1 + n\sqrt{PQ}\right\}^{\frac{1}{2}}}\right).$$

Similarly

$$s_2 = \tfrac{1}{2}n\left(1 - \frac{(Q-P)\left(\tfrac{1}{2}n + m + 1\right)}{\left\{\tfrac{1}{2}n + m + 1 - n\sqrt{PQ}\right\}^{\frac{1}{2}}\left\{\tfrac{1}{2}n + m + 1 + n\sqrt{PQ}\right\}^{\frac{1}{2}}}\right).$$

Or, writing $\frac{1}{2} + \frac{m+1}{n} = \epsilon$ for brevity,

$$s_1 = \tfrac{1}{2}n\left(1 + \frac{(Q-P)\,\epsilon}{(\epsilon^2 - PQ)^{\frac{1}{2}}}\right),$$

$$s_2 = \tfrac{1}{2}n\left(1 - \frac{(Q-P)\,\epsilon}{(\epsilon^2 - PQ)^{\frac{1}{2}}}\right),$$

$$b = cn\,(\epsilon^2 - PQ)^{\frac{1}{2}}.$$

$$b_1 = \tfrac{1}{2}\,\{b + nc\,(Q-P)\,\epsilon\},$$

$$b_2 = \tfrac{1}{2}\,\{b - nc\,(Q-P)\,\epsilon\}.$$

Distance of mode from start of hypergeometrical

$$= c\,\{P\,(m+1) - \tfrac{1}{2}\}.$$

Further distance from mode to mean

$$= (b_1 - b_2)/(n+2) = c\,(Q-P)\left(\frac{1}{2} + \frac{m}{n+2}\right).$$

(6) *Illustration.* I owe an illustrative example of this to Mr E. C. Rhodes, and the accompanying diagram to Miss A. Davin. Suppose 20 % of individuals in a sample of 1000 have been found to possess a given character, what will be the chances of such percentage arising in a further sample of 100 individuals?

This is the type of problem which arises every day in statistical work and which I term the *fundamental* problem of statistics. Only too often the first sample is treated as indefinitely great, and the probabilities calculated from the probability integral of the normal curve on the hypothesis that $\sigma_0 = \sqrt{mPQ}\,.\,c$. In Diagram I the results for the normal curve have been bettered by taking

$$\sigma_0 = \sqrt{(m+1)\,PQ\left(1 + \frac{m+1}{n}\right)}\,c.$$

It is still quite inadequate to provide the requisite percentages. These are plotted as the small full circles, and they are seen to lie most closely on the skew curve reckoned with the above constants. The integral of the curve was then obtained by the integraph, and the areas read off to the nearest tenth. The results are given in Table I, p. 13. See Diagram I.

It will be seen that the skew curve gives frequencies never differing more than a fraction per cent. from the series, while the deviations of the Gaussian are much larger. A still better result would have been obtained had we fitted the series by moments rather than by the geometrical relation. This will be illustrated in the next example.

(7) It may be of interest to see what does really happen when we suppose n infinite or the population "known."

Returning to p. 10 we have

$$\sigma_0^2 = PQ\,(m+1)\,c^2,$$

Diagram I

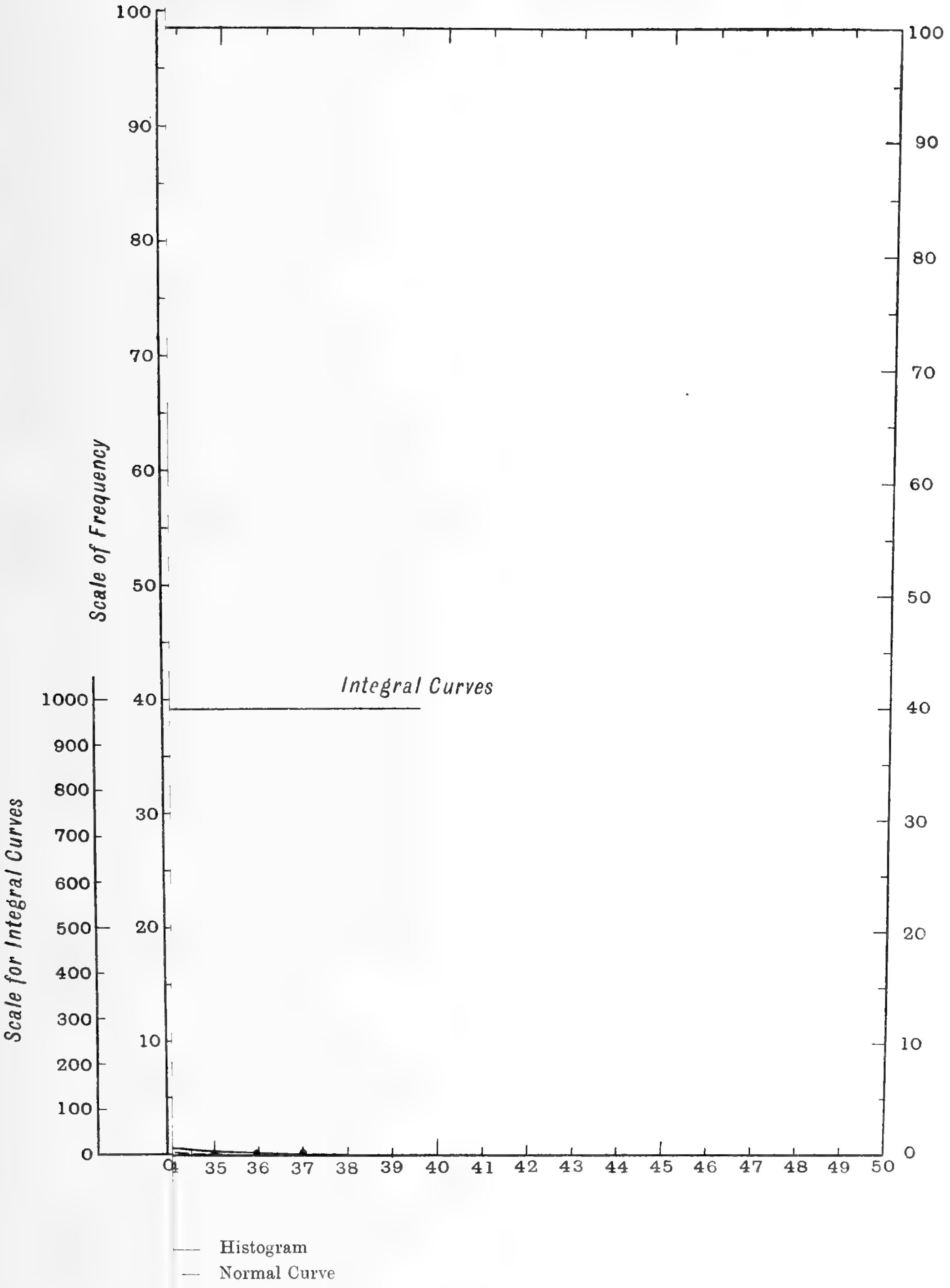
Scale of Frequency
Scale for Integral Curves
Integral Curves
100
90
80
70
60
50
40
30
20
10
0
1000
900
800
700
600
500
400
300
200
100
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
Histogram
Normal Curve

Diagram I

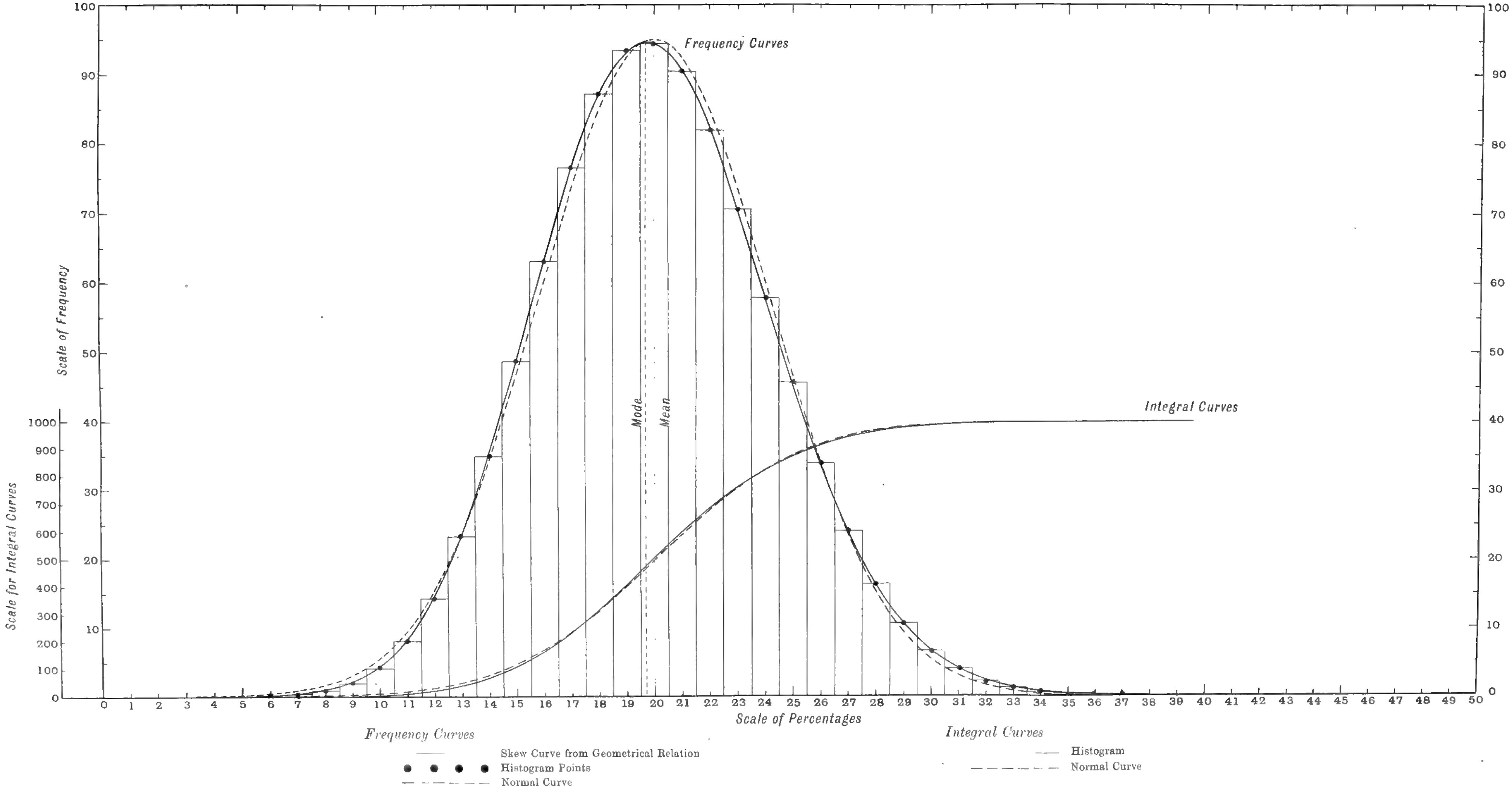
Frequency Curves
Integral Curves
Mode
Mean
Scale of Frequency
Scale for Integral Curves
Scale of Percentages
0
10
20
30
40
50
60
70
80
90
100
0
100
200
300
400
500
600
700
800
900
1000
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50
Frequency Curves
Skew Curve from Geometrical Relation
● ● ● ● Histogram Points
Normal Curve
Integral Curves
Histogram
Normal Curve

Diagram II

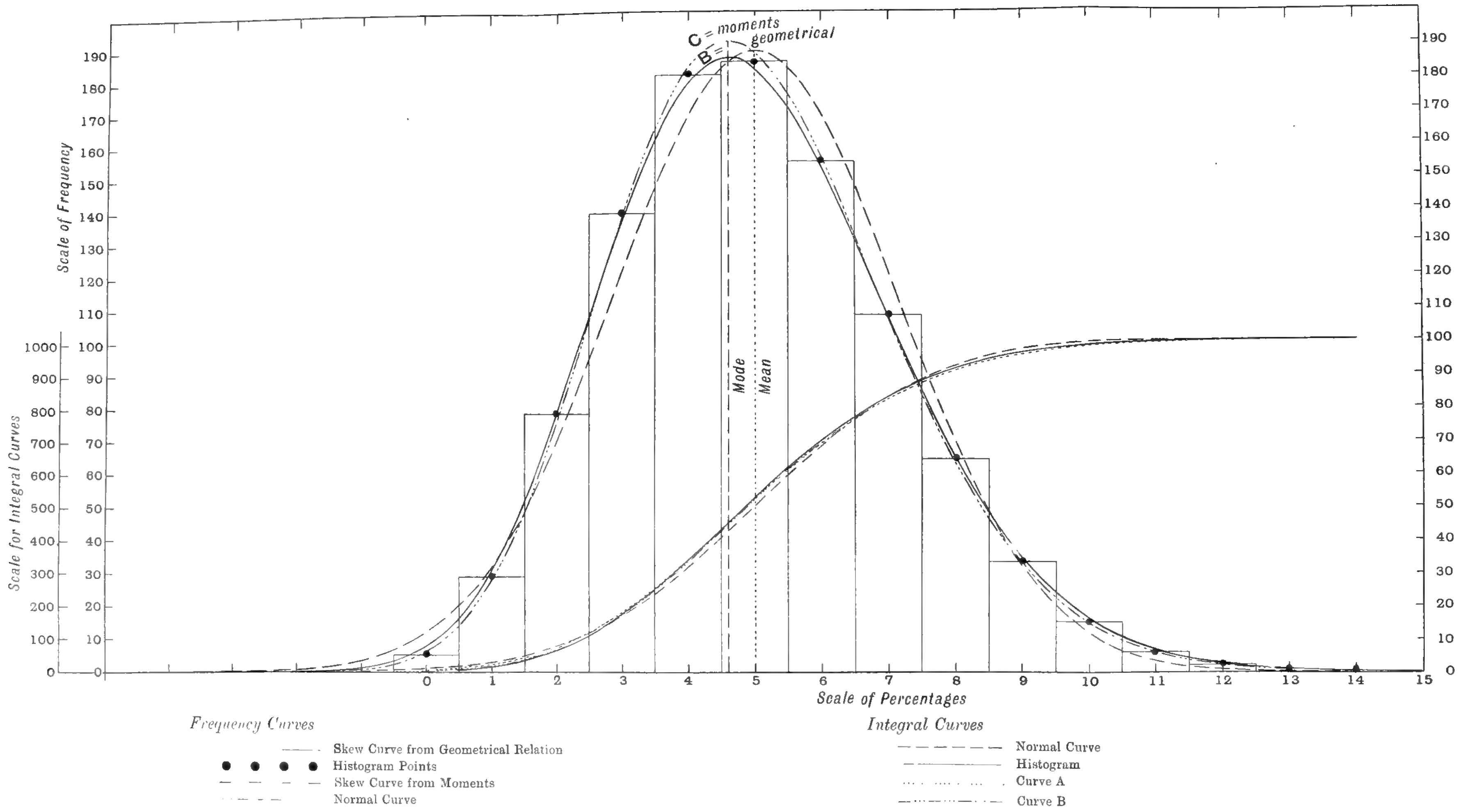
C = moments
B = geometrical
Mode
Mean
Scale of Frequency
Scale for Integral Curves
Scale of Percentages
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
190
0
100
200
300
400
500
600
700
800
900
1000
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Frequency Curves
Skew Curve from Geometrical Relation
● ● ● ● Histogram Points
Skew Curve from Moments
Normal Curve
Integral Curves
Normal Curve
Histogram
Curve A
Curve B

Diagram II

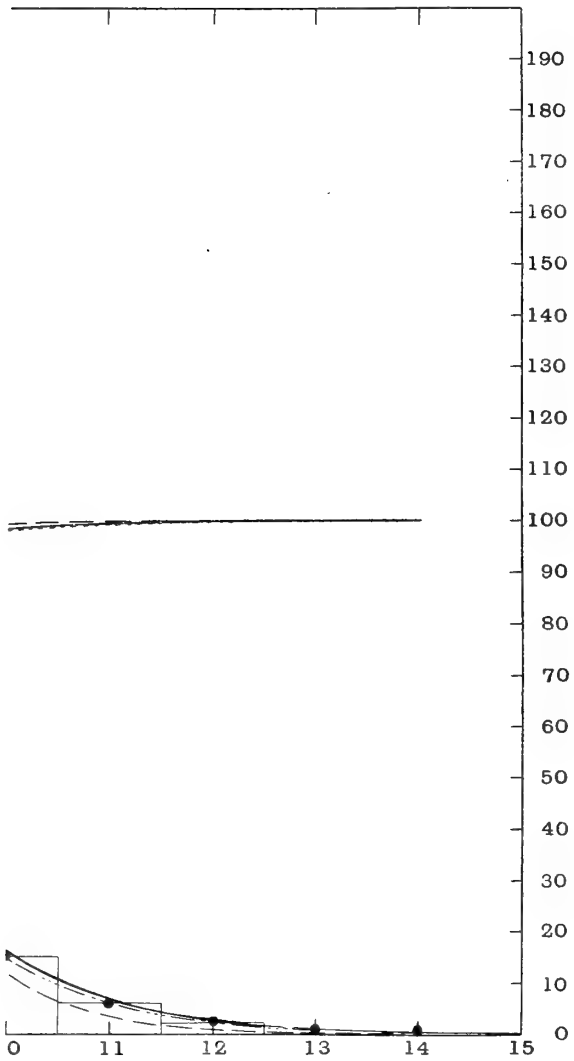

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TABLE I.

Percentages of Frequency from Mean to Deviations in First Column.

	In Excess of Mean				In Defect of Mean		
		Calculated				Calculated	
Up to	Actual	Gaussian	Skew Curve	Up to	Actual	Gaussian	Skew Curve
·5	4·73	4·74	4·73	− ·5	4·73	4·74	4·72
1·5	13·77	13·97	13·77	− 1·5	14·08	13·97	14·02
2·5	21·98	22·44	22·06	− 2·5	22·80	22·44	22·76
3·5	29·04	29·79	29·27	− 3·5	30·46	29·79	30·42
4·5	34·83	35·83	35·07	− 4·5	36·78	35·83	36·72
5·5	39·40	40·51	39·62	− 5·5	41·65	40·51	41·50
6·5	42·81	43·94	43·09	− 6·5	45·15	43·94	44·99
7·5	45·23	46·31	45·41	− 7·5	47·49	46·31	47·28
8·5	46·88	47·86	47·12	− 8·5	48·93	47·86	48·69
9·5	47·96	48·82	48·21	− 9·5	49·74	48·82	49·46
10·5	48·63	49·38	48·89	− 10·5	50·16	49·38	49·91
11·5	49·04	49·69	49·28	− 11·5	50·37	49·69	50·11
12·5	49·26	49·86	49·50	− 12·5	50·46	49·86	50·18
13·5	49·39	49·94	49·66	− 13·5	50·49	49·94	50·18
14·5	49·45	49·97	49·72	− 14·5	50·49	49·97	50·18
15·5	49·49	49·99	49·77	− 15·5	—	49·99	—
16·5	49·51	50·00	49·82	− 16·5	—	50·00	—

and

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{\sigma_0^2\left(1 + \frac{\frac{1}{2}(Q-P)}{\sqrt{QP(m+1)}}\frac{x}{\sigma_0}\right)}$$

will be the equation to the curve associated now with the binomial $(p+q)^m$. Let us write it in the form

$$\frac{1}{y}\frac{dy}{d\left(\frac{x}{a}\right)} = \frac{-\lambda\frac{x}{a}}{1+\frac{x}{a}}.$$

Here $$a = \frac{2\sqrt{QP(m+1)}}{Q-P}\sigma_0 = \frac{2PQ(m+1)}{Q-P}c, \text{ and } \lambda = \frac{4QP(m+1)}{(Q-P)^2},$$

and integrating $$y = y_0 e^{-\lambda\frac{x}{a}}\left(1+\frac{x}{a}\right)^\lambda,$$

where the origin is the mode.

The distance from the mode to the mean is

$$d = a/\lambda = \frac{1}{2}\frac{Q-P}{\sqrt{QP(m+1)}}\sigma_0 = \tfrac{1}{2}(Q-P)c$$

and $$y_0 = \frac{M}{a}\frac{\lambda^{\lambda+1}e^{-\lambda}}{\Gamma(\lambda+1)}.$$

Now the true standard deviation of the *curve* is not σ_0*, but is given by

$$\sigma = \frac{\sqrt{\lambda+1}}{\lambda}\, a,$$

or

$$\sigma^2 = \{QP\,(m+1) + \tfrac{1}{4}(Q-P)^2\}\, c^2 = \sigma_0^2 + \tfrac{1}{4}(Q-P)^2\, c^2$$
$$= \sigma_0^2 \text{ nearly,}$$

if m be fairly large and neither Q nor P small.

In both cases the (standard deviation)^2 of the curve exceeds that of the series by a term depending on the square of the distance between mean and mode. Unless either P or Q are small, this will be a small difference, a fraction of the square of the plotting unit c. As the curves are not reached by equating moments, but from a geometrical property common to the polygon of frequency and to the curve, this deviation is to be anticipated. It is the more to be anticipated as in the case of the curve we are dealing with the moments of continuous areas, and in the case of the series with concentrated lumps†.

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[*] Similarly for the limited range curve of p. 11:

$$\sigma_0^2 = (m+1)\, PQ\left(1 + \frac{m+1}{n}\right) c^2 = nPQ\,(\epsilon^2 - \tfrac{1}{4})\, c^2.$$

But the true standard deviation σ is given by

$$\sigma^2 = \frac{b^2\,(s_1 s_2 + s_1 + s_2 + 1)}{(s_1+s_2+2)^2\,(s_1+s_2+3)} = \frac{c^2}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\left\{nPQ\,(\epsilon^2 - \tfrac{1}{4}) + \left(1+\frac{1}{n}\right)(\epsilon^2 - PQ)\right\}$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\,\epsilon^2\,(P-Q)^2 c^2$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\, d^2,$$

where d = distance from mean to mode.

[†] The true (standard deviation)^2 of the binomial is $\sigma_1^2 = mPQc^2$. Thus

$$\sigma^2 = \{mPQ + PQ + \tfrac{1}{4}(Q-P)^2\}\, c^2 = \sigma_1^2 + \tfrac{1}{4}c^2,$$

or the (standard deviation)^2 of the binomial series is less than that of curve by $\tfrac{1}{4}c^2$.

The true standard deviation of the series when n is not indefinitely great is given by

$$\sigma_2^2 = c^2\left[\frac{mPQ\left(1+\frac{m+2}{n}\right)}{1+\frac{3}{n}} + \frac{\frac{m}{n}\left(1+\frac{1}{n}\right)\left(1+\frac{m+2}{n}\right)}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\,(P-Q)^2\right]$$

$$= \frac{1}{1+\frac{3}{n}}\,\sigma_0^2 + \frac{\left(1+\frac{1}{n}\right)\epsilon^2\,(P-Q)^2}{\left(1+\frac{2}{n}\right)^2\left(1+\frac{3}{n}\right)}\, c^2 - \tfrac{1}{4}c^2\,\frac{1+\frac{1}{n}}{1+\frac{3}{n}}$$

$$= \sigma^2 - \tfrac{1}{4}c^2\,\frac{1+\frac{1}{n}}{1+\frac{3}{n}}. \quad \text{(See preceding footnote.)}$$

Thus the (standard deviation)^2 of the series is less than that of the curve by a term of the order $\tfrac{1}{4}c^2$.

(8) I am able to illustrate by another example kindly worked by Mr E. C. Rhodes, the accuracy of this binomial curve. He has considered the case of an indefinitely large population containing 10 % of a given character, and worked out the Type III curve by both the geometrical relationship and the equality of moments to determine the frequency distribution of samples.

TABLE II.

Table of Percentage Frequencies up to given Distances from the Mean.

In Excess of Mean					In Defect of Mean				
Up to	Gaussian	Actual	B* Curve	C† Curve	Up to	Gaussian	Actual	B Curve	C Curve
·5	9·40	9·25	8·84	9·05	− ·5	9·40	9·25	9·29	9·61
1·5	26·24	24·66	24·01	24·52	− 1·5	26·24	27·34	26·76	27·55
2·5	38·30	35·42	34·71	35·40	− 2·5	38·30	41·19	40·25	41·09
3·5	45·22	41·85	41·39	41·96	− 3·5	45·22	48·99	48·03	48·57
4·5	48·40	45·18	44·89	45·35	− 4·5	48·40	51·85	51·13	51·40
5·5	49·57	46·70	46·68	46·72	− 5·5	49·57	52·37	52·00	52·10
6·5	49·91	47·31	47·38	47·53	− 6·5	49·91	52·37	52·17	52·27
7·5	49·99	47·53	47·72	47·73	− 7·5	49·99	52·37	52·17	52·27
8·5	50·00	47·61	47·78	47·73	− 8·5	50·00	52·37	52·17	52·27
9·5	50·00	47·63	47·80	47·73	− 9·5	50·00	52·37	52·17	52·27
10·5	50·00	47·63	47·83	47·73	− 10·5	50·00	52·37	52·17	52·27

* B is the Type III curve determined by the geometrical relation.

† C is the Type III curve determined by equality of moments.

It will be seen that the use of moments gives a better result than the geometrical property. But Type III curve obtained either way is far closer than the Gaussian to the actual distribution. See Diagram II.

(9) It would appear, therefore, that starting from an enlarged view of Bayes' Theorem, and approaching the matter from the standpoint of Laplace, we reach, both theoretically and of course in any case of actual numbers, a better result than the normal curve by using Type I or Type III for our hypergeometrical series, i.e. by using tables of the incomplete B- or Γ-functions rather than the table of the probability integral (areas of the Gaussian or normal curve). It must be insisted upon that Laplace reached the probability integral as an approximation to the sum of terms in a certain hypergeometrical series, and any better approximation to the sum of those terms has greater logical validity than Laplace's integral. Other deductions of the probability integral are, so to speak, after-thoughts. Its essential origin lies in Laplace's endeavour to solve the fundamental problem in statistics. The so-called Gaussian or normal curve was first introduced into statistics as a *rough and ready* solution for the sum of a certain number of terms in a definite hypergeometrical series, and the sacrosanct character of the "probability integral" and the "probable error" in the eyes of many physicists and astronomers is of the character of a dogma; it is based on authority rather than reason.

(10) It will be seen from this paper that the next stage forward in the development of the probability integral must be (i) the issue of a table of the incomplete Γ-function. Such a table is now completed and arrangements are being made for its publication; (ii) the issue of a table of the incomplete *B*-function. Such a table presents grave disadvantages. It must be a table of threefold entry and this presents more serious difficulties than even those we have encountered in the table of the incomplete Γ-function, because (*a*) of the great extent of the computations requisite and (*b*) the great cost involved in publication even if completed. I should like to draw the attention of pure mathematicians to the following problem: Can the incomplete *B*-function be expressed even approximately in terms of a limited number of incomplete Γ-functions?

Mr H. E. Soper, Miss M. Seegar and I have tried many solutions of this problem. None as yet with satisfaction to ourselves. It will not be easily solved, nor by the obvious methods of attack. Yet I am convinced that a solution is possible, if pure mathematicians of greater knowledge and larger inspiration will only deal with the matter.

I have to acknowledge my great indebtedness to Mr E. C. Rhodes for numerical work and Miss A. Davin for graphical work on the illustrations of this paper, and for help in mechanical integration of the frequency curves. Such integration of course cannot be correct to the last figure.

CORRELATION OF CEPHALIC MEASUREMENTS IN EGYPTIAN BORN NATIVES.

By MYER M. ORENSTEEN, F.S.S.
(*Ministry of Public Works, Cairo, Egypt.*)

1. *Object of the Paper.* A previous paper on the "Correlation of Anthropometrical Measurements in Cairo Born Natives*" pointed out that the coefficients of correlation resulting from the length and the breadth of the head vary much from one race to another.

The significance of this variability is suggestive and the writer accordingly undertook another investigation.

The present paper is intended to exhibit fresh coefficients of correlation for the cephalic measurements and is therefore a help to the study of the variation in question. The analysis into the variability will be confined to a separate paper already in hand.

2. *The Material.* Full particulars as to the nature of the raw material have been given in the above-mentioned paper. Suffice to repeat here that the statistics were drawn absolutely at random without any special selection whatever and that they are fairly representative of the ordinary community which they stand for. All persons measured are adult males, i.e. individuals of 20 years of age and above, and each has been classified according to his place of birth. The measurements were taken during the period 1901 to 1906.

3. *The Computations.* The total number of individuals involved in the computations is 9430. Considering the large amount of the data available, the observations have been grouped into frequencies where a standard unit of 3 millimetres was adopted.

This grouping was carried out throughout the statistics with the result that at least a dozen sub-groups was obtained for each character in each correlation-table.

It will be seen that the frequency curves have high contact at both ends, therefore Sheppard's adjustments were applied to the moments before calculating the standard deviations.

All constants have been computed twice and independently by removing the base-group by a unit of grouping.

4. *The Constants.* Table I is self explanatory. N is the number of individuals entering in each classification, A and σ are the mean and corresponding standard deviations of the group, and E_A and E_σ their respective probable errors.

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* *Biometrika*, Vol. XI. Nos. 1 and 2, November 1915.

TABLE I. *Means and Standard Deviations for the Cephalic Measurements.*

Group	N	Length of the Head		Breadth of the Head	
		$A \pm E_A$	$\sigma \pm E_\sigma$	$A \pm E_A$	$\sigma \pm E_\sigma$
Alexandria Governorate	643	189·77 ± 0·16	5·95 ± 0·11	144·64 ± 0·13	4·97 ± 0·09
Cairo ,,	802	190·52 ± 0·14	5·90 ± 0·10	144·45 ± 0·11	4·67 ± 0·08
Canal ,,	127	190·56 ± 0·34	5·78 ± 0·24	145·13 ± 0·34	5·66 ± 0·24
Beheira Province	526	191·17 ± 0·17	5·81 ± 0·12	143·69 ± 0·14	4·73 ± 0·10
Gharbîya ,,	1104	190·90 ± 0·12	5·93 ± 0·09	143·53 ± 0·10	4·79 ± 0·07
Minufîya ,,	717	191·06 ± 0·15	6·00 ± 0·11	143·59 ± 0·12	4·60 ± 0·08
Daqahlîya ,,	504	190·39 ± 0·20	6·67 ± 0·14	143·96 ± 0·15	4·83 ± 0·10
Sharqîya ,,	515	190·75 ± 0·18	6·16 ± 0·13	143·65 ± 0·14	4·82 ± 0·10
Qalyubîya ,,	295	190·82 ± 0·23	5·87 ± 0·16	143·69 ± 0·19	4·78 ± 0·13
Gîza ,,	326	191·79 ± 0·22	5·78 ± 0·15	143·17 ± 0·17	4·60 ± 0·12
Faîyûm ,,	413	191·20 ± 0·20	5·86 ± 0·14	141·84 ± 0·16	4·82 ± 0·11
Beni Sueif ,,	384	191·74 ± 0·19	5·65 ± 0·14	142·68 ± 0·17	4·90 ± 0·12
Minya ,,	491	191·72 ± 0·17	5·65 ± 0·12	142·61 ± 0·14	4·56 ± 0·10
Asyût ,,	887	190·91 ± 0·13	5·85 ± 0·09	142·52 ± 0·10	4·46 ± 0·07
Girga ,,	610	191·49 ± 0·16	6·05 ± 0·12	142·30 ± 0·12	4·41 ± 0·08
Qina ,,	824	191·23 ± 0·14	5·99 ± 0·10	142·30 ± 0·11	4·77 ± 0·08
Aswân ,,	262	190·50 ± 0·25	5·97 ± 0·18	144·26 ± 0·22	5·23 ± 0·15

It might be remarked that the means and the standard deviations given here are not exactly the same as those given by Mr J. I. Craig*. In both cases the mathematics have been carefully checked and found correct. The difference may be accounted for in the difference in the unit of the grouping method. In the former a standard unit of 3 millimetres was adopted giving at least a dozen sub-groups, whereas in the latter case the unit of grouping was invariably 2 or 3 millimetres giving about 20 sub-groups.

The difference however is insignificant (on the average three-hundredths of a millimetre) and in no case does it reach the corresponding probable error.

5. *The Coefficients of Correlation.* The following coefficients of correlations have been computed and are given in order of magnitude:

Beni Sueif Province	$r = +0·208 \pm 0·033$
Daqahlîya ,,	$r = +0·212 \pm 0·029$
Beheira ,,	$r = +0·213 \pm 0·028$
Minya ,,	$r = +0·227 \pm 0·029$
Girga ,,	$r = +0·228 \pm 0·026$
Minufîya ,,	$r = +0·230 \pm 0·024$
Alexandria Governorate	$r = +0·244 \pm 0·025$
Cairo ,,	$r = +0·244 \pm 0·025$
Sharqîya Province	$r = +0·248 \pm 0·028$
Aswân ,,	$r = +0·260 \pm 0·039$
Gîza ,,	$r = +0·277 \pm 0·034$
Qina ,,	$r = +0·286 \pm 0·022$
Qalyubîya ,,	$r = +0·303 \pm 0·026$

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* *Biometrika*, Vol. VIII. Nos. 1 and 2, July 1911.

Asyût Province . . . $r = +0·308 \pm 0·021$
Gharbîya „ . . . $r = +0·316 \pm 0·018$
Canal Governorate . . . $r = +0·330 \pm 0·054$
Faîyûm Province . . . $r = +0·369 \pm 0·029$

All the above coefficients of correlation are positive in sign and each exceeds three times its corresponding probable error. It will be seen from the above range that the differences are suggestive, therefore I am undertaking further investigations and hope to give the results shortly.

Table II, (*a*) to (*q*) gives the correlation-tables from which the constants have been calculated.

TABLE II. (*a*) *Alexandria.*

Length of Head.

Breadth of Head.	*170—172*	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	Totals
130—132	1	—	—	—	—	—	—	1	—	—	—	—	—	2
133—135	1	—	1	2	2	1	2	1	—	—	—	—	—	10
136—138	—	—	2	7	7	17	9	9	1	1	3	—	1	57
139—141	—	1	4	5	15	25	24	20	10	3	4	—	—	111
142—144	—	1	2	6	15	29	38	24	17	6	3	3	1	145
145—147	—	1	3	3	8	18	28	34	20	12	7	1	2	137
148—150	—	—	4	1	7	10	22	33	13	11	4	1	1	107
151—153	—	—	1	2	2	8	11	8	5	3	1	4	—	45
154—156	—	—	—	1	2	2	4	5	2	—	1	2	—	19
157—159	—	—	—	—	—	2	1	2	—	1	1	—	1	8
160—162	—	—	—	—	1	—	1	—	—	—	—	—	—	2
Totals	2	3	17	27	59	112	140	137	68	37	24	11	6	643

$r = +0·244 \pm 0·025.$

(*b*) *Cairo.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	Totals
130—132	—	1	—	1	1	1	1	—	1	—	—	—	6
133—135	—	—	—	2	2	2	2	—	—	—	—	—	8
136—138	3	3	10	9	9	10	15	8	8	—	—	—	75
139—141	—	1	5	14	17	29	35	8	4	4	3	—	120
142—144	—	2	9	16	24	54	31	24	19	8	1	2	190
145—147	1	1	9	17	39	40	41	29	12	11	1	1	202
148—150	—	—	3	7	19	32	22	13	18	8	1	2	125
151—153	—	—	5	1	2	6	11	13	7	5	1	4	55
154—156	—	—	—	2	1	1	2	2	2	1	—	1	12
157—159	—	—	—	—	—	2	—	1	4	1	—	—	8
160—162	—	—	—	—	—	—	—	—	—	1	—	—	1
Totals	4	8	41	69	114	177	160	98	75	39	7	10	802

$r = +0·244 \pm 0·025.$

(*c*) *Canal.*

Length of Head.

Breadth of Head.	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	Totals
133—135	2	—	1	—	2	—	—	—	—	—	5
136—138	—	3	1	1	3	—	2	1	—	—	11
139—141	—	1	3	4	4	5	1	1	—	—	19
142—144	—	—	2	4	4	7	1	—	2	1	21
145—147	—	—	5	4	6	8	4	4	2	—	33
148—150	—	1	3	—	4	2	5	2	—	—	17
151—153	—	—	—	1	—	2	2	3	2	1	11
154—156	—	—	—	1	—	1	1	3	—	—	6
157—159	—	—	—	—	1	1	—	—	—	—	2
160—162	—	—	—	1	—	1	—	—	—	—	2
Totals	2	5	15	16	24	27	16	14	6	2	127

$r = +0{\cdot}330 \pm 0{\cdot}054.$

(*d*) *Beheira.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	Totals
121—123	—	—	—	—	—	—	—	—	—	—	1	—	1
124—126	—	—	—	—	—	—	—	—	—	—	—	—	0
127—129	1	—	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	1	—	2	1	2	—	—	—	—	—	6
133—135	—	—	1	1	4	2	—	1	—	1	—	—	10
136—138	—	1	2	7	10	12	12	7	2	1	—	1	55
139—141	1	2	6	7	13	18	17	8	13	—	—	—	85
142—144	2	2	4	6	12	27	23	38	13	4	4	1	136
145—147	—	—	5	9	14	26	34	20	15	7	2	—	132
148—150	—	—	—	6	6	7	20	13	4	3	3	—	62
151—153	—	1	—	—	6	2	4	4	4	6	2	—	29
154—156	—	—	—	1	1	—	—	1	1	2	—	—	6
157—159	—	—	—	—	—	—	2	1	—	—	—	—	3
Totals	4	6	19	37	68	95	114	93	52	24	12	2	526

$r = +0{\cdot}213 \pm 0{\cdot}028.$

(*e*) *Gharbîya.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	*209—211*	*212—214*	*215—217*	*218—220*	Totals
127—129	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	2
130—132	1	—	—	1	1	2	2	—	—	1	—	—	—	—	—	—	8
133—135	1	—	6	5	7	9	5	3	2	—	—	—	—	—	—	—	38
136—138	1	2	9	13	23	27	20	9	10	—	—	—	—	—	—	—	114
139—141	1	2	14	18	42	48	40	29	18	6	1	1	—	—	—	—	220
142—144	—	2	9	25	26	62	45	44	16	6	1	1	—	—	—	—	237
145—147	1	6	2	14	41	53	51	48	31	22	6	1	1	—	—	—	277
148—150	—	1	3	4	20	20	27	19	21	12	6	—	—	—	—	1	134
151—153	—	—	—	3	2	7	12	14	9	2	2	—	—	—	—	—	51
154—156	—	—	—	—	—	3	2	4	2	6	—	—	—	—	—	—	17
157—159	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	2	4
160—162	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
163—165	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
166—168	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
Totals	5	13	43	83	163	232	206	170	110	56	16	3	1	0	0	3	1104

$r = +0{\cdot}316 \pm 0{\cdot}018.$

(*f*) *Minufîya.*

Length of Head.

Breadth of Head.	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
130—132	—	—	—	1	1	1	—	1	—	—	—	—	—	—	—	4
133—135	—	—	1	2	4	4	6	2	1	1	4	1	—	—	—	26
136—138	1	1	1	1	6	17	24	13	1	1	2	—	1	—	—	69
139—141	—	1	—	3	22	23	26	25	21	9	5	2	1	—	—	138
142—144	—	—	1	9	8	27	26	35	31	13	10	3	1	1	—	165
145—147	—	1	—	6	10	26	38	39	27	17	13	2	—	—	1	180
148—150	—	—	—	6	4	11	13	17	19	10	7	3	1	1	—	92
151—153	—	—	—	1	—	4	2	4	8	6	5	—	—	—	—	30
154—156	—	—	—	—	—	1	1	—	2	3	2	1	—	—	—	10
157—159	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	2
160—162	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	3	3	29	55	116	136	136	110	60	49	12	4	2	1	717

$r = +0{\cdot}230 \pm 0{\cdot}024$.

(*g*) *Daqahlîya.*

Length of Head.

Breadth of Head.	167—169	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	Totals
130—132	—	—	—	—	2	—	1	2	—	—	—	—	—	—	—	5
133—135	—	—	—	—	1	3	4	4	1	1	1	—	—	—	—	15
136—138	—	—	—	1	2	6	12	12	3	4	3	1	—	1	1	46
139—141	1	—	3	4	5	6	15	18	18	11	6	4	—	—	—	91
142—144	—	—	—	4	6	17	14	15	22	19	9	7	4	—	—	117
145—147	—	1	1	1	6	7	13	26	18	15	9	3	4	2	—	106
148—150	—	1	2	—	4	3	7	15	15	11	10	14	1	1	—	84
151—153	—	—	—	—	2	—	4	3	7	8	2	1	—	—	—	27
154—156	—	—	—	1	—	—	—	1	1	3	3	1	1	—	—	11
157—159	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	2
Totals	1	2	6	11	28	42	70	97	85	73	43	31	10	4	1	504

$r = +0{\cdot}212 \pm 0{\cdot}029$.

(*h*) *Sharqîya.*

Length of Head.

Breadth of Head.	164—166	167—169	170—172	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
127—129	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	3
133—135	—	—	—	—	1	1	4	2	4	2	4	2	1	—	—	—	—	21
136—138	—	—	—	—	1	3	6	5	11	7	5	8	—	1	1	—	—	48
139—141	—	—	—	—	1	10	3	11	32	19	9	5	1	3	—	1	—	95
142—144	—	—	—	—	1	4	12	26	24	22	17	9	4	2	—	—	—	121
145—147	1	—	—	—	2	2	11	20	17	27	18	10	3	2	1	1	—	115
148—150	—	—	—	—	2	1	3	5	18	17	23	9	4	3	—	—	—	85
151—153	—	—	—	—	—	—	1	2	3	1	4	1	2	—	—	—	1	15
154—156	—	—	—	—	—	—	—	—	—	2	2	—	2	—	—	—	—	6
157—159	—	—	—	—	—	—	—	—	—	—	1	2	—	1	—	—	—	4
160—162	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
163—165	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
166—168	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
Totals	2	0	0	1	8	21	40	72	110	97	83	46	17	12	3	2	1	515

$r = +0{\cdot}248 \pm 0{\cdot}028$.

(*i*) *Qalyubîya.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	Totals
130—132	—	—	—	1	—	—	1	—	—	—	—	—	2
133—135	—	1	1	2	2	2	1	2	—	—	—	—	11
136—138	—	—	1	4	7	7	5	2	2	—	—	—	28
139—141	—	1	1	6	8	12	7	7	4	2	—	—	48
142—144	1	1	3	7	11	18	14	9	9	5	2	—	80
145—147	—	1	2	5	10	17	15	13	4	3	3	1	74
148—150	—	—	—	5	3	4	7	9	2	3	—	—	33
151—153	—	—	—	—	—	1	3	2	—	—	1	—	7
154—156	—	—	—	—	—	1	2	2	1	2	—	—	8
157—159	—	—	—	—	—	—	1	—	—	1	1	1	4
Totals	1	4	8	30	41	62	56	46	22	16	7	2	295

$r = +0{\cdot}303 \pm 0{\cdot}026.$

(*j*) *Gîza.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	*209—211*	*212—214*	Totals
127—129	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
130—132	—	—	—	2	—	1	—	—	—	—	1	—	—	—	4
133—135	1	1	—	1	—	3	1	3	—	—	—	—	—	—	10
136—138	—	1	2	2	10	8	4	3	3	—	1	—	—	—	34
139—141	—	—	—	7	10	14	15	8	5	2	—	—	—	—	61
142—144	1	—	3	5	5	18	18	21	12	5	3	1	—	—	92
145—147	—	—	—	3	7	14	21	13	4	10	—	1	—	—	73
148—150	—	—	1	2	3	8	3	7	4	4	1	—	—	1	34
151—153	—	—	—	1	—	1	6	2	1	1	1	—	—	—	13
154—156	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
157—159	—	—	—	—	—	—	—	1	1	1	—	—	—	—	3
Totals	2	2	6	24	35	67	69	58	30	23	7	2	0	1	326

$r = +0{\cdot}277 \pm 0{\cdot}034.$

(*k*) *Faîyûm.*

Length of Head.

Breadth of Head.	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	*209—211*	*212—214*	Totals
130—132	—	—	2	2	4	—	—	—	—	—	—	—	—	8
133—135	2	3	3	9	6	7	2	3	—	—	—	—	—	35
136—138	1	5	6	6	15	12	8	3	—	—	—	—	—	56
139—141	1	4	10	11	19	21	15	7	3	1	—	—	—	92
142—144	—	1	8	14	26	16	21	13	6	1	3	—	—	109
145—147	1	1	2	8	12	16	18	10	3	—	—	—	—	71
148—150	—	—	1	4	1	2	7	3	3	1	3	—	—	25
151—153	—	—	1	1	—	3	4	1	1	—	—	—	—	11
154—156	—	—	—	—	1	1	—	—	—	1	—	—	—	3
157—159	—	—	—	—	—	—	—	—	—	—	—	—	2	2
160—162	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	5	14	33	55	84	78	76	40	16	4	6	0	2	413

$r = +0{\cdot}369 \pm 0{\cdot}029.$

(*l*) *Beni Sueif.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	Totals
124—126	—	—	—	—	—	—	—	—	—	—	—	1	1
127—129	—	—	—	1	—	—	—	—	—	—	—	—	1
130—132	—	—	—	1	2	1	—	—	—	—	—	—	4
133—135	1	—	2	5	4	4	3	1	—	—	—	—	20
136—138	2	5	7	4	12	7	5	2	3	2	—	1	50
139—141	1	2	5	9	14	17	15	10	2	1	1	—	77
142—144	—	1	7	12	23	24	17	12	6	—	—	—	102
145—147	—	—	3	8	12	15	10	6	5	2	—	—	61
148—150	—	—	3	3	6	12	11	5	7	2	—	—	49
151—153	—	—	—	1	2	4	—	4	2	—	—	—	13
154—156	—	—	1	—	—	2	1	1	—	—	—	—	5
157—159	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	4	8	28	44	75	86	62	42	25	7	1	2	384

$r = +0{\cdot}208 \pm 0{\cdot}033.$

(*m*) *Minya.*

Length of Head.

Breadth of Head.	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	215—217	Totals
127—129	—	—	—	1	—	1	—	—	—	—	—	—	—	—	2
130—132	—	—	—	1	1	—	—	2	—	—	—	—	—	—	4
133—135	1	—	1	6	5	4	1	1	1	—	—	—	—	—	20
136—138	2	3	8	12	8	15	10	2	5	1	—	—	—	—	66
139—141	—	3	4	19	21	28	13	10	3	1	—	—	—	—	102
142—144	1	5	8	13	25	33	25	13	11	2	1	—	—	—	137
145—147	—	3	6	6	18	23	18	9	8	2	1	—	—	—	94
148—150	—	—	2	6	6	9	9	2	4	3	—	—	—	1	42
151—153	—	—	—	—	2	6	6	1	3	—	1	—	—	—	19
154—156	—	—	—	—	—	2	1	—	—	1	—	—	—	—	4
157—159	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
Totals	4	14	29	64	86	122	83	40	35	10	3	0	0	1	491

$r = +0{\cdot}227 \pm 0{\cdot}029.$

(*n*) *Asyût.*

Length of Head.

Breadth of Head.	173—175	176—178	179—181	182—184	185—187	188—190	191—193	194—196	197—199	200—202	203—205	206—208	209—211	212—214	Totals
130—132	1	1	1	2	1	2	—	1	—	—	—	—	—	—	9
133—135	—	—	1	6	14	7	9	5	3	—	1	—	—	—	46
136—138	—	7	9	19	21	23	16	11	3	5	—	—	—	1	115
139—141	—	4	5	14	33	46	45	16	11	7	1	1	—	—	183
142—144	1	2	2	15	38	48	59	33	24	14	3	1	—	—	240
145—147	—	1	4	9	22	33	35	36	20	11	5	2	—	1	179
148—150	—	—	2	3	10	15	22	12	9	4	4	2	—	—	83
151—153	—	—	—	3	—	—	2	7	10	—	3	—	—	—	25
154—156	—	—	—	—	1	—	—	1	3	1	—	—	—	—	6
157—159	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	2	15	24	71	140	174	188	122	84	42	17	6	0	2	887

$r = +0{\cdot}308 \pm 0{\cdot}021.$

(*o*) *Girga.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	*209—211*	*212—214*	Totals
127—129	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
130—132	—	1	1	1	3	1	1	—	—	—	—	—	—	—	8
133—135	—	—	2	2	5	9	10	2	1	—	—	—	—	—	31
136—138	—	—	4	9	14	19	22	7	7	4	1	—	—	1	88
139—141	1	1	5	11	18	23	22	13	17	8	—	—	1	1	121
142—144	—	—	5	16	22	33	34	29	10	7	4	1	—	—	161
145—147	—	1	1	9	16	21	28	21	17	12	1	5	1	—	133
148—150	—	—	1	2	2	9	9	13	4	7	3	1	1	—	52
151—153	—	—	—	—	—	5	3	1	2	—	2	—	—	—	13
154—156	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
157—159	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	3	20	50	81	120	129	86	59	38	11	7	3	2	610

$r = +0{\cdot}228 \pm 0{\cdot}026.$

(*p*) *Qina.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	*209—211*	Totals
130—132	—	1	1	2	3	5	—	1	—	—	—	—	—	13
133—135	1	4	4	9	10	5	7	5	—	—	1	—	—	46
136—138	1	5	7	10	14	33	23	14	12	3	2	—	—	124
139—141	1	2	7	13	26	40	31	25	14	11	5	1	—	176
142—144	—	1	4	20	24	46	41	33	29	6	8	1	—	213
145—147	—	1	2	4	13	21	34	27	17	8	—	2	—	129
148—150	—	1	2	4	6	13	23	19	11	12	2	1	1	95
151—153	—	—	—	—	2	4	3	3	2	—	—	1	—	15
154—156	—	—	1	—	1	—	1	2	1	2	—	1	—	9
157—159	—	—	—	—	—	—	—	2	1	—	—	—	—	3
160—162	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	3	15	28	62	99	167	163	132	87	42	18	7	1	824

$r = +0{\cdot}286 \pm 0{\cdot}022.$

(*q*) *Aswân.*

Length of Head.

Breadth of Head.	*173—175*	*176—178*	*179—181*	*182—184*	*185—187*	*188—190*	*191—193*	*194—196*	*197—199*	*200—202*	*203—205*	*206—208*	Totals
127—129	1	1	—	—	—	—	—	—	—	—	—	—	2
130—132	—	—	1	1	—	—	—	—	—	—	—	—	2
133—135	—	—	2	—	2	—	1	—	1	—	—	—	6
136—138	—	1	—	2	2	9	3	5	—	—	1	—	23
139—141	—	1	4	7	5	11	5	5	4	2	—	—	44
142—144	—	1	1	3	16	21	9	9	5	5	1	—	71
145—147	—	—	—	3	2	14	4	8	3	—	2	—	36
148—150	—	—	3	4	8	6	8	5	6	5	—	—	45
151—153	—	—	—	1	4	5	5	2	1	3	1	1	23
154—156	—	—	—	1	2	1	2	—	1	1	1	—	9
157—159	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	1	4	11	22	41	67	37	35	21	16	6	1	262

$r = +0{\cdot}260 \pm 0{\cdot}039.$

NOTES ON THE HISTORY OF CORRELATION.

Being a paper read to the Society of Biometricians and
Mathematical Statisticians, June 14, 1920.

By KARL PEARSON, F.R.S.

(1) As I have often stated, Laplace anticipated Gauss by some 40 years. In his memoir of 1783, *Histoire de l'Académie*, pp. 423—467, he gives the expression for the probability integral

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}x^2}\,dx$$

and suggests (p. 433) its tabulation as a useful task. It is clear that to do this is to recognise the existence of the probability-curve

$$y=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2},$$

or in its doubly projected form

$$y=\frac{N}{\sqrt{2\pi}\sigma}\,e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}.$$

Laplace's investigation while not proceeding from the very simple axioms of Gauss, which lead directly to the above equation, is more satisfactory than Gauss' because we see better the nature of the approximations by which the curve is reached and get hints of how to generalise it. Many years ago I called the Laplace-Gaussian curve the *normal* curve, which name, while it avoids an international question of priority, has the disadvantage of leading people to believe that all other distributions of frequency are in one sense or another 'abnormal.' That belief is, of course, not justifiable. It has led many writers to try and force all frequency by aid of one or another process of distortion into a 'normal' curve.

Gauss starting with a normal curve as the law of distribution of errors reached at once the method of least squares. To understand the origin of the correlational calculus we must really go back to Gauss' fundamental memoirs on least squares, namely the *Theoria combinationis observationum erroribus minimis obnoxiae* of 1823 and the *Supplementum* of 1826.

We observe or measure *directly* a certain number of quantities a, b, c, d, ... Each of these quantities is supposed by Gauss to be independent and to follow the normal law. The combined probability of the system* is accordingly

$$P \propto e^{-\frac{1}{2}\left\{\left(\frac{a-\bar{a}}{\sigma_a}\right)^2+\left(\frac{b-\bar{b}}{\sigma_b}\right)^2+\left(\frac{c-\bar{c}}{\sigma_c}\right)^2+\dots\right\}},$$

or the product of the independent probabilities, where $\sigma_a, \sigma_b, \sigma_c, \dots$ are the variability in errors of $a, b, c, \dots$ and $\bar{a}, \bar{b}, \bar{c}, \dots$ the means. This probability will be a maximum when

$$u^2_{a,\, b,\, c,\, \dots} = S\left(\frac{a-\bar{a}}{\sigma_a}\right)^2$$

is a minimum. This is really the principle of weighted least squares. Its validity depends upon the normal law of distribution of error. Without this law holding it may be a utile method, but we have no means of proving it the 'best.'

The investigator in Gauss' case is, however, not interested in the quantities observed, but in certain indirectly ascertained quantities $x_1, x_2, \dots x_n$ which are functions of them. Thus

$$\begin{aligned} x_1 &= f_1(a, b, c, \dots), \\ x_2 &= f_2(a, b, c, \dots), \\ \dots &= \dots\dots\dots\dots\dots, \end{aligned}$$

where $f_1, f_2, \dots$ are known functions. Now Gauss cannot as a rule express from these general equations $a, b, c, \dots$ in terms of $x_1, x_2, \dots x_n$.

He assumes that all of them differ slightly from their mean or 'true' values and accordingly expands by Taylor's theorem and reaches the result†

$$\begin{aligned} x_1 - \bar{x}_1 &= \alpha_1(a-\bar{a}) + \beta_1(b-\bar{b}) + \gamma_1(c-\bar{c}) + \dots, \\ x_2 - \bar{x}_2 &= \alpha_2(a-\bar{a}) + \beta_2(b-\bar{b}) + \gamma_2(c-\bar{c}) + \dots, \end{aligned}$$

where the $\alpha, \beta, \gamma, \dots$ are $\frac{df}{da}, \frac{df}{db}, \frac{df}{dc}, \dots$ and can be ascertained *a priori*. Clearly Gauss supposes that a linear relationship is adequate, in other words he replaces statistical differentials by mathematical differentials, a step he does not really justify.

From these linear equations we can find the $a-\bar{a}$, $b-\bar{b}$, $c-\bar{c}$, ... in terms of the indirectly observed variables $x_1-\bar{x}_1$, $x_2-\bar{x}_2$, $x_3-\bar{x}_3$, ... by solution in determinantal form, say

$$\begin{aligned} a-\bar{a} &= A_1(x_1-\bar{x}_1) + B_1(x_2-\bar{x}_2) + C_1(x_3-\bar{x}_3)\dots, \\ b-\bar{b} &= A_2(x_1-\bar{x}_1) + B_2(x_2-\bar{x}_2) + C_2(x_3-\bar{x}_3)\dots. \end{aligned}$$

Substituting in u^2 we find

$$u^2_{x_1,\, x_2,\, \dots} = S\left(\frac{A_1^2}{\sigma_a^2}\right)(x_1-\bar{x}_1)^2 + S\left(\frac{B_1^2}{\sigma_a^2}\right)(x_2-\bar{x}_2)^2 + 2S\left(\frac{A_1B_1}{\sigma_a^2}\right)(x_1-\bar{x}_1)(x_2-\bar{x}_2) + \dots.$$

Hence the probability of $x_1, x_2, \dots$ occurring is

$$P \propto e^{-\frac{1}{2}u^2_{x_1,\, x_2,\, \dots}}.$$

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* I use throughout notation which I assume now-a-days to be more familiar than that of Gauss.

† $\bar{a}, \bar{b}, \bar{c}, \dots$ are actually in Gauss' method approximate or *guessed* solutions not means, but this does not affect the general nature of the discussion.

This is a normal surface which contains *the product terms*. As we now interpret it we say that the x's are *correlated* variates. And in this sense Gauss in 1823 reached the normal surface of n correlated variates. But he does not seek to express all his relations in terms of the S.D.'s σ_{x_1}, σ_{x_2}, σ_{x_3}, ... and the correlations r_{12}, r_{23}, ... of these variates. These x-variates are not for Gauss, nor for those who immediately followed him, the *directly* observed quantities. What he is seeking is the expression for σ_x, or the probable error of an indirectly observed variate in terms of

$$S\left(\frac{A_1^2}{\sigma_a^2}\right),\quad S\left(\frac{B_1^2}{\sigma_a^2}\right),\quad S\left(\frac{A_1 B_1}{\sigma_a^2}\right),\ \dots.$$

In this case A, B, C are ratios of minors and determinants of the α, β, γ, ... which are Gauss' known quantities. His object therefore is to express σ_x not from direct observations but in terms of α, β, γ, ... through the sums of determinantal terms.

Writers on Least Squares and Adjustment of Observations then take w any function of x_1, x_2, ... x_n, i.e.

$$w = F(x_1, x_2, \dots x_n),$$

express the relation in a linear form, i.e.

$$w - \bar{w} = \lambda_1 (x_1 - \bar{x}_1) + \lambda_2 (x_2 - \bar{x}_2) + \dots,$$

and then, to find σ_w^2, go through lengthy analysis to determine

$$\text{Mean } (x_1 - \bar{x}_1)^2, \quad \text{Mean } (x_2 - \bar{x}_2)^2, \quad \text{Mean } (x_1 - \bar{x}_1)(x_2 - \bar{x}_2), \text{ etc.}$$

in terms of the original α, β, γ, ... There is not a word in their innumerable treatises that what is really being sought are the mutual correlations of a system of correlated variables. The mere using of the notation of the correlational calculus throws a flood of light into the mazes of the theory of errors of observation. There is much more in the theory of least squares than I have stated; there are equations of conditions—the angle and side equations of geodesy, etc.—these only complicate the matter. The point is this: that the Gaussian treatment leads (i) to a non-correlated surface for the directly observed variates, (ii) to a correlation surface for the *indirectly* observed variates. This occurrence of product terms arises from the geometrical relations between the two classes of variates, and not from an organic relation between the indirectly observed variates appearing on our direct measurement of them.

It will be seen that Gauss' treatment is almost the inverse of our modern conceptions of correlation. For him the *observed* variables are independent, for us the observed variables are associated or correlated. For him the non-observed variables are correlated owing to their known geometrical relations with observed variables; for us the unobservable variables may be supposed to be uncorrelated causes, and to be connected by unknown functional relations with the correlated variables. In short there is no trace in Gauss' work of observed physical variables being—apart from equations of condition—associated organically which is the fundamental conception of correlation.

(2) The next important work to be considered is that of August Bravais. It is entitled "Sur les probabilités des erreurs de situation d'un point." It was published in the *Mémoires présentés par divers savants à l'Académie royale des Sciences de l'Institut de France,* T. IX. Paris, 1846, pp. 256—332. It appears, however, to have been reported favourably upon in 1838*. Bravais was in many respects a remarkable man. Essentially a geologist he wrote also on astronomy, physics, meteorology and the theory of probabilities. He made a voyage to Lapland for geodesic purposes and took the opportunity of measuring a number of Lapp skulls! He had a width of action most sympathetic to the biometrician.

Writing in 1895 of the history of correlation I said:

"The fundamental theorems of correlation were for the first time and almost exhaustively discussed by Bravais [Title as above of his memoir] nearly half a century ago. He deals completely with the correlation of two and three variables." Then speaking of Galton's coefficient of correlation I say: "This indeed appears in Bravais' work, but a single symbol is not used for it. It will be found of great value in the present discussion. In 1892 Professor Edgeworth, also unconscious of Bravais' memoir, dealt in a paper on 'Correlated Averages' with correlation for three variables (*Phil. Mag.* Vol. XXXIV. 1892, pp. 194—204). He obtained results identical with Bravais', although expressed in terms of 'Galton's functions'" [i.e. coefficients of correlation].

Again later, p. 287, in giving the fundamental equation for the correlation of three variates I wrote: "This agrees with Bravais' result, except that he writes for r_1, r_2, r_3 the values $\Sigma(yz)/(n\sigma_2\sigma_3)$ etc., which we have shown to be the best values (see *loc. cit.* p. 267)." Again on p. 301 I write before proving the general theorem of multiple correlation: "*Edgeworth's Theorem.* We may stay for a moment over the results above to deduce Professor Edgeworth's Theorem," with the footnote, "Briefly stated with some rather disturbing printer's errors in the 'Phil. Mag.' Vol. XXXIV. p. 201, 1892."

Now all these statements if they were correct would indicate that Bravais discovered correlation before Galton and that Edgeworth first published the form of the multiple correlation surface. They have been accepted by later writers, notably Mr Yule in his manual of statistics, who writes (p. 188):

"Bravais introduced the product-sum, but not a single symbol for a coefficient of correlation. Sir Francis Galton developed the practical method, determining his coefficient (Galton's function as it was termed at first) graphically. Edgeworth developed the theoretical side further and Pearson introduced the product-sum formula."

Now I regret to say that nearly the whole of the above statements are hopelessly incorrect. Bravais has no claim, whatever, to supplant Francis Galton as the discoverer of the correlational calculus. For the most part he is simply taking a very special case of the Gaussian analysis, and nowhere on p. 267 of his memoir can I now find that he has used the expressions for the correlation symbols without

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* *Comptes rendus*, T. VII. p. 77.

their names. Again Edgeworth did *not* obtain results identical with Bravais', he went on a route of his own to find the true multiple correlation surface and gave as I said in 1895 only doubtful results. But I fear they were not all due to printer's errors. On re-examining his memoir 25 years later I think he harnessed imperfect mathematical analysis to a jolting car and drove it into an Irish bog on his road, and that it was doubtful analysis not errors of printing which led to his obscure conclusions. I was scarcely justified in 1895 in calling the multiple regression result Edgeworth's Theorem. He had tried in 1892 to solve the problem, and he can hardly be said to have succeeded properly. It is very difficult to explain now how my errors of ascription came about, still less possible is it to understand why later writers have not corrected my false history, but merely repeated it.

As far as I can remember what happened at all, it was as follows. I know that I was immensely excited by Galton's book of 1889—*Natural Inheritance*—and that I read a paper on it in the year of its appearance. In 1891—2 I lectured popularly on probability at Gresham College, taking skew whist contours as illustrations of correlation. In 1892 I lectured on variation, in 1893 on correlation to research students at University College, the material being afterwards published as the first four of my *Phil. Trans.* memoirs on evolution. At this time I dealt with correlation and worked out the general theory for three*, four and ultimately n variables. The field was very wide and I was far too excited to stop to investigate properly what other people had done. I wanted to reach new results and apply them. Accordingly I did not examine carefully either Bravais or Edgeworth, and when I came to put my lecture notes on correlation into written form, probably asked somebody who attended the lectures to examine the papers and say what was in them. Only when I now come back to the papers of Bravais and Edgeworth do I realise not only that I did grave injustice to others, but made most misleading statements which have been spread broadcast by the text-book writers.

(3) Let us now examine Bravais' memoir. He commences by stating that he is going to measure the errors of the determination of the coordinates x, y, z of a point in space. These coordinates are not measured directly but are functions of the observed elements $a, b, c, \ldots$, and he puts

$$\begin{aligned} x &= \phi(a, b, c, \ldots), \\ y &= \psi(a, b, c, \ldots), \\ z &= \chi(a, b, c, \ldots). \end{aligned}$$

He then expands x, y, z linearly in terms of a, b, c assuming that mathematical differentials may be used for errors; thus he writes

$$\begin{aligned} \delta x &= A\delta a + B\delta b + C\delta c + \ldots, \\ \delta y &= A'\delta a + B'\delta b + C'\delta c + \ldots, \\ \delta z &= A''\delta a + B''\delta b + C''\delta c + \ldots. \end{aligned}$$

He tells us that the A, B, C are differential coefficients, i.e. of the known functions ϕ, ψ, χ, and that to justify the neglect of higher powers and products we must get

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* Published in the *R. S. Proc.* Vol. LVIII. p. 241, 1895.

rid of constant sources of error which arise chiefly from vices of method of observation, ignorance of physical laws, etc. That they can be removed by increasing the number of our observations, and in surveying—which he has essentially in mind—by using the repeating circle, which destroys the majority of constant errors and lessens the influence of variable causes by the fact itself of repeating the observed angles. It is clear that he is thinking solely of theodolite work, and that his x, y, z are Gauss' indirectly observed quantities, his directly observed quantities being angles and bases a, b, c, ...

He now changes his notation; he uses x, y, z for the errors δx, δy, δz, and m, n, p for δa, δb, δc, and takes equations

$$x = Am + Bn + Cp + \dots$$

and calls x, y, z the *dependent* variables, m, n, p the *independent* variables. He says that Laplace has shown that a variation of x between x and $x + \delta x$ will be of the form

$$\sqrt{\frac{h_x}{\pi}}\, e^{-h_x x^2}\, dx,$$

where h_x is given by

$$\frac{1}{h_x} = \frac{A^2}{h_m} + \frac{B^2}{h_n} + \frac{C^2}{h_p} + \dots.$$

It is therefore clear that he supposes that his *observed* quantities m, n, p, ... are *uncorrelated* in our sense of the word. In fact he gives for two and three variates the expressions

$$\sqrt{\frac{h_m h_n}{\pi . \pi}}\, e^{-(h_m m^2 + h_n n^2)}\, dm\, dn \quad \text{(p. 261)},$$

$$\sqrt{\frac{h_m h_n h_p}{\pi . \pi . \pi}}\, e^{-(h_m m^2 + h_n n^2 + h_p p^2)}\, dm\, dn\, dp \quad \text{(p. 264)}.$$

There is obviously not a single step, not a line in this, which does not occur in Gauss, except that Gauss would use

$$u^2 = h_m m^2 + h_n n^2 + h_p p^2$$

and not trouble to state that the probability was given by the exponential.

Now Gauss' problem was to express the variability of x in terms of the variability of the observed quantities a, b, c, ... or m, n, p, and of the differential coefficients A, B, C. This is absolutely the same as Bravais' problem, and Bravais' treatment goes very little further than Gauss'—indeed it is essentially narrower as while Gauss neither limits the number of his variables nor their nature, Bravais treats only of *position* in space.

I will now give the value of the expression Bravais reaches for his surface of two dimensions, expressing by d^2w the briquette of frequency on $dxdy$:

$$\frac{d^2w}{dx\,dy} = \frac{1}{\pi} \frac{1}{\left\{\Sigma \frac{(AB' - A'B)^2}{h_m h_n}\right\}^{\frac{1}{2}}} e^{-\frac{\left\{x^2 \Sigma \frac{A'^2}{h_m} - 2xy\, \Sigma \frac{AA'}{h_m} + y^2 \Sigma \frac{A^2}{h_m}\right\}}{\Sigma \left(\frac{(AB' - A'B)^2}{h_m h_n}\right)}} \quad \text{(p. 272)}.$$

Now, if you take

$$x = Am + Bn + \dots,$$

$$y = A'm + B'n + \dots,$$

$$\sigma_x^2 = A^2\sigma_m^2 + B^2\sigma_n^2 + \dots,$$

$$\sigma_y^2 = A'^2\sigma_m^2 + B'^2\sigma_n^2 + \dots,$$

$$\text{Mean}\,(\delta x\,\delta y) = r_{xy}\sigma_x\sigma_y = AA'\sigma_m^2 + BB'\sigma_n^2 + \dots.$$

Whence

$$\sigma_x^2\sigma_y^2\,(1 - r_{xy}^2) = \Sigma\,(A^2B'^2 + A'^2B^2 - 2AA'BB')\,\sigma_n^2\sigma_m^2$$

$$= \Sigma\,(AB' - A'B)^2\,\sigma_n^2\sigma_m^2,$$

whence, remembering $h_m = \dfrac{1}{2\sigma_m^2}$, $h_n = \dfrac{1}{2\sigma_n^2}$,

we easily deduce the

$$z = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - r_{xy}^2}}\, e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} - \frac{2r_{xy}xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\frac{1}{1 - r_{xy}^2}}$$

of our familiar notation.

But this is precisely what Bravais does *not* do, and for the simple reason that his x, y, z are not variables which he has directly determined and for which he can directly find σ_x, σ_y and r_{xy}. He is merely seeking to express the variability of x and y in terms of the directly determined constants and certain differential coefficients. This is one of the fundamental problems of the Method of Least Squares and had already been solved by Gauss. Bravais adds so far *nothing whatever* to Gauss' solution of 20 years earlier. If Bravais discovered correlation, then Gauss had done so previously.

As a matter of fact while the above expression shows how a hasty examination of Bravais' memoir might lead one to believe he had reached the correlation surface, he was in fact occupied with an entirely different problem, one which was really only a particular case of Gauss' earlier and more comprehensive work.

We cannot pass over, however, the really valuable portion of Bravais' memoir. It lies in this: Having got his coefficients of x and y in terms of the differential coefficients A, B, C, ... he writes the surface

$$\frac{K}{\pi}\, e^{-K^2(\alpha_1 x^2 - 2\beta_0 xy + \alpha_2 y^2)},$$

and then discusses the properties of a surface of which the contours are

$$ax^2 + 2cxy + by^2 = D,$$

i.e. the familiar ellipses of our normal surface. He gets the conjugate of x-axes as the locus of maximum y's and determines the probability of points lying in certain areas—bounded by similar ellipses or in angular sectors. He gets the line $x = -\dfrac{c}{a}y$, which corresponds to Galton's regression-line. But this is not a result of observing x and y and determining their association, but of the fact that x and y

are functions of certain independent and directly observed quantities. When he thinks of c and a at all, it is not in terms of observations on x and y but of the differential coefficients A, B, C of the geometrical relations between position in space and the angles by which that position is found.

Next we come to his surface of three variates and the treatment is identical. He writes

$$z = \frac{G}{\pi^{\frac{3}{2}}} e^{-(ax^2 + by^2 + cz^2 + 2ezy + 2fxz + 2gxy)} \quad \text{(p. 296)},$$

and his primary object is to determine a, b, c, e, f, g in terms of the differential coefficients A, B, C and the variabilities of the observed independent variates.

Thus he gives

$$a = G^2 \Sigma \frac{1}{h_m h_n} \{(A'B'')\}^2, \quad e = G^2 \Sigma \frac{1}{h_m h_n} \{(A''B)(A''B')\}, \quad \frac{1}{G^2} = \Sigma \frac{\{(AB'C'')\}^2}{h_m h_n h_p}.$$

There is throughout merely the standpoint of the Gaussian method of treating errors of observation, and if we are to attribute any discovery of the idea of correlation to Bravais we must with the same confidence assert that Gauss was the primary originator of the whole idea. To my mind this is absurd*. In the case of both these distinguished men the quantities they were observing were *absolutely independent*; they neither of them had the least idea of correlation between observed quantities. The product terms in their expressions—never analysed in the sense of correlation—arise solely not from organic relationships, but from the *geometrical* relationships which exist between their observed quantities and the indirectly observed quantities they deduce from them. Bravais himself (p. 331) says that the application of his results are narrowly circumscribed by the nature of his assumptions—astronomy and the great geodesic surveys alone provide sufficiently accurate material. As far as Gauss and Bravais are concerned we must, I think, hold that they contributed nothing of real importance to the problem of correlation, and that my statement of 1895 was a totally erroneous one.

The same criticism applies to all the treatment of the normal surfaces by later writers, which are described at very considerable length by Czuber in his *Theorie der Beobachtungfehler*, Leipzig, 1891. In all cases the variables are *indirectly* observed quantities and the product terms arise because they are mathematically supposed to be linear functions of the directly observed, but quite independent variables. That the directly measured quantities might themselves be correlated does not seem to have occurred to the many writers on the theory of observations.

As far as I am aware there is nothing to record on our subject beyond the work of the writers on the theory of observations referred to above until we reach Francis Galton himself. His first statement of his ideas was in a lecture at the

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* I feel quite certain that if any one had told either Gauss or Bravais that $\Sigma(ab)$ for their *observed* measurements need not be zero, they would have been laughed out of court, as the astronomers now laugh at us, when we assert that their measurements of different stellar magnitudes are very probably correlated!

Royal Institution, Feb. 9, 1877. He had found it very difficult to collect human material for two generations and after careful consideration selected sweet pea seeds. These seeds were both measured and weighed and actually observations were taken on foliage and length of pod although as far as I am aware Galton never published the reductions of the latter. As he himself writes in 1885: "It was anthropological evidence that I desired, caring only for the seeds as means of throwing light on heredity in man. I tried in vain for a long and weary time to obtain it in sufficient abundance, and my failure was a cogent motive, together with others, in inducing me to make an offer of prizes for family records, which was largely responded to, and furnished me last year with what I wanted*."

The title of Galton's R. I. lecture was *Typical Laws of Heredity in Man*. Here for the first time appears a numerical measure r of what is termed 'reversion' and which Galton later termed 'regression.' This r is the source of our symbol for the correlation coefficient, which was really the first letter of 'reversion' not of 'regression.' The main results are given in a mathematical appendix†. Galton works with the modulus—i.e. our $\sqrt{2}\sigma$—probably because the tables of the probability integral were then given in the modulus as argument. But we can at once convert into more customary notation. Thus we find the now familiar result

$$v = \sqrt{1 - r^2}\,c_1,$$

or, translating his symbols:

$$\text{Variability of family} = \sqrt{1 - r^2} \times \text{variability of general population.}$$

Galton had already reached the idea of homoscedasticity in the arrays of offspring. "I was certainly astonished to find the variability of the produce of the little seeds to be equal to that of the big ones; but so it was and I thankfully accept the fact, for had it been otherwise, I cannot imagine, from theoretical considerations, how the typical problem could be solved" (p. 10).

Next Galton supposes the mean taken of both parents and notes that the "variability of the parentage," what he would have called later the mid-parentage, $= \frac{1}{\sqrt{2}}$ variability of either parent. He has not yet reached the idea of reducing one sex to the standard of the other, and the result is only true, if we have to deal with characters not sexually differentiated.

Now we come to the test point‡:

"Reversion"—Galton tells us, p. 10—"is the tendency of the ideal mean filial type to depart from the parental type, reverting to what may be roughly and perhaps fairly described as the average ancestral type. If family variability had been the only process in simple descent that affected the characteristics of a sample the dispersion of the race from its mean ideal type would indefinitely increase with

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* Address to Anthropological Section, *B. A. Report*, 1885, p. 1207.

† Royal Institution of Great Britain, Friday, February 9, 1877.

‡ Let the reader remember that these words were spoken just 40 years ago, and that they waited 12 to bring forth fruit!

the number of generations, but reversion checks this increase, and brings it to a standstill."

Galton's proof assuming homoscedasticity is of a very simple nature. Let the reversion be λx, where x is the parental character. Then the mean variability of the offspring generation

$$= \sigma^2 (1 - r^2) + \lambda^2 (\text{mean } x^2)$$
$$= \sigma^2 (1 - r^2) + \lambda^2 \sigma^2.$$

Therefore unless $\lambda = r$ the population cannot remain stable. Or without any hypothesis as to normality, only on the basis of linearity of reversion, homoscedasticity and stability the Galton coefficient r of reversion must be equal to the r which gives the reduction of the 'family variability.' Thus the lecture of 1877, while it contains points which later work was to clear up, still in the main lines gives on the data for size in sweet pea seeds the fundamental properties of the regression line. I have worked out Galton's data for sweet peas and show you a diagram of the result which Miss A. Davin has prepared for me. The parent seed was of course selected seed, and Galton took 100 of each parental grade and determined the mean of the offspring, which of course were non-selected seed, i.e. not seedsman's seed. Galton fixes the regression in round numbers at $\frac{1}{3}$, I make it slightly larger. In any case the regression coefficient is small, if we consider the sweet pea, as Galton did, as self-fertilising. It has been so proclaimed in several botanical investigations on heredity in the sweet pea. But in 1907 I watched a row of sweet peas and observed *Megachile Willughbiela*, the leaf-cutting bee, in quite considerable numbers visiting the flowers. The Superintendent of the R. H. S.'s garden at Wisley also replied to an inquiry that he had no doubt some English insect cross-fertilised sweet peas because in trying new sorts the gardeners had to place the rows in different parts of the garden to reduce the risk of cross-fertilisation. Darwin's statement* that "in this country it"—the sweet pea—"seems invariably to fertilise itself," appears open to question. Galton's coefficient may therefore, although low, be not so low as it appears on the assumption of self-fertilisation.

The next few years Galton was occupied in collecting material for further investigation of regression and heredity. He had established his Anthropometric Laboratory at South Kensington and by offering prizes obtained his Records of Family Faculties. The first-fruits of these data are to be found in his Presidential Address to the Anthropological Section of the British Association at Aberdeen in 1885. The part of this Address dealing with regression was considerably extended in a paper read to the Anthropological Institution in the same year. Galton now deals with the inheritance of stature and transmutes female to male stature before determining his mid-parentages. He does this, not as we should do now by multiplying by the ratio of paternal and maternal standard deviations, but by the multiplying factor of mean statures 1·08. This is roughly permissible if the coefficients of variation for the two sexes are the same as they very nearly are for stature. In this paper we have the first published diagram of the *two* regression

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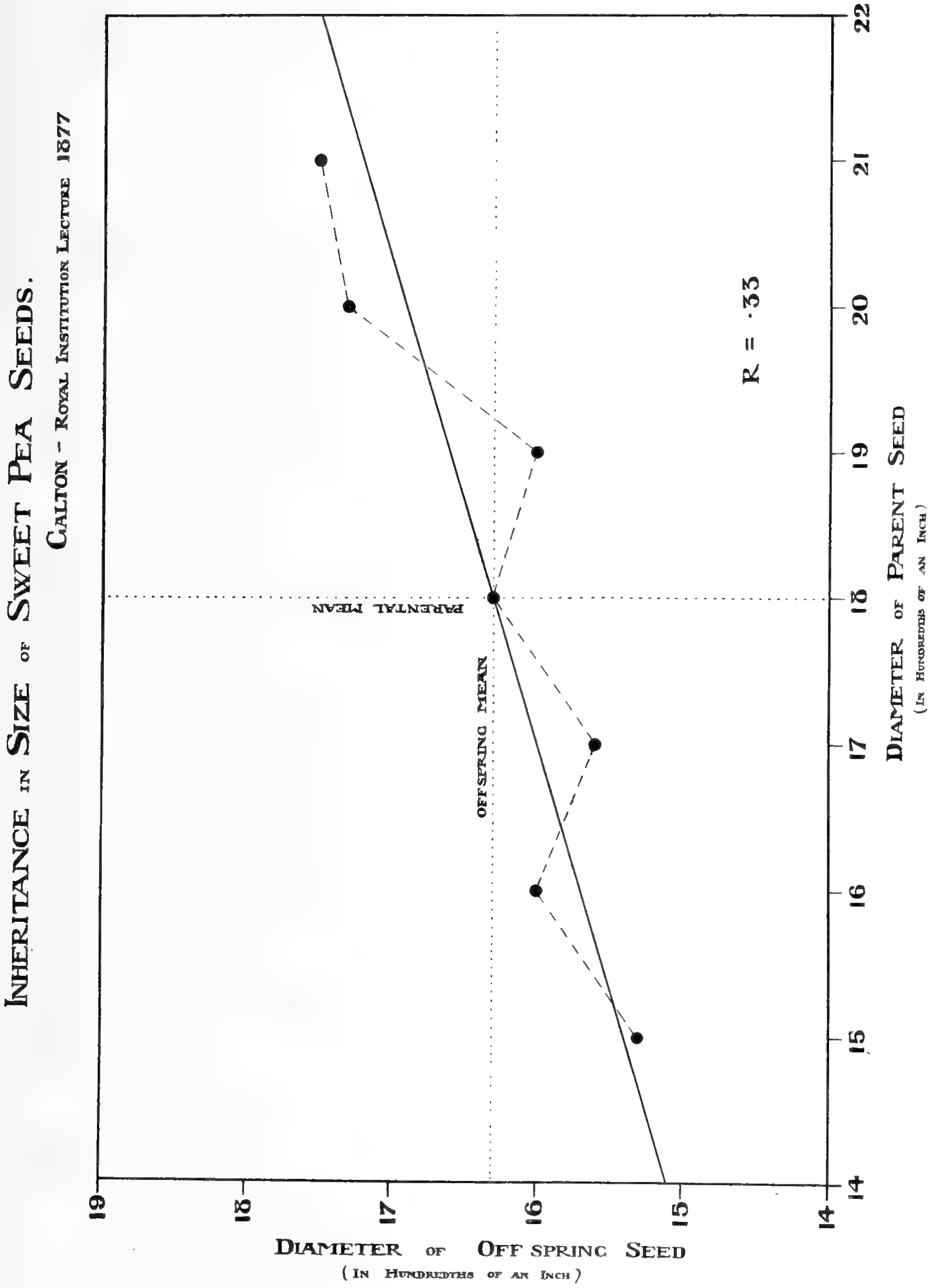
* *Cross and Self-fertilisation of Plants*, 1878, p. 153.



INHERITANCE IN SIZE OF SWEET PEA SEEDS.

GALTON - ROYAL INSTITUTION LECTURE 1877

lines and the first correlation table (of stature in parentage and offspring) as we should now call it.

Also Galton gives the diagram* which indicates how he discovered observationally the form of the normal frequency surface. He proceeded to smooth his correlation table by taking the mean of four adjacent cells, and then drawing contour lines through points of the same frequency. He found such contour lines were a system of concentric similar and similarly placed ellipsoids and that the regression lines

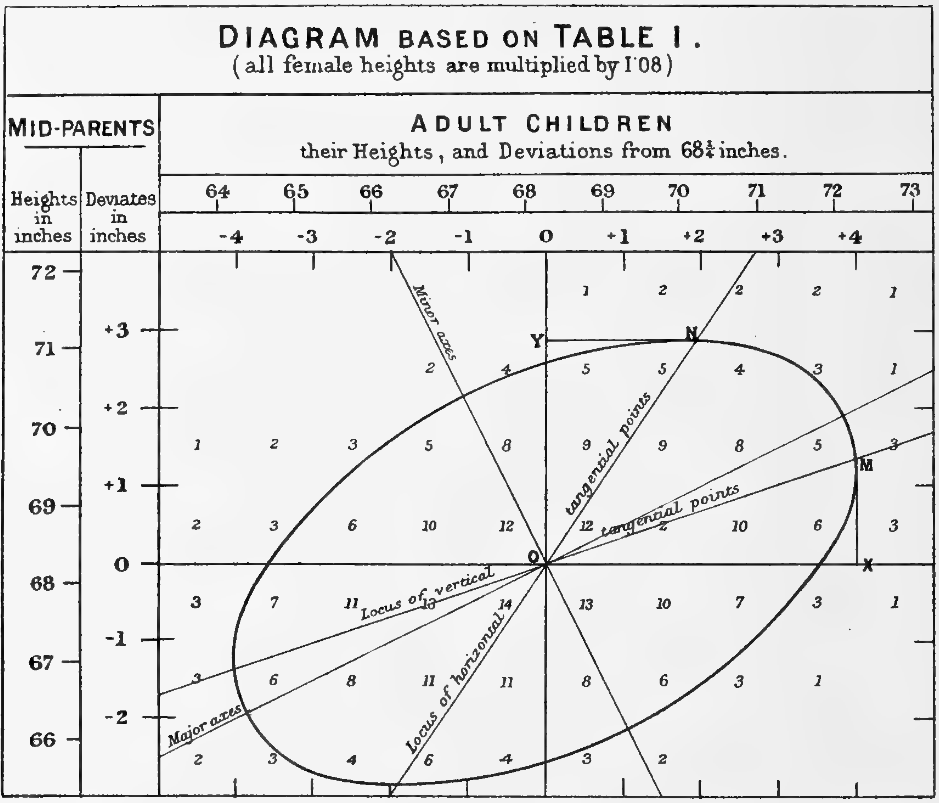
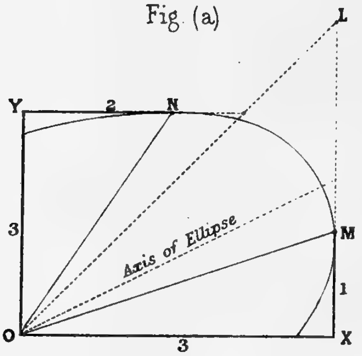


Fig. (a)



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* Reproduced here by permission of the Royal Anthropological Institute.

were what the mathematician terms the conjugate diameters of the variate axes. He discovered that the sections parallel to the variate axes were 'apparently' normal curves of equal S.D. but that this S.D. was reduced and bore a constant ratio to the S.D. of the general population. He knew 8 years earlier the relation of $\sigma\sqrt{1-r^2}$ to the 'reversion' coefficient r. That Galton should have evolved all this from his observations is to my mind one of the most noteworthy scientific discoveries arising from pure analysis of observations.

Why Galton did not at once write down the equation to his surface as

$$z \propto e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}} \times e^{-\frac{1}{2}\frac{1}{\sigma_x^2(1-r^2)}\left(x - r\frac{\sigma_x}{\sigma_y}y\right)^2}$$

has always been a puzzle to me. Actually he carried the problem, stated in the language of probability, to Mr J. D. Hamilton Dickson, a mathematician of Peterhouse, Cambridge, who after stating the wording of Galton's problem, wrote down the answer substantially as above in the fourth line of his memoir*! The fact is that Galton's statement of his problem, involving as it did the assumption of normal distribution, homoscedasticity and linear regression, provided the answer the moment his results were read in symbols. The explanation of Galton's action possibly lies in the fact that Galton was very modest and throughout his life underrated his own mathematical powers.

Thus in 1885 Galton had completed the theory of bi-variate normal correlation. The next stage in the theory of correlation, multi-variate correlation, was directly indicated by the general problem of ancestry. As is now well known the fundamental regression equation is

$$\frac{x_0 - \bar{x}_0}{\sigma_0} = -\Sigma\left(\frac{R_{0s}}{R_{00}}\frac{x_s - \bar{x}_s}{\sigma_s}\right) \quad \text{..........................(i)},$$

where R_{pq} is the p, q minor of the determinant

$$R = \begin{vmatrix} 1 & r_{01} & r_{02} & \dots & r_{0n} \\ r_{10} & 1 & r_{12} & \dots & r_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ r_{n0} & r_{n1} & r_{n2} & \dots & 1 \end{vmatrix}$$

and the variability of the array is

$$\sigma_0\sqrt{\frac{R}{R_{00}}} \quad \text{....................................(ii)}.$$

Galton endeavoured to reach this by a short cut, and thus evolved his law of ancestral heredity. This was a brilliant and suggestive step, but he was not able to state the conditions under which it is theoretically correct or bring forward data at that time to confirm its observational accuracy.

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* *R. S. Proc.* Vol. XL. p. 63, 1886. Galton himself writes (*B. A. Report*, 1885, p. 1211), "I may be permitted to say that I never felt such a glow of loyalty and respect towards the sovereignty and magnificent sway of mathematical analysis as when his answer reached me, confirming, by purely mathematical reasoning, my various and laborious statistical conclusions with far more minuteness than I had dared to hope, for the original data ran somewhat roughly, and I had to smooth them with tender caution."

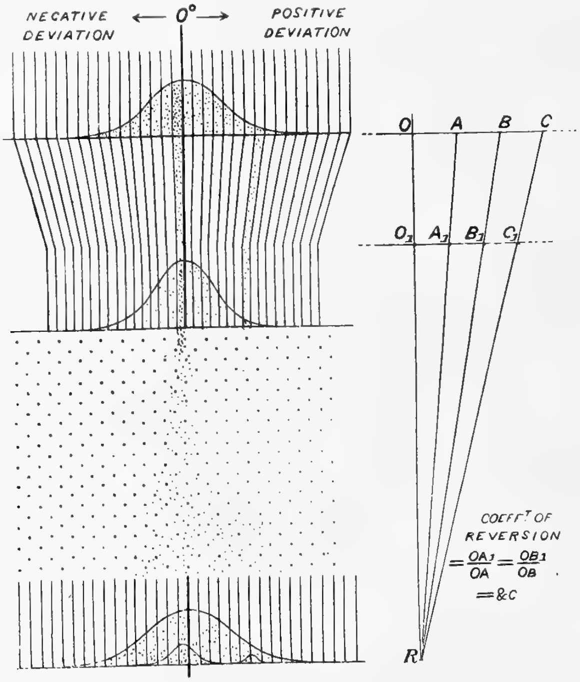
Another feature of Galton's work at this time must be noted. He worked with the median instead of the mean, and he used probable errors or quartile values instead of standard deviations. Further, to obtain r, he somewhat laboriously expressed both variates in terms of their quartile deviations: thus r became the slope of his regression line. It was then determined by graphically fitting a good line, or from certain chosen arrays. Thus he worked with somewhat primitive statistical tools, and the wonder is that he achieved as much by their aid as he did.

Given A and B with regression r_{ab}, B and C with regression r_{bc}, Galton assumed $r_{ac}=r_{ab}\times r_{bc}$ to obtain his kinship relations. A nephew is the son of a brother. Hence

$$r \text{ for uncle and nephew} = r \text{ for brothers} \times r \text{ for father and son.}$$

This of course is incorrect; it implies the vanishing of the corresponding partial correlation coefficient. Again, I think, his mid-parental correlation is not theoretically consonant with his parental correlations.

Another noteworthy point of the 1877 R. I. and the 1885 A. I. papers is the ample provision of mechanical models to illustrate by dropping shot or seeds the properties of bi-variate frequency. One wonders whether these elaborate quincunxes have been preserved and if so where they are at the present time. I reproduce one of them by permission from the *Journal of the Royal Institution*.



In 1886 Galton published a paper in the *Royal Society Proceedings** on "Family Likeness in Stature." This contains Hamilton Dickson's note and further data from Galton's *Family Records*.

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* Vol. XL. pp. 42—66.

He gives under the headings "Mean regression *w*," and "Quartile of individual variability" the coefficients of correlation of various pairs of relations: Midparent and Offspring, Brothers, Fathers and Sons, Uncles and Nephews, Grandparents and Grandsons, but he does not realise that on the theory of multiple regression there are certain inconsistencies in his values. I do not think that there is much additional contribution to theory in this paper.

In 1888, however, Galton took a great step forward. He recognised that the whole statistical apparatus he had evolved for the treatment of the problem of heredity had a vastly wider significance. In a paper read to the Royal Society on December 5, 1888*, entitled "Correlations and their Measurement chiefly from Anthropometric Data," the term correlation first appears in our subject. Thus Galton's opening lines run:

> "Co-relation or correlation of structure" is a phrase much used in biology, and not least in that branch of it which refers to heredity, and the idea is even more frequent than the phrase; but I am not aware of any previous attempt to define it clearly, to trace its mode of action, or to show how to measure its degree.
>
> Two variable organs are said to be correlated when the variation of the one is accompanied on the average by more or less variation of the other, and in the same direction (p. 135).

The last words seem to us now out of place, but Galton had not yet reached the idea of negative correlation. Also the balance is still swinging between 'co-relation' and 'correlation' although it has ultimately fallen to the more weighty word. How clearly Galton grasped the essence of correlation may be shown by the following sentences which might have saved many ingenious later investigators thinking they had made an important discovery. "It is easy to see that co-relation must be the consequence of the variations of the two organs being partly due to common causes. If they were wholly due to common causes, the co-relation would be perfect, as is approximately the case with the symmetrically disposed parts of the body. If they were in no respect due to common causes, the co-relation would be *nil*. Between these two extremes are an endless number of intermediate cases, and it will be shown how the closeness of co-relation in any particular case admits of being expressed by a single number" (p. 135). This single number it is needless to say is our present coefficient of correlation. Galton drops now the *w* of his 1886 work and returns to the *r* of his 1877 lecture, and the symbol *r* has remained to the present day.

Galton's process is the same as in the heredity problem. He used median and quartile and reduces the deviations to their respective quartiles as unit. He then smooths his means of arrays, draws a line to represent them and reads off its slope as *r*. He thus determines seven correlations which he here terms "indices of correlation†." They are between Stature and Cubit, Stature and Head Length, Stature and Middle Finger Length, Cubit and Middle Finger Length, Head Length and Head Breadth, Stature and Height of Knee, Cubit and Height of Knee. He fully

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* *R. S. Proc.* Vol. XLV. pp. 135—145.

† On p. 143 *r*, the index of co-relation, is identified with the 'regression' or 'reversion' of Galton's earlier papers.

realises (1) that r is the same when obtained from either variate as 'relative,' (2) that r is always less than unity, (3) that r measures the closeness of co-relation, and (4) provides the regression line (p. 145).

On p. 144 the term "partial co-relation" is used but hardly in our modern sense although Galton is feeling his way towards multiple correlation. One problem he gives on p. 144 perhaps deserves mention, namely, if the n variates be expressed in terms of their quartiles then the quartile variability of their sum is $\sqrt{n}$ if they are independent and n if they be "rigidly and perfectly co-related." "The actual value would be almost always somewhere intermediate between these extremes, and would give the information that is wanted."

What Galton needs is the "multiple correlation coefficient," i.e.

$$\sqrt{1-\frac{R}{R_{00}}},$$

but he is not yet on the right track for it.

In 1889 appeared Galton's book *Natural Inheritance* embodying most of the work we have discussed in the earlier memoirs of 1877 to 1888. Beyond this Galton did not carry the subject of correlation. He, in my opinion to-day, created it; there is nothing in the memoirs of Gauss or Bravais that really antedates his discoveries. They were dealing with the relatively narrow problem of determining the probable errors of indirectly observed quantities deduced from independent or uncorrelated directly observed quantities. The product-terms that arise in their investigations were expressed in terms of differential coefficients; they were not treated as a means of determining organic relationships between directly measured variates. Galton, starting from the organic relationship between parent and offspring, passed to the idea of a coefficient measuring the correlation of all pairs of organs, and thence to the 'organic' relationship of all sorts of factors. If you think Galton did not appreciate the width of his new methods you must turn to the last paragraph of his *Introduction* to the *Natural Inheritance*.

"The conclusions cannot, however, be intelligently presented in an introductory chapter. They depend on ideas that must first be well comprehended, and which are now novel to the large majority of readers and unfamiliar to all. But those who care to brace themselves to a sustained effort, need not feel much regret that the road to be travelled over is indirect, and does not admit of being mapped beforehand in a way they can clearly understand. It is full of interest of its own. It familiarizes us with the measurement of variability, and with curious laws of chance *that apply to a vast diversity of social subjects.* This part of the inquiry may be said to run along a road on a high level that affords wide views in unexpected directions, and from which easy descents may be made to totally different goals to those we have now to reach. I have a great subject to write upon..." (p. 2).

Galton realised as fully as any of us now the width of application that would open up to the new calculus of correlation, and what easy descents there would be from the "high level road" to strange goals. His notebooks of this period show that he was applying correlation and the regression line in a variety of ways thus to the relation between wing and tail length in birds, to fertility and to disease. His advance was chiefly hampered by the restriction of his data and the need for organised observers and computers.

The publication of *Natural Inheritance* provided Francis Galton with at least three recruits for the field of correlation: Weldon, Edgeworth and myself.

Weldon started in 1889 measuring the organs of shrimps at Plymouth and he was able to announce early in 1890—the letter is now in the glass case in our library here—the first correlation coefficients, or as he termed them "Galton Functions," between organs in shrimps. This was rapidly followed by his work on crabs, and the attempt to show that Galton functions were the same for all local races of the same species. In his first paper on shrimps Weldon writes[*]:

"In making this investigation I have had the great privilege of being constantly advised and helped in every possible way by Mr Galton. My ignorance of statistical methods was so great that without Mr Galton's constant help, given by letter at the expenditure of a very great amount of time and trouble, this paper would never have been written."

The pupil, however, was soon to outdistance the master in the width of his theoretical knowledge. A second paper on the shrimp followed in 1892[†], and this deals more closely with the correlations. Weldon now replaces medians by means in both marginal totals and arrays. He still uses probable errors or quartiles, and goes through the laborious process of reducing each deviation to the probable error. He uses r "in accordance with Mr Galton's notation" to represent the constant which measures the "degree of correlation" between organs. I think, but it is not quite clear, that he determined his probable error from the mean error, not from the quartile. He then determined r from each individual array and took the mean value of these r's as the true r. He accordingly introduced a greatly increased accuracy into the computing of correlation. He dealt with five local races of shrimps and found correlations for 22 pairs of organs. His regression diagram, p. 8, is still an admirable sample of this type of work. The correlations between post-spinous portion and total carapace length may be cited as illustrations of what Weldon and Galton were testing:

Plymouth (1000)	$r = 0{\cdot}81$
Southport (800)	$r = 0{\cdot}85$
Roscoff (500)	$r = 0{\cdot}80$
Sheerness (380)	$r = 0{\cdot}85$
Helder (300)	$r = 0{\cdot}83$

The suggestion that r has the same value for all races of the same species was supposed to be confirmed by these results. We now realise that without a knowledge of the probable error of r, such a statement is illusory. But it was this very series of values which led to the investigation of the probable error of r and so to the extension of the correlational calculus.

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[*] *R. S. Proc.* Vol. XLVII. p. 445, 1890.

[†] *R. S. Proc.* Vol. XLI. p. 2.

In this paper Weldon also published for the first time with due appreciation of their meaning *negative* correlation coefficients. In conclusion Weldon remarks: "A large series of such specific constants would give an altogether new kind of knowledge of the physiological connexion between the various organs of animals; while a study of those relations which remain constant through large groups of species would give an idea, attainable at present in no other way, of the functional correlations between various organs which have led to the establishment of the great subdivisions of the animal kingdom" (p. 11). In these lines we can read the starting-point of biometry as applied to other types of life than man.

I will not keep you longer over Weldon's contributions than to say that in 1893 appeared his third statistical paper* on "Correlated Variations in Naples and Plymouth shore Crabs." Weldon dealt with 23 pairs of organs in both Naples and Plymouth races. He proposes to call r "Galton's function"†. The paper shows that the 23 values of r at Plymouth and Naples are fairly close, but was again inconclusive because the significance of the differences could not be ascertained without a knowledge of the probable error of r.

We may next turn to Edgeworth, whose fundamental paper is that on "Correlated Averages" which appeared in the *Philosophical Magazine* of August, 1892, pp. 190—204. Edgeworth starts by referring to Galton's memoir of 1888 and Weldon's of 1892 on shrimps. He assumes for the probability that any particular values $x_1, x_2, \dots$ shall occur

$$\Pi = Je^{-R}dx_1dx_2dx_3 \dots,$$

where R is

$$= p_1(x_1 - \bar{x}_1)^2 + p_2(x_2 - \bar{x}_2)^2 + \dots + 2q_{12}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + \dots.$$

He does not justify this assumption but hopes to do so in a subsequent paper. He states that Galton by the happy device of measuring each deviation by the corresponding quartile had reduced in the case of two variates

$$R = \frac{x_1^2}{1-\rho^2} - \frac{2\rho x_1 x_2}{1-\rho^2} + \frac{x_2^2}{1-\rho^2}$$

to the discovery of a single constant ρ. This is hardly accurate; to reduce the expression R to the above it would be needful to measure not in terms of the quartile but of $\sqrt{2}$ S.D., which is I think sometimes termed the 'modulus‡. Edgeworth replaces Galton's "Index of Co-relation" and Weldon's "Galton's Function" by the term "coefficient of correlation." He then proceeds to weaken down Weldon's process of finding a mean r by suggesting that it will be adequate to find it by taking some of the ratios of 'subject' and mean 'relative' instead of the whole series. I look upon this suggestion as a distinctly retrogressive step.

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* *R. S. Proc.* Vol. LIV. pp. 318—329.

† "The importance of this constant in all attempts to deal with the problems of animal variation was first pointed out by Mr Galton...and I would suggest that the constant whose changes he has investigated and whose importance he has indicated, may fitly be known as 'Galton's function,'" p. 325.

‡ Edgeworth appears to realise this on p. 194, but he did not go back and correct his statement of p. 190.

Our object should be to find the 'best value' for r and not how it may be most easily determined at the obvious cost of accuracy.

Although I am unable to follow some of Edgeworth's notation, he undoubtedly reaches something like the correct value for the correlation surface of three variates. In his notation

$$R = \Delta \{(1 - \rho_{23}^2)\, x_1^2 + (1 - \rho_{31}^2)\, x_2^2 + (1 - \rho_{12}^2)\, x_3^2 \\ - 2x_1x_2(\rho_{12} - \rho_{13}\rho_{23}) - 2x_2x_3(\rho_{23} - \rho_{21}\rho_{31}) - 2x_3x_1(\rho_{31} - \rho_{32}\rho_{12})\},$$

where

$$\Delta^{-1} = \{(1 - \rho_{13}^2)(1 - \rho_{12}^2) - (\rho_{12}\rho_{23} - \rho_{13})^2\}$$

according to him, but the factor $1 - \rho_{13}^2$ should be replaced by $1 - \rho_{23}^2$ I think.

Even with this change I am unable to reach the value he gives in Galton's case of

$$\rho_{12} = {\cdot}8, \quad \rho_{13} = {\cdot}9, \quad \rho_{23} = {\cdot}8,$$

for these seem to give

$$\Delta = 9{\cdot}9305,$$

whereas Edgeworth's value is 16·129.

I do not grasp his equation at the foot of p. 196, nor follow how the equation at the top of p. 197 follows from it.

Lastly we come to p. 201 where we should expect to find the general regression equation. Edgeworth tells us that the reasoning is quite general and accordingly we ought to anticipate that his results whatever they are would give our accepted values

$$p_{ss} = \frac{R_{ss}}{R} \text{ and } p_{ss'} = \frac{R_{ss'}}{R},$$

where R is the determinant of the correlations. Instead of this simple rule Edgeworth sums up in the middle of the page with equations

$$\begin{aligned} \Delta\rho_{13} &= + \Delta^3(\rho_{24}\rho_{31}\rho_{42}), \\ \Delta\rho_{14} &= - \Delta^3(\rho_{21}\rho_{32}\rho_{43}), \\ \vdots \quad & \quad \vdots \quad \quad \vdots \end{aligned}$$

There is no explanation of what the symbolism means, and I cannot interpret it, so as to provide the requisite generalisation for n variates.

On the other hand while unable to interpret Edgeworth's general analysis I agree in the case of four variates with the only two terms I have taken the trouble to test in his numerical illustration of this case,

$$\rho_{12} = \frac{1}{\sqrt{2}}, \quad \rho_{13} = \sqrt{\frac{1}{3}}, \quad \rho_{14} = \sqrt{\frac{1}{4}}, \quad \rho_{23} = \sqrt{\frac{2}{3}}, \quad \rho_{24} = \sqrt{\frac{2}{4}}, \quad \rho_{34} = \sqrt{\frac{3}{4}},$$

namely 2 as the coefficient of x_1^2 and $-2\sqrt{2}$ as the coefficient of x_1x_2, my R being $\frac{1}{24}$, R_{11} being $\frac{1}{12}$ and R_{12} being $\frac{-\sqrt{2}}{24}$. Edgeworth does not provide the needful external constant of the frequency surface, i.e.

$$\frac{N}{(2\pi)^n \sigma_1 \sigma_2 \dots \sigma_n} \frac{1}{\sqrt{R}}.$$

I should sum up Edgeworth's work of 1892 by saying that he left the problem of multiple correlation at least in a very incomplete state. He probably knew what he was seeking himself, but he did not give the requisite attention to the wording or printing of his memoir to make it clear to others, and accordingly in looking back at the matter now I am very doubtful whether in 1895 I ought to have called the problem of multiple correlation, "Edgeworth's Problem." He certainly did not put the answer to it in a form in which the statistician with a customary amount of mathematical training could determine the form of the surface for n variates, as soon as their S.D.'s and correlations had been calculated. I think I am justified in saying this for I have not to my recollection come across any treatment of multiple correlation which starts from Edgeworth's paper or uses his notation.

It will be seen from what has gone before that in 1892 the next steps to be taken were clearly indicated. They were, I think,

(*a*) The abolition of the median and quartile processes as too inexact for accurate statistics.

(*b*) The replacement of the laborious processes of dividing by the quartiles and averaging the deduced values of r, by a direct and if possible 'best' method of finding r.

(*c*) The determination of the probable errors of r as found by the 'best' and other methods.

(*d*) The expression of the multiple correlation surface in an adequate and simple form.

These problems were solved by Dr Sheppard or myself before the end of 1897.

Closely associated with these problems arose the question of generalising correlation. Why should the distribution be Gaussian, why should the regression curve be linear?

As early as 1893 I dealt with quite a number of correlation tables for long series and was able to demonstrate

(i) by applying Galton's process of drawing contours of equal frequency that most smooth and definite systems of contours can arise from long series, obviously mathematical families of curves, which are (*a*) ovaloid, not ellipsoid, and (*b*) which do not possess—like the normal surface contours—more than one axis of symmetry,

(ii) that regression curves can be quite smooth mathematical curves differing widely from straight lines,

(iii) that in cases wherein (i) and (ii) hold, homoscedasticity is not the rule.

I obtained differential equations to such systems, but for more than 25 years while often returning to them, have failed to obtain their integration.

This seems to me the desideratum of the theory of correlation at the present time: the discovery of an appropriate system of surfaces, which will give bi-variate skew frequency. We want to free ourselves from the limitations of the normal surface, as we have from the normal curve of errors.

As early as 1897 Mr G. U. Yule*, then my assistant, made an attempt in this direction. He fitted a line or plane by the method of least squares to a swarm of points, and this has been extended later to n-variates and is one of the best ways of reaching the multiple regression equations and the coefficient of multiple correlation†. Now while these methods are convenient or utile, we may gravely doubt whether they are more accurate *theoretically* than the assumption of a normal distribution. Are we not making a fetish of the method of least squares as others made a fetish of the normal distribution? For how shall we determine that we are getting a 'best fit' to our system by the method of least squares?

If we are fitting a curve $y = f(x, c_1, c_2, c_3)$ to a series of observations we can only assert that least square methods are theoretically accurate on the assumption that our observations of y for a given x obey the normal law of errors. That is the proof which Gauss gave of his method and I personally know no other. *Theoretically* therefore to have justification for using the method of least squares to fit a line or plane to a swarm of points we must assume the arrays to follow a normal distribution. If they do not, we may defend least squares as likely to give a fairly good result but we cannot demonstrate its accuracy. Hence in disregarding normal distributions and claiming great generality for our correlation by merely using the principle of least squares, we are really depriving that principle of the basis of its theoretical accuracy, and the apparent generalisation has been gained merely at the expense of theoretical validity. Take other distributions of deviations for the arrays and the method of least squares is not the one which will naturally arise from making the combined probability a maximum. It is by no means clear therefore that Mr Yule's generalisation indicates the real line of future advance.

I have endeavoured to indicate in this paper the broad outline of the early history of correlation which has now a most extensive literature. It is a long step from Francis Galton's 'reversion' in sweet pea seeds to the full theory of multiple correlation, which we now know to be identical with the spherical trigonometry of high-dimensioned space, the total correlation coefficients being the cosines of the edges of the polyhedra and the partial correlation coefficients the cosines of the polyhedral angles. But to find the correlation of the health of a child with the number of people per room while you render neutral its age, the health of its parents, the wages of its father, and the habits of its mother, is no less vital a problem than Galton's correlation of character in parent and offspring. It requires indeed more mathematics, but the mathematics are not there for the joy of the analyst but because they are essential to the solution. It is the transition from the mill as pestle and mortar to the mill with steam driven grain crushing steel rollers. But the inventor of milling was the person who bruised grain between two stones, and Galton was the man who discovered the highway across this new country with what he aptly terms "its easy descents to different goals."

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* *Journal of Royal Statistical Society*, Vol. LX. Part iv, p. 3.

† *Biometrika*, Vol. VIII. p. 438. The method adopted in the paper is not that of fitting a generalised plane by least squares, but of making a generalised correlation coefficient take its maximum value. It appeals only to the rules of the differential calculus and not to the method of least squares, or indirectly to Gauss' law of errors.

HISTOLOGICAL EXAMINATION OF AN ADULT HUMAN ALBINO'S EYEBALL, WITH A NOTE ON MESOBLASTIC PIGMENTATION IN FOETAL EYES.

By C. H. USHER, M.B., B.C., Camb.

A. Introductory.

When some ten years ago the subject of albinism was taken up by Karl Pearson, the late E. Nettleship and myself, the following problems naturally arose from the then state of our knowledge of the subject.

(i) Is there or is there not a complete congenital absence of pigment in the eye of the albino? This problem follows at once on the definition which had several times been repeated, that an albino was a person with complete congenital absence of pigment in the eye. The answer to this first problem could not be reached by clinical methods but only by microscopical examination of excised eyes. At the time just referred to not a single human eye had been reported upon which was entirely devoid of pigment. An examination of a portion of the iris only, even if it is found pigmentless, cannot be conclusive for settling the question of entire absence.

(ii) The presence or absence of a fovea and of yellow colour at the macula in the case of the albinotic eye. Presence and absence of both have been asserted. But in some of the few available cases there is no evidence that a thorough microscopic examination of an adequate series of sections was made.

(iii) What truth is there in the theory originally propounded by Meckel and Mansfeld, that the source of albinism is to be sought in an arrest of development? Such a theory involves the extent of pigmentation in an albino corresponding to that of some foetal stage. Accordingly the study of the development of pigmentation in the foetal eye becomes of very great interest. It would be of greater interest still, if we knew or could find out anything about the pigmentation in the eye of the albinotic foetus.

The accompanying study is a contribution of data bearing upon the above problems. It consists in the microscopic examination of an albinotic eye and of the eyes of normal dark-raced foetuses.

B. THE EYE OF THE ADULT HUMAN ALBINO.

In literature there are few records of a microscopical examination of a human albino eye. In all, these relate to two pieces of iris, removed during operations for the extraction of senile cataract(6), and to the eyeballs of six other cases* (Manz, Nettleship, Fritsch, Adler and McIntosh, Coats, Elschnig). Of the two pieces of iris, one (Nettleship, Fig. F) contained no pigment, the other (Usher) had an abnormally small quantity of pigment in the two posterior epithelial layers and not any in the stroma (Fig. D). Of the six albinos where the whole eyeball was obtained, five were adults, and the sixth was an infant of ten weeks. All of the eyes from four of the adults contained some pigment, in the fifth case Fritsch(7) makes no mention of either the presence or absence of pigment in the eye. In the case of Manz(3) and Elschnig(16), there was some pigment present in the retinal epithelium, but none at all in the stroma of the iris and ciliary body or in the choroid. In the case described by Nettleship(6), the retinal epithelium was pigmented, but the pigment was in much less quantity than normally. There was some pigment in the choroid, more particularly at the macular region, but the stroma of iris and ciliary body was without pigment (Fig. C). Coats(15), in the eye of an albinotic negro (Fig. E), found that the iris pigment epithelium was brown, considerably lighter than in the normal blue European eye. The epithelial pigmentation of the ciliary body was as deep as in many European eyes. The pigmentation was very light in the whole of the posterior part of the globe. The iris stroma was pigmented, but not the stroma of the ciliary body or the choroid (Fig. E). In the infant's eyes, Adler and McIntosh(11) found no pigment. They make no special mention of the histological appearances of the macula. For two reasons an account of this region would have been of interest. In the first place no one has so far recorded the presence of a normal fovea in a human albino eyeball—true, in this albino baby a fovea is mentioned as having been seen macroscopically—and in the second place there is reason to believe that it is to this region particular attention should be given in a search for pigment, and that no eye should be pronounced free from pigment until this part has been carefully examined. It is well known that in the normal European eye the hexagonal cells of the retina are darker at the macula than in the parts immediately surrounding it. Also, the choroidal pigment is frequently dense in this region. Again, in an eye of one of the five adult albinos there is a small quantity of choroidal pigment principally located at the macula, and in the case to be described the only mesoblastic pigment present is situated at the macular region. Furthermore, from an examination of eyes from twelve dark-race foetuses, and new-born and very young infants (see below, pp. 51—4), I have found pigment present considerably before birth both in the iris stroma, and in the choroid especially at

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* Since this paper was written I have found in the *Muenchener Medizinische Wochenschrift*, Nr. 26, 26 June 1917, S. 845, Velhagen's account of a microscopical examination of the eyes of an albinotic man, age 23. No pigment was present in choroid, stroma of iris, and retinal epithelial layer as far as ora serrata. The ciliary body epithelium and posterior surface of iris were pigmented. The macula lutea was not differentiated, no fovea was recognised.

the posterior part of the eye. In a five-months' Malay-Tamil foetus pigment was present in the choroid at the posterior part of the eye (Fig. 1) when no pigment could be detected in the iris stroma. At the same region of the choroid, pigment was present in a seven-months' Cantonese foetus, whilst in the iris it was only with the aid of an oil immersion lens that a few chromatophores containing faint brown granules could be found. In a seven-months' Tamil foetus, the choroid contained round brown pigment granules in the bodies and processes of the chromatophores (Fig. 2). At birth there is much pigment in the iris stroma (Fig. 3) and choroid of these dark-raced eyes, but it falls far short of what obtains in the adult Tamil (Figs. 4 and 5), Chinese, and some other dark-race eyes. Therefore, there is increase of pigment in these structures after birth. In view of these observations and from what is known of the distribution of ocular pigment in albinotic lower animals, it is conceivable that a human albinotic eye with scanty pigmentation may yet be found in which the pigment is limited to the macular region and located in the retinal hexagonal cells and the choroidal cells, or else confined to either the one or the other of these structures. No case has so far been published of a human eyeball containing pigment in the choroid with entire absence of pigment in the hexagonal cells of the retina, though such a condition we have found in the Avian eye—that of a hedge sparrow (*Accentor modularis*). In the iris of man clinical examination, in a few cases, has shown that pigment may be absent from the epiblastic layers whilst present in the stroma (17), though a microscopical examination is still required in such cases to determine whether the pigment is completely absent from the posterior layers. It is probable, as Adler and McIntosh suggest in their case, that had she lived, this infant of ten weeks would have remained a complete albino. In normal European infants, however, the iris, stroma and choroid contain little if any pigment at birth*, and for some months afterwards (18).

The following are the notes of the case before us:

Alexina H., female, age 17, native of Banffshire, admitted to the Aberdeen Royal Infirmary on November 15th, 1918, under the care of Professor A. W. Mackintosh, to whom I am indebted for the particulars of her illness. Since the age of 12 she has been subject to epileptic fits†. In February 1918 she had rheumatic fever, and since that time has never been strong. The diagnosis

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* Rieke (4) states that the earliest deposition of pigment in the human choroid is in the seventh month of foetal life, but that this is by no means the case in all individuals as there is great variation.

Schreiber and Schneider (8) found no trace of pigment in the stroma cells of the uvea in a thirty-three weeks' foetus.

Lauber (9) examined the eyes of human embryos, and found that stroma pigment was completely absent in the eighth month. In an early-born foetus of the ninth month a pigmentation of the stroma cells of the iris could be made out, whilst the choroid was still completely pigmentless.

Treacher Collins and Mayou (12) give the seventh month as the average time at which pigment first makes its appearance in the branched cells of the choroid. The pigmentation of the stroma cells of the iris, they state, does not commence until after birth.

v. Szily (13) says that the pigment of mesodermal origin appears in man shortly before birth or even later, and that it is fully developed during the first year.

† She also had fits in infancy.

made was epilepsy, meningitis, and mitral incompetence. Her thyroid was definitely enlarged; knee jerks plus; plantar reflexes not elicited; no ankle clonus; difficulty in articulation suggestive of a bulbar lesion; reported difficulty in swallowing; patient now cannot walk. On November 18th had incontinence of urine and faeces. On November 20th lumbar puncture was made. The fluid was clear and under slight pressure. Professor Shennan's report of the cerebro-spinal fluid states that "Few gram-negative cocci which did not grow in culture medium were found present." The temperature gradually rose from normal on November 17th to 105° on November 22nd, when she died.

On November 18th I noted that the optic discs were normal. The choroidal vessels were markedly exposed except at the macular region. Each fundus was albinotic. No red reflex was visible in the pupils. The iris was blue, and no pigment could be detected in the stroma, even when it was viewed through a loupe. There was constant lateral nystagmus in each eye. The skin was blond, and the hair was pale yellow in artificial light. When seen again three days later, in good day-light, she was semi-conscious, and at first the eyeballs were motionless, but on being roused a little, the nystagmus commenced again though the movements were slow. In this light the hair was tow colour. When attending the Ophthalmic Department of the Infirmary on October 5th, 1911, the refraction of the eyes was estimated after the use of atropine. The right eye had 1·5 D. of hypermetropia in an oblique meridian and 3 D. of myopia in the opposite meridian; acuteness of vision with correction was $\frac{6}{36}$. The left eye had 1·5 D. of hypermetropia in one meridian, and 2·25 D. of myopia in the opposite meridian; acuteness of vision with correction was $\frac{6}{36}$. At the same date the pupils looked black, though occasionally a port-wine colour was visible. A description of this case, by Dr W. C. Souter, and the pedigree, is published in *Albinism in Man*, Pt. IV. No. 646. At this date, 1909, the hair was white and contained no granules and no diffused pigment. The eyes were less myopic and the pupils were red. No general post-mortem examination was obtained. The left eye was excised 17 hours after death and placed in Zenker's fluid. The eye was opened equatorially, embedded in celloidin and then sectioned.

Macroscopically the ciliary body was well pigmented, and definite brown pigmentation of the fundus was visible. A horizontal fold of retina passed outwards from the optic disc through the macula, and a smaller fold passed nasalwards from the optic disc. (Folds of the retina at the macula are of common occurrence in eyes removed after death.) No fovea was discernible. The retina at the macular region was of a pale yellow colour, such as is seen in the normal eye. A complete series of consecutive sections, cut horizontally, was made through the macula. No section was lost. The anterior portion of the eyeball was cut antero-posteriorly.

Microscopical Examination. The iris contains no stroma pigment. Sections both stained and unstained were searched for pigment with an oil immersion lens. The two epiblastic layers at the back of the iris are pigmented, but not so deeply as those of a normal blue or brown eye. The nuclei of a number of the cells are clearly visible (Fig. A), whilst in a normal eye (Fig. B) they cannot be distinguished because of the dense pigmentation. The pigmented derivatives of the epiblastic layers, V. Michel's and Fuch's spurs at the region of the sphincter muscle, and Grunert's spur at the root of the iris, are feebly represented, and no round clump cells are present. The ciliary body epithelial pigment layer contains less pigment than that in a normal eye, and the nuclei of some of its cells are visible. This layer contains, however, much pigment and terminates posteriorly at the ora serrata in the feebly pigmented hexagonal cells of the retina. The stroma of the ciliary body contains no pigment. The choroid is devoid of pigment, except at the macular region, where there are a few pigmented chromatophores. The hexagonal cells of the retina are considerably less pigmented than those in a normal eye. They are darker at the macula than in the region surrounding it. They contain elongated pigment granules. The retina has several layers of ganglion cells at the macula as in normal eyes, but in none of the sections is there a fovea visible*. The nuclear layers are both present in every section and neither of them disappears at any point, and the same holds good for the ganglion cell layers. The elements composing the bacillary layer have undergone post-mortem change, which makes it impossible to state what their condition was during life.

The distribution of pigment then in this eye as regards the retinal epithelium, is the same as in a normal eye, but it is less in quantity. Mesoblastic pigmentation is entirely absent in the iris, ciliary body and most of the choroid, only a few cells at the macular region contain any pigment. In all these respects this eye resembles the case published by Nettleship (6), in which, however, there was rather more choroidal pigment, some of it lying in the parts of the choroid away from the macular region. It resembles it also in that no proper fovea is found in the sections. Fritsch (7) found the fovea imperfectly developed in the retina of an albinotic Herero. And in Elschnig's case there was no fovea, rods as well as cones were present at the macula, and the hexagonal cells contained no needle-shaped granules.

These four cases appear to be the only ones in albinos in which the fovea was looked for microscopically, and in none of them was it found fully developed. Absence of the fovea has been recorded by Seefelder (10) in a case of aniridia with nystagmus. In a case of hereditary nystagmus (14), seen by myself, in a child aged 2 years, with blond skin, yellow brown hair, lashes and eyebrows, and with constant lateral nystagmus, no proper fovea could be found in either eye. The retinal epithelium was normally pigmented throughout, and pigmented chromatophores were present in the choroid and in small numbers in the iris. Since Fritsch and Elschnig published their cases, Ichikawa (19), with Elschnig, examined the macular region of six albinos with the ophthalmoscope and found that there was

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* A puzzling appearance seen in a few consecutive sections can be explained by a folding of the retina, and it is difficult to conceive how a normal fovea could escape detection even at this part.

no differentiation, corresponding to the appearance of the macula and fovea in a normal eye, and that the arrangement of the retinal blood vessels at this region was different from that seen in normal eyes, in that instead of converging regularly towards a fovea, they met and crossed each other irregularly.

When a normal human eyeball is opened and examined macroscopically, a diffuse pale yellow colour is visible in the retina at the macula. In albino eyes examined by Manz, Fritsch and Carron du Villards (2), no yellow was found at the macula, while in a case by Buzzi (1), and in the one already referred to by Nettleship, the yellow colour was present, as in this case. During life the yellow colour at the macula is not seen with the ophthalmoscope, when the usual sources of illumination are employed. Recently however Affolter (20), by using red free light, has been able to see the yellow colouration with the ophthalmoscope, but he failed to find it in two albinos' eyes.

The imperfect vision in albinos has been explained by attributing it principally to the absence of pigment, or defective pigmentation, in the retinal epithelium, but the results obtained by histological examination of the macula in these few human albino eyes, and the records of the two cases with nystagmus, already referred to (aniridia and hereditary nystagmus), in which the fovea was absent, suggest another explanation, and pending further investigation of the macular region in the human albino eye, there is reason to regard the imperfectly developed or absent fovea as possibly the chief cause of the defective vision and nystagmus in albinos.

C. ON THE OCULAR MESOBLASTIC PIGMENTATION OF THE IRIS AND CHOROID IN SOME DARK-RACE EYES.

The following eyes were examined:

1. Tamil seven-months' foetus. Eyes received from Dr H. Fraser, from the Federated Malay States (Kuala Lumpur), November 30th, 1915. Iris stroma contains some pigment, even with a low magnification it is evident that the anterior layers of the stroma are lightly pigmented, and with high powers numerous brown granules are seen in both bodies and processes of the chromatophores. None of the large round dark cells so conspicuous in the adult dark-race iris are present. Ciliary body contains lightly pigmented chromatophores. Choroid has well pigmented cells at macular region. Further forwards the cells are not so well pigmented.

2. Tamil seven-months' foetus. Eyes received from Dr H. Fraser, Federated Malay States (Kuala Lumpur), November 30th, 1915. The eyes are smaller than those of No. 1. In the iris stroma, only a few chromatophores are found containing pigment granules, and a few are present in the ciliary body. The choroid contains brown, round, pigment granules in the bodies and processes of the chromatophores, especially well marked at the macula (Fig. 2). None of the cells are so dark as those in an adult Tamil eye, and they become very lightly pigmented away from the macular region. The pigmentation is much greater in the choroid than in the iris.

3. Tamil full-time foetus. Eyes received in alcohol, after formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, May 10th, 1910. Antero-posterior and equatorial diameters of the eyes are both equal to 18 mm. Iris stroma is lightly pigmented. Pigment is most marked at the anterior surface of the iris. The branched cells are not so dark as in the adult Tamil's eye. Choroid contains many pigmented cells, but these are neither so numerous or quite so dark as in the adult Tamil eye. Choroid is darker at yellow spot than elsewhere.

4. Tamil foetal eyes received in spirit, after formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, March 10th, 1910. Eye measurements: right eye, antero-posterior diameter = 19·5 mm., equatorial diameter = 18 mm.; left eye in both diameters is equal to 18·5 mm. Iris contains much stroma pigment, which is most marked anteriorly and around blood vessels. No clump cells found. In front of the sphincter muscle the cells are branched. Ciliary body stroma is pigmented, but much less densely than in the adult. Choroid is pigmented most markedly at its outer parts and around blood vessels.

5. Tamil full-time foetus. Both parents are Tamils. Eyes received in 10% formaline, from Dr H. Fraser, Kuala Lumpur, Federated Malay States, October 21st, 1915. Antero-posterior diameter of each eye = 18·1 mm., equatorial diameter of each eye = 17·7 mm. The chromatophores in the iris stroma are well defined and contain brown pigment granules. As usual the pigmentation is most marked in the anterior layers of the iris which are heavily pigmented (Fig. 3). Ciliary body has pigmented chromatophores. The choroid contains many pigmented cells.

6. Malay-Tamil foetus (Father a Malay, Mother a Tamil), age 5—6 months. Eyes received in formaline from Dr George A. Finlayson, Singapore, July 13th, 1910. Both antero-posterior and equatorial diameters measure 9 mm. The iris and the ciliary body have no pigment in the stroma. The choroid contains a few branched cells with pigment granules (Fig. 1). These cells are fewer than those in a seven-months' Chinese foetal eye, No. 10, and are principally situated at the posterior part of the eye.

7. Dark-race infant (probably Malay, Chinese, or Tamil) thought to be still-born. Eyes received in formaline from the late Dr Robert D. Keith, Singapore, March 1st, 1915. Antero-posterior diameter of each eye = 18 mm. The iris chromatophores are pigmented with small brown granules that lie in the cell bodies and processes, some cells are more pigmented than others. The ciliary body has pigmented chromatophores as in the iris. The choroid is rather heavily pigmented, especially at its outer part, and at the posterior pole of the eye.

8 and 9. Two Chinese infants. The age of one is "about 2 weeks"; of the other, not stated. Eyes received from Dr George A. Finlayson, Singapore, March 14th, 1914. As the distribution and quantity of pigment in the iris and choroid is similar in these two cases, one description will serve for both. The iris stroma cells contain brown pigment granules. The amount of pigment in the

individual cells varies considerably. At the anterior part of the iris the chromatophores are packed together, giving a darker appearance to this part of the iris. No dark, round, unbranched cells are present, which are so numerous in the irides of adult Tamils and Chinese. (There are a few dark brown ones present in one case.) The ciliary body contains lightly pigmented chromatophores. There is much pigment in the choroid lying in branched cells, especially posteriorly.

10. Chinese foetus (Canton) "about 7th month, perhaps a little older." Eyes received in formaline from Dr George A. Finlayson, Singapore, January 15th, 1910. Eye measurements: antero-posterior diameter about 16 mm., equatorial diameter = 14 mm. The iris stroma is unpigmented, with the exception of a few pale brown chromatophores at the anterior surface of the iris, which contain pigment granules. In the choroid is a considerable quantity of pigment, lying in elongated and in branched cells, though very much less than in the full-time foetus No. 3, Tamil foetus No. 4, and in a negro foetus No. 11. The pigment is most marked in the outer layers of the choroid, and there is some around the blood vessels.

11. Fantis Negro, full-time twin foetus. Eyes received in formaline from Dr F. W. Moir, through Dr Alexander Low, February 7th, 1910. Eye measurements: both antero-posterior and equatorial diameters = 18 mm. The iris stroma contains a quantity of pigment, though much less than in an adult dark-race eye; there are comparatively few large pigmented branched cells; especially in anterior layers of iris and around the blood vessels are numerous round black cells. Many of these round cells show no evidence of branches, but on high magnification many of them, though roughly circular, exhibit different forms. In the ciliary body are comparatively few pigmented chromatophores. The choroid contains a considerable quantity of pigment in branched and elongated cells; many of the cells are only lightly pigmented, with brown pigment granules. The pigmentation is not nearly so dense as in a dark-race adult eye.

12. Tamil infant. Eyes received from the late Dr Robert D. Keith, Singapore, on July 29th, 1913. Eye measurements: antero-posterior and equatorial diameters are both equal to 18 mm. The iris contains much brown pigment in the branched chromatophores and some large, round, dark cells are present, but not in distinct groups. The epiblastic layers are dark and show well-marked spurs. The ciliary body contains many pigmented chromatophores. The choroid is well pigmented, and much pigment is present round the optic nerve entrance. Neither iris nor choroid are as deeply pigmented as those of an adult Tamil eye.

The pigmented retinal epithelium on the posterior surface of the iris, in ciliary body; and on the inner surface of the choroid, is exceedingly dark in every one of these twelve cases. This is to be expected, as the retinal pigment is fully formed by the twelfth week of foetal life (18). It is interesting to find mesoblastic pigmentation so early as is seen in some of these cases. The age of the foetus or infant has usually been given by the medical man who sent its eyes, and in most cases, measurements of the eyeball were made before opening. In case No. 6 where pigmented chromatophores are present in the choroid, the age of the foetus is given as 5—6 months and the eye measurements were 9 mm. E. Treacher Collins (5) measured the eye-balls of some human foetuses and found that at the fourth month, two eye-balls measured 8·5 mm. in the antero-posterior diameter, 8 mm. in the vertical, and 8 mm. in the lateral diameter. At the sixth month, two eye-balls measured 10·3 mm. in the antero-posterior diameter, 9·75 mm. in the vertical, and 10 mm. in the lateral diameter. According to these measurements, case No. 6 would certainly not be older than 5 months. It is quite obvious from an examination of these cases that the mesoblastic pigment in the eyes of dark races appears earlier and is laid down in foetal life in greater quantity than in European eyes. These notes refer only to the pigmentation of the choroid, ciliary body and iris, but mesoblastic pigment is conspicuous in some of the cases at other parts of the eye, such as along the course of the ciliary blood vessels in the sclerotic.

D. GENERAL CONCLUSIONS.

(*a*) Of the six albino eyeballs thus far examined, the albinos being determined clinically by the usual clinical characters, four are found to have pigment, of a fifth no statement was made by the examiner as to pigment. The sixth case was that of an *infant*, and here no trace of pigment was found in any of the structures of the eye. The remaining examinations of the human albino eye are concerned only with portions of the iris; in one no pigment, in the other some pigment, was observed. The former case provides no evidence as to whether the whole eyeball was or was not without pigment. We must conclude therefore that the total absence of pigment in the eye cannot be used as a definition of human albinism.

(*b*) As apart from absence of pigment, the absence or imperfect development of the *fovea centralis* shown to occur in albinotic eyes may possibly be the chief cause of defective vision in these cases.

(*c*) An examination has been made of the distribution of pigment in the eyes of a number of dark-raced individuals, adult, infant and foetal. The notes made indicate that in the dark-race eye mesoblastic pigment appears earlier in foetal life, especially in the choroid, and is in much larger quantity by the time of birth than in the European eye.

According to this we should expect, if albinism consists of "an arrest of development," more pigment in albinos of dark races than in albinos of white races. Clinical examination of such albinos appears to indicate that this is a fact, but it is a fact which might be accounted for on other than the Meckel-Mansfeld hypothesis. At the same time it renders any definition of human albinism as a complete absence of ocular pigment less valid for any practical service in the study of the heredity of albinism.

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DESCRIPTION OF PLATES.

PLATE I.

Fig. 1. Section through choroid and retinal pigment layer of a dark-race foetus, age 5—6 months. $\times 600 \times \frac{6}{7}$. Shows branched cells with pigment granules.

Fig. 2. Section through choroid and retinal pigment layer of a seven-months' Tamil foetus. $\times 600 \times \frac{6}{7}$. Shows pigmented chromatophores.

Fig. 3. Section of iris from full-time Tamil foetus. $\times 125 \times \frac{6}{7}$.

Fig. 4. Section of iris from adult Tamil, age 27. $\times 125 \times \frac{6}{7}$.

Fig. 5. Section through choroid and retinal pigment layer of adult Tamil, age 27. $\times 600 \times \frac{6}{7}$. (Compare with Figs. 1 and 2.)

PLATE II.

Fig. A. Section of iris from British female albino, age 17. $\times 105 \times \frac{6}{7}$. Nuclei of some epiblastic pigment cells are visible.

Fig. B. Section of a normal British iris. $\times 105 \times \frac{6}{7}$.

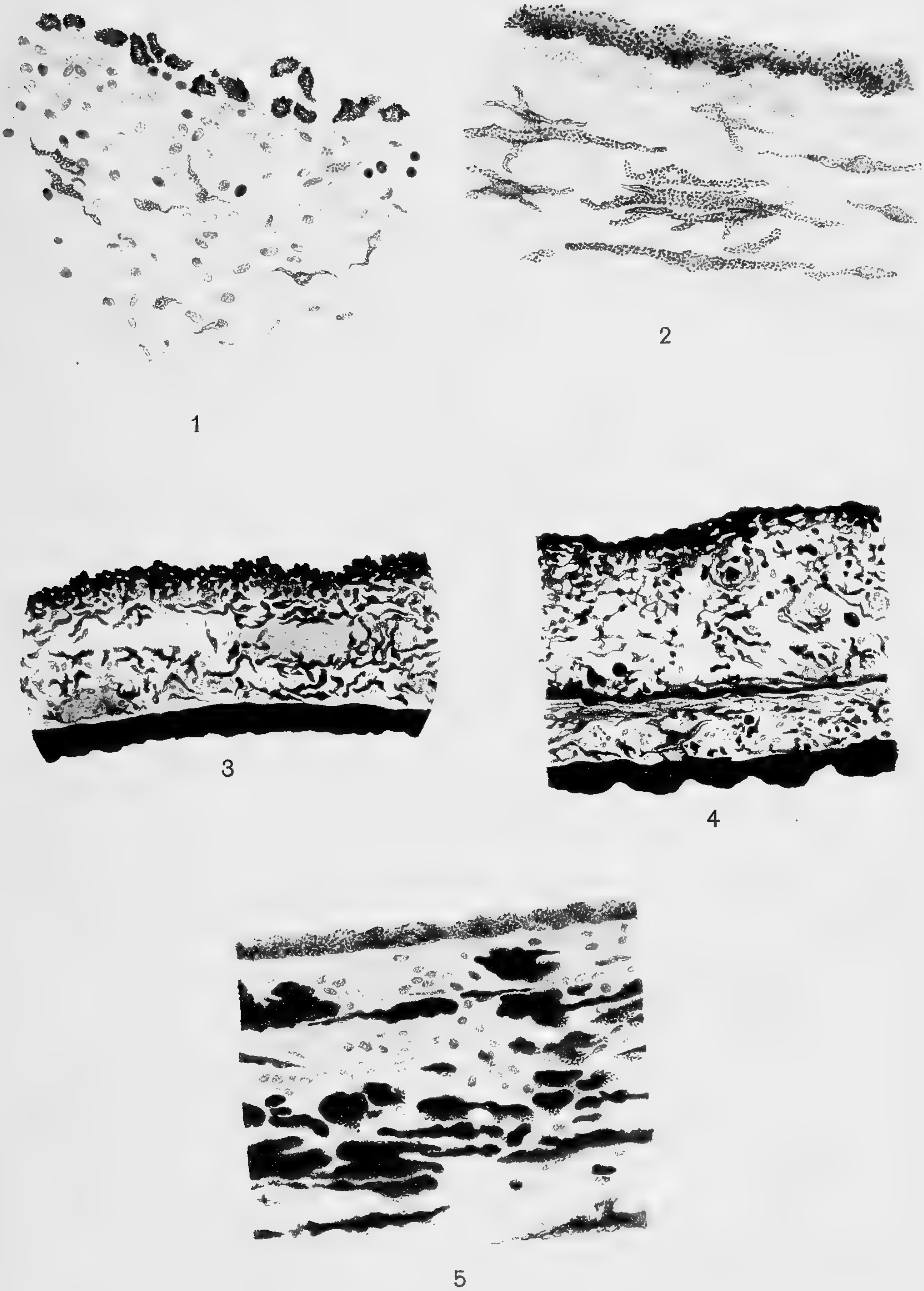
Fig. C. Section of iris from British male albino, age 67. $\times 105 \times \frac{6}{7}$. Brown pigment is present in both layers of the epithelium on the back of the iris.

Fig. D. Section of iris from British male albino, age 74. $\times 330 \times \frac{6}{7}$. A bit of iris was removed at an extraction of cataract. Pigment is present in the cells of both epithelial layers on the back of the iris.

Fig. E. Section of iris from Negro albino. $\times 105 \times \frac{6}{7}$. (Coat's case.) Shows pigmented chromatophores as well as brown pigment in posterior epithelial layers.

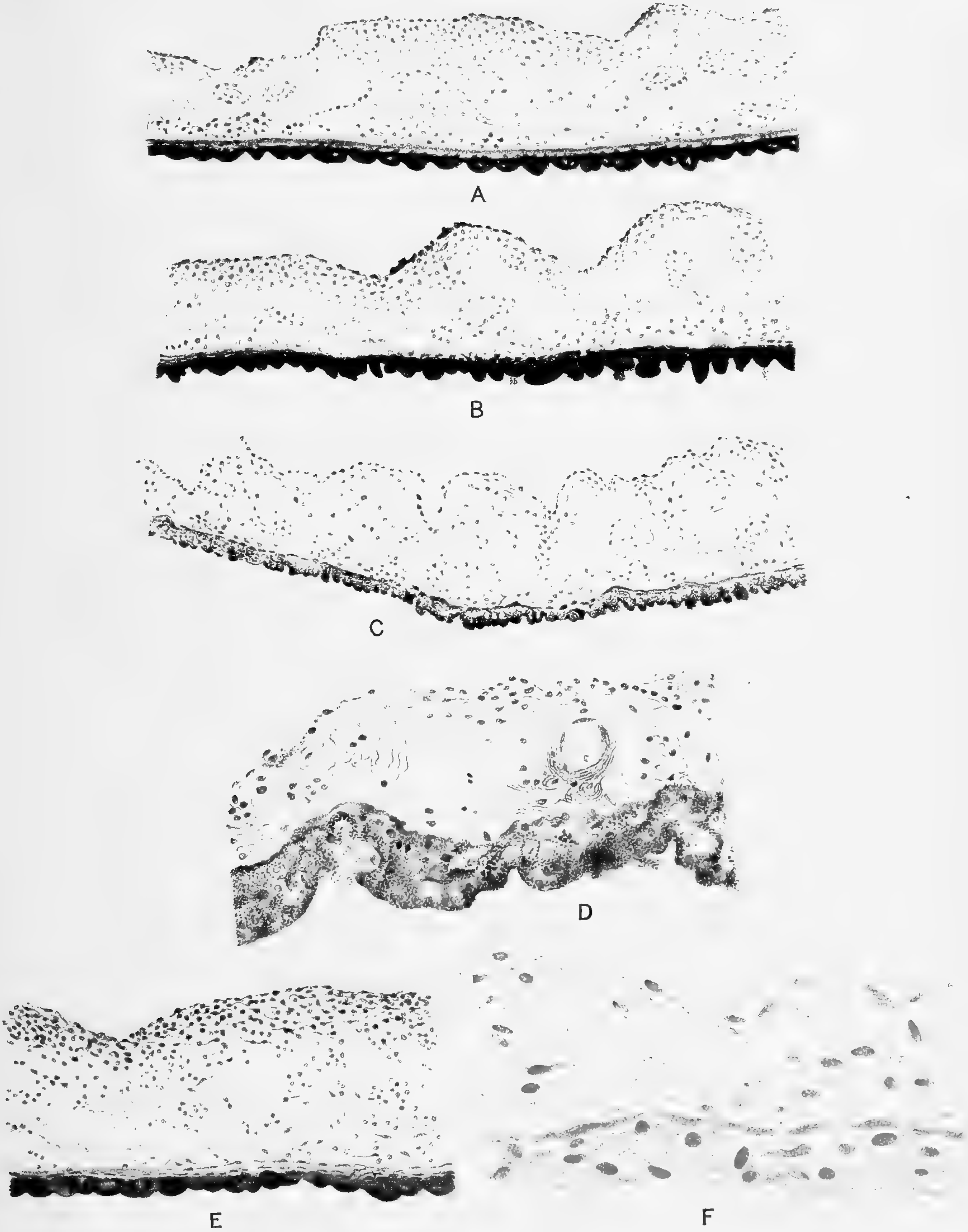
Fig. F. Section of iris of a remarkably perfect British male albino, age 74. Portion of iris removed during the operation for extraction of cataract. (Nettleship's case.) Iris is pigmentless.

The original drawings were reduced 7 : 6 for the plate reproduction.



1, 2, 3, Tamil foetus at different ages; 4, 5, Tamil adults.

Plate II



A, C, D and F, British Albinos ; E, Negro Albino.

ON THE INHERITANCE OF THE FINGER-PRINT.

By ETHEL M. ELDERTON, Galton Research Fellow.

FIRST PAPER.

(1) *Explanatory and Historical.*

The inheritance of finger-print types when once those types have been analysed and classified seems an almost ideal subject for the study of heredity in man. It would appear at first sight that they would present most excellent material for comparing various theories of inheritance and measuring the intensity of heredity between the different grades of kinship. Yet as far as I am aware nothing has hitherto been published on the subject either from the Biometric or the Mendelian standpoint. By 1903, data had been collected and reduced showing the degrees of inheritance for many physical characters in both animals and man and the collection and reduction of data for the heredity of mental and moral characteristics in man had been started. It would appear strange that finger-print types should remain unregarded,—the "arch," the "loop," the "whorl," the "composite" were familiar to many and seemed by their easy determination for the purpose of criminal indices to be most appropriate for the study of heredity. Two chief difficulties, however, stood in the way. First the great labour of collecting in adequate quantities the finger-prints of relatives; and secondly the still more important fact that while four or five broad categories were adequate for a criminal index, they were inadequate for scientific distinctions. Galton at first ran up his four or five categories to 53, and the continual appearance of transitional forms led him further and further in the sense of finger-print continuity. Without exaggeration one can say that he sought for years for quantitative measures of the finger-print*, which might be applied to any type, and that, when he had appreciated continuity, it was this failure to obtain quantitative measurement which ultimately led him to put the problem on one side. Nevertheless those who were intimate with him in the last ten years of his life know the prominent part which the inheritance of finger-print types played both in his thoughts and activities.

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* The Galton Laboratory possesses a great variety of cameras and other apparatus for enlarging, orientating and measuring finger-prints designed by Galton. Many of these, especially those for angular measurements on the finger-print, are now, failing any memorandum of purpose, hardly interpretable.

In 1903* Galton set about obtaining a collection of family finger-prints for the purpose of studying heredity. The persistence of finger-print type throughout life established by Herschel and Galton was to make the inquiry a simple one; it was unnecessary to collect adults only; classification could take place at any time during life.

As early as 1892 Galton had made a preliminary inquiry into the inheritance of finger-print types, but he was not satisfied with either the extent of his material, or with the classification of types, which he had at that time evolved.

We shall give the results he obtained below. They were published in 1892 in his work entitled *Finger-Prints.*

In this book Galton devotes a chapter to the question of inheritance but at that time there was no method of obtaining a correlation coefficient when the characters were not quantitative and Galton could only show that type of finger-print is inherited but could not measure the strength of inheritance nor compare it with the results obtained from the study of other characters. The method Galton used was the following: he found what he describes as (1) Random occurrences, (2) Observed occurrences, (3) Utmost possibilities. He used for the first part of the investigation three categories only, namely arch, loop, and whorl. The types of 101 (by mistake for 100) couplets of prints of the right forefingers of school children were taken from a large collection, the two members A and B being picked out at random and formed into a couplet. These were tabled and found to agree very well with calculated random couplets.

To study the fraternal relationship the observed occurrences were then taken from 105 fraternities and finally the greatest number of correspondences which could (without changing the total distribution of types) occur if the kinship were as close as possible were found. I now reproduce the results obtained by Galton which I take from p. 176 of his book. It will be seen that he only deals with the diagonal of his correlation table in this summary of the results.

	A and *B* both being		
	Arches	Loops	Whorls
Random	1·7	37·6	6·2
Observed	5·0	42·0	10·0
Utmost possible ...	10·0	61·0	25·0

We note that in all three cases the observed exceed the random but that this excess is not very marked, particularly in loops. Galton says that many other cases of this description were calculated all yielding the same general result. He

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* He made an appeal for material in the pages of *Biometrika*, Vol. II. p. 356, 1903, and the letters accompanying the data he received show how active he was in the matter of collecting during 1903 and 1904.

then sub-divided the patterns under 53 heads and an experiment was made with the fore, middle and ring fingers of 150 paternal couples by Mr F. Howard Collins. The resulting table shows the same thing but the multiplicity of sub-divisions, especially when the number of pairs used is comparatively small, makes it even more difficult to assign any degree of correlation though some undoubtedly exists. Seventeen sets of twins show an even closer resemblance, two sets agreeing in all their three couplets, four agreeing in two and five in one. There are instances of partial agreement in five others.

The knowledge of the inheritance of type of finger-print was at this stage in 1903 when Galton began to collect the material which forms the chief part of the data used in this paper. The total number of persons whose finger-prints were taken was 2,300 but many of these are isolated individuals and of no use for the study of inheritance, but when these are omitted we have over 400 cases in the tables dealing with maternal inheritance, over 300 in the paternal tables and between 800 and 900 in the fraternal tables. The most complete family schedules were those collected by Mr Perrycoste from the inhabitants of Polperro, Cornwall, and in this material we have an extraordinarily interesting series. In some cases we have the finger-print of some old man and his wife, the finger-prints of all their living children, of the husbands and wives of these children and of many of the old folks' grandchildren. The people of Polperro evidently in 1900 had not started restricting their birth-rate and we have families of eight and nine brothers and sisters with as many children again of the married ones. In Galton's later collection the finger-prints of the first fingers only were taken and the reason for this limitation is given in the schedule sent out from which I quote the following.

"It is proposed to confine the collection to prints of the two *forefingers* of many persons, to *rolled* impressions of them, and to ask for *four* prints of each.

The two forefingers are selected because their patterns are more varied than the rest in respect to 'radial' direction, and not less varied in other respects. Also because rolled impressions can be obtained most easily from them, requiring as they do the extension of the finger in use, while the others are tightly closed to be out of the way. Lastly, the degree of closeness of correlation between the two forefingers has been found to be very suitable to the inquiry."

Unluckily experience has shown us that it is more satisfactory to work with the prints of all the fingers of both hands, so I also considered the larger and earlier general finger-print collection made by Galton and I have found therein about 350 individuals who had a brother or a sister recorded; in this case all their finger-prints had been taken. It was on this collection that most of the published work done by Sir Francis on finger-prints was based, but I am not clear whether the special material he used for the early consideration of inheritance was taken from this data or not. If it were then, I think, only a selection of the available pairs of brothers was extracted unless the general collection was added to in more recent years after the publication of his book in 1892.

I have therefore had at my disposal two series of data (1) the earlier one, in which finger-prints of every finger were taken, consisting of about 370 pairs of brothers and sisters where only the fraternal relationship can be discussed, (2) the later series in which only the first finger was considered but from which we can find, not only the fraternal relationships but also the paternal and maternal and to a certain extent the relationship between cousins and the relationships between uncles and aunts and their nephews and nieces.

(2) *Classification of Finger-Prints.*

Nearly 800 of his later collection were classified by Galton and he attempted new methods of counting the ridges in an effort to obtain a numerical scale for all types of finger-prints, not only for loops. He failed, however, to find a quantitative scale that satisfied him and he put the material completely aside some years before his death. In the classification of types Galton used great detail and an examination of the detail is most suggestive. One of the questions that arises in a study of finger-prints is, as we have already indicated, whether the types of finger-print show a continuous variation; we are accustomed to think in the first place of three distinct patterns, arch, loop and whorl; but a minute examination of finger-prints at once reveals many intermediate steps and Galton in a very early stage of his work recognized many more groups than these and laid great stress on what he called "transitional cases." In 1892 he sub-divided patterns under 53 heads and the reproduction of these patterns forms a most interesting series. Of these transitional cases the most common occur among loops and all stages between loops and whorls can be found. Galton in his later work uses four types of loops (*a*) plain loops, (*b*) eyed loops "y," (*c*) invaded loops "v" and (*d*) hooked loops "k." Of these "y" and "v" are the more common and "y" is obviously the first stage in a whorl. Central pocket loops as Henry calls them seem to be another stage, but it is quite easy to find several finger-prints that might be considered either "y" loops or central pocket loops. Francis Galton wrote in his *Finger-Print Directories,* 1895, p. 100: "The transitional cases between loops and whorls are largely due to the presence in loops of a strongly marked 'y,' doubt arising whether its outline does or does not make a complete circuit." Galton used the word "compound" to cover these transitional cases, but the word has been superseded by "composite" which includes those cases compounded of loop and whorl but where I think whorl is uppermost, for composites do not include what I might describe as "not plain" loops.

By a study of the reproductions between pp. 74 and 75 in Galton's *Finger-Prints* we can see how arches pass into loops through "forked" and "tented" arches and how "eyed" loops (Galton used the term "eyeletted" in 1892) become central pocket loops and central pocket loops become whorls. There is no difficulty in making 53 or more groups of finger-prints; the difficulty arises when we have to agree where arches end and loops begin and where loops end and whorls begin. Galton found this a difficulty when Mr Howard Collins and he were working together. He writes: "living at a distance apart, it was not easy at the time they were made, to bring our respective interpretations of transitional and of some of the other invaded loops, into strict accordance, so I prefer to keep his work, in which I have perfect confidence, independent from my own."

On the basis of my own experience I should prefer to arrange finger-prints in five groups and in the following order, arch, plain loop, "not plain" loop, composite, whorl, and if I had to combine two of these groups I should put "not plain" loop with composite but I am not yet in a position to prove that this is the correct order and the difficulty of using such an order is that plain loops still remain too large a group and "not plain" loops and composites are both very small. I therefore decided to follow Dr Waite* and divide the loops in my table according to the number of ridges, i.e. into small and large loops. The method used for working out all the tables is mean square contingency, and in applying this method we must use the class index correction for grouping. This correction assumes that the categories under consideration are approximately normal and continuous and depends upon a knowledge of the "natural order." As I said before I have no doubt that the types of finger-print form a continuous variation, and that with a more extensive study of more ample data we should be able to determine this "natural order." I do not consider that we have yet found a wholly satisfactory one, but as we shall see presently the corrective factor for grouping in this data seems to depend rather more on the number in each group than on the exact order of the grouping and that we can apply this factor without a knowledge of the absolutely correct "natural order." This question of a natural order of finger-print type attracted Francis Galton's attention and on p. 178 of his *Finger-Prints* he writes:

"It would be essential in exact working that the mutual relations of the patterns should be taken into account; for example, suppose an arch to be found on the forefinger of one brother and a nascent loop on that of the other; then, as these patterns are evidently related, their concurrence ought to be interpreted as showing some degree of resemblance."

Thus the discovery of a "natural order" is not merely essential to the completeness of mathematical reduction; it arises at once when we begin to classify our characters from the standpoint of inheritance.

(3) *The provisional Natural Order of the present paper.*

We have next to investigate how far a natural order has been established if we divide loops into small and large loops† and have as our other groups, arches, composites and whorls. Certain types of patterns are associated with small and large loops respectively. Thus small loops are more nearly associated with arches and large loops with whorls. "Not plain" loops occur much more frequently in

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* *Biometrika*, Vol. x. p. 419.

† Dr Waite defines a small loop as one having twelve or fewer ridges, a large loop as one with more than twelve ridges.

large loops than in small. My tables will illustrate these points. I have worked on Dr Waite's material* and have separated "not plain" loops from loops.

If we consider the number of *small* loops in the fingers of the right hand when arches occur on 0, 1, 2, 3 and 4 fingers respectively we get the following:

If 0 arches on right hand on remaining 5 fingers we have 27·7 °/₀ small loops
1 ,, ,, ,, 4 ,, ,, 52·4 °/₀ ,,
2 ,, ,, ,, 3 ,, ,, 59·5 °/₀ ,,
3 ,, ,, ,, 2 ,, ,, 66·7 °/₀ ,,
4 ,, ,, ,, 1 ,, ,, 93·8 °/₀ ,,

If a right hand has four fingers with arches, then in 15 out of 16 cases the remaining finger has a small loop, i.e. 93·8 °/₀ of small loops, while if there are 0 arches on the hand the percentage of loops on the five fingers is 27·7 showing clearly a correlation between arches and small loops. If we consider whorls in connection with large loops we obtain the following:

If 0 whorls on right hand on remaining 5 fingers we have 27·7 °/₀ large loops
1 ,, ,, ,, 4 ,, ,, 37·6 °/₀ ,,
2 ,, ,, ,, 3 ,, ,, 43·4 °/₀ ,,
3 ,, ,, ,, 2 ,, ,, 49·1 °/₀ ,,
4 ,, ,, ,, 1 ,, ,, 43·1 °/₀ ,,

Here, the greater the number of fingers with whorls the greater the percentage of large loops on the remaining fingers. I have already noted that "not plain" (eyed, invaded and hooked) loops occur more frequently in large loops than in small. If we consider the later series collected by Francis Galton we find the following on the first finger of the left hand:

Number of ridges	Males				Females			
	Plain Loops	Not Plain Loops	Totals	Percent. Not Plain	Plain Loops	Not Plain Loops	Totals	Percent. Not Plain
1, 2, 3, 4 ...	112	10	122	8·2	185	10	195	5·1
5, 6, 7, 8 ...	113	13	126	10·3	105	18	123	14·6
9, 10, 11, 12	89	18	107	16·8	114	22	136	16·2
13, 14, 15, 16	68	19	87	21·8	87	22	109	20·2
17, 18, 19, 20	22	9	31	29·0	29	11	40	27·5
21 and over	4	4	8	50·0	5	5	10	50·0

Just as whorls are associated with large rather than small loops so are "not plain" loops, which is what we should expect if the theory that "not plain" loops are a stage on the way from loops to whorls be a true one.

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* I should like here to record my indebtedness to Dr Waite for allowing me to use material that he has classified and tabled for my experiments in this direction; its use has saved me much labour and I am most grateful to him.

Dr Waite correlated* the number of each type of finger-print on the whole hand with every other type and from his diagram it is clear that arches are positively correlated with small loops and negatively with every other type so that we must place arch next to loop. Large loops are positively associated with whorls and composites and negatively with small loops and arches. Composites and whorls have a positive association. Composites seem to be more nearly associated with middle sized loops than with large ones. In fact what seems to interfere with obtaining a true natural order is the arbitrary division at 12 ridges of large and small loops; the sign in some of these tables is difficult to ascertain, a difficulty that never occurs if we deal with loops with no sub-division into small and large, or with arches or whorls. As I said before, arch, loop, "not plain" loop, composite, whorl is the order I favour, but we should want larger numbers of finger-prints than are available now and even then "not plain" loops would be too small a group comparatively for satisfactory statistical treatment. Possibly in the end we shall have to combine "not plain" loops and composites, they certainly come very close together in any natural order. In the meantime there seem to be two possible practical orders (1) Arch, Small Loop, Large Loop, Composite, Whorl, and (2) Arch, Small Loop, Composite, Large Loop, Whorl. I worked out the corrective factors (class index correlations) for both orders and was pleased to discover that they were almost identical, and that the value of the factor in this data certainly depended on the number in each group rather than on the actual order. In six cases which I tried I obtained the following results:

	Arch, S. L., L. L., C., W.	Arch, S. L., C., L. L., W.
Son's Right Forefinger	·934	·934
Father's Left Forefinger	·936	·936
Mother's Left Forefinger	·941	·941
Daughter's Left Forefinger	·934	·934
Daughter's Right Forefinger	·944	·942
Mother's Right Forefinger	·943	·943

Thus in only one case is there any difference in the third decimal place so that if we are prepared to accept type of finger-print as a continuous character we can apply the corrective factor for grouping even though we have not established an entirely satisfactory natural order. I have chosen the first of the two orders to use in every case, arch, small loop, large loop, composite, whorl. In the original tables I kept central pocket loops in a class by themselves, and throughout I worked all contingency in two ways, (1) grouping C. P. L. with whorls, and (2) grouping C. P. L with composites, but the results are not significantly different

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* Dr Waite in his paper on the association of finger-prints in the same individual has endeavoured to form a scale of relationship on the basis of association as measured by contingency in the prints of the several fingers. I think he is on the right path to deduce such a scale, but I believe that it cannot be done by merely comparing the numerical magnitude of the associations. It is true that contingency has no sign, but if we are to form a scale of next relationship, we must consider sign, a positive and negative association of the same magnitude being not of the same significance in a scalar relationship. I have therefore felt compelled to give signs to Dr Waite's contingency coefficients based on other considerations than those of contingency, namely by actual examination of the contingency tables themselves. This has compelled me to modify his scale of relationship.

so I have decided to give here the result for the first grouping only, which is the one that I prefer on the whole as it makes composites a rather more satisfactory group in size. I have used mean square contingency throughout and considered the correction for the number of cells only when any doubt arose as to whether ϕ^2 was significant or not but have not subtracted this correction from ϕ^2 before finding C_2*.

(4) Before I applied biometric methods to the problem of finger-prints some endeavour was made to consider the inheritance of types from the Mendelian standpoint. The difficulties attending any attempt to consider finger-prints from this point of view consist not only in the existence of transitional forms, but from the fact that there may be inheritance with change of finger. It would be exceedingly difficult to collect cases only of both parents having all their finger-prints unique in type, for example a father with 10 whorls and a mother with 10 whorls, or a father with 10 whorls and a mother with 10 arches. Yet something of this kind seems needful, if we admit transference of type from one finger in parent to another in child. Galton's later data at any rate will not help us in this matter as we have only the prints for two fingers provided. If we assume inheritance is to the *same* finger we can obtain from Galton's data the following tables for interpretation on Mendelian lines:

Inheritance of Finger-Print Type on Left Forefinger.

Father's Type.

	Mother's Type	Arch				Loop				Composite				Whorl				Totals
		A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	
Child's Type.	Arch ...	1	19	2	1	18	14	6	7	1	2	—	—	1	4	—	—	76
	Loop ...	1	30	5	8	35	141	24	47	7	15	6	5	6	55	2	7	394
	Composite	—	1	2	—	6	9	6	11	1	4	2	5	1	5	1	3	57
	Whorl ...	—	6	—	4	4	28	3	27	—	1	4	8	1	18	4	11	119
	Totals	2	56	9	13	63	192	39	92	9	22	12	18	9	82	7	21	646

Inheritance of Finger-Print Type on Right Forefinger.

Father's Type.

	Mother's Type	Arch				Loop				Composite				Whorl				Totals
		A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	A.	L.	C.	W.	
Child's Type.	Arch ...	8	6	—	3	13	25	4	10	2	4	—	—	2	5	—	—	82
	Loop ...	5	27	—	14	31	147	23	32	5	15	—	3	12	46	2	9	371
	Composite	1	5	—	3	5	19	3	14	1	5	—	4	—	7	2	3	72
	Whorl ...	—	2	—	5	1	31	5	28	—	3	—	10	2	27	—	13	127
	Totals	14	40	—	25	50	222	35	84	8	27	—	17	16	85	4	25	652

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* The "correction for cells" is the mean value of the mean square contingency when the association is zero. It forms of course the proper and adequate test of zero association, but Professor Pearson tells me that he is now extremely doubtful whether it should be subtracted from the mean square contingency when the latter is really significant.

Now if we treat our four fundamental types as distinct we have the following results of crossings:

Arch × Arch = 9 Arches + 6 Loops + 1 Composite + 0 Whorls,
Arch × Loop = 56 Arches + 123 Loops + 17 Composites + 13 Whorls,
Arch × Composite = 5 Arches + 17 Loops + 4 Composites + 0 Whorls,
Arch × Whorl = 7 Arches + 40 Loops + 4 Composites + 12 Whorls,
Loop × Loop = 39 Arches + 288 Loops + 28 Composites + 59 Whorls,
Loop × Composite = 16 Arches + 77 Loops + 18 Composites + 12 Whorls,
Loop × Whorl = 26 Arches + 180 Loops + 37 Composites + 100 Whorls,
Composite × Composite = 0 Arches + 6 Loops + 2 Composites + 4 Whorls,
Composite × Whorl = 0 Arches + 12 Loops + 12 Composites + 22 Whorls,
Whorl × Whorl = 0 Arches + 16 Loops + 6 Composites + 24 Whorls.

These are the equations which the Mendelian has to interpret. Certain things are at once obvious:

(i) That both Arch × Arch and Arch × Composite appear to give no whorls.

(ii) That both Whorl × Whorl and Whorl × Composite appear to give no arches.

(iii) That Arch × Loop, Whorl × Loop, Composite × Loop and therefore not unnaturally Loop × Loop can reproduce all four types.

(iv) That the Composite is so relatively rare that it is difficult to lay stress on the numbers obtained for it. But Composite × Composite suggests on meagre evidence that this mating cannot produce arches.

(v) That Arch × Whorl can reproduce all types.

These results suggest that the loop is of a much more heterozygous character than either arch or whorl. If we throw composites in with whorls, the Composite × Whorl by no means gives the same result as Whorl × Whorl; and if we throw composite in with loop we do not find the equations for Composite × Loop, Arch or Whorl in satisfactory agreement with those for Loop × Loop, Arch or Whorl. The evidence, meagre as it is, seems to indicate a real genetic differentiation between Composite and Loop.

Still if we venture to put Composite with Loop, we obtain a much simplified series of equations, which admit of being expressed as follows in *round* numbers:

			Ratios
(i)	Arch × Whorl	= 6 Arches + 42 Loops + 12 Whorls,	(1 : 7 : 2)
(ii)	Arch × Arch	= 30 Arches + 30 Loops + 0 Whorls,	(1 : 1 : 0)
(iii)	Arch × Loop	= 16 Arches + 40 Loops + 4 Whorls,	(4 : 10 : 1)
(iv)	Loop × Loop	= 6 Arches + 45 Loops + 9 Whorls,	(2 : 15 : 3)
(v)	Loop × Whorl	= 4 Arches + 38 Loops + 18 Whorls,	(1 : 10 : 4)
(vi)	Whorl × Whorl	= 0 Arches + 30 Loops + 30 Whorls.	(0 : 1 : 1)

The symmetry of (ii) and (vi) and again of (iii) and (v) are possibly suggestive; while a comparison of (i) and (iv) seems to indicate that the whorl is to some degree dominant over the arch, though both whorl and arch must be heterozygous in their nature. We have not succeeded in putting any satisfactory interpretation on the above equations, although we have made a variety of attempts. It is possible that some one better versed than we are in handling Mendelian ratios will be able to throw more light on the matter. But we feel that the above equations are in themselves forced because:

(*a*) there is no real justification for putting the Composite with the Loop,

(*b*) there are no really clear-cut categories, but a great variety of types with transitional forms, and

(*c*) there exists good evidence to show that the type is not necessarily inherited from one finger of the parent to the same finger of the child.

Failing a Mendelian interpretation, which appeared to us at all satisfactory, we have discussed the inheritance in this paper solely from the biometric standpoint.

(5) *Parental Correlation in Finger-Print Types.*

We will first consider the later and larger series of data collected by Francis Galton in which prints were taken of the first finger only and deal with the parental tables.

TABLE I.

	Right first finger		Left first finger	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Father and Son ...	·291	·333	·379	·434
Father and Daughter	·321	·363	·236	·273
Mother and Son ...	·369	·423	·366	·416
Mother and Daughter	·366	·411	·341	·386
Mean Parental value	·337	·383	·334	·382

I have not worked out the true probable error of these contingency values. If we use the probable error of r it is of the order of ·03; contingency would be higher but as all the values of the coefficients are obviously significant, it seemed needless to undertake the labour of finding the true probable error. The mean parental value for both fingers is ·38 when corrected for grouping, which is a lower value than that generally found for the relationship between parent and child. The coefficient between the left finger of father and daughter is distinctly lower than any other of the coefficients in the table, and this difference exists whatever grouping I use. I was at first very interested to find this smaller value for the father and daughter as Francis Galton in his study of finger-prints thought he traced some preponderance of maternal influence in the transmission of finger patterns but the value found for left first finger of father and son upsets this theory, and later work by Mr Howard Collins casts some doubt on the earlier results he had obtained which Galton quoted*. At the same time I thought it would be of interest to compare the finger-print of the mother with all her children irrespective of sex and of the father with all his children.

The advisability of combining the sexes is discussed later in this paper; it is certainly legitimate for the first finger of the left hand but I am doubtful whether it is for the first finger of the right hand. I give the values I obtained in Table II.

TABLE II.

	Father		Mother	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Right first ...	·261	·297	·357	·405
Left first... ...	·289	·332	·338	·383

In both cases the value is higher for the mother than for the father; the probable error of the maternal coefficients is of the order ·020 and of the paternal ·023 and of the difference ·03, and from this we might infer that in the right first finger the difference was significant, but since in the right first finger I am very doubtful if I ought to have combined the sexes I think that we cannot assume on this data a preponderance of maternal influence in the inheritance of type of finger-print. A comparison of the results with those found for stature, span and forearm† shows a distinctly lower value for inheritance of finger-prints.

TABLE III.

Character	Father and		Mother and	
	Son	Daughter	Son	Daughter
Stature	·514 ± ·015	·510 ± ·013	·494 ± ·016	·507 ± ·014
Span	·454 ± ·016	·454 ± ·014	·457 ± ·016	·452 ± ·015
Forearm	·421 ± ·017	·422 ± ·015	·406 ± ·017	·421 ± ·015
Mean ...	·463	·462	·452	·460

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* *Finger-Prints*, p. 190.

† "On the Laws of Inheritance in Man," by K. Pearson and A. Lee, *Biometrika*, Vol. II. pp. 357 et seq. 1903.

The only physical character in parent and child with which the inheritance of type of finger seems to agree is that of general health where the following coefficients were found.

TABLE IV.

	Corrected for grouping only
Father and Son	·438
Father and Daughter ...	·314
Mother and Son	·401
Mother and Daughter ...	·393
Mean	·387

When no correction for the number of cells is applied, the parental inheritance of general health and of type of finger appear practically the same, but in the case of general health there is a reason for the low value found which does not apply in the other case, and the resemblance is apparent rather than real. The larger number of deaths among delicate children is probably sufficient to account for the lower parental value but the whole question is discussed in detail in *Biometrika**.

(6). *Fraternal resemblance.*

In this case we have two sets of data (1) the long series for the first finger only and (2) the series for the whole hand. In the long series which we will consider first, there are 840 pairs of brothers, 886 pairs of sisters and 869 pairs of brothers and sisters.

TABLE V.

	Right first finger		Left first finger	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Brother-Brother ...	·369	·424	·335	·385
Sister-Sister ...	·316	·356	·305	·348
Brother-Sister ...	·336	·382	·340	·387
Mean	·340	·387	·327	·374

The probable error, if we use the formula for the probable error of r, is of the order ·02 and the values are very similar to those found for the paternal and maternal correlation. The correlation between sisters is the lowest for both right

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* "On the Hereditary Character of General Health," by Karl Pearson and Ethel M. Elderton, *Biometrika*, Vol. IX. pp. 320 et seq.

and left finger but is certainly not significantly different for the left hand; it might be just significant if we compare sister-sister and brother-brother results for the right first finger; but I think not, considering that the probable error of a contingency coefficient is decidedly higher than the probable error of r. Before comparing these results with those found for other characters we will consider the results from the series in which the whole hand was taken. There are roughly 380 males and 230 females. I decided to put males and females together if possible and work with sibships. I had therefore to see first whether the distribution of the types of finger-prints differed sensibly, as, if it did, the sexes could not be combined. I used the method described by Professor Pearson in *Biometrika** and found the following values for χ^2 and P for the different fingers. There were five groups and I worked out χ^2 and P in the double classification, including Central Pocket Loops first with composites and secondly with whorls.

TABLE VI.

	Including C. P. L. with Composites		Including C. P. L. with Whorls	
	χ^2	P	χ^2	P
R_1	9·21	·06	9·93	·04
R_2	10·87	·03	12·08	·02
R_3	4·04	·40	4·72	·32
R_4	1·47	·83	2·34	·68
R_5	5·91	·21	4·59	·34
L_1	3·89	·42	3·87	·43
L_2	·97	·91	1·26	·87
L_3	4·03	·40	5·50	·24
L_4	3·70	·45	4·02	·41
L_5	6·57	·16	5·33	·26

It is clear from this table that, whichever grouping is used, there is no reason why the sexes should not be put together except for the thumb and first finger of the right hand. For the right thumb the odds are roughly 17 to 1 or 24 to 1 according to the grouping used against the males and females being random samples of the same population and for the right first finger the odds are roughly 59 to 1 or 33 to 1 against. This sex difference in the first finger of the right hand is confirmed by the series in which only the first finger was taken. I find from that series that for the first left finger when C. P. L.'s are grouped with composites $\chi^2 = 2{\cdot}96$ and $P = {\cdot}56$ but for the right first finger $\chi^2 = 9{\cdot}86$ and $P = {\cdot}04$ and the odds are about 23 to 1 against males and females being random samples of the same population in *these two series*. That two independent series should both support the suggestion that there is a sexual difference between the finger-print type of the right forefinger (possibly also of the right thumb) is very noteworthy, and the possibility of it should be borne in mind and retested on

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* "On the Probability that Two Independent Distributions of Frequency are really samples from the same Population," by Karl Pearson, F.R.S., *Biometrika*, Vol. VIII. pp. 250—254, July 1911.

larger numbers. The biological origin of such a difference in one (or two) out of the ten fingers is not obvious, but it is the sort of clue which should not be allowed to drop. At the same time I am doubtful if it will be confirmed eventually that there is a sex difference. In both series the higher value found for χ^2 is due mainly to a disproportion in the number of arches between males and females, but in the shorter and earlier series the males have a larger proportion of arches than the females and in the later and longer series the females have a larger proportion than the males! A recount, after an interval of three or four years, confirmed the original figures, and I think the discrepancy may be due to the fact that in both series we are dealing with family finger-prints and a predominance of a certain type of finger-print in a large family of males or females may be the source of the divergency. For the right thumb there is no other series with which to compare it, but I should certainly lay no stress on the sex difference in that case; again arches are mainly responsible for the high value found for χ^2, and as arches are very rare on the thumb the percentages of arches are based on 7 males and 11 females and with such small numbers are liable to a large error. This sex difference may or may not exist; on the available data I think it would be unwise to be dogmatic on the point. Whether or not there is a sex difference there is a difference in the right thumb and right first finger in this data and under the circumstances I was reluctant to put males and females together, but the number of pairs of sisters is so small that I decided not to keep them separate. I sought justification for this step in the following way. I worked out the correlation coefficients between each finger and every other finger for (*a*) males alone, (*b*) males and females together and found that in the tables involving the right thumb and the right first finger the differences between the coefficients found were not significant and I trust that the results for inheritance in the case of the right thumb and right first finger will not be upset by dealing with sibships and neglecting the difference in sex. Table VII gives the differences in the correlation coefficients and the probable error of the difference.

Again I have used the p.e. of r which is an underestimate of the probable error of a contingency table and if the differences are not significant when this

TABLE VII.

	Right Thumb	Right First Finger
Left Thumb ...	·001 ± ·021	·035 ± ·036
First ...	·034 ± ·036	·009 ± ·026
Second ...	·069 ± ·037	·031 ± ·027
Third ...	·016 ± ·035	·028 ± ·033
Fourth ...	·053 ± ·033	·001 ± ·036
Right Thumb ...	—	·049 ± ·036
First ...	·049 ± ·036	—
Second ...	·043 ± ·036	·057 ± ·028
Third ...	·037 ± ·034	·009 ± ·033
Fourth ...	·056 ± ·033	·026 ± ·033

probable error is used we can be quite sure they will not be significant if we used the true probable error.

I will now give the inheritance coefficients for each finger for the sibships; the results are summarized in the following table.

TABLE VIII.

Sibships	Right hand		Left hand	
	Uncorrected for grouping	Corrected for grouping	Uncorrected for grouping	Corrected for grouping
Thumb	·320	·379	·425	·477
First finger ...	:340	·389	·361	·411
Second finger ...	·331	·384	·443	·502
Third finger ...	·402	·477	·358	·411
Fourth finger ...	·368	·439	·366	·440
Mean ...	·352	·413	·391	·449

If we compare the values found for the first finger with the mean values found for the first finger in the longer series we find they are in close agreement, but the agreement is less close in the left hand ·374 and ·411 being the values; the difference between them is, however, not significant. It will be interesting to see whether there is any significant difference in inheritance in the different fingers. Again I have not worked out the p.e. of the contingency but taking the p.e. of r as a rough measure we find that all the probable errors in this table lie between ·020 and ·023 and that the probable error of the difference between any two coefficients or any one coefficient and the mean is about ·03. As this probable error is an underestimate of the true probable error it will be wiser to take three times the probable error and consider that unless differences in the coefficients exceed ·09 they are unlikely to be significant. No coefficient in either hand differs from its mean by more than ·064 when the correction for grouping has been made, and if we take all ten coefficients for both hands we find that the mean is ·431, and we see that no coefficient exceeds this mean by more than ·071 which is certainly not significant. The greatest difference in the coefficients found are in the right thumb and in the second left finger where the difference is ·123. A comparison of all the values for fraternal correlation shows that those for finger-prints are lower than we should expect. There is a table of fraternal resemblances on p. 25 in *Nature and Nurture, The Problem of the Future* by Karl Pearson* and the lowest fraternal coefficient is that of brother and sister for the Cephalic Index (·43). In certain fingers when the whole hand is considered one or two values approach to what we should expect, as for instance the third right finger, the second left finger and the left thumb, but on the whole the values are lower than we now expect to find when we consider fraternal resemblance.

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* *Eugenics Laboratory Lecture Series*, No. VI. 1910. Cambridge University Press.

(7) *Assortative Mating.*

In this connection it would be as well to point out that in the data we have there is no tendency for like to marry like and that the assortative mating coefficient for type of finger-print is nil. There are only 177 cases where the finger-prints of husband and wife are given, and the ϕ^2 becomes ·0615 and ·0536 for left and right first finger respectively and the correction for the number of cells is ·0904 for both coefficients; accordingly the ϕ^2 is insignificant.

TABLE IX.

	First Right Finger						First Left Finger					
	Husband						Husband					
Wife	Arch	S. Loop	L. Loop	Comp.	Whorl	Totals	Arch	S. Loop	L. Loop	Comp.	Whorl	Totals
Arch	1 **1·82**	8 **9·49**	6 **5·07**	1 **1·17**	7 **5·46**	23	1 **2·82**	14 **10·88**	3 **3·53**	3 **1·84**	4 **5·93**	25
S. Loop	6 **5·69**	30 **29·39**	16 **15·86**	3 **3·66**	17 **17·08**	72	9 **6·89**	23 **26·54**	9 **8·62**	3 **4·48**	17 **14·47**	61
L. Loop	4 **2·77**	12 **14·44**	8 **7 71**	1 **1·78**	10 **8·31**	35	3 **3·28**	13 **12·62**	4 **4·10**	2 **2·13**	7 **6·88**	29
Comp.	— **·47**	4 **2·47**	2 **1·32**	— **·31**	— **1·42**	6	1 **1·69**	7 **6·53**	1 **2·12**	3 **1·10**	3 **3·56**	15
Whorls	3 **3·24**	19 **16·91**	7 **9·03**	4 **2·08**	8 **9·73**	41	6 **5·31**	20 **20·45**	8 **6·64**	2 **3·45**	11 **11·15**	47
Totals	14	73	39	9	42	177	20	77	25	13	42	177

The black figures give the independent probability and it can be seen quite easily that the differences between the independent probability and the actual figures are small and the signs indiscriminate; there are no well marked positive differences along the diagonal and it is fairly obvious that there is no evidence of assortative mating in finger-prints. It will be clear therefore that assortative mating had no influence on our results.

(8) *Probable Explanation of the low values of the coefficients of inheritance in the case of Finger-Prints.*

Galton's material for parental heredity limits us to the consideration of what happens in the case of the forefingers of the two hands. But this limitation compels us to discuss only the case of inheritance from like finger to like finger. Now let us suppose that the finger-prints of both parents were entirely whorls, but that the offspring had composites on the two forefingers and the remaining fingers all whorls. There would clearly here be a marked inheritance of very anomalous distribution, but an investigation which confined itself to the forefingers would fail or largely fail to measure its intensity. If we deal solely with like fingers, we overlook the possibility that the inheritance of the type may not be from like finger to like finger alone, but that the type may be transferred to some other finger of the offspring. And it is fairly easy to demonstrate that there is an association not only between the types of like fingers in parent and child but also between the types of unlike fingers. In other words if a very rare type, say, occurs on the forefinger of the parent, it may more frequently appear on the same finger of the offspring, but should it appear on another finger of the offspring, we cannot pass it by and assume that its appearance is independent of heredity. Clearly we should not do so if polydactyly occurred in the right hand of a parent, but in the left hand or even either foot of the offspring! We must therefore suppose a correlation ρ_{ss} between like fingers in parent and offspring and another correlation $\rho_{ss'}$ between unlike fingers. Our treatment thus far would only be satisfactory provided $\rho_{ss'} = 0$. If this were true descent would be from like finger to like finger. If $\rho_{ss'} = \rho_{ss}$ descent would be indifferent as to finger. *A priori* we should anticipate that $\rho_{ss'}$ would not be zero but $< \rho_{ss}$, or there would be some bias in favour of the type being transferred to like finger. In collecting a large mass of material by voluntary help it was clearly needful to reduce to the utmost the *desiderata*. But in the present case Galton's restriction of his data to the simplicity of forefinger prints has sadly limited the possibility of our determining the real strength of finger-print inheritance. Like all preliminary inquiries, however, it has sufficed to indicate what we need at the next stage, namely complete sets of family finger-prints, and it is to the heavier task of such a collection that the Galton Laboratory is now addressing itself. The theory of the general inheritance of finger-prints may be attacked in the following manner.

Let u be a variable which depends on 10 other variables $x_1, x_2 \ldots x_{10}$ (i.e. the finger-print types of the ten fingers) and u' be the like variable in a second individual. Now if we suppose the variability in type of each finger to be the same—which is only approximately true—and each finger to contribute equally to u, we can take as appropriate values:

$$u = \tfrac{1}{10}(x_1 + x_2 + \ldots + x_{10}),$$
$$u' = \tfrac{1}{10}(x_1' + x_2' + \ldots + x'_{10}).$$

We will further suppose all our variates measured from their means.

Then $r_{uu'}$ will be the correlation between the characters u and u' in the two individuals.

Further if σ denote a standard deviation:

$$\sigma_u^2 = \tfrac{1}{100}\{\sigma^2_{x_1} + \sigma^2_{x_2} + \ldots + \sigma^2_{x_{10}} + 2S\sigma_{xs}\sigma_{xs'}r_{ss'}\}$$
$$= \frac{\sigma_x^2}{10}\left\{1 + \frac{2S(r_{ss'})}{10}\right\}.$$

If $\bar{r}_{ss'}$ be the mean value of $r_{ss'} = \tfrac{1}{45}S(r_{ss'})$, we have

$$\sigma_u^2 = \tfrac{1}{10}\sigma_x^2(1 + 9\bar{r}_{ss'}) = \sigma_{u'}^2,$$

supposing u and u' are individuals in a stable population.

Again if N be the total number of pairs

$$\frac{1}{N} S(uu') = \frac{\sigma_x^2}{100}\left\{S(\rho_{ss}) + S(\rho_{ss'})\right\},$$

where ρ_{ss} is the correlation of x_s and x'_s and $\rho_{ss'}$ of x_s and $x'_{s'}$. Accordingly if $\bar{\rho}_{ss} = \frac{1}{10}(\rho_{11} + \rho_{22} + \ldots + \rho_{10\,10})$, and $\bar{\rho}_{ss'} = \frac{1}{90} S(\rho_{ss'})$, i.e. are the mean correlations, we have

$$\frac{1}{N} S(uu') = \tfrac{1}{10}\sigma_x^2(\bar{\rho}_{ss} + 9\bar{\rho}_{ss'}).$$

Thus we obtain finally:

$$r_{uu'} = \frac{\bar{\rho}_{ss} + 9\bar{\rho}_{ss'}}{1 + 9\bar{r}_{ss'}}.$$

Here $\bar{r}_{ss'}$ is a mean of organic correlations in the same individual, and $\bar{\rho}_{ss}$ and $\bar{\rho}_{ss'}$ are mean direct and cross coefficients of heredity.

If inheritance be unassociated with likeness of finger, we have $\bar{\rho}_{ss'} = \bar{\rho}_{ss}$ and accordingly

$$r_{uu'} = 10\bar{\rho}_{ss}/(1 + 9\bar{r}_{ss'}).$$

If inheritance is only concerned with like fingers $\bar{\rho}_{ss'} = 0$ and

$$r_{uu'} = \bar{\rho}_{ss}/(1 + 9\bar{r}_{ss'}).$$

If there were equal heredity for any characters or function of characters then we must have $r_{uu'} = \bar{\rho}_{ss}$, which leads directly to

$$\bar{\rho}_{ss'} = \bar{\rho}_{ss} \times \bar{r}_{ss'}$$

the relation suggested by Pearson for cross-heredity, i.e. the correlation between two *different* characters in parent and offspring*. It seems, however, that there is extremely likely to be a closer relationship than that of cross-heredity between finger-prints on different fingers, and the only means of testing this point is to measure directly $\bar{\rho}_{ss}$ and $\bar{\rho}_{ss'}$.

Unfortunately Galton's later and longer series will not here help us. We are compelled to turn to his earlier series, not gathered for hereditary purposes, but containing—so to speak—accidentally some 600 pairs of siblings with all ten finger-prints given in most of our individual records. I decided to find $\bar{r}_{ss'}$ from that data; then assuming that inheritance of type was indifferent as to finger we could obtain limits between which the true correlation coefficient of type of finger-print might be said to lie. As before I included Central Pocket Loops in Composites. I combined males and females and used the correction for grouping. The results obtained are given in Table X. There are about 600 cases. The mean (taken from four figures) was found to be ·6076. We can now proceed to apply this corrective factor to the coefficients we have found using the coefficient corrected for grouping. We are assuming for the moment that $\bar{\rho}_{11} = \bar{\rho}_{12}$, then the coefficients already obtained must be multiplied by 1·54598. The results are given in Tables XI and XII. Now all

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* *Phil. Trans.* Vol. 197, p. 290, *R. S. Proc.* Vol. 62, p. 410, and *Biometrika*, Vol. II. p. 385.

TABLE X. *Galton's Earlier Data. Correlation of Different Finger Types in the same Individual.*

Males and Females.

	L. Thumb	L_1	L_2	L_3	L_4	R. Thumb	R_1	R_2	R_3	R_4
L. Thumb	1	—	—	—	—	—	—	—	—	—
L_1 ...	·559	1	—	—	—	—	—	—	—	—
L_2 ...	·566	·653	1	—	—	—	—	—	—	—
L_3 ...	·470	·560	·674	1	—	—	—	—	—	—
L_4 ...	·577	·532	·503	·708	1	—	—	—	—	—
R. Thumb	·825	·518	·511	·539	·565	1	—	—	—	—
R_1 ...	·510	·728	·694	·576	·547	·516	1	—	—	—
R_2 ...	·510	·686	·786	·653	·475	·514	·665	1	—	—
R_3 ...	·525	·541	·608	·772	·685	·598	·589	·619	1	—
R_4 ...	·606	·593	·577	·744	·815	·556	·593	·584	·717	1

TABLE XI.

	Right First Finger		Left First Finger	
	r	R	r	R
Father and Son	·333	·515	·434	·671
Father and Daughter ...	·363	·561	·273	·422
Mother and Son	·423	·654	·416	·643
Mother and Daughter ...	·411	·635	·386	·597
Mean	·383	·592	·382	·591

TABLE XII.

	Sibship		Brother-Brother		Sister-Sister		Brother-Sister	
	r	R	r	R	r	R	r	R
Right Thumb ...	·379	·586	—	—	—	—	—	—
First ...	·389	·601	·424	·655	·356	·550	·382	·591
Second ...	·384	·594	—	—	—	—	—	—
Third ...	·477	·737	—	—	—	—	—	—
Fourth ...	·439	·679	—	—	—	—	—	—
Mean ...	·413	·638	—	—	—	—	—	—
Left Thumb ...	·477	·737	—	—	—	—	—	—
First ...	·411	·635	·385	·595	·348	·538	·387	·598
Second ...	·502	·776	—	—	—	—	—	—
Third ...	·411	·635	—	—	—	—	—	—
Fourth ...	·440	·680	—	—	—	—	—	—
Mean ...	·449	·694	—	—	—	—	—	—

these new coefficients are distinctly higher than we should expect but depend on the assumption that inheritance of type is indifferent as to finger. Now if this be true, the correlation coefficient between any one finger of a sibling and all the other fingers in a second sibling should be the same; but if inheritance of type depends absolutely on the one finger as I have assumed by correlating each finger separately there would be no correlation between one finger and any other finger in a sibship. I decided to test this on the available data in two cases, namely for the left thumb and for the left first finger. I did not go further than this for several reasons, chiefly because the labour involved is very great, and since I am doubtful about the wisdom of combining males and females it seemed waste of time to continue till we had a larger series of data for the whole hand, and were not entirely dependent on about 360 pairs of brothers and sisters from Galton's earlier data. The cross tables were made so that each pair of siblings gave two entries. The numbers in each table averaged just over 700. The following coefficients were obtained.

TABLE XIII.

Left Thumb	of first Sibling	and Left	Thumb	of second	Sibling	·477
„	„	„	First	„	„	·368
„	„	„	Second	„	„	·309
„	„	„	Third	„	„	·235
„	„	„	Fourth	„	„	·314
Left Thumb	of first Sibling	and Right	Thumb	„	„	·410
„	„	„	First	„	„	·316
„	„	„	Second	„	„	·267
„	„	„	Third	„	„	·344
„	„	„	Fourth	„	„	·269

It will be seen at once that all these coefficients are significant but all are lower than the correlation between the same finger in siblings*; that is to say inheritance of type is partially indifferent as to finger but not wholly so, and to find the true measure of correlation between type of finger-print we should correct by using the formula $\frac{\bar{\rho}_{11} + 9\bar{\rho}_{12}}{1 + 9\bar{r}_{12}}$. Adopting this correction for the right thumb we find that $R = \frac{\cdot 477 + 2\cdot 832}{1 + 5\cdot 4684} = \cdot 512$ which is a very reasonable value for the coefficient of correlation between siblings.

I did the same thing for the left first finger, and found the results in Table XIV.

These results agree fairly well with the results previously found for the resemblance between brothers and I think it is reasonable to assume that when we can apply similar corrections to the parental tables we shall reach results corresponding equally well. In other words it is extremely probable that finger-prints are in-

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* It will be noted at once that like fingers have the highest, homologous fingers the next highest correlation.

herited at the same rate as other physical characters, but the type characteristic of any finger in the parent while mostly likely to pass to the like finger in the offspring, may easily pass to the homologous finger of the other hand or indeed to any other finger whatever of the offspring.

TABLE XIV.

Left First Finger of first Sibling and Left	First Finger	of second Sibling	·411
” ” ” ”	Thumb	” ”	·368
” ” ” ”	Second	” ”	·387
” ” ” ”	Third	” ”	·357
” ” ” ”	Fourth	” ”	·345
Left First Finger of first Sibling and Right	Thumb	” ”	·314
” ” ” ”	First	” ”	·399
” ” ” ”	Second	” ”	·342
” ” ” ”	Third	” ”	·384
” ” ” ”	Fourth	” ”	·354

and

$$R = \frac{\cdot 411 + 3\cdot 258}{1 + 5\cdot 4684} = \cdot 567.$$

(9) It is possible from the finger-prints in Galton's later series to obtain for the first fingers the correlation between grandparents and their grandchildren, between uncles and aunts and their nephews and nieces and between pairs of cousins. For the consideration of the grandparental relationship we have only 405 cases and to obtain even this small number we have had to combine the sexes. In the uncle and aunt tables we have rather over 500 cases in each table so we could keep the sexes separate but could not distinguish between paternal and maternal relations. We found 657 pairs of female cousins, 795 pairs of males and 1510 of male and female.

Until we have data for every finger it is not possible to deal adequately with the inheritance of type of finger-print as we cannot allow for the fact that inheritance of type of finger-print is not wholly indifferent as to finger but we give the results in Table XV; it seems especially needful to give them since they do not wholly confirm results previously reached; cousins show a decidedly lower value than we should expect.

For grandparents the values found for right and left forefingers are practically identical; the values given on p. 78 are below the true value and when we can make the necessary adjustment we shall probably find that the grandparental coefficient for the inheritance of type of finger-print is about ·3. The only grandparental data at present reduced for man are those for eye colour and the eight cases give a mean value of ·32 with which the value found for finger-prints agrees very well but both these are somewhat higher than the mean value found for other species. We are hoping soon to work out the grandparental coefficients for other characters.

In the uncle and aunt tables the coefficients found for the right forefinger are lower than those found for the left but the difference is not significant. The value

TABLE XV.

	First Left		First Right	
		Corrected for grouping		Corrected for grouping
Grandparent and Grandchild	·237	·281	·246	·285
Uncle and Nephew	·302	·342	·291	·337
Uncle and Niece	·259	·297	·211	·245
Aunt and Nephew	·266	·303	·229	·262
Aunt and Niece	·280	·323	·192	·217
Mean	—	·316	—	·265
Male Cousins...	·115	·131	·210	·244
Female Cousins	·152	·174	·130	·145
Male and Female Cousins ...	·109	·124	·137	·156
Mean	—	·143	—	·182

of ·316 is higher than we should expect from other work done and this will be increased when we can allow for the inheritance between the first finger and the other fingers. If we compare the mean of the two forefingers ·29 with that found for the grandparental relationship we see that they are almost the same and we had supposed from previous work that the grandparental relationship was slightly higher. At the same time considering the number of cases and the probable error there is really nothing in contradiction with former work in these results.

We cannot however say the same when we turn to cousins. Inheritance of type of finger-print in cousins is significantly lower than in the grandparental or avuncular relationship, a result not confirmed by any other work we have done except in the less easily measured characters such as temper. Though the values of the coefficients are very low the diagonals are positive and there is no doubt about the sign which is an important point to consider when the contingency values, uncorrected for grouping, are as small as they are in this table. Further the correction for the number of cells leaves ϕ^2 positive, but we must admit that we were surprised to find such low values and it is difficult to account for them. Family schedules accompanied the finger-prints and there seems very little chance that there could have been any selection of the cousins considered. When the anthropometric department at the Galton Laboratory can be fully equipped and opened we hope to obtain more extensive finger-print data for the whole hand for collaterals and until we have that material we can only give the results we have reached on Galton's own data. We have at present no explanation to offer for the low values obtained for the resemblance in finger-prints between cousins.

APPENDIX OF DATA.

GALTON'S LATER SERIES.

TABLE I.

Father's Left First Finger and Son's Left First Finger.

Father.

		A.	S. L.	L. L.	C.	W.	Totals
Son.	A.	13	16	3	—	1	33
	S. L.	18	62	23	8	26	137
	L. L.	1	25	10	4	12	52
	C.	1	11	6	7	4	29
	W.	8	25	11	9	25	78
	Totals	41	139	53	28	68	329

TABLE II.

Father's Right First Finger and Son's Right First Finger.

Father.

		A.	S. L.	L. L.	C.	W.	Totals
Son.	A.	8	15	8	1	3	35
	S. L.	19	64	22	10	25	140
	L. L.	6	21	12	4	14	57
	C.	3	10	4	6	9	32
	W.	2	28	12	11	27	80
	Totals	38	138	58	32	78	344

TABLE III.

Father's Left First Finger and Daughter's Left First Finger.

Father.

		A.	S. L.	L. L.	C.	W.	Totals
Daughter.	A.	14	22	8	3	2	49
	S. L.	28	68	15	16	24	151
	L. L.	7	28	10	6	12	63
	C.	2	12	6	5	8	33
	W.	5	18	6	4	9	42
	Totals	56	148	45	34	55	338

TABLE IV.

Father's Right First Finger and Daughter's Right First Finger.

Father.

		A.	S. L.	L. L.	C.	W.	Totals
Daughter.	A.	7	21	5	5	5	43
	S. L.	14	80	18	6	27	145
	L. L.	6	25	19	6	11	67
	C.	5	9	13	7	4	38
	W.	7	17	15	3	15	57
	Totals	39	152	70	27	62	350

TABLE V.

Mother's Left First Finger and Son's Left First Finger.

Mother.

		A.	S. L.	L. L.	C.	W.	Totals
Son.	A.	16	18	3	4	3	44
	S. L.	30	74	29	14	16	163
	L. L.	9	18	10	8	16	61
	C.	5	10	5	6	14	40
	W.	7	25	17	10	37	96
	Totals	67	145	64	42	86	404

TABLE VI.

Mother's Right First Finger and Son's Right First Finger.

Mother.

		A.	S. L.	L. L.	C.	W.	Totals
Son.	A.	13	19	1	3	8	44
	S. L.	31	72	21	6	25	155
	L. L.	4	21	15	8	12	60
	C.	3	17	9	4	13	46
	W.	5	27	16	6	43	97
	Totals	56	156	62	27	101	402

TABLE VII.

Mother's Left First Finger and Daughter's Left First Finger.

Mother.

Daughter.	A.	S. L.	L. L.	C.	W.	Totals
A.	19	19	9	4	6	57
S. L.	21	83	45	12	35	196
L. L.	6	27	14	8	17	72
C.	6	12	5	6	12	41
W.	3	18	7	6	28	62
Totals	55	159	80	36	98	428

TABLE VIII.

Mother's Right First Finger and Daughter's Right First Finger.

Mother.

Daughter.	A.	S. L.	L. L.	C.	W.	Totals
A.	17	22	7	5	5	56
S. L.	34	71	35	17	19	176
L. L.	7	25	19	9	17	77
C.	6	13	9	7	13	48
W.	2	16	16	5	33	72
Totals	66	147	86	43	87	429

TABLE IX.

Brothers' Left First Finger.

First Brother.

Second Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	22	53	10	9	6	100
S. L.	53	172	40	34	52	351
L. L.	10	40	18	12	38	118
C.	9	34	12	4	19	78
W.	6	52	38	19	78	193
Totals	100	351	118	78	193	840

TABLE X.

Brothers' Right First Finger.

First Brother.

Second Brother.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	49	12	6	5	96
S. L.	49	170	39	22	60	340
L. L.	12	39	28	17	39	135
C.	6	22	17	2	29	76
W.	5	60	39	29	92	225
Totals	96	340	135	76	225	872

TABLE XI.

Sisters' Left First Finger.

First Sister.

Second Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	30	54	10	8	10	112
S. L.	54	216	70	27	42	409
L. L.	10	70	26	14	31	151
C.	8	27	14	16	13	78
W.	10	42	31	13	40	136
Totals	112	409	151	78	136	886

TABLE XII.

Sisters' Right First Finger.

First Sister.

Second Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	38	46	10	12	8	114
S. L.	46	174	67	43	44	374
L. L.	10	67	30	18	23	148
C.	12	43	18	10	24	107
W.	8	44	23	24	46	145
Totals	114	374	148	107	145	888

TABLE XIII.

Brother-Sister Left First Finger.

Brother.

Second Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	39	36	7	9	14	105
S. L.	49	181	61	30	64	385
L. L.	4	45	31	14	37	131
C.	13	29	11	14	28	95
W.	9	51	25	15	53	153
Totals	114	342	135	82	196	869

TABLE XIV.

Brother-Sister Right First Finger.

Brother.

Second Sister.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	46	17	4	13	112
S. L.	43	173	45	29	64	354
L. L.	14	37	26	15	30	122
C.	15	32	24	12	32	115
W.	3	43	35	15	69	165
Totals	107	331	147	75	208	868

TABLE XV.

Siblings' Left Thumb.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	24	14	2	3	47
S. L.	24	44	61	13	14	156
L. L.	14	61	166	26	45	312
C.	2	13	26	22	13	76
W.	3	14	45	13	58	133
Totals	47	156	312	76	133	724

TABLE XVI.

Siblings' Right Thumb.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	—	8	11	3	4	26
S. L.	8	30	50	3	21	112
L. L.	11	50	170	17	82	330
C.	3	3	17	2	7	32
W.	4	21	82	7	116	230
Totals	26	112	330	32	230	730

TABLE XVII.

Siblings' First Left Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	38	44	18	1	8	109
S. L.	44	118	60	11	39	272
L. L.	18	60	34	9	36	157
C.	1	11	9	—	9	30
W.	8	39	36	9	66	158
Totals	109	272	157	30	158	726

TABLE XVIII.

Siblings' First Right Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	37	8	7	16	100
S. L.	37	114	50	15	44	260
L. L.	8	50	18	13	44	133
C.	7	15	13	2	8	45
W.	16	44	44	8	76	188
Totals	100	260	133	45	188	726

TABLE XIX.

Siblings' Second Left Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	32	41	17	2	2	94
S. L.	41	130	81	2	19	273
L. L.	17	81	142	4	25	269
C.	2	2	4	2	4	14
W.	2	19	25	4	28	78
Totals	94	273	269	14	78	728

TABLE XX.

Siblings' Second Right Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	27	13	—	5	53
S. L.	27	188	80	3	27	325
L. L.	13	80	114	3	36	246
C.	—	3	3	2	2	10
W.	5	27	36	2	14	84
Totals	53	325	246	10	84	718

TABLE XXI.

Siblings' Third Left Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	6	11	8	—	4	29
S. L.	11	68	56	9	29	173
L. L.	8	56	164	13	60	301
C.	—	9	13	4	12	38
W.	4	29	60	12	72	177
Totals	29	173	301	38	177	718

TABLE XXII.

Siblings' Third Right Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	2	8	2	—	5	17
S. L.	8	84	46	3	32	173
L. L.	2	46	118	9	79	254
C.	—	3	9	4	12	28
W.	5	32	79	12	126	254
Totals	17	173	254	28	254	726

TABLE XXIII.

Siblings' Fourth Left Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	15	3	1	2	25
S. L.	15	156	91	4	14	280
L. L.	3	91	192	6	21	313
C.	1	4	6	—	1	12
W.	2	14	21	1	12	50
Totals	25	280	313	12	50	680

TABLE XXIV.

Siblings' Fourth Right Finger.

First Sibling.

Second Sibling.	A.	S. L.	L. L.	C.	W.	Totals
A.	4	12	3	—	—	19
S. L.	12	150	97	3	20	282
L. L.	3	97	170	7	35	312
C.	—	3	7	—	6	16
W.	—	20	35	6	14	75
Totals	19	282	312	16	75	704

TABLE XXV*.

Left Thumb and Left First Finger.

Left Thumb.

Left First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	23	35	19	8	8	93
S. L.	11	69	101	22	18	221
L. L.	2	10	74	18	24	128
C.	—	1	14	5	8	28
W.	—	6	61	16	62	145
Totals	36	121	269	69	120	615

TABLE XXVI.

Left Thumb and Left Second Finger.

Left Thumb.

Left Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	23	14	9	8	78
S. L.	8	73	98	16	29	224
L. L.	3	22	126	31	37	219
C.	—	1	6	3	7	17
W.	1	2	22	9	37	71
Totals	36	121	266	68	118	609

TABLE XXVII.

Left Thumb and Left Third Finger.

Left Thumb.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	3	—	4	23
S. L.	17	50	55	13	8	143
L. L.	6	41	126	28	42	243
C.	3	5	16	5	7	36
W.	1	17	64	23	57	162
Totals	36	120	264	69	118	607

TABLE XXVIII.

Left Thumb and Left Fourth Finger.

Left Thumb.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	5	2	—	—	19
S. L.	20	72	98	11	21	222
L. L.	4	37	137	41	70	289
C.	—	1	3	3	4	11
W.	—	2	15	11	18	46
Totals	36	117	255	66	113	587

TABLE XXIX.

Left Thumb and Right Thumb.

Left Thumb.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	15	1	1	1	—	18
S. L.	21	55	11	5	2	94
L. L.	1	49	197	12	13	272
C.	—	4	10	17	4	35
W.	—	10	51	34	101	196
Totals	37	119	270	69	120	615

TABLE XXX.

Left Thumb and Right First Finger.

Left Thumb.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	24	25	20	8	6	83
S. L.	11	65	98	18	23	215
L. L.	1	13	64	12	27	117
C.	1	5	16	6	8	36
W.	—	13	70	25	56	164
Totals	37	121	268	69	120	615

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* Tables XXV—LXIX are organic correlation tables, i.e. correlation between the prints of the several fingers in the same person.

TABLE XXXI.

Left Thumb and Right Second Finger.

Left Thumb.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	17	15	7	5	2	46
S. L.	16	80	116	21	37	270
L. L.	3	20	116	29	48	216
C.	—	2	2	2	7	13
W.	—	3	29	12	16	60
Totals	36	120	270	69	110	605

TABLE XXXII.

Left Thumb and Right Third Finger.

Left Thumb.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	1	3	—	—	13
S. L.	17	49	50	14	10	140
L. L.	4	39	91	18	40	192
C.	1	10	15	4	3	33
W.	4	22	107	33	67	233
Totals	35	121	266	69	120	611

TABLE XXXIII.

Left Thumb and Right Fourth Finger.

Left Thumb.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	3	—	—	—	14
S. L.	21	79	98	13	15	226
L. L.	4	30	136	39	70	279
C.	1	2	5	1	6	15
W.	—	5	26	14	25	70
Totals	37	119	265	67	116	604

TABLE XXXIV.

Left First Finger and Left Second Finger.

Left First Finger.

Left Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	42	31	5	—	—	78
S. L.	38	128	34	7	17	224
L. L.	10	57	71	14	66	218
C.	1	1	5	3	6	16
W.	1	5	11	3	52	72
Totals	92	222	126	27	141	608

TABLE XXXV.

Left First Finger and Left Third Finger.

Left First Finger.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	16	5	2	—	—	23
S. L.	43	82	12	1	5	143
L. L.	21	88	64	11	57	241
C.	3	15	6	4	9	37
W.	10	27	42	12	71	162
Totals	93	217	126	28	142	606

TABLE XXXVI.

Left First Finger and Left Fourth Finger.

Left First Finger.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	15	4	—	—	—	19
S. L.	49	116	29	5	22	221
L. L.	26	82	73	22	86	289
C.	1	3	1	1	6	12
W.	1	7	15	—	22	45
Totals	92	212	118	28	136	586

TABLE XXXVII.

Left First Finger and Right Thumb.

Left First Finger.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	5	2	—	—	18
S. L.	34	51	3	—	6	94
L. L.	31	114	66	12	48	271
C.	5	10	10	1	7	33
W.	11	42	47	14	84	198
Totals	92	222	128	27	145	614

TABLE XXXVIII.

Left First Finger and Right First Finger.

Left First Finger.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	45	32	5	—	—	82
S. L.	39	137	29	3	6	214
L. L.	5	28	49	9	26	117
C.	3	8	9	7	8	35
W.	2	17	35	9	103	166
Totals	94	222	127	28	143	614

TABLE XXXIX.

Left First Finger and Right Second Finger.

Left First Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	35	8	2	—	—	45
S. L.	46	159	30	10	26	271
L. L.	13	47	78	9	68	215
C.	—	—	5	1	7	13
W.	—	6	13	8	43	70
Totals	94	220	128	28	144	614

TABLE XL.

Left First Finger and Right Third Finger.

Left First Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	—	—	—	13
S. L.	44	78	14	1	4	141
L. L.	16	73	48	11	38	186
C.	4	17	5	—	6	32
W.	18	48	57	16	95	234
Totals	92	219	124	28	143	606

TABLE XLI.

Left First Finger and Right Fourth Finger.

Left First Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	13	1	—	—	—	14
S. L.	58	121	27	9	10	225
L. L.	19	83	72	18	87	279
C.	—	4	2	—	7	13
W.	3	9	21	1	38	72
Totals	93	218	122	28	142	603

TABLE XLII.

Left Second Finger and Left Third Finger.

Left Second Finger.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	16	7	—	—	—	23
S. L.	46	79	14	—	1	140
L. L.	10	103	112	5	12	242
C.	4	11	18	6	4	43
W.	2	22	71	5	54	154
Totals	78	222	215	16	71	602

TABLE XLIII.

Left Second Finger and Left Fourth Finger.

Left Second Finger.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	6	—	—	—	18
S. L.	46	110	54	3	10	223
L. L.	16	85	134	10	41	286
C.	1	3	5	—	2	11
W.	2	8	16	2	17	45
Totals	77	212	209	15	70	583

TABLE XLIV.

Left Second Finger and Right Thumb.

Left Second Finger.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	3	1	—	1	17
S. L.	29	50	13	1	2	95
L. L.	24	108	112	5	19	268
C.	5	11	13	2	4	35
W.	8	52	80	8	46	194
Totals	78	224	219	16	72	609

TABLE XLV.

Left Second Finger and Right First Finger.

Left Second Finger.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	45	31	5	1	1	83
S. L.	27	124	56	3	2	212
L. L.	5	32	71	5	3	116
C.	—	16	16	1	4	37
W.	1	23	70	6	61	161
Totals	78	226	218	16	71	609

TABLE XLVI.

Left Second Finger and Right Second Finger.

Left Second Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	31	13	1	1	—	46
S. L.	46	167	46	2	6	267
L. L.	1	40	143	2	28	214
C.	—	2	3	6	2	13
W.	—	3	25	5	36	69
Totals	78	225	218	16	72	609

TABLE XLVII.

Left Second Finger and Right Third Finger.

Left Second Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	1	—	—	—	12
S. L.	46	77	14	1	1	139
L. L	11	79	83	4	14	191
C.	2	15	14	1	1	33
W.	7	52	106	10	55	230
Totals	77	224	217	16	71	605

TABLE XLVIII.

Left Second Finger and Right Fourth Finger.

Left Second Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	2	—	—	—	14
S. L.	48	119	50	4	4	225
L. L.	14	88	129	9	37	277
C.	—	2	8	1	4	15
W.	2	10	28	2	25	67
Totals	76	221	215	16	70	598

TABLE XLIX.

Left Third Finger and Left Fourth Finger.

Left Third Finger.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	6	1	1	1	19
S. L.	10	106	70	11	23	220
L. L.	3	24	157	25	80	289
C.	—	—	4	1	6	11
W.	—	2	3	2	39	46
Totals	23	138	235	40	149	585

TABLE L.

Left Third Finger and Right Thumb.

Left Third Finger.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	7	4	3	3	—	17
S. L.	7	49	21	7	10	94
L. L.	5	67	124	14	56	266
C.	—	7	9	4	17	37
W.	4	14	87	12	75	192
Totals	23	141	244	40	158	606

TABLE LI.

Left Third Finger and Right First Finger.

Left Third Finger.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	14	39	20	5	3	81
S. L.	9	79	90	12	22	212
L. L.	—	16	60	7	32	115
C.	—	2	12	5	15	34
W.	—	7	62	10	86	165
Totals	23	143	244	39	158	607

TABLE LII.

Left Third Finger and Right Second Finger.

Left Third Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	14	22	8	2	—	46
S. L.	8	104	108	20	26	266
L. L.	1	16	108	10	78	213
C.	—	—	6	—	7	13
W.	—	—	14	8	48	70
Totals	23	142	244	40	159	608

TABLE LIII.

Left Third Finger and Right Third Finger.

Left Third Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	3	—	1	—	12
S. L.	14	91	30	3	2	140
L. L.	1	29	129	9	21	189
C.	—	8	14	6	5	33
W.	—	11	68	21	130	230
Totals	23	142	241	40	158	604

TABLE LIV.

Left Third Finger and Right Fourth Finger.

Left Third Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	1	—	—	14
S. L.	11	110	71	11	21	224
L. L.	2	23	152	23	76	276
C.	—	—	2	4	9	15
W.	—	4	12	2	52	70
Totals	23	140	238	40	158	599

TABLE LV.

Left Fourth Finger and Right Thumb.

Left Fourth Finger.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	4	5	—	—	17
S. L.	5	59	22	—	4	90
L. L.	6	112	121	3	16	258
C.	—	7	20	1	6	34
W.	—	39	121	5	22	187
Totals	19	221	289	9	48	586

TABLE LVI.

Left Fourth Finger and Right First Finger.

Left Fourth Finger.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	14	45	20	—	2	81
S. L.	5	116	76	2	7	206
L. L.	—	32	69	1	9	111
C.	—	7	25	1	2	35
W.	—	21	100	5	28	154
Totals	19	221	290	9	48	587

TABLE LVII.

Left Fourth Finger and Right Second Finger.

Left Fourth Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	22	13	—	—	45
S. L.	9	133	99	5	11	257
L. L.	—	55	124	3	25	207
C.	—	3	8	—	2	13
W.	—	10	46	1	10	67
Totals	19	223	290	9	48	589

TABLE LVIII.

Left Fourth Finger and Right Third Finger.

Left Fourth Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	7	2	3	—	—	12
S. L.	8	104	25	—	3	140
L. L.	—	77	101	2	4	184
C.	1	6	20	2	—	29
W.	2	33	138	5	41	219
Totals	18	222	287	9	48	584

TABLE LIX.

Left Fourth Finger and Right Fourth Finger.

Left Fourth Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	8	4	1	—	—	13
S. L.	11	164	41	—	3	219
L. L.	—	45	203	5	14	267
C.	—	2	10	2	1	15
W.	—	2	31	2	30	65
Totals	19	217	286	9	48	579

TABLE LX.

Right Thumb and Right First Finger.

Right Thumb.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	25	25	6	14	82
S. L.	4	58	109	9	35	215
L. L.	1	6	61	6	42	116
C.	1	—	11	3	22	37
W.	—	6	64	11	83	164
Totals	18	95	270	35	196	614

TABLE LXI.

Right Thumb and Right Second Finger.

Right Thumb.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	18	12	3	4	46
S. L.	6	63	130	8	63	270
L. L.	1	12	107	17	79	216
C.	1	1	2	2	6	12
W.	—	1	20	5	44	70
Totals	17	95	271	35	196	614

TABLE LXII.

Right Thumb and Right Third Finger.

Right Thumb.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	7	3	3	—	—	13
S. L.	5	48	60	5	22	140
L. L.	3	22	105	8	53	191
C.	—	7	16	3	7	33
W.	2	14	84	19	114	233
Totals	17	94	268	35	196	610

TABLE LXIII.

Right Thumb and Right Fourth Finger.

Right Thumb.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	5	8	1	—	—	14
S. L.	8	61	120	4	32	225
L. L.	4	23	122	20	110	279
C.	—	—	3	3	8	14
W.	1	2	21	7	40	71
Totals	18	94	267	34	190	603

TABLE LXIV.

Right First Finger and Right Second Finger.

Right First Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	34	8	3	—	1	46
S. L.	38	154	33	12	35	272
L. L.	8	51	62	18	76	215
C.	1	2	5	—	5	13
W.	1	1	14	6	47	69
Totals	82	216	117	36	164	615

TABLE LXV.

Right First Finger and Right Third Finger.

Right First Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	3	—	—	—	13
S. L.	42	81	12	2	4	141
L. L.	12	78	47	12	41	190
C.	4	11	10	3	5	33
W.	13	42	46	20	113	234
Totals	81	215	115	37	163	611

TABLE LXVI.

Right First Finger and Right Fourth Finger.

Right First Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	2	—	—	—	14
S. L.	49	120	34	7	16	226
L. L.	17	81	64	23	95	280
C.	1	2	3	4	4	14
W.	3	9	11	2	46	71
Totals	82	214	112	36	161	605

TABLE LXVII.

Right Second Finger and Right Third Finger.

Right Second Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals.
A.	10	3	—	—	—	13
S. L.	24	100	16	1	—	141
L. L.	6	88	84	3	11	192
C.	4	15	11	1	1	32
W.	2	64	102	8	58	234
Totals	46	270	213	13	70	612

TABLE LXVIII.

Right Second Finger and Right Fourth Finger.

Right Second Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	10	4	—	—	—	14
S. L.	28	142	47	3	6	226
L. L.	6	103	127	5	39	280
C.	—	4	6	2	3	15
W.	1	14	31	2	22	70
Totals	45	267	211	12	70	605

TABLE LXIX.

Right Third Finger and Right Fourth Finger.

Right Third Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	7	5	1	—	1	14
S. L.	5	103	69	11	36	224
L. L.	1	29	111	16	120	277
C.	—	—	1	4	10	15
W.	—	1	5	1	63	70
Totals	13	138	187	32	230	600

TABLE LXX*.

Siblings' Left Thumb and Left First Finger.

Left Thumb.

Left First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	17	36	41	9	7	110
S. L.	20	70	123	22	35	270
L. L.	6	28	73	17	27	151
C.	—	1	15	6	8	30
W.	6	17	60	22	53	158
Totals	49	152	312	76	130	719

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* Tables LXX—LXXXVI are cross-hereditary tables, e.g. Left Thumb in First, Left First Finger in Second Sibling.

TABLE LXXI.

Siblings' Left Thumb and Left Second Finger.

Left Thumb.

Left Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	12	32	32	10	8	94
S. L.	19	65	123	26	45	278
L. L.	16	46	124	28	43	257
C.	1	1	3	1	8	14
W.	1	10	29	10	28	78
Totals	49	154	311	75	132	721

TABLE LXXII.

Siblings' Left Thumb and Left Third Finger.

Left Thumb.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	3	9	11	3	3	29
S. L.	21	49	74	14	21	179
L. L.	16	57	141	33	51	298
C.	2	7	15	4	10	38
W.	7	31	72	20	47	177
Totals	49	153	313	74	132	721

TABLE LXXIII.

Siblings' Left Thumb and Left Fourth Finger.

Left Thumb.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	2	12	9	1	1	25
S. L.	31	69	131	21	31	283
L. L.	11	58	142	40	75	326
C.	—	3	7	1	3	14
W.	3	7	16	8	16	50
Totals	47	149	305	71	126	698

TABLE LXXIV.

Siblings' Left Thumb and Right Thumb.

Left Thumb.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	—	11	9	2	2	24
S. L.	21	36	37	9	10	113
L. L.	18	71	172	25	45	331
C.	2	7	16	4	4	33
W.	8	29	79	35	71	222
Totals	49	154	313	75	132	723

TABLE LXXV.

Siblings' Left Thumb and Right First Finger.

Left Thumb.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	40	35	12	4	102
S. L.	21	59	112	25	44	261
L. L.	7	26	61	13	23	130
C.	2	6	19	2	18	47
W.	8	23	86	23	43	183
Totals	49	154	313	75	132	723

TABLE LXXVI.

Siblings' Left Thumb and Right Second Finger.

Left Thumb.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	6	15	23	6	2	52
S. L.	27	81	147	25	45	325
L. L.	11	43	109	30	53	246
C.	—	2	3	3	6	14
W.	5	13	31	10	25	84
Totals	49	154	313	74	131	721

TABLE LXXVII.

Siblings' Left Thumb and Right Third Finger.

Left Thumb.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	2	8	5	1	—	16
S. L.	23	57	68	15	14	177
L. L.	11	39	127	27	44	248
C.	1	11	12	5	14	43
W.	12	38	102	26	60	238
Totals	49	153	314	74	132	722

TABLE LXXVIII.

Siblings' Left Thumb and Right Fourth Finger.

Left Thumb.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	3	5	9	1	1	19
S. L.	29	72	129	24	31	285
L. L.	14	60	129	36	75	314
C.	—	4	8	2	5	19
W.	3	8	33	12	18	74
Totals	49	149	308	75	130	711

TABLE LXXIX.

Siblings' Left First Finger and Left Second Finger.

Left First Finger.

Left Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	35	39	6	4	9	93
S. L.	47	119	62	4	47	279
L. L.	24	96	61	13	62	256
C.	2	2	2	1	7	14
W.	2	16	21	7	32	78
Totals	110	272	152	29	157	720

TABLE LXXX.

Siblings' Left First Finger and Left Third Finger.

Left First Finger.

Left Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	15	2	1	—	29
S. L.	45	75	32	4	22	178
L. L.	31	117	70	11	67	296
C.	7	11	5	—	15	38
W.	16	51	43	14	52	176
Totals	110	269	152	30	156	717

TABLE LXXXI.

Siblings' Left First Finger and Left Fourth Finger.

Left First Finger.

Left Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	11	9	3	1	1	25
S. L.	55	126	57	5	40	283
L. L.	36	113	74	17	83	323
C.	—	5	4	1	4	14
W.	3	9	13	4	20	49
Totals	105	262	151	28	148	694

TABLE LXXXII.

Siblings' Left First Finger and Right Thumb.

Left First Finger.

Right Thumb.	A.	S. L.	L. L.	C.	W.	Totals
A.	6	12	3	—	2	23
S. L.	35	47	13	2	16	113
L. L.	45	133	77	14	63	332
C.	6	11	7	1	8	33
W.	19	69	51	13	69	221
Totals	111	272	151	30	158	722

TABLE LXXXIII.

Siblings' Left First Finger and Right First Finger.

Left First Finger.

Right First Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	36	42	8	4	9	99
S. L.	36	125	56	8	38	263
L. L.	14	50	26	6	33	129
C.	7	13	13	2	12	47
W.	18	40	49	10	66	183
Totals	111	270	152	30	158	721

TABLE LXXXIV.

Siblings' Left First Finger and Right Second Finger.

Left First Finger.

Right Second Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	16	22	8	1	4	51
S. L.	66	141	60	9	51	327
L. L.	22	82	65	13	63	245
C.	1	2	4	—	6	13
W.	6	24	15	7	31	83
Totals	111	271	152	30	155	719

TABLE LXXXV.

Siblings' Left First Finger and Right Third Finger.

Left First Finger.

Right Third Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	1	—	—	17
S. L.	46	81	29	3	18	177
L. L.	26	98	60	6	58	248
C.	2	18	7	1	13	41
W.	28	66	56	19	68	237
Totals	111	270	153	29	157	720

TABLE LXXXVI.

Siblings' Left First Finger and Right Fourth Finger.

Left First Finger.

Right Fourth Finger.	A.	S. L.	L. L.	C.	W.	Totals
A.	9	7	—	—	3	19
S. L.	57	124	62	11	32	286
L. L.	32	116	65	15	84	312
C.	3	5	3	2	6	19
W.	6	16	20	2	29	73
Totals	107	268	150	30	154	709

TABLE LXXXVII.

Grandparent's Left First Finger and Grandchild's Left First Finger.

Grandparent.

		A.	S. L.	L. L.	C.	W.	Totals
Grandchild.	A.	9	31	—	2	23	65
	S. L.	35	82	19	1	63	200
	L. L.	8	22	7	—	18	55
	C.	1	6	3	—	5	15
	W.	12	18	9	—	31	70
	Totals	65	159	38	3	140	405

TABLE LXXXVIII.

Grandparent's Right First Finger and Grandchild's Right First Finger.

Grandparent.

		A.	S. L.	L. L.	C.	W.	Totals
Grandchild.	A.	17	23	10	—	27	77
	S. L.	37	61	21	1	44	164
	L. L.	6	14	13	4	18	55
	C.	5	10	2	—	8	25
	W.	19	24	9	2	30	84
	Totals	84	132	55	7	127	405

TABLE LXXXIX.

Uncle's First Left Finger and Nephew's First Left Finger.

Uncle.

		A.	S. L.	L. L.	C.	W.	Totals
Nephew.	A.	21	39	4	3	8	75
	S. L.	44	77	20	16	28	185
	L. L.	11	32	14	11	23	91
	C.	3	13	4	6	11	37
	W.	10	31	23	13	28	105
	Totals	89	192	65	49	98	493

TABLE XC.

Uncle's First Right Finger and Nephew's First Right Finger.

Uncle.

		A.	S. L.	L. L.	C.	W.	Totals
Nephew.	A.	9	38	8	2	3	60
	S. L.	16	94	16	9	46	181
	L. L.	11	38	10	3	27	89
	C.	1	19	10	3	13	46
	W.	9	39	13	15	41	117
	Totals	46	228	57	32	130	493

TABLE XCI.

Uncle's First Left Finger and Niece's First Left Finger.

Uncle.

		A.	S. L.	L. L.	C.	W.	Totals
Niece.	A.	18	31	7	2	19	77
	S. L.	41	99	23	25	51	239
	L. L.	4	18	11	13	22	68
	C.	7	10	5	6	10	38
	W.	7	33	13	8	10	71
	Totals	77	191	59	54	112	493

TABLE XCII.

Uncle's First Right Finger and Niece's First Right Finger.

Uncle.

		A.	S. L.	L. L.	C.	W.	Totals
Niece.	A.	10	43	10	4	22	89
	S. L.	13	98	20	12	45	188
	L. L.	4	19	15	9	23	70
	C.	5	25	8	4	17	59
	W.	8	35	8	9	27	87
	Totals	40	220	61	38	134	493

TABLE XCIII.

Aunt's First Left Finger and Nephew's First Left Finger.

Aunt.

		A.	S. L.	L. L.	C.	W.	Totals
Nephew.	A.	5	25	8	4	11	53
	S. L.	24	91	28	20	29	192
	L. L.	9	44	17	9	16	95
	C.	3	20	4	14	4	45
	W.	7	50	39	20	13	129
	Totals	48	230	96	67	73	514

TABLE XCIV.

Aunt's First Right Finger and Nephew's First Right Finger.

Aunt.

		A.	S. L.	L. L.	C.	W.	Totals
Nephew.	A.	10	20	5	6	12	53
	S. L.	28	89	28	25	29	199
	L. L.	3	33	13	7	12	68
	C.	6	23	10	9	15	63
	W.	10	57	7	21	36	131
	Totals	57	222	63	68	104	514

TABLE XCV.

Aunt's First Left Finger and Niece's First Left Finger.

Aunt.

		A.	S. L.	L. L.	C.	W.	Totals
Niece.	A.	12	23	8	17	17	77
	S. L.	24	139	28	29	44	264
	L. L.	—	25	19	8	11	63
	C.	4	10	7	8	10	39
	W.	4	37	13	17	17	88
	Totals	44	234	75	79	99	531

TABLE XCVI.

Aunt's First Right Finger and Niece's First Right Finger.

Aunt.

		A.	S. L.	L. L.	C.	W.	Totals
Niece.	A.	17	41	10	9	20	97
	S. L.	24	103	27	27	31	212
	L. L.	8	30	16	7	15	76
	C.	5	18	9	10	16	58
	W.	9	32	14	12	21	88
	Totals	63	224	76	65	103	531

TABLE XCVII.

Male Cousins' First Left Finger.

First Cousin.

		A.	S. L.	L.L.	C.	W.	Totals
Second Cousin.	A.	48	96	39	26	56	265
	S. L.	96	148	69	53	129	495
	L. L.	39	69	38	28	75	249
	C.	26	53	28	8	44	159
	W.	56	129	75	44	118	422
	Totals	265	495	249	159	422	1590

TABLE XCVIII.

Male Cousins' First Right Finger.

First Cousin.

		A.	S. L.	L. L.	C.	W.	Totals
Second Cousin.	A.	52	100	19	13	36	220
	S. L.	100	230	65	50	159	604
	L. L.	19	65	38	13	61	196
	C.	13	50	13	16	35	127
	W.	36	159	61	35	152	443
	Totals	220	604	196	127	443	1590

TABLE XCIX.

Female Cousins' First Left Finger.

First Cousin.

		A.	S. L.	L. L.	C.	W.	Totals
Second Cousin.	A.	26	78	27	17	35	183
	S. L.	78	294	104	44	84	604
	L. L.	27	104	44	11	25	211
	C.	17	44	11	14	24	110
	W.	35	84	25	24	38	206
	Totals	183	604	211	110	206	1314

TABLE C.

Female Cousins' First Right Finger.

First Cousin.

		A.	S. L.	L. L.	C.	W.	Totals
Second Cousin.	A.	30	76	26	29	34	195
	S. L.	76	206	65	77	72	496
	L. L.	26	65	36	29	34	190
	C.	29	77	29	26	35	196
	W.	34	72	34	35	62	237
	Totals	195	496	190	196	237	1314

TABLE CI.

Male and Female Cousins' First Left Finger.

Male Cousin.

		A.	S. L.	L. L.	C.	W.	Totals
Female Cousin.	A.	27	68	37	23	58	213
	S. L.	115	247	114	64	153	693
	L. L.	24	66	42	16	69	217
	C.	22	36	23	14	32	127
	W.	41	86	40	29	64	260
	Totals	229	503	256	146	376	1510

TABLE CII.

Male and Female Cousins' First Right Finger.

Male Cousin.

		A.	S. L.	L. L.	C.	W.	Totals
Female Cousin.	A.	26	63	38	12	65	204
	S. L.	93	228	77	50	126	574
	L. L.	23	78	33	22	55	211
	C.	32	94	25	22	62	235
	W.	30	97	40	28	91	286
	Totals	204	560	213	134	399	1510

THE INHABITANTS OF THE EASTERN MEDITERRANEAN.

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THE ethnology of the Eastern Mediterranean was in ancient times a matter of keen interest to the Greeks themselves; since the revival of Greek learning it has concerned many scholars. Unfortunately owing, no doubt, to the wealth of cultural and documentary evidence, the physical affinities have been rather neglected and the earlier excavators either reburied or threw away the bones they discovered.

Professor J. L. Myres was good enough to take me as his assistant to Cyprus in 1913, and I was thus given an opportunity of studying the anthropology of that island at first hand. The evidence brought forward in this paper has been collected while studying my Cypriot material and is a natural corollary to work in Cyprus. I am indebted to Professor Myres and to Professor Arthur Thomson for continual advice and criticism during the progress of my work at Oxford, and to Professor Karl Pearson for initiating me into biometric methods. The photographs of crania were taken by Mr W. Chesterman, Assistant in the Anatomical Department, Oxford.

The method followed in this paper has been first to enumerate a few of the representative views on the population of the Eastern Mediterranean, secondly to discuss in detail the cephalic index, glabello-occipital length, greatest head breadth, upper facial index, nasal index, stature and pigmentation, to which characters we are limited by our lack of further evidence, and thirdly to attempt to summarize the tentative conclusions arrived at in dealing with racial problems in the Eastern Mediterranean.

While it is generally admitted by most writers that the population of the Eastern Mediterranean consists of an admixture of Mediterranean and Alpine—or alternatively Armenoid types, extremely diverse opinions have been held both about the original population and also about the degree of admixture which has taken place. v. Luschan in a paper of 1891 (1) when dealing with the early population of Lycia, declares that "there are clearly two types, one short headed found especially in the mountains and in the swamps and a second long headed found in the towns and on the coast, the latter probably do not represent a single unit. It would be difficult to bring the figures especially the high indices of Makri, Xanthos, Rekowa and Myra into line with those from the East Lycian coast." The cephalic indices of the Asiatic Greeks have two summits 75 and 88 with the lesser summit at 75. The broad high skulls belong to the old type who reach from the

Caucasus to the Euphrates. In a later paper ⁽²⁾ this view is expanded and v. Luschan expresses the belief that the earliest inhabitants of the Mediterranean were Armenoid and were the authors of the -issos, -andos, etc., terminations for place names. The Dorians were a kind of backwash of the same people going East instead of West. The long blond (!) skulls of the modern Kurds do not appear in Asia Minor before the second century B.C.; there is otherwise no trace of any other people, but the place names with the terminations in question do appear. He disagrees with Sergi, who believes that the Mediterranean race provided the earliest inhabitants of the Mediterranean, for he maintains that the Armenoid type appeared in Sicily and in Sardinia at the very end of the Neolithic period.

Ripley ⁽³⁾ in his survey of the peoples of Eastern Europe has three main divisions, Greek, Turk and Slav, the first, except in Thessaly, being essentially a littoral people, the last an inland people and the Turks a negligible factor. He states that all authorities agree that the ancient Hellenes were long-headed and of Mediterranean stock and quotes v. Luschan's curve. The modern Greeks are very brunet. In dealing with the people of Western Asia Ripley depends mainly on Chantre for his information. He calls attention to two types, Kurds and Armenians, the one dolichocephalic, the other brachycephalic. The Kurds are the oldest and have been quite unaffected by all invasions. The custom of shaping the heads of the children has accentuated the natural differences of these peoples. In regard to the question as to which is the earlier people, the Armenoid or the Mediterranean, Ripley maintains that v. Luschan's argument depends on the scattered nature of the Armenoid settlements and on their various religions; the long heads occur on the coast especially in Greek necropoles. Ripley believes that the Mediterranean people were the earliest arrivals, the Turks the latest. The Armenoids are of importance because they link up Europe and Asia, they are perhaps the Pelasgians. In Persia this great contrast disappears possibly owing to the great plains.

Sergi ⁽⁴⁾ states, "I am convinced that the primitive population of Lycia and the rest of Asia Minor as also of Syria is of the same type as the Egyptian and derived from the same centre of diffusion. This primitive population...could not have been of brachycephalic Armenoid type....It is probable that the immigrants encountered a population coming from Northern Arabia but as the skull characters of the two races were allied it is difficult to distinguish them."

Ridgeway ⁽⁵⁾ states that the original inhabitants of Greece were, as are the present inhabitants, a dark people. They already inhabited Greece in the Neolithic period. About 1500 B.C. the immigration of Teutonic people, the Achaeans, took place. The Illyrio-Thracian tribes belong to the same stock as the original race, but were conquered in many cases by Keltic from the Alps. The Dorians were like the Thessalians an Illyrian tribe, and Herodotus (VIII, 43) thought them Macedonian. The Spartans thought there was a difference of race between the Dorians and the Achaeans.

As far as physical characteristics are concerned the Homeric Achaeans were tall men with fair hair. There is no reference to the colour of the Spartans' hair so we imagine that it did not differ from that of the rest of the Greeks; we should have expected a mention of it both when before Thermopylae the Spartans are combing their long hair and in the Lysistrata.

Deniker (27) in general deals with the modern population. He states (28), however, "en somme si les crânes grecs d'aujourd'hui sont tous sous-brachycéphales, tandis que les anciens crânes mesurés jusqu'à présent sont pour la plupart sous-dolichocéphales, il ne faut pas oublier que l'étude des crânes connus de la Grèce classique très peu nombreux et appartenant presque tous à l'Attique, n'autoriserait aucune conclusion générale, et que d'après les temoignages mêmes des auteurs anciens et des monuments de l'art grec une très grande partie de l'ancienne population de la Grèce était brachycéphale et surtout sous-brachycéphale." In regard to the modern population he considers those west of 20° E. of Paris to belong to his Adriatico-Albanian type of tall stature and brachycephalic heads while those east, the Greeks of Thessaly and perhaps those of Attica, are lower in stature and cephalic index.

Pittard (6) in his paper on the Greeks strongly disagrees with this conclusion.

Myres (7) sums up his views on the subject: "In the Eastern basin of the Mediterranean, the hold of the Mediterranean race on the whole north shore is of the slightest...and there is growing reason to believe that the strong 'Alpine' strain in the Morea, which is certainly ancient, may even be primitive there. Even in the Aegean islands, and in Crete, which were admittedly occupied early and decisively by Mediterranean man, traces of continental intruders, of Alpine affinities, begin already at the close of the Stone Age, shewing that Alpine man was already present in force on the neighbouring mainlands. Mediterranean man, therefore, must be regarded as in all probability an intruder from the south; just as 'Alpine' man reveals himself more and more clearly now as a longitudinal immigrant from the east along the Mountain Zone. Both movements alike are very ancient."

MATERIAL USED.

Neolithic Period.

The only authenticated Neolithic skull from the north coast of the Eastern Mediterranean is reported by Velde (8) from the island of Leukas. This skull, which was very fragmentary and has not been thoroughly described, differed from other skulls dating from the second millenium B.C. found in the same island in the following particulars. It was brachycephalic with an index of 81, and the cranial bones were markedly thick. As far as can be judged from the photographs and the brief description of the author this skull did not differ essentially from the Bronze Age skulls found at Lapithos in Cyprus. It may be an unusual Mediterranean type, but in view of the definite statement of the finder that it differed from the Mediterranean skulls found in the island and of later date, we may provisionally suggest that the same mixed type which we shall shew is so characteristic of the area under discussion in later times, may have also been found as early as the Neolithic Age, but at present we have insufficient evidence to justify such a conclusion.

Bronze Age.

Five skulls of this period, but not apparently very exactly dateable, were found in Leukas during Dorpfeld's excavation (8). They were remarkable in their homogeneity, all having cephalic indices from 74—76. The cranial bones were reported to be noticeably thin, but no other details have been published.

We have a fairly complete series of Bronze Age skulls from Crete (2), (9), (10), (11), coming from Hagios Nikolaos, Palaikastro and Gournia.

The Palaikastro series numbered 64 (of which 46 were male). No Bronze Age skulls appear to have been excavated in Asia Minor. The Cypriot series from Lapithos number fifteen (18). The contrast on which Velde has insisted so strongly between thick and thin bones of the cranial vault could be observed in the Bronze Age skulls at Lapithos extending over a series of about a hundred, which could not be measured, but both types were present often in the same tombs.

Early Iron Age.

The early Iron Age appears to be represented only by three skulls from Amathus in Cyprus excavated by Myres (12). Two are now in the Department of Human Anatomy, University Museum, Oxford, and one was in 1913 in the Cyprus Museum.

Hellenic, Hellenistic and Roman.

There are nine skulls in the University Museum in Oxford, varying in date from early Hellenic to Hellenistic Four fourth century Athenians excavated by Schliemann were examined by Virchow (13). There are seven skulls of this period in the Cyprus Museum and one of Roman date from Leukas.

In addition to the above there are four series of "ancient" Greek crania in the Williamson Collection, whose date is uncertain. They include six from Samos, four from Corfu, and two from Myrma on the coast of Asia Minor.

There is a small series, presumably of Roman date, from Halicarnassus in Caria also in the Williamson Collection, but unfortunately they were not sufficiently well preserved for accurate measurement.

Modern skulls whose geographical provenances are authenticated are rare. Weisbach (31) published a large series but from so wide an area that it has been unfortunately necessary to reject them. The only series available for statistical treatment come from Crete and Adalia (1) in Asia Minor.

Measurements on the living are available for statistical treatment from the following peoples. A small number of men were examined by Velde (8) in Leukas (St Maura), an island off the west coast of Greece just north of the Gulf of Corinth. Schiff (14) measured 99 men from Mani on the Southern Peloponnese and a small series from Meligala in Messenia. A considerably larger series of Albanians were examined by Haberlandt and Lebzelter (15). Unfortunately these latter came from a rather large area. Measurements were made in Crete by Hawes, v. Luschan and Schiff (16). The first has unfortunately not yet published his figures. v. Luschan is the author of several exceedingly interesting papers on Lycia. The Cypriot measurements were made by the author. The figures used for comparative purposes have been drawn from the following sources: for Sardinia and Corsica, Duckworth (10), for Egypt, Craig (17).

Duckworth and Craig have reduced their own material to statistical form and Duckworth has published Hawes' figures.

Cultural conditions are of special importance in regard to the Bektasch and Tadchadsky, Lycian gypsies, who contrast very strongly with their Greek and Turkish neighbours not only in their measurements but what is still more important the standard deviations of these measurements. These people form a small endogamous religious community which claims to keep itself free from intermixture with either Christian Greeks or Moslems, a claim substantiated by the low standard deviations. Similar communities are found among the Druses and Maronites on Mount Lebanon and to a lesser degree among the Jews. The purity of the Lycian gypsies will be seen to be specially striking because they live amongst an unusually mixed population.

The measurements from Cyprus and Crete have been subdivided according to localities. In Crete the eparchies of Selinos and Sphakia, selected by v. Luschan as representing a special "type," have been worked out separately; in Cyprus (18) I have divided my own measurements into four groups. Group I is a miscellaneous group mostly from Nicosia and the neighbourhood, Group II the villages on the north coast, Group III the villages round the Bay of Salamis on the east coast and Group IV the village of Levkoniko in the Mesaoria, this division being purely geographical. In both cases the small subdivisions are included in a final summation which I have termed in the case of Cyprus "groups combined" and in Crete, "whole island." Schiff's measurements for Pyrgiótissa, which include nine men from Sphakia, have not been included in the Cretan total which is limited to v. Luschan's 320 men.

For the physical characters of the Jews a good deal of material is available· This has been very conveniently summarized by Fishberg (19), who includes, besides his own observations (a series of 500 measured in New York), series from Weissenberg (20), Lombroso (21), Blechman (22), Yakowenko (23) and others. Schiff (24) has summarized some figures and has treated them statistically; unfortunately most of his series which include Askenazim, Shephardim, Grusinians, and Jews from the Yemen, Aleppo and Morocco are except for the first two small, his probable errors have also been wrongly calculated. Weissenberg has published a number of valuable papers on the Jews. Use has been made of his figures from S. Russia (20), the Yemen (25), and Samarkand (26).

CEPHALIC INDEX.

The cephalic indices of living adult males have been tabulated in Table I. Measurements from Corsica and two Egyptian series have been added for comparative purposes. None of the means appear to be in agreement with the pure Mediterranean type represented in Corsica and in a less pure form in Egypt. If the means are compared by taking the difference between any two and the probable error of that difference and we accept the difference as being significant where it is greater than three times its probable error, we find that on v. Luschan's figures there is no significant difference between the Lycian Greeks and the Cretans although Hawes' figures, a very much larger series, suggest a possible difference; the Lycians and Maniots are very similar, there is a significant difference between the Lycians and both the men of Meligala and of Cyprus and a very marked difference between Cretans and Cypriots.

TABLE I.

Cephalic Index. Living Adult Males.

Provenance	Number	Mean	σ	V
Corsica	53	75·53 ± ·27	2·90 ± ·19	3·82 ± ·25
Albania	140	86·07 ± ·25	4·44 ± ·17	5·16 ± ·23
Leukas	42	84·55 ± ·40	3·86 ± ·28	4·60 ± ·34
Meligala	33	82·49 ± ·38	3·21 ± ·26	3·89 ± ·32
Mani	99	79·92 ± ·26	3·77 ± ·18	4·72 ± ·23
Crete (Hawes)	1600	79·20 ± ·07	4·10 ± ·05	5·17 ± ·06
„ (Schiff) Pyrgiótissa...	64	79·27 ± ·32	3·89 ± ·23	4·91 ± ·29
„ (v. Luschan) Selinos and Sphakia	51	81·68 ± ·37	3·91 ± ·26	4·80 ± ·32
„ Whole island ...	320	79·26 ± ·16	4·24 ± ·11	5·35 ± ·14
Lycia, Turks	187	81·29 ± ·26	5·21 ± ·18	6·41 ± ·22
„ Greeks	179	80·27 ± ·35	6·94 ± ·25	8·65 ± ·31
„ Gypsies	53	87·51 ± ·26	2·83 ± ·19	3·23 ± ·21
Cyprus, Group I	111	81·41 ± ·26	3·99 ± ·18	4·91 ± ·22
„ „ II	221	81·94 ± ·13	4·02 ± ·13	4·90 ± ·16
„ „ III	167	83·38 ± ·24	4·63 ± ·17	5·55 ± ·20
„ „ IV	87	84·16 ± ·31	4·25 ± ·22	5·03 ± ·26
„ Groups combined	586	82·54 ± ·11	4·09 ± ·08	4·96 ± ·38
Alexandria	643	76·29 ± ·08	3·13 ± ·06	4·10 ± ·08
Giza	326	74·75 ± ·11	2·83 ± ·08	3·79 ± ·10

TABLE II.

Differences between Cephalic Indices.

People	Compared	Difference	Remarks
Lycian Greeks ...	Cretans	1·01 ± ·38	v. Luschan's figures
„ „ ...	„	1·07 ± ·35	Hawes' figures
„ „ ...	Maniots	·35 ± ·44	—
„ „ ...	Meligala	2·45 ± ·51	—
„ „ ...	Cypriots	2·27 ± ·37	—
Cretans	„	3·34 ± ·13	Hawes' figures

It would appear from Hawes' figures that Lycians and Cretans possibly did not belong to the same group, but the excess of the difference over three times the error is so small that for the moment the matter can be left open. Grouping our mean values we have three classes:

Under 81	82	Over 84·5
Crete	Meligala	Leukas
Mani	Cyprus	Albania
Lycia (Greeks and Turks)		Lycian Gypsies

That such a grouping is of any real significance seems unlikely when we turn to the standard deviations. Here we have a striking contrast between Lycian Greeks and all the other Greek countries, suggesting a very big mixture of two races. Apart from the figure from Meligala in Messenia where the numbers measured are too small for any satisfactory review of the standard deviation it will be seen that in every case we have a much greater standard deviation than in the comparatively homogeneous Corsica and greater than in the cosmopolitan Alexandria. It should also be noticed that the smaller series have smaller standard deviations suggesting that the small areas from which the men were drawn are more homogeneous than the larger areas as for instance Crete or Cyprus. The evidence of four areas measured in Cyprus does not however bear this statement out, as the difference between the Cypriot standard deviations either between the villages themselves, or between the villages and the total number of men measured, is seldom of any significance. It may be concluded that we are dealing with a very mixed population. This fact is strikingly confirmed by two pieces of evidence. First, the local differences between different groups of villages in Cyprus. The villages on the north coast (221 adult males) have a cephalic index of 81·94 and 167 adult males from villages round Enkomi one of 83·38. Secondly, 60 adult males from five selected eparchies of Crete have a cephalic index of 76·6, and 72 adult males from another five eparchies a cephalic index of 81·8. The selection of eparchies by v. Luschan on the basis of cephalic index naturally increases the difference and tends to emphasize the third piece of evidence more strongly than it perhaps deserves. Even if we disregard it however there appears a good case for a great deal of admixture among the living "Greeks" and at the same time considerable local variation and the appearance of similar types at opposite ends of the Greek world.

One further point which tends to suggest mixture and proves the necessity of taking large series is the great contrast between Hawes' and v. Luschan's figures for Sphakia and Selinos. The cephalic index for the whole series of Cretans is the same in both cases and differs by only ·06 per cent., whereas however v. Luschan believes that the cephalic index of the Sphakiots (30 measured) and Selinots (21 measured) to be 81·6 in both cases. Hawes, on the evidence of 98 and 35 respectively, has a cephalic index of 85. The very close resemblance of their final mean suggests a similar technique, but this immense local divergence shews clearly the danger of elaborating racial theories on an insufficient number of cases. It also throws very considerable doubt on v. Luschan's cephalic index map of Crete and any conclusions that may be drawn from it.

While the high standard deviations of Lycia are noticeable, the extremely low standard deviation of the Lycian gypsies calls for special comment. We have already drawn attention to the fact that these people formed a strictly endogamous community and did not mix with their neighbours. The evidence of the low standard deviation, especially in contrast with the great variation shewn by people living in the same place but belonging to a different social *milieu*, supports the claim of these gypsies to be a pure blooded race.

The crania (see Table III) belong to very small series and it is difficult to argue with certainty. It appears that as a general rule the modern Greeks are slightly more brachycephalic than the ancient skulls from the small place, even after making allowance for the difference between the measurements of the living and the dead. This difference appears still more striking when the distribution into dolichocephalic, etc. is made. Even granting however that this difference exists, considering the small number of crania, it is perhaps more remarkable that the ancient skulls should be so extremely close considering the very large standard deviations which we have found. The comparison has been made in Table IV.

TABLE III.

Cephalic and Cranial Indices.

Provenance	Adult Living Males		Modern Crania			Ancient Crania			
	No.	Cephalic Index	No.	Sex	Cranial Index +2 units	Date	No.	Sex	Cranial Index +2 units
Leukas ...	42	84·55	—	—	—	Neolithic	1	?	83
—	—	—	—	—	—	Bronze Age	5	—	75
—	—	—	—	—	—	Late Roman	1	—	77
Corfu ...	—	—	—	—	—	Ancient	4	—	79·98
Crete ...	1600	79·20	52	♂	80	"Beginning of Bronze Age"...	6	—	77·1
—	—	—	17	♀	82·9	Early Middle Minoan ...	46	♂	75·10
Crete, Khania	—	—	13	♂	77·8	Middle Minoan	13	—	77·5
—	—	—	—	—	—	Early Late Minoan	5	—	78·5
—	—	—	—	—	—	Late Minoan III	7	—	81·1
Lycia (Greeks)	179	80·27	93	—	79·9	—	—	—	—
Samos ...	—	—	—	—	—	"Ancient"	6	♂	79·40
Cyprus ...	586	82·54	—	—	—	Bronze Age	14	—	80·35
—	—	—	—	—	—	Hellenic and Roman ...	7	—	79·20
Athens ...	60(?)	79·6	—	—	—	Early Hellenic to Hellenistic	13	—	79·38

There are however certain points of resemblance and difference which need special attention. First there is in general a closer approximation between the earliest cranial indices and modern ones than among those of later date. This holds true in every case. Secondly there is only one case of a considerable difference between the ancient crania and the modern. This occurs in Leukas. The one early (Neolithic) skull has a cranial index of 81, and this index is not very different from that of the modern men of Leukas. The evidence of the five later skulls with their apparent great homogeneity—for details have not yet been published—seems to suggest a possibility of some change in Bronze Age times.

TABLE IV.

Comparison of Ancient and Modern Cephalic Indices.

Bronze Age. Locality	Cranial Index + 2	Modern Living Males. Locality	Cephalic Index	Difference and Remarks	Whether difference significant
Lapithos	80·35 ± ·69	Lapithos	81·94 ± ·18	1·59 ± ·71	No
”	”	Enkomi	83·34 ± ·23	2·99 ± ·73*	Yes
”	”	Cyprus	82·54 ± ·11	2·19 ± ·70	No ?

* The difference between the modern inhabitants of the Lapithos and Enkomi areas is 1·40 ± ·29.

Only one ancient series is available for statistical treatment, Duckworth's series from Palaikastro. The figures are:

No.	Sex	*M*	σ	*V*
46	♂	73·10 ± ·38	4·03 ± ·28	5·5 ± ·39

These figures seem to suggest that the mixing of racial elements in Crete had reached in the Early Middle Minoan Period a similar state to modern times; a small series of skulls from Hagios Nikolaos, dated by Tod at the very beginning of the Bronze Age, also present the appearance of a mixed population. It has been argued that a change took place in the population of Crete in Minoan times. Such evidence as we have at present would seem to suggest that the mixed character of the population had already been established and that the variations which appear in the cephalic indices of the slender number of skulls at our disposal are such that might be expected in such a mixed population. We are inclined to believe that a similar early hybridisation had taken place in Cyprus though our numbers are insufficient for statistical treatment.

Let us next consider the two factors which make up the cranial index, namely, the glabello-occipital length (Table V) and the greatest head breadth (Table VI). The data are less than in cephalic indices as in many cases a complete series of measurements do not appear to have been published. There are several points of interest that appear from an examination of the data. First that in Cyprus the head breadth is very stable and the differences in cephalic index are due to variations in the glabello-occipital length, secondly, in Crete the differences in cephalic index are due to variations in the head breadth, the length being stable. The whole of the Lycian figures have not been published but two small groups of Bektasch and Tadchadsky have an exactly similar head breadth, the difference in length causing the variation in the index. The variation in cephalic index from the south to the north of the Peloponnese would appear to be due to both a shortening and a broadening of the head, but our figures are not sufficient to be conclusive. In spite of the fact that neither the Cypriot nor the Cretan indices

TABLE V.

Glabello-Occipital Length. Living Adult Males.

Provenance	Number	Mean	σ	*V*
Leukas	42	182·8 ± ·66	6·34 ± ·47	3·47 ± ·26
Meligala	33	185·38 ± ·57	4·88 ± ·41	2·63 ± ·22
Mani	99	187·09 ± ·41	6·01 ± ·29	3·32 ± ·16
Crete, Pyrgiótissa	64	186·06 ± ·57	6·80 ± ·41	3·67 ± ·22
„ Selinos and Sphakia	51	190·95 ± ·69	7·59 ± ·51	3·98 ± ·27
„ Whole island ...	320	191·76 ± ·52	7·24 ± ·19	3·78 ± ·10
Lycian Gypsies	53	176·33 ± ·53	5·73 ± ·38	3·25 ± ·21
Cyprus, Group I	111	183·12 ± ·40	6·18 ± ·28	3·38 ± ·15
„ „ II	221	182·41 ± ·31	6·90 ± ·22	3·78 ± ·12
„ „ III	167	178·43 ± ·34	6·42 ± ·24	3·60 ± ·13
„ „ IV	87	178·29 ± ·47	6·50 ± ·33	3·65 ± ·19
„ Groups combined	586	180·81 ± ·20	7·00 ± ·14	3·87 ± ·08
Alexandria	643	189·74 ± ·16	5·99 ± ·11	3·16 ± ·06
Giza	326	191·66 ± ·22	5·75 ± ·15	3·00 ± ·08

TABLE VI.

Head Breadths. Living Adult Males.

Provenance	Number	Mean	σ	*V*
Leukas	42	154·11 ± ·55	5·24 ± ·39	3·40 ± ·25
Meligala	33	152·59 ± ·67	5·72 ± ·47	3·75 ± ·31
Mani	99	150·99 ± ·35	5·15 ± ·25	3·43 ± ·18
Crete, Pyrgiótissa	64	147·91 ± ·51	6·03 ± ·36	4·08 ± ·25
„ Selinos and Sphakia	51	157·21 ± ·48	5·11 ± ·34	3·25 ± ·22
„ Whole island ...	320	151·76 ± ·23	6·07 ± ·16	4·00 ± ·11
Lycian Gypsies	53	150·67 ± ·38	4·12 ± ·27	2·73 ± ·18
Cyprus, Group I	111	148·76 ± ·38	5·99 ± ·27	3·38 ± ·15
„ „ II	222	149·24 ± ·24	5·19 ± ·17	3·48 ± ·11
„ „ III	167	148·54 ± ·28	5·39 ± ·20	3·63 ± ·13
„ „ IV	87	149·63 ± ·38	5·22 ± ·27	3·49 ± ·18
„ Groups combined	587	149·10 ± ·15	5·35 ± ·11	3·59 ± ·07

are the least, the Cypriots have the narrowest heads recorded and the Cretans the longest.

With these facts in mind it might have been conjectured that the coefficients of variation of the variable factor would be greater than the coefficients of variation of the more constant factor. This however does not prove to be the case. In no case in our series (see Table VII) is the difference between the coefficients of variation of any significance. The head does appear to have a slightly more variable length among brachycephalic people, and to have a slightly more variable breadth among dolichocephalic people, but statistical evidence is in our present series against such a conclusion being definitely drawn.

TABLE VII.

Differences between Coefficients of Variation. Head Lengths and Breadths.

Locality	Difference	Locality	Difference
Leukas	+ ·07 ± ·36	Lycian Gypsies ...	+ ·52 ± ·28
Meligala	− 1·12 ± ·38	Cyprus, Group I ...	·00 ± ·21
Mani	− ·11 ± ·24	,, ,, II ...	+ ·30 ± ·16
Crete, Pyrgiótissa	− ·41 ± ·34	,, ,, III ...	− ·03 ± ·18
,, Selinos and Sphakia	+ ·73 ± ·35	,, ,, IV ...	+ ·16 ± ·26
,, Whole island ...	− ·22 ± ·15	Groups combined	+ ·28 ± ·11

+ indicates that the coefficient of variation of the head length is greater, − that that of the head breadth is greater.

UPPER FACIAL INDEX.

The upper facial index has been reduced for a small series (see Table VIII), but after inspection the conclusion I arrived at was that on the whole as a racial test this index did not appear to be of great value, at least within the area under consideration. The reasons for this conclusion are that the indices fall into three groups:

TABLE VIII.

Upper Facial Index. Living Adult Males.

Provenance	Number	Mean	σ	V
Leukas	42	52·76 ± ·30	3·09 ± ·21	5·85 ± ·38
Meligala	33	56·85 ± ·47	4·02 ± ·33	7·08 ± ·59
Mani	98	57·45 ± ·25	3·73 ± ·18	6·48 ± ·31
Crete, Selinos and Sphakia ...	49	53·00 ± ·28	2·95 ± ·20	5·56 ± ·37
Albania	138	48·42 ± ·18	3·20 ± ·13	6·61 ± ·27
Lycian Gypsies	53	54·23 ± ·29	2·88 ± ·22	5·31 ± ·40
Cyprus, Group I	111	50·10 ± ·25	3·92 ± ·18	7·82 ± ·35
,, ,, II	221	48·63 ± ·12	2·69 ± ·09	5·53 ± ·18
,, ,, III	167	49·71 ± ·19	3·61 ± ·13	7·26 ± ·27
,, ,, IV	87	50·90 ± ·28	3·87 ± ·20	7·60 ± ·39
Groups combined ...	586	49·52 ± ·10	3·71 ± ·07	7·50 ± ·15

(1) Meligala and Mani.
(2) Crete, Lycia and Leukas.
(3) Cyprus.

We have previously found that on the basis of other measurements the extremes are Lycia on the one hand and the Sphakiots and Selinots on the other. It would appear therefore that some special factor was at work differing from those that we have previously met with.

It has been suggested by Pearson and Crewdson Benington (29) that the personal equation in measurement may be responsible for this divergence. Until Hawes' series are fully worked out or other data are available from elsewhere this is difficult to test on the series under examination. For small series the difference between v. Luschan's measurements on 49 men and Hawes' Sphakiots and Selinots (98 and 33 men respectively) is not great. The latter are worked out on *absolute measurements* and we give below v. Luschan's absolute measurements index. Comparing the absolute measurements Hawes' and v. Luschan's means of upper facial height agree to within a millimetre and the divergencies occur in the bizygomatic width, but even here the difference—under 3 mm.—is not too great to be the result of chance. The figures are:

Place	Authority	Bizygomatic Width	Upper Facial Height	Upper Facial Index
Sphakia	Hawes ...	140·6	76·7	54·56
"	v. Luschan	143	76	54·39
Selinos	Hawes ...	140·9	74·9	53·20
"	v. Luschan	143	75	52·51

Pearson's criticisms are based on skull measurements and are more than borne out by Martin's table (30) where the extreme range for the human race is given—by groups—as 49·0—56·0, but his remarks apply with still greater force to measurements on the living.

Summarizing them there are two reasons for rejecting the upper facial index as a racial test, first the possibility of error in measurements and secondly the small variation occurring in the human race. In regard to the first we have shewn that as far as can be judged from the limited data at our disposal independent observers have obtained similar results in Crete. In regard to the second the differences obtained in our area are sufficient to warrant a careful inquiry into the causes of this difference.

We still have a marked distinction between our two big groups Crete and Cyprus, but the conditions of the smaller groups are different. Crete is in an intermediate position, a group of nearly 100 men from the Peloponnese—sufficient at any rate to point the way to the true condition in Mani—at one end and Cyprus at the other.

NASAL INDICES.

The nasal indices (see Table IX) all shew as is usual a high degree of variation. They fall into two groups, the Meligalots, Maniots, Selinots and Sphakiots on the one hand and Cypriots on the other. The third Cyprian subgroup (Enkomi area) has a low index, the Leukadians occupy an intermediate position, the remainder are broader nosed. It would appear that the narrow-nosed groups are rather more homogeneous than the remainder, but the difference between the standard deviations, except in the case of Cyprus, is not sufficient to be of any significance. The fact that the means for Crete and Cyprus, each island taken as a single unit, are almost in exact agreement is striking, but the difference between the standard deviations

(1·28 ± ·24) must be considered before any conclusion can be adopted. Finally it should be noted that the standard deviations in Cyprus are higher than elsewhere.

TABLE IX.

Nasal Indices.

Place	Number	Mean	Standard Deviation	Coefficient of Variation
Albania	133	67·36 ± ·44	7·53 ± ·31	11·18 ± ·47
Leukas	42	67·07 ± ·75	7·22 ± ·53	10·76 ± ·80
Meligala	33	65·95 ± ·66	5·64 ± ·47	8·55 ± ·75
Mani	99	62·36 ± ·41	6·02 ± ·29	9·65 ± ·48
Crete, Whole island	320	68·42 ± ·26	6·87 ± ·18	10·44 ± ·28
„ Selinos and Sphakia ...	51	65·76 ± ·51	5·45 ± ·36	8·29 ± ·56
Cyprus, Nicosia	111	69·59 ± ·55	8·66 ± ·39	12·45 ± ·56
„ Lapithos	221	69·47 ± ·36	7·94 ± ·25	11·42 ± ·37
„ Enkomi	167	64·25 ± ·36	6·91 ± ·25	10·75 ± ·40
„ Levkoniko	87	69·25 ± ·55	7·50 ± ·38	10·82 ± ·55
„ 4 groups combined ...	586	68·08 ± ·23	8·15 ± ·16	11·98 ± ·24

STATURE.

There is little evidence of the stature in ancient Greece. v. Luschan's estimate of the Minoan stature is 1550 mm., 1600 mm. and 1650 mm. for three individuals; this is based on a humerus and two femora, neither the sex nor the method employed is stated. Duckworth's estimate for the Middle Minoan from Palaikastro is 1625.

Using Pearson's method of estimating stature for the Bronze Age stature in Cyprus, four females, two of which were reconstructed on the evidence of four long bones, one on the femur and humerus and one on the femur and tibia, had a mean stature of 1505 and one male 1602. The evidence is too slender to be of great value.

TABLE X.

Stature. Living Adult Males.

Place	Number	Mean	Standard Deviation	Coefficient of Variation
Crete, Whole island	318	170·61 ± ·26	6·75 ± ·18	3·96 ± ·12
„ Eparchies of Selinos and Sphakia only	50	175·26 ± ·54	5·71 ± ·39	3·26 ± ·22
Albanians	140	169·32 ± ·37	6·57 ± ·26	3·88 ± ·16
Cyprus, Whole island	585	168·77 ± ·17	6·16 ± ·12	3·64 ± ·07
„ Nicosia and neighbourhood ...	—	167·88 ± ·39	6·05 ± ·27	3·60 ± ·16
„ Lapithos	221	168·00 ± ·25	5·47 ± ·18	3·25 ± ·10
„ Enkomi	167	169·05 ± ·32	6·08 ± ·22	3·59 ± ·13
„ Levkoniko	87	168·98 ± ·46	6·37 ± ·33	3·77 ± ·19
„ Leukas	42	166·80 ± ·67	6·43 ± ·47	3·86 ± ·33
Lycian Gypsies	53	166·02 ± ·44	4·78 ± ·31	2·88 ± ·20
Alexandria	643	166·62 ± ·16	5·97 ± ·11	3·59 ± ·07
Giza	326	167·80 ± ·22	5·88 ± ·16	3·50 ± ·09

The living stature is available from Crete and Cyprus only in large numbers (see Table X); in both cases the stature is practically identical, 1690 and 1688 respectively. Three other small series are available all of which fall into a single shorter group, Leukas (1668), Mount Parnon (1662) and Lycians (1660).

The modern stature seems to be slightly greater than the ancient. In Cyprus this can be tested; two of the females were below the average in the measurements of the cranium, a third although she appeared to have been an adult when she died was certainly young, the fourth and shortest female had a head of average size, the male had the longest skull measured and a head breadth of average size. It may fairly be argued that the females were below the average stature and the male might be expected to be about average stature. Three of the females however came from one tomb, and one male and one female from a second. They therefore do not represent an average selection from the population.

Among the modern peoples we find first an unusual difference between the men from Selinos and Sphakia compared with the rest of the population of Crete. This difference is greater than any other difference in our series but until Hawes' figures are published cannot be satisfactorily explained. The differences between the various groups in Cyprus are small and do not give a clue to possible wide divergencies locally. The Cretans are however noticeably taller than the rest of the series. The significant differences may be tabulated.

TABLE XI.

Differences in Stature. Living Adult Males.

People compared		Difference
Leukas	Cyprus	1·97 ± ·69
,,	Crete	3·81 ± ·72
Crete	Cyprus	1·84 ± ·31

The figures suggest what had previously been deduced from the cephalic index, namely that in Crete we have an extreme type, and among the Lycian gypsies another extreme type, Leukas being more allied to the latter than to any of the other intermediates. We have again a general similarity in variation except among the Lycian gypsies who have a lower standard deviation than any of the other peoples.

PIGMENTATION.

Exact information in regard to hair and eye colour among the Greeks is scanty. Pittard remarks that brown eyes are in the majority, grey eyes frequently occur and blue are not rare. Ornstein* out of a total of 1767 men found 170 with blond hair, 1561 with brown hair and 36 with black. Weisbach's figures are very different: out of 47 cases 20 had black hair, two fair hair and the remainder various shades of brown; two out of the 47 had blue eyes. Schiff believed that he

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* *Verhandlungen der Berliner anthrop. Gesellschaft*, 1877, S. 39 u. 1879, S. 305.

could make a distinction between the Southern and Northern Maniots. If our figures are correct we get three groups, (1) Albania and Cyprus, (2) Meligala and Mani, (3) Crete. In order to make the figures comparable they have been reduced to percentages removing all white haired people.

Hair Colour Percentages.

	Albania	Cyprus	Crete	Meligala	Mani
Fair ...	5·1	3·20	10	15	12
Brown ...	54·0	56·16	17	36	30·5
Black ...	40·9	40·64	74	48	57·5

A point which deserves special attention is that although Cyprus and Albania contain the least amount of dark hair they also contain the least fair hair; the Cretans, although the greater proportion of them have dark hair (Braun-schwarz and Schwarz), do not contain the least amount of fair hair. The personal equation of the observers should perhaps be considered responsible for the divergency but the Albanian observer also belonged to v. Luschan's school as well as the observers for Crete and the Peloponnese and Messenia.

Eye Colour Percentages.

	Albania	Cyprus	Crete	Meligala	Mani
Blue	10·20	9·58	5·3	9·1	6
Intermediate ...	48·20	39·02	25·4	33·3	36·5
Brown	41·60	51·40	69·3	57·6	57·5

Taking the eye colour we get rather a similar grouping, Albania and Cyprus being the most clearly allied and the Cretans who had the largest number of persons with dark hair have the smallest proportion of blue eyes.

There are two points of special importance to which attention should be drawn. First the number of blue-eyed persons occurring in some parts of the Greek world. In Albania and Cyprus about one man in ten has blue eyes and even in dark Crete one man in twenty. To take a practical example one would meet a blue-eyed man in Crete or the Peloponnese rather more frequently than one would meet a red-headed man in Oxford. The fact however that there appears to be such a difference in the distribution of blue eyes would possibly account for the blue-eyed tribes of classical times.

The second point of importance is that pigmentation does clearly divide up the two branches of the Alpine peoples. Among v. Luschan's Lycian gypsies every person examined has brown eyes and dark hair. How far this is a correct estimate it is difficult to decide as it is not quite in agreement with Twarjanowitsch's results from Armenia where he finds a population with a cephalic index of 86·89, and with 70 °/₀ having the eyes brown and the hair black—a close resemblance to Crete!

It is extremely difficult then to come to any conclusion about the pigmentation of the Greeks, but as far as our evidence goes at present we can affirm that blue eyes locally are not uncommon and that pigmentation does not appear to bear a definite correlation to cephalic index.

Our figures as a whole appear to shew that we have in the Greek world considerably different local types which conform as far as our evidence goes at present more to a short brachycephalic race, than to any other type; it is possible to form certain groups which on the whole with such limited evidence as is at our disposal conform, the Cretans and especially v. Luschan's Selinots and Sphakiots being at one end of the scale, i.e. tallest and most long-headed, the Lycian gypsies at the other. We are not inclined to lay much stress on the grouping which results from a study of the means alone: taken in conjunction with the question of variation it would appear to be of great importance.

If then we compare the variations as a whole, certain points of importance appear (see Table XII in which the standard deviations of indices and the coefficients of variation of absolute measurements have been tabulated). Unfortunately few measurements have been published on Lycian Greeks and Turks and they cannot therefore be included in our Tables. There is a general agreement in variation but the Cretans appear to be slightly more variable than the rest, this variation dating—on the evidence of cranial indices alone—at least from Middle Minoan times. On the other hand the series from Selinos and Sphakia measured by v. Luschan shew a remarkably low variation; we have shewn however that in some measurements Hawes obtained very different means, although he has not published his figures so that comparison of variations is impossible. We are inclined to believe therefore that v. Luschan did not obtain a normal sample of the population. It is not possible to discover whether his sample was due to some

TABLE XII.

Provenance	σ			V		
	Cephalic Index	Upper Facial Index	Nasal Index	Stature	Glabello-Occipital Length	Head Breadth
Albania	4·44 ± ·17	3·20 ± ·13	7·53 ± ·31	3·85 ± ·18	—	—
Leukas	3·86 ± ·28	3·09 ± ·21	7·22 ± ·53	3·86 ± ·33	3·51 ± ·29	3·44 ± ·28
Meligala	3·21 ± ·26	4·02 ± ·33	5·64 ± ·47	—	2·63 ± ·23	3·75 ± ·31
Mani	3·77 ± ·18	3·73 ± ·18	6·02 ± ·29	—	3·11 ± ·23	3·40 ± ·18
Crete (Pyrgiótissa) ...	3·89 ± ·23	—	—	—	3·71 ± ·25	4·09 ± ·25
„ (Selinos and Sphakia)	3·91 ± ·26	2·95 ± ·20	5·45 ± ·36	2·95 ± ·22	3·98 ± ·31	3·27 ± ·22
„ Whole island ...	4·24 ± ·11	—	6·87 ± ·18	3·96 ± ·12	3·76 ± ·11	4·00 ± ·11
Lycian Gypsies	2·83 ± ·19	2·59 ± ·17	—	2·88 ± ·20	3·25 ± ·23	2·73 ± ·18
Cyprus, Group I	3·99 ± ·18	3·92 ± ·18	8·66 ± ·39	3·60 ± ·16	3·38 ± ·15	3·38 ± ·15
„ „ II	4·02 ± ·13	2·69 ± ·09	7·94 ± ·25	3·25 ± ·10	3·78 ± ·12	3·48 ± ·11
„ „ III	4·63 ± ·17	3·61 ± ·13	6·91 ± ·25	3·59 ± ·13	3·60 ± ·13	3·63 ± ·13
„ „ IV	4·25 ± ·22	3·87 ± ·20	7·50 ± ·38	3·77 ± ·19	3·65 ± ·19	3·49 ± ·18
„ Groups combined	4·09 ± ·08	3·71 ± ·07	8·15 ± ·16	3·64 ± ·07	3·87 ± ·08	3·59 ± ·07

artificial selection—his men are tall and may have been picked for this quality—or whether in those eparchies there does exist an unmixed strain.

The Lycian gypsies represent a consistently homogeneous strain with low variations in every case except in the glabello-occipital length where they have a normal coefficient of variation (i.e. one similar to the rest). In this latter case we have a curious exception in the low variation occurring among the people of Meligala, but no stress should be laid on this owing to the very small sample.

If we admit the Lycian gypsies to be a homogeneous people, and take into consideration their social *milieu*, it becomes of importance to find if possible another people who practised endogamy for similar religious reasons. The communities of the Druses and Maronites present an exact parallel, but no measurements appear to be available for statistical purposes. There exist however a number of anthropological measurements of Jews. Table XIII summarizes some of these. It will be seen that with very few exceptions they present features which are remarkably consistent. The data have been collected by different observers, and the countries represented are world wide. Standard deviations below 3 in the cephalic index occur in two cases only, in one of which however so small a number was measured that the figure is uncertain, the other case from South Russia, in which 100 individuals are included, is noticeable; all the other cases in the table fall within the limits of 3—4, except 71 Grusinians whose standard deviation is 4·2 and the Jews from Samarkand (σ 4·09). It is impossible to discuss at length here the vexed question of the Jewish race, but it would appear that we have in the above table a consistent degree of variation in a race which has on the whole been endogamous although exposed to many vicissitudes of climatic conditions, suggesting that such

TABLE XIII.

Measurements of Jews.

Provenance	No.	Cephalic Index		Nasal Index		Stature	
		M	σ	*M*	σ	*M*	*V*
Askenazim	202	83·7 ± ·14	3·0 ± ·10	—	—	—	—
Shephardim	149	81·5 ± ·18	3·2 ± ·12	—	—	—	—
Grusinians	71	86·7 ± ·34	4·2 ± ·24	—	—	—	—
Yemen	57	79·0 ± ·29	3·2 ± ·20	—	—	—	—
Aleppo	43	84·5 ± ·32	3·1 ± ·23	—	—	—	—
Morocco	30	80·2 ± ·28	2·3 ± ·20	—	—	—	—
Various Countries ...	500	81·42 ± ·10	3·16 ± ·07	—	—	—	—
Italy	112	82·45 ± ·25	3·87 ± ·17	—	—	—	—
Russian	139	81·09 ± ·20	3·58 ± ·14	—	—	—	—
European (? Germany)	100	80·3 ± ·21	3·14 ± ·15	—	—	—	—
South Russia ...	100	81·94 ± ·17	2·48 ± ·12	—	—	—	—
Persian Jews ...	57	81·32 ± ·35	3·92 ± ·25	60·37 ± ·56	6·07 ± ·40	164·35 ± ·52	3·53 ± ·22
Yemen	78	74·49 ± ·25	3·27 ± ·18	57·41 ± ·42 (50)	4·35 ± ·29	159·40 ± ·46	3·76 ± ·20
Samarkand	100	83·69 ± ·28	4·09 ± ·19	58·66 ± ·55 (50)	5·67 ± ·38	166·42 ± ·39	3·52 ± ·17

hybridisation as has occurred is not recent but of very early date; their original centre of dispersion is the Eastern shore of the Mediterranean and it would appear that their racial origin is similar to that which we shall shew later appears probable for the other inhabitants of the Eastern Mediterranean.

In attempting to decide the racial position of the people under discussion we are clearly faced with the difficulty that on general grounds we cannot place a greater (or a less) reliance on the cephalic index than on other features. Our figures however shew that of all the material available the cephalic index is less variable and therefore should prove a good guide even if it does not possess the ideal attributes assigned to it by Ripley. Moreover, as we have already shewn, most of the other characters would lead us to establish a grouping not unlike the grouping on which we should have to depend if the cephalic index were our only guide. Clearly then the group-complex of what may be called "racial characters" expresses a definite entity by which one group may be distinguished from another. We are however met by the great difficulty that these groups are in our area by no means equally dispersed although this is possibly due to the incomplete nature of our data. If we adopt such a method of classification we cannot admit any of the inhabitants of the Eastern Mediterranean to homogeneity with the "Mediterranean Race" as exemplified in Corsica, Sardinia and Egypt, but must consider them as a mixed race.

The only clear representatives of the "Armenoid" race are the Bektasch and Tadchadsky of Lycia, but the Leukadians have certain points in common with them. We have a series of groups which do not appear to be homogeneous occupying an intermediate position. There are some remarkable factors which need attention. In the homogeneous groups and even among the heterogeneous groups we may go great distances and meet little variation. It is a far cry from Giza to Corsica, yet the cephalic index of two series from these two groups is closely akin; again from Lycia to the Peloponnese is within our area a comparatively long journey, yet the resemblance between the Lycians and the Maniots is in some respects most striking. On the other hand if we cross the mountains and travel from the Bay of Salamis in Cyprus to the North coast—a leisurely day's journey—we find that the two populations are dissimilar. We have shewn however that there are two racial types of comparative homogeneity at either end of the scale and that those peoples who present these local divergencies are very variable, having in every case a high standard deviation. As far as our present evidence goes then the subdivision into numerous local types would appear to serve no useful purpose. When we find two sets of peoples whose cephalic index differs by as much as 10 units but who appear to be comparatively homogeneous we are justified in considering them to belong to different groups, but when we find numerous smaller populations differing no doubt from the limiting groups and from each other but all possessing considerable heterogeneity we can most easily explain this heterogeneity by admitting racial admixture. This admixture does not appear to have been similar in every case, although in a few it has been. We have not at present sufficient evidence to speak of Greece proper, but such evidence as we have would appear to justify the assertion that the numerous small communities of the ancient Mediterranean differed physically; thus we may say that there was a physical background to the struggles between Amathus and Salamis, and probably between the cities of Greece. To suppose that it is possible to establish a Greek type and to distinguish between Hellene and Barbarian *φύσει* does not appear justifiable.

It has been also suggested that there are two "types" in Greece—a brachycephalic Alpine and a long-headed Mediterranean—and that these can be distinguished. Now we have shewn that we do actually find such populations in a comparatively pure state in certain parts of the Mediterranean. That they exist in Greece is certain and it is not difficult to pick out of a large number of examples of both types. To say however that the population is made up of two types is unsatisfactory, because though it is possible and indeed probable that both types have participated in forming the present population, the latter seems at present to be so inextricably mixed that its separate elements are no longer distinguishable.

It may be argued that a large degree of variability is not due to admixture but is rather an attribute of the race. At present the series of crania of ancient date are so rare that no statistical evidence of variation in ancient times is available, it would appear however a more reasonable supposition to admit racial admixture when at either end of the scale it is possible to select specimens both of the Mediterranean and Alpine type, especially when we find a race that is admittedly pure has a low degree of variation.

It has been held by some classical scholars that the Nordic race has contributed to the population of the Eastern Mediterranean. It has been felt necessary to account for elements in the population who had fair hair and light eyes. It must however be remembered that colour terms are necessarily relative and among a population so dark as the Mediterranean peoples the standard is necessarily different from the standard of Northern Europe. A small proportion of blue-eyed persons does still occur in the Eastern Mediterranean and is certainly commoner in some areas than others. In Cyprus the blue-eyed persons appeared as far as a careful examination could determine to be akin to the "Armenoid" race and no trace of Nordic types was found. Individuals, or groups with a number of individuals, possessing features of this type, would sufficiently account for the frequent references in classical writers to fair peoples, without presuming Nordic blood.

We have so far limited the discussion to race in the older sense, that is apart from environment. It is possible that environmental conditions have played a certain part in determining the numerous local variations which are to be met with in the area we are examining, but we have not at present sufficient evidence upon which to generalize.

In regard to the historical ethnology again evidence is lacking, people akin to the Mediterranean race were inhabiting Egypt in predynastic times. v. Luschan's argument (quoted on p. 93 above) that the earliest inhabitants of Lycia were

Armenoids can be combated by the equally plausible, but perhaps also invalid, argument that these curious endogamous religious communities are the remains of later invaders who refused to ally themselves with the people of the land. The mixture of population had taken place in Cyprus—or among the inhabitants of Cyprus in an earlier home—before the beginning of the Bronze Age and it appears to have been taking place in Crete at about the same time. Elsewhere evidence is entirely lacking.

It is impossible on physical grounds at present to come to any conclusions in regard to the nature of the Pelasgians*. In regard to the Achaeans we have shewn that there appears to be no good ground for suspecting the presence of Nordics.

Ripley (p. 410) speaking of the classical Greeks says: "The facial features seem to be well demonstrated in the classic statuary although it is curious as Stephanos observes that these ideal heads are distinctly brachycephalic. Either the ancient sculptors knew little of anthropology or else we have again a confirmation of our assertion that however conscious of their peculiar facial traits a people may be, the head form is a characteristic whose significance is rarely recognized." We have attempted to shew that brachycephaly certainly did occur among the Greeks of the classical period. The small series of crania at our disposal contain both long-headed and short-headed types and we cannot agree with Ripley's dictum that the primitive Hellenic type was of pure Mediterranean stock. It is possible that the earliest dwellers in Greece may have been Mediterranean people—evidence is lacking on that point—but long before Hellenic civilization had developed the inhabitants of the Greek world appear, as far as our scanty data go, to have been similar generally to the inhabitants of that area today.

Their further history we cannot exactly follow. We know of movements of peoples, of irruption of Slavs, and of Turks. It is not improbable that these late comers were physically akin to the people who were already in occupation. With the Turk and possibly before came a large admixture of Negroid elements. Exactly how far the population has been affected by this influence it is difficult to say, but it forms an interesting example of a definite introduction of alien blood in the population, which has up to the present not been absorbed.

In smaller quantities and at various times, notably under the Byzantine Empire and during the Crusades, it is not unlikely that Nordic influence made itself felt in the Eastern Mediterranean but this influence cannot be detected at present.

While admitting then the presence of numerous minor differences, sufficiently great to make it necessary to know the exact provenance of any anthropological data we may wish to examine, it would not seem possible to assign any definite racial position to the greater part of the people of the Eastern Mediterranean but

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* Cf. Treidler, "Alte Völker der Balkenhalbinsel," *Archiv für Anthropologie*, XL. S. 97: "In einem Punkte muss man Beloch recht geben die ganze thrakische Frage ist insofern belanglos als zwischen Thrakern und Hellenen in physischer Beziehung kein wesentlicher Unterschied bestanden hat." The same author continues: "no doubt the 'Urpelasger' were identical with the Thracians."

rather to class them as representing a combination probably of comparatively early date of Alpine and Mediterranean both of which stocks are found sporadically in a comparatively unmixed state in some parts of the area.

DESCRIPTION OF PLATE III.

Figs. 1, 2. Athenian cranium Fourth Century B.C.
Figs. 3, 4. Modern Greek (Cyprus).
Figs. 5, 6. Athenian cranium Third Century B.C.

The crania are in the University Museum, Oxford.

Measurements.

Fig. 1.	Fig. 2.
Naso-occipital length	Minimum Frontal Diameter
181	99·5
Fig. 5.	Fig. 6.
Naso-occipital length	Minimum Frontal Diameter
186	94

No measurements available for Figs. 3 and 4.

LITERATURE.

(1) v. Luschan. *Archiv für Anthropologie*, 1891, xix. S. 31.
(2) —— *Zeitsch. für Ethnologie*, Jahrg. xlv. 1913, S. 307.
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Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.

Modern and Ancient Greek Heads.

For Description, see p. 112.

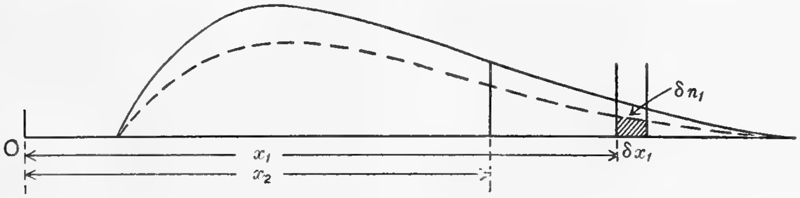
ON THE PROBABLE ERRORS OF FREQUENCY CONSTANTS.

PART III.

EDITORIAL.

(1) THE two previous parts* of this *résumé* of the theory of probable errors dealt with the probable errors of moments, and deduced the probable error of any frequency constant by considering that constant as a function of the moments, the probable errors of which had earlier been investigated. This third section proceeds from a different standpoint; it treats of the probable errors of constants supposed to be determined by a knowledge of the ranges in which certain proportions of the frequency lie. The fundamental papers on the subject are Edgeworth's "Exercises in the Calculation of Errors" (*Phil. Mag.* Vol. 36, pp. 98 *et seq.* 1893) and Sheppard's "On the Application of the Theory of Error to Cases of Normal Distribution and Correlation" (*Phil. Trans.* Vol. 192, pp. 130 *et seq.* 1898). We are unable to follow Edgeworth's reasoning, particularly that portion of it in the footnote on p. 99, where he states that the displacements of the two quartiles and the median are independent. They have, as we shall see, considerable correlation. With Dr Sheppard's numerical conclusions we are generally in agreement.

(2) We start from a population distributed according to any law of frequency, the ordinate at x_1 being y_1', and m_1 giving the total frequency beyond y_1' on the right, say. The total population M is supposed very large, and a sample N is taken from it, small as compared to M, but such that $1/\sqrt{N}$ is small as compared to unity. Let $\bar{n}_1 = m_1 \times N/M$ be the mean quantity that would be found beyond y_1' in many samples, then in any individual sample we shall not find $\bar{n}_1$ but $n_1 = \bar{n}_1 + \delta n_1$ beyond y_1 or x_1 of the sampled population. Accordingly the ordinate which cuts off $\bar{n}_1/N$ of the sample frequency will not be at the point x_1 but at some distance δx_1 from it and the area of the dotted curve which represents the sample on δx_1 will be δn_1. Thus if $y_1 = y_1' N/M$ be the ordinate at x_1 of the sample we shall have to



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* *Biometrika*, Vol. II. pp. 273—281 and Vol. IX. pp. 1—10. As in the earlier papers, this paper has been reproduced from Lecture Notes.

a first approximation, i.e. if we may neglect the terms in $\frac{1}{N}$ as compared with those in $\frac{1}{\sqrt{N}}$:

$$y_1 \delta x_1 = \delta n_1 \quad \text{..................................(1).}$$

Similarly for a second ordinate

$$y_2 \delta x_2 = \delta n_2 \quad \text{................................(1 \textit{bis}).}$$

Such equations connect the change in the grade with the change in the frequency and provide at once the relations

$$\sigma_{x_1} = \sigma_{n_1}/y_1, \quad \sigma_{x_2} = \sigma_{n_2}/y_2 \quad \text{..........................(2).}$$

If it be suggested that we shall have for a given sample a whole series of very irregular distributions of the n_1 individuals beyond y_1, the reply is that we are seeking to find σ_{x_1} for all samples, and we are justified to our degree of approximation in considering that δx_1 is the average grade change for *all* distributions which give an excess δn_1 beyond y_1, and that such average grade change may be looked upon as resulting from an area change and not from a scheme of isolated individuals.

Now if N be large as compared to M we have at once

$$\left.\begin{aligned} \sigma^2_{n_1} &= N\frac{m_1}{M}\left(1 - \frac{m_1}{M}\right) \\ \sigma^2_{n_2} &= N\frac{m_2}{M}\left(1 - \frac{m_2}{M}\right) \end{aligned}\right\} \quad \text{..............................(3),}$$

which lead to

$$\left.\begin{aligned} \sigma^2_{x_1} &= \frac{N}{y_1^2}\frac{m_1}{M}\left(1 - \frac{m_1}{M}\right) = \frac{N}{y_1^2}\frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_1}{N}\right) \\ \sigma^2_{x_2} &= \frac{N}{y_2^2}\frac{m_2}{M}\left(1 - \frac{m_2}{M}\right) = \frac{N}{y_2^2}\frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_2}{N}\right) \end{aligned}\right\} \quad \text{.....................(4).}$$

To our order of approximation $\left(\text{terms in } \frac{1}{\sqrt{N}}\right)$ we write finally

$$\sigma_{x_1} = \frac{1}{\sqrt{N}}\left(\frac{N}{y_1}\right)\sqrt{\frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_1}{N}\right)}, \quad \sigma_{x_2} = \frac{1}{\sqrt{N}}\left(\frac{N}{y_2}\right)\sqrt{\frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_2}{N}\right)} \quad \text{......(5).}$$

Again
$$\text{Mean}(\delta x_1 \delta x_2) = \frac{1}{y_1 y_2}\text{Mean}(\delta n_1 \delta n_2).$$

Now if we agree that x_2 shall be less than x_1,

$$n_2 = n_1 + n_2 - n_1 = n_1 + n_3 \text{ say,}$$
$$\delta n_2 = \delta n_1 + \delta n_3,$$
$$\text{Mean}(\delta n_1 \delta n_2) = \text{Mean}(\delta n_1)^2 + \text{Mean}(\delta n_1 \delta n_3).$$

Now n_1 and n_3 are frequencies having no part of their ranges in common and accordingly if N be large as compared with M,

$$\begin{aligned} \text{Mean}(\delta n_1 \delta n_3) &= -N\frac{m_1}{M}\frac{m_3}{M} \\ &= -N\frac{\bar{n}_1}{N}\frac{\bar{n}_3}{N} \quad \text{...........................(6),} \end{aligned}$$

with the same approximation as before.

Accordingly

$$\text{Mean}\,(\delta n_1 \delta n_2) = N\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) - N\frac{\bar{n}_1}{N}\left(\frac{\bar{n}_2-\bar{n}_1}{N}\right),$$

or

$$\text{Mean}\,(\delta n_1 \delta n_2) = N\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right) \quad\text{..........................(7).}$$

Thus

$$\text{Mean}\,(\delta x_1 \delta x_2) = \frac{N}{y_1 y_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right) \quad\text{........................(8).}$$

Equations (5) and (8) are the fundamental equations of our present subject. They give the standard deviations and correlations of the errors in any lengths measured along the x-axis as determined by the frequencies of the corresponding ranges.

If we have

$$z = f(x_1, x_2, \dots x_p),$$

then to our degree of approximation,

$$\sigma_z^2 = S\left\{\left(\frac{df}{dx_s}\right)^2 \sigma^2_{x_s}\right\} + 2S'\left\{\left(\frac{df}{dx_s}\frac{df}{dx_{s'}}\right)\times \text{Mean}\,(dx_s dx_{s'})\right\} \quad\text{.........(9),}$$

where S denotes summation of s from 1 to p and S' summation of every pair of different s and s' from 1 to p.

With our degree of approximation, i.e. to first order terms in $1/\sqrt{N}$, the above results are true whatever be the law of frequency.

For the special case $z = x_2 - x_1$ we find

$$\sigma^2_{x_2-x_1} = \frac{1}{N}\left\{\frac{N^2}{y_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) + \frac{N^2}{y_2^2}\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right) - 2\frac{N^2}{y_1 y_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)\right\} \quad\text{......(10).}$$

For the special case that $x_2 - x_1 = l_{q_R q_L}$, the intraquartile range, $\frac{\bar{n}_1}{N} = \frac{1}{4}$, $\frac{\bar{n}_2}{N} = \frac{3}{4}$,

$$\sigma^2_{l_{q_R q_L}} = \frac{1}{N}\left\{\frac{3}{16}\left(\frac{N^2}{y^2_{q_R}} + \frac{N^2}{y^2_{q_L}}\right) - \frac{2}{16}\frac{N^2}{y_{q_R} y_{q_L}}\right\} \quad\text{..................(11).}$$

For a symmetrical distribution $y_{_R} = y_{_L} = y_q$ say, and

$$\sigma_{l_{qq}} = \frac{\sigma}{\sqrt{2N}}\frac{N}{\sqrt{2}\,\sigma y_q} \quad\text{..........................(11 \textit{bis}).}$$

For the special case that $x_2 - x_1 = l_{mq_R}$, the median to right quartile range,

$$\sigma^2_{l_{mq_R}} = \frac{1}{N}\left(\frac{3}{16}\frac{N^2}{y^2_{q_R}} + \frac{1}{4}\frac{N^2}{y^2_m} - \frac{1}{4}\frac{N^2}{y_{q_R} y_m}\right) \quad\text{...................(12),}$$

and for $x_2 - x_1 = l_{q_L m}$, the left quartile to median range,

$$\sigma^2_{l_{q_L m}} = \frac{1}{N}\left(\frac{1}{4}\frac{N^2}{y^2_m} + \frac{3}{16}\frac{N^2}{y^2_{q_L}} - \frac{1}{4}\frac{N^2}{y_{q_L} y_m}\right) \quad\text{...............(12 \textit{bis}).}$$

(12) and (12 *bis*) will only give the same result provided $y_{q_R} = y_{q_L}$, which is of course satisfied by symmetrical curves but might be satisfied by other curves as well.

(3) We may now examine some of the resulting error correlations. If $r_{x_1x_2}$ be the correlation in errors of x_1 and x_2 we have by (8)

$$r_{x_1x_2} = \sqrt{\frac{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)}{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_1}{N}\right)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

For special cases:

(*a*) Quartile and Quartile: $r_{q_Lq_R} = \frac{1}{3} = {\cdot}3333,$

(*b*) Median and Quartile: $r_{mq_L} = r_{mq_R} = \frac{1}{\sqrt{3}} = {\cdot}5774,$

(*c*) Median and Decile: $r_{md_L} = r_{md_R} = \frac{1}{3\sqrt{3}} = {\cdot}1925,$

(*d*) Decile and Decile: $r_{d_Ld_R} = \frac{1}{9} = {\cdot}1111,$

(*e*) Quartile and Decile: $r_{d_Rq_R} = r_{d_Lq_L} = \frac{1}{\sqrt{3}} = {\cdot}5774,$

(*f*) Quartile and Decile: $r_{d_Rq_L} = r_{d_Lq_R} = \frac{1}{3\sqrt{3}} = {\cdot}1925,$

(*g*) Median and Quarto-decile: $r_{mqd_R} = r_{mqd_L} = \sqrt{\frac{2}{3}} = {\cdot}8165.$

It is clearly impossible to treat such correlations as zero*. It is of interest to note that they are absolutely independent of the nature of the frequency distribution, and approach unity as the correlated grades approach each other.

Now if we consider the determination of a range from the mean and mean square aspect we shall have

$$x_1 = \bar{x} + \lambda_1\sigma, \quad x_2 = \bar{x} + \lambda_2\sigma,$$

where λ_1 and λ_2 are quantities to be determined from the form of the frequency distribution when we know its nature.

Thus $$\sigma^2_{x_1} = \sigma_{\bar{x}}^2 + \lambda_1^2\sigma_\sigma^2 + 2\lambda_1 \times \text{Mean}\,(\delta\bar{x}\,\delta\sigma),$$

$$\sigma^2_{x_1-x_2} = (\lambda_1 - \lambda_2)^2\sigma_\sigma^2.$$

Now $$\sigma_{\bar{x}}^2 = \sigma^2/N, \quad \sigma_\sigma^2 = \frac{1}{4N}(\beta_2 - 1)\,\sigma^2$$

and $$\text{Mean}\,(\delta\bar{x}\,\delta\sigma) = \tfrac{1}{2}\sqrt{\beta_1}\,\sigma^2/N,$$

where β_1 and β_2 are the usual fundamental frequency constants. Accordingly

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}}\{1 + \lambda_1\sqrt{\beta_1} + \tfrac{1}{4}\lambda_1^2(\beta_2 - 1)\}^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots(14),$$

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}}(\lambda_1 - \lambda_2)\sqrt{\tfrac{1}{2}(\beta_2 - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

$$\text{Mean}\,(\delta x_1\,\delta x_2) = \frac{\sigma^2}{N}\{1 + \tfrac{1}{2}(\lambda_1 + \lambda_2)\sqrt{\beta_1} + \tfrac{1}{4}\lambda_1\lambda_2(\beta_2 - 1)\} \quad \ldots\ldots(16).$$

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* See remark, p. 133.

(14), (15) and (16) give the magnitudes to be compared with (5), (10), and (8) respectively when results obtained by mean and standard deviation methods are to be compared for accuracy with those obtained by grade methods.

We may now consider what these results reduce to when the frequency follows the simple normal distribution.

We shall suppose

$$y_s = \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x_s-\bar{x})^2}{\sigma^2}} \quad \text{.........................(17)},$$

and write $$h_s = (x_s - \bar{x})/\sigma \quad \text{and} \quad H_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2},$$

thus $y_s = NH_s/\sigma$, and H_s may be found directly from the *Tables for Statisticians*, when h_s is known, and h_s will be known when $\bar{n}_s$ is known. Again, $\lambda_s\sigma = x_s - \bar{x}$, and therefore $\lambda_s = h_s$.

We obtain the following results:

For (5):

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}}\frac{1}{H_1}\sqrt{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right)}, \quad \sigma_{x_2} = \frac{\sigma}{\sqrt{N}}\frac{1}{H_2}\sqrt{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right)} \quad \text{.........(18)}.$$

For (14):

$$\sigma_{x_1} = \frac{\sigma}{\sqrt{N}}(1+\tfrac{1}{2}h_1^2)^{\frac{1}{2}}, \quad \sigma_{x_2} = \frac{\sigma}{\sqrt{N}}(1+\tfrac{1}{2}h_2^2)^{\frac{1}{2}} \text{..............................(19)}.$$

For (10):

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}}\left\{\frac{2}{H_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right) + \frac{2}{H_2^2}\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_2}{N}\right) - \frac{4}{H_1H_2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)\right\}^{\frac{1}{2}}$$

$$\text{......(20)}.$$

For (15)*:

$$\sigma_{x_1-x_2} = \frac{\sigma}{\sqrt{2N}}(h_1 - h_2) \quad \text{...(21)}.$$

For $r_{x_1x_2}$ from (13):

$$r_{x_1x_2} = \sqrt{\frac{\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_2}{N}\right)}{\frac{\bar{n}_2}{N}\left(1-\frac{\bar{n}_1}{N}\right)}} \quad \text{...(22)}.$$

For $r_{x_1x_2}$ from (18):

$$r_{x_1x_2} = \frac{1+\frac{1}{2}h_1h_2}{\sqrt{1+\frac{1}{2}h_1^2}\sqrt{1+\frac{1}{2}h_2^2}} \quad \text{..(23)†}.$$

We may illustrate (23) first.

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* The reader must bear in mind that with the present notation $h_1 > h_2$ and $n_1 < n_2$.

† h_1 and h_2 may be of opposite sign.

Correlation of errors of grades as found by mean and standard deviations.

(*a*) Quartile and Quartile: $r_{q_L q_R} = \cdot 6294$,
(*b*) Median and Quartile: $r_{m q_R} = \cdot 9026$,
(*c*) Median and Decile: $r_{m d_R} = \cdot 6152$,
(*d*) Decile and Decile: $r_{d_L d_R} = \cdot 0982$,
(*e*) Quartile and Decile: $r_{q_R d_R} = \cdot 9579$,
(*f*) Quartile and Decile: $r_{q_L d_R} = \cdot 3798$,
(*g*) Median and Quarto-decile: $r_{m\, qd_R} = \cdot 9694$.

It will be seen that the correlation of grade errors as thus deduced is far higher than when the grades are found directly from ranking.

We now turn to the position of the grade and investigate the S.D. of its error as found in the two different methods by (18) and (19):

TABLE I. *Accuracy as found from Ranking and from Moments.*

n	Grade	Position from ranking	Position from Mean and S.D.	Ratio *
·9	1st Decile	$1{\cdot}7094\, \sigma/\sqrt{N}$	$1{\cdot}3495\, \sigma/\sqrt{N}$	1·2667
·8	2nd Decile	$1{\cdot}4288\, \sigma/\sqrt{N}$	$1{\cdot}1637\, \sigma/\sqrt{N}$	1·2278
·75	Left Quartile	$1{\cdot}3626\, \sigma/\sqrt{N}$	$1{\cdot}1079\, \sigma/\sqrt{N}$	1·2299
·7	3rd Decile	$1{\cdot}3180\, \sigma/\sqrt{N}$	$1{\cdot}0665\, \sigma/\sqrt{N}$	1·2358
·6	4th Decile	$1{\cdot}2680\, \sigma/\sqrt{N}$	$1{\cdot}0159\, \sigma/\sqrt{N}$	1·2482
·5	Median	$1{\cdot}2533\, \sigma/\sqrt{N}$	$1{\cdot}0000\, \sigma/\sqrt{N}$	1·2533
·4	6th Decile	$1{\cdot}2680\, \sigma/\sqrt{N}$	$1{\cdot}0159\, \sigma/\sqrt{N}$	1·2482
·3	7th Decile	$1{\cdot}3180\, \sigma/\sqrt{N}$	$1{\cdot}0665\, \sigma/\sqrt{N}$	1·2358
·25	Right Quartile	$1{\cdot}3626\, \sigma/\sqrt{N}$	$1{\cdot}1079\, \sigma/\sqrt{N}$	1·2299
·2	8th Decile	$1{\cdot}4288\, \sigma/\sqrt{N}$	$1{\cdot}1637\, \sigma/\sqrt{N}$	1·2278
·1	9th Decile	$1{\cdot}7094\, \sigma/\sqrt{N}$	$1{\cdot}3495\, \sigma/\sqrt{N}$	1·2667

It will be seen that increase of inaccuracy at all grades is singularly nearly constant lying between 23 °/ₒ and 27 °/ₒ and for rough purposes may be taken as 25 °/ₒ throughout the usual range of grading in deciles.

We shall now work out two further series of values, namely $\sigma_{x_1 - x_2}$ as found (*a*) from left to right deciles or quartiles, and (*b*) from the median to these grades.

(*a*) Turning to (20) we put

$$H_2 = H_1, \text{ and } \frac{\bar{n}_2}{N} = 1 - \frac{\bar{n}_1}{N},$$

---

* The values in columns 3 and 4 were found to 6 decimals and the ratio of the S.D.'s found from these. Columns 3 and 4 must be multiplied by ·67449 if the probable error instead of the standard deviation be required.

and deduce, since $x_2 = -x_1$,

$$\sigma_{x_2 x_1} = \frac{\sigma}{\sqrt{2N}} \frac{2}{H_1} \sqrt{\frac{\bar{n}_1}{N}\left(1 - \frac{2\bar{n}_1}{N}\right)} \quad \text{.......................(24).}$$

TABLE II. *Accuracy of Determination of σ from Ranking symmetrically.*

$\bar{n}_1/N$	Range $2x_1$	σ_{2x_1}	σ_σ from σ_{2x_1}	Values from Mean and S.D.
$\frac{1}{40}$	$3{\cdot}92007\ \sigma$	$5{\cdot}2737\ \sigma/\sqrt{2N}$	$1{\cdot}3453\ \sigma/\sqrt{2N}$	The S.D. of the ranges are the coefficients of 2nd column × $\frac{\sigma}{\sqrt{2N}}$. The ratios are therefore the coefficients of the 4th column. The S.D. of σ is of course $\sigma/\sqrt{2N}$, and the ratios are again coefficients of the 4th column
$\frac{1}{30}$	$3{\cdot}66787\ \sigma$	$4{\cdot}7514\ \sigma/\sqrt{2N}$	$1{\cdot}2954\ \sigma/\sqrt{2N}$	
$\frac{1}{20}$	$3{\cdot}28975\ \sigma$	$4{\cdot}1137\ \sigma/\sqrt{2N}$	$1{\cdot}2505\ \sigma/\sqrt{2N}$	
$\frac{1}{15}$	$3{\cdot}00219\ \sigma$	$3{\cdot}7178\ \sigma/\sqrt{2N}$	$1{\cdot}2384\ \sigma/\sqrt{2N}$	
$\frac{1}{14}$	$2{\cdot}93050\ \sigma$	$3{\cdot}6290\ \sigma/\sqrt{2N}$	$1{\cdot}2383\ \sigma/\sqrt{2N}$	
$\frac{1}{13}$	$2{\cdot}85219\ \sigma$	$3{\cdot}5358\ \sigma/\sqrt{2N}$	$1{\cdot}2397\ \sigma/\sqrt{2N}$	
$\frac{1}{12}$	$2{\cdot}76602\ \sigma$	$3{\cdot}4378\ \sigma/\sqrt{2N}$	$1{\cdot}2429\ \sigma/\sqrt{2N}$	
$\frac{1}{10}$	$2{\cdot}56310\ \sigma$	$3{\cdot}2233\ \sigma/\sqrt{2N}$	$1{\cdot}2576\ \sigma/\sqrt{2N}$	
$\frac{2}{10}$	$1{\cdot}68324\ \sigma$	$2{\cdot}4747\ \sigma/\sqrt{2N}$	$1{\cdot}4702\ \sigma/\sqrt{2N}$	
$\frac{1}{4}$	$1{\cdot}34898\ \sigma$	$2{\cdot}2252\ \sigma/\sqrt{2N}$	$1{\cdot}6495\ \sigma/\sqrt{2N}$	
$\frac{3}{10}$	$1{\cdot}04880\ \sigma$	$1{\cdot}9926\ \sigma/\sqrt{2N}$	$1{\cdot}8999\ \sigma/\sqrt{2N}$	
$\frac{4}{10}$	$\cdot50669\ \sigma$	$1{\cdot}4642\ \sigma/\sqrt{2N}$	$2{\cdot}8897\ \sigma/\sqrt{2N}$	

It will be clear from this table that the quartiles are not the best ranks from which to find the standard deviation with least error. The range should be that corresponding to $\frac{1}{14}$ to $\frac{1}{15}$ from each end of the series in the ranking, say $\frac{1}{14}$. This quatuordecimal range will provide σ with about 24 °/ₒ greater probable error than the moment method. The quartiles give it with 65 °/ₒ greater inaccuracy. The first and last deciles as a convenient divisor give only 26 °/ₒ less accuracy. The use of the quartiles was undoubtedly adopted because of their relation in the normal curve to the probable error. But theoretically they are very inferior to the quatuordecimals.

(*b*) The table on the following page provides the data for the probable errors of σ when found from ranges whose terminals are not symmetrical.

It will be clear that the inaccuracy of the determination of σ from ranking increases with great rapidity as we cause our asymmetrical ranks to approach each other. Further the range taken on one side of the median gives a worse result than the same range placed symmetrically about the median. Or again, looked at in another way, the range corresponding to $\frac{4}{10}$ of the frequency on one side the median gives $\sigma_\sigma = 1{\cdot}93\sigma/\sqrt{2N}$, but the same frequency symmetrically placed about the median gives $\sigma_\sigma = 1{\cdot}47\sigma/\sqrt{2N}$. It might be supposed at first that our Table II could be derived from Table III by simply supposing that in doubling the frequency from which our determinations are made we have increased the accuracy by $\sqrt{2}$. But the figures deduced in this way from the fifth column of Table III have for multipliers of $\sigma/\sqrt{2N}$

1·3660, 1·6036, 1·7887, 2·0406, 3·0150

instead of the multipliers

1·2576, 1·4702, 1·6495, 1·8999, 2·8897

TABLE III. *Accuracy of Determination of σ from Asymmetrical Ranks.*

$\bar{n}_2/N$	$\bar{n}_1/N$	Range $x_1 - x_2$	$\sigma_{x_1-x_2}$	σ_σ from $\sigma_{x_1-x_2}$	From Mean and S.D.
·5	·1	1·28155 σ	2·4757 $\sigma/\sqrt{2N}$	1·9318 $\sigma/\sqrt{2N}$	The S.D. of the ranges will be the coefficients of column 3 × $\sigma/\sqrt{2N}$. The S.D. of the S.D. will be throughout $\sigma/\sqrt{2N}$
·5	·2	·84162 σ	1·9087 $\sigma/\sqrt{2N}$	2·2678 $\sigma/\sqrt{2N}$	
·5	·25	·67449 σ	1·7062 $\sigma/\sqrt{2N}$	2·5296 $\sigma/\sqrt{2N}$	
·5	·3	·52440 σ	1·5134 $\sigma/\sqrt{2N}$	2·8859 $\sigma/\sqrt{2N}$	
·5	·4	·25335 σ	1·0803 $\sigma/\sqrt{2N}$	4·2639 $\sigma/\sqrt{2N}$	
·9	·3	1·80595 σ	2·7114 $\sigma/\sqrt{2N}$	1·5014 $\sigma/\sqrt{2N}$	
·8	·1	2·02317 σ	2·8808 $\sigma/\sqrt{2N}$	1·4239 $\sigma/\sqrt{2N}$	
·9	$\frac{1}{14}$	2·74680 σ	3·4327 $\sigma/\sqrt{2N}$	1·2497 $\sigma/\sqrt{2N}$	
·9	$\frac{1}{20}$	2·92643 σ	3·6974 $\sigma/\sqrt{2N}$	1·2634 $\sigma/\sqrt{2N}$	

of the last five entries of the fourth column of the table on p. 119. The explanation of this result is the relatively high correlation between x_1 and x_2 which reduces the higher values above resulting from a supposed independence of errors in the determination of our two quartile distances. Physically this signifies that if we were to determine one median to quartile distance from a first sample of N, and the other median to quartile distance from a second sample, we should get a less accurate result than determining both quartiles on the *same* sample. For the principle of correlation shows us that in the same sample if we get an excess in one quartile distance we shall on the average get a defect in the other. We thus may actually lose in accuracy when we combine measurements drawn from different samples in estimating the constants of a sampled population.

(4) The next question to be raised is whether the median can be obtained more accurately than by halving our total frequency. We have seen that the error of this process is $1{\cdot}2533\sigma/\sqrt{N}$. We might, however, obtain it as the half of any two grades on either side the median at equal distances from it, i.e.

$$m = \tfrac{1}{2}(x_1 + x_2).$$

Let each of the grades cut off $\bar{n}_1$ from the tails of the frequency, then

$$\begin{aligned} \sigma_m &= \tfrac{1}{2}\sigma_{x_1+x_2} \\ &= \frac{\sigma}{\sqrt{2N}}\left\{\frac{1}{H_1^2}\frac{\bar{n}_1}{N}\left(1-\frac{\bar{n}_1}{N}\right)+\frac{1}{H_1^2}\frac{\bar{n}_1}{N}\frac{\bar{n}_1}{N}\right\}^{\frac{1}{2}} \\ &= \frac{\sigma}{\sqrt{N}}\frac{1}{H_1}\sqrt{\frac{\bar{n}_1}{2N}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25). \end{aligned}$$

To find when this is a maximum, we must differentiate with regard to $\bar{n}_1$ and put $d\sigma_m/d\bar{n}_1 = 0$ or

$$\frac{1}{2\bar{n}_1} = \frac{dH_1}{d\bar{n}_1} = \frac{1}{H_1}\frac{dH_1}{dh_1}\frac{dh_1}{d\bar{n}_1} = \frac{1}{N}\frac{h_1}{H_1}.$$

Thus

$$\frac{\bar{n}_1}{N} = \frac{H_1}{2h_1} \qquad \text{.................................(26).}$$

Putting

	$H_1/2h_1$	$\bar{n}_1/N$
$h_1 = \cdot 60$	·277,687	·274,253
·61	·271,487	·270,931
·62	·265,471	·267,629
·63	·259,629	·264,347

we see that h_1 lies between ·61 and ·62 and thence reach by interpolation ·612,0014 as the required value of h_1. This leads to

$$\bar{n}_1/N = \cdot 270{,}268,$$

or the right value of $\bar{n}_1$ is a little more than the quartile, or $x_1 - \bar{x}$, instead of being $\cdot 67449\sigma$, $= \cdot 61200\sigma$. This leads us to

$$\sigma_m = 1\cdot 1112\sigma/\sqrt{N} \qquad \text{.............................(27),}$$

or the median found from the $\bar{n}_1/N = \cdot 270268$ grades is more than twice as accurate as when found directly.

As these grades do not differ very much from $\bar{n}_1/N = \cdot 25$ or the quartiles we note that the σ_m of the median found from the two quartiles is

$$\sigma_m = 1\cdot 1126\sigma/\sqrt{N} \qquad \text{.............................(28).}$$

The difference in accuracy between (27) and (28) is too small to be of any importance for most statistical purposes, and accordingly the quartiles may be used to determine the median, and this with double the accuracy of direct investigation of the mid-individual. We can put into a table the results for finding the median from each pair of the following series:

TABLE IV. *Accuracy of Determination of Median from Pairs of Symmetrical Grades.*

$\bar{n}_1/N$	σ_m	Symbol for median
$\frac{1}{14}$	$1\cdot 3858\,\sigma/\sqrt{N}$	m_7
$\frac{1}{10}$	$1\cdot 2741\,\sigma/\sqrt{N}$	m_6
$\frac{2}{10}$	$1\cdot 1295\,\sigma/\sqrt{N}$	m_5
·25	$1\cdot 1126\,\sigma/\sqrt{N}$	m_4
max. ·270268	$1\cdot 1112\,\sigma/\sqrt{N}$	m_3
$\frac{3}{10}$	$1\cdot 1139\,\sigma/\sqrt{N}$	m_2
$\frac{4}{10}$	$1\cdot 1576\,\sigma/\sqrt{N}$	m_1
$\frac{5}{10}$	$1\cdot 2533\,\sigma/\sqrt{N}$	m_0

(5) The question now arises as to whether we could still further reduce the above inaccuracies by combining these results in pairs in the best proportions, say λ_s and $\lambda_{s'}$ where $\lambda_s + \lambda_{s'} = 1$. Let the corresponding median value be $m_{ss'}$, then

$$m_{ss'} = \lambda_s m_s + \lambda_{s'} m_{s'},$$

$$\sigma^2_{m_{ss'}} = \lambda^2_s \sigma^2_{m_s} + \lambda^2_{s'} \sigma^2_{m_{s'}} + 2\lambda_s \lambda_{s'} [\delta m_s \delta m_{s'}]^*,$$

and making this a minimum for varying λ_s, $\lambda_{s'}$ we find

$$\left.\begin{aligned} \lambda_s &= \frac{\sigma^2_{m_{s'}} - [\delta m_s \delta m_{s'}]}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \\ \lambda_{s'} &= \frac{\sigma^2_{m_s} - [\delta m_s \delta m_{s'}]}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \\ \sigma^2_{m_{ss'}} &= \frac{\sigma^2_{m_s}\sigma^2_{m_{s'}} - [\delta m_s \delta m_{s'}]^2}{\sigma^2_{m_s} + \sigma^2_{m_{s'}} - 2[\delta m_s \delta m_{s'}]} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(29).$$

Table IV supplies the needful values of the σ_m's and what we require are the mean products

$$\begin{aligned} [\delta m_s \delta m_{s'}] &= \tfrac{1}{4}[(\delta x_1 + \delta x_1')(\delta x_2 + \delta x_2')] \\ &= \tfrac{1}{4}([\delta x_1 \delta x_2] + [\delta x_1 \delta x_2'] + [\delta x_1' \delta x_2] + [\delta x_1' \delta x_2']) \\ &= \frac{1}{4}\frac{\sigma^2}{NH_sH_{s'}}\left\{\frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_2}{N}\right) + \frac{\bar{n}_1}{N}\left(1 - \frac{\bar{n}_2'}{N}\right) + \frac{\bar{n}_2}{N}\left(1 - \frac{\bar{n}_1'}{N}\right) + \frac{\bar{n}_2'}{N}\left(1 - \frac{\bar{n}_1'}{N}\right)\right\}. \end{aligned}$$

But $\bar{n}_1 = N - \bar{n}_1'$, $\bar{n}_2 = N - \bar{n}_2'$, thus finally

$$[\delta m_s \delta m_{s'}] = \frac{\sigma^2}{N}\frac{1}{H_sH_{s'}}\frac{\bar{n}_1}{2N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30),$$

where $\bar{n}_1$ corresponds to x_1 the outermost of the paired grades.

This leads us at once to

$$[\delta m_s \delta m_{s'}] = \sigma^2_{m_s} H_s / H_{s'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

which much simplifies the calculations, if we remember that σ_{m_s} and H_s correspond to the grade further from the median.

Hence for correlation of m_s, $m_{s'}$ we have

$$r_{m_s m_{s'}} = \frac{\sigma_{m_s} H_s}{\sigma_{m_{s'}} H_{s'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

Tables V and VI give first the mean product-moment $[\delta m_s \delta m_{s'}]$ and secondly the standard deviation $\sigma_{m_{ss'}}$ and the coefficients λ_s, $\lambda_{s'}$ of $m_{ss'} = \lambda_s m_s + \lambda_{s'} m_{s'}$.

It is clear from Table VI that the lowest value of the probable error of the median will be obtained by deducing the median from the two 1st deciles and from the two 3rd deciles and taking ·34388 of the first plus ·65612 of the second value; the value of $\sigma_{m_{ss'}}$ will then be $1{\cdot}0468\sigma/\sqrt{N}$, i.e. only about 5 °/。 more inaccurate than that found by the arithmetic mean. We may search about this value for

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* It is convenient to write $[\delta m_s \delta m_{s'}]$ to signify mean value of the product $\delta m_s \delta m_{s'}$.

TABLE V. *Values of* $[\delta m_s \delta m_{s'}] \Big/ \frac{\sigma^2}{N}$.

s'

s	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	·25	·270268	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{14}$	1·920,540	1·492,315	·935,478	·824,158	·791,690	·753,246	·677,891	·656,482
$\frac{1}{10}$	—	1·623,401	1·017,657	·896,557	·861,238	·819,416	·737,441	·714,152
$\frac{2}{10}$	—	—	1·275,856	1·123,943	1·079,666	1·027,238	·924,472	·895,274
·25	—	—	—	1·237,845	1·189,081	1·131,339	1·018,159	·986,004
·270268	—	—	—	—	1·234,835	1·174,871	1·057,336	1·023,944
$\frac{3}{10}$	—	—	—	—	—	1·240,793	1·116,663	1·081,398
$\frac{4}{10}$	—	—	—	—	—	—	1·339,936	1·297,619
$\frac{1}{2}$	—	—	—	—	—	—	—	1·570,796

TABLE VI. *Values of* $\sigma_{m_{ss'}} \Big/ \frac{\sigma}{\sqrt{N}}$ *and* λ_s, $\lambda_{s'}$.

s'

s	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	·25	·270268	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{14}$	1·3858	1·2620	1·0902	1·0604	1·0535	1·0475	1·0535	1·0895
λ_s	—	·23437	·25680	·27395	·28190	·29462	·34759	·41972
$\lambda_{s'}$	—	·76563	·74320	·72605	·71810	·70538	·65241	·58018
$\frac{1}{10}$	—	1·2741	1·0948	1·0624	1·0545	1·0468	1·0469	1·0748
λ_s	—	—	·29886	·31952	·32894	·34388	·40478	·48511
$\lambda_{s'}$	—	—	·70114	·68048	·67106	·65612	·59522	·51489
$\frac{2}{10}$	—	—	1·1295	1·0904	1·0800	1·0687	1·0559	1·0671
λ_s	—	—	—	·42850	·44162	·46207	·54178	·63964
$\lambda_{s'}$	—	—	—	·57150	·55838	·53793	·45822	·36036
·25	—	—	—	1·1126	1·0965	1·0809	1·0718	1·0780
λ_s	—	—	—	—	·48408	·48683	·59427	·69898
$\lambda_{s'}$	—	—	—	—	·51592	·51317	·40573	·30102
·270268	—	—	—	—	1·1112	1·0983	1·0800	1·0845
λ_s	—	—	—	—	—	·52367	·61422	·72169
$\lambda_{s'}$	—	—	—	—	—	·47633	·38578	·27831
$\frac{3}{10}$	—	—	—	—	—	1·1139	1·0938	1·0962
λ_s	—	—	—	—	—	—	·64269	·75432
$\lambda_{s'}$	—	—	—	—	—	—	·35731	·24568
$\frac{4}{10}$	—	—	—	—	—	—	1·1576	1·1551
λ_s	—	—	—	—	—	—	—	·86587
$\lambda_{s'}$	—	—	—	—	—	—	—	·13413
$\frac{1}{2}$	—	—	—	—	—	—	—	1·2533
λ_s	—	—	—	—	—	—	—	—
$\lambda_{s'}$	—	—	—	—	—	—	—	—

a lower grade of inaccuracy, but there is little advantage to be gained. Thus for $s' = \frac{1}{12}$, $s' = \frac{3}{10}$ we find

$$\sigma_{m_{ss'}} = 1\cdot0466 \frac{\sigma}{\sqrt{N}}, \quad \lambda_s = \cdot31660, \quad \lambda_{s'} = \cdot68340 \quad \text{...........(33)},$$

only a very slightly less inaccuracy with a considerable change in the values of the λ's.

It is desirable in such a case to take round numbers easily remembered, and thus we may propose

$$m_{\frac{1}{10}, \frac{3}{10}} = \tfrac{1}{3} m_{\frac{1}{10}} + \tfrac{2}{3} m_{\frac{3}{10}} \quad \text{..........................(34).}$$

In this case

$$\sigma_{m_{\frac{1}{10}, \frac{3}{10}}} = 1\cdot0469 \frac{\sigma}{\sqrt{N}} \quad \text{..........................(35).}$$

The practical difference of 1·0466 and 1·0469 is not worth considering and the coefficients are most easy to remember.

If we wish to better this at all substantially we must take three pairs of decile-determined medians. For example, suppose we take

$$m_{\frac{1}{10}, \frac{3}{10}, \frac{5}{10}} = \lambda_{\frac{1}{10}} m_{\frac{1}{10}} + \lambda_{\frac{3}{10}} m_{\frac{3}{10}} + \lambda_{\frac{5}{10}} m_{\frac{5}{10}}.$$

We find

$$\sigma_{m_{\frac{1}{10}, \frac{3}{10}, \frac{5}{10}}} = 1\cdot0321 \frac{\sigma}{\sqrt{N}},$$

where

$$\left.\begin{aligned} \lambda_{\frac{1}{10}} &= \cdot33426 \\ \lambda_{\frac{3}{10}} &= \cdot44795 \\ \lambda_{\frac{5}{10}} &= \cdot21779 \end{aligned}\right\} \quad \text{..............................(36).}$$

Or very approximately,

$$\lambda_{\frac{1}{10}} = \tfrac{3}{9}, \quad \lambda_{\frac{3}{10}} = \tfrac{4}{9}, \quad \lambda_{\frac{5}{10}} = \tfrac{2}{9} \quad \text{........................(37),}$$

which give also to four decimals

$$\sigma_m = 1\cdot0321 \frac{\sigma}{\sqrt{N}} \quad \text{..............................(38).}$$

Whether this 2 °/。 gain on using three instead of two median values is worth the extra labour, which is not, however, very great, may be discussable. Probably with the use of four to five medians we could make the accuracy substantially that of an arithmetic mean. The difficulties lie not so much in the reduction of the inaccuracy as in two other matters: (i) the whole superstructure of these probable errors of decile medians is based on the frequency distribution approaching that of the normal curve. We shall get in many cases larger errors than those of random sampling, due to definite deviations from such normality in our frequency; and (ii) the decile values, if there has been grouping in classifying the frequency, can only be determined by use of the probability integral tables. We cannot pick out the character of the individual who stands n_1 individuals from the top. He will lie somewhere in a group of p individuals, and unless these individuals be arranged in order of character, we cannot find the character of the n_1th and $(n_1 + 1)$th individual. Their characters would have to be found from those of the individuals who bound their group by using the probability integral and actually determining from them the mean and standard deviation of the total frequency, the very quantities we are seeking to find by a more accurate process.

It may be said why not arrange the material in order of magnitude? The answer is that this is not an easy task if the observations run to hundreds or thousands. And further, while a mean may give accurately a result to $\frac{1}{100}$th of an inch, it may be far from desirable or possible to measure to more than $\frac{1}{2}''$, e.g. in the case of stature. Hence the decile individuals will be only known to $\frac{1}{2}''$ and the median found from a pair of deciles will only at best be known to the $\frac{1}{4}''$. The whole theory indeed supposes the characters of the decile individuals measured to very near the same accuracy as we need the mean. There are none of the advantages which arise from grouping after measuring in fairly coarse units, and then averaging up to get the mean. The application of the decile method should therefore be confined to cases where the numbers are not too large for ranking, where the distribution is approximately normal and where the character has been determined with considerable exactness at least for individuals in the region of the deciles. These limitations take much of our material out of the field of grade treatment.

(6) We can now adopt the same process as we have applied to the determination of the median to find if possible a more accurate value for the standard deviation by aid of ranking.

We take as before $$\sigma_{ss'} = \lambda_s \sigma_s + \lambda_{s'} \sigma_{s'},$$

where λ_s and $\lambda_{s'}$ are to be chosen so that

$$\sigma^2_{\sigma_{ss'}} = \lambda^2_s \sigma^2_{\sigma_s} + \lambda^2_{s'} \sigma^2_{\sigma_{s'}} + 2\lambda_s \lambda_{s'} [\delta\sigma_s \delta\sigma_{s'}]$$

is a minimum.

Accordingly we have for the minimum value of $\sigma_{\sigma_{ss'}}$ the equation

$$\left.\begin{aligned} \sigma^2_{\sigma_{ss'}} &= \frac{\sigma^2_{\sigma_s}\sigma^2_{\sigma_{s'}} - [\delta\sigma_s\delta\sigma_{s'}]^2}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]}, \\ \lambda_s = \frac{\sigma^2_{\sigma_{s'}} - [\delta\sigma_s\delta\sigma_{s'}]}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]}, &\quad \lambda_{s'} = \frac{\sigma^2_{\sigma_s} - [\delta\sigma_s\delta\sigma_{s'}]}{\sigma^2_{\sigma_s} + \sigma^2_{\sigma_{s'}} - 2[\delta\sigma_s\delta\sigma_{s'}]} \end{aligned}\right\} \quad \ldots(39).$$

We already know (see p. 119) the values of σ_{σ_s}, $\sigma_{\sigma_{s'}}$ as found from any symmetrical pair of grades, i.e.

$$\sigma^2_{\sigma_s} = \frac{\sigma^2}{2N}\frac{1}{H_s^2 h_s^2}\frac{\bar{n}_s}{N}\left(1 - \frac{2\bar{n}_s}{N}\right).$$

It is easy to show that

$$[\delta\sigma_s\delta\sigma_{s'}] = \sigma_{\sigma_s}\sigma_{\sigma_{s'}} \sqrt{\frac{\frac{2\bar{n}_s}{N}}{1 - \frac{2\bar{n}_s}{N}}} \sqrt{\frac{1 - \frac{2\bar{n}_{s'}}{N}}{\frac{2\bar{n}_{s'}}{N}}} \quad \ldots\ldots\ldots\ldots\ldots(40)$$

where $\bar{n}_s$ is always less than $\bar{n}_{s'}$.

It will be seen that the first requisite is to table $r_{\sigma_s \sigma_{s'}}$. We have

TABLE VII. *Correlations of Errors in* σ_s *and* $\sigma_{s'}$, *or values of* $[\delta\sigma_s\delta\sigma_{s'}]/(\sigma_{\sigma_s}\sigma_{\sigma_{s'}})$.

$\bar{n}_{s'}/N$

$\bar{n}_s/N$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{1}{2}$
$\frac{1}{40}$	1·000,000	·906,083	·726,483	·593,171	·484,322	·296,586	·242,161	·197,724	·121,081	·000,000
$\frac{1}{30}$	—	1·000,000	·801,784	654,654	·534,522	·327,327	·267,261	·218,218	·131,815	·000,000
$\frac{1}{20}$	—	—	1·000,000	·816,497	·666,667	·408,248	·333,333	·272,166	·166,667	000,000
$\frac{1}{14}$	—	—	—	1·000,000	·816,497	·500,000	·408,248	·333,333	·204,124	·000,000
$\frac{1}{10}$	—	—	—	—	1·000,000	·612,372	·500,000	·408,248	·250,000	·000,000
$\frac{2}{10}$	—	—	—	—	—	1·000,000	·816,497	·666,667	·408,248	·000,000
$\frac{1}{4}$	—	—	—	—	—	—	1·000,000	·816,496	·500,000	·000,000
$\frac{3}{10}$	—	—	—	—	—	—	—	1·000,000	·612,372	·000,000
$\frac{4}{10}$	—	—	—	—	—	—	—	—	1·000,000	·000,000
$\frac{1}{2}$	—	—	—	—	—	—	—	—	—	1·000,000

TABLE VIII. *Standard Deviations in* $\sigma_{\sigma_{ss'}}$, *or mean square error in* σ *as found from two pairs of grades* n_s *and* $n_{s'}$ *properly weighted.*

$\bar{n}_{s'}/N$

$\bar{n}_s/N$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{14}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{4}{10}$	
$\frac{1}{40}$ $\sigma_{\sigma_{ss'}}$	1·3453	1·2839	1·2037	1·1471	1·1181	1·1292	1·1582	1·1933	1·2709	$\times \sigma/\sqrt{2N}$
λ_s	0	·30024	·36974	·41434	·43488	·56284	·63180	·70412	·85474	—
$\lambda_{s'}$	1·00000	·69976	·63026	·58566	·56512	·43716	·36820	·29588	·14526	—
$\frac{1}{30}$ $\sigma_{\sigma_{ss'}}$	—	1·2954	1·2061	1·1503	1·1175	1·1176	1·1410	1·1701	1·2340	$\times \sigma/\sqrt{2N}$
λ_s	—	0	·41128	·43490	·46821	·59323	·66010	·72921	·86898	—
$\lambda_{s'}$	—	1·00000	·58872	·56510	·53179	·40677	·33990	·27079	·13102	—
$\frac{1}{20}$ $\sigma_{\sigma_{ss'}}$	—	—	1·2505	1·1858	1·1448	1·1254	1·1401	1·1606	1·2073	$\times \sigma/\sqrt{2N}$
λ_s	—	—	0	·47329	·50849	·63438	·69886	·76518	·88961	—
$\lambda_{s'}$	—	—	1·00000	·62671	·49151	·36562	·30114	·23482	·11039	—
$\frac{1}{14}$ $\sigma_{\sigma_{ss'}}$	—	—	—	1·2383	1·1889	1·1516	1·1594	1·1732	1·2070	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	0	·54211	·66755	·72955	·79042	·90467	—
$\lambda_{s'}$	—	—	—	1·00000	·45789	·33245	·27045	·20958	·09533	—
$\frac{1}{10}$ $\sigma_{\sigma_{ss'}}$	—	—	—	—	1·2576	1·2021	1·2036	1·2117	1·2352	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	—	0	·69611	·75568	·82034	·91706	—
$\lambda_{s'}$	—	—	—	—	1·00000	·30389	·24432	·17966	·08294	—
$\frac{2}{10}$ $\sigma_{\sigma_{ss'}}$	—	—	—	—	—	1·4702	1·4580	1·4552	1·4614	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	—	—	0	·80332	·85376	·93936	—
$\lambda_{s'}$	—	—	—	—	—	1·00000	·19668	·14624	·06064	—
$\frac{1}{4}$ $\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	1·6495	1·6429	1·6440	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	—	—	—	0	·86640	·94646	—
$\lambda_{s'}$	—	—	—	—	—	—	1·00000	·13360	·05354	—
$\frac{3}{10}$ $\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	—	1·8999	1·9051	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	—	—	—	—	0	·94934	—
$\lambda_{s'}$	—	—	—	—	—	—	—	1·00000	·05066	—
$\frac{4}{10}$ $\sigma_{\sigma_{ss'}}$	—	—	—	—	—	—	—	—	2·8897	$\times \sigma/\sqrt{2N}$
λ_s	—	—	—	—	—	—	—	—	0	—
$\lambda_{s'}$	—	—	—	—	—	—	—	—	1·00000	—

As in the case of the discovery of the best value of the median from two balanced pairs of grades, it will be seen again that there are two factors at work. When the pairs of grades are close to each other then the correlation is high in the errors of σ, and we lose the advantage of duplicating our determinations of σ. On the other hand when the grades are taken far apart the correlation is low and the gain in duplicating our determinations should be greater. But in this case one or other pair of grades must be far removed from the $\frac{1}{14}$ grade which gives the lowest probable error of any determination of σ.

Our second table shows us that we must push one pair of grades considerably out from the $\frac{1}{14}$ pair, and the other pair considerably in. The lowest value occurs at a little under 12 °/。 increase of inaccuracy above mean and moment values of σ. The pairs of grades are approximately $\frac{1}{30}$ and $\frac{1}{10}$ or $\frac{2}{10}$. As the latter gives very closely the proportions ·6 and ·4 the following rule was tested. Take ·6 times the value of σ as found from the interval between the $\frac{1}{30}$ grades and ·4 times the value of σ as found from the $\frac{1}{5}$ grades (i.e. the second deciles from either end). The mean square error of σ is then found to be $1{\cdot}1175 \dfrac{\sigma}{\sqrt{2N}}$ exactly. This is close to the minimum value obtainable by the use of two pairs of grades.

Of course by using three pairs of grades we can still further increase the accuracy of grade methods, but the choice of the three *best* pairs would require lengthy investigations for their determination, and every increase of complexity in grade investigations reduces their already small advantage over mean and moment methods. We may illustrate three pairs on the supposition that $\frac{1}{30}$, $\frac{1}{14}$ and $\frac{2}{10}$ are suitable grade pairs.

We have
$$\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}} = \lambda_{\frac{1}{30}}\sigma_{\frac{1}{30}} + \lambda_{\frac{1}{14}}\sigma_{\frac{1}{14}} + \lambda_{\frac{2}{10}}\sigma_{\frac{2}{10}}$$
leading to
$$\sigma^2_{\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{20}}} = \lambda^2_{\frac{1}{30}}\sigma^2_{\sigma_{\frac{1}{30}}} + \lambda^2_{\frac{1}{14}}\sigma^2_{\sigma_{\frac{1}{14}}} + \lambda^2_{\frac{2}{10}}\sigma^2_{\sigma_{\frac{2}{10}}}$$
$$+ 2\lambda_{\frac{1}{30}}\lambda_{\frac{1}{14}}[\delta\sigma_{\frac{1}{30}}\delta\sigma_{\frac{1}{14}}] + 2\lambda_{\frac{1}{14}}\lambda_{\frac{2}{10}}[\delta\sigma_{\frac{1}{14}}\delta\sigma_{\frac{2}{10}}] + 2\lambda_{\frac{2}{10}}\lambda_{\frac{1}{30}}[\delta\sigma_{\frac{2}{10}}\delta\sigma_{\frac{1}{30}}].$$

Substituting the numerical values from Tables VII and VIII we have the following equation from which to find $\lambda_{\frac{1}{30}}$, $\lambda_{\frac{1}{14}}$, $\lambda_{\frac{2}{10}}$ subject to the condition
$$\lambda_{\frac{1}{30}} + \lambda_{\frac{1}{14}} + \lambda_{\frac{2}{10}} = 1:$$
$$\sigma^2_{\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}}} = \lambda^2_{\frac{1}{30}}\,1{\cdot}6780{,}6116 + \lambda^2_{\frac{1}{14}}\,1{\cdot}5333{,}8689 + \lambda^2_{\frac{2}{10}}\,2{\cdot}1614{,}8804$$
$$+ 2\lambda_{\frac{1}{30}}\lambda_{\frac{1}{14}}\,1{\cdot}0501{,}2591 + 2\lambda_{\frac{1}{14}}\lambda_{\frac{2}{10}}\,{\cdot}9102{,}7433 + 2\lambda_{\frac{2}{10}}\lambda_{\frac{1}{30}}\,{\cdot}6233{,}9299.$$

We determine
$$\lambda_{\frac{1}{30}} = {\cdot}38325, \quad \lambda_{\frac{1}{14}} = {\cdot}32446, \quad \lambda_{\frac{2}{10}} = {\cdot}29229 \quad \ldots\ldots\ldots\ldots\ldots(41),$$
leading to
$$\sigma_{\sigma_{\frac{1}{30},\,\frac{1}{14},\,\frac{2}{10}}} = 1{\cdot}0798\,\frac{\sigma}{\sqrt{2N}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42).$$

The values of λ are so nearly one-third that without much greater inaccuracy we can assume them $\frac{1}{3}$, we have then

$$\sigma_{\frac{1}{30},\, \frac{1}{14},\, \frac{2}{10}} = \tfrac{1}{3}\left(\sigma_{\frac{1}{30}} + \sigma_{\frac{1}{14}} + \sigma_{\frac{2}{10}}\right) \quad \text{..................(43)},$$

with

$$\sigma_{\sigma_{\frac{1}{30},\, \frac{1}{14},\, \frac{2}{10}}} = 1{\cdot}0822\,\frac{\sigma}{\sqrt{2N}} \quad \text{..........................(44)}.$$

It is quite conceivable that another choice of the three pairs of grades might lead to an even smaller increase of inaccuracy over the mean and moment method of finding σ. But it suffices here to have shown that the excess inaccuracy by discarding the use of quartiles for quatuordecimals and then these for the $\frac{1}{30}$ and $\frac{2}{10}$ pairs of grades and lastly the latter for three pairs of grades can be reduced from 65 °/₀ to 24 °/₀, from 24 °/₀ to 12 °/₀ and ultimately to 8 °/₀.

(7) In the course of this paper we have seen that it is quite possible to better vastly the determination of median and quartile deviation by not determining the median and quartile directly but from pairs of appropriately chosen grades. We have further seen, however, that either these chosen grades will not be those of classification, or on the other hand if the individual ranking has been obtained as in small series, the measurement of the ranked individuals is hardly likely to be fine enough to give the mean and standard deviation to an adequate number of decimal points. In short while the disadvantages in accuracy are great, the advantages in brevity of treatment will not be compensatory in the majority of cases, and mean and moment methods should be always given the preference. Why then is it needful to discuss at length the probable errors of grade methods? The answer lies in the fact that in all cases of classification by broad qualitative categories no other method of solution is feasible. Further it is certain that hitherto too little attention has been paid to the best means of deducing the constants of such distributions. In such cases it is futile to attempt to obtain directly median quartiles or deciles. Such attempts throw us back on interpolations of an unsatisfactory character. We are compelled to accept the grades given by the data themselves, and these grades will usually not be symmetrical and are not capable of satisfactory readjustment. From every pair of grades we can find the standard deviation in terms of a chosen variate interval, and from every grade we can find the position of the mean in terms of this standard deviation. The standard deviations as found from each pair of grades will not, however, be independent; if there be p broad categories only $p-2$ values of the standard deviation in terms of the variate sub-ranges are *a priori* open to choice and only $p-1$ determinations of the median. Let us consider how this, the really practical problem of grading, is associated with the methods discussed in this paper.

We start with the sampled population, the mean of which may be treated as a fixed origin for measurements in the samples. This population is divided up by dichotomic ordinates into broad categories; the distances between these dichotomic ordinates will be the same in the sample as for the sampled population, and although their values have to be found from the sample they are really constant differences depending on the classification chosen. The essential feature of the sample is that the frequency found between these dichotomic ordinates is not exactly the proportion to be expected on the average, i.e. between x_s and x_{s+1} it is not $\bar{n}_{s+1} - \bar{n}_s$ but differs from this by $\delta n_{s+1} - \delta n_s$, where δn_s, δn_{s+1} are the deviations of random sampling*.

Let $\bar{h}_r$, $\bar{x}_r$, $\bar{\sigma}$ be quantities of the sampled population and h_r, x_r, σ refer to the sample. Further let σ_{rs} be the σ of the sample as found from the rth and sth dichotomic interval. Thus if $x_r > x_s$ or $r < s$, measuring x positive to the right-hand, we have

$$(h_r - h_s)\,\sigma_{rs} = x_r - x_s = \bar{x}_r - \bar{x}_s = \sigma\,(\bar{h}_r - \bar{h}_s).$$

Accordingly to a first approximation:

$$\frac{\delta\sigma_{rs}}{\bar{\sigma}} = -\frac{\delta h_r - \delta h_s}{\bar{h}_r - \bar{h}_s},$$

or, $$\sigma^2{}_{\sigma_{rs}} = \frac{\bar{\sigma}^2}{(\bar{h}_r - \bar{h}_s)^2}\{\sigma^2{}_{h_r} + \sigma^2{}_{h_s} - 2\,[\delta h_r \delta h_s]\}.$$

Thus

$$\sigma_{\sigma_{rs}} = \frac{\sigma}{\sqrt{2N}}\,\frac{1}{h_r - \bar{h}_s}\left\{\frac{2}{\bar{H}_r{}^2}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_r}{N}\right) + \frac{2}{\bar{H}_s{}^2}\frac{\bar{n}_s}{N}\left(1 - \frac{\bar{n}_s}{N}\right) - \frac{4}{\bar{H}_r\bar{H}_s}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right\}^{\frac{1}{2}} \qquad \ldots\ldots(45).$$

Now it is convenient to put this in a somewhat different form. Let two dichotomic ordinates be taken and suppose the range between them to represent 100 units of the character in question. It is convenient to call this unit in general a *notace*, while *mentace, sanitace, munditace* represent special units of intelligence, health, cleanliness, etc. Let these dichotomic ordinates be p and q, $p < q$.

Then $100 = \bar{\sigma}(\bar{h}_p - \bar{h}_q)$ and accordingly:

$$\sigma_{\sigma_{rs}} = \frac{100}{\sqrt{2N}}\,\frac{1}{(\bar{h}_p - \bar{h}_q)(\bar{h}_r - \bar{h}_s)}\left\{\frac{2}{\bar{H}_r{}^2}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_r}{N}\right) + \frac{2}{\bar{H}_s{}^2}\frac{\bar{n}_s}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right.$$
$$\left. - \frac{4}{\bar{H}_r\bar{H}_s}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right\}^{\frac{1}{2}} \qquad \ldots\ldots(46).$$

In actually calculating this value of the $\sigma_{\sigma_{rs}}$ as found from the rth and sth dichotomic ordinates we must of course substitute for $\bar{h}_p$, $\bar{h}_q$, $\bar{h}_r$, $\bar{h}_s$, $\bar{H}_p$, $\bar{H}_q$, $\bar{n}_r$, $\bar{n}_s$ the values provided by the sample in the usual way.

Now it will be clear that it is not easy to determine which pair of dichotomic ordinates will give the best, i.e. lowest, value of $\sigma_{\sigma_{rs}}$. For while it is disadvantageous to select s near to r owing to the presence of $\bar{h}_r - \bar{h}_s$ in the denominator, the increasing correlation of h_r and h_s as s approaches r makes the radical smaller and smaller.

Our previous investigations, however, indicate to us the neighbourhood where $\sigma_{\sigma_{rs}}$ is a minimum and we shall endeavour to choose our pair of grades out of the

---

* We suppose the subscript s to refer to the sth dichotomic ordinate numbered from the right-hand end of the distribution.

predetermined dichotomic ordinates to approach these as far as the material permits. As a rule $\bar{h}_s$ (and $\bar{h}_q$) will have the opposite sign to $\bar{h}_r$ (and $\bar{h}_p$), but it will not be possible to make them of equal magnitude.

It will probably be desirable to work with several pairs of dichotomic lines. In this case $\sigma_{\sigma_{r's'}}$ can be written down at once from the value of $\sigma_{\sigma_{rs}}$ by proper interchange of subscripts. But we need $[\delta\sigma_{rs}\delta\sigma_{r's'}]$. If we suppose the r' and s' dichotomic ordinates to fall on the median sides of r and s respectively, we easily find in the manner of earlier work in this paper:

$$[\delta\sigma_{rs}\delta\sigma_{r's'}] = \frac{100^2}{(\bar{h}_p - \bar{h}_q)^2(\bar{h}_r - \bar{h}_s)(\bar{h}_{r'} - \bar{h}_{s'})}\left\{\frac{1}{\bar{H}_r\bar{H}_{r'}}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_{r'}}{N}\right) - \frac{1}{\bar{H}_r\bar{H}_{s'}}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_{s'}}{N}\right)\right.$$
$$\left. - \frac{1}{\bar{H}_s\bar{H}_{r'}}\frac{\bar{n}_{r'}}{N}\left(1 - \frac{\bar{n}_s}{N}\right) + \frac{1}{\bar{H}_s\bar{H}_{s'}}\frac{\bar{n}_{s'}}{N}\left(1 - \frac{\bar{n}_s}{N}\right)\right\} \quad \ldots\ldots(47).$$

Further any range like $x_r - x_s$ is to be found from

$$x_r - x_s = 100(\bar{h}_r - \bar{h}_s)/(\bar{h}_p - \bar{h}_q) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48),$$

and the error in $x_r - x_s$ will arise from our using h_r, h_s, h_p, h_q instead of $\bar{h}_r, \bar{h}_s, \bar{h}_p, \bar{h}_q$. In other words:

$$\frac{\delta(x_r - x_s)}{x_r - x_s} = \frac{\delta h_r - \delta h_s}{\bar{h}_r - \bar{h}_s} - \frac{\delta h_p - \delta h_q}{\bar{h}_p - \bar{h}_q}$$
$$= \frac{\delta\sigma_{pq}}{\bar{\sigma}} - \frac{\delta\sigma_{rs}}{\bar{\sigma}},$$

or

$$\sigma^2_{(x_r - x_s)} = (\bar{h}_r - \bar{h}_s)^2\left\{\sigma^2_{\sigma_{pq}} + \sigma^2_{\sigma_{rs}} - 2[\delta\sigma_{pq}\delta\sigma_{rs}]\right\}.$$

The terms in the curled brackets are all known from equations (46) and (47) above, and we can thus determine the probable error of each portion $x_r - x_s$ of our notace scale as found from equation (48).

We can now consider the position of the mean. The distance of the mean from the rth dichotomic line is

$$\bar{x}_r = \frac{\bar{h}_r 100}{\bar{h}_p - \bar{h}_q},$$

and accordingly

$$\frac{\delta x_r}{\bar{x}_r} = \frac{\delta h_r}{\bar{h}_r} - \frac{\delta h_p - \delta h_q}{\bar{h}_p - \bar{h}_q},$$

or

$$\delta x_r = \bar{\sigma}\delta h_r + \bar{x}_r\delta\sigma_{pq}/\bar{\sigma}.$$

Thus

$$\sigma^2_{x_r} = \bar{\sigma}^2\sigma^2_{h_r} + \bar{x}_r^2\sigma^2_{\sigma_{pq}} + 2\bar{x}_r\bar{\sigma}[\delta h_r\delta\sigma_{pq}] \quad \ldots\ldots\ldots\ldots\ldots\ldots(49).$$

Now this may be written:

$$\sigma^2_{x_r} = \bar{\sigma}^2\left\{\sigma^2_{h_r} + \bar{x}_r^2\frac{\sigma^2_{\sigma_{pq}}}{\sigma^2} + 2\bar{h}_r\left[\delta h_r\frac{\delta\sigma_{pq}}{\sigma}\right]\right\} \quad \ldots\ldots\ldots\ldots(49\ bis).$$

Now

$$\sigma^2_{h_r} = \frac{1}{NH_r^2}\frac{\bar{n}_r}{N}\left(1 - \frac{\bar{n}_r}{N}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50).$$

$\sigma^2_{\sigma_{pq}}$ is given in equation (46) and we easily find

$$\left[\delta h_r \frac{\delta\sigma_{pq}}{\sigma}\right] = -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p\bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_p}{N}\right) - \frac{1}{\bar{H}_q\bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_q}{N}\right)\right\}$$

if $\bar{n}_r < \bar{n}_p$ and $< \bar{n}_q$,

$$= -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p\bar{H}_r}\frac{\bar{n}_p}{N}\left(1-\frac{\bar{n}_r}{N}\right) - \frac{1}{\bar{H}_q\bar{H}_r}\frac{\bar{n}_r}{N}\left(1-\frac{\bar{n}_q}{N}\right)\right\}$$

if $\bar{n}_r > \bar{n}_p$ and $< \bar{n}_q$,

$$= -\frac{1}{N}\frac{1}{\bar{h}_p - \bar{h}_q}\left\{\frac{1}{\bar{H}_p\bar{H}_r}\frac{\bar{n}_p}{N}\left(1-\frac{\bar{n}_r}{N}\right) - \frac{1}{\bar{H}_q\bar{H}_r}\frac{\bar{n}_q}{N}\left(1-\frac{\bar{n}_r}{N}\right)\right\}$$

if $\bar{n}_r > \bar{n}_p$ and $> \bar{n}_q$ ....(51).

$\sigma^2_{x_r}$ can always be found, and the error of our determination of the mean from the rth dichotomic line determined. But unless we are given some lengths on our variate line corresponding to dichotomic lines we can only choose (i) the p, q range to give σ, (ii) the rth dichotomic line to determine the mean. All other scale lines are determined by these, but the degree of accuracy of their determination depends on the fitting choice of p, q and r.

(8) As illustration let us determine the probable errors of the following distribution of Intelligence in Schoolboys on the assumption that the distribution may be taken as normal.

Quick Intelligent	279·5		$n_6/N =$ 1·000,000
		(x_5, h_5)	
Intelligent	788		$n_5/N =$ ·883,005
		(x_4, h_4)	
Slow Intelligent	771·5		$n_4/N =$ ·553,160
		(x_3, h_3)	
Slow	369		$n_3/N =$ ·230,222
		(x_2, h_2)	
Slow Dull	139·5		$n_2/N =$ ·075,764
		(x_1, h_1)	
Very Dull	41·5		$n_1/N =$ ·017,371
	2389		

$H_1 =$ ·042,944, $H_2 =$ ·142,542, $H_3 =$ ·303,847, $H_4 =$ ·395,391, $H_5 =$ ·196,487

$h_1 =$ 2·111,366, $h_2 =$ 1·434,202, $h_3 =$ ·737,957, $h_4 = -$ ·133,651, $h_5 = -$ 1·190,144

Taking the range of Intelligent to be 100 mentaces we have:

$$\sigma(h_5 - h_4) = 100, \quad \text{or} \quad \sigma = 94{\cdot}653 \text{ mentaces.}$$

Whence by (46), p. 129, putting $r = p$, $s = q$, we have

$$\sigma_{\sigma_{5,4}} = 2{\cdot}4594 \text{ mentaces,}$$

and the probable error of $\sigma_{5,4} = 1{\cdot}6589$ mentaces or is less than two mentaces, which is amply accurate for this type of work.

The position of the mean is at $h_4\sigma$ from the border of Intelligent and Slow Intelligent into the latter $= 12{\cdot}6504$ mentaces from the 4th dichotomic line.

We want the probable error of such a value. It equals $\dfrac{h_4 100}{h_5 - h_4} = x_4$, whence

$$\frac{\sigma^2_{x_4}}{\bar{x}_4^2} = \frac{\sigma^2_{h_4}}{\bar{h}_4^2} + \frac{1}{(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_5} + \sigma^2_{h_4} - 2\,[\delta h_5 \delta h_4]) - \frac{2}{\bar{h}_4(\bar{h}_5 - \bar{h}_4)}([\delta h_4 \delta h_5] - \sigma^2_{h_4})$$

$$= \frac{1}{\bar{h}_4^2(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_4}\bar{h}_5^2 + \sigma^2_{h_5}\bar{h}_4^2 - 2\bar{h}_4\bar{h}_5\,[\delta h_4 \delta h_5]).$$

Thus

$$\sigma_{x_4} = \frac{100}{(\bar{h}_5 - \bar{h}_4)^2}(\sigma^2_{h_4}\bar{h}_5^2 + \sigma^2_{h_5}\bar{h}_4^2 - 2\bar{h}_4\bar{h}_5\,[\delta h_4 \delta h_5])^{\frac{1}{2}} \quad \text{..........................(52)},$$

$$= \frac{1}{\sqrt{N}}\frac{100}{(\bar{h}_5 - \bar{h}_4)^2}\left\{\frac{\bar{h}_5^2}{\bar{H}_4^2}\frac{\bar{n}_4}{N}\left(1 - \frac{\bar{n}_4}{N}\right) + \frac{\bar{h}_4^2}{\bar{H}_5^2}\frac{\bar{n}_5}{N}\left(1 - \frac{\bar{n}_5}{N}\right) - \frac{2\bar{h}_4\bar{h}_5}{\bar{H}_4\bar{H}_5}\frac{\bar{n}_4}{N}\left(1 - \frac{\bar{n}_5}{N}\right)\right\}^{\frac{1}{2}}$$

$$\text{......(52 \textit{bis})}.$$

Applying this to our case we have

$$\sigma_{x_4} = 2{\cdot}759{,}657,$$

or the probable error of the position of the mean from the 4th dichotomic line is 1·8614 mentaces. Symmetry of the result, as we might also anticipate from *a priori* reasons, shows us that this is also the probable error of the distance of the mean from the 5th dichotomic line.

We may conclude this investigation by considering the error in the position of the 2nd dichotomic line, or that which divides the Slow and Slow Dull categories.

We have $$\bar{x}_2 = \frac{\bar{h}_2 100}{\bar{h}_5 - \bar{h}_4} = 135{\cdot}7512 \text{ mentaces},$$

and using (49) and (51) obtain

$$\sigma_{x_2} = 4{\cdot}4241,$$

or the probable error of x_2 is 2·9840 mentaces.

Thus in terms of our Intelligent Range as 100 mentaces we find in mentaces

$$\begin{aligned} x_2 &= 135{\cdot}7512 \pm 2{\cdot}9840, \\ x_4 &= -12{\cdot}6504 \pm 1{\cdot}8614, \\ x_5 &= -112{\cdot}6504 \pm 1{\cdot}8614, \\ \sigma_{4,5} &= 94{\cdot}653 \pm 1{\cdot}659. \end{aligned}$$

Any other dichotomic ordinate can have its probable error determined in like manner.

It is clear that the positions of the dichotomic lines can be found with probable errors of 2 to 3 mentaces. The whole range of intelligence being about 600 mentaces we see that with 2389 individuals, we can determine the positions of the category divisions within ranges of about 5 to 8 mentaces in a total range of variation of 600 mentaces.

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CONTENTS

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Index. Table of Contents to Vol. L.

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ON THE SESAMOIDS OF THE KNEE-JOINT.

BY KARL PEARSON AND ADELAIDE G. DAVIN.

PART I. MAN.

CONTENTS

(1) *Introductory.*

During the course of recent work* on the femur of the primates undertaken in the Biometric Laboratory, much material accumulated bearing on the sesamoids of the knee-joint. It was found impossible to incorporate this in that work itself, and as it was very largely due to a third investigator it is only fitting that the paper should be not only published separately, but with a different authorship. Some of those acquainted with the literature of the subject may consider that all that is necessary to be known on such an apparently small topic has been fully dealt with already in the voluminous memoirs of Gruber† and Pfitzner‡. Both of these memoirs have their individual excellences, but we still venture to think that they have no more exhausted the subject than we ourselves are likely to do. Indeed merely a résumé of their work in a form easily accessible to English readers would be of service; but we hope to have added at least something of value to it, having trusted to what we believe is the only safe method of research, namely the first-hand investigation of the phenomena themselves before starting any study of the observations and opinions of others§.

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* Pearson and Bell, *A Study of the Longbones of the English Skeleton.* Part II, The Femur of Man, with special reference to the Primate Femora, see p. 512. *Drapers' Company Research Memoirs*, Cambridge University Press.

† For the full title of Gruber's classical memoir (hereinafter cited merely as Gruber) see our p. 156.

‡ For the full title of Pfitzner's memoir (hereinafter cited merely as Pfitzner) see our p. 163.

§ Some time may be lost by postponing the study of the literature of a topic until at least a late stage of one's own researches, but on the other hand a preliminary study will too often check one's own individual line of research either by emphasising others' methods of approach, as if they were the sole roads to truth, or by crowding the mind with a mass of other people's details, often best described as debris.

Before we can properly talk about our subject it is really needful to have some terminology. The sesamoid may be a true ossification of hyaline cartilage, or it may be a thickening or nodule of cartilage without ossification. In the young it is open to us to consider the latter as a stage of growth towards the former. In the adult this is not possible. We can, of course, assert, as many anatomists have done, that all sesamoids are produced by intensive stress acting on cartilage, but we shall then have to account for the absence of sesamoids in regions where such stress undoubtedly exists. Or we may look upon the non-ossified nodule as an attempted sesamoid, or better still as a vestige or trace in the species of where a sesamoid once has been. Whatever account of the matter we choose to accept, it is needful to distinguish between the two cases. Pfitzner proposed to call the bone a *sesamum* and the nodule a *sesamoid.* This upsets the historic origin of the word, for the Greeks most probably knew only the *sesamum* and called it a *sesamoid,* and the name sesamoid remained as a name for the bone through many centuries. We might speak of sesamoid bone and sesamoid nodule or cartilaginous sesamoid, but the terms are cumbersome. We shall accordingly reserve the term sesamoid or *orthosesamoid* for a true bone and adopt an independent term for the thickened but unossified element of cartilage, calling it a *hemisesamoid.* Further we shall term all pathological ossifications, exostoses, or again, broken off and not reunited portions of bone occurring in or near the positions of sesamoids *pseudosesamoids.*

In the knee-joint itself we have come across no less than ten sesamoids, some of which are, however, more frequent or better established than others. To these sesamoids we have given the following distinctive names:

We retain the names of lateral *fabella* and mesial *fabella* for the sesamoids found in the tendons of *M. gastrocnemius* above the lateral and mesial condyles respectively.

We introduce the term *cyamella** for the sesamoid which is to be found in the tendon of *M. popliteus.* It may be termed the *lateral cyamella* to distinguish it from a *mesial cyamella* which Pfitzner asserts he observed in two cats†. Gruber, notwithstanding his wide knowledge, seems to us to have confused the lateral *cyamella* in man with an unusual position of the lateral *fabella.* The existence in the cat of both lateral *fabella* and lateral *cyamella* ought to have warned him that both might appear in man.

Lastly a number of sesamoids occur in the borders of the semi-lunar cartilages. We term these *lunulae*‡. They may occur in six positions which we describe as follows: antero-lateral, antero-sagittal, antero-mesial, postero-lateral, postero-sagittal, and postero-mesial. Of these the sagittal are doubtful and may be anomalous in all species. Thus we have two *fabellae,* two *cyamellae* and possibly six *lunulae* as sesamoids of the knee-joint. In most cases some only of these will be present; in certain species some will always be present as true sesamoids, or will be present as

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* *Cyamella*, a small bean, the bean-stone, an ornament worn by Roman women.

† Pfitzner, S. 584, without identification of an association with any particular muscle.

‡ *Lunula*, a little moon, a crescent, or crescent form of ornament worn by Roman women.

anomalies. In other species some will only appear as hemisesamoids; while in a third species, if they have been reported at all, it is probable that they were really pseudosesamoids.

Now to obtain a large number of knee-joints for dissection in the case of a particular species is practically impossible. In the case of dry material and skeletons the articulator may, perhaps generally does, preserve the *fabellae*, far more rarely the *cyamellae* or *lunulae**. Thus the actual presence of *cyamella* or *lunula* conveys definite information, their absence practically conveys nothing. It is extremely hard to demonstrate an invariable absence, especially in the case of an anomaly which may have a very small percentage of occurrences like the lateral *cyamella* in man. When one anatomist asserts the existence of a sesamoid which nine others deny, there is no grave difficulty in reconciling their statements, if the anomaly be present in only a small percentage of cases, and none of these anatomists have had the chance of examining more than a few individual cases of the species in question. This is, we think, the quite adequate explanation of the contradictions we find in the early history of the subject, and forms the chief difficulty in applying our analysis of the sesamoids of the knee-joint to the problems of evolution.

These problems are of the following character. Suppose we can classify those families in which certain sesamoids are (i) invariably present in the knee-joint, (ii) invariably absent, (iii) exist only as anomalies of a given order of rarity, will not such classifications throw light on evolutionary order? For example if we confine our attention to the primates, there are families in which the *fabellae* are both invariably present, others in which they are both invariably absent, others in which both are anomalous, and still another group in which the *fabella lateralis* is a true or orthosesamoid when it appears, and the *fabella mesialis* a hemisesamoid. Again the *cyamella lateralis* is invariable in certain monkeys, never appears in the gibbon, but is constant in the orang, and a rare anomaly in man. A similar series of statements can be made with regard to the presence or absence of *lunulae* in the primates. May it not be possible to throw light on primate evolution from such considerations?

The problems with regard to sesamoids are not only rendered difficult of solution by reason of the amount of material requisite for adequate classification, but by the longstanding existence of opinions and dogmas with regard to them.

The definition of the medical student that a sesamoid is any bone which it is unnecessary to study for examination purposes is, of course, an echo of the treatment of sesamoids in the current anatomical text-books, and that treatment has its historic evolution—broken only and for a short period by the discovery of the *fabellae* in man by Vesalius—from Galen to the present day; the text-book anatomists from his day through the centuries to our own have echoed his phrase as to sesamoids—that of those bones "which are termed sesamoids from their

---

* In 15 dog skeletons in the Biometric Laboratory, the *fabellae* are almost invariable, the *cyamella* has not been preserved in a single case.

likeness to sesame" *non est necessarium hoc opere pertractare**. If only Galen had told us which sesamoids he knew, what sesame was†, and how and why they were like the seeds of it, our task would have been simplified! The comparison may have been obvious to the Greek anatomists because the sesame seed that they were referring to was familiar to their readers and because they had special sesamoid bones in mind, it is far less obvious nineteen centuries later.

But we are not exact in stating that sesamoids have as a rule been dismissed in the language of Galen in the text-books. The opinion that it was unneedful to discuss them has been frequently accompanied by the dogma that sesamoids are the product of intensive stress acting upon cartilage. This dogma appeared to receive confirmation from the undoubted existence of hemisesamoids in regions where orthosesamoids are usually or on occasion discoverable. A hemisesamoid was an orthosesamoid in the making. This dogma appealed to that tendency in anatomy —largely the product of pathological study—which attributes innumerable characters of individual bones to specific use. The bone was not adapted by evolution to its use, but use was the source of its characters, and indeed if these were observed in the foetus or at birth, they were in themselves evidence of the inheritance of acquired characters. The *fossa hypotrochanterica* and the crural trough were the product of muscular action on the bone, even if they were to be found more frequently in female than male femora and were discoverable in infants and non-robust or even diseased bones. The "intensive stress" origin of sesamoids had indeed been questioned by Dr Robert Nesbit before the middle of the 18th century. He wrote as follows in his *Human Osteogeny*, London, 1736, p. 136:

"The only parts of a foetal skeleton, which remain to be taken notice of, are the *ossa sesamoidea* which all the writers, I have met with on this subject, have wholly passed over unobserved. The number of them in *foetuses* are, as in adults, very different in different subjects. Those, which are the most constantly found, are two in the bottom of the foot, fixt in the ligament of the articulation of the first bone of each great toe, with its *os metatarsi*. In all *foetuses*, from three months after conception to birth, the places of these *ossa sesamoidea* are always filled with cartilages of nearly the same shape those *sesamoide* bones usually have, when they are arrived at perfect maturity, *Plate* IV [actually Plate VI] *fig.* 18 *a. a. fig.* 19 *a. a.* In one subject at birth I found, in each of the *sesamoide* bones of one foot, a very small point of ossification.

"In like manner those *ossa sesamoidea* which are sometimes found at the beginning of the *musculi gastrocnemii*, are to be seen in *foetuses*.

"By the descriptions I have now given of all the different parts of foetal skeletons, it manifestly appears, that there is not one single bone, except the teeth, or one *epiphysis* in an adult skeleton, which is not to be found in a full grown *foetus*, or in its place a cartilage of nearly the same shape: consequently, the account cannot be right, which the ingenious professor *Monro* gives of the *ossa sesamoidea* being

---

* *Liber de ossibus Galeni Librorum Pars Quinta*, Basileae, 1538, p. 727.

† See, however, below, p. 141.

'nothing else than the ligaments of the articulations, or the firm tendons of strong 'muscles, or both, become bony by the violent compression they suffer in the 'situation they are.'" (*Anatomy of the Bones*, Edinburgh, 1726, p. 337.)

It is true that this statement of Nesbit's, notwithstanding its definiteness and perspicacity, has been questioned somewhat cavalierly by Gruber[*]. He says that for many years he examined the cadavers of embryos in last month, of new-born children in considerable number and of several children one to seven years old, and never found in the tendons of *M. gastrocnemius* a *cartilago hyalina* or *fibro-cartilago*. He does not state the exact numbers and appears to hold that Nesbit asserted the existence of orthosesamoids, whereas Nesbit as we read him is speaking of prefigurations. Yet Gruber did himself find the hyaline cartilage elements in the tendons of puppies at birth where the *fabellae* later develop, and notes that the lateral ossifies first (at about six months)[†]. He does not state how many embryos and children he examined, and it is clear that as at least 90 °/₀ of adults exhibit no *fabellae*, it is quite conceivable that Nesbit may have found the prefigurations and Gruber not. But the fact that prefigurations are found in puppies and kittens, and the additional fact that not all sesamoids are associated with intensive stress while many regions of intensive stress are not associated with sesamoids[‡] should be enough to dismiss this hypothesis, were it not that its supporters can appeal, as has been done in the case of femoral facets, to the principle that acquired characters are inherited, a haven ready for the supporters of production by use, and one where the difficulty of demonstrating a negative—other than by Occam's Razor methods—leaves them in comparative security.

Perhaps the most effective ejectant is the appeal, such as we hope to make in this paper, to the facts of the evolution of the sesamoids. Pfitzner, who was a strong opponent of the intensive stress hypothesis, not only cites Nesbit's results, but cites the case of the *Marsupialia*, and figures the Wombat, of which he writes:

> Wenn man annehmen müsste dass Gebilde von so reicher Formenentfaltung wie diese Epiperone (so möchte ich sie nennen) beim Wombat jederzeit als "Product entzündlicher Reizung" oder dergl. neu entstehen könnten, so würde man bald jede wissenschaftliche Bearbeitung des Skeletsystems als ein aussichtloses Unterfangen erklären müssen[§].

The reader who wishes to appreciate the full force of Pfitzner's argument should examine our Plate V, Fig. 13. It seems unanswerable as far as the Marsupials are concerned. Can we look for any evolutionary links between Pfitzner's "Epiperone," the "parafibula" of the Marsupials, and the sesamoids of the knee-joint of higher types?

For Pfitzner the source of the sesamoids is the same as that of any other bone of the skeleton, i.e. the force of heredity[||]. He distinguishes as we have seen between

---

[*] Gruber, p. 13.

[†] Gruber, p. 57. We have ourselves found them, as well as the *cyamella lateralis*, prefigured in puppies at birth, and the two *fabellae* and the *cyamella* in kittens at birth.

[‡] Why should man and the greater anthropoids have lost their *fabellae* while they are universal in the lesser apes? The "stress" has hardly disappeared.

[§] Pfitzner, S. 586.

[||] Pfitzner, S. 535 and S. 563.

orthosesamoids ("sesambeine") and hemisesamoids ("sesamoide"); the latter he considers develop in fibrous cartilage ("faseriges Gewebe") and do not ossify[*]. Hemisesamoids are abortive sesamoids; we may term them vestiges of past phases of evolutionary development. Pfitzner does not, however, attempt to follow up the view here suggested by the examination of sesamoids in a large number of types other than man. Man is of course the subject of his paper, and the sesamoids of the knee-joint with which we are at present interested form only one element in his monograph.

The keynote, however, of Pfitzner's suggestion that hemisesamoids are not necessarily orthosesamoids in the making, but may be the evolutionary debris of what have been orthosesamoids, vestiges of ossificatory centres where no ossification now sets in, is well worth bearing in mind. It is not incompatible with sesamoids appearing in regions of intensive stress, for it is possible that in such regions the sesamoid or its precursor would be of special value to the organism.

(2) *Origin of the term Sesamoid.*

The term sesamoid is not to be found in any author previous to Galen[†], but he uses it in the sense of a word familiar to his readers. It is probable that Galen only knew the sesamoids of hand and foot. He does not directly refer to those of the knee-joint. Eustachius in 1564 argued that Galen must have dissected the human cadaver and not those of monkeys[‡], because he does not refer to the *fabellae*, but as Galen clearly considered sesamoids of no great importance, he may simply have omitted reference to them. If he dissected carefully a considerable number of human bodies, it is singular that he should not have met with the *fabellae* occasionally, but the same criticism applies to all anatomical writers up to Vesalius (1555).

We have been able to discover four passages, three in the accepted, one in the spurious books of Galen, referring to sesamoids. We cite from the Latin translations.

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[*] This does not seem in accordance with Gruber's view (see our pp. 157 and 175), that sesamoids are invariably prefigured by hyaline and not fibrous cartilage.

[†] Galen was born at Pergamum A.D. 131. He went to Rome and was known to Marcus Aurelius. He died aged 70.

[‡] "Admiratione denique obstupui, cur anatomiae instauratores, in secunda libri de hominis fabrica editione, ubi administrationem musculorum pedis docent, duo ossicula magnitudine ciceris describant, quae supra utrumque femoris tuberculum reperiuntur, qua nimirum primi duo pedem moventes oriuntur musculi: nam & si ea in simijs & in canibus semper occurrant, in hominibus tamen adeo raro inueniuntur, ut non modo (quod posteriores addiderunt) saepius alterum, nunc internum, nunc externum deficiat, uerum etiam ut plurimum utrumq: ipsorum desit, miraculum enim existimabo si duo simul occurrent, & non mediocri diligentiae ascribam, si post multorum cadaverum sectionem alterum illorum quis inveniat. ideo que tantum abest, ut aequum sit eiusmodi ossicula inter hominis partes numerare, ut etiam liceat suspicari ea a Galeno consulto fuisse pretermissa, indeque sumere argumentum tantum anatomicae artis praeceptorem homines non simias nobis descripsisse. Addo sine aliquo fructu laborare illos, qui commentitias quasdam utilitates horum ossiculorum confingunt: neque enim natura, quae provida est & omnium opportunitatum moderatrix, veretur ne in cruris directione principia paulo ante memoratum musculorum comprimantur & atterantur. Si enim hic esset eius scopus in illis conformandis multo magis in homine, quam in simia & cane reperirentur, quando horum uterque longe minus, quam homo tibiam ambulando extendit."

See Bartholomaeus Eustachius, *Opuscula Anatomica*, Venice, 1564, p. 180.

The first reference is in the *Liber de ossibus*, and contains the passage to which we have already referred:

> Quod si quod ossiculum alibi utpote in corde, in naso, in gutture, itemque in aliquo ex digitis, ut quae a sesami similitudine sesamoide vocant, aliudque ejus generis deprehenditur, non est necessarium hoc opere pertractare*.

It will be seen that while the name sesamoid bone is of very early origin, the existence of the *fabellae*, the first sesamoids of the knee-joint to be talked about, if known was not recorded until 1555 and then not accurately, not indeed before Eustachius.

The second passage occurs in the *De usu partium corporis humani*† and runs as follows:

> Nonum aliquis potuit hoc carpi os numerare, sed non est numeratum ab anatomicis, sicut nec aliquod eorum, quae *σησαμοειδῆ*, a sesami leguminis minuti specie, vocantur, quae multis manuum et pedum articulis natura ex abundanti velut auctarium circumponet securitatis gratia.

Another reference occurs in the *De usu partium*, Lib. III, Cap. viii, where Galen mentions the two metatarsial phalangeal sesamoids of the great toe. It runs as follows:

> Quinetiam ei parti plantae, que magno digito praeposita est, duo [ossa *or* sesamoidea?] videntur subjecta esse inferiori regione, veluti sustentacula quaedam & stabilamenta, ut prius os magni digiti praedictae plantae parti, omnino jam in terram defixae conjugatur, undique, opinor, huic parti pedis natura suppeditante securitatem, ceu multum laboraturae propter praepositam sibi concavitatem & velut fornicem ossium. [Basel edition, 1561, *Prima Classis*, p. 259.]

Neither the Latin translation, nor the original Greek [Basel edition, 1538, p. 397], use the term sesamoid. But the word *ossa* seems omitted. It is supplied in the French translation (*De l'usage des parties du corps humain*, Paris, 1659, p. 120).

Now it is probable that these were the special sesamoid bones of the Greek anatomists, and they are bean-like rather than irregular or even lens-shaped. It is difficult to find a resemblance in them to the seed of the existing forms of sesame, as the reader will see by examining our Plate I, Fig. 2.

It will be noted that these passages from the real writings of Galen make the comparison with the sesame directly and not with its seed. It is otherwise in our third passage taken from the section *De anatomia oris* of the *Liber de anatomia virorum* ascribed to Galen‡:

> Veniunt autem duo musculi ei ab additamentis sesaminis, id est quibusdam parvis ossiculis ad modum seminis sesami, et sunt supplementa spondylorum colli.

Thus in the earliest traces we have of sesamoids the name seems to have stood for any small bone of no recognised importance, which in some way was like sesame, a small species of pulse. The earlier references may, the later does indicate com-

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* *Galeni Librorum Pars Quinta*, Basileae, 1538, p. 727. We can hardly expect that Galen would have anticipated the great evolutionary importance which the "sesamoidea gutturis" would attain to in later ages!

† Liber II, Cap. xii, *Galeni Librorum Prima Classis*, Basileae, 1561, p. 248.

‡ Galen's *Works*, Basileae, 1561, p. 49 c.

parison to the seed of this plant. These passages have been followed by later anatomists, and in each case we are told that certain bones are termed sesamoids because they resemble the seed of the sesame. What they themselves meant by "sesame," or what the Greeks understood by sesame in Galen's time, seems rarely to have entered the minds of these writers! Through the courtesy of the Director of Kew Gardens we have received samples of *Sesamum indicum* or *orientale,* the sesame as at present known in the East. It is almost impossible to believe that the Greek anatomists, who at least were familiar with the sesamoids "ex digitis," could have called these after their resemblance to the seed of sesame as now known*.

* The seed is crescent rather than lentil-shaped, and extremely small; only its coloration is in its favour. The most obvious sesamoids for the Greeks were the sesamoids of the big toe (see our Plate I, Fig. 2) and there is nothing really comparable between the seed and these bones. In size and shape, if less in coloration, they are remarkably like the seeds of *Ricinus.* We ventured to put our difficulties before the Director of the Royal Gardens, Kew, and he has most kindly sent us the following memorandum by Dr O. Stapf. It seems more than ever probable from this that Galen in his account of the plants was not speaking from personal experience, but following Theophrastus and Dioscorides. If the seed of the sesame was known to Galen he appears to have had rather a poor imagination for comparisons. Perhaps he was familiar with a variety having a much larger seed. The comparison would be excellent had the sesamoids first observed been the *lunulae* of the squirrel.

Dr Stapf's *Memorandum on Sesamum and Sesamoids.*

1. I have no doubt that the *SESAME* of Galen was *Sesamum indicum.* The Greeks knew it as far back as the days of Herodotus, and the Greeks of Asia Minor were probably quite familiar with it.

2. The history of the plant or drug mentioned by Greek writers under the name Σησαμοειδές has been shrouded in mystery from the earliest times. We meet with it first in one of the few fragments which have been preserved to us from the writings of Diocles of Karystos, the most prominent physician of the fourth century B.C., the "sectator Hippocratis quem Athenienses iuniorem Hippocratem vocarunt" (see Wellmann in *Festgabe f. Franz Susemihl*, 1898, S. 23). Διοκλῆς, says Erotianos in his Glossary of Hippocrates, οὕτω φησὶ καλεῖσθαι τὸν ἐν 'Αντικύρᾳ ἐλλέβορον—ἕτεροι δὲ πόαν τινὰ ἕτεραν (Wellmann, *l.c.* 26). Thus it seems the people of Anticyra, a town in Phokis reputed for its Hellebore cures, used the word as a synonym of ἐλλέβορος, whilst others applied it to another herb. Theophrastus (about 370—285 B.C.) somewhat later speaks of the fruit of the sesamoid Hellebore being administered by the people of Anticyra, adding in the way of explanation that the fruit is similar to sesame (οἱ ἐν 'Αντικύρᾳ τοῦ σησαμώδους ἐλλεβόρου διδόασιν ὅτι ὁ καρπὸς ὅμοιος σησάμῳ, ed. Loeb, *Class. Libr.* with transl. by A. Hort, ii. 288); whilst in another place (*l.c.* 260) he says that the fruit of the Hellebore which at Anticyra is used as a purge contains the well-known σησαμώδης. Here then the conception of "sesamoides" as a part of Hellebore is quite unequivocal.

Pliny refers in two places to Sesamoides; but his account is confused. In the 22nd book, chap. 25, he introduces it after *Sesame* and contrasts it with it. Sesamoides, he says, has its name because it resembles Sesame, but its grain is bitter, its leaves are smaller and it grows in gravelly ground; but there is also another kind which grows in Anticyra and is called Anticyron by some and this has leaves like those of groundsel. The seeds of both are taken as a purgative and the latter kind has white Hellebore root added to enhance its action. He once more reverts to "Sesamoides" in the 25th book, chap. 5, where he says that the people of the island of Anticyra add Sesamoides to their Hellebore (the black Hellebore is meant) to render it safe to take. There are thus two kinds of Sesamoides indicated, just as in Diocles, the Anticyrian and another. The Anticyrian, however, is not any longer a Hellebore or part of a Hellebore, but a different plant with leaves like those of Groundsel, whilst the other is also a distinct plant but of a different kind. If we now turn to Dioscorides who has largely drawn on the same sources as Pliny, we find the two "Sesamoides" still more sharply divided as μέγας (lib. iv, chap. 149) and μικρόν (lib. iv, chap. 163; Wellmann, *Dioscor.* pp. 292 and 309). It is the greater Sesamoides that stands for Hellebore in Anticyra and is added to the white Hellebore in preparing the purgative. It is described as resembling Groundsel (ἠριγέρων) or Peganum, and as having large leaves, white flowers, a slender inefficacious root and bitter seeds similar to those of sesamum. The alternative Groundsel or Peganum is puzzling. It can clearly not be similar to both, but from Wellmann's annotations it appears

The natural thing appears to be to consult Galen himself as to what was the nature of the plant he termed sesame*. Galen in several passages refers to the sesame or to plants of a sesamoid nature. We will give here the references we have been able to discover. In his *De simplicum medicamentorum facultatibus*† we read:

> *Sesamon.* Sesamon non panem in se continet viscosum et pingue; quare emplasticum est & emolliens ac modice calidum. Ejusdem facultatis est, quod ex eo conficitur oleum; et herbae quoque decoctum similem vim obtinet.

This is immediately followed by a description of the sesamoid plants‡.

> *Sesamoides.* Sesamoides magnum, quod & anticyricos helleboros appellatum est, propterea videlicet quia semen eius ut helleborus purget; hoc & reliqua facultate helleboro simile est. Nam & in abstergendo, excalfaciendo, desiccandoque, similem illi vim obtinet.
>
> *Sesamoides album.* Sesamoides albi semen nonnullum item acrem qualitatem continet; sed multum amarum est; excalfacit itaque, rumpit, extergit.

In Liber I, Cap. XXX, "De Sesamis et Erysimo" of the *De alimentorum facultatibus Libri tres*:

> Sesami semen pingue est ideoque repositum celerrime fit oleosum; quamobrem eos, qui ipso

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that the words *ἠριγέρωντι ἢ* are missing in Oribasius' manuscript of Dioscorides, whilst the codices of Constantinople, Naples and Munich lack the words *ἢ πηγάνῳ*. The latter reading is supported by the figure in the Constantinopolitan Codex, which represents faithfully a specimen of *Senecio lividus.* This figure is very like that of *ἠριγέρων* in the same code, the latter being unmistakably *Senecio vulgaris.* Compared with this *S. lividus* has actually larger leaves, whilst it shares with it the white pappus (flower) and slender inefficacious root; on the other hand it is difficult to see where the resemblance of the "seeds" (achenes) with Sesame seeds comes in. Nor are the achenes of either *Senecio* likely to be physiologically active.

The lesser Sesamoides (also called White Sesamoides or Wild Sesam) is said to have stems a span high, leaves like those of *κορωνόπους*, but rougher and smaller and to bear on the tips of the stems heads of purplish flowers with a white centre and in them bitter yellowish seeds like sesam; the root is slender and the plant grows in rough places. Of this too there is a fair figure in the Constantinopolitan Codex which can be identified with *Dorycnium hirsutum*, whilst *κορωνόπους* with which it is compared is from the table in the same Codex *Lotus arnithopodioides.* The representation of the leaves of the lesser *Sesamoides* and the *Coronopus* in the figures of the Code is indeed very similar. The white "centre" may mean the pale pods, whilst the seeds might very roughly be compared with those of sesame. As in the case of *ἠριγέρων* it is difficult to understand how small doses of the seeds, mixed with honey, as prescribed, could have been very effective. At the same time it is true that *Dorycnium hirsutum* was at one time in use as Herba Loti anti-haemorrhoidalis. Thus the plant was not unknown in the old herbalists' shops; but it long ago disappeared from the market.

To summarize briefly, the name *Sesamoides* designated early different plants or drugs. One of them was identical with the Hellebore of Anticyra or at least with its fruit. The others were entirely different species, in Pliny's and Dioscorides' times *Senecio lividus* and *Dorycnium hirsutum.* As there is no doubt that the (black) Hellebore of Anticyra was *Helleborus cyclophyllus*, the comparison of its fruit (follicles) with that of Sesame was not very far fetched, however little the two plants have in common in other respects.

To me the question of *Sesamoides*, whether magnum or parvum, seems to be irrelevant as far as the problem of the sesamoid bones is concerned. The sesamoid bones of Galen were no doubt in shape similar to the seeds of *Sesamum indicum*, hence "sesamoid," quâ forma, whilst the latter were similar to the seeds of *Ricinus* in yielding a fat oil, that is "sesamoid," quâ usu. Their relation to the sesamoid bones on one side and the plants called Sesamoides on the other is therefore merely that of "tertium comparationis."

* Pfitzner, S. 520, appears to confuse the sesame with the two sesamoides.

† Basileae, 1561, Liber VIII, p. 122.

‡ *Opera*, Secunda Classis, Basileae, 1561, p. 29, Greek Edition, Basileae, 1538, p. 317.

> vescuntur celeriter implet; stomachumque subvertit, ac tarde concoquitur, pingueque corpori praebet alimentum. Liquet ergo, quod ventriculi partibus vigorem ac robur addere nequit, quemadmodum neque aliud quodvis pingue. Est autem crassi succi, ideoque non propere pervadit. Ipso aut solo non admodum vescuntur, sed cum melle crudo quas vulgo *σισμίδας* vocant effingentes. Panibus etiam inspergitur[*]. Porro, quemadmodum milio panicum (quod etiam *μελίνης* diximus appellari) adsimile quidem quodammodo est, verum undequaque deterius; ad eundem modum & sesamo erysimum corporis substantia quodam pacto est affine, sed in cibo est insuavius, corporique alimentum parcius exhibit, praedictoque omnino est deterius. Caeterum utrumque temperamento est calidum, ob eamque causam sitem etiam excitat.

A further reference to Sesamoides occurs in Galen's commentary on Hippocrates' book *De victus ratione in morbis acutis*[†].

> Sesamoides superiorem ventrem purgat sesquidrachmae pondere in oxymelete tritum et potui datum. Miscetur autem veratris ita ut tertiam efficiat partem. Nempe id hoc pacto minus suffocat.

Lastly in Galen's book *De substitutis medecinis*[‡] we read that the substitute:

> "pro sesamo" is *Linisemen* (λινόσπορος), i.e. linseed, "pro sesamoide" is *Amaranthi expressio* (*ἀμαράντου πίεσμα*), "pro Linisemine" is *Fabae medulla.*

There can be small doubt that for Galen sesame was an emollient, *Sesamoides magnum* a purgative, and *Sesamoides album* an emetic. The plants and their seeds have so little real resemblance that it is not impossible that the oil-giving property was the source of their common name[§].

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[*] Sesame cakes or puddings were known to Xenophon and Aristophanes.

[†] *Opera*, Basileae, 1561, Septima Classis, p. 296, com. III, cap. 123.

[‡] *Opera*, Basileae, 1561, Quinta Classis, pp. 173—4. It is clear that Galen distinguished linseed definitely from sesame, although he notes the emollient properties of sesame. See also the *De alimentorum facultatibus*, Liber I, cap. xxxii.

[§] On p. 295 of the *Rariorum Plantarum Historia* 1601 of Carolus Clusius are given cuts of *Sesamoides magnum Salmantic.* and *Sesamoides parvum Salmantic.*, which indicate what was the type of plant denoted by these names in the latter half of the sixteenth century. Dr Stapf has kindly examined Clusius' figures for us. He says they are excellent figures, that the *Sesamoides magnum* is *Silene Otites*, Sm. and that *Astrocarpus Clusii*, J. Gay, is drawn for *Sesamoides parvum.* He has further found figures of the two *Sesamoides* in the *Codex Vindobonensis* (± a. 500 A.D.). *Senecio lividus* is figured as *S. magnum* and *Dorycnium hirsutum* as *S. minus.* Thus in Galen's time, in 500 A.D. and at the end of the sixteenth century we find different plants named as Sesamoides, and neither of the sets of figures here referred to are in the least helpful in interpreting what Galen understood by these plants, nor do the seeds of these plants throw any additional light on the adoption of the word by the anatomists—they are not more like a sesamoid bone than sesame itself. Dr Stapf throws light on the matter in the following note: "I have also gathered much information with respect to the attempts made by the writers of the renaissance to explain the two Sesamoides of Dioscorides, and particularly their reference to species of *Reseda.* I thought there might have been some tradition linking up their interpretation with the classic writers. So far as the Arabic literature is available to me there does not seem to be any such connection, but I have not seen Ibn Baithar who I understand quotes from an Arab author. As far as I can see the interpretations of the herbalists were nothing but more or less ingenious guesses. The *Reseda* theory can be traced back to Luca Ghini who laid out and was in charge of the Botanic Garden at Pisa from 1544—1555. Amatus Lusitanicus (1554) also speaks of having been told by a nobleman of Padua that the *Sesamoides maius* of Dioscorides has been found in Italy, but does not say what it is. Amatus Lusitanicus was a Salamanca student, and it is just possible that he was instrumental in starting the *Reseda* theory in his University where Clusius seems to have picked it up subsequently. There was no prominent botanist at Salamanca at the time of Clusius' short visit (1565)—we know in fact that he had a very low opinion of the 'viri docti' of the Spain which he knew—and no importance can be attached to his interpretation of the two Sesamoides."

There appears to be some probability that *Sesamoides magnum* was *Ricinus communis**, the castor-oil plant, although Candolle identifies *Ricinus* with *Sesamoides parvum* or *album*. Sprengel states that *Sesamoides* was a kind of *Reseda*. The seeds of *Reseda*, which we have examined, are hardly comparable with sesamoid bones, but those of the castor-oil plant might well pass muster. This would suggest that the bones were called from a sesamoid plant and not from the sesame itself, did not Galen so clearly distinguish between *sesamum* and *sesamoides* and associate the bones with the former.

Both plants were largely grown in Egypt for the sake of their oils (Dioscorides, Sprengel, Lib. I, Cap. xxviii and Lib. II, Cap. cxxi), but we are informed that the root is Arabic and not Egyptian. The familiar words "Open Sesame" of the *Arabian Nights* might certainly suggest that the root SM SM was connected with a purgative. Cameron's *Arabic-English Vocabulary* gives *simsim* as sesame, millet. We have already seen that Galen associates its method of use with *panicum*. Owing to the courtesy of the Director of the Royal Gardens, Kew, we have been able to examine seeds of four kinds of millet†, of which three might pass for reasonable comparatives, but the fourth, great millet or guinea corn, is less plausible. We cannot discover, however, that *simsim* was used in early Arabian for millet. On the contrary, Socrates Spiro (*An Arabic-English Vocabulary*, London, 1895) renders it as *coriander* seed, probably following Freytag's *Lexicon Arabico-Latinum*, Halle, 1830, who gives *simsim* = *semen coriander*.

Thus far these references might only signify a modern use of sesamum for coriander, but on consulting Lane's great *Arabic Lexicon* (p. 1420) we find that the sense of coriander is very ancient indeed. It occurs in Fairuzabadi, 1329–1414, and in Ibn Sīda's *Al-Qāmūs* of 1066. The passage from Lane cited below does not throw very much light on the matter, but indicates that either the Arabian writers copied the Greeks or the Greeks the Arabians, which alternative considering the origin of the word is, perhaps, the more likely.

["Sesame; sesamum orientale" of Linn.; applied in the present day to the plant and its grain;] "a well-known grain"; (Mṣb;) it is called in Pers. كُنْجُدْ; (MA, KL;) i.q. جُلْجُلَانٌ, (M, Ḳ,) said by AḤn to be "abundant in the Saráh (السَّرَاة), and El-Yemen," and to be "white"; (M;) [by this is evidently here meant "sesame," or the "grain thereof," or "both"; though it also signifies the "fruit of the coriander"; for otherwise, the most commonly-known meaning of سِمْسِمٌ would be unmentioned in the M;] the "grain of the حَلّ"; [i.e. the "grain from which the oil called حَلّ is expressed";] (Ṣ, Ḳ; [by the author of the latter of which, this was evidently understood to be different from the جُلْجُلَان, which is mentioned by him after the description of properties here following;]) "it is glutinous, corruptive to the stomach and the mouth; but is rendered good by honey; and when it is digested, it fattens; and the washing of the hair with the water in which it has been cooked lengthens and improves it: the wild sort thereof is known by the name of جَلَبَهَنْك," (Ḳ, TA,) thus, with fet-ḥ to the ج and ب and

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* *Sesamum silvestre*, of Pliny, which at any rate would seem to indicate that *sesamum* could be used for *sesamoides*.

† *Letaria italica* (Italian Millet), *Panicum miliaceum* (Indian Millet), *Pennisetum typhoideum* (African Millet), and *Sorghum vulgare* (Great Millet).

ه, and sukoon to the ل and ن, [but written in the CḲ جَلْبَهَنَكْ,] a Pers. word, [originally جَلْبَهَنْكَ,] arabicized; (TA;) "its action is nearly like that of the خَرْبَق [or "hellebore"]; and sometimes from half a drachm to a drachm is administered to him who is affected with palsy, and he is cured thereby, (Ḳ, TA,) speedily; (TA;) but a drachm thereof is dangerous, (Ḳ, TA,) in a great degree."

On the other hand the ribbed fruit of *Coriandrum sativum*, a sample of which we owe to the courtesy of the Director of the Royal Botanic Gardens, does not seem very suggestive of the sesamoid bones of the human digits[*]. We are thus thrown back on the following problems:

(*a*) Galen distinguishes *Sesamum* from *Sesamoides magnum*, and directly tells us that the sesamoid bones are called from the seeds of the former. But the seeds of the latter if identical with *Ricinus* are far more like sesamoid bones[†].

(*b*) It is conceivable that the sesamoid bones were given their name long before Galen's day, and that like his successors he simply repeated a tag with regard to them—a commonplace of the Greek anatomists. The name being Arabic (or even possibly Persian) it is conceivable that it was applied to the sesamoid bones before it reached the Greeks.

Johannes Riolanus has a section *De sesamoideis ossiculis*[‡] in which he refers to the sesamoids of hand and foot, but not to the *fabellae*. He especially notices the sesamoid bone of the great toe which he tells us the Magi called *Albadaran*, it was of the magnitude of a small pulse (*cicer*); it could not perish, and from it as from a seed the whole body would regenerate on the Day of Judgment. This tale also occurs in Cornelius Agrippa's *Liber de occulta Philosophia*, Cap. xx, with the Hebrews substituted for the Magi:

> Est in humano corpore os quoddam minimum, quod Hebraei *Luz* vocant magnitudine ciceris mundati, quod nulli corruptioni obnoxium, ne igne quidem vincitur, sed semper conservatur illaesum, ex quo veluti planta ex semine, in resurrectione mortuorum nostrum corpus animale repullulabit[§].

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[*] Memorandum of Dr O. Stapf in reply to a letter from K. Pearson with regard to Arabic use of SM SM for coriander (Royal Botanic Gardens, Kew, 10 May 1920):

I cannot find anything in our books containing Arabic plant names that would suggest that SM SM (semsem) was ever used in Arabic for *Coriandrum sativum*. The latter is Kuzbarah in the Arabic of Egypt, Syria and India. The Greeks knew it well and the *Codex Vindobonensis* (c. 500 A.D.) of Dioscorides contains a very good figure of it. It is practically certain that the Greeks took the word sesame from the Semitic peoples of Asia, but they would hardly have mistaken it for *Coriandrum*, which has seeds (or rather fruits) of a very different appearance and taste and yields practically no fat oil. Dioscorides deals rather fully with its properties, but there is no direct allusion to a similarity with sesame, nor do his observations suggest it. Prof. Pearson may judge for himself whether there is any similarity between *Coriandrum* fruits and the sesamoid bones of man.

[†] While Galen uses *σησαμίς* as equal to sesame there is some evidence that it was also used for *σησαμοειδὲς μέγα*, see Liddell and Scott's *Greek-English Lexicon* who quote Dioscorides, Noth. 4, 152.

[‡] *Osteologia*, Paris, 1614, p. 483. In this work Jacob Sylvius' commentary on Galen's *De ossibus* is given and on p. 477 we read: "In pollicibus et primis digitorum articulis & in simiarum poplite sesamoidea plura, pauciora extensionem immodicam, et que luxationem minaretur, prohibentia." Sylvius (1478–1555) antedates Vesalius (b. 1514); he clearly knew of the *fabellae* in apes, and may well have started Vesalius on the discovery of those in man.

[§] See also Hyrtl, *Das Arabische und Hebräische in der Anatomie*, S. 165 *et seq.* Further Joannis Munnics, *De re anatomica*, Treves, 1697, p. 213.

Thus the seedlike character of the sesamoid bone is preserved in both the Arabian *Albadaran* and the Hebrew *Luz* with their mythical associations*. It would be of great interest to know the age of these myths. If the Semites before our epoch had discovered the sesamoid bones and attached these legends to them, there might be no weight at all in Galen's explanation of sesamoid as like to the sesame-seed—the whole idea and terminology may be Semite and not Greek, and Galen may have become acquainted with it in Asia Minor.

(3) *History of the Sesamoids of the Knee-Joint. First Period—that of isolated Records of Anatomical Writers.*

From the Greeks down through the Middle Ages the study of anatomy was essentially the study of human anatomy; references to dogs or apes are few and far between and then very superficial in their associations†. Probably Jacob Sylvius' reference to the *fabellae* in apes is the first note we have of the existence of sesamoids of the knee-joint (see our footnote ‡, p. 144). There is no evidence that the Greeks or Arabs were acquainted with them. Vesalius was the first to draw attention to their existence in man. In his *De corporis humani fabrica,* Lib. I, Cap. xxviii (Basel edition, 1555, p. 153, Leyden edition, 1725, Tom. I, p. 107) he writes:

> Deinde bina recensebit in poplite occurrentia ossicula, quae duorum primorum pedem moventium musculorum innascuntur capitibus, mox in illorum ex femoris osse principio. Ossicula enim haec laevi sua et lubrica superficie, qua extra musculorum substantiam prominent, elatiorum spectant sedem posterioris regionis inferiorum femoris capitum, quorum impetum illa solvunt & sustinent, hoc privatim sibi vindicantia, quod musculorum exortibus, non vero aliorum fere omnium ossiculorum sesamo comparatorum modo tendinibus innectantur.

It is curious that in this passage Vesalius speaks as though the *fabellae* were constant in man, and this although as we shall see the mesial *fabella* is very rare indeed, and the lateral *fabella* does not exist in more than seven to ten per cent. of

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* The word *Luz* signifies a tree producing a small nut or the nut itself. In Genesis xxx. 37 it occurs and is rendered in the English version by "almond." Rashi, prince of commentators, explains it as chestnut, but it has been interpreted of any tree bearing small nuts, or almonds. Its secondary meaning in Rabbinic literature is a bone or cartilage resembling the almond said to be in the vertebral column of man. It was quite possibly the word used by Jewish anatomists for the sesamoids.

Commenting on the words "And the almond tree shall blossom" we read in the *Midrash* (Ecclesiastes xii. 5: see also Genesis, Ralba § 28, and Leviticus, Ralba § 18) the following:

R. Levi says, This is the "Luz" (i.e. almond-shaped bone) of the spinal column. Hadrian once asked R. Joshua b. Hananiah from which part of the body of man shall he blossom forth in the life of the future? And he answered "From the 'Luz' of the spinal column." He then said: How so? They then brought such a "Luz," put it in water, and it did not dissolve finally; into fire, and it was not burnt; into the mill, and it was not ground; they put it on a block and beat upon it with a hammer, and the block was split and the hammer broke, and all was of no avail [to destroy it].

The *Midrash Ralba* is a very old collection of Hebrew lore, parts of which are probably antecedent to the Talmud recension. We have to thank our colleague Dr Hermann Gollancz for the above references. We think it probable, however, that Jewish anatomy was non-existent before the appearance of the Greek followed by the Arabian medical schools.

† The first *Simiae Osteologia* is due to the elder Riolanus and edited by the younger in the *Osteologia,* Paris, 1614. Caput xxvi is entitled: *De sesamoideis* and we read (p. 535), "Duo ossicula magnitudine ciceris supra utrumque tuberculum femoris in origine gemellorum reperiuntur."

cadavers. It is further noteworthy that he does not state that he has definitely in a given cadaver discovered sesamoids in the heads of *M. gastrocnemius.* He speaks as he might have spoken of the sesamoid of the great toe with fifteen centuries of history behind it. There is no statement of a new discovery, nor that the occurrence is not universal. He writes as he might write of the patella itself! We have not succeeded in finding any reference to the *fabellae* in Vesalius' work of 1543*.

The next reference we have been able to find to the *fabellae* occurs in Eustachius' *Opuscula Anatomica* issued in 1564, but the permission to print which dates from 1562. We have quoted the relevant passage in the footnote to our p. 138. We seem on reading Eustachius' words to be in the atmosphere of a man who had really examined the matter in many cases, and we are almost inclined to wonder whether, if Eustachius had been able to get his works printed when they were written†, Vesalius would have been the reputed discoverer of the *fabellae.* At any rate Eustachius' statements are correct, and here as elsewhere he not only criticises Vesalius, but does so legitimately‡.

Fallopius in his *Observationes Anatomicae* of 1561 (fol. 113) says that the *fabellae* are frequent in apes and that he has seen them, but that only Vesalius has seen them in man.

At the end of the 16th and beginning of the 17th centuries we have Caspar Bauhin. He has in his works a good deal about sesamoid bones. In his *De corporis humani fabrica* (Basel, 1590) there are references to the sesamoids on pp. 356, 372, 395, but we have failed to examine a copy, and cannot say whether he refers to the *fabellae,* but he probably does. In his *Anatome* (Basel, 1597), after enumerating (p. 259) *sesamina duodecim* of hand and foot, he continues "Hic addere possimus sesamina duo in poplite§." Again in his *Theatrum Anatomicum* (Frankfurt, 1603) he has considerable references to sesamoid bones (pp. 1169—1171, 1207, 1277—80), and for the *fabellae* (p. 1278). He gives, Tab. xv, Fig. 11, diagrams of the sesamoid bones of the hand and foot, even the very small ones, but not of the *fabellae.* This leads us to believe that he had never seen the latter. Confirmation for this view arises from the fact that what he says is principally paste-and-scissors-work from Galen and Vesalius. Speaking generally of sesamoids in his Cap. xxix, he does say, however, "solida sunt, rotunda, aliquantulum depressa, aliquando cartilaginosa, aliquando ossea." This may be original, but is most probably due to an earlier writer in a passage which has escaped us.

Lastly in Bauhin's *Institutiones anatomicae,* Basel, 1609, he refers to sesamoids in each head of origin of *M. gastrocnemius* (p. 247).

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* The first edition of his anatomy which appeared at Basle in this year.

† As to Eustachius' difficulties in publication see *Biographie Universelle,* Paris, 1815, Tom XIII, p. 534.

‡ Eustachius' writings seem to indicate some deep ground for dislike of Vesalius. Can it be that unpublished discoveries of Eustachius were talked about and made use of by Vesalius in his great work? At any rate Eustachius clearly knew more about the *fabellae* in man than Vesalius did.

§ This statement of the *fabellae* being *in poplite* must not be taken as referring to the popliteal muscle's origin, but simply when met with in the early anatomists as indicating the neighbourhood of the popliteal surface.

It will be clear that Bauhin does little more than confirm the error of Vesalius by speaking as if the *fabellae* were constant in man and occurred equally in both heads of origin of *M. gastrocnemius.* To this nearly universal overlooking of Eustachius, and to the parrot repetition of the views of Vesalius we owe most of the obscurity about the *fabellae* which followed their discovery. In Riolanus, the Elder, *Osteologia*, Paris, 1614 (pp. 69, 121, 123, etc. of the *Isagogica de ossibus Tractatio*), we have not been able to find references to the *fabellae.* There is only the passage already cited (see our p. 144, footnote ‡) with regard to the "in simiarum poplite sesamoidea" in Sylvius' commentary on Galen's *De ossibus* (p. 475) and the reference to the *fabellae* in the apes in the *Simiae Osteologia*, p. 535. The younger Riolanus in his *Anthropographia*, Liber V, Paris, 1650, Cap. xliii, p. 333, writes:

> Externus gastrocnimius (sic!) ab externo condylo nascitur...singulis eorum principiis primus Vesalius observavit singula ossiculis sesamena apposita, saepius alterum deficit, authore Fallopio. Ut laevi lubricaque sua superficie, dum ossi & musculo interjecta sunt, impediant ne in cruris directione musculi atterantur aut ab osse laedantur.

The direct attribution to Vesalius is here made and Fallopius is cited, not Eustachius, as casting some doubt on the constant appearance of at least *one* of the *fabellae.*

Gorraeus in 1601, Paaw, 1615, and Laurentius in the same year (also in 1628) confine their attention to the sesamoids of hand and foot*.

In 1632 we find published at Frankfurt Adrian Spiegel's *De humani corporis fabrica*, and on p. 78 under *Gastrocnemius externus* we read:

> Huius musculi duobus capitibus, quemadmodum Vesalius notat non procul ab exortu sesamoidea ossicula tributa sunt.

In the *Tabulae anatomicae* of J. Casserius, edited by D. Bucretius (Frankfurt, 1632), which accompany Spiegel's work, Tabella XXXVIII, Fig. 11, professes to give a diagram of *gastrocnemius.* The figure represents the lower limb from knee downwards, the muscle appears to be cut at the two points of insertion and turned back from the knee, and S, S, two sesamoids, in it appear to rest on the ground. The figure is somewhat obscure, but it is the first professed representation of the *fabellae*, probably largely a product of the imagination (see our Plate I, Fig. 1). Dominicus de Marchetti returns to the healthy statement of his own personal experience. He rightly holds that, while he himself has not found the *fabellae* in the many bodies he has dissected, he cannot deny their existence. But that they are certainly not found in all bodies as Vesalius and Riolanus have asserted†.

Thomas Bartolinus in his *Anatomia reformata*, Hagae Com., 1655, returns very nearly to the Vesalius legend. In Liber IV, Cap. xxii, *De ossibus sesamoideis*, p. 528, we read:

> Item bina ossicula in poplite juxta os femoris, musculorum (duorum priorum pedum moventium) non tendinibus, sed principiis uniata, quae in senibus reperiunter, & animalibus siccis, ut cervis, canibus et leporibus.

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* There is also no reference that we have been able to discover in the Sylvius of 1635.

† *Compendium anatomicum*, Patavii, 1652, p. 167, and *Anatomia*, Padua, 1652, p. 153.

How these mediaeval anatomists used each others' very words! There is, however, the qualifying phrase soon to become a part of the myth, that the *fabellae* occur in the aged. Further the deer and the hare are added to the list, man, ape and dog, that own *fabellae*. This reference to the aged is emphasised again in the edition printed at Leyden, 1686, p. 756.

Diemerbroeck, *Anatome corporis humani*, Utrecht, 1672, p. 944, is content to return to the complete Vesalius legend, and finds two *fabellae* and these always occurring. Stephen Blancard's *Anatomia reformata*, Leyden, 1695, p. 729, and Gerhardus Blasius' *Anatome hominis*, Amsterdam, 1673, p. 62, give also the complete Vesalian story. But the 17th century is not to close without one or two cases of men who report what they have actually seen. Thus Schrader in 1674 reports that he had found a *fabella* in the anatomy of a *cadaver virile**, and that sturdy old surgeon-anatomist William Cowper, in his *Myotomia reformata*, London, 1694, p. 206, states first the opinion of Vesalius and Riolanus:

> that in the two beginnings of this muscle there are two *Ossicula Sesamoidea*, which we must acknowledge with Marchetti have hitherto escaped our Observation though it is likely it may be so in Aged Bodies, as it appeared in a subject I lately dissected on one side only.

We might almost say that from the reported discovery of Vesalius, only Eustachius, Schrader and Cowper had really seen a *fabella* up to the end of the 16th century.

Even in the case of Cowper we are to find later that the dogmas as to the appearance of *fabellae* in the aged, and as to their object, are accepted. Thus in his *The Anatomy of Human Bodies*, 2nd edition, Leyden, 1737, T. 103, F. 2, under *Ossa sesamoidea*, we read:

> In some Bodies especially Aged, we find Two Ossa Sesamoidea on the Superior Parts of the Two Lower Heads of the Thighbone D, E: The Office of which is to Defend the Bending Tendons of the Tibia from too great a Collision on those Heads of the Bone which they would else be subject to.

At the very end of the century Joannis Munnics, in his *De re anatomica*, Treves, 1697, p. 213, sums up the pure Vesalian doctrine:

> In censu quoque ossiculorum sesamoidum poni debent duo, quae in poplite inferioribus femoris appendicibus apponuntur, duorum priorum pedem moventium musculorum principiis inhaerent.

For him all sesamoids also start as cartilaginous and become solid with the progress of age.

Thus far the history of the *fabellae* has been fully representative of an oft-occurring scientific tragedy. Authority dominates inquiry and dogma fills up the space with its story of how a non-existing phenomenon (the constant appearance of both *fabellae* in man) arises and what purpose it serves. The time was therefore ripe for a little comedy, and very early in the 18th century we find this in a double fashion, the Trew-Heister incident. In 1715 C. J. Trew wrote an Altorf Dissertation entitled *De chylosi foetus in utero*. Tabella II contains a large but crude drawing of a lateral *fabella* resting in a cavity on the condyle. There is a separate drawing

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* *Observationes anatomico-medicae*, Amsterdam, 1674, p. 193.

of the sesamoid itself (FVI) (see our Plate II, Fig. 4). Trew's thesis was republished by Haller, *Disputationum anatomicarum selectarum*, Volumen v, Göttingen, 1750, p. 475. Figs. v and vi should give the *fabella*, but the plate is missing in the two copies of Haller that we have examined, possibly the plate never existed in Haller. The description to Fig. v, C is "C est os sesamoideum insidens cavitati condyli exterioris in omnibus femoribus conspicuae." This sesamoid (if only a lateral *fabella*), and the cavity conspicuous in all femora, were something extraordinary, and Trew seems to have thought so, for he writes as follows:

> *Observatio II. Ossicula sesamoidea in femore ac minimo manus digito.*
>
> Cum aliquando occupatus eram in ossibus humanis a carne sordibusque mundandis, atque accurate in praeparatione artuum in numerum ossiculorum inquisiverim, cum primis deprehendi numerum sesamoideorum auctum. Primum mihi antea nondum cognitum se offerebat in femoris parte inferiori, ubi cum tibia articulatur, in condylo exteriori: ubi in fovea quadam deprehendi ossiculum rotundum figura & magnitudine Fig. VI. quod instar parvae patellae loco C. Fig. V. insidebat in utroque femore. Putabam tunc temporis, id extraordinarium fuisse; dum vero illud postmodum inquisivi in aliis femoribus, luculenter semper deprehendi foveam in loco, cui insidet, ita ut deinceps ulterius illius existentiam perscrutari excitarer; quod & mihi nunquam non in tot, quotquot deinceps cadavera inspicere poteram, reperire contigit; siquidem brevi in quatuor illud inveni, atque etiam forsan cuilibet inquirenti patebit.

It will be observed that Trew does not say that he has ever discovered a mesial *fabella*. But such a big discovery was too much for Trew's anatomical instructor, Lawrence Heister. In the first edition of his *Compendium anatomicum*, Nürnberg, 1719, he magnified Trew's lateral *fabella* into two *fabellae*, he forgot to mention Trew's name and he proclaimed the discovery of two new bones! One might at least have supposed an acquaintance with the text-books of his day. The English translation of Heister's work, *The Compendium of Anatomy*, London, 1752, appears to have been made from the earliest edition of Heister, and does not contain the long apologetic note of the Nürnberg edition of 1732 (Vol. II, p. 47) or the Amsterdam edition of 1748 (Vol. II, p. 48). We read on p. 65:

> There remain now to be considered, the *ossa sesamoidea*. These are small bones, and are most conspicuous in old subjects; they somewhat resemble the seeds of the sesamum, whence they have their name. Their most usual situation is...[details of sesamoids of hand and foot] one frequently in each external condyle of the *os femoris* (Tab. I. figs. 2, 3, 4)...... These are usually found in adults, or in elderly people: sometimes though more rarely there is one also in the internal condyle of the *os femoris*...... Upon the whole there are very rarely found more than sixteen of them often fewer. Those anatomists, therefore err greatly who reckon more than forty of them......too many, however, have given in to this error.
>
> All these bones, except those that are found in the condyles of the femur, adhere to the tendons of the muscles: those excepted, are connected to the origin of the muscles. Their size and shape are various and irregular: they are cartilaginous in young subjects, but they grow hard and boney by age: and it is therefore that they are so much the more easily found in old people. They serve as a kind of trochleae to the muscles and increase their power.

On p. 463 Heister gives an "Explanation of the Figures" (see our Plate II, Fig. 5):

> Fig. 2. Represents the lower part of the *os femoris* with two sesamoid bones not described by other authors. A, The *os femoris*. B, The internal condyle. C, The external condyle. D, The larger of the two new sesamoide bones, situated in a considerable cavity in the external condyle. This, however, in some subjects, is much larger than it is represented here. E, The lesser of the

two new sesamoide bones, situated in the internal condyle. This is but very seldom met with in dissection, most subjects wanting it: but the other is seldom wanting.

Fig. 3. Represents the larger of these two sesamoide bones, separated from the cavity in the external condyle, in which it is generally lodged.

Fig. 4. Shews the lesser sesamoide bone, of the internal condyle, separate.

Apparently Heister was not left long to enjoy his triumph at the discovery of "two new sesamoide bones." In the long and somewhat abject note of 1732, he admits that the "rediscovery" was due to Trew; he cites a few authors of the past who did not refer to the *fabellae*, but he does not adequately explain why he had apparently overlooked Vesalius, Eustachius, Riolanus, Cowper, etc. Indeed "rediscovery" is hardly the right description for the *fabellae* in 1715, when W. Cheselden* could write in *The Anatomy of the Human Body*, 1713, pp. 22—23:

> *Ossa Sesamoidea*...and sometimes one in the lower end of each Thigh-bone at the beginning of the *Plantaris* muscle. Their use is the same with the Patella.

A brief and pregnant sentence almost the most suggestive since that of Eustachius. Another remarkable point is that Heister should have "rediscovered" the suppositious influence of old age, one of the points the 17th century writers had much emphasised.

Gruber† throws complete discredit on Heister's statements and especially on his plate. He says that Heister has placed the *fabellae* where they do not usually exist, and apparently Gruber considers that if they occurred at all they must have been pathological osteomata. Yet Heister says that he has found them in four corpses, one after another, and that they can always be found except in youthful cadavers, where they are cartilaginous‡! We are by no means certain that Heister's drawing, as far as the lateral *fabella* is concerned, is not a rough copy of the crude diagram in Trew's Dissertation of 1715; both exhibit the lateral *fabella* resting in a hollow, not of the articular surface, but of the lateral condylar eminence in its proximal part. As for the mesial *fabella*, which Trew does not provide, Heister rests it, the femur being vertical, on the commencement of the popliteal area as it springs from the articular surface of the mesial condyle: see our Plate II, Fig. 5. We believe the whole drawing to be imaginary copied from no actual subject, and totally out of accordance with all skiagrams of the lateral *fabella* in man with which we are acquainted. Not one of the first three drawings, Casserius (1632), Trew (1715) and Heister (1717), appears to us to represent in any way the *fabella* as it exists in man. The comedy of Heister on the *fabella* might well be preserved in the history of science as an illustration of the process of concocting a text-book.

Saltzmann§ (*Decas observationum illustrium anatomicarum*, p. 6), Strassburg, 1725, reports that he found no *fabellae* on the condyles of a given cadaver. Bass (*Observationes anat.-chirurg.-med.*, p. 220), Halle, 1731, and Albinus (*Historia*

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* He did not apparently return to the subject in his *Osteographia or the Anatomy of Bones*, London, 1733.

† Gruber, p. 11, footnote.

‡ *Academiae Cassareo-Leopoldinae Naturae Curiosorum Ephemerides Centuria* VI. Obs. XXIX. pp. 245—6, Ossa sesamoidea in femore...Francofurti et Lipsiae, 1717. See also *Centuria* VII. Obs. XXIII. pp. 49—51, 1719.

§ For fuller details of the following eight references: see Pfitzner's bibliography.

musculorum hominis and other works), 1734—1757, have no reference to the *fabellae.* Palfin (*Anatomie chirurgicale...du corps humain*, Vol. II, p. 159), 1726—1734, says they are not always to be found as Vesalius believed, while Winslow (*An Anatomical Exposition of the Structure of the Human Body*, London, 1749, and *Exposition anatomique de la structure du corps humain*, Paris, 1732) apparently considers they are always to be found, and in cartilage-lined hollows. George Thomson (*Anatomy of Human Bones*, London, 1734, p. 104) asserted the existence of both, which was the view taken by Kulmus in his *Tabulae anatomiae* (p. 62) of 1732 and supported by the authority of Haller in 1745. In such a sea of wearisome positive and negative assertions unaccompanied by personal examination, it is almost a comfort to meet such a bare statement as that of Drake in his *Anthropologia nova*, 3rd edition, 1750, p. 434:

> There are sometimes found two *ossa sesamoidea* in the two beginnings of the *Gastrocnemius externus* muscle, but they are rarely met with and only in aged Bodies.

He at least knew that they did not always occur or fail to occur, even if he supposed no difference in frequency of the pair. Still more comforting is Morgagni's statement (*Adversaria anatomica*, T. II, Leyden, 1723, p. 64*) that he had found the lateral *fabella* several times, but the mesial *fabella* only once. It goes at any rate a little way to balance the statements of Lieutaud (1742) that he had found *fabellae*, more often mesial than lateral, of Disdier (1745) that one usually exists in the groove behind the lateral condyle, of Lauth (1798) that he had never found any *fabellae*, and of Bertin (1783) that he had found two on each femur!

How strikingly barren these 18th century anatomists seem to have been, at least in this matter of the knee-joint! It is not till the very end of the century that with Sömmering and Peter Camper we get some progress in our problem, and we begin again to grasp the difference between big and little naturalists. We will take the former first because his contribution is less important than those of Camper.

Sömmering (*Vom Bau des menschlichen Körpers*, Leipzig, 1791, Bd. III, S. 295) recognised that a *fabella* is not infrequent in the lateral head of *M. gastrocnemius.* He continued to study the subject, however, and in his *Lehre von den Muskeln* of 1841, S. 347—351, he recognised that the lateral *fabella* is often only a hemisesamoid, i.e. cartilaginous, and that the mesial *fabella* is very rare, and when it does occur is far more likely to be a hemisesamoid than an orthosesamoid. Further like Cheselden (see our p. 150) he recognises a relation of the lateral *fabella* to *M. plantaris*:

> Der Ursprung des *M. plantaris* hängt mit der Ursprungsehne des lateralen *Gastrocnemius*-Kopfes, namentlich auch mit dessen Sehnenbeinchen eng zusammen.

We are clearly in touch with a man, who is observing and thinking on his own.

Peter Camper's first contribution occurs in his *Dissertatio de fractura patellae et olecrani*, Haag, 1789, translated as *Abhandlung von Bruch der Kniescheibe und*

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* His words are: "Nos certe quod alterum eorum in nonnullis cadaveribus et in uno internum nominatim invenisse meminimus." They do not occur in the edition of 1714 but will be found in the Venice edition of 1782.

des Olecranon, reprinted in the German translation in the *Vermischte Schriften,* Lengen, 1801, S. 68, and Plate I, Fig. i. K. We read (see our Plate III, Fig. 6):

> A, K, I, L die Sehne des äusseren Bauchs des Wadenmuskels (*Gemelli seu gastrocnemii externi*) worin sich ein Knöchelgen (*ossiculum sesamoideum*) f. g. findet welches man in den inneren Bauch derselben nie sieht. Es wird bei den Affen, den Orang ausgenommen, bei dem Hunde allzeit in beiden Portionen, auch beim Fuchse, bei der Katze, dem Igel, in dem Agu und allen Thieren gefunden.

The statements are too sweeping, but they are those of a man who is stating what he knows of his own observation, and are not the mere reiterations of his predecessors. We are beginning also to get a wider range of comparative observation—comparative anatomy is starting its career.

We have reproduced a sketch of Camper's diagram of the "Knöchelgen" near K on Plate III, it is far from an ideal representation, but it is a great advance on those of Casserius, Trew and Heister. We next turn to Camper's *Naturgeschichte des Orang-utangs,* Düsseldorf, 1791, S. 126, and note that Camper, after citing Eustachius' view about Galen and the *fabellae* in man, continues:

> Hier muss ich aber Eustach aus eigener Erfahrung widersprechen, weil ich dieselben in sehr vielen Körpern sowohl von Männern, als Weibern, doch allein in äussern Kopfe des Wadenmuskels oder Gastrocnemius, gefunden habe. In meiner Sammlung habe ich eine grosse Menge derselben aufbewahrt. Am meisten wundere ich mich, dass Albin derselben gar nicht erwähnt, da Eustach sehr deutlich sagt, dass er diese Beinchen nicht selten bei Menschen, in Affen und Hunden aber immer gefunden habe. Nicht allein Eustach hat dieses wahrgenommen, sondern auch Coiter (59) und Sylvius dem Blasius (60) zufolge, kannten den Sitz derselben in den Affen sehr wohl, dass sie nämlich in beiden Köpfen der Gastrocnemii in den Affen, bisweilen in der Sehne des Peroneus Longus und Tibialis zu finden sein: dieses kann ich in verschiedenen Affengerippen, und selbst ein Sesambeinchen in der Tibialis eines weiblichen Gerippes zeigen.
>
> Im Orang, auch im Rumpfe des Orangs, der im Thiergarten des Prinzen von Oranien gelebt hat, habe ich keine gefunden—ich bin auch gewiss, dass sie nicht da waren, weil ich sonst den Knorpel müsste angetroffen haben, so wie ich das deutlich in der Sehne des Muskels Popliteus fand, und noch aufbewahre. Tyson spricht auch nicht davon, und bei dem allen findet man sie immer in den geschwänzten Affen, im Pithecus oder Aegyptischen Affen, in den Hunden, Katzen, Füchsen, Stachelschweinen und dergleichen, und zwar doppelt.

Further on S. 187 we read:

> Von dem Knie werden wir nichts mehr sagen, als dass ich in der Sehne Kniescheibenmuskels (popliteus) ein sehr grosses, doch knorplichtes Sesambeinchen gefunden habe. Nicht allein im Rumpfe, sondern auch in dem Orang des Hoffmann, das ist, in meinem ersten traf ich ein solches Bein an.
>
> Dieses Sesambeinchen konnte meiner Aufmerksamkeit destoweniger entwischen, weil es nie beim Menschen, sondern immer in den Pavianen, Hunden, Fuchsen, Katzen u. s. w. angetroffen wird.
>
> Ich sahe also gleich das Werk des Tyson nach, weil Cowper, der sich durch die Zergliederung der Muskeln so berühmt gemacht hat, auch die des Pigmy beschrieben hatte.—Doch anstatt hier einiges Licht anzutreffen, fand ich diese offenherzige Erklärung, dass ihm dieser Muskel entgangen sei (70). Auch suchte ich vergebens in der uns vom Daubenton mitgetheilten Zergliederung des Gibbon nach demselben.

According to Peter Camper we have accordingly:

(*a*) Both *fabellae,* the lateral and mesial, in a variety of apes, in dogs, cats, hedgehogs, etc.

(*b*) In man only a single *fabella* occasionally, and this on the lateral condyle.

(*c*) A sesamoid in the tendon of the popliteus (which we term the *cyamella lateralis*) in the orang, but no *fabellae.*

(*d*) No *cyamella lateralis* in man, but it does occur in baboons, dogs, foxes, cats, etc.

Most of the above statements of Camper's can be confirmed, but there are certain exceptions, notably the *cyamella lateralis*, the sesamoid of the popliteal tendon, has been found in man, but it is very rare. Thus while it required 14 centuries after Galen to establish the existence of the first two sesamoids of the knee-joint, the *fabellae,* in man and nearly 250 years after Vesalius to reach the third sesamoid, the *cyamella lateralis,* in the orang, it took 80 years after Camper to establish the *cyamella* as an occasional anomaly in man. Such is the slow growth of scientific knowledge as evidenced in a single anatomical detail like the present. It must not, however, be supposed that with this rapid increase of knowledge in the last years of the 18th century, the 19th century would see rapid progress in our anatomical knowledge of the knee-joint. On the contrary the inertia and follow-the-other-fellow policy of the first half of the 18th century reappears in the 19th.

Cloquet in 1816 (*Traité d'anatomie descriptive,* Paris, I, p. 205) assures us that the *fabellae* appear fairly persistently on both condyles of the femur in man. Bourgery (*Traité complet de l'anatomie*, Paris, 1831, II, p. 103) after describing the *M. gastrocnemius* continues:

> Parfois chez les vieillards des os sésamoïdes se développent dans leurs tendons fémoraux, surtout dans celui du jumeau interne (!).

Indeed no treatise of anatomy could be complete without these "vieillards"; we have followed them through the centuries. The doubting Thomas, however, is always with us. Sappey as late as 1876 (*Traité d'anatomie descriptive,* Paris, II, p. 430) denies the very existence of *fabellae* in man! Even the modern English text-books of anatomy often do not exhibit a knowledge of the sesamoid of the knee-joint comparable with that of Eustachius, or as suggestive as the remark in Humphry (*A Treatise on the Human Skeleton*, 1858, p. 537):

> Where this band (posterior ligament of the knee-joint = *lig. popliteum auct. ab semi-membranoso*) joins the *gastrocnemius* upon the outer condyle of the femur is commonly a thick mass of fibrous tissue and sometimes a sesamoide bone (Pl. 51, Fig. 2D[*]) is developed.

Humphry (see our Plate IV, Fig. 7) is clearly adding to the accumulating knowledge of the mass of muscular attachments associated with the lateral *fabella*;

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[*] This is we think the *fifth* representation of a *fabella.* Gruber (Gruber, p. 7) suggests that Weitbrecht (*Syndesmologia*, 1742, Fig. 57, *h*) shows a *fabella* : "hat es vom Ursprunge bedeutend entferut (etwas 2 cent.) im *Gastrocnemius externus* abgebildet, aber un Texte nicht bezeichnet." Pfitzner (Bibl. Pfitzner, S. 758) denies this. It is possible that Weitbrecht or his artist indicated something they did not understand, but the plate is too obscure to allow us to come to any conclusion whatever, and as Weitbrecht does not refer to the point in his text, the matter is not of the slightest real importance. There could not be another rediscovery of the *fabella* in 1742!

we have already cited in this direction Cheselden and Sömmering (see our pp. 150 and 151). Quain, however, in 1867 (*Elements of Anatomy*, 7th edition, London, Vol. I, p. 284) can tell us nothing more than:

> A sesamoid fibro-cartilage is sometimes met with over the outer condyle and occasionally over the inner, it is rarely ossified—

a statement which is apparently incorrect in at least two points;—while Gray as late as 1905 (*Anatomy, Descriptive and Surgical*) is content with the assertion that there is a sesamoid fibro-cartilage (rarely) osseus in the tendon of the outer head and one occasionally in the tendon of the inner head—another statement too brief to be in any way accurate.

The German anatomists have been somewhat more precise. Hyrtl in 1838 ("Physiologische anatomische Bemerkungen über die Kniegelenksknorpel," *Medicinisches Jahrbuch d. K. K. Oesterr. States.* Bd. XX, Wien, S. 31—32) tells us that the *fabella lateralis* is more frequent than the *fabella mesialis*, that they may be hemisesamoids, and that he has found them in both heads of *M. gastrocnemius.* Again in his *Lehrbuch der Anatomie der Menschen*, 12 Aufl., Wien, 1873, S. 420, after stating that Camper had only seen *fabellae* in the external head, he continues: "Nach meinen Beobachtungen kommt es in beiden Köpfen vor, obwohl in äusseren ungleich häufiger." Lastly we may note that Gegenbauer in 1890 (*Lehrbuch der Anatomie der Menschen*, Leipzig, S. 457) has for a text-book a fairly complete and adequate statement.

Before we come to the important memoirs of the last quarter of the 19th century which have made the sesamoids a special study we ought to note an important memoir by Macalister of 1872 (*Muscular Anomalies in Human Anatomy*, Dublin, 1872*). It is important for our purposes for three reasons:

(*a*) Because it provides evidence for the existence of the mesial *fabella* in man:

> Gastrocnemius—1. May have a sesamoid bone in its inner head: I have found it: 2. Or in its outer—this is more common. (p. 118.)

Macalister clearly speaks here of an orthosesamoid, not a hemisesamoid.

(*b*) Because it gives the first statement of the existence of the *cyamella lateralis* as an anomaly in man.

(*c*) Because it records anomalies of the popliteus muscle which have considerable bearing on both the *fabella lateralis* and the *cyamella lateralis.*

> Popliteus muscle has been described as double (Fabricus ab Aquapendente: "De Motu locali Animalium" in *Opera Anatomica et Physiologica*, Lipsiae, 1687, p. 359). The same has been seen by Professor Bevan, of the Royal College of Surgeons, Ireland....A small superior popliteus, arising from the outer condyle, and above the popliteus muscle, to which it was united, and inserted into the upper part of the tibia, was seen by Calori†. A sesamoid bone has been found in its tendon. Slips from the semi-membranosus have been found inserted into the fascia over this muscle.

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* Also as *Transactions Royal Irish Academy*, Vol. XXV, 1872. J. F. Knott, "Muscular Anomalies," *Proc. R. Irish Academy*, Second Series, Vol. III, p. 639, 1882, notes a case of bilateral orthosesamoid in tendon of *Popliteus.*

† *Mem. della Accad. delle Scienze dell' Istituto di Bologna*, 2 series, Vol. VI, p. 143, refers only to the supernumerary popliteal muscle not to sesamoids.

The bearing of the whole of this paragraph on Gruber's later work is important, and it is regrettable that he did not give more consideration to its statements.

The exact position of affairs with regard to the sesamoids of the knee-joint in man in 1872 was as follows:

(i) The erroneous statement of Vesalius, at once corrected by Eustachius, had been more or less discarded, even in the text-books.

(ii) The existence of the lateral *fabella* in man was recognised as a not uncommon occurrence and it might be an orthosesamoid or a hemisesamoid. If the latter it was described as "fibro-cartilaginous."

(iii) The existence of the mesial *fabella* was realised to be something much rarer, and less often an orthosesamoid. It was, however, vouched for by such credible observers as Eustachius, Morgagni, Sömmering, Hyrtl and Macalister, and by the less credible Heister.

(iv) There was a growing realisation that in man the *fabella lateralis* had relationship not only to the tendon of the outer head of *gastrocnemius*, but to *M. plantaris* (Cheselden and Sömmering) and to the posterior ligament of the knee-joint (Humphry).

(v) Camper had discovered the *cyamella lateralis* in the orang as a persistent feature, although the *fabellae* are absent. He had found it accompanied by the *fabella lateralis* in certain of the lesser apes, and such association is invariable in the cat*. There was really no excuse in 1872 for confusing the lateral *cyamella* with the lateral *fabella*. While Camper denied the existence of the *cyamella* in man, Macalister had discovered it as a rare anomaly.

(4) *History of the Sesamoids of the Knee-Joint. Second Period—that of Monographic Literature.*

The three chief memoirs to be now considered are those of Gillette (1872), Gruber (1875) and Pfitzner (1892). Of these authors Gillette and Pfitzner consider all the sesamoids of the human skeleton, Gruber only those of the knee-joint, and of these he realises only the *fabellae*.

(*a*) Gillette's paper is entitled "Des os sesamoides chez l'homme," *Journal de l'Anatomie et de la Physiologie*, pp. 506—538, Plate XX, Paris, 1872. Gillette commences by dividing sesamoids into *peri-articulaires* and *intra-tendineux*—this distinction is not without its uses. He says, however, that the first are true bones and the second are not (p. 535). He admits, however, when he comes to the *fabellae*, which clearly are intratendinous, that they, apparently because they are more voluminous, have greater resemblance to the periarticular sesamoids. He gives in his Fig. 16 the first really good illustration of the *fabella lateralis in situ*. We have reproduced it in our Plate I, Fig. 3.

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* See *Mammalian Anatomy* of Horace Jayne, London and Philadelphia 1898. Part I. The Skeleton of the Cat, pp. 704—5.

Gillette (p. 533) asserts that in the foetus and the child there are no traces of the *fabellae.* This is, of course, opposed to Nesbit's conclusion, but as in perhaps nine cases out of ten no trace of the *fabellae* can be found in the adult, his assertion is like Gruber's by no means conclusive*. Gillette is a strong supporter of the intensive stress theory of the origin of sesamoids. He asserts that little by little:

> par le frottement le tendon s'élargit, s'aplatit, devient plus dense. Il se produit à ce niveau et dans son épaisseur un épaissement d'abord fibro-cartilagineux, c'est *l'os sesamoide intra-tendineux cartilagineux*, mais sous certaines influences toujours de nature mécanique, il se dépose des molecules osseuses dans l'intérieur de ce noyau qui se constitue *os sesamoide osseux* (p. 534).

The argument seems to us of precisely the same character as the argument that the environment produces the plant. In a certain sense it does, but the seed contains the potentiality of the plant. We might argue that nourishment and exercise produce the human skeleton, and it is only in this sense, that the *fabella* is no more the product of mechanical causes than the femur itself, that we can accept Gillette's statement. We believe, however, that both he and other supporters of the theory of intensive stress mean a great deal more by it than we are willing to admit. Some decision might be made by the examination of the puppy in various stages of growth, we should be much surprised, if the intense activity and restlessness of a young puppy could be subdued, to find it growing up without or even with much diminished *fabellae*!

The classification of sesamoids into intratendinous and periarticular is convenient, but to identify these with hemisesamoids and orthosesamoids seems to us extremely doubtful, and to state that the former pass into the latter owing to mechanical causes we do not find at all helpful.

(*b*) *Gruber's Monograph*†: This memoir exhibits all the surpassing merits and some of the small but disturbing faults of this eminent anatomist. It presents Gruber's usual wealth of new material and his extensive knowledge of earlier literature. We have at once a sweeping clearance out of all the old anatomical myths, and the forcible statement of new and sounder ideas based on wider ranges of observation. For the first time in the history of our subject we have adequate material directly examined and recorded. Again for the first time we have a real attempt made to record the appearance or non-appearance of the *fabellae* in various living forms. We begin to realise at once that as in the case of man, so in the case of many other living animals the contradictions of earlier writers chiefly arise from want of adequate material; the problem is not whether *fabellae* do or do not exist, but in what proportion of cases they will be found. In

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* It is needful to take a case where a *fabella* always occurs, e.g. the dog, and when Gruber dissected puppies he did find the *fabellae* prefigured. We have repeated this observation, and we have also found that the kitten at birth prefigures the lateral *fabella* and *cyamella* by nodules of *hyaline*, not fibrous cartilage.

† "Monographie über die aus wahren (hyalinschen) cartilagines praeformirten *Ossicula sesamoidea* in den Ursprungssehnen der Köpfe des *M. gastrocnemius* bei den Menschen und bei den Säugethieren." *Mémoires de l'Académie impériale des Sciences de St Pétersbourg*, VII^e Série, Tome XXII, 1875, No. 4.

other words it is not the "always-present" nor "always-absent" categories that are the difficult ones, but the "sometimes present" category—needing statistical treatment—which is the hardest to ascertain and yet is, perhaps, the most valuable for evolutionary suggestion.

Wenzel Gruber starts by telling us (1875) that the *fabellae* in man have been known for 319 years, which gives us closely the date of the 1555 edition of Vesalius' work, and that those of the mammals have been known for 313 years, which is a close approximation to the date of Eustachius' work (permission to print 1562, issued 1564). There is no reference to Sylvius' commentary on Galen's *De ossibus* (see our p. 144). Gruber at once sweeps away the statements that *fabellae* occur always in man, that they occur only or occur occasionally on the mesial side, that they are produced by stress or by friction, or that they occur most frequently in the aged or in males*. While an osteoma may and does occur in the head of both *M. gastrocnemii* it is far rarer than the true sesamoid and does not occur in the same situation. The origin of the *fabellae* is normal and perfectly analogous in the mammals and in man. Gruber then gives the statistics of his observations, to these we shall refer later; they are rather obscurely expressed, Gruber not having a keen sense for figures. We shall refer here only to a few points which bear on his general conclusions. He says that for many years he examined the cadavers of human embryos in last month (total *not* stated), new-born children "in considerable number," "several" children aged one to seven and never in any of these nor in two five year old boys or in one seven year old girl did he find in the tendons of *M. gastrocnemius*, a *cartilago hyalina*, or *fibro-cartilago*. The wording does not strike one as referring to very copious material and possibly the material was inadequate for a *negative* conclusion. In 44 cadavers 10 to 17 years Gruber found only the *hyalina cartilago* as hemisesamoid; in 426 cadavers 18 to 83 years he found only ossified *fabellae*, the orthosesamoids. In the external head he found only *ossicula sesamoidea* and hyaline *cartilagines sesamoideae* (our orthosesamoids and hemisesamoids), never *fibro-cartilagines* or pathological ossifications (pseudosesamoids†). In the internal head none of these occurred except a single pathological ossification, which he figures in Table III, Fig. 3 *g* (see our Plate IV, Fig. 8) and this would never be confused with a true *fabella*. From these data he concludes that (i) neither hemisesamoid nor orthosesamoid occur in the mesial head of *M. gastrocnemius* (p. 67), (ii) there is no trace of a sesamoid in the lateral head in the case of man before 10 years of age (p. 70), and (iii) orthosesamoids originate in a hyaline *cartilago* never a *fibro-cartilago* (p. 68). In other words our hemisesamoid according to Gruber is always hyaline and not fibro-cartilaginous. Here we think Gruber's lesser faults appear. He dogmatically asserts the non-appearance of hemisesamoid or orthosesamoid on the mesial side. He simply discards the

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* The existence of *fabellae* in man is wholly independent of age and occupation and does not arise from friction (p. 69).

† Pathological ossifications in *M. gastrocnemius* are very rare in man and when they do occur are never in the neighbourhood of articular condylar surfaces, but in head of tendon close to femur and so removed far from site of *fabella* in man (p. 68).

records of other anatomists because his own large but necessarily limited experience has not confirmed theirs*.

Again in new-born mammals he admits that hyaline cartilaginous *fabellae* are to be seen (pp. 69—70) "as small faint points or streaks macroscopically visible," and yet he asserts that no traces of the *fabellae* are to be found in man before 10 years of age, although elsewhere he emphasises the point that the *fabellae* in man and the lower animals are "völlig analog." We believe that sparsity of material and the comparative ease with which when they do occur (as anomaly) they would escape notice probably account for the discrepancy. The statement that orthosesamoids are always prefigured by hyaline cartilaginous hemisesamoids is very definite and emphasised in the very title of Gruber's paper. All statements as to sesamoids arising from ossification of fibro-cartilage are Gruber tells us incorrect (p. 68). If Gruber be right then we have another and striking instance of how erroneous statements are still perpetuated in the history of sesamoids. For Pfitzner writing only 17 years later, and referring to Gruber's memoir† as a classic of the subject still speaks of our hemisesamoids as developing from "faserigen Gewebe" (S. 563), and even as late as 1897 in the *Seventh Report of the Committee of Collective Investigation of the Anatomical Society of Great Britain* we find sesamoids divided into two classes: "firstly those in which the bodies are osseous, and secondly those in which they were of some other material (fibrous, fibro-cartilaginous, or cartilaginous)‡. Thus if Gruber's statement be correct, it has not got current in the course of 20 years in English anatomy; if incorrect, not even as accurate a German as Pfitzner thought it desirable apparently to issue a formal contradiction! Yet Gruber wrote in 1875 that true sesamoids are always prefigured by the existence of hyaline cartilage, never by fibro-cartilage (p. 65). This prefiguration, he tells us, had not been previously noted in man, and had not been adequately emphasised in mammals generally. All statements as to sesamoids arising from ossification of fibro-cartilage are incorrect (p. 68). Surely such a statement from such an authority deserved at least a contradiction if incorrect?

Yet the anatomists still talk glibly of the origin of sesamoids by intensive stress in fibro-cartilage, and it is left to biometricians to confirm or confute Gruber! We were not able to make an intensive investigation, but we pursued two lines of research.

(i) We had sections taken of the *fabellae* and the *cyamella*, or rather of the nodules where we should suppose them to be, of a kitten at birth. The lateral *fabella* and the *cyamella* exhibited a kernel of hyaline cartilage. We were less successful in our sections of the mesial *fabella*, either it was absent, or else the very

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* All statements during 319 years of the existence of a mesial *fabella* in man are "durchaus irrige" (p. 67), Gruber even goes so far as to assert that owing to the *Bursa mucosa supracondyloidea interna* (discovered by him) the internal *fabella* in man would be rather harmful than valuable if it occurred in the neighbourhood of this *bursa*, and if some 2 cms. higher than the external *fabella* would be useless (p. 68). This appeal to "use" is odd.

† Pfitzner gives the wrong volume T. XXIV instead of T. XXII of the St Petersburg memoirs as the locus of Gruber's monograph.

‡ See also Humphry, Quain and other English anatomists for the fibro-cartilaginous sesamoid.

small prefiguration of it had been ejected in the sectioning as one or two cavities were visible in the section. Either alternative is possible as the nodule is much harder than its environment and the knife may slip on it, and again, not only is the mesial *fabella* not infrequently absent in the cat, but it develops far less rapidly than the lateral.

(ii) We asked Professor Elliot Smith if he could provide any information as to the origin of sesamoids in general from a human foetus. The *fabella* and *cyamella* not being universal in man the patella and a metatarsial sesamoid were selected. He placed the matter in the hands of Mr R. B. Green, who has most kindly sent us the accompanying report.

Note on the Histology of Sesamoids in the full-term human Foetus.

Longitudinal sections through the developing sesamoid bone in the tendon of the *Flexor brevis Hallucis* show the cartilage to be essentially of the hyaline variety. At the periphery of the cartilaginous nodule a narrow transition zone between the cartilage and the surrounding young connective tissue may be seen. In this zone fine fibres are interspersed in and gradually fade away into the hyaline matrix. Other fibres pass round the periphery of the cartilage from one extremity to the other.

Longitudinal sections of the patella in the full-term foetus show a similar arrangement in which those fibres which are a continuation of the proximal to the distal tendinous attachment on the surface of the cartilage are more marked.

It will thus be seen that independent lines of investigation confirm Gruber's statement and there is little doubt that if it holds from cat to man, we are fairly safe in asserting that sesamoids are always prefigured by hyaline cartilage*.

The importance of this conclusion is very great. If sesamoids are always prefigured by hyaline cartilage, and do not appear without this origin, and further if this origin can be detected in the foetal state, it is clear that the presence or absence of hyaline cartilage in the foetal state is the condition for the presence or absence of either hemisesamoid or orthosesamoid; and the theory that they are produced by intensive stress or friction falls to the ground, unless it, indeed, be asserted that such factors have produced antenatally the hyaline cartilage nodule! Unless fibrous cartilage could first be converted by stress into hyaline, it is hopeless to maintain that such stress could produce sesamoids. But a study of the regions intermediate between fibrous and hyaline cartilaginous areas appears to indicate that, while there is a form of cartilage which is neither and yet may become either, the one is not an antecedent of the other. In our Plates XVII, XVIII are shown (i) the section of the kitten's *cyamella* and the enlargement of the hyaline cartilage, (ii) the section of the kitten's lateral *fabella,* (iii) the section of the metatarsial sesamoid in the human foetus and (iv) the section of the foetal human *patella.* A more complete study of such material would we believe be of great interest. At present we are only concerned with the fact that the master-craftsman was correct and the text-book writers, who still cling to mediaeval traditions and hypotheses, are wrong.

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* Since the above words were written, we have been able to confirm Gruber's statement in the case of the dog, the rabbit, and other mammals. The results will be discussed in the next instalment of this memoir.

While Gruber asserts the universal absence of the mesial *fabella* in man, he draws attention to the fact that in other mammals the nature of the mesial is differentiated from that of the lateral. Lateral *fabellae* may always occur without mesial; or the mesial may occur occasionally; but there is no type of life in which the mesial always occurs without a lateral *fabella*. This differentiation both in man and other mammals between mesial and lateral *fabellae* is very significant and should find its expression in any theory of origin*. Among other points dealt with by Gruber are the shapes and sizes of the *fabellae*. He divides them into 20 shapes to which he gives mathematical terminology, the most common being tetrahedra, cones, half ovals, four-sided pyramids, oval or circular plates or discs. There is no mathematical accuracy in these descriptions, see for example his Table IV, Figs. 9 and 10 (which he calls a three sided pyramid form!), pp. 30–31. All that we can say is that the *fabellae* are very variable in shape, and that they may be pyramidal as in the marsupial type, or lens-form even to spherical as in the apes, and this only in the roughest resemblance. The "base" is usually smooth. It seems to us that the shape largely depends on the number of muscle-attachments of which the *fabella* is in any case the node, but this point is not emphasised by Gruber. On pp. 35–38 certain measurements are given. The greatest weight of any human *fabella* was 0·847 gr. and the smallest weight 0·009 gr. In the greatest tetrahedral form the height was 13·5 to 14 mm. and the base is given as 10 mm. diameter.

We now come to a most interesting consideration of the *locus* of the external *fabella* in man. Gruber gives three situations for the gastrocnemic ossicles in man. The first of these (p. 38) is the *sulcus popliteus externus*! Now if the reader will examine the case described by Gruber on his p. 38 and illustrated in Fig. 1 of his Table I† (see our Plate V, Fig. 10) I think he will agree that it is no *fabella* at all, but a true *cyamella* and probably most of the pyramidal *fabellae* described by Gruber are really *cyamellae*. Now both *fabella* and *cyamella* occur simultaneously in the knee-joints of many genera and are frequent in the apes. Hence it is remarkable that Gruber should have overlooked this important distinction, and confused a sesamoid occurring in the *depth* of the *sulcus popliteus externus* (p. 71) with a *fabella* of the tendon of the *M. gastrocnemius*. Gruber even tells us that the hump due to this sesamoid can be so large that it can be mistaken for an exostosis; in the extended leg the hump can be so great as to be felt or even *seen*‡ (p. 71). Now there is, perhaps, more reason for this confusion than

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* Confirmation of this differentiation of origin of the *fabellae* is to be found (i) in the fact that the two *fabellae* are not equal in size. It is possible that in the apes the mesial may occasionally be larger, but in all other species it is smaller and frequently much smaller, (ii) the position of the *fabellae* is often different, (iii) ossification of the external *fabella* begins and ends sooner, (iv) the mesial more often than the lateral fails to ossify, and (v) the microscopic structure may be different.

† This figure represents what is almost certainly a *cyamella* embedded in *M. gastrocnemius*, *Ligamentum popliteum* and short external lateral ligament.

‡ "Der Höcker von dem Ossiculum oder der Cartilago wird 2·6—4·2 cms. (in medium 3·325) über der Spitze der mittleren, höchsten Zacke als dem höchsten Punkte des *capitulum fibulae* gefühlt oder sichtbar" (p. 39).

appears if we loosely speak of the *fabella* as lying in the tendon of *M. gastrocnemius* and the *cyamella* as in the tendon of *M. popliteus.*

We have already prepared the reader by citing Cheselden and Humphry (see our pp. 150, 153) for such not being the case; both *cyamella* and *fabella* are or may be the node, "Knotenpunkt," or junction of a whole network of fibrous structures, tendons and ligaments. Thus Gruber himself tells us (p. 69) that the lateral *fabella* is at the junction of *M. gastrocnemius externus* and the *Ligamentum popliteum* at the point before *M. gastrocnemius* is free of the knee-capsule and where the short lateral ligament takes its departure. For this reason Gruber asserts that the external *fabella* serves (i) to strengthen this junction, (ii) to keep the "Knotenpunkt" on the articular surface of the condyle, and (iii) to prevent *M. gastrocnemius externus* slipping on to the lateral face of the external condyle. This explanation does not seem to us of great validity for if the *fabella* were of such great service, how would nine-tenths of the human population get on without it? But this conception of the *fabella* as a node to which there are many muscular attachments not only accounts largely for its variety of form, but also leads us to seek for its origin in something to which such muscular attachments were peculiar.

Another most suggestive point here is that Gruber noticed occasionally a supernumerary head[*] of *M. popliteus* (see our p. 154 and our Fig. 9, Plate IV) springing directly from the *fabella.* Thus the two heads of *popliteus* are brought into touch with the two lateral sesamoids and this division of the head is not inconsistent with the division of an original structure. Gruber's confusion of lateral *fabella* and *cyamella* has in itself a slight contributory significance.

More than once Gruber speaks of the controlling power of the *fabella* whether orthosesamoidal or hemisesamoidal over the ligaments and tendons. Thus on p. 72 he notes that in man its movements control: *M. gastrocnemius externus, M. plantaris, M. semimembranosus,* and *M. popliteus biceps* by means of its supernumerary head supposing this abnormal muscle present[†]. Whenever both *M. plantaris* and the *fabella lateralis* exist in man, a portion of the former, or the whole of it, originates in this sesamoid (p. 45). In short in the case of man in the extended position of

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[*] On seven occasions this supernumerary head sprang from the *fabella*; on four occasions without a sesamoid this anomalous head originated from the head of the oblique popliteal ligament, or and also from the knee-capsule, pp. 45—6. We think, but it is not quite clear, that Macalister (see our p. 154) also had associated the supernumerary head of *M. popliteus* with a sesamoid. See also J. F. Knott, *Proc. R. Irish Academy*, Second Series, Vol. III, p. 639, 1882.

[†] Again p. 48 we read: "Da nun das Ossiculum und die Cartilago gerade im Knotenpunkte der Vereinigung der Kniekapsel, der Sehne des *Gastrocnemius externus*, des *Lig. popliteum* und des *Lig. laterale externum breve genu* sitzen, da von denselben der *Gastrocnemius externus* mit Fleischbündeln abgehen kann, immer der *Plantaris* davon entspringt, oder falls er fehlt, durch von da kommende Fleischbündel des *Gastrocnemius externus* ersetzt wird, bisweilen von denselben der *Popliteus* mit einem supernumerären Kopfe entspringt; so werden auch diese Muskeln die Verschiebung des *Ossiculum* und der *Cartilago* dirigiren, und es werden durch denselben Apparat, durch den die im Bereiche des *Condylus externus* laxere Kniekapsel vor Einklemmung geschützt wird, auch das *Ossiculum* und die *Cartilago* vor Einklemmung bewahrt. Beide werden bei diesem Manöver, und bevor es zur grössten Spannung kömmt, schräg ein—und aufwärts an dem *Condylus externus*, wenn auch nur eine mässige Strecke, gleiten müssen, um der Einklemmung zu entgehen, und bei der Streckung des Kniegelenkes wieder auf ihren Platz auf demselben Wege zurückkehren."

the limb tendons (*gastrocnemius, popliteus, plantaris*) and ligaments (oblique popliteal, and short external lateral) all united to the sesamoid hold it *in situ* on the posterior surface of the external condyle, near, or on, or in rare cases projecting over the outer condylar border (pp. 46—71).

The reader will recognise what an enormous stride the master-craftsman made in thus investigating the muscular attachments of the *fabella lateralis* in man! Yet even to-day the anatomical text-books content themselves by describing this sesamoid as occasionally occurring in the external head of *M. gastrocnemius,*—a rough description as old as the mid 16th century—and suggesting as Galen did that, as a sesamoid, it is scarcely worthy of study. If we disregard, as we think we must, Gruber's theory of the purpose of the *fabella lateralis* and his identification of the *cyamella* with this sesamoid, we have still to trace the origin of these muscular attachments and the undoubtedly close relations of these two sesamoids. If we discard the first *locus*, i.e. the popliteal sulcus, which Gruber gives to the *fabella lateralis* in man, we find him recording two alternatives*.

(i) Embedded in fibrous structures. The place of the embedment is the posterior wall of the knee-capsule where the tendon of *M. gastrocnemius externus* unites with the oblique popliteal ligament and with the origin of the short external lateral ligament or the partial union of some of these and the knee-capsule itself (pp. 39—40).

(ii) In a facet or groove on the condylar articular surface generally near the border. This facet is either flat or slightly concave, sometimes saddle-shaped† (p. 40).

In the case of the *fabella* lying in a facet the base in contact with the facet has according to Gruber no cartilaginous covering. In the case of the *fabellae* of all mammals—*Gulo vittalus* excepted—the base where they articulate has a cartilaginous covering (p. 71).

Accordingly it will be seen that it is hardly accurate to say of the *fabella* and the *cyamella* that the former is found in the tendon of *M. gastrocnemius* and the latter in the tendon of *M. popliteus*. It is more accurate to assert that the latter will be found in the popliteal sulcus, and the former on the posterior articular surface of the condyle, either somewhat removed from it, or in the close contact of a facet. The muscular attachments of both are found to be in part closely similar. Thus Gruber's error in failing to distinguish between the *fabella* and *cyamella* in man,

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* It is singular how very closely Gruber's description of the sesamoid of the popliteal sulcus (our *cyamella*) agrees in its muscular attachments with those of the *fabella lateralis* in its embedment locus. Thus he writes: referring to Fig. No. 4 of his Tafel (see our Plate V, Fig. 10) "Das *ossiculum* und die *Cartilago* liegen im *Sulcus popliteus externus* von fibrösen Gebilden (a, b, d [i.e. Knee capsule, oblique popliteal ligament and short external lateral ligament]) mit welchen sie verwachsen sind, und von Muskelbündeln (k, l, [i.e. *M. gastrocnemius externus, M. plantaris*]) welche von ihnen entspringen umlagert." It will be seen that when both sesamoids are not simultaneously present—as they are in the case of the cat which has both lateral *fabella* and lateral *cyamella*,—there is some reason for confusing the two.

† A flat facet occurs in $\frac{4}{7}$ of such cases, a really deep groove is very infrequent (p. 41).

while noteworthy, if he had started from other primates where both coexist, is not so remarkable when he started from the musculature in man, where both rarely coexist.

The bulk of the remainder of Gruber's monograph is taken up with a discussion of the *fabellae* in other living forms than man. He made a long study of the literature of comparative anatomy and added many observations of his own. To these matters we shall return in the following section of this memoir.

Before we leave the topic of the *locus* of the *fabella lateralis* in man we think some study of our Plates I and II is desirable. In Plates I and II, Figs. 1, 4, 5 represent the strange positions of the *fabella* provided by Trew, Heister, and Casserius. Fig. 4 gives the position as envisaged by Camper, while Fig. 5 first shows some daylight as cast on the subject by Humphry*. In Plate I, Fig. 3 we provide a copy of Gillette's representation. It is clearly a case of *Locus II*, the *fabella* resting in a flat facet on the condyle. Plate VI, Figs. 14 and 15 give a knee-joint from the Dissecting-Room at University College, London, to which our attention was drawn by Dr D. E. Derry. It is again a case like Gillette's of a facet *fabella*. The facet lies somewhat high on the articular surface of the condyle. Plate V, Fig. 10 gives Gruber's representation of a sesamoid in the popliteal sulcus. Fig. 12 is a good representation from Gruber of a "Knotenpunkt" *fabella*. Plate IV, Fig. 9 is an illustration of the *M. popliteus biceps* with the supernumerary head attached to the sesamoid; it also shows the short lateral ligament likewise attached and the relation to *M. gastrocnemius externus*. Plate V, Fig. 11 shows the relationship of the *fabella* to *M. plantaris* as pointed out by Cheselden and Humphry, while Plate IV, Fig. 8 gives the only mesial ossicle observed by Gruber, an obviously pathological ossification as he states.

We may say that both *locus* and muscular attachments had been fully studied as far as the lateral *fabella* is concerned by 1876.

(*c*) *Pfitzner's Monograph.* "Die Sesambeine des Menschen." Schwalbe's *Morphologische Arbeiten*, Bd. I. S. 517—762, Jena, 1892. We have already referred to many points of Pfitzner's work (see our pp. 134, 137, 158). He rejected the intensive stress hypothesis and carried the subject a step beyond Gruber by asserting a vestigial rather than a "use" origin. Sesamoid bodies (hemisesamoids) which he considers develop in "faserigen Gewebe" are "Abortivzustande" of sesamoid bones (orthosesamoids), p. 571. He places the *fabella lateralis* on the top and edge of condyle (pp. 568 and 578)—a position in accord with some of Gruber's observations, but differing widely from our own skiagram series. Only in one case (p. 581) did Pfitzner find the *fabella* in a facet on the condyle, which he speaks of as being its original place. While Gruber uses the word "gleiten" = glide, for the action of *fabella* on articular surface of condyle, Pfitzner states that the sesamoid is incapable of moving on the condyle (p. 548), although it is never united (verschmolzen) to the femur.

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* Gruber considers that Weitbrecht and Humphry first placed the lateral *fabella* in its true position. We have already (p. 153, footnote) referred to the grave doubts which arise about what the former intended.

Pfitzner is less dogmatic than Gruber as to the existence of a mesial fabella in man allowing a certain amount of weight to the records of Heister, Morgagni, Hyrtl and Macalister (see our p. 155), but he states that he has never seen it himself (p. 580). On the other hand he has always found it in dog, fox and hare, while in 52 cats he found the *fabella lateralis* always present, but the *fabella mesialis* was absent in 29 out of the 52 cases.

Turning to the *cyamella lateralis*, which Pfitzner describes as in the tendon of *M. popliteus* and working on the curved continuation of the articular surface of the *Condylus lateralis tibiae*, he states that it is constant in *Felidae* and *Leporidae*, and that he has himself *once* found it in a dog. He says that it has never been noticed in man. If we can trust the observations of Macalister, Krause and Knott, and if we have not misinterpreted those of Gruber this statement is inaccurate (pp. 583—584). Lastly as far as we are aware Pfitzner is the only person who has recorded a *cyamella mesialis*, and this only in two cats (p. 584). He was unable to note the muscular attachments. We have gone through a considerable number of dog skeletons in the Biometric Laboratory, in which as a rule the *fabellae* are preserved, but have met with no cases of *cyamellae* being preserved.

The literature of the sesamoids of man is fairly comprehensively treated by Pfitzner, and we have found his references very helpful in our search into the history of our knowledge of the sesamoids in man. We shall refer to his statistics of occurrence in man later. On the whole while Pfitzner's memoir is of great value—in particular he recognised the *cyamella*—it is not epoch-making in the sense of Gruber's.

(5) *Frequency of the Fabellae in Man.*

Putting aside the vague statements of the mediaeval anatomists we have five sources from which this frequency may be determined (i) Gruber, (ii) Ost, (iii) Pfitzner, (iv) the *Seventh Report of the Collective Investigation Committee*, (v) Skiagram Data.

We can dismiss Ost at once; his paper* is very brief and refers to only 30 cases in which he found a *fabella* five times or 16·7 $^{\circ}/_{\circ}$; sex is not given. Pfitzner observed hemisesamoids or orthosesamoids in 30 out of 290 cases or in 10·34 $^{\circ}/_{\circ}$. He never met with a *fabella mesialis*. While Pfitzner presents little analysis of his cases Gruber provides a considerable amount, but it is occasionally far from clear as to what exactly he is referring to.

Thus Gruber tells us (p. 70) that without regard to size or sex it appears in about $\frac{1}{6}$ of the cases, i.e. 16·7 $^{\circ}/_{\circ}$. On p. 26, however, he tells us that in 2340 limbs he found it in 400 instances or 17·1 $^{\circ}/_{\circ}$. Paying regard to sex Gruber states, again on p. 70, that he found it in about $\frac{1}{5}$ of the male cases and $\frac{1}{3}$ to $\frac{1}{4}$ of the female; according to this the lateral *fabella* occurs in 20 $^{\circ}/_{\circ}$ of cases in man and

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* Ueber das Vorkommen eines Sesambeines in den Ursprungssehnen des *Gastrocnemius* beim Menschen. *Zeitschrift für Anatomie und Entwickelungsgeschichte*, Bd. II, S. 309, Leipzig, 1881.

25°/o to 33°/o in woman. Turning back to p. 26 we find that Gruber makes the following statements:

Ages 10 *to* 83* *without regard to sex:*

220 cadavers or 440 limbs, 66 lateral *fabellae*, i.e. 15 °/o.

Sexed Cases:

400 male limbs, lateral *fabellae* in 77 cases,
100 female limbs, lateral *fabellae* in 22 cases,
i.e. Males 19·25 °/o, Females 22·0 °/o.

Extremities, Age unknown. Both sexes:

1400 with 235 lateral *fabellae*, i.e. 16·8 °/o.

Without Age, but sexed:

Male limbs 294 with 61 lateral *fabellae*,
Female limbs 98 with 29 lateral *fabellae*,
i.e. Males 20·8 °/o, Females 29·6 °/o.

Returning to the 2340 limbs, Gruber tells us, however, that (p. 26)

940 male limbs gave 165 lateral *fabellae*,
840 female limbs gave 143 lateral *fabellae*,
i.e. Males 17·6 °/o, Females 17·0 °/o.

Finally we read (p. 27) that

1155 right limbs gave 212 lateral *fabellae*,
1185 left limbs gave 188 lateral *fabellae*,
i.e. Right Limbs 18·4 °/o, Left Limbs 15·8 °/o.

We find it extremely difficult to deduce anything like definite percentages from this tangle of figures, and it would seem that this huge mass of material has been largely wasted for statistical purposes. We can only suppose that the difference of percentages arises from the proportion of young limbs which may be included in the different series.

We may, perhaps, conclude:

(*a*) That there is no significant sexual difference.

(*b*) That there may be preponderance of right limbed lateral *fabellae*.

(*c*) That for all ages and sexes the percentage (judged from 2340 limbs) is 17·1 °/o.

Why this percentage should run up to 22·0 °/o in the case of sexed femora of unknown age we are unable to say; or, why it should fall to 15·0 °/o in the case of the 220 cadavers, unless the younger ages have been omitted in the previous result, we cannot explain. Possibly the latter form an earlier part of the record, and that observation became more stringent and accurate as the investigation

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* The maximum frequencies at the decades indicate that these records of age are very rough.

progressed. The *fabellae* enumerated include both hemisesamoids and orthosesamoids. The results are in rough agreement with Ost's short series, but shew a far higher frequency than Pfitzner's record.

We now turn to the interesting *Seventh Report of the Committee of Collective Investigation of the Anatomical Society of Great Britain and Ireland for the year* 1896—1897*. We have already commented on the high percentage of lateral *fabellae* reached by Gruber, but these percentages are at least doubled in the statistics of this *Report*. Over 40 °/₀ of occurrence of *fabellae* are here recorded, and if this rate of increase of percentage continues we shall soon have to return to the standpoint of Vesalius! One wonders (i) if the distinction between limb and cadaver has always been maintained, (ii) if the distinction between condyle and limb has also been maintained†, and lastly supposing the results do indicate the number of *fabellae* per 100 condyles, (iii) if the "Knotenpunkt" itself in which the lateral *fabella* is to be sought has not been occasionally mistaken for a sesamoid‡.

In 287 limbs of both sexes examined, 81 lateral *fabellae* and 39 mesial *fabellae* were found. These figures provide 28·2 °/₀ of lateral *fabellae* and 13·6 °/₀ of mesial *fabellae*, i.e. about twice as many lateral as mesial. Now Pfitzner, Gruber and ourselves have not discovered in man any mesial *fabellae*. Hence we must put against Pfitzner's 10·34°/₀ and Gruber's 17·1 °/₀ the 41·8 °/₀ of the present investigation. But the matter takes a different form if we distinguish between osseous and non-osseous *fabellae*§. Of the 81 lateral *fabellae* only 28 were orthosesamoids; and of the 39 mesial *fabellae* only three were reported as osseous. In other words only 9·75 °/₀ of orthosesamoidal lateral *fabellae* occur, and 1·05 °/₀ of orthosesamoidal mesial *fabellae*. These results are very luminous for the controversies of the mediaeval anatomists. Supposing those three mesial orthosesamoids were actually and correctly diagnosed then if the mediaeval anatomists had, as they probably had an osseous body in view they might easily have examined a hundred or more limbs without finding a mesial *fabella*. It is even conceivable that Pfitzner might not find one in 290 limbs. It is less conceivable that they should have escaped Gruber in the case of 2340 limbs, and in the case of both Pfitzner and Gruber who were well acquainted with hemisesamoids, it is surprising that neither found a mesial hemisesamoid in 2630 limbs examined, when they ought to have found 331 on the scale of the *Collective Investigation Report*. It must be quite clear that these different investigators had a totally different conception of the nature of a non-osseous sesamoid, and that this is the origin of the surprising percentage differences we encounter in these investigations as to the presence of *fabellae*

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* *Journal of Anatomy and Physiology*, Vol. XXXII, p. 182, London, 1898.

† A condyle gives only one chance, a limb two chances, and a cadaver four chances of a *fabella*.

‡ The *Report* says: "With regard to the constitution of the bodies we have evidence that in many of the investigations no microscopical examination was made. We shall therefore content ourselves with dividing the cases in which sesamoid bodies occur into two classes,—first those in which the bodies were osseous; secondly those in which they were of some other materials (fibrous, fibro-cartilaginous, or cartilaginous)," p. 184.

§ It is not, perhaps, adequate to treat these as orthosesamoids and hemisesamoids, as they may contain fibrous structures, which Gruber and probably we ourselves should not class as hemisesamoids.

in man*. We believe that until the nature of the hemisesamoid has been standardised (for example, it is defined as a hyaline *cartilago* after Gruber, or otherwise), it is unwise to lay stress on any other statistics than those of orthosesamoids.

Out of a total of 79 *fabellae* found in male subjects by the *Collective Investigation* 22 were orthosesamoids, and of 28 *fabellae* found in female subjects 10 were orthosesamoids. Accordingly less than a third of the *fabellae* were osseous, or more exactly the chance of a *fabella* in man being orthosesamoidal is almost ·30. Thus we should expect in the 79 male *fabellae* 24 orthosesamoids and 55 hemisesamoids (as against 22 and 57 observed), and in the 28 female *fabellae* 8 orthosesamoids and 20 hemisesamoids (as against 10 and 18 observed). With a standard deviation in the males of about 4·0 and in the females of about 2·4, the differences of observed and computed are non-significant, or we cannot assert a sex-differentiation in ossification. This *pro tanto* is against an intensive stress theory of origin, for if orthosesamoids are produced by stress from hemisesamoids, we might reasonably expect greater stresses in the male, and a larger proportion of orthosesamoids in their total.

Again in a total of 146 right limbs 61 *fabellae* were found or 41·8 per 100 limbs. In 141 left limbs 59 sesamoids were found or again 41·8 per 100 limbs. It is therefore difficult to accept the statement of the *Report* (p. 186, (2)) that *fabellae* are rather more common on right than left limbs. In 157 male limbs 76 *fabellae* were found or 48·2 per 100 limbs; in 84 female limbs 27 *fabellae* were found or 32·1 per 100 limbs. This shows a greater preponderance of *fabellae* of all sorts in the male, although both sexes have the same ratio of orthosesamoids to hemisesamoids. This result contradicts that of Gruber (see our p. 165, (*a*)) and appears opposed to the conclusion drawn by the framer of the *Report* (p. 186) himself, who writes: "sex has little influence on their [i.e. the *fabellae's*] frequency."

The chief value of the statistics† in this *Report* is the confirmation they

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* Again on the scale of the *Collective Investigation Report*, Gruber and Pfitzner between them ought to have found between 26 and 27 mesial orthosesamoidal *fabellae*. They found *none*. Hence we are compelled to suppose either racial differences exist in this respect, or that one or other set of observers is in error.

† A further point is worth recording, namely the correlation between the appearance of *fabellae* in both heads. Clubbing together both sexes and both sides we have 241 limbs showing the following table:

Presence of *Fabellae* in the two Heads of *M. gastrocnemius.*

Outer Head.

		Present	Absent	Totals
Inner Head.	Present	13 (10)	20 (23)	33
	Absent	57 (60)	151 (148)	208
	Totals	70	171	241

provide of the occasional occurrence of mesial orthosesamoidal *fabellae* as asserted by Hyrtl, Macalister and others. But even assuming the correctness of the observation we are forced to the conclusion, that mesial orthosesamoidal *fabellae* are very rare in man, that mesial hemisesamoids are also much rarer than lateral, and that generally man confirms the view that there is a marked differentiation in origin between the lateral and mesial seasamoids of the knee-joint.

(6) *Skiagraphic Data for the Occurrence of the Sesamoids of the Knee-Joint in Man.*

Our data in this case consists of (i) 102 skiagrams of the knee-joint taken at a military orthopaedic hospital and examined by the kindness of Dr Stanley Melville, and (ii) 352 skiagrams of the knee-joint, which were found among the thousands of skiagrams taken at the Fulham Military Hospital and now deposited at the Royal College of Surgeons. We owe to the kindness of Professor Keith our access to this material. Dr Melville found seven cases of the presence of lateral *fabellae*, we found 22 certain cases and a doubtful four further cases. The results 6·9 °/₀ and 6·2 °/₀ to 7·4 °/₀ are in very good agreement. We think it safe to conclude that skiagraphically visible *fabellae* occur in about 7 °/₀ of human cases. If we suppose (i) that hemisesamoids would not show on the skiagrams and that (ii) they are twice as numerous as orthosesamoids we should reach about 21 °/₀ as the occurrence of the lateral *fabella* in man, i.e. a percentage approximating to Gruber's and that of the *Collective Investigation Report*, but much in excess of Pfitzner's. Dr Stanley Melville was not able to report a single case of the appearance of two *fabellae*, although there is a shadow on one of his plates (see our Plate XV, Fig. 33) which by a great stretch of the imagination might possibly be interpreted as a second *fabella*, much more distally situated than is customary. In the skiagrams from the Fulham Military Hospital we have not succeeded in finding any trace of a mesial *fabella*. Yet if the data of the *Collective Investigation Report* are to be trusted we ought to have come across five to six mesial orthosesamoids in the total of 454 limbs; we do not refer to the hemisesamoids for we assume they would not appear in skiagrams. Now while we cannot disregard the statements of such qualified observers as Hyrtl and Macalister, and dogmatically assert with Gruber that a mesial *fabella* has no existence in man, we feel bound to say that we feel it exceedingly elusive, and that we must emphasise the importance of the next recorder preserving an anatomical

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If we apply tetrachoric computation to this table—and the variety in size and constitution of the *fabellae* roughly justifies it—we find for the correlation

$$r = \cdot 1790 \pm \cdot 0885.$$

We see at once that there is no close and significant relation between the appearance or absence of *both fabellae* simultaneously, i.e. there is no close organic relation between them in man. In fact if distributed purely at random we should anticipate 10 cases in which mesial and lateral *fabellae* would appear simultaneously. We give the numbers which might be anticipated if the appearances were absolutely independent in brackets. This approach to independence in origin is certainly singular in the case of man, considering the universality of *fabellae* in so many lower types of life.

preparation, and further urging all able to do so to examine carefully skiagrams of the knee-joint, recording not only presence but absence of one or both *fabellae*[*].

The skiagrams give very accurately the position of the *fabella* with extended limb. That position is opposing the middle third of the posterior face of the lateral condyle. Generally we may say that while it is sometimes more proximal or more distal, it stands midway opposite to the vertical wall of the articular surface. As our skiagrams are not stereoscopic it is impossible to hazard any statement how far the *fabellae* are removed from the midplane of the condyle towards the lateral border of the articular surface. Our diagrams show what great variation there is in the size of these sesamoids.

In seven of our illustrations (Plates VII, VIII, IX, XV, XVI, Figs. 16, 17, 19, 20, 32, 33, 34) the *fabella* shows complete separation from the articular surface of the condyle; it is extremely unlikely that a facet could exist in these cases, and certainly not a grooved facet. In possibly three cases only (Plates VIII, IX, XVI, Figs. 18, 21, 35) is it reasonable to suppose that facets may have existed; in these three cases, however, it is hardly reasonable to suggest that the sesamoid is in the least depressed below the articular condylar surface. It would thus appear that the first of Gruber's positions for the *fabella*, i.e. embedded in the node of fibrous structures, is about, very roughly, two to three times as frequent as his second position, i.e. situated in a facet on the condylar articular surface.

Plate XV, Fig. 33 is the case we have already referred to (p. 168) in which there is some sign, still very elusive, of a second *fabella*. We are unable to lay any stress on it. It will be at once recognised that the drawings of Trew, Heister and Casserius hardly describe anything which appears in our skiagrams.

We may now draw attention to sesamoids of the knee-joint appearing in somewhat anomalous positions. Plate XIII, Fig. 29 shows apparently two small sesamoids opposing the lateral face of the external condyle, but somewhat removed from its surface. Plate XIV, Fig. 31 gives with less certainty a single sesamoid in a somewhat similar situation. The roundness of these individuals appears to preclude any pathological origin. They are too high up to represent in any way normal *cyamellae*. They might almost be described as intermediates in position between *cyamella* and *fabella*. Their almost vertical position over the head of the fibula is not without suggestiveness. But without evidence from the dissecting-room, or confirmatory skiagraphic records, we wish neither to assert that there is a third position possible for the *fabella*, or that occasionally an additional sesamoid of the knee-joint may occur in a position intermediate between that of *fabella* and *cyamella*. We know of no such sesamoids in lower types of life; for the time being we may call them x-sesamoids and if in the future no further evidence is forthcoming with regard to them, they may be dropped. We did not, however, like to pass them by without a word of record.

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[*] We shall be extremely grateful for any material with regard to sesamoids of the knee-joint which may be sent to us (Biometric Laboratory, University College, London, W.C. 1).

Cyamella in Man. It is very much less easy to differentiate the *cyamella* by aid of skiagraphic analysis. As it lies deep in the popliteal sulcus, it can hardly be distinguished by a *norma lateralis* of the knee-joint. A *norma frontalis* provided the *cyamella* is of considerable size appears to give the best chance. Thus Plate XI, Fig. 25 provides an illustration of what is probably a *cyamella lateralis.* Plate X, Figs. 22 and 23 shows another *cyamella lateralis* from posterior and frontal aspects. Plate XI, Fig. 24 gives our nearest approach to a *cyamella mesialis.* There is something similar in Figs. 22 and 23 of Plate X. We decline to interpret them and should refuse entirely to be dogmatic about them. But taken in conjunction with Pfitzner's statement as to what he had found in two cats (see our p. 164), we think it important that in the dissecting-room, the skiagraphic studio and the laboratory of comparative anatomy the investigator should bear in mind the slight possibility of a *mesial cyamella**. We have found two probable and one doubtful lateral *cyamellae* in our 352 cases, or in 0·57 °/₀ of cases probably and in 0·85 °/₀ possibly. Thus the *cyamella lateralis* in man first recorded by Macalister appears to be of great rarity.

Lunulae. Our search for these sesamoids so common in lower types was not very successful. In the first place the skiagrams do not exhibit ossifications of the semi-lunar cartilages at all clearly, and in the next pathological ossifications of the semi-lunar cartilages appear to be far from infrequent. Plate XII, Figs. 26, 27 and Plate XIII, Fig. 28 indicate what may be true ossicles of the semi-lunar cartilages. In the case of Fig. 27 the *lunula* is accompanied by what may possibly be an ossicle on the antero-lateral border of the tibia immediately above the *capitulum fibulae.* On the whole we found nine *lunulae* ranging through various shades of doubtfulness. Thus the occurrence of *lunulae* in man does not exceed 2·5 °/₀ of knee-joints *and is probably far less.*

We have not included the *patella* in our consideration of the knee-joint sesamoids, but in examining these skiagrams we have found apart from obvious pathological cases five or six of double *patella*†. We are prepared indeed to be told that even these six are pathological ossifications of the tendon of the quadriceps. But we are not equally certain that those who will take this view are fully aware of the regular occurrence of the double *patella* in certain lower types. Tillmans and Bernays‡ have found in man in several cases a hemi-sesamoidal body above the true *patella* and Bernays notes the same conditions in some of the *Felidae* and the rodents (Cat, Mouse, Rabbit). Pfitzner found such a double *patella* in the *Leporidae* (*loc. cit.* S. 577). In this consideration it is worth noting that *Macropus giganteus* has a hemisesamoid instead of an orthosesamoid

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* In the first and last cases an anatomical preparation should certainly be preserved of any discovery or supposed discovery. Our own case is not so definitely an osteome as Gruber's: see our Plates X and XI, Figs. 22—24.

† No percentage is possible as the anterior and posterior views of the knee-joint give no power of discriminating a double *patella.*

‡ "Die Entwicklungsgeschichte des Kniegelenkes," *Morphologisches Jahrbuch,* IV, Supplement, 1878, S. 442.

for its *patella*. Our Plate XIII, Fig. 28 and Plate XIV, Fig. 30 show typical cases of such double *patellae* in man. We suggest that in discussing the pathological nature of these cases, the possibility that they have vestigial associations should not be entirely overlooked*.

(7) *Summary of our Knowledge as to the Sesamoids of the Knee-Joint in Man.*

After 19 centuries of consideration, 18 of which were little better than fumbling, we can sum up our present knowledge of these sesamoids as follows:

(*a*) An orthosesamoidal lateral *fabella* occurs in man in about 7 °/₀ of knee-joints.

(*b*) In two to three times this number of cases there is a hemisesamoid, probably if indeed not almost definitely a hyaline *cartilago*.

(*c*) The situation of this *fabella* in about three-fifths of the cases is roughly opposed to the articular surface of the condyle near its most posterior point, but separated from it by 3 to 7 mms. It is the "Knotenpunkt" of a variety of muscles. In about two-fifths of its occurrences the *fabella* lies on a facet upon the articular surface of the condyle.

(*d*) There is some evidence for the far rarer existence of a mesial orthosesamoidal *fabella*, but both the evidence and this *fabella* itself are very elusive. The *Collective Investigation Report* records it in about 1 °/₀ of knee-joints. Gruber did not find a single case in 2340 knee-joints, and we have only found a single and very doubtful trace of it in 1 out of 454 skiagrams of the knee-joint. The same *Report* asserts that there are 13 times as many mesial hemisesamoidal *fabellae* as orthosesamoidal. Gruber again, and Pfitzner also, record none whatever. Macalister and Hyrtl and others have recorded their presence.

(*e*) The lateral *cyamella* has been reported by Macalister in man. We believe that we have found it in under 1 °/₀ of skiagrams of the knee-joint. If we interpret Gruber's statements correctly, he has erroneously termed the sesamoid he found in the popliteal sulcus a *fabella*, it should be a *cyamella*, and this is further evidence of its rare appearance as an anomaly.

(*f*) The existence of a mesial *cyamella* in man cannot be said to be established by the few doubtful skiagrams we have reproduced (Plates X and XI). Mesial *cyamellae* do not occur in any family of mammals universally. Pfitzner reports two cases in the cat, which always has a lateral *cyamella*. The sole interest of the anomalous appearance of a mesial *fabella* would be to strengthen the possibility that an organ which is persistent on one side of a limb may anomalously reduplicate itself on the other.

(*g*) There is some slender evidence of the rare occurrence of *lunulae* in man. They have not as far as we are aware been systematically sought for in the dissecting-room. We should expect them as rare anomalies in man as they exist universally in certain lower forms.

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* In the Fulham series of skiagrams also occurred four well marked cases of Gruber's *Processus supracondyloideus ossis femoris*: see Pearson and Bell, *Monograph on the English Longbones*, Part I, The Femur, Text pp. 85—89, and Atlas, Plates XXVI—LXVIII.

(*h*) Three hypotheses have been provided for the existence of the sesamoids of the knee-joint, namely:

(i) The theory of their manufacture from cartilage by intensive stress or even by friction. Against this hypothesis it is adequate to cite the cartilaginous *patella* of the giant kangaroo, the persistent *fabellae* of the smaller primates, and the relatively rare occurrence in man, the persistent *lunulae* of the squirrel and the absence of *lunulae* except as anomalies in man; the fact that the male has no higher percentage of *fabellae* than the female, and the general absence of *fabellae* in the greater and more massive anthropoids.

(ii) The theory that the lateral *fabella* serves a useful purpose as strengthening the "Knotenpunkt" of a congeries of muscular attachments. Such a hypothesis of use value might be valid, if man were a living form developing for the first time a *fabella*. On the contrary more primitive types of the primates have it universally and the higher types appear to have lost it or at least to be losing it. If the lateral *fabella* in man was there because it was of great service it would be difficult to explain why 7 °/$_{\circ}$ need it and 93 °/$_{\circ}$ get on very well without it!

(iii) The theory that the *fabellae* and the *cyamellae* are vestiges of some structure of earlier form, which has disappeared, and that hemisesamoids are not stages towards orthosesamoids, but the last steps in a degenerative process. Any such theory of the sesamoids of the knee-joint if valid will have to account for the marked differentiation in frequency, size and constitution of lateral and mesial sesamoids, and it will have to give some account of the structure of which they may be supposed to be the debris. It will also have to account for the muscular attachments of the sesamoids which we now find associated with the knee-joint in man.

This third hypothesis seems to us the only reasonable one hitherto proposed for the existence of the sesamoids of the knee-joint, and we shall therefore endeavour to trace the evolutionary history of the *fabella* and *cyamella*. If we observe that in the earlier primate forms the sesamoids of the knee-joint are osseous, we can dismiss the question of hemisesamoids; the only account we need give of them is that they are a measure of the extent of degeneracy, signposts to where an orthosesamoid once existed, and not the beginnings of sesamoids awaiting old age or intensive stress to make them osseous. We turn now to a study of the sesamoids of the knee-joint in living forms other than man, endeavouring to combine our own observations with those occasionally contradictory *inter se* of other observers.

(*To be continued*)

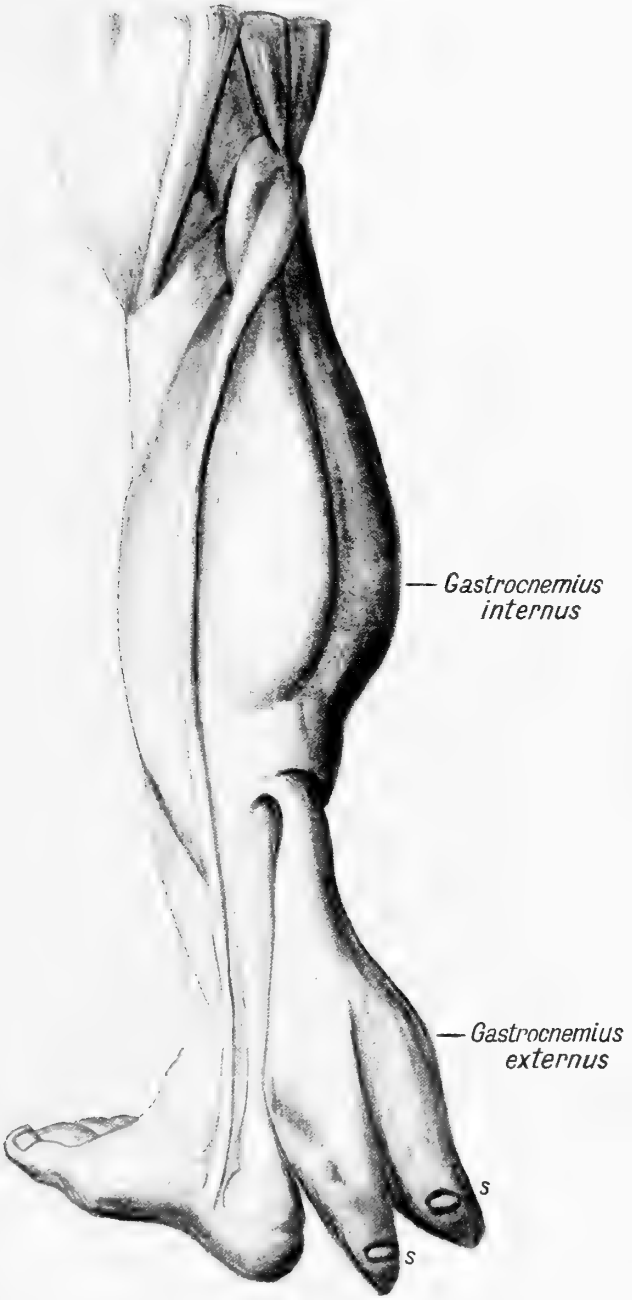


Fig. 1. First attempt at illustration of the *fabellae* in the two heads of *gastrocnemius*. From Casserius' *Tabulae anatomicae*, 1632.

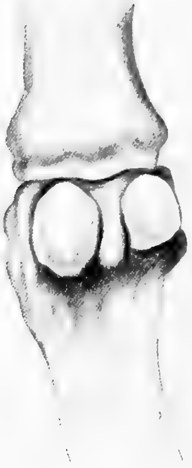


Fig. 2. (c. nat. size.) Sesamoids of first metatarsal of foot in man. Sesamoids known to the Greeks. After Pfitzner.

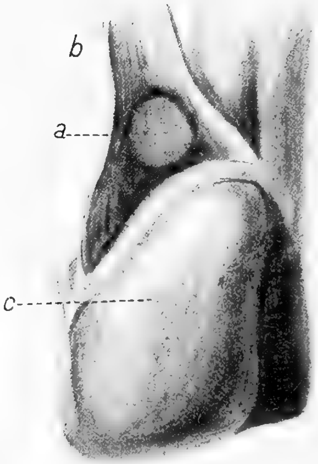


Fig. 3. *Fabella lateralis* as figured by Gillette, 1872. See description of plates.

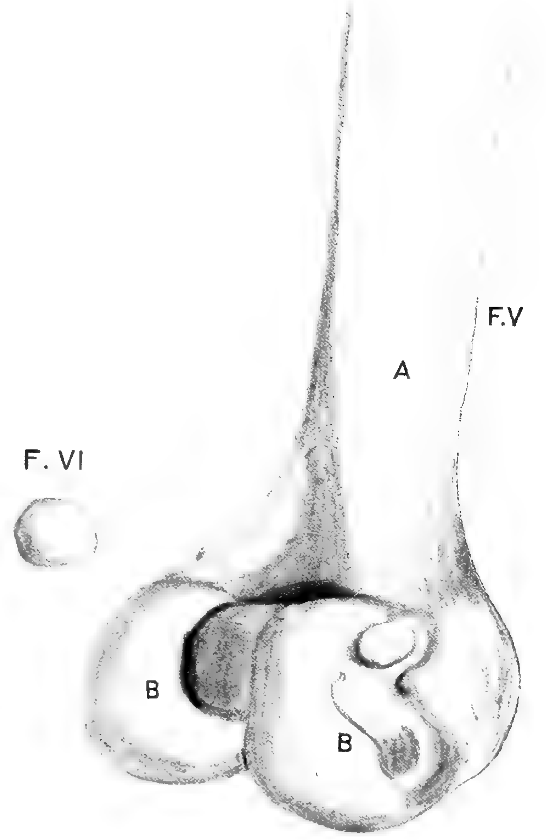


Fig. 4. Trew's rediscovery of the *fabella lateralis*, 1715. Note the unusual position given to sesamoid *C*. (The letter *C* should have been inserted above right hand *B*.)

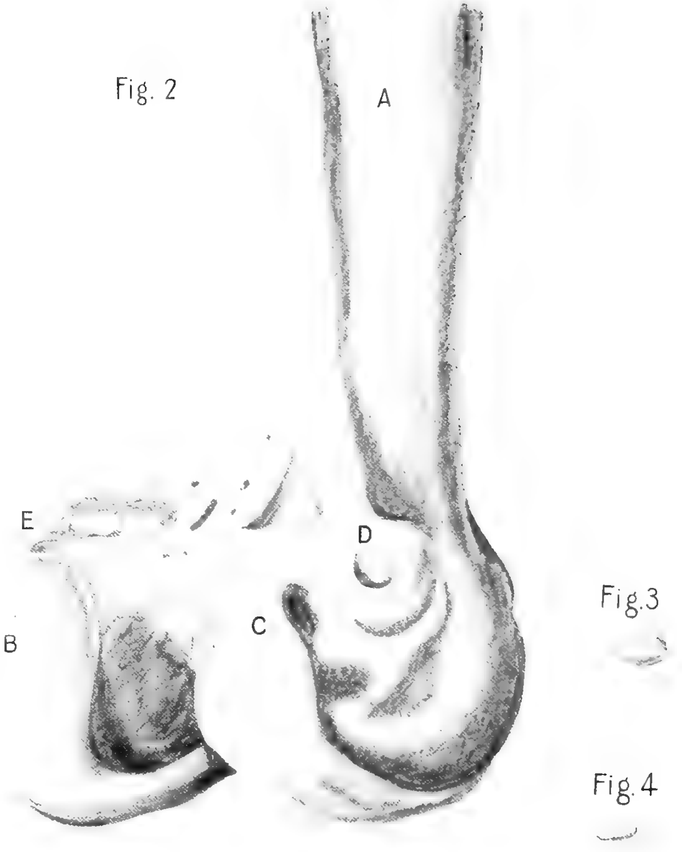


Fig. 5. Heister's "improvement" on Trew; he provided a small *fabella mesialis* in 1732. Both his sesamoids are in very unusual if not impossible places. See description of plates.

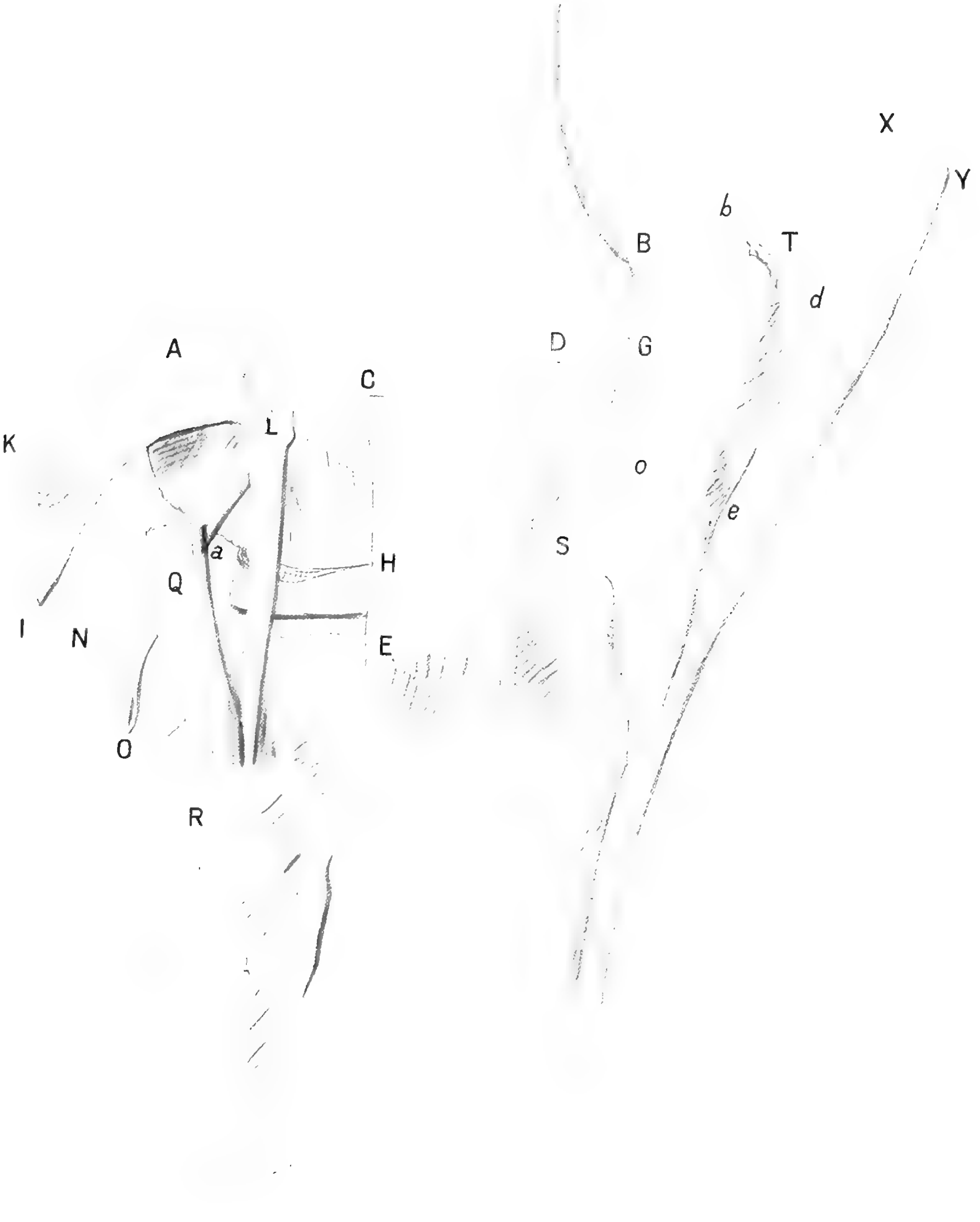


Fig 6. (c. nat. size.) Peter Camper's illustration of the external *fabella* (near *K*) in the tendon of *gastrocnemius*, 1754.

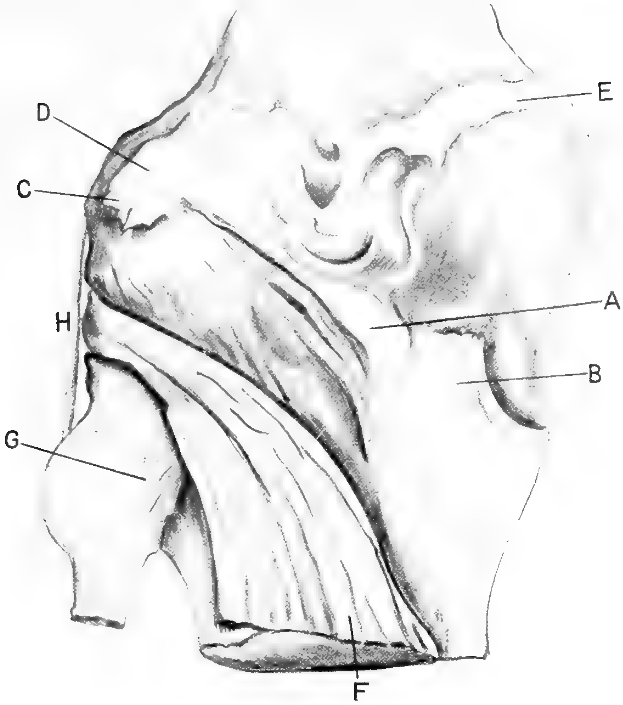


Fig. 7. (c. $\frac{4}{5}$ nat. size.) Sir George Humphry's illustration of the *fabella lateralis*. The sesamoid in correct position, 1858. See description of plates.

Fig. 8. (c. $\frac{1}{2}$ nat. size.) Pathological ossification at head of internal *gastrocnemius*, Gruber, 1876.

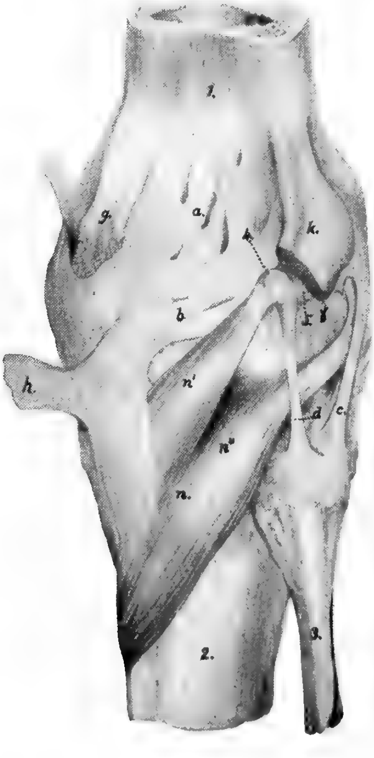


Fig. 9. (c. $\frac{1}{2}$ nat. size.) *M. popliteus biceps*, the additional head as well as the *ligamentum posticum* and *M. gastrocnemius* joining the *fabella lateralis*, Gruber, 1876.

Fig. 10. (c. $\frac{5}{8}$ nat. size.) Ossicle (4) at "Knotenpunkt" of *M. gastrocnemius* (*n*), *Ligamentum popliteum* (*b*), and *Ligamentum posticum* (*d*), Gruber, 1876.

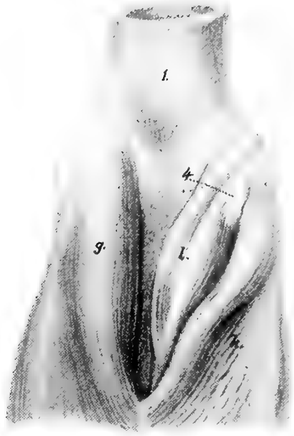


Fig. 11. (c. $\frac{3}{8}$ nat. size.) External *fabella* (4) in relation to *M. gastrocnemius* (*k*), and *M. plantaris* (*l*), Gruber, 1876.

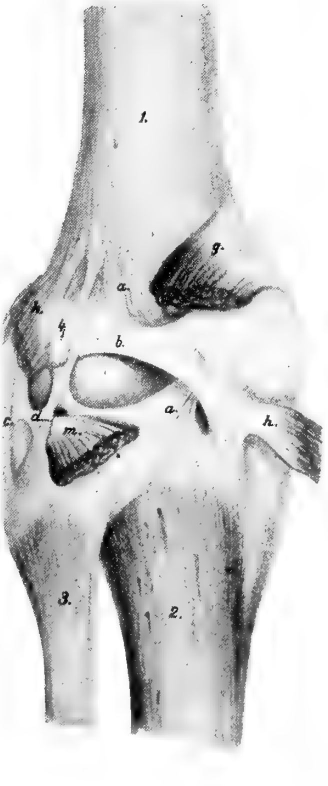


Fig. 12. (c. $\frac{3}{8}$ nat. size.) External *fabella* (4) in relation to *M. gastrocnemius* (*k*), *Ligamentum popliteum* (*b*), and *Ligamentum posticum* (*d*), Gruber, 1876.

Fig. 13. (c. nat. size.) Sesamoid on the head of the fibula in the case of the Wombat, Pfitzner, 1892. See description of plates.

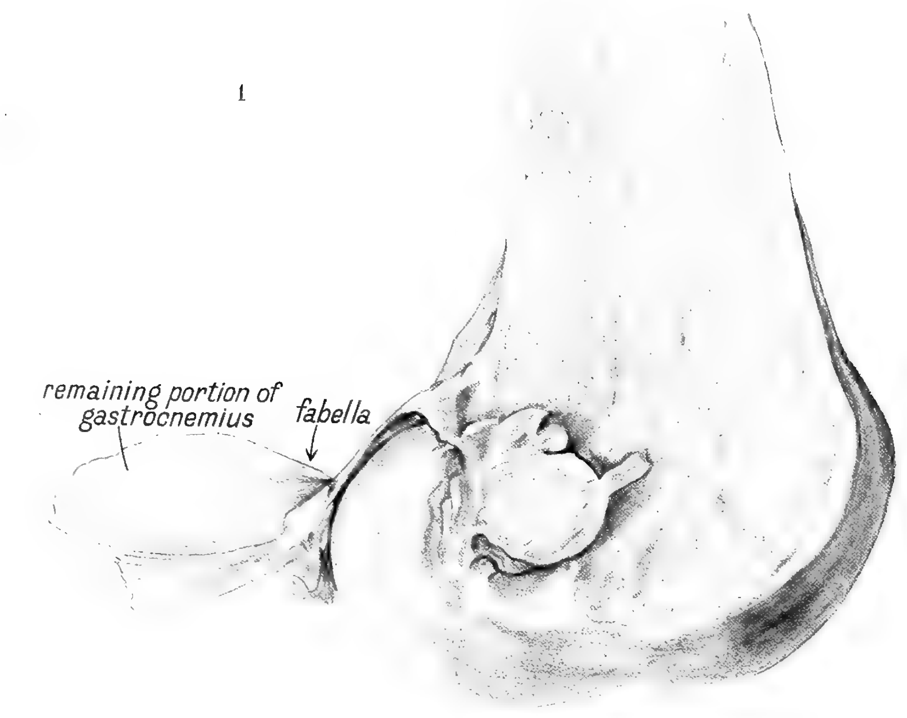


Fig. 14.

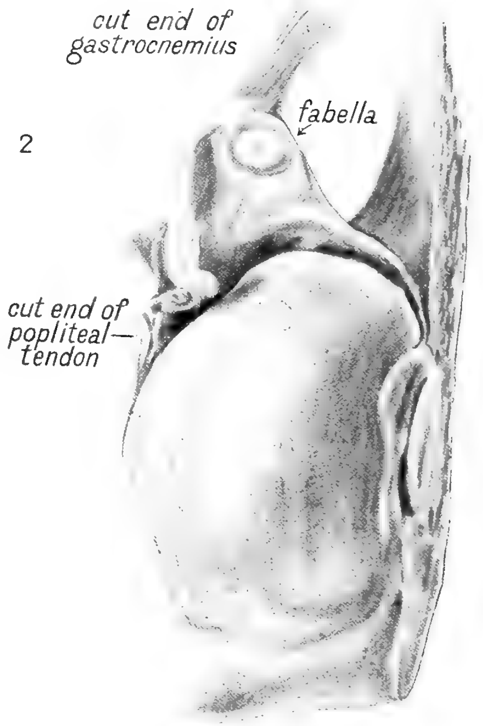


Fig. 15.

Drawings illustrating the position of the external *fabella* in man from a case occurring in the Dissecting-Room at University College, 1919. This is a relatively rare position: compare the skiagrams of the knee-joint. See description of plates. Circa $\frac{5}{4}$ nat. size.

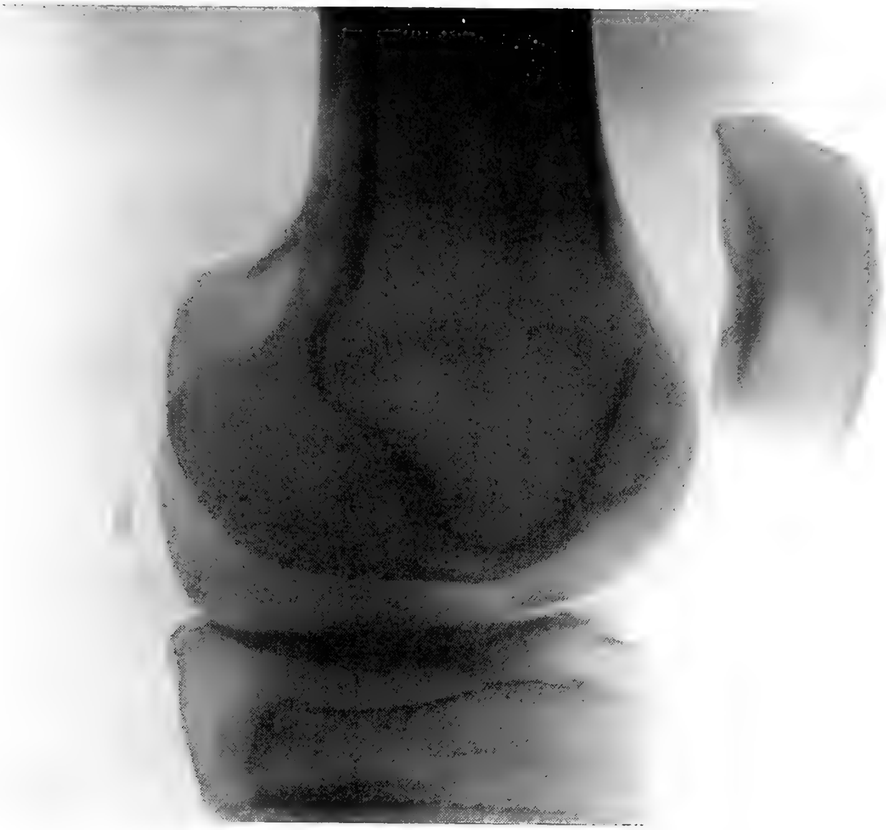


Fig. 16. Small lateral *fabella*.

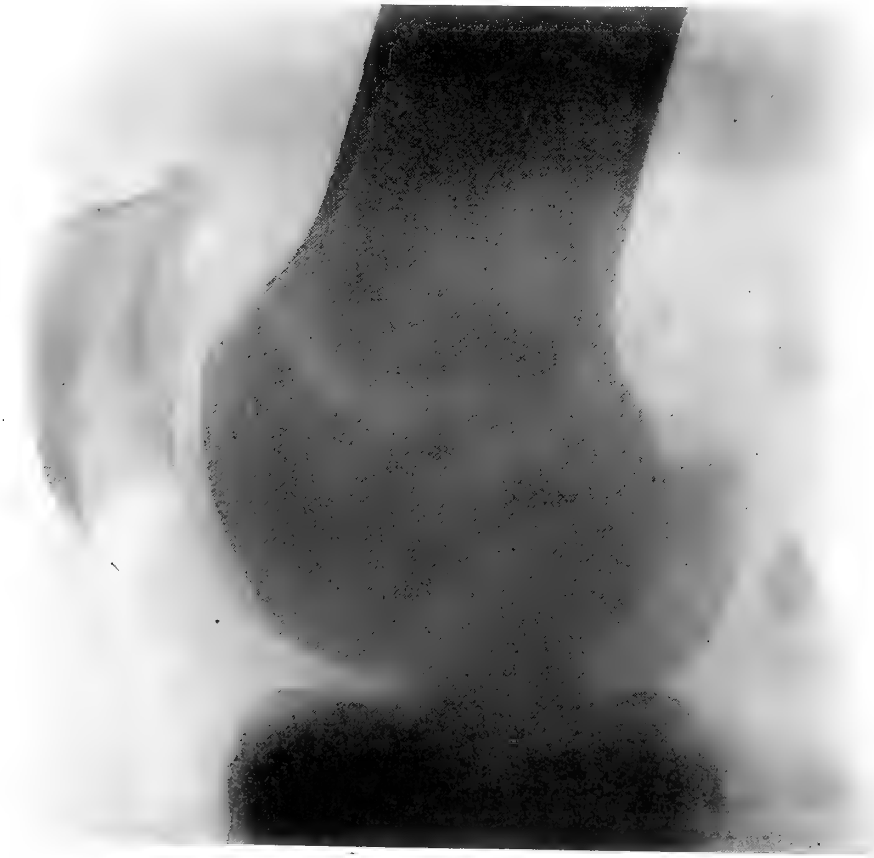


Fig. 17. Large lateral *fabella*.

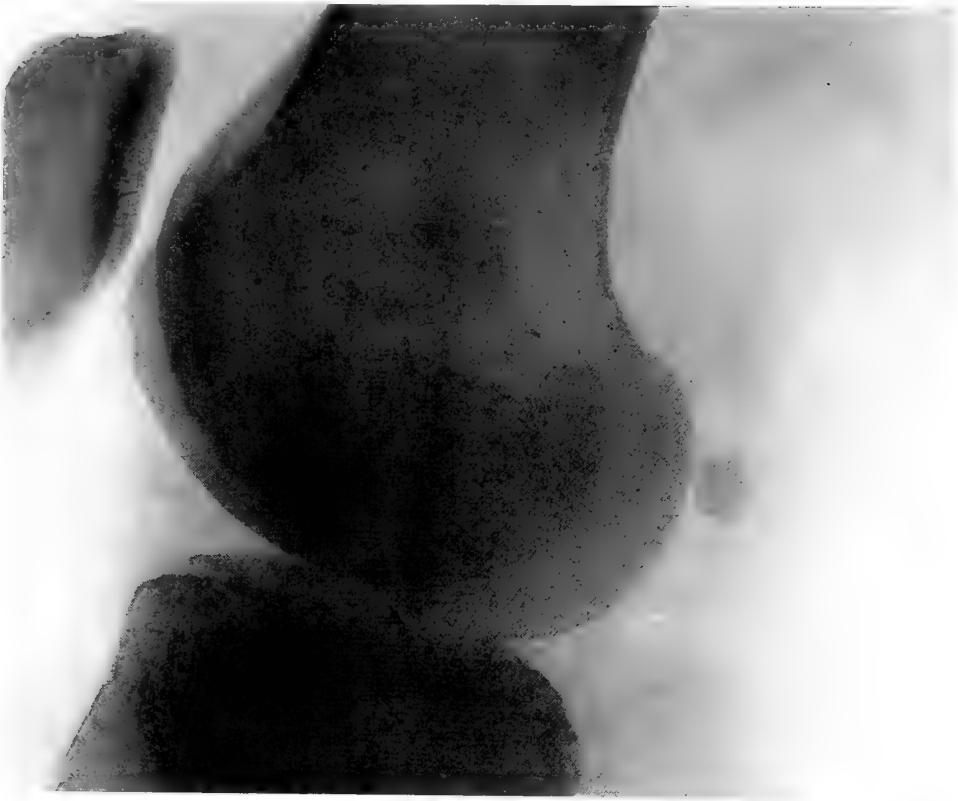


Fig. 18. Large lateral *fabella*.

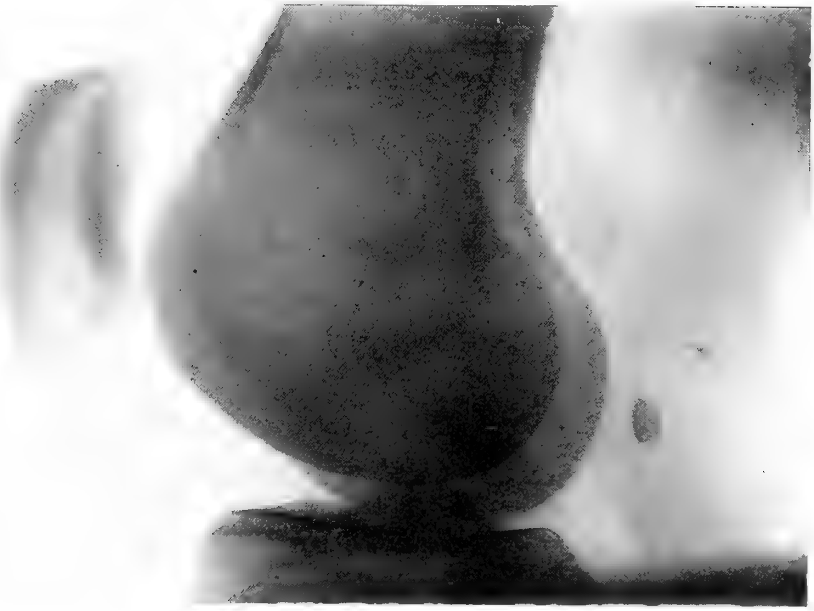


Fig. 19. Lateral *fabella*.

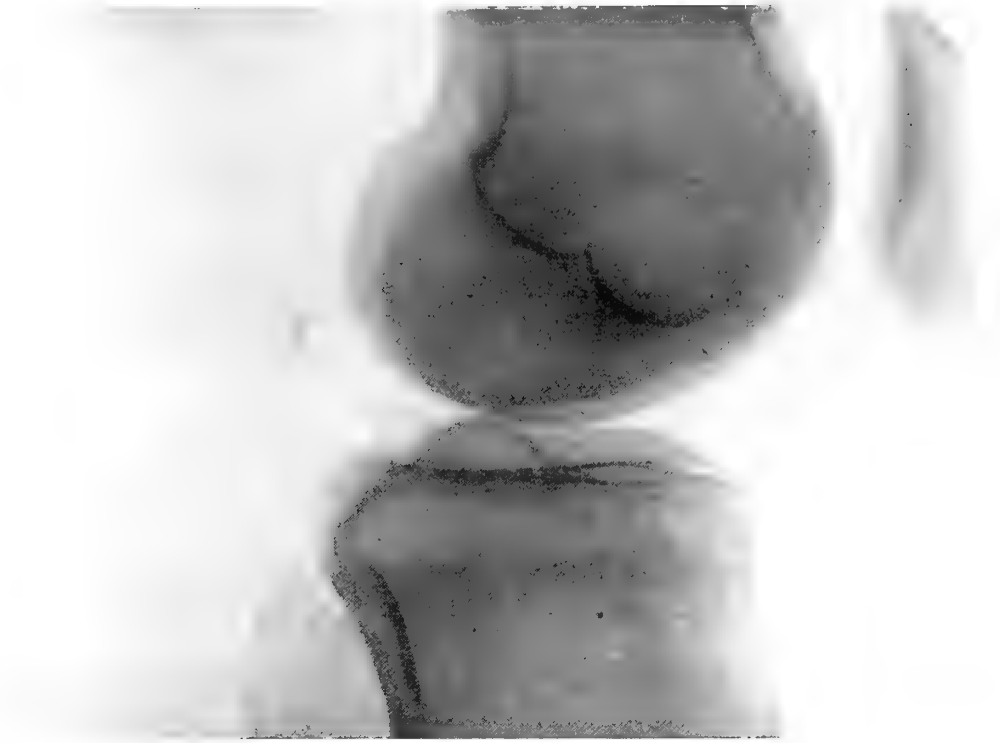


Fig. 20. Small lateral *fabella*.

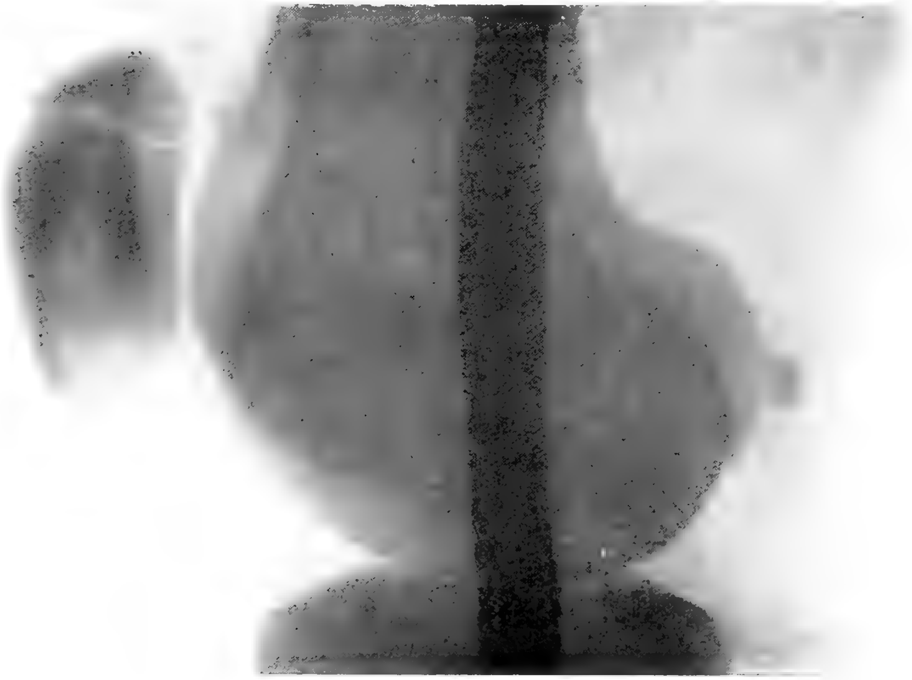


Fig. 21. Large lateral *fabella*.

Fig. 22. Lateral *cyamella* in popliteal sulcus in posterior aspect.

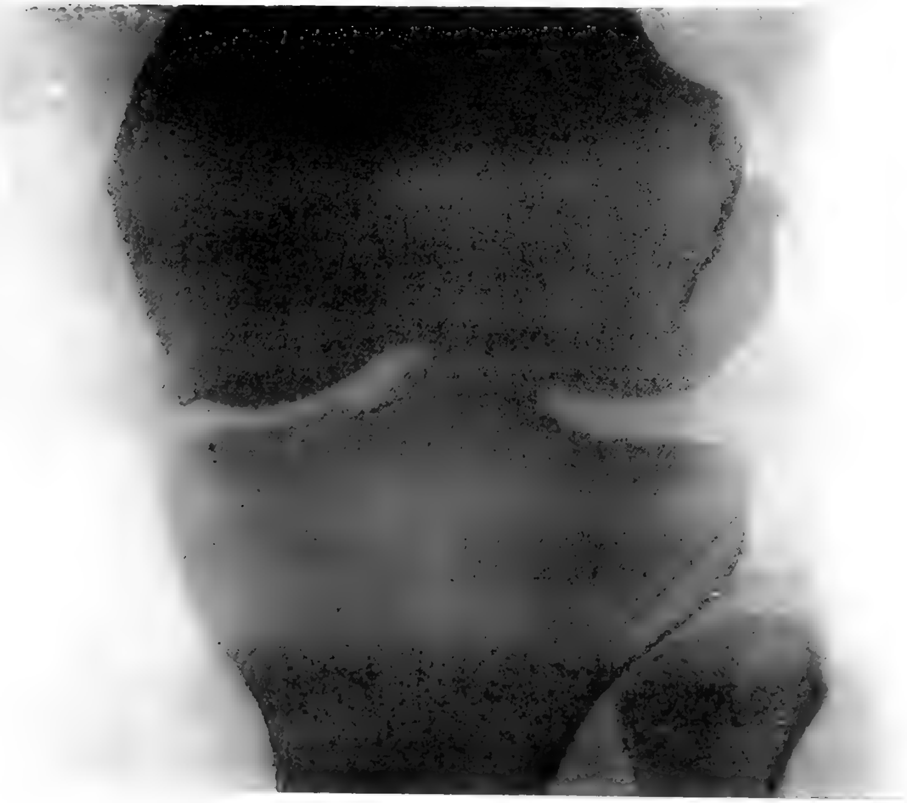


Fig. 23. The same sesamoid in anterior aspect.
Note in both figures the possibility of a mesial *cyamella*.

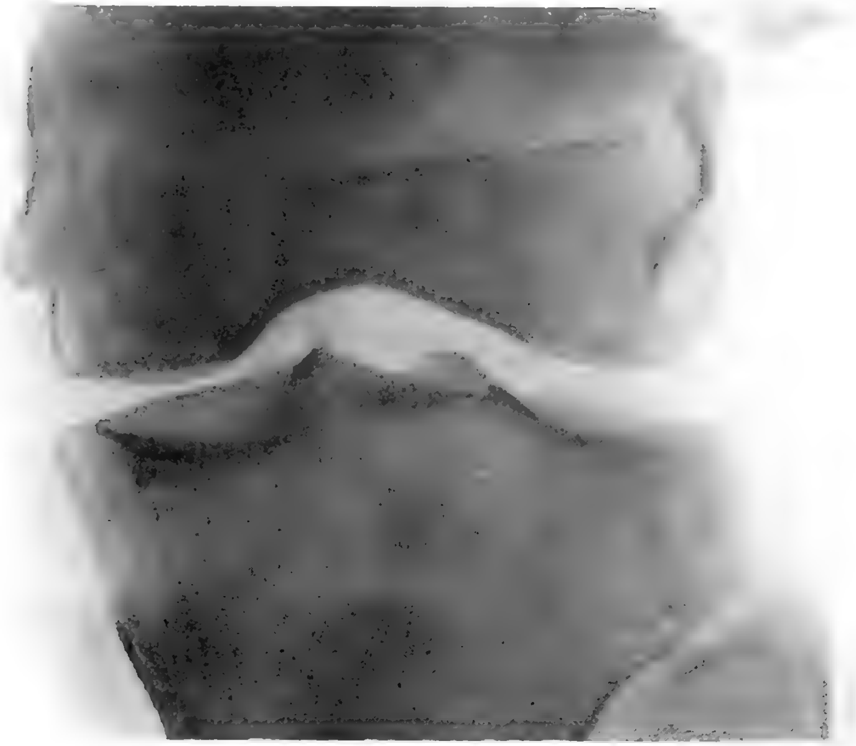


Fig. 24. Occurrence of what might possibly be interpreted as a mesial *cyamella*.

Fig. 25. Lateral *cyamella* in popliteal sulcus.

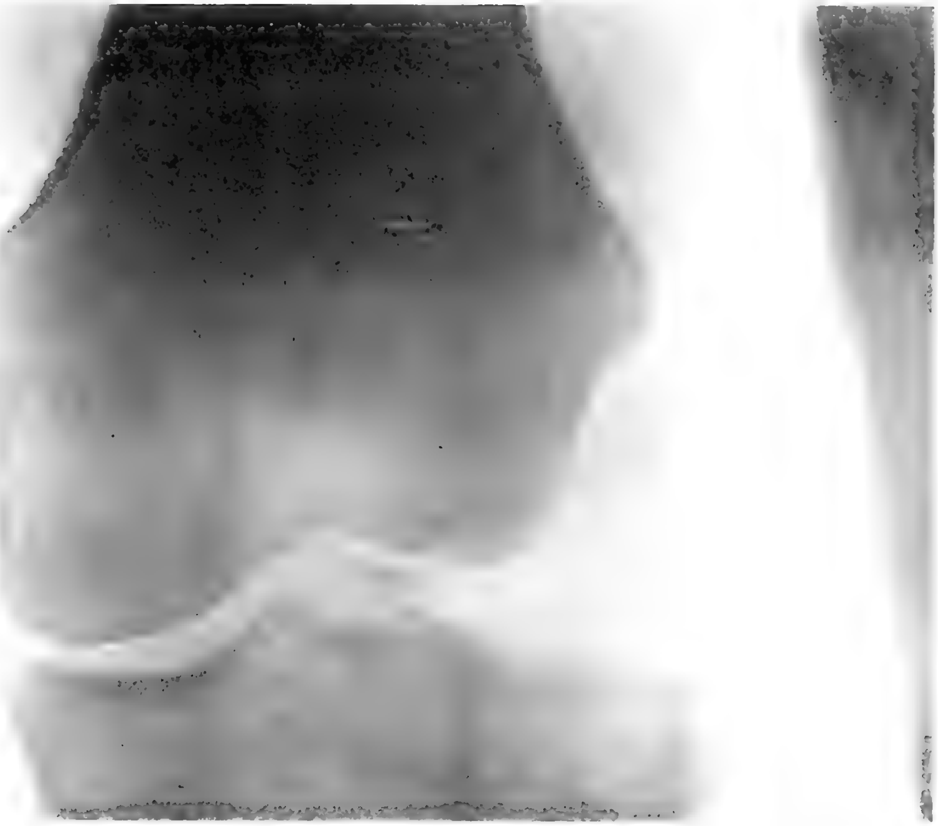


Fig. 26. Lateral *lunula*.

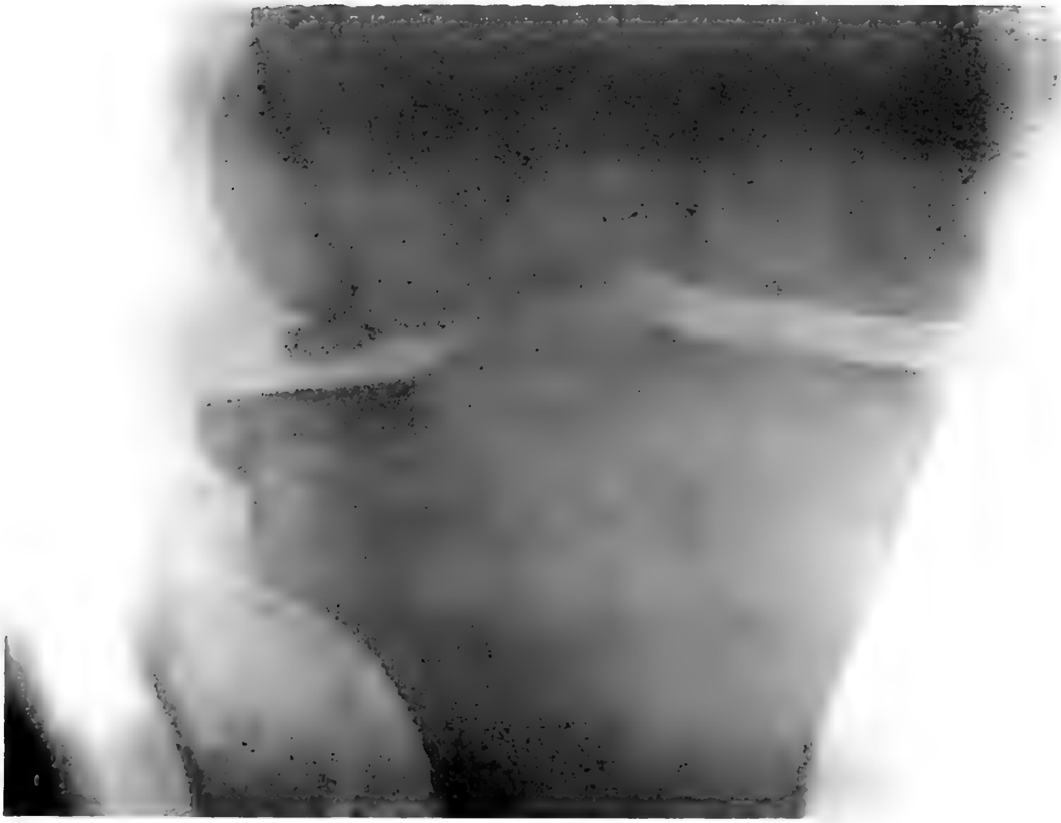


Fig. 27. Another case of lateral *lunula*. Also a sesamoid above the head of the fibula (? displaced *fabella*).

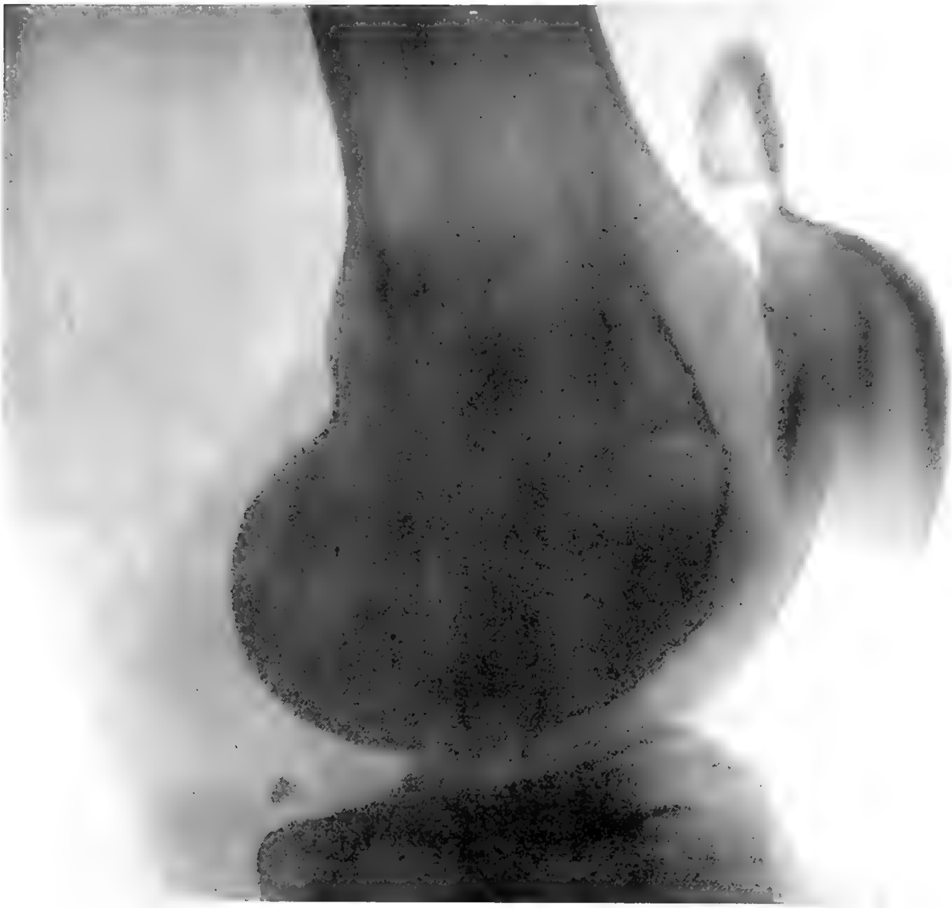


Fig. 28. Double *patella* and posterior *lunula* (?).

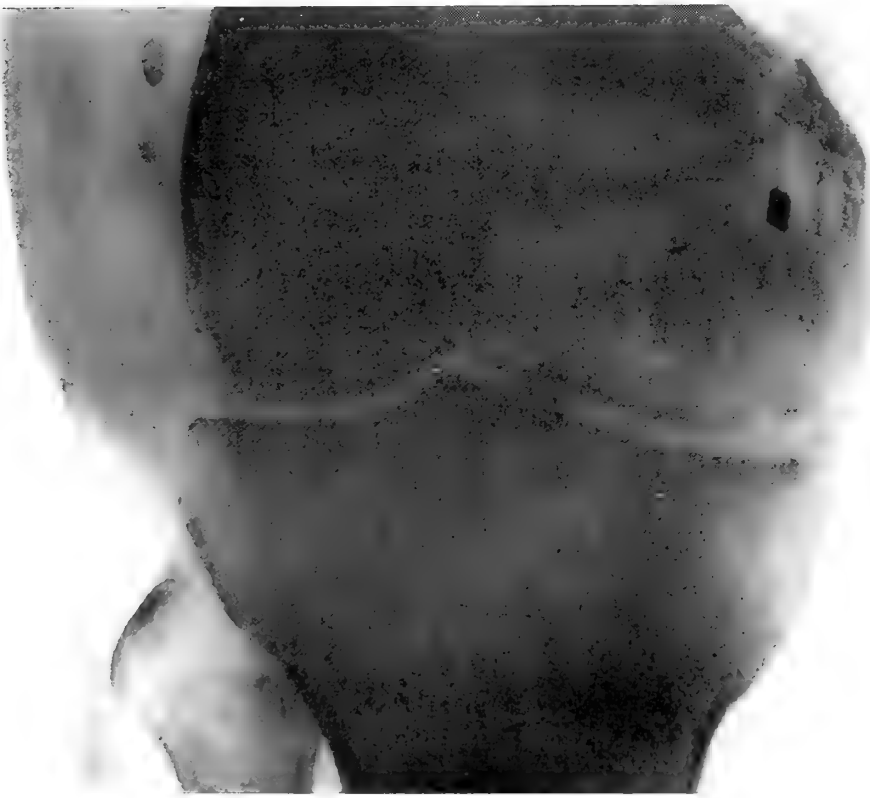


Fig. 29. Two small lateral sesamoids in unusual positions, possibly associated with the lateral femoro-fibular ligament.

Fig. 30. Double *patella*. (? Small external posterior *lunula*.)

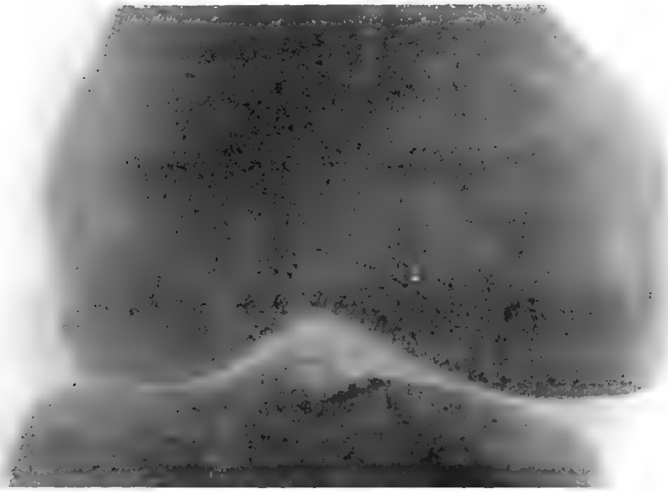


Fig. 31. Sesamoid in femoro-fibular ligament? Cf. Plate XIII, Fig. 29.

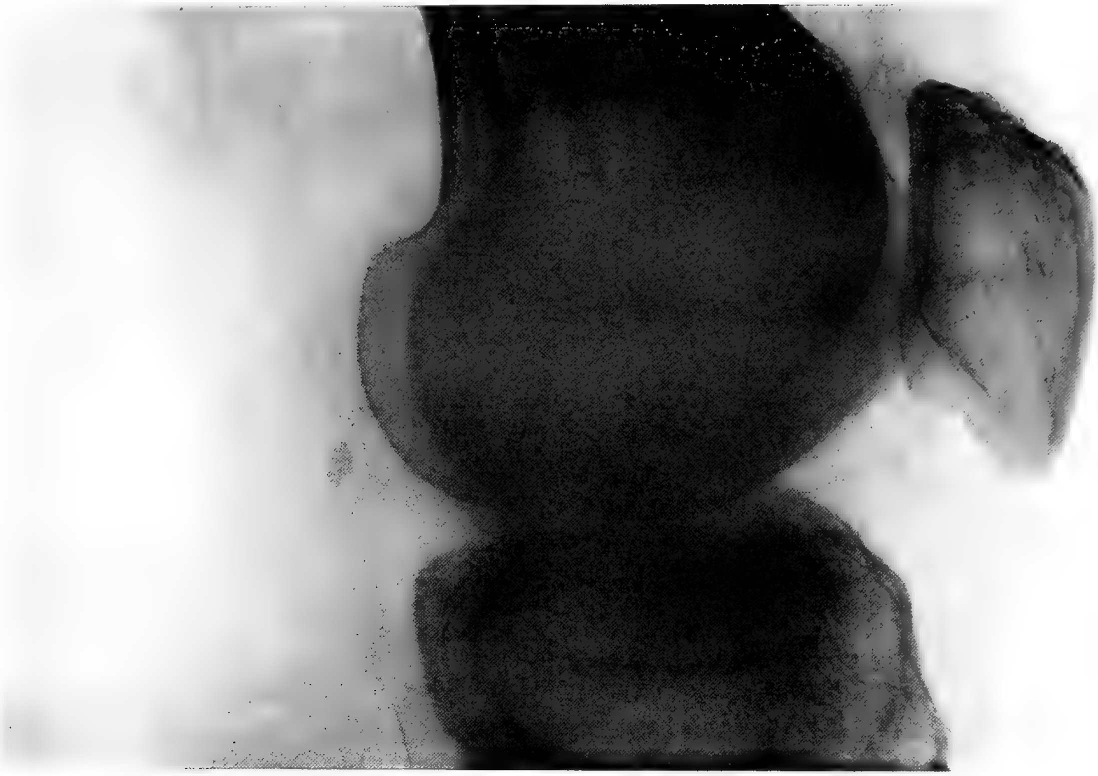


Fig. 32. Lateral *fabella*.

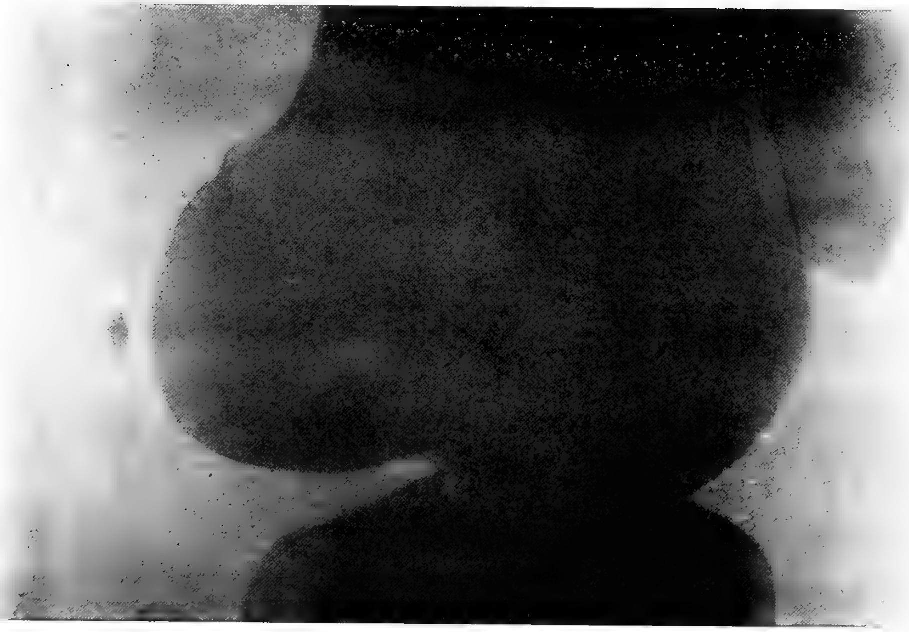


Fig. 33. Lateral *fabella*, with some trace of a second sesamoid beneath it.

Figs. 32 and 33 are reproduced from Atlas, Part II, to the *Monograph on The Long Bones of the English Skeleton*, by Pearson and Bell.

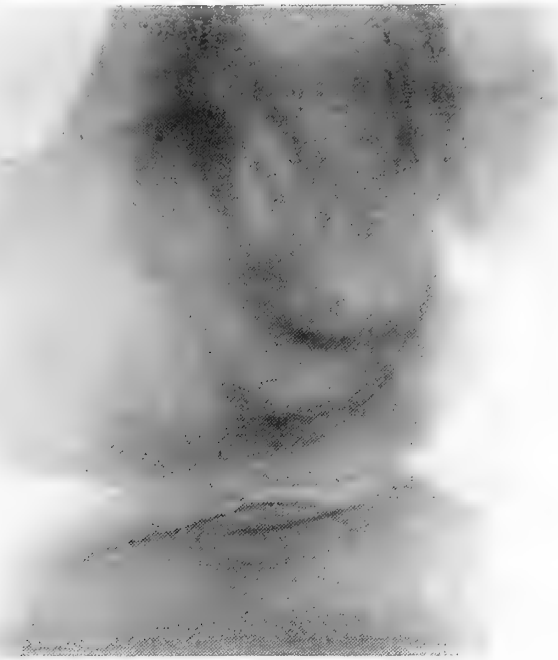


Fig. 34. Lateral *fabella.*

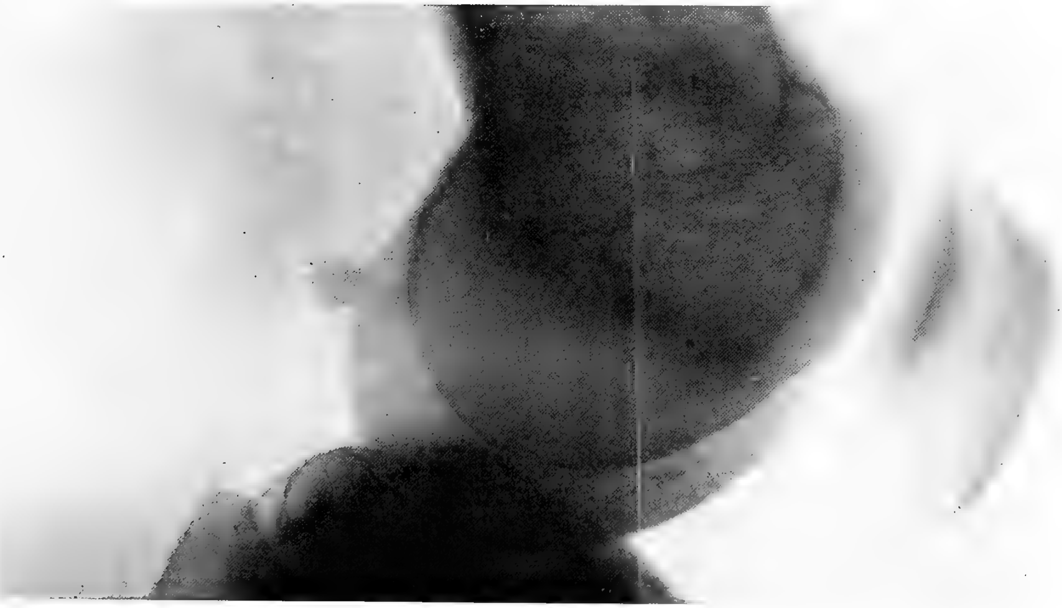


Fig. 35. Lateral *fabella.*

Figs. 34, 35 are reproduced from Atlas, Part II, to the *Monograph on The Long Bones of the English Skeleton*, by Pearson and Bell.

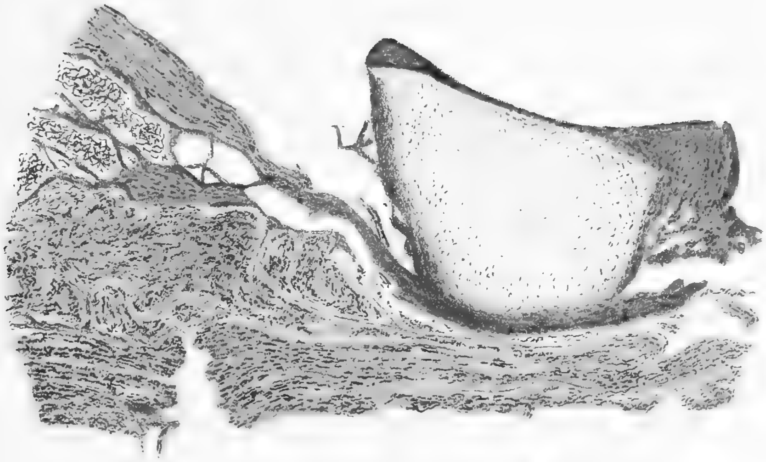


Fig. 36 (i).

× 45·5

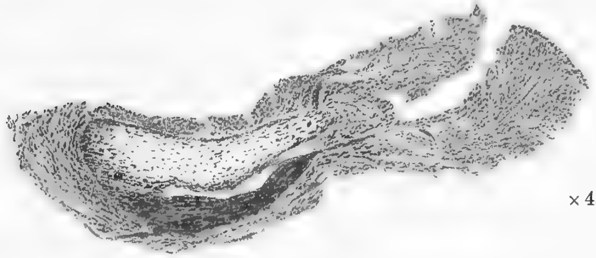
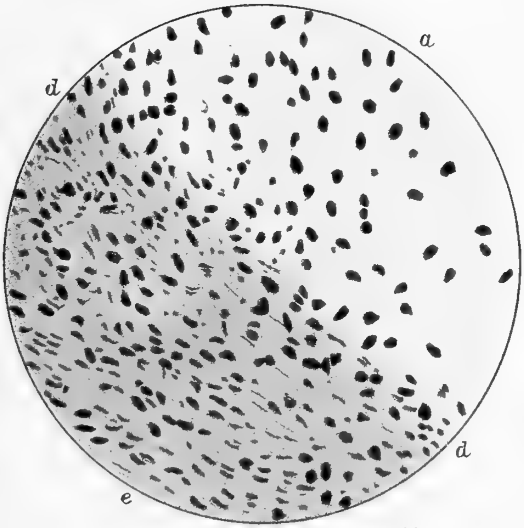


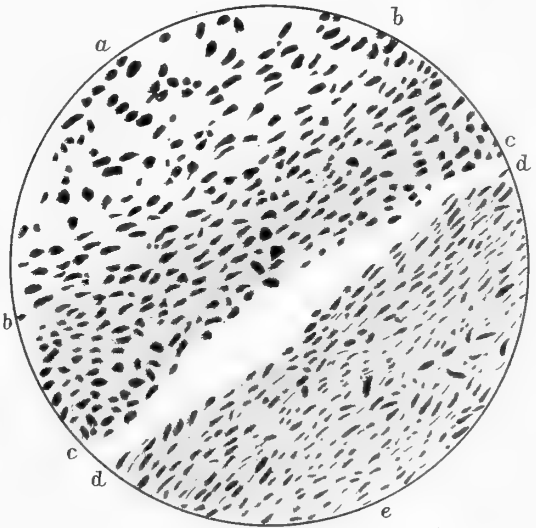
Fig. 37 (i).

× 45·5



× 300

Fig. 36 (ii).



× 300

Fig. 37 (ii).

Sections of the Sesamoids of a Newly-born Kitten.

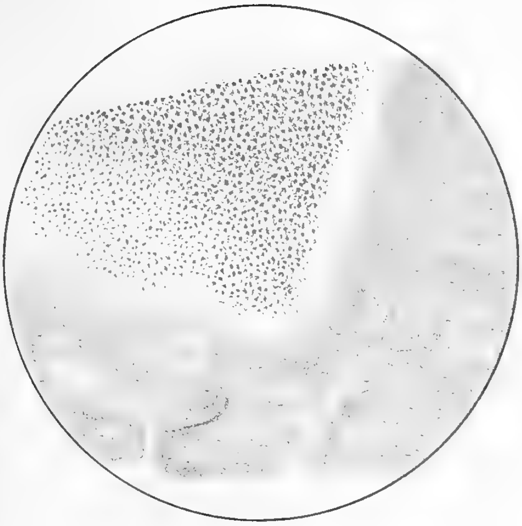


Fig. 38 (i). × 45·5

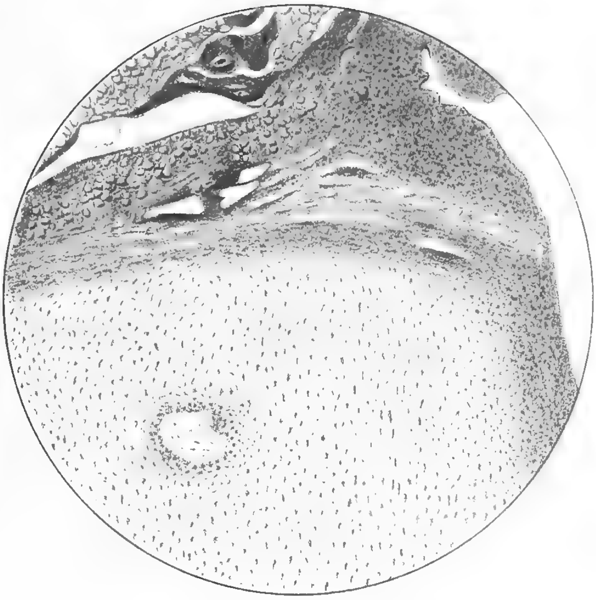


Fig. 39 (i). × 45·5

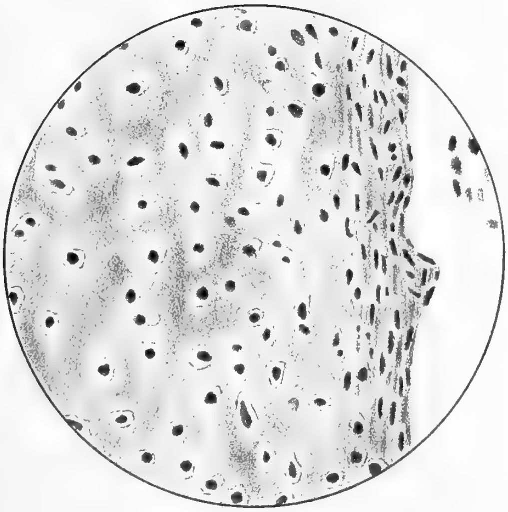


Fig. 38 (ii). × 300

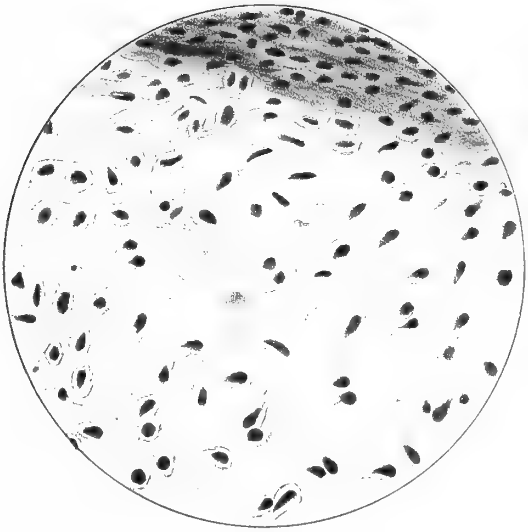


Fig. 39 (ii). × 300

Sections of the Sesamoids of a full-term Human Foetus.

DESCRIPTION OF PLATES.

The first five plates represent historically the development of the anatomists' conception of the sesamoids of the knee-joint.

Plate I. Fig. 2 shows after Pfitzner the sesamoids of first metatarsal of the foot in man. Such sesamoids of the digits were certainly known to the Greeks. They are the most conspicuous and we cannot demonstrate that the Greeks knew those of the knee-joint. It is reasonable to suppose that the seed of the plants the Greeks called sesamé should somewhat resemble these sesamoids.

Fig 1. First drawing of the *fabellae.* It occurs in Casserius' *Tabulae anatomicae* of 1632 which illustrate Adrian Spiegel's *De humani corporis fabrica.* The latter following Vesalius asserts the existence of sesamoids in *both* heads of the external *gastrocnemius*, and they are thus somewhat naively exhibited by Casserius. There is no trace of the "Knotenpunkt" of actuality.

Fig. 3. See under Plate IV.

Plate II, Fig. 4. Diagram from the Altorf 1715 dissertation, *De chylosi Foetus in utero*, of C. J. Trew. The diagram shows none of the muscular relationships of the external *fabella.* F. V shows the sesamoid resting in what Trew thought its customary situation, "os sesamoideum insidens cavitate condyli exterioris in omnibus femoribus conspicuae." F. VI exhibits "os sesamoideum femoris magnitudine naturali extra situm." Trew makes no reference to an internal *fabella* in his supposed discovery. The "conspicuous cavity in all femora" appears to be the roundish impression for the insertion of the lateral head of *M. gastrocnemius* on the external epicondyle, a situation where the sesamoid is not in our experience ever found.

Fig. 5. This drawing is taken from Laurence Heister's *Compendium Anatomicum*, Edition 1748. There is much in it to indicate that Heister notwithstanding his paper of 1717 (see our p. 150 ftn.) copied Trew's illustration, but added the mesial *fabella*, which he rests on the horizontal surface of the condyle in immediate association with the supracondylar tubercle.

Plate III, Fig. 6. Peter Camper's drawing of the external *fabella* from his 1754 memoir: "Vom Bruch der Kniescheibe und des Olecranon" (S. 68 of the *Vermischte Schriften* of 1801). *AKIL* is the tendon of the outer head of *gastrocnemius* "worin sich ein Knöchelgen (*ossiculum sesamoideum*) *f. g.* findet, welches man in dem inneren Bauch derselben nie sieht." *f. g.* is not marked on the figure, but is presumably the shaded portion. If so, the relation of this sesamoid to the other muscles and ligaments is far from clearly indicated.

Plate IV, Fig. 7. A drawing after Sir George M. Humphry's figure in his *A Treatise on the Human Skeleton*, 1858 (Plate LI, Fig. 2). Posterior view of left knee-joint, described as follows: *A*, *ligamentum posticum*; the thickest portion of it ascends from *B* the *semimembranosus* to *C* the outer head of *gastrocnemius. D* a sesamoid bone at its junction with the latter. *E* inner head of *gastrocnemius*, also connected with *ligamentum posticum. F* popliteus muscle, raised a little to show *G* the posterior peroneo-tibial ligament. *H* external lateral ligament. Humphry thus indicated the relation of the sesamoid to the *ligamentum posticum.*

Fig. 3 of Plate I (placed there for economy of space) is from the historical order to be considered here. It is copied from Gillette's memoir of 1872 (see our p. 155). *a* is the external *fabella in situ*, *b* the *gastrocnemius* and *c* the situation where it rests on the posterior surface of the condyle. The situation is fairly in accordance with that suggested by Humphry, but Gillette does not indicate the relationship of the sesamoid to other ligaments and muscular attachments.

Figs. 8 and 9 of Plate IV and 10, 11, 12 of Plate V are all taken from W. Gruber's great memoir of 1876, which is the first to illustrate carefully the muscular attachments to the sesamoid.

In all these figures 8 to 12, except Figs. 9 and 10 where by an unfortunate slip of the engraver 5 has been placed for δ, the system of lettering is the same, namely:

1 = femur, 2 = tibia, 3 = fibula, 4 = sesamoid. *a* = knee-capsule, *b* = *ligamentum popliteum*, *c* = long external lateral ligament or *ligamentum anticum*, *d* = short external lateral ligament, or *ligamentum posticum*, *e* = external *meniscus genualis*, *f* = great adductor muscle, *g* = *M. gastrocnemius internus*, *h* = *M. semimembranosus*, *i* = *M. biceps femoris*, *k* = *M. gastrocnemius externus*, *l* = *M. plantaris*, *m* = *M. popliteus*,

$n = $ *M. popliteus biceps* (Fig. 9), $o = $ internal supracondylar tubercle, $p = $ *processus tuberositatis* of the internal condyle for the insertion of the tendon of the adductor muscle, $\gamma = $ superficial and δ (engraved 5 in Figs. 9 and 10) = deeper layers of the under "strängformig" portion of the tendon of origin of *M. gastrocnemius externus.*

Fig. 8 shows the only mesial *fabella*, 5, observed by Gruber; it was a pathological ossification, α being the upper vertical, and β the transversal limb of this pseudosesamoid; it appears to be associated with a somewhat emphasised tuberosity of the internal condyle, *p*.

Fig. 9. A most interesting illustration of the anomaly of *M. popliteus biceps*, *n*, of the two branches n'' is the outer deeper head and n' the supernumerary inner and superficial head attached to the external *fabella*. This *fabella* (4) appears as a "Knotenpunkt" of tendon of *M. gastrocnemius* k $(5 = \delta)$ of the short external lateral ligament (*d*), and its posterior face is shown by the removal of *M. gastrocnemius* (*k*) and *M. plantaris* (*l*).

Plate V, Fig. 10. Left knee-joint seen in posterior aspect. We see the ossicle exposed with its upper inner and outer surfaces, the head of origin of *M. gastrocnemius* being cut through and drawn back over the ossicle (4). We see the ossicle attached to the *ligamentum popliteum* (*b*) and the short external lateral ligament (*d*). The position of the sesamoid and its shape are unusual for a *fabella*. It appears to be low down on the external face of the lateral condyle, and approaching the popliteal sulcus. Its position seems at least transitional from that of *fabella* to that of *cyamella*.

Fig. 11. Right knee-joint in posterior aspect. The *fabella lateralis* (4) is seen wedged in so to speak between *gastrocnemius* (*k*) and *plantaris* (*l*). This is the position given for it by Cheselden in 1733, and more fully recognised by Sömmering in 1841: see our pp. 150 and 151.

Fig. 12. Left knee-joint in posterior aspect. The *fabella lateralis* (4) has been freed from *gastrocnemius* (*k*) and seen in relation to the *ligamentum popliteum* (*b*), the short external lateral ligament (*d*), and by the opened knee-capsule (*a*) in relation to its position on the external condylar surface.

It will be seen how great an advance Gruber made in our knowledge of the muscular and ligamental attachments of the *fabellae.*

Fig. 13. Sesamoid of the external condyle in the case of the Wombat (after Pfitzner). This sesamoid rests on the head of the fibula, and appears to be directly associated with the *parafibula* of other Australian types, and again the direct articulation of fibula with the external condyle in other species. These points are discussed in Part II of the present memoir.

Plate VI. Drawings of an actual case of the *fabella lateralis* in man from a cadaver in the Dissecting-Room at University College.

Fig. 14 is a sagittal section of the left femur showing the position of the *fabella* before *M. gastrocnemius* was turned back. Fig. 15 shows the *fabella* in the tendon of the muscle after it has been turned back and the facette on the articular surface of the condyle in which it rested. The closeness of contact differs considerably from the usual position as indicated in the skiagrams: see Plates VII—IX.

Plate VII, Fig. 16 and Fig. 17. Lateral *fabellae.*

Plate VIII, Fig. 18 and Fig. 19. Lateral *fabellae.*

Plate IX, Fig. 20. Small lateral *fabella.* Fig. 21, Large lateral *fabella*, probably differing from the previous illustrations in resting in a facette on the articular surface of condyle. Cf. Plate VII.

Plate X. Most probably a lateral *cyamella* in the neighbourhood of the popliteal sulcus in posterior (Fig. 22) and anterior (Fig. 23) aspects. Is there evidence of a mesial *cyamella*?

Plate XI. There is some evidence in favour of a mesial *cyamella* placed in the groove on the internal epicondyle in Fig. 24. Fig. 25 gives a fair example of a lateral *cyamella* in the popliteal sulcus.

Plate XII, Fig. 26. There is a small bone in the semi-lunar, which may be a posterior lateral *lunula.* Fig. 27 gives more definite evidence of an anterior lateral *lunula.* Both of these *lunulae* are fairly smoothly shaped and do not resemble pathological ossifications: at the same time the damaged condition of lateral condyle in Fig. 26 prevents any stress being laid on the former case. In Fig. 27 there appears to be a well shaped sesamoid at the head of the fibula in close proximity to the *lunula.*

Plate XIII, Fig. 28. This must be taken in conjunction with Plate XIV, Fig. 30. In both cases we have an ossification of the tendon of the quadriceps extensor muscle. In both cases it *may* be considered as pathological, but at the same time its general resemblance to the double or "tandem" *patella*, which occurs regularly in the hare, should be noted. The cases of double *patella* recorded by Gruber (*Virchow's Archiv*, Bd XCIV, S. 358, 1883) and Wright (*Journal of Anatomy and Physiology*, Vol. XXXVIII. p. 65, 1904) are comparable with each other, but not atavistic in character like those of our skiagrams*. Both our Figs. 28 and 30, seem to suggest the existence of posterior *lunulae*.

Plate XIII, Fig. 29 shows two sesamoids on the face of the lateral epicondyle, possibly related to the lateral femoro-fibular ligament.

Plate XIV, Fig. 31 gives an indication of a single sesamoid in much the same position.

Plate XV, Fig. 32. Lateral *fabella*.

Fig. 33. Lateral *fabella* with a faint trace of a second sesamoid a little below it—possibly but improbably a hemisesamoid of the mesial condyle.

Plate XVI, Fig. 34. Lateral *fabella*. This knee-joint exhibits considerable pathological ossification of the semilunar cartilage.

Fig. 35. Lateral *fabella*.

Figs. 32—35 are from skiagrams taken by Dr Melville at the Orthopaedic Military Hospital, and are reproduced from the *Monograph on the Longbones of the English Skeleton*, by Pearson and Bell, Atlas II, CI A and CI B.

Plates XVII and XVIII. Sections of the sesamoids in Foetal Forms. These confirm Gruber's view that a hyaline *cartilago* is the necessary precursor of a sesamoid. Neither hemisesamoid nor orthosesamoid appears to arise from fibrous cartilage. Fig. 36. Section of lateral *fabella* of newly-born kitten, (i) Magnification 45·5, (ii) Magnification 300; showing hyaline cartilage area (*dad*) bordered by fibro-cartilage (*dcd*). Fig. 37. Section of the *cyamella* of a newly-born kitten, (i) Magnification 45·5, (ii) Magnification 300; *a—bb* hyaline cartilage; *bb—cc* intermediate area showing typical rounded nuclei of hyaline cartilage and elongated spindle nuclei of fibro-cartilage, but no threadlike nuclei of fibrocytes; *dd—e* fibro-cartilage showing spindle nuclei and threadlike nuclei. Fig. 38. Longitudinal section of metatarsial sesamoid of full-term human foetus, (i) Magnification 45·5, (ii) Magnification 300, see our p. 159. The metatarsial sesamoid has clearly a hyaline *cartilago* as precursor. Fig. 39. Longitudinal section of *patella* of full-term human foetus, (i) Magnification 45·5, (ii) Magnification 300, see our p. 159. The precursor of the *patella* in man is clearly hyaline.

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* Tillmans and Bernays had found hemisesamoidal supernumerary *patellae* in cadavers (*Morphologisches Jahrbuch*, IV. Supplement, 1878, S. 442). Presumably these double *patellae* were arranged tandem fashion for Bernays refers to those in cats, mice, and rabbits as similar.

A FIRST STUDY OF THE BURMESE SKULL.

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Biometric Laboratory, University College, London.

CONTENTS

1. INTRODUCTORY.

THE collection of skulls which forms the subject of the present study was procured from the neighbourhood of Moulmein, in the southern part of Burma, by the late Colonel P. H. Caster, I.M.S., at the request of Professor Karl Pearson. The territory which stretches from the Namkin mountains in the north down the whole length of the Irawadi river and still further down the coast towards the Malay Peninsula, and is administered under the name Burma, embraces many racial units which might be classed together generally as Burmese, but of which the Burman proper is only one. The request was for purely Burman skulls, but it was evidently difficult to ensure this, and the collection of 142 skulls includes some that are of a different racial type from the majority. The series is on the whole well-preserved, though some few are rather badly damaged, and cannot yield a full series of measurements.

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* The author wishes to acknowledge the very valuable aid she has received from the Department of Industrial and Scientific Research. Without this aid it would not have been possible for her to devote the past two years to craniological training. We have also to acknowledge the assistance of a grant made by the Royal Society Government Grant Committee towards the cost of the plates of this memoir.

Although our material, when divided up into sexes and groups, is far from ample enough to allow us to form decisive conclusions as to the racial constants, and as to the exact relation in which the Burman race stands to other types, it is still by far the largest collection of Burmese skulls yet submitted to craniometrical methods, and until a fuller series shall be available to supplement these, a preliminary study of the type by computing means only will not be labour spent in vain.

2. MEASUREMENTS AND METHODS OF MEASUREMENT.

The following direct measurements* have been taken: C = capacity taken with mustard-seed tightly packed in skull, and then weighed, the worker having previously performed this operation on one of the "crânes étalons" described by Macdonnell† in order to bring her personal equation into line with that of her predecessors. The contents of the crâne étalon were packed into the measuring glass, as well as weighed, and the water capacity was also measured in the glass, in order to obtain the constants for the calculation of capacity. F = Flower's ophryo-occipital length. L' = glabellar projective horizontal length‡. L = maximum length from glabella to occiput§. B = maximum horizontal breadth on parietal bones. B' = least forehead breadth from one temporal crest to the other. H' = basio-bregmatic height. H = height measured on craniophor, from basion to the point vertically above it (generally a little behind the bregma). OH = auricular height, as measured on the craniophor, that is, the height of the skull vertically above that part of the auricular orifice which rests on the top edge of the ear-plugs‖. LB = length from basion to nasion. Q = transverse arc perpendicular to the horizontal plane, passing through the "apex," and terminating on both sides at the top of the ear-rods, this measurement being taken when the skull is on the craniophor. In this measurement the tape is carried on to the ear-rods after passing over the zygomatic ridges. Q' = a similar measurement taken off the craniophor, and terminating at the "auricular points ¶". S = sagittal arc from nasion to opisthion. S_1 = arc from nasion to bregma. S_2 = arc from bregma to lambda. S_3 = arc from lambda to opisthion. S_3' = chord from lambda to opisthion. U = horizontal circumference measured directly above the superciliary ridges and round the most projecting mesial part of the occiput, i.e. the most projecting part between the inion and lambda. PH = premaxillary height, measured from alveolar point to the tip of the anterior nasal spine, if such exists, and, where it does not, to the sagittal point of the lower edge of the pyriform aperture. $G'H$ = upper face height from nasion to alveolar point. GB = face breadth from the lower end of one zygomatic-maxillary suture to that of the other, where the suture crosses the lower front rim of the cheek-bone. J = zygomatic breadth, from the most lateral point of one zygomatic arch to that of the other. NH (R) and NH (L) = nasal

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* Major lengths have been given to the nearest ·5 mm., the smaller measurements to the nearest ·1 mm.

† *Biometrika*, Vol. III. p. 203.

‡ *Ibid.* Vol. I. p. 415.

§ I have thought it best, on account of a good deal of asymmetry in the series, to take L always in the median plane, and consequently it may occasionally be less than L'.

‖ See p. 182.

¶ See p. 181

height from nasion to lowest edge, right and left, of pyriform aperture. NB = nasal breadth, greatest breadth of pyriform aperture, wherever it may be. DS = dacryal subtense, being the shortest subtense in the sagittal plane from the bridge of the nose to the dacryal chord, measured by Mérejkowsky's simometer. DC = dacryal chord, i.e. length from dacryon to dacryon. DA = dacryal arc, i.e. the shortest arc over the bridge of the nose from dacryon to dacryon, measured with the tape. SS = simotic subtense, the shortest subtense in the sagittal plane from the nasal bridge to the simotic chord, as measured by the simometer. SC = simotic chord, i.e. the minimum chord between the two naso-maxillary sutures. O_1 = greatest breadth of orbit, R and L, from side to side, using Fawcett's curvature method* to determine the margin on the nasal side†. O_2 = greatest height of orbit, R and L, taken perpendicular to O_1. G_1 = length of palate, from the point of the spina nasalis posterior to an imaginary line tangential to the inner rim of the alveoli of the middle incisors. G_1' = similar measurement to G_1 but taken from base of spine. G_2 = breadth of palate between inner alveolar walls at second molars ‡. GL = profile length from basion to alveolar point. fml = length of foramen magnum from basion to opisthion. fmb = greatest breadth of foramen magnum. $G'H$ = upper face height from alveolar point to nasion. $P\angle$ = profile angle, or angle made by the line from the nasion to alveolar point with the horizontal, found by means of Ranke's goniometer when the skull is in horizontal position on the craniophor, as described by Fawcett (*Biometrika*, Vol. I. p. 414). From these direct measurements there were also calculated the various indices: occipital (*Oc. I.*), cephalic (B/L', H/L, H/L', B/H), facial ($G'H/GB$), nasal (NB/NH, R and L), orbital (O_1R/O_2R; O_1L/O_2L), palatal (G_2/G_1), foraminal (fmb/fml), dacryal (DS/DC), simotic (SS/SC). The angles ($N\angle$, $A\angle$, $B\angle$) of the fundamental triangle whose apices are the nasion, alveolar point and basion, were calculated from the measured lengths LB, GL, $G'H$ by the aid of Pearson's Trigonometer in the manner described by Fawcett (*Biometrika*, Vol. I. p. 418). θ_1, the basio-nasal horizontal angle, was obtained by the subtraction of $N\angle$ from the supplement of $P\angle$; θ_2, the basio-alveolar horizontal angle, was arrived at by the subtraction of $A\angle$ from $P\angle$.

Of these various measurements, one or two are new; most are similar to those made by previous craniometricians.

C. The measurements, and methods of measurement, described by Fawcett (*Biometrika*, Vol. I. pp. 416–419) have been adopted exactly, with one exception, viz. that of C, the capacity. Macdonnell's labour-saving device of weighing the mustard-seed to determine its volume has been adopted wherever possible by all later workers in this Laboratory.

The only available one of the three "crânes étalons" used by Macdonnell (*Biometrika*, Vol. III. p. 204) was skull δ. After some preliminary practice in tight

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* See *Biometrika*, Vol. I. p. 430. Also *Biometrika*, Vol. VIII. pp. 311, 312.

† Another measurement O_1' (= breadth of orbit measured from dacryon) had to be taken later for comparison with the data obtained for other racial series.

‡ This may occasionally be the same as the measurement between the inner rims of the alveoli of the second molars, but is more frequently rather less than these.

packing, the skull was filled five times in succession by me, and the seed packed again each time in the measuring-glass; after which it was weighed. This operation was also performed by Professor Pearson.

Prepared Skull.

	Cm.^{3} of seed	Weight of seed in grams
K. Pearson	1325	1065·45
M. L. Tildesley ...	1325	1071·00
	1325	1062·55
	1320	1063·50
	1325	1064·30
	1325	1060·30
Totals ...	7945	6387·10
Mean ...	1324·17	1064·52

The skull was then filled with water and the water measured in the measuring-glass: this, which should be the true capacity of the skull, came out at 1365 cm^3.

We thus arrived at the equation:

$$\text{Volume of 1000 grs. of seed} = \frac{1365 \times 1000}{1064{\cdot}52}\,\text{cm.}^3 = \frac{1000}{779{\cdot}87}\,\text{cm.}^3$$

The result obtained by Macdonnell on this skull was that 1000 cm.^{3} contain 764·63 grs. of seed. The seed we used had been kept for a large number of years, and was probably slightly more shrunken than that used by him, and would therefore give a denser packing.

H and *H'*. The height of the skull from the basion was measured by Macdonnell to the bregma, not vertically as by Fawcett. In this he was followed by Benington* and Thomson†. His *H* is also the *H* used by Flower and Turner. I have therefore taken, for the purposes of comparison with their results, both these measurements, calling Macdonnell's *H*, *H'*.

Q and *Q'*. We may note that Macdonnell has also measured his *Q* over the bregma, instead of in the vertical plane, following, he says, Virchow and Turner, and Benington does the same, but also takes the vertical *Q* under the title Q_1. Thomson abandoned the bregmatic *Q*, but added *Q'*‡, taken off the craniophor, from the upper margin of one auricular passage to the upper margin of the other. I have done as Thomson has done. With regard to the determination of the position of the apex in defective skulls where the orbits or auricular passages were missing, so that the skull could not be adjusted accurately to the Frankfurt horizontal on the craniophor: Macdonnell said he put it at 2–3 cms. behind the bregma. This evidently varies for different races, since the mean distance for both males and females of our series was just under 15 mms. I have therefore estimated the apex at 15 mms. behind the bregma where it could not be properly determined, and if the orbits

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* *Biometrika*, Vol. VIII. p. 296.

† *Ibid.* Vol. XI. p. 84.

‡ *Ibid.* Vol. XI. p. 85.

alone were missing have settled the Frankfort plane by means of this apex. All measurements dependent on this approximation are of course queried.

O_1R and O_1L. Of the two methods—the "curvature" method and the "geodesic*"—of completing the orbital margin where it becomes indefinite on the nasal side, I have used the curvature method, which Fawcett herself used and which was followed by Thomson, although Macdonnell and Benington used the geodesic.

G_1'. This was a measurement first made by Macdonnell, and has continued to be used by his successors in this Laboratory.

NH. In all the papers of this school until Thomson's this measurement was defined as taken to the lowest point of the margin of the pyriform aperture, whether this may be to the right or to the left. Thomson says she has taken it to the left, unless this is damaged, and has also measured the right if this be sensibly different; but from the fact that she gives no separate measurement of the right side in her data we infer that her interpretation of the term "sensible difference" applied only to marked nasal asymmetry and that this feature in the Moriori Series was much more regular than among the Burmese, whose maximum difference between nasal height *R* and *L* is 2·7 mms. I have measured both sides in all cases.

S_1, S_2, S_3, S_3', *Oc. I.*, *fmb*, *fml*, *fmb*/*fml* are all measurements added to the series by Macdonnell†, and have continued to be made by his successors here.

Oc. I. was called by him the Cerebellar Index, but the more accurate title of Occipital Index was adopted later.

DA and *DC* were first measured by Benington‡, and *DS* was obtained from them by calculation.

SA, *SS*, *SC* were also measured by him§ later, when he had the use of the instrument invented by Mérejkowsky, by which I also have obtained *DS* as well as *SS*.

Sex. The sex of the skulls was determined by Professor Pearson, when classifying them into the three types to be described later (p. 217).

Mandibles. There were no skulls with mandibles in my series.

3. ON THE DIFFICULTIES ATTACHED TO THE DEFINITION OF CERTAIN CRANIAL "POINTS."

Basion. In Thomson's paper it was suggested that the basion and opisthion should be marked off as points before any measurements are made from them‖, it having been noticed that slightly different points may be selected on the different occasions on which use is made of them, e.g. in measuring the basio-bregmatic height, and the foraminal length, and in drawing the sagittal contour. I have adopted the suggestion and find it very advantageous. In selecting basion and opisthion on the foraminal border it is very desirable to bear in mind that the foraminal length will be determined probably by the dividers and the basio-

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* *Biometrika*, Vol. I. p. 431.

† *Ibid.* Vol. V. p. 91.

‡ *Ibid.* Vol. VIII. p. 297.

§ *Ibid.* Vol. VIII p. 317.

‖ *Ibid.* Vol. XI. p. 129.

bregmatic height by a large pair of callipers. The points on the border should accordingly be so chosen that they provide a resting point for the vertical or horizontal measuring instrument, i.e. the points should be about central on the curved rim where the surface changes its direction from horizontal to vertical.

Auricular Point. The ambiguities attaching to this "point" have been fruitful in the provision of difficulties, and reduce the value for comparative purposes of a good deal of material. The difficulties have been much discussed already, but I cannot forbear to make my own contribution to the discussion, in the hope of assisting somewhat in the solution of the working problem. The main fact, of course, is that the "auricular point" like most other cranial points is not a point, but an area, only it happens to be nearly the worst of its kind and less susceptible of a definition by which different workers will identify approximately the same position for it. I will summarise the history of this in our own Laboratory.

From the beginning of craniometric work here the Frankfurt Concordat was taken as the basis of our method of measurement, though it has had to be modified in some details—as few as possible—and more exactly defined in others. The Concordat gives the auricular point as that point on the upper rim of the auricular orifice which lies immediately above the centre of the orifice when the skull is adjusted to the Frankfurt horizontal plane. When the skull is placed on a Ranke's craniophor its auricular height is taken from the ear-plugs on which it rests; naturally to achieve stability it is the highest point in the vertical section* through the point of contact with the roof of the orifice upon which the skull rests. But this point may vary with the depth to which the ear-plugs are inserted; and in any case it is not possible to guarantee that this point is immediately above the centre of the orifice, whatever the centre of an irregular trumpet-shaped cavity may be: the section of the orifice might be an oblique oval (such as Le Double refers to†, and I have myself found). Probably, however, the difference, if any, due to this cause would be slight. Fawcett found it needful to drop "the centre of the orifice" and modified the definition of the auricular points to "the highest points of the upper rims of the auricular passages‡." The application of this definition—and probably of any definition by which one could attempt to indicate exactly this exceedingly difficult point—is not simple in a considerable number of skulls, at any rate in my series, owing to the fact that their auricular orifice has no very definite rim. The auricular passage is roofed by the squamous portion of the temporal bone, and the curve upwards and outwards towards the zygomatic ridge is sometimes so smooth that it is hard to say at what point there is a greater change in direction than at any other: this leaves a good deal of room for personal equation in determining the point, and unavoidably so. Of course in many other skulls there *is* a definite break in direction, giving a distinct upper rim, and all stages are to be found between these two extremes. I do not claim to add anything to the definitions, which I have tried to apply in determining my "auricular points." In this connection I may, however, mention one

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* By "vertical section" is here to be understood one parallel to the sagittal plane of the skull.

† *Variations des Os du Crâne*, p. 325.

‡ *Biometrika*, Vol. I. p. 413.

feature which I was tempted sometimes to think answered best to the description "upper rim." This is a notch on the postero-superior side whose lower lip, projecting somewhat, suggests a border to the orifice, especially where this notch, as may happen, is carried forward above the opening. Although the edge of this lip, or the curve continued from it where it fails to extend right over the opening, might in many cases be a possible interpretation of the words (one also perhaps giving less room for personal equation between different workers and different schools of workers), there were others in which the upper rim was exceedingly well defined and certainly did not coincide with the curve continued from the notch. I therefore disregarded this feature in all the skulls, endeavouring to identify the break in direction of the upward curve of the bone apart from this.

So much for the determination of the auricular point as defined. In the use of this point for craniophor measurements, however, another difficulty presented itself in a considerable number of cases in my series. The skull is supported on the craniophor by means of ear-plugs, and it is obvious that the ear-plugs must enter a little distance into the orifices for the skull to be supported safely by them: their tips cannot stop short at the auricular points. Now, in some skulls, the roof of the auricular orifice does not rise immediately after leaving the auricular point, nor yet take a horizontal direction, but continues with a slight downward direction for a short distance. When this is so, our auricular points do not actually come in contact with the ear-plugs, which touch a lower part of the roof a little distance inside. From an examination of the transverse contours I find that in 129 cases out of the 259 auricular passages (right and left) which were traced, the auricular point rested on the ear-plugs; in the remaining 130 it was slightly raised. The mean distance of auricular point from top of ear-rod in the whole series was 0·5 mm., and the maximum distance—reached in two cases—was 2·8 mms. Thus the auricular height *OH*, defined as height of skull above the auricular points, when measured by the craniophor, actually takes the height of skull above the point of contact with the ear-rods, and thereby increases by 0·5 mm. in my series the auricular height as defined: an amount which would doubtless vary with every racial series measured.

Certainly this might be reduced or increased by the personal equation of the worker inserting the plugs. I have therefore examined my transverse contours to see to what extent this factor operated in my series. One might take 5 mms. as the minimum distance consistent with safety to which the tops of the plugs could be pushed in beyond the auricular points. In practice I estimate that I myself pushed the plugs in further than this, roughly to 15 mms. beyond the vertical from the zygomatic ridge, which would be between 11 and 12 mms. beyond the auricular point (the difference between the mean ordinates to auricular point and zygomatic ridge in the transverse contours being 3·4 mms. for males, 3·3 mms. for females). What difference might this make? My transverse contours show that the lowest point of the part of the vertical section of the roof of the orifice which lies within 11–12 mms. of the perpendicular through the auricular point—and would thus be the point of contact of ear-plug with the skull as I inserted it—

occurred at a distance from this perpendicular of 5 mms. or less in 239 out of my 259 cases; in the remaining 20 it did exceed that. Except in the case of these 20, therefore, no difference would have been made by my inserting the ear-plugs less far, and the difference made for these few individuals would have a very slight effect on the mean value for my series. In another series it might have more, in others less.

The other direct measurement involving the auricular points was the transverse arc Q which Fawcett* defines as the "cross circumference of the skull measured in a vertical plane from the upper rim of one auricular passage to that of the other," a definition which is repeated by Benington (his Q_1†) and by Thomson‡. We find, however, a note to Thomson's definition which says that "Q is taken with the tape over the skull, and measured from the top of one ear-rod to the top of the other."

Q', on the other hand, "the same measure taken when the skull is off the craniophor," is terminated on each side by the selected auricular points. As the note states, "Q'—owing to the recessing of the margins of the auricular passages—is not quite the same as Q," i.e. in the individual measurements, where the difference is seen to be as much as 4 or 5 mms. in some cases (the excess sometimes being in favour of Q, sometimes of Q'); though the mean values of Q and Q' obtained for the Moriori skulls are extremely close together, there being only 0·2 mm. difference between them for both males and females.

In my own series the individual differences were of the same order, though more numerous, and the mean value of Q (the craniophor measurement) was 1·8 mms. shorter for males and 2·2 mms. for females. These differences can be practically accounted for by the recessing of the margins of the auricular passages: the transverse contours show the zygomatic ridges at a mean height of 3·0 mms. (♂ and ♀) above the auricular points (selected) and projecting 3·4 mms. beyond them in the males, 3·3 mms. in the females, so that the mean distance between the two on the skull is 4·5 mms. (♂ and ♀). For Q', taken *off* the craniophor, the tape traverses this 4·5 mms. distance; I have interpreted Thomson's note to mean a vertical drop to the ear-rod from the zygomatic ridge, which gives a mean distance of 3·0 mms. This would lead us to expect mean difference of 1·5 mms. for my series between Q and Q', a difference quite reasonably close to what I have obtained. In the case of the Moriori crania the constants for this region of the skull would probably differ from those of the Burmese; indeed the peculiar conformation of the Moriori skull in the region above the auricular orifice was one of its distinguishing racial characters. On the other hand, it is possible that there may also have been some difference between the interpretation put by Thomson and myself on the definition of the auricular point, as we have already seen that in many skulls there is room for a considerable personal equation in identifying the "upper rim of the auricular passage."

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* *Biometrika*, Vol. I. p. 416.

† *Ibid.* Vol. VIII. p. 296.

‡ *Ibid.* Vol. XI. p. 85.

4. DETERMINATION OF THE INION.

The determination of the inion has been discussed by more than one distinguished craniometrician and anatomist; it has been dealt with in our own biometric school in a paper by M. A. Lewenz and K. Pearson[*] in the course of an argument which demonstrated its unsuitability as terminal for an arc measurement on the living, and which therefore was concerned to emphasise the very real difficulties in identifying this point, rather than to make any constructive suggestions as to the method of doing so.

If I venture to take up this subject again, the circumstance is accounted for by the fact that the position of the inion still had to be marked on my sagittal contours, in spite of the difficulties.

To recapitulate briefly: the original definition of the inion by Broca and in the Frankfurt Verständigung, as the meeting-place of the lineae nuchae superiores, was abandoned generally by later scientists, Merkel[†] pointing out that the inion corresponded to the meeting-place of the lineae nuchae supremae.

I was able to identify a point which accorded with this later definition in 53 of my male skulls; in seven it was vague; and of one I have only the skull-cap sawn off above the inionic region. In female skulls the inion is of course usually less pronounced, and the point as defined could be determined in only 58 out of 74 cases. These numbers may be compared with those given by Professor Thane[‡] where the inion could be determined in 41 out of 43 ♂ and 43 out of 60 ♀ English skulls. The seven infant skulls in the Burmese collection naturally have little to indicate its position.

I was thus left with 7 ♂, 16 ♀ and 7 infant skulls, for which the position corresponding to this same anatomical point had to be inserted in my contours, within as narrow limits of accuracy as could be achieved.

As a guide in doing this, I examined the skulls of my series in which the inion is to be found by Merkel's definition, and noted the following facts with regard to them:

a. The lineae supremae are frequently very vague, even untraceable, although their juncture may be clear and prominent.

b. The lineae superiores are usually, though by no means invariably, more clearly marked; their junction is generally recognisable, and in all cases in my collection where this is so, they are seen to curve downwards into a **V** before meeting.

c. The lineae supremae, when descending to form a **U** about the median plane, as is often the case, occasionally short-circuit as well, giving an upper ridge with a slight depression between that and the lower junction which is our inion. This may be misleading at first if the inion be not boldly marked.

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[*] *Biometrika*, Vol. III. p. 375.

[†] "Die Linea nuchae Suprema," Leipzig, 1871. Quoted by Schwalbe, *Zeitschrift für Morphologie und Anthropologie*, Vol. I. p. 24, 1899.

[‡] *Biometrika*, Vol. III. p. 376.

d. Of the 53 male skulls, where the inion as defined by Merkel could be determined, it lay in 21 on the common horizontal tangent* to the lineae superiores, in 17 rather below, in 15 above.

In the 58 female skulls, it lay on the common tangent in 37 cases, in 9 below, in 12 above. Its average position in my series is evidently therefore at the point where the median plane cuts the common horizontal tangent to the lineae nuchae superiores.

e. The lineae superiores also short-circuited sometimes, and formed a misleading ridge along their common tangent, as well as being continued V-wise to a point below in the normal manner.

f. Where I found an inion projecting from the skull in a downward direction to form a hook, into which both the lineae supremae and lineae superiores are continued, it was possible to detect the spot at which the lineae supremae stopped short of the others. When such a skull was regarded in profile, the curve downwards towards the tip of the hook showed a slight break at this point.

Having made these general observations upon my 53 ♂ and 58 ♀ skulls, I made use of any that might serve to elucidate the system of lines upon my doubtful skulls, determining the inion by *d* where there was nothing to indicate for it another position.

With reference to the very considerable variations in form of the inion, I quote Le Double†: "Il peut être remplacé par une dépression, faire défaut, ou constitué par une éminence, lisse ou rugueuse, mamelonnée ou recourbée en crochet à sommet inférieur," adding that only in one case in our whole series of 141 possessing the inionic region is the inion clearly replaced by a depression.

5. CONTOURS.

A study was made of the three chief contours, Transverse, Horizontal, Sagittal. The contours were drawn by means of the Klaatsch contour tracer in the manner described by Dr Benington in his paper on Cranial Type Contours‡. From certain average measurements on these were constructed type contours for each of the three racial classes we were dealing with. The method employed was as follows:

(*a*) *The Transverse* (*vertical or auricular coronal*) *Section*. The skull being placed on the craniophor and adjusted in the usual manner to the standard horizontal plane, the horizontal rod of the craniophor determines the "apex" of the skull, which is marked with a pencil stroke. Short lines are also drawn on the skull just above the auricular passages, and in the same vertical plane as the middle of the ear-rods, leading into the auricular passages, and meeting the little horizontal lines by which the position of the auricular points has been marked. The skull being now transferred to the Klaatsch apparatus, the vertical plane is made horizontal, and the contour traced from as far into the orifice as the pointer

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* That is, the tangent perpendicular to the sagittal plane.

† *Variations des Os du Crâne*, p. 12.

‡ *Biometrika*, Vol. VIII. p. 123.

will go, through the auricular points and the apex line*. The line joining auricular point to auricular point is the auricular line of the contour, and the line perpendicular to it through its mid-point is the vertical axis MA of the transverse vertical contour. This vertical axis is divided into ten equal parts by aid of the proportional compasses, and lines parallel to the auricular line are drawn through the points of division. These lines are numbered from below upwards—1 (the auricular line), 2, 3, 4, 5, 6, 7, 8, 9, 10; numbers 2 to 10 will not necessarily be bisected by the vertical axis owing to the asymmetry of the skull. Another line ($A\frac{1}{4}$) is taken one quarter of the last section from the apex, and yet another ($M\frac{1}{4}$) one quarter of the first section from the auricular line. Ordinates are also drawn from the points ZR (R) and ZR (L) where the zygomatic ridges are crossed (that is, from the line of the ridge which is a continuation of the upper edge of the zygomatic arch). These last three lines were added to what had been done previously, in order to give an idea of the shape of the contour just above the auricular points. (When they are omitted, and the terminals of mean 1 and 2 splined up in the type contour, an impression is given that the contour is traced only from the zygomatic ridges.) A tangent is drawn to that part of the contour which traces the roof of the auricular orifice, at its lowest point within about $1\frac{1}{2}$ cms. from the mouth of the opening, i.e. at the points which will have rested on the horizontal ear-rods on the craniophor, the rods being inserted at not more than about $1\frac{1}{2}$ cms.† This tangent will not necessarily be parallel to the auricular line. Finally, a line parallel to the tangent is drawn touching the skull in the region of the apex. These two tangents aim at reproducing on the transverse contour the measurement taken on the craniophor, where a horizontal rod (i.e. parallel to the ear-rods) is brought down touching the skull from above, the height of it giving the OH of the direct measurements. The importance should be realised of adjusting the skull very carefully so that the pointer passes exactly along the lines leading into the auricular orifices, which give the plane in which the ear-rods lie. If, when this vertical plane is made horizontal, the pointer goes slightly above or below these lines, the outline of the roof of the orifice may be considerably altered (since it offers a rather irregular surface), and consequently also the position of the tangent.

The following measurements are now taken: length of vertical axis MA, length to R and L of each of the horizontals (the sum of $1R$ and $1L$ being the interauricular diameter), distance between the two tangents, and the co-ordinates of the two zygomatic points. They are entered in the columns of the record for each contour, in the order above given, as follows:

MA	$1R=1L$	$M\frac{1}{4}R$	$M\frac{1}{4}L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$

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* As a matter of fact I drew as much of the vertical section of the auricular passage both inferior and superior as my instrument permitted and these have been preserved for future study when comparative material is available.

† This is the distance to which I inserted the rods. It may be advisable (see p. 182) to shorten it

											ZR (R)		ZR (L)	
$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$	$A\frac{1}{4}R$	$A\frac{1}{4}L$	TH	y	x	y	x

The means of the columns give the corresponding lengths for the type skulls of each of our own groups, and from these we can plot the points in the type section. A line is now drawn passing through the plotted points, and we thus reach the transverse type contour. Since the skull is resting face upwards during this tracing, the contour as drawn gives the *norma facialis*, and this is the aspect that Benington has chosen. I myself, like Thomson, have reversed on the type contour, the right and left of the individual contours, thus giving the *norma occipitalis*. (See Figs. I–VI.)

(*b*) *The Glabellar Horizontal Section*. This is obtained by placing the skull on the craniophor and by aid of the scriber marking points on the same horizontal plane as the nasion: one, the gamma, in the occipital region where it is crossed by the sagittal plane, the other in the region above the left auricular passage*.

The skull is now adjusted on the Klaatsch support until these three points are found to be in the same horizontal plane. The horizontal contour is then drawn parallel to this plane, but through the glabella; it will fall below the lambda. The points in the median plane immediately above the nasion and below the lambda on this contour are specially marked with the Klaatsch tracer. They will be spoken of as F and O, the glabellar and occipital points of the horizontal contour. The points (T_R and T_L) at which the contour crosses the temporal lines, right and left, are separately marked with the tracer as Thomson has done: it is found more accurate to mark them in this way, as there is a certain liability to error in turning a sharp corner such as that at the temporal lines, unless special care is exercised. This completes the tracing of the contour. The fronto-occipital line FO is now drawn, and is divided into ten equal parts by the proportional compasses; lines perpendicular to FO are drawn through the points of division and numbered 2 to 10, travelling from F to O; one at $O\frac{1}{4}$ to obtain the occipital curvature; lines through $F\frac{1}{4}$ and $F\frac{1}{2}$ to give the frontal curvature, and to complete this, ordinates from T_R and T_L.

We may note that when these temporal lines are not marked, as in Dr Benington's contours, the resultant effect in the frontal region of the type contour is much more rounded, and suggests at first sight that the plane in which the contour was taken is higher than the glabella, where the temporal ridge is much less prominent.

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* It is not really necessary to make such a point above both auricular passages, since three points suffice to determine a plane; though a fourth point might act as a check on the orientation of the skull upon the Klaatsch holder

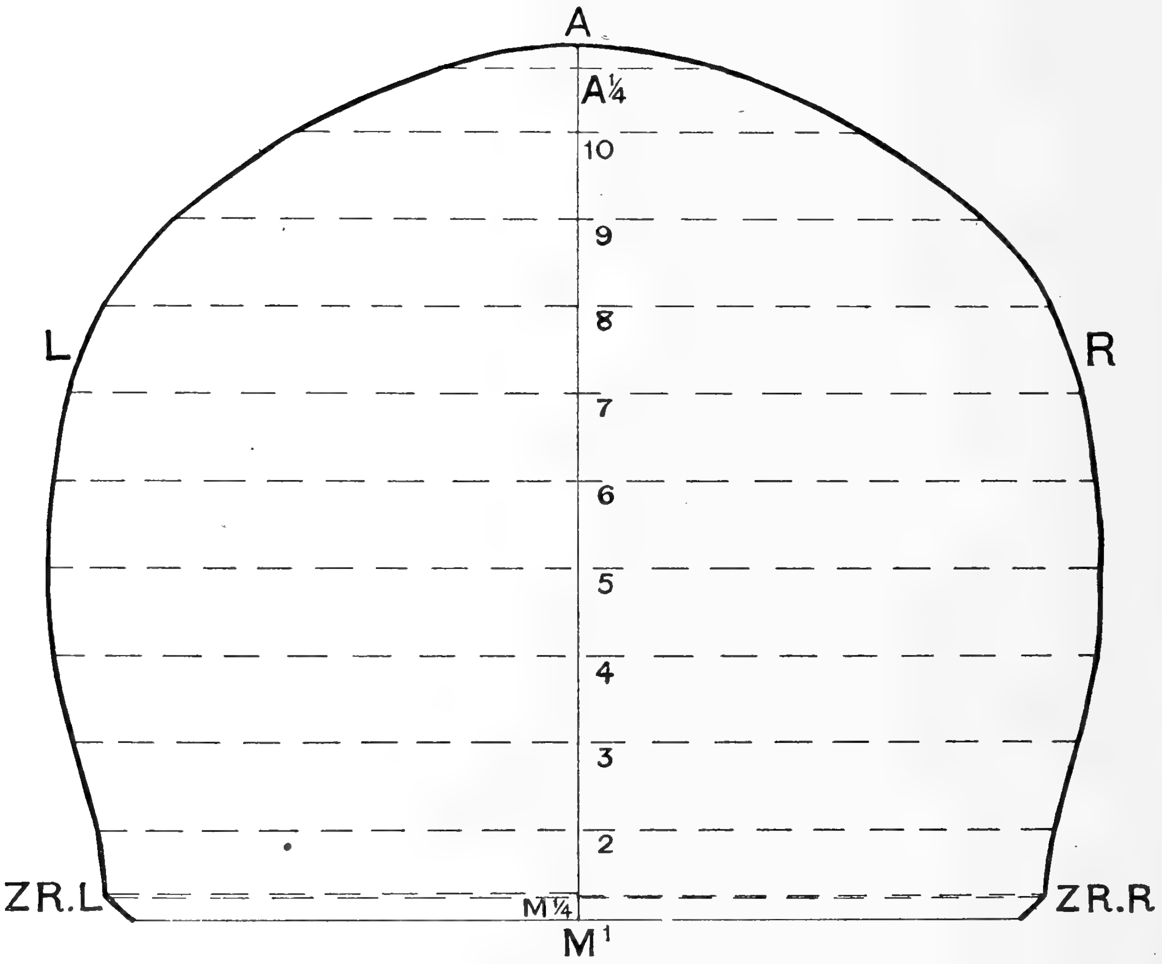

FIG. I. Burmese Crania. Type A (Burmans). ♂. Vertical Contour. (44 crania.)

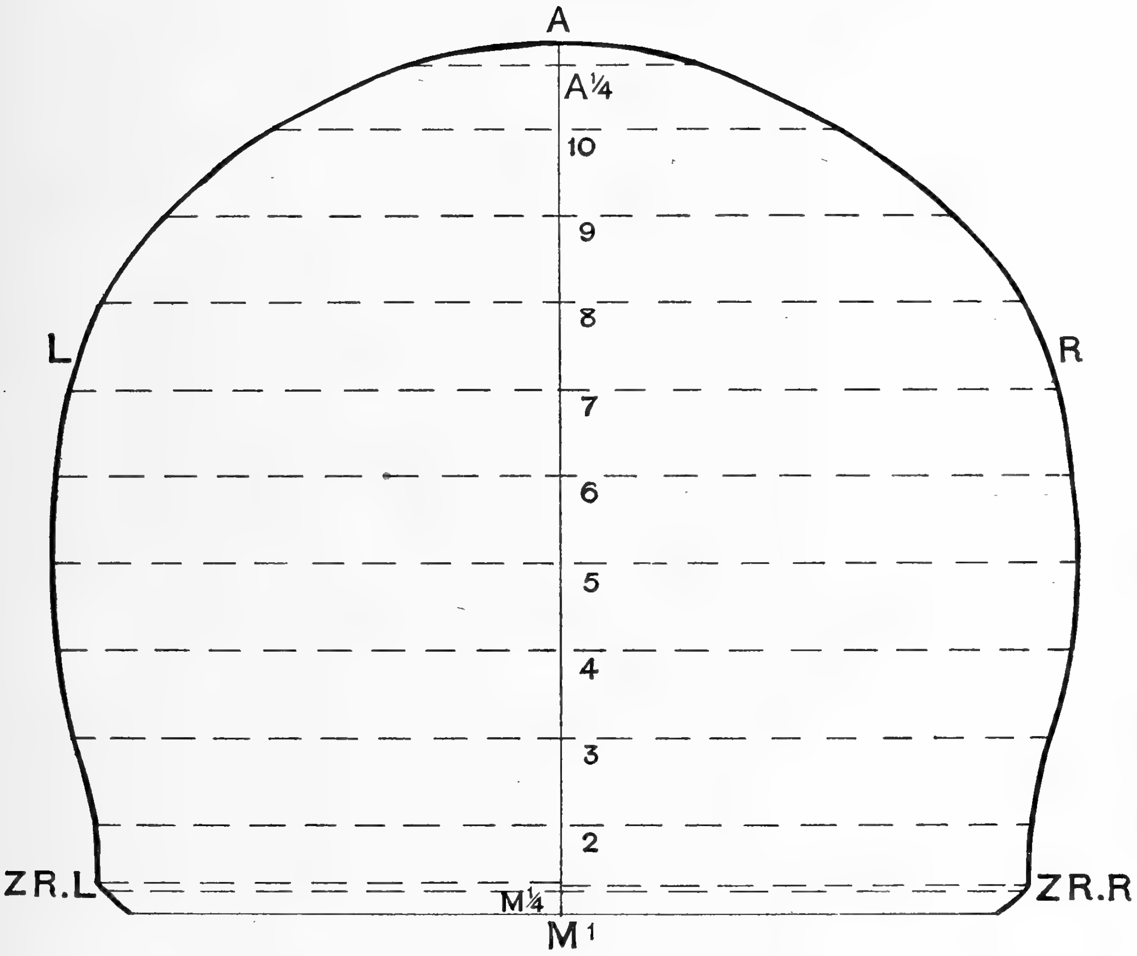

FIG. II. Burmese Crania. Type B (? Hybrids). ♂. Vertical Contour. (7 crania.)

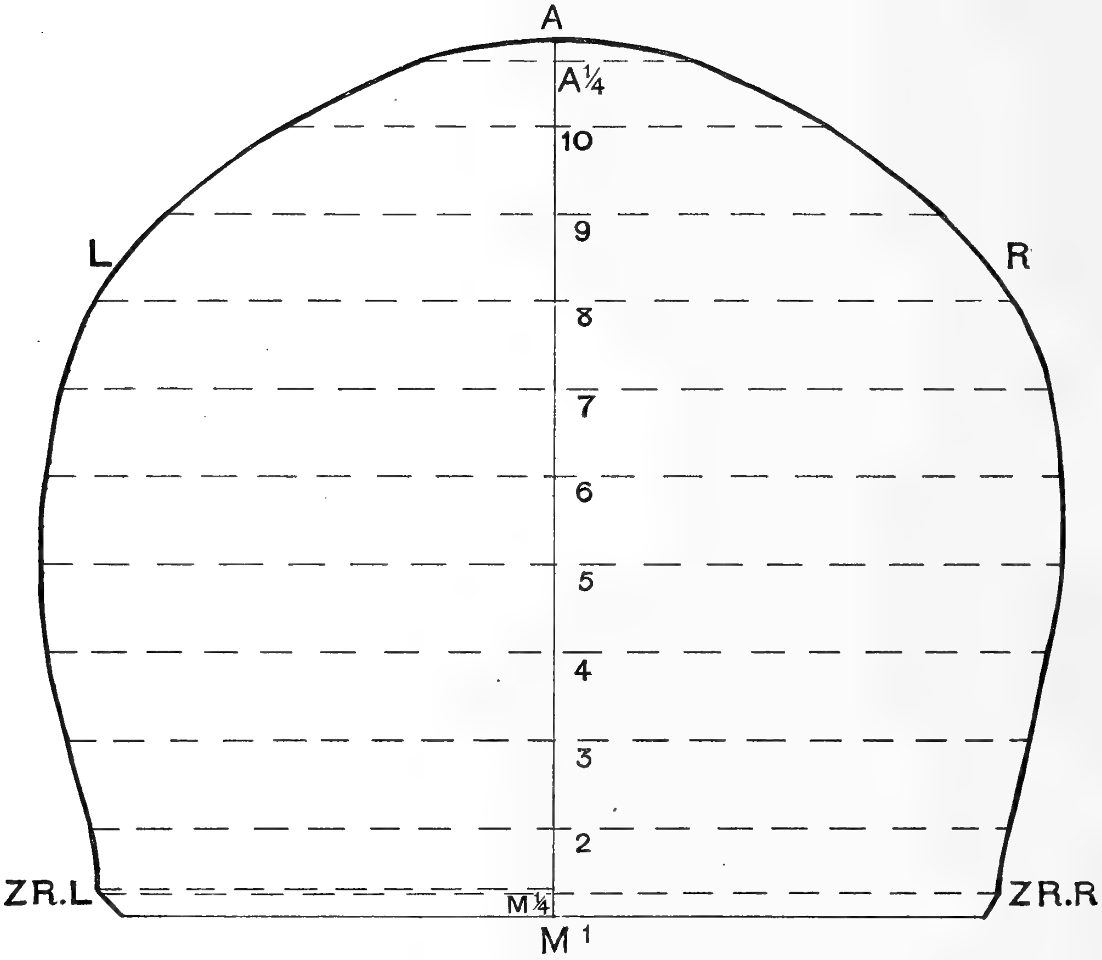

FIG. III. Burmese Crania. Type C (? Karens). ♂. Vertical Contour. (8 crania.)

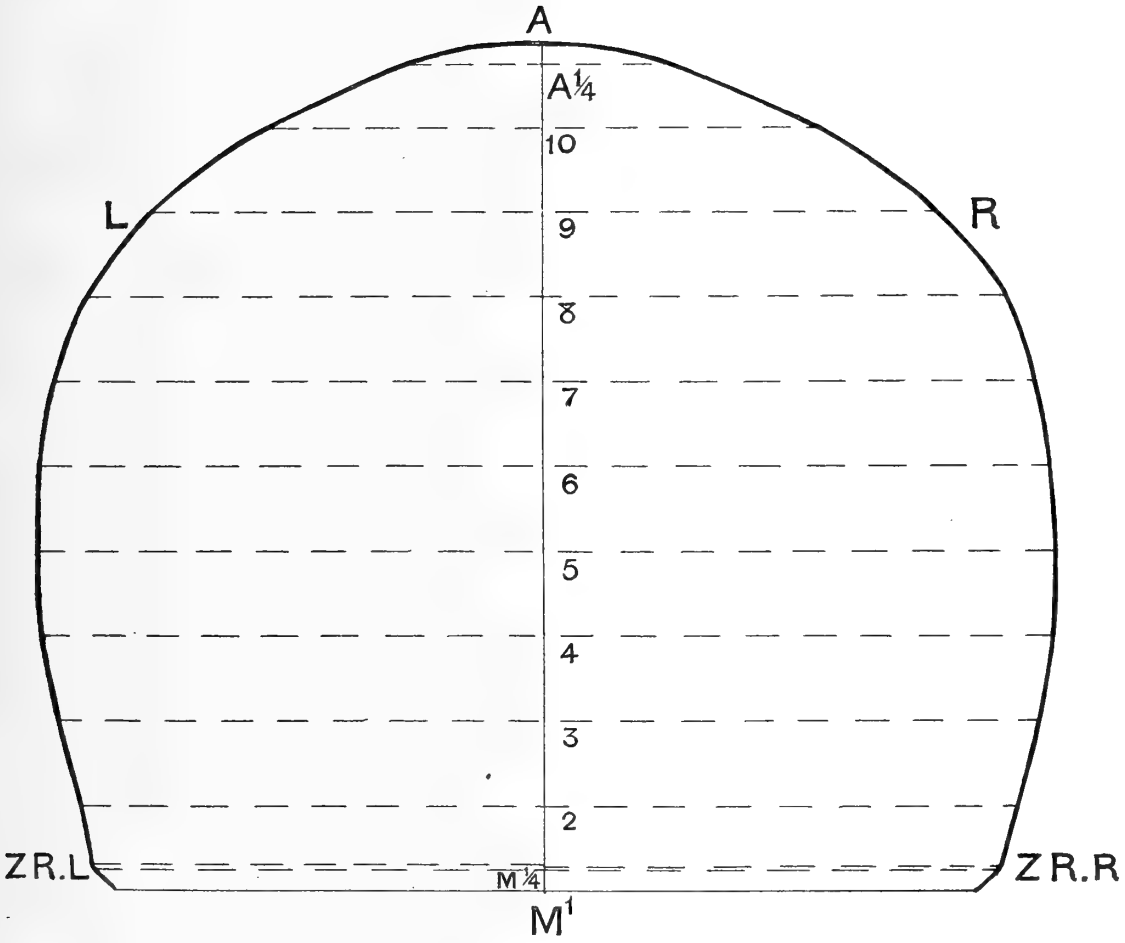

FIG. IV. Burmese Crania. Type A (Burmans). ♀. Vertical Contour. (38 crania.)

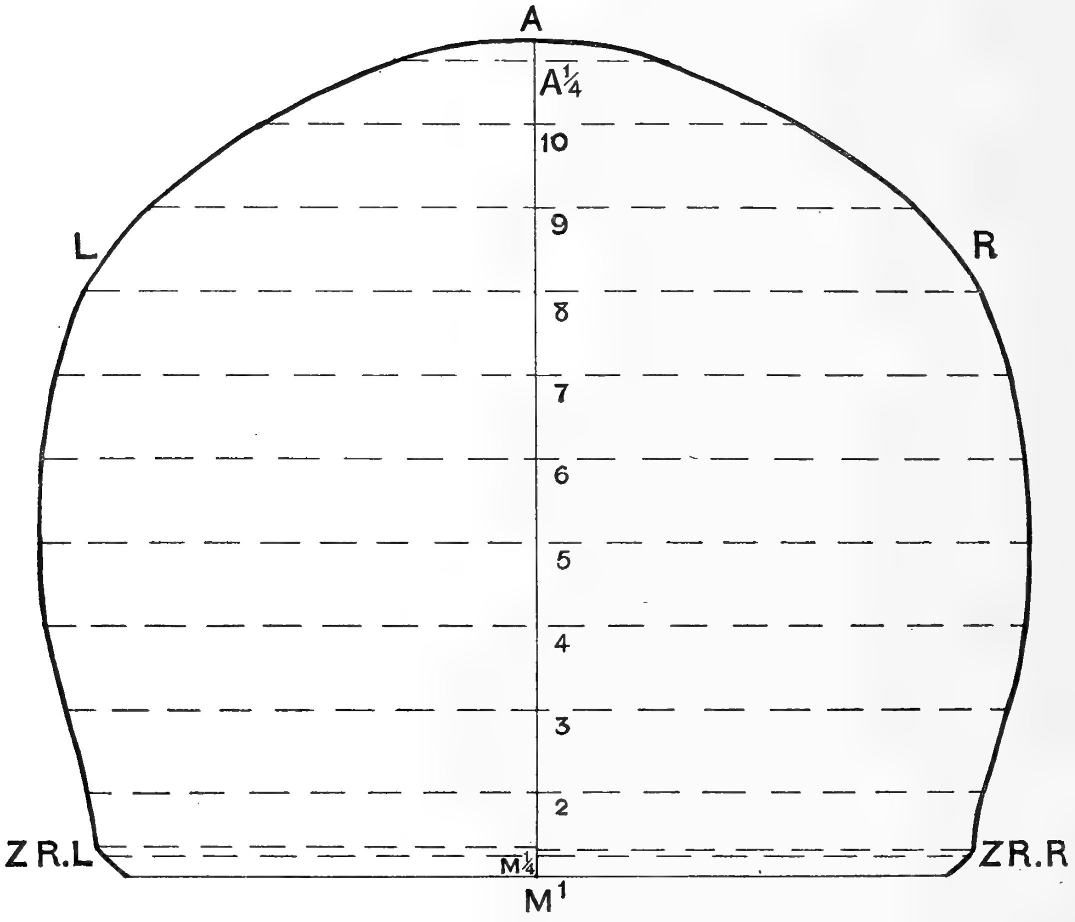

FIG. V. Burmese Crania. Type B (? Hybrids). ♀. Vertical Contour. (17 crania.)

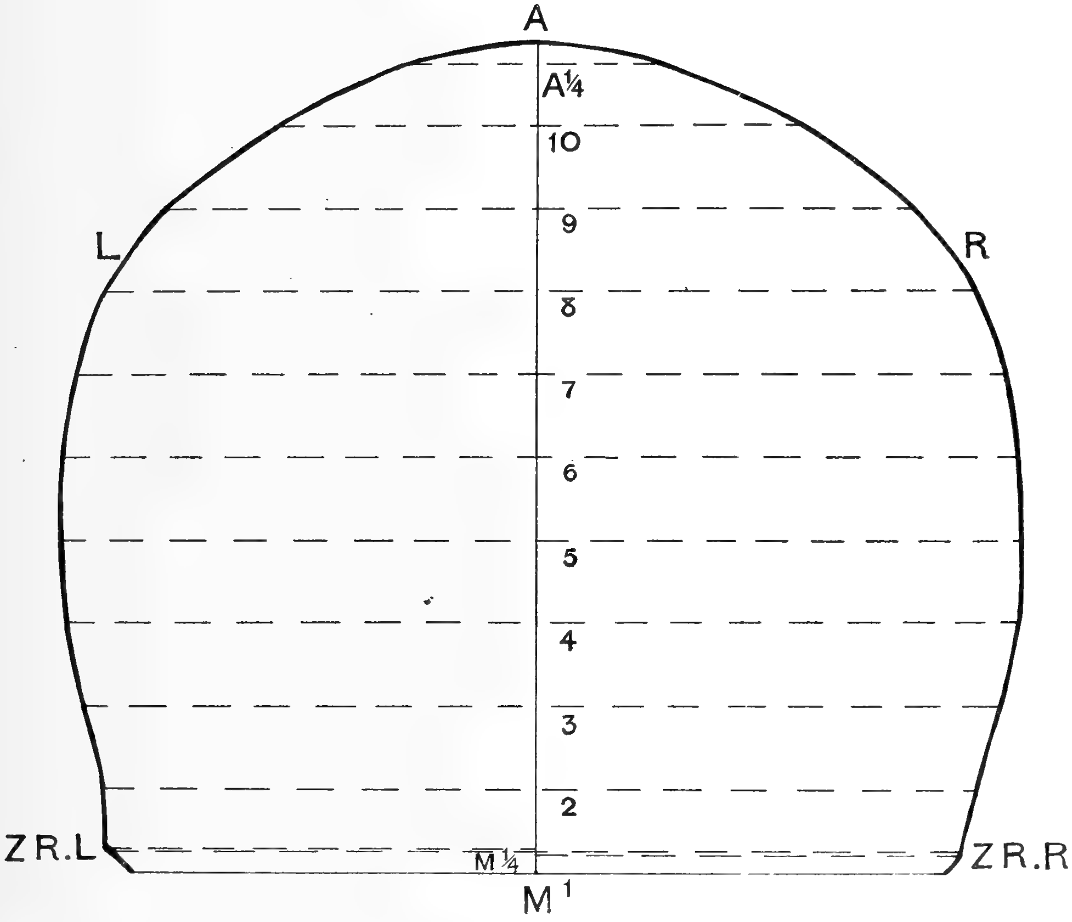

FIG. VI. Burmese Crania. Type C (? Karens). ♀. Vertical Contour. (18 crania.)

We have the following measurements recorded for each available skull:

FO	$F\frac{1}{4}R$	$F\frac{1}{4}L$	$F\frac{1}{2}R$	$F\frac{1}{2}L$	$2R$	$2L$	$3R$	$3L$	$4R$	$4L$	$5R$	$5L$	$6R$	$6L$

										T_R		T_L	
$7R$	$7L$	$8R$	$8L$	$9R$	$9L$	$10R$	$10L$	$O\frac{1}{4}R$	$O\frac{1}{4}L$	y	x	y	x

The means are again found for all the columns, and the type contours constructed as before.

When a horizontal contour is being traced the skull is lying with its apex downwards. In plotting the type contours we reverse the measurements to the right and left in the individual contours, in order to give the horizontal sections as looked at from above when the skull is in its normal position apex uppermost; thus we obtain the *norma verticalis*. (See Figs. VII–XII.)

(*c*) *The Sagittal or Median Section.* The point γ and the apex having been already marked on the skull, this is placed in the Klaatsch support right side downwards, and adjusted so that nasion, bregma and lambda all lie in the same horizontal plane. The contour is then drawn from the point where the basi-occipital bone unites with the sphenoid, to the basion; from the opisthion round to the nasion and along the nasal ridge; from the nasal spine to the alveolar point and on to the end of the palate spine. The alveolar point, the nasion, the bregma, the apex, the lambda, the gamma, the inion, the opisthion, the basion, the point where the left half of the transverse palatine suture meets the sagittal palatine suture (all lying approximately in this plane), are marked, as also the projection of the auricular* and sub-orbital points upon it.

It will be noted that when we have determined our sagittal plane for contouring, by the nasion, bregma, lambda, this plane is not bound to pass exactly through the inion, the opisthion, basion and alveolar point, nor exactly along the ridge of the nose, the mesial ridge of the premaxilla, and the middle of the palate to the tip of the nasal spine. Various slight asymmetries of the skull will throw these points out, and our plane will miss them. It is these things themselves, however, that we want represented in our sagittal contour—it will not, for instance, give a measure of nasal prominence, if, the nose being skewed as often happens, we pass the pointer along the nasal bones but off the nasal ridge. I therefore admit that, having fixed my plane, I have not kept rigidly to it, but have slightly depressed

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* See p. 181.

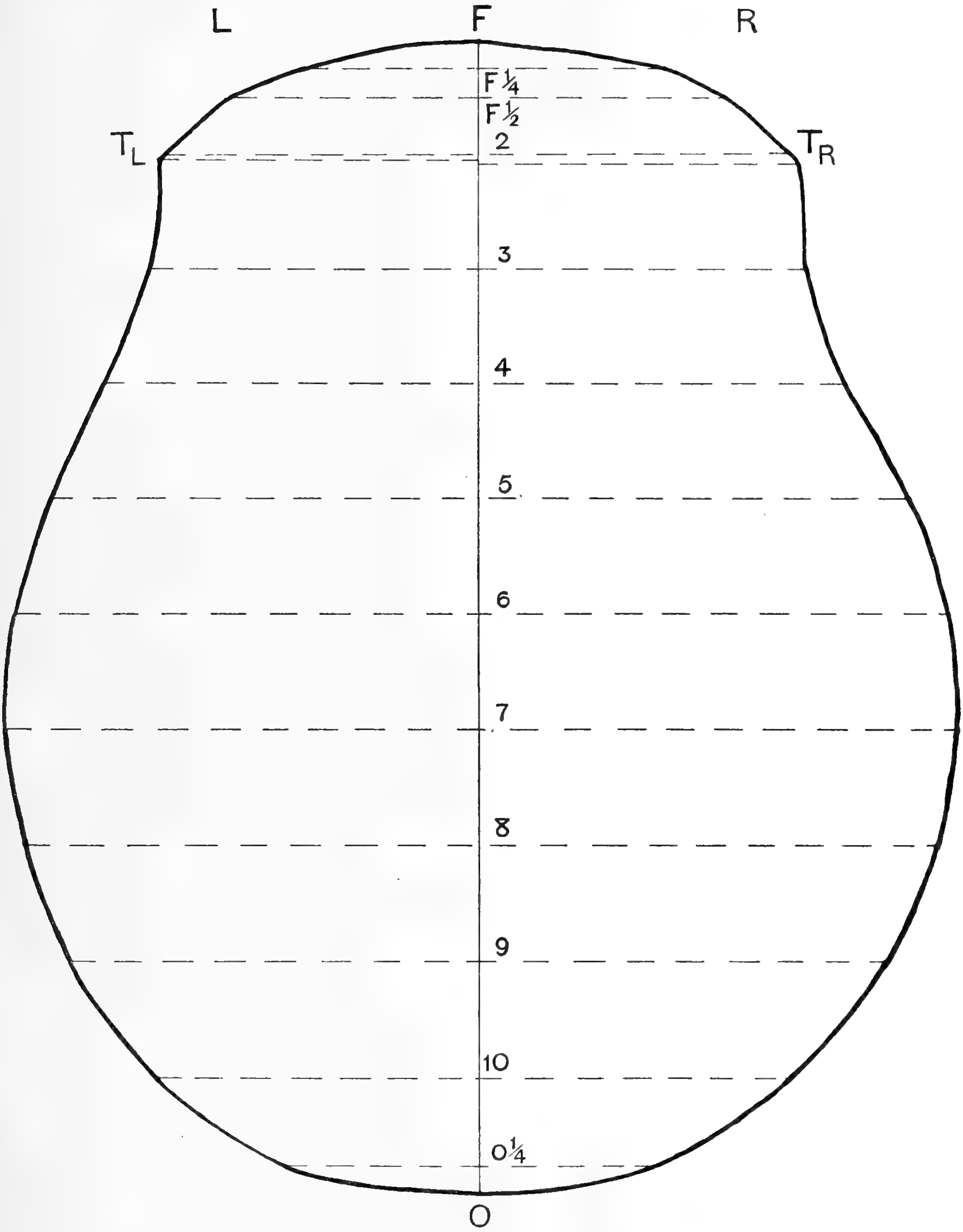

FIG. VII. Burmese Crania. Type A (Burmans). ♂. Horizontal Contour. (44 crania.)

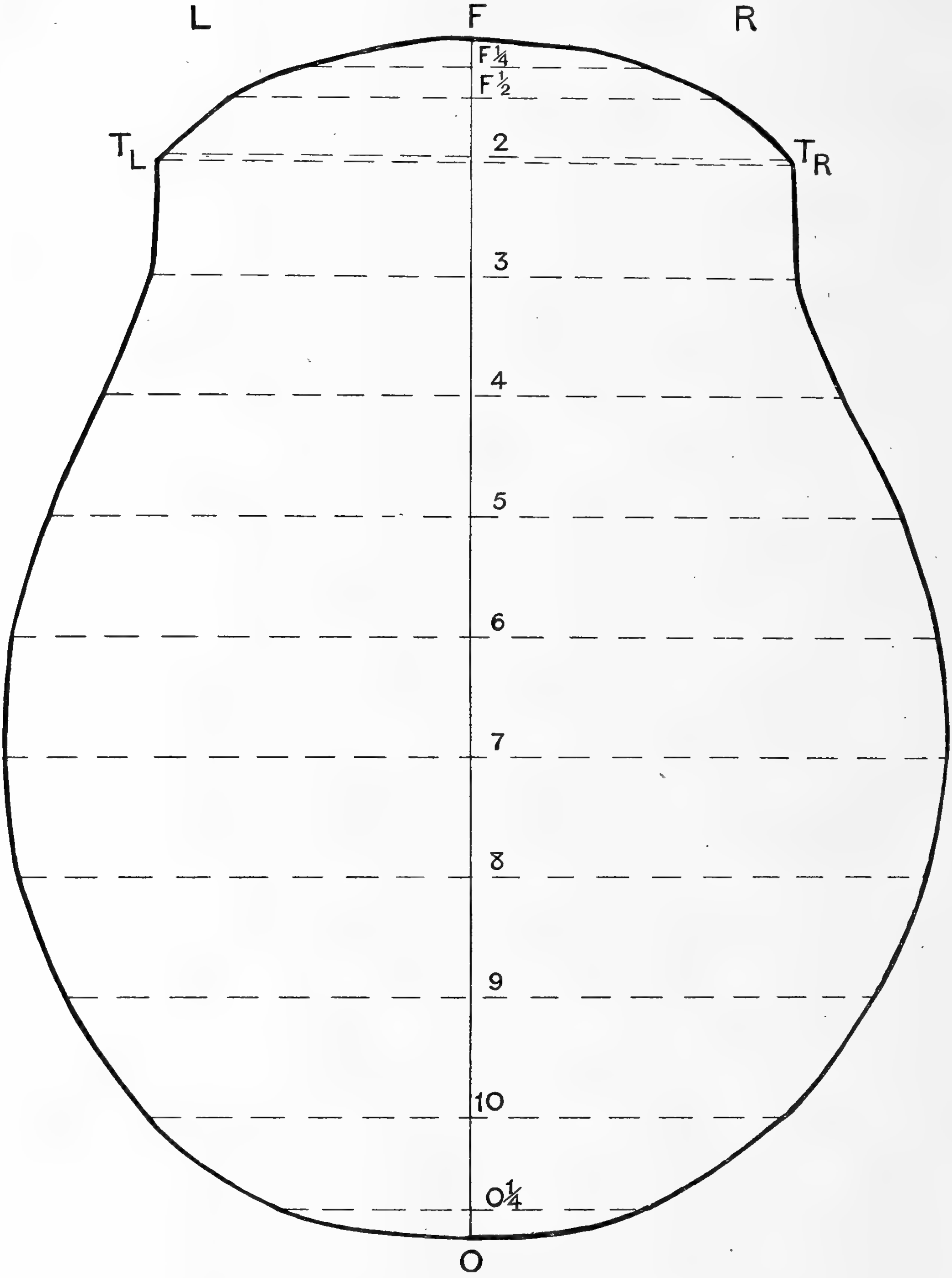

Fig. VIII. Burmese Crania. Type B (? Hybrids). ♂. Horizontal Contour. (7 crania.)

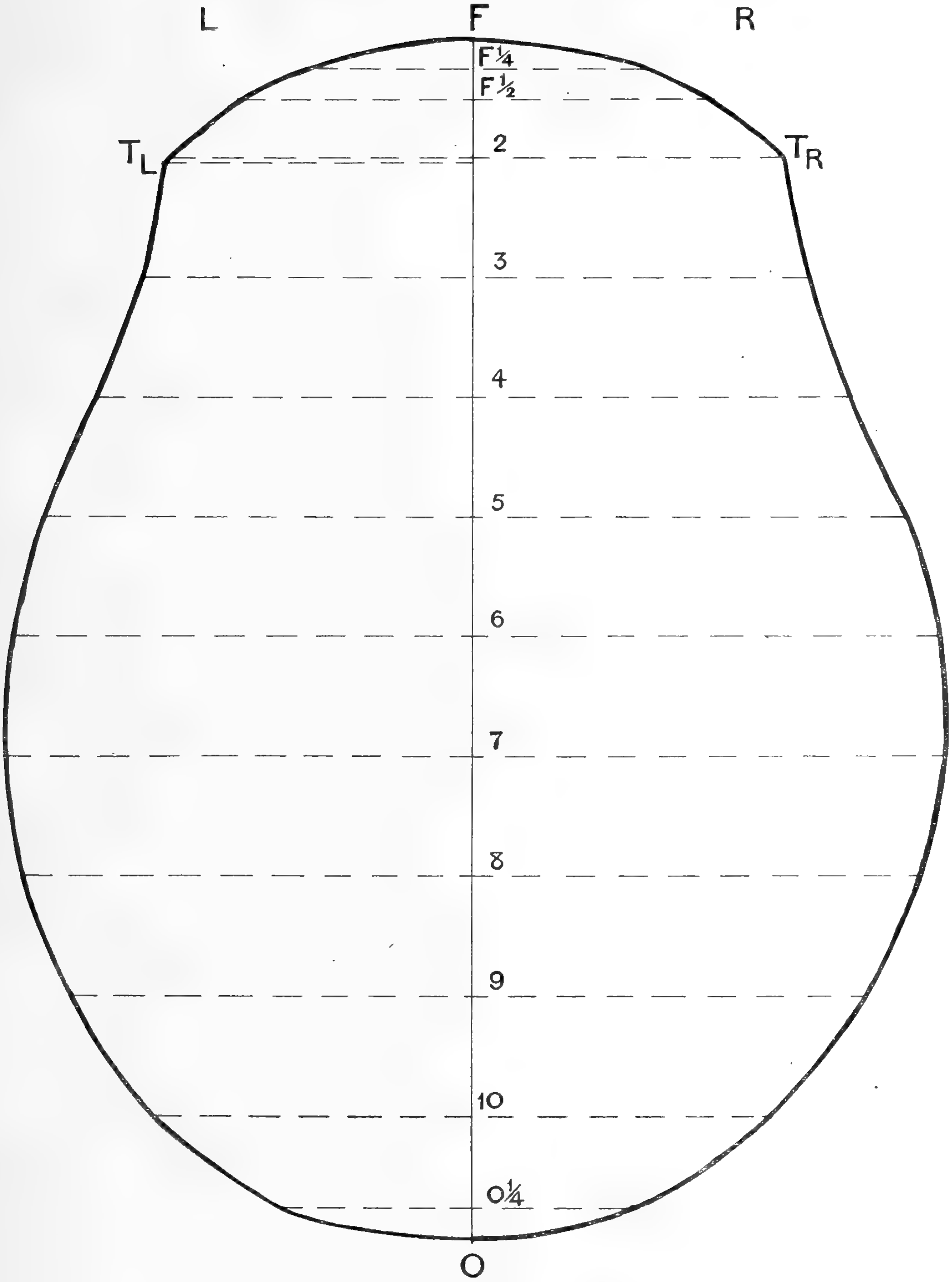

FIG. IX. Burmese Crania. Type C (? Karens). ♂. Horizontal Contour. (8 crania.)

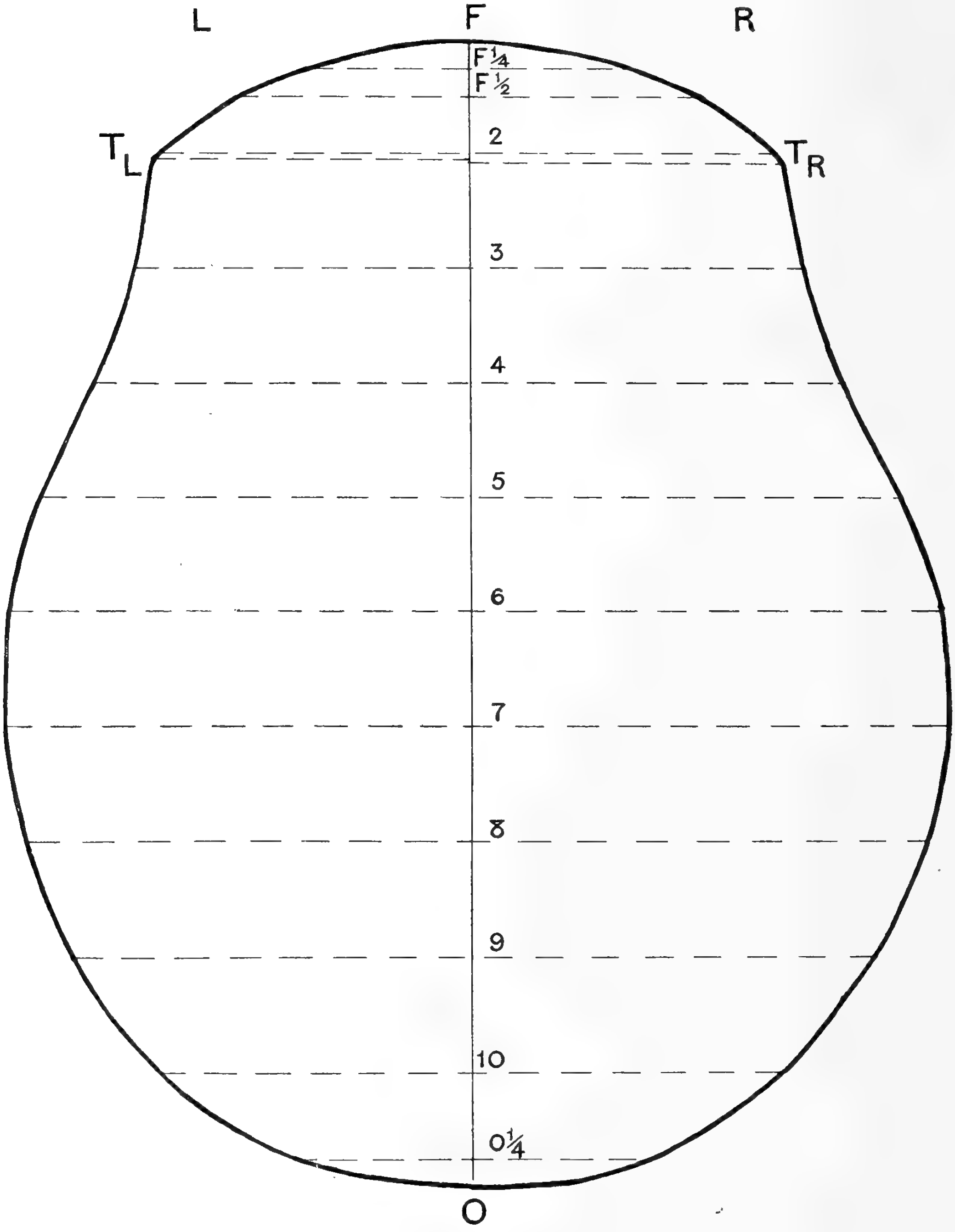

Fig. X. Burmese Crania. Type A (Burmans). ♀. Horizontal Contour. (39 crania.)

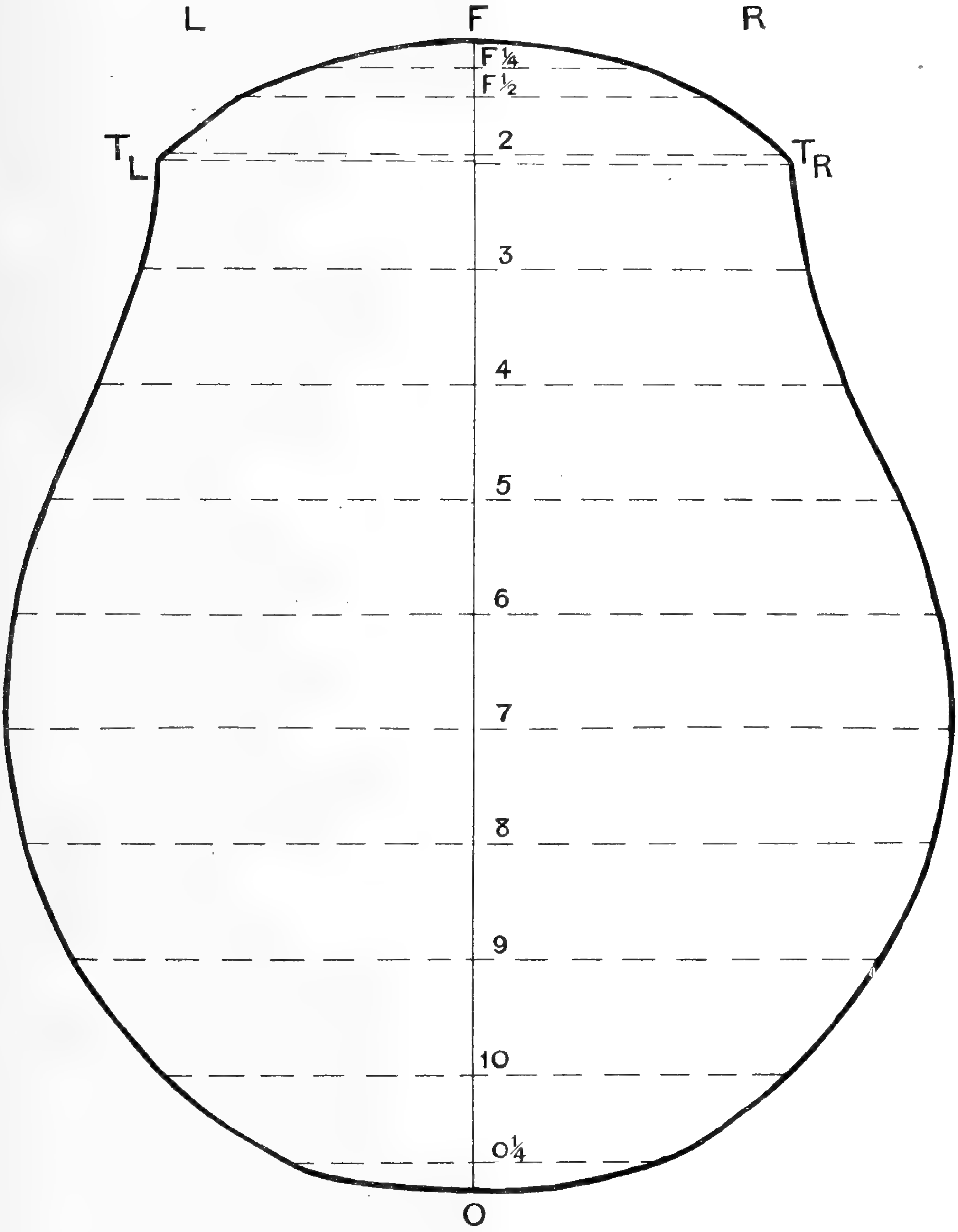

FIG. XI. Burmese Crania. Type B (? Hybrids). ♀. Horizontal Contour. (17 crania.)

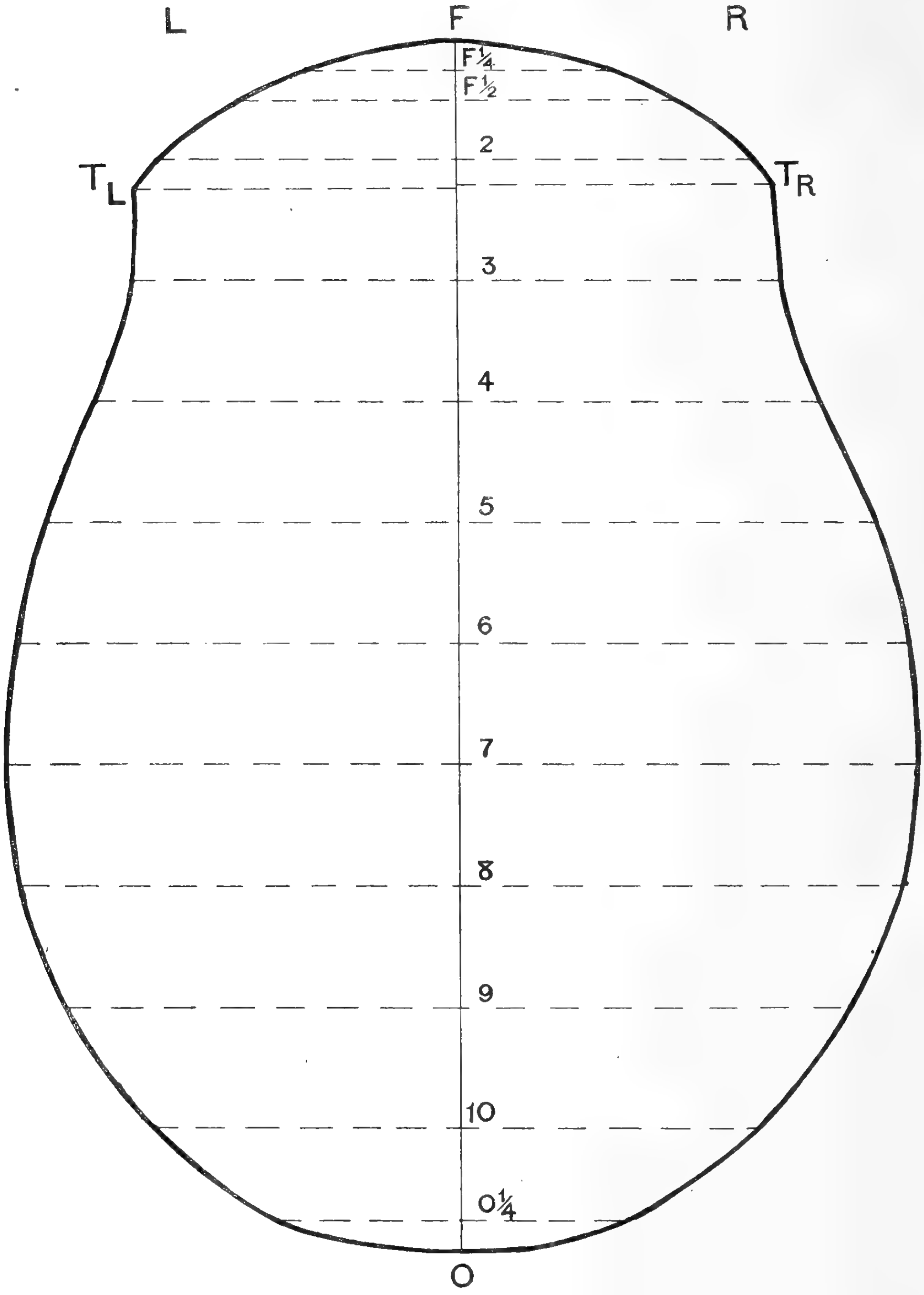

Fig. XII. Burmese Crania. Type C (? Karens). ♀. Horizontal Contour. (18 crania.)

or elevated the pointer if necessary when approaching one of these characters, in order to pass right over or along it, and represent it properly in the tracing*.

We now take the horizontal line through the nasion, i.e. the line linking the nasion to the gamma (γ), as our horizontal axis, and divide it into ten equal pieces with the proportional compasses. The nasion counting as 0 and the other divisions from nasion to gamma as 1, 2, ... 9, ordinates above are drawn through these ten points perpendicular to the axis, also through a point ($N\frac{1}{4}$) at a quarter of the distance of 1 from N, and two points ($\gamma\frac{1}{4}$ and $\gamma\frac{1}{8}$) a quarter and an eighth of the distance of 9 from gamma respectively. Ordinates 8, 9 and $\gamma\frac{1}{4}$ are produced below the axis, to the contour line, and another ordinate is added below it from a point $\gamma\frac{1}{2}$, halfway between 9 and γ. Ordinates are drawn from the bregma, lambda, opisthion, and sub-orbital point perpendicular to the axis. Vertical tangents to the most projecting points of the glabellar and occipital regions are drawn to $N\gamma$ produced, the points of contact being named G and Oc respectively.

The basion is joined to the nasion and gamma; so also is the inion. The alveolar point is joined to the nasion and basion, giving the fundamental triangle. To determine the vertex a horizontal tangent is drawn to the top of the skull, and an ordinate drawn to $N\gamma$ from the point of contact. To give some measure of nasal characteristics, the tip (L) of the nasal contour has been joined to the nasion, so that NL gives the nose length, and the angle between NL and $N\gamma$ is the angle of nasal prominence. Some nasal ridges exhibit a double curve, turning downward towards the tip so that for a little distance NL either cuts the outline or coincides with it. The point where NL first meets the outline of the ridge is marked L', and NL' is also measured; where there is no double curve NL of course is equal to NL'. The maximum subtense is drawn from NL to the curve above L'.

It was desired to take some measurement of the proportions of the palate. A horizontal was therefore drawn through the alveolar point and a vertical line drawn to it through the point where the left half of the transverse palatine suture meets the sagittal palatine suture. Another horizontal was drawn through the highest point† in the contour taken along the palatine process of the maxilla, and the junction of this with the vertical already drawn gave the point P of which I have measured the co-ordinates. It will be seen that I have taken measurements only of the palatine process, the ordinate of P giving its greatest vertical height, and the abscissa its horizontal length.

Finally, a line is drawn through the sub-orbital point parallel to the $N\gamma$ line, and a perpendicular dropped to this from the apex gives the auricular point which is to be used for the contours. Now if the plane passing through the nasion, bregma and lambda, which we have used as our sagittal plane, were exactly at right angles to the horizontal plane defined as passing through the left sub-orbital point and

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* Our Klaatsch tracer is provided with a vertical knife-edge as well as a tracing pointer to meet these difficulties, but as the pointer must be used for horizontal and transverse sections the labour of attachment of the knife-edge for the sagittal is against its use. Duplicate instruments would be desirable.

† It very frequently occurs that the point where the sutures meet is itself the highest point.

the two auricular points (in practice the "craniophor auricular points*"), then when the sagittal plane is adjusted in a horizontal position for this contour, the interauricular axis will be in a plane vertical to it, and the projections of the two auricular points on our paper will be in the same straight line as the sub-orbital point, this line being parallel to the horizontal axis $N\gamma$. If, in addition, our vertical plane (which determines the apex and has been made perpendicular to the horizontal plane) be also exactly perpendicular to the sagittal plane, the projections of our two auricular points will coincide. They will also coincide with the perpendicular drawn in the sagittal contour from the apex to the horizontal line through the sub-orbital point. All this, however, supposes a high degree of symmetry in the skull, such as I have found comparatively few skulls to possess. We must also take into account the fact that the "craniophor auricular points"—i.e. the points which actually touch the ear-rods and from which the horizontal plane is determined—are frequently not accessible by the pointer of the Klaatsch when the skull is lying on its side, so that it is the "auricular points by definition" which are actually used for projection on the paper. We thus see that it would be possible, but for the limitations of our contour tracer, to obtain five "auricular points" on every sagittal contour, and while they might all coincide, the odds are against such coincidence. As it is, we have three, and of these I have used that obtained by the perpendicular from the apex on the sub-orbital horizontal, for my measurements; but shall discuss later the differences of position between this and the other two.

This completes the measurements which were used in the construction of the type sagittal contours. (See Figs. XIII–XVIII.)

I have to place on record my thanks to Miss A. G. Davin for the preparation of all the graphs of my type contours.

Our record-sheet entries are:

$N\gamma$	Vertex		β		Glabella	
	x from N	y	x from N	y	Gx	Gy

Ordinates above $N\gamma$													
0	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	γ

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* See above, p. 181

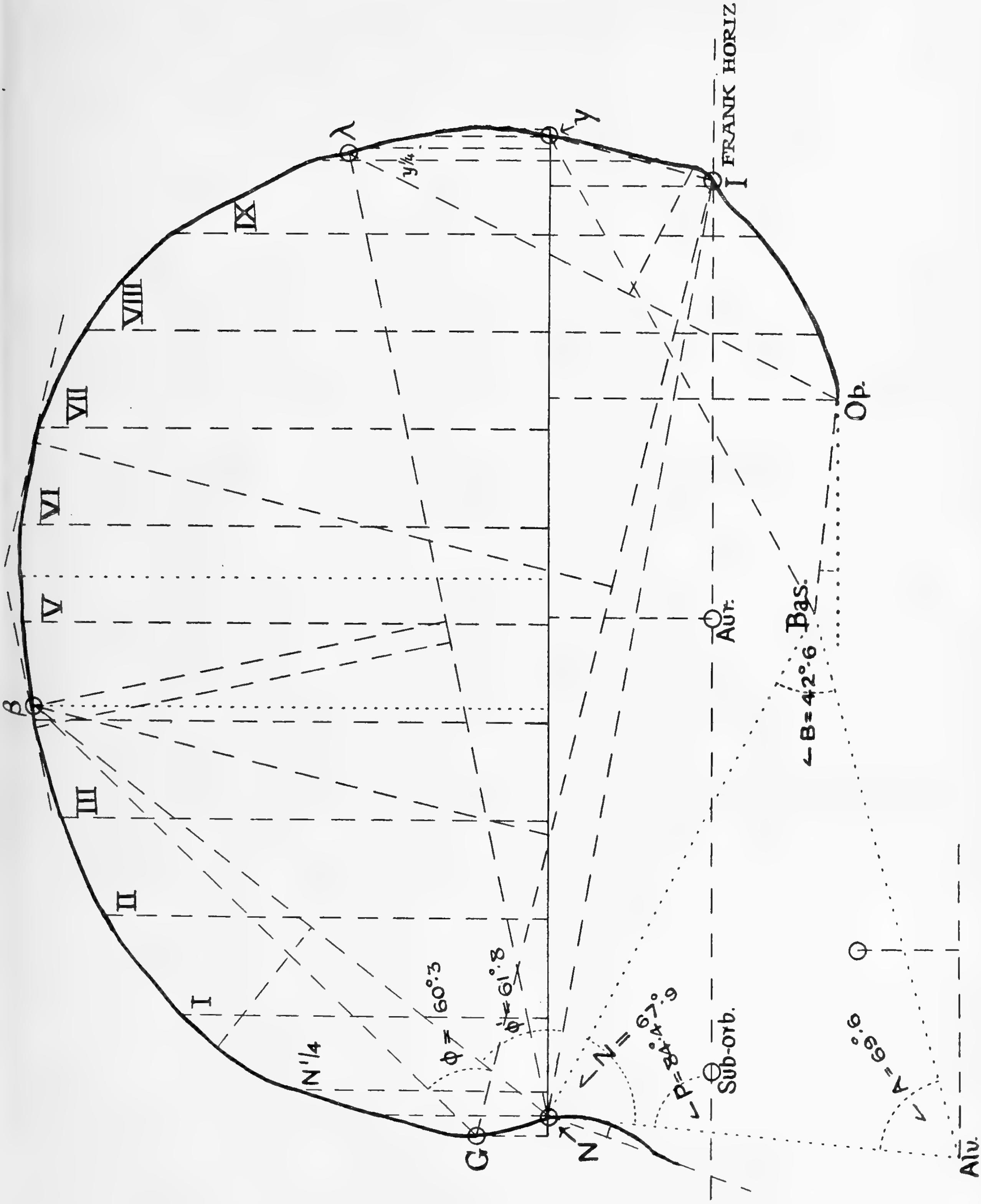

FIG. XIII. Burmese Crania. Type A (Burmans). ♀. Sagittal Contour. (44 crania.)

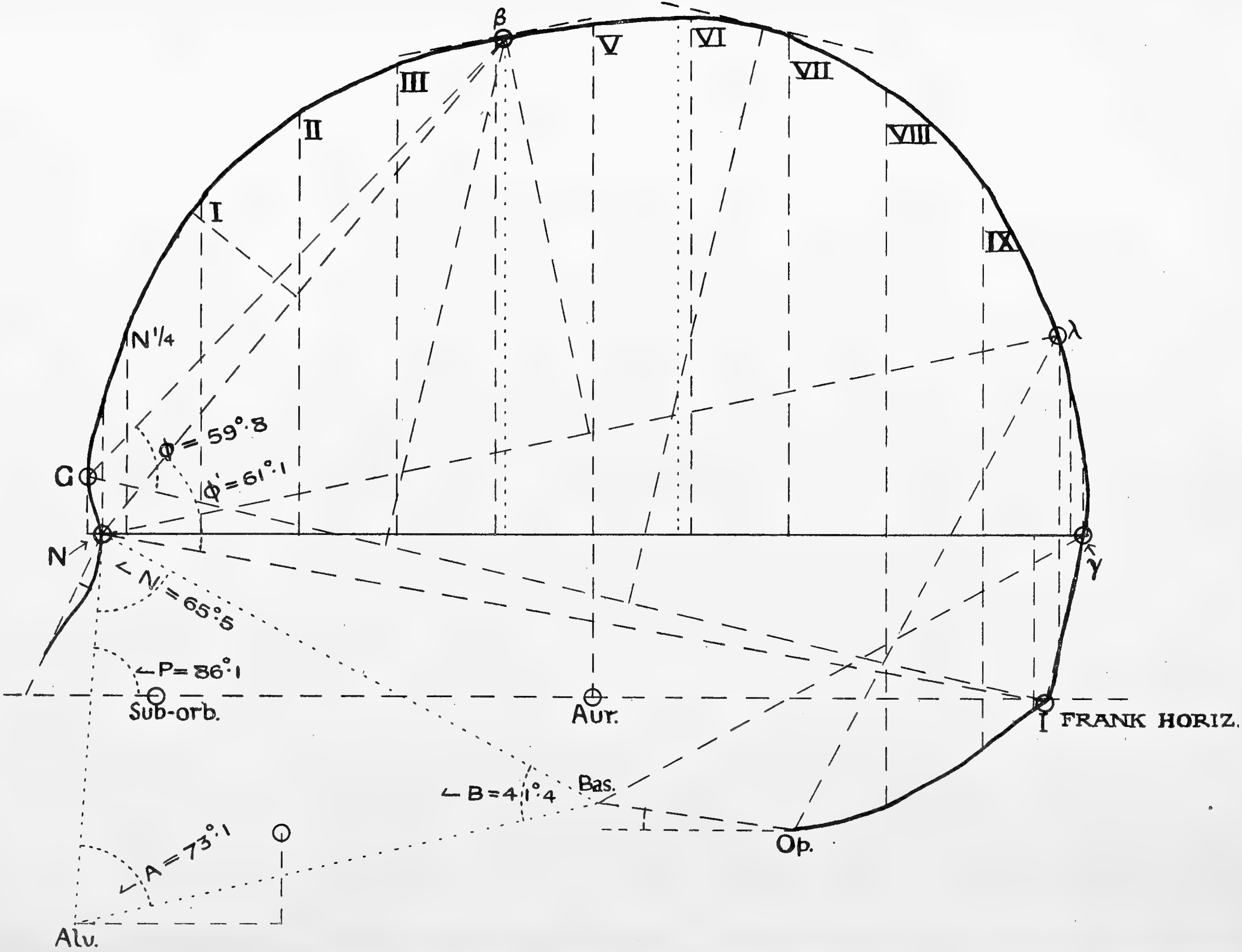

Fig. XIV. Burmese Crania. Type B (? Hybrids). ♂. Sagittal Contour. (7 crania.)

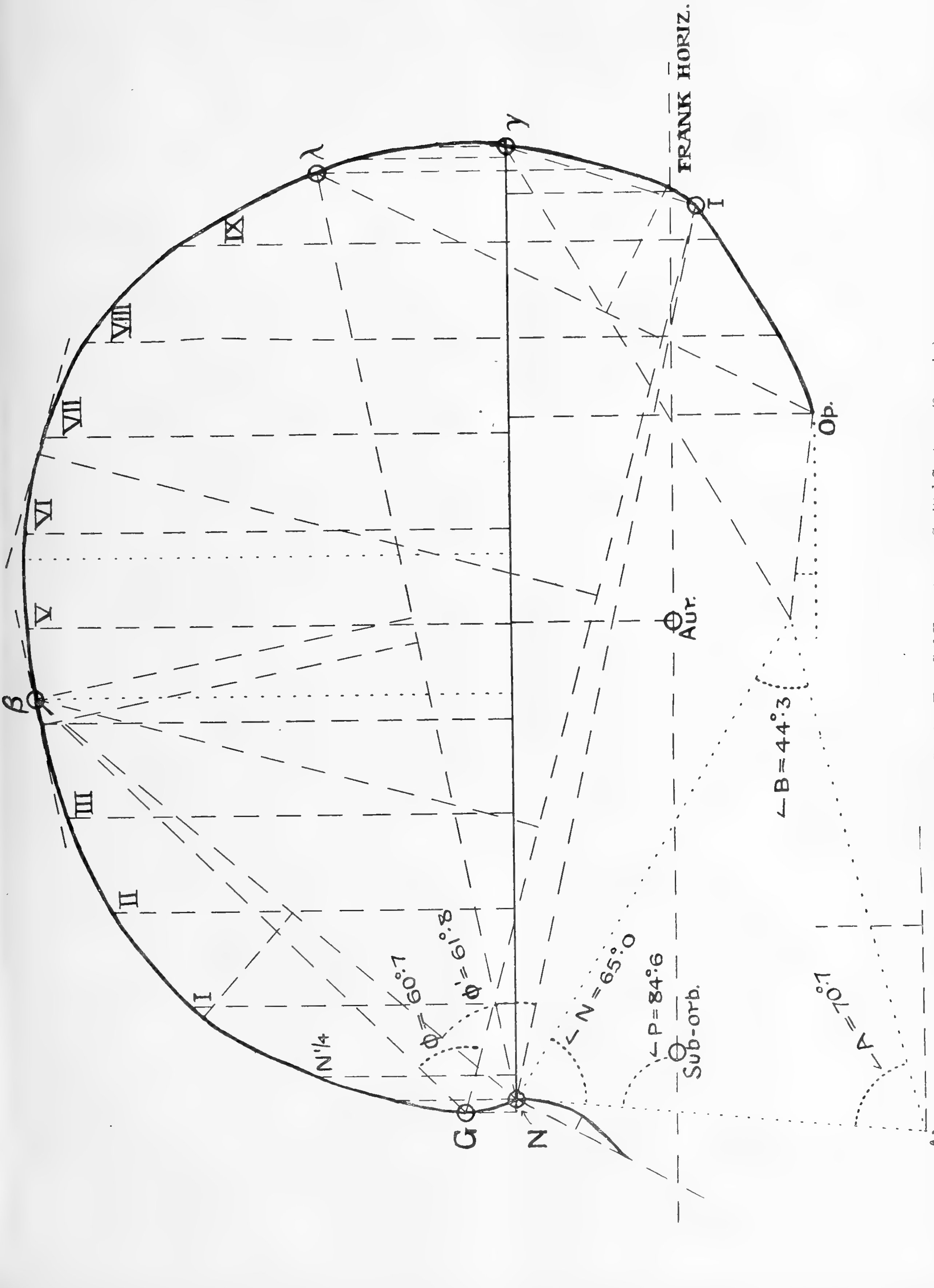

Fig. XV. Burmese Crania. Type C (? Karens). ♂. Sagittal Contour. (8 crania.)

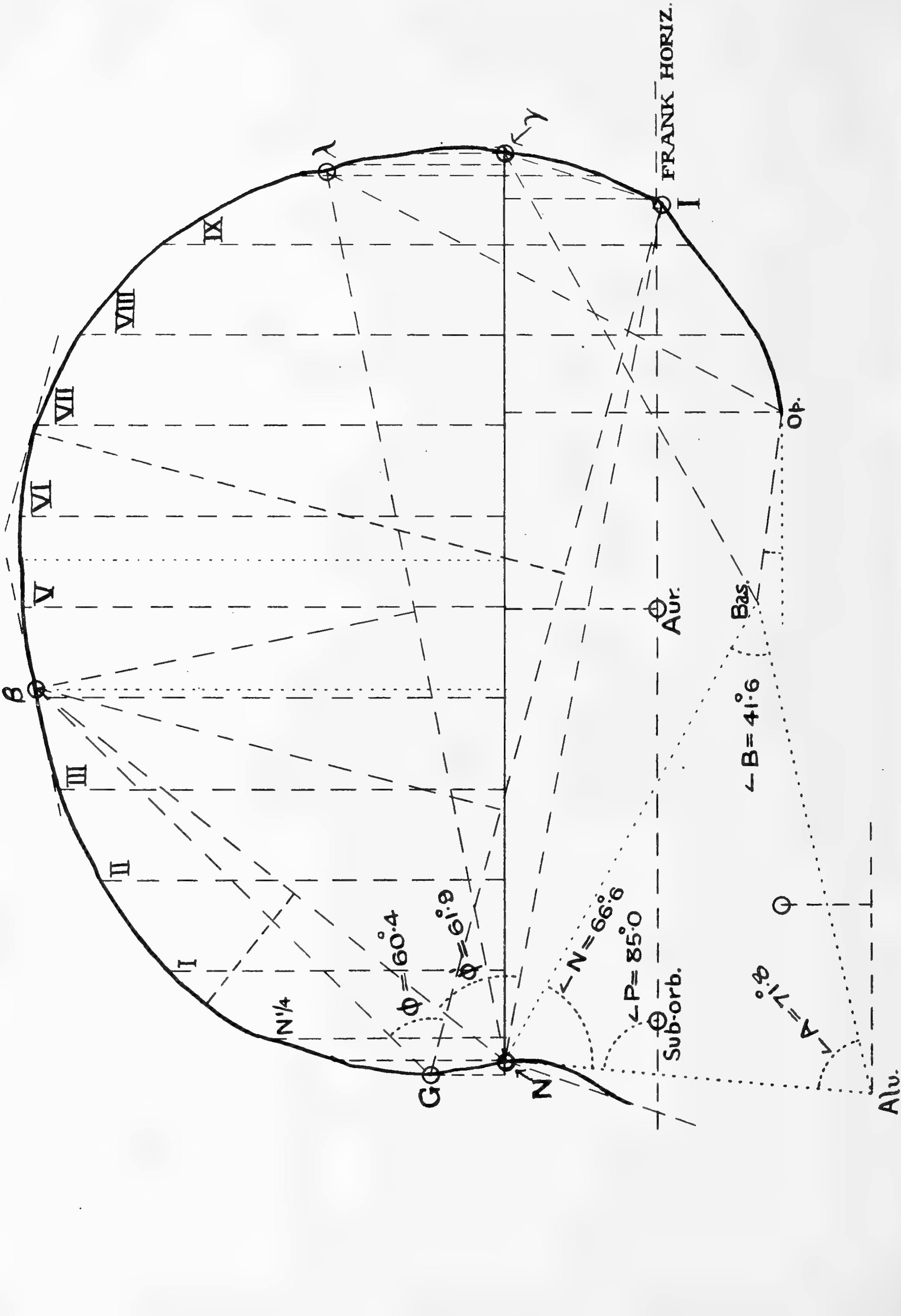

Fig. XVI. Burmese Crania. Type A (Burmans). ♀. Sagittal Contour. (38 crania.)

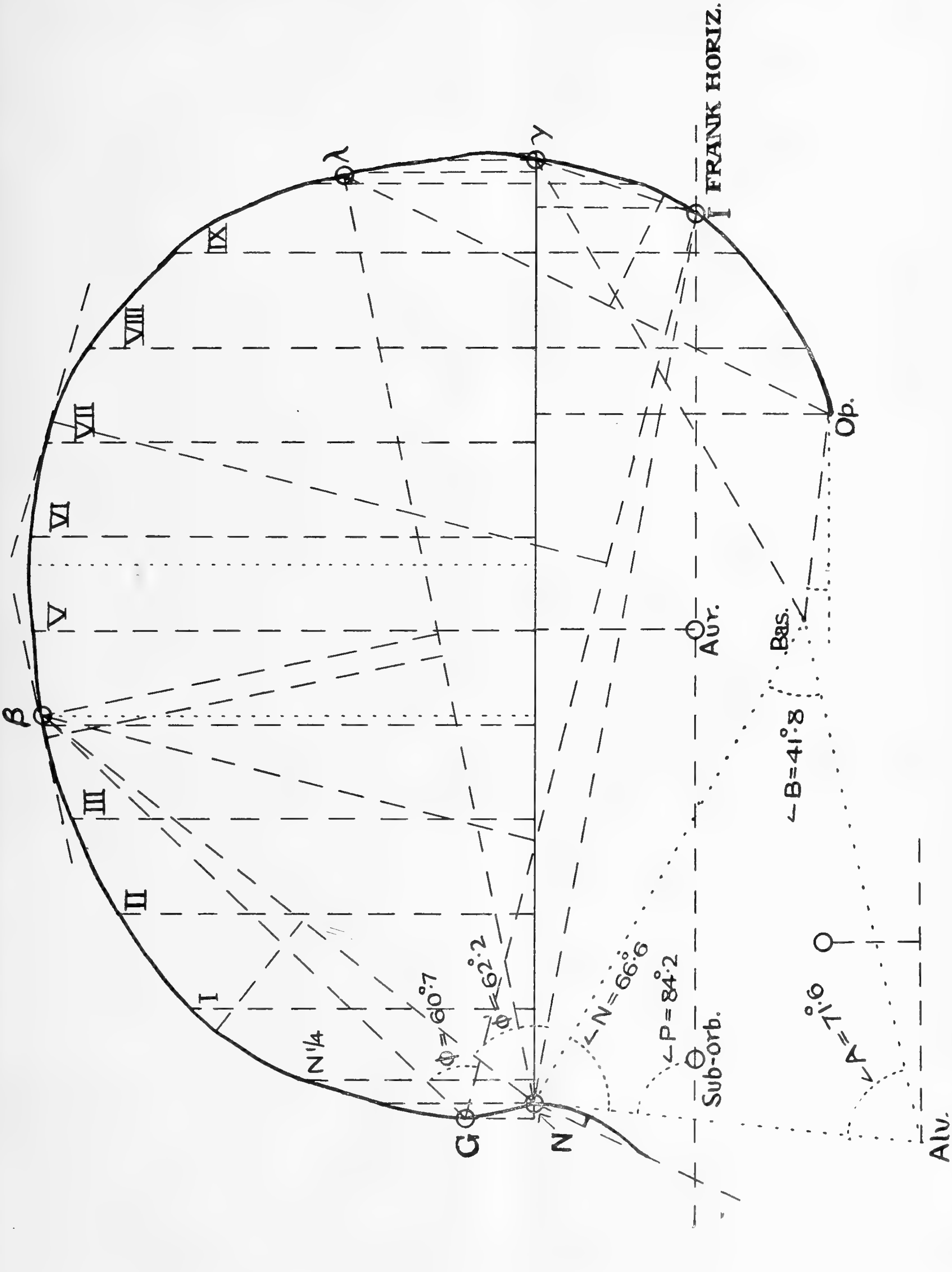

Fig. XVII. Burmese Crania. Type B (? Hybrids). ♀. Sagittal Contour. (17 crania.)

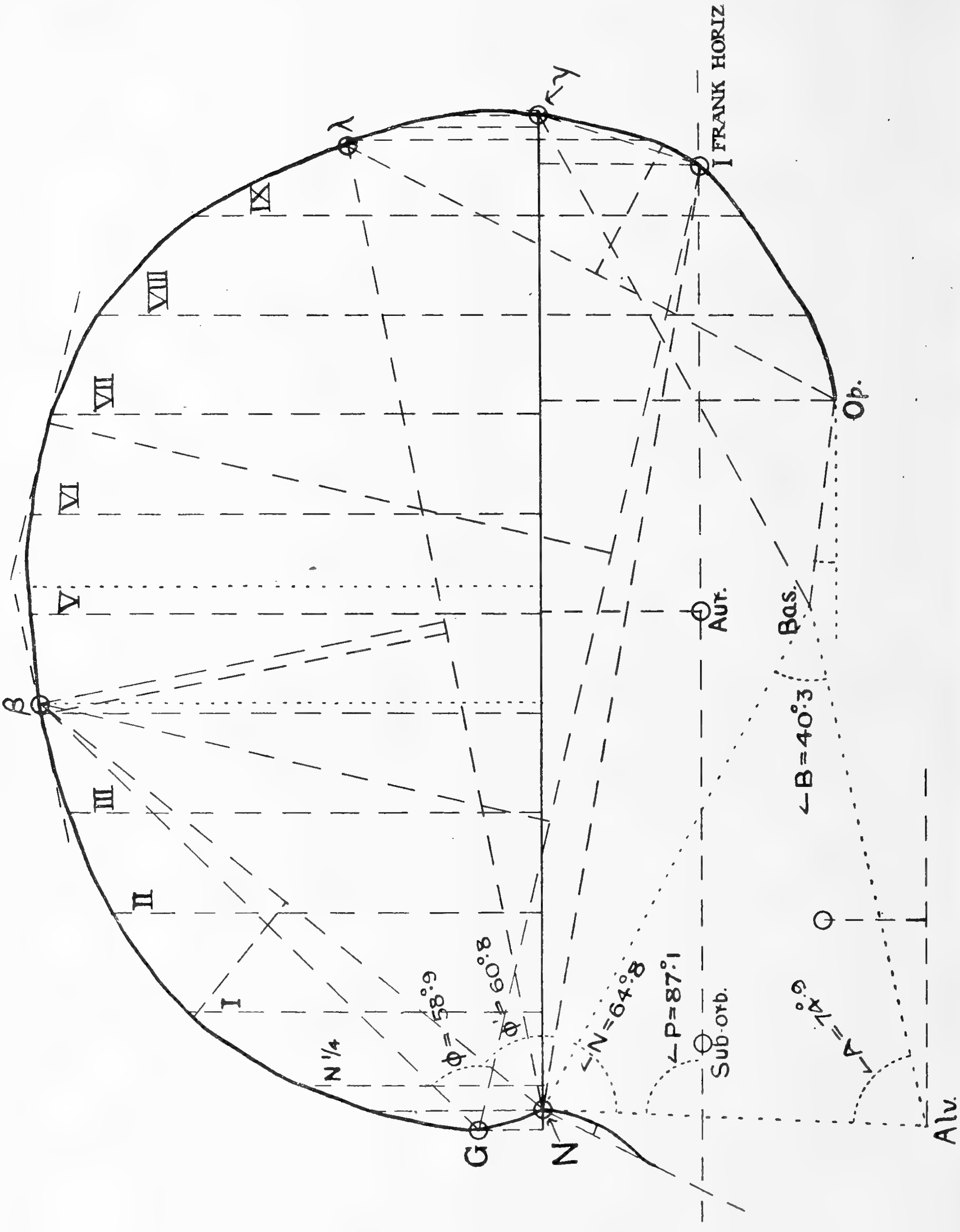

Fig. XVIII. Burmese Crania. Type C (? Karens). ♀. Sagittal Contour. (18 crania.)

Ordinates below $N\gamma$				Occipital point		λ		Sub-orbital point	
8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from γ	y	x from γ	y	x from N	y

Auricular point		Opisthion		Inion		Basion	
x from γ	y	x from γ	y	γI	NI	γ Bas.	N Bas.

Alveolar point		Nose				Palate	
				Max. nasal subtense		Max. subtense	
N to A.P.	Bas. to A.P.	NL	NL'	x from N	y	x from A.P.	y

In addition to these measurements I took a number of others which were not used in construction of the type. This was done with a double object: partly to compare their means with the corresponding measurements that were read off from the type skull and thus test the accuracy with which extra values are given by the type (theoretically they are not bound to be exactly the same); partly for comparative purposes. They are as follows:

Lengths.

$N\beta$	$\lambda Op.$	$N\lambda$	β subtense to $N\lambda$		GI	β subtense to GI	
			x from N	y		x from G	y

Angles.

$\phi = \angle \beta GI$	$\phi' = \angle \beta NI$	$A\angle$	$N\angle$	$B\angle$	$P\angle$	$fml\angle$

i.e. nasion to bregma, lambda to opisthion, nasion to lambda, co-ordinates of bregmatic subtense to $N\lambda$, glabella to inion, co-ordinates of bregmatic subtense to GI; angles at glabella and nasion respectively, between chords to bregma and inion; angles of the fundamental triangle, at alveolar point, nasion and basion; profile angle or inclination to the horizontal of the line from nasion to alveolar point; and, lastly, the inclination of the basio-opisthionic line to the horizontal.

The mean position in the type contour of the terminals of all these measurements had already been determined in some other way, so these were not used in building up the type contour.

There were, however, four maximum subtenses also measured on the individual crania, and inserted in the type contour wherever the outline could reasonably be made to pass through them, which was generally the case; as with the last-named measurements, points, whose mean position is obtained in one way, are not theoretically bound to lie on a "mean" outline obtained in another, though they will be very close to it.

The maximum subtenses of which I have obtained the mean length and position are: maximum frontal subtense to nasio-bregmatic chord; maximum occipital subtense to opisthio-lambda chord; and maximum calvarial subtense, using nasio-lambda and glabella-inion lines respectively as base-lines.

Max. frontal subt. to $N\beta$		Max. occip. subt. to λOp.		Max. subt. to $N\lambda$		Max. subt. to GI	
x from N	y	x from λ	y	x from N	y	x from G	y

6. ON THE ACCURACY OF CONTOUR VALUES FOR THE DETERMINATION OF CRANIAL CONSTANTS.

It was promised in the last craniometrical paper published by this Laboratory* that the causes of discrepancies between contour and direct measurements should be investigated more closely with a view to the improvement of technique.

I will deal first with those measurements alone which are given by the means of characters in the individual contours, not with those which are read off from the type contour and not used in its construction.

I was hampered at the outset in my attempt to make the two methods of measurement give equal results by the fact that our contour tracer was no longer in quite good condition. Having been in frequent use for several years, the point of the scriber which is passed round the bone had been gradually worn down; and since one usually traces from left to right, it was not only slightly shortened, but rubbed away more on the right side (looking along the scriber towards the tip). The point

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* *Biometrika*, Vol. XI. p. 131 footnote.

of the scriber is therefore not immediately above the point of the ink-container; but a little behind and to the left of it; with the result that a point on the skull will be projected a little in front of and to the right of its true position: 0·3 mm. in front and 0·4 mm. to the right, the hypotenuse of the triangle giving us 0·5 mm. displacement of the actual point.

When the scriber is moved round the skull in the usual direction, i.e. left to right, making an angle of about 60° with the direction in which it is travelling, the whole 0·5 mm. displacement operates to throw the tracing *within* its true position. Thus, in taking a measurement from one side to another of the tracing or wherever the positions of the scriber have been approximately parallel but in the opposite sense, one gets a result which is about 1 mm. less than the same measurement taken direct on the skull.

That the Klaatsch contour tracer had to be used with this defect, was a state of affairs due to the war. It was impossible to get the instrument over to the makers, and skilled instrument-makers here were too much occupied with other work to do this repair properly.

In comparing contour and direct measurements, however, I was aware of a certain discrepancy for which this instrumental defect did not wholly account. Further investigation revealed another source of error in the drawing-board on which the tracing was done. This was found to be slightly warped, in such a way that when the Klaatsch was tracing the top and base of the skull in the sagittal and transverse contours, and the sides in the horizontal contours, the instrument was tilted slightly away from the skull. As nearly as I can estimate it this tilt is only about 0·5° along each of the two long sides of the board, but at the height at which the scriber is placed during most of the contour tracing, this suffices to throw its tip about 0·5 mm. backwards, and consequently the line traced about 0·5 mm. *within* its true position*. Where this operates on two opposite sides of the tracing, we get maximum shortening of about 1 mm. on this score.

We will now proceed to the comparison of our actual results.

Burmese mean Glabellar-Occipital Length.

Types:	Males			Females		
	A	B	C	A	B	C
From direct measurements	173·5 (44)	173·8† (8)	176·7 (8)	166·5 (38)	163·8 (17)	170·2 (18)
From sagittal contours	172·0 (44)	174·4 (7)	175·5 (8)	165·7 (39)	162·8 (17)	169·1 (18)

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* I fully admit that the possibility of such a defect should have occurred to me before the board was used. As, however, it was not realised till afterwards, when the long process of contour-drawing, ruling, measuring and averaging the measurements was complete and could not be repeated, it has had to remain uncorrected.

† For the seven skulls contoured, mean of direct measurements is 175·8.

In the position of the skull on the board when the scriber was passing over the glabella and occiput the warping of the board would have no effect. If we add to the contour measurements 1 mm. which the defective Klaatsch was responsible for docking, the agreement is seen to be very close indeed.

The length as obtained from horizontal contours does not tally with the above, of course, since their plane does not pass through the occipital point. It is on the average 2 mms. less.

Burmese Mean Auricular Height, OH.

Types:	Males			Females		
	A	B	C	A	B	C
Measured on craniophor ...	117·7 (44)	116·7 (7)	116·9 (8)	111·7 (38)	111·9 (17)	110·1 (17)
Measured on sagittal contours	116·3 (44)	114·9 (7)	115·4 (8)	110·6 (38)	110·0 (17)	109·5 (18)
Measured on transverse contours ...	117·1 (44)	116·1 (7)	116·1 (8)	111·5 (38)	111·4 (17)	109·4 (18)

In the case of the auricular height as given by the sagittal contours, we have to allow for some shortening due to both contour tracer and drawing-board defects. The tracer passes over the bregma inclined at about 60°: the bregma is therefore thrust about 0·5 mm. nearer to the auricular point; in marking the auricular point the tracer is not held aslant, and the shortening of the distance in question at this point will be about 0·3 mm.; i.e. about 0·8 mm. in all through the defect in the tracer. The warping of the board would affect both terminals and further shorten the line by about 1 mm. The addition of 1·8 mms. to the sagittal contour means brings them into closer agreement with the craniophor means except in the case of C ♀; and the difference of 1·2 mms. there obtained is not serious. With the transverse contours, neither source of error displaces the mid-point of the auricular line which is one terminal, but they combine to displace the apex, our estimate (already used) being about ·8 mm.

Burmese Mean Upper Face Height, G′H.

Types:	Males			Females		
	A	B	C	A	B	C
Direct measurements	71·4 (39)	68·2 (8)	74·5 (7)	66·0 (34)	66·1 (17)	63·9 (16)
From sagittal contours	70·7 (38)	67·9 (7)	74·2 (7)	65·7 (34)	65·7 (15)	63·9 (16)

In marking the terminals of this measurement (nasion and alveolar point), the positions of the Klaatsch were parallel and pointing in the same direction. The warp of the board would not affect the nasion, but might have a slight effect on the position of the alveolar point, shortening the line slightly in some cases.

Burmese Mean Nasio-Basion Length, LB.

Types:	Males			Females		
	A	B	C	A	B	C
Direct measurements	98·5 (43)	98·8 (8)	100·5 (7)	93·9 (39)	94·7 (17)	95·7 (17)
From sagittal contours	97·4 (43)	98·0 (7)	100·2 (8)	93·0 (39)	93·3 (17)	94·7 (18)

In marking the basion, defects in both board and tracer would operate to thrust the point about 1 mm. in the direction of the obelion; for the nasion we need only allow for the defect in the tracer which would carry the point about ·5 mm. towards the occiput. If we consider the angle at which these two directions cut each other, the shortening shown of about 1 mm. in the contour measurements seems very reasonable.

Mean Length of Foramen Magnum, fml.

Types:	Males			Females		
	A	B	C	A	B	C
Direct measurements	36·7 (43)	35·4 (8)	37·3 (8)	34·7 (39)	35·1 (16)	35·1 (18)
From sagittal contours	36·8 (42)	35·1 (8)	37·1 (8)	34·7 (39)	35·3 (16)	35·1 (18)

The contour tracer occupied parallel positions in marking these points, and was pointing in the same sense. The defects in board and tracer displace their true positions to the same extent and in parallel directions, thus having no effect on the distance between them. The correspondence between the two sets of results is seen to be remarkably close.

Burmese Mean Basio-Bregmatic Height, H′.

Types:	Males			Females		
	A	B	C	A	B	C
Direct measurements	136·8 (43)	136·6 (7)	140·1 (8)	131·4 (39)	131·9 (17)	129·9 (18)
From sagittal contours	134·1 (43)	133·6 (7)	137·5 (8)	128·8 (39)	129·3 (17)	128·3 (18)

As in the case of the auricular height allowance must be made for shortening due to both defects, which we estimated at about 2 mms.

The addition of this again gives us reasonably close results.

Burmese Mean Length from Basion to Alveolar Point, GL.

Types:	Males			Females		
	A	B	C	A	B	C
Direct measurements	96·6 (39)	93·8 (8)	95·9 (7)	90·7 (34)	91·5 (17)*	89·2 (16)
From sagittal contours	96·5 (38)	93·6 (7)	95·6 (7)	90·3 (34)	90·4 (15)	88·8 (16)

* 91·0 for 15 skulls.

Although these two points are displaced by both the defects in our apparatus, the warping displaces each at right angles to the line joining them, and though the Klaatsch would tend to push the alveolar point rather towards the basion, this is partly compensated by the direction in which it moves the basion, so that we should expect a very slight error of less than 0·5 mm. from this source.

It will be seen that the results are good.

We now turn from lengths to angle measurements.

Comparison between Mean Angles obtained from Direct Measurements, and Mean Angles from Individual Contours.

Types:		Males			Females		
		A	B	C	A	B	C
$A\angle$	Direct measurements ...	70°·5 (38)	73°·4 (8)	71°·2 (7)	72°·1 (34)	72°·0 (17)	75°·2 (16)
	From sagittal contours ...	69°·6 (38)	72°·6 (7)	70°·4 (7)	71°·8 (34)	71°·6 (15)	75°·0 (16)
$N\angle$	Direct measurements ...	66°·8 (38)	65°·2 (8)	64°·4 (7)	66°·2 (34)	66°·5 (17)	64°·5 (16)
	From sagittal contours ...	68°·0 (38)	66°·1 (7)	64°·8 (7)	66°·8 (34)	66°·9 (15)	64°·7 (16)
$B\angle$	Direct measurements ...	42°·6 (38)	41°·4 (8)	44°·5 (7)	41°·8 (34)	41°·5 (17)	40°·3 (16)
	From sagittal contours ...	42°·5 (38)	41°·2 (7)	44°·8 (7)	41°·4 (34)	41°·6 (15)	40°·3 (16)
$P\angle$	Direct measurements ...	86°·0 (39)	85°·9 (8)	84°·0 (7)	85°·8 (33)	84°·0 (17)	86°·4 (16)
	From sagittal contours ...	84°·4 (38)	86°·1 (7)	83°·6 (7)	85°·0 (34)	83°·8 (15)	87°·2 (16)

Of these angles the first three are angles of the fundamental triangle, whose sides (*LB*, *G′H*, *GL*) we have already discussed, and have seen to be very little affected by the errors due to defective apparatus. Only one of the differences exceeds 1°: the correspondence is therefore good.

Contouring errors should not affect the direction of the lines ($N\gamma$ and *N* Alv.) by which the profile angle ($P\angle$) is determined. The direct measurements were taken by means of Ranke's goniometer when the skull was on the craniophor. As this is a measure which we find it difficult to take accurately*, I am very satisfied to have obtained so good a degree of correspondence in the two sets of figures. All the measurements of which use has been made in this enquiry into the accuracy of contour tracing are means of individual contour measurements; they are not read off from type contours, since such readings are not bound theoretically to be identical with the corresponding means of the individual contours. Thus any discrepancy due to this cause has been obviated. The conclusion is that contours may be used to replace direct measurements and can do so with great accuracy. The results are, however, more subject to error through the conditions under which the work is done, and what seems to be a very slight defect in the instrumental conditions may be considerably magnified in the tracings.

Comparisons between Measurements on Type Contours and Means of Individual Contour Measurements which are not used in Construction of Type Contours.

Types:		Males			Females		
		A	B	C	A	B	C
$N\beta$	Mean contour value	112·6 (44)	111·3 (7)	113·1 (8)	106·7 (39)	106·0 (7)	108·2 (18)
	Type contour ...	112·6	111·5	113·1	106·5	105·9	107·5
$\lambda Op.$	Mean contour value	93·7 (43)	97·3 (7)	100·2 (8)	92·8 (39)	91·8 (17)	91·2 (18)
	Type contour ...	93·9	97·5	100·0	91·8	91·5	91·1
$N\lambda$	Mean contour value	167·9 (44)	169·2 (7)	171·4 (8)	162·0 (39)	160·3 (17)	163·4 (18)
	Type contour ...	168·0	169·4	171·3	161·4	160·2	163·1
β subt. to $N\lambda$	Mean contour value	72·0 (44)	70·2 (7)	70·0 (8)	68·5 (39)	68·0 (17)	68·1 (18)
	Type contour ...	72·4	70·4	69·7	68·5	68·2	68·4
GI	Mean contour value	168·7 (44)	171·1 (7)	169·3 (8)	160·7 (39)	157·7 (17)	164·1 (18)
	Type contour ...	168·9	171·4	169·3	160·7	157·7	164·3
β subt. to GI	Mean contour value	91·2 (44)	90·8 (7)	93·6 (8)	85·3 (39)	85·7 (17)	86·5 (18)
	Type contour ...	91·4	91·2	93·8	85·4	86·1	86·7
β bas. (H')	Mean contour value	134·1 (43)	133·6 (7)	137·5 (8)	128·8 (39)	129·3 (17)	128·3 (18)
	Type contour ...	134·7	134·4	137·5	129·3	129·7	128·7
fmb	Mean contour value	36·8 (42)	35·1 (8)	37·1 (8)	34·7 (39)	35·3 (16)	35·1 (18)
	Type contour ...	36·7	34·4	37·3	34·2	35·1	34·5
Glabella to occiput (L)	Mean contour value	172·0 (44)	174·4 (7)	175·5 (8)	165·7 (39)	162·8 (17)	169·1 (18)
	Type contour ...	172·3	174·3	175·4	165·3	163·0	169·2

We have here 54 pairs of measurements. In only nine of these is the discrepancy as much as 0·5 mm., and the largest is 1 mm. As any difference between the results of mean individual contour values and type contour values would tend to show itself more in small series than in larger ones a study of this table encourages us to rely with some confidence on mean values as read from the type contours,

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* This difficulty was commented on by Fawcett and Macdonnell. (See *Biometrika*, Vol. III. p. 202.)

in comparing race with race. In this way, the method of contouring can be used to save the taking of a considerable number of individual length measurements.

We will now compare angle-measurements in similar fashion.

Comparison between Angle-Measurements on Type Contours, and Means of Individual Contour-Measurements.

Types:		Males			Females		
		A	B	C	A	B	C
ϕ	Mean contour value	60°·1 (44)	59°·7 (7)	60°·5 (8)	60°·6 (39)	60°·6 (17)	59°·1 (18)
	Type contour ...	60°·3	59°·8	60°·7	60°·4	60°·7	58°·9
ϕ'	Mean contour value	61°·7 (44)	61°·1 (7)	61°·6 (8)	62°·2 (39)	61°·9 (17)	60°·8 (18)
	Type contour ...	61°·8	61°·1	61°·8	61°·9	62°·2	60°·8
$A\angle$	Mean contour value	69°·6 (38)	72°·6 (7)	70°·4 (7)	71°·8 (34)	71°·6 (15)	75°·0 (16)
	Type contour ...	69°·6	73°·1	70°·7	71°·8	71°·6	74°·9
$N\angle$	Mean contour value	68°·0 (38)	66°·1 (7)	64°·8 (7)	66°·8 (34)	66°·9 (15)	64°·7 (16)
	Type contour ...	67°·9	65°·5	65°·0	66°·6	66°·6	64°·8
$B\angle$	Mean contour value	42°·5 (38)	41°·2 (7)	44°·8 (7)	41°·4 (34)	41°·6 (15)	40°·3 (16)
	Type contour ...	42°·6	41°·4	44°·3	41°·6	41°·8	40°·3
$P\angle$	Mean contour value	84°·4 (38)	86°·1 (7)	83°·6 (7)	85°·0 (34)	83°·8 (15)	87°·2 (16)
	Type contour ...	84°·4	86°·1	84°·6	85°·0	84°·2	87°·1
$fm\angle$	Mean contour value	7°·3 (42)	8°·8 (7)	7°·6 (8)	8°·8 (39)	7°·8 (17)	8°·2 (18)
	Type contour ...	6°·0	7°·8	7°·2	7°·3	6°·6	7°·2

In investigating the causes of the discrepancy between contour values and direct measurements, I found the explanation was the difference between auricular height as obtained by the two methods, which showed itself both in Thomson's and Crewdson Benington's work, and which Thomson was at a loss to account for*.

Ranke's craniophor which is used in the Biometric Laboratory was designed by the late Professor J. Ranke to provide means of obtaining easily the cranial measurements prescribed by the Frankfurt Concordat, among others the auricular height. This craniophor holds the skull in position supported on ear-plugs, and the auricular height should be measured from the top of the ear-plugs, the Frankfurt definition being "vertical" height of skull above the auricular points. A vertical rod to which a movable horizontal bar is attached is marked with a scale, which begins only at 38 mms., and from this the auricular height is read off. I have discovered that this scale, by an error of the instrument-maker, is actually graduated from the middle of the ear-plug, instead of from the top, as had always been assumed; and the diameter of the rod being 5 mms., the reading on the scale has always been 2·5 mms. in excess of the true auricular height. This error, which only became obvious when there were contour measurements to compare with direct

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* *Biometrika*, Vol. XI. p. 127.

measurements, has no doubt been made by all users of Ranke's craniophor. It will be necessary therefore to deduct 2·5 mms. from all direct measurements of auricular height given by Fawcett*, Macdonnell*, Benington† and Thomson‡.

Auricular Height. (*First five corrected.*)

Race	Male	Female
Naqada	113·1	110·6
Whitechapel English	112·1	106·7
Liverpool St English	111·3	106·9
Congo Negroes ...	111·4	106·5
Moriori	114·6	112·6
Burmese A	117·7	111·7
,, B	116·7	111·9
,, C	116·9	110·1

The above table may be of use as giving the corrected values of the auricular height. I have added my Burmese for comparison. It will be seen that it makes these results still *more* incompatible with those of Ranke for the "Altbayerisch" (♂ 120·8, ♀ 114·2), or Parson's for Hythe (♂ 120, ♀ 116) and Rothwell (♂ 120, ♀ 116).

7. DESCRIPTION OF MATERIAL AND COMPARISON OF DIRECT MEASUREMENTS OF THE THREE SERIES.

It was obvious in examining the whole series of 142 skulls that they were of rather mixed type, although one type predominated. It seemed desirable therefore to class together in separate groups those skulls which qualitative appreciation led us to suppose of the same type, reserving the right to combine these groups afterwards if the quantitative values obtained did not support this judgment. The procedure was as follows. Seven children's skulls, which could not be sexed, nor used to obtain any means, were withdrawn from the rest. The remaining skulls were then put into three groups: A and C, with an intermediate group B, which combined some of the features of both. Finally, the skulls in the various groups were sexed. It was necessary to leave the sexing until the types were separated, since some factors which affect the determination of sex, such as weight, size, and general boldness of shape, vary considerably from type to type, and some young males of a less robust racial type might easily be mistaken for females of a heavier and coarser one. For all this classification of material I have to thank Professor Pearson.

We thus had our series divided up into seven sub-series: three male and three female sub-series and one composed of children of various ages. The grouping was determined by the general appearance, more particularly of the face. As contributing factors to the general resemblance of those classed as Group A, we might single out as most noteworthy: the nose, broad at the bridge and flattish, with the pyriform aperture also wide, and its lower edge very rounded or even forming a double ridge with a negroid groove in between; marked sub-nasal prognathism; projecting cheek-bones; head of a massive and rugged type, etc.

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* *Biometrika,* Vol. I. pp. 426–7, Vol. III. pp. 208–9 and Vol. V. pp. 92–3. I have tested eight each of the Naqada and Whitechapel series and have satisfied myself that the extra 2·5 mms. were added to their measurements by the craniophor defect and must therefore be deducted.

† *Ibid.* Vol. VIII. pp. 208–9. ‡ *Ibid.* Vol. XI. p. 90.

In Group C less projecting cheek-bones, narrower noses with higher bridges, a sharp edge to the lower lip of the pyriform aperture; and again considerable prognathism; the whole head of a less bold and heavy character.

Sexing divided up Group A fairly evenly into 45 males and 39 females*; in Groups B and C the females preponderated, there being only eight males in each to 18 females in C and 17 in B. A number of the skulls had to be classed ♂ or it might be ♀ with a query, as is bound to happen in any series, but these dubious ones would not be numerous enough, if put into the wrong class, to modify the means to any considerable extent.

It was unfortunate that two out of six adult groups should contain as few as eight skulls, since these numbers are far too slight to base any reliable conclusions upon, or even to give any very helpful suggestions.

The whole series was accordingly examined once more some months afterwards, to see whether it would not be possible to divide the adults into two main types instead of three, putting the bulk of the intermediate group B along with C, so as to strengthen the numbers of the latter, and make the main categories less unequal. It was found that only two of the eight intermediate males could possibly have been put with the eight in group C, making ten in all; and though six of the seventeen B females might have been classed with the eighteen in C, this grouping was not very convincing since the C's were of a distinctly smaller and finer build.

In short, the small gain in numbers by the broader grouping would have been more than counterbalanced by the heterogeneity of the groups, and it was decided to leave the original divisions as they were.

It will not, however, be wise to attach significance to the results based on the two small male categories B and C, except in so far as they confirm the figures for the corresponding females.

After these conclusions had been arrived at upon the data themselves, we put our doubt as to the collection being purely Burman before Colonel Caster, and he reported that after further consideration there might be a certain number of Hindus or of Karens in the material, but that it was unlikely, owing to the source from which it was drawn, to include any Chinese, or any Mohammedans from India. In any case, the bulk of the skulls were Burman.

We therefore had little hesitation in applying the term Burman proper to our large groups A ♂ and A ♀. Since comparison with Hindu material showed little affinity between our C groups and the Hindus, as will be seen later, and taking into account the large proportion of Karen population around Moulmein, we suspect our C groups to be of Karen origin, though we have not enough definite evidence to establish the supposition. The few isolated Karen crania at the Royal College of Surgeons do not suffice to fix a type. Our B groups we suppose hybrids.

The following are the mean values obtained by direct measurement in the six groups:

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* Not all the crania could be used for each contour, and this accounts for the different totals given in Table I and Figs. I–XVIII.

TABLE I. *Mean Characters of Burmese Crania.*

Types:	Males			Females		
	A	B	C	A	B	C
Capacity	1406·9 (27)	1415·0 (4)	1442·2 (5)	1267·9 (27)	1232·4 (11)	1231·4 (7)
W. in grs.	656·8 (38)	595·6 (7)	578·0 (6)	530·6 (31)	501·5 (13)	425·2 (11)
F	172·4 (44)	172·4 (8)	175·5 (8)	166·5 (39)	163·2 (17)	169·4 (18)
L′	174·2 (44)	176·7 (7)	177·2 (8)	167·4 (38)	164·7 (17)	170·7 (18)
L	173·5 (44)	173·8 (8)	176·7 (8)	166·5 (38)	163·8 (17)	170·2 (18)
B	143·7 (45)	141·1 (7)	140·4 (8)	138·0 (39)	135·6 (17)	130·6 (18)
B′	94·3 (44)	89·7 (8)	90·4 (8)	90·7 (39)	88·5 (16)	88·7 (18)
H	136·8 (43)	136·6 (7)	140·1 (8)	131·4 (39)	131·9 (17)	129·9 (18)
H′	136·0 (43)	134·7 (7)	139·1 (8)	130·4 (39)	130·8 (17)	129·4 (18)
OH	117·7 (44)	116·7 (7)	116·9 (8)	111·7 (38)	111·9 (17)	110·1 (17)
LB	98·5 (43)	98·8 (8)	100·5 (7)	93·9 (39)	94·7 (17)	95·7 (17)
Q	323·7 (44)	319·2 (7)	317·8 (8)	309·1 (36)	307·4 (17)	300·7 (16)
Q′	325·8 (43)	319·5 (7)	319·4 (8)	312·0 (37)	308·8 (17)	302·1 (16)
S	363·7 (44)	367·0 (7)	367·4 (8)	350·2 (39)	340·0 (17)	350·9 (18)
S_1	128·6 (44)	124·9 (7)	127·9 (8)	121·9 (39)	119·5 (17)	122·6 (18)
S_2	124·0 (45)	124·4 (7)	122·5 (8)	120·3 (39)	119·6 (17)	119·6 (18)
S_3	111·2 (44)	117·7 (7)	117·0 (8)	107·9 (39)	104·9 (17)	108·7 (18)
S_3'	94·7 (44)	98·1 (7)	99·7 (8)	92·8 (39)	92·0 (17)	91·4 (18)
U	505·7 (44)	497·7 (7)	503·1 (8)	485·1 (39)	476·9 (16)	479·6 (18)
PH	19·9 (38)	17·2 (7)	20·1 (7)	17·3 (33)	17·6 (16)	16·8 (16)
G′H	71·4 (39)	68·2 (8)	74·5 (7)	66·0 (34)	66·1 (17)	63·9 (16)
GB	101·9 (40)	100·9 (8)	94·3 (7)	96·7 (34)	93·6 (15)	91·8 (15)
J	134·0 (40)	131·7 (8)	126·7 (7)	126·7 (35)	121·9 (15)	118·7 (13)
NHR	53·4 (41)	52·3 (8)	55·1 (7)	50·8 (34)	49·4 (17)	47·6 (14)
NHL	53·5 (41)	52·4 (8)	55·2 (7)	50·6 (34)	49·4 (17)	47·3 (16)
NB	28·1 (41)	26·6 (8)	25·5 (7)	26·8 (34)	25·0 (17)	24·0 (14)
DS	10·3 (42)	10·6 (8)	11·6 (7)	9·4 (36)	9·6 (17)	11·2 (13)
DC	22·8 (43)	21·3 (8)	18·7 (7)	20·9 (37)	19·7 (17)	18·8 (15)
DA	32·6 (41)	30·9 (8)	30·7 (7)	29·8 (35)	29·3 (17)	30·7 (13)
SS	3·0 (41)	3·5 (8)	3·6 (7)	2·5 (37)	2·9 (17)	3·6 (13)
SC	9·0 (42)	7·8 (8)	7·2 (7)	8·4 (38)	7·7 (17)	7·7 (14)
O_1R	44·2 (41)	44·3 (8)	44·4 (7)	42·0 (34)	42·2 (17)	42·2 (15)
O_1L	43·6 (41)	43·2 (8)	44·1 (7)	41·8 (34)	41·5 (14)	42·2 (16)
O_2R	35·0 (41)	33·6 (8)	35·6 (7)	34·5 (34)	34·0 (16)	33·7 (14)
O_2L	35·0 (41)	34·3 (8)	35·9 (7)	34·6 (34)	34·6 (15)	33·2 (16)
G_1	49·9 (37)	49·2 (8)	50·0 (7)	46·7 (26)	47·0 (14)	46·4 (13)
G_1'	45·4 (37)	44·8 (8)	46·2 (7)	42·7 (29)	43·1 (16)	42·8 (13)
G_2	39·6 (40)	40·1 (6)	40·6 (7)	38·0 (32)	36·4 (15)	38·1 (14)
GL	96·6 (39)	93·8 (8)	95·9 (7)	90·7 (34)	91·5 (17)	89·2 (16)
fml	36·7 (43)	35·4 (8)	37·3 (8)	34·7 (39)	35·1 (16)	35·1 (18)
fmb	30·7 (42)	30·6 (8)	30·8 (8)	29·4 (39)	30·0 (16)	29·8 (17)
Oc. I.	62·8 (44)	61·5 (7)	62·8 (8)	64·1 (39)	66·1 (17)	61·4 (18)
100 *B/L′*	82·4 (44)	80·0 (7)	79·4 (8)	83·1 (38)	83·0 (17)	76·6 (18)
100 *H/L′*	78·2 (43)	76·3 (7)	78·7 (8)	78·0 (38)	79·5 (17)	75·9 (18)
100 *B/L*	82·9 (44)	80·4 (7)	79·5 (8)	83·1 (38)	82·9 (17)	76·9 (18)
100 *H/L*	78·5 (43)	76·7 (7)	78·9 (8)	78·5 (38)	79·9 (17)	76·1 (18)
100 *B/H*	105·8 (43)	104·9 (7)	101·0 (8)	105·8 (39)	103·7 (17)	101·0 (18)
100 $(B-H)/L$	+4·0 (43)	+2·6 (7)	+0·1 (8)	+3·9 (38)	+2·4 (17)	+0·9 (18)
100 *G′H/GB*	69·8 (38)	67·8 (8)	79·0 (7)	68·6 (34)	69·8 (15)	70·2 (15)
100 *NB/NHR*	52·8 (41)	51·2 (8)	46·4 (7)	52·9 (34)	50·8 (17)	50·4 (14)
100 *NB/NHL*	52·7 (41)	51·0 (8)	46·3 (7)	53·0 (34)	50·7 (17)	50·4 (14)
100 O_2/O_1L	80·0 (41)	79·4 (8)	81·5 (7)	82·8 (34)	83·0 (14)	78·8 (16)
100 O_2/O_1R	79·1 (41)	75·9 (8)	80·3 (7)	82·4 (34)	80·4 (16)	79·9 (14)
100 G_2/G_1	79·8 (37)	82·2 (6)	81·3 (7)	80·5 (25)	76·8 (12)	83·0 (13)
100 *fmb/fml*	83·8 (42)	86·6 (8)	82·9 (8)	84·8 (39)	85·9 (16)	85·0 (17)
100 *DS/DC*	45·4 (42)	49·8 (8)	61·9 (7)	45·1 (36)	48·7 (17)	60·3 (13)
100 *SS/SC*	32·7 (41)	45·2 (8)	54·0 (7)	29·8 (37)	38·7 (17)	48·5 (13)
N ∠	66°·8 (38)	65°·2 (8)	64°·4 (7)	66°·2 (34)	66°·5 (17)	64°·5 (16)
A ∠	70°·5 (38)	73°·4 (8)	71°·2 (7)	72°·1 (34)	72°·0 (17)	75°·2 (16)
B ∠	42°·6 (38)	41°·4 (8)	44°·5 (7)	41°·8 (34)	41°·5 (17)	40°·3 (16)
θ_2	15°·4 (38)	12°·9 (8)	12°·8 (7)	13°·5 (33)	12°·0 (17)	11°·2 (16)
θ_1	27°·2 (38)	28°·8 (8)	31°·7 (7)	28°·3 (33)	29°·5 (17)	29°·1 (16)
P ∠	86°·0 (39)	85°·9 (8)	84°·0 (7)	85°·8 (33)	84°·0 (17)	86°·4 (16)

Let us now compare the characteristics of the extreme groups A and C.

In order to do this we need to compare any difference in the means with the probable error of that difference, to see whether it is significant or not. And for the probable error we need to know the standard deviation of each character within the race to which it belongs. Our numbers are altogether too slight to furnish us with this standard deviation; I have therefore adopted as next best thing the standard deviations as found in some other race, on the assumption, confirmed by previous experience*, that though the S.D.'s vary from race to race, they are at least of about the same order. Our probable error of means based on these borrowed S.D.'s will necessarily be only approximate, and we shall not be able to test the likely significance of a difference in means by asking definitely whether it is more or less than about 2·5 times the probable error. More latitude must be allowed than usual, especially when one sex seems to show a significant difference and the other not; a truer standard deviation might alter the probable error so as to make the two sexes indicate the same thing.

The standard deviations of the Egyptians (Naqada Race†) have been borrowed for as many characters as are given there; this is our longest measured series, and its standard deviations are therefore subject to the smallest probable errors. For characters not measured by Fawcett, I have taken from the Moorfields English Crania studied by Macdonnell‡ the S.D.'s of such characters as he added to Fawcett's list, and from Benington's Congo series§ the S.D.'s of the nose measurements which were his new contribution. The standard deviation of weight|| (W. in grs.) was taken from an Egyptian series of which 50 of each sex were weighed by A. G. Davin for comparison with Moriori crania. Finally, that of $100\,(B-H)/L$ was calculated from the S.D.'s of the component indices, and that of the premaxillary height (PH), which is approximately equal to the upper-face height ($G'H$) minus height of nose (NH), by means of the S.D.'s of these two characters.

On the following page is a table showing the excess or defect of Group C means as against those of Group A, with the approximate probable errors of the differences.

The characteristics which influenced our merely appreciative grouping are well brought out by this table. The greater width of the cheek-bones in Group A is indicated by the longer bi-zygomatic breadth (J) and the greater upper-face breadth (GB), the differences exceeding five times the probable error in both cases for both sexes. Again, as regards the nose, the width of the pyriform aperture in Group A exceeds that in Group C by more than five times the probable error in males, and by seven times in females; about the nasal height there is no such clear verdict, with a barely significant difference in the males, the C's exceeding, while the A's have a very definite excess for the females. In any case the relative proportions of the nose are not left in doubt, since the A's have a significantly higher nasal index in both.

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* See Tables of Interracial Variability in Benington's Paper, *Biometrika,* Vol. VIII. pp. 323 and 324.

† Fawcett's paper in *Biometrika,* Vol. I. p. 438.

‡ *Biometrika,* Vol. V. pp. 92, 93.

§ *Biometrika,* Vol. VIII. pp. 298, 299.

|| Not yet published.

The flatness of the nasal bridge in Group A relative to Group C is brought out by the absolute values of the simotic and dacryal subtenses (*SS* and *DS*), and by their ratio to the corresponding chords (*SS/SC*, *DS/DC*), the lowest of these index differences in either sex being nine times the probable error. The width of the bridge from dacryon to dacryon (*DC*) is much greater in Group A, being six to seven times the probable error; the width of the nasal bones themselves (*SC*) is not so emphatically greater, but as was pointed out in K. V. Riley's and J. Bell's study of the Nasal Bridge*, a physiognomic factor such as the whole bridge of the nose is probably of much more importance from the evolutionary point of view than a simple anatomical unit (such as the nasal bones), which forms only one part of it.

TABLE II. *Difference between Mean Characters of Burmese Group A (Burmans) and Group C (Karens?)*.*

	Males A–C	Females A–C		Males A–C	Females A–C
Capacity	−35·3 ± 35·1	+ 36·5 ± 25·4	O_1L	− 0·5 ± 0·6	− 0·4 ± 0·3
W. in grs.	+78·8 ± 33·1	+105·4 ± 18·0	O_2R	− 0·6 ± 0·6	+ 0·8 ± 0·4
F	− 3·1 ± 1·6	− 2·9 ± 1·2	O_2L	− 0·9 ± 0·6	+ 1·4 ± 0·4
L'	− 3·0 ± 1·6	− 3·3 ± 1·2	G_1	− 0·1 ± 1·0	+ 0·3 ± 0·6
L	− 3·2 ± 1·5	− 3·7 ± 1·1	G_1'	− 0·8 ± 1·0	− 0·1 ± 0·9
B	+ 3·3 ± 1·2	+ 7·4 ± 0·9	G_2	− 1·0 ± 1·0	− 0·1 ± 0·7
B'	+ 3·9 ± 1·2	+ 2·0 ± 0·8	GL	+ 0·7 ± 1·3	+ 1·5 ± 0·9
H	− 3·3 ± 1·4	+ 1·5 ± 0·9	fml	− 0·6 ± 0·7	− 0·4 ± 0·5
H'	− 3·1 ± 1·4	+ 1·0 ± 0·9	fmb	− 0·1 ± 0·5	− 0·4 ± 0·5
OH	+ 0·8 ± 1·2	+ 1·6 ± 0·8	*Oc. I.*	0·0 ± 0·6†	+ 2·7 ± 0·6†
LB	− 2·0 ± 1·3	− 1·8 ± 0·9	100 B/L'	+ 3·0 ± 0·7	+ 6·5 ± 0·4
Q	+ 5·9 ± 2·6	+ 8·4 ± 2·3	100 H/L'	− 0·5 ± 0·7	+ 2·1 ± 0·5
Q'	+ 6·4 ± 2·6	+ 9·9 ± 1·6	100 B/L	+ 3·4 ± 0·7	+ 6·2 ± 0·6
S	− 3·7 ± 3·1	− 0·7 ± 2·5	100 H/L	− 0·4 ± 0·7	+ 2·4 ± 0·6
S_1	+ 0·7 ± 1·4	− 0·7 ± 1·2	100 B/H	+ 4·8 ± 1·2	+ 4·8 ± 0·9
S_2	+ 1·5 ± 2·0	+ 0·7 ± 1·4	100 $(B-H)/L$	+ 3·9 ± 0·7	+ 3·0 ± 0·5
S_3	− 5·8 ± 2·2	− 0·8 ± 1·7	100 $G'H/GB$	− 9·2 ± 1·3	− 1·6 ± 0·9
S_3'	− 5·0 ± 1·5	+ 1·4 ± 1·2	100 NB/NHR	+ 6·4 ± 1·2	+ 2·5 ± 0·8
U	+ 2·6 ± 3·4	+ 5·5 ± 2·2	100 NB/NHL	+ 6·4 ± 1·2	+ 2·2 ± 1·0
PH	− 0·2 ± 0·8	+ 0·5 ± 0·7	100 O_2/O_1R	− 1·2 ± 1·4	+ 2·5 ± 1·2
$G'H$	− 3·1 ± 1·1	+ 2·1 ± 0·9	100 O_2/O_1L	− 1·5 ± 1·4	+ 4·0 ± 1·2
GB	+ 7·6 ± 1·4	+ 4·9 ± 0·9	100 G_2/G_1	− 1·5 ± 2·0	− 2·5 ± 1·1
J	+ 7·3 ± 1·4	+ 8·0 ± 1·2	100 fmb/fml	+ 0·9 ± 1·8	− 0·2 ± 1·2
NHR	− 1·7 ± 0·8	+ 3·2 ± 0·6	100 DS/DC	−16·5 ± 1·7	−15·2 ± 1·2
NHL	− 1·7 ± 0·8	+ 3·3 ± 0·7	100 SS/SC	−21·3 ± 2·2	−18·7 ± 2·1
NB	+ 2·6 ± 0·5	+ 2·8 ± 0·4	$N\angle$	+2°·4 ± 1°·0	+1°·7 ± 0°·6
DS	− 1·3 ± 0·4	− 1·8 ± 0·3	$A\angle$	−0°·7 ± 1°·2	−3°·1 ± 0°·5
DC	+ 4·1 ± 0·7	+ 2·1 ± 0·3	$B\angle$	−1°·9 ± 0°·8	+1°·5 ± 0°·5
DA	+ 1·9 ± 1·3	− 0·9 ± 0·7	θ_2	+2°·6 ± 1°·1	+2°·3 ± 0°·5
SS	− 0·6 ± 0·2	− 1·1 ± 0·2	θ_1	−4°·5 ± 0°·9	−0°·8 ± 0°·4
SC	+ 1·8 ± 0·6	+ 0·7 ± 0·5	$P\angle$	+2°·0 ± 0°·8	−0°·6 ± 0°·5
O_1R	− 0·2 ± 0·6	− 0·2 ± 0·3			

* Probable errors only approximate.

† The standard deviation for *Oc. I.* varies very markedly in different races (see *Biometrika*, Vol. VIII. p. 324) being nearly three times as great in the Negro as in the English. I have used the small standard deviation of the English in writing down the probable error. If I had used that of the Gaboon series, I should have got ±1·8 for ♂, ±1·6 for ♀.

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* *Biometrika*, Vol. IX. p. 423.

Finally, as regards the general appearance of greater massiveness and ruggedness in Group A: this is confirmed to some extent by the weight (W. in grs.). Weight, it is true, does not depend only on size, but also on thickness of the bone*; but here, again, since exterior unevennesses and protuberances are by no means bound to correspond with interior excavations, it is probable that the smoother skulls are also the thinner on the whole, and that a more rugged appearance is associated to some extent with thickness.

Let us turn now to other characters than those which were considered in grouping, and first the main proportions of the skull. The Burman Group is definitely broader-headed than Group C for ♂ and ♀, the difference in maximum breadth (B) being greater in the case of the females (seven times the probable error) than with the males (three times). It is also shorter-headed, as is seen by the three length measurements L, L' and F. The result of both facts is that Group A has a higher cephalic index ($100B/L$ or $100B/L'$) in both sexes, being brought well within the brachycephalic category, while Group C is merely mesocephalic. The difference between the indices in both sexes ranges from about four to sixteen times its probable error.

Associated with the greater maximum breadth in Group A, we find in both ♂ and ♀ greater forehead breadth (B'), and the greater breadth of face, nose and nasal bridge to which we have already referred. It is not, however, accompanied by any significant difference in breadth of orbit (O_1, R and L), of palate (G_2), or of foramen magnum (*fmb*). Nor, on the other hand, is the shorter head-length in Group A associated with a significantly shorter foramen magnum (*fml*) or palate (G_1' and G_1).

The indices for these two (*fmb*/*fml* and G_2/G_1) likewise give no definite sign of differentiation.

When we consider the measurement for height (H and H') we do not find any such marked differences between the two groups as in length. Such difference as there is, is in opposite sense for the two sexes, but in neither case is this 2·5 times the probable error, and cannot be counted as significant. Nor do we find any significance in the differences between the means of the two groups for height above the auricular passages (OH); in height of orbit (O_2R, O_2L), upper-face height ($G'H$), and height of nose (NH, R and L), Group C shows a definite advantage, but for females only.

The orbital index (O_2/O_1, R and L) indicates that the Burman women are rounder-eyed than those of Group C. The men of the two groups are much alike.

As to the cranial indices in which height is a factor, Group A has a higher value for breadth over height (B/H), as we should expect; but the females alone of Group A have a significantly higher value for height over length (H/L and H/L'), the difference in males being negligible. The combined index $(B-H)/L$ shows a difference of about six times the probable error in both ♂ and ♀. This index places

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* And again on the manner in which the skull has been preserved. In the present case the same average conditions of preservation apply to the different series.

the Burmans between the modern Europeans and the primitive races dealt with in the table compiled by Thomson[*], whereas Group C comes in the middle of the primitive list; this will be discussed later[†]. Of the remaining features that exhibit significant differences in the means of the two groups, I need only refer to the transverse arc from auricular point to auricular point over the apex (Q' and Q), which is greater for Group A, doubtless owing largely to the greater width of skull; and to the angle formed by the inclination to the horizontal of the line joining alveolar point to basion (θ_2): this is significantly greater in Group A, while the distance between these two points is about the same in the two groups.

8. COMPARISON OF TYPE CONTOURS.

We can now turn to the type contours and see what further points of difference or affinity are shown there to exist among our three types.

The next three tables give the mean contour values from which type contours were constructed.

TABLE III. *Burmese Skulls. Means of Transverse Contours.*

Types:	Males			Females		
	A	B	C	A	B	C
MA	116·5 (44)	115·7 (7)	115·6 (8)	111·0 (38)	111·0 (17)	108·9 (18)
1*R*	59·1 (44)	57·3 (7)	57·3 (8)	56·4 (38)	54·6 (17)	53·3 (18)
1*L*	59·1 (44)	57·3 (7)	57·3 (8)	56·4 (38)	54·6 (17)	53·3 (18)
$1\frac{1}{4}R$	62·1 (44)	61·2 (7)	59·1 (7)	59·4 (36)	57·7 (17)	55·7 (17)
$1\frac{1}{4}L$	62·1 (44)	61·2 (7)	60·2 (7)	59·1 (38)	57·8 (17)	55·7 (17)
2*R*	63·5 (44)	62·0 (7)	60·3 (7)	61·5 (36)	59·5 (17)	57·7 (17)
2*L*	63·7 (44)	62·2 (7)	61·7 (8)	60·9 (38)	59·8 (17)	57·2 (17)
3*R*	66·6 (44)	64·9 (7)	63·3 (7)	64·4 (36)	62·8 (17)	60·7 (17)
3*L*	66·8 (43)	64·7 (7)	64·8 (8)	63·9 (38)	62·6 (17)	59·8 (17)
4*R*	69·3 (44)	66·8 (7)	65·8 (7)	66·6 (36)	65·3 (17)	63·1 (17)
4*L*	69·4 (43)	67·4 (7)	67·1 (8)	66·1 (38)	65·2 (17)	61·9 (17)
5*R*	69·7 (44)	67·4 (7)	67·4 (8)	66·9 (38)	65·8 (17)	63·6 (18)
5*L*	70·0 (43)	68·3 (7)	67·8 (8)	66·5 (38)	65·8 (17)	62·7 (18)
6*R*	69·1 (44)	66·9 (7)	67·3 (8)	66·2 (38)	65·1 (17)	63·1 (18)
6*L*	69·3 (43)	67·9 (7)	67·2 (8)	66·1 (38)	65·3 (17)	62·4 (18)
7*R*	67·2 (44)	65·2 (7)	65·6 (8)	64·3 (38)	63·2 (17)	61·4 (18)
7*L*	67·3 (43)	66·1 (7)	65·2 (8)	64·2 (38)	63·4 (17)	60·7 (18)
8*R*	62·9 (44)	60·8 (7)	61·0 (8)	60·4 (38)	59·3 (17)	57·7 (18)
8*L*	62·8 (43)	61·7 (7)	60·4 (8)	60·0 (38)	59·4 (17)	56·9 (18)
9*R*	54·3 (44)	52·2 (7)	51·7 (8)	51·7 (37)	50·8 (17)	49·2 (18)
9*L*	53·4 (43)	52·4 (7)	51·2 (8)	51·3 (38)	50·9 (17)	48·6 (18)
10*R*	38·1 (44)	37·7 (7)	36·5 (8)	36·3 (37)	36·0 (17)	35·0 (18)
10*L*	37·1 (43)	37·1 (7)	35·6 (8)	36·1 (38)	36·0 (17)	34·1 (18)
$A\frac{1}{4}R$	19·5 (44)	19·9 (7)	19·0 (8)	17·1 (37)	17·8 (17)	17·0 (18)
$A\frac{1}{4}L$	18·4 (44)	19·4 (7)	17·1 (8)	17·8 (38)	18·7 (17)	16·9 (18)
TH	117·1 (44)	116·1 (7)	116·1 (8)	111·5 (38)	111·4 (17)	109·4 (18)
ZR (*R*) *y*	62·3 (44)	61·6 (7)	59·2 (7)	59·8 (36)	58·2 (17)	55·7 (17)
x	3·0 (44)	3·7 (7)	2·6 (7)	3·1 (36)	3·5 (17)	2·2 (17)
ZR (*L*) *y*	62·5 (44)	61·9 (7)	60·7 (8)	59·6 (38)	58·4 (17)	56·4 (17)
x	3·4 (44)	4·1 (7)	3·4 (8)	3·3 (38)	3·8 (17)	3·0 (17)

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[*] *Biometrika*, Vol. XI. p. 95.

[†] See pp. 245–6.

TABLE IV. *Burmese Horizontal Contours. Mean Values.*

Types:	Males			Females		
	A	B	C	A	B	C
FO	170·9 (44)	173·7 (7)	174·4 (8)	164·1 (39)	161·6 (17)	168·2 (18)
$F\frac{1}{4}R$	27·2 (44)	26·8 (7)	25·6 (8)	23·4 (39)	24·0 (16)	21·9 (17)
$F\frac{1}{4}L$	25·7 (44)	24·0 (7)	23·5 (8)	22·9 (39)	22·4 (17)	20·8 (18)
$F\frac{1}{2}R$	36·7 (44)	36·7 (7)	35·0 (8)	33·8 (39)	33·3 (16)	30·9 (17)
$F\frac{1}{2}L$	37·3 (44)	35·4 (7)	34·3 (8)	33·7 (39)	33·6 (17)	30·0 (18)
2*R*	46·2 (44)	45·9 (7)	44·7 (8)	44·0 (39)	43·1 (16)	41·5 (18)
2*L*	46·6 (44)	44·4 (7)	43·7 (8)	44·1 (39)	43·2 (17)	41·7 (18)
3*R*	48·6 (44)	47·5 (7)	48·9 (8)	47·8 (39)	46·9 (17)	44·9 (18)
3*L*	48·9 (44)	46·6 (7)	48·2 (8)	47·8 (39)	47·0 (17)	45·2 (17)
4*R*	54·3 (44)	54·1 (7)	55·1 (8)	53·5 (39)	52·3 (17)	50·3 (18)
4*L*	55·7 (44)	53·7 (7)	54·8 (8)	53·8 (39)	53·1 (17)	50·3 (18)
5*R*	63·6 (44)	63·1 (7)	63·3 (8)	62·2 (39)	60·6 (17)	58·3 (18)
5*L*	63·7 (43)	61·4 (7)	62·4 (8)	61·3 (39)	59·9 (17)	57·2 (17)
6*R*	69·3 (44)	68·1 (7)	67·9 (8)	67·4 (39)	65·6 (17)	62·7 (18)
6*L*	68·8 (43)	66·5 (7)	66·9 (8)	65·9 (39)	64·7 (17)	61·7 (17)
7*R*	70·7 (44)	69·6 (7)	69·0 (8)	68·5 (39)	67·3 (17)	64·1 (18)
7*L*	70·2 (43)	67·6 (7)	68·0 (8)	66·6 (39)	65·7 (17)	62·9 (18)
8*R*	67·9 (44)	66·5 (7)	65·1 (8)	65·3 (39)	64·6 (17)	61·4 (18)
8*L*	67·0 (44)	65·5 (7)	65·3 (8)	63·6 (39)	62·8 (17)	60·9 (18)
9*R*	60·3 (44)	59·0 (7)	56·9 (8)	57·9 (39)	57·3 (17)	54·4 (18)
9*L*	60·5 (44)	58·7 (7)	58·4 (8)	56·8 (39)	56·1 (17)	54·6 (18)
10*R*	46·5 (44)	45·7 (7)	43·1 (8)	44·5 (39)	44·1 (17)	41·4 (18)
10*L*	47·9 (44)	46·5 (7)	46·0 (8)	44·5 (39)	43·2 (17)	42·3 (18)
$O\frac{1}{4}R$	26·2 (44)	25·9 (7)	23·0 (8)	25·4 (39)	26·5 (17)	22·9 (18)
$O\frac{1}{4}L$	29·2 (44)	28·0 (7)	27·1 (8)	25·4 (39)	25·7 (17)	25·9 (18)
T (*R*)	47·6 (44)	46·8 (7)	45·3 (8)	45·2 (39)	44·6 (16)	44·0 (18)
T (*L*)	47·7 (44)	45·6 (7)	45·1 (8)	45·2 (39)	44·5 (17)	44·8 (18)
xR	18·3 (44)	17·9 (7)	17·3 (8)	17·6 (39)	17·3 (16)	20·3 (18)
xL	17·8 (44)	17·9 (7)	18·0 (8)	16·9 (39)	17·0 (17)	20·7 (18)

TABLE V. *Means of Contour Measurements used in plotting Sagittal Type Contours.*

Types:		Males			Females		
		A	B	C	A	B	C
$N\gamma$	... ...	167·4 (44)	170·9 (7)	172·3 (8)	161·8 (39)	159·3 (17)	165·1 (18)
Vertex	x from N	91·8 (44)	100·5 (7)	98·4 (8)	89·1 (39)	90·9 (17)	87·4 (18)
	y	90·7 (44)	90·5 (7)	88·0 (8)	86·3 (39)	85·8 (17)	84·8 (18)
β	x from N	69·5 (44)	70·1 (7)	73·3 (8)	66·1 (39)	65·1 (17)	67·5 (18)
	y	88·2 (44)	86·5 (7)	86·0 (8)	83·5 (39)	82·9 (17)	83·0 (18)
Glabella	Gx	3·4 (44)	2·7 (7)	2·3 (8)	2·6 (39)	2·4 (17)	3·2 (18)
	Gy	12·2 (44)	10·0 (7)	9·2 (8)	13·2 (39)	11·6 (17)	10·8 (18)
Ordinates above $N\gamma$	O	29·7 (44)	22·0 (7)	21·7 (8)	29·4 (39)	25·5 (17)	28·1 (18)
	$N\frac{1}{4}$	44·1 (44)	36·0 (7)	36·0 (8)	42·9 (39)	40·2 (17)	40·4 (18)
	1	62·4 (44)	58·5 (7)	58·0 (8)	59·7 (39)	58·0 (17)	58·8 (18)
	2	75·5 (44)	73·1 (7)	72·2 (8)	71·6 (39)	70·4 (17)	70·9 (18)
	3	83·5 (44)	81·7 (7)	80·6 (8)	78·8 (39)	78·0 (17)	78·2 (18)
	4	88·1 (44)	86·2 (7)	85·3 (8)	83·3 (39)	82·8 (17)	82·8 (18)
	5	90·2 (44)	89·1 (7)	87·4 (8)	85·5 (39)	84·8 (17)	84·2 (18)
	6	90·3 (44)	90·1 (7)	87·5 (8)	85·8 (39)	85·4 (17)	84·0 (18)
	7	87·4 (44)	86·9 (7)	84·3 (8)	82·8 (39)	82·7 (17)	80·8 (18)
	8	79·5 (44)	77·9 (7)	76·5 (8)	75·2 (39)	75·7 (17)	72·9 (18)
	9	64·0 (44)	61·2 (7)	59·9 (8)	60·2 (39)	61·1 (17)	56·6 (18)
	$\gamma\frac{1}{4}$	39·4 (44)	34·8 (7)	34·3 (8)	36·3 (39)	38·0 (17)	31·3 (18)
	$\gamma\frac{1}{8}$	31·8 (44)	27·1 (7)	28·7 (8)	28·8 (39)	30·9 (17)	24·3 (18)
	γ	21·8 (44)	14·1 (7)	15·7 (8)	16·7 (39)	16·7 (17)	14·2 (18)
Ordinates below $N\gamma$	8	46·4 (44)	47·2 (7)	49·3 (8)	45·2 (39)	46·1 (17)	44·4 (18)
	9	35·8 (44)	37·7 (7)	38·0 (8)	33·6 (39)	34·5 (17)	33·9 (18)
	$\gamma\frac{1}{2}$	27·7 (44)	30·2 (7)	30·7 (8)	25·3 (39)	25·4 (17)	26·1 (18)
	$\gamma\frac{1}{4}$	19·0 (44)	21·7 (7)	19·6 (8)	17·5 (39)	17·8 (17)	18·7 (18)
Occipital	x from γ	1·2 (44)	0·8 (7)	0·8 (8)	1·0 (39)	1·0 (17)	0·9 (18)
	y	11·7 (44)	8·0 (7)	12·4 (8)	8·8 (39)	9·6 (17)	7·7 (18)
λ	x from γ	3·2 (44)	5·4 (7)	4·6 (8)	3·4 (39)	2·7 (17)	5·1 (18)
	y	34·6 (44)	34·6 (7)	33·7 (8)	31·8 (39)	32·2 (17)	31·7 (18)
Sub-orbital point	x from N	7·8 (42)	9·3 (7)	8·2 (7)	7·0 (34)	7·6 (17)	11·0 (16)
	y	28·1 (42)	28·4 (7)	29·3 (7)	27·3 (34)	27·1 (17)	26·6 (16)
Auricular point	x from γ	82·6 (44)	85·3 (7)	86·6 (8)	81·5 (39)	79·4 (17)	82·1 (18)
	y	28·1 (44)	28·4 (7)	29·4 (8)	27·1 (39)	27·1 (17)	26·5 (18)
Opisthion	x from γ	45·0 (43)	50·1 (7)	49·6 (8)	46·2 (39)	43·0 (17)	47·1 (18)
	y	49·6 (43)	51·6 (7)	55·5 (8)	49·5 (39)	49·8 (17)	49·0 (18)
Inion	γI	28·9 (44)	29·8 (7)	36·2 (8)	29·4 (39)	28·8 (17)	28·0 (18)
	NI	163·0 (44)	166·6 (7)	164·9 (8)	155·4 (39)	152·9 (17)	159·1 (18)
Basion	γ bas.	93·1 (43)	96·5 (7)	100·2 (8)	91·7 (39)	90·2 (17)	92·7 (18)
	N bas.	97·4 (43)	98·0 (7)	100·2 (8)	93·0 (39)	93·3 (17)	94·7 (18)
Alveolar point	N to A.P.	70·7 (38)	67·9 (7)	74·2 (7)	65·7 (34)	65·7 (15)	63·9 (16)
	Bas to A.P.	96·5 (38)	93·6 (7)	95·6 (7)	90·3 (34)	90·4 (15)	88·8 (16)
Nose	$\angle$ of nasal prominence	110°·0 (31)	115°·4 (7)	118°·5 (6)	108°·9 (22)	115°·1 (15)	116°·8 (12)
	NL	22·9 (29)	22·7 (6)	26·4 (5)	21·8 (16)	21·3 (13)	21·2 (9)
	NL'	20·8 (29)	21·0 (6)	22·4 (5)	20·3 (16)	19·2 (13)	17·9 (9)
	Max. nasal subt. x from N	10·4 (29)	9·6 (6)	11·9 (5)	9·9 (16)	9·2 (13)	9·1 (9)
	y	2·6 (29)	2·6 (6)	3·2 (5)	2·6 (16)	2·5 (13)	2·0 (9)
Palate	Max. Subt. x from A.P. ...	35·5 (38)	35·9 (7)	37·0 (7)	33·5 (33)	33·9 (15)	34·5 (16)
	y ...	17·7 (38)	15·8 (7)	19·8 (7)	15·9 (33)	16·4 (15)	17·2 (16)
Frontal	Max. subt. to $N\beta$ x from N	51·8 (44)	53·5 (7)	53·1 (8)	47·9 (39)	48·4 (17)	48·9 (18)
	y	26·1 (44)	24·0 (7)	25·0 (8)	25·2 (39)	23·9 (17)	24·8 (18)
Occipital	Max. subt. to $\lambda Op.$ x from λ	53·9 (43)	55·5 (7)	58·6 (8)	49·2 (39)	50·8 (17)	49·5 (18)
	y	24·6 (43)	28·4 (7)	25·8 (8)	23·4 (39)	21·2 (17)	24·7 (18)
$N\lambda$ base line	Max. Subt. x from N ...	82·6 (44)	88·1 (7)	84·5 (8)	81·1 (39)	81·2 (17)	79·1 (18)
	y ...	72·9 (44)	71·1 (7)	70·7 (8)	69·0 (39)	68·7 (17)	68·8 (18)
GI base line	Max. Subt. x from G ...	96·7 (44)	96·8 (7)	96·4 (8)	90·9 (39)	91·9 (17)	91·8 (18)
	y ...	102·1 (44)	103·4 (7)	104·5 (8)	97·0 (39)	97·3 (17)	96·1 (18)

In addition I append a table of mean measurements taken on the sagittal contours, but not used in construction of the sagittal type contour, as the position of the terminals had already been fixed by the means of other measurements. I have discussed already the degree of exactness with which these means from individual contours correspond to the values read off from the type contour.

TABLE VI. *Burmese Mean Sagittal Contour Values not used in Construction of Type.*

Types:		Males			Females		
		A	B	C	A	B	C
Lengths	$N\beta$	112·6 (44)	111·3 (7)	113·1 (8)	106·7 (39)	106·0 (17)	108·2 (18)
	$\lambda Op.$	93·7 (43)	97·3 (7)	100·2 (8)	92·8 (39)	91·8 (17)	91·2 (18)
	$N\lambda$	167·9 (44)	169·2 (7)	171·4 (8)	162·0 (39)	160·3 (17)	163·4 (18)
	β subtense to $N\lambda$ x from N	86·2 (44)	86·2 (7)	89·0 (8)	81·5 (39)	80·4 (17)	82·5 (18)
	y	72·0 (44)	70·2 (7)	70·0 (8)	68·5 (39)	68·0 (17)	68·1 (18)
	GI	168·7 (44)	171·1 (7)	169·3 (8)	160·7 (39)	157·7 (17)	164·1 (18)
	β subtense to GI x from G	53·0 (44)	53·4 (7)	53·5 (8)	49·8 (39)	48·3 (17)	54·2 (18)
	y	91·2 (44)	90·8 (7)	93·6 (8)	85·3 (39)	85·7 (17)	86·5 (18)
Angles	ϕ $(= \angle \beta GI)$	60°·1 (44)	59°·7 (7)	60°·5 (8)	60°·6 (39)	60°·6 (17)	59°·1 (18)
	ϕ' $(= \angle \beta NI)$	61°·7 (44)	61°·1 (7)	61°·6 (8)	62°·2 (39)	61°·9 (17)	60°·8 (18)
	$A \angle$	69°·6 (38)	72°·6 (7)	70°·4 (7)	71°·8 (34)	71°·6 (15)	75°·0 (16)
	$N \angle$	68°·0 (38)	66°·1 (7)	64°·8 (7)	66°·8 (34)	66°·9 (15)	64°·7 (16)
	$B \angle$	42°·5 (38)	41°·2 (7)	44°·8 (7)	41°·4 (34)	41°·6 (15)	40°·3 (16)
	$P \angle$	84°·4 (38)	86°·1 (7)	83°·6 (7)	85°·0 (34)	83°·8 (15)	87°·2 (16)
	$fml \angle$	7°·3 (42)	8°·8 (7)	7°·6 (8)	8°·8 (39)	7°·8 (17)	8°·2 (18)
Indices	Frontal index i_f	23·1 (44)	21·6 (7)	22·1 (8)	23°·6 (39)	22·5 (17)	22·9 (18)
	Occipital index i_{oc}	26·2 (43)	29·1 (7)	25·8 (8)	25·2 (39)	23·0 (17)	27·1 (18)

Before comparing our type contours we need to get some measure of the range of variation which would be covered by means based on other samples of the same size as our own, drawn from the same population. On pp. 143, 145, 147 of Crewdson Benington's paper on Type Contours* we have graphical representations of the limits between which the bulk of these variations would lie for the three contours representing a series of 100 English skulls. Assuming the standard deviations of English and Burmese characters to be approximately the same (and they will certainly be of the same order), and taking into consideration the fact that the probable error of a mean varies directly in proportion to the inverse square root of the number of skulls on which it is based, we see that we must allow for our largest series of forty-four (Group A ♂) a range half as big again as that shown for the English contour; while for our smallest series of seven (Group B ♂) the range must be made about 3·8 times as big. This demonstrates once more the unreliability of definite conclusions based upon short series.

Below are given the proportions in which the range of 100 English type must be increased for our six series; and for the convenience of those who have not the Benington paper for reference we give a few measurements to indicate the extent of the range.

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* *Biometrika*, Vol. VIII.

Type contours based on 100 English skulls:

Transverse Contour. Range of variation from 1·4 mms. at apex tapering down to 0·8 mm. by the auricular points.

Horizontal Contour. Range about 1·2 mms. all round.

Sagittal Contour. About 1·6 mms. from glabella to lambda, tapering to 0·7 mm. at gamma and 0·9 mm. at nasion (below gamma not given).

Extent to which this range should be increased for Burmese groups:

Group	No.	Multiplying coefficient	Group	No.	Multiplying coefficient
♂ A	44	1·5	♀ A	39	1·6
♂ B	8	3·5	♀ B	18	2·4
♂ C	7	3·8	♀ C	17	2·4

Type Contours of Groups A and C (*Burmese*) *compared.*

One rather noticeable characteristic of the Group which we take to be Burman —Group A—is an impression of flatness at the back of the skull. Our photograph of skull No. 110 (Plate I, Fig. 1) in profile shows an extreme example of this flattening; it will also, however, be remarked in No. 4 (Plate III, Fig. 8) which has been photographed as a type (with all the characteristics rather marked) of our Group A males, and here we can see it also as looked at from above. The effect is helped by the fact that the parietal eminences are further back in proportion than are those of the Chinese for example (Plate VI, Fig. 19), or the Hindu (Plate V, Fig. 16), or to go outside the various series we have specially used for comparison in this paper, further back than in the Moriori skull (Plate XI, *Biometrika*, Vol. XI. after p. 135). This is a feature which is not brought out in our direct measurements, nor indeed very clearly in our contours, since the vertical contours are taken in a plane well in front of these eminences while in the horizontal contours the pointer of the contour tracer passes well below them. We shall, however, expect to find some confirmatory indication in the horizontal contour even though this is not drawn at the height where it would most clearly illustrate our point.

In so far as the effect is due to the actual flattening of the upper part of the occipital bone, that is, to its steeper fall from the lambda to the inion, the sagittal contours should serve quite well to demonstrate it.

Let us first take up the transparent-paper tracings of the ♂ and ♀ Horizontal Type contours for Group A (Burmans) and place them upon the corresponding contours for Group C, F to F, and FO falling along FO. We notice at once the slimmer proportions of Group C, and the more flattened occipital region in Group A. It will, however, also be noticed that the frontal curve of the C's is more concave, and it may be this that has thrust the occipital point O of Group C type contour too far out when we made our mid-frontal points F to coincide. We will therefore make the points where the contours indicate our temporal ridges lie in the same straight line, *FO* lying along *FO* as before. The occipital part of the Group C contour no longer projects so far beyond that of the Group A, but still enough to show quite clearly the greater flatness of the latter; a flatness which clearly characterises both occipital and frontal regions. The swelling curves to left and right in Type A bring out in a striking manner the greater width which our direct measurements revealed.

We now turn to the transverse type contours, and lay first the tracing for Burman males over that of Group C males. If we make the mid-points of our auricular axes coincide and their perpendicular bisectors lie along one another, Group C contour is seen to fall entirely within Group A. Though the chief difference is in width, not in height.

But now, in order to compare the shape of the lateral curves, let us slide the auricular axis of the Group A tracing along that of the other until the *R* auricular points come together. The contours now coincide to a great extent on the right side; but they diverge in two places. In one of these (that about ordinate nine) we can perhaps identify the beginning of that swelling which, carried further back, culminates in the marked parietal eminence to which we have referred. We say perhaps, because the differences noted here are well within the limits of probable error. In the other region, that lying just above the auricular axis, we see indicated the greater massiveness of the Burman about the zygomatic ridge and in the area above it; this again is within our range of variation for the small Group C.

The ♀ type contours for Groups A and C demonstrate the same kind of difference, though here there is a greater difference in height and less massiveness in proportion in the part about the zygomatic ridge.

Lastly, the sagittal type contours.

We are spared the necessity of considering the relative values of the nasio-lambda line and the standard horizontal through the nasion ($N\gamma$), as the best base-line for comparison, since the angle between these two at the nasion is practically the same in the two male types: the two female types also agree together very nearly.

The question of the superiority of either of these to Schwalbe's glabella-inion line or Klaatsch's glabella-lambda line was examined in considerable detail in Thomson's Moriori paper* with a verdict in favour of the $N\gamma$ line, which is parallel to the standard horizontal plane.

Three statements were made about the standard horizontal plane through the sub-orbital point in the case of type crania.

1. That it passes very nearly through, but a little below the inion.

In five of our six type contours it does pass practically through the inion; in the sixth, that for the males of Group C, it is 6 mms. above. This, however, is our smallest series—seven only, and might be expected to vary.

The above important statement is therefore fully supported by the evidence of our Burmese.

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* *Biometrika*, Vol. XI. pp. 105–111.

2. That it is approximately parallel to the plane of the foramen magnum.

On the Moriori ♀ type contours it is actually parallel; for the males there is only a difference of 2°·5 between the two lines representing the planes, that of the foramen magnum being so tilted as to bring the opisthion slightly nearer to the N line than is the basion.

The inclination of these two lines to one another has been measured on our Burmese individual contours. In our type contours the positions of basion and opisthion were determined by other measurements and the angle as shown on the type is not bound to correspond exactly with the mean of individual contours.

The following are our double values for this angle:

Types:	Burmese ♂			Burmese ♀		
	A	B	C	A	B	C
Mean contour value	7°·3	8°·8	7°·6	8°·8	7°·8	8°·2
From type contours	6°·0	7°·8	7°·2	7°·3	6°·6	7°·2

and in all the above, the tilt of the basio-opisthion line was such as to bring the *basion* nearer to $N\gamma$.

We see therefore that this second argument in support of $N\gamma$ as base-line for comparison is much less potent in the case of the Burmese than in that of the Moriori. The Burmese basio-opisthionic line is in fact about equally inclined to the $N\gamma$ and glabella-inion lines in our type contours, and would contribute nothing to a decision between these two.

3. That the standard horizontal is approximately parallel to the maximum length, i.e. the glabella-occipital line. This is quite true for our series, in all six types.

We will try each of these lines in turn as base-lines in comparing our own Group A and Group C type contours. Putting nasion on nasion and $N\gamma$ over $N\gamma$, we get very markedly in the males and to a less extent in the females, an impression of flattened frontal in Group C: the ♀ type contour for the C's has an outline which is decidedly more re-entrant at the nasion than that of the A's, so that the glabellar region of Group A ♀ seems more prominent—a feature not shown in the male contours, where in fact Group A shows slightly greater prominence.

Now if we move our Group A ♀ contour further forward so that the glabellar regions are together—while still keeping $N\gamma$ along $N\gamma$—we get a very similar contrast between the two groups, in the two sexes; the forehead less sloping in Group A, and the back of the head descending more abruptly from the crown. In the males the occipital bone of Group C ends at a point some 5 mms. lower than that of Group A, but this is not so with the females, and we must remember that the range of variation in Group A males is nearly four times that of Benington's type contour for 100 skulls, so that even this divergence might be accounted for by the smallness of the Group C sample. It is, in fact, impossible to be dogmatic about any difference in the type which our Group C represents, as suggested by contour comparison, especially in the males; our conclusions can be but tentative. This applies, it need not be repeated, also to what follows:

We will now proceed to examine in how far the shape of the frontal bone itself has a share in giving the impression of greater frontal flatness in Group C, we superpose the nasio-bregmatic lines, nasion covering nasion as at first. The two chords are of practically the same length, and though there is slightly greater concavity in Group A above the ophryon, in the ♂ and ♀, this is so slight as to be negligible. The impression of flatness in Group C therefore is due very little to the actual flattening of the frontal but chiefly to its position.

If we use the glabella-lambda line as base-line, the characteristics we have already remarked are brought out quite well: in this position we get rather less the effect of frontal flatness, and more of the difference accounted for by the back of the head.

Finally, the glabella-inion line: this has the advantage in both sexes of making the opisthions coincide; and in the female contours brings out quite well the characteristics of the two groups. In the males, however, the much greater length of the occipital bone for the short series of Group C, with the corresponding lowering of the inion, makes Group A type fall entirely within Group C, from the middle of the frontal right away to the inion, so that here it is Group A that would seem to have a more flattened frontal, and about equal downward slope for the back of the head; an effect that is suggested neither by the actual appearance of the skulls nor by the other methods of contour comparison.

Whichever of these various base-lines is used, however, there is one feature so pronounced that it is brought out by all of them. This is the much less prominence of the nose in Group A, for both male and female.

The result of our comparison of these two groups by means of direct measurements and contours may be summed up as follows: Group A, the Burman Group, is broad and high in proportion to its length (in the brachycephalic and hypsicephalic categories), with marked frontal eminences, and is more vertical than Group C from the crown of the head down to the inionic region; it is of a rugged build, has prominent cheek-bones, nose wide and rather flat at the bridge, wide and rather depressed from the bridge downwards (nasal index classifies it as platyrhine), orbits flat rather than round, but not markedly so (mesoconchic, to use another of these cumbersome class designations), given to sub-nasal prognathism, as the "Remarks" in Appendix II testify, but face fairly vertical above this point; palate not specially broad in proportion to its length (on the borders of the leptostaphyline and mesostaphyline categories).

Group C is both longer and narrower in the main skull measurements but again high (mesocephalic and hypsicephalic); it is of smoother and lighter build with less outstanding cheek-bones, higher nasal bridge, and more prominent nose, the nose being narrower both at the bridge and lower down (mesorhine); orbits slightly flatter than those of Group A (on the border between the chamaeconchic and mesoconchic categories); frequently characterised by sub-nasal prognathism, but fairly vertical above; palate also similar to that of Group A.

One of the measurements which has been taken, I believe, for the first time on the sagittal contours of my series, is that of height of palate (see p. 201). The term "high palate" figures frequently in literature dealing with mental defect, and it is generally assumed that a connection exists between these two phenomena. It seemed, accordingly, to be of some interest to discover whether any association existed between height of palate and capacity of skull in the present series: of mental capacity I naturally had no record. Miss Margaret Moul of this Laboratory was kind enough to make out for this purpose a correlation table, using the palate height index* as one variate, and the capacity of the skull as measured by mustard-seed (C of the direct measurements) as the other. The two sexes were kept separate; and as there were only 31 male skulls and 38 female on which it had been possible to take both measurements, the numbers were too few for grouping, and the correlation coefficient was worked out on the individual measurements.

The following results were arrived at:

Coefficient of Correlation between palate index and capacity of skull:

$$♂: r = -\cdot 3402 \pm \cdot 1071,$$
$$♀: r = -\cdot 3283 \pm \cdot 0976.$$

In spite of the large "probable error," due to the small series of skulls, the value of r is seen to exceed three times that of the probable error for both sexes, and must be accepted as significant: that is to say, the figures show a tendency for the higher palate to be associated with smaller *skull* capacity, in both male and female. It is not easy to conceive how a relatively high palate could connote physically a reduced capacity for the skull although it might mark a reduced nasal cavity. The point is of much interest, since the opinion of those dealing with mental defectives as to their relatively high palates has not been confirmed by the American investigators, who took casts of the mouth of 300 normal and 300 mentally defective children†. The present relation will be shortly tested on a wider series, a special instrument having recently been designed for the measurement of the palate. But if this correlation should hold, it does not follow that the relative palate height would be correlated with intelligence. A recent series of 88 mentally defective children of ages eight to fourteen collected for age showed no sensible correlation between their mental ages judged by the Binet-Simon tests and their cranial circumference. Although no great weight can be placed on this series, it tends *pro tanto* to confirm the results reached by Pearson in 1906‡ that the correlation between size of head and intelligence is very small.

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* By palate-height index I understand 100 × height of palate ÷ length of palate. The latter was the G_1 of my direct measurements, while the height of the palate was obtained as described on p. 201 from the sagittal contour.

† *American Journal of Insanity*, Vol. XVI. pp. 687–697. "Comparative Measurements of the Hard Palate in Normal and Feeble-minded Individuals," by Drs Walter Channing and Clark Wissler.

‡ *Biometrika*, Vol. V. pp. 119–125.

9. On the "Remarks" as to the Cranial Anomalies.

Lastly, having examined direct measurements and type contours we turn to the Cranial "Remarks" which accompany the table of individual measurements in Appendix II.

The re-examination of the series of skulls for anomalies or other points which seemed worthy of comment, was carried out on the following system—lacking which, any contribution which these customary comments may make has extremely little value.

A list was made of characters and anomalies for which each skull should be examined, a list partly suggested by the comments made by previous workers in the school. Of the features included in this list, it may be rightly assumed that when they are not mentioned in the "Remarks," they do not exist in the skull under consideration.

The points are as follows:

Age and signs of age. Where no comment is made the skull is adult, with sutures joined but not obliterated; basilar synchondrosis is noted where it exists; also any teeth that are in process of coming through, or, on the other hand, any considerable loss of teeth during life and the consequent absorption of the alveolus: this of course may occur in the younger skulls, but is more associated with old age; any falling-in or thinning of the calvarium, due to old age, has been looked for and noted.

Teeth. Any case of undeveloped or imperfectly developed third molars (as when they have but a single fang and resemble incisors in shape and size) has been commented on. The only cases in which it would be impossible to detect this would be those in which the alveolus was absorbed in the molar region. I have also noted cases of extreme attrition of the teeth, where they remain. Many teeth show signs of very rough usage, presumably due to betel chewing, and are so worn, especially on the inner side, that they present a surface with a steep slope towards the palate. Very often they are considerably blackened also by this habit.

Palate. Each was examined for the existence of bony ridges across the palatine grooves leading from the pterygo-palatine canals, and referred to by Le Double*. Where such a one is thrown over the inner groove on either side, it is described as an inner palate bridge, left or right; over the outer groove as an outer palate bridge.

On thirteen of our skulls the palate was missing; on the remaining 129 we found a bridge in seven cases over the inner palatine groove (five times on *R*, twice on *L*) and in one case over the outer groove (on *L*).

Thus of 129 Burman and Burmese-hybrid skulls, 5·4 per cent. have an inner palate-bridge, 0·8 per cent. an outer. We may insert these figures into the table quoted by Le Double from Buntaro Adachi†.

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* *Variations des Os de la Face*, p. 266.

† B. Adachi, *Zeitschrift für Morphologie und Anthropologie*, Band II, S. 202. Stuttgart, 1900.

Race	No. of skulls	Inner palate bridge	Outer palate bridge
Burmese	129	5·4 %	0·8 %
Europeans	1193	2·3 %	0·5 %
Asiatics	688	12·8 %	2·6 %
Australians and South Sea Islanders	64	7·8 %	4·7 %
American Indians (N. and S.)	122	1·6 %	0·0 %
Africans	123	9·7 %	7·3 %
Ancient Egyptians	384	4·7 %	0·0 %

The conclusion which Adachi himself draws is as follows: "Jedenfalls geht aus meinen Ausgaben vor, dass die Canalbildungen der Gefässfurchen am harten Gaumen bei Europäern und Amerikanern überhaupt selten, dagegen bei Asiaten, Australiern, und Südsee-Insulanern und Afrikanern (Altägypter ausgenommen) viel häufiger vorkommen†."

It will be seen that the Burmese figures are not of the same order as those given by Adachi for "Asiatics," and do not support the inference which he makes.

The palate was also examined for palatine torus, a thickening of the bone on both sides of the median suture; only two cases were found, and neither extreme.

Base of the pyriform aperture. The moulding of the base of the pyriform aperture has been mentioned as being one of the characteristics which influenced our classification of the skull. There is therefore included in the "Remarks" a brief description of this character for each skull in which the bones of the face were not broken away. For brevity this feature has been indicated by the letters P.B., and the description has reference first to that part of the floor of the nasal cavity which lies immediately behind the lower edge of the aperture, and secondly to the edge itself. The floor of the cavity is variously described as "flat"—in which case it may be assumed to be approximately horizontal as well—or "sloping upwards and outwards," or "downwards and outwards"; the edge itself is described as sharp, blunt or rounded, and if it is double, with a groove between its two parts, this also is mentioned. In some cases the floor of the cavity is so rounded as to form part of an almost unbroken curve with the rounded lip: such a case is described shortly as P.B. rounded, or V. rounded.

Taking into consideration the form of the edge alone, one finds the following distribution among our groups:

Type of edge	A ♂	A ♀	B ♂	B ♀	C ♂	C ♀
Sharp	5½*	8	4½*	8	3	9
Blunt	6	16	2	6	1	3
Rounded ...	18½*	5	1½*	1	3	4
V. rounded ...	9	3	1	1	—	—

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* The "½" is due to skulls in which the lower border of the pyriform is "rounded" on the *R*, "sharp" on the *L*.

† B. Adachi, *loc. cit.* p. 203.

a table which suggests that this feature tends to be finer in the female than in the male; also that it is on the whole coarser among the Burmans proper than in our B and C groups.

As regards the "double edge" above referred to, we have noted seven cases of this among the ♂ Burmans, only one in the corresponding ♀, and one in each of the B groups.

Asymmetry. In going through the series I noted all cases of asymmetry, both occipital and frontal; also the existence of a Sylvian depression, and whether greater on one side or the other; also the relative sizes of the jugular foramina. As, however, these observations have been used to estimate the degree of association between different forms of asymmetry they have been omitted from the printed "Remarks."

*Interparietal bones**. The tripartite interparietal in its complete form of os pentagonale and *R* and *L* os triangulare* occurred in one case only (♀) in my series, and was associated with marked bathrocephaly. There were three cases (1 ♂, 2 ♀) in which one os triangulare was found, each time on the *R* side. In one of the females it was accompanied by considerable asymmetry of the occipital region, the greater prominence being on the same side as the interparietal bone. In the other two cases, there was a fairly prominent occipital region, but no marked asymmetry.

Ossicles in the sutures. Where these exist they have been noted, and included in the "Remarks."

Conformation at the pterion. All cases of epipteric bones have been mentioned; also instances of the thrusting of a process of the squama temporalis between the parietal and sphenoid bones to join the frontal.

Fourteen of our 61 ♂ skulls, and 21 of our 75 ♀ gave evidence of such peculiarities, 5 ♂ and 8 ♀ having them both *R* and *L*. There was an epipteric on the *R* in 11 ♂, 11 ♀ skulls; on the *L* in 6 ♂, 9 ♀; frontal process to temporal bone on the *R* in 6 ♀, on the *L* in 2 ♂, 3 ♀.

Malar bones. There are in our series five cases of a horizontal suture right across the malar bone (out of 127 skulls where face-bones exist) separating off what is frequently termed the os japonicum, two of these existing in the same skull.

In several other cases part of the suture is indicated, but it is incomplete.

Any specially heavy malar marginal processes have been remarked upon.

Pre-condyles. The existence of one or a pair of these has been recorded in the case of nine skulls. They represent all stages from a single small bony growth on the *R* or *L* between the condyles, in front of the foramen magnum, to a pair of well-developed separate characters and from that to a pair of pre-condyles partly fused and forming a small bridge, and, lastly, to a pair entirely fused and forming a conspicuous central eminence.

Metopism. There are two cases of metopism in our 142 skulls.

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* See *Biometrika,* Vol. III. p. 220.

Other features. In addition to the above, all cases of flattening at the obelion, metopic ridge, and tympanic perforation have been recorded; and notes have been added on any other singularities that seemed worthy of comment.

10. COMPARATIVE MATERIAL.

When we turn to the consideration of the Burmese skull in its relation to other races, we are confronted with a problem that is among the most difficult and interesting of those many that await the study of the craniologist.

As with the lower types of life, so with man, we are, in Indo-China or Further India, on the border-line of two great divisions, the Oriental and Euro-Asian. Much work remains to be done still on the Crania of the Burmese races, and much on those of Thibet and Siam, before a full light can be thrown on the exact relations of these borderland peoples. Nor have we as yet any comprehensive study of the skulls of the great races that lie to the north, south, and west of our borderline, although a considerable amount of cranial material has been collected and is scattered throughout the museums of Europe. The measurements that have been taken vary very considerably, both as to number of characters chosen and as to the convention by which they are measured; and practically all were left in the records as mere measurements, not yet reduced to the constants of racial types.

In so far as the material served, I have used it to compile tables on which the means of between twenty and thirty characters can be based for four neighbouring types: Malayan, Chinese, Hindu and Dravidian.

In preparing these tables it was necessary to reject a considerable number even of the rather scanty measurements which were all that most craniologists supplied: either because the origin of the skull was stated too vaguely, or because the character measured was not one usually measured by other craniometricians; or because, if the same measurement were attempted it was taken by a non-comparable method. The necessity for ascertaining exactly what had been measured in the case of each worker made the compilation of my own series of means from theirs rather a laborious process: I have, however, taken all the care I possibly could to avoid errors arising from this source.

The following is the comparative material obtained:

Malayans. I was fortunate enough to find in Dr Emil Schmidt's Catalogue of of his own collection* a series of over 80 Malayan skulls. Of these, one was a child, and one a badly deformed hydrocephalic male: of the remainder, six only were ♀s, the rest ♂s. The ♀ skulls are of course too few alone to be of any use; the ♂ skulls, however, not only form a long series, but have had a large number of measurements taken on them, and these mostly in accordance with the Frankfurt Concordat, so that we have means of no less than 32 characters based on 69 to 78 skulls, with which to make comparisons.

♀ Malayans had to be collected from various sources, and were brought up to a total of 56; of these 38 were recorded in *Die Anthropologischen Sammlungen*

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* *Die Anthropologischen Privat-Sammlungen Deutschlands*, Abtheilung I.

Deutschlands (Emil Schmidt's Collection 6, Breslau 1*, Munich 5*, Darmstadt 8*, Frankfurt 3*, Berlin 6*, Göttingen 6*, Bonn 3*), and 18 from Barnard Davis' Catalogue.

The skulls of the Malayan series have been assembled from the various islands of the Malay Archipelago, which we are aware contain a considerable Indonesian element. We may, however, surely dismiss the possibility that so experienced a craniologist and collector as Dr Emil Schmidt (to whom we owe all the ♂ data), when out to collect Malayans, would accept Indonesians and classify them under the title "Malay." There can be equally little doubt, however, that by "Malay" is not meant the pure Malay race, but any of the kindred races scattered upon those islands which are now more usually grouped under the broader term Malayan. That we are not dealing here with a pure race is revealed by the standard deviation of the cephalic index B/L. For the Malayan series it is $5{\cdot}13 \pm {\cdot}28$; and this may be compared with the corresponding standard deviations for other races quoted by Fawcett†. Here we see that the fairly homogeneous collections of Naqadas, Negroes, Panjabi Low Castes, Ainos, and Row Grave Germans show a standard deviation of the cephalic index below 3·0 for male skulls; for the ♂ Bavarians, Whitechapel English and Egyptian Mummies it lies between 3·0 and 4·0 and only in the collection of ♂ Modern Egyptians where it rises as high as 5·42 is our Malayan standard deviation exceeded.

There is therefore doubtless some heterogeneity in the Malayan material, but pending further study of the crania of the individual islands, we are compelled to use the means obtained from it, as a working approximation to a Malayan cranial type.

Chinese. No single series of Chinese skulls in the various German museums was long enough alone to supply anything like the number desirable, so it was again necessary to collect data scattered throughout the publication, and we have before us from German sources measurements taken in 69 male Chinese skulls (Schmidt's 19, Strassburg 2, Breslau 2, Heidelberg 2, Munich 13, Göttingen 11, Bonn 1, Frankfurt 7, Darmstadt 12), but about half of these, having been measured before 1882, and therefore not in accordance with the Concordat, yielded comparatively few measurements of which use could be made.

In addition there are sixteen skulls from Flower's catalogue of the collection at the Royal College of Surgeons, and seventeen from Barnard Davis' collection. Total for Chinese ♂ skulls 102, the numbers on which the various means are based ranging from 102 downwards.

Drawing on all these sources I only managed however to collect measurements of 11 ♀ Chinese skulls all told (3 from Flower, 1 from Schmidt, Munich 1, Bonn 1, Barnard Davis 5). The ♀ Chinese skull is evidently a rarity, and any further specimens that can be secured will form a very valuable addition to the existing collections of crania‡.

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* Parts XII, X, IX, VI, V (i), II, I, respectively.

† *Biometrika*, Vol. I. p. 440.

‡ The Biometric Laboratory would be very grateful indeed for the gift or loan of Chinese crania with locus of origin. Recent attempts to procure such directly have failed owing to the strong Chinese religious prejudices backed by legal ordinances.

The material from which our female means have had to be obtained is much too slender to base any definite conclusions upon, and is only interesting in so far as it may confirm the results obtained for males in the comparisons which follow.

Another weakness which has to be admitted is that this material is probably heterogeneous, being drawn from widely separated districts in China. The catalogues frequently omit to state from which part of China the skulls were obtained: 47 out of our 102 ♂ skulls are simply given as Chinese. Of the rest, 25 are from S.E. China, and 16 from the West Indies—presumably settlers from the S.E. rather than the north. 12 are from Shanghai and Ningpo district, and only two are from as far north as Pekin.

The females are distributed in much the same proportions: 5 from unknown districts, 4 from South China, 1 from Batavia, 1 from Shanghai. Thus we can say that the large majority of those whose origin is known are from the southern part of China. If the unknowns which form less than half our collection are divided in somewhat the same proportions between north and south, our means will be approximately those of the South China type, which is the one we need for comparison with our Burmese. And the standard deviation of the cephalic index B/L for males which is 4·52 ± ·29 encourages us to believe that the sample is not more heterogeneous than our Malayan sample.

Hindus. We turn now to find a Caucasian neighbour from the west. Here again the difficulty arose that in many cases no details beyond the term Hindu, which, as is well known, is not racial but religious, and includes an agglomeration of types of non-Caucasian origin, as well as highly diversified Caucasian types. The term "Hindu" was therefore still less satisfactory than the bare term "Chinese," and the material to hand had to be rigorously sorted out.

The final table has been confined to skulls from the North-Eastern part of India, mainly Bengal, but including a few from round Benares, Allahabad being our western limit; further, all from within this area which had not belonged to persons of the Hindu religion were struck out as being more likely to contribute heterogeneous elements; I also omitted individuals belonging to such castes as I could discover by means of Risley's *Tribes and Castes of Bengal* to be of Dravidian origin or admixture. Again testing the homogeneity of my material by means of the standard deviation of the cephalic index, I obtained the value 4·53 ± ·38 for ♂ Hindus, so that judged by this single constant the heterogeneity is of the same order as that of the Chinese sample.

In this Laboratory there is a short series of skulls of Hindus (9 ♂ and 6 ♀) from Bengal, from the Lower Ganges. These I have measured and pooled with the rest.

The total stands at 69 ♂ and 39 ♀ Hindus and is made up as follows: *Males*, Royal College of Surgeons 36, Barnard Davis' collection 23, Munich 1, Biometric Laboratory 9. *Females*, Royal College of Surgeons 8, Barnard Davis 23, Munich 1, Göttingen 1, Biometric Laboratory 6.

Dravidians. Finally, I took as the material for a Dravidian type the series of 40 skulls brought over by Dr Shortt from Madras in 1878, and belonging chiefly to the Maravar tribe, a Dravidian stock from the Madura district of the Madras Presidency. It was unfortunate that Dr Shortt saw fit to divide his material into two halves, one of which went to the museum of the Anthropological Society of Paris, while the other came to our own Royal College of Surgeons. This second series of 20 was sexed and measured by Flower. Three are given as ♀; three are indeterminate, and if we may class these with the males we have seventeen of the latter. The 20 in France were measured by E. Callamand* by Broca's method, but were unfortunately not sexed. Comparing, however, the means obtained by him with those obtained by Flower I have come to the conclusion that the 20 in France must be either all, or practically all, males, and have put them with Flower's 17, making 37 in all. This has been done with the more confidence as I have seen, on examining the series in the Royal College of Surgeons by the kindness of Professor Keith, that the skulls are very closely alike and form a distinct type with no marked exceptions, so that the means may be obtained with as great accuracy from a score of them as from threescore of a more mixed race.

With regard to the whole of our comparative material, I may say that I am probably more conscious than anyone else of how much work remains to be done on all the races dealt with. The tables of means on pp. 239–240 are such as the data available could supply.

11. Comparison of Burmese with adjacent Racial Series.

We will begin by comparing our largest group, the Burmans proper, with the Malayans, basing of course our judgment of any difference between their respective means for any character, on its relation to the probable error of that difference in samples of the size used†.

We notice how extremely close the two means are in the majority of characters; in fact, a significant difference is only revealed by both sexes in the following points: the Malayan, though brachycephalic, has a smaller cephalic index (B/L) than the

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* "Le crâne des Noirs de l'Inde," *Revue d'Anthropologie*, Vol. I. 4th series, pp. 607–625, 1878.

† The probable errors can be roughly estimated from those given in Table II, where the Burman (Type A) is compared with Type C. The means of this latter type are based on about eight individual measurements for males, and about seventeen for females. As these numbers vary in the different series which we shall compare with our Burmans, so the probable errors will be modified, and in the following proportions:

Males.

For means based on 14 skulls, the probable error of the difference will be ·75 times that given in Table II.
„ „ 21 „ „ „ „ „ ·65 times.
„ „ 54 „ „ „ „ „ ·5 „
„ „ 266 „ „ „ „ „ ·4 „

Females.

For means based on 6 skulls the probable error will be 1·5 times that given in Table II.
„ „ 50 „ „ „ ·75 times.
„ „ 278 „ „ „ ·6 „

TABLE VII*. *Comparative Table.* (*Males.*)

Types:	Burmese ♂			Malayan ♂	Chinese ♂	Maravar ♂ (?)	Hindu ♂
	A	B	C				
Capacity ...	1406·9 (27)	1415·0 (4)	1442·2 (5)	[1424·4 (76)]	[1467·6 (46)]	[1289·7 (17)]	[1319·9 (34)]
F	172·4 (44)	172·4 (8)	175·5 (8)	—	179·6 (16)	175·6 (17)	175·9 (45)
L'	174·2 (44)	176·7 (7)	177·2 (8)	—	179·2 (4)	—	173·2 (9)
L	173·5 (44)	173·8 (8)	176·7 (8)	174·7 (78)	177·1 (84)	175·6 (21)	175·4 (33)
B	143·7 (45)	141·1 (7)	140·4 (8)	142·2 (77)	139·5 (102)	131·4 (38)	132·3 (69)
B'	94·3 (44)	89·7 (8)	90·4 (8)	93·4 (77)	93·9 (49)	93·2 (21)	92·4 (10)
H	136·8 (43)	136·6 (7)	140·1 (8)	136·2 (76)	136·6 (37)	132·5 (38)	132·1 (43)
H'	136·0 (43)	134·7 (7)	139·1 (8)	137·4 (76)	136·9 (69)	—	131·5 (10)
H''† ...	(136·8 (44))	(138·7 (8))	(139·3 (8))	—	136·6 (17)	—	137·8 (23)
OH	117·7 (44)	116·7 (7)	116·9 (8)	118·1 (77)	119·2 (38)	—	111·4 (10)
LB	98·5 (43)	98·8 (8)	100·5 (7)	99·5 (76)	99·1 (66)	98·8 (38)	99·2 (44)
Q'	325·8 (43)	319·5 (7)	319·4 (8)	319·0 (77)	321·2 (39)	—	302·9 (10)
S	363·7 (44)	367·0 (7)	367·4 (8)	365·7 (73)	370·3 (78)	—	363·5 (33)
S_1	128·6 (44)	124·9 (7)	127·9 (8)	—	126·6 (57)	127·0 (21)	127·1 (33)
S_2	124·0 (45)	124·4 (7)	122·5 (8)	—	126·0 (56)	—	126·5 (33)
S_3	111·2 (44)	117·7 (7)	117·0 (8)	—	116·1 (56)	—	110·1 (33)
S_3'	94·7 (44)	98·1 (7)	99·7 (8)	98·0 (73)	98·7 (19)	—	92·3 (9)
U	505·7 (44)	497·7 (7)	503·1 (8)	505·7 (77)	508·5 (82)	488·7 (38)	493·5 (69)
$G'H$ ...	71·4 (39)	68·2 (8)	74·5 (7)	70·1 (73)	71·2 (49)	—	63·8 (9)
GB	101·9 (40)	100·9 (8)	94·3 (7)	99·3 (76)	99·9 (38)	—	95·2 (9)
J	134·0 (40)	131·7 (8)	126·7 (7)	133·2 (75)	131·8 (65)	124·4 (21)	126·8 (32)
NH ...	53·5 (41)	52·4 (8)	55·2 (7)	51·9 (75)	53·1 (54)	46·5 (38)	49·0 (45)
NB	28·1 (41)	26·6 (8)	25·5 (7)	26·1 (73)	25·5 (60)	24·0 (38)	24·2 (43)
DC	22·8 (43)	21·3 (8)	18·7 (7)	—	—	—	—
O_1R ...	44·2 (41)	44·3 (8)	44·4 (7)	—	—	—	42·6 (10)
O_1L ...	43·6 (41)	43·2 (8)	44·1 (7)				
O_1'	39·6 (41)	39·1 (8)	39·4 (7)	39·1 (73)	38·3 (54)	36·6 (17)	37·6 (46)
O_2R ...	35·0 (41)	33·6 (8)	35·6 (7)	33·5 (74)	33·8 (54)	31·7 (38)	32·4 (45)
O_2L ...	35·0 (41)	34·3 (8)	35·9 (7)				
G (G_1) ...	49·9 (37)	49·2 (8)	50·0 (7)	—	54·5 (4)	50·5 (21)	46·4 (5)
G_1'	45·4 (37)	44·8 (8)	46·2 (7)	47·3 (69)	45·8 (32)	—	45·2 (7)
G_2	39·6 (40)	40·1 (6)	40·6 (7)	41·2 (70)	39·2 (35)	—	35·9 (9)
GL	96·6 (39)	93·8 (8)	95·9 (7)	98·2 (76)	97·7 (58)	96·0 (36)	95·1 (39)
fml	36·7 (43)	35·4 (8)	37·3 (8)	35·5 (73)	36·0 (23)	33·9 (21)	35·5 (8)
fmb ...	30·7 (42)	30·6 (8)	30·8 (8)	30·0 (74)	30·9 (23)	28·2 (21)	27·4 (8)
$100\,B/L'$ ...	82·4 (44)	80·0 (7)	79·4 (8)	—	77·8 (13)	—	76·2 (9)
$100\,B/L$ ...	82·9 (44)	80·4 (7)	79·5 (8)	81·7 (77)	78·9 (73)	74·6 (21)	75·8 (33)
$100\,H/L'$ ...	78·2 (43)	76·3 (7)	78·7 (8)	—	—	—	75·5 (9)
$100\,H/L$ ...	78·5 (43)	76·7 (7)	78·9 (8)	78·2 (76)	77·4 (21)	75·2 (21)	75·6 (9)
$100\,H'/L$ ...	(78·4 (43))	(77·5 (7))	(78·7 (8))	—	77·4 (35)	—	75·8 (10)
$100\,H''/L$ ...	(78·8 (44))	(79·8 (8))	(78·8 (8))	—	78·5 (17)	—	78·3 (23)
$100\,B/H$ ...	105·8 (43)	104·9 (7)	101·0 (8)	104·6 (76)	103·2 (37)	(99·1 (21))	100·0 (43)
$100\,B/H'$ ...	(105·7 (43))	(104·8 (7))	(100·9 (8))	—	102·4 (46)	—	99·8 (10)
$100(B-H)/L$	+4·0 (43)	+2·6 (7)	+0·1 (8)	+3·2 (76)	+1·3 (69)	(−0·6 (38))	−0·2 (44)
$100\,G'H/GB$	69·8 (38)	67·8 (8)	79·0 (7)	70·7 (73)	71·1 (36)	—	68·0 (8)
$100\,NB/NH$	52·7 (41)	51·0 (8)	46·3 (7)	50·4 (73)	48·9 (53)	51·7 (38)	49·7 (43)
$100\,O_2/O_1R$	79·1 (41)	75·9 (8)	80·3 (7)	—	—	—	77·2 (9)
$100\,O_2/O_1L$	80·0 (41)	79·4 (8)	81·5 (7)				
$100\,O_2/O_1'$ ...	(88·4 (41))	(87·7 (8))	(91·1 (7))	86·0 (73)	88·4 (54)	84·4 (17)	86·3 (45)
$100\,G_2/G_1$ ...	79·8 (37)	82·2 (6)	81·3 (7)	—	78·4 (4)	—	77·8 (5)
100 *fmb*/*fml*	83·8 (42)	86·6 (8)	82·9 (8)	84·6 (73)	85·9 (23)	83·3 (21)	77·3 (8)
$P\angle$ ...	86°·0 (39)	85°·9 (8)	84°·0 (7)	—	84°·9 (17)	—	86°·1 (9)

* Square brackets outside entries and frequencies indicate that the measurements have been taken by possibly non-comparable methods. Round brackets outside entries and frequencies indicate that the values may be only approximate, not having been obtained directly from individual crania; for example in the case of indices the figures give the ratio of the means of two measurements.

† H'' is here used to denote Barnard Davis' measure of cranial height, namely the greatest height above the plane of the foramen magnum. For the Burmese I have merely read the values off from the type contours.

TABLE VIII*. *Comparative Table.* (*Females.*)

Types:	Burmese ♀			Malayan ♀	Chinese ♀	Hindu ♀
	A	B	C			
Capacity	1267·9 (27)	1232·4 (11)	1231·4 (7)	1288·9 (16)]	[1265·0 (4)]	[1135·1 (8)]
F	166·5 (39)	163·2 (17)	169·4 (18)	160·0 (1)!	170·3 (3)	167·4 (14)
L'	167·4 (38)	164·7 (17)	170·7 (18)	—	—	169·2 (6)
L	166·5 (38)	163·8 (17)	170·2 (18)	167·6 (55)	172·3 (8)	174·2 (31)
B	138·0 (39)	135·6 (17)	130·6 (18)	136·6 (55)	132·3 (11)	128·4 (39)
B'	90·7 (39)	88·5 (16)	88·7 (18)	93·0 (18)	92·0 (2)	87·9 (8)
H	131·4 (39)	131·9 (17)	129·9 (18)	129·0 (6)	130·2 (4)	126·3 (14)
H'	130·4 (39)	130·8 (17)	129·4 (18)	130·1 (38)	130·0 (3)	126·8 (7)
H''†	(132·1 (39))	(131·7 (17))	(129·8 (18))	133·5 (18)	136·1 (5)	130·6 (23)
OH	111·7 (38)	111·9 (17)	110·1 (17)	112·6 (12)	112·5 (2)	107·6 (7)
LB	93·9 (39)	94·7 (17)	95·7 (17)	94·6 (14)	97·3 (6)	92·8 (15)
Q'	312·0 (37)	308·8 (17)	302·1 (16)	312·1 (12)	300·5 (2)	297·0 (7)
S	350·2 (39)	344·0 (17)	350·9 (18)	351·8 (52)	356·6 (7)	359·1 (30)
S_1	121·9 (39)	119·5 (17)	122·6 (18)	122·9 (45)	120·5 (7)	124·6 (31)
S_2	120·3 (39)	119·6 (17)	119·6 (18)	123·4 (44)	126·5 (7)	124·6 (31)
S_3	107·9 (39)	104·9 (17)	108·7 (18)	107·9 (45)	112·4 (6)	109·8 (30)
S_3'	92·8 (39)	92·0 (17)	91·4 (18)	92·5 (6)	94·0 (1 !)	92·8 (6)
U	485·1 (39)	476·9 (16)	479·6 (18)	489·1 (35)	488·4 (10)	482·3 (38)
$G'H$	66·0 (34)	66·1 (17)	63·9 (16)	65·1 (18)	66·0 (2)	60·8 (7)
GB	96·7 (34)	93·6 (15)	91·8 (15)	95·7 (12)	85·0 (1 !)	89·5 (6)
J	126·7 (35)	121·9 (15)	118·7 (13)	125·6 (35)	122·7 (7)	119·3 (28)
NH	50·6 (34)	49·4 (17)	47·3 (16)	47·3 (12)	49·4 (5)	44·9 (14)
NB	26·8 (34)	25·0 (17)	24·0 (14)	25·5 (15)	23·6 (5)	22·6 (14)
DC	20·9 (37)	19·7 (17)	18·8 (15)	—	—	18·4 (6)
O_1R	42·0 (34)	42·2 (17)	42·2 (15)	41·0 (1)!	—	38·6 (5)
O_1L	41·8 (34)	41·5 (14)	42·2 (15)		—	39·3 (6)
O_1'	38·0 (34)	37·0 (17)	38·0 (16)	37·2 (11)	36·4 (5)	36·1 (15)
O_2R	34·5 (34)	34·0 (16)	34·2 (15)	31·9 (12)	35·4 (5)	32·0 (15)
O_2L	34·6 (34)	34·6 (15)	33·2 (16)			
G_1	46·7 (26)	47·0 (14)	46·4 (13)	48·6 (5)	—	44·3 (5)
G_1'	42·7 (29)	43·1 (16)	42·8 (13)	45·2 (6)	43·0 (1)!	41·1 (5)
G_2	38·0 (32)	36·4 (15)	38·1 (14)	39·0 (12)	35·5 (2)	34·8 (6)
GL	90·7 (34)	91·5 (17)	89·2 (16)	95·1 (20)	96·0 (5)	89·6 (13)
fml	34·7 (39)	35·1 (16)	35·1 (18)	33·0 (6)	35·0 (1)!	34·2 (6)
fmb	29·4 (39)	30·0 (16)	29·8 (17)	28·5 (6)	28·0 (1)!	28·7 (6)
100 B/L'	83·1 (38)	83·0 (17)	76·6 (18)	—	—	76·3 (6)
100 B/L	83·1 (38)	82·9 (17)	76·9 (18)	81·8 (54)	76·4 (8)	73·8 (31)
100 H/L'	78·0 (38)	79·5 (17)	75·9 (18)	—	—	74·7 (6)
100 H/L	78·5 (38)	79·9 (17)	76·1 (18)	80·1 (6)	74·4 (1)!	74·9 (6)
100 H'/L	(78·3 (39))	(79·8 (17))	(76·0 (18))	78·2 (38)	76·1 (2)	75·4 (7)
100 H''/L	(79·3 (39))	(80·4 (17))	(76·3 (18))	79·1 (17)	78·9 (5)	74·3 (23)
100 B/H	105·8 (39)	103·7 (17)	101·0 (18)	107·5 (6)	101·8 (4)	102·2 (14)
100 B/H'	(105·8 (39))	(103·7 (17))	(100·9 (18))	104·1 (38)	101·3 (3)	101·4 (7)
100 $(B-H)/L$	+3·9 (38)	+2·4 (17)	+0·9 (18)	+4·9 (37)	+1·1 (6)	+1·5 (15)
100 $G'H/GB$	68·6 (34)	69·8 (15)	70·2 (15)	67·2 (12)	68·3 (1)!	67·5 (6)
100 NB/NH	52·7 (34)	50·7 (17)	50·5 (14)	53·6 (12)	48·1 (5)	50·4 (14)
100 O_2/O_1R	82·4 (34)	80·4 (16)	81·2 (15)	78·0 (1)!	—	83·3 (5)
100 O_2/O_1L	82·8 (34)	83·0 (14)	78·8 (15)		—	83·5 (6)
100 O_2/O_1'	(91·1 (34))	(93·0 (16))	(87·4 (16))	86·0 (11)	97·3 (5)	88·7 (15)
100 G_2/G_1	80·5 (25)	76·8 (12)	83·0 (13)	78·6 (5)	—	78·9 (5)
100 fmb/fml	84·8 (39)	85·9 (16)	85·0 (17)	86·5 (6)	80·0 (1)!	84·1 (6)
$P\angle$	85°·8 (33)	84°·0 (17)	86°·4 (16)	81°·2 (6)	—	83°·6 (6)

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* See note * p. 239.

† See note † p. 239.

Burman; he has also a smaller orbital index (O_2/O_1'), and an examination of the means of the absolute measurements on which this index is based shows that this is due to the greater height of the Burman orbit, the width being about the same in the two races. The nose again is larger in the Burman.

Both sexes exhibit greater height of nose (NH); this is accompanied by greater width (NB) a feature however which is much more marked in the male than in the female, so that while the resultant index for female Burmans is not markedly different from that for female Malayans, this is not true of the male Burman whose nasal index (NB/NH) is considerably higher than the male Malayan's. Both sexes agree in giving a longer palate to the Malayan (G_1'), and as this is accompanied by greater face length (GL), i.e. greater projection of alveolar point from basion, we are led to infer that the Malayan is the more prognathous. Unfortunately, our data supply no measurement of the profile angle ($P\angle$) for male Malayans; for females we have it in six cases only, but these give the Malayan a considerably smaller angle, the difference being about six times the probable error. It will be remembered that the Burmese are frequently characterised by sub-nasal prognathism: of true prognathism the examples are rare among them.

It has frequently been remarked in the study of other races, that racial characteristics are more emphasised in the male than in the female. We shall accordingly be quite prepared for the discovery of characters in which the females of our two races manifest no significant differences, while males do so. Thus we find in the male Burman a considerably greater transverse arc (Q'—measured from auricular point to auricular point vertically over the top of the head) than in the Malayan. The difference is about five times the probable error. An examination of the figures shows that this is not due to greater height of skull above the auricular axis (OH). It may be due in part to greater width at this point, but our two width measurements, namely, that for maximum width (B) and for width of forehead (B'), show no difference that can be regarded as significant. We have, however, noted in the Burman the heavy ridges carried back from the zygomatic arches above the auricular passages. It is possible that these have contributed somewhat to the length of the curve which crosses over them. Or, what is perhaps still more likely, the room for personal equation in determining the "auricular point" according to Frankfurt definition may have caused Dr Emil Schmidt, who measured the Malayan ♂ series, to select his terminals for this arc rather higher than I have done, thus shortening his transverse arc. Again, in the male Burman, we find a shorter occipital chord (S_3') than in the Malayan. Whether the corresponding arc from lambda to opisthion (S_3) is also less, and, accordingly, how the occipital indices of the two races compare together, we are unable to discover, since the arc measurement was one that Dr Schmidt did not take. Another racial difference is to be found in the male in the greater Burman breadth of face (GB) as measured by the width of the upper maxillary bone at the lower end of the malar-maxillary suture; associated with this is the greater nose-breadth which has already been noted. The palate, on the other hand, is wider in the Malayan, and we have already seen that it is longer and more projecting. The heavier development about the nose in the Burman, and about the mouth in the Malayan, would serve to mark a distinct difference between the appearance of the two races.

Finally, we find that the foramen magnum is significantly larger in our male Burman skull, both longer and broader.

In one point only do the females of the two races show a significant difference that is not confirmed by the data for the males. The Burman women have rather narrower heads than the Malayan, both as to maximum breadth (B) and breadth of forehead (B').

Burmans and Chinese. The slenderness of our data for ♀ Chinese makes it idle to use them for detailed comparison with the Burmans. I will therefore confine myself to a consideration of racial differences as shown by the males of the two series, pausing, however, to remark that the ♀ figures, for the little they are worth, are in quite good agreement as to the main racial characters.

As in the case of the Malayans, so with the Chinese, one is struck by the number of characters in which no significant difference is indicated between the two races. In the main proportions of the skull, however, the Chinaman is more markedly differentiated from the Burman: his skull is both longer (L) and narrower (B), the effect of both differences being to bring the Chinaman within the mesocephalic category (B/L) and to make his skull capacity somewhat greater. In height of skull (H) as also in auricular height (OH) there is no significant difference between the two. Taking these main linear measurements into consideration, we are not surprised to find a shorter transverse arc (Q') in the Chinaman, and with this a longer sagittal arc (S), whose greater length is due to greater occipital development: the distance from nasion to lambda ($S_1 + S_2$) is identical in the two means; from lambda to opisthion is considerably greater in the Chinese, whether arc-length (S_3) or chord-length (S_3') be taken. Unfortunately, we have not the data for calculating the occipital indices for the Chinese, to compare with those of the Burmans.

Turning now to the facial characters, we find the Chinese have definitely smaller orbits, both as to width (O_1') and height (O_2), the proportions, however, (O_2/O_1'), remaining approximately the same.

The great zygomatic breadth which was found among the Burmans is here reduced, though not to a very large extent, the degree of difference being about three times its probable error. The breadth of the upper maxillary (GB) is reduced in about the same proportion; and, height of face remaining practically unchanged, the upper-face index ($G'H/GB$) undergoes a similar reduction.

The width of nose (NB) is, however, reduced to a much greater extent, and since the height again is not changed significantly, the proportions of the nose (NB/NH) bring the Chinese into the middle nasal category, in contrast to the platyrhine Burmans. In the remaining features of which our data supply us with a measure—palate, foramen, etc.—we find no racial divergence indicated that may not be accounted for by random sampling.

Burmans and Hindus. We next compare our Burmans with the Hindu material that has been gathered together. Here we find more characters exhibiting differences which cannot be other than racial. The major measurements reveal at once the comparative smallness of the Hindu skull, which, though not losing anything in length* (L), shows a markedly smaller maximum breadth (B) especially for males, and less height (H), and, as we should expect, smaller capacity (C). The height above the auricular points (OH), as also the arc (Q') connecting these two in the vertical plane, are correspondingly reduced; it may be noted, however, that the breadth of forehead (B'), unlike the maximum breadth, remains practically the same. These figures bring out the fact that the cranial box of the Hindu differs from that of the Burman not only in size but in shape. Thus the Hindu skull is mesocephalic (B/L) against the brachycephaly of the Burman; its relative height and length (H/L) bring it to the border-line between hypsicephaly and orthocephaly, and its mean maximum breadth (B) is exactly equal to its mean height (H) while the Burman breadth exceeds the height.

When we examine the features of the Hindu face, we find again a considerable diminution in size. It is less high ($G'H$) than that of the Burman, less broad, both as regards zygomatic (J) and maxillary (GB) breadth; the nose is both shorter (NH) and narrower (NB); so also are the orbits (O_1' and O_2). Its palate is rather narrower, but is approximately as long as that of the Burman; the difference, however, is not so great as to put them in different palate categories.

The finer proportions of the Hindu nose classify it as mesorhine, one class higher than that of the Burman nose; and the orbit is less round.

All these racial characteristics are exhibited by both male and female, in their mean values.

As before, we find some racial differences which can be considered significant in the males alone: a smaller horizontal circumference (U) in the Hindu, and a

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* It will be noted that this inference is based upon the data for (L) from Hindu skulls alone. The bulk of the data for this character for Hindu females was obtained from the catalogue of Barnard Davis' collection, and a comparison of this mean measurement both with the corresponding male value, and with that of the females in our Hindu group at the Biometric Laboratory, suggests some doubt as to the accuracy of these particular figures.

Mean maximum length of Hindu skulls

	♂	♀
B. Davis' Collection	176·1 (23)	175·7 (23)
Biometric Laboratory Collection	172·9 (9)	168·6 (6)

Our own series is too small for any real stress to be laid upon the means obtained from them alone; but that the mean length of the male and female head should be practically the same in any one race, as the figures from the Barnard Davis' catalogue would indicate, is a condition paralleled by the constants of no other race that has been studied: the average reduction for the female mean is about one twenty-fourth of the male value (mean of 20 races).

narrower foramen magnum both in its absolute measurement (*fmb*) and in relation to its length (*fmb*/*fml*).

Burmans and Dravidians (*Maravars*). The number of characters which are available for comparison are somewhat fewer in number in the case of the Maravar, and, as already remarked, we have only data for male skulls at our disposal.

As in the case of the Hindus, we are confronted with considerable divergencies from our Burman constants, and frequently in the same direction.

Again we find a smaller capacity (C) and height (H) and a much smaller maximum breadth (B), while the maximum length (L) and minimum forehead breadth (B') are approximately the same; again a reduction in all the cephalic indices (B/L, H/L, B/H); also in the horizontal circumference (U), zygomatic breadth (J) and height of nose (NH), though to a greater extent than in the Hindu; the width of the nose (NB) is also reduced, but not quite so much as in the Hindu, so that the nasal index of the small nose of the Maravar does not differ significantly from that of the Burman's large one. The orbit (O_1', O_2) too is small in comparison with the Burman's and flatter; still flatter than the Hindu orbit. Unlike the Hindu, the Maravar has a foramen magnum which is shorter (*fml*) than the Burman's as well as being narrower (*fmb*), so that its proportions do not differ materially from those of the latter.

In examining columns of means such as are supplied by the comparative tables here discussed, it is possible to gain some idea of the extent to which a cranial character of one race is modified in another race. But is it possible to answer from them the much larger and more important question of how nearly the various races characterised by the multiplicity of racial constants are related together? One may certainly gain some vague appreciation of relative affinities between the various races, as here one gathers that the Burman seems more nearly related to the Malayan and Chinaman, than either to the Hindu or the Maravar.

But to which of the first is he more closely akin? to which of the last?

Would it be possible to get a measure of the relationship by taking a large number of characters in combination instead of considering each individually?

And if so, could we by this means test whether the larger group and the two smaller groups of our Burmese series belong indeed to one and the same population? or whether we were justified in considering them distinct?

These are such problems as must often have troubled other craniologists, and I have to thank Professor Pearson for providing a provisional solution*, in the form of a coefficient which shall express numerically to a first approximation the degree of relationship existing between any two racial groups.

Before passing, however, to the consideration of this coefficient, there is one feature in our comparative tables of which I have reserved mention until now;

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* Provisional in the sense that a more complete definition will only be possible when wider study of the type constants of races provides material for the determination of inter-racial variabilities and correlations.

this is the Index $(B-H)/L$, which measures the excess of breadth over height in proportion to length, and gives us the following suggestive order:

Values of 100 $(B-H)/L$ *for Burmese and neighbouring Races.*

Malayan Women	+4·9	(37)
Burmese Group A, Men	+4·0	(43)
,, ,, Women ...	+3·9	(38)
Malayan Men	+3·2	(76)
Burmese Group B, Men	+2·6	(7)
,, ,, Women ...	+2·4	(17)
Hindu (Bengal) Women	+1·5	(15)
Chinese Men	+1·3	(69)
Chinese Women	+1·1	(6)
Burmese Group C, Women ...	+0·9	(18)
,, ,, Men	+0·1	(8)
Hindu (Bengal) Men	−0·2	(44)
Maravar (Dravidians) Men ...	[−0·6]	(38)*

In this table it will be noted that the Hindu women are placed considerably higher than the Hindu men, but having already seen that the data obtained for Hindu females may not be altogether reliable, it may be well if we reserve any conclusions that might be drawn from this until their place in the list has been confirmed or altered by further evidence.

With this exception, the table gives a not unreasonable order: the Burman proper (Group A) is nearest to the Malayan, between that and the Chinese; Group C (Karens?) lies between Chinese and Hindus; and Group B is between Groups A and C in order, which would be expected if the surmise that it is hybrid be correct.

The Dravidian type is the most remote of any, from the Burman type.

Inserting these various values into the table of this index for a long series of races given by Thomson in the Moriori paper† we get the following:

Values of 100 $(B-H)/L$.

Würtemberg Women	+9·6	Congo Men	+2·6
,, Men	+9·5	Burmese (Group B) Men	+2·6
Bavarian Women	+9·2	,, ,, Women	+2·4
,, Men	+9·0	Australian Women	+2·4
English Women (Moorfields)	+7·9	Long Barrow Women	+2·3
French Men	+7·2	,, ,, Men	+2·2
English Men (Moorfields)	+7·1	Fuegian Men	+2·0
,, Women (Whitechapel) ...	+5·6	Congo Women	+1·9
Malayan Women	+4·9	Theban Mummies, Women	+1·8
English Men (Whitechapel)	+4·2	Hindu (Bengal) Women	+1·5‡
Burmese (Group A) Men	+4·0	Chinese Men	+1·3
,, ,, Women	+3·9	,, Women	+1·1
Moriori Men	+3·5	Prehistoric Egyptian (Naqada) Women	+1·0
,, Women	+3·2	Aino Women	+0·9
Malayan Men	+3·2	,, Men	+0·9

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* As the individual measurements were not given for the 21 skulls measured by Callamand, this index for the Maravars has had to be calculated from their mean value and from the individual measurements of Flower's seventeen skulls.

† *Biometrika*, Vol. XI. p. 95.

‡ See remark higher upon this page.

Values of 100 $(B - H)/L$—*(continued)*

Gaboon Men (1880)	+0·9
Burmese (Group C) Women	+0·9
Maori Men	+0·7
„ Women	+0·5
Gaboon Women (1864)	+0·5
Coptic Women	+0·5
Theban Mummies, Men	+0·4
Burmese (Group C) Men	+0·1
Gaboon Men (1864)	+0·0
Kaffir Men	+0·0
Hindu (Bengal) Men	−0·2
Coptic Men	−0·2
Australian Men	−0·2
Prehistoric Egyptian (Naqada) Men	−0·3
Gaboon Women (1880)	−0·4
Maravar (Dravidian) Men	−0·6
Zulu Men	−0·7
Angoni Men	−1·2
Negro Men (N. Africa)	−1·3

Here we find the Malayan women, highest of our new series, intruding into the European group, which previously remained exclusive, at the top of the list: that they really are higher up in this respect than English Men (Whitechapel) is not absolutely proved, since the difference between them, 0·7, would not be equal to three times its probable error*. In any case, both Malayan series and both Burman series approach more nearly to the Modern European values than any of the non-European races cited with the exception of the Moriori; while the Burmese hybrids do not lag far behind. The Dravidian series finds its place towards the bottom of the list.

Another feature which deserves special notice is the group of nasal measurements and indices. The comparative study of the nasal bridge to which Benington† devoted some attention in our own Laboratory was later made the subject of extensive research there by Kathleen V. Ryley and Julia Bell‡.

To the list of races arranged in order for values of the Simotic and Dacryal Indices§ I can now add my three Burmese Groups.

Simotic and Mesodacryal Indices. Mean Values.

Simotic Index SS/SC				Mesodacryal Index DS/DC			
♂		♀		♂		♀	
Congo	26	Nubians	19	Congo	39	Congo	37
Moluccas	27			Philippinos	42	Nubians	40
Gaboon (1880)	29	Gaboon (1880)	23	Borneo	42	Borneo	41
Sumatra	29	Congo	26	Moluccas	43	Javanese	43
Celebes	30	Gaboon (1864)	28	Gaboon (1864)	43	Gaboon (1880)	44
Borneo	30			Gaboon (1880)	43	Gaboon (1864)	44
Philippinos	30	Philippinos	29	Malays	44	Philippinos	45
Gaboon (1864)	31			Celebes	45		
Malays	32			Sumatra	45		
Burmans	33	Burmans	30	Javanese	45		
Javanese	34	Javanese	33	Burmans	45	Burmans	45
Nubians	34	Borneo	35	Nubians	48		
Ainos	43			Ainos	50		
Veddahs	44	Veddahs	37	Burmese Group B (? Hybrids)	50	Burmese Group B (? Hybrids)	49
Egyptians	44	Egyptians	37	Hindus	55	Hindus	49
Burmese Group B (? Hybrids)	45	Burmese Group B (? Hybrids)	39	English	57	English	53
Hindus	45	Hindus	39	Veddahs	57	Egyptians	53
English	51	English	47	Egyptians	58	Veddahs	59
Burmese Group C (? Karens)	54	Burmese Group C (? Karens)	49	Burmese Group C (? Karens)	62	Burmese Group C (? Karens)	60

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* Compare Table II, p. 221.

† *Biometrika*, Vol. VIII. pp. 315–320.

‡ *Ibid.* Vol. IX. pp. 391–445.

§ *Ibid.* Vol. IX. p. 404.

Here again we find our Burmans in close proximity to the Malayan Groups of the West Indian Archipelago. For each index of the nasal bridge the Burmese Group C gives a higher value than any other race for which it has been obtained, English and Egyptian included. It is regrettable that at present we have not these indices for the bridge of the nose in the case of more Caucasian races, nor for Chinese or Dravidians. It is clear that a complete table—especially with rather longer series for some of the races here provided—would be of the greatest suggestiveness. We need also to discover, if possible, the values for palaeolithic and neolithic man. The nasal bridge index and the differential cephalic index $[(B-H)/L]$ seem to me perhaps the more capable than any other cranial indices of providing adequate racial characterisations.

12. COEFFICIENT OF RACIAL LIKENESS.

We turn now to the problem already stated (p. 244) of obtaining a measure of the relationship between different races. For the general solution of this problem, an extensive knowledge of the degree of correlation between different cranial characters is required. We have two types of correlation in craniometry.

First, the intra-racial, that is, the correlation of characters in the individual, the measure of this relation being obtainable from a group of individuals to which he belongs. A good deal is known about such correlations already; more will be known when the long series of Egyptian skulls—1800 of one period—now in hand in the Biometric Laboratory, is completely reduced. But we do know quite enough to assert that the correlation is never very high between cranial characters which do not have any portion in common, and which are not right and left measurements of homologous characters. It is indeed often wholly negligible.

The *second* form of correlation is that between the *means of characters* in different races, and to obtain a measure of this we must have a large number of races whose means have been established. This may be conveniently termed inter-racial correlation to distinguish it from intra-racial.

There are at present, however, so few races for which we have a knowledge of the mean values of forty or more characters, based upon a series of adequate dimensions, that the principal inter-racial correlations remain largely unknown. We do know that inter-racial and intra-racial correlations differ considerably in a number of cases: thus, for example, the value of the relation between orbital and nasal index in the individuals of one race is considerably different from that between their mean values in a group of races. There is, however, on the basis of our present knowledge, no reason to believe that inter-racial correlations are likely to be at all intense. Accordingly, when we consider the difference in cephalic index of two races, and then proceed to consider the difference in zygomatic breadth, any inference of significant difference in the first will not be weakened by the fact that we find a significant difference in the second: in other words, the two judgments may be considered independent in the present state of our knowledge of inter-racial correlation. That is to say, we shall obtain at least a first approximation to an appreciation of racial resemblance or racial difference by treating each character as an independent judgment, i.e. omitting the influence of correlation.

Accepting this principle as a *temporary* working condition, we can reach for every pair of races, characterised by m measured characters, a single numerical coefficient; and can at the same time give its probable error. This coefficient will measure the probability that the two groups of crania under consideration are samples of one and the same population: it may be termed the *Coefficient of Racial Likeness* (C.R.L.). Such a coefficient will enable us to compensate for the smallness of a cranial series by measuring a large number of characters. The advantage to the craniologist will be considerable. This coefficient is defined as follows:

Let m characters be measured, and let the sth character in the first race have M_s for mean and σ_s for standard deviation, these two constants being based on n_s individual measurements. Let the corresponding characters for the second race be M_s', σ_s' and n_s'.

Then if the two races were really samples of the same population, the following expression should be sensibly zero:

$$\frac{1}{m}\left\{\frac{S\,(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}}\right\} - 1,$$

and the probable error of its deviation from zero will be $\cdot 67449/\sqrt{2m}$.

Now, for most cranial samples we have not adequate numbers to enable us to determine σ_s and σ_s': but there is not a very great difference in variability between one race and another. It will therefore suffice, for a first approximation, to put $\sigma_s = \sigma_s' =$ the value found for the standard deviation of this character in the longest series of homogeneous crania available. It is only needful to use the appropriate σ for the appropriate sex.

Thus our coefficient of racial likeness becomes:

$$\text{C.R.L.} = \frac{1}{m}\,S\left\{\frac{n_s n_s'}{n_s + n_s'}\left(\frac{M_s - M_s'}{\sigma_s}\right)^2\right\} - 1 \pm \frac{\cdot 67449}{\sqrt{2m}}.$$

I have calculated the C.R.L. for the Whitechapel Plague Pit Crania of the Biometric Laboratory which were measured by Macdonnell*; also for the Long Barrow British of the Oxford University Museum as measured by Schuster†. I have also compared the Whitechapel Crania with the measurements of the skulls of French prisoners dying in Germany in 1871*. I have compared Bavarians* with Würtembergers* and Prehistoric Egyptians as measured by Fawcett‡ with Congo negroes measured by Crewdson Benington§, using for these comparisons male skulls only.

The table on p. 249 gives us at once a much better appreciation of the degree of resemblance between the seven races they represent than could have been obtained from a comparison of seven columns of means for twenty to thirty individual characters.

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* *Biometrika*, Vol. III. p. 208.

† *Ibid.* Vol. V. p. 104.

‡ *Ibid.* Vol. I. p. 426.

§ *Ibid.* Vol. VIII. p. 298.

The following are the results:

Races ♂	No. of Characters	C.R.L.	Probable Error
17th Century Londoners and Long Barrow British ...	23	3·8	±·10
Bavarians and Würtembergers	24	12·1	±·10
17th Century Londoners and Modern French	28	24·5	±·09
Prehistoric Egyptians and Congo Negroes	27	27·3	±·09

Here we find the justification of Macdonnell's statement that the Long Barrow British have a close resemblance to seventeenth century Londoners*.

It is also clear that the critics who asserted a large negroid element in our Prehistoric Egyptians had little basis for their criticism. French and English are shown to be almost as far apart racially as Egyptians and Negroes. And if Bavarians and Würtembergers are not as racially distinct as French and seventeenth century Londoners, they are much further apart than are the latter from the Long Barrow British.

I turn now to the question with which this paper is more immediately concerned, and see how far this method will throw light upon my Burmese mixed population, which mere *appreciative* analysis of characters led us to differentiate into three types: Type A, the Burman proper; C, another type—possibly Karen; and Type B, a supposed hybrid.

Races	Sex	No. of characters	C.R.L.	Probable error
A and C	♂	30	3·12	±·09
,,	♀	30	6·44	±·09
A and B	♂	30	0·47	±·09
,,	♀	30	1·16	±·09
C and B	♂	30	1·30	±·09
,,	♀	30	2·87	±·09

These results indicate that A and C are as racially distinct as Long Barrow British and seventeenth century Londoners, but not as distinct as Bavarians and Würtembergers; they also show that both A and C are closer to the supposed hybrid than to each other.

The method therefore provides a numerical evaluation of a purely appreciative judgment, a judgment, however, which anyone who had examined a large number of cranial groups from the racial standpoint would be likely to make.

Before making use of this coefficient to elucidate the problem of the relative affinities between the Burmese groups and the adjacent racial series with which they have been compared, it will be well to measure the relation of these groups to one another.

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* *Biometrika*, Vol. III. pp. 243–4; Vol. v. p. 104.

Race	Sex	No. of characters	C.R.L.	Probable error
Chinese and Malayan	♂	30	3·15	±·09
,, ,,	♀	30	2·15	±·09
Chinese and Hindu	♂	30	15·10	±·09
,, ,,	♀	30	1·50*	±·09
Malayan and Hindu	♂	30	19·50	±·09
,, ,,	♀	30	11·85	±·09

* See remark on p. 245.

These results show several interesting points:

1. The Chinese are closely related to the Malayan.

2. Neither race is closely related to the Hindu, but the Chinese is nearer to a Caucasian type than the Malayan.

3. In so far as we can trust our rather slender data for women, the women are rather more alike than the men, that is, characteristic racial features are more pronounced in man than in woman.

Finally, the coefficient gives an answer to the fundamental problems of our Burmese series. The first is: Do either A or C or the assumed hybrid B approach the Caucasian or Dravidian types?

Race	Sex	No. of characters	C.R.L.	Probable error
Burmese A and Hindu	♂	30	23·94	±·09
,, ,,	♀	30	16·36	±·09
Burmese A and Maravar	♂	30	31·15	±·10
Burmese C and Hindu	♂	30	6·30	±·09
,, ,,	♀	30	1·44	±·09
Burmese C and Maravar	♂	30	7·92	±·10
Burmese B and Hindu	♂	30	3·79	±·09
,, ,,	♀	30	8·56	±·09
Burmese B and Maravar	♂	30	5·59	±·10

This table shows that Type A, the Burman proper, is widely divergent from both Caucasian and Dravidian races. The Burmese Type C (possibly Karens) is less markedly distinguished, though quite adequately distinguished, from the Caucasian and Dravidian types in the male, and may come closer to the Hindu in the female*. But on the whole, I think, we can safely say that our Type C is not a Hindu or Dravidian type. Except in the case of the females of Burmese Type B and Hindu* we see that the rule remains that the male shows the more marked racial characters.

From the more reliable male figures we see that Hybrid B stands closer in order of resemblance than either of the purer types A and C.

Lastly, we turn, after rejecting the Caucasian and Dravidian as direct sources of C, to the inter-relation of the Burmese series to Malayans and Chinese. Here

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* It would not be wise to attach too much importance to exceptional results obtained from the Hindu female data on the ground already stated on p. 245.

it is seen at once that much higher orders of racial resemblance are reached. The following table presents the closest results on the whole that have yet been obtained:

Race	Sex	No. of characters	C.R.L.	Probable error
Burmese A and Malayans	♂	30	2·69	±·09
,, ,,	♀	30	1·71	±·09
Burmese A and Chinese	♂	30	6·38	±·09
,, ,,	♀	30	2·66	±·09
Burmese B and Malayans	♂	30	2·35	±·09
,, ,,	♀	30	4·47	±·09
Burmese B and Chinese	♂	30	0·16	±·09
,, ,,	♀	30	0·53	±·09
Burmese C and Malayans	♂	30	0·03	±·09
,, ,,	♀	30	2·39	±·09
Burmese C and Chinese	♂	30	0·29	±·09
,, ,,	♀	30	1·54	±·09

Here the main results are at once apparent. Burmese A are closer for both sexes to the Malayans than to the Chinese. Burmese C are closer to the Chinese than to the Malayans. And again we find Burmese B (Hybrids) closer to both Malayans and Chinese than either of the purer types is. Their greater resemblance to both prototypes emphasises the probability that they are hybrids: it is a case of throwing back after mixture.

It seems reasonable accordingly to conclude as follows:

1. The Burmese population in the neighbourhood of Moulmein consists of at least two distinct races and a probable hybrid race.

2. One of these races, the pure Burman, is more closely allied to the Malayan than to the Chinese.

3. The second of these, probably the Karen, is more closely allied to the Chinese than to the Malayan; and as the Chinese is considerably closer to the Caucasian than is the Malayan, this section of the population, although quite distinct from the Caucasian, is somewhat closer to that type than the Burman proper.

4. The group we have spoken of as hybrid is closer to *both* Malayan and Chinese than either purer element of the population is to Malayan or Chinese. This is a not improbable result of hybridisation.

I have to thank Miss Noel Karn of the Biometric Laboratory for her help in working out the troublesome calculations on which these conclusions are based.

13. THE ASYMMETRY OF THE SKULL.

The fact of cranial asymmetry is of course a commonplace to anyone who has handled skulls, and reference has already been made* to the pitfalls it prepares if we assume that the "horizontal plane" as determined by certain features of the skull will, in the individual, be perpendicular to the "sagittal plane" as determined by others, and both of them perpendicular to the "vertical plane."

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[*] See p. 202.

I have made an attempt to examine in some degree the frequency with which certain kinds of asymmetry occur, and the extent to which they tend to be associated with one another.

Dr D. E. Derry, whom I have to thank for giving me the benefit of his anatomical experience in examining my series for anomalies, pointed out the frequent asymmetry in the occipital region, and suggested that this was usually accompanied by a slightly greater prominence of the frontal bone on the opposite side. I have, accordingly, made observations as to frontal and occipital asymmetry in this series of skulls, and submitted the data to mathematical processes, to see what association, if any, existed between the two.

Most of the observations made were merely appreciative: if some quantitative measure of the degree of asymmetry could be taken, the association could of course be determined with much greater accuracy.

The skull was held in such a manner that the frontal bosses and that portion of the squama occipitalis which lies above the region of the inion were silhouetted against the light, care being taken that what seemed to be asymmetry was actually so, and was not due to any fault of position. Three categories were made, classifying the material according to whether there was greater prominence on the left or right, or equally great on both. The following are the numbers which fell into the various classes.

	Greater on right	Equal	Greater on left	Total
Occipital prominence	38	34	69	141
Frontal ,,	41	63	37	141

These figures confirm the suggestion that the occipital region is usually asymmetrical, and that in this case it is more often the left side which manifests the greater development.

To test the degree of association between the two asymmetries two methods were open to me, neither, however, an ideal treatment for the data. One could throw the "Greater on Right" and "Equal" together, thus reducing the number of categories to two, and use the method of "tetrachoric r" to obtain a coefficient of correlation. This method, however, implies linearity of regression between the two variates. Or one could apply the method of mean squared contingency, which makes no such assumption, but which would give a much better result had there been five categories or more instead of three. I therefore used both methods.

The results were as follows:

Measure of association between greater occipital development on one side and greater frontal development on the opposite side:

Coefficient of correlation (Tetrachoric) $r = + \cdot 6169 \pm \cdot 0734$.

Coefficient of mean squared contingency corrected with class index corrections $C_2 = \cdot 6374$*.

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* Before the class index correction is applied, the value obtained for uncorrected C_2 is ·4965.

Table for obtaining Tetrachoric r.

Greater Frontal Prominence

Greater Occipital Prominence

	Right and equal	Left	Totals
Right and equal	17	21	38
Left	87	16	103
Totals	104	37	141

Contingency Table.

Greater Frontal Prominence

Greater Occipital Prominence

	Right	Equal	Left	Totals
Right	3	14	21	38
Equal	3	20	11	34
Left	35	29	5	69
Totals	41	63	37	141

The mean value of C_2 (uncorrected) in 3 × 3 tables of samples of this size drawn from a population where there is *no* contingency will be ·1661 ± ·0568. The value of C_2 obtained may be taken as significant as compared with this, and that of r is distinctly so. We can therefore conclude, on the basis of our material, that these two forms of asymmetry do tend to be associated together, though the association is by no means perfect.

A feature that has been frequently noted in the Cranial "Remarks" of previous workers, is the existence of a depression slightly above the region of the pterion, called the Sylvian depression. Noticing the frequency with which this occurred on one side only, or on one side more markedly than the other, it occurred to me to subject this depression to the same investigation as was applied to the occipital and frontal bones. For this purpose I have not concerned myself with the depth of the depression, but merely with the question whether it was equally represented (or equally unrepresented) on both sides. One hundred and thirty of my skulls were undamaged in that region and available for data. They gave the following results:

Sylvian depression greater on R in 90 skulls.
" " equal R and L in 33 "
" " greater on L in 7 "

In my series, therefore, there is a considerable preponderance in favour of the right side.

To test, by the method of mean squared contingency, whether this kind of asymmetry is associated with either of the other two I made out the following tables:

Greater Frontal Prominence

		Right	Equal	Left	Totals
Greater Sylvian Depression	Right	29	41	20	90
	Equal	10	15	8	33
	Left	1	2	4	7
	Totals	40	58	32	130

Greater Occipital Prominence

		Right	Equal	Left	Totals
Greater Sylvian Depression	Right	23	17	50	90
	Equal	7	11	15	33
	Left	4	2	1	7
	Totals	34	30	66	130

These gave the following results:

Greater Sylvian Depression on one side with Greater Frontal Prominence on same side: $C_2 = \cdot 1794$.

Greater Sylvian Depression on one side with Greater Occipital Prominence on opposite side: $C_2 = \cdot 2348$.

For 130 in a sample, with this number of categories, mean C_2 when there is no contingency is $\cdot 1728 \pm \cdot 0592$. As even the greater of our results differs from this by hardly more than the probable error, the results cannot be said to indicate any certain association between Sylvian asymmetry and asymmetry of the frontal and occipital bones.

Another point in which there is frequently a marked difference between the two sides of the cranium is to be found in the grooves on the inner side of the occipital bone which accommodate the lateral sinuses. Le Double* found that out of 200 European skulls, the superior longitudinal sinus was directly continued into the right lateral sinus in 137 cases, into the left lateral sinus in 29 cases, and in the rest neither could claim any advantage over the other.

One hundred and thirty-five of our Burmese crania were available for examination on this point. There were 94 in which the right lateral groove was the more obvious continuation of the superior longitudinal groove (and in this case usually

* *Traité des Variations des Os du Crâne*, p. 16 (1903).

larger than the left), 28 where it was the left lateral sinus, and 13 in which it was impossible to decide between them.

It seems reasonable to suppose that where the main stream of blood is directed down one side rather than the other, the outlet for the blood-vessel on that side will be larger than the one on the opposite side; and there was indeed a very noticeable inequality in the size of the two jugular foramina in a large proportion of skulls. Owing to the difference in shape of the two foramina, it was sometimes difficult to decide which was the greater when they approached each other in size. Our figures would, however, give a good approximation to the truth; they are: 82 skulls in which the right jugular foramen was the greater, 32 the left, 13 equal.

It will thus be seen that it is the right side which has much the greater number both of larger grooves and larger foramina. Can we rely upon the jugular foramina to tell us what is the arrangement of the sinuses? The answer is given by the coefficient of mean squared contingency, as obtained from the following table, which is ·6541 for uncorrected C_2 and ·7036 with the class index correction applied.

Greater Jugular Foramen

Greater Groove	Right	Equal	Left	Totals
Right	79	6	4	89
Equal	2	5	6	13
Left	1	2	22	25
Totals	82	13	32	127

This is for 127 skulls, and mean C_2 where no contingency exists will be of the same order as that for 130, which has been quoted already. The difference between this and the result obtained is roughly 8 times the probable error, and we may therefore conclude that there is a very close relation between the two (though not a perfect one). This means that we can accept the evidence of the foramina as to the arrangement of the sinuses, if the more direct evidence of the grooves be unobtainable.

In connection with this subject my attention was directed to an interesting paper by Professor Elliot Smith* in which a relation is suggested between exterior occipital asymmetry and the direction of the main sinus. The argument briefly is as follows: The occipital poles of the cerebral hemispheres are frequently asymmetrical. Professor Elliot Smith estimates that in roughly 80 per cent. of Egyptian brains, the area of cortex containing Gennari's stria (the area striata) extends much further on to the outer aspect of the *left* hemisphere than on to the corresponding surface of the right hemisphere: often as far as 3 or even 4 cms. from the median line on the left, whereas on the right side the area striata barely crosses the edge of the hemisphere. "As a rule, this lateral part of the area striata

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* *Anatomischer Anzeiger*, Band xxx. pp. 574–578, Jena, 1907.

is on a different plane from the rest of the surface of the occipital region, as though it were a mushroom-like cake placed upon the surface. Hence the caudal pole of the left hemisphere projects much further backward in such cases than that of the right, and a distinct depression is formed within the superior (cerebral) fossa of the *squama occipitalis.* Corresponding to this inner depression there is in such cases a marked projection of the external surface of the left side of the occipital bone."

"The fulness of the left occipital pole seems to be the reason for the dextral bending of the superior longitudinal sinus, the flattening of the right pole allowing more room for the bigger sinus on that side. That this is really the case, is shewn by the fact that in those cases in which the *right* area striata spreads on to the caudal surface to a greater extent than it does on the left side, the *right* occipital pole and the right occipital squama become more prominent and the superior longitudinal sinus turns to the *left.* Again in those cases in which the occipital poles are symmetrical, the superior longitudinal sinus usually bifurcates, one half going to each lateral sinus."

It seemed possible that the data I had already obtained might serve to give some measure statistically to the interesting associations thus suggested. It was naturally impossible to obtain from our material any measure of the association of extended area striata on one side, with main lateral sinus on the other. It is unlikely that the size of the area striata will be perfectly represented by a depression in the inner surface of the occipital (fossa cor. striatae): as Professor Elliot Smith is careful to say, "*as a rule* this lateral part of the area striata is on a different plane from the rest of the surface of the occipital region," and where it does *not* "resemble a mushroom-like cake," there will be no corresponding fossa. Probably, however, there would be a correlation between the two. Similarly, the occipital grooves are not bound to be an exact reflex of the longitudinal and lateral sinuses, though perhaps manifesting correlation. Again, the occipital bone is of very varying thickness, and an interior depression is not necessarily accompanied by an exterior projection within limits of say 1 to 2 mms. though again there might be a certain degree of correlation between them over a series. Lacking perfect association between these different stages, it would be impossible to form unassailable conclusions as to the conditions obtaining in the brain and sinuses from an examination of the exterior of the skull. Again, assuming correlation, but not almost perfect correlation, to exist between these different variates, it is by no means needful, as the theory of correlation shows, when C is correlated with B and B again with A and when D is correlated with E, for a correlation between C and D to involve a correlation between A and E; but it is worth examining whether A and E (exterior occipital asymmetry and interior direction of grooves) which we can observe are really correlated. The presence or absence of such correlation unless it were almost perfect would not, however, enable us to predict a relationship of C and D (area striata to direction of superior longitudinal sinus).

It is with data concerning exterior occipital asymmetry, and the direction of the main inner groove that I was furnished, and on these based a contingency table of the same kind (9-celled) as the previous ones. I also connected by a table the data relating to occipital asymmetry and size of jugular foramen, though one naturally expected the association to be rather less in this case.

Greater Occipital Prominence

		Right	Equal	Left	Totals
Greater Groove	Right	24	18	52	94
	Equal	6	1	6	13
	Left	8	9	11	28
	Totals	38	28	69	135

Greater Occipital Prominence

		Right	Equal	Left	Totals
Greater Jugular Foramen	Right	23	17	42	82
	Equal	5	3	5	13
	Left	6	8	18	32
	Totals	34	28	65	127

The results were as follows:

Association of greater occipital prominence on one side, with main lateral groove on the other side: coefficient of mean squared contingency $C_2 = \cdot 2035$. Association of greater occipital prominence on one side with greater jugular foramen on the other side: coefficient of mean squared contingency $C_2 = \cdot 1319$. The number of cases is 127, and the mean value of C_2 where there is no contingency in the sampled population is again of the order $\cdot 17 \pm \cdot 06$.

One has to conclude therefore that in this series there is no significant association between outer occipital asymmetry and the direction in which the main occipital sinus is turned when its course is changed from longitudinal to lateral.

The remaining point dealt with in this investigation concerned the asymmetry, not of the cranium, but of the face; and for this I was able to use actual measurements instead of merely appreciative methods to provide the data. It has been the custom among the workers of this school to take measurements of both orbits, not of one alone as did the older craniometricians (usually without specifying which), but in the case of the nose the nasal height has previously been taken to the lowest part of the edge of the pyriform aperture on one side only, the left, unless this was missing, in which case the measurement was taken on the right side. I have, however, taken the measurements on both sides.

It is obvious on comparing the means of the various series for height of orbit and height of nose, that symmetry is the normal condition: there is no tendency for either side to exceed the other. Thus, taking all the adults of the collection together, we get 51·45 mms. as the mean nasal height on the right side against 51·36 mms. on the left; 34·51 mms. is mean height of right orbit, 34·59 mms. mean of left: in both cases a difference of only about ·085 mm. between right and left.

In the bones of the individual face, however, as also in the lines of the living face, exact symmetry is by no means a rule. Is the lack of symmetry in one feature usually associated with lack of symmetry in another? Is inequality in height of orbits generally accompanied by inequality in nose-height measured right and left? and, if so, is the higher orbital height on one side associated with greater height of nose on that same side or the other side?

This question it was possible to answer with considerable exactness as far as my own series was concerned, by means of a Correlation Table in which one variate was the excess or defect of left nasal height over right, divided by *right* nasal height; and the other was the corresponding measure of orbital inequality. The range of difference in absolute height was from + 2·7 mms. to − 2·1 mms. for the nose and from + 2·2 mms. to − 2·2 mms. for the orbit. This gave for the nasal index $100\left(\frac{NH(R)-NH(L)}{NH(R)}\right)$ a range of + 5·3 mms. to − 4·4 mms.; and for the orbital index $100\left(\frac{O_2(R)-O_2(L)}{O_2(R)}\right)$ a range of + 5·9 mms. to − 6·6 mms., these varieties being divided into 11 and 13 categories respectively.

Correlation Table between Nasal and Orbital Asymmetry.

$$\frac{NH(R)-NH(L)}{NH(R)}\times 100$$

$\frac{O_2(R)-O_2(L)}{O_2(R)}\times 100$	−4·95	−3·95	−2·95	−1·95	− ·95	+ ·05	+1·05	+2·05	+3·05	+4·05	+5·05	Totals
−6·95	—	—	1	1	2	—	—	—	—	—	—	4
−5·95	—	—	2	1	1	1	—	—	—	—	—	5
−4·95	—	—	—	—	3	2	—	—	—	—	1	6
−3·95	—	—	1	2	—	3	—	—	1	—	—	7
−2·95	1	—	1	2	7	4	3	—	—	—	—	18
−1·95	—	2	1	1	5	—	1	—	—	—	—	10
− ·95	—	—	3	3	8	5	4	—	—	—	—	23
+ ·05	—	—	—	1	—	4	2	1	—	—	—	8
+1·05	—	—	—	—	4	3	3	1	1	—	—	12
+2·05	—	—	—	2	8	1	1	—	—	—	—	12
+3·05	—	—	2	—	4	1	1	2	—	—	—	10
+4·05	—	—	—	—	—	—	—	—	—	—	—	—
+5·05	—	—	—	—	—	—	1	—	1	—	—	2
Totals	1	2	11	13	42	24	16	4	3	—	1	117

The standard deviation of the difference in nasal height R and L was found to be 1·53, that of the difference in orbital height 2·76; and we obtain a coefficient of correlation between greater orbital height on one side and greater nasal height on the same side of the value $r = +\cdot 2160 \pm \cdot 595$.

This is more than three times as great as the probable error, and may therefore be deemed significant, though the correlation is not high. Hence the conclusion that where the orbits are of unequal height, there is a significant but slight tendency for the pyriform aperture to descend lower on that side where the higher orbit exists.

My thanks are due to Miss Noel Karn and Miss Margaret Moul of the Biometric Laboratory for assistance in working out the above Contingency and Correlation Tables. I am also deeply indebted to Professor Arthur Keith for the permission to photograph Chinese, Karen, Maravar and Malay crania in the collection of the Royal College of Surgeons.

DESCRIPTION OF PLATES.

The skulls selected to illustrate the difference in type in our Burmese series are all male, since the characteristic features are more marked in the male skull. The supposed hybrids (Type B) are unrepresented: I have contented myself with illustrations of the two extreme groups.

Plates I–III show two skulls of Type A (Burman) from various aspects:

Plate I, Fig. 1, Burman ♂, No. 110, Norma lateralis.
,, Fig. 2 ,, ,, ,, verticalis.
,, Fig. 3 ,, ,, ,, facialis.
Plate II, Fig. 4, Burman ♂, No. 4, Norma occipitalis.
,, Fig. 5 ,, ,, ,, verticalis.
,, Fig. 6 ,, ,, ,, facialis.
,, Fig. 7 ,, ,, ,, basalis.
Plate III, Fig. 8 ,, ,, ,, lateralis.

Plate III also shows a skull of Type C (? Karen) from two points of view:

Plate III, Fig. 9, Burmese Type C, ♂, No. 80, Norma lateralis.
,, Fig. 10 ,, ,, ,, ,, facialis.

In Plates IV, V and VI we have typical Karen*, Maravar, Hindu, Chinese and Malay skulls, for comparison:

Plate IV, Fig. 11, Karen ♂, Norma facialis.
,, Fig. 12 ,, ,, lateralis.
,, Fig. 13, Maravar ♂, Norma facialis.
,, Fig. 14 ,, ,, lateralis.
Plate V, Fig. 15, Hindu ♂, Norma lateralis.
,, Fig. 16 ,, ,, verticalis.
,, Fig. 17 ,, ,, facialis.
Plate VI, Fig. 18, Chinese ♂, Norma facialis.
,, Fig. 19 ,, ,, lateralis.
,, Fig. 20, Malay ♂, Norma facialis.
,, Fig. 21 ,, ,, lateralis.

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* The series from which the Karen skull was chosen was, perhaps, too slight (five in number) to enable me to assert that this one is "typical." It is however Karen.

The remaining plates illustrate anomalies of conformation found in the Burmese cranial series:

Plate VII, Fig. 22, skull No. 23: right temporal bone divided into three separate parts.
,, Fig. 23, skull No. 136: peculiar ossification in temporal suture.
,, Fig. 24, skull No. 95: two pre-condyles fused.
,, Fig. 25, skull, No. 13: two pre-condyles.
Plate VIII, Fig. 26, skull No. 113: very deep canine fossae.
,, Fig. 27, skull No. 82: os japonicum on L.
,, Fig. 28, skull No. 8: L. upper canine thrust in lateral direction.
,, Fig. 29, skull No. 92: fossa at anterior border of foramen magnum.
Plate IX, Fig. 30, skull No. 121: R. os triangulare of interparietal, and ossicle at lambda.
,, Fig. 31, skull No. 74: long epipteric bone on R.
,, Fig. 32, skull No. 103: complete tri-partite interparietal.

APPENDIX I

TABLE OF THE OCCIPITAL INDEX

For the convenience of future craniometrists I have tabled the occipital index as obtained from the formula

$$Oc.\ I. = \frac{S_3}{S_3'}\sqrt{\frac{S_3}{24\,(S_3 - S_3')}}$$

for all values of S_3/S_3' from 1·050 to 1·500; S_3/S_3' being the ratio of the occipital arc from lambda to opisthion, to the chord connecting these two points.

This index measures the convexity of the occipital bone from lambda to opisthion, giving the ratio of the radius of curvature (supposing the curvature to be that of an arc of a circle, which is only roughly the case), to the chord S_3'. The index obviously decreases as greater convexity shortens the radius.

The value given by our formula is very nearly equal to the index required. Its error for a normal proportion like $S_3/S_3' = 1{\cdot}250$, say, is only 3 per cent. At $S_3/S_3' = 1{\cdot}500$ it reaches 5 per cent., and after this point would give a value which increases instead of decreasing, with increased convexity. I have therefore made $S_3/S_3' = 1{\cdot}500$ the limit of my table, and it would indeed be a very abnormal skull for which this limit did not suffice.

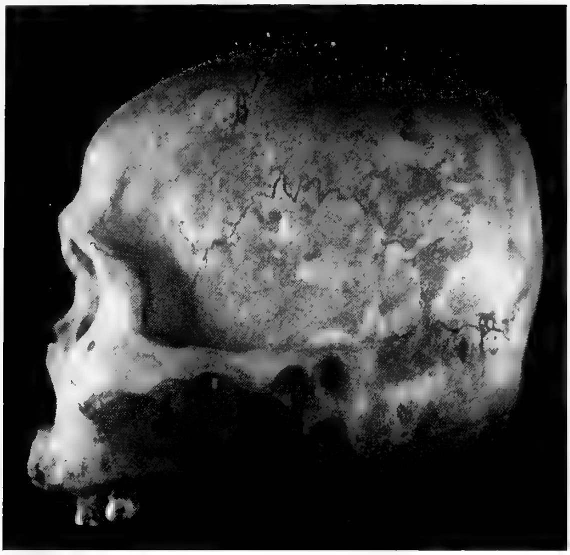

Fig. 1.

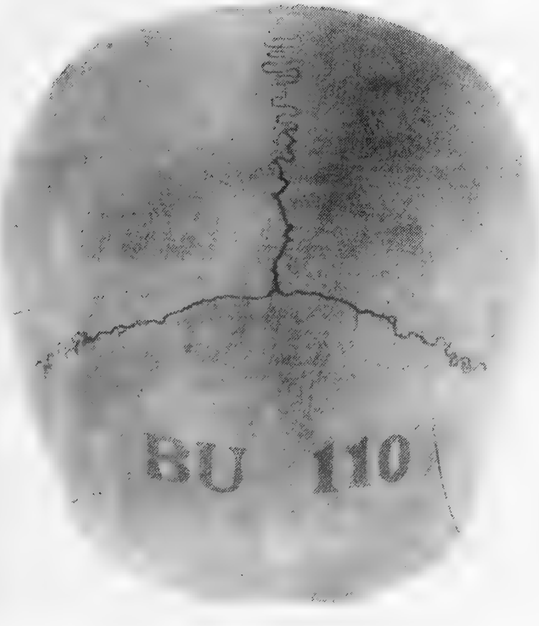

Fig. 2.

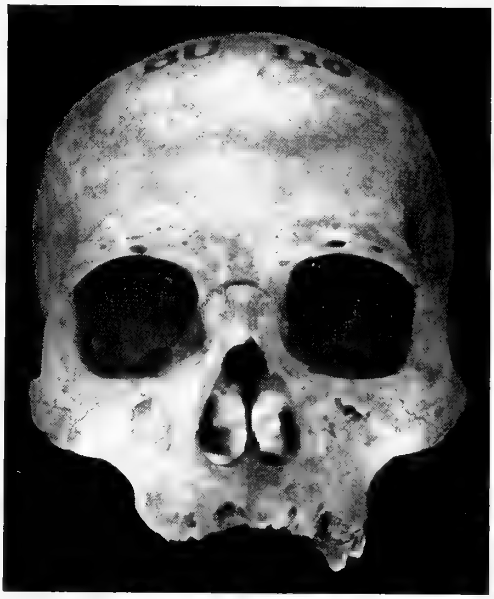

Fig. 3.

Cranium of Typical ♂ Burman, No. 110.
Fig. 1. N. lateralis. Fig. 2. N. verticalis. Fig. 3. N. facialis.

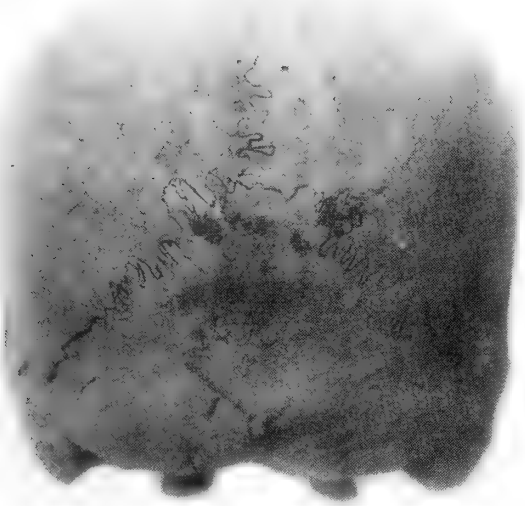

Fig. 4.

Fig. 5.

Fig. 6.

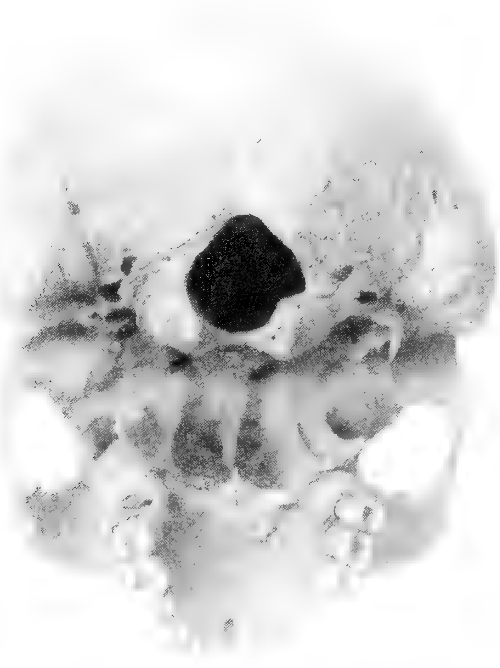

Fig. 7.

Cranium of Typical ♂ Burman, No. 4.

Fig. 4. N. occipitalis. Fig. 5. N. verticalis. Fig. 6. N. facialis. Fig. 7. N. basalis.

Fig. 8.
Typical ♂ Burman, No. 4. Fig. 8. N. lateralis.

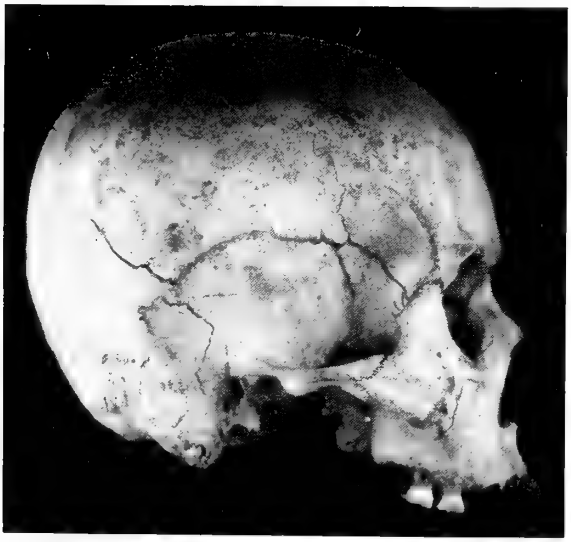

Fig. 9.

Fig. 10.

Typical ♂, Type C (Karen ?).
Fig. 9. N. lateralis. Fig. 10. N. facialis.

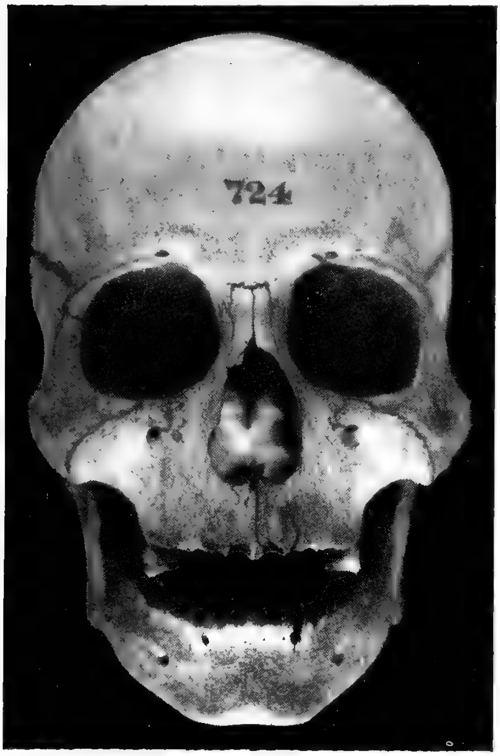

Fig. 11.

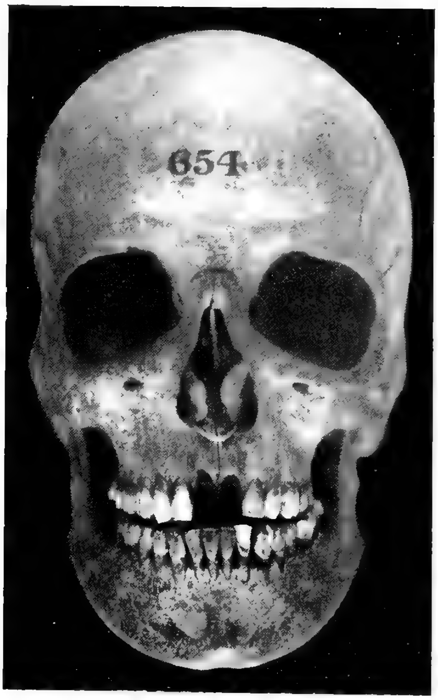

Fig. 13.

Fig. 12.

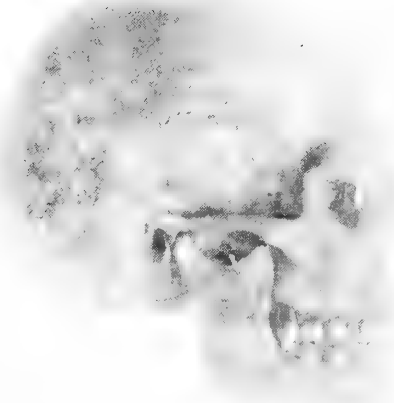

Fig. 14.

Karen ♂ Skull, No. 724.
Fig. 11. N. facialis. Fig. 12. N. lateralis.

Maravar ♂ Skull, No. 654.
Fig. 13. N. facialis. Fig. 14. N. lateralis.

From Crania in the Museum of the Royal College of Surgeons.

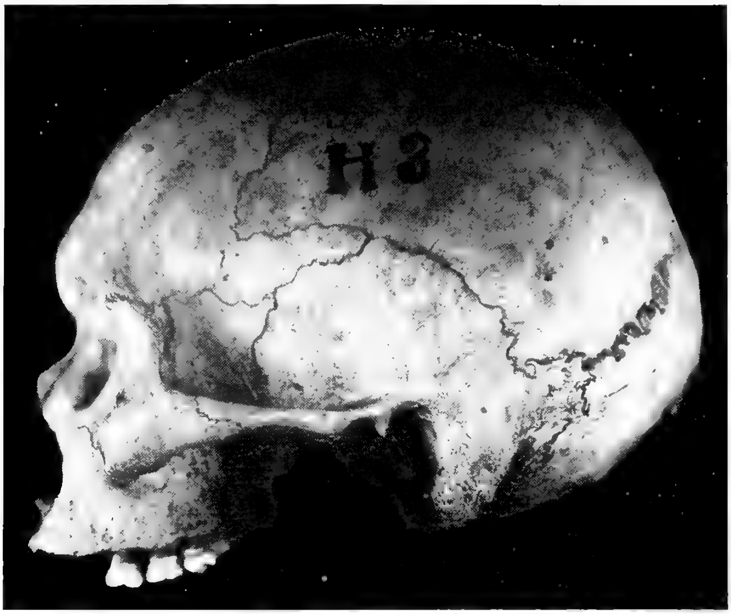

Fig. 15.

Fig. 16.

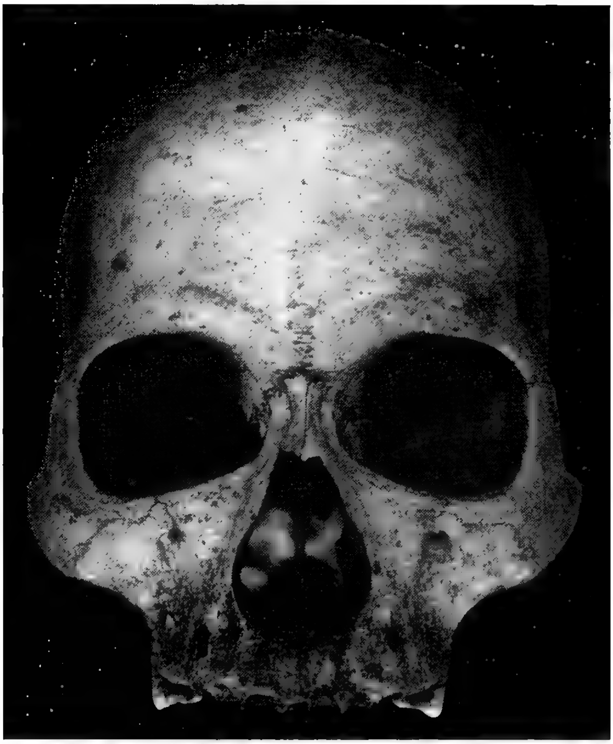

Fig. 17.

Typical ♂ Hindu (Lower Bengal). No. 3 of Hindu Series in Biometric Laboratory.

Fig. 15. N. lateralis. Fig. 16. N. verticalis. Fig. 17. N. facialis.

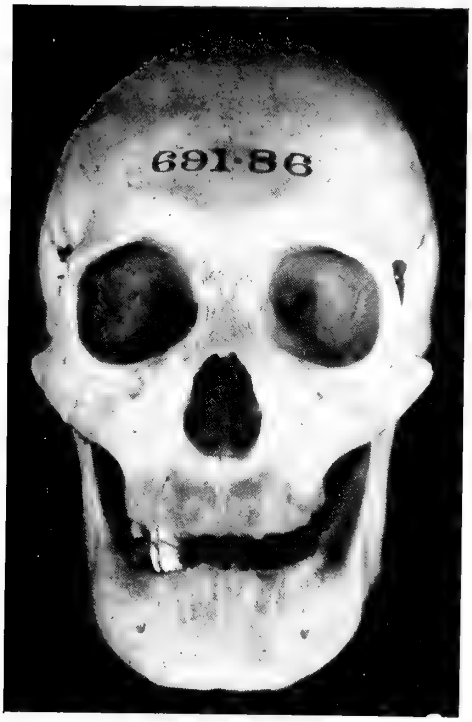

Fig. 18.

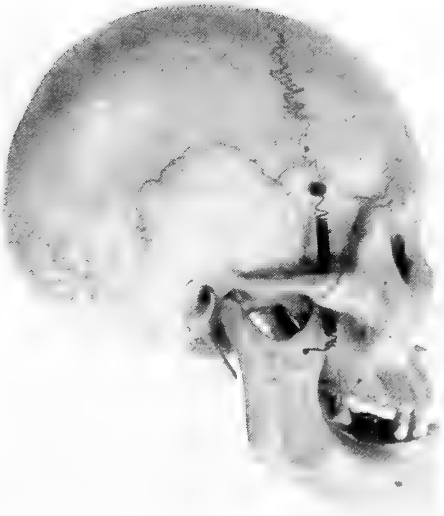

Fig. 19.

Young ♂ Chinaman.

Fig. 18. N. facialis. Fig. 19. N. lateralis.

Fig. 20.

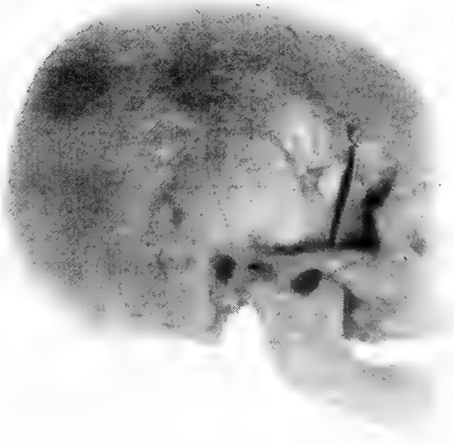

Fig. 21.

Typical ♂ Malay.

Fig. 20. N. facialis. Fig. 21. N. lateralis.

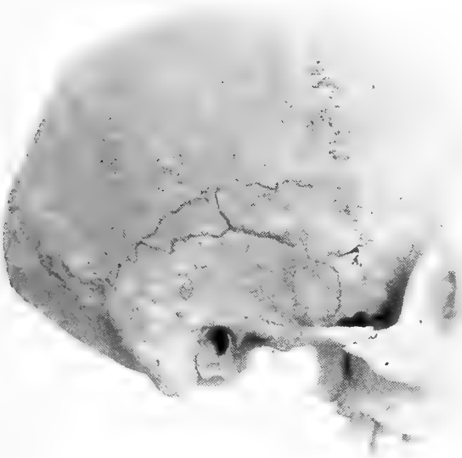

Fig. 22 (No. 23).

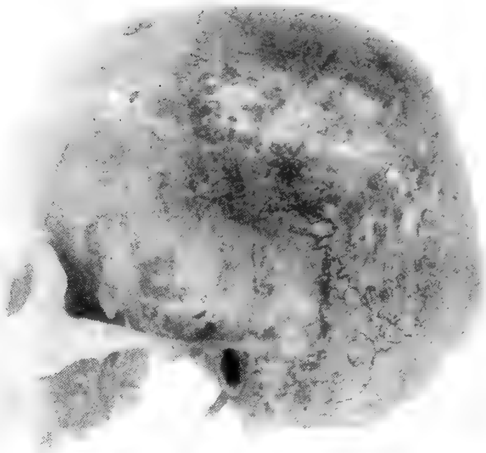

Fig. 23 (No. 136).

Fig. 24 (No. 95).

Fig. 25 (No. 13).

Special Burmese Crania, illustrating abnormalities.

Fig. 22. Right temporal bone divided into three separate parts. Fig. 23. Peculiar ossification of temporal suture. Fig. 24. Two pre-condyles fused. Fig. 25. Two pre-condyles.

Fig. 26 (No. 113).

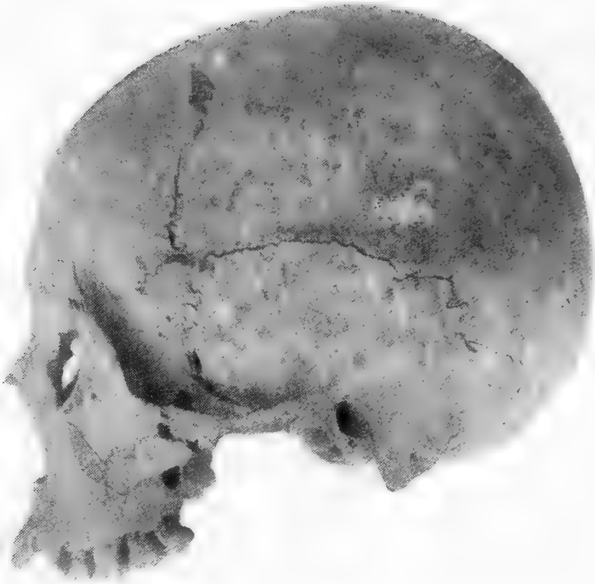

Fig. 27 (No. 82).

Fig. 28 (No. 8).

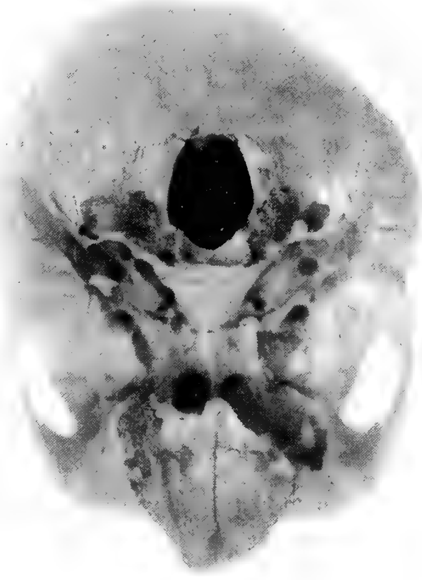

Fig. 29 (No. 92).

Special Burmese Crania, illustrating abnormalities.

Fig. 26. Very deep canine fossae. Fig. 27. *Os japonicum* on left.
Fig. 28. Left upper canine thrust in lateral direction. Fig. 29. Unusual fossa at anterior border of *foramen magnum*.

Fig. 30 (No. 121).

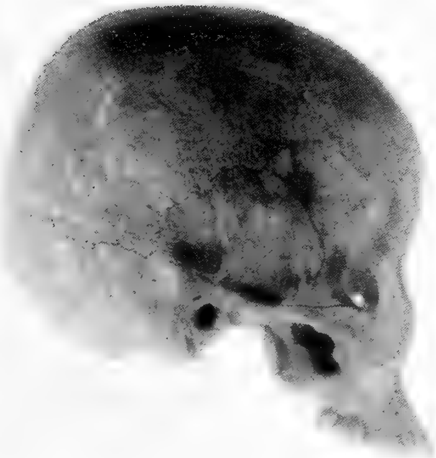

Fig. 31 (No. 74).

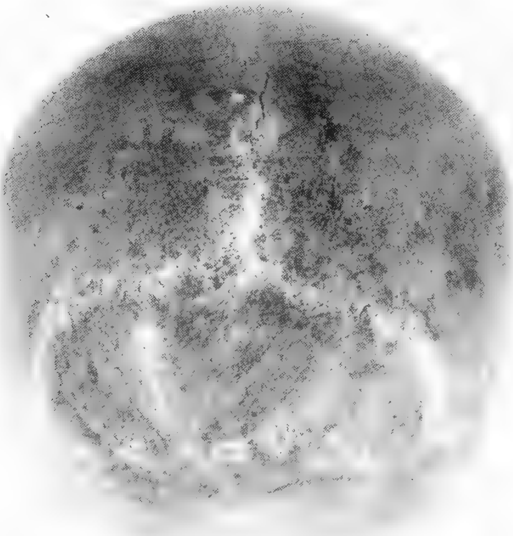

Fig. 32 (No. 103).

Special Burmese Crania, illustrating abnormalities.

Fig 30 R. *os triangulare* of interparietal and ossicle of lambda.
Fig. 31. Long epipteric bone on right. Fig. 32. Complete tri-partite interparietal.

Table for Calculating Occipital Index (Oc. I.) from Occipital Arc (S_3) and Chord (S_3').

S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*
1·050	98·22	1·100	74·47	1·150	65·00	1·200	60·00	1·250	57·05 +
1·051	97·39	1·101	74·20	1·151	64·87	1·201	59·92	1·251	57·01
1·052	96·59	1·102	73·94	1·152	64·74	1·202	59·85 +	1·252	56·96
1·053	95·81	1·103	73·68	1·153	64·61	1·203	59·78	1·253	56·92
1·054	95·05 +	1·104	73·42	1·154	64·48	1·204	59·71	1·254	56·88
1·055	94·32	1·105	73·17	1·155	64·36	1·205	59·63	1·255	56·83
1·056	93·60	1·106	72·92	1·156	64·23	1·206	59·56	1·256	56·79
1·057	92·91	1·107	72·68	1·157	64·11	1·207	59·49	1·257	56·75
1·058	92·24	1·108	72·44	1·158	63·99	1·208	59·42	1·258	56·70
1·059	91·58	1·109	72·21	1·159	63·87	1·209	59·36	1·259	56·66
1·060	90·95 −	1·110	71·98	1·160	63·76	1·210	59·29	1·260	56·62
1·061	90·32	1·111	71·74	1·161	63·64	1·211	59·22	1·261	56·58
1·062	89·72	1·112	71·52	1·162	63·52	1·212	59·15 +	1·262	56·54
1·063	89·13	1·113	71·30	1·163	63·41	1·213	59·09	1·263	56·50
1·064	88·56	1·114	71·08	1·164	63·30	1·214	59·02	1·264	56·46
1·065	88·00	1·115	70·87	1·165	63·19	1·215	58·96	1·265	56·42
1·066	87·45 −	1·116	70·66	1·166	63·08	1·216	58·89	1·266	56·38
1·067	86·92	1·117	70·45 −	1·167	62·97	1·217	58·83	1·267	56·34
1·068	86·40	1·118	70·25 −	1·168	62·86	1·218	58·77	1·268	56·30
1·069	85·89	1·119	70·04	1·169	62·76	1·219	58·71	1·269	56·26
1·070	85·39	1·120	69·84	1·170	62·65 +	1·220	58·64	1·270	56·22
1·071	84·91	1·121	69·65 −	1·171	62·55 +	1·221	58·58	1·271	56·19
1·072	84·43	1·122	69·46	1·172	62·45 −	1·222	58·52	1·272	56·15 −
1·073	83·97	1·123	69·26	1·173	62·35 −	1·223	58·46	1·273	56·11
1·074	83·52	1·124	69·08	1·174	62·25 −	1·224	58·40	1·274	56·08
1·075	83·08	1·125	68·89	1·175	62·15 −	1·225	58·35 +	1·275	56·04
1·076	82·64	1·126	68·71	1·176	62·05 +	1·226	58·29	1·276	56·00
1·077	82·22	1·127	68·53	1·177	61·95 +	1·227	58·23	1·277	55·97
1·078	81·80	1·128	68·35 +	1·178	61·86	1·228	58·17	1·278	55·93
1·079	81·40	1·129	68·18	1·179	61·77	1·229	58·12	1·279	55·90
1·080	81·00	1·130	68·01	1·180	61·67	1·230	58·06	1·280	55·86
1·081	80·61	1·131	67·84	1·181	61·58	1·231	58·01	1·281	55·83
1·082	80·23	1·132	67·67	1·182	61·49	1·232	57·95 −	1·282	55·80
1·083	79·85 +	1·133	67·50	1·183	61·40	1·233	57·90	1·283	55·76
1·084	79·49	1·134	67·34	1·184	61·31	1·234	57·84	1·284	55·73
1·085	79·13	1·135	67·18	1·185	61·22	1·235	57·79	1·285	55·70
1·086	78·78	1·136	67·02	1·186	61·13	1·236	57·74	1·286	55·67
1·087	78·43	1·137	66·86	1·187	61·05 −	1·237	57·69	1·287	55·64
1·088	78·09	1·138	66·71	1·188	60·96	1·238	57·64	1·288	55·60
1·089	77·76	1·139	66·55 +	1·189	60·87	1·239	57·58	1·289	55·57
1·090	77·43	1·140	66·40	1·190	60·79	1·240	57·53	1·290	55·54
1·091	77·11	1·141	66·25 +	1·191	60·71	1·241	57·48	1·291	55·51
1·092	76·80	1·142	66·11	1·192	60·63	1·242	57·43	1·292	55·48
1·093	76·49	1·143	65·96	1·193	60·55 −	1·243	57·38	1·293	55·44
1·094	76·18	1·144	65·82	1·194	60·46	1·244	57·34	1·294	55·41
1·095	75·88	1·145	65·68	1·195	60·38	1·245	57·29	1·295	55·38
1·096	75·59	1·146	65·54	1·196	60·31	1·246	57·24	1·296	55·35 +
1·097	75·30	1·147	65·40	1·197	60·23	1·247	57·19	1·297	55·32
1·098	75·02	1·148	65·26	1·198	60·15 +	1·248	57·15 −	1·298	55·30
1·099	74·74	1·149	65·13	1·199	60·07	1·249	57·10	1·299	55·27
1·100	74·47	1·150	65·00	1·200	60·00	1·250	57·05 +	1·300	55·24

Table for Calculating Occipital Index (continued).

S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*	S_3/S_3'	*Oc. I.*
1·300	55·24	1·350	54·12	1·400	53·46	1·450	53·13
1·301	55·21	1·351	54·10	1·401	53·45$^+$	1·451	53·13
1·302	55·18	1·352	54·09	1·402	53·44	1·452	53·12
1·303	55·16	1·353	54·07	1·403	53·44	1·453	53·12
1·304	55·13	1·354	54·06	1·404	53·43	1·454	53·11
1·305	55·10	1·355	54·04	1·405	53·42	1·455	53·11
1·306	55·07	1·356	54·02	1·406	53·41	1·456	53·11
1·307	55·05$^-$	1·357	54·01	1·407	53·40	1·457	53·10
1·308	55·02	1·358	53·99	1·408	53·39	1·458	53·10
1·309	55·00	1·359	53·98	1·409	53·38	1·459	53·09
1·310	54·97	1·360	53·96	1·410	53·37	1·460	53·09
1·311	54·94	1·361	53·94	1·411	53·36	1·461	53·09
1·312	54·92	1·362	53·93	1·412	53·35$^+$	1·462	53·09
1·313	54·89	1·363	53·91	1·413	53·35$^-$	1·463	53·08
1·314	54·87	1·364	53·90	1·414	53·34	1·464	53·08
1·315	54·84	1·365	53·88	1·415	53·33	1·465	53·08
1·316	54·82	1·366	53·87	1·416	53·32	1·466	53·08
1·317	54·79	1·367	53·85$^+$	1·417	53·32	1·467	53·08
1·318	54·77	1·368	53·84	1·418	53·31	1·468	53·07
1·319	54·74	1·369	53·82	1·419	53·31	1·469	53·07
1·320	54·72	1·370	53·81	1·420	53·30	1·470	53·07
1·321	54·70	1·371	53·80	1·421	53·29	1·471	53·07
1·322	54·68	1·372	53·78	1·422	53·28	1·472	53·07
1·323	54·65$^+$	1·373	53·77	1·423	53·28	1·473	53·06
1·324	54·63	1·374	53·75$^+$	1·424	53·27	1·474	53·06
1·325	54·61	1·375	53·74	1·425	53·26	1·475	53·06
1·326	54·59	1·376	53·73	1·426	53·25$^+$	1·476	53·06
1·327	54·57	1·377	53·72	1·427	53·25$^-$	1·477	53·06
1·328	54·54	1·378	53·70	1·428	53·24	1·478	53·05$^+$
1·329	54·52	1·379	53·69	1·429	53·24	1·479	53·05$^+$
1·330	54·50	1·380	53·68	1·430	53·23	1·480	53·05$^-$
1·331	54·48	1·381	53·67	1·431	53·22	1·481	53·05$^-$
1·332	54·46	1·382	53·66	1·432	53·22	1·482	53·05$^-$
1·333	54·44	1·383	53·64	1·433	53·21	1·483	53·05$^-$
1·334	54·42	1·384	53·63	1·434	53·21	1·484	53·05$^-$
1·335	54·40	1·385	53·62	1·435	53·20	1·485	53·04
1·336	54·38	1·386	53·61	1·436	53·20	1·486	53·04
1·337	54·36	1·387	53·60	1·437	53·19	1·487	53·04
1·338	54·34	1·388	53·59	1·438	53·19	1·488	53·04
1·339	54·32	1·389	53·58	1·439	53·18	1·489	53·04
1·340	54·30	1·390	53·57	1·440	53·18	1·490	53·04
1·341	54·28	1·391	53·55$^+$	1·441	53·17	1·491	53·04
1·342	54·26	1·392	53·54	1·442	53·17	1·492	53·04
1·343	54·25$^-$	1·393	53·53	1·443	53·16	1·493	53·04
1·344	54·23	1·394	53·52	1·444	53·16	1·494	53·04
1·345	54·21	1·395	53·51	1·445	53·15$^+$	1·495	53·03
1·346	54·19	1·396	53·50	1·446	53·15$^-$	1·496	53·03
1·347	54·17	1·397	53·49	1·447	53·14	1·497	53·03
1·348	54·16	1·398	53·48	1·448	53·14	1·498	53·03
1·349	54·14	1·399	53·47	1·449	53·13	1·499	53·03
1·350	54·12	1·400	53·46	1·450	53·13	1·500	53·03

MALES

APPENDIX II. TABLES OF THE ABSOLUTE AND INDICIAL MEASUREMENTS OF THE INDIVIDUAL CRANIA (ABSOLUTE LENGTHS IN MMS.)

					Lengths									Arcs								Face				Nose								Orbits				Palate and Profile				Foramen		Indices															Angles						
No.	Type	Sex	C	W. in grs.	F	L'	L	B	B'	H	H'	OH	LB	Q	Q'	S	S_1	S_2	S_3	S_3'	U	PH	G'H	GB	J	NH R	NH L	NB	DS	DC	DA	SS	SC	O_1R	O_1L	O_2R	O_2L	G_1	G_1'	G_2	GL	fml	fmb	Oc. I	B/L'	H/L'	B/L	H/L	B/H	G'H/GB	NB/NH R	NB/NH L	O_2/O_1 R	O_2/O_1 L	G_2/G_1	fmb/fml	DS/DC	SS/SC	N∠	A∠	B∠	θ_2	θ_1	P∠	REMARKS*
3	A	♂	—	655	174	174	174	138	97	139	139·5	125·5	102	326	327·5	375	137	124	114	97	503	18·0?	73	99	127	57·2	56·5	24·4	10·5	19·0	31	4·7	11·0	44·5	44·9	36·8	35·7	—	—	—	96·5	34·0	30·5	62·4	79·2	80·1	79·3	80·2	98·9	74·0	42·7	43·2	82·7	78·2	—	89·7	55·3	42·7	64°·2	72°·6	43°·2	17°·4?	25°·8?	90°·0?	P.B. sloping forward and downward with very rounded double edge.
4	A	♂	1496	828	175	177	176	153	92·5	140·5	139	125·5	92	340	340	383	137	128	118	97	525	20·0	75·5	107·5	141	57·7	58·0	29·5	9·0	20·8	29	2·0	6·3	45·1	44·9	36·2	37·2	47·6	43·7	42·3	89·5	37·0	33·4	58·9	86·4	78·5	86·8	78·9	110·0	70·1	51·1	50·9	80·3	82·9	88·9	90·3	43·3	31·8	63°·6	67°·4	49°·0	21°·6	27°·4	89°·0	Sutures becoming obliterated; P.B. very rounded and double with large depressions between; very large mastoids and heavy supra mastoid ridges; wormians in R. lambdoid and at R. asterion; lipping of margin of L. occipital condyle.
14	A	♂	1264	685	185	189·5	188·5	131	91·5	135	135·5	115·5	107	304·5	307	370	130	128	112	92	506·5	19·5	66·5	105·5	133	48·3	48·3	31·4	14·5	27·5	43	6·1	14·0	43·3	43·0	33·0	33·0	53·1	48·0	41·0	103·5	37·0	31·7	58·8	69·2	71·4	69·6	71·9	96·9	62·7	65·0	65·0	76·2	76·7	77·2	85·7	52·7	43·6	68°·7	74°·8	36°·5	7°·7	28°·8	82°·5	P.B. rounded; sub-nasal prognathism.
16	A	♂	1613·5	647	185·5	186·5	185·5	142	91	142	142	120·5	97·5	321	323	387	137	130	120	102·5	526	20·2	70	105	130	52·5	53·9	28·0	9·0	20·8	28·5	3·2	7·5	43·5	43·0	31·2	34·6	49·0	44·0	40·0	98	41·0	34·0	62·6	76·0	76·0	76·5	76·5	99·9	66·7	53·3	51·9	78·6	80·5	81·6	82·9	43·3	42·7	69°·3	68°·6	42°·1	8°·5	33°·6	77°·1	P.B. rounded; sub-nasal prognathism; wormians in L. lambdoid; fair malar marginal processes.
20	A	♂	1382	—	169	171	170	143	94·5	130	129	118	91·5	322	319	351?	120	124	107?	88?	503	—	—	—	—	—	—	—	—	26·0	—	—	—	—	—	—	—	—	—	—	—	38·5?		58·9	83·6	75·5	84·2	76·0	110·8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Marked depression of occipital bone behind foramen magnum; wormians R. and L. lambdoid; epipteric on R.; foramen magnum cut round.
22	A	♂	—	716	173	175·5	174·5	148	92	139·5	137·5	125	97·5	331·5	332	373	125	130	118	99	508	21·0?	69·5	104	133	52·0	52·0	28·2	11·6	24·5	35·5	3·9	11·3	45·0	44·0	33·9	36·1	49·8	45·3	37·0	95·5	35·4	30·0	60·8	84·1	78·4	84·9	78·9	107·6	66·8	54·2	54·2	75·3	82·0	75·5	84·7	47·3	34·5	67°·1	70°·8	42°·1	14°·2	27°·9	85°·0	Sutures open; teeth worn; torus palatinus; very asymmetric incisive foramen; receding forehead; P.B. rounded; wormians nearly all along lambdoid; fronto-squamous process on L.; small pre-condyle on R.
43	A?	♂	—	—	—	—	—	142	—	—	—	—	—	—	—	—	—	133	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Skull cap only; healed wound on L. lambdoid, leaving depression 3 cm. × 2½ cm. × 8 mm. deep; wormian in R. lambdoid.
44	A	♂	—	679	182	182	181·5	154	105	146	143·5	127	106·5	345	349	380	131·5	129·5	119	98	539	23·3	87	107·5	149	63·5	63·1	28·4	10·0	23·0	32·5	2·6	10·2	48·8	48·9	36·1	37·1	60·2	53·3	47·0	103	34·1	28·7	59·0	84·8	78·8	84·9	79·0	107·5	81·9	44·7	45·0	74·0	75·9	78·1	84·2	43·5	25·5	63°·2	67°·6	49°·2	17°·4	31°·8	85°·0	Teeth worn; large alveolar abscesses at base of wisdom teeth R. and L.; P.B. flat with blunt edge; slight flattening at obelion; small epipteric on R.; face asymmetrical; alveolar point 4·2 mm. to R. of sagittal plane.
52	A	♂?		734	169·5	173	171·5	144	87·5	135	133	117·5	97	322·5	323	357	125	124	108	95	503·5	21·0	75·5	102	138	56·3	57·2	28·9	10·6	23·2	34	2·4	10·1	41·4	41·1	37·5	38·0	55·2	51·1	39·0	99	38·2	32·6	66·9	83·3	77·0	83·9	77·6	108·2	73·9	51·3	50·5	90·6	92·5	70·7	85·3	45·7	23·8	68°·8	66°·0	45°·2	17°·5	27°·7	83°·5	Molars much worn; left inner palate bridge; P.B. flat with rounded edge; sub-nasal prognathism; very asymmetric, with L. parietal, L. occipital, R. frontal bulges; 3 large ossicles and depressions about lambda and wormians in L. lambdoid; ossicles in squamo-mastoid angle L. and R.; marked malar marginal prominence on L.
53	A	♂	1415·5	840	173·5	175·5	174·5	146	90·5	147·5	145·5	122	106	327	326	364	128	113	123	107	509	20·1	73·5	104·5	135·5	54·5	55·0	27·0	10·8	22·8	33	3·6	9·6	46·2	45·2	35·2	35·2	49·3	45·0	39·0?	96·5	36·8	30·3	65·3	83·1	83·0	83·7	83·5	100·2	70·2	49·5	49·1	76·2	77·9	79·1?	82·3	47·4	37·5	61°·8	76°·1	42°·1	9°·9	32°·2	86°·0	Ageing; sutures becoming obliterated; remaining tooth much worn; alveolar rim partly absorbed; receding forehead; P.B. flat with rounded edge; sub-nasal prognathism; flattened obelion; 2 pre-condyles, the larger on L.
54	A	♂?	–	600	170	171	170·5	141·5	103	139	136	119	101·5	321·5	320	353	122·5	125·5	105	93·5	505	20·3	74	103	136·5	55·0	54·7	27·3	8·4	25·5	31·5	—	9·5	45·0	43·5	34·0	32·7	51·2	47·7	40·1	97·5	39·0	31·8	69·6	72·7	79·5	83·1	79·9	104·0	71·5	49·6	49·9	75·6	75·2	78·3	81·5	33·0	—	65°·2	71°·1	43°·7	12°·5	30°·2	84°·6	P.B. flat with rounded edge; metopic; ossicle at R. squamo-mastoid angle; ossicle at bregma; suggestion of os japonicum R. and L.
55	A	♂	—	568	166	168·5	167	140·5	85	140·5	137·5	122	92	327·5	329	356·5	128	125·5	103	90	493	15·3	64	101	127	51·9	51·1	26·0	10·2	23·9	31	2·5	10·0	39·6	38·8	33·9	34·2	47·2	44·2	43·2	89	37·2	30·0	65·8	83·5	81·8	84·1	82·4	102·6	63·6	51·3	51·8	85·6	88·1	91·5	80·7	42·7	25·0	66°·9	71°·7	41°·4	13°·3	28°·1	85°·0	Young adult; P.B. flat with rounded edge; very prominent malar marginal process on R.
56	A	♂		823	179·5	182·5	181·5	153	108·5	157·5	155·5	133	108·5	357	358	387	140	139	108	95·5	535·5	24·6	77	114·5	146	57·0	56·0	27·2	10·2	25·0	35	3·2	10·1	47·2	47·3	36·2	36·0	46·7	43·0	46·0?	98·5	36·1	33·0	67·8	83·9	85·4	84·3	85·9	98·2	67·5	47·7	48·6	76·7	76·1	98·5?	91·5	39·8	31·7	61°·2	75°·4	43°·4	13°·6	29°·8	89°·0	Old; sutures partly obliterated; surface of teeth much worn to inward-upward slant; ossicle at R. asterion; very asymmetric, with L. parietal, L. occipital and R. frontal bulges; P.B. flat with sharp edge on L., rounded on R.
60	A	♂		478	168	167·5	168·5	142	98	137	138	115·5	100	315	315·5	354	124	112	118	103	497	16·3	68	103	137	55·4	56·2	29·9	9·9	23·0	33	3·7	13·6	41·8	42·0	33·7	33·0	42·7	41·0	41·5	89·5	35·5	31·8	53·2	84·7	82·2	84·4	81·9	103·0	66·3	54·0	53·2	80·6	78·6	97·2	89·6	43·0	27·2	60°·7	77°·4	41°·9	14°·6	27°·3	92°·0	P.B. with outward-upward slant and rounded edge; flattened obelion; ossicle at obelion and at lambda; ossicles in R. and L. lambdoid; small epipteric on R.
61	A	♂	1359	669	172	172·5	171·5	137	97·5	138	142	120	108	322	326	360	124	128	108	89	490	22·0?	75	99·5	134	55·6	56·0	27·0	11·9	24·0	38	3·6	9·6	45·6	44·0	31·8	33·1	50·9	46·1	39·0	107	31·0	25·6	58·9	79·5	82·5	79·8	82·9	96·3	75·6	48·6	48·2	69·7	75·2	76·6	82·6	49·6	37·5	69°·0	70°·0	41°·0	15°·4	25°·6	85°·4	P.B. with outward-upward slant and very rounded edge; wormians R. and L. lambdoid.
65	A	♂	—	733	171·5	172	173	149	102	148	145·5	126	103·5	336·5	339	362	125	132	105	88·5	511	18·6	75	115	146·5	57·5	58·6	29·0	10·6	25·0	35	1·5	7·0	46·4	48·0	35·5	35·8	54·2	46·5	47·3	101·5	38·1	30·6	63·1	86·7	84·5	86·1	84·0	102·5	65·0	50·4	49·5	76·5	74·6	87·1	80·3	42·4	21·5	67°·1	69°·9	43°·0	16°·1	26°·9	86°·0	P.B. flat with very rounded edge; ageing; sutures partly obliterated.
68	A	♂	1458·5	645	169·5	172	171	144·5	102	135·5	134·5	121	96	327	329	366	134	120·5	111·5	93	510	22·1?	72	107	139	51·0	52·2	27·4	14·5	24·5	41	2·5	9·0	47·4	47·1	38·2	38·1	50·7	47·0	40·2	98	41·0	38·3	60·1	84·0	78·1	84·5	78·5	107·6	67·4	53·7	52·4	80·6	80·9	79·3	93·4	59·2	27·8	69°·7	66°·8	43°·5	15°·9	27°·6	82°·7	Old; sutures becoming obliterated; P.B. with steep upward-outward slope and rounded edge; sub-nasal prognathism; suggestion of os japonicum on both sides; very large foramen magnum.
69	A (B?)	♂	1429·5	713	169	170·5	171	147	95·5	137	137·5	124	103	333·5	336	358	128	125	105	95	506	21·5?	73·5	106	131	53·5	53·0	26·0	12·1	22·0	35	3·6	8·5	46·6	44·0	34·2	34·2	54·5	49·2	34·5	103·5	37·2	32·0	72·9	86·2	80·5	86·0	80·3	107·1	69·4	48·6	49·1	73·4	77·7	63·3	86·0	55·0	42·4	69°·4	68°·9	41°·7	18°·1?	23°·6?	87°·0?	Aged; 3rd molars never cut; sutures closed; P.B. flat with sharp edge; sub-nasal prognathism; very asymmetric.
71	A	♂	1412·5	867	171	173	172·5	145·5	88	146	144·5	125	101	331·5	334	362	132·5	127	102·5	89·5	501·5	20·0	73	104·5	136	55·2	54·5	28·6	9·2	20·2	31	2·6	5·9	42·5	42·0	37·3	37·2	48·0	44·5	38·5	96·5	41·8	30·5	65·7	84·0	83·4	84·2	83·6	100·8	70·0	51·8	52·5	87·8	88·6	80·2	73·0	45·5	44·1	65°·0	71°·6	43°·4	13°·1	30°·3	84°·7	Sutures open; teeth worn; P.B. flat with rounded edge; flattened obelion; ossicle R. and L. lambdoid and at R. asterion, also in L. occipito-mastoid.
72	A	♂?	1421	566	163	164·5	163·5	148	90	135	135	122	98	330	331	354	128	116	110	94	494	—		95	126·5?	51·5	51·0	27·3	8·2	19·6	28	1·5	7·0	42·3	43·6	34·3	33·5	—	—	38·0?	89·5?	32·2	28·5	62·7	90·0	82·2	90·5	82·6	109·5	—	53·0	53·5	80·4	76·8	—	88·5	41·8	21·5	—	—	—			—	Old; sutures partly obliterated, alveolar rim absorbed; P.B. flat with rounded edge: flattened obelion; ossicle R. lambdoid.
74	A	♂	1380	555	166·5	167	166	142·5	94·5	135	135	120	92	319·5	320	358	129·5	120·5	108	93·5	489·5	21·4?	69	95·5	132·5	49·5	49·2	29·8	8·0	22·0	30	1·6	9·1	43·6	42·0	33·3	32·8	50·7?	45·8	38·4	95	36·2	29·0	64·6	85·2	80·8	85·6	81·2	105·4	72·3	60·2	60·6	76·4	78·1	75·7?	80·1	36·4	17·6	70°·7	66°·0	43°·3	16°·0	27°·3	82°·0	Teeth much worn; 3rd molars small; P.B. flat with rounded edge; marked sub-nasal prognathism; ossicles R. and L. lambdoid, long epipteric on R.
76	A	♂	1329·5	584	171	169	170	144	97	140	140	123·5	97·5	327	331	364	132	123	109	95	497	19·3	69	96·5	124	52·3	53·1	28·5	9·4	24·3	32·5	2·9	9·5	43·0	42·8	36·6	38·0	48·7	46·1	40·7	95·5	32·7	27·8	65·4	85·3	82·8	84·9	82·4	103·0	71·4	54·5	53·7	85·1	88·8	83·6	85·0	38·7	30·5	67°·6	70°·4	42°·0	11°·6	30°·4	82°·0	Teeth much worn; P.B. sloping upward and outward with very rounded edge; wormians R. and L. lambdoid; fronto-squamous process on L.; epipteric on R.; prognathous; ivory exostosis just above R. mastoid.
78	A	♂??	—	—	170·5	171·5	170·5	130?	94·5	130	130	115·5	100	302·5?	—	349·5	119	122·5	108	91	492?	19·7	68	97	130·5?	48·0	49·1	25·0	10·8	23·8	34	4·0	12·0	45·0	44·0	34·8	34·5	45·5	42·0	39·1	98	35·1	29·5	60·9	75·8	75·8	75·9?	76·3	100·0?	69·9	52·1	50·9	77·3	77·4	85·9	84·0	45·4	33·3	68°·5	71°·4	40°·1	14°·6	25°·5	86°·0	Ageing; sutures becoming obliterated; teeth much worn; P.B. flat with sharp edge; sub-nasal prognathism.
86	A	♂	1473·5	837	178·5	178·5	178	151·5	97·5	131	130·5	122	93	332	333	383	132·5	125	125·5	99	520	23·0?	71?	105	129·5	51·3	50·4	28·3	9·1	22·5	31·5	2·5	10·6	45·0	44·3	38·0	38·8	53·4	48·8	37·0?	100·5?	34·0	28·3	56·3	84·7	73·1	85·0	73·4	115·8	67·6?	55·2	50·2	84·4	87·6	69·3?	83·2	40·4	23·6	74°·0?	63°·0?	43°·0?	17°·0?	26°·0?	80°·0?	Alveolus almost completely absorbed; P.B. flat with rounded edge: prognathous; flattened obelion; epipterics R. and L.; heavy inionic ridge.
90	A	♂?	1338	551	170	170	170	141·5	95	132·5	130·5	115	95·5	313	315	363	120	131	112	92·5	499	20·0	76	101	135·5	57·0	57·2	29·0	6·7	23·0	29·5	1·4	9·9	46·0	45·2	36·2	36·8	50·0	46·0	38·2	94	31·4	31·4	59·2	83·2	76·8	83·3	76·9	108·4	75·1	50·9	50·7	78·7	81·4	76·4	100·0	29·1	14·1	65°·1	67°·6	47°·3	14°·5	32°·8	82°·1	Ageing: sutures nearly obliterated; teeth much worn; P.B. flat with very rounded edge; sub-nasal prognathism; large dint near apex to R. of sagittal suture.
91	A	♂?	1389	446	174	180	179	144	95	130·5	130	116·5	98·5	316	318	368	135·5	116·5	116	97	518	18·0?	68·5?	98	132	51·7	51·7	29·2	10·2	24·0	33·5	5·2	11·0	44·8	45·6	34·8	35·2	50·1?	45·5?	38·5?	92·5?	33·3	25·1	60·3	79·9	72·3	80·4	72·8	110·4	69·8?	56·5	56·5	77·7	77·2	76·8?	75·4	42·5	47·3	64°·1?	73°·9?	42°·0?	26°·1?	15°·9?	100°·0?	Senile; sutures obliterated; alveolus almost completely absorbed; P.B. flat with blunt edge; old injury to L. superciliary ridge; very large mastoids; pronounced inion.
94	A (B?)	♂?	1379·5	604	166	168·5	168·5	142	90	138	136	120	98·5	321·5	322·5	350	128	126·5	95·5	86·5	488	25·4?	76·5	99	128	56·8	55·8	25·8	8·8	20·5	31·5	1·8	6·0	43·8	43·6	37·2	36·0	51·0	46·4	41·3	96·5	37·0	30·8	73·7	84·1	80·7	84·2	80·8	104·2	77·2	45·4	46·2	84·9	82·6	81·0	83·2	43·0	30·0	65°·5	68°·3	46°·2	16°·9	29°·3	85°·2	P.B. flat with rounded edge; flattened obelion.
96	A	♂?	1365·5	610	170	172·5	170·5	140·5	97·5	131	132	117·5	99·5	317·5	319	353	132	118	103	88	500	16·6	65	95·5	131·5	51·0	50·9	26·7	10·4	22·0	31·5	2·8	8·4	42·7	43·2	33·0	33·0	51·0	47·3	37·7	97·5	34·6	31·0	62·9	81·5	76·6	82·4	77·4	106·4	68·1	52·4	52·5	77·3	76·4	74·0	89·6	47·3	33·3	69°·0	72°·5	38°·5	13°·7	24°·8	86°·2	Sutures partly obliterated; P.B. flat with very rounded edge; sub-nasal prognathism; ossicles in occipito-mastoid sutures R. and L.; metopic ridge; epipteric on L.
97	A	♂	1353·5	545	164	167·5	165·5	145	98·5	128·5	126	117·5	92	319	321	352·5	119	121	112·5	96	499	21·0?	64	97	125	46·0	46·0	28·5	10·1	23·0	34	2·1	10·0	45·0	44·1	33·6	33·3	49·5	45·1	33·4	91	33·6	29·0	62·4	80·5	75·2	87·6	76·1	115·0	65·8	62·0	62·0	74·7	65·5	67·5	86·3	43·9	21·0	68°·6	70°·6	40°·8	12°·4	28°·4	83°·0	Basilar synchondrosis; teeth worn; P.B. flat with very rounded edge; beginnings of suture across occipital bone from R. and L. asterion.
98	A	♂?	—	—	176·5	176	176	143·5	99	130	130·5	116·5	98	321·5	322	367	136	119	112	92	508·5	28·2	80	—	—	57·3	57·3	29·0	12·0	21·5	34·5	2·2	7·6	43·2		37·5	—	48·0	45·2	37·0?	—	34·2	30·4	59·0	81·5	74·0	81·6	74·1	110·1	—	50·6	50·6	86·8	—	77·1?	88·9	55·8	29·0	—	—	—	—	—	—	Ageing; sutures partly obliterated; teeth worn; P.B. with outward-upward slope and very rounded double edge; perforation of tympanic plates R. and L.; L. outer palate bridge; abscess cavity in 1st L. bi-cuspid.
105	A	♂	1458·5	600	179	180	178·5	146	94	139	138·5	121·5	104·5	325	328	373	134·5	123	115·5	99	514	14·5	67	102	142	53·7	54·0	26·4	9·8	21·0	30	1·9	8·1	45·6	43·0	32·0	34·1	46·8	42·5	41·4?	99·5	38·0	32·0	63·0	81·0	77·0	81·7	77·7	105·2	65·8	49·2	48·9	70·2	79·3	88·5?	84·2	46·7	23·5	66°·6	75°·2	38°·2	15°·8	22°·4	91°·0	Ageing; sutures beginning to be obliterated; alveolus partly absorbed; P.B. flat with blunt edge; whole base of skull much inflamed; bulging obelion; bi-parietal depressions; wormians R. and L. lambdoid; beginning of suture across occipital bone from R. and L. asterion; heavy supra-orbital ridges; receding forehead, marked malar marginal processes.
106	A	♂?	1291	649	169·5	175	174·5	142·5	93·5	125·5	126	111·5	94·5	306·5	311	350	125·5	113·5	111	97	499	13·2?	68?	97·5	129·5	54·7	54·7	26·9	11·0	22·0	29	3·6	8·9	45·0	43·7	31·0	33·0	46·7	37·8?	41·4	89?	40·8	29·8	65·4	81·3	72·1	81·6	72·4	112·8	69·8?	49·2	49·2	75·6	75·5	88·7	73·0	50·0	40·4	64°·1?	72°·4?	43°·5?	18°·6?	24°·9?	91°·0	Teeth much worn; P.B. flat with very rounded edge; flattened obelion; perforated tympanic plate on L.; very large post-condyloid foramina.
109	A	♂	—	591	168	167·5	168	144	86·5	131·5	132·5	110·5	100·5	313·5?	314	338	127	95	116	97·5	490	19·2	68	100	134?	52·0	51·3	28·0	12·7	21·0	35	2·9	7·6	43·0	43·0	32·3	31·9	49·0	43·7	36·8	88	37·7	32·2	60·8	87·0	79·0	85·7	78·8	108·8	68·3	53·8	54·6	75·1	74·2	75·1	85·4	60·5	38·2	59°·0	79°·2	41°·8	16°·8	25°·0	96°·0	Ageing; sutures becoming obliterated; P.B. with upward-outward slope and double edge, inner edge rounded, outer sharp; flattened obelion; perforated tympanic plates R. and L.; deep symmetrical depressions behind foramen magnum.
110	A	♂	1492·5	785	166·5	170	168·5	149·5	100	146·5	146	123	99·5	333	333·5	395	138	124	103	91	510	17·6?	72?	108	143	56·3	56·7	31·3	8·6	25·0	31	3·3	7·4	44·4	43·8	37·8	37·2	51·5	47·0	41·9?	96·5?	41·5	34·0	67·7	88·0	86·0	88·6	86·5	102·4	66·6	55·6	55·2	85·1	84·9	81·4?	81·9	34·4	44·6	66°·0?	71°·0?	43°·0?	11°·7?	31°·3?	82°·7?	Teeth much worn on inside; P.B. with outward-upward slope and double edge, both edges sharp; ossicles R. lambdoid and at R. squamo-mastoid angle; epipterics R. and L.; fairly heavy inion; marked supra-orbital ridges.
112	A	♂	1488·5	653	179·5	182	182	142	95·5	137·5	136	122	103·5	325	330	369	121	115	133	113	513	—	64·5?	103·5	132	52·7	52·9	30·6	8·9	24·0	30	2·8	7·2	45·0	43·5	34·7	33·5	47·5?	42·5?	41·5?	95?	41·5	31·7	62·0	77·9	74·6	78·1	74·8	104·4	62·2	58·1	57·8	77·1	77·0	87·4?	76·4	37·1	38·9	64°·0?	78°·2?	37°·8?	12°·5?	25°·3?	90°·7?	Old; sutures nearly obliterated; alveolar rim absorbed; ossicle at lambda; P.B. flat with blunt edge.
113	A	♂	1406	707	162	167·5	165·5	151	96·5	137·5	137	118	96·5	323·5	320·5	349	131	112	106	94·5	503	20·0	78	98	141	58·3	56·3	26·3	10·9	20·6	31·5	4·4	10·2	44·7	43·7	38·1	37·4	52·4	47·6	34·2	96	39·6?	34·0?	69·5	90·2	81·9	91·3	82·9	110·2	79·7	45·1	46·7	85·2	85·1	65·3	85·9?	52·9	43·1	63°·9	67°·7	48°·4	18°·3	30°·1	86°·0	Ageing; sagittal suture nearly obliterated; abscess cavity in L. 3rd molar: P.B. with outward-upward slant and sharp edge; sub-nasal prognathism; very asymmetrical with R. frontal, L. occipital, L. parietal bulges; very deep canine fossae; wormians R. lambdoid.
117	A	♂	—	–	175·5	180·5	178·5	148	91	—		120	—	326·5?	320?	376	136·5	126	113·5	92	498	—	—	—	—	—	—	—	11·4	19·5?	–	4·7	9·6	—	40·2	—	35·0	—	—	—	—	—	—	57·8	82·1	—	82·9	—	—	—	—	—	—	87·1	—	—	58·5	49·0	—	—	—	—		—	Fairly well marked supra-orbital ridges.
119	A	♂	—	562	160·5	163·5	162	142	87	130	128·5	115	90	313·5	313	339·5	120·5	115	104	93	481	16·0?	64	94·5	127	48·0	48·7	28·1	8·0	19·7	26	2·5	5·1	43·8	42·1	36·5	37·0	41·3?	39·7?	35·5	88	37·4	34·0	70·2	86·8	78·6	87·0	79·2	110·5	67·0	58·5	57·7	83·3	87·9	86·0?	90·9?	40·6	49·0	67°·1	70°·8	42°·1	17°·2	24°·9	88°·0	P.B. flat with rounded edge; sub-nasal prognathism; flattened obelion; small ossicle at R. asterion and in L. occipito-mastoid; epipterics R. and L.; L. tympanic plate perforated; large round foramen magnum.
121	A	♂	1481·5	640	183·5	183·5	183	143	90·5	133	130	118·5	96·5	322·5	325	384·5	125	140·5	119	95·5	521	16·8	65	101	128	48·8	49·2	31·5	11·6	27·0	37	3·5	10·1	42·7	41·2	32·5	32·8	55·5	49·1	38·9	101·5	33·0	24·9	57·1	78·0	70·8	78·3	71·1	110·1	64·1	64·5	64·0	76·1	79·6	70·1	75·5	43·0	34·7	75°·0	67°·0	38°·0	16°·0	22°·0	83°·0	Flattening round lambda and at obelion; large ossicles at lambda and R. inter-parietal associated with bathrocephaly; very asymmetrical with R. frontal and L. parietal and L. occipital bulges; wormian at obelion and in L. lambdoid; beginning of horizontal malar sutures R. and L.; P.B. flat with rounded edge; very small asymmetrical foramen magnum.
123	A?	♂	—		178·5	180·5	179	138	90	126	126	115	93	312	312	368·5	134	129·5	105	86	503	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·6	29·5	58·8	76·3	69·9	77·1	70·5	109·4	—	—	—	—	—	—	80·6	—	—	—	—	—	—	—	—	Ageing; coronal and sagittal sutures nearly obliterated.
124	A	♂?	1211	605	166·5	166·5	167	136	92·5	135	133	112·5	95·5	304	307	341·5	118	121	102·5	89	489·5	26·0	76	99·5	129·5	51·2	51·0	29·0	9·1	22·5	31	1·6	6·9	42·6	42·4	35·6	35·5	49·3	45·9	39·1	95·5	38·0	27·8	65·0	81·9	80·0	81·6	79·7	102·3	76·5	56·6	56·9	83·6	83·7	79·3	73·2	40·4	23·2	66°·4	66°·6	47°·0	18°·0	29°·0	84°·6	Teeth much worn; abscess cavity in 1st molar on L.; P.B. flat with blunt edge: prognathous; R. inner palate bridge.
130	A	♂	—	—	173	177	174·5	154	94	136	135	121·5	101·5	335	341	360·5	124·5	119	117	100·5	516	17·3?	71·5	96·5?	132·5	56·2	55·8	27·7?	12·4?	23·0?	36·5?	3·2	9·3	43·1	42·5?	35·1	34·4?	—	—	40·5	96·5?	37·0	30·2	63·3	87·2	76·4	88·2	77·3	114·1	73·7?	49·3?	49·6?	81·4	81·0?	—	81·6	54·0?	34·4	65°·1?	72°·7?	42°·2?	16°·0?	26°·2?	88°·7?	Teeth much worn; P.B. flat with rounded edge; very asymmetrical with R. parietal and R. occipital bulges; flattened obelion; R. and L. tympanic plates perforated.
136	A	♂	1593	701	185	184·5	183·5	137·5	91	142	141	129	96	337·5	343	392·5	132	147	113·5	96	517	—		99	127	49·0	48·7	25·0	10·7	23·5	33	3·2	9·8	42·0	43·3	32·5	33·2	—		40·3	—	40·8	34·6	61·2	74·5	76·5	74·9	76·9	97·4	—	51·0	51·3	77·4	76·7	—	84·8	45·5	32·7	—	—	—	—	—	87°·5?	3rd molars undeveloped; P.B. flat with sharp edge; flattened lambdoid region with 2 ossicles at lambda; bulging occiput; lower half lambdoid complicated with numerous large ossicles continuing right round temporals to asterions; flat circular condyles projecting over foramen magnum.
139	A	♂?	—	685	183	184	184	139	88·5	135·5	134·5	120·5	99	316	316	381	129·5	130·5	115	95	512	19·8?	74	103	134·5	53·9	55·0	29·1	10·2	20·5	29	2·7	6·5	43·9	43·5	32·5	32·8	51·6	45·6	38·9?	108·5	36·1	29·0	59·2	75·4	73·0	75·5	73·1	103·3	71·8	54·0	52·9	74·0	75·4	75·4?	80·3	49·8	41·5	76°·1	62°·4	41°·5	18°·4	23°·1	80°·8	Old; sutures partly obliterated; P.B. flat with blunt edge; prognathous; metopic ridge and sagittal ridge; wormians R. lambdoid; ossicles at R. and L. asterion; R. epipteric.
142	A	♂	1305·5	602	164	166·5	166·5	141·5	90	134·5	132·5	120·5	94	317·5	324	352·5	120·5	128	104	90	495	22·0?	68·5	104	139	49·5	49·5	29·1	9·3	23·0	32	1·2	7·1	44·0	42·5	33·2	34·0	46·3?	42·0	39·2	98·5	37·0	29·1	64·6	85·0	79·5	85·1	79·6	106·9	65·9	58·8	58·8	75·5	80·0	84·7?	78·6	40·5	16·9	72°·9	65°·6	41°·5	18°·1	23°·4	83°·7	Old; sutures becoming obliterated; teeth worn; receding forehead; sagittal crest and heavy inionic ridge; P.B. flat with rounded edge; sub-nasal prognathism; os japonicum on R.; pre-condyle on R.; marked temporal and supra-orbital ridges.
45	B	♂	1416·5	662	182·5	185·5	184·5	136	89	142	140	121	106	320	322	375	123	125·5	120·5	103	512	16·6?	68·5?	101	130	54·0	53·2	26·2	13·4	23·1	37	4·0	7·8	44·5	43·0	34·6	34·2	53·2	47·2	—	102?	35·0	30·7	58·2	73·5	75·6	73·8	75·9	97·2	67·8	48·5	49·2	77·8	79·5	—	87·8	58·0	51·3	67°·6	74°·0	38°·4	11°·0	27°·4	85°·0	Alveolar rim absorbed except near middle incisors; P.B. flat with blunt edge.
49	B	♂?	—	—	162?	—	160?	—	95·5	—	—	—	100·5	—	—	—	—	—	—	—	475·5?	—	65·5?	103·5?	136·5	49·3	49·3	27·9?	10·2	21·8	31·5	2·8	8·3	44·2	43·5	34·1	34·1	48·0?	43·8?	—	89·5	35·5	30·3	—	—	—	—	—	—	63·2?	56·6?	56·6?	77·1	78·4	—	85·4	46·8	33·7	60°·9?	79°·5?	39°·6?	8°·5?	31°·1?	88°·0	Crown sawn off and missing; senile; alveolus almost entirely absorbed; P.B. slopes upward and outward and has sharp edge.
50	B	♂	1354	569	168·5	171·5	170·5	137·5	87	142·5	141	119	100·5	320	318	359	126	121	112	97·5	483	20·0	69	98	133	54·9	54·9	26·0	11·5	21·5	33	3·1	7·1	45·0	44·0	31·0	32·6	47·7	44·0	41·6	92	36·1	33·2	65·0	80·2	82·4	80·5	82·8	97·2	70·4	47·4	47·4	68·9	74·1	87·2	92·0	53·5	43·7	62°·3	76°·0	41°·7	10°·7	31°·0	86°·7	Old; sutures becoming obliterated; teeth worn: P.B. flat with double edge; upper edge blunt, lower edge rounded; heavy superciliary ridges; flattening at obelion; wormians at L. asterion; tympanic plates perforated R. and L.
63	B?	♂	—	563	165	169·5	169	148·5	82	132·5	131·5	116·5	99·5	321·5	321	350	124·5	120	105·5	95·5	497	17·1	71·5	91	128·5	55·6	55·6	23·5	9·4	16·9	21·5	4·1	7·9	42·5	41·5	35·8	36·2	46·1	41·3	36·2	94	33·6	30·9	72·7	87·4	77·6	87·8	77·9	112·6	78·4	42·3	42·3	84·2	87·2	78·5	92·0	55·6	55·7	64°·0	72°·5	43°·5	11°·2	32°·3	83°·7	Teeth worn; P.B. flat with rounded edge; L. tympanic plate perforated.
66	B	♂?	—	537	168·5	172	170·5	136	84	132	130·5	116	96	308	309	361	119·5	134·5	107	87	484·5	15·9	64	104·5	133	50·0	50·0	29·3	9·0	21·8	29	3·4	8·0	43·2	42·1	32·1	31·7	49·5	44·7	42·0	90·5	34·2	29·9	58·0	79·2	75·8	79·7	76·4	104·4	61·2	58·6	58·6	74·3	75·3	84·8	87·4	41·3	42·5	65°·1	74°·8	40°·1	17°·2	22°·9	92°·0	Teeth worn; P.B. flat with rounded edge R. and sharp edge L.; small ossicle R. lambdoid; perforated tympanic plates R. and L.
75	B	♂	1596·5	508	175	177·5	176·5	148	99	142·5	139	126	99	335·5	336·5	375	130	131	114	97·5	520	18·0?	71·5?	106·5	133?	55·9	57·0	27·4	11·5	22·9	33·5	3·2	8·8	44·7	45·0	35·0	37·0	50·2	47·5	40·8?	92	37·3	30·8	62·8	83·5	78·3	84·1	78·8	106·6	67·1	49·0	48·1	78·3	82·2	81·3?	82·6	50·2	36·4	63°·0	73°·1	43°·9	14°·9?	32°·0?	85°·0?	Old; sutures nearly obliterated; alveolus partly absorbed; parietal protrusions near vertex corresponding to twin depressions on inner surface of skull, probably Pacchionian depressions; flattened obelion and lambda; ossicles in squamo-mastoid angles; perforated tympanic plates R. and L.; P.B. flat with sharp edge.
79	B?	♂	—	689	177·5	181	179·5	142·5	95·5	132	131	117	99	315·5	316	366	126	119·5	120·5	95·5	512	14·0	68·5	107	139	52·4	52·4	26·1	11·6	23·0	35·5	4·2	7·9	45·5	43·3	33·3	34·2	51·1	47·1	41·0	98·5	32·6	29·5	56·5	78·7	72·4	79·5	73·1	108·7	64·1	49·8	49·8	73·2	79·0	80·2	90·5	50·5	53·2	69°·1	70°·2	40°·7	13°·3	27°·4	83°·5	Old; teeth worn; sutures mostly obliterated; P.B. flat with sharp edge.
88	B	♂?	1293	641	180	180	180	139·5	86	132·5	130	119	90	314	314	383	125	119·5	138·5	110·5	498	19·0	67	96	120·5	46·2	47·2	26·5	8·3	19·6	26	2·9	6·4	44·6	43·5	32·6	34·6	48·0	43·0	39·0?	92	30·0	29·2	57·0	77·5	72·1	77·6	72·2	107·5	69·8	57·4	56·1	73·1	79·5	81·3?	74·9	42·3	45·3	69°·9	67°·0	43°·1	16°·6	26°·5	83°·6	Old; sutures nearly obliterated; alveolus about molars absorbed; P.B. flat with sharp edge; flattened obelion; R. inner palate bridge; epipteric on L.
24	C	♂?	1450	580	163·5	166	165	144	87·5	147	145	122·5	100	326·5	333	357	128	115	114	103	486	19·2?	72	95	133·5	54·7	54·5	26·6	11·3	19·2	31	3·9	7·5	45·0	44·5	36·6	37·4	49·4	44·0	39·0	92	41·5	33·5	73·2	86·7	87·4	87·2	87·9	99·2	75·8	48·6	48·8	81·3	84·0	79·0	80·7	59·0	52·0	62°·1	74°·1	43°·8	8°·9	34°·9	83°·0	Remaining tooth worn; receding forehead; P.B. flat with blunt edge; ossicles in occipito-mastoid sutures R. and L.; large post-condyloid depression on L.; small pre-condyles R. and L.
28	C	♂?	1578·5	560	185	184·5	184·5	144	91	141·5	140·5	118	102	317·5	319	380	133	130	117	95·5	[illegible]	14·4	73	99	136·5	58·0	58·0	26·5	12·2	18·8	32	3·6	6·5	43·6	43·0	35·8	34·4	52·7	47·3	45·0	95	38·7	32·0	58·4	78·0	76·3	78·0	76·0	102·3	73·5	45·7	45·7	82·1	80·0	85·4	82·7	64·9	55·4	63°·4	73°·5	43°·1	12°·0	31°·1	85°·5	Abscess cavity in outer alveolus L., and in 2nd L. molar; inflammation along alveolar margin; slight torus palatinus; bulging occiput; P.B. flat with sharp edge; wormians R. and L. lambdoid; epipteric on R.
31	C or outlier	♂?	1470·5	556	179	181	180	130·5	89·5	143·5	142·5	120	100	309	309	383·5	132	133	118·5	100	500	22·0	72	91	117·5	49·2	49·0	23·4	14·4	20·7	36	3·7	7·8	46·0	46·4	33·6	34·7	48·7	48·0	36·3	95·5	35·3	29·0	61·4	72·2	78·6	72·5	79·0	91·8	79·1	47·6	47·8	73·0	74·8	74·5	82·2	69·6	47·4	65°·0	71°·9	43°·1	9°·1	34°·0	81°·0	Sutures partly obliterated; P.B. has steep upward-outward slant with sharp edge; sub-nasal prognathism; sagittal crest; prominent malar marginal process on L.
51	C	♂	—	526	178	183	182	141·5	88	136·5	134·5	120·5	97	320	323·5	377·5	128·5	117	132	113	509·5	21·0	76·5	98·5	126·5	56·4	57·1	27·6	12·1	18·5	30·5	3·1	9·5	44·2	44·1	34·2	34·0	50·0	49·0	44·0	101	36·9	29·4	62·9	77·3	73·5	77·7	73·9	105·2	77·7	48·9	48·3	77·4	77·1	88·0	79·7	65·4	32·6	70°·0	64°·5	45°·5	18°·5	27°·0	83°·0	Old; teeth badly worn, with surface slanting inwards and upwards; big abscess cavity in 1st R. molar; marked depression at obelion; large ossicle in R. lambdoid; very large post-condyloid foramina especially on R.; suggestion of os japonicum; P.B. flat with rounded edge.
67	C	♂	1322·5	607	172	170	171	133·5	90	140	139·5	117	101·5	308·5	309	351·5	120	117	114·5	92·5	491	20·5	76	90·5	130·5?	54·9	54·6	23·5	9·8	19·0	28·5	3·3	7·4	45·1	45·6	35·9	37·0	48·7	43·9	39·3	93	38·8	30·0	57·5	78·6	81·9	78·0	81·4	95·9	83·0	42·8	43·0	79·6	81·1	80·7	77·3	51·6	44·6	61°·2	73°·0	45°·8	12°·2	33°·6	85°·2	Hole through frontal near L. stephanion probably due to abscess; 1st molars worn; abscess cavity 1st and 2nd R. molars; P.B. with outward-upward slant and sharp edge; flattened obelion; ossicles R. and L. lambdoid, and at R. asterion.
80	C	♂	1389·5	639	169·5	171·5	171	142·5	91	142·5	141	124	100	327	329	362	119·5	128·5	114	101	492	24·0?	77	98	127·5	55·0	55·0	25·5	10·5	18·5	29	3·6	7·8	42·0	41·4	37·1	37·1	53·0	47·2	40·0	99	33·6	29·2	67·8	83·1	82·4	83·2	82·5	100·9	78·5	46·4	46·4	88·3	89·6	75·5	86·9	56·8	46·2	66°·6	67°·9	45°·5	14°·1	31°·4	82°·0	P.B. flat with roundish edge; small ossicle R. lambdoid; perforation of R. tympanic plate.
111	C	♂	—	—	171	174·5	173·5	152	90	142·5	142	118	103	320?	320?	360	130	119	111	98·5	510	19·4	75	88	—	57·5	58·3	25·6	10·7	16·2	28	4·2	4·2	45·0	43·6	36·3	36·5	47·4	43·7	40·6	95·5	37·5	32·8	62·9	87·1	81·4	87·6	81·8	107·0	85·2	44·5	43·9	80·7	83·7	85·7	87·5	66·0	100·0	62°·4	73°·2	44°·4	14°·8	29°·6	88°·0	Teeth worn; hooked inion; sutures beginning to be obliterated; P.B. flat with slightly rounded edge; prognathous.
135	C	♂?	—	—	186	187·5	187	135	96	127	128	115·5	—	314	313	367·5	132	120·5	115	94·5	512	—		—	115?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35·8	30·8	58·6	71·9	67·9	72·2	68·4	105·5		—	—	—	—	—	86·0	—	—	—	—	—	—	—	—	Old; sutures nearly obliterated; depressed obelion.

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* See Section 9 of Text.

FEMALES

APPENDIX II (*continued*). TABLES OF THE ABSOLUTE AND INDICIAL MEASUREMENTS OF THE INDIVIDUAL CRANIA (ABSOLUTE LENGTHS IN MMS.)

					Lengths									Arcs								Face				Nose								Orbits				Palate and Profile				Foramen		Indices															Angles						
No.	Type	Sex	C	W. in grs.	F	L′	L	B	B′	H	H′	OH	LB	Q	Q′	S	S_1	S_2	S_3	S_3'	U	PH	G′H	GB	J	NH R	NH L	NB	DS	DC	DA	SS	SC	O_1R	O_1L	O_2R	O_2L	G_1	G_1'	G_2	GL	fml	fmb	Oc I	B/L′	H/L′	B/L	H/L	B/H	G′H/GB	NB/NH R	NB/NH L	O_2/O_1 R	O_2/O_1 L	G_2/G_1	fmb/fml	DS/DC	SS/SC	N ∠	A ∠	B ∠	θ_2	θ_1	P ∠	Remarks*
1	A (B?)	♀	1276·5	550	168	168·5	169	142·5	87	127	127	111	93·5	312	312·5	350	124	118	108	93	495	16·1?	65·5	91	125	49·7	49·2	25·0	8·0	17·9	25	1·7	5·2	40·0	39·7	36·6	36·5	46·0	41·6	37·0	90	37·0	31·0	63·8	84·6	75·5	84·3	75·3	112·0	72·2	50·3	50·8	91·5	92·0	80·4	83·8	44·7	32·7	66°·2	71°·9	41°·9	16°·1?	25°·8?	88°·0	P.B. flat with blunt edge.
2	A	♀	1248	643	159·5	159	159	147·5	90	139·5	137·5	119·5	95	325·5	324	344	121	123	100	90·5	485	20·0?	71?	95·5	130·5	53·1	52·5	24·9	10·1?	22·6?	32?	3·3	9·8	37·3?	39·2	34·0	35·0	50·0	44·1	39·5	91?	33·9	29·0	72·7	92·7	86·1	92·8	86·4	107·4	74·3?	46·9	47·4	91·2?	89·3	80·6	85·3	45·0?	33·7	64°·6?	70°·6?	44°·8?	12°·9?	31°·9?	83°·5	Flattened obelion; P.B. flat with sharp edge; small triangular epipteric on R.; alveolus absorbed in molar region R. and L.; perforation of tympanic plates R. and L.; right interior palate bridge.
5	A	♀	—	—	160	161	159·5	142	84	130	128	114·5	86·5	315	310·5	359	127?	129	103	91	480	—	—	—	117?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	32·0	29·9	67·7	88·4	79·7	88·4	80·5	110·8	—	—	—	—	—	—	93·8	—	—	—	—	—	—	—	—	Flattening in lambdoid region; ossicles in lambdoid suture R. and L.; squamo-mastoid ossicle on R.; small epipteric on R.
7	A	♀	1085·5	574	159·5	161	160	132	85·5	126·5	126·5	110·5	92·5	295·5	295·5	334	115	115	104	89·5	465	22·1	70·5	92	119	51·5	51·5	24·3	11·7	21·5	33	2·4	7·9	38·8	37·9	32·5	31·4	50·0	47·4	38·2	93·5	32·5	28·2	63·9	81·9	78·5	82·4	78·9	104·4	76·6	47·2	47·2	83·8	82·9	76·4	86·8	54·4	30·4	68°·5	66°·9	44°·6	15°·1	29°·5	82°·0	Young adult; sutures open; P.B. rounded; prognathous; frontal process to squamous on R.; squamo-mastoid ossicle on L.; small pre-condyle on L.
8	A	♀	—	497	173	176	175	139·5	91·5	136	132·5	116	96	308·5	312	360	117	129	114	94	502	18·7	73·5	99	134·5	56·4	57·5	27·2	8·5	21·0	29	2·9	10·8	41·3	41·8	36·2	35·0	44·0	39·5	39·2	88·5	37·0	31·2	59·1	79·3	75·3	79·7	75·8	105·2	74·2	48·2	47·3	87·7	83·7	89·1	84·3	40·5	26·9	60°·8	72°·4	46°·8	14°·1	32°·7	86°·5	Sutures closed; L. canine never cut, but grown in lateral direction across 2nd incisor; P.B. flat with blunt edge; squamo-mastoid ossicle on L.; epipteric on L.; glenoid fossa perforated on R.
10	A	♀	1090·5	500	163	165·5	164	133	97	130	129·5	111	97·5	296·5	298	339	125·5	118	95·5	85	474	18·0?	65	102	129	51·8	51·8	30·2	12·4	22·0	35	2·6	9·0	44·8	43·7	30·8	30·8	52·5	50·5	40·0?	99	34·8	25·2	69·6	80·6	78·3	81·1	78·9	102·9	63·6	58·3	58·3	68·8	70·5	76·2?	72·4	56·4	28·9	72°·0	69°·4	38°·6	10°·6	28°·0	80°·0	Ageing; sutures round lambda nearly obliterated; P.B. very rounded and double; sub-nasal prognathism; slight pre-condyles on R. and L.
11	A	♀	—	555	175	175	174	133·5	88·5	139	139	118·5	96	311?	310	372	125	130	117	100	489	16·3?	65·5	95·5	125?	49·5	49·6	25·8	8·0	18·8	26·5	2·1	6·5	40·0	41·0	31·0	30·5	46·0	43·5	41·0	93·5	34·4	27·6	62·7	76·1	79·3	76·6	79·9	95·9	68·7	52·1	52·0	77·5	74·4	89·1	80·2	42·6	32·3	67°·6	72°·0	40°·4	12°·0	28°·4	84°·0	Adolescent; sutures open; P.B. flat with blunt edge; ossicle in R. occipito-mastoid suture; small pre-condyle on R.
17	A	♀	1215·5	472	166	167	166	132·5	87·5	127	126·5	110	95	302·5	302·5	339	125	112	102	88	477	11·6?	58	92	122·5	46·9	46·9	26·5	10·1	20·6	30	3·2	11·0	41·4	42·3	33·5	32·4	42·5	40·0	29·0	87	34·6	29·0	64·0	79·3	75·7	79·7	76·1	97·1	63·0	56·5	56·5	80·9	76·6	68·2	83·8	49·0	29·1	64°·0	79°·0	37°·0	9°·7	27°·3	88°·7	Old; coronal and sagittal sutures becoming obliterated; os japonicum both sides but suture partly obliterated; P.B. rounded; perforation of tympanic plates R. and L.
21	A	♀?	1442·5	487	166	167·5	166·5	142	94	143·5	140·5	118	97	315	311·5	360	121	125	114	98·5	493	13·2	69·5	99	136	57·9	56·8	28·0	10·7	19·2	31	3·3	9·2	43·5	42·8	36·0	36·6	47·0	42·5	40·8	87?	36·4	31·0	63·8	84·9	83·8	85·5	84·4	101·3	70·3	48·4	49·3	82·8	85·5	86·8	85·2	55·7	35·9	60°·2?	75°·8?	44°·0?	8°·3?	35°·7?	84°·1	Aged; sagittal suture partly obliterated; abscess cavity in 1st R. molar and in alveolus; sloping forehead; P.B. flat with blunt edge; sub-nasal prognathism; sagittal crest and slight metopic crest.
26	A	♀	1417·5	527	172	172	172	135·5	96·5	141	140	117·5	100·5	311·5	312·5	357·5	121·5	127	109	93·5	494	14·7	65·5	105	139·5?	53·1	54·8	29·0	7·6	23·0	28·5	2·6	8·5	44·1	45·2	38·4	37·8	48·8	45·0	40·0	95	35·0	30·6	63·3	78·6	81·3	78·7	81·4	96·6	62·3	54·6	52·9	87·1	83·6	82·0	87·4	33·0	30·6	66°·1	74°·9	39°·0	3°·1	35°·9	78°·0	P.B. flat with blunt edge; bony spicule for pulley of superior oblique muscle in R. orbit; R. inner palate bridge.
29	A	♀	1342·5	578	168	171	170	141	91·5	133	131	114	102·5	309·5	309	343	117	108	118	103	491	21·0	71	108·5	132	52·0	51·0	29·5	9·3	24·4	34·5	1·5	11·4	43·4	43·5	33·2	33·5	52·5	48·9	42·5	105·5	35·0	29·0	65·5	82·5	76·6	82·9	77·1	107·6	65·6	56·7	57·8	76·5	77·0	81·0	82·9	38·1	13·2	72°·1	67°·9	40°·0	10°·8	29°·2	78°·7	P.B. flat with blunt edge; tympanic plates perforated R. and L.
32	A	♀	1411·5	515	170·5	171·5	171	142	93·5	133·5	134·5	112	99·5	310·5	312	354	130·5	122	101·5	89	498	18·8	70	104	135	52·2	52·9	26·3	—	20·4	—	—	6·6	44·4	44·0	39·0	38·1	—	—	41·5?	91	37·2	33·0	66·4	82·8	78·4	83·0	78·6	105·7	67·3	50·4	49·7	87·8	86·6	—	88·7	—	—	62°·2	74°·8	43°·0	10°·8	32°·2	85°·6	Old; sagittal suture nearly obliterated; 3rd molars never cut; P.B. flat with blunt edge; slight flattening at obelion; ossicle at R. asterion.
33	A	♀	—	—	162	—	162·5	145	92·5	133·5	131	—	85	—	—	370	132·5	133	104·5	91·5	491	17·2?	63	95	—	48·0	47·9	25·5	11·8	20·8	33	3·9	8·7	41·6	41·1	30·8	31·8	43·8	42·3	38·2	91	34·1	27·0	66·1	—	—	89·2	80·6	110·8	66·2	53·1	53·3	74·0	77·4	87·2	79·2	56·7	44·8	74°·0	64°·2	41°·8	—	—	—	Sagittal suture open; molars much worn; slight torus palatinus; P.B. flat with blunt edge; sub-nasal prognathism; suggestion of os japonicum R. and L.
35	A	♀	1322·5	560	179·5	181·5	181	131	93	128·5	126·5	111·5	98	300	299	365	124	125	116	95	504	21·9	69·5	99	126·5	49·9	49·9	26·4	7·0	20·8	26	1·4	5·5	42·5	42·7	34·0	33·0	46·2	42·0	39·4	91	33·5	30·0	58·6	72·2	69·5	72·5	69·8	103·9	70·4	52·9	52·9	80·0	77·3	85·3	89·6	33·7	25·5	63°·0	73°·9	43°·1	16°·0	27°·1	89°·9	Aged; sagittal suture nearly obliterated; wormian at R. squamo-mastoid angle; P.B. having outward-upward slope with blunt edge.
37	A (B?)	♀	1254	554	172·5	172	171·5	129	88·5	125	124	110	89	297	295·5	365	123	134	108	91·5	486	14·2?	59	98	123	45·0	46·0	24·5	11·1	21·1	32	2·6	8·5	42·0	40·5	32·3	34·1	—	37·2?	37·8	82·5	33·5	30·0	61·5	75·1	72·1	75·4	72·4	104·1	60·2	54·4	53·3	76·9	84·2	—	89·6	52·6	30·6	63°·8	76°·3	39°·9	13°·2	26°·7	89°·5	P.B. flat with sharp edge; triangular epipteric on L.; R. tympanic plate perforated; 2 marked pre-condyles; slight exostosis on L. parietal near epipteric.
42	A	♀?	—	478	180·5	181	—	134	95·5	122	123	108	96·5	294·5	293	364	121	126	117	89·5	508	24·0?	73	105·5	142	51·5	51·5	27·6	10·7	27·5	36	2·9	9·3	46·3	45·5	36·3	36·8	52·1	45·6	39·0	95	32·0	28·7	55·0	74·0	68·0	—	—	108·9	69·0	54·0	54·0	78·4	80·9	74·9	89·7	38·9	31·2	66°·3	69°·0	44°·7	17°·0	27°·7	86°·0	Large hole in 1st L. molar; P.B. having outward-upward slant and rounded edge; suggestion of os japonicum R. and L.
64	A	♀	—	455	164	167·5	166	142	92	141·5	139	115·5	91	310·5	313	365	127	126	112	102	490	—	65?	100·5	137	51·9	51·4	24·3	7·8	18·1	25·5	2·2	7·0	42·7	41·4	36·1	35·9	43·2	38·5	34·8	75?	38·9	33·1	75·1	84·9	83·2	85·5	83·[illegible]	102·0	64·7?	46·8	47·3	84·5	86·7	80·6	85·1	43·1	31·5	54°·4?	80°·6?	45°·0?	13°·4?	31°·6?	94°·0?	P.B. flat with blunt edge; hooked inion; ossicle in R. lambdoid.
70	A	♀	1335·5	570	176	176·5	176	136	95·5	132	130	119	97	316	321	362	125·5	121	115·5	99	499	15·0	64·5	97·5	124?	51·2	52·0	28·9	10·4	24·2	36	2·5	8·8	44·0	46·5	33·8	34·6	—	—	37·0	90·5	32·8	29·5	63·0	77·0	73·6	77·3	73·0	104·6	66·1	56·4	55·6	76·8	74·4	—	89·9	43·0	28·2	64°·4	75°·5	40°·1	13°·5?	26°·6?	89°·0?	P.B. flat with rounded edge.
73	A	♀?	1404·5	570	165	169	168·5	145	92·5	136	135	118	98	318	319	356	131·5	121·5	103	94	495·5	18·4	69·5	99·5	135	54·7	53·7	26·0	12·1	18·2	28	5·1	9·5	45·5	45·0	35·0	34·6	43·2	38·2	41·2	90	36·0	32·5	76·3	86·0	80·0	86·2	80·[illegible]	107·5	69·8	47·5	48·4	76·9	76·9	95·4	90·3	66·5	53·7	62°·0	74°·8	43°·2	13°·7	29°·5	88°·5	Old; sutures nearly all obliterated; teeth worn and the whole crown of 1st L. molar destroyed by abscess; 3rd molars have one fang only and are shaped like incisors; P.B. flat with very rounded edge; sub-nasal prognathism.
81	A	♀	—	—	161	162·5	163	136	86	125·5	123·5	111·5	88·5	—	—	339	119	113	107	95·5	467	18·8	68	92	118·5?	52·0	52·0	27·4	8·9	18·0	28	2·5	9·0	39·0	39·9	33·1	34·0	48·1	44·6	34·2	93	34·0	30·1	70·2	83·6	76·1	83·4	75·9	109·9	73·0	52·5	52·5	84·9	85·2	71·1	88·5	49·4	27·8	71°·5	64°·5	44°·0	21°·0	23°·0	85°·5	Young adult; L. 3rd molar coming through; sutures open; P.B. flat with very rounded edge; sub-nasal prognathism; small pre-condyle on R.
83	A	♀	—	473	168	170	169	134·5	88	120·5	120	109·5	94	302?	300	344·5	119	115	110·5	91	479·5?	17·0?	60	88	112?	45·0	45·0	25·6	8·9	17·8	26·5	2·1	7·0	40·0	40·5	32·7	32·0	43·2?	39·5	33·8	97·5	30·9	27·0	59·0	79·1	70·6	79·5	70·9	112·1	68·0	56·9	56·9	81·8	79·0	78·2?	87·4	49·9	30·0	74°·8	68°·8	36°·4	16°·8	19°·6	85°·6	P.B. flat with sharp edge; marked sub-nasal prognathism; ossicles R. and L. lambdoid; small epipterics R. and L.
84	A	♀	1169·5	588	153	152	152·5	143	92·5	131	129	118·5	92·5	320	318	332	116·5	122	93·5	81	469	14·0	59	97·5	129	48·2	48·0	27·9	9·3	19·8	28·5	2·4	7·9	42·5	40·9	33·1	33·2	43·8	39·0	35·3	86	32·2	31·0	64·5	94·1	84·9	93·9	84·7	110·9	60·6	57·9	58·1	77·9	81·2	80·6	96·3	47·0	30·8	65°·2	76°·4	38°·4	15°·1	23°·3	91°·5	P.B. flat with blunt edge; wormians R. and L. lambdoid; L. tympanic plate perforated.
85	A	♀	—	527	155	155·5	154	146·5	90	129	128	115	91	321	325	328	116·5	108	103·5	90·5	479	13·0?	64·5	90·5	128·5	51·2	51·8	24·4	6·5	17·7	23·5	1·0	7·6	42·1	41·1	36·7	37·5	44·1	40·1	35·8	86	37·5	29·0	65·8	94·1	82·3	94·9	83·0	114·4	71·3	47·7	47·1	87·2	91·2	81·2	77·3	36·7	21·1	64°·6	73°·1	42°·3	11°·4	30°·9	84°·5	P.B. flat with blunt edge; large anterior alveolar abscess above 2nd L. molar; large epipteric on R.; beginning of horizontal malar suture R. and L.; L. tympanic plate perforated; prognathous.
87	A (B?)	♀	1239	503	161·5	162	161·5	139·5	87·5	133·5	131·5	115	90	308·5	306	345	116	124	105	91·5	474	22·0?	70	98·5	127	49·9	50·8	29·0	8·1	20·2	29	2·4	9·0	43·3	42·0	37·3	37·5	49·5	45·2	36·7?	89	33·5	31·0	65·5	86·0	81·2	86·4	81·6	105·9	70·0	58·1	57·1	86·1	89·3	74·1?	92·5	40·1	26·7	66°·3	67°·6	46°·1	13°·4	32°·7	81°·0	P.B. flat with sharp edge; prognathous; posterior half of sagittal suture depressed; R. and L. lambdoid almost entirely formed of large wormians and ossicles; ossicle R. and L. asterion and R. and L. squamo-mastoid angle; fronto-squamous process R. and L.; R. tympanic plate perforated; pre-condyle on L.; parieto-squamous suture straight and nearly horizontal.
89	A	♀	1345·5	527	171·5	172	171·5	132·5	91	134·5	134	115	96·5	302·5	302	353	120·5	122·5	110	93	486	18·5?	71?	96	124	53·2	53·2	27·5?	7·6	23·6	30	2·3	9·4	42·2	42·8	35·5	34·5	—	—	41·8	90·5?	40·4	30·2	61·4	77·4	77·8	77·4	78·2	99·0	84·0?	51·7?	51·7?	84·1	80·6	—	74·8	32·2	24·5	63°·2?	72°·1?	44°·7?	14°·4?	30°·3?	86°·5?	P.B flat with blunt edge; wormian L. lambdoid; ossicle in L. occipito-mastoid; perforated tympanic plate on R.
93	A	♀	1111·5	256	158·5	158·5	158	134	89	121	120	107	83	289·5	292	332	115	106	111	93·5	467	17·0?	63?	92	120·5	50·5	47·8	30·7	6·8	22·2	29	1·6	11·5	39·2	39·0	31·4	32·8	43·5	40·6	36·5	78·5?	34·0	27·5	61·2	84·6	75·7	85·0	76·0	111·8	68·3?	60·8	64·2	80·1	84·1	83·9	79·5	30·6	14·0	63°·3?	71°·0?	45°·7?	14°·0?	31°·7?	85°·0?	Senile; sutures nearly all obliterated; alveolus absorbed near L. incisors; P.B. with outward-upward slope, steeper on L., with blunt edge; sub-nasal prognathism; depressed obelion and parietal depressions to R. and L. of obelion; epipteric on L.; perforation of R. and L. tympanic plates.
99	A	♀	1481·5	549	169	171	170	147	91	129	127·5	116	92	317·5	320	352	124·5	118·5	109	96	501·5	16·3?	65·5	95·5	131	50·2	50·0	26·2	8·4	22·0	29	0·9	4·8	43·8	43·0	35·6	37·0	48·6	45·6	36·8	95	40·5	33·5	67·5	86·1	74·6	86·6	75·0	115·5	68·5	52·2	52·4	81·3	86·0	75·7	82·7	38·2	18·8	71°·5	67°·5	41°·0	14°·3	26°·7	81°·8	L. 3rd molar coming through; P.B. flat with sharp edge; wormians in L. lambdoid; ossicles in R. and L. occipito-mastoid; large epipteric on R.
103	A	♀	1184	621	171	167·5	169·5	133	95	131·5	132·5	114	97	305	306	348·5	117·5	119	112	93	479	15·7	63·5	96	122	49·0	48·0	28·3	8·9	22·0	30·5	1·4	9·5	43·8	43·0	35·0	35·2	46·7?	42·5	36·5	95	35·8	29·4	59·8	79·5	79·1	78·6	78·1	100·5	66·1	57·8	59·0	79·9	81·9	78·2?	82·1	40·5	14·7	68°·9	72°·6	38°·5	8°·4	30°·1	81°·0	Bathrocephalic with tri-partite inter-parietal; flattened obelion; large perforations of R. and L. tympanic plates; P.B. flat with sharp edge.
107	A	♀	1373	420	174	175	174	140·5	93·5	135	134	118·5	97·5	318·5	318	354·5	127	117	110·5	96	501·5	18·6?	64·5	94	130	50·2	48·8	27·5	10·1	20·5	30	2·6	5·9	42·1	40·6	35·5	35·0	42·4	40·1	—	90	35·0	29·5	64·9	80·3	76·5	80·7	76·9	105·0	68·8	54·8	56·4	84·3	86·2	—	84·3	49·3	44·1	63°·5	76°·5	40°·0	16°·0	24°·0	92°·5	Alveolus partly absorbed; P.B. flat with blunt edge; epipterics R. and L.; symmetrical parietal depressions.
115	A	♀	—	—	181·5	182·5	182	133	91	124	127	108	100	—	398?	359·5	124·5	123	112	91·5	502?	13·5?	60·5	93?	—	47·4	47·8	28·0	11·6	23·0	36·5	2·5	8·2	41·0	41·1	34·0	34·0	—	45·0?	39·6	94	35·9	27·2	58·4	93·0	69·7	73·1	69·7	104·8	64·8	59·1	58·6	82·9	82·7	—	75·8	50·4	34·1	66°·5	77°·5	36°·0	10°·5	25°·5	88°·0	Sagittal suture becoming obliterated; surface of teeth worn down, with inward-upward slant; P.B. flat with rounded edge; nasal spine in 2 parts, R. and L. divided by large groove; ossicle in sagittal near lambda.
118	A?	♀	1061·5	—	163	165·5?	165	132·5	91·5	125·5?	125	111?	89·5	301·5?	299?	339	122	106	111	96	472	—	—	—	—	—	—	—	—	—	—	3·5?	9·0?	—	—	—	—	—	—	—	—	34·0	27·2	64·2	79·8?	75·3?	80·2	75·7	106·0	—	—	—	—	—	—	80·0	—	36·7?	—	—	—	—	—	—	Flattened obelion; small foramen magnum.
120	A	♀	1194·5	—	168	165·5	167	134	93·5	132?	132	116?	93	314	313	352	123	119·5	109·5	91·5	477·5	—	—	—	123·5?	—	—	—	10·1	24·3	33	2·[illegible]	11·2	—	—	—	—	—	—	—	—	33·0	27·3	60·4	80·9	79·6	80·2	79·0	101·6	—	—	—	—	—	—	82·7	41·6	17·9	—	—	—	—	—	—	Oldish; depressed obelion; tympanic plate perforated R. and L.
125	A	♀	1355·5	697	170	169·5	169·5	143·5	93	134	134	117	92	318	320	363	126	127	110	95	493·5	16·2?	66·5	99	123·5	51·8	50·6	26·3	9·9	18·3	29	2·7	8·0	39·8	39·2	35·0	34·8	49·0	45·5	38·8	94	33·0	27·1	64·1	84·7	79·0	84·7	79·1	107·1	67·2	50·8	52·0	87·9	88·8	79·2	82·1	54·1	33·8	70°·4	67°·6	42°·0	15°·4	26°·6	83°·0	Teeth worn; P.B. flat with rounded edge; sub-nasal prognathism; ossicle at L. asterion; fronto-squamous process on L.; pterion in K. on R.
126	A	♀	1130	538	154·5	154	154·5	138·5	86	126·5	126	111	91	302·5	301·5	321·5	112	111	98·5	86	462	19·3	69	96	116	51·5	53·0	23·8	7·4	20·3	27	2·6	9·0	40·9	39·2	35·0	35·0	—	42·0	41·0	92	33·8	28·5	53·2	89·9	81·7	89·8	81·7	110·0	71·5	46·2	44·9	85·6	89·3	—	84·3	36·5	26·9	68°·4	67°·4	44°·2	17°·1	27°·1	84°·5	Beginning of cleft palate; depressed obelion; P.B. flat with slightly rounded edge; sub-nasal prognathism; small ossicle R. lambdoid; R. tympanic plate perforated.
129	A	♀	1126	640	164·5	164·5	164·5	136·5	88	132·5	132	112·5	92·5	308	310	350·5	121·5	118·5	110·5	96	475·5	13·5	59	95·5	120·5	48·0	47·5	26·8	7·0	19·7	24·5	1·4	6·2	40·0	41·8	36·0	37·0	—	—	—	89·5	33·2	26·4	65·0	83·0	80·1	83·1	80·3	103·6	61·6	55·8	56·4	90·0	88·5	—	79·5	35·5	22·6	68°·7	73°·4	37°·9	11°·6	26°·3	85°·0	Alveolus absorbed in molar region; P.B. flat with blunt edge.
131	A	♀	—	—	160·5	159?	160	145	84	133	132	118·5?	97?	314?	316?	342·5	116	121	105·5	91	483·5	—	—	—	—	—	—	—	11·9	21·0	33	2·[illegible]	9·0	—	—	—	—	—	—	—	—	33·0?	30·0	63·9	91·3	83·0	90·8	82·4	110·1	—	—	—	—	—	—	90·9?	56·7	32·2	—	—	—	—	—	—	Old; sutures becoming obliterated; sagittal suture from vertex to obelion depressed between bi-parietal ridges; L. tympanic plate perforated.
132	A?	♀?	1295·5	—	163	163·5	163	135·5	86·5	135·5	135	117·5	94·5	311·5	320	348·5	126·5	124	98	84·5	473	—	—	—	128?	—	—	—	9·8?	21·9?	—	2·3	6·8	—	—	—	—	—	—	—	—	34·0	28·0	63·8	82·8	82·5	83·2	82·8	100·4	—	—	—	—	—	—	82·4	44·7?	33·8	—	—	—	—	—	—	Young adult; flattened obelion; small ossicles at R. and L. asterion, and in L. masto-parietal suture; small epipteric on R; L. tympanic plate perforated.
137	A	♀	1320·5	471	163	164	163	139·5	93	134·5	133	122	89·5	324·5	327	353	124	124	105	92·5	485·5	17·7	65	92	128	49·6	49·7	24·0	8·7	20·1	29	2·4	8·9	41·9	41·2	33·5	33·6	48·0	43·2	36·8	89	36·0	30·5	66·7	85·0	81·0	85·5	81·5	105·0	70·5	48·4	48·3	80·0	81·6	76·7	84·7	43·3	27·0	68°·3	68°·9	42°·8	19°·1	23°·7	88°·0	Remaining tooth worn; P.B. flat with blunt edge; sub-nasal prognathism; depressed sagittal; ossicles R. and L. lambdoid; R. tympanic plate perforated; 3rd L. molar single-fanged.
141	A	♀	—	555	164	163·5	164	133·5	89·5	131	130	111·5	96·5	300	301	340	112·5	111·5	116	98	474·5	17·0	67	94?	119	52·0	50·8	27·2	10·1	19·0	27	3·8	8·6	41·9	42·5	35·5	34·2	—	—	35·0	88·5	34·3	32·0	61·3	81·8	79·5	81·5	78·3	102·8	71·1?	52·3	53·5	84·7	80·5	—	93·3	53·2	44·2	62°·3	77°·5	42°·2	15°·3	26°·9	90°·8	P.B. slanting upward and outward with sharp edge; wormian near obelion, and at R. squamo-mastoid angle; large tympanic perforation R. and L.
12	B	♀	—	274	156·5	158	157·5	126·5	70·5	123	121	104·5	87·5	281	279	326	114·5	120·5	91	81·5	451	14·2	59·5	87·5	111·5	46·4	46·4	21·5	6·9	14·7	21	1·8	4·0	38·7	37·4	34·0	34·3	41·2	36·8	32·9	82·5	38·5	30·0	71·1	80·1	70·6	80·3	76·8	104·5	67·[illegible]	46·3	46·3	87·9	91·7	79·9	77·9	47·0	45·0	64°·7	74°·6	40°·7	8°·4	32°·3	83°·0	Adolescent; basilar synchondrosis; no 3rd molars and no room for them; P.B. flat with blunt edge; flattened obelion; ossicle at lambda, and in L. occipito-mastoid suture.
13	B?	♀	1320	518	166·5	171·5	170·5	138	93·5	139·5	138	123	102	321·5	321	354	123	122	109	98·5	490	18·1	69	97	—	52·4	52·0	27·0	10·0	24·8	34·5	3·5	10·3	42·3	—	36·2	36·5	48·4	44·3	39·8	99·5	35·3	33·4	72·2	80·5	80·5	80·9	80·9	100·0	71·1	51·5	51·9	85·6	—	82·2	94·6	40·3	34·0	68°·0	72°·0	40°·0	11°·0	29°·0	83°·0	Old sutures closing; teeth worn; 3rd molars not cut; R. inner palate bridge; sloping forehead; P.B. flattish with sharp edge; flattened obelion; very marked pre-condyles nearly meeting to form bridge; symmetrical anterior parietal flattening.
15	B?	♀	1333	561	179	179	179·5	131·5	99	138	138	119·5	104·5	311·5	313	363	127	113	123	105	498	17·0?	65?	94	126	51·0	51·0	28·0	13·1	23·1	37	4·7	11·5	45·8	44·5	35·0	34·0	—	—	41·0?	100?	39·0	34·0	62·2	73·4	77·1	73·1	76·9	95·1	69·[illegible]	54·9	54·9	76·4	76·4	—	87·2	56·7	40·9	67°·8?	75°·2?	37°·0?	8°·8?	28°·2?	84°·0?	Old; sutures closed and partly obliterated; alveolar margin absorbed except at L. canine; P.B. flat with blunt double edge; large wormian at lambda; malar marginal processes; R. tympanic plate perforated; healed fracture of L. superciliary ridge and L. zygomatic arch; sagittal suture depressed about vertex.
19	B?	♀	1149	418	164	166·5	166	127	91	133	131·5	112	96	294	293	344	110·5	136·5	97	84·5?	465	8·5?	59·5?	87	120·5	47·5	47·5	23·0	13·4	19·5	36	[illegible]	9·6	43·8	43·8	29·0	30·2	46·1	42·4	34·7	87?	—	—	64·9	76·1	79·1	76·5	79·4	96·3	68·[illegible]	48·4	48·4	66·2	69·0	75·3	—	68·7	38·5	62°·9?	79°·5?	37°·6?	6°·3?	31°·3?	85°·8?	P.B. rounded; alveolar abscess near 1st R. molar; inflammatory condition about L. foramen ovale of sphenoid and petrous portion of L. temporal; mutilation of R. nasal bone (due to syphilis?); foramen magnum cut; asymmetric palate.
34	B?	♀	1112·5	523	156·5	158·5	157	134	81	130	128	113·5	94	307	308	330·5	111	106	113·5	98	458	17·9?	68	96	116·5	51·0	51·0	25·7	9·5	19·4	29·5	[illegible]	7·7	40·0	40·4	32·0	32·8	50·8	46·6	34·8	93·5	32·2	25·5	63·6	84·8	80·9	85·5	81·5	104·8	70·[illegible]	50·1	50·4	80·0	81·2	68·5	79·2	49·0	33·8	68°·3	69°·7	42°·0	11°·6	30°·4	81°·3	P.B. rounded.
36	B	♀	—	—	171·5	170·5	169	125·5	—	125·5	125·5	107	89·5	286	292	355	120	127·5	107·5	90·5	—	21·4?	67·5	90·5	120·5	46·5	46·8	27·6	10·0	19·0	31	[illegible]	9·0	40·6	41·9		34·5	—	46·3	41·0	86	37·3	30·2	61·0	73·6	73·5	74·2	74·1	100·1	74·[illegible]	59·3	59·0		82·3	—	81·0	52·6	53·3	65°·0	69°·9	45°·1	12°·6	32°·5	82°·5	Old; sutures closing; teeth much worn; P.B. flat with sharp edge; sub-nasal prognathism.
41	B (A?)	♀	1369·5	368	164	165·5	164·5	139·5	90·5	137	137	116	93	315	315	347	124	123	100	90	482	20·0?	70·5	94	125	51·0	49·2	23·9	8·5	20·0	28	[illegible]	8·1	42·9	41·8	37·5	35·3	48·0	44·3	—	91	38·1	32·8	71·7	84·4	82·8	84·9	83·3	101·9	65·[illegible]	46·9	48·6	87·4	84·4		86·1	42·5	37·0	65°·8	69°·2	45°·0	10°·8	34°·2	80°·0	Adolescent; basilar synchondrosis; 3rd molars undeveloped; P.B. flat with blunt edge; wormian at R. squamo-mastoid angle.
47	B?	♀	1289·5	537	167	168	166·5	134·5	90	136·5	136	117	95	308·5	310	353	128	117	108	94·5	486	16·8?	67?	93	124·5	49·9	50·5	25·0	9·5	23·0	31·5	[illegible]	8·2	40·7	40·0	31·0	32·0	48·7	45·0	32·8	91·5	35·1	29·9	65·8	80·0	80·9	80·9	81·8	99·0	72·[illegible]	50·1	49·5	76·2	80·0	67·4	85·2	41·3	28·0	66°·1?	71°·8?	42°·1?	11°·9?	30°·2?	83°·7	Metopic; teeth worn; P.B. flat with sharp edge; flattened obelion.
58	B	♀?	—	—	161	161·5	161	144	94	133·5	132·5	114	100	315	316	334	119	116·5	98·5	89·5	488	22·8	72·5	—	—	50·3	49·5	26·1	9·8	23·2	31	[illegible]	8·7	43·9	—	34·4	—	51·1	47·2	41·2	100·5	34·7	31·3	74·2	89·1	82·1	89·4	82·4	108·5	—	51·9	52·7	78·4	—	80·6	90·2	42·2	41·4	69°·0	68°·7	42°·3	12°·3	30°·0	81°·0	Teeth much worn giving surface an inward-upward slope; P.B. flat with sharp edge; face asymmetrical; cranium very asymmetrical.
62	B	♀?	—	422	157·5	159·5	159	148	92·5	134·5	134	114	96	318·5	321	335	123·5	92·5	119	104·5	480	19·3?	68·5?	97·5	131	49·5	49·5	24·4	8·1	16·8	24	[illegible]	5·1	42·0	41·8	34·5	35·4	46·8	41·0	35·0	91?	35·4	31·1	66·6	92·8	84·1	93·1	84·3	110·4	70·2?	49·3	49·3	82·1	84·7	74·8	87·9	48·2	47·0	64°·6?	72°·7?	42°·7?	11°·8?	30°·9?	84°·5?	P.B. flat with sharp edge.
82	B	♀?	1257·5	576	164·5	166·5	165·5	140	90	130	127	116	93·5	315	315	349	118·5	120	110·5	94	492·5	20·4?	71·5	96	123·5?	52·0	52·7	23·3	8·5	18·0	27	2·8	7·0	43·5	43·6	33·2	35·2	44·0	41·0	38·0	93	36·3	30·5	62·1	84·0	76·1	84·5	76·6	110·3	74·5	44·8	44·2	76·3	80·7	86·4	84·0	47·2	40·0	67°·2	67°·6	45°·2	16°·8	28°·4	84°·4	Young adult; P.B. flat with rounded edge; sub-nasal prognathism; wormians R. and L. lambdoid; epipteric on R.; os japonicum on L.; suggestion of same on R.
92	B	♀	—	555	163	166·5	164	139·5	86	132?	130·5	118?	96	313·5?	312·5?	345	117·5	121	106·5	94	482	16·4?	66·5?	94	125·5	51·0	50·9	24·0	11·2	18·5	31	2·6	9·3	44·9	41·0	34·5	35·5	48·5	45·0	—	92·5?	35·0	25·0	67·2	83·7	78·2	85·0	79·4	107·0	71·0?	47·1	47·2	76·8	86·6	—	71·4	60·5	28·0	66°·2?	72°·4?	41°·4?	14°·6?	26°·8?	87°·0	Alveolus absorbed except about R. incisors; P.B. flat with blunt edge; epipteric on L.; fronto-squamous process on R.; R. tympanic plate perforated; glenoid cavity on L. flattened out by disease; long narrow foramen magnum with deep depression at fore-end walled in by a ridge.
95	B	♀?	1233	544	157	157	157	138	89	137	134	118	91	313·5	320	344	115	124	105	90	468	21·5?	70·5	94·5	125·5	52·5	52·5	25·8	9·8	19·5	30	[illegible]	9·6	42·5	41·6	35·5	35·6	50·0	46·5	35·8	86	33·8	30·3	62·8	88·0	85·4	87·9	85·4	103·0	74·5	49·1	49·1	83·5	85·6	71·6	89·6	50·3	29·2	62°·7	70°·5	46°·8	15°·5	31°·3	86°·[illegible]	Aged; sutures partly obliterated; teeth worn; P.B. flat with sharp edge; very asymmetrical; large ossicle in L. squamo-mastoid angle; 3rd condyle formed by fusion of 2 pre-condyles.
100	B	♀	1161·5	557	159	160	159	137	89	127	127	111	93·5	301	305	334	120	118	96	85	467	17·0?	65	94	116	48·1	50·2	25·6	7·5	19·0	26	2·6	7·0	42·0	42·8	35·9	36·0	—	40·1	32·6	87	32·8	28·8	68·2	85·5	79·2	86·0	79·7	107·9	60·0	53·2	51·0	85·5	84·1	—	86·9	39·5	41·4	63°·5	74°·5	42°·0	12°·0	30°·0	86°·[illegible]	R. 3rd molar diseased and much inflammation of maxilla above this; metopic ridge; P.B. flat with sharp edge; ossicle in R. and L. squamo-mastoid angles; very marked malar marginal processes; os japonicum on L.; horizontal malar suture nearly halfway on R.
102	B	♀	1134	—	160	160	158·5	132·5	84·5	129·5	130	117	96	314	315	345	123	118	104	90·5	472·5	—	62?	—	120	47·5	48·0	24·0	9·0	19·6	28	2·7	6·5	40·9	—	30·7	—	46·2?	41·0?	33·5?	93·5?	28·0	28·5	67·2	82·8	81·1	83·6	82·0	101·9	—	50·5	50·0	75·1	—	72·5?	102·0	45·9	41·5	68°·6?	73°·4?	38°·0?	12°·6?	25°·4?	86°·[illegible]	Senile; sutures becoming obliterated; alveolus almost all absorbed; P.B. flat with blunt edge; perforated tympanic plate on L.
116	B	♀	—	—	167·5	169	168	134	87	124·5	124·5	108	94	297·5	299·5	344	124	127	93	82·5	480	13·0	58	93·5	123?	45·8	45·6	25·0	9·6	18·2	26	1·6	4·1	41·4	40·6	33·0	35·0	43·7	39·8	35·3	89	35·6	29·4	57·4	79·2	73·7	79·9	71·3	107·5	61·8	54·0	54·8	81·2	86·2	80·8	82·6	52·7	46·3	67°·0	76°·4	36°·6	12°·6	24°·0	89°·[illegible]	3rd molars coming through; P.B flat with sharp edge; flattened obelion; ossicle R. lambdoid; beginning of horizontal malar suture R. and L.; inner L. palate bridge.
128	B	♀	1192	666	160·5	162·5	162	136	88	131	129	116	88	313	314	345·5	113	130·5	102	92	470	17·0	64	95	120	46·9	46·9	25·8	8·3	19·4	26	2·6	6·0	41·8	40·5	36·8	36·0	45·0	41·8	37·0	92	34·0	29·9	66·0	93·7	79·5	83·8	79·7	105·3	67·0	55·0	55·0	88·0	88·9	82·2	88·0	42·8	33·3	72°·6	65°·8	41°·6	14°·5	27°·1	80°·[illegible]	P.B. flat with blunt edge; flattened obelion; wormians L. lambdoid; prognathous; perforated tympanic plate on L.
9	C	♀	1128·5	261	166·5	166	166·5	126	88·5	119	119	105·5	91·5	279	281	343	122	114	107	90	469	17·5?	64?	92	112·5	47·0	48·7	26·1	10·2	17·0	27	[illegible]	1·9	43·0	43·7	31·9	32·3	42·4	41·5	36·0	87?	37·3	30·8	60·6	75·8	71·7	75·6	71·5	105·7	69·6?	55·5	53·5	74·2	73·9	84·9	82·6	60·0	61·2	65°·1?	72°·9?	42°·0?	13°·6?	28°·4?	86°·[illegible]	Senile; sutures closing; double parietal thinning and depression especially on R.; arthritic condition of palate, of posterior part of upper maxillary on L. and of anterior part on R.; hole into maxillary omtrum on R. near 2nd molar, due to abscess; P.B. has outward and upward slope and rounded edge.
18	C	♀	—	375	167	171·5	169	122	88	131·5	131	112	97?	295·5	295	345·5	122?	121	102·5	87	469	21·0?	67?	91·5	—	47·0?	47·0?	24·0?	—	21·8			8·6	41·0	42·2	37·0	36·1	51·0	46·7	40·4	92·5?	37·0	28·0	61·9	71·2	76·6	72·0	77·5	92·9	73·1?	50·9?	50·9?	90·2	85·5	79·2	75·7	—	—	65°·8?	73°·0?	41°·2?	13°·5?	27°·7?	80°·[illegible]	Young adult; basilar synchondrosis; sutures open; marked R. occipital bulge associated with R. os triangulare; P.B. flat with blunt edge; sub-nasal prognathism; fronto-squamous process on R. and L.
23	C	♀	—	348	177	178	178	125	87	127·5	127·5	111	96	291·5	293·5	357	127	122	108	85	489	17·0	64·5	89	120	47·5	48·2	22·5	—	17·0			—	43·6	44·3	37·2	37·3	50·2	46·0	38·5	92	34·8	32·0	56·3	70·2	71·6	70·2	71·6	97·3	72·2	47·4	46·7	85·3	84·2	76·7	92·0	-	—	66°·9	73°·1	40°·0	11°·9	28°·1	85°·[illegible]	Very bulging occiput; P.B. flat with blunt edge; sub-nasal prognathism; flattened obelion; small wormians R. and L. lambdoid near asterion; upper part of right squamous divided off by transverse suture; frontal process to upper half, meeting frontal bone in irregular suture; perforation of tympanic plate both sides; small L. pre condyle.
25	C	♀	—	—	165	165?	166·5	127	88	128	127·5	—	98	—	—	331	116	112	103	86·5	465?	15·0	62	92	—	46·9	46·0	22·8	9·0	19·0	27	[illegible]	8·7	41·1	43·6	37·0	35·0	48·0	44·0	38·0	93·5	35·1	31·0	61·0	76·8	77·3	77·3	76·7	99·7	67·4	48·6	49·6	83·9	80·3	79·2	88·3	47·4	32·2	67°·0	75°·2	37°·8	6°·4?	31°·4?	81°·[illegible]	Old; sagittal and part of coronal nearly obliterated; teeth much worn; arthritic condition of palate; P.B. flat with sharp edge; injury during life to right side of frontal.
27	C	♀?	—	400	177·5	180	180	127·5	84	129·5	128·5	112	98	294	294	359·5	121·5	127	111	90·5	486	23·1	71	97·5	127?	50·0	49·0	27·3	12·6	20·5	35	[illegible]	9·1	40·0	42·0	31·2	31·2	52·5	48·0	43·4	93·5	35·7	27·0	58·3	70·9	71·5	70·8	71·5	99·1	73·0	51·6	55·7	78·0	74·3	82·7	75·6	61·5	46·2	64°·9	71°·7	43°·4	15°·3	28°·1	87°·[illegible]	Sutures partly obliterated; teeth, especially incisors, much worn, surface having an inward-upward slant; receding forehead; P.B. flat with sharp edge; ossicles R. and L. lambdoid; epipterics R. and L.; septic condition of teeth; enamel worn off outer surface.
30	C	♀	—	—	162·5	164	163·5	129·5	93	128	128	107	100	—	—	327	115	117	95	82·5	468	16·8	62	91	—	45·8	45·8	20·6?	10·4	17·8	28·5	[illegible]	8·0	44·0	44·2	36·0	35·2	40·5	36·0	36·5	83	38·0	30·0	64·5	78·8	78·2	79·1	78·4	100·9	68·0	45·0?	45·0?	81·8	79·6	90·1	78·9	58·4	57·5	56°·0	86°·0	38°·0	3°·5	34°·5	89°·[illegible]	Young adult; 3rd L. molar about to be cut; 3rd R. molar undeveloped; P.B. flat with sharp edge; small epipteric on L.
38	C	♀	1199·5	477	165	164·5	164·5	134·5	86·5	125·5	126	108·5	90·5	292·5	297	347	120	118	109	88	471	16·7	65	87·5	116	46·6	46·5	20·5	10·8	17·3	29	[illegible]	7·5	39·4	39·4	33·1	33·0	47·2	44·0	30·3	81	35·0	28·8	57·6	81·7	76·6	81·7	76·5	100·8	74·3	44·0	44·1	84·0	83·8	76·9	82·3	62·4	45·3	60°·5	75°·5	41°·0	13°·0	31°·0	88°·[illegible]	P.B. flat with sharp edge; ossicle L. lambdoid; epipteric on R.; L. tympanic plate perforated.
39	C	♀	—	—	180·5	181·5	181	128·5	89·5	137	138·5	111	103	295	294	360	135	106	119	96	492·5	15·0	61·5	95?	120	—	47·7	—	10·3	20·1	30	[illegible]	7·8	42·1	41·9	—	33·2	—	—	—	92	37·0	32·0	57·6	70·6	76·2	71·0	76·5	92·8	64·7	—	—	—	79·2	—	86·5	51·2	47·4	62°·2	81°·6	36°·2	6°·6	29°·0	88°·2	Bulging occiput; large wormians at lambda; P.B. flat with sharp edge; flattened obelion.
40	C	♀	1266	554	175	176·5	175·5	130·5	91·5	130·5	129	114	96	303	305	357	124	123·5	109·5	92·5	497	21·8	70·5	101	119	49·6	50·8	24·1	15·4	17·8	36	[illegible]	8·0	41·0	40·3	33·3	34·0	45·5	42·0	39·0	89	36·5	29·5	61·4	74·1	73·2	74·5	73·6	101·2	69·4	48·6	47·4	81·2	84·4	85·7	80·8	86·5	49·4	62°·6	72°·8	44°·6	15°·2	29°·4	88°·[illegible]	Old; sagittal partly obliterated; surface of teeth much worn with inward-upward slant; P.B. flat with sharp edge; flattened obelion; pre-condyles R. and L
46	C	♀	1207·5	505	155·5	156·5	155·5	135·5	89·5	140	138·5	121	95·5	320·5	324	348·5	121·5	124	103	91	465	7·7	63	88	118·5	50·7	50·7	25·0	10·8	18·3	29	[illegible]	6·5	41·8	42·0	34·5	34·0	41·6?	38·6	37·1	86	31·2	26·5	67·7	86·7	88·5	87·3	89·1	98·0	71·7	49·3	49·3	82·5	81·0	89·2?	85·0	59·0	44·6	61°·9	77°·8	40°·3	10°·7	29°·6	88°·5	P.B. flat with blunt edge; traces R. and L. of suture across occipital from asterion.
48	C	♀	—	—	170·5	173·5	173	128·5	88	134	132·5	114	93	302	297	357	118	123	116	96·5	478	12·3?	55·5	—	121	—	44·1	—	—	—	—	[illegible]	—	—	41·1	—	27·8	—	—	—	87	33·3	27·0	59·9	74·2	76·3	74·3	76·4	97·3	—	—	—	—	67·6	—	81·1	—	—	66°·1	78°·1	35°·8	8°·1	27°·7	86°·[illegible]	Ageing; sagittal suture partly obliterated; 3rd molars uncut; P.B. flat with sharp edge; epipterics R. and L.
57	C	♀	—	—	162	162·5	161	136·5	86·5	135	132	116·5	91·5	313·5	313	344·5	121	117·5	106	92	470	19·4	68	94	123	50·3	50·0	28·9	9·2	19·6	28	[illegible]	8·0	41·1	41·0	35·3	34·8	46·1	41·6	36·3	89·5	33·3	30·0	64·7	83·9	81·3	84·6	81·9	103·3	72·4	57·5	57·8	85·9	84·9	78·7	90·1	47·0	41·3	66°·2	69°·5	44°·3	15°·5	28°·8	85°·[illegible]	P.B. flat with rounded edge; ossicle in L. lambdoid.
59	C	♀	1392·5	474	179	179	179	135	93	137·5	138	113	97	305	308	370	130	126	114	95·5	499	16·9	58·5	92	119	42·0	41·6	25·0	13·0	22·1	38	[illegible]	7·5	41·9	44·4	28·0	20·4	44·0	42·5	38·9	93	33·3	31·3	60·4	75·4	77·0	75·4	77·0	98·0	63·8	59·5	60·1	62·4	66·2	88·4	94·0	58·8	65·3	68°·3	75°·8	35°·9	4°·6	31°·3	80°·[illegible]	P.B. flat with slightly rounded edge; inter-parietal os triangulare on R.; ossicles in R. lambdoid.
77	C?	♀	—	395	169·5	172·5	171·5	139	90	126	124·5	117·5	91	312	315	357	117	119	121	99·5	492	17·2?	67·5?	88	111·5	51·4	51·0	26·0	9·7	17·5	27·5	[illegible]	7·5	43·8	44·0	35·2	35·0	—	—	33·5	87?	33·7	27·5	59·0	80·6	72·1	81·1	72·6	111·7	76·3?	50·6	51·0	80·4	79·5	—	81·6	55·4	21·5	64°·5?	71°·1?	44°·4?	15°·1?	29°·3?	86°·2?	Old; sutures nearly obliterated; teeth much worn; P.B. flat with rounded edge; fronto-squamous process on R.; tympanic plate perforated R. and L ; flattening at lambda.
108	C?	♀?	—	—	176	180?	179	137·5	91·5	132?	130·5	120?	98	320?	320?	372	132	128	112	96·5	502	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36·2	31·0	63·9	76·4?	72·4?	76·9	72·8	105·5	—	—	—	—	—	—	85·6	—	—	—	—	—	—	—	—	Ageing; sutures becoming obliterated; flattened obelion.
122	C	♀	1243	501	169·5	170·5	169·5	133·5	91	133	133	115	94·5	306·5	308	356	124	126·5	105·5	91·5	482	15·7	63·5	93	119	47·3	47·0	21·9	13·4	20·0	34·5	5·4	7·0	42·1	40·2	30·5	32·0	49·2	45·6	40·1	93	34·1	32·8	64·6	78·2	77·8	78·8	78·4	100·5	68·3	46·3	46·6	72·4	79·6	81·5	96·2	67·0	77·1	68°·7	71°·9	39°·4	14°·1	25°·3	86°·0	3rd R. molar undeveloped; P.B. flat with sharp edge; ossification of trochlea of superior oblique muscle R. and L. orbit; wormians R. and L. lambdoid.
133	C	♀	1182·5	387	168·5	169	168·5	123·5	88·5	126	124·5	108	96	286	291	343	120	123·5	99·5	84·5	474	15·2?	59	86	117	43·7	43·0	20·8	11·1	16·0	30	3·2	7·8	40·8	40·4	31·0	31·0	45·0	40·5	38·9	89	35·0	31·3	61·4	73·1	73·6	73·3	73·9	99·3	68·8	47·6	48·4	76·0	76·7	86·4	89·4	69·4	41·0	65°·0	78°·0	37°·0	12°·0	25°·0	90°·0	R. parieto-temporal suture obliterated, other sutures have hardly begun to disappear; sagittal depressed, and very complex throughout; teeth worn; 3rd molars single-fanged; P.B. with outward-upward slope and very sharp edge; R. tympanic plate perforated; opening behind L. post-condyloid foramen (due to disease?).
138	C	♀	—	—	162	162·5	162	130·5?	83·5	119	120	109	—	295?	297·5?	341	120·5	104	116·5	101	464?	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34·8	—	64·5	80·4?	73·9	80·6?	74·1	108·7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	Depressed obelion; wormians R. and L. lambdoid; ossicle R. squamo-mastoid angle; R. and L. tympanic plates perforated.

NON-ADULTS

6	A	?	—	—	153·5	—	153·5	136·5	86·5	117?	116·5	-	78·5	—	—	331	119	111	101	85·5	459	12·8?	57	83	—	43·7	43·7	23·0	8·0	18·3	25·5	[illegible]	8·9	40·0	39·6	32·0	32·0	-	37·2	31·8	77	34·7	27·2	61·4	—	—	88·9	75·8	117·3	68·0	52·6	52·6	80·0	80·8	—	78·4	43·7	38·2	66°·9	69°·9	43°·2	13°·6?	29°·6?	83°·5?	First permanent molars coming through; basilar synchondrosis; sutures open; P.B. rounded; small wormians in lambdoid R. and L.
101	A	?	1242·5	279	153·5	154·5	153·5	141	85	119·5	118·5	113·5	84	318	322	327·5	120	105	102·5	88·5	462	13·3	52	79	103	39·8	40·2	21·2	7·5	16·7	23	2·2	8·1	36·6	36·6	32·2	32·4	38·0?	35·5	28·0	79·5	30·0	25·6	64·2	91·1	76·6	91·9	77·2	118·9	66·0	53·3	52·7	88·0	88·5	73·7?	71·1	44·9	27·2	66°·8	76°·4	36°·8	14°·8	22°·0	91°·2	2nd permanent L. incisor coming through; basilar synchondrosis; P.B. flat with sharp edge; ossicles R. and L. lambdoid and R. and L. asterion; sutures open.
104	A??	?	—	—	157	156?	157	139	83·5?	132?	132	117?	85·5	—	—	339	125	115	99	89	468	—	—	—	—	—	—	—	—	—	—	—	—	34·1	—	—	—	—	—	—	—	38·8	28·4	71·1	89·1?	84·6?	88·6	84·3	105·3	—	—	—	—	—	—	73·2	—	—	—	—	—	—	—	—	Sutures open; basilar synchondrosis.
114	A	?	—	—	153·5	153·5	152·5	140	85	122?	120·5	105·5?	89	303?	299	317·5	111	108	98·5	87	459	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	32·4	27·8	67·7	91·2	78·4	91·9	79·1	116·3	—	—	—	—	—	—	85·8	—	—	—	—	—	—	—	—	Sutures open; flattened lambdoid region, and obelion; large perforations in tympanic plates R. and L.
127	A	?	—	220	140·5	141	140·5	128·5	79	118·5	116·5	108·5	75	298·5	300	315·5	105	111	99·5	85·5	422	14·8?	51·5?	78·5	99	37·3	37·5	19·1	7·3	15·2	22	1·2	4·8	35·7	35·7	30·0	29·0	—	31·0	30·7?	73?	28·9	24·5	63·5	90·9	82·4	91·4	82·9	108·3	65·0?	51·2	50·9	84·0	81·2	—	84·8	48·0	25·0	67°·7?	71°·8?	40°·5?	16°·2?	24°·3?	88°·0?	Sutures open; basilar synchondrosis; 1st R. permanent incisor coming through; P.B. rounded; beginning of sutures across occiput starting from R. and L. asterion; perforated tympanic plates R. and L.
134	A	?	1286	295	149·5	149·5	149	149	88·5	125·5	125·5	112	87	320	323	319	121	98·5	99·5	88	470	14·6	56·5	84	114	42·5	42·5	21·9	7·1	18·9	24·5	1·9	5·7	37·5	37·1	36·0	36·2	36·5	35·1	34·0	83·5	33·1	28·4	67·5	99·6	83·8	100·0	84·2	118·8	67·0	51·5	51·5	96·0	97·6	93·2	86·1	37·6	35·2	67°·2	74°·1	38°·7	15°·7	23°·0	89°·8	Sutures open; basilar synchondrosis; 1st permanent molars coming through R. and L.; P.B. flat with slightly rounded edge; ossicle R. lambdoid; tympanic plates perforated R. and L.
140	A	?	1294·5	316	150·5	151	151	141	84	137	136·5	118·5	90·5	322·5	327	336	121·5	122·5	92	85·5	459	15·2	60·5	84·5	113	46·9	46·9	23·8	—	17·0	—	-	—	39·2	38·3	33·0	32·0	39·0	36·0	34·0	81·5	35·0	28·3	83·6	93·2	90·3	93·2	90·3	103·2	71·8	50·7	50·7	84·2	83·6	87·2	80·9	—	—	61°·6	77°·6	40°·8	16°·2	24°·6	93°·8	Sutures open; basilar synchondrosis; cutting bi-cuspid; slight torus palatinus; P.B. flat with rounded edge; wormians R. lambdoid.

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* See Section 9 of Text.

AN EXPERIMENTAL DETERMINATION OF THE PROBABLE ERROR OF DR SPEARMAN'S CORRELATION COEFFICIENTS.

[Being the paper read to the Society of Biometricians and Mathematical Statisticians, December 13th, 1920.]

BY "STUDENT."

IN the *British Journal of Psychology*, Vol. II. p. 96, 1906*, Dr Spearman suggested two methods of determining correlation, based on replacing actual measurements by ranks.

As an illustration we may take the following purely imaginary example:

TABLE I.

Individual	Height	Length of middle finger mm.	Rank in height	Rank in length of finger
A	6′ 0″	12·8	2	1
B	5′ 3″	11·5	4	3
C	5′ 7″	10·0	3	4
D	6′ 1″	12·4	1	2

Instead of correlating the figures in the second and third columns of the above table Dr Spearman proposed to use the figures in the fourth and fifth columns, and to determine one or other of two coefficients: of these the first (ρ) gives the ordinary correlation coefficient between the figures representing the ranks, and the second (R) was described as a 'footrule' for correlation, i.e. a rough instrument which could be used by the unskilled. Dr Spearman also proposed to use R in cases where it was thought advisable to weight mediocre observations more heavily than extremes.

The method of determining ρ and R was to take the difference D between the numbers representing the ranks, e.g. for A in Table I

$$D = 2 - 1 = 1.$$

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* [Dr Spearman's results were first given in a paper entitled "The Proof and Measurement of Association between Two Things" in the *American Journal of Psychology*, Vol. xv. pp. 72—101, 1904. The dogmatic statements as to the accuracy of his methods in that paper are, I think, erroneous, and he does not lay adequate stress on the fact that correlation of ranks is not a correlation of variates and may differ very considerably from it. The suggestion of considering the correlation of ranks is due to A. Binet and V. Henri: see *La Fatigue Intellectuelle*, Paris, 1898, p. 252, also *L'Année Psychologique*, T. IV. p. 155, Paris, 1898. Their process is very obscure and they also do not appear to have realised that the correlation of variates is not that of ranks. Ed.]

Then
$$\rho = 1 - \frac{S(D^2)}{\frac{n(n^2-1)}{6}} \quad \text{.......(i)},$$

and
$$R = 1 - \frac{S(D)}{\frac{n^2-1}{6}} \quad \text{.......(ii)},$$

where n is the number in the sample: in the case of R, $S(D)$ denotes the summation of positive differences only.

Dr Spearman gave an empirical formula connecting R and ρ, viz. $\rho = \sin\left(\frac{\pi}{2} R\right)$; but I do not suppose that he attached any very great importance to this.

He further gave the probable errors of ρ and R for the case of no correlation as $\frac{\cdot 6745}{\sqrt{n}}$ and $\frac{\cdot 4266}{\sqrt{n}}$.

In his memoir 'On further methods of determining correlation' Prof. Pearson investigated these coefficients for the case of the normal correlation surface and found the relations between ρ and R and r the ordinary correlation coefficient to be
$$r = 2 \sin\left(\frac{\pi}{6} \rho\right) \quad \text{.......(iii)},$$

and
$$r = 2 \cos \frac{\pi}{3}(1 - R) - 1 \quad \text{.......(iv)}.$$

Pearson further found the standard error of ρ to be for large samples
$$\frac{1-\rho^2}{\sqrt{n}} \{1 + \cdot 086\rho^2 + \cdot 013\rho^4 + \cdot 002\rho^6 + \ldots\} \quad \text{.......(v)},$$

and of r_ρ, i.e. r determined from ρ by (iii), to be
$$1 \cdot 0472 \frac{1-r^2}{\sqrt{n}} \{1 + \cdot 042r^2 + \cdot 008r^4 + \cdot 002r^6 + \ldots\} \quad \text{.......(vi)}.$$

He did not succeed in evaluating the error of R or of r_R (i.e. of r determined by (iv)), but pointed out that just as in the case of r the $\sqrt{n}$ in the denominator is really $\sqrt{n-1}$. He also pointed out that R can only take values between $+1$ and $-\cdot 5$ and that Spearman's $\frac{\cdot 4226}{\sqrt{n-1}}$ does not imply that R is more accurate than ρ or r with their probable error of $\frac{\cdot 6745}{\sqrt{n-1}}$ since R itself is smaller than ρ or r in about the same proportion.

Since that time the use of ρ and R has become general among psychologists, especially in America, where they are preferred to r on account of the ease and speed with which they can be determined for small samples.

For example in a note on correlation in *Employment Psychology*, by H. C. Link*, a book written to urge the claims of Psychology on the devotees of 'Scien-

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* Macmillan, 1919.

tific Management,' the author mentioned three methods of determining correlation, ρ which is to be used for samples smaller than 30, R for samples over 30 and r which though acknowledged to be rather more accurate is not to be used at all since it takes four times as long to calculate as the others.

Now to save time at the expense of accuracy is justifiable when, and only when, the time saved can be devoted to increasing the number of observations so as to obtain greater accuracy on the whole series, otherwise it will take longer to get equally trustworthy conclusions, and it seems to be of interest to investigate the probable errors of ρ and R for samples of the size that the employment psychologist is contemplating. And here we may note that the saving of time only occurs when the sample is comparatively small; as it increases, the labour of grading becomes more and more severe till at some point in the neighbourhood of 40 it becomes quicker to use the ordinary product moment r if that be possible.

It should perhaps be pointed out that there are many cases where it is possible to grade a sample for some character which is not capable of being measured on a scale and it might be thought that in this case large samples could profitably be dealt with by the ρ or R method, but in fact it is just these scaleless characters which present the greatest difficulty in grading.

We have then to consider the variability of ρ and R and of the derivatives r_ρ and r_R, determined from small samples, and it seemed worth while to use the material of a former sampling experiment so as to get an idea of how small samples depart from the results obtained by Prof. Pearson for ideally large samples. The material in question consists of 750 samples of four drawn from a population of 3000 criminals whose height and left middle finger length give an approximately normal correlation surface with correlation ·66.

These are capable of being combined easily to give 375 samples of eight and in addition there are 100 samples of 30, which may be taken to be a size of sample which is no longer quite 'small.'

Accounts of the former results were given in *Biometrika,* VI. p. 1 and p. 302, since which time the frequency distributions of the correlation coefficients of small samples drawn from normally correlated populations have been very thoroughly investigated by Soper, Fisher and the authors of the cooperative paper in Vol. XI. p. 328 of *Biometrika*: it is hoped that some mathematician may be interested in the general solution of the problems raised in the present paper which may then afford material for checking his results.

When I came to apply the methods to my samples I found that owing to the rather coarse grouping, there were a large number of ties, so that it became necessary to find out the right correction for ties.

Prof. Pearson had discussed the question of ties and had suggested two ways of dealing with them. One way was to rank them all as if they were the highest number of the tie which he called the bracket-rank method and the other was to rank them all half-way down the tie which he called the mid-rank method. Thus the first way would rank 1, 2, 2, 4 while the second would rank 1, $2\frac{1}{2}$, $2\frac{1}{2}$, 4 if the second and third of four individuals constituted a tie.

Now the first would give different results according as we read the scale forwards or backwards and also alter the mean of the set of numbers so I have only tried to use the mid-rank method, for which I have found the correction which follows.

Correction of ρ *for ties.*

If $D = x - y$, when x and y are any two variables measured from their means, then

$$D^2 = x^2 + y^2 - 2xy.$$

Summing for all n samples and dividing by n

$$\frac{\Sigma(D^2)}{n} = \sigma_x^2 + \sigma_y^2 + 2r_{xy}\sigma_x\sigma_y,$$

$$\therefore\ r_{xy} = \left(\sigma_x^2 + \sigma_y^2 - \frac{\Sigma(D^2)}{n}\right)\Big/ 2\sigma_x\sigma_y \quad \text{.....................(vii).}$$

If now x and y are the first n numbers, then

$$\sigma_x^2 = \sigma_y^2 = \frac{1}{n} \times \text{sum of squares of 1st } n \text{ numbers} - \left(\frac{\text{sum of 1st } n \text{ numbers}}{n}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2-1}{12} \quad \text{..(viii).}$$

Substituting in (vii) we find

$$\rho = r_{xy} = \left(\frac{n^2-1}{6} - \frac{\Sigma(D^2)}{n}\right)\Big/\frac{n^2-1}{6}$$

$$= \frac{\dfrac{n(n^2-1)}{6} - \Sigma(D^2)}{\dfrac{n(n^2-1)}{6}} \quad \text{..............................(ix).}$$

Now suppose that there is on the x side a tie of t in number from q to $q+t-1$. Using the mid-rank method we substitute for each of the numbers

$$q,\ q+1,\ \ldots\ q+t-1 \text{ their mean } \frac{2q+t-1}{2}.$$

Hence in finding σ_x^2 the mean is unaltered but in the sum of the squares $q^2 + (q+1)^2 + \ldots + (q+t-1)^2$ is replaced by $\dfrac{t(2q+t-1)^2}{4}$.

Hence σ_x^2 is smaller by

$$\frac{1}{n}\left\{q^2 + (q+1)^2 + \ldots + (q+t-1)^2 - \frac{t(2q+t-1)^2}{4}\right\}.$$

This is equal to

$$\frac{1}{n}\left\{tq^2+2q\,\{1+2+\dots+(t-1)\}+\{1^2+2^2+\dots+(t-1)^2\}\right.$$
$$\left.-\,tq^2-qt\,(t-1)-\frac{t\,(t-1)^2}{4}\right\}$$
$$=\frac{1}{n}\left\{\frac{(t-1)\,t\,(2t-1)}{6}-\frac{t\,(t-1)^2}{4}\right\}$$
$$=\frac{t\,(t^2-1)}{12n},$$
$$\therefore\ \sigma_x^{\ 2}=\frac{n^2-1}{12}-\frac{t\,(t^2-1)}{12n}.$$

This is clearly additive for any number of ties so that if $T_x=\Sigma\left(\frac{t\,(t^2-1)}{12}\right)$ summing for all the ties on the x side and similarly T_y for the y side

$$\sigma_x^{\ 2}=\frac{n^2-1}{12}-\frac{T_x}{n}\ \text{and}\ \sigma_y^{\ 2}=\frac{n^2-1}{12}-\frac{T_y}{n},$$

and substituting in (vii)

$$\rho=r_{xy}=\frac{\left\{\frac{n^2-1}{6}-\frac{1}{n}(T_x+T_y)-\frac{\Sigma\,(D^2)}{n}\right\}}{\sqrt{\left(\frac{n^2-1}{6}-\frac{2T_x}{n}\right)\left(\frac{n^2-1}{6}-\frac{2T_y}{n}\right)}}$$

$$=\frac{\frac{n\,(n^2-1)}{6}-(T_x+T_y)-\Sigma\,(D^2)}{\sqrt{\left(\frac{n\,(n^2-1)}{6}-2T_x\right)\left(\frac{n\,(n^2-1)}{6}-2T_y\right)}}\qquad\dots\dots\dots\dots\dots\dots\text{(x)},$$

$$=\frac{\frac{n\,(n^2-1)}{6}-(T_x+T_y)-\Sigma\,(D^2)}{\left\{\frac{n\,(n^2-1)}{6}-(T_x+T_y)\right\}\sqrt{1-\frac{(T_x-T_y)^2}{\left\{\frac{n\,(n^2-1)}{6}-(T_x+T_y)\right\}^2}}}.$$

So that if T_x and T_y do not differ appreciably

$$\rho=1-\frac{\Sigma\,(D^2)}{\frac{n\,(n^2-1)}{6}-(T_x+T_y)}\qquad\dots\dots\dots\dots\dots\dots\dots\dots\text{(xi)}.$$

In estimating T_x or T_y each pair contributes $\frac{1}{2}$,
triplet ,, 2,
quartet ,, 5,
quintet ,, 10,

and so on. For example, if the x ranks for a sample of 10 were

$$1,\ 2\tfrac{1}{2},\ 2\tfrac{1}{2},\ 5,\ 5,\ 5,\ 8\tfrac{1}{2},\ 8\tfrac{1}{2},\ 8\tfrac{1}{2},\ 8\tfrac{1}{2},$$

T_x would be $\frac{1}{2}+2+5=7\frac{1}{2}$ and if there were no ties in the y ranks ρ would be

$$\frac{165-7\frac{1}{2}-S(D^2)}{\sqrt{(165-15)\,165}}=\frac{157\frac{1}{2}-S(D^2)}{\sqrt{150\,.\,165}},$$

and if we were to take it as $1-\dfrac{S(D^2)}{157\frac{1}{2}}$ the error would come in the third significant place of decimals.

In determining ρ for my 375 samples of eight I found that much tied samples usually gave low values of ρ and it occurred to me that although undoubtedly equation (x) gives the true value of the correlation of ranks, yet it might be that the loss of precision due to ties would give low values for the correlation. To test this I doubled the width of my unit of grouping first for one variable and then for the other so that I got three values of ρ for each sample:

(i) Converting the original figures into ranks,

(ii) Using coarser grouping on one side and the original grouping on the other before converting into ranks,

(iii) Using coarser grouping on both sides.

An example will make my meaning clearer.

(1) Original figures		(2) x grouped coarsely		(3) Both grouped coarsely		Ranks					
						(1)		(2)		(3)	
x	y	Putting +1 and 0 as $\frac{1}{2}$, &c.	y	x	y	x	y	x	y	x	y
0	+3	$+\frac{1}{2}$	+3	$+\frac{1}{2}$	$+2\frac{1}{2}$	$5\frac{1}{2}$	3	$4\frac{1}{2}$	3	$4\frac{1}{2}$	$3\frac{1}{2}$
−2	0	$-1\frac{1}{2}$	0	$-1\frac{1}{2}$	$+\frac{1}{2}$	8	7	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$6\frac{1}{2}$
+3	+3	$+1\frac{1}{2}$	+3	$+2\frac{1}{2}$	$+2\frac{1}{2}$	$1\frac{1}{2}$	3	$1\frac{1}{2}$	3	$1\frac{1}{2}$	$3\frac{1}{2}$
−1	−2	$-1\frac{1}{2}$	−2	$-1\frac{1}{2}$	$-1\frac{1}{2}$	7	8	$7\frac{1}{2}$	8	$7\frac{1}{2}$	8
+1	+3	$+\frac{1}{2}$	+3	$+\frac{1}{2}$	$+2\frac{1}{2}$	$3\frac{1}{2}$	3	$4\frac{1}{2}$	3	$4\frac{1}{2}$	$3\frac{1}{2}$
+1	+2	$+\frac{1}{2}$	+2	$+\frac{1}{2}$	$+2\frac{1}{2}$	$3\frac{1}{2}$	5	$4\frac{1}{2}$	5	$4\frac{1}{2}$	$3\frac{1}{2}$
+3	+4	$+1\frac{1}{2}$	+4	$+2\frac{1}{2}$	$+4\frac{1}{2}$	$1\frac{1}{2}$	1	$1\frac{1}{2}$	1	$1\frac{1}{2}$	1
0	+1	$+\frac{1}{2}$	+1	$+\frac{1}{2}$	$+\frac{1}{2}$	$5\frac{1}{2}$	6	$4\frac{1}{2}$	6	$4\frac{1}{2}$	$6\frac{1}{2}$
			Pairs			3	—	2	—	2	1
			Triplets ...			—	1	—	1	—	—
			Quartets ...			—	—	1	—	1	1

Here originally $T_x=1\frac{1}{2}$ and $T_y=2$ and $\dfrac{n(n^2-1)}{6}-(T_x+T_y)=80\frac{1}{2}$.

After grouping x coarsely $T_x=6$ and $T_y=2$ " $=76$.

After grouping both coarsely $T_x=6$ and $T_y=5\frac{1}{2}$ " $=72\frac{1}{2}$,

and ρ will be found to take the values ·833, ·869 and ·834 in succession. Working in this way I obtained three values of ρ for each of the 375 samples and determined the mean, standard deviation and mean (T_x+T_y) for each of the three series of 375. These results are given in Table II.

TABLE II.

	Mean ρ	σ_ρ	Mean T_x	Mean T_y	Mean (T_x+T_y)
Original series	·5798	·2887	1·92	1·90	3·82
x grouped coarsely ...	·5798	·2903	4·67	1·90	6·57
x and y grouped coarsely	·5696	·2874	4·67	4·37	9·04

Here an increase in the correction to be made for ties from 3·82 to 9·04 has made a difference of ·01 in the mean value of ρ, the probable error being about ·015, and a still less appreciable difference in the Standard Deviation. It is, I think, a fair inference that the correction is applicable to the series in question, and the reason for the observed low values of ρ in much tied samples is to be sought elsewhere*. But it will be asked 'what if no correction be made for ties?' The answer is that the mean value of ρ will rise as the ties become more numerous and the S.D. will fall. Thus Table II would become Table III if no corrections were made.

TABLE III.

	Mean ρ	σ_ρ	Mean (T_x+T_y)
Original series	·602	·2677	3·82
x grouped coarsely ...	·616	·2887	6·57
x and y grouped coarsely	·622	·2414	9·04

At first sight this may appear to be highly advantageous since the mean value approximates more nearly to the value which would be obtained from a large sample and the S.D. is smaller. A little reflection will show however that the means of the ρ's of all populations would be subject to the same rise and that in fact the ρ of one population is no more differentiated from the ρ of another population than it is when corrected, while the mean value when corrected is constant over a fairly wide range of ties. If the correction is not made ρ can be cooked up to any required value by increasing the ties.

The fact is that as soon as there is a single tie, uncorrected ρ can no longer take all values between $+1$ and -1 and if one of the scales be reversed the correlation instead of being $-\rho$ becomes $-\rho + \dfrac{(T_x+T_y)}{\dfrac{n(n^2-1)}{6}}$. We are therefore forced to use the correction which after all gives us the distribution of ρ that we should get from ideal material containing no ties.

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* The low value of ρ for much tied samples is due to the fact that a much tied sample is as a rule one in which the S.D. of the original variables is low.

Now as a matter of experience I find that of samples drawn from a normally distributed population those with S.D. above the average tend to give high and stable values of the correlation coefficient, while those with S.D. below the average tend to give low and variable values.

The form of the correlation surface for variables σ_x and r_{xy} is of considerable interest to those who have to deal with small samples and merits the attention of mathematicians. I hope to deal with the experience obtained from my samples at some later time.

To see what happens when ties are carried to an extreme I determined ρ from the original table of 3000 entries (*Biometrika*, I. p. 216) and from the same table condensed to six groups each way by using a 4″ scale of height and ·8 mm. scale of finger lengths.

In the first case $\rho = {}\cdot 637$ giving r_ρ ·655 and in the second ·557 with r_ρ ·575. There seems therefore in extreme cases to be a tendency for the correction to give too low a value of ρ.

Correction of R for grouping.

In Dr Spearman's original paper R is defined as $1 - \dfrac{S(D)}{\dfrac{n^2-1}{6}}$ when $\dfrac{n^2-1}{6}$ is taken as the average value which $S(D)$ assumes.

The simplest way to see that this is the average value is to write down all the possible D's thus:

$$\begin{array}{cc|cccccc}
1 & 1 & & & & & & \\
2 & 2 & 1 & & & & & \\
3 & 3 & 2 & 1 & & & & \\
4 & 4 & 3 & 2 & 1 & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
(n-1) & (n-1) & (n-2) & (n-3) & (n-4) & & 1 & \\
n & n & (n-1) & (n-2) & (n-3) & \dots & 2 & 1
\end{array}$$

Here the two columns on the left are composed of the first n numbers. The third column is formed by subtracting the top number of the first column from all the numbers in the second column in turn, the fourth by subtracting the second number from all numbers which give a positive remainder and so on.

Thus the numbers in the second column could be arranged opposite the numbers of the first column in $n!$ ways.

And in $(n-1)!$ of these arrangements any given pair will occur.

Hence the average value of $S(D)$ will be

$$\frac{(n-1)!}{n!}\{[1+2+\dots+(n-1)]+[1+2+\dots+(n-2)]+\dots+(1+2)+1\},$$

$$\therefore \text{ average value of } S(D) = \frac{1}{n}\left\{\frac{n(n-1)}{2}+\frac{(n-1)(n-2)}{2}+\dots+\frac{2\,.\,3}{2}+\frac{1\,.\,2}{2}\right\}$$

$$= \frac{1}{2n}S_1^{\,n}(n^2-n) = \frac{1}{2n}\left\{\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right\}$$

$$= \frac{(n+1)}{12}\{2n+1-3\} = \frac{n^2-1}{6} \quad \dots\dots\dots\dots\dots\dots(\text{xii}).$$

If we now substitute in the second column ties instead of consecutive numbers we can find out what effect ties will have on the average value of $S(D)$. As I can see no general way of proving the results I propose merely to state my results as follows:

(1) A tie of t on one side which is opposed by no ties on the other side will diminish $\frac{n^2-1}{6}$ by $\frac{t(t^2-1)}{24n}$ if t be odd and by $\frac{t(t^2-4)}{24n}$ if t be even.

(2) Overlapping ties on opposite sides interfere with the above simple rule, the total to be subtracted from $\frac{n^2-1}{6}$ being increased or decreased according to Table IV.

TABLE IV.

		Distance between centres of ties												
x tie	y tie	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
2	2	+ 1		0 →										
3	2		0 →											
3	3	+ 2		− 1		0 →								
4	2	+ 2		0 →										
4	3		+ 1		− 1		0 →							
4	4	+ 6		0		− 1		0 →						
5	2		0 →											
5	3	+ 4		− 1		− 1		0 →						
5	4		+ 3		− 2		− 1		0 →					
5	5	+10		0		− 4		− 1		0 →				
6	2	+ 3		0 →										
6	3		+ 2		− 1		− 1		0 →					
6	4	+10		+1		− 2		− 1		0 →				
6	5		+ 7		− 2		− 4		− 1		0 →			
6	6	+19		+4		− 4		− 4		− 1		0 →		
7	2		0 →											
7	3	+ 6		− 1		− 1		− 1		0 →				
7	4		+ 5		− 2		− 2		− 1		0 →			
7	5	+16		+2		− 5		− 4		− 1		0 →		
7	6		+13		− 1		− 7		− 4		− 1		0 →	
7	7	+28		+7		− 6		− 10		− 4		− 1		0 →

etc., etc.

As an example of the use of Table IV, suppose a set of eight ranks to contain on the x side a tie of 5 centred at 3, i.e. let the x ranks be 3, 3, 3, 3, 3, 6, 7, 8, and let the y ranks have a tie of 4 centred at $2\frac{1}{2}$, i.e. let the y ranks be $2\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$, 5, 6, 7, 8. Then the amount to be subtracted from $\frac{8^2-1}{6}$ is firstly $\frac{5}{8}$ (for the 5 tie) $+\frac{2}{8}$ (for the 4 tie) $+\frac{3}{8}$ (from Table IV) $=1\frac{1}{4}$. Had the y ranks been 1, $3\frac{1}{2}$, $3\frac{1}{2}$, $3\frac{1}{2}$, $3\frac{1}{2}$, 6, 7, 8, the correction would be the same, but if the y ranks were 1, 2, $4\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{1}{2}$, 7, 8, the correction would be $\frac{5}{8}+\frac{2}{8}-\frac{2}{8}=\frac{5}{8}$,

and if 1, 2, 3, $5\frac{1}{2}$, $5\frac{1}{2}$, $5\frac{1}{2}$, $5\frac{1}{2}$, 8 $\frac{5}{8}+\frac{2}{8}-\frac{1}{8}=\frac{3}{4}$,

and if 1, 2, 3, 4, $6\frac{1}{2}$, $6\frac{1}{2}$, $6\frac{1}{2}$, $6\frac{1}{2}$ $\frac{5}{8}+\frac{2}{8}=\frac{7}{8}$.

It is only with very small and much tied samples that the correction is appreciable.

GRAPH FROM TABLE V.

Giving Frequency Distributions of various Correlation Coefficients from 375 samples of 8

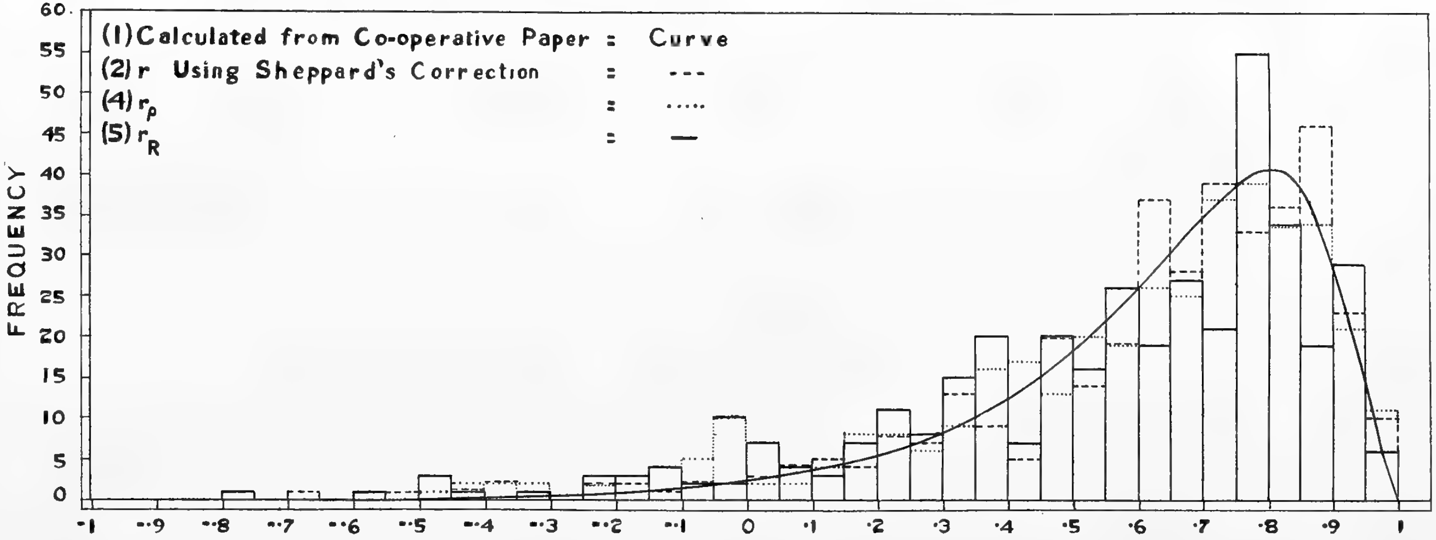


TABLE V. *Giving Frequency Distributions of Various Correlation Coefficients from* 375 *samples of* 8.

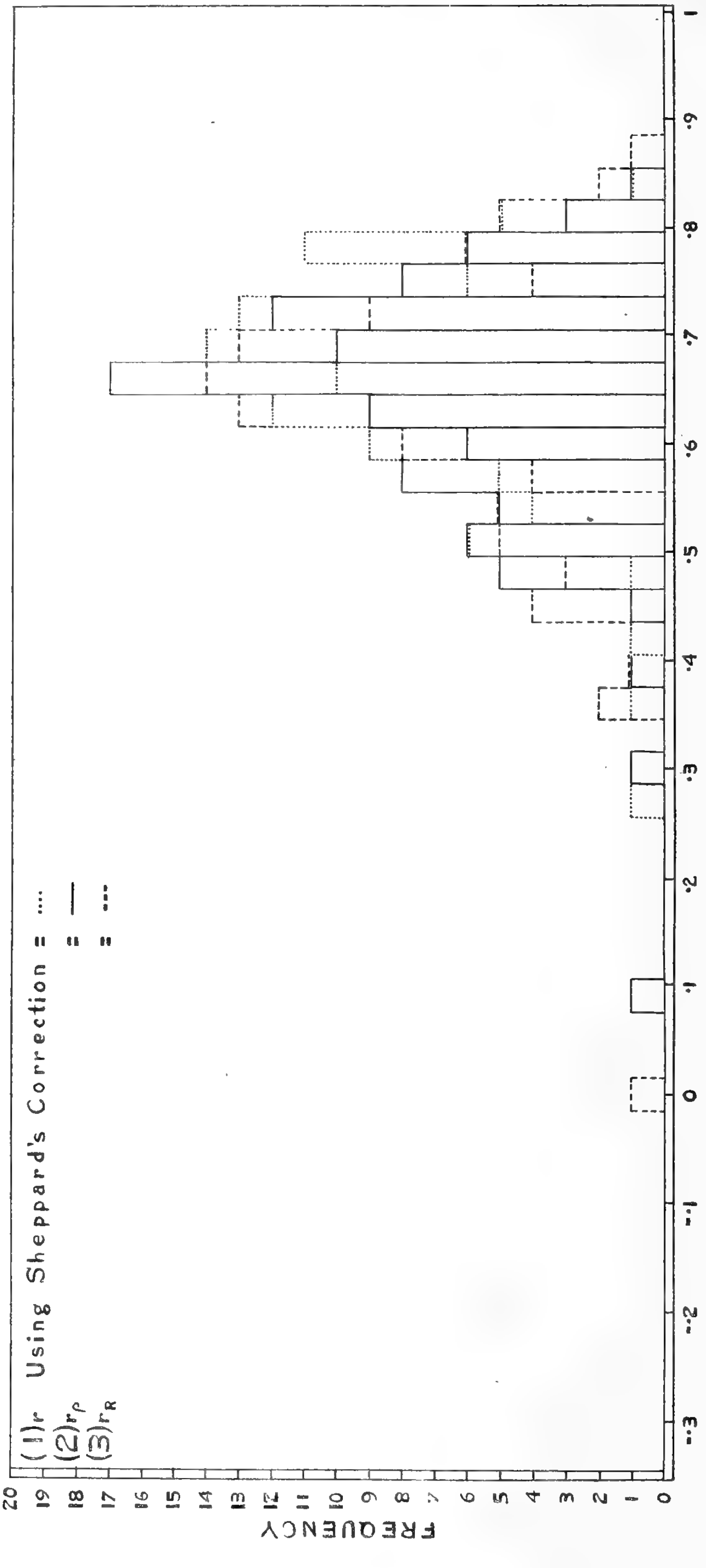
	−·8005 to −·7505	−·7505 to −·7005	−·7005 to −·6505	−·6505 to −·6005	−·6005 to −·5505	−·5505 to −·5005	−·5005 to −·4505	−·4505 to −·4005	−·4005 to −·3505	−·3505 to −·3005	−·3005 to −·2505	−·2505 to −·2005	−·2005 to −·1505	−·1505 to −·1005	−·1005 to −·0505	−·0505 to −·0005	−·0005 to ·0495	·0495 to ·0995	·0995 to ·1495	·1495 to ·1995	·1995 to ·2495	·2495 to ·2995	·2995 to ·3495	·3495 to ·3995	·3995 to ·4495	·4495 to ·4995	·4995 to ·5495	·5495 to ·5995	·5995 to ·6495	·6495 to ·6995	·6995 to ·7495	·7495 to ·7995	·7995 to ·8495	·8495 to ·8995	·8995 to ·9495	·9495 to ·9995
(1) Calculated from the cooperative paper	—	—	·1	·1	·1	·2	·3	·3	·4	·5	·7	·9	1·1	1·4	1·7	2·2	2·7	3·3	4·0	5·0	6·1	7·5	9·2	11·2	13·7	16·6	20·0	23·7	28·2	32·8	37·0	40·3	40·5	35·3	22·1	5·9
(2) r actual using Sheppard's correction	—	—	1	0	0	1	0	1	2	0	0	2	2	1	2	0	3	4	5	4	8	7	13	9	5	20	14	19	37	28	39	33	36	46	23	10
(3) r actual using no corrections	—	—	—	1	0	1	0	0	2	1	0	1	3	1	2	0	3	4	5	5	7	7	13	11	9	18	15	22	39	28	38	31	39	40	22	7
(4) r_ρ actual... ...	—	—	—	—	—	—	1	2	2	2	0	2	0	0	5	10	2	2	4	8	8	6	9	16	17	13	20	19	26	25	37	39	34	34	21	11
(5) r_R ,, ...	1	0	0	0	1	0	3	1	0	1	0	3	3	4	2	2	7	4	3	7	11	8	15	20	7	20	16	26	19	27	21	55	34	19	29	6
(6) ρ ,, ...	—	—	—	—	—	—	—	3	1	3	0	1	1	0	5	10	2	2	5	8	8	6	11	16	18	15	22	21	25	24	39	36	35	29	21	8
(7) R ,, ...	—	—	—	—	—	—	—	—	1	0	1	4	1	4	6	4	11	9	12	22	21	27	16	28	24	29	43	31	27	19	19	10	2	3	1	—

TABLE VI. *Giving Frequency Distributions of Various Correlation Coefficients from* 100 *samples of* 30.

	−·165 to −·135	−·135 to −·105	−·105 to −·075	−·075 to −·045	−·045 to −·015	−·015 to ·015	·015 to ·045	·045 to ·075	·075 to ·105	·105 to ·135	·135 to ·165	·165 to ·195	·195 to ·225	·225 to ·255	·255 to ·285	·285 to ·315	·315 to ·345	·345 to ·375	·375 to ·405	·405 to ·435	·435 to ·465	·465 to ·495	·495 to ·525	·525 to ·555	·555 to ·585	·585 to ·615	·615 to ·645	·645 to ·675	·675 to ·705	·705 to ·735	·735 to ·765	·765 to ·795	·795 to ·825	·825 to ·855	·855 to ·885	·885 to ·915
(1) r actual using Sheppard's correction	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	0	0	1	0	1	0	1	6	4	5	9	12	10	14	13	6	11	5	1	—	—
(2) r_ρ actual	—	—	—	—	—	—	—	—	1	0	0	0	0	0	0	1	0	0	1	0	1	5	6	5	8	6	9	17	10	12	8	6	3	1	—	—
(3) r_R ,, ...	—	—	—	—	—	1	0	0	0	0	0	0	0	0	0	0	0	2	1	0	4	3	5	5	4	8	13	14	13	9	4	6	5	2	1	—
(4) r actual by median four-fold division $=\cos\frac{\pi B}{A+b}$	1	0	0	0	0	0	0	0	1	1	0	0	1	1	0	1	3	0	0	3	7	0	10	5	5	5	9	8	6	9	5	1	9	6	1	2
(5) ρ actual	—	—	—	—	—	—	—	1	0	0	0	0	0	0	0	1	0	1	0	1	2	7	4	8	5	10	13	13	8	12	6	5	2	1	—	—
(6) R ,, ...	—	—	—	—	—	1	0	0	0	0	0	0	2	1	3	6	5	7	9	15	17	12	8	4	6	3	0	0	1	—	—	—	—	—	—	—

GRAPH FROM TABLE VI.

Giving Frequency Distributions of various Correlation Coefficients from 100 samples of 30



Discussion of the Frequency Distributions obtained.

Tables V and VI give the frequency distributions of r, determined with Sheppard's corrections for grouping, of ρ and of R and their derivatives (from equations (iii) and (iv)) r_ρ and r_R.

In addition we have, in Table V, r determined without Sheppard's corrections and the theoretical distribution of r calculated from the table (*Biometrika*, XI. p. 384), of the cooperative paper by interpolating between $\rho = \cdot 65$ and $\rho = \cdot 70$, drawing the frequency curve and estimating the areas by counting the squares: it is probably not very accurate, but fairly close to the truth.

In Table VI is included r calculated from the fourfold table taken through the medians by Sheppard's formula $r = \cos \frac{\pi B}{A+B}$ where B is the frequency in the 'small' cells. This probably suffers a good deal from the coarse grouping which makes it necessary to divide the centre groups in an arbitrary manner.

The most remarkable thing about these tables is the very wide spread of all the distributions. There is of course nothing new in this, but I cannot help thinking that an examination of these tables may be beneficial for all who try to work with very small samples.

Besides this there is not very much to be found in these tables which is not seen to greater advantage in Tables VIII and IX of the means and S.D.'s, but as a matter of interest I have compared lines 2—4 of Table V with line 1 by the χ^2 test with the following results:

TABLE VII.

	25 groups		16 groups	
	χ^2	P	χ^2	P
r with Sheppard's corrections ...	30·10	·18	20·49	·17
r without Sheppard's corrections	20·43	·67	12·39	·66
r_ρ actual	60·25	·000,064	30·63	·01
r_R "	74·10	·000,002	55·65	(say) ·000,002

The 25 groups were the 24 groups on the right of the table and the tail, which includes all groups which are less than 1·0 in line 1: the 16 groups were taken so that no group in line 1 was less than 10.

With such a small sample as 375 the χ^2 test is only decisive for considerable departures from the theoretical and the regular excess over the theoretical for all groups less than ·40 avoids detection.

At the same time it is interesting to note that judged by the χ^2 test Sheppard's corrections do not seem to have improved the calculation of r.

TABLE VIII.

Certain Constants of the Frequency Distributions of Various Correlation Coefficients derived from 375 Samples of 8.

	Mean	S.D.	Coefficient of Variation	Number of samples required to give as great accuracy as 100 samples of (1)	Number of samples required to give as great accuracy as 100 samples of (2)
(1) r calculated from co-operative paper	·631	·250	39·6	100	—
(2) r actual using Sheppard's corrections	·624 ± ·010	·274 ± ·007	43·9 ± 1·3	120 ± 5·9	100
(3) r actual using no correction for grouping	·614 ± ·010	·271 ± ·007	44·1 ± 1·3	117 ± 5·8	98
(4) r_ρ actual	·586 ± ·010	·291 ± ·007	49·7 ± 1·5	135 ± 6·7	113
(5) r_R "	·566 ± ·011	·309 ± ·008	54·6 ± 1·7	153 ± 7·5	127
(6) ρ "	·580 ± ·011	·289 ± ·007	49·8 ± 1·5	—	—
(7) R "	·407 ± ·008	·237 ± ·006	58·2 ± 1·9	—	—

TABLE IX.

Certain Constants of the Frequency Distributions of Various Correlation Coefficients derived from 100 *Samples of* 30.

	Mean	S.D.	Coefficient of Variation	Number of samples required to give as great accuracy as 100 samples of (1)	Number of samples required to give as great accuracy as 100 samples of (2)
(1) r calculated from co-operative paper	·653	·109	16·7	100	—
(2) r actual using Sheppard's corrections	·661 ± ·007	·101 ± ·005	15·3 ± ·7	86 ± 8·2	100
(3) r_ρ actual	·639 ± ·008	·113 ± ·005	17·7 ± ·9	108 ± 10·3	125
(4) r_R "	·638 ± ·008	·122 ± ·006	19·1 ± ·9	125 ± 11·9	146
(5) r actual from median 4-fold division $= \cos \frac{\pi B}{A+B}$	·609 ± ·012	·183 ± ·009	30·1 ± 1·6	282 ± 25·1	328
(6) ρ actual	·624 ± ·008	·116 ± ·006	18·6 ± ·9	—	—
(7) R "	·428 ± ·007	·100 ± ·005	23·4 ± 1·2	—	—

Tables VIII and IX give the means, S.D.'s and coefficients of variation of the frequency distributions in Tables V and VI and in addition the calculated constants for the samples of 30.

As well as this I have calculated the number of samples which would be required to give as great accuracy by the less accurate methods as 100 samples determined (1) on the theoretical basis of normal correlation, and (2) on the actual samples by the product moment method using Sheppard's corrections.

The object of this is to get an idea of how much time must be saved in order to gain by using the rank methods. First however we may note in Table VIII the marked difference between the theoretical S.D. and that actually obtained by the product moment method.

I attribute this almost entirely to the grouping which was unfortunately rather coarse and which cannot be corrected by Sheppard's corrections in small samples. The slight divergency of the population from normal correlation may have helped to a very small extent but for the most part the excess in the lower values of r which cause the mean to be low and the S.D. to be high is due to those samples which have low S.D.'s and I incline to believe that if the grouping can be chosen so that the S.D.'s are not less than 3 the actual distribution of r will be found to be very close to the calculated for samples drawn from normally correlated material.

In Table IX on the other hand the actual has a higher mean and lower S.D. than the calculated, but as the differences are in each case less than twice the probable error, I think we may put them down to the error of random sampling which is of course large in such a small sample as 100.

Next we may note that Prof. Pearson's formulae, no doubt because they are correct for grades, do not enable us to correct rank correlations for small samples. The means of both r_ρ and r_R are too low for samples of eight, and for samples of 30 probably so.

As for the S.D.'s of ρ and r_ρ the values found are in the case of samples of eight much higher than those calculated from equations (v) and (vi) which are ·258 and ·243 respectively. The samples of 30 however give values which agree sufficiently well, for the calculated S.D. is in each case ·114, well within the P.E.

Line 5 in Table IX shows that as determined in this investigation Sheppard's median division formula gives a mean value of r well below the population value and a very high standard deviation *. While this is not unlikely to be the case for small samples the arbitrary division of the central groups makes it impossible to say that this is not due to the fact that we have only used an approximation to median division in this case.

The chief point of interest however in Tables VIII and IX lies in column 4 showing the number of samples which we must have to get the same accuracy by the various methods as that given by 100 samples in which r is determined with sufficiently fine grouping by the product moment method.

Column 5 is put in in case there are any who do not accept my explanation of the difference between the calculated and actual distribution of r namely that it is due to the coarse grouping. I have not been able to estimate the P.E.'s of the figures in column 5 as they are complicated by correlation between the numerator and denominator of the fractions from which the figures are calculated. They must however be larger than the P.E.'s in column 4.

---

* The S.D. calculated from the formula $\sigma_r = \frac{2\pi\sqrt{1-r^2}}{\sqrt{N}} \left\{\frac{ab}{N^2}\right\}^{\frac{1}{2}}$ is however rather higher, being ·191 if r be taken as ·66 and ·207 if r be taken as ·609. Miss Elderton kindly looked up this formula for me, but I cannot find that it has been published. [See, however, *Biometrika*, Vol. IX. p. 23. It is also involved in the early paper by Sheppard, *Phil. Trans.* Vol. 192, A, p. 147 etc. Ed.]

In any case there is a strong indication that with samples of eight the loss of accuracy due to the use of r_ρ instead of r will practically always more than counterbalance the gain of time in calculation. Either method is however so little to be depended upon for a single sample of very small size, except as the merest indication, that very little is lost by the use of r_ρ. If however a number of small samples can be averaged so as to obtain a coefficient of some value, the product moment method should be used when possible.

With samples of 30 the 8 °/₀ more samples required compares fairly with Prof. Pearson's 10 °/₀ more for large samples, but seeing that the particular sample of 100 gave too low a value for σ_r, the value of σ_{r_ρ} which must be correlated with it is likely to be low also and the 8 °/₀ may easily be 18 °/₀ or more.

In any case it would very seldom pay to have to collect 8 °/₀ more samples of 30 even if one could save 8 °/₀ of the time on samples of that size.

In both tables there is a considerable loss from the use of r_R instead of r_ρ, since from 13 °/₀ to 16 °/₀ more samples would be required of the former to give the same accuracy as the latter. The gain in calculation is not very appreciable since most of the time is spent in ranking the samples. Dr Spearman prefers R to ρ at times because less importance attaches to outlying samples, but as the extremes of small samples tend to be outliers even in normally correlated material owing to the phenomenon to which attention was drawn in Galton's Difference problem*, it seems to me that as much weight as possible should be given to them.

To Combine two Methods of Determination.

At an early stage in the investigation I hoped to be able to combine r and r_ρ to get a value less subject to error than either. Curiously enough Prof. Pearson in his editorial in the last number of this Journal gives the equations which I proposed to use for the purpose (p. 7 (29)).

As they are perfectly general I will state them in a slightly more general form.

If x and y be two estimates of any quantity obtained in different ways then a quantity z can always be found which will have a lower error than either of them, unless x and y are perfectly correlated.

Thus
$$z = \frac{\sigma_y^2 - r_{xy}\,\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2 - 2r_{xy}\,\sigma_x\sigma_y} \cdot x + \frac{\sigma_x^2 - r_{xy}\,\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2 - 2r_{xy}\,\sigma_x\sigma_y} \cdot y \quad \text{.........(xiii)},$$

and
$$\sigma_z^2 = \frac{\sigma_x^2\,\sigma_y^2\,(1 - r_{xy}^2)}{\sigma_x^2 + \sigma_y^2 - 2r_{xy}\,\sigma_x\sigma_y} \quad \text{..(xiv)}.$$

---

* *Biometrika*, Vol. I. pp. 385—399. In this connection it is of interest to note that the correlation surface of ranks is not an elliptical hill as is the normal correlation surface but two comparatively steep ridges joined by a saddle, the ridges having a skew section.

In the case of the samples of eight x may be taken as r without Sheppard's corrections and y as r_ρ, when we have

$$\sigma_x^2 = (\cdot 271)^2 = \cdot 073{,}441, \quad \sigma_x\sigma_y = \cdot 078{,}861,$$

$$\sigma_y^2 = (\cdot 291)^2 = \cdot 084{,}681,$$

$$r_{xy} = \cdot 885,$$

and hence from (xiii)

$$z = \cdot 804r + \cdot 196r_\rho,$$

and

$$\sigma_z = \cdot 270,$$

i.e. there is no appreciable gain in our case since σ_r is ·271. It may be that with a lower value of the population correlation the gain would be greater, but on the other hand if r had been determined for very fine grouping σ_r^2 would have been ·0625, the contribution of r_ρ to z would have been practically negligible, and the gain in accuracy by the use of z less than that found. There is however another case where the above formulae might be applied, namely to the values of ρ obtained from the original grouping and those from coarse grouping.

These are given in Table II from the first and third lines of which it appears that σ_{ρ_1} and σ_{ρ_3} may both be taken as ·288.

In this case σ_z^2 reduces to $\dfrac{\sigma_\rho^2(1 + r_{\rho_1\rho_3})}{2}$,

and as

$$r_{\rho_1\rho_3} = \cdot 903, \quad \sigma_2 = \cdot 281.$$

This is somewhat more encouraging, but the process is rather troublesome and could only be applied to cases where there is a proper scale. If however there is a proper scale greater accuracy could be obtained by the product moment method with very little more trouble (since we have now to make two calculations to find ρ).

We may therefore conclude that as far as this sampling experiment may be taken as typical:

(1) Where the unit of grouping is small (say $< \frac{1}{3}$ the standard deviation) the product moment method should be used if the most is to be made of the time and statistics at our disposal, however small the sample.

(2) Where a coarse grouping has to be used, the mean value of r will fall below that calculated from the cooperative paper (*Biometrika*, XI. p. 328 et seq.) and the S.D. will rise. For small samples Sheppard's corrections will approximately correct the former but will increase the latter still further. Indeed it is possible that for very coarse grouping ρ might vary less than r.

(3) For this, or any other, purpose ties should be dealt with by one or other of the formulae in equations (x) and (xi) of this paper.

(4) Where one or both variables can be ranked but not scaled, as frequently happens in some kinds of work, or for what Professor Pearson has called "purposes of assay," ρ can be determined with advantage and may be considered the natural method to adopt.

TABLE X.

Giving Correlation (·885) between r (no corrections) and r_ρ for samples of 8.

r_ρ

r	− ·5005 to − ·4505	− ·4505 to − ·4005	− ·4005 to − ·3505	− ·3505 to − ·3005	− ·3005 to − ·2505	− ·2505 to − ·2005	− ·2005 to − ·1505	− ·1505 to − ·1005	− ·1005 to − ·0505	− ·0505 to − ·0005	− ·0005 to ·0495	·0495 to ·0995	·0995 to ·1495	·1495 to ·1995	·1995 to ·2495	·2495 to ·2995	·2995 to ·3495	·3495 to ·3995	·3995 to ·4495	·4495 to ·4995	·4995 to ·5495	·5495 to ·5995	·5995 to ·6495	·6495 to ·6995	·6995 to ·7495	·7495 to ·7995	·7995 to ·8495	·8495 to ·8995	·8995 to ·9495	·9495 to ·9995	Totals
− ·6505 to − ·6005	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·6005 to − ·5505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·5505 to − ·5005	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·5005 to − ·4505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·4505 to − ·4005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·4005 to − ·3505	—	1	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
− ·3505 to − ·3005	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·3005 to − ·2505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·2505 to − ·2005	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·2005 to − ·1505	—	—	1	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
− ·1505 to − ·1005	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·1005 to − ·0505	—	—	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
− ·0505 to − ·0005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·0005 to ·0495	—	—	—	—	—	—	—	—	—	—	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
·0495 to ·0995	—	—	1	—	—	—	—	—	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
·0995 to ·1495	—	—	—	—	—	1	—	—	—	2	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	5
·1495 to ·1995	—	—	—	—	—	—	—	—	1	1	—	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	5
·1995 to ·2495	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2	2	1	—	—	—	—	—	—	—	—	—	—	—	—	7
·2495 to ·2995	—	—	—	—	—	—	—	—	—	2	—	—	1	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
·2995 to ·3495	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	4	1	1	2	—	1	—	—	—	—	—	—	—	—	13
·3495 to ·3995	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	4	2	—	—	—	1	—	—	—	—	—	—	—	11
·3995 to ·4495	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	—	—	1	1	2	1	—	—	—	—	—	—	1	—	—	9
·4495 to ·4995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	—	3	3	1	5	1	1	1	—	—	—	—	—	—	18
·4995 to ·5495	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	4	—	2	2	3	—	—	—	—	—	—	15
·5495 to ·5995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	1	4	5	5	2	1	—	1	—	—	—	22
·5995 to ·6495	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	3	—	5	4	8	7	5	2	2	—	—	—	39
·6495 to ·6995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	4	2	2	9	5	2	1	—	—	28
·6995 to ·7495	—	—	—	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	1	3	1	—	3	4	9	7	7	—	—	—	38
·7495 to ·7995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	—	2	2	4	8	5	5	2	—	31
·7995 to ·8495	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	7	12	9	5	4	—	39
·8495 to ·8995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	1	1	5	7	15	5	2	40
·8995 to ·9495	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	6	9	4	22
·9495 to ·9995	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	5	7
Totals	1	2	2	2	0	2	0	0	5	10	2	2	4	8	8	6	9	16	17	13	20	19	26	25	37	39	34	34	21	11	375

TABLE XI.

*Giving Correlation (·903) between ρ Calculated in Original Grouping (ρ_1) and ρ Calculated in Groups twice as coarse (ρ_3)**.

ρ_1

ρ_3	− ·4995 to − ·4495	− ·4495 to − ·3995	− ·3995 to − ·3495	− ·3495 to − ·2995	− ·2995 to − ·2495	− ·2495 to − ·1995	− ·1995 to − ·1495	− ·1495 to − ·0995	− ·0995 to − ·0495	− ·0495 to + ·0005	+ ·0005 to ·0505	·0505 to ·1005	·1005 to ·1505	·1505 to ·2005	·2005 to ·2505	·2505 to ·3005	·3005 to ·3505	·3505 to ·4005	·4005 to ·4505	·4505 to ·5005	·5005 to ·5505	·5505 to ·6005	·6005 to ·6505	·6505 to ·7005	·7005 to ·7505	·7505 to ·8005	·8005 to ·8505	·8505 to ·9005	·9005 to ·9505	·9505 to 1·0005	Totals
− ·4995 to − ·4495	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·4495 to − ·3995	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
− ·3995 to − ·3495	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
− ·3495 to − ·2995	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·2995 to − ·2495	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·2495 to − ·1995	—	—	—	2	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
− ·1995 to − ·1495	—	1	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
− ·1495 to − ·0995	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
− ·0995 to − ·0495	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
− ·0495 to + ·0005	—	—	—	—	—	—	—	—	1	1	—	1	2	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	8
+ ·0005 to ·0505	—	—	1	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	5
·0505 to ·1005	—	—	—	—	—	—	—	—	—	2	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
·1005 to ·1505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
·1505 to ·2005	—	—	—	—	—	—	—	—	1	1	—	1	1	—	1	—	2	1	1	—	—	—	—	—	—	—	—	—	—	—	9
·2005 to ·2505	—	—	—	—	—	—	—	—	1	—	—	—	1	5	—	—	2	3	—	—	—	—	—	—	—	—	—	—	—	—	12
·2505 to ·3005	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	2	—	2	1	1	1	—	—	—	—	—	—	—	—	—	10
·3005 to ·3505	—	—	—	—	—	1	—	—	—	1	—	—	2	2	—	1	—	—	2	—	—	2	—	—	—	—	—	—	—	—	11
·3505 to ·4005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	4	2	1	2	1	—	—	—	—	—	—	—	14
·4005 to ·4505	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	—	2	4	3	2	3	—	1	—	—	—	—	—	—	—	18
·4505 to ·5005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	3	4	3	2	—	2	1	—	—	—	—	—	19
·5005 to ·5505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	5	6	6	2	—	—	—	1	—	—	—	24
·5505 to ·6005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	5	2	8	5	5	2	1	—	—	—	29
·6005 to ·6505	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	1	3	5	9	4	1	2	—	—	—	27
·6505 to ·7005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	6	4	2	8	6	2	2	—	—	33
·7005 to ·7505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	3	9	6	6	1	1	—	29
·7505 to ·8005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	4	3	3	10	6	—	4	—	31
·8005 to ·8505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	6	6	4	3	1	23
·8505 to ·9005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	3	9	14	4	—	35
·9005 to ·9505	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	4	2	1	11
·9505 to 1·0005	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	5	1	8
Totals	1	2	1	3	0	2	0	0	6	10	1	3	6	10	9	5	12	16	18	15	23	24	26	27	35	35	36	27	19	3	375

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* The reader is requested to note that the subranges here are not the same as in Tables V, VI, and X. This is the only table which contains a perfect correlation coefficient which occurred in the ρ_3 series.

(5) In such cases it should be borne in mind that for small samples the distribution of ρ is similar to the distribution of r, but that the mean, even of r_ρ, is lower than that of r and the S.D. greater, by amounts which doubtless depend on the population correlation.

(6) R and r_R are not worth determining in serious work; their use should therefore be confined to the elementary statistics for which its author intended R.

(7) It is interesting to observe that Sheppard's median division fourfold table has given for small samples a mean value very much below the population value. While this is only what one might have expected, it may in this case be due to the coarse grouping which prevented me from making an accurate median division.

(8) The following problems might be of interest to mathematicians:

(*a*) The determination of the form of the Rank Correlation Surface.

(*b*) The determination of the frequency distribution of ρ for small samples drawn from a normally distributed population.

(*c*) The determination of the nature of the correlation surface when a standard deviation is taken as one variable and the correlation coefficient as the other, both being determined from small samples drawn from a normally distributed population.

ON THE MATHEMATICAL EXPECTATION OF THE MOMENTS OF FREQUENCY DISTRIBUTIONS.

PART II.

BY PROFESSOR AL. A. TCHOUPROFF of Petrograd*.

CHAPTER I

I

LET $X_1, X_2, \ldots X_N$ be N variates *each following its own law of frequency distribution* and let N experiments be performed,—the first on the variate X_1, the second on the variate X_2, and so forth, the last on the variate X_N, where the N experiments are mutually independent. Put:

$$EX_i = m_1^{(i)}\,;\ EX_i^2 = m_2^{(i)}\,;\ EX_i^r = m_r^{(i)}\,;$$

$$E\,[X_i - m_1^{(i)}]^2 = m_2^{(i)} - [m_1^{(i)}]^2 = \mu_2^{(i)}\,;\ E\,[X_i - m_1^{(i)}]^r = \mu_r^{(i)}\,;$$

$$\frac{1}{N}\sum_{i=1}^{N} m_1^{(i)} = m_{[1,\,N]}\,;\ \frac{1}{N}\sum_{i=1}^{N} m_r^{(i)} = m_{[r,\,N]}\,;$$

$$\frac{1}{N}\sum_{i=1}^{N} \mu_2^{(i)} = \mu_{[2,\,N]}\,;\ \frac{1}{N}\sum_{i=1}^{N} \mu_r^{(i)} = \mu_{[r,\,N]}.$$

Let X_i' be the chance value obtained by the experiment on the variate X_i, and put further

$$\frac{1}{N}\sum_{i=1}^{N} X_i' = X_{(N)},$$

$$EX^r_{(N)} = m_{r,\,(N)}\,;\ E\,[X_{(N)} - m_{1,\,(N)}]^r = \mu_{r,\,(N)}.$$

Noting, that $$m_{1,\,(N)} = \frac{1}{N}\sum_{i=1}^{N} EX_i' = m_{[1,\,N]},$$

we see that

$$\mu_{r,\,(N)} = E\,[X_{(N)} - m_{[1,\,N]}]^r = \sum_{h=0}^{r} (-1)^h\, C_r^h\, m^h_{[1,\,N]}\, m_{r-h,\,(N)}.$$

Conversely,

$$m_{r,\,(N)} = E\,[X_{(N)} - m_{[1,\,N]} + m_{[1,\,N]}]^r = \sum_{h=0}^{r} C_r^h\, m^h_{[1,\,N]}\, \mu_{r-h,\,(N)}.$$

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* This paper was most kindly translated by Dr L. Isserlis from the Russian and read by him with Professor Tchouproff's permission to the Society of Biometricians and Mathematical Statisticians, June 7, 1921.

On the other hand,

$$\left.\begin{aligned}\mu_{[2,\,N]} &= \frac{1}{N}\sum_{i=1}^{N}\left[m_2^{(i)} - (m_1^{(i)})^2\right] = m_{[2,\,N]} - \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)}\right]^2 \\ &= m_{[2,\,N]} - m^2_{[1,\,N]} - \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,\,N]}\right]^2 \\ \mu_{[r,\,N]} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{h=0}^{r}(-1)^h C_r^h (m_1^{(i)})^h m_{r-h}^{(i)} \\ &= m_{[r,\,N]} + \sum_{h=1}^{r-2}(-1)^h C_r^h \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)}\right]^h m_{r-h}^{(i)} \\ &\quad + (-1)^{r-1}(r-1)\frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)}\right]^r \end{aligned}\right\}\ldots(1).$$

II

Now,
$$\left[\sum_{i=1}^{N} X_i\right]^r = \sum_j \sum_{i_1} \sum_{i_2} \ldots \sum_{i_j} \sum \frac{r\,!}{r_1!\, r_2! \ldots r_j!} X_{i_1}^{r_1} X_{i_2}^{r_2} \ldots X_{i_j}^{r_j},$$
where the summation for j is for all integer values from 1 to either r or N whichever is the smaller, the summation with regard to $i_1, i_2, i_3, \ldots i_j$ extends to all integer and mutually unequal values of $i_1, i_2, \ldots i_j$ from 1 to N, and the last summation to all integer values of $r_1, r_2, \ldots r_j$ satisfying the equation

$$r_1 + r_2 + \ldots + r_j = r.$$

Hence,

$$\left.\begin{aligned} m_{r,\,(N)} = EX^r_{(N)} &= \frac{1}{N^r} E\left[\sum_{i=1}^{N} X_i\right]^r \\ &= \frac{1}{N^r}\sum_j \sum_{i_1}\sum_{i_2} \ldots \sum_{i_j}\sum \frac{r\,!}{r_1!\, r_2! \ldots r_j!} m_{r_1}^{(i_1)} m_{r_2}^{(i_2)} \ldots m_{r_j}^{(i_j)} \end{aligned}\right\}\ldots\ldots\ldots(2),$$

or, when $r \leqq N$,

$$\left.\begin{aligned} m_{r,\,(N)} = \frac{1}{N^r}\Bigg\{ &\sum_{i=1}^{N} m_r^{(i)} + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N}\sum_{r_1=1}^{r-1}\frac{r\,!}{r_1!\,(r-r_1)!} m_{r_1}^{(i_1)} m_{r-r_1}^{(i_2)} \\ &+ \sum_{i_1=1}^{N-2}\sum_{i_2=i_1+1}^{N-1}\sum_{i_3=i_2+1}^{N}\sum_{r_1=1}^{r-2}\sum_{r_2=1}^{r-r_1-1}\frac{r\,!}{r_1!\, r_2!\,(r-r_1-r_2)!} m_{r_1}^{(i_1)} m_{r_2}^{(i_2)} m_{r-r_1-r_2}^{(i_3)} + \ldots \\ &+ \sum_{i_1=1}^{N-r+2}\sum_{i_2=i_1+1}^{N-r+3} \ldots \sum_{i_{r-1}=i_{r-2}+1}^{N}\sum_{r_1=1}^{2}\sum_{r_2=1}^{3-r_1} \ldots \sum_{r_{r-2}=1}^{r-r_1-r_2-\ldots-r_{r-3}-1}\frac{r\,!}{2} m_{r_1}^{(i_1)} m_{r_2}^{(i_2)} \ldots \\ &\qquad \ldots m_{r_{r-2}}^{(i_{r-2})} m_{r-r_1-r_2-\ldots-r_{r-2}}^{(i_{r-1})} \\ &+ \sum_{i_1=1}^{N-r+1}\sum_{i_2=i_1+1}^{N-r+2} \ldots \sum_{i_r=i_{r-1}+1}^{N} r\,!\, m_1^{(i_1)} m_1^{(i_2)} \ldots m_1^{(i_r)}\Bigg\} \\ = m^r_{[1,\,N]} &+ \frac{1}{N} C_r^2 m^{r-2}_{[1,\,N]}\mu_{[2,\,N]} \\ &+ \frac{1}{N^2}\left\{1\,.\,3\, C_r^4 m^{r-4}_{[1,\,N]}\mu^2_{[2,\,N]} + C_r^3 m^{r-3}_{[1,\,N]}\mu_{[3,\,N]}\right\} + \ldots \end{aligned}\right\}(3)^*.$$

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* Cf. "On the Mathematical Expectation of the Moments of Frequency Distributions" (*Biometrika*, Vol. XII, Chap. 1, § II, equation (10), p. 151).

When $r = 2$ we find

$$m_{2,(N)} = m_{[1,N]}^{2} + \frac{1}{N}\mu_{[2,N]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Similarly we find*

$$\left.\begin{aligned} m_{3,(N)} &= m_{[1,N]}^{3} + \frac{3}{N} m_{[1,N]}\mu_{[2,N]} + \frac{1}{N^2}\mu_{[3,N]} \\ m_{4,(N)} &= m_{[1,N]}^{4} + \frac{6}{N} m_{[1,N]}^{2}\mu_{[2,N]} + \frac{1}{N^2}\left\{4m_{[1,N]}\mu_{[3,N]} + 3\mu_{[2,N]}^{2}\right\} \\ &\quad + \frac{1}{N^3}\left\{\frac{1}{N}\sum_{i=1}^{N}\left[\mu_4^{(i)} - 3\left(\mu_2^{(i)}\right)^2\right]\right\} \end{aligned}\right\} \quad (5).$$

Comparing these relations with the corresponding relations established in (11) of Chapter I of the first Part of my paper on the "Mathematical Expectation of the Moments of Frequency Distributions†," we see that when each variate follows its own law of distribution we may obtain $m_{2,(N)}$ and $m_{3,(N)}$ by replacing m_1, μ_2 and μ_3 in equations (11) by the corresponding arithmetical averages for all the N variates. But in the case of $m_{4,(N)}$ this method would only produce the first three terms; the term of order $\frac{1}{N^3}$ is not equal to $\mu_{[4,N]} - 3\mu_{[2,N]}^{2}$ but to the arithmetic mean of the differences $\mu_4^{(i)} - 3\left[\mu_2^{(i)}\right]^2$.

III

If in (2) we put $m_1^{(i)} = 0$ and replace the m's by the corresponding μ's, we find

$$\mu_{r,(N)} = \frac{1}{N^r}\sum_{j}\sum_{i_1}\sum_{i_2}\ldots\sum_{i_j}\sum \frac{r!}{r_1!\,r_2!\ldots r_j!}\mu_{r_1}^{(i_1)}\mu_{r_2}^{(i_2)}\ldots\mu_{r_j}^{(i_j)},$$

where the summation with regard to j extends to all integer values from 1 to r or N whichever is the smaller, the summation with regard to $i_1, i_2, \ldots i_j$ extends to all mutually unequal integer values of $i_1, i_2, \ldots i_j$ from 1 to N, and the last summation to all integer values, not less than 2, of $r_1, r_2, \ldots r_j$ satisfying the relation $r_1 + r_2 + \ldots + r_j = r$.

Hence,

$$\left.\begin{aligned} \mu_{2r,(N)} &= \frac{1}{N^{2r}}\left\{\sum_{i=1}^{N}\mu_{2r}^{(i)} + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N}\sum_{r_1=2}^{2r-2}\frac{[2r]!}{r_1!\,[2r-r_1]!}\mu_{r_1}^{(i_1)}\mu_{2r-r_1}^{(i_2)}\right. \\ &\quad + \sum_{i_1=1}^{N-2}\sum_{i_2=i_1+1}^{N-1}\sum_{i_3=i_2+1}^{N}\sum_{r_1=2}^{2r-4}\sum_{r_2=2}^{2r-r_1-2}\frac{[2r]!}{r_1!\,r_2!\,[2r-r_1-r_2]!}\mu_{r_1}^{(i_1)}\mu_{r_2}^{(i_2)}\mu_{2r-r_1-r_2}^{(i_3)} \\ &\quad \left. + \ldots + 1.3.5\ldots(2r-1)\,r!\sum_{i_1=1}^{N-r+1}\sum_{i_2=i_1+1}^{N-r+2}\ldots\sum_{i_r=i_{r-1}+1}^{N}\mu_2^{(i_1)}\mu_2^{(i_2)}\ldots\mu_2^{(i_r)}\right\} \\ &= 1.3.5\ldots(2r-1)\left\{\frac{1}{N^r}\mu_{[2,N]}^{r} + \frac{1}{N^{r+1}}\left[\tfrac{1}{6}r^{[-2]}\mu_{[2,N]}^{r-2}\mu_{[4,N]}\right.\right. \\ &\quad \left.\left. + \tfrac{1}{9}r^{[-3]}\mu_{[2,N]}^{r-3}\mu_{[3,N]}^{2} - \tfrac{1}{2}r^{[-2]}\mu_{[2,N]}^{r-2}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\mu_2^{(i)}\right)^2\right)\right] + \ldots\right\} \end{aligned}\right\} \quad (6)‡,$$

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* Cf. Tschuproff, "Zur Theorie der Stabilität statistischer Reihen" (*Skandinavisk Aktuarietidskrift*, 1919, p. 82).

† *Biometrika*, Vol. XII, p. 151.

‡ Cf. *Biometrika*, Vol. XII, p. 154, equation (22).

$$\left.\begin{aligned}\mu_{2r+1,\,(N)} = {} & \frac{1}{N^{2r+1}}\left\{\sum_{i=1}^{N}\mu_{2r+1}^{(i)} + \sum_{i=1}^{N-1}\sum_{i_2=i_1+1}^{N}\sum_{r_1=2}^{2r-1}\frac{[2r+1]!}{r_1!\,[2r+1-r_1]!}\mu_{r_1}^{(i_1)}\mu_{2r+1-r_1}^{(i_2)}\right. \\ & + \ldots + \frac{1.3.5\ldots(2r+1)\,r!}{3}\sum_{i_1=1}^{N-r+1}\sum_{i_2=i_1+1}^{N-r+2}\ldots\sum_{i_r=i_{r-1}+1}^{N}\sum_{r_1=2}^{3}\sum_{r_2=2}^{5-r_1}\ldots \\ & \left.\ldots\sum_{r_{r-1}=2}^{2r-1-r_1-r_2-\ldots-r_{r-2}}\mu_{r_1}^{(i_1)}\mu_{r_2}^{(i_2)}\ldots\mu_{r_{r-1}}^{(i_{r-1})}\mu_{2r+1-r_1-\ldots-r_{r-1}}^{(i_r)}\right\} \\ = {} & 1.3.5\ldots(2r+1)\left\{\frac{1}{N^{r+1}}\tfrac{1}{3}r\,\mu_{[2,\,N]}^{r-1}\mu_{[3,\,N]}\right. \\ & + \frac{1}{N^{r+2}}\left[\tfrac{1}{30}r^{[-2]}\mu_{[2,\,N]}^{r-2}\mu_{[5,\,N]} + \tfrac{1}{18}r^{[-3]}\mu_{[2,\,N]}^{r-3}\mu_{[3,\,N]}\mu_{[4,\,N]}\right. \\ & + \tfrac{1}{81}r^{[-4]}\mu_{[2,\,N]}^{r-4}\mu_{[3,\,N]}^{3} - \tfrac{1}{3}r^{[-2]}\mu_{[2,\,N]}^{r-2}\left(\frac{1}{N}\sum_{i=1}^{N}\mu_2^{(i)}\mu_3^{(i)}\right) \\ & \left.\left. - \tfrac{1}{6}r^{[-3]}\mu_{[2,\,N]}^{r-3}\left(\frac{1}{N}\sum_{i=1}^{N}(\mu_2^{(i)})^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\mu_3^{(i)}\right)\right] + \ldots\right\}\end{aligned}\right\}\ldots(7)^*.$$

Putting $r = 1, 2, \ldots$ we find †

$$\left.\begin{aligned}\mu_{2,\,(N)} &= \frac{1}{N}\mu_{[2,\,N]} \\ \mu_{3,\,(N)} &= \frac{1}{N^2}\mu_{[3,\,N]} \\ \mu_{4,\,(N)} &= \frac{3}{N^2}\mu_{[2,\,N]}^{2} + \frac{1}{N^3}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_4^{(i)} - 3(\mu_2^{(i)})^2]\right\} \\ \mu_{5,\,(N)} &= \frac{10}{N^3}\mu_{[3,\,N]}\mu_{[2,\,N]} + \frac{1}{N^4}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_5^{(i)} - 10\mu_3^{(i)}\mu_2^{(i)}]\right\}\end{aligned}\right\}\ldots\ldots(8).$$

IV

Relations (6) and (7) show that as N increases $\dfrac{\mu_{2r,\,(N)}}{\mu_{2,\,(N)}^{r}}$ tends to the limit $1.3.5\ldots(2r-1)$ and $\dfrac{\mu_{2r-1,\,(N)}^{2}}{\mu_{2,\,(N)}^{2r-1}}$ tends to the limit zero provided that

$$\frac{\sum_{i=1}^{N}\mu_{2h}^{(i)}}{\left[\sum_{i=1}^{N}\mu_2^{(i)}\right]^h} = \frac{N\mu_{[2h,\,N]}}{[N\mu_{[2,\,N]}]^h} \quad\text{and}\quad \frac{\left[\sum_{i=1}^{N}\mu_{2h-1}^{(i)}\right]^2}{\left[\sum_{i=1}^{N}\mu_2^{(i)}\right]^{2h-1}} = \frac{N^2\mu_{[2h-1,\,N]}^{2}}{[N\mu_{[2,\,N]}]^{2h-1}}$$

tend, as N increases, to the limit zero, for $h = 2, 3, \ldots r$. Hence, if

$$\frac{\left[\sum_{i=1}^{N}\mu_r^{(i)}\right]^2}{\left[\sum_{i=1}^{N}\mu_2^{(i)}\right]^r} \quad\text{or}\quad \frac{N\mu_{[r,\,N]}}{[N\mu_{[2,\,N]}]^{r/2}}$$

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* Cf. *Biometrika*, Vol. XII, p. 154, equation (23).

† *Vide* Tschuproff, "Zur Theorie der Stabilität statistischer Reihen" (*Skandinavisk Aktuarietidskrift*, 1919, p. 82) and *Biometrika*, Vol. XII, p. 155, equation (26).

tends, as N increases, to the limit zero, for $r = 3, 4, 5, \dots \infty$, then the law of distribution of the values of $X_{(N)}$ tends to become the Laplace-Gauss law as N becomes infinitely great, whatever may be the law of distribution of the individual variates[*].

V

If all the variates follow the Laplace-Gauss law then for all positive integral values of r,

$$\mu_{2r+1,\,(N)} = 0,$$

$$\begin{aligned}\mu_{2r,\,(N)} = \frac{1}{N^{2r}} \Bigg\{ & 1\,.\,3\,.\,5 \dots (2r-1) \sum_{i=1}^{N} [\mu_2^{(i)}]^r + \sum_{i_1=1}^{N-1} \sum_{i_2=i_1+1}^{N} \sum_{r_1=1}^{r-1} \frac{[2r\,!]}{[2r_1]!\,[2r-2r_1]!} \\ & \times 1\,.\,3\,.\,5 \dots (2r_1-1)\,.\,1\,.\,3\,.\,5 \dots (2r-2r_1-1)\,[\mu_2^{(i_1)}]^{r_1}\,[\mu_2^{(i_2)}]^{r-r_1} \\ & + \dots + 1\,.\,3\,.\,5 \dots (2r-1)\,.\,1\,.\,2\,.\,3 \dots r \sum_{i_1=1}^{N-r+1} \sum_{i_2=i_1+1}^{N-r+2} \dots \sum_{i_r=i_{r-1}+1}^{N} \mu_2^{(i_1)} \mu_2^{(i_2)} \dots \mu_2^{(i_r)} \Bigg\} \\ = \frac{1\,.\,3\,.\,5 \dots (2r-1)}{N^{2r}} \Bigg\{ & \sum_{i=1}^{N} [\mu_2^{(i)}]^r + \sum_{i_1=1}^{N-1} \sum_{i_2=i_1+1}^{N} \sum_{r_1=1}^{r-1} \frac{r\,!}{r_1!\,(r-r_1)!} [\mu_2^{(i_1)}]^{r_1} [\mu_2^{(i_2)}]^{r-r_1} \\ & + \dots + \sum_{i_1=1}^{N-r+1} \sum_{i_2=i_1+1}^{N-r+2} \dots \sum_{i_r=i_{r-1}+1}^{N} r\,!\,\mu_2^{(i_1)} \mu_2^{(i_2)} \dots \mu_2^{(i_r)} \Bigg\} \\ = \frac{1\,.\,3\,.\,5 \dots (2r-1)}{N^{2r}} & \left[\sum_{i=1}^{N} \mu_2^{(i)} \right]^r = \frac{1\,.\,3\,.\,5 \dots (2r-1)}{N^r} \mu_{[2,\,N]}^r ,\end{aligned}$$

and consequently
$$\frac{\mu_{2r,(N)}}{[\mu_{2,\,(N)}]^r} = 1\,.\,3\,.\,5 \dots (2r-1).$$

Thus in this case, the arithmetic mean of the random values taken by the variates follows the Laplace-Gauss law whatever may be the constants of the laws of distribution of the individual variates[†].

CHAPTER II

I

Putting[‡]
$$\frac{1}{N} \sum_{i=1}^{N} [X_i' - m_1^{(i)}]^r = \mu'_{[r,\,N]},$$
$$\frac{1}{N} \sum_{i=1}^{N} [X_i' - m_{[1,\,N]}]^r = \mu''_{[r,\,N]},$$

we find

$$\left.\begin{aligned} E\mu'_{[r,\,N]} &= \frac{1}{N} \sum_{i=1}^{N} \mu_r^{(i)} = \mu_{[r,\,N]} \\ E\mu''_{[r,\,N]} &= \frac{1}{N} \sum_{i=1}^{N} \sum_{h=0}^{r} C_r^h\, \mu_{r-h}^{(i)} [m_1^{(i)} - m_{[1,\,N]}]^h \\ &= \mu_{[r,\,N]} + r \frac{1}{N} \sum_{i=1}^{N} \mu_{r-1}^{(i)} [m_1^{(i)} - m_{[1,\,N]}] \\ &\quad + \dots + \frac{r(r-1)}{2} \frac{1}{N} \sum_{i=1}^{N} \mu_2^{(i)} [m_1^{(i)} - m_{[1,\,N]}]^{r-2} + \frac{1}{N} \sum_{i=1}^{N} [m_1^{(i)} - m_{[1,\,N]}]^r \end{aligned}\right\} \dots (1).$$

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[*] Cf. *Biometrika*, Vol. XII, pp. 156—157.

[†] Cf. *Biometrika*, Vol. XII, p. 158.

[‡] *Vide* Tschuproff, "Zur Theorie der Stabilität statistischer Reihen" (*Skandinavisk Aktuarietidskrift*, 1919, pp. 82—83).

It follows that if all the quantities $m_1^{(i)}$ are equal to one another, then $\mu''_{[r,N]} = \mu'_{[r,N]}$, whatever the frequency distributions of the separate variates may be. If however the quantities $m_1^{(i)}$ are unequal then $\mu''_{[r,N]} \neq \mu'_{[r,N]}$ and, in general, $E\mu''_{[r,N]}$ is also not equal to $E\mu'_{[r,N]}$.

On the other hand

$$\begin{aligned} E\mu''_{[r,N]} &= \frac{1}{N} \sum_{i=1}^{N} \sum_{h=0}^{r} (-1)^h C_r^h m_{[1,N]}^h m_{r-h}^{(i)} = \sum_{h=0}^{r} (-1)^h C_r^h m_{[1,N]}^h m_{[r-h,N]} \\ &= m_{[r,N]} + \sum_{h=1}^{r-2} (-1)^h C_r^h m_{[1,N]}^h m_{[r-h,N]} + (-1)^{r-1} (r-1) m_{[1,N]}^r. \end{aligned}$$

When $r = 1$ we get

$$\frac{1}{N} \sum_{i=1}^{N} [X_i' - m_{[1,N]}] = \frac{1}{N} \sum_{i=1}^{N} [X_i' - m_1^{(i)}] = X_{(N)} - m_{[1,N]};$$

so that the quantities $\mu'_{[1,N]}$ and $\mu''_{[1,N]}$ are identically equal. If in the relation (1) we put r equal to 2, 3, 4 successively, we find,

$$\left.\begin{aligned} E\mu''_{[2,N]} &= \mu_{[2,N]} + \frac{1}{N} \sum_{i=1}^{N} [m_1^{(i)} - m_{[1,N]}]^2 \\ E\mu''_{[3,N]} &= \mu_{[3,N]} + \frac{3}{N} \sum_{i=1}^{N} \mu_2^{(i)} [m_1^{(i)} - m_{[1,N]}] + \frac{1}{N} \sum_{i=1}^{N} [m_1^{(i)} - m_{[1,N]}]^3 \\ E\mu''_{[4,N]} &= \mu_{[4,N]} + \frac{4}{N} \sum_{i=1}^{N} \mu_3^{(i)} [m_1^{(i)} - m_{[1,N]}] + \frac{6}{N} \sum_{i=1}^{N} \mu_2^{(i)} [m_1^{(i)} - m_{[1,N]}]^2 \\ &\qquad + \frac{1}{N} \sum_{i=1}^{N} [m_1^{(i)} - m_{[1,N]}]^4. \end{aligned}\right\} \ldots (2).$$

II

(1) Noting that $\mu'_{[r,N]}$ is the arithmetic mean of the mutually independent quantities $[X_1' - m_1^{(1)}]^r, [X_2' - m_1^{(2)}]^r, \ldots [X'_N - m_1^{(N)}]^r$ and that $E[X_i' - m_1^{(i)}]^r = \mu_r^{(i)}$, while $E[X_i' - m_1^{(i)}]^{kr} = \mu_{kr}^{(i)}$, we find from relations (3) of Chapter I, on replacing $m_{[1,N]}$ by $\frac{1}{N} \sum_{i=1}^{N} \mu_r^{(i)} = \mu_{[r,N]}$, $\mu_{[2,N]}$ by $\frac{1}{N} \sum_{i=1}^{N} [\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]$ and so forth,

$$\left.\begin{aligned} E[\mu'_{[r,N]}]^m &= \mu_{[r,N]}^m + \frac{1}{N} \frac{m(m-1)}{2} \mu_{[r,N]}^{m-2} \left(\frac{1}{N} \sum_{i=1}^{N} [\mu_{2r}^{(i)} - (\mu_r^{(i)})^2] \right) \\ &+ \frac{1}{N^2} \left\{ \frac{m^{[-4]}}{8} \mu_{[r,N]}^{m-4} \left(\frac{1}{N} \sum_{i=1}^{N} [\mu_{2r}^{(i)} - (\mu_r^{(i)})^2] \right)^2 \right. \\ &\qquad \left. + \frac{m^{[-3]}}{6} \mu_{[r,N]}^{m-3} \left(\frac{1}{N} \sum_{i=1}^{N} [\mu_{3r}^{(i)} - 3\mu_{2r}^{(i)} \mu_r^{(i)} + 2(\mu_r^{(i)})^3] \right) \right\} + \ldots \end{aligned}\right\} \ldots (3)^*.$$

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* Cf. *Biometrika*, Vol. XII, p. 160, equation (6).

Similarly we find

$$\left.\begin{aligned}
E[\mu'_{[r,N]}]^2 &= \overset{2}{\mu}_{[r,N]} + \frac{1}{N}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right\}\\
E[\mu'_{[r,N]}]^3 &= \overset{3}{\mu}_{[r,N]} + \frac{3}{N}\mu_{[r,N]}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right\}\\
&\qquad + \frac{1}{N^2}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{3r}^{(i)} - 3\mu_{2r}^{(i)}\mu_r^{(i)} + 2(\mu_r^{(i)})^3]\right\}\\
E[\mu'_{[r,N]}]^4 &= \overset{4}{\mu}_{[r,N]} + \frac{6}{N}\overset{2}{\mu}_{[r,N]}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right\}\\
&\quad + \frac{1}{N^2}\left\{4\mu_{[r,N]}\left(\frac{1}{N}\sum_{i=1}^{N}[\mu_{3r}^{(i)} - 3\mu_{2r}^{(i)}\mu_r^{(i)} + 2(\mu_r^{(i)})^3]\right)\right.\\
&\qquad\qquad \left. + 3\left(\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right)^2\right\}\\
&\quad + \frac{1}{N^3}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{4r}^{(i)} - 4\mu_{3r}^{(i)}\mu_r^{(i)} + 6\mu_{2r}^{(i)}(\mu_r^{(i)})^2 - 3(\mu_r^{(i)})^4]\right.\\
&\qquad\qquad \left. - 3\left(\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]^2\right)\right\}
\end{aligned}\right\}\dots(4).$$

(2) Introduce the notation,

$$\left.\begin{aligned}
Z_{(kr,N)}^{(i)} &= E[X_i' - m_{[1,N]}]^{kr} = \sum_{h=0}^{kr}(-1)^h C_{kr}^h \overset{h}{m}_{[1,N]} m_{rk-h}^{(i)}\\
&= \sum_{h=0}^{kr} C_{kr}^h [m_1^{(i)} - m_{[1,N]}]^h \mu_{rk-h}^{(i)}\\
Z_{(kr,N)} &= \frac{1}{N}\sum_{i=1}^{N} Z_{(kr,N)}^{(i)} = \sum_{h=0}^{kr}(-1)^h C_{kr}^h \overset{h}{m}_{[1,N]} m_{[kr-h,N]}
\end{aligned}\right\}\quad(5).$$

Noting that $\mu''_{(r,N)}$ is the arithmetic mean of the mutually independent quantities

$$[X_1' - m_{[1,N]}]^r,\ [X_2' - m_{[1,N]}]^r, \dots [X'_N - m_{[1,N]}]^r,$$

we find

$$\left.\begin{aligned}
E[\mu''_{[r,N]}]^m &= [Z_{(r,N)}]^m + \frac{1}{N}\frac{m(m-1)}{2}[Z_{[r,N]}]^{m-2}\left[Z_{(2r,N)} - \frac{1}{N}\sum_{i=1}^{N}(Z_{r,N}^{(i)})^2\right]\\
&\quad + \frac{1}{N^2}\left\{\frac{m^{[-4]}}{8}[Z_{(r,N)}]^{m-4}\left[Z_{(2r,N)} - \frac{1}{N}\sum_{i=1}^{N}(Z_{(r,N)}^{(i)})^2\right]^2\right.\\
&\quad + \frac{m^{[-3]}}{6}[Z_{(r,N)}]^{m-3}\left[Z_{(3r,N)} - 3\frac{1}{N}\sum_{i=1}^{N}Z_{(2r,N)}^{(i)} Z_{(r,N)}^{(i)}\right.\\
&\qquad\qquad \left.\left. + 2\frac{1}{N}\sum_{i=1}^{N}(Z_{(r,N)}^{(i)})^3\right]\right\} + \dots
\end{aligned}\right\}\quad(6).$$

III

(1) Put

$$\sum_{h=0}^{k}(-1)^h C_k{}^h [\mu_r{}^{(i)}]^h \mu_{(k-h)r}^{(i)} = \chi_k{}^{(i)}, \quad \frac{1}{N}\sum_{i=1}^{N}\chi_k{}^{(i)} = \chi_{[k,N]}.$$

If in equations (6) and (7) of Chapter I we replace $\mu_k{}^{(i)}$ by $\chi_k{}^{(i)}$ and $\mu_{[k,\,N]}$ by $\chi_{[k,\,N]}$ we find

$$\left.\begin{aligned}
E\,[\mu'_{[r,\,N]} - \mu_{[r,\,N]}]^{2m} &= 1\,.\,3\,.\,5\,\ldots\,(2m-1)\left\{\frac{1}{N^m}(\chi_{[2,\,N]})^m\right.\\
&+\frac{1}{N^{m+1}}\Big[\tfrac{1}{6}m^{[-2]}(\chi_{[2,\,N]})^{m-2}\chi_{[4,\,N]} + \tfrac{1}{9}m^{[-3]}(\chi_{[2,\,N]})^{m-3}(\chi_{[3,\,N]})^2\\
&\left.-\tfrac{1}{2}m^{[-2]}(\chi_{[2,\,N]})^{m-2}\left(\frac{1}{N}\sum_{i=1}^{N}(\chi_2{}^{(i)})^2\right)\Big]+\ldots\right\}
\end{aligned}\right\}\quad\ldots\ldots(7),$$

$$\left.\begin{aligned}
&E\,[\mu'_{[r,\,N]} - \mu_{[r,\,N]}]^{2m+1}\\
&\quad= 1\,.\,3\,.\,5\,\ldots\,(2m+1)\left\{\frac{1}{N^{m+1}}\tfrac{1}{3}m\,(\chi_{[2,\,N]})^{m-1}\chi_{[3,\,N]}\right.\\
&\qquad+\frac{1}{N^{m+2}}\Big[\tfrac{1}{30}m^{[-2]}(\chi_{[2,\,N]})^{m-2}\chi_{[5,\,N]} + \tfrac{1}{18}m^{[-3]}(\chi_{[2,\,N]})^{m-3}\chi_{[3,\,N]}\chi_{[4,\,N]}\\
&\qquad+\tfrac{1}{81}m^{[-4]}(\chi_{[2,\,N]})^{m-4}(\chi_{[3,\,N]})^3 - \tfrac{1}{3}m^{[-2]}(\chi_{[2,\,N]})^{m-2}\left(\frac{1}{N}\sum_{i=1}^{N}\chi_2^{(i)}\chi_3^{(i)}\right)\\
&\qquad\left.-\tfrac{1}{6}m^{[-3]}(\chi_{[2,\,N]})^{m-3}\left(\frac{1}{N}\sum_{i=1}^{N}(\chi_2^{(i)})^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}\chi_3^{(i)}\right)\Big]+\ldots\right\}
\end{aligned}\right\}\quad(8).$$

Hence*

$$\left.\begin{aligned}
E\,[\mu'_{[r,\,N]} - \mu_{[r,\,N]}]^2 &= \frac{1}{N}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right\}\\
E\,[\mu'_{[r,\,N]} - \mu_{[r,\,N]}]^3 &= \frac{1}{N^2}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{3r}^{(i)} - 3\mu_{2r}^{(i)}\mu_r^{(i)} + 2(\mu_r^{(i)})^3]\right\}\\
E\,[\mu'_{[r,\,N]} - \mu_{[r,\,N]}]^4 &= \frac{3}{N^2}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]\right\}^2\\
&\quad+\frac{1}{N^3}\left\{\frac{1}{N}\sum_{i=1}^{N}[\mu_{4r}^{(i)} - 4\mu_{3r}^{(i)}\mu_r^{(i)} + 6\mu_{2r}^{(i)}(\mu_r^{(i)})^2 - 3(\mu_r^{(i)})^4]\right.\\
&\quad\left.-3\,\frac{1}{N}\sum_{i=1}^{N}[\mu_{2r}^{(i)} - (\mu_r^{(i)})^2]^2\right\}
\end{aligned}\right\}\quad(9).$$

When $r = 2$, for the case in which the frequency distributions of all the variates follow the Laplace-Gauss law, we have

$$\left.\begin{aligned}
E\,[\mu'_{[2,\,N]} - \mu_{[2,\,N]}]^2 &= \frac{2}{N}\left(\frac{1}{N}\sum_{i=1}^{N}(\mu_2^{(i)})^2\right)\\
E\,[\mu'_{[2,\,N]} - \mu_{[2,\,N]}]^3 &= \frac{8}{N^2}\left(\frac{1}{N}\sum_{i=1}^{N}(\mu_2^{(i)})^3\right)\\
E\,[\mu'_{[2,\,N]} - \mu_{[2,\,N]}]^4 &= \frac{12}{N^2}\left(\frac{1}{N}\sum_{i=1}^{N}(\mu_2^{(i)})^2\right)^2 + \frac{48}{N^3}\left(\frac{1}{N}\sum_{i=1}^{N}(\mu_2^{(i)})^4\right)
\end{aligned}\right\}\quad\ldots(10).$$

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* Cf. *Biometrika*, Vol. XII, pp. 162—163, equations (14), (15) and (16).

(2) Putting

$$\sum_{j=0}^{s}(-1)^j C_s{}^j \left(Z^{(i)}_{(r,\,N)}\right)^j Z^{(i)}_{([s-j]\,r,\,N)} = \phi_s{}^{(i)}, \quad \frac{1}{N}\sum_{i=1}^{N}\phi_s{}^{(i)} = \phi_{[s,\,N]},$$

we find

$$\left.\begin{aligned} E[\mu''_{[r,\,N]} - Z_{[r,\,N]}]^{2m} = 1\,.\,3\,.\,5 \ldots (2m-1) \Bigg\{ & \frac{1}{N^m}(\phi_{[2,\,N]})^m \\ & + \frac{1}{N^{m+1}}\Big[\tfrac{1}{6}m^{[-2]}(\phi_{[2,\,N]})^{m-2}\phi_{[4,\,N]} + \tfrac{1}{9}m^{[-3]}(\phi_{[2,\,N]})^{m-3}(\phi_{[3,\,N]})^2 \\ & - \tfrac{1}{2}m^{[-2]}(\phi_{[2,\,N]})^{m-2}\Big(\frac{1}{N}\sum_{i=1}^{N}(\phi_2{}^{(i)})^2\Big)\Big] + \ldots\Bigg\} \end{aligned}\right\} \quad (11).$$

$$\left.\begin{aligned} E[\mu''_{[r,\,N]} - Z_{[r,\,N]}]^{2m+1} = 1\,.\,3\,.\,5 \ldots (2m+1) \Bigg\{ & \frac{1}{N^{m+1}}\tfrac{1}{3}m(\phi_{[2,\,N]})^{m-1}\phi_{[3,\,N]} \\ & + \frac{1}{N^{m+2}}\Big[\tfrac{1}{30}m^{[-2]}(\phi_{[2,\,N]})^{m-2}\phi_{[5,\,N]} + \tfrac{1}{18}m^{[-3]}(\phi_{[2,\,N]})^{m-3}\phi_{[3,\,N]}\phi_{[4,\,N]} \\ & + \tfrac{1}{81}m^{[-4]}(\phi_{[2,\,N]})^{m-4}(\phi_{[3,\,N]})^3 - \tfrac{1}{3}m^{[-2]}(\phi_{[2,\,N]})^{m-2}\Big(\frac{1}{N}\sum_{i=1}^{N}\phi_2{}^{(i)}\phi_3{}^{(i)}\Big) \\ & - \tfrac{1}{6}m^{[-3]}(\phi_{[2,\,N]})^{m-3}\Big(\frac{1}{N}\sum_{i=1}^{N}(\phi_2{}^{(i)})^2\Big)\Big(\frac{1}{N}\sum_{i=1}^{N}\phi_3{}^{(i)}\Big)\Big] + \ldots\Bigg\} \end{aligned}\right\} \quad (12).$$

Hence

$$\left.\begin{aligned} E[\mu''_{[r,\,N]} - Z_{[r,\,N]}]^2 &= \sigma^2{}_{\mu''_{[r,\,N]}} \\ &= \frac{1}{N}\Bigg\{\frac{1}{N}\sum_{i=1}^{N}\Bigg([\mu^{(i)}_{2r} - (\mu^{(i)}_r)^2] + [m^{(i)}_1 - m_{[1,\,N]}][2r\mu^{(i)}_{2r-1} - 2r\mu^{(i)}_r\mu^{(i)}_{r-1}] \\ &\quad + \sum_{h=2}^{r-2}[m^{(i)}_1 - m_{[1,\,N]}]^h\Big[C^h_{2r}\mu^{(i)}_{2r-h} - \sum_{j=0}^{h}C^j_r C^{h-j}_r\mu^{(i)}_{r-j}\mu^{(i)}_{r-h+j}\Big] \\ &\quad + (m^{(i)}_1 - m_{[1,\,N]})^{r-1}\Big[C^{r-1}_{2r}\mu^{(i)}_{r+1} - \sum_{j=1}^{r-2}C^j_r C^{r-1-j}_r\mu^{(i)}_{r-j}\mu^{(i)}_{1+j}\Big] \\ &\quad + (m^{(i)}_1 - m_{[1,\,N]})^r\Big[C^r_{2r}\mu^{(i)}_r - 2\mu^{(i)}_r - \sum_{j=2}^{r-2}C^j_r C^{r-j}_r\mu^{(i)}_{r-j}\mu^{(i)}_j\Big] \\ &\quad + \sum_{h=1}^{r-2}[m^{(i)}_1 - m_{[1,\,N]}]^{r+h}\Big[C^{r+h}_{2r}\mu^{(i)}_{r-h} - \sum_{j=0}^{r-h}C^{r-j}_r C^{h+j}_r\mu^{(i)}_j\mu^{(i)}_{r-h-j}\Big]\Bigg)\Bigg\} \end{aligned}\right\} \quad (13).$$

When $r = 2$, we find

$$\left.\begin{aligned} \sigma^2{}_{\mu''_{[2,\,N]}} = \frac{1}{N}\Bigg\{\frac{1}{N}\sum_{i=1}^{N}\Bigg(& [\mu_4{}^{(i)} - (\mu_2{}^{(i)})^2] + 4[m_1{}^{(i)} - m_{[1,\,N]}]\mu_3{}^{(i)} \\ & + 4[m_1{}^{(i)} - m_{[1,\,N]}]^2\mu_2{}^{(i)}\Bigg)\Bigg\} \end{aligned}\right\} \quad (14).$$

In the case for which all the variates are distributed according to the Gauss-Laplace law,

$$\sigma^2{}_{\mu''_{[2,\,N]}} = \frac{2}{N}\left\{\frac{1}{N}\sum_{i=1}^{N}\left(\mu_2{}^{(i)}[\mu_2{}^{(i)} + 2(m_1{}^{(i)} - m_{[1,\,N]})^2]\right)\right\} \quad \ldots\ldots\ldots(15).$$

(3) Noting that

$$E[\mu''_{[r,N]} - \mu_{[r,N]}]^2 = E[\mu''_{[r,N]} - Z_{[r,N]}]^2 + [Z_{(r,N)} - \mu_{(r,N)}]^2,$$

we find (cf. supra equation (5) and Chapter I, equation (1))

$$\left.\begin{aligned} E[\mu''_{[2,N]} - \mu_{[2,N]}]^2 &= \left\{\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\right\}^2 \\ &+ \frac{1}{N}\left\{\frac{1}{N}\sum_{i=1}^{N}\Big([\mu_4^{(i)} - (\mu_2^{(i)})^2] + 4[m_1^{(i)} - m_{[1,N]}]\,\mu_3^{(i)} \right. \\ &\qquad\qquad\left. + 4[m_1^{(i)} - m_{[1,N]}]^2\mu_2^{(i)}\Big)\right\} \end{aligned}\right\}\ldots(16).$$

But (see equation (2) above)

$$E[\mu''_{[2,N]} - \mu_{[2,N]}] = \frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2.$$

We thus see that

$$\left.\begin{aligned} \sigma^2_{\mu''_{[2,N]} - \mu_{[2,N]}} &= E\left\{[\mu''_{[2,N]} - \mu_{[2,N]}] - \frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\right\}^2 \\ &= \frac{1}{N}\left\{\frac{1}{N}\sum_{i=1}^{N}\Big([\mu_4^{(i)} - (\mu_2^{(i)})^2] + 4[m_1^{(i)} - m_{[1,N]}]\,\mu_3^{(i)} \right. \\ &\qquad\qquad\left. + 4[m_1^{(i)} - m_{[1,N]}]^2\mu_2^{(i)}\Big)\right\} \end{aligned}\right\}\ldots(17).$$

(4) From the relations (2) and (1) we get

$$E[\mu''_{[2,N]} - \mu'_{[2,N]}] = \frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2.$$

Noting, that

$$\begin{aligned} \mu''_{[2,N]} - \mu'_{[2,N]} &= \frac{1}{N}\sum_{i=1}^{N}[X_i' - m_{[1,N]}]^2 - \frac{1}{N}\sum_{i=1}^{N}[X_i' - m_1^{(i)}]^2 \\ &= \frac{1}{N}\left\{2\sum_{i=1}^{N}X_i'[m_1^{(i)} - m_{[1,N]}] - \sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\right\}, \end{aligned}$$

we find

$$E[\mu''_{[2,N]} - \mu'_{[2,N]}]^2 = \frac{4}{N^2}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\mu_2^{(i)} + \left(\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\right)^2 \quad (18),$$

and

$$\left.\begin{aligned} \sigma^2_{\mu''_{[2,N]} - \mu'_{[2,N]}} &= E\left\{[\mu''_{[2,N]} - \mu'_{[2,N]}] - \frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\right\}^2 \\ &= \frac{4}{N^2}\sum_{i=1}^{N}[m_1^{(i)} - m_{[1,N]}]^2\mu_2^{(i)} \end{aligned}\right\} \quad (19).$$

CHAPTER III

I

Let us put

$$\frac{1}{N}\sum_{i=1}^{N}[X_i' - X_{(N)}]^r = \nu'_{r,(N)},$$

$$E\nu'_{r,(N)} = \nu_{r,(N)}.$$

When $r=1$, we have the identities,

$$\nu'_{1,(N)}=0 \text{ and } \nu_{1,(N)}=0.$$

Noting that

$$\begin{aligned}X_i'-X_{(N)}&=[X_i'-m_1^{(i)}]+[m_1^{(i)}-m_{[1,N]}]-\frac{1}{N}\sum_{j=1}^{N}[X_j'-m_1^{(j)}]\\&=\frac{N-1}{N}[X_i'-m_1^{(i)}]+[m_1^{(i)}-m_{[1,N]}]-\frac{N-1}{N}\,\frac{1}{N-1}\sum_{j\neq i}[X_j'-m_1^{(j)}],\end{aligned}$$

and putting

$$E\left\{\frac{1}{N-1}\sum_{j\neq i}[X_j'-m_1^{(j)}]\right\}^g=\mu_{g,(N-1)/i},$$

we find

$$\begin{aligned}\nu_{r,(N)}&=\frac{1}{N}\sum_{i=1}^{N}\sum_{g=0}^{r}\sum_{h=0}^{r-g}(-1)^g C_r^g C_{r-g}^h\left[\frac{N-1}{N}\right]^{r-h}[m_1^{(i)}-m_{[1,N]}]^h\mu_{r-g-h}^{(i)}\mu_{g,(N-1)/i}\\&=\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^r+\frac{1}{N}\sum_{i=1}^{N}C_r^{r-2}\left[\frac{N-1}{N}\right]^2[m_1^{(i)}-m_{[1,N]}]^{r-2}[\mu_2^{(i)}+\mu_{2,(N-1)/i}]\\&\quad+\frac{1}{N}\sum_{i=1}^{N}C_r^{r-3}\left[\frac{N-1}{N}\right]^3[m_1^{(i)}-m_{[1,N]}]^{r-3}[\mu_3^{(i)}-\mu_{3,(N-1)/i}]\\&\quad+\frac{1}{N}\sum_{i=1}^{N}\sum_{h=1}^{r-4}C_r^h\left(\frac{N-1}{N}\right)^{r-h}[m_1^{(i)}-m_{[1,N]}]^h\\&\qquad\times\left[\mu_{r-h}^{(i)}+\sum_{g=2}^{r-h-2}(-1)^g C_{r-h}^g\mu_{r-h-g}^{(i)}\mu_{g,(N-1)/i}+(-1)^{r-h}\mu_{r-h,(N-1)/i}\right]\\&\quad+\frac{1}{N}\sum_{i=1}^{N}\left[\frac{N-1}{N}\right]^r\left[\mu_r^{(i)}+\sum_{g=2}^{r-2}(-1)^g C_r^g\mu_{r-g}^{(i)}\mu_{g,(N-1)/i}+(-1)^r\mu_{r,(N-1)/i}\right].\end{aligned}$$

Hence, noting that

$$\mu_{2,(N-1)/i}=\frac{1}{(N-1)^2}\sum_{j\neq i}\mu_2^{(j)}=\frac{N}{(N-1)^2}\mu_{[2,N]}-\frac{1}{(N-1)^2}\mu_2^{(i)},$$

$$\mu_{3,(N-1)/i}=\frac{N}{(N-1)^3}\mu_{[3,N]}-\frac{1}{(N-1)^3}\mu_3^{(i)} \text{ and so forth,}$$

we find*

$$\left.\begin{aligned}\nu_{r,(N)}&=\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^r+C_r^2\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^{r-2}\mu_2^{(i)}\\&\quad+C_r^3\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^{r-3}\mu_3^{(i)}+\sum_{h=0}^{r-4}C_r^h\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^h\mu_{r-h}^{(i)}\\&\quad+\frac{1}{N}\left\{C_r^2\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^{r-2}[\mu_{[2,N]}-2\mu_2^{(i)}]\right.\\&\qquad\qquad-3C_r^3\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^{r-3}\mu_3^{(i)}\\&\quad+\sum_{h=0}^{r-4}C_r^h(r-h)\frac{1}{N}\sum_{i=1}^{N}[m_1^{(i)}-m_{[1,N]}]^h\\&\qquad\left.\times\left[\frac{r-h-1}{2}\mu_{[2,N]}\mu_{r-2-h}^{(i)}-\mu_{r-h}^{(i)}\right]\right\}+\dots\end{aligned}\right\}\quad(1).$$

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* Cf. *Biometrika*, Vol. XII, p. 186, equation (6).

When $r = 2, 3, 4$ we find similarly*

$$\left.\begin{aligned}
\nu_{2,(N)} &= \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^2 + \frac{N-1}{N}\mu_{[2,N]}\\
\nu_{3,(N)} &= \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^3\\
&\qquad + 3\,\frac{N-2}{N}\,\frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]\mu_2^{(i)} + \frac{N^2-3N+2}{N^2}\mu_{[3,N]}\\
\nu_{4,(N)} &= \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^4 + \frac{6}{N}\mu_{[2,N]}\,\frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^2\\
&\quad + 6\,\frac{N-2}{N}\,\frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^2\mu_2^{(i)}\\
&\qquad + 4\,\frac{N^2-3N+3}{N^2}\,\frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]\mu_3^{(i)}\\
&\quad + \frac{(N-1)(N^2-3N+3)}{N^3}\mu_{[4,N]} + 3\,\frac{2N-3}{N^2}\mu^2_{[2,N]}\\
&\qquad - 3\,\frac{(2N-3)}{N^3}\,\frac{1}{N}\sum_{i=1}^{N}\left(\mu_2^{(i)}\right)^2
\end{aligned}\right\}\quad (2).$$

II

Noting that

$$\begin{aligned}
\left\{\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^2\right\}^2 &= \left\{\sum_{i=1}^{N}\left(m_1^{(i)}\right)^2 - \frac{1}{N}\left[\sum_{i=1}^{N} m_1^{(i)}\right]^2\right\}^2\\
&= \frac{1}{N^2}\left[\sum_{i=1}^{N} m_1^{(i)}\right]^4 - \frac{2}{N}\left[\sum_{i=1}^{N} m_1^{(i)}\right]^2\sum_{i=1}^{N}\left(m_1^{(i)}\right)^2 + \left[\sum_{i=1}^{N}\left(m_1^{(i)}\right)^2\right]^2\\
&= \frac{1}{N^2}\left[\sum_{i=1}^{N} m_1^{(i)}\right]^4 + \left[1 - \frac{2}{N}\right]\left[\sum_{i=1}^{N}\left(m_1^{(i)}\right)^4 + \sum_{i_1=1}^{N}\sum_{i_2\neq i_1}\left(m_1^{(i_1)}\right)^2\left(m_1^{(i_2)}\right)^2\right]\\
&\quad - \frac{2}{N}\left[2\sum_{i_1=1}^{N}\sum_{i_2\neq i_1}\left(m_1^{(i_1)}\right)^3\left(m_1^{(i_2)}\right) + \sum_{i_2=1}^{N}\sum_{i_2\neq i_1}\sum_{i_3\neq i_2\neq i_1}\left(m_1^{(i_1)}\right)^2\left(m_1^{(i_2)}\right)\left(m_1^{(i_3)}\right)\right],
\end{aligned}$$

we find, after some fairly tedious but quite straightforward transformations†,

$$\begin{aligned}
E\left[\sum_{i=1}^{N}\left(X_i' - X_{(N)}\right)^2\right]^2 &= \left\{\Sigma\left[m_1^{(i)} - m_{[1,N]}\right]^2\right\}^2 + \frac{(N-1)^2}{N}\mu_{[4,N]}\\
&\quad + \frac{N^2-2N+3}{N^2}\left\{\left[\sum_{i=1}^{N}\mu_2^{(i)}\right]^2 - \sum_{i=1}^{N}\left[\mu_2^{(i)}\right]^2\right\}\\
&\quad + 4(N-1)\left\{\frac{1}{N}\sum_{i=1}^{N}\mu_3^{(i)} m_1^{(i)} - \mu_{[3,N]}\, m_{[1,N]}\right\}\\
&\quad + \frac{2N-1}{N}\left(\sum_{i=1}^{N}\mu_2^{(i)}\right)\left(\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,N]}\right]^2\right)\\
&\quad + 4\sum_{i=1}^{N}\mu_2^{(i)}\left[m_1^{(i)} - m_{[1,N]}\right]^2,
\end{aligned}$$

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* *Vide* Tschuproff, "Zur Theorie der Stabilität statistischer Reihen" (*Skandinavisk Aktuarietidskrift*, 1919, p. 84, equations (1) and (2)) and *Biometrika*, Vol. XII, p. 186, equation (7).

† Cf. *Biometrika*, Vol. XII, p. 192, equation (19).

and hence

$$\begin{aligned}\sigma^2_{\nu'_{2,\,(N)}} &= E\left[\nu'_{2,\,(N)} - \nu_{2,\,(N)}\right]^2 \\ &= E\left\{\frac{1}{N}\sum_{i=1}^{N}\left[X_i' - X_{(N)}\right]^2\right\}^2 - \left(\frac{N-1}{N}\mu_{[2,\,N]} + \frac{1}{N}\sum_{i=1}^{N}\left[m_1^{(i)} - m_{[1,\,N]}\right]^2\right)^2 \\ &= \frac{(N-1)^2}{N^3}\frac{1}{N}\sum_{i=1}^{N}\left[\mu_4^{(i)} - 3\left(\mu_2^{(i)}\right)^2\right] + 2\,\frac{N-1}{N^2}\frac{1}{N}\sum_{i=1}^{N}\left[\mu_2^{(i)}\right]^2 \\ &\quad - \frac{2}{N^2}\frac{1}{N}\sum_{i=1}^{N}\left[\mu_2^{(i)} - \mu_{[2,\,N]}\right]^2 + \frac{4}{N}\frac{1}{N}\sum_{i=1}^{N}\mu_2^{(i)}\left[m_1^{(i)} - m_{[1,\,N]}\right]^2 \\ &\quad + 4\,\frac{N-1}{N^2}\left\{\frac{1}{N}\sum_{i=1}^{N}\mu_3^{(i)} m_1^{(i)} - \mu_{[3,\,N]}\, m_{[1,\,N]}\right\}^{*\dagger}.\end{aligned}$$

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[*] *Vide* Tschuproff, "Zur Theorie der Stabilität statistischer Reihen" (*Skandinavisk Aktuarietidskrift*, 1919, p. 85, equation (4)) and *Biometrika*, Vol. XII, p. 193, equation (23).

[†] The problems discussed above can also, as regards most of them, be attacked by the methods employed in the fourth chapter of the first part of my paper "On the Mathematical Expectation of the Moments of Frequency Distributions" (*Biometrika*, Vol. XII, pp. 194—210). Suppose that the aggregate of all the possible values that may be assumed by $X_1, X_2, \dots X_N$ reduces to $\xi_1, \xi_2, \dots \xi_k$. Denote by $p_j^{(i)}$ the probability that the variate X_i may take the value ξ_j, where $p_j^{(i)} = 0$, if ξ_j does not occur among the possible values of X_i; denote by n_j the number of times, that among the N experiments performed, one or other of the variates takes the value ξ_j.

Putting
$$p_j = \frac{1}{N}\sum_{i=1}^{N} p_j^{(i)};$$
we find

$$m_r^{(i)} = \sum_{j=1}^{k} p_j^{(i)}\,\xi_j^r; \quad m_{[r,\,N]} = \sum_{j=1}^{k} p_j\,\xi_j^r;$$

$$\mu_r^{(i)} = \sum_{j=1}^{k} p_j^{(i)}\left[\xi_j - m_1^{(i)}\right]^r; \quad \mu_{[r,\,N]} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k} p_j^{(i)}\left[\xi_j - m_1^{(i)}\right]^r;$$

$$X_{(N)} = \frac{1}{N}\sum_{j=1}^{k} n_j\,\xi_j = m'_{[1,\,N]}, \quad \text{where } m'_{[r,\,N]} = \frac{1}{N}\sum_{j=1}^{k} n_j\,\xi_j^r;$$

$$\mu''_{[r,\,N]} = \frac{1}{N}\sum_{j=1}^{k} n_j\left[\xi_j - m_{[1,\,N]}\right]^r;$$

$$\nu'_{r,\,(N)} = \frac{1}{N}\sum_{j=1}^{k} n_j\left[\xi_j - X_{(N)}\right]^r = \sum_{h=0}^{r}(-1)^h C_r^h\left(\mu''_{[r,\,N]}\right)^{r-h}\omega^h, \quad \text{where } \omega = X_{(N)} - m_{[1,\,N]}.$$

Of the various quantities discussed above, $\mu'_{[r,\,N]}$ is the only one that cannot be expressed in terms of n_j and ξ_j. The others can be studied by means of relations analogous to those numbered (1) to (6) in the fourth chapter of the first part of my paper (*Biometrika*, Vol. XII, pp. 195—6). This method is however of little practical use, as the resulting formulae are too cumbrous. Thus with $En_j = Np_j$, we get for the mathematical expectation of the square of n_j,

$$\begin{aligned}En_j^2 &= N^2p_j^2 + Np_j - \sum_{i=1}^{N}\left(p_j^{(i)}\right)^2 \\ &= N^2p_j^2 + Np_j(1 - p_j) - \sum_{i=1}^{N}\left[p_j^{(i)} - p_j\right]^2.\end{aligned}$$

For the mathematical expectation of the cube of n_j we find

$$En_j^3 = N^3p_j^3 + 3N^2p_j^2(1 - p_j) + Np_j\left[1 - 3p_j + 2p_j^2\right] - 3\left[Np_j - 2p_j + 1\right]\sum_{i=1}^{N}\left[p_j^{(i)} - p_j\right]^2 + 2\sum_{i=1}^{N}\left[p_j^{(i)} - p_j\right]^3$$

and so forth.

MISCELLANEA.

I. On a General Method of determining the successive terms in a Skew Regression Line*.

BY KARL PEARSON, F.R.S.

Let x and y be the two variates. Suppose the x-range divided up into any not necessarily equal intervals h_x giving arrays of y of which the mean of those which centre at x (lying somewhere in h_x) is $\bar{y}_x$ and the array total n_x. Let $\bar{x}$, $\bar{y}$, σ_x, σ_y be the means and standard deviations of the total population N of the two variates. Assume the form of the regression line of y on x to be:

$$\frac{\bar{y}_x-\bar{y}}{\sigma_y}=a_0\psi_0+a_1\psi_1+\dots+a_n\psi_n \quad\dots\dots\dots\dots\text{(i)},$$

where a_0, $a_1 \dots a_n$ are absolute constants to be determined and ψ_s is an orthogonal function of $\frac{x-\bar{x}}{\sigma_x}$, i.e. is subject to the condition that:

$$S(n_x\psi_s\psi_{s'})=0, \quad s \text{ and } s' \text{ unequal} \quad\dots\dots\dots\dots\text{(ii)},$$

if the summation S be taken for all values of x corresponding to the arbitrary system of arrays.

Further let us suppose† that:

$$q_{1t}=\frac{S\{n_x(\bar{y}_x-\bar{y})(x-\bar{x})^t\}}{N\sigma_y\sigma_x^{\,t}}=\frac{\Sigma\{n_{xy}(y-\bar{y})(x-\bar{x})^t\}}{N\sigma_y\sigma_x^{\,t}}=p_{1t}/(\sigma_y\sigma_x^{\,t})$$

in the usual product-moment notation.

Clearly if the ψ_s's can be determined we must have by virtue of (ii) from (i)

$$S\left(\frac{n_x(\bar{y}_x-\bar{y})}{N\sigma_y}\psi_n\right)=a_n\frac{S(n_x\psi_n^{\,2})}{N} \quad\dots\dots\dots\dots\text{(iii)}.$$

Now if ψ_n be determined as an integral function of $(x-\bar{x})/\sigma_x$ of the nth degree, the left-hand side of (iii) is expressible in terms of the q_{1t}'s, or the product-moments of the correlation distribution. Thus a_n will be determined. Let us write:

$$\left.\begin{aligned}\kappa_n&=\frac{S\{n_x(\bar{y}_x-\bar{y})\psi_n\}}{N\sigma_y}\\ \text{and}\qquad \lambda_n&=\frac{S(n_x\psi_n^{\,2})}{N}\end{aligned}\right\}\quad\dots\dots\dots\dots\text{(iv)}.$$

Then

$$a_n=\kappa_n/\lambda_n \quad\dots\dots\dots\dots\text{(iii) bis}$$

is known, if the ψ_s's have once been determined, from the product-moments of the system of the correlated variates, and the moment coefficients of the x-variate.

Square (i), multiply by n_x/N and sum for all arrays, we have:

$$S\left(\frac{n_x(\bar{y}_x-\bar{y})^2}{N\sigma_y^{\,2}}\right)=\eta^2_{y,x}=a_0^{\,2}\lambda_0+a_1^{\,2}\lambda_1+\dots+a_n^{\,2}\lambda_n \quad\dots\dots\dots\dots\text{(v)}.$$

Here $\eta_{y,x}$ is the well-known correlation ratio of y-variate on x-variate, and must always lie between 0 and 1. The series on the right-hand side of (v) consists of a system of squares, and accordingly every additional term we take in our series for the regression line must carry us nearer this value of $\eta^2_{y,x}$. Unless λ_n tends to zero the a_n's must grow smaller and smaller, or we have considerable anticipation of the convergency of the series, but this does not amount of course to definite proof.

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[*] Reproduced from lecture notes.

[†] S denotes summation of all arrays, Σ denotes summation for each individual point.

Now suppose we had to determine our a_s's from (i) by the method of least squares, each array being weighted by its frequency. We should have to make

$$u^2 = S\left\{\frac{n_x}{N}\left(\frac{\bar{y}_x - \bar{y}}{\sigma_y} - a_0\psi_0 - a_1\psi_1 - \dots - a_n\psi_n\right)^2\right\}$$

a minimum.

But
$$\frac{du^2}{da_s} = 0 = 2S\left(\frac{n_x(\bar{y}_x - \bar{y})}{N\sigma_y}\psi_s\right) - 2a_s\frac{S(n_x\psi_s^2)}{N},$$

or
$$a_s = \kappa_s/\lambda_s,$$

in agreement with the result obtained in (iii) bis above as a result of the ψ_s's being orthogonal functions. Now the fit by least squares to the means of the arrays is precisely the same as the fit by least squares to the whole swarm of points. In other words if we fit our regression line as above indicated to the whole population it will be "the best" fitting curve, if we make the assumption that least squares does give the best fit*. Anyhow it is likely to be a good fit, and that suffices for our present purposes.

We will now write $X = (x - \bar{x})/\sigma_x$ and $Y_x = (\bar{y}_x - \bar{y})/\sigma_y$, whence it follows that $q_{1t} = S(n_x Y_x X^t)/N$. We shall further write:

$$\mu_s = S(n_x X^s)\,\sigma_x^s/N,$$

and
$$\beta_{2s} = \mu_{2s+2}/\sigma_x^{2s+2} = \frac{S\{n_x(x-\bar{x})^{2s+2}\}}{N\sigma_x^{2s+2}} = \frac{S(n_x X^{2s+2})}{N},$$

$$\beta_{2s-1} = \frac{\mu_{2s+1}\mu_3}{\sigma^{2s+4}}.$$

We may rewrite (i)
$$Y = a_0\psi_0 + a_1\psi_1 + \dots + a_n\psi_n \quad \text{.................................(i) bis.}$$

Multiply both sides by $n_x\psi_0/N$ and sum:

$$\frac{S(n_x Y\psi_0)}{N} = a_0\frac{S(n_x)}{N}\psi_0^2 = a_0,$$

if we assume $\psi_0 = 1$ as we are at liberty to do. But the left-hand side is zero. Therefore $a_0 = 0$.

Now assume $\psi_1 = X - c_{10}\psi_0$ and multiply both sides by $n_x\psi_0/N$ and sum:

$$\frac{S(n_x\psi_0\psi_1)}{N} = 0 \text{ by hypothesis} = \frac{S(n_x X\psi_0)}{N} - c_{10}\frac{S(n_x\psi_0^2)}{N}.$$

But
$$\frac{S(n_x X\psi_0)}{N} = 0, \quad \therefore\ c_{10} = 0, \text{ and } \psi_1 = X.$$

Multiply both sides by $n_x X/N$ and sum:

$$\frac{S(n_x YX)}{N} = r = a_1\frac{S(n_x X\psi_1)}{N} = a_1\frac{S(n_x X^2)}{N} = a_1.$$

Hence $a_1 = r$.

Now assume $\psi_2 = X^2 - c_{21}\psi_1 - c_{20}\psi_0$. Multiply by $n_x\psi_0/N$ and sum:

$$0 = \frac{S(n_x X^2)}{N}\psi_0 - 0 - c_{20} = 1 - c_{20}.$$

Thus $c_{20} = 1$.

Multiply by $n_x\psi_1/N$ and sum:

$$0 = \frac{S(n_x\psi_1 X^2)}{N} - c_{21}\frac{S(n_x\psi_1^2)}{N},$$

$$\frac{S(n_x X^3)}{N} = \sqrt{\beta_1} = c_{21}\frac{S(n_x X^2)}{N} = c_{21}.$$

Thus $c_{21} = \sqrt{\beta_1}$.

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* In order that this statement should be absolutely true our arrays would have to follow the normal or Gaussian distribution. This produces, however, an unnecessary limitation. It is known, but possibly not well known, that expansions in orthogonal functions as defined by (ii) give least square fits. I am rather inclined to think that this is an argument in favour of least square fits, rather than a justification of the expansion, i.e. that the method of least squares has a wider validity than Gauss' proof provides.

$\sqrt{\beta_1}$ must of course be given the same sign as μ_3. Accordingly we have:

$$\psi_2 = X^2 - \sqrt{\beta_1}\, X - 1.$$

Square, multiply by n_x/N and sum and we have:

$$\frac{S\,(n_x \psi_2{}^2)}{N} = \frac{S\,(n_x \psi_2 X^2)}{N} = \beta_2 - \beta_1 - 1.$$

It remains to find:

$$\frac{S\,(n_x Y \psi_2)}{N} = S\,\{n_x Y\,(X^2 - \sqrt{\beta_1}\, X - 1)/N\} = q_{12} - r\sqrt{\beta_1}.$$

Thus it follows from (iii) that $\quad a_2 = (q_{12} - r\sqrt{\beta_1})/(\beta_2 - \beta_1 - 1)$.

We now proceed to find ψ_3:

$$\psi_3 = X^3 - c_{32}\,\psi_2 - c_{31}\,\psi_1 - c_{30}\,\psi_0.$$

Multiply by $n_x \psi_0/N$ and sum:

$$0 = \frac{S\,(n_x X^3)}{N} - c_{30}, \quad \text{or} \quad c_{30} = \sqrt{\beta_1}.$$

Multiply by $n_x \psi_1/N$ and sum:

$$0 = \frac{S\,(n_x X^4)}{N} - c_{31}\,\frac{S\,(n_x X^2)}{N}, \quad \text{or} \quad c_{31} = \beta_2.$$

Multiply by $n_x \psi_2/N$ and sum:

$$0 = \frac{S\,\{n_x X^3\,(X^2 - \sqrt{\beta_1}\, X - 1)\}}{N} - c_{32}\,\frac{S\,(n_x \psi_2{}^2)}{N},$$

$$\frac{\mu_5}{\sigma_x{}^5} - \sqrt{\beta_1}\,\beta_2 - \sqrt{\beta_1} = c_{32}\,(\beta_2 - \beta_1 - 1),$$

$$\frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}} = c_{32}\,(\beta_2 - \beta_1 - 1),$$

$$c_{32} = \frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1)}.$$

Accordingly:

$$\psi_3 = X^3 - \frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1)}\,\psi_2 - \beta_2\,\psi_1 - \sqrt{\beta_1}\,\psi_0,$$

or

$$\psi_3 = X^3 - \frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1)}\,X^2 + \frac{\beta_3 - \beta_2{}^2 + \beta_2 - \beta_1}{\beta_2 - \beta_1 - 1}\,X + \frac{\beta_3 - 2\beta_1\beta_2 + \beta_1{}^2}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1}.$$

We have next to determine κ_3:

$$\begin{aligned}
\kappa_3 &= \frac{S\,(n_x Y \psi_3)}{N} \\
&= q_{13} - \frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1)}\,q_{12} + \frac{\beta_3 - \beta_2{}^2 + \beta_2 - \beta_1}{\beta_2 - \beta_1 - 1}\,r \\
&= q_{13} - \beta_2 r - \frac{\beta_3 - \beta_1\beta_2 - \beta_1}{\sqrt{\beta_1}\,(\beta_2 - \beta_1 - 1)}\,(q_{12} - r\sqrt{\beta_1}).
\end{aligned}$$

Let us write:

$$\epsilon_{12} = q_{12} - r\sqrt{\beta_1}, \quad \epsilon_{13} = q_{13} - \beta_2 r,$$

$$\phi_2 = \beta_2 - \beta_1 - 1, \quad \phi_3 = (\beta_3 - \beta_1\beta_2 - \beta_1)/\sqrt{\beta_1}.$$

Thus

$$\kappa_3 = \epsilon_{13} - \frac{\phi_3}{\phi_2}\,\epsilon_{12}.$$

We have next to find λ_3:

$$\begin{aligned}
\lambda_3 &= \frac{S\,(n_x \psi_3{}^2)}{N} = S\left\{\frac{n_x X^3}{N}\,\psi_3\right\} \\
&= \beta_4 - \frac{(\beta_3 - \beta_1\beta_2 - \beta_1)\,\beta_3}{\beta_1\,(\beta_2 - \beta_1 - 1)} + \frac{(\beta_3 - \beta_2{}^2 + \beta_2 - \beta_1)\,\beta_2}{\beta_2 - \beta_1 - 1} + \frac{\beta_3 - 2\beta_1\beta_2 + \beta_1{}^2}{\beta_2 - \beta_1 - 1} \\
&= \beta_4 - \beta_2{}^2 - \beta_1 - \frac{(\beta_3 - \beta_1\beta_2 - \beta_1)^2}{\beta_1\,(\beta_2 - \beta_1 - 1)} \\
&= \phi_4 - \phi_3{}^2/\phi_2 \quad \text{if } \phi_4 = \beta_4 - \beta_2{}^2 - \beta_1.
\end{aligned}$$

Accordingly:
$$a_3 = \frac{\kappa_3}{\lambda_3} = \frac{\epsilon_{13}\phi_2 - \epsilon_{12}\phi_3}{\phi_2\phi_4 - \phi_3^2}.$$

We now pass to the fourth order regression line function:

$$\psi_4 = X^4 - c_{43}\psi_3 - c_{42}\psi_2 - c_{41}\psi_1 - c_{40}\psi_0,$$

and we find:

$$\frac{S(n_x X^4 \psi_3)}{N} = c_{43}\frac{S(n_x \psi_3^2)}{N} = c_{43}\lambda_3$$

$$= \frac{\beta_5}{\sqrt{\beta_1}} - \frac{(\beta_3 - \beta_1\beta_2 - \beta_1)\beta_4}{\sqrt{\beta_1}(\beta_2 - \beta_1 - 1)} + \frac{(\beta_3 - \beta_2^2 + \beta_2 - \beta_1)\beta_3}{\sqrt{\beta_1}(\beta_2 - \beta_1 - 1)} + \frac{(\beta_3 - 2\beta_1\beta_2 + \beta_1^2)\beta_2}{\sqrt{\beta_1}(\beta_2 - \beta_1 - 1)}$$

$$= \{\beta_5(\beta_2 - \beta_1 - 1) - \beta_4(\beta_3 - \beta_1\beta_2 - \beta_1) + \beta_3(\beta_3 - \beta_2^2 + 2\beta_2 - \beta_1) + \beta_1\beta_2(\beta_1 - 2\beta_2)\} \div \sqrt{\beta_1}(\beta_2 - \beta_1 - 1).$$

Let
$$\phi_5 = \frac{\beta_5(\beta_2 - \beta_1 - 1) - \beta_4(\beta_3 - \beta_1\beta_2 - \beta_1) + \beta_3(\beta_3 - \beta_2^2 + 2\beta_2 - \beta_1) + \beta_1\beta_2(\beta_1 - 2\beta_2)}{\sqrt{\beta_1}}.$$

Thus:
$$c_{43} = \frac{\phi_5}{\phi_4\phi_2 - \phi_3^2}.$$

Again:
$$\frac{S(n_x X^4 \psi_2)}{N} = c_{42}\lambda_2,$$

$$\beta_4 - \beta_3 - \beta_2 = c_{42}(\beta_2 - \beta_2 - 1),$$

or
$$\phi_4 - \phi_3\sqrt{\beta_1} + \phi_2\beta_2 = \phi_4', \text{ say, } = c_{42}\phi_2.$$

Thus:
$$c_{42} = \frac{\phi_4'}{\phi_2}.$$

Further:
$$\frac{\beta_3}{\sqrt{\beta_1}} = c_{41},$$

and
$$\beta_2 = c_{40}.$$

Thus:
$$\psi_4 = X^4 - \frac{\phi_5}{\phi_4\phi_2 - \phi_3^2}\psi_3 - \frac{\phi_4'}{\phi_2}\psi_2 - \frac{\beta_3}{\sqrt{\beta_1}}\psi_1 - \beta_2\psi_0.$$

To find a_4 we must determine κ_4 and λ_4:

$$\lambda_4 = \frac{S(n_x X^4 \psi_4)}{N}$$

$$= \beta_6 - \frac{\phi_5^2}{(\phi_4\phi_2 - \phi_3^2)\phi_2} - \frac{\phi_4'}{\phi_2}(\beta_4 - \beta_3 - \beta_2) - \frac{\beta_3^2}{\beta_1} - \beta_2^2$$

$$= \phi_6 - \frac{\phi_5^2}{\phi_2(\phi_4\phi_2 - \phi_3^2)} - \frac{\phi_4'^2}{\phi_2},$$

where
$$\phi_6 = \beta_6 - \frac{\beta_3^2}{\beta_1} - \beta_2^2,$$

$$\kappa_4 = \frac{S(n_x Y \psi_4)}{N} = q_{14} - \frac{\phi_5}{\phi_4\phi_2 - \phi_3^2}\left(\epsilon_{13} - \frac{\phi_3}{\phi_2}\epsilon_{12}\right) - \frac{\phi_4'}{\phi_2}(q_{12} - r\sqrt{\beta_1}) - \frac{\beta_3}{\sqrt{\beta_1}}r.$$

Let us write $q_{14} - \frac{\beta_3}{\sqrt{\beta_1}}r = \epsilon_{14}$, then

$$a_4 = \frac{\epsilon_{14} - \dfrac{\phi_5}{\phi_4\phi_2 - \phi_3^2}\left(\epsilon_{13} - \dfrac{\phi_3}{\phi_2}\epsilon_{12}\right) - \dfrac{\phi_4'}{\phi_2}\epsilon_{12}}{\phi_6 - \dfrac{\phi_5^2}{\phi_2(\phi_4\phi_2 - \phi_3^2)} - \dfrac{\phi_4'^2}{\phi_2}}.$$

We have thus obtained the regression orthogonal functions up to the fourth order. Higher order terms can also be found, but their expressions become very complicated and such expressions involving fifth product-moments and eighth marginal total moments will be subject to very large probable errors.

We can now by aid of (v) express $\eta_{y,x}$. We have:

$$\eta^2_{y,x}=r^2+\epsilon_{12}^2/\phi_2+\left(\epsilon_{13}-\frac{\phi_3}{\phi_2}\epsilon_{12}\right)^2\Big/(\phi_4-\phi_3^2/\phi_2)$$
$$+\frac{\left\{\epsilon_{14}-\frac{\phi_4'}{\phi_2}\epsilon_{12}-\frac{\phi_5}{\phi_2^2}\left(\epsilon_{13}-\frac{\phi_3}{\phi_2}\epsilon_{12}\right)\Big/(\phi_4-\phi_3^2/\phi_2)\right\}^2}{\phi_6-\phi_4'^2/\phi_2-\frac{\phi_5^2}{\phi_2^2}\Big/(\phi_4-\phi_3^2/\phi_2)}+\text{etc.}\ldots\ldots$$

The conditions therefore for linear regression, or $\eta_{rx}=r$, are:

$$\epsilon_{12}=\epsilon_{13}=\epsilon_{14}=\text{etc.}=0\ldots\ldots$$

That is: $$q_{12}=r\sqrt{\beta_1},\quad q_{13}=r\beta_2,\quad q_{14}=r\frac{\beta_3}{\sqrt{\beta_1}},\ \text{etc.}\ldots\ldots$$

For parabolic regression: $$\epsilon_{13}=\frac{\phi_3}{\phi_2}\epsilon_{12},\quad \epsilon_{14}=\frac{\phi_4'}{\phi_2}\epsilon_{12},\ \text{etc.}\ldots\ldots,$$

or $$q_{13}-\frac{\phi_3}{\phi_2}q_{12}=r\left(\beta_2-\frac{\phi_3}{\phi_2}\sqrt{\beta_1}\right),$$

$$q_{14}-\frac{\phi_4'}{\phi_2}q_{12}=r\left(\frac{\beta_3}{\sqrt{\beta_1}}-\frac{\phi_4'}{\phi_2}\sqrt{\beta_1}\right),\ \text{etc.}\ldots\ldots$$

And lastly for cubical regression:

$$\epsilon_{14}-\frac{\phi_4'}{\phi_2}\epsilon_{12}-\frac{\phi_5}{\phi_2^2}\left(\epsilon_{13}-\frac{\phi_3}{\phi_2}\epsilon_{12}\right)\Big/\left(\phi_4-\frac{\phi_3^2}{\phi_2}\right)=0,\ \text{etc.}\ldots\ldots$$

Such conditions, especially with regard to their probable errors, become less and less manageable as we proceed.

The general principle involved in the present paper has been discussed by Tchebycheff*, and more adequately by J. P. Gram†, but the former had in view the fitting or graduating of curves. He calculated quantities which correspond to our μ_s's on the assumption that $n_x=1$, i.e. that the weight of the $\bar{y}_x$'s are all the same or that the marginal total is a *rectangle*. He was thinking of fitting a curve to a curve and not fitting a curve to a swarm of points. In his case each μ_s and accordingly each β and each ϕ is expressible in terms of the total number m of subranges which he takes of equal length. There are I think simpler methods of calculating the equation to a higher order parabola in such cases‡. As far as I am aware these orthogonal regression functions have not hitherto been dealt with and they throw a good deal of light on the original equations I provided in 1905 for skew regression. I had not recognised at that time that my expressions of each order were true orthogonal functions. It will be seen that my solution does not involve equality of subranges and is not limited to any special frequency distribution.

II. Note on the "Fundamental Problem of Practical Statistics."

(*Biometrika*, Vol. XIII, p. 1.)

Some misunderstanding has arisen with regard to my paper under the above title in the last issue of this *Journal*. I believe it to be due to the critics not having read Bayes' original theorem as given by Price in the *Phil. Trans.*, Vol. LIII. Bayes takes a ball and places it at random on a table, say of breadth unity, and its distance from one side being x, its chance of falling between x and $x+\delta x$ is δx. x is thus not a chance, but a *variate*. He now calls a "success," the chance that any other ball placed at random on the table will be nearer to the same side than the first

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* *Mémoires de l'Académie de Saint-Pétersbourg.* Memoirs in 1854 and 1859. A *résumé* by R. Radau: *Bulletin Astronomique*, T. VIII, Paris, 1891, pp. 350, 376 *et seq.* See also Liouville's *Journal*, 2ᵉ Série, T. III (1858), p. 289 *et seq.*

† *Thesis*: "Om Rækkeudviklinger bestemte ved Hjaelp af de mindste Kradvaters Methode." Kjøbenhavn, 1879.

‡ *Biometrika*, Vol. II, pp. 12—16.

placed ball and a "failure" that it should be greater. The chance therefore of p successes and q failures now happening is

$$x^p (1-x)^q \times dx \times \frac{\underline{|p+q}}{\underline{|p}\,\underline{|q}}.$$

It is solely the fact that all possible values of the variate x are made *a priori* equally likely that makes the chance of a success x, equal to the variate itself. Those who criticise Bayes after reading his actual paper, say that he ought not to have made the chances of a ball being placed anywhere on the table equally likely. He makes in fact his distribution of the variate x a straight line—a somewhat unusual form of frequency distribution*. My answer to that objection to Bayes' work was that you can make the distribution of that variate—i.e. position on the table—any continuous curve you please as Bayes' Theorem with Bayes' results will flow from it equally well. Against this position my critics raise the cry that the chance is no longer x of a success and $1-x$ of a failure. Of course not, because that depends on horizontality of frequency distribution and it was merely fortuitous that for that case Bayes' variate x corresponded to a chance. In other cases the chance is a function of the variate x and not x itself. But if the critics say: Then this is not what we mean by Bayes' Theorem, I would reply: Quite so, but it is what Bayes meant by his own Theorem, and it probably fits much better the type of cases to which we are accustomed to apply it than what you mean by Bayes' Theorem.

Let me illustrate this point. An event has happened p times and failed q times; what is the chance that in $r+s$ further trials it will occur r times and fail s? The critics say: this is *our* Bayes' Theorem, not your Bayes' Theorem. I reply that it is both, but that your way of solving it is not Bayes'. Perhaps Bayes saw further than some of the critics who have not troubled to read his original paper. What Bayes said was this, the event will happen when there is an excess (or it may be defect) of a certain variate, but I do not know what is the limiting value of this variate *a priori*. Look at Bayes' billiard table from a more modern standpoint. Men will sicken from a disease when their resistance falls below x an *a priori* unknown value of the variate. Bayes took the chance of this limiting value lying between x and $x+\delta x$ to be dx, if the total range be taken as unity. He ought to have taken it $\phi(x)\,dx$, where $\phi(x)$ is arbitrary. He took the chance of occurrence of the disease to be x, when he ought to have taken it $\int_0^x \phi(x)\,dx = P_x$; and of failure $1-x$ instead of $1-P_x$.

But had he taken these better values he would have reached finally precisely the same result as he did by his equal distribution of ignorance. Bayes made every value of his variate x equally likely. He ought to have given them a perfectly arbitrary frequency distribution. *A priori* all degrees of immunity are not equally likely to be the limiting value in the case of a disease. The generalised Bayes as thus envisaged has a very wide application to vital statistics; in fact it seems to me to entirely replace the other sort of Bayes' Theorem suggested by his critics.

Nay I would go further, and say that it is Bayes' Theorem in Bayes' sense that we need in most questions of prediction of the future from the past. If, for example, two men play a set of games and A has won p and B q games and we consider the chance in the following $r+s$ games of A's winning r and B's winning s, then I believe that A's winning may be accurately considered as depending on the excess of a certain variate x which is a function of A's skill relative to B's. *A priori* we do not know what the value of x is for which A will win. All we can say is that when relative to B he exhibits a certain excess of skill he will win, but that we must not assume with Bayes that all limit-values of x are equally likely. We must take any frequency curve for the possible distribution of x.

I believe that in most cases such a variate may be hypothecated and if it can the objection to Bayes that he made all positions of his balls on the table "equally likely" can be removed, and if removed one fundamental objection to his theorem as he stated it, i.e. in terms of excess or defect of a variate, disappears.

K. P.

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* It is in fact the "rectangle point" $\beta_1=0$, $\beta_2=1{\cdot}8$, just as limited a distribution as the Gaussian $\beta_1=0$, $\beta_2=3$.

III. Second Note on the Coefficient of Correlation as determined from the Quantitative Measurement of one Variate and the Ranking of a Second Variate (see *Biometrika*, Vol. X, p. 416).

By KARL PEARSON, F.R.S.

In a previous note I pointed out that if $\rho_{g_x y}$ be the correlation of the grade of a variate x with the quantitative value of a second variate y, then the *variate* correlation between x and y would be

$$r_{xy} = \sqrt{\frac{\pi}{3}}\,\rho_{g_x y} = 1{\cdot}023{,}278\,\rho_{g_x y}\text{*}.$$

If we use ranks instead of grades, i.e. σ_{ν_1} instead of σ_{g_1}, we have

$$r_{xy} = \sqrt{\frac{\pi}{3}}\,\frac{\sqrt{N^2-1}}{N}\,\rho_{\nu_x y},$$

which gives a slight further correcting factor depending on N, but as a rule of very little importance.

It follows from the above that on the assumption of a normal distribution the correlation of variate and grade for the same character will be ·9773. Accordingly when we are testing correlations of rank and variate we are actually sampling a population of which the correlation of the two characters is ·9773. But we know from a paper in *Biometrika*, Vol. XI, p. 401, that even for $\rho = {\cdot}9$ in the sampled population the curve of distribution of r in samples is markedly skew and accordingly deviations in defect can occur, which are not possible in excess.

If we take samples of 25 the mean value of r found from (xxv), p. 336 of the above mentioned memoir is

$$\bar{r} = {\cdot}9763,$$

and for samples of 35:

$$\bar{r} = {\cdot}9766.$$

We should thus anticipate that the mean of small samples of this order would come out slightly less than ·9773.

Accordingly with the help of my colleagues Miss Elderton, Miss Karn and Miss Moul, the correlations between rank and marks were worked out for 14 Civil Service Examinations and for 16 examinations in a Technical School most kindly provided by my friend Dr Ritchie Scott. These examinations furnished the correlation coefficients given in the following table.

It will be seen that the English results of the Technical School are somewhat erratic. This is an experience similar to that which I had some years ago, when drawing up a report on the Matriculation Examination of the University of London, the language marking in that case being peculiarly subject to personal equation of an erratic kind.

Taking the 14 Civil Service Examinations we find for the weighted mean,

$$\text{Correlation} = -{\cdot}9666 \pm {\cdot}0086.$$

The result to be anticipated, using $(\text{xx})^{bis}$ to obtain the probable error of the mean is

$$\text{Correlation} = -{\cdot}9760 \pm {\cdot}0093.$$

The difference is therefore slightly less than once the probable error of the last result and on the basis of this alone could not be considered significant.

If we take the eight Second Year Examinations of the Technical Schools *including English* the result is

$$\text{Correlation} = -{\cdot}9601 \pm {\cdot}0086,$$

while the theoretical result is $\text{Correlation} = -{\cdot}9766 \pm {\cdot}0123.$

The difference is about 1·3 times the probable error and is not in itself significant.

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[*] Actually the sign may be positive or negative according as the variate in one case rises or falls with the rank of the other variate.

Correlation of Rank and Examination Marks.

Examination	Numbers	Correlation and P.E.
Civil Service Junior Examinations:		
Liverpool Messengers, 1901	27	− ·9739 ± ·0067
Glasgow Messengers, 1901	24	− ·9831 ± ·0046
Manchester Messengers, 1901	21	− ·9323 ± ·0196
Liverpool Postmen, 1901	19	− ·9861 ± ·0043
Glasgow Postmen, 1901	21	− ·9359 ± ·0183
Manchester Postmen, 1901	18	− ·9647 ± ·0110
Female Learners, Post Office, Nottingham, 1900	26	− ·9416 ± ·0150
„ „ „ Leeds, 1900 ...	46	− ·9694 ± ·0060
„ „ „ Bristol, 1900 ...	33	− ·9795 ± ·0048
„ „ „ Manchester, 1899	21	− ·9784 ± ·0064
„ „ „ Leeds, 1899 ...	20	− ·9736 ± ·0079
Female Skilled Telegraphists, December, 1899 ...	18	− ·9684 ± ·0099
„ „ „ May, 1899 ...	17	− ·9238 ± ·0240
Female Learners, Post Office, Birmingham, 1899	29	− ·9951 ± ·0012
Technical School:		
English, Second Year	37	− ·8690 ± ·0271
Mechanics „	35	− ·9286 ± ·0157
Metalwork „	38	− ·9676 ± ·0070
Gymnastics „	36	− ·9728 ± ·0060
Machine Drawing, Second Year	36	− ·9831 ± ·0038
Art, Second Year	36	− ·9845 ± ·0035
Science „	33	− ·9876 ± ·0030
Mathematics, Second Year...	38	− ·9896 ± ·0023
English, First Year	33	− ·9957 ± ·0010
Mechanics „	31	− ·9725 ± ·0066
Metalwork „	33	− ·9877 ± ·0029
Gymnastics „	31	− ·9694 ± ·0073
Machine Drawing, First Year	33	− ·9508 ± ·0113
Art, First Year	32	− ·9838 ± ·0038
Science „	30	− ·9905 ± ·0023
Mathematics, First Year	33	− ·9742 ± ·0060

Now taking the eight First Year examinations we have

Correlation = − ·9780 ± ·0032,

while the theoretical result is − ·9766 ± ·0123.

The difference is well under the probable error.

If we omitted the erratic English markings we obtain for the two cases − ·9735 and − ·9754 both fairly good representations of − ·9766.

If we include the English results and pool both series we obtain

Correlation = − ·9685 ± ·0070

against the theoretical result Correlation = − ·9766 ± ·0087,

the difference being just below the probable error of the latter result.

The impression formed upon my mind by these results is that the result of correlating rank and marks of the candidates in an examination will give a result slightly less than that to be anticipated by the theory of a normal distribution; but so little less that an effective control for detecting erratic markings such as that of the English in the technical school returns (or possibly the Female Skilled Telegraphists, May 1899 of the Civil Service Examinations) may be obtained by this correlation.

As a second illustration I will take the following data from a Civil Service Examination for Messengers. There were 10 posts to be filled and 27 candidates. The subjects were Arithmetic, Orthography, Handwriting, Geography and English Composition. The total marks were 300 in the first subject and 200 in each of the others—grand total 1100. I give the results below as far as I need them.

Rank in Total	Total Marks	Rank in Arithmetic	Marks in Arithmetic
1	907	1	230
2	764	9	158
3	748	2	228
4	746	10	154
5	724	8	162
6	718	5	182
7	710	14	129*
8	703	7	164
9	677	3	187
10	665	4	186
11	645	11	151
12	643	6	167
13	634	20	103
14	628	12	146
15	580	13	131
16	561	15	128
17	560	18	116
18	532	22	82
19	529	16	125
20	526	17	122
21	515	19	114
22	484	21	93
23	463	25	61
24	444	26	38
25	386	27	37
26	369	23	63
27	288	24	62
a	*b*	*c*	*d*

I first correlated ranks a with ranks c. This leads to

$$\rho_{ac} = \cdot 8834$$

by the formula (xx) of my memoir "On Further Methods of Determining Correlation" (*Drapers' Company Research Memoirs*, IV, p. 13, Cambridge University Press).

Using formula (xviii) of the same memoir, we obtain for the correlation of variates, i.e. general capacity and arithmetical capacity:

$$r'_{bd} = \cdot 893.$$

I worked out directly by the product-moment method the same correlation and found

$$r_{bd} = \cdot 896 \pm \cdot 026.$$

The agreement between r_{bd} and r'_{bd} in this case is excellent.

I next investigated the correlation of total marks and ranks in arithmetic and found

$$\rho_{bc} = - \cdot 8712\dagger,$$

whence we deduce by the fundamental equation of the present paper,

$$r''_{bd} = \cdot 892.$$

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* I have altered this figure by one mark to avoid the trouble of dealing with a single bracket, the only one involved.

† See footnote on p. 302 as to change of sign between ρ_{bc} or ρ_{ad} and r''_{bd} or r'''_{bd}.

The next step was to find the correlation of rank in the total examination with marks in arithmetic. There resulted

$$\rho_{ad} = -\cdot 8805^*,$$

whence we deduce $r'''_{bd} = \cdot 901.$

All four correlations r_{bd}, r'_{bd}, r''_{bd} and r'''_{bd} differ by less than ·01, or less than half the probable error of r_{bd}. The differences therefore have no significance, or the values are interchangeable.

Equally clearly r''_{bd} and r'''_{bd} are as stated above substantially equal which involves the equality of ρ_{bc} and ρ_{ad} to the same degree as stated above.

Lastly I correlated total marks with rank in total examination and found

$$\rho_{ab} = -\cdot 9739.$$

Multiplying this by 1·023,278 we have $r_{bb} = \cdot 997,$

which is as near the desired result of perfect correlation as we can expect to get on the assumption of a normal distribution of the variate.

As far therefore as this isolated example is concerned the method appears to be satisfactory.

The results reached in both our illustrations suggest that the correlation of rank and variate for the same character may be of considerable value for control purposes in dealing with examination results.

IV. A Study of Women Delinquents in New York State by Mabel R. Fernald, Mary H. S. Hayes and Almena Dawley with a Statistical Chapter by Beardsley Ruml and a preface by Katherine Bement Davis. Published by The Century Co., New York City, 1920.

By ETHEL M. ELDERTON.

At the present time there seems to be an increasing demand for statistics in connection with social problems and a desire to collect data sometimes without a very clear idea as to how the material when collected can be used but with the pious hope that it will be useful in some way. The interesting memoir before us if it had done nothing else would be extremely valuable as showing what is required before any social problem can be adequately studied from the statistical aspect, how investigators are hampered on all sides by a lack of comparative data and how after years of work and a most careful compilation of facts the main problem may be still unsolved. The authors of this book on women delinquents in New York State fully realize the necessity of such a study and the difficulties in the way. They start by pointing out that though there is abundant literature on the subject of the criminal population much of it is useless because it rests "at its best, upon the most casual and superficial observations, and at its worst upon what the writer thinks he would find on observation." The writers point out that when the object of imprisonment was simply the two-fold one of punishment for the crime and protection of the state from further crimes an exact study of the causes of criminal acts was of less importance than it is to-day when the object of the imprisonment of criminals is of a more humanitarian nature, namely not only to protect the community but to attempt to readjust the criminal to society. As far as we know this is the earliest extensive study of women delinquents using modern statistical methods. The first efforts of the investigators were directed towards obtaining information as to the distinguishing characteristics of women convicted in New York State, of their mental capacity and of the main facts of their personal and environmental histories. Physical and medical facts had to be disregarded almost entirely since adequate medical data for most of the groups of women studied could not be obtained. The first comparison which the authors wished to make was between the women delinquents and women in general: Are the delinquent women a special selection out of the female population or are they a random sample of the total differentiated only by their criminal career? Do they differ significantly in their mentality, in their environment, in their education? Here the difficulty is, that as in so much literature dealing with medical and social questions only the special group is studied and not the general population, and though the United States census gives some information it is too general to be of much use, and the authors have to fall back on data collected for special groups of women other than criminals who may not be representative of women as a whole. A second comparison which might be made is with men criminals, but the study of men is almost entirely confined to men convicted of relatively serious offences and does not deal with those committed to workhouses, county penitentiaries and reformatories. Dr Goring's study of English criminals would cover practically none of this group; the criminals he considered were felons while the

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* See footnote on p. 302 as to change of sign between ρ_{bc} or ρ_{ad} and r''_{bd} or r'''_{bd}.

bulk of these women delinquents are misdemeanants. A third comparison can be made within the delinquent group dividing them according to class of offence, nationality, mental capacity, etc.

The women considered are those who have a legal age of 16 years and over and were convicted in the courts of New York State. The investigators would have preferred to secure the women at the time of conviction as they came through the various courts but this was not possible and the plan followed, though necessary, is open to objections. To avoid selection consecutive commitments to the different institutions were taken whenever possible but even this could not always be carried out, as for example in the workhouse. The groups studied were (*a*) New York State Reformatory for Women at Bedford Hills; this includes women convicted of felonies and misdemeanors. Women convicted of murder in the first or second degree or who had previously committed a felony are not committed to Bedford Hills and very few women over 30 are found there. There were 102 examined for the main part of the enquiry. (*b*) Auburn, for felons only; 88 were examined and the enquiry was carried on for over two years. (*c*) The New York Magdalen Home under private management; includes women between the ages of 16 and 35; felons are not generally admitted. (*d*) The New York County Penitentiary; there is no organized training provided and one would expect unpromising cases, indeterminate sentences and older women to be sent here. (*e*) The New York City Workhouse which has only misdemeanants. The types of women who are committed here are (1) general misdemeanants who seem unpromising or have failed before, (2) new and relatively hopeful cases who have a sentence of only a few days, (3) intoxication cases. Only the first group could be studied since the second group were not in the workhouse long enough and the group of inebriates seemed to the investigators so senile and untruthful and their dwelling places were so shifting that information could not be obtained*. (*f*) Probation Cases from the Women's Night Court of Manhattan and the Bronx; these women were all misdemeanants and generally first offenders. It is quite true that all offenders are represented except inebriates, but the author's material is not a true sample of all women delinquents even when the results are pooled and this seems to me to invalidate much of the work on age for instance. The period over which the women were examined varies considerably in the different institutions, as for example in Auburn, where the enquiry was carried out for over two years and we shall therefore get a larger proportion of the type of women committed to Auburn than we should have in the general delinquent population; in this case we shall have more felons and probably more older felons among these delinquent women than really exist. We are inclined to think that consecutive cases should have been examined over the same length of time in each institution or that the institutions should never have been pooled. Generally each institution is kept separate but not always.

Correlation coefficients and ratios have been largely employed and in all cases an "index of reservation is given"; this index is the probable error of more general use without the constant ·67449. We are not sure that an η from three groups is very satisfactory but the only alternative is to use a bi-serial "r" but we believe that this method is never used throughout the book.

The mental condition of these women has been studied very fully. Four tests were used, Binet-Simon Scale 1911, Yerkes-Bridges Point Scale, Stanford-Binet Scale and Woolley Series, and finally educational tests were given to the Bedford women. Great care was taken in scoring and all the answers were taken down verbatim so that a second examiner could check the marks given. Also the women were judged by Professor Pearson's scale and generally speaking those who judged by this scale did not know the result of the test examination. The final tests selected were those which gave the highest correlation between intelligence as judged by estimation and the result of tests applied to the Bedford group. The social investigation was carried out by six field workers. Information from the offender was followed by visits to the home, to other relatives, to the employers and to social agencies who had been in touch with the offender. This investigation and that of the mental capacity of these women must have entailed much work and seems to us an excellent model for future investigations which one would like to see carried out on an unselected female population. The relationship between age and number of previous convictions is discussed and compared with Dr Goring's results for English convicts but one doubts whether such comparisons are very useful. Dr Goring was dealing only with felons and he omitted first offenders while nearly half these women delinquents are first offenders and more than half are not felons, and as Dr Goring pointed out in England men and women differ considerably in the annual number of convictions for crime and in the type of crime committed. The larger number of previous convictions among the users of alcohol is probably significant but we think a correction for age should be applied, since use of alcohol is correlated with age in the data as here presented and there is a significant though small correlation between the number of convictions and the age of the women. One would also like to know whether, if

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* To eliminate the inebriates, however, is to eliminate a considerable element of mental defectives, for to judge by English experience it is largely the mentally defective who take to alcohol.

convictions for intoxication were omitted the user of alcohol is more often guilty of misdemeanors and felonies than the non-user. It is interesting to note that foreign white women have a smaller percentage in each institutional group than they have in the population as a whole in view of the prevailing opinion that foreigners in New York are largely responsible for crime in that city. The foreign-born are however more prone to commit serious offences than the native-born. We do not consider that much can be gained by discussing the numbers age, etc. of native coloured delinquents in the institutions since some of them, such as the Magdalen, take no negro women, and as we said before, a combination of institutions does not seem to be justifiable.

A very careful record was made of the home conditions, combining the economic condition, moral standard and adequacy of parental supervision, and it was found that there was a significant correlation between the condition of the home and the age at first conviction (·31), the women from the poorest homes being first convicted at an earlier age, but there seems to be no correlation between the number of previous convictions and the home conditions though here also we should like to see a correction for age. It is very unfortunate, as the authors point out, that there is no control series, so that no comparison with the normal population can be made. Neither, for the same reason, is the hereditary aspect considered, and we feel that we must protest against the form in which the data available are given. On p. 241 it is stated that in 15·9 per cent. of families, which means here parents and siblings, some member had been convicted of crime, but there is no statement as to the number of siblings, and one criminal in a family of eleven for instance is a different matter from one criminal in a family of four. This information about the number of the family is available and we should have thought that by taking estimates of criminality in New York State the hereditary side could have been dealt with. A study of the educational opportunities of these women is practically limited to a study of their age at leaving school and the years they were at school, and here again only a comparison within the group can be carried out. There seems to be no clear relationship between age at leaving school and age at first conviction. As far as educational attainments are concerned, school attainment does not seem to have kept pace with length of time at school but we do not know how far the two are connected in the non-delinquent population. The older women were found to have reached a lower standard and to have been fewer years at school than the younger women, but probably this would be the same in the general population. There is no correlation between number of convictions and the grade reached even when age is made constant.

As far as the occupation of these women is concerned there seem to be more domestic workers among the delinquent women than in the general population not necessarily as the investigators state that domestic work causes delinquency but it may be that those who go into domestic service represent the unskilled untrained women from whom the delinquents are drawn. The domestic workers appear to have reached a lower grade than the women employed in other occupations. There was no evidence of a connection between earning capacity and number of convictions. The women seem to be earning a low and irregular wage but they belong largely to unskilled workers and judging by the low grade reached at school they were probably for the most part incapable of better work. The authors do not think that the delinquency is the result of the low and irregular wage; an elaborate investigation would be needed to enable us to discuss this point; judging by the accounts of the individuals one wonders that some of the women were ever employed at all and in the worst cases work and character were so unsatisfactory that dismissal occurred soon after employment.

In connection with intelligence there is a good deal of comparison with Dr Goring's work but we are doubtful as to how far such a comparison is useful considering the differences in sex, type of offence, etc. The women delinquents' intelligence is judged by the test aggregate and no figures are given showing the percentages found in the grades which correspond to those used by Dr Goring. As judged by the test aggregate, delinquent women are 1·6 to 2·0 years behind the army group. If we take Table 165 and consider army men and delinquent women as samples of the population and work out the correlation coefficient we find it is ·38 ± ·02 (where ·02 is the probable error not the index of reservation) between delinquency and a low intelligence. The actual figures are not given so that I have had to restore the table to its original form by multiplying the columns by 447 and 653 respectively, never a very satisfactory business.

In a work of this kind it would be an enormous advantage if the summarized data for each individual were given as Dr Goring gave it in Part II of his work for the criminals he studied; this would enable those interested to work out other correlations and test fresh points for themselves. More than half of these delinquent women are guilty of sex offences and since the correlation between mental defect and sexual crime was found by Dr Goring to be ·46 we do not feel that there is any great disagreement in these conclusions though we know that we are not comparing data that are strictly comparable. There are certainly some differences between English male criminals and these delinquent women but these may be due to a different definition of degree of criminality. Dr Goring uses the number of convictions per year of freedom and number of months imprisoned per year of freedom but since in New York the indeterminate sentence is given for hopeful cases instead of a short sentence "number of convictions" was used among these delinquent women as a measure of criminality. Dr Goring found frequency of conviction associated with the relatively weak-minded but shorter periods of imprisonment, while though delinquent women tended in the same direction the values were barely significant. Property offenders seem to be more intelligent than offenders against chastity which, if property offenders includes burglars and thieves, agrees with Dr Goring's conclusions. In discussing wage earned and mental capacity those who "live in," chiefly domestic servants, have had to be separated from the other groups of employment. Among domestic servants the authors find no correlation between mental capacity and wage but they do find a correlation between wage and mental capacity among workers other than domestic servants. We can hardly assume however that domestic service differs from other occupations in that mental capacity does not vary with wage in this occupation while it does in others, since no comparison within one group of occupations has been undertaken and except in factory workers the numbers would not be large enough for this to be worth doing. There is certainly a correlation between mental capacity and type of occupation and between wage and type of occupation and between mental capacity and wage when different occupations are considered, but the wage within any one occupation need not necessarily vary with intelligence any more than it does with domestic service though we may expect it to do so.

We have briefly indicated the lines on which this enquiry has proceeded and throughout the reading we have been much impressed by the caution with which opinions have been expressed and it is therefore the more astonishing to find in the last paragraph that these guarded expressions of opinion have gone to the winds. We find the sentence "We disagree (i.e. with Dr Goring) however in the preëminence attached to such a constitutional factor as defective intelligence in contrast with economic factors." We fail to find this disagreement in the text of the book. Dr Goring finds that 10 to 20 per cent. of the criminals were feeble minded, the authors of this book find 17 per cent. of the delinquent women mentally poorer than the 296 working girls who "should make a fairly good average for the whole community." If these working girls are a sample of the whole community probably one or two would be feeble minded and all those below the worst of these girls in intelligence would certainly be feeble minded and the agreement with Dr Goring's estimate is very striking. On p. 244 the authors write "How much of a factor the element of poor home conditions is, as causative of delinquency, it is difficult to judge, since we have no similar estimates of the general population." This sentence and the number of feeble minded among the delinquent women do not seem to agree with the sentence from the last paragraph of the book quoted above. It is true that within the group crime begins at an earlier age in the poorer homes but there is no correlation between poor homes and the number of convictions. In this connection we might mention that in 1902 when Dr Goring started collecting his data the work of Binet and Simon had not been published and also that in estimating the amount of feeble mindedness in the general population Dr Goring was guided not only by the report of the Royal Commission, as stated in this work, but also by a knowledge of the number of feeble minded children in the schools of London and Liverpool. Personally we prefer to draw our conclusions from a study of the book as a whole rather than from the summary and that study seems to us to emphasize the necessity of an investigation into the economic conditions, home environment and mental capacity of a sample of the non-criminal women of New York State without which it seems to us no comparison as to the relative importance of constitutional and environmental factors as determinants of crime can be adequately discussed. It is more difficult among these women to obtain an estimate of the amount of criminality owing to the existence of the indeterminate sentence than it was among the criminals in England and many apparently contradictory facts emerge when the two estimates of criminality, age at first conviction and number of convictions, are taken. In every country we need a very full control series with which we can compare our special groups and we wish that such an investigation could be carried out on the lines followed in this interesting study of women delinquents.

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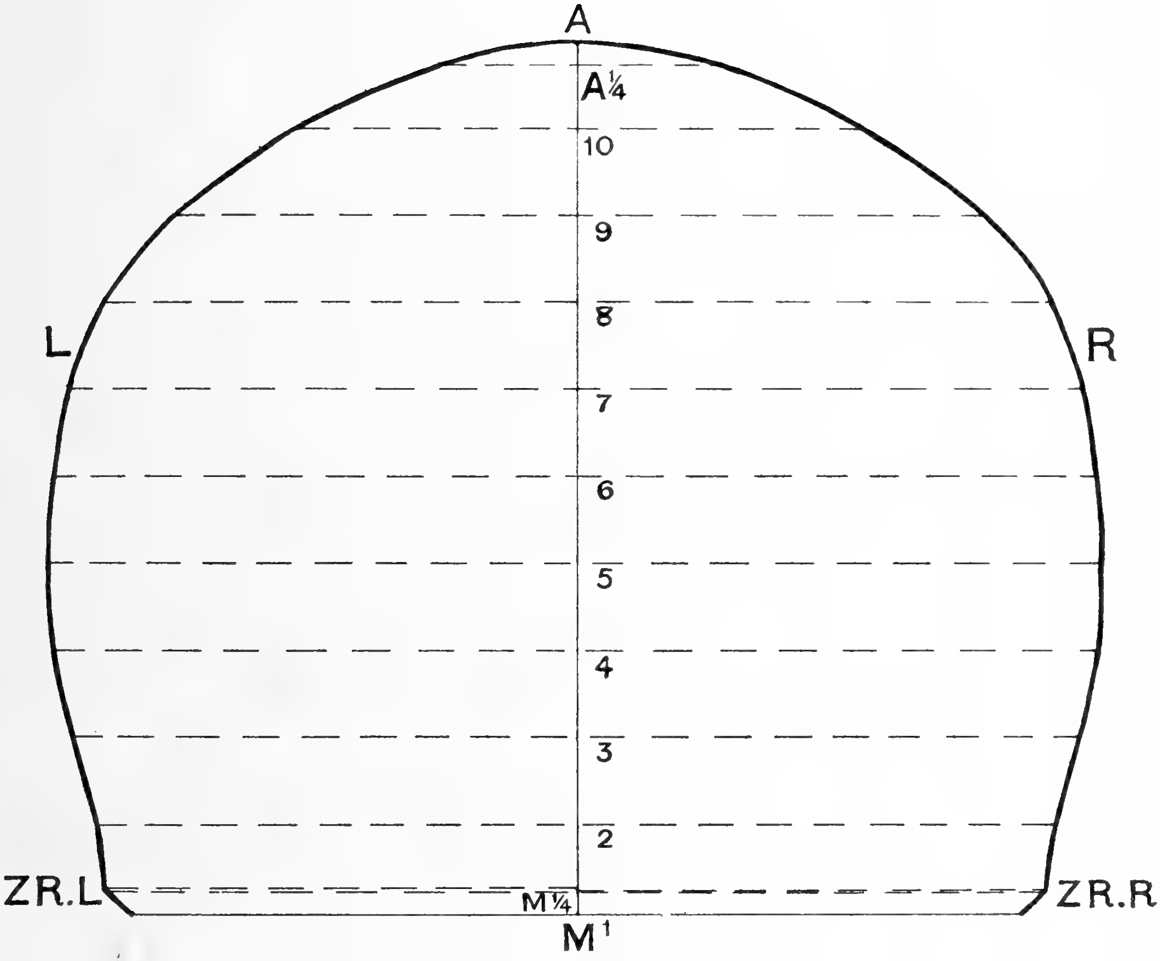


FIG. I. Burmese Crania. Type A (Burmans). ♂. Vertical Contour. (44 crania.)

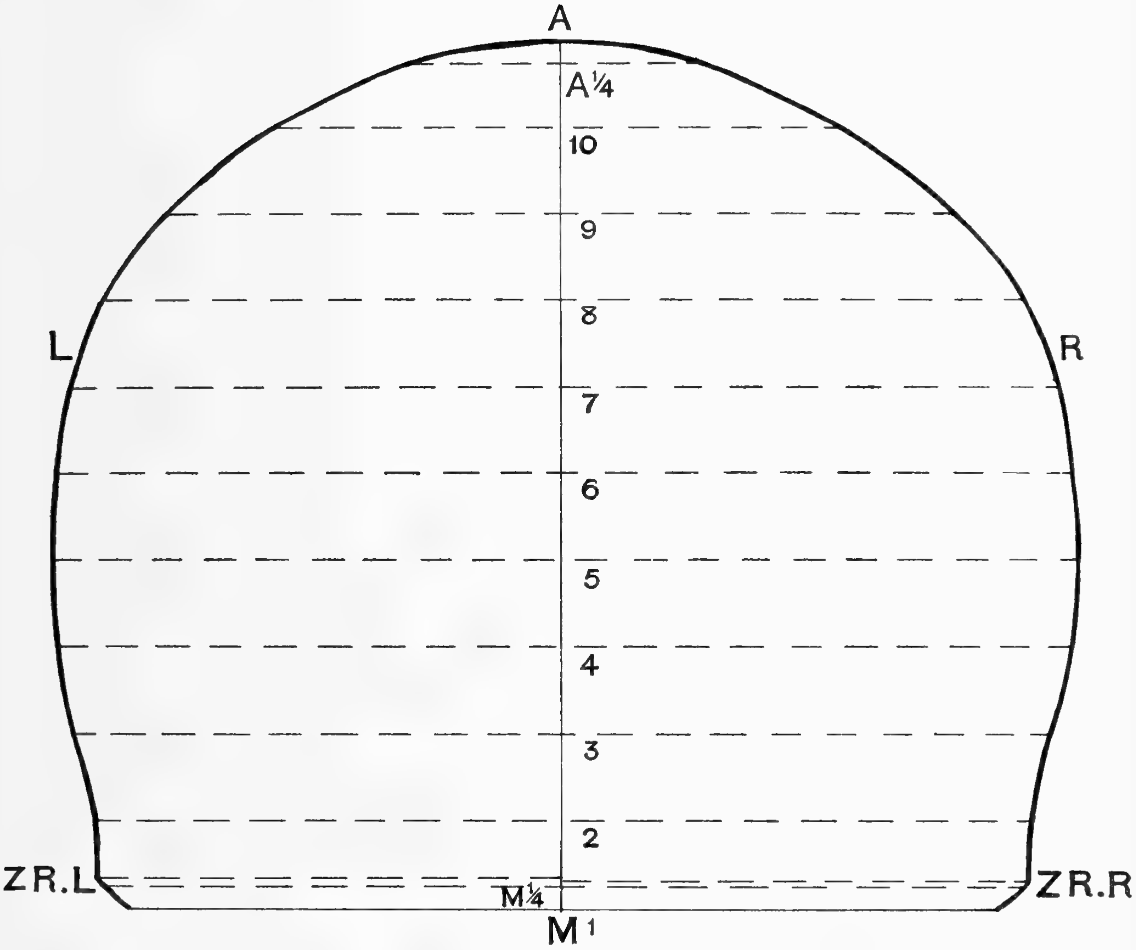
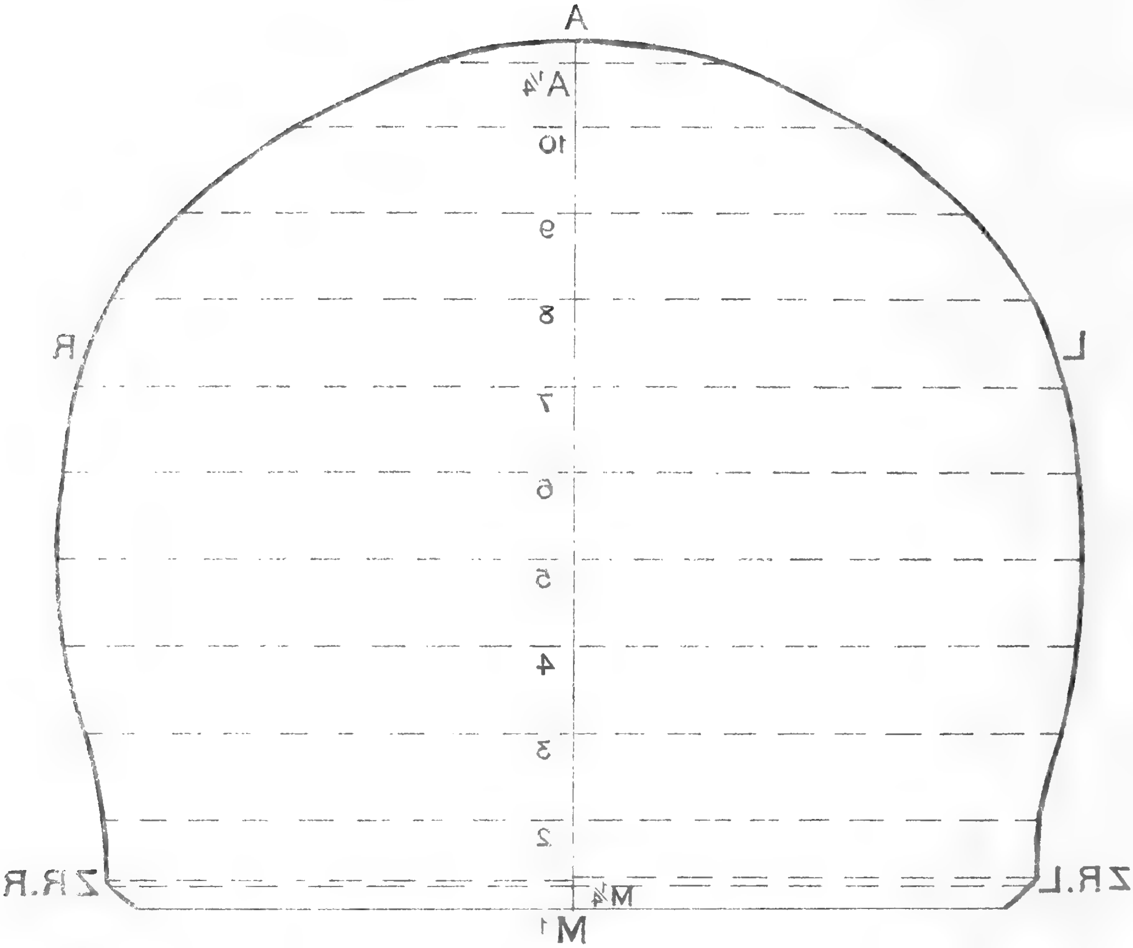


FIG. II. Burmese Crania. Type B (? Hybrids). ♂. Vertical Contour. (7 crania.)



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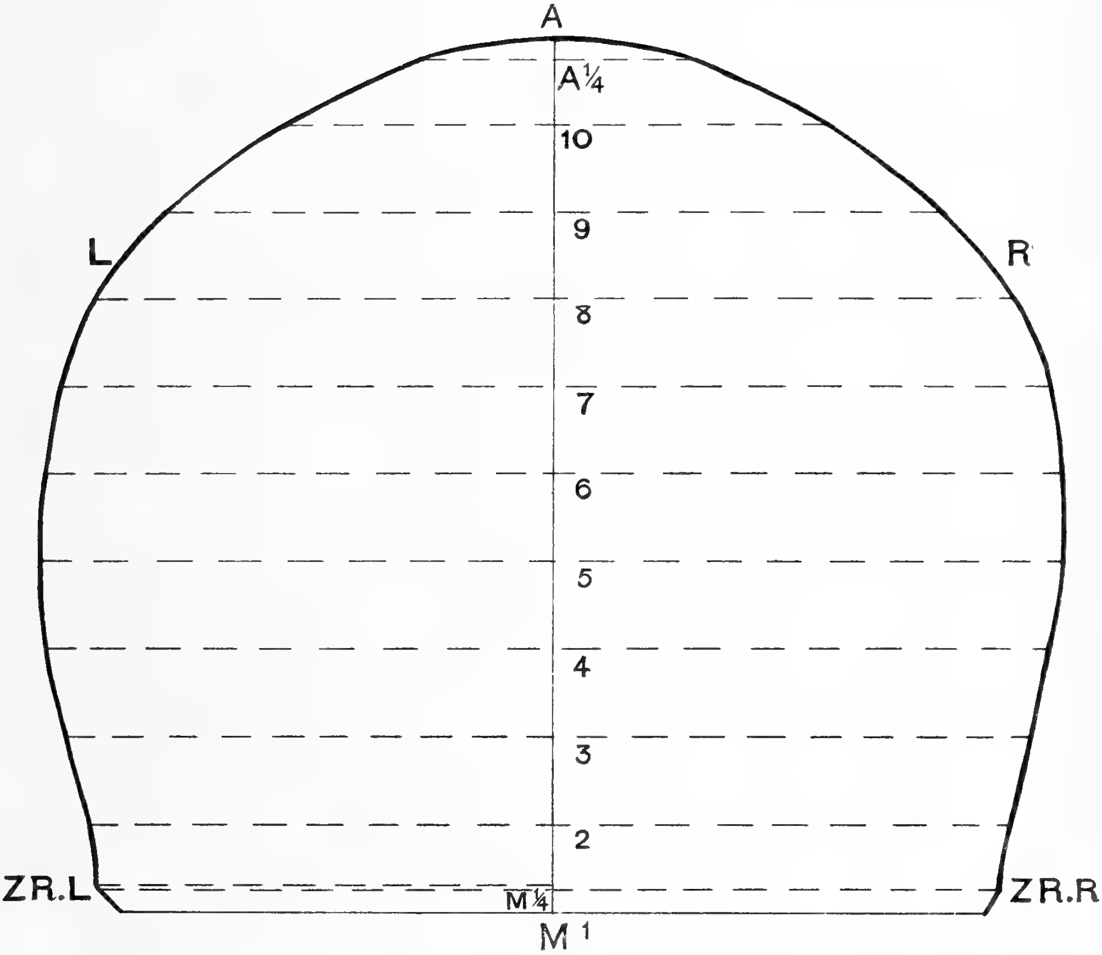
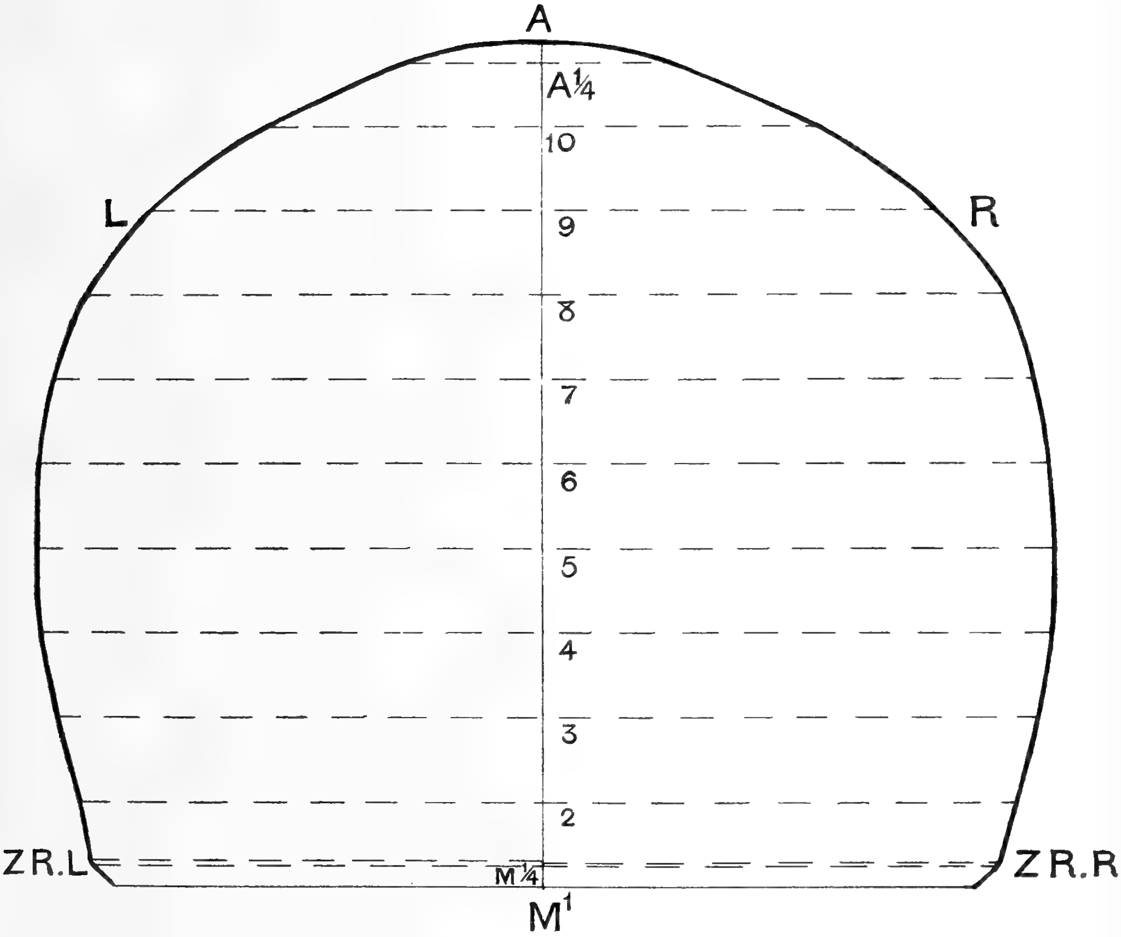


Fig. III. Burmese Crania. Type C (? Karens). ♂. Vertical Contour. (8 crania.)



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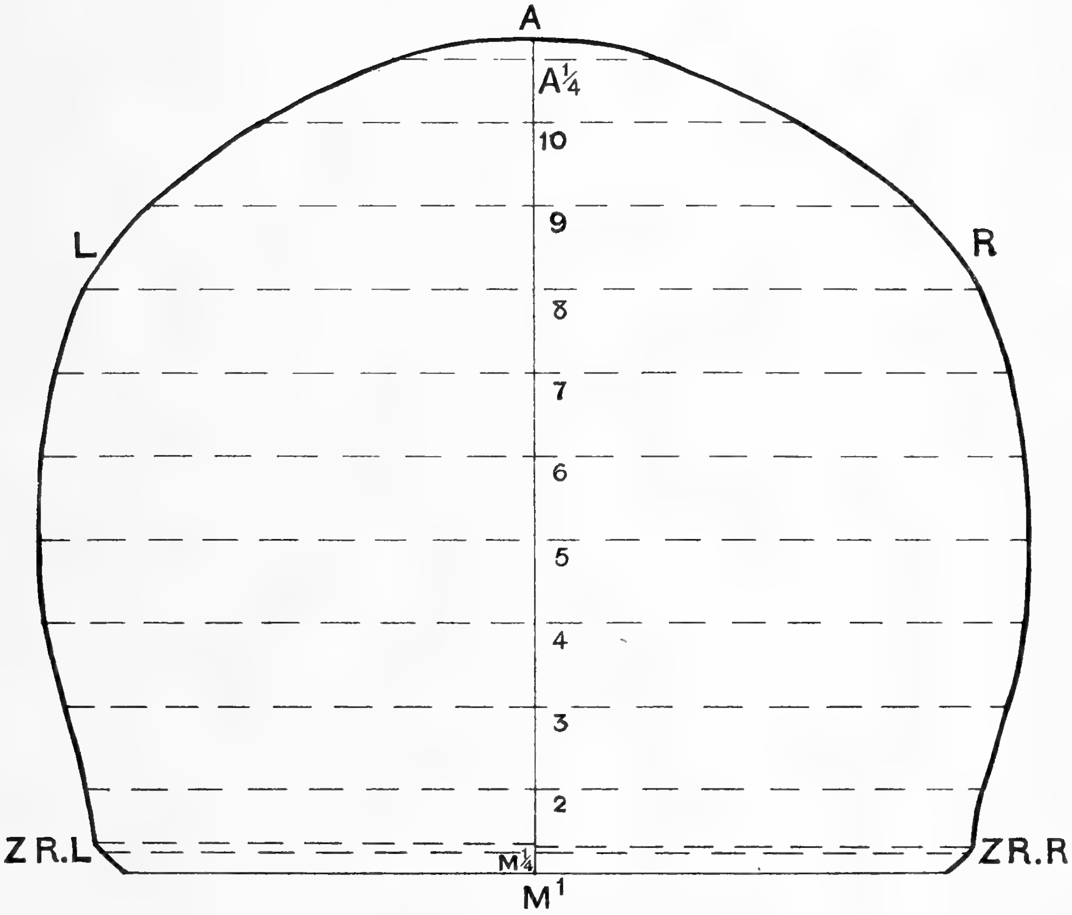
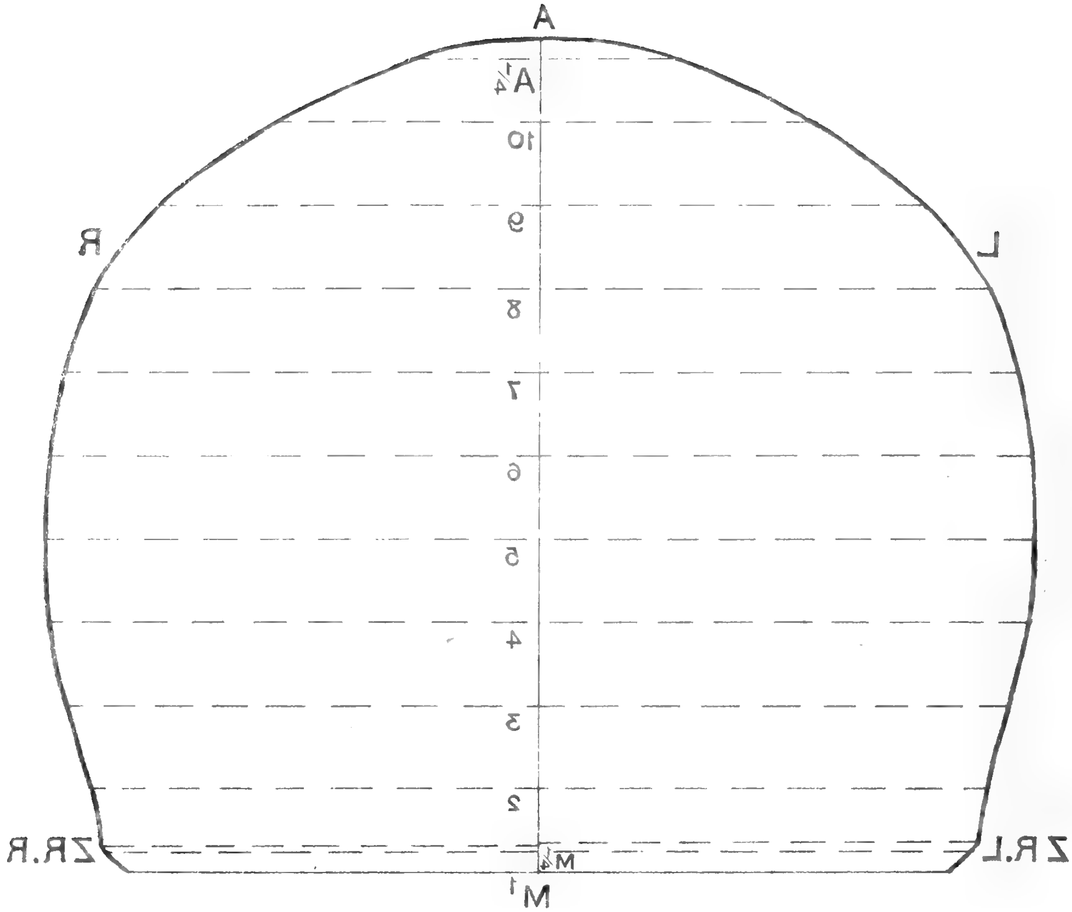


Fig. V. Burmese Crania. Type B (? Hybrids). ♀. Vertical Contour. (17 crania.)



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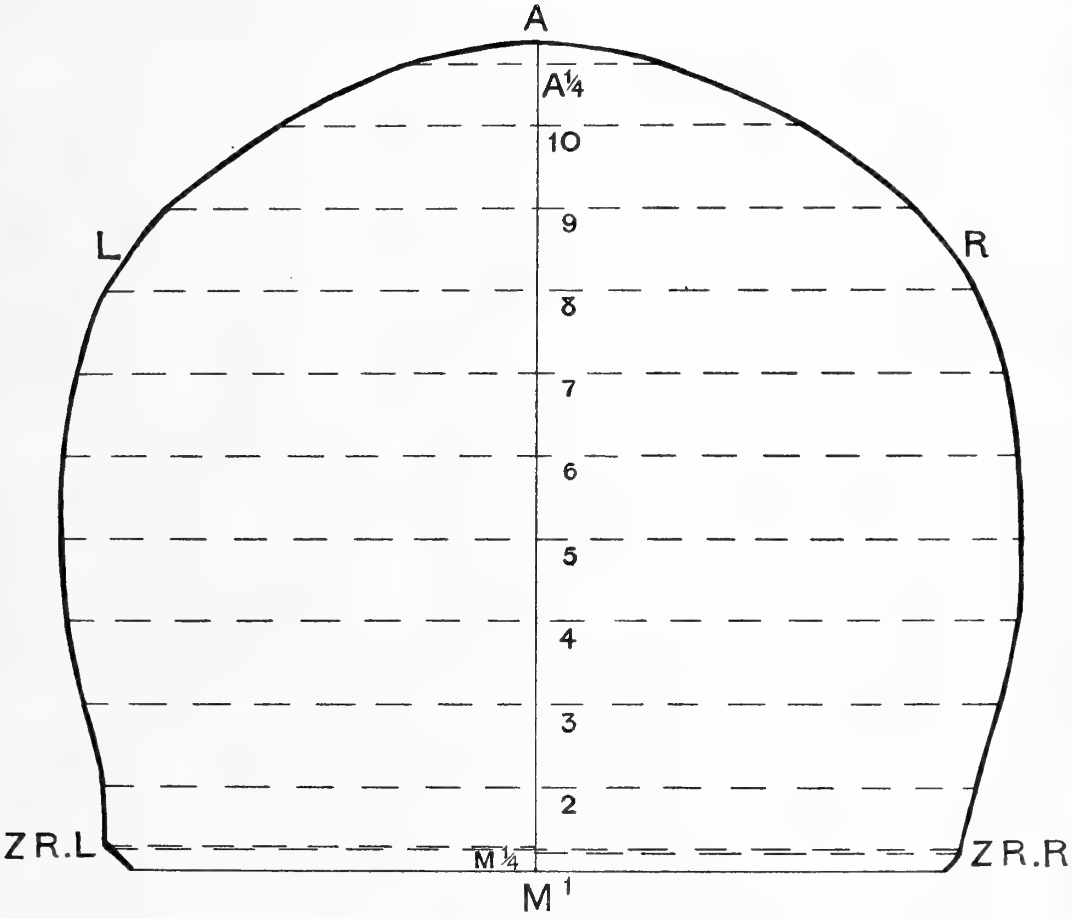


Fig. VI. Burmese Crania. Type C (? Karens). ♀. Vertical Contour. (18 crania.)

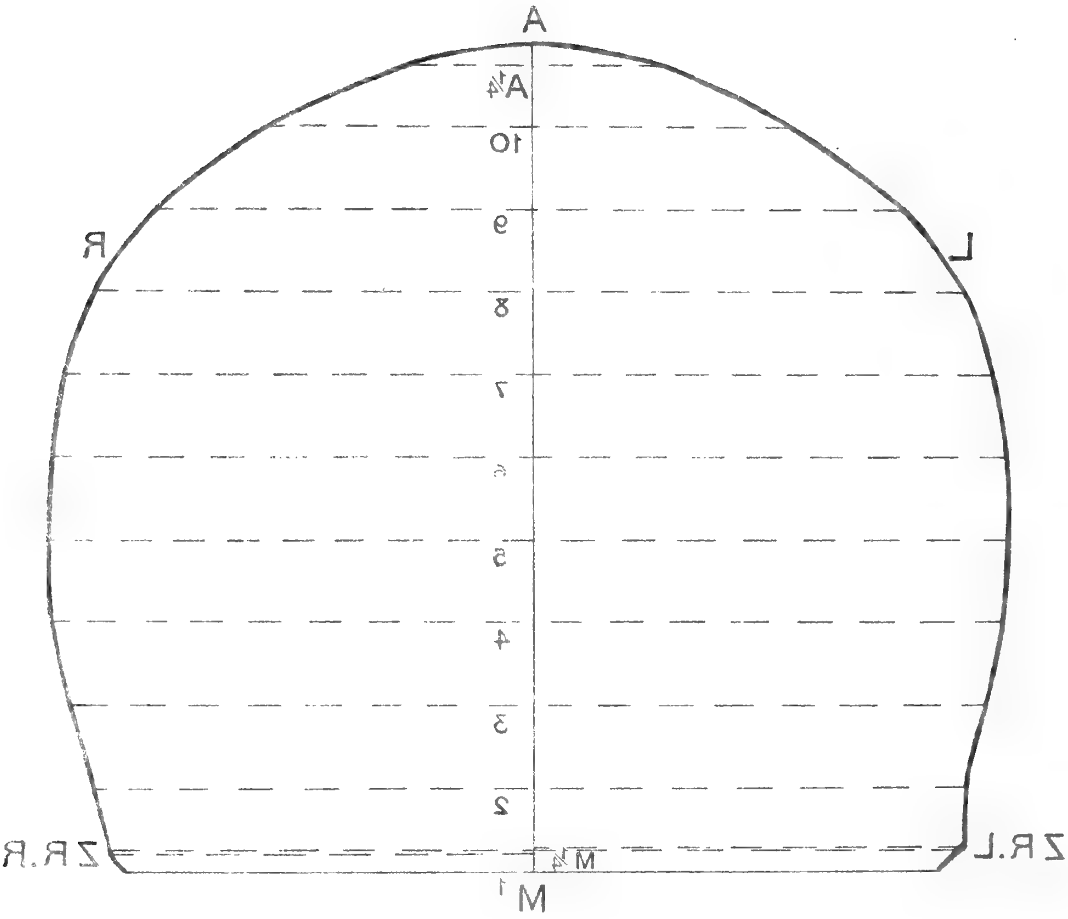


Fig. VI. Burmese Crania. Type C (? Karens). ♀. Vertical Contour. (18 crania.)

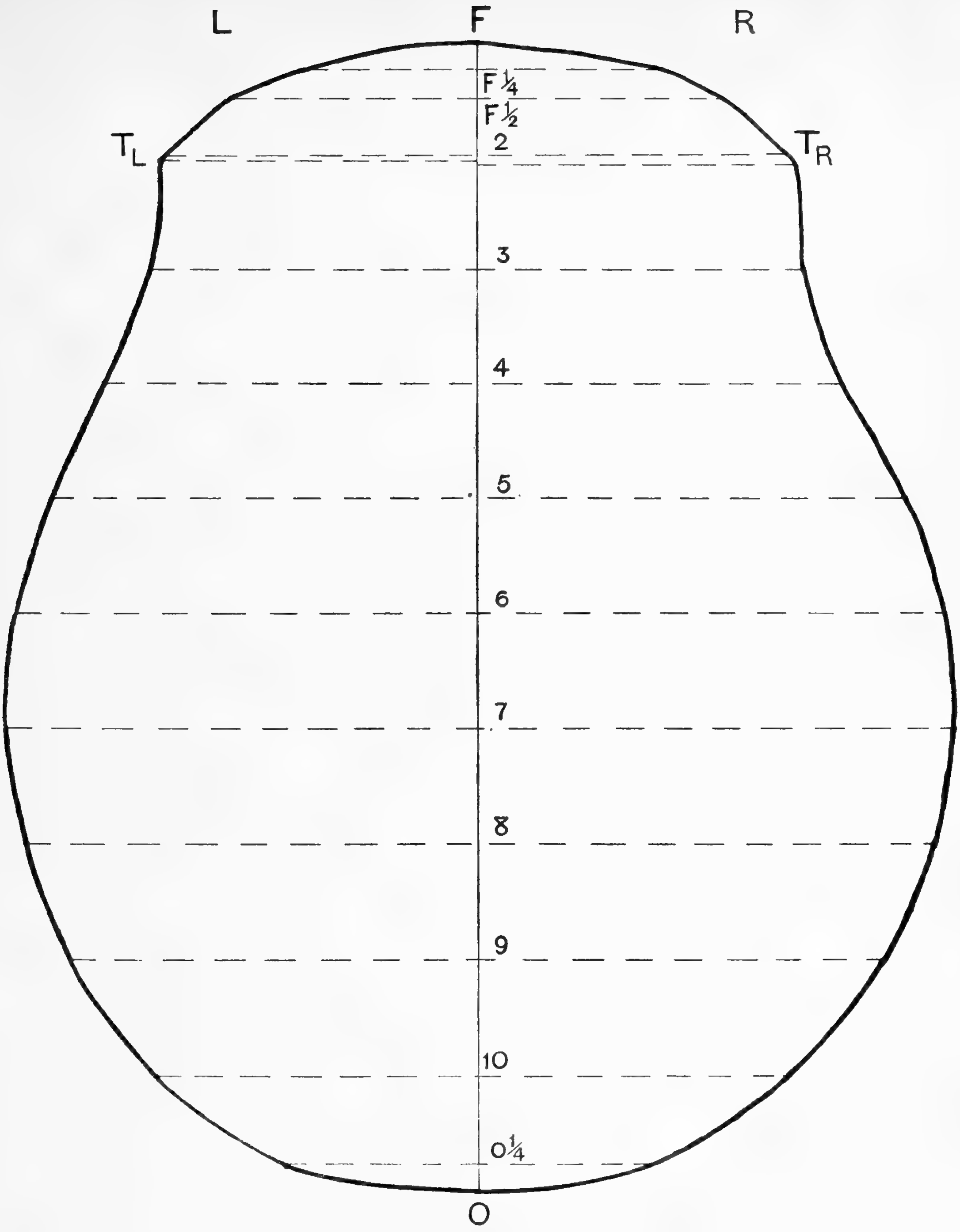


Fig. VII. Burmese Crania. Type A (Burmans). ♂. Horizontal Contour. (44 crania.)

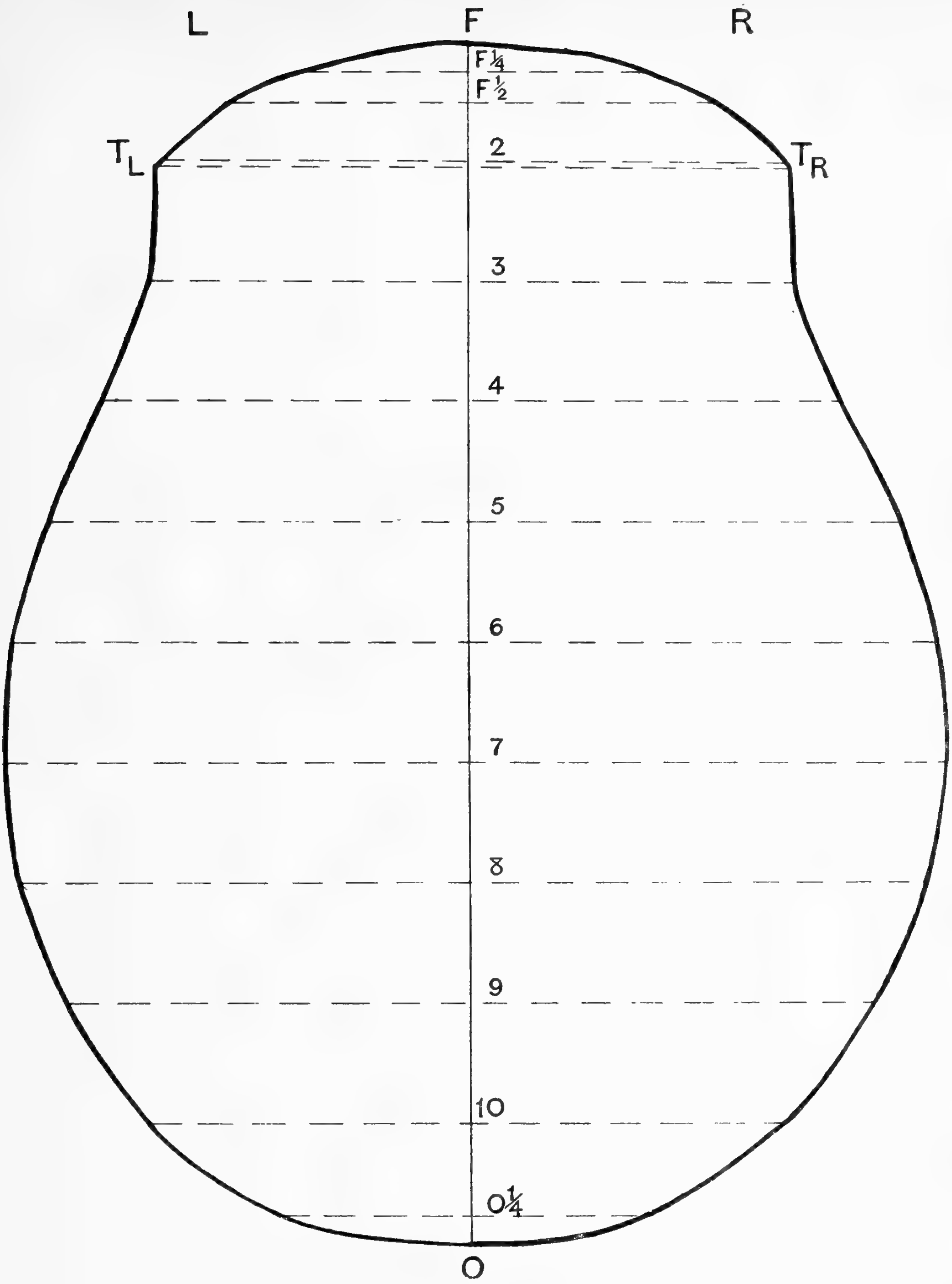


FIG. VIII. Burmese Crania. Type B (? Hybrids). ♂. Horizontal Contour. (7 crania.)

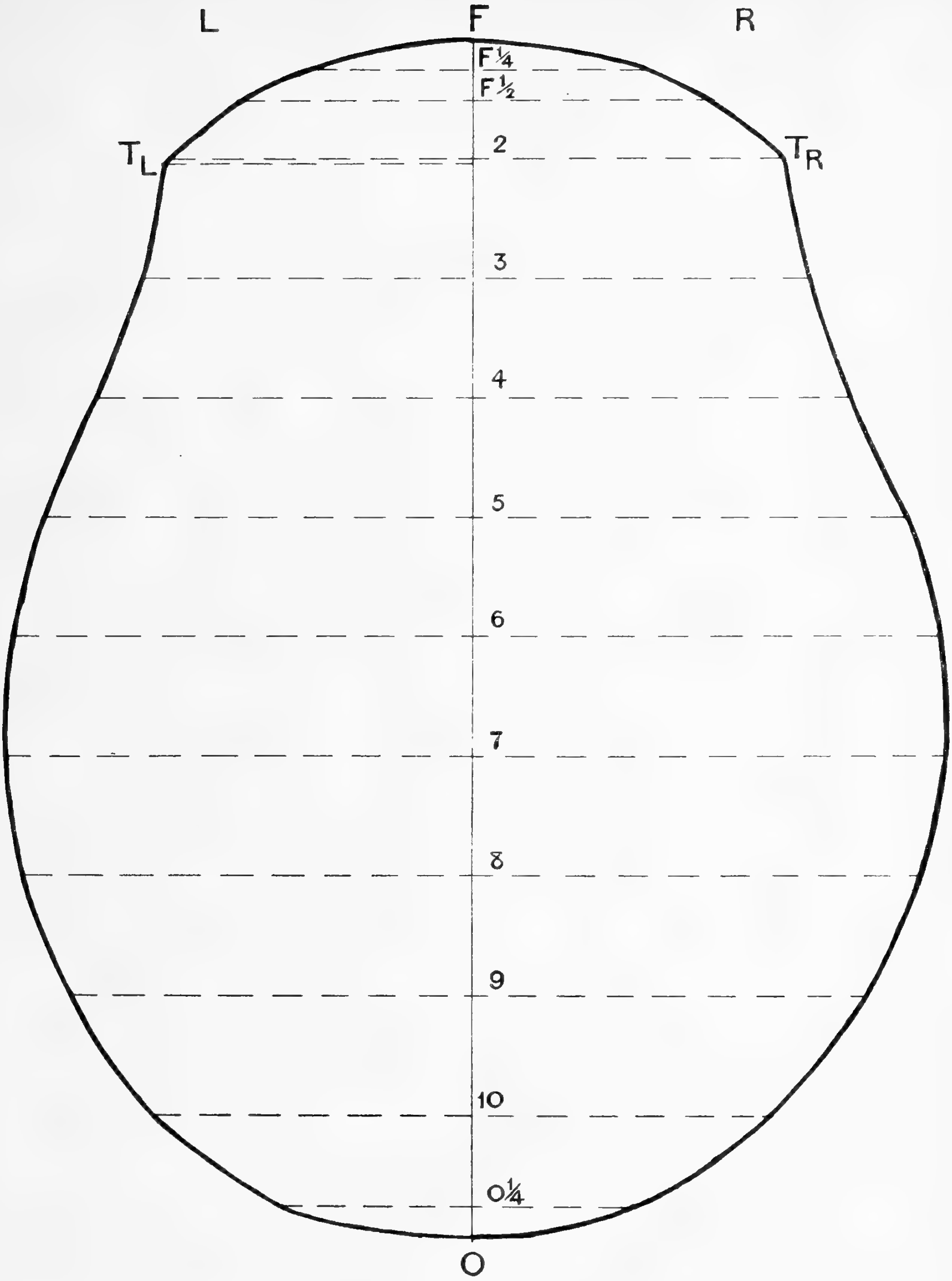


FIG. IX. Burmese Crania. Type C (? Karens). ♂. Horizontal Contour. (8 crania.)

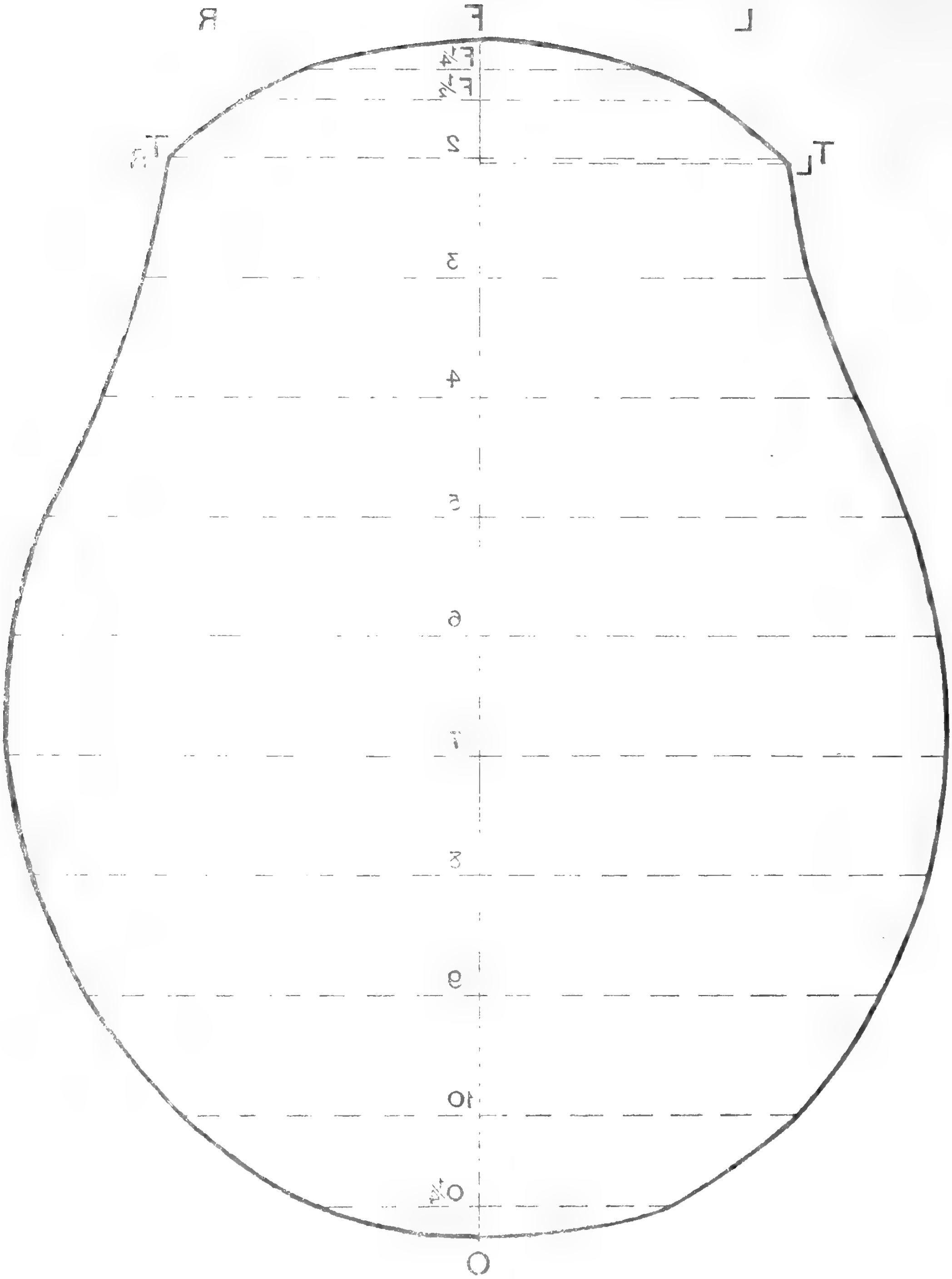
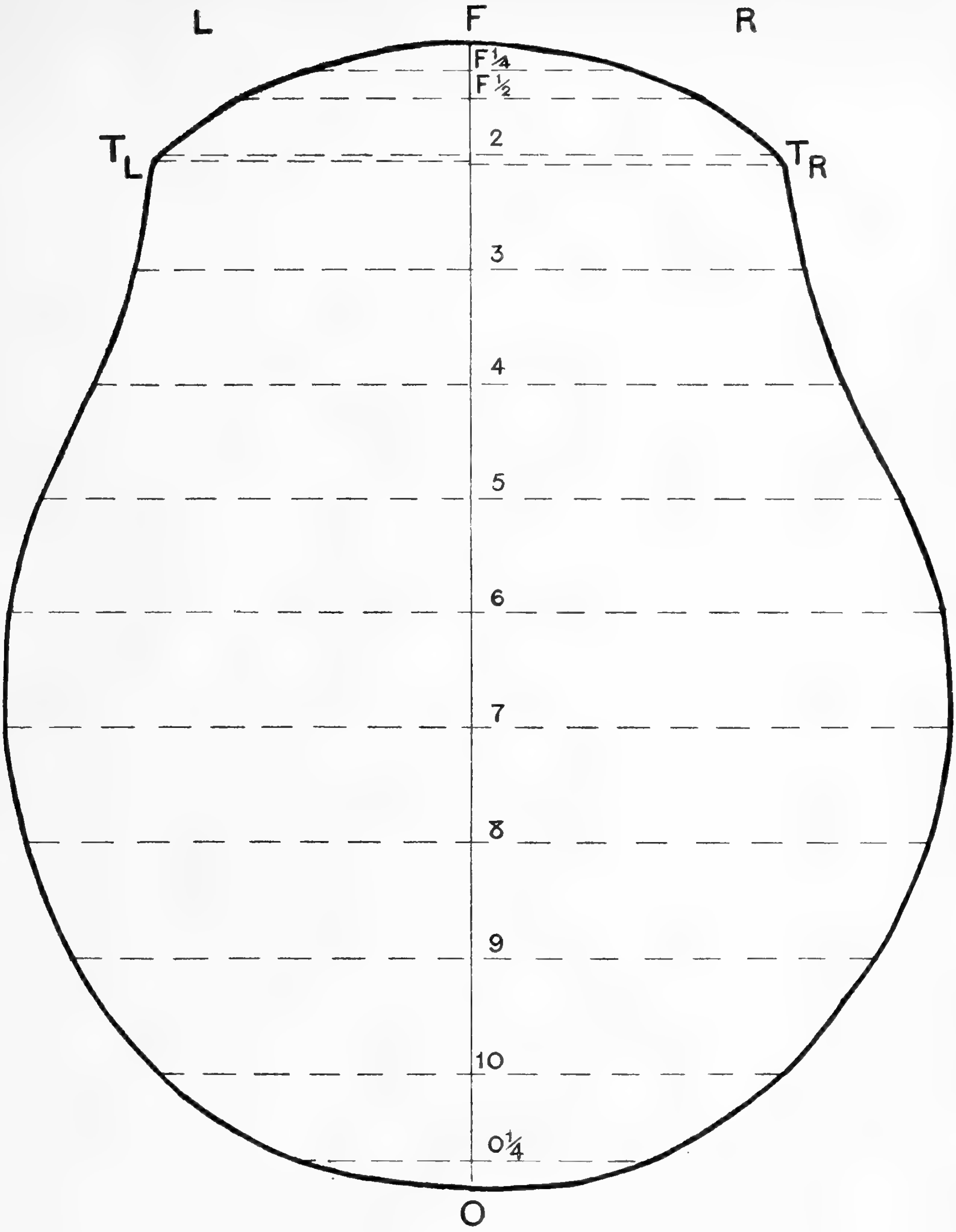


Fig. IX. Burmese Crania. Type C (? Karens). ♂. Horizontal Contour. (8 crania.)



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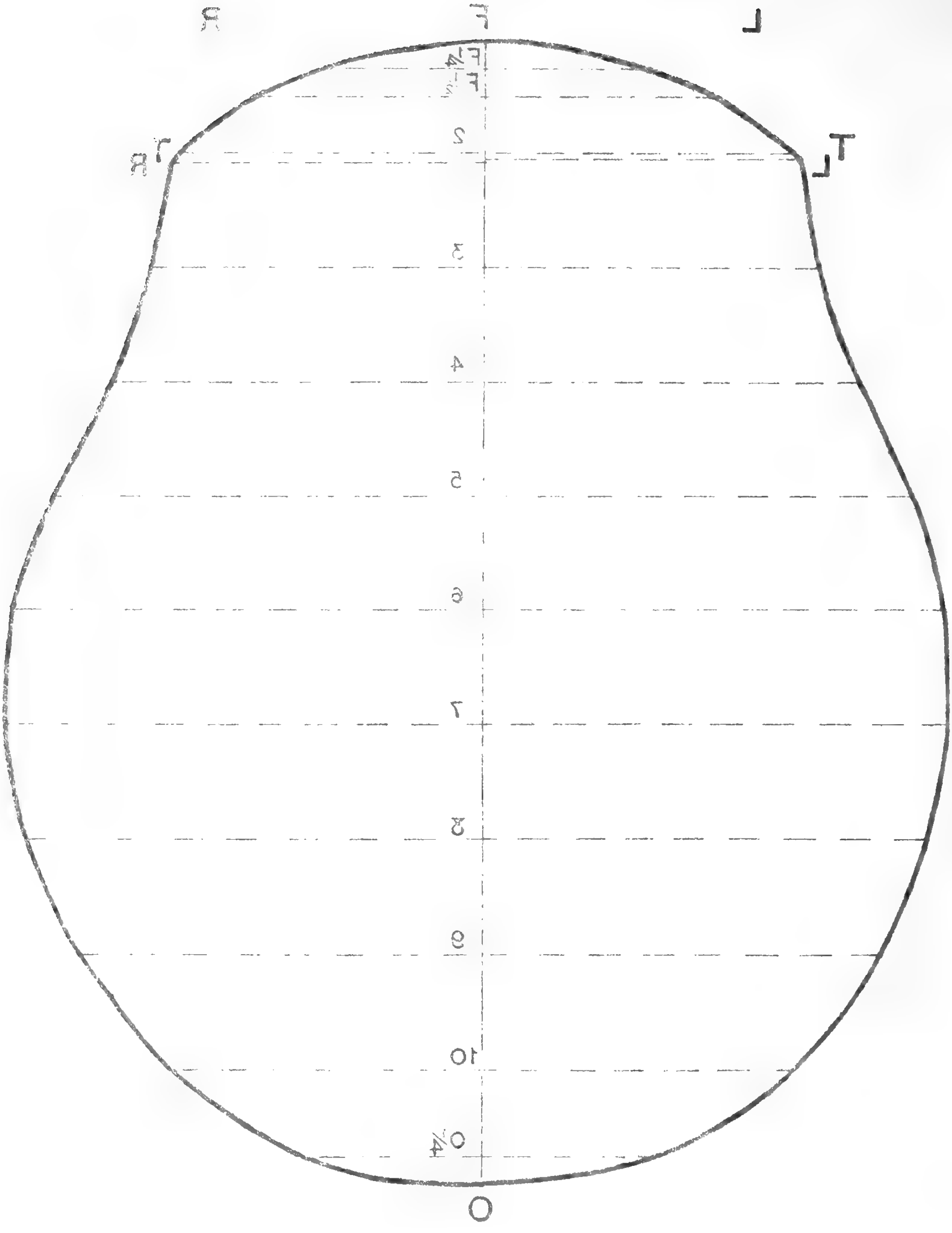


Fig. X. Burmese Crania. Type A (Burmans). ♂. Horizontal Contour. (39 crania.)

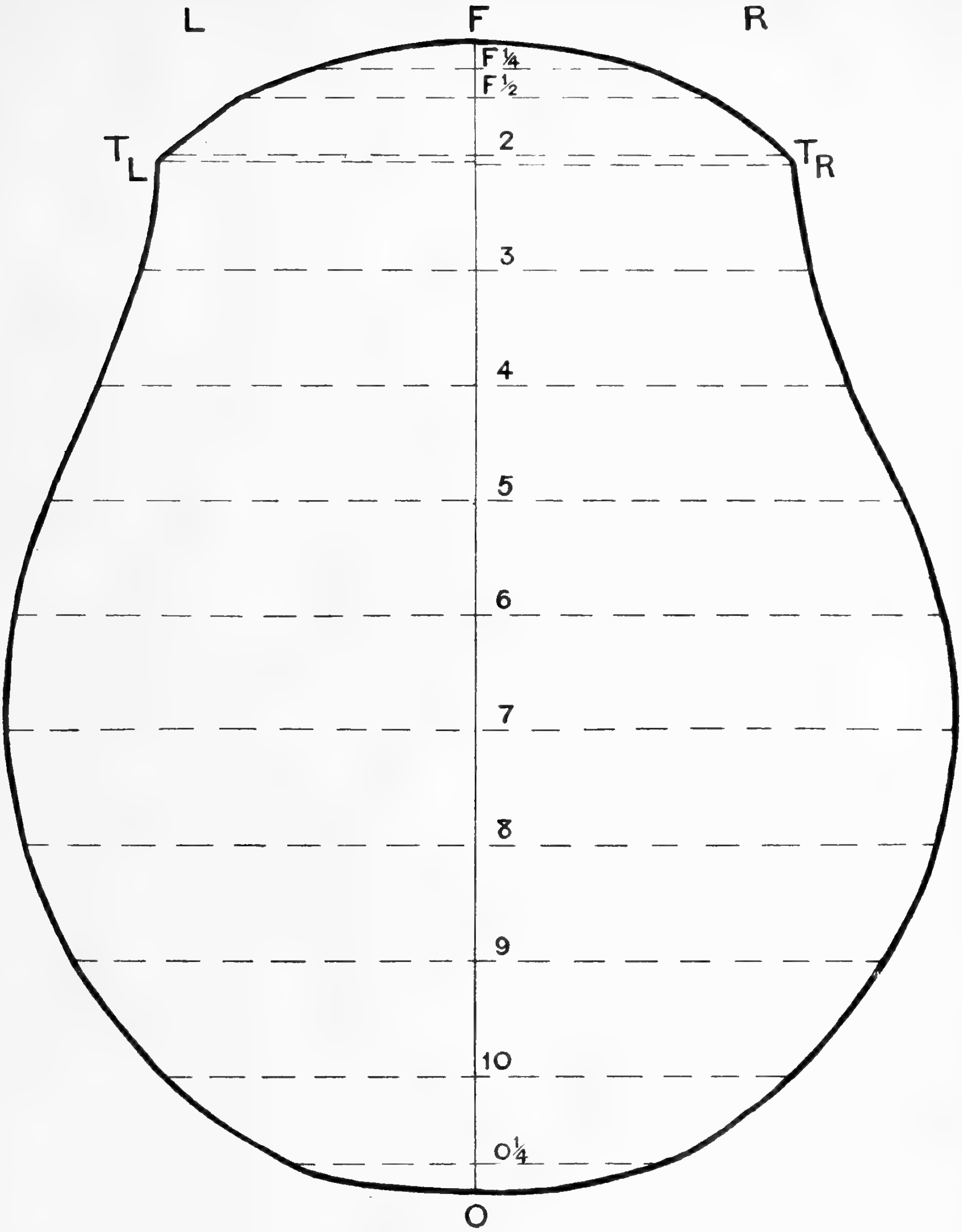


Fig. XI. Burmese Crania. Type B (? Hybrids). ♀. Horizontal Contour. (17 crania.)

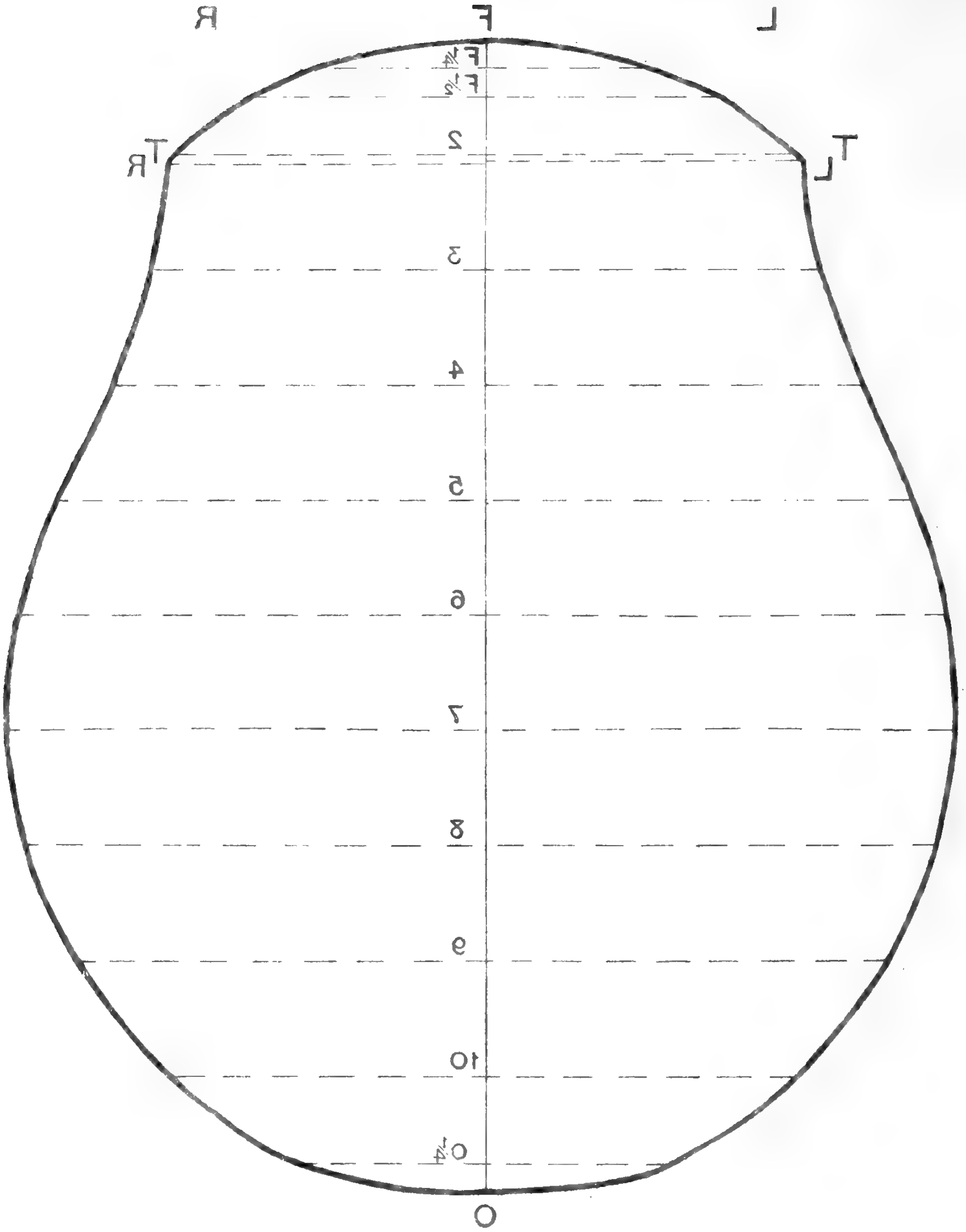


Fig. XI. Burmese Crania. Type B (? Hybrids). ♀. Horizontal Contour. (17 crania.)

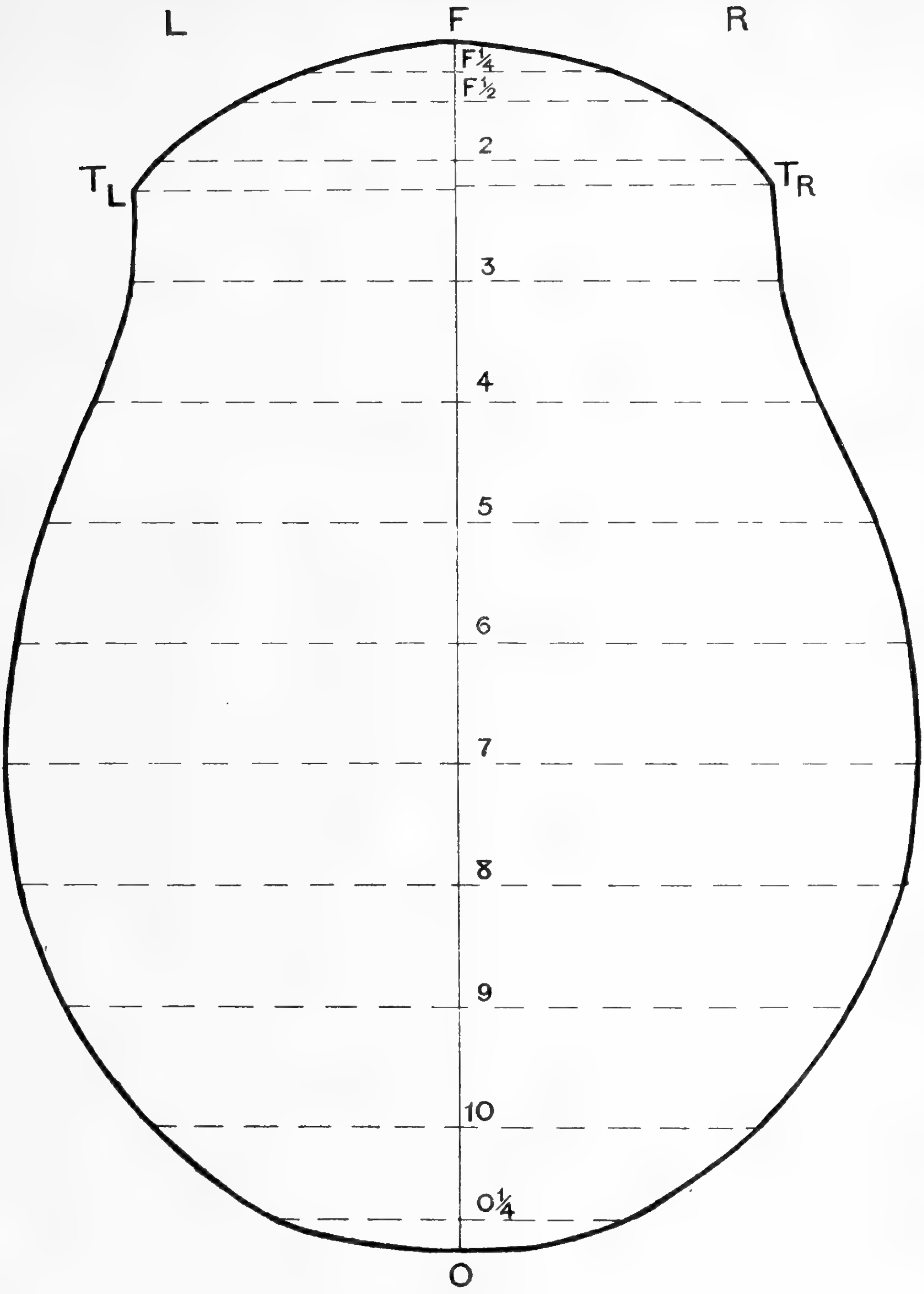


FIG. XII. Burmese Crania. Type C (? Karens). ♀. Horizontal Contour. (18 crania.)

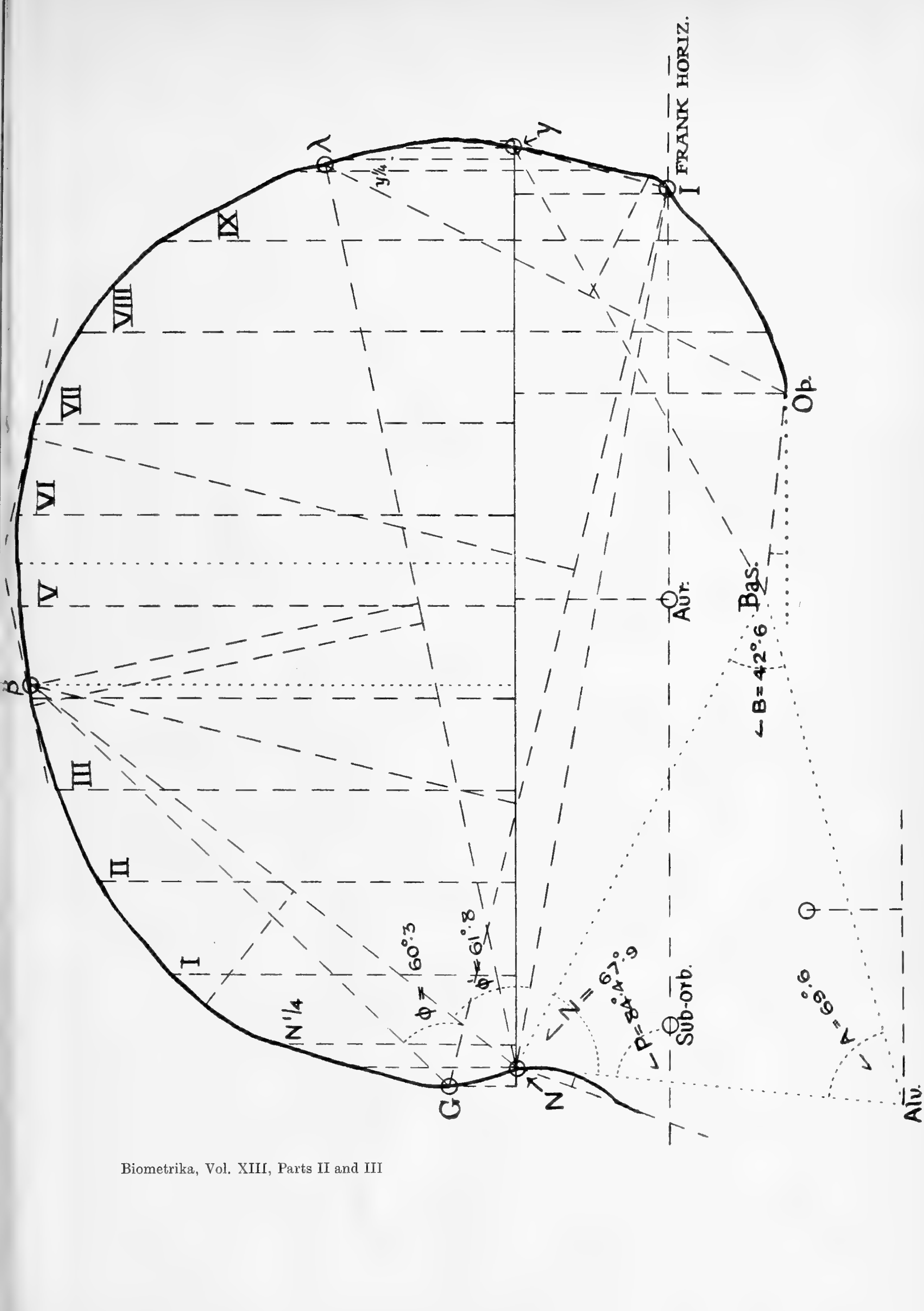


Fig. XIII. Burmese Crania. Type A (Burmans). ♂. Sagittal Contour. (44 crania.)

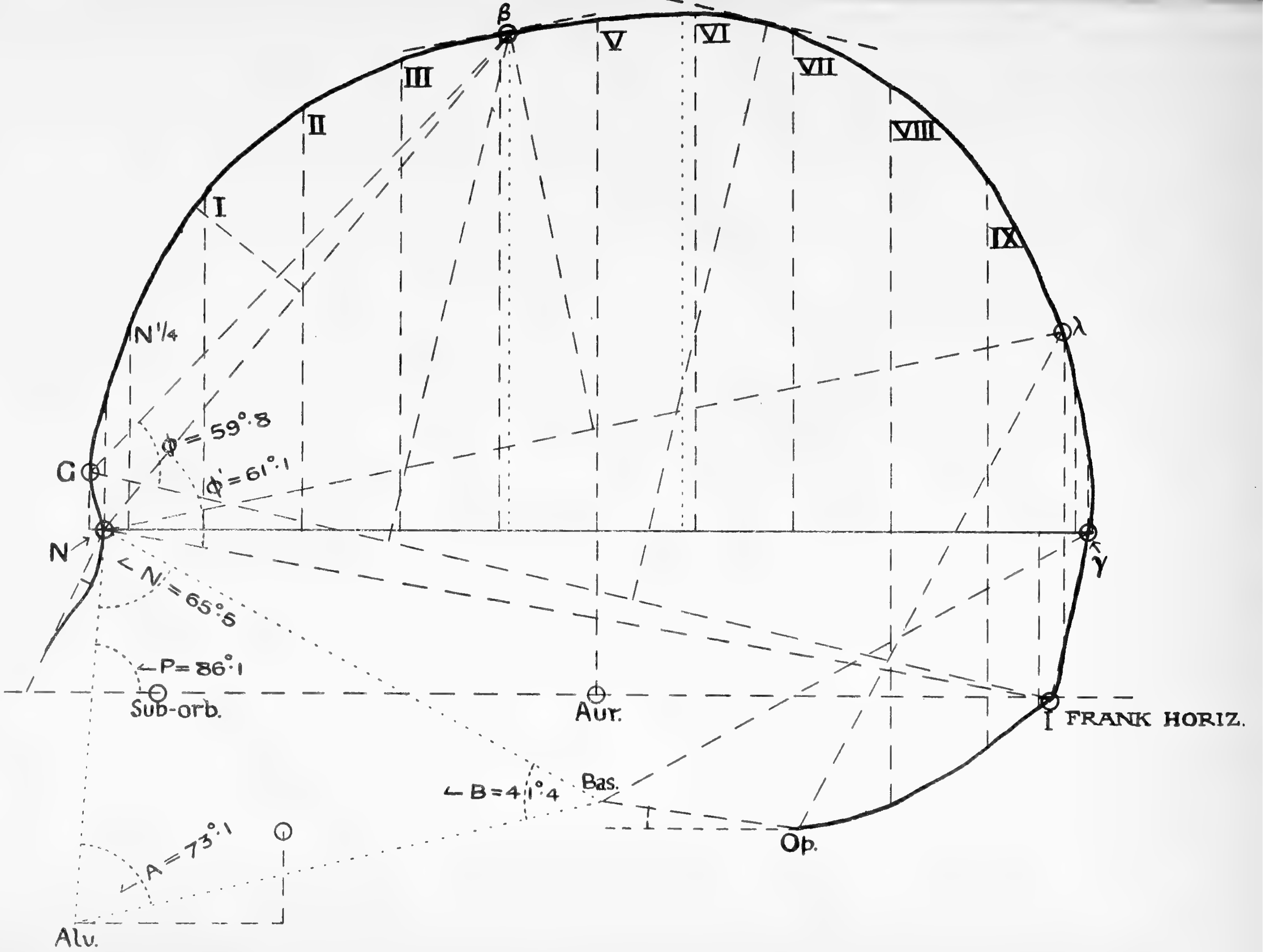


Fig. XIV. Burmese Crania. Type B (? Hybrids). ♂. Sagittal Contour. (7 crania.)

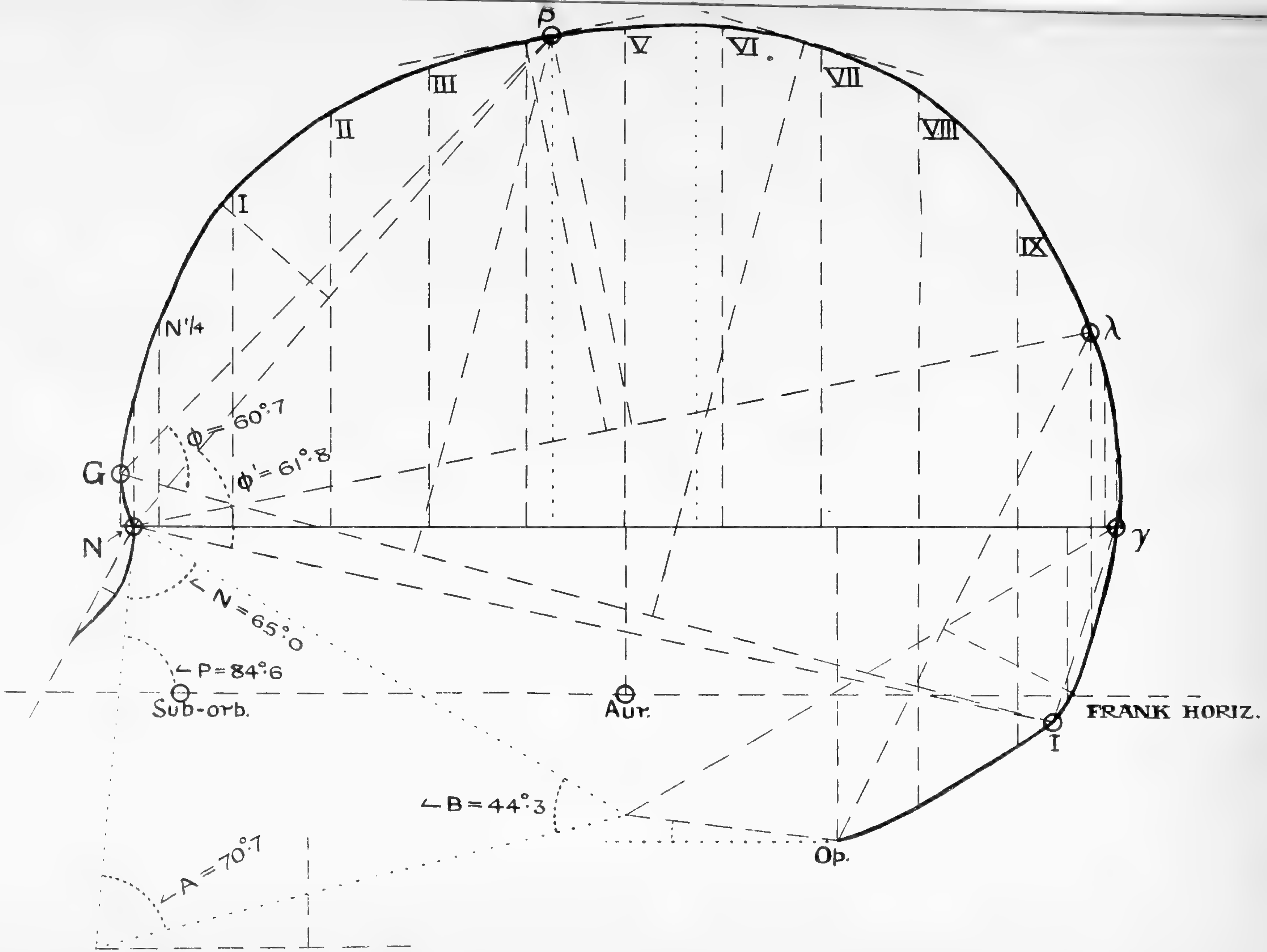
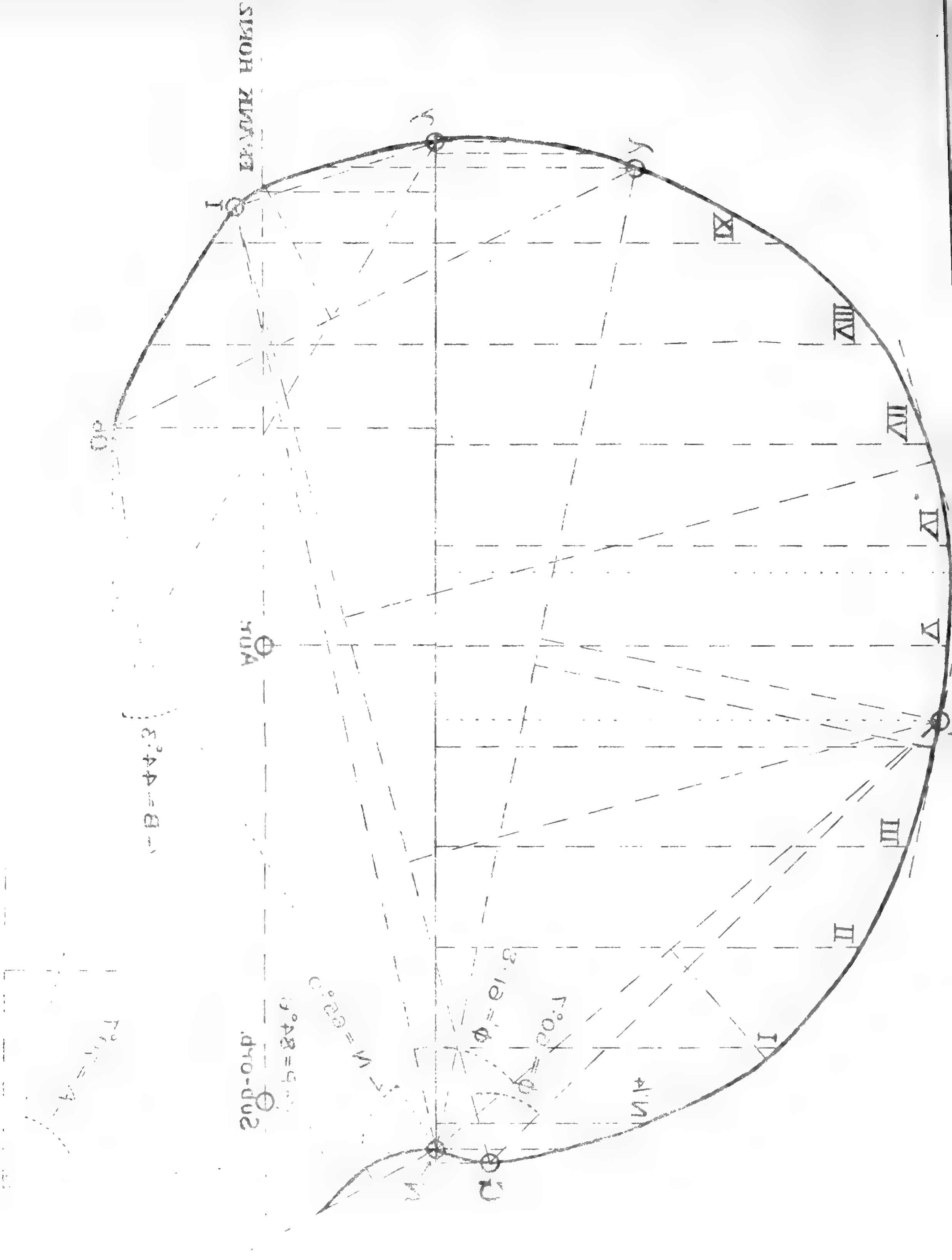


Fig. XV. Burmese Crania. Type C (? Karens). ♂. Sagittal Contour. (8 crania.)


Frank. Horiz.
Aur.
Sub-orb.
Alv.
Op.
I
II
III
IV
V
VI
VII
VIII
IX
N+1
$\phi = 61.8$
$\phi = 100.1$
$N = 65.0$
$P = 84.6$
$B = 44.3$

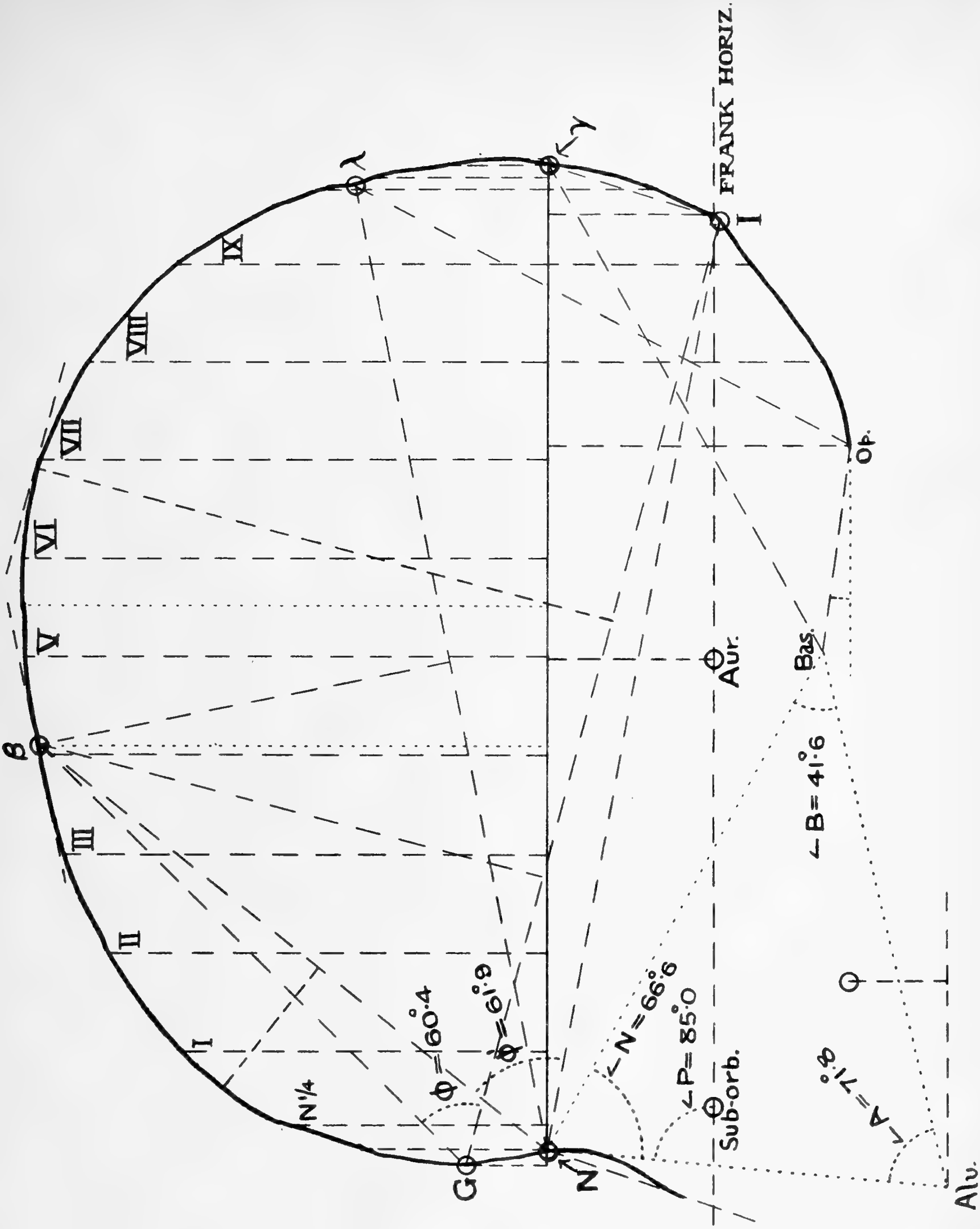


Fig. XVI. Burmese Crania. Type A (Burmans). ♀. Sagittal Contour. (38 crania.)

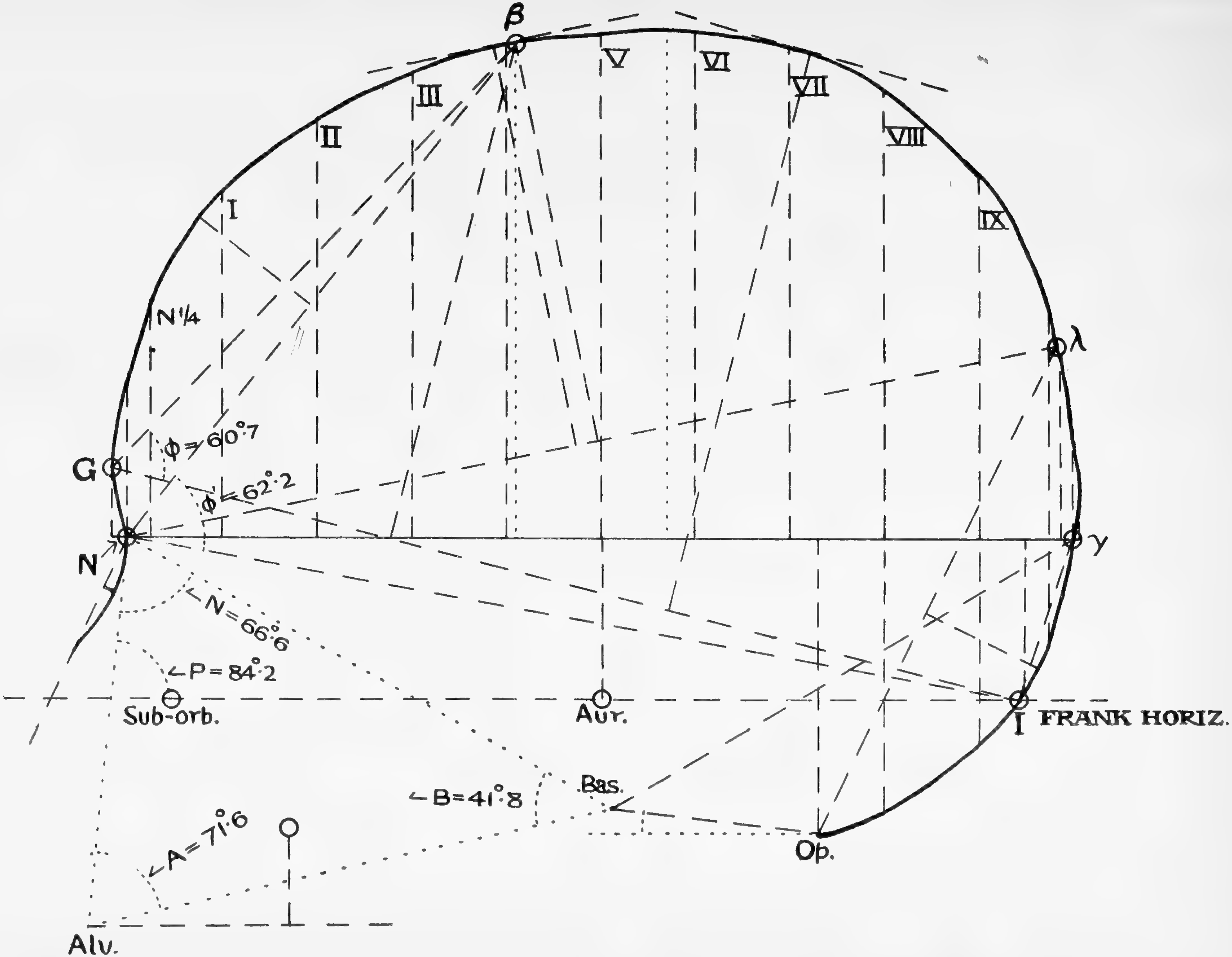
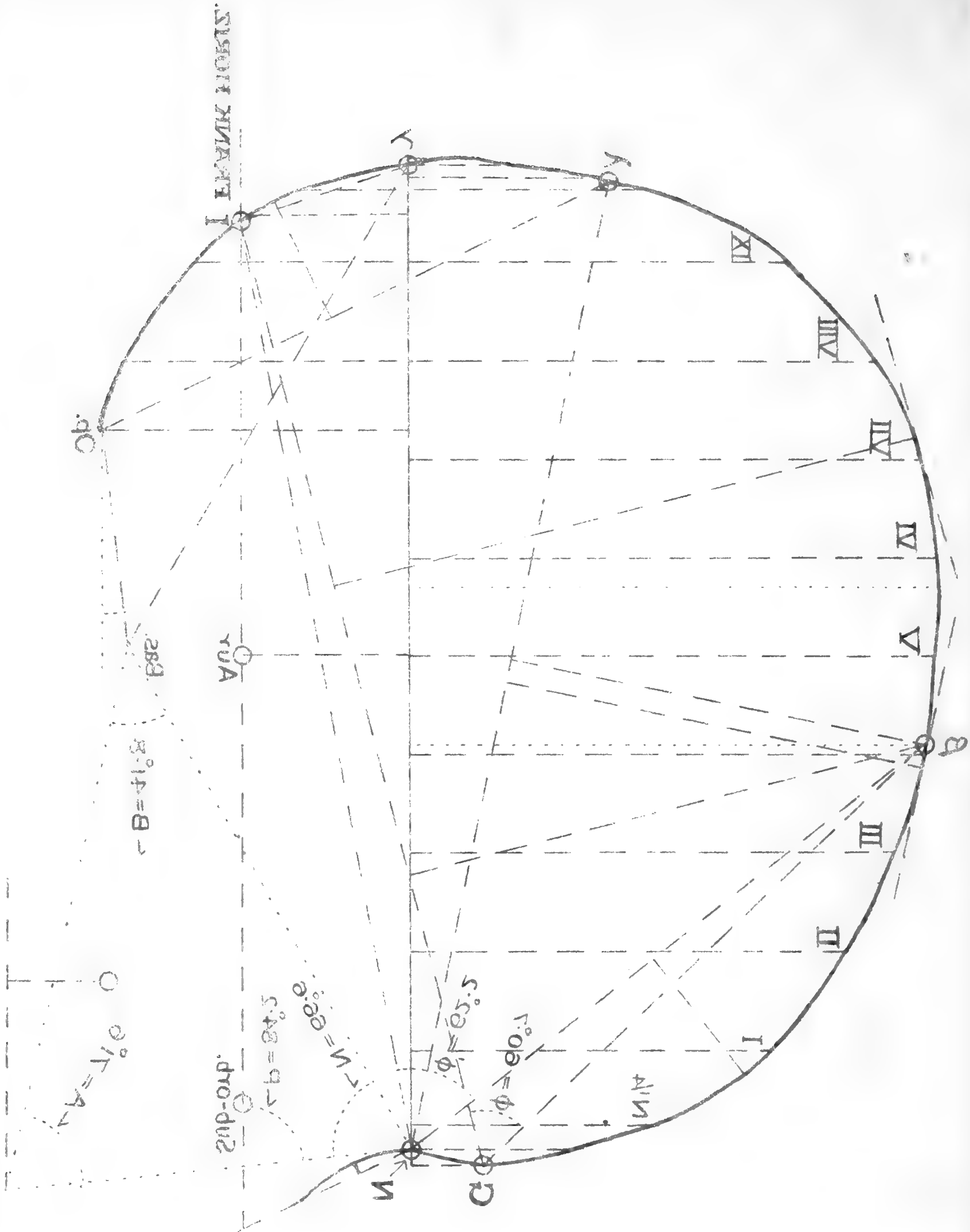


Fig. XVII. Burmese Crania. Type B (? Hybrids). ♀. Sagittal Contour. (17 crania.)



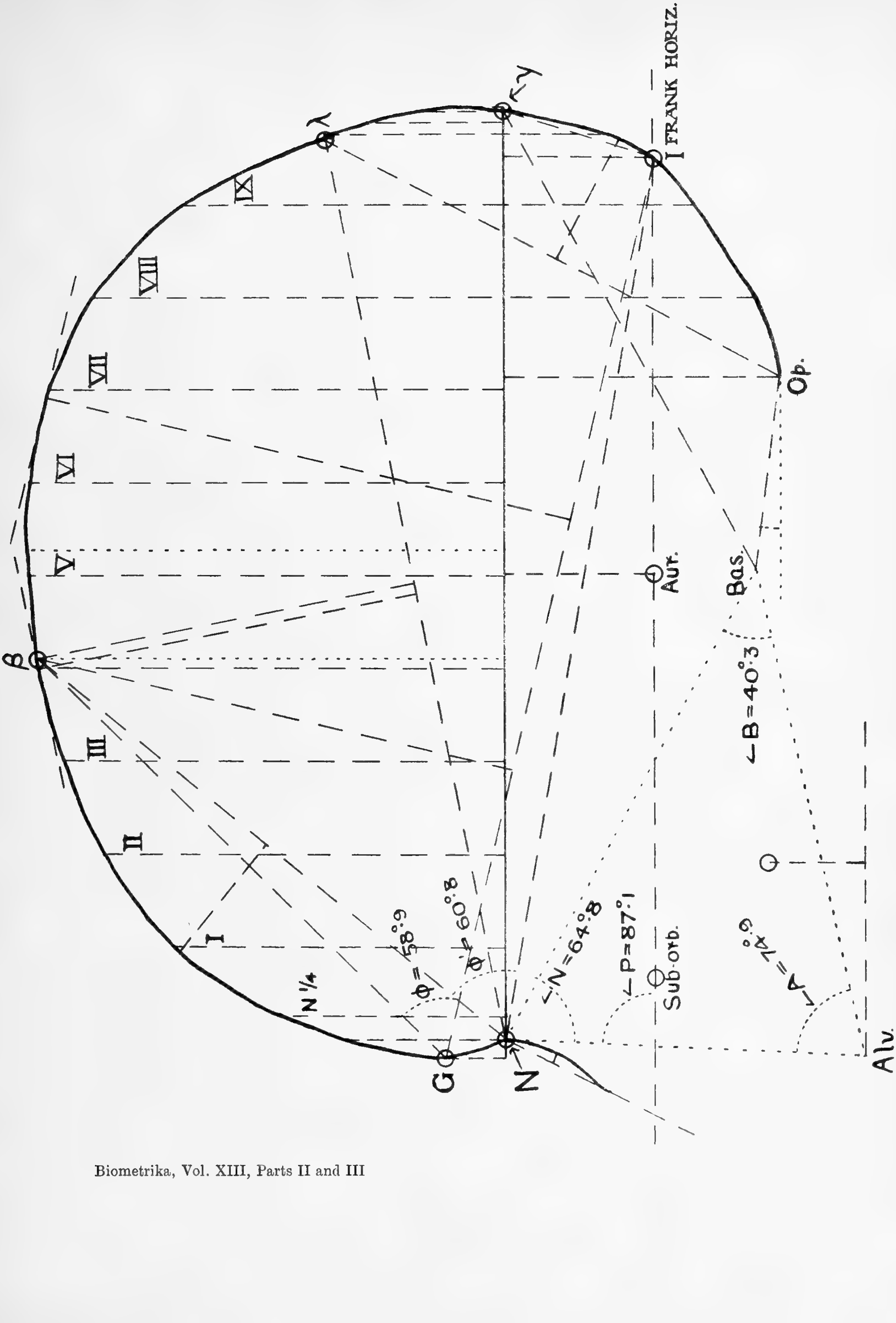


Fig. XVIII. Burmese Crania. Type C (? Karens). ♀. Sagittal Contour. (18 crania.)

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CONTENTS

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Index

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ON RANDOM OCCURRENCES IN SPACE AND TIME, WHEN FOLLOWED BY A CLOSED INTERVAL.

BY G. MORANT, B.Sc.

(Being a paper read to the Society of Mathematical Statisticians and Biometricians, 1921.)

I. *Introductory.*

The object of the present paper is to investigate a little more fully than has hitherto been done the formulae for random occurrences in space and time, and to test their adequacy on the basis of experimental data.

The fundamental formula, which was first stated and proved by Whitworth*, is obtained as follows: An event happens at random once in a period m, therefore its chance of occurring in an interval of time or space δt is $\frac{\delta t}{m}$, and of its failing to occur $\left(1-\frac{\delta t}{m}\right)$. If we now take n such intervals, its chance of not occurring in time $n\delta t$ is $\left(1-\frac{\delta t}{m}\right)^n$, accordingly if we make $t=n\delta t$ and suppose n to become indefinitely large and δt indefinitely small, we have

$$\operatorname*{Lt}_{n=\infty}\left(1-\frac{t}{nm}\right)^n=e^{-\frac{t}{m}},$$

which is therefore the chance of non-occurrence during an interval t, when the occurrence measured on a very long period averages once in the interval m.

Whitworth gives no experimental data to confirm this theory and apparently did not fully recognise its immense importance, if confirmed, for medical and sociological "events." That is to say, as a means of distinguishing between random and associated occurrences, for which it offers, with some expansions, a most valuable criterion.

At first sight it might seem an easy task to justify or refute such a law of distribution by actual experiment. But in endeavouring to find how such a series of events may be obtained by observation or experiment there are serious difficulties to be encountered. Dr M. Greenwood suggested dealing experimentally with the matter, and he drew numbered counters from a bag and considered the differences between successive numbers to be intervals between events. But such a case does not fully accord with theory, for the counters being numbered by units the intervals will be clustered in masses differing by a unit. The chief difficulty of any experimental demonstration lies in allowing for a large number of simultaneous occurrences. For example, if we take taxi-cabs passing a given lamp-post in a given street in one direction, it is perhaps only possible for one or two to pass at one

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* *Choice and Chance*, Cambridge, 1901, 5th Edition, p. 200.

instant of time. If only one is able to pass, there will be a "closed-time" β, measured by the length of the taxi plus a certain margin of safety, both varying with the size of the taxi and the bump of precaution of the second driver. In considering experiments likely to exhibit the law, we were repeatedly met by the existence of such a "closed-time" β intervening between event and event. Some of the most important cases to which the theory may be applied, socially or medically, involve such a closed time between occurrences. Accordingly it seems desirable to extend the theory to cases in which there is of necessity a "closed-time" β within which no event can follow a given occurrence. As most experimental and many social and vital phenomena involve a β as well as an m, it becomes necessary to devise an experiment which will give some approach to a constant β and a constant m.

If a chronograph be running and marks seconds, and an observer taps on the occurrence of random events, he will be unable to tap any number of simultaneous events; they will be separated by at least the reaction time requisite for noting the event and tapping the key. But it is by no means certain that the β and m in such a case remain really constant; in fact it will be shown that experiments suggest that they do not. There may be slight irregularities due to the apparatus, such as in the time of swing of the pendulum or the rate at which the tape is moving, but more important than any of these will be the fluctuating "personal equation" of the observer.

A suitable series of random intervals seemed to be those between the successive occurrences of the figure "5" in the units place of columns of numbers, each containing at least four figures, in a Census Report. The unit figures were read down as far as possible at a constant rate, tapping the key of the chronograph on coming to a "5." Six half-hour readings were taken in this way and it was assumed that the values of m and β for the six tapes were not significantly different from each other, so that the whole might be added to give a homogeneous distribution. It will be shown later how far this assumption was justified.

A constant β, such as the theory postulates, would be given if the length of the interval between two "5's" were measured by the number of intervening figures. But this case would be similar to that of drawing numbered counters from a bag; the intervals could only differ by units. By introducing the personal factor we obtain a continuous distribution but the constants β and m are sure to fluctuate slightly throughout the experiment. It was hoped that this would not materially affect the results.

II. *The Frequency of Intervals in an indefinitely long Time**.

The first case to be considered is that of the frequency of intervals which elapse between two successive occurrences of the event when the period in which the observations are taken is continuous and indefinitely long. We have seen that the

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* We shall use the terminology of "time" throughout, but the intervals may always be interpreted as intervals of space and the occurrences as random marks on a line.

chance of non-occurrence during an interval t, when the occurrence measured in a very long period averages once in time m, is $e^{-\frac{t}{m}}$. If there are N intervals, it follows that the frequency of intervals of length lying between t and $t+dt$ will be given by $f_n\,dt = \frac{N}{m}e^{-\frac{t}{m}}\,dt$, or the frequency curve is $y_t = \frac{N}{m}e^{-\frac{t}{m}}$. Now we have to consider the case in which an occurrence is always followed by a "closed-interval" β, so the form of the curve will be $y_t = \frac{N}{m}e^{-\frac{(t-\beta)}{m}}$.

Thus, having observed the frequency of intervals in an indefinitely long time, it remains to find the values of β and m which will fit this curve best. The beginning of the range, being at the point $t=\beta$, is not known *a priori*, so that the moments of the observed distribution cannot be found in the usual way. The following method is used to determine suitable values of β and m, applying for fitting, as now usual, the method of moments.

The curve starts at β and runs theoretically to $t=\infty$. Let

$$I_{s,r} = \frac{N}{m}\int_{\beta}^{t_r} t^s e^{-\frac{(t-\beta)}{m}}\,dt;$$

or, integrating by parts,

$$I_{s,r} = N\{\beta^s + sm\beta^{s-1} + s(s-1)m^2\beta^{s-2} + \ldots\}$$
$$- Ne^{-\frac{(t_r-\beta)}{m}}\{t_r^s + smt_r^{s-1} + s(s-1)m^2t_r^{s-2}\} \ldots\text{(i)}.$$

Accordingly:

$$I_{1,\infty} = N(\beta+m),$$
$$I_{2,\infty} = N(\beta^2 + 2m\beta + 2m^2),$$
$$I_{0,t_r} = N\left(1 - e^{-\frac{(t_r-\beta)}{m}}\right) = n_{t_r} = N\tilde{n}_{t_r}, \text{ say,}$$
$$I_{1,t_r} = N\{\beta + m - e^{-\frac{(t_r-\beta)}{m}}(t_r+m)\}$$
$$= N\{\beta - t_r + \tilde{n}_{t_r}(t_r+m)\},$$
$$I_{2,t_r} = N\{\beta^2 + 2m\beta + 2m^2 - e^{-\frac{(t_r-\beta)}{m}}(t_r^2 + 2mt_r + 2m^2)\}$$
$$= N\{\beta^2 - t_r^2 + 2m(\beta - t_r) + \tilde{n}_{t_r}(t_r^2 + 2mt_r + 2m^2)\} \quad\ldots\ldots\ldots\text{(ii)}.$$

Now $\qquad I_{1,\infty} - I_{1,tr} =$ 1st moment of all observations about $t=0$, excluding the frequency up to intervals t_r.

$I_{2,\infty} - I_{2,t_r} =$ 2nd moment of all observations about $t=0$, excluding the frequency up to intervals t_r.

Let these be respectively M_1 and M_2.

Then we have

$$M_1 = Ne^{-\frac{(t_r-\beta)}{m}},$$

$$M_2 = Ne^{-\frac{(t_r-\beta)}{m}}(t_r^2 + 2mt_r + 2m^2),$$

$$\frac{M_2}{M_1} = \lambda, \text{ say} = \frac{t_r^2 + 2mt_r + 2m^2}{t_r + m},$$

or $$2(m + t_r)^2 - (\lambda + t_r)(m + t_r) + t_r^2 = 0 \text{ .... (iii).}$$

This is a quadratic to find $(m + t_r)$ and therefore m when λ is known.

Now the curve is exponential and we start with a finite ordinate at $t = \beta$, and therefore in dealing with M_1 and M_2 from the observations we are bound to use careful adjustments, i.e. the abruptness coefficients for the moments*. Not knowing β, however, we cannot evaluate the moments of the first group. But we do know an approximate value of β, because we can note what the least interval observed is, and we may assume t_r to be something greater than this. In actual observation work Equation (iii) did not give good results and the reason for this is fairly obvious. In determining the moments we neglect a very important factor, the moments of the first group, and throw all the weight of determining β and m on the later frequencies.

A start may be made in the following manner. Take approximate values of β and m, say β_0 and m_0, and calculate the position of the mean and the moments of the first group on the assumption that these values are correct, and add them to the observed moments of the remainder to obtain the total observed moments for equating against our theoretical results for the determination of a new β and m. In doing this we may obtain β_0 and m_0 from the area value and the first moment of all the data without the first group (observed frequency n_1). Let the limit of the first group be $t = \gamma$; then we shall take γ as close to β as is safe in our ignorance as to the exact value of β in order to make the influence of this first group as small as possible. For example, if β be about ·3, we shall take $\gamma = {\cdot}5$ rather than 1, and reckon, say, our groupings from ·5 to 1·5, 1·5 to 2·5 etc., rather than from 1 to 2, 2 to 3 etc. of our units of time.

Now suppose we find the first moment of $N - n_1$ of the observations after $t = \gamma$ (this being M_1) and also $N - n_1$. These give

$$N - n_1 = Ne^{-\frac{(\gamma-\beta_0)}{m_0}},$$

$$M_1 = Ne^{-\frac{(\gamma-\beta_0)}{m_0}}(\gamma + m_0).$$

Then $$m_0 = \frac{M_1}{(N - n_1)} - \gamma \text{ .... (iv),}$$

and $$\beta_0 = \gamma - \frac{m_0\{\log_{10} N - \log_{10}(N - n_1)\}}{\log_{10} e} \text{ .... (v).}$$

Thus our preliminary values m_0 and β_0 are determined.

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* See *Biometrika*, Vol. XII, p. 240.

In finding these values M_1 should first be found with great accuracy, remembering the difficulties with regard to even the first moment of curves with a finite initial ordinate and an abrupt departure from that ordinate. M_2 must be found with the same accuracy.

Now n_1 being the observed number in the first group and μ_1', μ_2' its first and second moment coefficients about $t=0$; μ_1'', μ_2'' the first and second moment coefficients of the remainder, $(N-n_1)$ about $t=0$, we have

$$Nc_1 = n_1\mu_1' + (N-n_1)\,\mu_1'',$$
$$Nc_2 = n_1\mu_2' + (N-n_1)\,\mu_2'',$$

whence c_1 and c_2 will be determinable from the observations: for μ_1' and μ_2' are to be given the values they would have for β_0 and m_0, and $(N-n_1)\,\mu_1''$ and $(N-n_1)\,\mu_2''$ are M_1 and M_2 respectively.

Let $\tilde{n}_1$ be the *theoretical* frequency in the range $t=\beta_0$ to γ, then

$$\tilde{n}_1 = \int_{\beta_0}^{\gamma} \frac{N}{m_0} e^{-\frac{(t-\beta_0)}{m_0}}\,dt = N\left(1-e^{-\frac{(\gamma-\beta_0)}{m_0}}\right),$$

$$\tilde{n}_1\,\mu_1' = \int_{\beta_0}^{\gamma} \frac{N}{m_0} e^{-\frac{(t-\beta_0)}{m_0}}\,t\,dt = N\left\{\beta_0 + m_0 - (\gamma+m_0)\,e^{-\frac{(\gamma-\beta_0)}{m_0}}\right\},$$

$$\tilde{n}_1\,\mu_2' = \int_{\beta_0}^{\gamma} \frac{N}{m_0} e^{-\frac{(t-\beta_0)}{m_0}}\,t^2dt = N\left\{(\beta_0+m_0)^2 + m_0^2 - [(\gamma+m_0)^2+m_0^2]\,e^{-\frac{(\gamma-\beta_0)}{m_0}}\right\}.$$

Thus

$$\mu_1' = \frac{\beta_0 + m_0 - (\gamma+m_0)\,e^{-\frac{(\gamma-\beta_0)}{m_0}}}{1-e^{-\frac{(\gamma-\beta_0)}{m_0}}} \quad \text{......(vi)},$$

$$\mu_2' = \frac{(\beta_0+m_0)^2 + m_0^2 - \{(\gamma+m_0)^2+m_0^2\}\,e^{-\frac{(\gamma-\beta_0)}{m_0}}}{1-e^{-\frac{(\gamma-\beta_0)}{m_0}}} \quad \text{......(vii)}.$$

Now knowing c_1 and c_2 from the observations in this way, we have from the whole theoretical curve $c_1 = \beta + m$ and $c_2 = (\beta+m)^2 + m^2$, and accordingly

$$m^2 = c_2 - c_1^2 \text{ or } m = \sqrt{c_2 - c_1^2} \quad \text{......(viii)},$$

$$\beta = c_1 - m \text{ or } \beta = c_1 - \sqrt{c_2 - c_1^2} \quad \text{......(ix)}.$$

These give the second approximations to β and m, and if necessary we can use these values to redetermine μ_1' and μ_2' and proceed to a third approximation.

We will now illustrate the process on the experimental data for the frequency of intervals in an indefinitely long time. The smallest interval recorded was ·39 sec., and it seemed that β was fluctuating slightly throughout the experiment. To make the influence of the first group as small as possible, γ was taken to be ·5 so that the first group would be from β to ·5 and succeeding ones, ·5 to 1·5, 1·5 to 2·5 and so on. To find β and m, the first and second moment coefficients of the distribution excluding the first group are found about ·5, using abruptness coefficients. These are transferred to $t = 0$, giving

$$\mu_1'' = 2{\cdot}929{,}7114, \qquad \mu_2'' = 14{\cdot}781{,}9089,$$

which give M_1 and M_2 respectively. Using these values the first approximations to β and m (i.e. β_0 and m_0) are, by equations (iv) and (v),

$$m_0 = 2{\cdot}429{,}7114, \qquad \beta_0 = {\cdot}428{,}8488.$$

Then the first and second moment coefficients of the first group about $t = 0$ (μ_1' and μ_2') are given the values they would have for β_0 and m_0, and these are, by (vi) and (vii),

$$\mu_1' = {\cdot}464{,}2481, \qquad \mu_2' = {\cdot}215{,}9042,$$

leading to $$c_1 = 2{\cdot}858{,}5599, \qquad c_2 = 14{\cdot}361{,}5450,$$

and finally by (viii) and (ix) the second approximations to β and m.

If it be thought necessary, these values can be used to redetermine μ_1' and μ_2' and hence third approximations to β and m. The results were in our case:

	β	m
2nd approximations	·370,552,	2·488,007,
3rd approximations	·368,867,	2·488,841.

The second approximations would be quite sufficiently accurate for most purposes.

The areas of the curve $y = \frac{N}{m} e^{-\frac{(t-\beta)}{m}}$ corresponding to the observed frequencies can be easily found when m and β are known.

The usual χ^2 test of goodness of fit* gave a very low value for P and it was clear that this was largely due to the very bad agreement of the first two groups (Table I). Clubbing these together (i.e. taking the first group from β to 1·5) gave the much higher value $P = {\cdot}222$. It seemed possible that this disagreement at the beginning of the range might be due to the fact that too few observations had been included in the first group and to test this the material was regrouped on the bases β to 1, 1 to 2 etc. This arrangement gave the constants

	β	m
2nd approximations	·362,918,	2·495,527,

the intermediate steps being

$$\begin{aligned} \mu_1'' &= 3{\cdot}390{,}0644, & \mu_2'' &= 17{\cdot}848{,}5430, \\ m_0 &= 2{\cdot}390{,}0644, & \beta_0 &= {\cdot}468{,}3801, \\ \mu_1' &= {\cdot}724{,}3450, & \mu_2' &= {\cdot}548{,}1725, \\ c_1 &= 2{\cdot}858{,}4446, & c_2 &= 14{\cdot}398{,}3602. \end{aligned}$$

Even without a knowledge of the probable errors of β and m we can safely say that the differences between these values and those found for the other grouping

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* *Tables for Statisticians and Biometricians*, Table XII.

TABLE I.

Lengths of Intervals in an indefinitely long time, starting with a $\beta-{\cdot}5$ grouping.

Frequency	Lengths of intervals in seconds																					Totals
	β—·5	*·5—1·5*	*1·5—*	*2·5—*	*3·5—*	*4·5—*	*5·5—*	*6·5—*	*7·5—*	*8·5—*	*9·5—*	*10·5—*	*11·5—*	*12·5—*	*13·5—*	*14·5—*	*15·5—*	*16·5—*	*17·5—*	*18·5—*	*19·5—* 	Over *20·5*	
Observed	106	1,217	845	475·5	353·5	221·5	143·5	95·5	65·5	59	29·5	17·5	11	7	8	8	5	1	2	·5	·5	1	3,673
Calculated	186·2	1,154·0	772·1	516·5	345·6	231·2	154·7	103·5	69·2	46·3	31·0	20·7	13·9	9·3	6·2	4·2	2·8	1·9	1·2	·8	·6	1·1	3,673

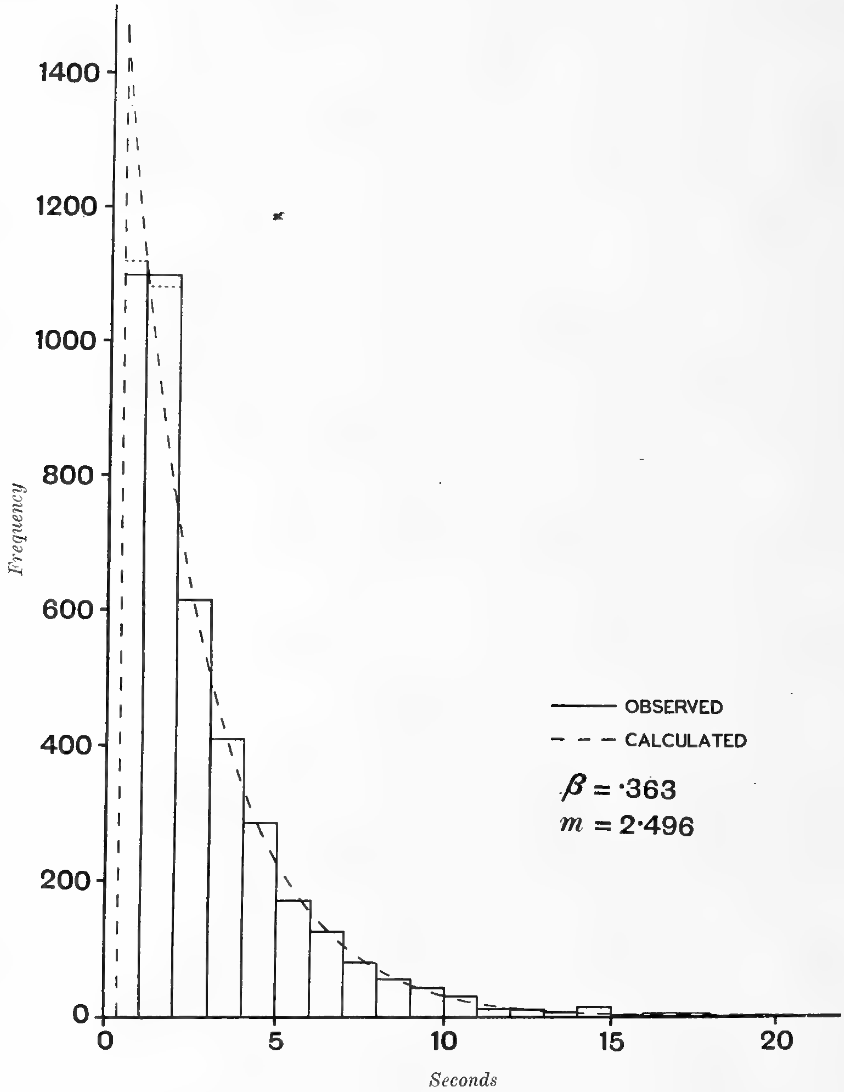
are not significant. The curve is shown fitted to the observed frequencies in the case when the first group is from β to 1 (Fig. I) and the areas are compared in Fig. II. Again there is a marked disagreement between the theoretical and observed frequencies of the first two groups; combining these, as before, gives for goodness of fit $P = {\cdot}193$, so that regrouping the material has not given a better fit. The curve does not fit the observations well, but there can be little doubt that this is due to the conditions of the experiment and not to the form of the curve or the methods of finding β and m. β may have fluctuated considerably throughout the experiment and this would account for the bad fit at the beginning of the range; the longer intervals not being affected.

By finding the constants β and m from a sample only of the whole material we may get some idea of the way in which they were fluctuating. Taking two of the six tapes gave

β	m
·354,020,	2·587,989.

These values differ appreciably from those found from all the data, but as we have at present no knowledge of the probable errors of β and m, it is not possible to say how significant these differences are. It would be more satisfactory if the above law of distribution of random intervals could be justified empirically with material which was more homogeneous and for which the "closed-time" β was more nearly constant.

Fig. I. Lengths of intervals in an indefinitely long time.



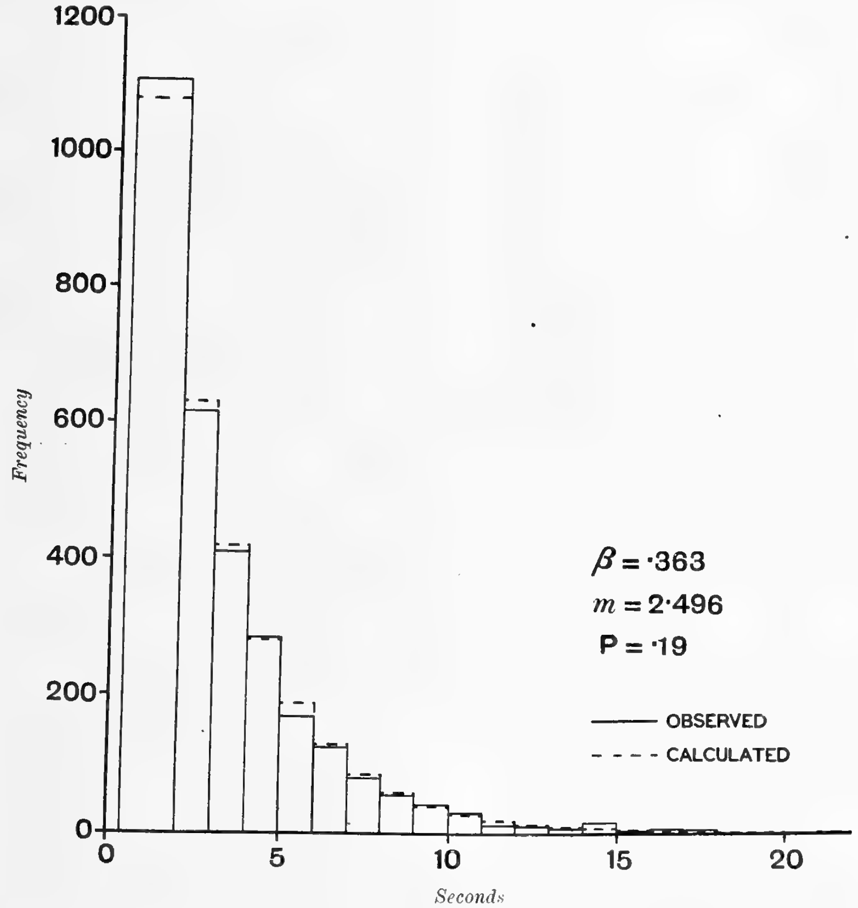
III. *Frequency of Occurrences in limited Periods of Time.*

The second question to be considered is that of the frequency of occurrences of the event in a limited period T. From a practical point of view this is, perhaps, a more important one than the preceding.

(*a*) It will be shown that the hypothesis has to be made that there was no occurrence within the "closed-time" β before the beginning of the period T, so that the material used is only a portion of that which can be used when considering events occurring in an indefinitely long time. Now, in the mathematical theory

m and β are rigidly constant quantities, but in actual practice they fluctuate slightly during the experiments or observations, and also the material is probably subject to fairly large errors of random sampling. Thus it by no means follows that the values of β and m found from the whole of the data will agree very closely with those obtained in another phase of the investigations when only a portion of the data can be used. It is therefore very essential to obtain m and β from each phase of the work independently, and to test the accordance of the values thus obtained. If the material given refer only to a limited period, the moment method of finding β and m, from the frequency of intervals in an indefinitely long time, cannot be used.

Fig. II. Lengths of intervals in an indefinitely long time.



We have to investigate the probability of n occurrences in a limited time T on the hypothesis that there was not an occurrence within the "closed-time" β before the beginning of the period T. Let the times of occurrence be $t_1, t_2 \dots t_n$ from the beginning of the period and let $dt_1, dt_2 \dots dt_n$ be indefinitely small intervals round

$t_1, t_2 \dots t_n$. Let $f(\beta)$ be the chance of no occurrence happening during the interval β before the start of the period T. Then the chance of the observed event is

$$\delta C = f(\beta)\, e^{-\frac{t_1}{m}} \frac{dt_1}{m} e^{-\frac{(t_2-t_1-\beta)}{m}} \frac{dt_2}{m} e^{-\frac{(t_3-t_2-\beta)}{m}} \frac{dt_3}{m} \dots \frac{dt_n}{m}\, q(t_n),$$

where $q(t_n)$ is the chance of no event occurring between t_n and T. If $t_n + \beta < T$, this is $e^{-\frac{(T-t_n-\beta)}{m}}$, but if $t_n + \beta \geqslant T$, this is certainty, for the time between t_n and T will be closed. Accordingly there is discontinuity in the form of the function $q(t_n)$ and we leave it for the moment as $q(t_n)$. We have therefore for the chance

$$\delta C = f(\beta) \frac{e^{+\frac{(n-1)\beta}{m}}}{m^n}\, dt_1 dt_2 \dots dt_n q(t_n)\, e^{-\frac{t_n}{m}}.$$

We have now to integrate this expression for all possible values of the t's. The limits of t_n are clearly $t_{n-1} + \beta$ to T and our first integral

$$= \int_{t_{n-1}+\beta}^{T} dt_n q(t_n) = \int_{t_{n-1}+\beta}^{T-\beta} q(t_n)\, e^{-\frac{t_n}{m}} dt_n + \int_{T-\beta}^{T} e^{-\frac{t_n}{m}} \times 1 \times dt_n$$

$$= \int_{t_{n-1}+\beta}^{T-\beta} e^{-\frac{(T-\beta)}{m}}\, dt_n + m\left(e^{-\frac{(T-\beta)}{m}} - e^{-\frac{T}{m}}\right)$$

$$= e^{-\frac{(T-\beta)}{m}} \left\{(T - t_{n-1} - 2\beta) + m\left(1 - e^{-\frac{\beta}{m}}\right)\right\}.$$

Having thus freed ourselves from t_n our chance now becomes

$$\delta C' = f(\beta) \frac{e^{-\frac{(T-n\beta)}{m}}}{m^n}\, dt_1 dt_2 \dots dt_{n-1} \left\{(T - t_{n-1} - 2\beta) + m\left(1 - e^{-\frac{\beta}{m}}\right)\right\}.$$

Now t_{n-1} must be integrated from $t_{n-2} + \beta$ up to β of the final position of t_n; this was $(T - \beta)$ for the first part of the integral of t_n, i.e. $(T - t_{n-1} - 2\beta)$, and T for the second part, i.e. $m\left(1 - e^{\frac{\beta}{m}}\right)$.

Thus our δt_{n-1} integral is

$$\int_{t_{n-2}+\beta}^{T-2\beta} (T - t_{n-1} - 2\beta)\, dt_{n-1} + m\left(1 - e^{-\frac{\beta}{m}}\right) \int_{t_{n-2}+\beta}^{T-\beta} dt_{n-1}$$

$$= \frac{1}{2\,!}(T - t_{n-2} - 3\beta)^2 + m\left(1 - e^{-\frac{\beta}{m}}\right)(T - t_{n-2} - 2\beta).$$

Continuing this process we reach finally

$$C = f(\beta) \left\{\frac{e^{-\frac{(T-n\beta)}{m}}}{n\,!} \left(\frac{T - n\beta}{m}\right)^n + \left(1 - e^{-\frac{\beta}{m}}\right) \frac{e^{-\frac{(T-n\beta)}{m}}}{(n-1)\,!} \left(\frac{T-(n-1)\beta}{m}\right)^{n-1}\right\} \quad \dots\dots(\mathrm{x}).$$

Now it is clear that this is a very complicated expression for the frequency of n occurrences and will be hard to deal with if m and β are as a rule unknown and have to be found from the observations. The expression will be still more complicated if we introduce the frequency of n intervals in T, when an event has occurred in the interval β before T starts. For this case we shall have to replace $f(\beta)$ by $1-f(\beta)$, multiply by $\frac{dT}{m}$ and integrate from $T-\beta$ to T, or this chance will be

$$C''=\{1-f(\beta)\}\left\{\int_{T-\beta}^{T}\left[\frac{e^{-\frac{(T-n\beta)}{m}}}{n\,!}\left(\frac{T-n\beta}{m}\right)^{n}+\left(1-e^{-\frac{\beta}{m}}\right)\frac{e^{-\frac{(T-n\beta)}{m}}}{(n-1)\,!}\left(\frac{T-(n-1)\beta}{m}\right)^{n-1}\right]dt\right\}$$

$$=\{1-f(\beta)\}\left\{\int_{\frac{T-(n+1)\beta}{m}}^{\frac{T-n\beta}{m}}\frac{e^{-x}x^{n}}{n\,!}\,dx+\int_{\frac{T-n\beta}{m}}^{\frac{T-(n-1)\beta}{m}}\left(e^{\frac{\beta}{m}}-1\right)\frac{e^{-x}x^{n-1}}{(n-1)!}\,dx\right\}\quad\text{.........(xi).}$$

These integrals could be integrated out by parts, but the process would be lengthy.

Now in a long time the mean interval between events is $\beta+m$, and the mean occupied space is β. Hence the chance that we start our epoch at an occupied space is $\frac{\beta}{\beta+m}$ and at an unoccupied space is $\frac{m}{(\beta+m)}=f(\beta)$.

(*b*) The mathematical theory is thus complete but is clearly of too complicated a character for much serviceable application. To test its accuracy experimentally it is desirable to throw out from our observations all sets of T in which an event occurs within β before the start of T or within β before the end of T. If β be relatively small both $1-f(\beta)$ and $1-e^{-\frac{\beta}{m}}$ will be small and this will not involve the rejection of a large number of T-periods.

When this is done, our formula reduces to

$$C=\frac{m}{\beta+m}\,\frac{e^{-\frac{(T-n\beta)}{m}}}{n\,!}\left(\frac{T-n\beta}{m}\right)^{n}.$$

The curve of frequency of periods T with n occurrences will accordingly be of the form

$$f_n=\frac{N\dfrac{e^{-\frac{(T-n\beta)}{m}}}{n\,!}\left(\dfrac{T-n\beta}{m}\right)^{n}}{\overset{n_0}{\underset{0}{S}}\dfrac{e^{-\frac{(T-n\beta)}{m}}}{n\,!}\left(\dfrac{T-n\beta}{m}\right)^{n}}\quad\text{..................................(xii),}$$

where $N=$ total number of periods of length T and $n_0=$ largest integer in $\frac{T}{\beta}$.

To simplify this further we ought to find a finite value for the series

$$\overset{n_0}{\underset{0}{S}}\frac{(a-nb)^{n}}{n\,!},\ \text{where } n_0=\text{integer part of }\frac{a}{b}.$$

There appears to be no discussion of such a series, nor any likelihood of its being expressible in a simple form.

To find the β and m we proceed as follows:

$$f_0 = Ne^{-\frac{T}{m}}, \text{ and write } a = \frac{T}{m} e^{\frac{\beta}{m}}, \quad b = \frac{\beta}{m} e^{\frac{\beta}{m}}.$$

Accordingly
$$\frac{f_n}{f_0} = \frac{(a - nb)^n}{n\,!},$$

and
$$\chi_n = \left(\frac{n\,! f_n}{f_0}\right)^{\frac{1}{n}} = a - nb.$$

The weight w_n of χ_n will be inversely as the square of its standard deviation. But

$$\frac{\delta\chi_n}{\chi_n} = \frac{1}{n}\left(\frac{\delta f_n}{f_n} - \frac{\delta f_0}{f_0}\right),$$

and f_n and f_0 being frequency groups,

$$\sigma^2_{f_n} = f_n\left(1 - \frac{f_n}{N}\right), \quad \sigma^2_{f_0} = f_0\left(1 - \frac{f_0}{N}\right),$$

$$\text{Mean } (\delta f_n \delta f_0) = -\frac{f_n f_0}{N}.$$

Thus approximately

$$\frac{\sigma^2_{\chi_n}}{\chi_n^2} = \frac{1}{n^2}\left\{\frac{1}{f_n}\left(1 - \frac{f_n}{N}\right) + \frac{1}{f_0}\left(1 - \frac{f_0}{N}\right) + \frac{2f_n f_0}{N} \cdot \frac{1}{f_n f_0}\right\}$$
$$= \frac{1}{n^2}\left(\frac{1}{f_n} + \frac{1}{f_0}\right) \quad \text{...(xiii).}$$

Accordingly we may take

$$w_n = \frac{n^2}{\chi_n^2} \frac{f_0 f_n}{(f_0 + f_n)} \quad \text{..............................(xiv).}$$

To find the best values of a and b we have thus to make

$$u^2 = \overset{n_0}{\underset{0}{S}} (a - nb - \chi_n)^2 w_n \text{ a minimum,}$$

or we reach

$$\left.\begin{aligned} a\overset{n_0}{\underset{0}{S}}(w_n) - b\overset{n_0}{\underset{0}{S}}(nw_n) &= S(\chi_n w_n) \\ a\overset{n_0}{\underset{0}{S}}(nw_n) - b\overset{n_0}{\underset{0}{S}}(n^2 w_n) &= S(n\chi_n w_n) \end{aligned}\right\} \quad \text{.....................(xv).}$$

These equations have not been found too laborious in practice. When a and b are determined, we have

$$\beta = \frac{b}{a} T \text{.......................................(xvi)}$$

and then if $z = \dfrac{\beta}{m}$, we must find z and eventually m from

$$b = ze^z \quad \text{...................................(xvii).}$$

(*c*) The above method of finding β and m cannot be used when the observed frequency f_0 is zero, and we should not expect it to give good results if f_0 were small. A more general method is required to cover these cases.

We have, by (xii),

$$f_n = C\frac{e^{-\frac{(T-n\beta)}{m}}}{n\,!}\left(\frac{T-n\beta}{m}\right)^n,$$

where C is a constant.

Accordingly

$$\frac{f_n^2}{f_{n-1}f_{n+1}} = \frac{n+1}{n}\,\frac{(T-n\beta)^{2n}}{(T-(n-1)\,\beta)^{n-1}\,(T-(n+1)\,\beta)^{n+1}},$$

or

$$\chi_n' = \log_e\left(\frac{f_n^2 n}{f_{n-1}f_{n+1}\,(n+1)}\right)$$

$$= 2n\log_e(T-n\beta)-(n-1)\log_e(T-(n-1)\,\beta)-(n+1)\log_e(T-(n+1)\,\beta).$$

Taking the weight w_n' of χ_n' to be inversely as the square of its standard deviation,

$$\delta\chi_n' = \frac{2\delta f_n}{f_n}-\frac{\delta f_{n+1}}{f_{n+1}}-\frac{\delta f_{n-1}}{f_{n-1}}.$$

Then

$$\sigma^2_{\chi_n} = \frac{4}{f_n^2}\,\sigma^2_{f_n}+\frac{\sigma^2_{f_{n+1}}}{f^2_{n+1}}+\frac{\sigma^2_{f_{n-1}}}{f^2_{n-1}}-\frac{4\,[\delta f_n\,\delta f_{n+1}]}{f_n f_{n+1}}-\frac{4\,[\delta f_n\,\delta f_{n-1}]}{f_n f_{n-1}}+\frac{2\,[\delta f_{n+1}\,\delta f_{n-1}]}{f_{n+1}f_{n-1}},$$

the square brackets denoting mean values.

Thus approximately

$$\sigma^2_{\chi_n'} = \frac{4}{f_n}\left(1-\frac{f_n}{N}\right)+\frac{1}{f_{n+1}}\left(1-\frac{f_{n+1}}{N}\right)+\frac{1}{f_{n-1}}\left(1-\frac{f_{n-1}}{N}\right)+\frac{4}{N}+\frac{4}{N}-\frac{2}{N}$$

$$= \frac{4}{f_n}+\frac{1}{f_{n+1}}+\frac{1}{f_{n-1}},$$

or

$$w_n' = \frac{f_n\,f_{n+1}\,f_{n-1}}{f_n f_{n-1}+f_n f_{n+1}+4f_{n-1}f_{n+1}} \qquad \text{.....................(xviii).}$$

To find the best value of β we have to make

$$u^2 = \overset{n_0}{\underset{1}{S}}\,w_n'\,\{2n\log_e(T-n\beta)-(n-1)\log_e(T-(n-1)\,\beta)$$
$$-(n+1)\log_e(T-(n+1)\,\beta)-\chi_n'\}^2$$

a minimum, giving

$$0 = \overset{n_0}{\underset{1}{S}}\Bigg\{w_n'\,[2\log_e(T-n\beta)-(n-1)\log_e(T-(n-1)\,\beta)$$
$$-(n+1)\log_e(T-(n+1)\,\beta)-\chi_n']$$
$$\times\left(\frac{2n^2}{T-n\beta}-\frac{(n-1)^2}{T-(n-1)\,\beta}-\frac{(n+1)^2}{T-(n+1)\,\beta}\right)\Bigg\} \qquad \text{......(xix).}$$

β can be found from this equation by making successive approximations, but the process is a lengthy one. Logarithms can be taken to the base 10 since this change will not affect the relative weights.

We have next to find m when β is known.

We have

$$\psi_n = \log_{10}\frac{f_n}{f_{n-1}} = \frac{\beta}{m}\log_{10} e - \log_{10} n - \log_{10} m + n\log_{10}(T - n\beta)$$
$$- (n-1)\log_{10}(T - (n-1)\beta).$$

Proceeding as before, the weight w_n'' of ψ_n will be $\frac{f_n . f_{n-1}}{f_n + f_{n-1}}$ and for the best value of m,

$$u^2 = \overset{n_0}{\underset{1}{S}} w_n'' \left\{\frac{\beta}{m}\log_{10} e - \log_{10} n - \log_{10} m + n\log_{10}(T - n\beta)\right.$$
$$\left. - (n-1)\log_{10}(T - (n-1)\beta) - \psi_n\right\}^2$$

has to be made a minimum. Thus we reach

$$0 = \overset{n_0}{\underset{1}{S}} \left\{ w_n'' \left(\frac{\beta}{m}\log_{10} e - \log_{10} n - \log_{10} m + n\log_{10}(T - n\beta)\right.\right.$$
$$\left. - (n-1)\log_{10}(T - (n-1)\beta) - \psi_n\right)$$
$$\left.\times\left(\frac{\beta\log_{10} e}{m} + \frac{1}{\log_e 10 . m}\right)\right\} \text{......(xx)},$$

from which m can be found since β is known.

(*d*) We have not succeeded in finding the mean number of occurrences or the standard deviation of occurrences, when they are limited terminally as above described.

Corollary. If we take $\beta = 0, f(\beta) = 1$, then our formula reduces to

$$f_n = N\frac{e^{-\frac{T}{m}}}{n!}\left(\frac{T}{m}\right)^n \text{ ....(xxi)},$$

the well-known Poisson expansion, which gives the frequency of n occurrences in time T.

(*e*) We come now to the application of the above to the experimental data. The tapes were divided into 10-second periods and all those were rejected which were either preceded by an occurrence in time less than β, or in which there was an occurrence within β of the end of the period. But at this stage the value of β which the sample being dealt with would give was, of course, not known. That found from the distribution of lengths of intervals in an indefinitely long time was used, since the difference between the two would certainly be small, but usually in practice such a close approximation would not be available.

The constants β and m have first to be found. Of a total of 931 periods there were 22 in which there was no occurrence of the event, so the $\chi_n = \left(\frac{n! f_n}{f_0}\right)^{\frac{1}{n}}$ method could be safely used. The values of χ_n calculated for all possible values of n and weighted by (xiv) lead to the two equations (xv) and hence

$$a = 4{\cdot}444{,}441, \quad b = {\cdot}157{,}272.$$

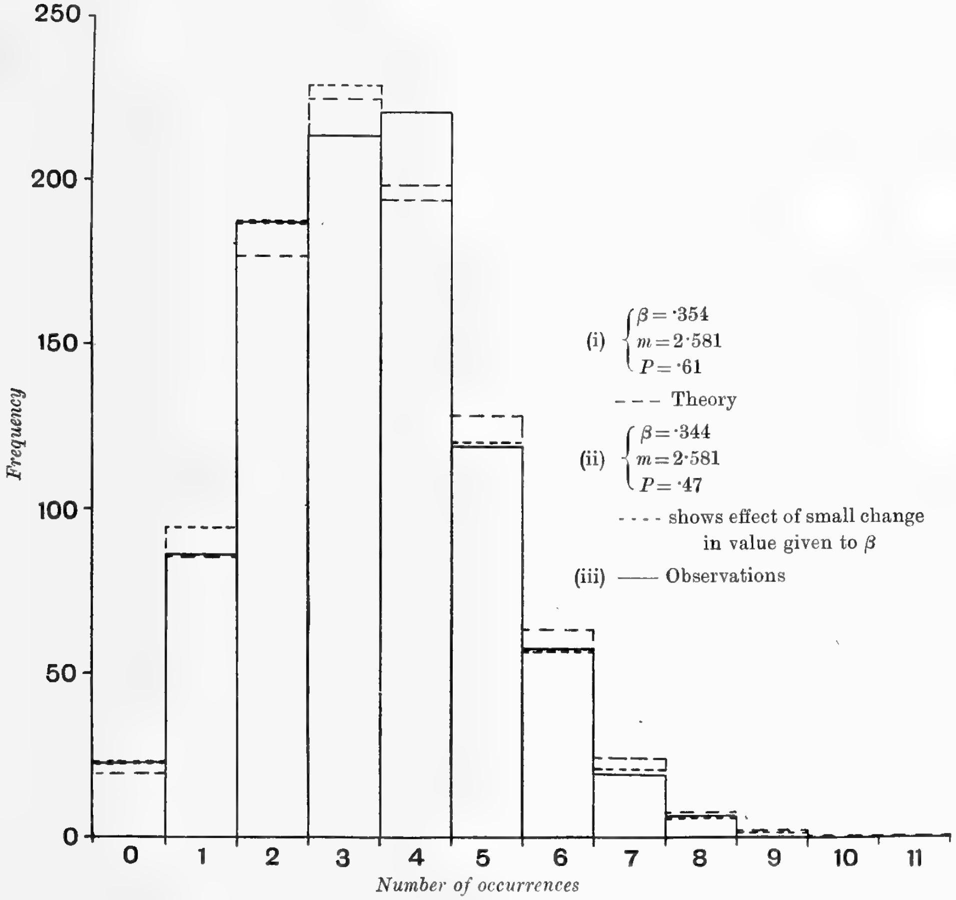
These by (xvi) give $\beta = \cdot353{,}862$ and equation (xvii) is

$$\bar{1}\cdot196{,}6514 = \log z + (z \times \cdot434{,}2945),$$

where $z = \frac{\beta}{m}$, and hence $m = 2\cdot580{,}641$.

These values of β and m differ appreciably from those found from the whole material; this possibly being due to the fact that the selected portion was not a random sample. The frequency of periods T containing n occurrences is given by (xii) and Fig. III shows that the theoretical fits the observed distribution quite satisfactorily. The P for goodness of fit is ·614.

Fig. III. Number of occurrences in periods of 10 seconds.



The alternative $\chi_n' = \log\left(\frac{f_n^2}{f_{n-1}f_{n+1}}\frac{n}{n+1}\right)$ method gave

$$\beta = \cdot393{,}772, \quad m = 2\cdot495{,}831,$$

and the following distribution.

For goodness of fit this table gives $P = \cdot80$ which is a considerable improvement on the previous value found ($P = \cdot61$), but the first method is very much less

TABLE II.

Frequency of Periods of 10 *seconds with n occurrences.*

Frequency	$n=0$	1	2	3	4	5	6	7	8	9 and over	Totals
Observed	22·0	86·0	187·0	214·0	221·0	119·0	57·0	19·0	6·0	0·0	931·0
Calculated	19·5	87·7	189·3	229·6	197·8	123·1	57·2	20·2	5·5	1·4	931·3

laborious than the second and it will generally be quite accurate enough. Using the values of β and m found from the frequency of intervals in an indefinitely long time with the β to ·5 grouping (i.e. $\beta = {}$·370,552, $m = 2{\cdot}488{,}007$) to give the frequency of periods of 10 secs. with n occurrences, gave for goodness of fit to the observed distribution $P = {}$·67, which shows that it may not be always necessary in practice to recalculate the constants when they have been found for one phase of the work. But it will always be necessary to do so unless, as in the case considered, β is small relative to the length of the period taken, and there are thus few periods having an occurrence within β of their beginning or end.

IV. *Lengths of Intervals in limited Periods of Time.*

We now pass to the consideration of the number of intervals which will occur in a limited period of time T, and also to the distribution of the lengths of intervals within that period when there are, say, n intervals in the period. In this case one occurrence and no occurrences will not give an interval. If $n' =$ number of intervals, then they involve $n'+1$ occurrences, and if $F_{n'}$ be the frequency of n' intervals, we shall have $F_{n'}$ proportional to

$$\frac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\frac{T-(n'+1)\beta}{m}\right)^{n'+1},$$

or for the frequency of intervals N',

$$F'_{n'} = \frac{N'\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}}{\mathop{S}\limits_{1}^{n_0'}\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}} \quad \text{...............(xxii)},$$

where $n_0' =$ integer part of $\dfrac{T}{\beta} - 1$.

If $N''=$ total frequency when we count the periods T in which there are *no* intervals, then since no intervals may arise from 0 or 1 occurrence, we shall have

$$F_0' = \frac{N''e^{-\frac{T}{m}}\left(1 + e^{\frac{\beta}{m}}\dfrac{T-\beta}{m}\right)}{\mathop{S}\limits_{-1}^{n_0'}\dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}} \quad \text{...............(xxiii)},$$

while

$$F'_{n'} = \frac{N'' \dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}}{\overset{n_0'}{\underset{-1}{S}} \dfrac{e^{-\frac{(T-(n'+1)\beta)}{m}}}{(n'+1)!}\left(\dfrac{T-(n'+1)\beta}{m}\right)^{n'+1}} \ldots\ldots\ldots\ldots (\text{xxiii})^{\text{bis}}.$$

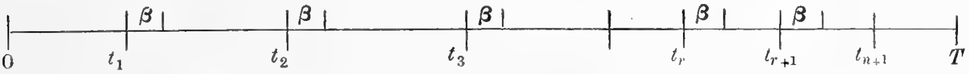
If we are only given F_0' as a whole, and the several $F'_{n'}$, it is difficult to see any ready solution of the problem. If we are given the part F_0'' of F_0' which covers no occurrences in T we can proceed exactly as in the previous problem, writing

$$\chi'_{n'} = \left\{\frac{F'_{n'}(n'+1)!}{F_0''}\right\}^{\frac{1}{n'+1}} = a - (n'+1)\,b,$$

and the solution proceeds just as before. If we are given F_0''', the number of cases of a single occurrence, but not F_0'', the equation seems as intractable as when we only know F_0' as a whole. Yet this seems a not impossible case in practice, namely one in which there would be no record unless at least one occurrence happened.

We shall now determine the frequency distribution for intervals of size t occurring in the period T which we will suppose to contain n' intervals.

We enquire first what is the chance that the rth interval of these n' intervals is of magnitude t. We shall again exclude the possibility of an occurrence having happened at an interval less than β before the start of T.



Then, as before, the probability of the system being as in the diagram above is

$$f(\beta)\, e^{-\frac{t_1}{m}} \frac{dt_1}{m} e^{-\frac{(t_2-t_1-\beta)}{m}} \frac{dt_2}{m} e^{-\frac{(t_3-t_2-\beta)}{m}} \frac{dt_3}{m} \ldots \frac{dt_r}{m} e^{-\frac{(t_{r+1}-t_r-\beta)}{m}} \frac{dt_{r+1}}{m} \ldots$$

$$\ldots e^{-\frac{(t_{n+1}-t_n-\beta)}{m}} \frac{dt_{n+1}}{m} q(t_{n+1}).$$

Here $t_{r+1} - t_r = t$ and is to remain constant while we integrate for all the $t_1 \ldots t_{n+1}$ but t_{r+1}. Clearly the above probability can be written

$$f(\beta)\, \frac{1}{m^{n+1}} e^{-\frac{(T-n\beta)}{m}} dt_1 dt_2 \ldots dt_{n+1}\, e^{-\frac{(t_{n+1}-T)}{m}} q(t_{n+1}).$$

We first integrate t_{n+1} from $t_{n+\beta}$ to T, i.e. we determine

$$\int_{t_n+\beta}^{T} e^{-\frac{(t_{n+1}-T)}{m}} q(t_{n+1})\, dt_{n+1}$$

$$= \int_{t_n+\beta}^{T-\beta} e^{-\frac{(t_{n+1}-T)}{m}} e^{-\frac{(T-t_{n+1}-\beta)}{m}} dt_{n+1} + \int_{T-\beta}^{T} e^{-\frac{(t_{n+1}-T)}{m}} dt_{n+1}$$

$$= e^{\frac{\beta}{m}} (T - t_n - 2\beta) + m\left(e^{\frac{\beta}{m}} - 1\right).$$

Thus as the result of our first integration we have

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\dots dt_n\left\{(T-t_n-2\beta)+m\left(1-e^{-\frac{\beta}{m}}\right)\right\}.$$

We have next to integrate t_n from $t_{n-1}+\beta$ up to $t_{n+1}-\beta$, where t_{n+1} is given in the first part of the subject of integration the value $T-\beta$ and in the second T. Accordingly the integration leads to

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\dots dt_{n-1}\left\{\frac{(T-t_{n-1}-3\beta)^2}{2!}+m\left(1-e^{-\frac{\beta}{m}}\right)(T-t_{n-1}-2\beta)\right\}.$$

If we continue this process up to dt_{r+1} we have

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\dots dt_r dt_{r+1}\left\{\frac{[T-t_{r+1}-(n-r+1)\beta]^{n-r}}{(n-r)!}\right.$$
$$\left.+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{[T-t_{r+1}-(n-r)\beta]^{n-r-1}}{(n-r-1)!}\right\}.$$

Now we do not integrate with regard to dt_{r+1} but put $t_{r+1}=t+t_r$ and proceed to integrate with regard to t_r from $t_r=t_{r-1}+\beta$ to $T-t-(n+r+1)\beta$ in the first subject of integration and to $T-t-(n-r)\beta$ in the second subject. We have accordingly

$$f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\,dt_1dt_2\dots dt_{r-1}\left\{\frac{[T-t_{r-1}-t-(n-r+2)\beta]^{n-r+1}}{(n-r+1)!}\right.$$
$$\left.+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{[T-t_{r-1}-t-(n-r+1)\beta]^{n-r}}{(n-r)!}\right\}dt_{r+1}.$$

Continuing this process we have finally after writing dt for dt_{r+1}

$$C_t=f(\beta)\frac{e^{-\frac{(T-n(+1)\beta)}{m}}}{m^{n+1}}\left\{\frac{(T-t-n\beta)^n}{n!}+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{(T-t-(n-1)\beta)^{n-1}}{(n-1)!}\right\}dt$$
$$\text{......(xxiv)}$$

for the chance of an interval between t and $t+dt$. Now if we integrate this from $t=\beta$ to $t=T-n\beta$ for the first subject and to $t=T-(n-1)\beta$ for the second subject we have

$$\int C_t\,dt=f(\beta)\frac{e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}}\left\{\frac{(T-(n+1)\beta)^{n+1}}{(n+1)!}+m\left(1-e^{-\frac{\beta}{m}}\right)\frac{(T-n\beta)^n}{n!}\right\},$$

which agrees with (x), if we remember that n intervals signify $n+1$ occurrences. In precisely the same way we get a form like (xi) (with n changed to $n+1$) by integrating with regard to T from $T-\beta$ to T and replacing $f(\beta)$ by $1-f(\beta)$. The result expresses the chance of any interval t occurring in the series of intervals when the period starts somewhere in a closed interval β. The great complexity of the resulting formula compels us again to simplify matters if β and

m are not *a priori* known by selecting those intervals only in which no event occurs in the period β before T starts or in the period β before the end of T.

In this case the chance C_t of a definite interval in an n interval period being between t and $t+dt$ is

$$C_t = f(\beta)\, e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}\, n\,!}\, dt \ldots\ldots\ldots\ldots\ldots\ldots\text{(xxv)}.$$

This chance is clearly the same whether the given interval occurs between the rth and $r+1$th events or between any other pair of events. It therefore represents the chance of an interval of the required length being in any position. But if we are going to tabulate the frequency of intervals there will be n cases in which such a length of interval could occur, and we shall have for our frequency surface

$$f_{n,t} \text{ varies as } e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)\,!}.$$

To obtain the constant of variation, C, we must find

$$N' = \mathop{S}\limits_{1}^{n_0} \int_{\beta}^{T-n\beta} f_{n,t}\, dt = \mathop{S}\limits_{1}^{n_0} \left\{ C \times e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\},$$

or,

$$C = \frac{N'}{\mathop{S}\limits_{1}^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\}},$$

where N' is the total number of *intervals*.

Thus finally

$$f_{n,t} = \frac{N' e^{-\frac{(T-(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)\,!}}{\mathop{S}\limits_{1}^{n_0} \left\{ e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)\,!}\, n \right\}} \ldots\ldots\ldots\ldots\text{(xxv)}^{\text{bis}}.$$

We proceed now to deduce some consequences from this result.

(i) If $\beta = 0$, $f(\beta) = 1$, and we no longer have difficulties about terminal conditions, and the chance that a given interval shall be of length between t and $t+dt$ in an n-interval period is

$$C_t = e^{-\frac{T}{m}} \frac{(T-t)^n}{m^{n+1}\, n\,!}\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxvi)}.$$

Integrating t from 0 to T

$$\int_0^T C_t dt = e^{-\frac{T}{m}} \frac{T^{n+1}}{m^{n+1}(n+1)\,!}.$$

This is the total chance of n intervals regardless of their size, and summing from $n=1$ to ∞,

$$\mathop{S}\limits_{1}^{\infty} \int_0^T C_t dt = e^{-\frac{T}{m}} \mathop{S}\limits_{1}^{\infty} \left\{ \frac{T^{n+1}}{m^{n+1}(n+1)\,!} \right\} = e^{-\frac{T}{m}} \left(e^{\frac{T}{m}} - 1 - \frac{T}{m} \right)$$

$$= 1 - e^{-\frac{T}{m}} - \frac{T}{m} e^{-\frac{T}{m}}$$

$$= \text{certainty} - \text{chance of } \textit{no} \text{ event} - \text{chance of one event}$$

which it clearly should be.

(ii) Let us return to our fundamental equation (xxv)$^{\text{bis}}$, and let us first find the total frequency in an array corresponding to a given n. This will equal (if λ represent a constant)

$$N_n = \int_\beta^{T-n\beta} f_{n,t}\,dt = \lambda e^{-\frac{(T-(n+1)\beta)}{m}} \frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!}\, n.$$

Next, let $\bar{t}_n$ be the mean interval for the array. This will be given by

$$N_n\bar{t}_n = \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n-1)!}\int_\beta^{T-n\beta} t\,(T-t-n\beta)^n\,dt$$

$$= \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n-1)!}\left\{\frac{\beta(T-(n+1)\beta)^{n+1}}{n+1} + \frac{1}{(n+1)(n+2)}(T-(n+1)\beta)^{n+2}\right\}$$

$$= \frac{\lambda e^{-\frac{(T-(n+1)\beta)}{m}}}{m^{n+1}(n+1)!}\cdot\frac{n(T+\beta)}{n+2}(T-(n+1)\beta)^{n+1}.$$

Dividing out by the value first found for N_n we have

$$\bar{t}_n = \frac{T+\beta}{n+2} \quad\text{..............................(xxvii).}$$

The means of the arrays are therefore points on a rectangular hyperbola of which $t=0$ and $n=-2$ are the asymptotes, and that curve is one of the regression lines of the surface.

Again, if $\sigma_{t,n}$ be the standard deviation of the n array of intervals

$$\sigma^2_{t,n} + \bar{t}_n^2 = \frac{(n+1)\int_\beta^{T-n\beta} t^2(T-t-n\beta)^n}{(T-(n+1)\beta)^{n+1}}$$

$$= \beta^2 + \frac{2\beta}{n+2}(T-(n+1)\beta) + \frac{2(T-(n+1)\beta)^2}{(n+2)(n+3)},$$

whence, substituting for $\bar{t}_n$, we find after some reductions

$$\sigma^2_{t,n} = \frac{(T-(n+1)\beta)^2(n+1)}{(n+2)^2(n+3)} \quad\text{.......... ....(xxviii).}$$

We see accordingly that the system is markedly heteroscedastic, the variability of the array decreasing rapidly with n.

For the special case of $\beta = 0$,

$$\bar{t}_n = \frac{T}{n+2}, \qquad \sigma_{t,n} = \frac{T}{n+2}\sqrt{\frac{n+1}{n+3}} \quad\text{.................(xxix).}$$

Thus the mean and standard deviation of any array tend to equality.

We have further for the mean interval

$$\bar{t} = (T+\beta)\times\frac{\mathop{S}\limits_1^{n_0}\left\{e^{-\frac{(T-(n+1)\beta)}{m}}\frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!}\frac{n}{n+2}\right\}}{\mathop{S}\limits_1^{n_0}\left\{e^{-\frac{(T-(n+1)\beta)}{m}}\frac{(T-(n+1)\beta)^{n+1}}{m^{n+1}(n+1)!}n\right\}} \quad\text{........(xxx).}$$

And doubtless σ_t could be found in a similar if more complicated form; but such expressions do not suggest any great hope of our being able to derive β and m from the mean and standard deviations of the intervals occurring in periods T.

In the same way the equations for the mean $\bar{n}_t$ and the standard deviation σ_t of the array of n's for values of t from t to $t+dt$ do not look very likely to lead anywhere in the case of a finite β, i.e.

$$\bar{n}_t = \frac{\overset{n_0}{\underset{1}{S}} \left\{ e^{\frac{(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!} n \right\}}{\overset{n_0}{\underset{1}{S}} \left\{ e^{\frac{(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!} \right\}} \quad \text{.....................(xxxi)},$$

$$\sigma^2_{t,n} + \bar{n}_t^2 = \frac{\overset{n_0}{\underset{1}{S}} \left\{ e^{\frac{(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!} n^2 \right\}}{\overset{n_0}{\underset{1}{S}} \left\{ e^{\frac{(n+1)\beta}{m}} \frac{(T-t-n\beta)^n}{m^{n+1}(n-1)!} \right\}} \quad \text{.....................(xxxii)}.$$

(iii) All these results, however, admit of comparatively easy development when there is no "closed-time" β.

In this case for the frequency distribution of N intervals

$$f_{n,t} = \frac{\dfrac{N(T-t)^n}{m^{n+1}(n-1)!}}{\left\{1 + e^{\frac{T}{m}}\left(\dfrac{T}{m} - 1\right)\right\}} \quad \text{.......................(xxxiii)}$$

$$= \frac{\lambda'(T-t)^n}{m^{n+1}(n-1)!}, \text{ say.}$$

We have already seen what $\bar{t}_n$ and $\sigma_{t,n}$ are. We will now find $\bar{n}_t$ and $\sigma_{n,t}$.

$$\overset{\infty}{\underset{1}{S}} \left\{ \frac{\lambda'}{m} \frac{(T-t)^n}{m^n} \frac{dt}{(n-1)!} \right\} = \frac{(T-t)\lambda'}{m^2} \overset{\infty}{\underset{1}{S}} \left\{ \frac{(T-t)^{n-1}}{(n-1)!} \frac{dt}{m^{n-1}} \right\}.$$

Or, $$N_t = \frac{\lambda'(T-t)}{m^2} dt\, e^{-\frac{T-t}{m}} *,$$

$$\bar{n}_t N_t = \lambda' \overset{\infty}{\underset{1}{S}} \left\{ \frac{n(T-t)^n}{m^{n+1}(n-1)!} \right\} dt$$

$$= \lambda' dt \left[\frac{(T-t)^2}{m^3} \overset{\infty}{\underset{2}{S}} \left\{ \frac{(T-t)^{n-2}}{(n-2)!\, m^{n-2}} \right\} + \frac{(T-t)}{m^2} \overset{\infty}{\underset{1}{S}} \left\{ \frac{(T-t)^{n-1}}{(n-1)!\, m^{n-1}} \right\} \right]$$

$$= \lambda' \left[\frac{(T-t)^2}{m^3} e^{\frac{T-t}{m}} + \frac{(T-t)}{m^2} e^{\frac{T-t}{m}} \right] dt.$$

---

* This shows us that the frequency curve for intervals t is

$$y = \frac{\dfrac{T-t}{m^2} e^{\frac{T-t}{m}}}{1 + e^{\frac{T}{m}}\left(\dfrac{T}{m} - 1\right)} \quad \text{.................................(xxxiii)}^{\text{bis}}.$$

If we make T infinite, this reduces to the well-known form $y = \frac{1}{m} e^{-\frac{t}{m}}$. We have determined its mean and standard deviation in the text.

Thus $$\bar{n}_t = \frac{T-t}{m} + 1 \quad\text{(xxxiv).}$$

There should be a direct proof of this since $\left(\frac{T-t}{m}\right)$ is clearly the mean number of occurrences in the interval $(T-t)$, or we ought to be able to show that the mean number of occurrences is one *less* than the mean number of intervals*.

The regression of n on t is thus seen to be linear, while that of t on n is hyperbolic.

Again,

$$N_t(\sigma^2_{n,t} + \bar{n}_t^2) = \lambda' \mathop{S}\limits_1^\infty \left\{\frac{n^2(T-t)^n}{m^{n+1}(n-1)!}\right\} dt$$

$$= \lambda' \left\{\frac{(T-t)^3}{m^4} \mathop{S}\limits_3^\infty \left(\frac{(T-t)^{n-3}}{m^{n-3}(n-3)!}\right) + \frac{3(T-t)^2}{m^3} \mathop{S}\limits_2^\infty \left(\frac{(T-t)^{n-2}}{m^{n-2}(n-2)!}\right) + \frac{(T-t)}{m^2} \mathop{S}\limits_1^\infty \left(\frac{(T-t)^{n-1}}{m^{n-1}(n-1)!}\right)\right\} dt$$

$$= \lambda' \frac{e^{\frac{T-t}{m}}}{m} \left\{\left(\frac{T-t}{m}\right)^3 + 3\left(\frac{T-t}{m}\right)^2 + \frac{T-t}{m}\right\} dt.$$

Accordingly, $$\sigma^2_{n,t} + \bar{n}_t^2 = \left(\frac{T-t}{m}\right)^2 + \frac{3(T-t)}{m} + 1,$$

and $$\sigma_{n,t} = \sqrt{\frac{T-t}{m}} \quad\text{(xxxv).}$$

The arrays are accordingly heteroscedastic.

Clearly, $$N\bar{n} = \lambda' \int_0^T \frac{(T-t)}{m^2}\left(\frac{T-t}{m} + 1\right) e^{+\frac{T-t}{m}} dt,$$

whence integrating and substituting the value for λ' we find

$$\bar{n} = \frac{\dfrac{T^2}{m^2}}{\dfrac{T}{m} - 1 + e^{-\frac{T}{m}}} - 1 \quad\text{(xxxvi).}$$

To find σ_n we have

$$N(\sigma_n^2 + \bar{n}^2) = \mathop{S}\limits_1^\infty \left\{\lambda' \int_0^T \frac{(T-t)^n n^2}{m^{n+1}(n-1)!} dt\right\}$$

$$= \lambda' \mathop{S}\limits_1^\infty \left\{\frac{T^{n+1}}{m^{n+1}} \frac{n^2}{(n+1)(n-1)!}\right\} = \lambda' \mathop{S}\limits_1^\infty \left\{\frac{n^3}{(n+1)!} \frac{T^{n+1}}{m^{n+1}}\right\}$$

$$= \lambda' \left\{\mathop{S}\limits_2^\infty \left(\frac{T^3}{m^3} \frac{T^{n-2}}{m^{n-2}(n-2)!}\right) + \mathop{S}\limits_1^\infty \left(\frac{T}{m} \frac{T^n}{m^n n!}\right) - \mathop{S}\limits_1^\infty \left(\frac{T^{n+1}}{m^{n+1}(n+1)!}\right)\right\}$$

$$= \lambda' \left\{e^{\frac{T}{m}}\left(\frac{T^3}{m^3} + \frac{T}{m} - 1\right) + 1\right\}.$$

---

* The number of intervals is always one less than the number of occurrences, but then there are cases of no or one occurrence (giving no intervals) which affect the mean number of occurrences.

Thus
$$\sigma_n^2 + \bar{n}^2 = \frac{\left(\frac{T}{m}\right)^3 + \frac{T}{m} - 1 + e^{-\frac{T}{m}}}{\frac{T}{m} - 1 + e^{-\frac{T}{m}}}$$
$$= 1 + \frac{\left(\frac{T}{m}\right)^3}{\frac{T}{m} - 1 + e^{-\frac{T}{m}}},$$

whence we deduce
$$\sigma_n = \frac{\frac{T}{m}}{\frac{T}{m} - 1 + e^{-\frac{T}{m}}}\sqrt{\frac{T}{m}\left(1 + e^{-\frac{T}{m}}\right) - 2\left(1 - e^{-\frac{T}{m}}\right)} \quad \text{.........(xxxvii).}$$

We may now deduce $\bar{t}$,
$$N\bar{t} = \lambda' \overset{\infty}{\underset{1}{S}} \left\{\int_0^T \frac{(T-t)^n\, t}{m^{n+1}(n-1)!}\,dt\right\}$$
$$= \lambda' \left\{\overset{\infty}{\underset{1}{S}}\left(\frac{T^{n+2}n}{m^{n+1}(n+1)!}\right) - \overset{\infty}{\underset{1}{S}}\left(\frac{n(n+1)\,T^{n+2}}{m^{n+1}(n+2)!}\right)\right\}$$
$$= \lambda' m \left\{\frac{T^2}{m^2}\overset{\infty}{\underset{1}{S}}\left(\frac{T^n}{m^n\, n!}\right) - \frac{T}{m}\overset{\infty}{\underset{1}{S}}\left(\frac{T^{n+1}}{m^{n+1}(n+1)!}\right) - \frac{T^2}{m^2}\overset{\infty}{\underset{1}{S}}\left(\frac{T^n}{m^n\, n!}\right)\right.$$
$$\left. + 2\frac{T}{m}\overset{\infty}{\underset{1}{S}}\left(\frac{T^{n+1}}{m^{n+1}(n+1)!}\right) - 2\overset{\infty}{\underset{1}{S}}\left(\frac{T^{n+2}}{m^{n+2}(n+2)!}\right)\right\}$$
$$= \lambda' m \left\{\frac{T}{m}\left(e^{\frac{T}{m}} - 1 - \frac{T}{m}\right) - 2\left(e^{\frac{T}{m}} - 1 - \frac{T}{m} - \frac{1}{2}\frac{T^2}{m^2}\right)\right\}$$
$$= \lambda' m \left\{e^{\frac{T}{m}}\left(\frac{T}{m} - 2\right) + \frac{T}{m} + 2\right\}.$$

Accordingly,
$$\bar{t} = m\left\{\frac{e^{\frac{T}{m}}\left(\frac{T}{m} - 2\right) + \left(\frac{T}{m} - 2\right)}{1 + e^{\frac{T}{m}}\left(\frac{T}{m} - 1\right)}\right\} \quad \text{..................(xxxviii).}$$

Similarly,
$$N(\sigma_t^2 + t^2) = \lambda' \left\{\overset{\infty}{\underset{1}{S}}\left(\int_0^T \frac{(T-t)^n\, t^2}{m^{n+1}(n-1)!}\,dt\right)\right\}$$
$$= \lambda' \overset{\infty}{\underset{1}{S}}\left\{\int_0^T \frac{(T-t)^n}{m^{n+1}(n-1)!}\left(T^2 - 2T(T-t)(T-t)^2\right)dt\right\}$$
$$= m^2\lambda' \overset{\infty}{\underset{1}{S}}\left\{\frac{T^{n+3}}{m^{n+3}}\frac{n}{(n+1)!} - \frac{2T^{n+3}}{m^{n+3}}\frac{n(n+1)}{(n+2)!} + \frac{T^{n+3}}{m^{n+3}}\frac{(n+2)(n+1)^n}{(n+3)!}\right\}$$
$$= m^2\lambda' \overset{\infty}{\underset{1}{S}}\left\{\frac{2T^{n+3}}{m^{n+3}}\frac{1}{(n+2)!} - \frac{6T^{n+3}}{m^{n+3}}\frac{1}{(n+3)!}\right\}$$
$$= m^2\lambda' \left\{\frac{2T}{m}\left(e^{\frac{T}{m}} - 1 - \frac{T}{m} - \frac{1}{2}\frac{T^2}{m^2}\right) - 6\left(e^{\frac{T}{m}} - 1 - \frac{T}{m} - \frac{1}{2}\frac{T^2}{m^2} - \frac{1}{6}\frac{T^3}{m^3}\right)\right\},$$

or

$$\sigma_t^2 + \bar{t}^2 = \frac{m^2 \left\{ e^{\frac{T}{m}} \left(\frac{2T}{m} - 6 \right) + \frac{T^2}{m^2} + \frac{4T}{m} + 6 \right\}}{1 + e^{\frac{T}{m}} \left(\frac{T}{m} - 1 \right)} \quad \text{...........(xxxix).}$$

From this we deduce

$$\sigma_t = \frac{m \sqrt{e^{\frac{2T}{m}} \left(\frac{T^2}{m^2} - \frac{4T}{m} + 2 \right) + e^{\frac{T}{m}} \left(\frac{T^3}{m^3} + \frac{T^2}{m^2} + \frac{4T}{m} - 4 \right) + 2}}{1 + e^{\frac{T}{m}} \left(\frac{T}{m} - 1 \right)} \quad \text{......(xl).}$$

Lastly we may consider the correlation of length of intervals and number of intervals. We have just seen that the regression is not linear in the case of t on n, it would therefore be desirable to have $\eta_{t,\,n}$ and $\eta_{n,\,t}$, the two correlation ratios, as well as the correlation coefficient $r_{n,\,t}$.

To determine the latter we must first find

$$\begin{aligned} Np_{11}' &= \overset{\infty}{\underset{1}{S}} \left\{ \int_0^T \lambda' \frac{nt\,(T-t)^n}{m^{n+1}\,(n-1)\,!}\,dt \right\} \\ &= \overset{\infty}{\underset{1}{S}} \left\{ \lambda' \frac{n^2 T^{n+2}}{m^{n+1}\,(n+2)\,!} \right\} \\ &= \overset{\infty}{\underset{1}{S}} \left\{ \lambda' \left(\frac{T^{n+2}}{m^{n+1}\,n\,!} - \frac{3T^{n+2}}{m^{n+1}\,(n+1)\,!} + \frac{4T^{n+2}}{m^{n+1}\,(n+2)\,!} \right) \right\} \\ &= \lambda' m \left\{ \frac{T^2}{m^2} \left(e^{\frac{T}{m}} - 1 \right) - \frac{3T}{m} \left(e^{\frac{T}{m}} - 1 - \frac{T}{m} \right) + 4 \left(e^{\frac{T}{m}} - 1 - \frac{T}{m} - \frac{T^2}{2n^2} \right) \right\}. \end{aligned}$$

Thus

$$p_{11}' = \frac{m \left\{ e^{\frac{T}{m}} \left(\frac{T^2}{m^2} - \frac{3T}{m} + 4 \right) - \frac{T}{m} - 4 \right\}}{1 + e^{\frac{T}{m}} \left(\frac{T}{m} - 1 \right)} \quad \text{..................(xli).}$$

We must now transfer by aid of (xxxvi) and (xxxviii) this product moment to the means. We have

$$p_{11} = - \frac{m \left\{ e^{\frac{2T}{m}} \left(\frac{T^2}{m^2} - \frac{4T}{m} + 2 \right) + e^{\frac{T}{m}} \left(\frac{T^3}{m^3} + \frac{T^2}{m^2} + \frac{4T}{m} - 4 \right) + 2 \right\}}{\left\{ 1 + e^{\frac{T}{m}} \left(\frac{T}{m} - 1 \right) \right\}^2} \quad \text{...(xli)}^{\text{bis}}.$$

Accordingly we have, by aid of (xxxvii) and (xl),

$$r_{t,\,n} = \frac{p_{11}}{\sigma_t \sigma_n} = - \frac{m \sqrt{e^{\frac{2T}{m}} \left(\frac{T^2}{m^2} - \frac{4T}{m} + 2 \right) + e^{\frac{T}{m}} \left(\frac{T^3}{m^3} + \frac{T^2}{m^2} + \frac{4T}{m} - 4 \right) + 2}}{T \sqrt{e^{\frac{2T}{m}} \left(\frac{T}{m} - 2 \right) + e^{\frac{T}{m}} \left(\frac{T}{m} + 2 \right)}}$$

......(xlii).

Clearly the regression coefficient

$$r_{t,n}\frac{\sigma_n}{\sigma_t}=\frac{p_{11}}{\sigma_t^2}=-\frac{1}{m},$$

which agrees with (xxxiv).

It is further easy from (xxxviii) and (xxxvi) to verify that

$$\bar{n}+\frac{\bar{t}}{m}=\frac{T}{m}+1 \quad \text{..............................(xliii)},$$

which completes the checking of (xxxiv).

As the mean n_t's for arrays of t are not hard to find*, it would be fairly easy to deduce m from a correlation table by the slope of this regression line. I have not so far succeeded even in this simple case in finding either correlation ratios in a simple form; but as $\bar{t}_n$ and $\bar{t}$ as well as N_n and σ_t are known theoretically, it would be simple to calculate their theoretical values from the given values of m and T arithmetically, using

$$\eta^2{}_{t,n}=\overset{n_0}{\underset{1}{S}}\frac{(\bar{t}_n-\bar{t})^2 N_t}{N\sigma_t^2}$$

and to compare this value with that found from the observed value of these quantities. Proceeding in the same way we, of course, find

$$\eta_{n,t}=\frac{1}{m}\left\{S\left(\frac{N_t(t-\bar{t})^2}{N\sigma_n^2}\right)\right\}^{\frac{1}{2}}=\frac{1}{m}\frac{\sigma_t}{\sigma_n}=r_{n,t},$$

or the other η is equal to the correlation coefficient, owing to the linearity of the regression.

(iv) We will now apply the process of finding the frequency of intervals in periods T supposing the periods to contain n intervals. The fundamental formula for this is $(\text{xxv})^{\text{bis}}$ which is of the form $f_{n,t}=C_n(T-t-n\beta)^n$, where C_n is a constant dependent on n. The frequencies of lengths of intervals for different values of n are shown in Table III, which is thus a correlation table for the number of intervals in periods of 10 secs. and the lengths of the intervals. No satisfactory method of finding the constants m and β from such a table of observed frequencies has been discovered so we are obliged to use the values found from the frequency of occurrence of the event in period T, i.e. the ratio of frequency method. The P for goodness of fit for the whole table is ·233, but, as in the case of intervals in an indefinitely long time, this low value is due to the bad fit of the first two groups (β to 1 and 1 to 2) for the different values of n. Taking the first group in each case to be β to 2 gives the much improved value $P=\cdot 707$. The P for the right marginal total of Table III (Fig. IV), that being the distribution of lengths of intervals in 10 secs. irrespective of the number of intervals in that period, when the first group is from β to 2, is ·882.

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* The intervals of time, subranges in time, must be small. There is, however, none of the difficulty due to the abruptness and unknown value of β (in the general case) which accompanies the finding of the t_n's.

TABLE III.

$\beta = \cdot354$
$M = 2\cdot581$

$P' = \cdot22$
$P'' = \cdot71$

TABLE OF
LENGTHS OF INTERVALS AND NUMBER OF INTERVALS.

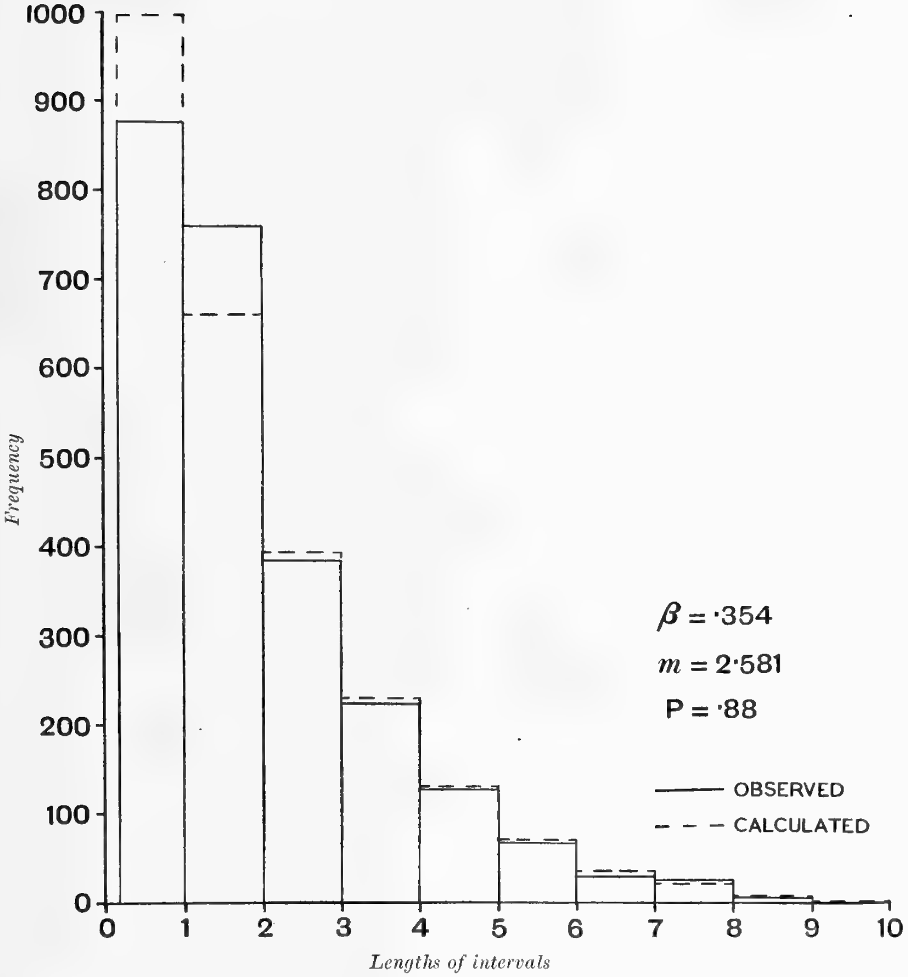
LENGTHS OF INTERVALS	NUMBER OF INTERVALS								TOTALS
	1	2	3	4	5	6	7	8 & OVER	
β - 1	27 25·5	82 93·9	144 158·6	140·5 163·3	103 114·4	48·5 57·8	22 21·8	0 7·9	567 643·2
1 - 2	49 35·9	107·5 118·9	233·5 179·7	182·5 163·9	125·5 100·7	47·5 44·1	16 14·2	0 4·2	761·5 661·6
2 - 3	20 31·5	82 90·4	139·5 116·7	86·5 89·5	39·5 45·3	14 16	4 4·1	0 ·9	385·5 394·4
3 - 4	25·5 27·1	64·5 65·8	75·5 70·5	43·5 43·9	11 17·6	4 4·8	0 ·9	0 ·1	224 230·7
4 - 5	18·5 22·7	45·5 45·1	43·5 38·5	17 18·5	5 5·5	0 1·1	0 ·1		129·5 131·5
5 - 6	20 18·3	23·5 28·3	19 18·1	5·5 6·1	1 1·2	0 ·1			69 72·1
6 - 7	10 13·8	13 15·4	8 6·6	·5 1·5	0 ·1				31·5 37·4
7 - 8	13 9·4	9 6·4	0 1·5						22 17·3
8 - 9	4 5·1	1 1·2							5 6·3
9 & OVER	0 ·6	0 ·2							0 ·8
TOTALS	187 189·9	428 465·6	663 590·2	476 486	285 284·7	114 123·9	42 41·1	0 13·1	**2195** 2195·3

The theoretical mean length of interval for a given n is shown in the following table together with experimental mean found as for the lengths of intervals in an indefinitely long time.

TABLE IV.

Number of Intervals in Periods of 10 seconds (n)	$\frac{T+\beta}{n+2}$	Experimental Means	β from Experimental Means
1	3·4513	3·3191	negative
2	2·5885	2·6219	·4876
3	2·0708	2·0926	·4630
4	1·7256	1·7903	·7418
5	1·4791	1·4240	negative
6	1·2941	1·2733	·1864
7	1·1504	1·0952	negative

Fig. IV. Lengths of intervals in periods of 10 seconds.

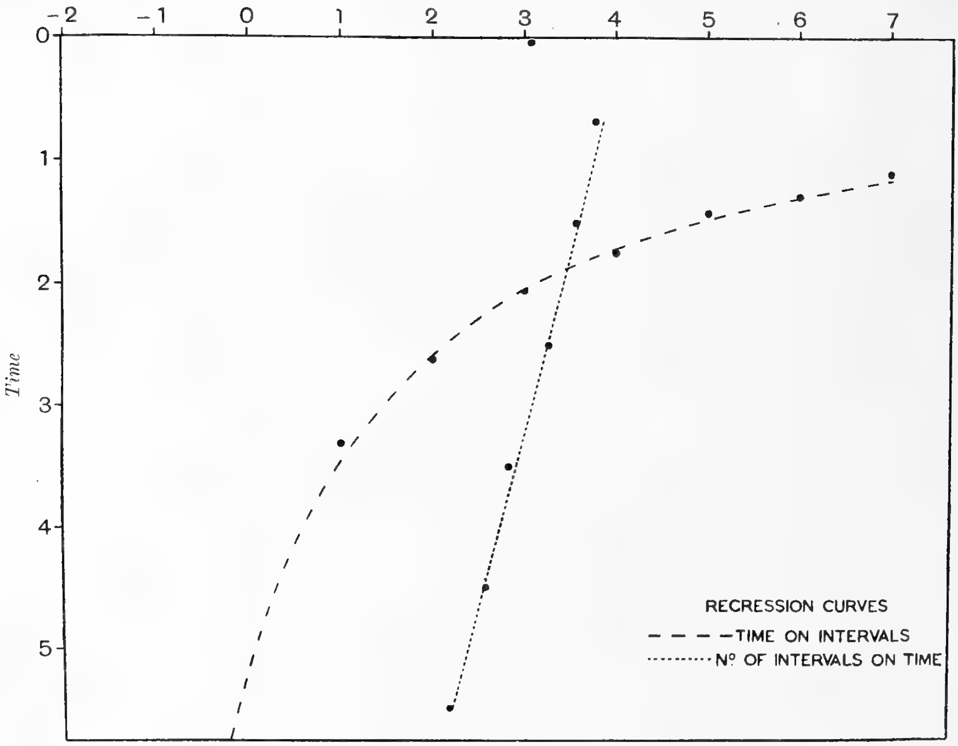


The β's deduced from these latter values are so erratic that it is clear that we cannot hope to get any reliable value from them.

These means are shown plotted in Fig. V, thus providing one regression line of the surface. The other, giving the mean number of intervals for lengths of intervals between t and $(t+dt)$, is calculated theoretically from (xxxi) and it fits the observed means quite satisfactorily; the curve appears to be linear, but we have been unable to deduce a proof even of its approximate linearity beyond the fact shown above that it is linear for $\beta = 0$.

Fig. VI shows the distribution of lengths of intervals in 10 secs. when there are three intervals in that period; one of the theoretical frequencies being that calculated by $(\text{xxv})^{\text{bis}}$ while making the total frequency for $n = 3$ equal to the observed. This method would have to be used if only one column of the table were known.

Fig. V. Number of intervals.



Below are given the values of the constants β and m found by the different methods; it should be remembered that the last two are not found from precisely the same material as the others, only a portion of the whole being used.

Various methods of finding β and m.

A. From duration of intervals in an indefinitely long time.

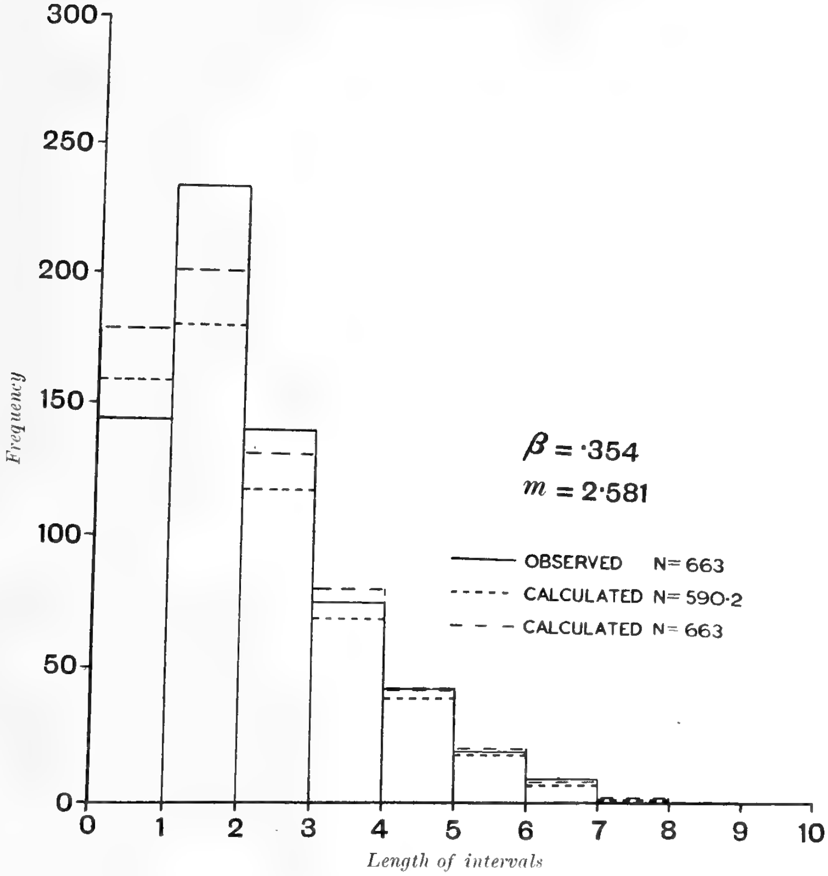
	β	m
(i) First Group β to ·5	·370,552,	2·488,007,
(ii) First Group β to 1	·362,918,	2·495,527.

B. From frequency of occurrences in periods T.

	β	m
(iii) $\chi_n = \left(\frac{n!\,f_n}{f_0}\right)^{\frac{1}{n}}$ method	·353,862,	2·580,641,
(iv) $\chi_n' = \left(\frac{f_n^2}{f_{n-1}f_{n+1}}\frac{n}{n+1}\right)$ method ...	·393,772,	2·495,831.

These values agree moderately well with each other and we may fairly safely predict that although their probable errors are at present unknown, they will be found small enough to make the methods sufficiently reliable for practical purposes. The tests of goodness of fit are reasonably satisfied and the experimental data may

Fig. VI. Lengths of intervals in periods of 10 seconds when there are 3 intervals in each period.



be held to confirm the above theory of random occurrences, if it is supposed that the discrepancies observed in dealing with short intervals in the neighbourhood of β are due to fluctuations throughout the experiment of the least interval between occurrences. More satisfactory evidence of the value of the theory might be given if experimental or observational data could be found for which β was more adequately constant.

I am indebted to Professor Karl Pearson for suggestion and assistance especially in the algebraic portion of this paper and to Miss Ida McLearn for the preparation of the diagrams.

WAS THE SKULL OF THE MORIORI ARTIFICIALLY DEFORMED?

By KARL PEARSON, F.R.S.

Professor V. Giuffrida-Ruggeri, in a recent paper, has criticised the memoir on the Moriori crania published by Eveline Y. Thomson in this Journal[*]. It is, perhaps, needless to say that anything produced by the Biometric School is likely to be anathema to an old-fashioned anthropologist of Professor Giuffrida-Ruggeri's type. He views the more exact calculations on which a biometrician would lay stress in much the same spirit as a biometrician regards the old school anthropologist who divides data of a character obtained from various sources for 19 or 20 crania into three groups and thinks a scientific racial differentiation can be based upon comparing the percentages thus found with those of another and similar grouping! The questions of different methods of measurement, the influence of random sampling, the futility of small samples are not difficulties to this type of anthropologist, they are merely shibboleths of the mathematician. Further he is quite sure the mathematician does not look at anything but his figures, and so overlooks the obvious in his material, which he, the old school anthropologist, sees at first glance—not because he has handled more material, but because he is the Simon pure—the genuine anthropologist. Let us admit for the occasion that the mere mathematician might well handle 65 Moriori crania and not find out that that race was in the habit of deforming its skull! But is it not somewhat strange that other observers with at least some claim to be genuine anthropologists—i.e. old-school anthropologists—should have examined Moriori crania and overlooked this deformation?

Welcker[†] was acquainted with two Moriori crania. He does not refer to any artificial deformation in them, although he does in the case of Peruvian skulls. Flower knew eight Moriori crania and does not refer to their deformation[‡], although he was fully aware of the deformation of New Hebridean skulls.

De Quatrefages and Hamy[§] handled three male crania of the Moriori and figured one. They do not note any deformation in these crania. They are quite conversant, however, with deformed skulls and figure one. Those who will take

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[*] "A Study of the Crania of the Moriori, or Aborigines of the Chatham Islands, now in the Museum of the Royal College of Surgeons," *Biometrika*, Vol. XI. pp. 82—135. The writer of the paper was not a mathematician but a biologist.

[†] *Archiv für Anthropologie*, Bd. I. S. 135, 154, 157, etc. 1866.

[‡] *Journal of the Anthropological Institute*, Vol. XXVI. p. 295, 1896, and *Catalogue of Royal College of Surgeons Museum*, Part I. Man, pp. 128—130.

[§] *Crania ethnica*, *Text*, pp. 460, 461, *Atlas*, Pl. LV. Figs. III and IV.

the trouble to compare the artificial deformed skull on p. 475 of their *Text* with the Moriori skull on Plate LV of their *Atlas* will see at once that the Moriori skull has a naturally receding frontal and not an artificial one; they will see the difference between the post-coronal depression of the Moriori and that of the Peruvian deformed skull. The main difference between Thomson's photograph reproduced (without permission!) by the Italian Professor and Quatrefages and Hamy's lithograph is the tilt in the former produced by adjusting it to the Frankfurt horizontal plane. In the *norma lateralis* of a deformed skull the lowest point of the rim of the orbit joined to the pterion gives a line meeting the posterior part of the sagittal contour in about its most posterior point. In the natural skull it will meet the sagittal contour in the neighbourhood of the apex--i.e. in the *norma verticalis* and not the *norma occipitalis.* See our Plate I.

Fridolin[*] has described and measured one Moriori adult male cranium. He has not referred to any deformation, yet he was fully conscious of this possibility for he figures and describes a New Guinea skull, "wahrscheinlich künstlich deformirt."

Broesike[†] has examined two Moriori crania from the Chatham Islands. Of the second he says nothing as to deformation. Of the first he tells us that it is that of a youth, much weathered and very porous with the sutures gaping. He says—which is not to be wondered at—that the skull appears to be misshaped[‡] ("verunstaltet"). His description of the asymmetrical "Verunstaltung" accords well with the not unusual post-mortem deformation of the thin cranium of a young person. Anyhow, whether this be the fact or not, he does not describe the skull as artificially deformed, and this is the only reference to deformation of Moriori crania that we have come across in the literature of the subject till Professor Giuffrida-Ruggeri's memoir.

Poll, whose training in the *Anatomische-biologisches Institut* in the University of Berlin should at least have taught him to recognise an artificially deformed skull, had ten male and three female Moriori, with two skulls indeterminable as to sex and five children's crania to work on. He gives almost a page of minute qualitative description to each skull, but he describes none as being "künstlich deformirt." Nor does he tacitly reject lengths or indices which would be meaningless if his crania had been artificially deformed. He discusses them all as racial characters[§].

Duckworth, who is not only an anatomist by profession but Lecturer on Physical Anthropology at Cambridge, reported in 1900 on ten Moriori crania at Cambridge[||], but he did more, he examined very carefully for anomalies the 65 Moriori crania at the Royal College of Surgeons afterwards studied by Thomson.

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[*] *Archiv für Anthropologie,* Bd. XXVI. S. 696 and Tafel XIII. 1900.

[†] *Die Anthropologischen Sammlungen Deutschlands,* Theil Berlin, Erster Theil, S. 51, 52.

[‡] He does not use the technical term "künstlich deformirt" but a term which would apply to earth pressure on a thin skull. This is probably the skull which Poll terms Bl. 2 and says is that of a non-adult of 15 years of age; he says it is "porotisch," "zerbrochen" and "eingedrückt," but he says nothing about artificial deformation.

[§] *Zeitschrift für Morphologie und Anthropologie,* Bd. V. S. 7, *et seq.* 1903.

[||] *Journal of the Anthropological Institute,* Vol. XXX. p. 141 *et seq.* 1900.

He failed to note any deformation in either the Cambridge or the London series. If the Moriori practised as a race artificial deformation, it entirely escaped him.

Lastly we may note that Scott, the first and one of the best investigators in this field, with a collection of 46 Moriori crania at his command, and acquainted with the practice of deformation in the New Hebrides, as well as close to the men who had studied the Moriori customs at first hand, makes absolutely no mention of his material being artificially deformed!

Up to the time when E. Thomson undertook her examination of the 65 London Moriori crania, nearly 170 Moriori skulls had been handled by anthropologists of considerable distinction, but they had one and all failed to realise that artificial deformation was practised by the Moriori.

Duckworth, I regret, gives no reproductions.

Poll has some excellent plates, unfortunately not orientated to the Frankfurt horizontal*, but they show exactly the same features as Thomson's series, the receding frontal, the pentagonal *norma occipitalis*, the post-coronal depression, the sagittal crest, and the extraordinary massiveness and muscularity of the type.

The mean frontal index† of the male Moriori cranium as given by Thomson‡ is 19·4, while the frontal indices of Poll's Br. 2 and Dr. 1 crania are 18·9 and 19·3 respectively as determined from the photographs, Thomson's 765^{45} has a frontal index of 18·4 of the same order as Poll's Br. 2, or if one skull is artificially flattened, so are the others judged by this test. In Quatrefages and Hamy's illustration—not photographic—this index appears to be 19·4, but it would be difficult to lay stress on what might be very largely influenced by the uncertainties of a lithographed drawing.

Now what scientific line of criticism was open to Professor Giuffrida-Ruggeri if he suspected that the Moriori were a race who practised artificial deformation of the skull?

The best line of conduct would be to have waited before writing his memoir till he had been able to examine at first hand the collections at Bremen, Cambridge or London—I will not suggest that he should have travelled to New Zealand. The next best line would have been to have examined carefully the whole series of photographs—particularly those of the lateral aspect—already published. Thomson gives ten *normae laterales*, Poll two, and Quatrefages and Hamy's drawing might

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* Poll writes, S. 124 "Die Photographien der Schädel sind in der halben Grösse angefertigt. Die Schädel sind mit Hilfe der Waldeyer'schen Stativs in die deutsche Horizontale eingestellt; dies gelingt ausserordentlich leicht und bequem." The German horizontal plane should make the plane through the lowest points of the orbits and the highest points of the auricular passages horizontal. It is hard to imagine this is so in Poll's *norma lateralis* in Fig. 3, still less in Fig. 6 of Tafel II. We think there must have been some error of orientation either in the original photograph, or in the trimming of the border by the engraver to make these photographs fit his plate. But the question of the tilt has an all-important bearing on the aspect of the *norma lateralis* and influences the judgment of a superficial observer of the photograph as to whether a cranium looks "deformed."

† Frontal Index = 100 × subtense of nasio-bregmatic arc divided by the nasio-bregmatic chord.

‡ *Biometrika*, Vol. XI. p. 95.

be taken into account. If, after doing this and comparing these representations with any long typical series of Peruvian and New Hebridean skulls, he had not concluded with anatomists and anthropologists like Scott, Poll, Flower, Turner and Duckworth, that there is no question of deformation—I will not suggest that he should have analysed numerically the extraordinary differences in absolute lengths as well as indices between deformed and natural series—he should have criticised in the first place the obtuseness of these anatomical craniologists* who could overlook such a fundamental point!

Professor Giuffrida-Ruggeri does not adopt such a course. He takes *one* out of *ten* of Thomson's *normae laterales* and says this skull is artificially deformed *ergo* the series as a whole is artificially deformed, *ergo* all these elaborate biometrical investigations here (and no doubt elsewhere) are idle. To point the moral as strongly as possible, the inference drawn from a single photograph of a single skull is carried to a sweeping conclusion in the title of the Professor's memoir. It runs "A proposito della leptorrinia dei Moriori e della loro deformazione cranica†." On such a slender basis the Moriori race, according to Professor Giuffrida-Ruggeri, henceforth is to be looked upon—not as one with most noteworthy cranial characters—but merely as a race with a self-distorted cranium.

He does indeed say that Poll—without suspecting the deformation—has drawn attention to the markedly receding frontal, but this had also been done by Scott and Duckworth. Thomson, however, was the first to show that the Moriori have a smaller frontal index than any other measured race, and this is how Giuffrida-Ruggeri sums up the biometric work on the point after referring to p. 95 of her memoir:

> Altri calcoli piu precisi—ma inutili—si leggone a p. 130. Anche a p. 110 altre valutazioni dell' appiattimento dell' osso frontale—molto complicate, ma fuori di posto, essendo viziate della stessa inavvertenza morfologica—danno risultati sfavorevoli ai Moriori e non potrebbe essere diversamente (p. 7 of offprint).

Why indeed must the Moriori be an artificially deformed race? Simply because a flat frontal is "not a character of the South Oceanic races," and therefore its very existence must be interpreted as a deformation. But this is to beg the fundamental question of whether the Moriori were a "South Oceanic race" or whether, pure or hybridised, they were a still earlier race being pushed to the extremes by a South Oceanic racial invasion. The question of deformation has to be settled quite apart from whether a flat frontal is or is not a character of the "South Oceanic races." And it has to be settled appreciatively or quantitatively by comparing a series of the Moriori against a series where deformation is admitted to be practised. Postponing for a time the question of quantitative measurement‡, I examined the Peruvian and New Hebridean series in the Royal College of Surgeons Museum as a whole against the Moriori series, and have no doubt that the Moriori as a race

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* More than one of them had actually examined in whole or part the series studied by Thomson!

† *Rendi della R. Accademia delle Scienze Fisiche e Mathematiche di Napoli*, Ser. 3^a, Vol. XXVII, 1921.

‡ For example the cephalic index of the male Moriori is 76·15 with a standard deviation of 2·58, but that of 47 Peruvian male crania is 89·15 with a standard deviation of 8·25!!

at the time these crania were those of living persons did *not* practise deformation. I think this is definitely indicated not only by the bulk in all aspects of Thomson's plates, but by her figures, in particular her mean sagittal contours (pp. 113 and 115 of her paper). It is also indicated by Poll's cranial photographs and photographs of living representatives of the Moriori*. However, some general impression of what a deformed skull looks like and what it does not look like may be obtained if the reader will examine the accompanying plate in which I have reproduced the *normae laterales* of two specimens practically taken at random from each of the Moriori, New Hebridean and Peruvian series. Apart from this I think the pentagonal *norma occipitalis,* the curvature of the posterior portion of the sagittal contours, the sagittal crest and the extraordinarily rugous and muscular character of the whole skull are quite sufficient to put deformation out of the question.

Is the skull 765^{45} singled out by Professor Giuffrida-Ruggeri really a case where there may have been some deformation? I personally think not, it only shows the racial characters of the Moriori in a possibly somewhat emphasised manner†. If the view be taken that the bulk of the Royal College series of crania are normal and this particular one artificially deformed, then the continuous graduation of the frontal index throughout the series will have to be met. And further, are we to term artificially deformed all seventeenth century English crania—and they are fairly numerous—with a lower frontal index than the mean Moriori skull? Plates XII, XIV and XVI of *Biometrika,* Vol. V exhibit such crania. I think there is very little doubt that the anatomical anthropologists will agree that the artificial deformation of the Moriori has remained undiscovered up till to-day—simply because it does not exist. The Moriori have a minimum frontal index, not because they deformed their skulls, but because they contained a racial element more primitive than the bulk of the "South Oceanic races"‡.

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* These photographs unfortunately in no case give a profile, and it is accordingly difficult to appreciate the retreating frontal. Giuffrida-Ruggeri dismisses these photographs by saying that the later Moriori may well have given up the practice of deformation. But our type contours do not suggest bad fits to the living head.

† Professor Giuffrida-Ruggeri wrote apparently to Sir Arthur Keith asking him to verify—not whether the Moriori series was as a whole artificially deformed but—whether 765^{45} was artificially deformed "come io sospettaro." According to Professor Giuffrida-Ruggeri the reply fully confirmed the view that it is a case of artificial deformation. I hardly interpret Sir Arthur Keith's reply to sanction the hypothesis of a *general artificial deformation* of the Moriori crania. He wrote: "I have just examined the skull Chatham Islander 765^{45} and compared it with deformed skulls from the New Hebrides. You are right: it shows exactly the same frontal features as the deformed Hebridean—differing from them in the great parietal width." While it shows as well as other Moriori crania do a flat frontal, a character of the Hebridean deformed crania, they show entirely different features not only of the frontals, but of the parietals and occipitals from those of the New Hebrides, the features in the latter case being markedly correlated with the artificial frontal changes.

‡ Professor Giuffrida-Ruggeri speaks of the postbregmatic depression ("insellatura retrobregmatica"), noted by both Poll and Thomson, as in his opinion evidence of artificial deformation. We found it equally conspicuous in many seventeenth century English crania, but were unable to associate it with any "legatura trasversale al vertico," which we could trace in the customs of the day; it occurred in over 19°/₀ of 412 English crania. It can therefore hardly be regarded as in any way evidence of artificial deformation.

Passing from this fundamental conception of our Italian professor—i.e. that the Moriori only differ from other "South Oceanic races" because they deformed their skulls—we turn to his discussion of the nasal index. We have seen that he is content to divide nasal indices into the percentages falling into three classes and does not trouble himself about the probable errors of these percentages even when based on 19 crania. He compares Poll's distribution with Thomson's and seems suspicious of the latter's results, because her differences for the two sexes are very considerable. Now Thomson states very clearly how she measured her nasal height, i.e. from the nasion to the lowest point of the edge of the left pyriform border, or if this were damaged, to the lowest point of the edge of the right aperture. Poll says he has taken his measurements according to the Frankfurt Concordat. He may have measured as Thomson did, but actually that "Verständigung" runs on this point as follows: "*Nasenhöhe*, Fig. 2, wNH: von der Mitte der Sutura naso-frontalis bis zur Mitte der oberen Fläche des Nasenstachels resp. zum tiefsten Rand der Apertura pyriformis" (*Archiv für Anthropologie*, Bd. xv, S. 4, 1884). This would justify a measurement to the nasal spine. It is therefore not absolutely clear what Poll may have done. But, in the results cited by Giuffrida-Ruggeri, Poll has pooled with his own data results obtained by Scott, Turner and others who certainly did not know anything of the Frankfurt Concordat *. It is not therefore justifiable to place Thomson's distribution against Poll's without further consideration of this point. The sexed crania of Duckworth and Poll are so few that the means are subject to large probable errors, but it is still suggestive to consider the differences in the results reached by various authorities. The numbers upon which each depend are given in brackets in the following Table.

Measurement of the Nose.

	MALES			FEMALES		
	Nasal Height	Breadth	Index	Nasal Height	Breadth	Index
Poll	54·4 (6)	26·8 (6)	49·1 (6)	50·4 (5)	26·3 (5)	52·0 (5)
Scott	57·5 (32)	26·5 (32)	46·1 (32)	50·9 (10)	24·8 (10)	48·8 (10)
Thomson ...	57·3 (34)	25·3 (34)	43·9 (34)	52·5 (21)	25·3 (21)	48·2 (21)
Duckworth ...	56·2 (6)	24·2 (6)	43·8† (6)	52·3 (3)	24·3 (3)	46·5 (3)

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* Perhaps the most illusory description of nasal height is that given by Sir W. H. Flower (Royal College of Surgeons' Catalogue, Man, p. xviii), "A vertical line between the nasion and lower border of the nasal aperture." The height will not be "vertical" in any case. Flower's figure (p. xii) shows he did not intend it to be vertical. If he meant in the vertical or sagittal plane of the skull then the measurement must be to the nasal spine, rather than to the border of the pyriform aperture. If he meant to the lowest point on the border of the pyriform aperture, then his nasal height is neither vertical, nor in the vertical sagittal plane, and his figure can only represent not the real length but a projection of the real length! Unfortunately many English craniologists have only had Flower's definition before them.

† Duckworth himself gives for his mean index for *seven* male crania 44·3 (p. 145) but his table p. 150 shows only seven male crania in all and one has no nasal measurements.

In all these cases we see that the male is markedly more leptorrhine than the female*, and that Thomson's mean for males agrees with Duckworth's and her mean for females with Scott's. Her results differ widely from Poll's, but so do Duckworth's and, to a considerable but lesser extent, Scott's. It seems to me that if the paucity of numbers be not regarded (the probable error of Poll's ♂ mean is about 0·83), we must conclude that Poll measured his nasal height somewhat differently to Thomson, and accordingly Professor Giuffrida-Ruggeri is not justified in contrasting his results directly with hers.

If, instead of working with means, we use the loose threefold category apparently approved by Professor Giuffrida-Ruggeri, we find:

Distribution of Nasal Indices in Percentages.

	Male				Female			
Author...	Poll	Scott	Thomson	Duckworth	Poll	Scott	Thomson	Duckworth
Leptorrhine	0·0	59·4	88·3†	83·3	20·0	20·0	35·7†	33·3
Mesorrhine	100·0	31·2	11·7	16·7	20·0	60·0	31·0	33·3
Platyrrhine	0·0	9·4	0·0	0·0	60·0	20·0	33·3	33·3
Number	6	32	34	6	5	10	21	3

Results at once flow from this table which bear strongly on Professor Giuffrida-Ruggeri's classificatory table (p. 4, *loc. cit.*). In the first place Duckworth's own measurements absolutely confirm Thomson's, although with his few cases it was hardly to be expected that so close an agreement would arise, and it must be of course somewhat fortuitous. Poll's results differ widely from those cited by Giuffrida-Ruggeri from Poll, simply because he has taken from Poll the data Poll gives for

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* This sexual difference is fairly general. Thus we have:

Nasal Index.

	Naquada	Modern Egyptian	Alt Bayerisch	Aino	Würtemberger
♂	51·1	49·3	48·7	50·7	46·6
♀	52·3	54·6	49·2	50·5	50·3

	17th Century English	Maori	Congo Bantus	Long Barrow	Burmese
♂	47·6	47·9	55·2	49·0	52·8
♀	47·8	49·1	55·8	49·1	52·9

	Kareens	Malayans	Chinese	Hindus
♂	46·4	50·4	48·9	49·7
♀	50·4	53·6	48·1	50·4

and the general tendency is for the female to be more platyrrhine than the male.

† These results do not accord with those Professor Giuffrida-Ruggeri attributes to Thomson. Thomson has only 21 available nasal indices for females, not 22.

his own measurements *combined* with other craniologists' results. But, as we have already stated, Poll's data show that he did not probably take the nasal height in the same way as Thomson. If we suppose him to have measured to the nasal spine this will be about 5% shorter than Thomson's measurement (·9485 and ·9595 factors respectively for ♂ and ♀). We then obtain results:

	Poll	
	♂	♀
Leptorrhine ...	50·0 %	20 %
Mesorrhine ...	50·0 %	40 %
Platyrrhine ...	0·0 %	40 %

which more nearly approach those of other writers, especially when we consider the smallness of the numbers Poll had at his command.

If we turn to Scott we find that in his memoir he measured for nasal height the nasio-spinal length, i.e. he did not take the length to the lowest point of the pyriform border. He also classified leptorrhine, mesorrhine and platyrrhine according to Flower* and did not use the *Frankfurt Verständigung* divisions. Thus his nasal index and statements as to class of nasal index are not directly comparable with Thomson's. Accordingly they ought not to be compared directly with Thomson's results; it is difficult, however, to correct for the difference between nasio-spinal and nasio-pyriform border heights in the case of Scott's males, for Thomson's and his mean values of the heights are here the same. If we assume that the multipliers for the indices may be given the values found above—say ·95 and ·96 in round numbers for male and female respectively,—we obtain

	Scott	
	♂	♀
Leptorrhine ...	84·3 %	50 %
Mesorrhine ...	15·7 %	40 %
Platyrrhine ...	0·0 %	10 %

Thus Scott falls quite reasonably into line with the results of Thomson and Duckworth, and all three show precisely the sex difference to which our Professor takes exception. The differences in the matter of nasal index in the results of various writers, over which Professor Giuffrida-Ruggeri seems to be puzzled, could have been largely cleared up had he investigated whether the writers had measured the same character or classified its values in the same way.

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* Leptorrhine below 48, mesorrhine from 48 to 53, platyrrhine above 53. This is very different from the 47·0 and lower, 47·1—51·0, and 51·1 and above of the Verständigung.

According to Professor Giuffrida-Ruggeri, the leptorrhiny, the receding frontal and the massiveness and vertical ramus of the mandible of the Moriori are not characteristics marking them off as an extremely primitive race, but the product of climate, artificial deformation and diet. In short, are due to environmental conditions, which also in part have influenced Fuegians and Eskimos, and thus account for various common features of these "fringe" races. He does not explain whether this adaptation is due to long ages of selection by which the skull has retrogressed in many respects to a more primitive form, or is due to the inheritance of acquired characters, in which case the flat frontal ought to have become a permanent racial character and would not need to be artificially deformed. If we prefer the view after a close examination of the crania—which is an advantage Professor Giuffrida-Ruggeri has not had—and after a careful measuring and analysis of the measurements to look upon this race as forming one of the most primitive that have survived to recent times—and not as a product of environment*, does this justify Professor Giuffrida-Ruggeri in stating that:

> La Thomson, essendo della scuola del Pearson, naturalmente non s' interessa del materiale umano che nel senso biometrico (p. 9).

We can assure the worthy professor that from its founder, Francis Galton, downwards our school did not take up anthropology because its members were interested in biometric methods, but because, studying "human material," they found real progress was impossible unless new methods were invented and applied. And it is because of that desire to understand human development that they do not hesitate totally to condemn as nugatory and profitless the descriptive and appreciative processes in which Professor Giuffrida-Ruggeri puts his trust.

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* Or as, "un singolare prodotto della domesticazione"!! (p. 9).

Plate I

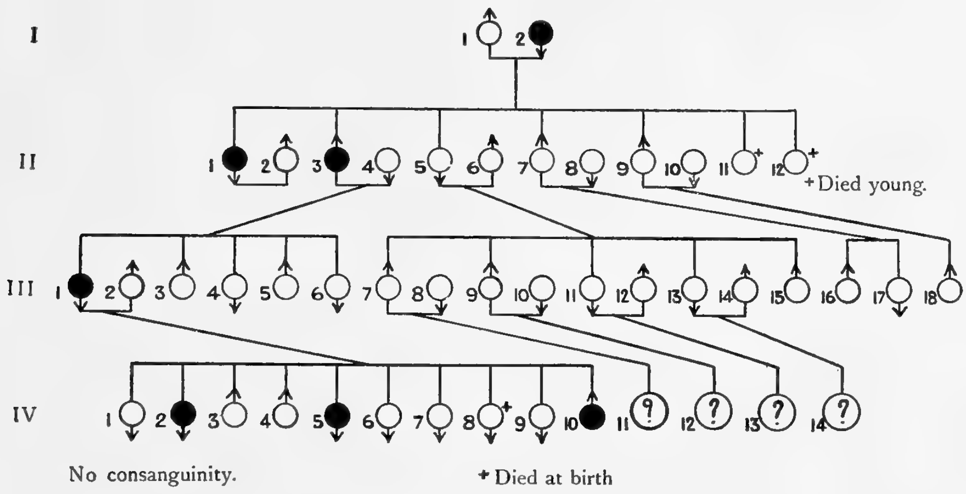
(i) Peruvian Crania, Nos. 999, 1000, Royal College of Surgeons. (ii) New Hebridean Crania, Nos. 1161^{11}, 1161^{7}, Royal College of Surgeons. (iii) Moriori Crania, Nos. 764, 765^{3}, Royal College of Surgeons.
(i) and (ii) Artificially deformed; (iii) Random specimens of a race asserted by Prof. Ruggeri to deform their skulls.

A NEW WHITE-LOCK FAMILY (BARTON'S CASE).

BY KARL PEARSON, F.R.S.

The hereditary nature of the "flare"—or pigmentless forehead patch accompanied by a white or yellow white-lock of hair—in man is now well known. Several illustrative pedigrees were published by Nettleship, Usher and Pearson in their *Monograph on Albinism**, and a good pedigree was also published by Dr E. A. Cockayne in 1914 under the title of "A Piebald Family†." That family embraced six generations of the hereditary flare. In the present case we have only been able to trace four generations with a total of seven affected members. Dr Cockayne's pedigree involved nineteen affected members, while Rizzoli's pedigree of the Bianconcini family extends to six generations with twenty affected members. I owe the case to Dr E. R. Barton, of University College Hospital, who came across III. 1 and IV. 10 in the Midwifery Department of the hospital and most courteously put me in touch with the family. The pedigree should accordingly be referred to under Dr Barton's name.

There is no consanguinity. The family goes grey early and there has been a certain amount of tuberculosis.



I. 1. T. H. Normal, died at about 40.

I. 2. J. F. Married I. 1 and is the earliest known ascendant to have the flare. She had both white forehead patch and white lock. She died at about 80.

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* *A Monograph on Albinism in Man*, Part I, p. 254, Cambridge University Press.

† *Biometrika*, Vol. x. pp. 197—200, with eight plates.

II. 1. R. H. White forehead patch and white lock. She married but has had no children.

II. 3. E. H. Married L. L. (II. 4). E. H. died at about 40 and had white forehead patch and white lock; not known whether he had any body patches.

II. 5. J. H. Normal, married II. 6, normal. She has had five normal children. One of these, III. 15, died at 25, the other four III. 7, 9, 11 and 13, all married and had again normal families, IV. 11, 12, 13 and 14.

II. 7. H. H. Normal, married a normal woman, II. 8, and had a boy and girl, III. 16 and 17, both normal.

II. 9. A. H. Normal, married a normal woman, II. 10, and has had a normal son, III. 18.

II. 11 and 12 died young, but are said to have been normal.

The family of II. 3 consisted of three daughters and two sons of whom only one was affected.

III. 1. R. J. H. Has a yellow-white lock and white forehead patch, said to have no white body patches. She married III. 2, W. S. R., a normal.

III. 3. A. W. H. Died owing to an accident at 24—25 years. He was normal.

III. 4. L. H. Was normal and died at 8 years, of meningitis.

III. 5. Normal, died as a baby.

III. 6. D. H. Normal, is alive at 28 years and unmarried. Thus the trait could only be carried on through III. 1. She has so far had only ten children, but three of them are affected.

IV. 1. R. R. Now 18 years old, is normal.

IV. 2. L. C. R. Died at 15½ years. She had the white forehead patch, the white lock and white patches on the skin.

IV. 3. W. R. Aged 14, and IV. 4, G. R., aged 11½, are both normal.

IV. 5. A. R. Aged 9, has yellow-white lock, white forehead patch, but no skin patches elsewhere.

IV. 6. D. R. Aged 7½, and IV. 7, G. R., aged 5, are normal.

IV. 8. Died at birth, and nothing is known of her.

IV. 9. B. R. Aged 1½, is normal.

IV. 10. T. C. R. Aged four weeks when seen, has quite a marked forehead patch and white lock on his fairly profuse dark hair. No white body patches.

There are three marriages—non-consanguineous—of affected with apparent normals, each of which produce affected. The affected in each case produce affected. The character therefore cannot be recessive. We are compelled to treat it as dominant, but as I. 2 has some normal offspring, she must have been heterozygous. It will be clear that none of the affected can be looked upon as pure dominants, and in their families there ought to be 50 °/₀ affected.

We must leave out II. 11 and II. 12 as there is no record, and IV. 8 who died at birth although III. 1 considered the infant normal. We have accordingly 2 in 5, Biometrika, Vol. XIII, Part IV

Plate I

Barton's *White-Lock Family*

III. 1 with IV. 10.

III. 1 with IV. 10.

IV. 10.

IV. 5.

1 in 5 and 3 in 9 in generations II., III. and IV. affected or 6 in 19. This is a fair approximation to a third, but rather a poor one to a half. There is in fact a deviation of 3·5 with a standard deviation of $\sqrt{19 \times \frac{1}{2} \times \frac{1}{2}} = 2{\cdot}179$, or the deviation is 1·61 times the S.D. The odds against such a defect are more than 17·5 to 1. Of course not impossible but improbable.

On the other hand there is no case in which an apparent normal marrying a normal has transmitted the character. In the case of IV. 11, 12, 13 and 14 there are considerable families, exact numbers not available, but it is known that none are affected. Thus it would appear that transmission through the unaffected either does not ever, or at least commonly, occur.

This rule holds also for Cockayne's family and for the Bianconcini.

IV. 2 is the only one with body pigmentless patches. As III. 1 mentioned this fact without special enquiry, I think we must take it that her statement with regard to IV. 5 and IV. 10 and to herself that they and she have no pigmentless body patches can be accepted. She believed that II. 1 had no such patches, but did not know about II. 3 or I. 2. It would seem therefore that in this family at least conspicuous body patches are not frequent. It should not therefore be looked upon as a piebald family in the ordinary sense. It exhibits the inheritance of a "flare."

ON THE SESAMOIDS OF THE KNEE-JOINT.

By KARL PEARSON, F.R.S. AND ADELAIDE G. DAVIN, B.Sc.

PART II. EVOLUTION OF THE SESAMOIDS.

(8) *Further Remarks on Origin of Sesamoids.*

In the first part of this paper attention was drawn to Gruber's statement that the sesamoid invariably arises from hyaline cartilage. We cited our own work as supporting this view and a considerable number of further sections made since entirely confirm it. We do not think it needful to publish the drawings made from these sections—they all show the same hyaline structure; and we now hold with Gruber that no thickening in the tendons of *M. gastrocnemius* or *M. popliteus* should be classed as a hemisesamoid unless it shows this hyaline structure*. Every

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* Another matter of some interest is elucidated by our sections. Lunghetti (*Internationale Monatschrift für Anatomie und Physiologie*, Bd. XVI. § 47) following Gillette in his classification of sesamoids into peri-articular and intratendinous claims to have shown that the origin of the peri-articular sesamoids is totally different from that of the intratendinous. We have already (p. 156) criticised Gillette's classification; it does not seem to us in accordance with the evolutionary origin of the *fabella lateralis* and the *cyamella*, both are ultimately peri-articular in origin, and our sections seem to show that Lunghetti's view is erroneous. He asserts that the peri-articular are prefigured by hyaline cartilage which can be noticed in the foetus with the form of the definite bone. With this we are in agreement; the sections of the metatarsal sesamoid in the human foetus (p. 159) demonstrate it. Lunghetti further asserts that the intratendinous only appear in the adult in the middle of a fibrous tissue which chondrifies a short time before ossification. This assertion may be true of pseudo-sesamoids such as the osteomata of riders. It is entirely erroneous in the case of *fabellae* and *cyamella* which can be detected in embryos and the young shortly after birth by the presence of patches of hyaline cartilage as we have indicated: see our pp. 158—159 and Plates XVII and XVIII.

orthosesamoid will be found to exhibit this prefiguration if the knee-joint be examined in the embryonic or early post-natal states.

Two points of much bearing on what follows later were reached as the outcome of these investigations:

(*a*) The *cyamella* of the rabbit was found near the head of the fibula, there was no marked popliteal groove for it to lie in.

(*b*) The lateral and mesial *fabellae* while both prefigured by hyaline cartilage in kittens were very unequal the former being far more developed. Pfitzner* says that he found the *fabella medialis* absent in 29 cases out of 52 in the cat. He states that it always occurs in dog, fox and hare. Meckel† states that both in cat and dog only the external *fabella* occurs. Gruber‡ places *Canis familiaris* (as well as *C. vulpes* and *C. lupus*) and *Felix domesticus* in the group which have both *fabellae*. Thus he tells us that in puppies at birth he found hyaline cartilage and later orthosesamoids; the external ossified first at about six months. The external *fabella* articulated with the femur in a trough ("Grube") with projecting rim on the upper part of the external condyle of the femur; the internal *fabella* was in a less marked trough on the upper part of the internal condyle. With regard to newborn kittens both *fabellae* were prefigured by hyaline cartilages: the external ossified first and the internal was smaller and of a different shape.

(9) *Carnivores.*

While our experience with kittens accords with that of Gruber, and not with that of Pfitzner, and while we have found the two *fabellae* in many adult dogs, we have not invariably found the mesial *fabella* after most careful sectioning of the tendon of the internal head of *gastrocnemius* in puppies of a fortnight old (failed in two cases). On a general balance of the evidence therefore we are inclined to believe that cases do occur in the smaller carnivores, in which both *fabellae* are not present, although Meckel's view that only one occurs in dog, cat and fox is certainly incorrect.

While dealing with the carnivores we may note that Gruber examined a female, aged one year, of *Felis leo,* and found an external hemisesamoid, the internal was represented by a pad ("Wulst") with no cartilage cells (*loc. cit.* p. 59). Further in *Gulo vittalus* there was an external *fabella* only, which as in man did not articulate with the femur, but lay in the tendon of *gastrocnemius*, and between that and the knee capsule (p. 57). Further the external *fabella* and that only has been found by Gruber in *Nasua, Ursus arctos, Mustela alpina* and *Lutra marina.* Blainville found none in *Ursus*, Meckel found the external only. The latter found the external only in *Felix lynx,* the raccoon, and the hyaena§, while Blainville and Owen found both in the hyaena; and Meckel found both in *Mustela* and *Lutra*. Davis‖ and

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* See work cited (*Biometrika*, Vol. XIII. p. 163), S. 582.

† *System der vergleichenden Anatomie*, S. 634, 635, Halle, 1828.

‡ See work cited (*Biometrika*, Vol. XIII. p. 156), pp. 57, 58.

§ *Loc. cit.* S. 634—635.

‖ *Journal of Anatomy and Physiology*, 1888, p. 215.

again Young* found the external *fabella* only in *Viverra civetta*†; and Macalister found none in the Laughing Otter (*Aonyx*)‡.

Lack of material has hindered our undertaking a considerable number of independent dissections on the carnivores; we have been largely compelled to examine prepared skeletons, and these can never prove a negative. They may, however, give positive evidence, and often confirm or refute earlier statements. The chief difficulty is one we have already referred to—the articulator preserves the *fabellae*, but far more rarely, when it exists, the *cyamella*.

We have found both *fabellae* and *cyamella* in: *Proteles cristatus* (Aard-Wolf), *Canis vulpes* (fox), *Canis domesticus* (not invariably) and *Felis domestica*§ (invariably: see Plate XX, Fig. 44).

We have found both *fabellae*, no *cyamella* in *Felis catus* (wild cat), the two-spotted Paradoxure, the *Hyaena crocuta* (spotted Hyaena), *Otocyon lalandii* (long-haired fox), the Java otter, *Cryptoprocta ferox* (fossa) where the external was much larger than the internal, *Felis nebulosa* (clouded leopard), *Canis melitus* (American wolf), *Cyon sumatrensis* (Malayan wild dog).

We have found the external *fabella* only in *Lutra canadensis* (Canadian otter), *Aelurus fulgens* (Panda), *Lutra vulgaris* (otter), *Enhydra marina* (sea otter), where the external *fabella* was markedly large and there was a much emphasised excavation corresponding to it above the external condyle on the shaft of the femur itself (see Plate XIX, Fig. 40), *Felis tigris* (2 cases)—*Eupleres goudoti* (a Madagascar

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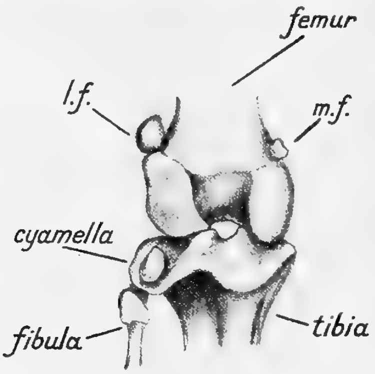
* *Journal of Anatomy and Physiology*, 1879, p. 174.

† Macalister (*Proc. Royal Irish Acad.* Vol. I. 2nd series, p. 512), states that there is a hemisesamoidal *cyamella*.

‡ *Proc. Royal Irish Acad.* Vol. I. *Science*, 2nd Series, p. 545.

§ The cat is a very good example of the complete system in the carnivores (see accompanying drawing after Plate XXV, Fig. 6 of Pfitzner's memoir). In it we see the intimate relation of the *cyamella* to both head of fibula and head of tibia, with the latter of which it articulates. The close association of external *fabella* with the *cyamella* when the limb is flexed is evident and also the dominance of the former over the mesial *fabella*.



viverrid), cheetah (2 cases), *Herpestes ichneumon* (the Egyptian ichneumon) with both lateral *fabella* and *cyamella*, *Canis dingo*[*].

We have not been able to confirm Gruber's result for the lion or Meckel's for bears. In *Felis leo* we have found a deep cavity on the articular surface of the external condyle which might well be for an external *fabella*. On a skeleton of *Ursus horribilis* (grizzly bear) we found nothing, but on the fossil *Ursus spelaeus* (great cave bear) there were depressions on the femur which might well have served for *fabellae*. The same remark applies to the fossil *Arctotherium bonariensi*, which we also examined.

We have purposely discussed the carnivores in the first place as a group marked by sesamoidal characters intermediate between the Primates and Ungulates. The higher primates have lost wholly or in part both *fabellae* and *cyamella*; the higher carnivores are in the same stage. Either they have never developed them, or they have lost them and comparison with the primates suggests that the latter is the better account. But one great feature comes out of any examination of the carnivores: There is no single species in which the mesial *fabella* is present without the external. If one *fabella* alone occurs, it is the external. The mesial *fabella* seems to be a pale reflex of the external *fabella*. It is often absent, often much smaller, oftener remains a hemisesamoid, and when it does ossify, ossifies later. Its development in the individual suggests a different evolutionary origin, and one of less importance than that of the external *fabella*.

(10) *Ungulates*.

Let us now pass to the Ungulates and note first what others have observed. Among the true ungulates, the *Perissodactyla*, tapirs, horses, etc., no *fabellae* have been seen. Among the *Artiodactyla* the swine[†] are without them and nearly all the ruminants. The only exception that we have seen noted is that of the stag, which is said by Meckel to have an external *fabella*. This is supported by Gruber, who says he found the external *fabella* in *Cervus tarandus* but otherwise in none of the ruminants. It fails in other members of the *Ungulata*, horse, elephant, hippopotamus, rhinoceros, giraffe, etc., etc.

While we have found sesamoids preserved in mounted skeletons of nearly all the lower primates, the rodents and certainly fifty per cent. of the carnivores, we have drawn a complete blank with the exception of *Hyrax* in the case of all ungulates. There seems no reason why the articulator should have made a practice of removing the *fabellae* in the case of the ungulates.

We have examined skeletons of: *Ovis aries*[‡], *Bos indicus*, *Capra hircus* (common goat), *Capra tartarica* (Saiga antelope, 3 cases), *Tetraceros chiliana* (four horned antelope, 4 cases), *Moschus moschiferus* (musk deer), *Cervus dama* (fallow deer),

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[*] Only one case, which had solely the external *fabella*. But the dogs are very difficult unless directly dissected. We have noted both *fabellae* in articulated skeletons of Saint Bernard, French Bloodhound and Chow; external only in Bulldog, Newfoundland and Deerhound, but this proves nothing.

[†] Gruber found a pad (Wulst) in *Sus scrofa* in the external head of *gastrocnemius* but no hemisesamoid, much less an orthosesamoid.

[‡] Several knee-joints of sheep on dissection showed no traces of *fabellae* or *cyamella*.

Rangifer tarandus (reindeer, 2 cases), *Tragulus javonicus*[*] (moose deer, 3 cases), *Cervus elaphus* (red deer), *Connochaetis gnu* (white tailed gnu), *Alces machlis* (moose or elk), *Cervus davidianus* (David's deer), *Cervus puda* (Puda deer), *Hyomoschus aquaticus* (African chevrotain), *Cervus capreolus* (roe, 2 cases), *Hydropotes inermis* (Chinese water deer), *Strepsiceros kudu* (koodoo), *Auchenia pacos* (alpaca), *Auchenia vicugna* (vicugna), *Sus scrofa* (Indian wild boar, 3 cases), *Dicotyles labiatus* (white lipped peccary, 2 cases), *Phacochaerus africanus* (Aelian's wart hog), *Hippopotamus amphibius* (2 cases), Baird's tapir, *Equus zebra, Equus asinus, Camelus bactrianus, Camelopardalis giraffa* (Nubian giraffe), *Elephans indicus, Rhinoceros unicornis, Rhinoceros sumatrensis, Bos taurus* and also *Equus caballus,* Orlando (the thoroughbred race-horse) and numerous other ungulates.

In the face of the large number of skeletons of *Cervidae* examined, it seems hard to believe that the external *fabella* has been removed in all these cases, i.e. that it can be, as Meckel asserted, universal in *Cervus*[†].

We have dissected a specimen of a young *Tragulus,* probably from Malasia, but found no *fabellae* nor indeed any sesamoids of the knee-joint. In our opinion an indication of the ancestral history.

Apart from *Cervus, Hyrax* is the only ungulate with sesamoids of the knee-joint. According to Blainville both *fabellae* are present, but Meckel and Brandt credit it only with the external *fabella.* We first examined two skeletons. On one *Hyrax dorsalis,* there was no trace of a *fabella* left, if there ever had been any; on the other *Hyrax capensis,* there was a small external *fabella.* Murie and Mivart in their "Myology of *Hyrax capensis*[‡]" describe *gastrocnemius, plantaris* and *popliteus,* but do not refer to any sesamoids. We were able to examine further one knee-joint of *Hyrax* with certain of the muscular attachments (see Plate XX, Fig. 45). Unfortunately the popliteal tendon had been cut short before it passed over the fibula so it is impossible to say whether a *cyamella* ever existed. The external *fabella* was present, but *gastrocnemius* had shrunk in drying, so that the sesamoid was lifted off the external condyle. There is a slight depressional facet on the articular surface of the internal condyle suggesting the possibility of a mesial *fabella.* There is, however, no doubt about the existence of the external *fabella.* In another mounted specimen of *Procavia,* we have found not only the external *fabella,* but an external anterior *lunula.* These facts provide at least suggestions for the reclassification of the *Hyracoidae.* Apart from the *Cervidae* and *Hyracoidae*—where the presence of the external *fabella* may throw light on their evolutionary history—we are forced to the conclusion that the large (and mixed!) order of the ungulates presents no sesamoids of the heads of *gastrocnemius* or *popliteus.* The absence of the *fabellae* in ungulates has been frequently noticed[§], but so far we believe no stress laid on its evolutionary importance.

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[*] We failed to find any sesamoid of the knee-joint on dissecting a young *Tragulus*: see below.

[†] On the fossil *Megaceros hibernicus* (Irish deer) we were unable to find any trace of a *fabella.*

[‡] *Proc. Zool. Society*, 1865, p. 350.

[§] Meckel, Gruber, etc. See also *Journal of Anatomy and Physiology*, Vol. XXXII. p. 750, 1898.

(11) *Pinnipedia.*

No sesamoids of the *gastrocnemius* were found by Gruber[*] in *Phoca* sp.? He thus confirmed an earlier statement of Meckel[†] that the *fabellae* were absent in seals. Again reminding the reader of the unreliability of evidence drawn from articulated skeletons, we may note that the following articulated skeletons were examined without finding traces of *fabellae*: *Otaria stelleri* (Steller's sea lion), *Stenorhynchus serridens* (saw-toothed seal), *Cystophora cristata* (bladder-nosed seal), *Otaria falklandina* (Falkland fur seal), *Phoca vitulina* (common seal), *Phoca greenlandica* (the harp seal), *Phoca hispida* (ringed seal), *Balaenoptera rostrata* (lesser fin whale), *Odoboenus rosmarus* (walrus), *Macrorhinus leoninus* (elephant seal), *Otaria jubata* (southern sea lion).

In *Otaria californiana* there is small process or upward growth where articular surface of external condyle meets popliteal area, but it is not sesamoidal in character. We dissected, however, a knee-joint of *Otaria australis* (South American fur seal) and found no such process. There is nothing, therefore, in our observations inconsistent with the view of Meckel that the *Pinnipedia* have no *fabellae.*

(12) *Chiroptera.*

Meckel[‡] asserts that the bats have only the external *fabella.* Owen[§] says that *Vespertilio murinus*, *Pteropus*, and *Galeopithecus* have both external and internal *fabellae,* while Gruber says that *Galeopithecus* has neither *fabella.* Blainville[||] asserts the existence of a sesamoid in the tendon of *popliteus* in the case of *Vespertilio L.*, i.e., of a *cyamella.* Humphry[¶] dealing with the myology of limbs of *Pteropus* and treating of *gastrocnemius* writes: "There is a minute ossicle in the outer head. I cannot find one in the inner head." The not unusual statement is that the sesamoids, the patella and even the semi-lunars are wanting. The above statements suggest that *fabellae* and *cyamella* exist occasionally or in minute forms. Parsons[**] writing of the *Chiroptera* says no movement except extension and flexion is allowed in fruit bats and there are no traces of the semilunar cartilages; the patella is absent. "In the long-eared bat (*Plecotus*) which will serve as an example of the insectivorous bats the knee-joint does allow a certain amount of rotation and in it semilunar cartilages are found as very delicate rings. This bat resembles the fruit bat, however, in the absence of any trace of a patella."

On the other hand the bat *Cynopterus marginatus* appears in the specimen (R. C. of S. Museum A. 3396) we were able to examine to have an ossified patella, and there might be other ossifications in the knee-joint, but it was not possible to say from this specimen. Gruber[††] also found the external *fabella* in *Phyllostoma hastatum*, but failed to find it in *Pteropus* sp.? Besides the above *Cynopterus* we have obtained and examined a considerable number of knee-joints of bats—principally

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[*] *Loc. cit.* p. 60.

[†] *Loc. cit.* S. 634.

[‡] *Loc. cit.* S. 634.

[§] *Anatomy of Vertebrates*, Vol. II. p. 358.

[||] *Ostéographie des Cheiroptères*, Tom. I. Fasc. 5, p. 30, 1840.

[¶] *Journal of Anatomy and Physiology*, 1869, p. 313.

[**] *Ibid.* Vol. XXXIV. p. 310, 1900.

[††] *Loc. cit.* p. 55.

Pteropus—having the muscles attached, but we have failed to find any sesamoids. It is probable that a microscopic examination of sections from moist material would lead to more conclusive results, and, perhaps, indicate minute patches of hyaline cartilage hardly appreciable by ordinary dissection. For our present purposes, however, such an investigation would be very unlikely to change the significance which can be drawn from the results for *Chiroptera*. In this order the sesamoids of the knee-joint appear occasionally, as in the higher primates, or in minute forms and are vestiges of a complete system of patella, *fabellae*, and *cyamella*, which belonged to their evolutionary ancestors. Even in this case which is merely vestigial, it is worth while noting the greater importance of the external *fabella*.

(13) *Insectivores.*

With this order we reach vestiges of a more primitive state of affairs in the sesamoids of the knee-joint. The *lunulae* common in the Rodents and carrying us back to reptilean forms are frequently preserved. In the carnivores and the higher primates they occur only as anomalies or have disappeared entirely. According to Blainville* *all* insectivores have both *fabellae*. Owen† qualified this by saying that *most* insectivores have both. Meckel‡ had already remarked that the mole has both *fabellae*, but the hedgehog he reported as having only the external *fabella*. Gruber§ on the contrary found the mole to have an external *fabella* only. He reports that both *fabellae* are wanting in *Erinaceus europaeus* and *E. auritus*, and again in the shrew mouse, both *Sorex vulgaris* and *S. fodius*. He examined three cases of *Myogale moschata*, the Russian desman. In the first case there were no *fabellae*; in the other two cases no internal but external *fabellae*.

> In den Fällen aber mit Vorkommen des Ossiculum externum sass dieses in einem Ausschnitte am Ende des scharfen Angulus externus über dem Condylus externus und daneben in einer kleiner Grube an der hinteren Fläche der Femur||.

In the case with no *fabellae* there was a "platter Fortsatz" on the same spot. Gruber found this also in *Myogale pyrenaica* where there were no ossicles. We dissected two moles, *Talpa europaeus*, and found the external *fabella* only; thus our specimens agreed with Gruber's and not Meckel's. The hedgehog (*E. europaeus*) examined by us had both *fabellae* and both lateral and mesial anterior *lunulae* (see Plate XXIII, Figs. 56*a* and 56*b*). Thus Gruber's statement is not universal. In a specimen of *Myogale moschata* we found only a large external *fabella* of unusual shape and size, and in *Myogale pyrenaica* no *fabellae*; these two latter in mounted specimens only. In mounted specimens we found only a large external *fabella* in *Talpa europaeus*, *Macrosceledes* (elephant shrew), *Solenodon cubanus* and *Tupaia javanica*. In a specimen of *Tupaia picta* in our laboratory both *fabellae* are present, and both occur in *Tupaia tana*, the Borneo tree shrew. We have also found both *fabellae* in

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* *Ostéographie des mammifères*, Tom. I. Fasc. 4, p. 47, Paris, 1841.

† *Anatomy of Vertebrates*, Vol. II. p. 393.

‡ *Loc. cit.* S. 634.

§ *Loc. cit.* p. 56.

|| Gruber notices (*loc. cit.* p. 64, that the external *fabella* in *Myogale* is sometimes fused to femur in a special *Fortsätzchen*. This is the only reported case of ankylosis of a sesamoid we have come across, except in the Marsupialia.

Hylomys suillus dorsalis (see Plate XXIV, Figs. 62 *a* and 62 *b*) and *Rhynchocyon cerni* (long-nosed jumping shrew). Two mounted specimens of the golden mole (*Chrysochloris aurea*) provided no signs of *fabellae*. A mounted specimen of *Potamogale* provided an external *fabella*, and a dissection of *P. velox* confirmed this. In *Ericulus setosus* we found no *fabellae*, but the *cyamella* and two anterior *lunulae*, an unusual combination.

Plate XXIV, Fig. 59, provides a drawing of the lateral aspect of the left knee-joint of *Potamogale velox*. There was no sign of ossification in the popliteus tendon and only the external *fabella* was present. Fibula and tibia are fused at their distal ends.

In *Hylomys suillus dorsalis* (Plate XXIV, Figs. 62 *a* and 62 *b*) both *fabellae* are present and fibula and tibia are again distally fused. There was no *cyamella*, and no *lunulae* could be found.

In the mole only the external *fabella* was present and there was no sign of ossification in the popliteal tendon. No *lunulae* could be discovered, but in such small animals absolute certainty could only be reached by cutting a series of sections of the semilunars.

Parsons* in his paper on the limb myology of *Gymnura rafflesii* notes the presence of an orthosesamoid in the external head of *gastrocnemius*; he does not refer to the existence of *lunulae* or *cyamella*. To sum up, the more primitive forms of the *Insectivora* probably possessed both *fabellae*, a *cyamella*, and mesial and lateral anterior *lunulae*. The external *fabella* has been generally preserved, but the complete system as illustrated in the hedgehog shows signs of disappearance by the loss of one or more members as we also find in the higher Primates or Carnivores. It is, however, very far from being vestigial as in the case of the *Chiroptera*.

(14) *Rodents.*

According to Meckel the marmot, the common hamster, the guinea-pig, the agouti, the jerboa (*Mus sagitta*) and the hare have both *fabellae*, but the beaver, the squirrel, the rat, the dormouse and the Cape mouse (*Aorychus capensis*) only the external *fabella*†. Bartolinus‡ had before Meckel referred to the two *fabellae* in the hare and Tyson to those in the guinea-pig. Owen§ found both *fabellae* in the squirrel as well as a *lunula*. He also found both *fabellae* in *Mus rattus*, the water vole (*Arvicola amphibia*), *Hydromys chrysogaster* (golden-bellied rat) and *Hystrix cristata* (common porcupine). Gruber made a fairly long investigation as to the *fabellae* in rodents. He found both *fabellae* in the following: *Myoxus murinus* (the fat dormouse), *Sciurus vulgaris* (common squirrel), external larger; *Pteromys volans* (? American flying squirrel), external larger; *Tamias striatus* (= *Tamias asiaticus* ?); *Spermophilus citillus* (European souslik); *Mus musculus* (common mouse); *Mus rattus* (common rat), external larger; *Hypudaeus* (? *Mus*

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* *Journal of Anatomy and Physiology*, 1898, p. 322.

† *Loc. cit.* S. 604—5.

‡ See Part I of this memoir, p. 147.

§ *Anatomy of Vertebrates*, p. 384.

amphibius, the water vole); *Lemmus obensis* (Siberian, ? Norwegian, lemming), internal somewhat larger! *Fiber zibethicus* (musquash), external large and internal small; *Dipus jaculus* (? *Alactaga jaculus*); *Spalax typhlus* (mole rat); *Lepus timidus** (hare), external in facet on external condyle, internal in groove, but this may happen with external; *Lepus cuniculus* (rabbit), internal small as compared with external; *Cavia* (guinea-pig), external larger; *Dasyprocta aguti,* both as hemisesamoids† (hyaline cartilages). On the other hand in a *young* specimen of *Castor fiber,* he found neither and this was also the case in *Cercolabes prehensilis* (tree porcupine), although *gastrocnemius* was very fully developed. In *M. glis* Gruber found the external *fabella* only, articulating on condyle. He examined three kinds of *Cricetus* (hamster) and found in one both *fabellae,* in a second the external only, and in a third neither.

We now come to our own investigations. In the authorities cited above we have no records of the presence of the *cyamella* and only the single reference of Owen to the presence of a *lunula* in the squirrel.

A dissection of the common rabbit showed both *fabellae* and the *cyamella* but no *lunula* (see Plate XXIV, Fig. 58). In a mounted skeleton, however, we found both *fabellae,* the *cyamella* and both anterior *lunulae.* A dissection of the common squirrel (two specimens) showed both *fabellae,* the *cyamella* and an anterior mesial *lunula* (see Plate XXIII, Fig. 55 *b*). In a third very young squirrel the lateral *fabella* was found, but no mesial had yet developed (no microscopic examination made); the *lunula* was well developed‡. In the African squirrel (*Paraxerus jacksoni*) we found both *fabellae,* the *cyamella* and the anterior mesial *lunula* (see Plate XXIV, Figs. 60 *a* and 60 *b*).

In the domestic mouse both *fabellae* and both anterior *lunulae* but no *cyamella* were found; the smallness, however, of the knee-joint renders (without microscopic examination of sections) the discovery of the sesamoids rather precarious.

In the musk rat (*Fiber zibethicus*) we found both *fabellae,* two anterior *lunulae,* of which the mesial is under the ridge of the patellar depression, and one posterior lateral *lunula.* There was no trace of a *cyamella,* and it is possible that it does not exist in the musk rat or is replaced by this posterior *lunula* (see Plate XXIV, Figs. 61 *a* and 61 *b*).

Turning to mounted specimens we found both *fabellae* in the following: *Trichys guentherai* (Gunther's porcupine), *Lagostomus trichodactylus* (Viscacha§) in two specimens, *Thrynomys swindernianus* (cave rat), *Mus fuscipes* (brown-footed rat), *Mesembriomys hirsutus* (Gould's jerboa rat), *Microtus amphibius* (water vole), *Alactaga decumana* (Kirchig jerboa), *Alactaga acontium* (Siberian jerboa), *Spermophilus guttatus* (a souslik), *Sciuropterus volucella* (North American flying

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* Mivart and Murie, *Proc. Zool. Society*, 1866, p. 413, state that the hare differs from the guinea-pig by having neither *fabella* (!).

† We have found orthosesamoids in our specimen.

‡ We also found only the lateral *fabella* in a mounted specimen of the common squirrel, but this is hardly evidence even of the occasional absence of the mesial. Cf. Meckel's statement p. 357.

§ There is a double-sized lateral and a smaller mesial *fabella*.

squirrel), *Dasyprocta aguti* (golden agouti)*, *Cavia porcellus* (guinea-pig), *Myopotamus coypus* (coypu), in a second specimen only the external *fabella* was preserved, *Anomalurus fraseri*, in this case the *cyamella* had been preserved.

In a skeleton of *Dolichotis patachonica* (Patagonian cavy) no *fabellae* remained, but there were exceedingly well-marked depressions for the *fabellae* on the condyles†, and in another skeleton of *Dasyprocta aguti* where the two *fabellae* had disappeared their positions were well marked by depressions on the condyles. There were well developed pits also on the condyles, almost certainly for *fabellae*, in *Hystrix leucura*. Occasionally when the *fabellae* were not traceable, *lunulae* had been preserved; thus in *Hydromys chrysogaster* (golden-bellied rat) we found both anterior *lunulae*. In another specimen, however, we found both *fabellae* and the anterior *lunulae in situ*. These are reproduced in Plate XXIII, Fig. 57 *a*. The *lunulae* of the anterior border of the semilunar cartilages existed in another skeleton of *Dolichotis patachonica*†. In a mounted specimen of *Hystrix cristata* (crested porcupine) the *fabellae* and *cyamella* failed but there appeared to be an external anterior *lunula*, while in *Synetheres insidiosus* (a South American porcupine) there were, if there ever had been any, no remaining sesamoids, not even the *lunulae*.

Parsons, who has dealt at length with the myology of the rodents, wrote as follows in 1894‡ of the *Hystriomorphinae*:

> The presence or absence of the *fabellae* does not seem to depend on the affinities of the animal, as they are large in *Aulacodus* on both sides, while in *Myopotamus*, only the outer one is present. In *Dasyprocta* they are both present, in *Calogenys* both absent.

We have found in a specimen of *Myopotamus* both *fabellae* present.

Again:

> In the *Sciuromorphinae*, however, they were found in every case except that of *Castor canadiensis* (*Sciurus*, *Pteromys*, *Xerus*, *Spermophilus*, *Arctomys*, *Castor*).

In a paper of 1896§ on the "Myology of the Rodents," Parsons writes that all except fam. *Spalacidae* (*Rhizomys*, *Georychus*, *Bathyergus*) have two *fabellae*.

It will be seen that a good deal of work remains to be done on the rodents. It is not easy to obtain adequate material to determine an isolated problem of the knee-joint like the present. It is to be hoped that myologists in the future may pay attention to the existence of *cyamella* and *lunulae* as well as of the *fabellae*. But our experience shows us that without a most careful examination, which in many cases ought to be microscopic, it is by no means easy to ascertain the presence or absence of these sesamoids even in most specimens of the smaller rodents. Further, it is quite clear from the material we have gathered together that persistent presence or absence in the same species cannot be asserted from the examination of isolated specimens.

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[*] Windle found both *fabellae* in *Dasyprocta isthmica*, *Journal of Anatomy and Physiology*, 1896—97, p. 352, and Mivart and Murie, *Proc. Zool. Society*, 1866, found both in *Dasyprocta cristata*.

[†] Windle, *Journal of Anatomy and Physiology*, 1896—1897, p. 352, says he found only the lateral *fabella*.

[‡] "Myology of the *Sciuromorphinae* and *Hystriomorphinae*," *Proc. Zool. Society*, 1894, p. 291.

[§] *Proc. Zool. Society*, 1896, p. 181.

The *Rodentia* undoubtedly show, however, a greater persistency of the *fabellae* than the *Insectivora*, and what is more a greater prevalence of *cyamella* and anterior *lunulae*. Had these been directly searched for by others as well as ourselves, we believe their prevalence would have been much more emphasised in the reader's mind. The appearance in a few cases of the external posterior *lunula* is also of much suggestiveness. We have not, however, yet come across a case of this *lunula* co-existing with the *cyamella**. In the rodents we have further evidence of the difference in origin of the two *fabellae*, partly in the fact that when both are present the lateral is larger and more important than the mesial, it being as in man an attachment of more than one muscle, and partly in the second fact that, in those species which have a single *fabella* only, it is invariably the mesial which fails.

Before we pass to the *Edentates* as linking up the groups we have already discussed we must consider the Primates themselves and ascertain whether they obey the same general law, namely that the lower and more primitive forms are multi-sesamoidal with regard to their knee-joints, while the higher types have retained fewer, or if they exhibit any at all it is only as anomalies.

(15) *Primates.*

We have already discussed Man. The lateral *fabella* occurs as an anomaly and may be either a hemisesamoid or an orthosesamoid. The mesial *fabella* as an orthosesamoid has not yet been satisfactorily demonstrated†. It is clearly far rarer than the lateral *fabella*. There is evidence for the *cyamella* as a very rare anomaly and possibly still more rarely of *lunulae*. The comparative frequency of these anomalous sesamoids of the knee-joint in men of different races would be a study of considerable interest and might give valuable hints. It should not be hard by means of skiagrams to obtain something like comparative figures for, say, Japanese, Negroes‡, Australians and American Indians which might be set against our English work.

Turning to the *Anthropoidea* we meet the difficulty so often referred to, namely, that the larger skeletons are apt to be too much cleaned for present purposes.

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* The reader will have noted that we have dropped the adjective "lateral" in this latter portion of our work. So far we have no evidence beyond Pfitzner's for cats and our own still more doubtful for man of the existence of the mesial *cyamella* even as an anomaly.

† In 1909 Lunghetti asserted that he had found the mesial *fabella* in two cases, but he does not give evidence that they were orthosesamoidal, *Internationale Monatschrift für Anatomie und Physiologie*, Bd. XXVI. S. 71. L. Stieda in 1902 found and exhibited to the German Anatomical Society (*Anatomischer Anzeiger. Verhandlungen der Anatomischen Gesellschaft. Ergänzungsheft zum Bd.* XXI. 1902, S. 127) a mesial *fabella* from a human cadaver. It articulated with the mesial condyle, and was enclosed in the knee capsule. It could be clearly seen and felt. Stieda says it was 1 cm. about in diameter, and therefore a very large size for even a lateral *fabella*. He did not apparently extract the *fabella* and so make himself absolutely certain that it was ossified. He had possibly not fully studied the difficulties of the subject for he describes (S. 128) Heister's figure as "eine sehr gute und lehrreiche Abbildung"! See our p. 150 and Plate II.

‡ Chudzinski according to Poirier holds that the sesamoid bone of the *inner* head of *gastrocnemius* is invariable in the Negro! We have been unable to verify the reference and very strongly doubt the truth of the assertion. Dr E. C. Derry kindly examined for us two negro cadavers in his dissecting room and found in both cases neither mesial nor lateral *fabellae*.

Meckel does not seem to have examined the *Anthropoidea* although he deals fairly fully with the lesser apes. Gruber has also no data. Owen's paper of 1866 contains no reference as to the sesamoids of the knee-joint*. Macalister in his paper on "The Muscular Anatomy of the Gorilla"† writes that

> The popliteal muscle had a sesamoid bone or cartilage; it [the muscle?] was about as large proportionately as in man.

Thus Macalister seems the first to have noted the *cyamella* in the gorilla. We have already drawn attention to the fact that Camper had found it in the orang in 1791. Macalister in a further paper‡ dealing with the chimpanzee, remarks:

> The popliteus was small and had no sesamoid nodule in its tendon thereby agreeing with Wilder's and differing from Vrolik's specimen. Trail did not find it [? *popliteus*] present.

Wilder in his paper on "The Comparative Myology of the Chimpanzee"§ states, when dealing with *popliteus*, that he "could not find the cartilaginous nodule in the external lateral ligament where the muscle arises." It is not clear that this really does refer to the *cyamella* Wilder continues: "Trail and Tyson could not find the muscle, but it was present in Vrolik's chimpanzee and in the gorillas of Wyman and Duvernoy" (*l.c.* p. 375). The "it" of Macalister's sentence just cited should refer to *popliteus*, otherwise Macalister has misread Wilder, who is referring to the muscle and not to the sesamoid||.

It would seem then that of the anthropoids the gorilla, chimpanzee and orang have no *fabellae* and that gorilla and orang have the *cyamella*. The chimpanzee on the weight of the evidence published above does not seem to have the *cyamella*. Reports that the chimpanzee has it arise from a not unnatural reading of Macalister's loose wording, at most it can only be as an anomaly. Thus even in this respect man approaches more closely to the chimpanzee than to the gorilla or orang.

By the courtesy of Sir Arthur Keith we were able to dissect the knee-joint of a chimpanzee. We found no *fabellae*, and not a *cyamella*, but a scarcely perceptible thickening of the popliteal tendon¶. We procured and dissected the knee-joints of two orangs. In neither case was either *fabella* present but in both cases there was a large *cyamella*: see Plate XXVI, Figs. 68 and 69. Sir Arthur Keith kindly allowed one of us to examine his manuscript thesis of 1894 entitled: "The Myology of the Catarrhini. A Study in Evolution." It is a most valuable work of 600 pages presented to the Library of the Royal College of Surgeons in 1919. In this work he notes that he has not found the *fabellae* in gorilla or orang, but he found the *cyamella* (described as sesamoid in the tendon of *popliteus* above

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* "Osteological Contributions to the Natural History of the Anthropoidal Apes, No. VII." *Trans. Zool. Soc. London*, Vol. V. The femora of gorilla, chimpanzee and the orang are discussed on p. 14, *et seq*.

† *Proc. Royal Irish Acad.*, Vol. I. 2nd series, p. 505. Dublin, 1870, p. 74.

‡ "Myology of the Chimpanzee and other Primates," *Annals of Natural History*, 1871, p. 9.

§ *Boston Journal of Natural History*, Vol. VII. p. 375, 1861.

|| Gruber says Vrolik did *not* find the sesamoid; and we think Gruber naturally read Macalister's sentence to indicate that Vrolik had found the sesamoid and he wished to contradict Macalister.

¶ Our chimpanzee was like those of Wilder, Macalister and Vrolik in the fact that *popliteus* was present.

external tibial tuberosity) in both, but not in the chimpanzee (p. 307). On the other hand he states that he did find in a chimpanzee (p. 297) a mesial *fabella*—i.e. the inner tendon of head of *gastrocnemius* contained a sesamoid. This result is of considerable importance. The consensus of opinion is against the occurrence of any *fabellae* in the chimpanzee but the occasional anomalous appearance of a *fabella*, especially a mesial hemisesamoid, would be exactly what we might anticipate with the close relationship that exists between man and the chimpanzee.

Hylobatinae. Hepburn in his memoir on "The Muscles and Nerves of the Anthropoid Apes" (*Journal of Anatomy and Physiology*, 1892, p. 337), found a sesamoid bone developed in each tendon of origin; in dealing with *popliteus* he makes no mention of the existence of a sesamoid, which he would have been pretty certain to have done had it existed. We dissected the knee-joint of a gibbon (*Hylobates muelleri*) and found the mesial *fabella* only, no *cyamella** and no *lunulae*: see Plate XXVII, Fig. 72. This result is in full accord with Sir Arthur Keith's expressed in his thesis: "Gibbon: sesamoids extremely variable even in adult animals. There may be one in each head or both may be absent or either may be absent" (p. 297). In Diagram 75 of Sir Arthur Keith's thesis figures the knee-joint of a gibbon, the internal head of *gastrocnemius* with large, the external head with minute sesamoid. He has never observed the *cyamella* in the gibbon. The existence of individual gibbons in which the mesial *fabella* only appears, while it may seem like the exception which proves the rule, is not so really, for the rule is that no species occurs in which the mesial *fabella* is invariably present and the lateral *fabella* invariably absent. The invariable presence of the lateral and invariable absence of the mesial *fabella* is as we have seen characteristic of certain species†.

The mounted skeletons of *Anthropoidea*, e.g. those at the British Museum (Natural History) and at the Royal College of Surgeons, are all "too clean" to show the sesamoids of the knee-joint. It may be hoped that in future the *cyamella*, where it occurs, will be preserved and mounted as well as the *fabellae*, both in these cases and those of the Old and New World Monkeys, although very often the *fabellae* are preserved in the latter.

Simiadae (Old World Monkeys). We have already seen that the existence of both *fabellae* in the Old World Apes was known to Sylvius and Riolanus (see our pp. 144—145). Meckel gives a fairly extensive list of lemuroids, New World and Old World Monkeys in which he had observed both *fabellae*‡. Gruber notes that he had found both in the genera *Cercopithecus*, *Inuus*, *Cynocephalus*§, and *Cebus*

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* The tendon of *popliteus* was "horny," but there was no sesamoid.

† It is worth noting that even in the gibbon Keith (*Thesis*, p. 298) found the lateral *fabella* more frequently present than the mesial.

‡ *Loc. cit.* S. 634—635.

§ We have found both *fabellae* in *Macacus leoninus*, *M. pileatus*, *M. rhesus* (several specimens), *M. inuus* (Barbary ape), *M. cynomolgus* (several specimens), *Cynocephalus anubis*, *M. nemestrinus* (two specimens), *M. speciosus* (Japanese ape), *Cercopithecus lalandii* (two specimens), *Colobus ursinus*, *C. vellerosus*, *Nasalis larvatus* (two specimens), *Semnopithecus orientalis*, *S. entellus*, *Papio hamadryas*, etc. etc. Burdach, Olg and others have noticed other cases from 1838 onwards.

(New World Monkey). In *Cynocephalus* he found sometimes the mesial sometimes the lateral the larger. In the remainder the lateral was invariably the larger. Neither Meckel nor Gruber* refers to the *cyamella* as existing in either *Simiadae* or *Cebidae.*

Keith in the above-mentioned thesis takes *Semnopithecus* as his type. He thus describes the relation of the muscles in his type to the sesamoids:

Gastrocnemius. Origin. *External head*; arises by musculo-tendinous fibres from a sesamoid embedded in the general capsule of the knee-joint and plying over the upper convexity of the external femoral condyle, and from the shaft of the femur immediately above and external to the femoral condyle (p. 296).

Internal Head: from a sesamoid situated similarly to that of the external head and from the shaft of the femur immediately above the internal condyle.

Plantaris. Origin; from the sesamoid in the external head of *gastrocnemius* (p. 304).

Popliteus. Origin. By a rounded tendon from the lateral aspect of the external femoral condyle beneath the external lateral ligament and within the capsule of the knee-joint....There is a sesamoid in the tendon over the external tibial tuberosity (p. 307).

The arrangement of the muscles relative to the sesamoids in *Semnopithecus* as given by Keith is curiously like what we have learnt of the relation of the muscles to the *fabellae* and *cyamella* when they occur in man. This is peculiarly true of *plantaris*, and is evidence, similar to that provided in the memoir on the Femur by Pearson and Bell, that in a number of respects man is closer to the *Simiadae* than to the *Anthropoidea* whether by retaining more primitive characters, or by reverting to earlier types as better suiting his differentiated development.

The *cyamella* as well as both *fabellae* are present not only in *Semnopithecidae*, but in macaques, *Cynocephali* and practically all the *Simiadae*, we have been able to investigate†. Our drawing (Plate XXVII, Figs. 71 *a* and 71 *b*) shows the two *fabellae* and the *cyamella* in *Mycetis palliatus*, both in position and after extraction.

Lastly we note that we have not been able to find *lunulae* in any knee-joint we have examined of the *Simiadae.* Should they occur, we believe their appearance is very rare.

Cebidae (New World Monkeys). Here the problem is somewhat easier than in the *Simiadae.* The bulk of the species being smaller in size our purchased specimens were less "clean," i.e. the muscular attachments remained. We were thus better able to give positive evidence as to the non-existence of *lunulae.*

(i) We found both *fabellae* in *Hapale aurita* (white-eared marmoset), *Midas oedipus, M. midas, M. ursulus* (two specimens), *M. auritus* (golden marmoset),

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* It is the more noteworthy in Gruber's case for he does refer to a large *cyamella* in *Stenops tardigradus.*

† We have noted *cyamella* in *Nasalis larvatus*, in *Macacus* (the Bonnet monkey), *M. inuus*, etc. and believe that it is general in the *Simiadae*, although usually cleaned off in mounted specimens.

Hapale jacchus (common marmoset), *Jacchus auritus*, and two other specimens (sp. ?) of *Jacchus*, *Cebus capuchinus* (two specimens), *Cebus* (sp. ?) (two specimens), *Lagothrix humboldtii* (two specimens), *Pithecia satanas* (Black Sahi), *Pithecia* (sp. ?), *Nyctipithecus*, *Chrysothrix sciurea* (three specimens).

(ii) In none of these were any *lunulae* to be found.

(iii) We found the *cyamella* definitely in *Hapale jacchus*, *Jacchus auritus*, in two other specimens of *Jacchus* (sp. ?), in *Midas ursulus* (two specimens), *M. midas*, *M. auritus*.

We could not definitely assert its absence in *Cebus* or *Pithecia*, or the squirrel monkeys, as the material was less complete.

The case of *Ateles* is also doubtful; two specimens were examined, in neither was a *cyamella* found; one possessed no *fabellae* and the other only an internal one. This may be due to rough usage, or it might emphasise the correspondence with the gibbons which Pearson and Bell noticed in considering the femur of *Ateles*.

While we should have anticipated finding *lunulae* in the more primitive types of *Cebidae*, we have not done so, and conclude that the *lunulae* were lost to the Primates except as occasional anomalies before the separation of *Simiadae* and *Cebidae*. The chief difference between the apes of the New and Old Worlds is that the *Cebidae* (except in the possible case of *Ateles*) have never progressed to the stage of the *Hylobatinae* with loss of *cyamella* and to the stage of the *Anthropoidea* with loss of *fabellae* as well.

Lemuroidea. We naturally turn with much interest to the prosimian group; for the sesamoids of the knee-joint in their case ought to link up with those of the Rodents and Insectivores and the test of this will clearly be the occasional or persistent occurrence of *lunulae* in one or more species.

Here again the expense of procuring moist material, especially when it is to be investigated only for a single point in as wide a range of species as possible, has much handicapped our work*.

We turn first to the lemurs proper and notice the following remarkable results†: Both *fabellae*, the *cyamella* and the lateral anterior *lunula* were found in *Lemur varia*, *L. catta*, *L. melanocephalus*, *L. coronatus*, *L. macaco leucomystax*‡. There is no doubt therefore that the *Prosimia* add the lateral anterior *lunula* to the *fabellae* and *cyamella* of the *Simiadae*. See Plate XXVII, Figs. 70 and 73.

Turning to *Galago* we have found: *Galago alleni*, both *fabellae*, the external very large, the *cyamella* and the lateral anterior *lunula*. *G. crassicaudata* ex-

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* The aid given by the Government Grant Committee to the Study of Pearson and Bell on the femur, for the purchase of primate material, has of course aided us.

† Meckel notes merely the existence of both *fabellae* in *L. mongoz*, and *L. albifrons*. Gruber states that *Stenops tardigradus* lacks both *fabellae* but has a large articulating sesamoid in the tendon of *M. popliteus*.

‡ Femora without the semilunars of *L. rubiventer*, *L. niger*, and *L. macao varius* showed both *fabellae* and in the first case the *cyamella* also remained.

hibited also the two *fabellae**, *cyamella* and lateral anterior *lunula.* Another *Galago* (sp. ?) gave the same result, while a series of mounted specimens provided sometimes (as oversights of the articulator!) the *fabellae*, or the *cyamella* and once a *lunula.*

In *Lemur macao niger*, both *fabellae*, the *cyamella* and apparently *both* anterior *lunulae* were present. *Nycticebus javanicus* had also *fabellae*, *cyamella* and lateral anterior *lunula*; a mounted specimen of *Nycticebus tardigradus* retained only the *cyamella*†. Going further we find *Loris gracilis* with *fabellae*, *cyamella* and lateral anterior *lunula*, apparently but difficult to decide definitely without a section there was also a lateral posterior *lunula*; a mounted specimen showed more definitely two *lunulae.*

Perodicticus potto gave us *fabellae*, *cyamella* and lateral anterior *lunula*, and the *cyamella* was confirmed in a mounted specimen of the Calabar potto.

We now reach cases in which we can be more or less definite as to absence of *lunulae* because the semilunars were preserved.

Chiromys (Aye-Aye) dissected gave both *fabellae* and *cyamella*, there were no *lunulae*: see Plate XXVIII, Fig. 74*a*. A second specimen in the Biometric Laboratory was investigated with precisely the same result no *lunulae.* Mounted specimens show the *fabellae* as a rule; in one we found the *cyamella* preserved. The existence of the *cyamella* in the Aye-Aye was, we believe, first noted by Owen. He makes the important point that "there is a sesamoid in the external lateral ligament of the knee-joint at its insertion into the head of the fibula." (See our Plate XXVIII, Fig. 74*b*.) He also refers to the *fabellae* in the Aye-Aye. ("On the Aye-Aye." *Trans. Zool. Soc. London*, Vol. v. p. 53, 1866.)

In *Avahi laniger* and *Hapalemur griseus* we found both *fabellae*, definitely no *lunulae*, the *cyamella* failed but might possibly have been removed. A mounted specimen of *Avahi laniger* as well as one of *Propithecus diadema* showed only the *fabellae.* In a mounted specimen of *Lepidolemur microdon* we found only *cyamella* and external *fabella.*

Chirogaleus and *Indris brevicaudata* gave only the *fabellae* and *cyamella.*

Lastly *Tarsius spectrum* has both *fabellae* and the *cyamella.* Curiously enough as in *Galago crassicaudata* the mesial *fabella* may be larger than the lateral, although we have seen specimens mounted having solely the lateral *fabella.* We can only say we have not yet found *lunulae* in *Tarsius.* This is not very strong evidence against their existence, because the semilunars are so small that without fresh moist material carefully sectioned and microscopically examined it would be difficult to give a strong negative. When we resume our results for the *Lemuroidea*

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* Mesial larger than lateral.

† Mivart and Murie (*Proc. Zool. Soc. London*, 1865, p. 129) in their "Anatomy of *Nycticebus tardigradus*" note of *M. popliteus* that it is inserted by a very strong tendon containing a large sesamoid bone...into the depression outside the lateral condyle. They do not mention when discussing *M. gastrocnemius* the existence of sesamoids therein.

we see their great importance. The non-appearance of the *lunulae* in the marmosets is more or less accounted for, because in the lower stage of the lemuroids while they are found, they are not universal; the *lunulae* have already begun to disappear, although the type lemuroid may be said to be that which has both *fabellae*, the *cyamella* and the lateral anterior *lunula*. A further point which will develop its interest as we proceed is that noted by Owen in the Aye-Aye, namely the situation of the *cyamella*. We reproduce, Plate XXVIII, Fig. 74 *b*, Owen's drawing. The reader will note how in a marvellous manner the *cyamella* and the lateral *fabella* are being brought into close association. The bearing of this on the origin of both of them will be seen in the sequel. The *cyamella* articulates with the head of the fibula and the possibility of the lateral *fabella* articulating with the *cyamella* is seen to exist*. Fig. 74 *a* shows our actual dissection; the head of the *fibula h* is in contact with the *cyamella f*. The two branches of *gastrocnemius a* and *b* have been deflected to show the *fabellae c* and *d*, as dark patches; the positions in which they rest on the condyles are also indicated.

Plate XXVIII, Fig. 77 gives the dissection of the knee-joint of *Indris brevicaudata* and shows the *fabellae* and *cyamella* in position.

Plate XXVII, Figs. 70 *a* and 70 *b* show in *Lemur varia* the type knee-joint of the lemurs. *a* is the external, *b* the internal *fabella*, *c* the *cyamella* on the head of the fibula and *d* is the lateral anterior *lunula*. Figs. 75 and 73 show with the same notation the same four sesamoids in *L. catta* and *L. melanocephalus*.

Plate XXVIII, Fig. 76 gives the lateral aspect of the dried knee-joint of *Perodicticus potto*. In this case no *fabellae* were present, having most probably been knocked off. The *cyamella* (*cy.*) is seen resting on the head of the fibula, the popliteal tendon sends a branch down to the head of the fibula. On the posterior and upper face the *cyamella* is concave, and this hollow seems designed for the *fabella* to rest in. The anterior external *lunula* is in its usual position, i.e. in the semi-lunar cartilage just in front of the popliteal tendon.

The main point of interest in the lemuroids is undoubtedly the link that the *lunula* presents with the Rodents and Insectivores. But the close relationship which the forms and positions of *cyamella* and lateral *fabella* in the lemuroids bear to a continuous articulating system connecting the head of the fibula with the lateral condyle of the femur is full of suggestiveness. This relationship of lateral *fabella* and *cyamella* explains we think to some extent Gruber's confusion of the two when as in man only one occurs and then as an anomaly.

(16) *Edentates.*

Before we discuss the very peculiar sesamoidal knee-joint characters of this order it seems desirable to be clear on a point which is rather obscure in some of the earlier writers, who are liable to confuse the *cyamella* with the lateral *fabella*.

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* It is worth while comparing the drawing of the cat's knee-joint on our p. 352 with Owen's of the Aye-Aye's knee-joint. The manner in which with flexed limb the lateral *fabella* and *cyamella* come in contact will be obvious.

The following rules appear to us to hold for the somewhat heterogeneous material classed as *Edentata*.

(i) *Lunulae* may occur.

(ii) There are no cases at all of the mesial *fabella* (with or without the lateral *fabella*) occurring[*].

(iii) When the *cyamella* occurs it is very extensive and when it occurs there is no lateral *fabella*.

(iv) Conversely when the lateral *fabella* occurs there is no *cyamella*.

(v) There are cases in which no sesamoids whatever of the knee-joint have been found.

We have accordingly reached a stage in which the mesial *fabella* has not been evolved, for it does not seem to exist in any earlier types. Further we have reached a stage where a non-sesamoidal type of *Monodelphia*—a type afterwards so widely developed in the *Ungulata*—is seen alongside a sesamoidal type afterwards so widely developed in *Rodentia, Insectivora, Primata* and *Carnivora*. It would seem as if the *Edentata* as modern representatives of these primitive *Monodelphia* must be from the knee-joint sesamoidal standpoint most suggestive from their very heterogeneity. Meckel speaks of the external *fabella* only as existing in the Edentates, and cites as examples the Anteater and Sloth (Aï). He does, however, say that this external *fabella* "indessen mehr dem Kneekehlmuskel gehört[†]." He also found this sesamoid very large in the Edentates, much larger than in other animals, a remark we can fully confirm.

Gruber[‡] found in *Bradypus tridactylus* (the three-toed sloth) and in *Dasypus sexcinctus* (six-banded armadillo) no sesamoids in the heads of *gastrocnemius* or in the tendon of *popliteus*, but in *Myrmecophaga* there was in the tendon of *popliteus* a sesamoid articulating with the *capitulum fibulae*.

Gruber's result for *Bradypus* is somewhat surprising, for while there are no *fabellae* there exists, as Blainville first noted, a great *cyamella*; there is in addition a *lunula*. The point is well put by Mackintosh in his paper "On the Myology of the Genus *Bradypus*"[§] that while there are no *fabellae* in *gastrocnemius*: "There is a large wedge-shaped *fabella* [i.e. *cyamella*] in the origin of this muscle [i.e. *popliteus*], the tibial surface of which is the larger and articulates both with the femur and the tibia."

We are, we think, reaching a stage of evolution in which lateral *fabella* and *cyamella* have coalesced and form a single large sesamoid. This large sesamoid may articulate with both femur and tibia as in *Bradypus*, or form a link between the femur and the *capitulum fibulae* as in the Anteater[||]. Gruber, who never

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[*] Compare what is stated later as to *Orycteropus*, p. 369.

[†] *Loc. cit.* p. 635.

[‡] *Loc. cit.* p. 62.

[§] *Proc. R. Irish Acad.* Vol. I. *Science*, 2nd Series, p. 528.

[||] This alternative, tibio-femoral or fibulo-femoral, link of the compound sesamoid might conceivably have bearing on the appearance of *fabellae* in *both* heads of *gastrocnemius*.

clearly distinguished between lateral *fabella* and *cyamella*, remarks* that in animals in which the *ossiculum* in *gastrocnemius externus* articulates on the head of the fibula, *popliteus* springs from the knee-capsule and from the *capitulum fibulae* not from the femur itself. We have indeed lost the distinction between the two sesamoids although we shall sometimes continue to speak of the *cyamella* as the single large sesamoid of the Edentates, because it is peculiarly associated with the *popliteus* †. The transition stage of the lemuroids in which *fabella* and *cyamella* were coming into close contact now provides much suggestiveness, which will be again emphasised when we come to the Marsupials.

In Plate XXIII, Fig. 55 *a* the reader will find a photograph of *Tamandua tetradactyla*‡ with the *cyamella* attached to the popliteal tendon originating in the popliteal groove on the lateral face of the external condyle. It will be easy to see how a not too careful articulator can mount this sesamoid on the surface of the external condyle. The *cyamella* itself articulates on the lateral side with external articular surface of the tibial head. This head projects over and downwards on the head of the fibula, which does not articulate with the femur. The large *cyamella* bears some trace of being a possible compound, i.e. *cyamella* and *fabella*. We have found this *cyamella*—without *fabella*—in:

Bradypus tridactylus (two mounted specimens), *Bradypus arctopithecius*, *Choloepus hoffmanni*, *Choloepus didactylus* (two specimens where the *cyamella* appears to be almost in the position of an anterior *lunula*), *Myrmecophaga tamandua*, *Manis dalmanni* and *Manis javanica*.

There is, we think, no doubt that in *Manis* we are dealing with a *cyamella* and not a lateral *fabella*. Owen, however, speaks of this sesamoid in *Manis* as an external *fabella* (*loc. cit.* p. 409). Windle and Parsons in their paper "Myology of the Edentates"§ remark under *gastrocnemius*: "It is interesting to notice that all observers have recorded the absence of *fabellae* except in the *Orycteropidae*" (p. 1004). This statement seems to overlook Owen and Gruber's statements as to *Manis*, and Macalister's with regard to *Tamandua* and *Myrmecophaga didactyla*.

In his paper|| on the "Myology of *Bradypus tridactylus*" Macalister writes of the heads of *gastrocnemius*, and says that there are no sesamoids in the origins of the muscle as there are in *Tamandua* and *Myrmecophaga didactyla*. He then continues that Meckel refers to the Aï as possessing one of these in its origin and in the *Megatherium* one of these appears to have existed for the outer head of the *gastrocnemius*. Later referring to *popliteus* Macalister states correctly that the

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[*] *Loc. cit.* p. 65, (15).

[†] It would be better to speak of it as the parafibula, except that in certain cases it has changed its position so as to articulate with the tibia.

[‡] Macalister in his "Myology of *Bradypus tridactylus*" (*Annals of Natural History*, 1869, p. 15) speaks of the large sesamoid in the tendon of *popliteus* and states that it does not seem to occur in *Tamandua* (!).

[§] *Proc. Zool. Soc. London*, 1899, p. 990.

[||] *Annals of Natural History*, 1869, p. 15.

sesamoid, a large one, of *Bradypus* lies in its tendon*, and then says that *popliteus* contains no sesamoid in *Tamandua*, *Orycteropus* and *Dasypus*. We feel sure he is wrong in the first case, the sesamoid of *Tamandua* is a *cyamella*. He is right about *Dasypus*. Owing to the kindness of Sir Arthur Keith it was possible for us to dissect a knee-joint of *Dasypus sexcinctus*; no *fabellae* were found, and only a cartilagenous thickening of the popliteal tendon, which it was not even possible to consider as a hemisesamoid†. There is no doubt, we think also, that the sesamoid in the fossil *Megatherium cuvieri* in the museum of the Royal College of Surgeons is a *cyamella* not a *fabella*. This fossil possesses also a very fine lateral anterior *lunula*: see Plate XXV, Fig. 66. Owen makes no reference to it, and we have not found any mention of it elsewhere. This sesamoid like the *cyamella* rests on the fibular articulating surface, but we consider that it would have been in the anterior border of the lateral semilunar. The so-called "fabella" is a parafibular sesamoid, it does not lie in the popliteal sulcus, but the tendon of popliteus would pass directly to it; it is not on the posterior surface but the lateral surface of the external condyle. It represents really something anterior to the differentiation of parafibula into *cyamella* and *fabella lateralis*.

Windle and Parsons' statements as to the sesamoids of the Edentates, namely that the

(*a*) *Bradypodidae* have no *fabellae* but a *cyamella* (p. 1010),
(*b*) *Myrmecophagidae* have no *fabellae* but a *cyamella* (p. 1011),
(*c*) *Manidae* have no *fabellae* but a *cyamella* (p. 1013),
(*d*) *Dasypodidae* have neither *fabellae* nor *cyamella* (p. 1012),

seem to us to represent the facts.

With regard to the *Orycteropidae* there appears to be some diversity of opinion. Humphry in his paper on the "Myology of *Orycteropus capensis*‡ (Aard Vark)" states that the three heads of *gastrocnemius* arise from the outer and inner condyles of the femur and from the head of the *fibula*, [? *soleus*], and that the two former have both sesamoids, i.e. *fabellae*. Galton in his "Myology of *Orycteropus capensis*"§ mentions only a sesamoid in the outer head of *gastrocnemius*, and cites Owen (*Anatomy of Vertebrates*, Vol. II. p. 409). He speaks of this *fabella* as "behind the outer condyle of the femur," and seems to think it corresponds exactly to what Meckel observed in *Choloepus didactylus* "though not mentioned by him in connection with this [*gastrocnemius*] or any other muscle." This is overlooking what Meckel himself wrote about the closer association of the sesamoid with the *popliteus* (see our p. 367)‖. And if *Orycteropus* really resembled *Choloepus*, then it would be a

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* The drawing of the skeleton of *Bradypus tridactylus* in Parker and Haswell's *Text-book of Zoology*, p. 529, shows the *cyamella* and the *lunula*.

† In one mounted specimen of *Dasypus sexcinctus* facets on the condyles were observed which some might interpret as vestiges of *fabellae*.

‡ *Journal of Anatomy and Physiology*, Vol. II. p. 313.

§ *Trans. Linnean Soc.* Vol. XXVI. p. 594.

‖ Mackintosh ("Muscular Anatomy of *Choloepus didactylus*," *Proc. Royal Irish Academy*, Vol. II. 1875—77, p. 75) states definitely that *popliteus* has a large *fabella* [i.e. *cyamella*] in the tendon of origin.

cyamella and not a *fabella* with which we are concerned*. The maximum confusion seems to be introduced by Macalister's paper of 1875†. Writing of *popliteus* in *Chlamydophorus* he says: "I found no sesamoid bone in its tendon, as it lies under the external lateral ligament, nor does Hyrtl mention the existence of one. In the three-toed Sloth there is a large sesamoid bone. It is similar in *Choloepus*. There is no sesamoid bone in the *Orycteropus*, but there is in *Cyclothurus* and *Rholidotus*. There does not appear to be one in *Myrmecophaga* or in *Tamandua*" (p. 266). The latter statement is incorrect. The statements as to *Bradypus* and *Choloepus* are correct, but Macalister adds *Chlamydophorus* to *Orycteropus* as without *cyamella*.

Turning to *gastrocnemius*‡ he says that *Chlamydophorus*§ has the usual two heads neither of which possesses a sesamoid. There are no sesamoids in *Bradypus*, but Meckel found one in his specimen [we have already seen that Meckel found a popliteal not a *gastrocnemic* sesamoid]. No sesamoids exist in *Choloepus didactylus*. In *Orycteropus* there is a sesamoid bone in the outer head. [Humphry says he found a sesamoid in *both* heads!] A similar one exists in *Cyclothurus* and *Tamandua*. [This last statement is incorrect with regard to *Tamandua*, it may be correct but we have found only the *cyamella* in three limbs of *Cyclothurus*. If it were correct *Cyclothurus* would be the one Edentate that has both *cyamella* and lateral *fabella*.] As *Chlamydophorus* has neither *fabellae* nor *cyamella*, it must be classed with *Dasypus sexcinctus* rather than with *Orycteropus*.

Putting aside Humphry's statement with regard to *Orycteropus* which may have referred to an anomalous specimen, we have no case in which both *fabellae* occur in the Edentates. Even if Macalister's statement as to *Cyclothurus* be correct, we have only one case in which both *cyamella* and lateral *fabella* are present, and only two cases in which lateral *fabellae* have been recorded at all. There are two cases, *Dasypus* and *Chlamydophorus*, in which neither *fabellae* nor *cyamella* appear.

Unfortunately we have been able to examine relatively few moist knee-joints of the Edentates, and have had to trust largely to other investigators and mounted specimens. This is especially undesirable in the case of *lunulae*. We know, however, that they certainly exist in the Edentates, for example in *Megatherium cuvieri*, and in *Bradypus sexcinctus*, which have anterior *lunulae*, and in *Prodontes gigas* which has a mesial posterior *lunula*. We also think it safe to say that the specimen of *Orycteropus* examined by us (there were alas! no sesamoids *in situ*) had purely cartilaginous semi-lunars, i.e. no *lunulae*.

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* The mounted specimen of *Orycteropus capensis* in the Museum of the Royal College of Surgeons has what seems to us a *cyamella* or parafibular sesamoid, but it is mounted on top of the external condyle! In the mounted specimen of *Manis aurita* the *cyamella* or parafibular sesamoid rests in what we think a much more natural position, i.e. between lateral condyle and head of fibula. It has been slightly separated from the latter for the sake of photography: see our Plate XXV, Fig. 67.

† "Anatomy of *Chlamydophorus truncatus* with Notes on the structure of other species of *Edentata*," *Trans. Royal Irish Academy*, Vol. xxv. p. 219.

‡ *Loc. cit.* p. 267.

§ We have examined two specimens of *Chlamydophorus* and find no *fabellae* or *cyamella*. The tibia and fibula are distally fused, and the latter is the more important member.

The typical Edentate may be said to be characterised by a large *cyamella* with probably one or more *lunulae*. The mesial *fabella* has practically no existence for this order, and in the few cases where the lateral *fabella* appears, it seems interchangeable with the *cyamella*. There is one group, however, *Dasypodidae* in which neither *cyamella* nor *fabellae* appear. The order of Edentates therefore *from our present standpoint* contains two fundamental groups one of which consists of species with non-sesamoidal knee-joint and the other of species with a sesamoidal knee-joint. The characteristic of the latter is the absence of the mesial *fabella* and the coalescence of lateral *fabella* and *cyamella* to form a large sesamoid which may act as an articulating link between the femur and one or other of the lower members of the limb.

(17) *Marsupials.*

It will be clear to the reader that the views expressed above on the affinities of genera, and orders based on the changes in the sesamoids of the knee-joint must receive strong confirmation, or failing it the *coup de grâce*, when we come to the Marsupials.

Historically we find the same vagueness as to the knee-joint sesamoids as in the Edentates. Meckel talks about the sesamoid of the kangaroo as if the kangaroo presented an "external fabella" only, and he makes a similar statement as to the opossum. Before Meckel Blumenbach had already noticed the external sesamoid in the *Didelphia* generally. Other fairly early notices of the external sesamoid are Pandir and d'Alton in *Didelphis cancrivora*, *Halmaturus elegans* and *H. giganteus* (Meckel also noticed it in *Halmaturus*), Cuvier and Laurillard in *Phalangista carifrons*, and Owen in *Myrmecobius* and some other marsupials. Owen describes the sesamoid as above and behind the external condyle of the femur*.

Gruber while still speaking of the lateral sesamoid of the Marsupials, as if it were directly comparable with the external *fabella* in other orders, gives useful information on its muscular relations. In *Didelphis philander* he tells us it is in the tendon of *gastrocnemius externus*, but articulates with the head of the fibula. The same statement applies to the sesamoid in *Dasyurus viverrinus*. Again in *Phalangista vulpina*, the external sesamoid articulates with the *capitulum fibulae* and is united to the fibula "durch eine schlaffe Gelenkkapsel und durch ein starkes Ligament." It is "fiddle-shaped" and with its thicker upper end the tendon of origin of *gastrocnemius externus* is united ("verwachsen"). In these three genera the external ossicle has no relation to the knee capsule and *popliteus* does not spring from the external condyle of the femur, but from the knee capsule and the *capitulum fibulae*†. None of the above has any internal *fabella*. These three species give very typical marsupial sesamoidal systems. They are cases really in which *fabella* and *cyamella* have coalesced and we do not believe it possible to speak of merely the "lateral" *fabella* existing. The remaining two cases, if Gruber has interpreted them rightly, seem to us more or less aberrant variants of the main marsupial type. They are :

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* *Anatomy of Vertebrates*, Vol. II. p. 358.

† *Loc. cit.* p. 59.

Hypsiprymnus, which has no mesial *fabella*, but (according to G.) a large lateral *fabella* articulating with external condyle*, and no sesamoid in the *popliteus*; and:

Halmaturus where (according to G.) *both fabellae* exist; the lateral articulates with external condyle and is united to tendon of *gastrocnemius*, but it had also a short ligament to the external semilunar. It was "hornformig."

We failed to find a mesial *fabella* in two specimens of *Halmaturus* (sp. ?). The species of *Halmaturus* is also not stated by Gruber.

According to Meckel *Halmaturus* (sp. ?) has only the lateral *fabella* and the same statement is made by Pandir and d'Alton for the species *H. elegans* and *H. giganteus*.

The only other record of a mesial *fabella* in the marsupials that we have come across is due to Windle and Parsons†, discussing the anatomy of *Macropus rufus*. They state that the inner head of *gastrocnemius* had also in the older specimen a *fabella* of much smaller size than that of the outer head and differently situated. "This *fabella* lies below the origin of the muscle from the back of the condyle of the femur and between the muscular fibres and that bone upon which therefore it plays."

We have examined a considerable number of mounted specimens of *Macropus*, the lateral sesamoid was almost invariably preserved but we have failed in any case to find a mesial orthosesamoid preserved‡. It is possible that the articulator in all these cases overlooked the smaller mesial *fabella*, but it is equally conceivable that the mesial *fabella* in Windle and Parsons' case was a hemisesamoid§.

Before discussing further other contributions to our knowledge of the knee-joint in the Marsupials, we wish to draw the attention of the reader to Plate XXIX, Fig. 78. The photograph on the right exhibits the lower limb bones of *Phascolomys*, the wombat. In this specimen there are no sesamoids at all, the large lateral sesamoid is seen as a fibular crest and is fused to the bone. The femur articulates with both tibia and fibula, and the head of the fibula comes in contact with the femur above the lateral condyle where the lateral *fabella* is often placed by the articulator. The limb to the left of the plate is that of *Sarcophilus*, the Tasmanian Devil. Here again the femur articulates with both tibia and fibula. The fibula has a parafibula precisely like that of *Phascolomys*. The summit of the fibula touches the femur above the lateral condyle, and there may even be a pit (as in the sea-otter; see our Plate XIX, Fig. 40) for its reception‖. The reader may say: "But this is the large external *fabella* of the marsupial myologists." It is and it is not; it really consists of two sesamoids, the lateral *fabella* and the *cyamella*; *they are in this case fused together and fused to the head of the fibula.*

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* Confirmed by us on an articulated specimen of *Hypsiprymnus murinus*.

† *Journal of Anatomy and Physiology*, 1897, p. 128.

‡ *Macropus rufus*, *Macropus* (sp.), *Macropus agilis*, *Macropus major*, *Macropus bennettii*, etc.

§ At the same time we note that without finding a mesial *fabella*, we have found a cartilaginous thickening—hardly a hemisesamoid—in a specimen of *Macropus agilis* and in one of *Dendrolagus*, the tree kangaroo.

‖ As for example in the mounted specimen in the British Museum (Natural History).

Plate XXIX, Fig. 79, gives a second wombat; here the big sesamoid is not fused to the head of the fibula, but it nevertheless consists of the associated *lateral fabella* and the *cyamella* fused together, a true parafibula. The limb has had to be extended to separate the sesamoid from articulating on the femur. The reader should now examine our drawing after Pfitzner in Plate V, Fig. 13, of this memoir, where the muscular attachments have been entirely removed; he will (as a careful study will show him in the present plates) be able to detect the two portions of the associated bone, which become respectively the lateral *fabella* and *cyamella*. These two portions of the great marsupial sesamoid of the knee-joint are also quite distinguishable in *Phalangista*. See our Plate XXX, Fig. 80.

While the hypothesis that lateral *fabella* and *cyamella* are the product of the breakdown of the parafibula finds much support from the fact that specimens occur in which these sesamoids are actually fused together and to the fibula*, we have to consider how far this hypothesis fits the myological facts and tends to throw light on Gruber's description of the lateral *fabella* in man as a muscular "Knotenpunkt." We have above all to ask whether the Marsupials themselves provide evidence in favour of this break-up of a parafibula. One of the most noteworthy facts in this respect is the occurrence in certain marsupials of a hemisesamoidal *cyamella*, this articulates on the head of the fibula, and upon this *cyamella* articulates the lateral *fabella*, which in its turn is in contact with the upper part of the condyle of the femur.

A specimen of *Macropus* (sp. ?) sent to us from Australia shows a lateral *fabella* —an orthosesamoid—and a hemisesamoid *cyamella*. The specimen consisted only of the lateral branch of the *gastrocnemius* with its orthosesamoid, the hemisesamoidal *cyamella* and the external semilunar cartilage all attached, these for the left hind limb. It was thus not possible to draw the knee-joint showing these in their natural position. It is not, however, difficult to understand their relations to other structures in the knee-joint, if a comparison be made between these drawings and those of other members of the *Marsupialia*. See Plate XXXIII, Figs. 83 *a* and 83 *b*.

In a museum specimen of *Macropus agilis* we found also this hemisesamoidal *cyamella*, while the lateral *fabella* was a large orthosesamoid. In the mesial head of *gastrocnemius* there was a cartilaginous nodule, possibly a hemisesamoid.

In another mounted specimen of *Macropus rufus* the large lateral sesamoid was simply wired on to the femur, a process which completely fails to indicate the sesamoid system as a link between femur and *capitulum fibulae*. Our Plate XXXIV, Figs. 87 and 88, shows museum specimens of *Macropus bennettii* and *Perameles lagotis* in which this sesamoid is simply pinned on to the femur†. Our Plate XXXV, Fig. 90, gives what appears to us to be the position of this sesamoid in Bennett's Wallaby; it has very clearly no relation to the position as indicated in Figs. 87 and 88.

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* We have never in examining several hundred knee-joints found a case in which the sesamoids were fused to the femur, and the only recorded case of ankylosis that we have met with is that reported by Gruber in a specimen of *Myogale*.

† We have also found the parafibular sesamoid treated in this manner in the case of *Dendrolagus ursinus*, where the sesamoid is generally similar to that of *Macropus bennettii*, and again in *Macropus major*.

But a preparation in the Museum of the Royal College of Surgeons* shows far more clearly the whole arrangement of a knee-joint of this character. We owe photographs of it in anterior and lateral aspects to Mr S. Steward (see our Plate XXXIV, Figs. 89 *a* and 89 *b*). They show the sesamoid mounted vertically on the top of the fibula and connected proximally by a tendon to the femur above the lateral condyle. The sesamoid itself is a true parafibula and does not articulate with the femur at all, as is well indicated by the anterior aspect†. In the cases of *Bettongia gaimardi*, *Hypsiprymnus murinus*, and *Myrmecobius fasciatus*, we think there is little doubt that we are dealing with a parafibular sesamoid and not with a simple lateral *fabella*.

Windle and Parsons‡, who have treated at length of the anatomy of *Macropus rufus*, state that

> the tendon of the *popliteus* contains a very strong fibro-cartilaginous *fabella*, which is placed above the head of the fibula to which it is attached, and on which it moves freely. It is with this *fabella* that the large gastrocnemic *fabella*, to be more fully described in another section, articulates.

And again:

> The outer head of the *gastrocnemius* possesses a remarkably large *fabella* already mentioned as articulating with that of the *popliteus*. This was crescentic in shape [see our Plate XXXV, Fig. 90] and in the older specimen measured 2·5 cms. from one corner to the other.

To the muscular attachments as noted by our authors we shall return later. We are here only concerned with making the point that the *cyamella* portion of the associated femoro-fibula sesamoidal link of the Marsupials may in certain species have degenerated into a cartilaginous§ mass—a hemisesamoid. In cases where this hemisesamoidal *cyamella* disappears entirely the whole purpose of the *fabella-cyamella* system is lost to sight, and we are provided with a detached sesamoid on the lateral condyle associated with a system of muscular attachments which formerly were associated with the *capitulum fibulae*.

It seems to us that in the *Marsupialia* there has been written the complete history of the lateral sesamoids of the knee-joint. We have:

(i) No sesamoids at all but a fibular crest, a continuous part of the fibula (some specimens of Wallaby).

(ii) The parafibula is still fused with the fibula but shows signs of splitting into two parts, *fabella* and *cyamella* (certain specimens of Tasmanian Devil).

(iii) The parafibula separates off as a large lateral sesamoid being an articulating link between femur and fibula. The two parts *fabella* and *cyamella* are still united (this is the general rule in the wombat).

(iv) The large lateral sesamoid breaks up into two components, the *fabella* and the *cyamella* (this is the case in *Didelphys*), or:

(v) The *cyamella* becomes a mere hemisesamoid (*Macropus rufus*), or:

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[*] See note †, p. 373.

[†] This parafibular sesamoid is not shown in the figured skeleton of the wallaby on p. 130 of Vol. x. of the *Cambridge Natural History*, nor is it referred to in the text; even with Galen of such bones "non est necessarium hoc opere pertractare"!

[‡] "On the anatomy of *Macropus rufus*," *Journal of Anatomy and Physiology*, 1897, p. 123 and p. 128.

[§] We much doubt its being "fibro-cartilaginous" as Windle and Parsons state. A section of the *cyamella* in *Macropus* should be examined as soon as possible microscopically.

(vi) The *cyamella* disappears entirely or almost entirely, being merely represented by a thickening of the popliteal tendon at the position where definite ossification occurs; noted in some cases of *Didelphys*.

We have already illustrated (i) and (ii) from individual specimens of the wombat and the Tasmanian Devil. (iii) is the more general case of both Tasmanian Devil and wombat. It is also the case of *Dasyurus macrourus, Dasyurus ursinus, Perameles nasuta* and in some instances of *Phalangista vulpina**.

Didelphys virginiana is typical for (iv). Our sketch, Plate XXXIII, Fig. 84, shows the relative position of *fabella* and *cyamella*—they form together when the cut ends of *gastrocnemius* are reunited an articulating link between the head of the fibula and the external condyle. The reader must remember that in the natural position the limb is much more flexed than in the sketch.

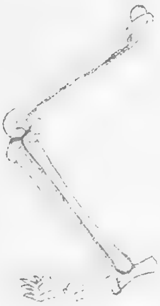
(v) Our drawing, Plate XXXIII, Figs. 83 *a* and 83 *b*, indicates in *Macropus* the *fabella* with hemisesamoidal *cyamella*.

Our drawings of *Phalangista vulpina* and *Phascolarctus* indicate the nature of the lateral side of the knee-joint when there is a single parafibular sesamoid which has often been described as a *fabella lateralis* (see Plate XXXIII, Figs. 85 and 86).

In mounted specimens of *Bettongia gaimardi, Hypsiprymnus murinus, Peragalea lagotis*†, *Myrmecobius fasciatus*, we found only a sesamoid attached to the lateral condyle.

Fossil Marsupials showing Parafibulae.

Prothylacinus patagonicus



Cladosictis lustratus

From the Santa Cruz Beds, after Sinclair

Diprotodon australis
after Shaw and Parker

The fossil *Diprotodon australis*‡ belongs apparently to class (iii), and shows according to Professor Stirling's restoration a large parafibula which would only

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* Our photograph, Plate XXXIII, Fig. 85, shows the combined sesamoid in *Phalangista*.

† In specimens of *Peragalea lagotis* and *P. leucura* the fibula did not articulate with femur, there was no ossification in the popliteal tendon, but the lateral *fabella* was present.

‡ See Parker and Haswell, Vol. II. p. 602, Fig. 1229. The fossil Patagonian Marsupials reconstructed by Sinclair show fibular crests (or parafibulae?). See his plates of *Cladosictis lustratus* and *Prothylacinus patagonicus*. He gives no details, however, on this point. He does say that the patella is ossified in the latter. See *Proc. American Phil. Soc.* Vol. XLIV. pp. 73—81, Philadelphia, 1905.

come in contact with the femur on extreme flexure. This figure throws much light on the position of the large lateral sesamoid in the Tasmanian Devil (see our Plate XXXIV, Figs. 89*a* and 89*b*) and in *Halmaturus* (see our Plate XXXV, Fig. 90).

In *Pseudochirus peregrinus*, one of the phalangers, the lateral condyle of the femur articulates with the fibula and both lateral *fabella* and *cyamella* are present. The fibula was prolonged upwards and posteriorly into a wing on the top of which the lateral *fabella* was found. The *cyamella* is a small nodule or inner notch of fibula. The arrangement is somewhat similar to that of *Didelphys*.

If the hypothesis developed in this paper be correct, namely that the lateral *fabella* and *cyamella* are vestiges of the break-up for the parafibula of an early mammalian evolutionary stage, we ought to find the lateral head of *gastrocnemius*, in whole or part, and the tendon of *popliteus* transferred to the *capitulum fibulae*. We shall turn shortly to the myology of the Marsupials and inquire whether there is any support for such a view.

One of the noteworthy points of the Marsupials and Monotremes is the apparent absence of *lunulae* which are so definitely present in many reptilian forms. We have failed to find them in *Ornithorhynchus*, or in the koala, wombat and a variety of kangaroos. The alternatives before us are: to suppose that both Monotremes and Marsupials have lost original *lunulae*, although they occur regularly or at least as anomalies in many placentals, or to suppose both to have originated in a reptilian form possessing a fibular crest or a parafibula, but no *lunulae*. A similar problem arises in the case of the Ungulates; we must suppose them to have completely lost all sesamoids of the knee-joint, and so thoroughly that they do not even appear as in man as anomalies, or we must conceive them descended from an ultimate form which has never possessed *parafibula* or *lunulae*.

(18) *On changes in the muscular attachments of the knee-joint.*

If we look upon the lateral *fabella* and the *cyamella* in Man as vestiges of an earlier structure which has now disappeared, it becomes of importance to consider the evolutionary history of the muscles attached to these sesamoids. Does it indicate a transition of their attachments from one bone to a second which is compatible with a transfer of the sesamoids from one bone of the lower limb to a second? We venture to think it does, and shall endeavour to place before the reader in this section our reasons for believing that the transfer of muscular attachments from the head of the fibula to the distal epiphysis of the femur has been accompanied by a transfer of the remnants of the parafibula to the lateral condyle of the femur.

The muscles with which we are principally concerned are *M. gastrocnemius* (in particular the lateral branch), *M. popliteus* and *M. plantaris*. Gruber has gone so far as to assert that the lateral *fabella* serves a useful purpose as a "Knotenpunkt" directing on the posterior surface of the lateral condyle the movements of *gastrocnemius externus, plantaris, semimembranosus* and occasionally *M. popliteus biceps*, when this anomalous supernumerary head of *popliteus* is present. Without accepting this interpretation of "usefulness" in a vestigial structure, we would suggest that the "Knotenpunkt" nature of the lateral *fabella* is a phenomenon to be accounted for. We must seek for an important structure to which these muscles could have been originally attached, and which in a pre-vestigial stage had a use value*. The supernumerary head of *popliteus* arising in the *fabella*, connects the lateral *fabella* at once with the *cyamella* in the principal head of *popliteus* as possible parts of the same original structure. The next point is to consider what changes have taken place in the attachment of *popliteus*. There is no doubt that in the course of evolution *popliteus* has migrated from the head of the fibula to the lateral face of the external condyle. Take the *cyamella* with the popliteal tendon attached still both to the condyle and to this sesamoid and place it at the head of the fibula; place the lateral *fabella* on top of the *cyamella*; fuse them together and we have by no means a bad representation of the muscular attachments of the highly flexed marsupial knee-joint and their relations to the parafibular sesamoid. Fuse this parafibula to the head of the fibula and we have a rough representation of the muscles of the knee-joint of the Monotremes.

Let us examine first the Monotremes. *Popliteus* in *Echidna* arises from the extremity of the process at the top of fibula instead of from the external condyle. *Gastrocnemius* is a slender and delicate muscle arising from the internal condyle; this is really the internal head, and the external head appears to be replaced by *Soleus* which "arises by tendinous fibres from the posterior and outer margin of the singular process at the summit of the fibula." This is Mivart's account† of these muscles; he does not refer to *plantaris*.

Manners-Smith has dealt with *Ornithorhynchus*‡, and describes the knee-joint musculature as follows:

He tells us that *gastrocnemius* is a large muscle arising by two distinct heads, a larger from the fibular crest and a smaller from "the lower end of the femur." It does not fuse with the *soleus*. The femoral head, he states, is supplied by the external popliteal. *Soleus* arises from the fibular crest and from the shaft of the fibula. A *few* fibres only arise from the tibia. Manners-Smith contradicts Owen's assertion that a large proportion of the fibres arise from the tibia and asserts that it is certainly a mistake. He gives no information about *popliteus* or *plantaris*. Our drawing (Plate XX, Fig. 42) shows the external head of *gastrocnemius* having no attachment to the external condyle and the internal head attached to the internal condyle.

In Reptiles *popliteus* arises from the fibula§, probably also from a groove on the external condyle and when it exists from the parafibular sesamoid‖. According to

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* As we have previously remarked we cannot consider a structure which has only an occasional and anomalous occurrence to have real "use" value; it can only be degenerating, i.e. vestigial.

† "Anatomy of *Echidna hystrix*," *Trans. Linnean Society, London*, 1866, p. 394.

‡ "Anatomy of *Ornithorhynchus*," *Proc. Zoological Society, London*, 1894, p. 707.

§ Mivart, "Myology of *Iguana tuberculata*," *Proc. Zoological Society, London*, 1867, p. 796. *Popliteus* arises from tibial aspect of head of fibula.

‖ See our p. 389, and the drawings of *Varanus*, Plate XXIV, Figs. 64 *a* and 64 *b*.

Macalister* *popliteus* in birds may be femoro-tibial or fibulo-tibial in origin, and the origin of *soleus* is also variable. There is of course the possibility that *popliteus* was not in the earlier stage largely fibular in its attachment, but acquired an attachment to the fibula in the marsupials. This appears to be the view of Parsons who writes†:

> In the koalas, the phalanger, the opossum and the Tasmanian devil, the great mobility of the fibula has produced the same effect on the *popliteus* that we noticed in the lemur, the fleshy part acquires an attachment to the fibula, while the tendon is converted into an extra external lateral ligament.

The question arises, however, as to whether the popliteal attachment to the fibula is not the earlier stage and thus the arrangement we find in the lemur would be vestigial. When the parafibula broke up, it is conceivable that *popliteus* carried off with the *cyamella* the whole or a portion of a fibulo-femoral ligament, i.e. it is not the tendon of *popliteus* converted into a ligament, but a ligament which has become the tendon of *popliteus.*

The point has been studied also by Gordon Taylor and W. V. Bonney‡. They take the view that the ultimate origin of the *popliteus* was fibular; that as movement between the tibia and fibula ceased to exist the upper origin of *popliteus* ascended to the external femoral condyles, but that in the lemur where movement between tibia and fibula has reappeared the *popliteus* appears to be descending to its primitive fibular attachment. The lemur problem is a question either of reversion or of vestige. The question of whether the fibula attachment of *popliteus* is the original attachment or an acquired one seems worthy of much more study especially in relation to the origin of the lateral knee-joint sesamoids, in particular to the muscular attachments of the parafibula in Reptiles.

Passing from the Monotremes to the Marsupials it becomes difficult to appreciate the contributions of various writers to the subject, for they invariably speak of the parafibular sesamoid as if it were a large *fabella* in the lateral head of *gastrocnemius.* Thus we find it hard to interpret the following statement of Macalister with regard to the myology of the Marsupials§:

> The *gastrocnemius externus* in the Wombat, Giant Kangaroo, Sarcophilus and Bennett's Kangaroo arises from the sesamoid bone at the back of the external condyle of the femur and is inserted into the tendo Achilles, etc.

It seems to us extremely difficult to speak of the parafibular sesamoid, which may even be fused to the fibular head, in the Wombat and Tasmanian Devil (see our Plate XXIX, Fig. 78) as a sesamoid at the back of the external condyle! The same difficulty occurs in the Wallaby (Bennett's kangaroo, or Halmaturus): see

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* "On some Points in Bird Myology," *Proc. R. Irish Academy*, Vol. II. 1875—77, p. 57.

† "Joints of Mammals compared with those of Man," *Journal of Anatomy and Physiology*, Vol. XXXIV. p. 313, 1900.

‡ "Homology and Morphology of Popliteus," *Journal of Anatomy and Physiology*, Vol. XXXIX. p. 35.

§ "Myology of the Wombat (*Phascolomys wombata*) and the Tasmanian Devil (*Sarcophilus ursinus*)," *Annals of Natural History*, 1870, p. 19 of our offprint.

our Plate XXXV, Fig. 90. We are dealing in all these cases with something much more closely allied to fibula than to femur.

Again:

> The *popliteus* in the Wombat and Sarcophilus is very large but thin, arising from the upper third of the back of the fibula, and inserted into the lower two-thirds of the back of the tibia. A few fibres of this muscle in Sarcophilus are attached to the sesamoid bone in the outer head of *gastrocnemius*. This muscle is smaller in the Giant Kangaroo, but in this and *Macropus bennettii* its origin is purely sesamoid*.

These two passages taken together suggest that both in the Great Kangaroo and the Wallaby *gastrocnemius externus* and *popliteus* arise from a *fabella* or sesamoid at the back of the external condyle. They do not suggest that they arise from a parafibular structure, which may be opposed rather laterally than posteriorly to the femur (see our Plate XXXIV, Figs. 89 *a* and 89 *b*), and which is itself united by a stout band to the femur above the external condyle, this band corresponding in the placentals to the tendon of *gastrocnemius externus*.

Macalister's statements about other muscles are clear. He tells us that *gastrocnemius internus* arises from the internal condyle and joins the tendo Achilles... it is larger than the *externus* in the Wombat, but smaller in the Tasmanian Devil and in all species examined was without sesamoid.

Soleus has a fibular small origin in the Wombat, but as usual has no tibial heads. The *soleus* is fibular also in Tasmanian Devil, Opossum, Phalanger and Giant Kangaroo.

The *plantaris* in the Opossum and Perameles is small but separate from the outer condyle to the outer side of the head, but no *plantaris* exists in the Wombat. A distinct small muscle in the Tasmanian Devil passes from the back of the external lateral ligament and from the head of the fibula down the internal side of the calcaneum and into the plantar fascia†.

In a *Macropus sp.* examined by one of the present writers the parafibular structure consisted proximally of an orthosesamoid from which *gastrocnemius externus* arose, this sesamoid articulated on a hemisesamoid which was linked by a tendon to the lateral face of the external condyle and from which *popliteus* arose; this hemisesamoid rested on the *capitulum fibulae*. *Plantaris* was not identifiable in our specimen. The orthosesamoidal part of this parafibular structure is what is termed by most writers the *gastrocnemial* lateral *fabella*, and the hemisesamoid is what we term the *cyamella* portion of the parafibular structure. *Popliteus* is not attached to the orthosesamoidal part of the structure at all. Interpreted in our own terms we see that this practically agrees, except with regard to *plantaris*, with what Windle and Parsons report of *Macropus rufus*‡:

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* *Loc. cit.* p. 17 of our offprint.

† Macalister, *loc. cit.* p. 19 of our offprint.

‡ *Journal of Anatomy and Physiology*, Vol. XXXII. p. 123, 1897. They say of *plantaris*, that the upper cornu of gastrocnemial *fabella* as well as external condyle give rise to *plantaris* which is thus a double-headed muscle.

> The tendon of the *popliteus* contains a very strong fibro-cartilaginous fabella, which is placed above the head of the fibula to which it is attached, and on which it moves freely. It is with this fabella that the large gastrocnemial fabella articulates.

In another paper* Parsons notes that in the koala, the phalanger, the opossum and the Tasmanian Devil the fleshy part of *popliteus* has acquired an attachment to the fibula while the tendon is converted into an extra external lateral ligament.

According to the views expressed in this paper it may be anticipated that the primitive marsupial would have a fused parafibula, and accordingly the later tendon of *popliteus* would be a femoro-fibular ligament and *popliteus* would arise from the fibula. From this standpoint it is not surprising that on the break-up of the fibular crest certain portions of *popliteus* should still be attached to the fibula rather vestigially than as an acquirement.

We dissected a specimen of Koala (*Phascolarctus cinereus*) and found the parafibular sesamoid articulating with the fibula: see our Plate XXXII, Figs. 82 *a* and 82 *b*. The fleshy portion of *popliteus* was attached to the fibula and the popliteal tendon ran from the antero-proximal corner of fibula to popliteal groove of external condyle. Also tendinous fibres from the fleshy portion on fibula cross the posterior surface of the fibula and mingle with the tendinous fibres of *gastrocnemius*, which run from the sesamoid to the external condyle.

No *plantaris* was discoverable. Young in his paper on the "Muscular Anatomy of the Koala†" does not refer to *popliteus* or *plantaris*, but states that *gastrocnemius* is well developed and arises by two heads in the external of which a *fabella* exists. We again urge that this parafibular sesamoid should not be described as a *fabella*. We think that for the whole genus *Macropus* we ought to consider that we are dealing with a parafibular structure and not a lateral gastrocnemial *fabella*. We have not been able to examine a case of *Petrogale xanthopus*, but Parsons' description seems in accordance with the above interpretation, if we replace his term *fabella* by parafibular sesamoid. Thus he says‡:

> The knee-joint like the hip is never fully extended, the capsule is strengthened by strong lateral ligaments ; the external lateral runs downwards and backward from the external condyle to the front of the head of the fibula ; its anterior fibres are continuous with the origin *extensor longus digitorum*. The *popliteus* also performs the function of an external lateral ligament, since it is attached to the femur and fabella, and below to the external semilunar cartilage and the head of the fibula. During extreme flexion which is the usual position of the joint, the *fabella* articulates with the head of the fibula.

Judging from the knee-joints of other specimens of *Macropus* with which we are more familiar, we should believe that the parafibular sesamoid was always in contact with the fibula and that in cases of extreme flexure its head may come in contact with the postero-lateral border of the femur. We do not think that like an

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* "Joints of Mammals compared with those of Man," *Journal of Anatomy and Physiology*, Vol. xxxiv. p. 313, 1900.

† *Journal of Anatomy and Physiology*, 1882, p. 237.

‡ "On the Anatomy of *Petrogale xanthopus* compared with that of other Kangaroos," *Proceedings Zoological Society, London*, 1899, p. 690.

ordinary *fabella*, it will be found to articulate with the articular surface of the lateral condyle. Turning to the muscular attachments Parsons states:

> The *gastrocnemius* has the usual two heads, the outer of which has three origins: (1) a small head from the outer side of the patella, (2) larger from the same place and separated from the last by the external popliteal nerve, (3) from the large *fabella* over the external condyle and from the external semilunar cartilage. The inner head is normal in origin and has no *fabella* developed in it (p. 207).
>
> The *soleus* is absent unless the origin of the *gastrocnemius* from the semi-lunar cartilage represents it. Meckel also describes it as wanting.
>
> *Plantaris* comes from the external *fabella**.
>
> *Popliteus* arises from the external condyle, partly from in front of the groove, and partly from the groove itself, which is deep and narrow and not adapted for the tendon to lie in†. The muscle also has an origin from the external semi-lunar cartilage and from the head of the fibula.

We see in this typical description of a marsupial knee-joint, *gastrocnemius*, *popliteus* and *plantaris* all associated with the parafibular sesamoid. We see the muscles which largely arise from the fibular crest in the Monotremes now attached to the parafibular sesamoid, and the fragments of the parafibula will be carried with them as *fabella* and *cyamella* to their ultimate tendons of insertion on the lateral femoral condyle. We begin to see now why the early human anatomists associated the lateral *fabella* with numerous muscles, why *popliteus* (with its anomalous head) passes to *fabella* as well as to *cyamella*; how Gruber could confuse these two sesamoids, and why he emphasised so strongly the "Knotenpunkt" character of the lateral *fabella*.

While *Macropus* seems to us to represent a later evolutionary stage of the parafibular sesamoid than *Phascolomys*, for in the Kangaroos we have various stages of disruption of the parafibula, while in the Wombat it may still be fused, the Phalangers form a more or less intermediate stage. They have a parafibular sesamoid of much the same form as the wombat's—quite different from that of the wallaby—and its form is seen to be clearly that of a fibular crest: see our Plate XXXIII, Fig. 85.

As Cunningham has shewn, *gastrocnemius* in *Phalangista maculata* springs from the posterior aspect of the fibula in its upper two-thirds, and *plantaris* from the parafibular sesamoid. Cunningham suggests that *gastrocnemius* is a compound muscle containing *soleus*‡. The musculature is much the same as in the Koala (*Phascolarctus cinereus*), except that in the latter we did not identify *plantaris*§, which is also said not to exist in the Wombat.

In *Perameles nasuta* we have the parafibular sesamoid, but the *Peramelidae* are transitional to the *Didelphyidae*. Parsons has pointed‖ out that "the popliteus

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* That is the parafibular sesamoid.

† Any marked popliteal sulcus has disappeared in *Macropus*, see our drawing of the Wallaby, Plate XXXV, Fig. 90.

‡ *Challenger Report*, Vol. v. p. 40.

§ In the Koala there is a collatero-fibular ligament—the ultimate popliteal tendon connecting the parafibular sesamoid with the lateral condyle of the femur.

‖ *Journal of Anatomy and Physiology*, Vol. XXXIV. p. 313, 1900.

tendon instead of being converted into an extra lateral ligament is continuous with its fleshy belly as usual and patella is ossified*." According to Macalister† the *plantaris* in *Perameles* as in *Didelphys* is small and arises from the outer condyle.

Of the *Dasyuridae* Cunningham has dealt with *Thylacinus cynocephalus*: he states that *gastrocnemius* arises by two very large and powerful heads: (1) The outer head which contains the *soleus* is much the larger of the two and it arises by two parts, (*a*) by a fleshy process fixed to a large sesamoid bone placed upon the posterior aspect of the head of the fibula—clearly our parafibular sesamoid, and from a powerful ligament which binds the upper part of this bone to the back of the external condyle of the femur, (*b*) by a tendinous slip attached to the outer aspect of the external condyle of the femur. (2) The inner head springs not only from the back of the inner condyle but also from the whole breadth of the popliteal surface at the same level. *Plantaris* in *Thylacinus* arises in common with the outer head of *gastrocnemius* and remains fused with its under surface for a considerable distance.

We have already noted that in *Dasyurus macrourus* and *D. ursinus* the sesamoid is parafibular (see our p. 375). MacCormick‡ has studied the myology of the limbs of *D. viverrinus*. He states that the outer head of *gastrocnemius* "arises directly from the sesamoid bone over the upper end of the fibula, and indirectly by two strong ligaments which bind this bone to the femur, one being attached to the outer border of the popliteal space, and the other passing over the femur to be attached to the upper border of the patella§ and to the outer part of the *vastus externus* tendon." He does not refer to *popliteus* and *plantaris*.

Didelphys is the last marsupial to which we shall refer. It represents the later stage of the break-up of the parafibula. *Gastrocnemius externus* arises from external condyle and contains a sesamoid—the lateral *fabella*. *Popliteus* follows the course of this muscle in higher mammals, arising from a groove in the lateral face of the external condyle, and contains a *cyamella*. We found no association between *popliteus* and the gastrocnemial *fabella*. But the *cyamella* of the Opossum is wedged between the under surface of the lateral condyle and the head of the fibula (see our Plate XXXIII, Fig. 85), and when the *gastrocnemius* is replaced the *fabella* lies on the external condyle immediately above the *cyamella*. According to Parsons||, however, the fleshy part of *popliteus* in the Opossum acquires an attachment to the fibula, while the tendon is converted into an extra lateral ligament. This did not seem to be the case in our specimen, the tendon of *popliteus* passed to the *cyamella*, which indeed might be looked upon as a parafibula, but unless that sesamoid was fused the tendon could hardly be treated

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* It is not ossified in Kangaroos and Wallabies and Opossums, although a small bony element has been reported in certain cases.

† *Annals of Natural History*, 1870, p. 17.

‡ *Journal of Anatomy and Physiology*, Vol. xxi. p. 213, 1887.

§ The reader must remember that the patella is cartilaginous in all Marsupials but *Perameles* although the existence in certain cases of a small ossified kernel has been asserted. The attachment the fibular crest to the patella in the case of Platypus may also be borne in mind here.

|| *Journal of Anatomy and Physiology*, Vol. xxxiv. p. 313, 1900.

as an extra lateral ligament in our case. We have already seen that the fusing* of the parafibular sesamoids is a possibility in the Marsupials, rare as it appears to be in the higher Mammals†. It is possible that in certain individuals, or even in one or other species of opossum the *cyamella* may as a vestige be fused to the fibula, and so the popliteal tendon become an extra lateral ligament.

Plantaris arises in *Didelphys* from the lateral femoral condyle‡.

It will be seen from the above that *Didelphys*, as far as the sesamoids of the knee-joint are concerned, is among the most advanced of the Marsupials. It stands at the opposite extreme to the Tasmanian Devil with its fused parafibula forming a fibular crest. The range marks the transfer of the *gastrocnemius*, *popliteus* and *plantaris* from fibula to femur and with them pass both lateral *fabella* and *cyamella* to the lateral femoral condyle.

(19) *Monotremes.*

We now come to the Monotremes and have to bear in mind the fundamental differences we have noted between the *Ornithodelphia* and the *Monodelphia*, in particular the existence of the parafibula, and the rearrangement of the muscular insertions of the posterior aspect of the knee-joint. We turn first to *Ornithorhynchus* (Platypus). Plate XX, Figs. 43 *a* and 43 *b* provide drawings of the femur and distal portions of tibia and fibula. The extraordinary development of the process at the distal end of the fibula will be at once noticed. From this fibular crest (see our Plate XX, Fig. 42) arises the lateral head of *gastrocnemius*, *plantaris* and *popliteus*. The patella is also attached to this expanded head of the fibula. This is well shown in the prepared specimen in the Museum of the Royal College of Surgeons. The internal head of *gastrocnemius* is attached above the internal condyle of the femur. There are no *fabellae*§, no *cyamella*, and what is more remarkable no *lunulae* (i.e. because the latter are very conspicuous in the Reptiles). Thus platypus verifies what we have observed in the Marsupials: namely that, with the appearance of the fibular crest, there is associated a disappearance of lateral *fabella* and *cyamella*. Accordingly we see in the Monotremes the three muscular attachments which are associated with the lateral *fabella* in Man (see our Part I, p. 161); those of the external head of *gastrocnemius*, of *popliteus* and of *plantaris*, are now associated with the fibular crest. Manners-Smith tells us that‖ *soleus* also arises from the expanded head of the fibula, but partly too from the shaft of the same bone.

By the kindness of Sir Arthur Keith one of us was able to dissect the knee-joint of *Echidna*. There were no signs of any *fabellae*. The popliteal muscle followed

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* We must not be understood as suggesting that this "fusing" is something which may occur anomalously as in an adult individual; it is specific in the case of the two parts of the parafibula, and vestigial when the parafibula is ankylosed with the fibula.

† See our p. 373, first footnote.

‡ This was so in our specimen, and Macalister says so also: see *Annals of Natural History*, 1870, p. 17.

§ Meckel, *loc. cit.* pp. 634—635, had already noticed the absence of the *fabellae* in Platypus ("Schnabelthier"). Gruber has not investigated the Monotremes.

‖ "Anatomy of Ornithorhynchus," *Proc. Zool. Soc. London*, 1894, p. 707. He also notes the parafibular origin of *gastrocnemius externus*, but says nothing of *popliteus* or *plantaris*.

much the same course as that shown in the drawing of *Didelphys*, but it lacked any *cyamella* where it passed over the head of the fibula; it arises as in *Ornithorhynchus* from the fibular crest and not from the external condyle of the femur. The expansion of the fibula is not quite so marked as in *Ornithorhynchus*, but from it arises not only *popliteus*, but the larger or external head of *gastrocnemius*. The internal head is much smaller and arises from the internal condyle*.

(20) *Centre of Ossification of Fibular Crest in Monotremes.*

In order to grasp how the fibular crest can be separated from the fibula and ultimately be found disintegrated as vestigial sesamoids attached to *gastrocnemius*, *popliteus* and *plantaris*, it is of some importance to determine whether this process at the head of the fibula has a separate centre of ossification.

Parker and Haswell† appear to have no doubt on the matter; writing of the Monotremes they say:

> The fibula has at its proximal end a remarkable compressed process which ossifies from a separate centre and greatly resembles the olecranon of the ulna.

Through the great kindness of Dr Wm. Colin Mackenzie of the Australian Institute of Anatomical Research, Melbourne, we were able to obtain sections of the fibula both of an embryonic and a young platypus. He also sent us the section of the parafibular sesamoid of a young wombat which had just left the pouch. Fig. A, on p. 385, indicates the bone of the young platypus set against that of an adult platypus to give the relative sizes. The length of the former animal from tip of bill to cloacal sphincter was 31 cms. as against the similar length of a fully developed adult of 44 cms. The length of the embryonic platypus was 13 cms. Fig. B shows the position of the sections Dr Mackenzie made of the embryonal and young platypus.

Taking the section A of the embryo (see our Plate XXXVII, Fig. 96), we have periosteum at the proximal end of the fibular process (top of plate), then follows a small area staining yellow and suggesting ossification, then an area of cartilage with every appearance of ossification starting, and then cartilage again. This certainly seems to accord with a centre of ossification in this fibular process or crest. In the young platypus in the corresponding section A we found no cartilage at all but bone covering the areas where in the embryo we had found cartilage and ossification taking place. It was therefore probably not young enough to show an intermediate partly ossified stage. On the other hand the section towards the top of the crest (Plate XXXVII, Fig. 95) shows fairly numerous Haversian canals which become less frequent or cease altogether as we pass down the section, to become frequent

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* Mivart ("Anatomy of *Echidna hystrix*," *Trans. Linnean Soc.* 1866, p. 394) mentions only what appears to be the inner head of *gastrocnemius*, and probably our outer head is his *soleus*, which like his *popliteus* arises from the process at the top of the fibula.

† *A Text-book of Zoology*, Vol. II. p. 489, London, 1897. The comparison of the olecranon with the tibial crest seems more reasonable, the fibular crest should be homologous with a radial rather than an ulnar process. The fibular crest of the Monotremes or that of certain Marsupials has little general resemblance to the olecranon, while the tibial crest in birds (*Colymbus*) or the tibia plus patella (as in *Hesperornis*) is very similar to the olecranon in appearance.

again and in a different direction as we reach the shaft. Their arrangement seems to us not inconsistent with a separate centre of ossification for the fibular crest.

Our view differs from that of Dr Mackenzie and Mr W. Owen, histologist to the Australian Institute. They speak of section A of the embryo as if it were wholly cartilaginous. It seems to us as if a portion of this superstructure of the fibula does show ossification starting (Plate XXXVII, Fig. 96). The articular process of fibula, section B of Fig. B below, has a distinct centre of ossification in the embryo. Further the section at C also shows ossification and Dr Mackenzie tells us that there is one at the distal end (section not forwarded). There would thus appear to be *four* centres of ossification, of which only three are to be found in the higher Mammals. The suggestion is that the fourth centre of ossification is at the basis of the parafibular sesamoid of the Marsupials, or the later *cyamella* and lateral *fabella*.

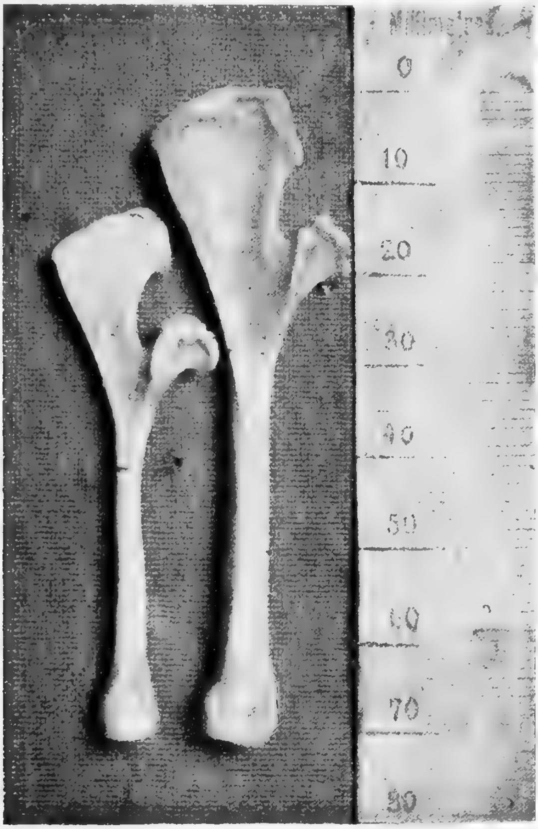

Fig. A.

Fig. B.

If the above view be correct that the Mammals with lateral sesamoids of the knee-joint are descended from a primitive mammalian form with a crested fibula, we should anticipate that still lower in the evolutionary scale, e.g. in the *Aves* or *Reptilia*, no lateral *fabella* or *cyamella* will be discoverable. It may be objected that the present hypothesis provides no account of the origin of the mesial *fabella*. That is quite true, but it has appeared over and over again in the course of our investigation that, whatever the origin of the mesial *fabella*, that origin must be differentiated from that of the lateral *fabella*. Not only has the mesial come later in most orders and disappeared in the higher types much sooner than the lateral, but it is far less significant in size and far rarer as an anomaly. Further, while the lateral head of *gastrocnemius* in the earlier forms is attached not to the lateral condyle but to the head of the fibula, the mesial head retains its association with the internal condyle even in the Monotremes. It would demand much further study than we have been able at present to give to the mesial *fabella* to trace satisfactorily its origin and the source of its frequently imperfect condition. We would note, however, that if the lateral *fabella* and *cyamella* arise from a crested fibula, one might seek for a similar origin of the mesial *fabella* (and the reputed mesial *cyamella ?*) in a like extension of the head of the tibia. Perhaps in this the cnemial crest of the tibia in certain birds—the *Tubinares* and still more the *Pygopodes*—might be borne in mind*. According to Shufeldt it has (like the fibular crest in Platypus) a separate centre of ossification and may be regarded as representing the olecranon of the ulna. Vicq d'Azyr as early as 1774, and he was followed by Owen, considered the patella itself to be homologous with the olecranon†. It is difficult to accept the view of the more modern writers that fibular or tibial crests are only extensions of the upper epiphysis of these bones if we find they have separate centres of ossification. Still less does it help us to be told that the patella is only a sesamoid, unless we hold the view that a sesamoid can be produced by some mechanical process when and wherever it is likely to be of service‡. At any rate if it can be, there seems no reason why we should stop at a sesamoid—the production of which would be at present as inexplicable as that of any new organ, and would require at least an acceptance of Lamarckism to ensure its perpetuation.

Besides the cnemial crest of birds as a vague suggestion§, there is a faint possibility of the transfer of a mesial posterior *lunula*, but we are unable to cite a

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* See W. P. Pycraft, *Proc. Zool. Soc. London*, 1899, pp. 401 and 1037.

† "Mémoire sur les Rapports qui se trouvent entre les usages et la structure des quatre extrémités dans l'Homme et dans les Quadrupèdes," *Histoire [et Memoires] de l'Académie Royale des Sciences*, 1774, pp. 254—270, Paris, 1778. Vicq d'Azyr associates tibia with the ulna and fibula with the radius. He does not, however, claim the association of patella and olecranon to have originated with himself, although many later writers have attributed it to him,—"la rotule qui tient lieu d'olécrâne comme plusieurs anatomistes l'ont démontré" is how he writes on pp. 260—261.

‡ "I could imagine if further strength were required a sesamoid would be called into being"—as an anatomist once wrote to one of the present authors.

§ There are evidences that the cnemial crest is not a specialised development reached only in a small group of birds. Thus Shufeldt found it as a vestige—disappearing with growth in the young of *Centrocercus*, and states that in the present state of our knowledge this is incomprehensible, "for in the old and mature birds of any of the Grouse the epicnemial crest is never very prominently produced, nor is it in any of their near kin. As age advances this segment becomes thoroughly confluent with the tibia, and leaves no trace of its early existence." "Osteology of the North American Tetraonidae," *Bulletin of the U.S. Geological Survey*, Vol. VI. p. 344, 1881.

case of any fibres of the mesial head of *gastrocnemius* being attached to the semilunar. Such cases may, however, exist.

(21) *Birds.*

We have not examined a large number of birds' pelvic limbs, but we have not found in any of the cases we have examined nor in mounted skeletons either *fabellae* or *cyamella*. On the other hand we have found *lunulae*, for example, in *Syrnium stridula* (the Wood Owl) and *Dendrocygna* (the Tree Duck), the latter having in our specimen at least three posterior *lunulae*. The tibia of this bird also possesses a marked cnemial crest (see Plate XXII, Figs. 52, 53 and 54) apparently articulating with a small patella; this crest reaches to the level of the top of the patella. In Divers, Grebes, Albatrosses (see Plate XXII, Figs. 50 and 51), etc., this crest formed from an extension of the middle and internal crests of the tibia becomes a still more emphasised process and can even be as long as the femur itself*.

The homologue to the tibial crest appears to be the olecranon of the ulna†. Owen has noted in certain birds an ossiculum belonging to the ulna which he states is "essentially the separated olecranon of that bone. This detached sesamoid bone is found attached (like the patella of the knee-joint) to the capsular ligament and the tendons of the extensor muscles in many of the *Raptores* and in the Swifts. In the Penguins it is double‡." It would seem more justifiable to compare the cnemial crest of the tibia with the olecranon than the latter with the fibular process of the Monotremes, and not unreasonable to suppose that if the olecranon can give rise to sesamoids, a similar change can take place in the cnemial crest. In *Apteryx australis* there is a semi-cnemial crest surmounted by a cartilaginous patella. It is true that in some birds we know with marked cnemial crests the patella still exists, partly articulating with the femur and partly with the cnemial crest, but in *Colymbus* it is reduced to a "flake-like bone§." The patella, as we have remarked, has been associated with the olecranon by Vicq d'Azyr‖. Shufeldt, while associating the olecranon with the cnemial crest, looks upon the patella as not arising from its partial break-up, but treats the patella as a sesamoid.

We have carefully studied the evidence given by Shufeldt¶, but it seems to us quite as capable of interpretation in favour of the patella being originally the part or whole of the cnemial crest, as of both having quite distinct and separate origins. He first cites the great diversity of patellar forms in birds, from its absence in certain species, as *Haemotopus niger*, to the small double almost cartilaginous

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* According to Owen its importance arises from the provision it affords for extensive attachments by way of insertions for the extensors of the tibia and by way of origin to the extensor of metatarsus. Thus the power of the back stroke of the foot is increased: see Todd's *Cyclopaedia of Anatomy and Physiology*, Vol. I. p. 287.

† Coues writes of *Colymbus torquatus* (*loc. cit.* on our p. 392) that the tibial crest exactly resembles the olecranon of the human species (p. 158).

‡ Owen's article *Aves* in Todd's *Cyclopaedia*, Vol. I. p. 286.

§ Of *Colymbus septentrionalis* (Red-throated Diver) we have examined two skeletons, but found no signs of a *patella*, although the semi-lunars were *in situ*.

‖ *Histoire de l'Académie Royale des Sciences*, Paris, 1774, p. 261.

¶ "Concerning some of the forms assumed by the Patella in Birds," *Proceedings U.S. National Museum*, Vol. VII. pp. 324—331, Washington, 1886.

patella of *Mergus servator,* to the flake-like patella of *Colymbus,* up to the huge quadrate patella of *Aptenodytes pennantii.* But he does not seem to lay any stress on the fact that where the cnemial crest is large as in *Colymbus* and *Fulmarus rodgersii* the patella is small, and where the patella is large as in *Podiceps* the cnemial crest is relatively small. Yet this result seems all important and easily interpretable on the assumption of the patella arising from a tibial crest. Lastly Shufeldt gives no explanation of the condition in the cormorant *Phalocrocorax bicristatus* of which he writes that it is almost unique; that on the face of it it almost looks as if a patella had developed of a size equal to that of the tibial crest and subsequently the two became thoroughly united and formed one large patella. This suggestion appears to us as an inversion of the real state of affairs. It also seems to be D'Arcy W. Thompson's view*. He treats the large patella of *Hesperornis* as homologous with the tibial crest of *Colymbus.* He actually speaks of this crest as a "co-ossified patella." He writes: "The existence of a small additional sesamoid in the knee-joint of *Colymbus* (Owen, *Comp. Anat.* II. p. 83) does not invalidate the homology here adopted of the long 'rotular process of the tibia' with the patella."

If the cnemial crest be identified with the patella, then it seems to us more reasonable to look upon the patella as a released tibial crest, than the tibial crest as a co-ossified patella†. It has not yet been suggested that the olecranon is a co-ossified sesamoid. Perhaps one of the chief difficulties in the idea of the tibial crest as a "co-ossified" patella is the large free patella of the cormorant; here the pro- and ecto-cnemial ridges of the tibia are carried right up the anterior face of the patella itself! (See Plate XXXVI, Fig. 93 *b.*) Why should an independent sesamoid have developed these ridges? Unless we accept the view that a sesamoid can always arise where it is a convenience, this leaves us still without an explanation of the origin of either the sesamoid or the patella. We have at any rate to bear in mind the very close association of patella and cnemial crest where they co-exist‡.

We note also that whereas a true patella, an orthosesamoid, exists in all placental mammals, it is absent in many Marsupials, it is not invariable in Birds and has disappeared when we get down to the Reptiles. Somewhere between the Reptiles and the branchpoint of the Birds a patella had to arise, somewhere between the Reptiles and the branchpoint of the Monotremes a fibular process had to arise§,

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* "On the systematic position of *Hesperornis,*" *Studies from the Museum of Zoology in University College, Dundee,* Vol. I. p. 108, Dundee, 1890.

† In both *Hesperornis* and *Ichthyornis* the tibia has a considerable epicnemial crest, in the former as well as the massive patella, in the latter no patella is figured or referred to; whether none has been found or it is assumed not to exist, we cannot say. See Marsh, "*Odontornithes,*" *U. S. Geological Exploration of the* 40*th Parallel,* Vol. VII. Plate XXXIV, 1880.

‡ In the fossil *Hesperornis regalis* which we have examined the patella so far accords with Professor D'Arcy Thompson's view, that it has at least a superficial resemblance to a liberated cnemial crest. See also Plate XXXVI, Fig. 92.

§ An approach to a fibular crest occurs in some birds; thus in the *Cereopsis Goose* (see Plate XXI, Figs. 48 and 49), the head of the fibula rises above the proximal articulating surface of the tibia, and works in a groove of marked character on the posterior face of the lateral condyle. There is also a substantial cnemial crest.

and somewhere between that branchpoint and that of the Marsupials the fibular process had to be replaced by a parafibula, and lastly somewhere between the branchpoint of the Marsupials and the early placental mammals a *cyamella* and a lateral *fabella* had to replace the parafibula. We can of course take the view that a process will disappear without vestiges, if it ceases to be useful, and that a sesamoid will appear whenever and wherever it is likely to be useful*. But on philosophical grounds there is need of a connected account of the evolutionary origin and subsequent history of even the sesamoids of the knee-joint.

(22) *Reptiles.*

The chief feature of the knee-joint sesamoids of the Reptiles consists in the presence usually of a considerable number of *lunulae.* Parker and Haswell† speaking of the young condition of certain Marsupials state that there is an accessory element situated outside the fibula at its proximal end; "this apparently corresponds to a bone known as the parafibula which occurs in some *Lacertilia.*" When the *cyamella* has left the popliteal groove and articulates with the head of the fibula, it becomes very difficult to say whether we are speaking of a *cyamella* or a parafibula. In our opinion the parafibula ultimately is divided into two elements; the upper portion becomes associated with the lateral head of *gastrocnemius* and is the lateral *fabella* of the placentals, the lower portion associated with *popliteus* is the *cyamella* proper. Hence when there is no *fabella* as in the Reptiles and Marsupials it seems a more reasonable course to term the single large sesamoid, which articulates with the head of the fibula and usually at the same time with the under surface of the lateral condyle, the parafibula. At the same time *popliteus* in most of these cases arises from the fibula, it may also be united with the semilunar, and it is not always easy to distinguish this form of the *cyamella*, a small parafibula, from a lateral anterior *lunula.* Our drawings, Plate XXIV, Figs. 64 *a* and 64 *b*, show a typical reptilian knee-joint, that of *Varanus niloticus* (Nile monitor). The anterior aspect shows the *cyamella* as a parafibula. The *popliteus* has a tendon in the popliteal groove of the femoral condyle, it is also attached to the superior posterior corner of the fibula, just internal to attachment of the external lateral ligament shown in the drawing. There are two anterior *lunulae.* The posterior aspect shows two posterior *lunulae*, a small mesial and a much larger lateral *lunula*‡. In *Varanus* the *cyamella* has become entirely anterior; it thus differs widely in position from the *cyamella* in higher mammalian forms. Cf. the Orang in our Plate XXVI, Figs. 68 and 69. This position of the *cyamella* is not universal in reptiles. Thus in *Trachysaurus rugosus*, the Australian stump-tailed lizard, the *cyamella* (see our Plate XXIV, Figs. 63 *a* and 63 *b*) is seen from *both* posterior and anterior aspects, and it is more easily confused with a *lunula* of the lateral semilunar. The knee-joint of this lizard has in addition two anterior *lunulae* and two posterior *lunulae*, of which the internal is larger than the external—exactly the opposite to what occurs in *Varanus.*

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* We have already indicated the difficulty of arguing as Gruber does the value of the knee-joint sesamoids to man in face of the fact that 90°/₀ of mankind do just as well without them.

† *Text-book of Zoology*, Vol. II. p. 524.

‡ In the specimen of *Varanus niloticus* mounted in the British Museum (Natural History) only the parafibula and the two anterior *lunulae* are visible.

Our third dissection was of a knee-joint of *Amphibolurus barbata.* Here we have no longer the parafibula, but posterior mesial and lateral *lunulae,* mesial anterior and lateral (or rather sagittal or trochlear) anterior *lunulae* (see our Plate XXIV, Figs. 65*a* and 65*b*). In this case the lateral posterior *lunula* is larger than the mesial, and there is a groove on the trochlear portion of the femur—exaggerated in the drawing to indicate its existence—which probably articulated with the sagittal *lunula.* In a very young specimen of *Iguana tuberculata* (*Lacertilia*) we were unable to find any parafibula; the posterior lateral *lunula* and the anterior sagittal *lunula* were present*. Mivart, who has studied the myology of *Iguana tuberculata*†, states that the outer head of *gastrocnemius* in this case arises from above the external condyle, that *plantaris* is very intimately connected with *gastrocnemius* and that *popliteus* arises from the *fibula.* He makes no mention of a parafibular sesamoid. In considering the absence of the parafibula this arrangement of the muscles should be borne in mind.

Specimens of *Agama stellio* and of a common lizard (*Lacerta vivipara*) dissected in our laboratory confirm the arrangement in *Varanus* of a parafibula with two anterior and two posterior *lunulae.*

In a specimen of *Spenodon punctata* examined no parafibula was found but only an anterior sagittal (mid-trochlear) *lunula.* This was confirmed from a mounted specimen of *Spenodon* seen at the South London Botanical Institute.

Mounted skeletons of the *Reptilia* do not as a rule afford much information, either when large on account of being over clean, or when small the knee-joints being often inaccessible even with the use of a lens to minute examination. They require dissection when moist. We may note the following observations made chiefly on specimens at the Royal College of Surgeons.

We examined three chameleons. In *Chamaeleo vulgaris,* we found a sesamoid at the top of the fibula and between this and external face of lateral condyle; there were no *lunulae* (or, of course, *fabellae*); the patella was ossified. In a specimen of *Chamaeleo verrucosus* there were cavities from which it looked as if parafibular sesamoids had fallen out. In a specimen of *Chamaeleo bifurcatus* nothing was distinguishable. In *Zonosaurus madagascarensis* there was an ossified patella, but no other visible knee-joint sesamoid. In *Eumeces algeriensis* there was a parafibula wedged between fibula and tibia; we also found a single *lunula.*

In *Corucia zebrata* there was a somewhat noteworthy system of knee-joint sesamoids. There was a sesamoid articulating with the head of the fibula and a second, directly connected by a ligament with the first, articulating with the head of the tibia; it would be easy to interpret either or both as cases of *lunulae*: see drawing, Plate XXXV, Fig. 91. There was a small ossified patella. Parafibular and paratibial sesamoids also occur in *Lacerta ocellata* together with an internal anterior *lunula*‡.

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* In a mounted specimen of *I. tuberculata,* we again found no parafibular sesamoid, but the anterior *lunula* was rather lateral than sagittal.

† *Proc. Zool. Society of London,* 1867, p. 795.

‡ This lizard had, but on the left limb only, what looked like a possible lateral *fabella.*

In *Tiliqua scincoides*, *Heloderma suspectum* (the Gila monster), *Amieva surinamensis* and *Callopestes* sp. we were only able to note the parafibular sesamoid.

In *Amblyrhynchus cristatus* only lateral posterior and anterior *lunulae* were visible.

In *Physignathus lesueuri* there were all four *lunulae.* In *Uromastix spinipes* there was a small lateral and large mesial posterior *lunulae* and possibly very small lateral and mesial anterior *lunulae.*

In *Jacare punctata* we found what was possibly a mesial posterior *lunula*, but this was doubtful. In no mounted specimens of crocodiles or alligators of a large size have we found parafibular sesamoid or *lunulae*[*].

(23) *Amphibia.*

Among the Batrachians we dissected two species of frogs without discovering any sesamoids at all in the knee-joint. We examined a large number of skeletons with the same result. There were, however, two possible exceptions. In *Leptodactylus pentadactylus* we believe that a mesial posterior *lunula* may exist, and in *Docidophryne gigantea* with more doubt a lateral posterior *lunula.*

Finally we may note that we have dissected specimens of *Trionyx gangetica* and *Testudo graeca* and found no ossifications at all in the knee-joint, even the patella being cartilaginous.

It will thus be seen that the *Amphibia* and *Reptilia* present a great variety of sesamoidal conditions in the knee-joint. We may start from no ossifications at all, even the patella being cartilaginous; we may have a single anterior sagittal *lunula*[†], or again two anterior and two posterior *lunulae*; or we may find in *Lacertilia* all four *lunulae* associated with a parafibula.

We have no real evidence of any lateral *fabella* in this order. The origin of this must be sought in the break-up of the parafibula into *cyamella* and lateral *fabella* in a stage between marsupial and primitive placental. One point appears to us clear, that the appearance of *lunulae* in the higher orders is a vestige of a reptilian ancestry, even in the case where *lunulae* appear in the Primates (from Lemuroids to Man). Further the appearances of *cyamella* and lateral *fabella* are vestiges (even in Man) of an ancestry of primitive ornithodelphian character which had a parafibula, or possibly a fibular crest.

The origin of the mesial *fabella* seems to us shrouded in far greater mystery; it is always of secondary importance, often merely cartilaginous, still more often absent; it is not a "Knotenpunkt" of important muscle attachments like the lateral *fabella.* The cnemial crest of the tibia is relatively rare and has been very rarely indeed

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[*] In a mounted specimen of *Crocodilus americanus* we noted what looked like a very tiny external *fabella*!

[†] The possibility of an anterior sagittal *lunula* being the source of a patella is possibly worth bearing in mind.

recorded in reptilian forms*, and we do not know to what extent the inner head of *gastrocnemius* is associated with this crest†. The possibility of the cnemial crest having given rise to either or both patella and mesial *fabella* may indeed be borne in mind, but it is not of a high order of probability. One feels inclined to propound some general ontogenetic principle by which what occurs by evolutionary origin on one side of a member will be "sympathetically" (or cytologically?) shadowed on the other. Thus *lunulae*, *fabellae* or even condyles occur in pairs, and this is true also of muscular attachments and tuberosities. To propound such a principle would, however, be too venturesome‡, and all we can assert is that the mesial *fabella* has certainly a different origin to the lateral. It is later in arriving and earlier in degenerating and never has the anatomical importance of its lateral colleague.

(24) *Conclusions.*

The main point that seems to us to arise from this study of the sesamoids of the knee-joint is that no parts of the mammalian skeletal form, however apparently insignificant, are without exceeding interest if time can be devoted to their study. If we take the whole evolution of the sesamoids of the knee-joint from their entire absence in the Amphibia to their culminating point in the *Tupaiidae* and *Sciuridae* and to their entire disappearance in certain *Pteropodidae*, we are compelled to admit the feebleness of the argument for use value in any case but that of the patella. Even in the case of the patella we have to admit that the Amphibia, many reptiles, the Marsupials, and some birds seem to have got on remarkably well

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* The occurrence of reptilian forms with even moderate tibial crests should be of a very exciting character. The only case we have so far been able to discover is that of *Ceratosaurus nasicornis* Marsh, a carnivorous Dinosaur of the American Jurassic. It is figured by Marsh and the tibia has a large cnemial crest. It shows in certain respects other avian resemblances as in the form of the pubis and in the co-ossifications of all the pelvic and of the metatarsal bones: see "The Dinosaurs of North America," by O. C. Marsh, *Sixteenth Annual Report of the U.S. Geological Survey*, Part I. pp. 156—162, and Plate XIV, 1896.

† E. Coues in his paper on "The Osteology of *Colymbus torquatus*, with notes on its Myology" (*Memoirs of the Boston Society of Natural History*, Vol. I. 1863, pp. 131—172) deals with muscles of the knee-joint. In this case the tibial crest is as long as the femur itself (2″): see our Plate XXXVI, Fig. 94, and the author makes no mention of an independent patella. Indeed he writes (p. 159): "Just at the border of the joint there is a very small projecting process of bone, which is generally regarded as the true analogue of the patella." Of the muscles he states that *Cruraeus* has a very extensive muscular attachment to the posterior face of the whole length of the tibial spine which projects above the knee-joint (p. 168). *Semimembranosus* is inserted by a broad membranous tendon into the crest of the tibia (p. 170). *Gastrocnemius* might, it seems, almost be described as having four heads. Part of the inner arises fleshy from the apex of the tibial spine and from the anterior face of the upper two-thirds of the bone, the other portion of the inner arises from the inner condyle. The outer head arises from the *linea aspera* for nearly two-thirds of its length and from the outer condyle; it also receives attachments from the *rectus femoris* (p. 171). But the *rectus femoris* instead of going to the patella as in man is inserted in the head of the fibula (p. 167). It would thus appear that the inner head arises from inner condyle and tibial crest, the outer head from outer condyle but is linked also to head of the fibula. The femur articulates with tibia and fibula; the articulation of inner condyle is tibial; that of outer condyle partly tibial, but most extensively fibular (p. 159). Without being conclusive this description is most suggestive for the possibility of the origin of the patella from the tibial crest and to a lesser extent of the origin of a mesial fabella from the same source.

‡ The tendency, however, to reach greater complexity by doubling may, perhaps, be traced in various organs, e.g. in the olfactory organ, or the evolutionary history of the ventricle of the heart.

without ossified patellae. If it be argued that this depends on a difference of mode of progression, a similar counter can hardly be made to the arguments adopted in favour of a use value for the *fabella*; these arguments are too strong, the *Anthropoidea* ought never to have lost their *fabellae*! If we take what appears to us the more rational view that all the sesamoids of the knee-joint are vestigial, and are debris of one or more structures originally of use value, we are bound to consider some such scheme as that of the accompanying diagram, and then the chief difficulty of linkage is seen to be at once apparent in the question as to how far the sesamoids in the knee-joint of any existing species are to be accounted for by acquisition or lapse. For example, the bear and the sheep possess the patella alone: are we to assume their ancestors have never possessed or have lost a complete system of knee-joint sesamoids? The frog and some species of bat possess no sesamoids at all of the knee-joint, but the former has never acquired them, and in the latter they have been suppressed in a long course of evolution. So it is with sheep and bear; the former belongs to the Ungulates who give no evidence of ever having developed a knee-joint sesamoid* beyond the patella, and the latter to the Carnivores with an elaborate evolutionary history of knee-joint sesamoids as complex as that of the Rodents or Insectivores. Again those who like Abel assume the Pinnipedia to have sprung from the *Ursidae*† seem to overlook the relationship of the latter to the full knee-joint sesamoid system of the Carnivores—especially to the *Canidae*.

Or, to go a stage further, we have so far not come across any Insectivores with patella only. Edentates of this character exist, but the Insectivore with only a patella has yet to be found. It may have existed, but the Insectivores seem to have sprung from a multi-sesamoidal form. Chiroptera and Rodentia can well have sprung from the Insectivores. To suggest as Kükenthal‡ that the Ungulates sprung from the Insectivores and the latter from Marsupials, is to neglect the fundamental contribution of the Marsupials to the knee-joint sesamoid system, namely, the *cyamella* and the lateral *fabella*. The parafibula is known in reptilian forms and the Edentates may have got their parafibula from these. But if the Ungulates sprung from the Marsupials it is astonishing that no cases even as anomalies of the *cyamella* in the Ungulates have been recorded.

We may note also a very special point of difficulty about the Marsupials themselves. An ossified patella is an extreme rarity. Are we to assume that the patella was lost in the Marsupials while *cyamella* and *fabellae* were developing? The ossified patella exists in the Monotremes, and this alone would differentiate them largely from the Marsupials. But if the patella was obtained in the reptilian stage and handed on to the Monotremes and certain birds, while it was lost in the Marsupials, it is hard to believe in a direct descent of Insectivores from Marsupials, at any rate it would point either to the *Peramelidae* or to some other Marsupial differing widely—as far as the knee-joint is concerned—from those hitherto known.

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* The lateral *fabella* of *Hyrax* we take to be evidence, not of a lost sesamoidal system in the Ungulates, but of a differentiated origin of *Hyrax*. We have not been able to confirm Meckel's and Gruber's statements as to the *Cervidae*.

† *Handwörterbuch der Naturwissenschaften*, Bd. VIII. S. 718, 1913.

‡ *Ibid.* Bd. VIII.

A study of either Carnivores or Primates indicates what a long course of evolution is involved in the loss of a sesamoidal system, and what a wide range of changing forms usually intervenes. The persistency of the same type of sesamoidal system for generations in the same species, and the occurrence of atavisms, such as *lunulae* in man, form strong evidence that the minor sesamoids of the knee-joint are as hereditary in character as the patella.

To assert that a sesamoid can be produced by "intensive stress" when and where it is required is to assert in its baldest and most futile form the inheritance of acquired characters. The most futile form because a parafibula, say, is not a merely formless mass, it is an organised link for a system of muscles, usually of a very elaborate kind. And if it could be acquired by "intensive stress" when and where needed, there would seem no reason to despair of any species producing a new bone of the skeleton, when and where needed. It is conceivable that a mutation could produce such a bone by an anomalous disturbance of the germ cells—but it would certainly not be due to "intensive stress" at the required point, and it would only be by the rarest chance that it would be "when and where needed." The whole mechanism of heredity is disregarded by such anatomical suggestions.

In forming the scheme of our diagram we wish to emphasise the fact that while there are factors of acquisition and suppression at work, we yet consider the primary factor to be heredity—the form of the sesamoidal system is inherited as the form of the teeth are inherited, and the change of form will in each case be a lengthy evolutionary process*.

In our diagram we have purposely refrained from providing any descent links, partly because there is such an immense amount of work still to be done on the sesamoid systems especially of reptilian and avian forms, and partly because to do so would be sure to give rise to the criticism that we had supposed one illustrative type to have sprung directly from a second illustrative type; whereas in fact both illustrative types may have been only *in cunabula* at the time of the link. The link lies between the sesamoidal systems, not necessarily between their present representatives.

We have really four origins to study:

(*a*) That of the patella; it may have arisen from the tibial crest or from an antero-sagittal *lunula*, or perhaps have had both origins. At present we have not come across any evidence of a tibial crest in the Reptilia except in the Dinosaurs.

(*b*) That (or better, perhaps, those) of the *lunulae*. We have no suggestion to make here. It needs a far greater study of the knee-joint of both Amphibia and Reptilia than we have been able to make.

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* That in an early period of evolution the germ cells of various allied species were not so highly specialised and differentiated as in a later period, is a possibility that we have not seen emphasised, and which would, perhaps, render by hybridisation the earlier evolutionary processes quicker. Even now we have very vague ideas of the limits of possible hybridisation. How far are the Primates *inter se* fertile? The fox and dog are natural enemies, but the cross is a feasible one, and will provide a second generation. Much of importance for evolution might arise by crossing—if needful by artificial impregnation—forms apparently fairly diverse. Divergence of habits and temperament and somatic differences may have more to do with the problem of infertility than any factor of the germ cells themselves.

PATELLA).

No Sesamoids
(Most *Amphibia*)

←—*Acquisition*

No Sesamoids
(Some *Reptilia*) (S

No Sesamoids
(Sc

Tibial Crest Til

Tibial Crest
(Some *Aves*, e.g. Albatross)

Patella
(Some Edentates, *Dasy*

Patella
(Some Insectivores??) *ıga*)

Patella, Cyamella and (or fused with) L. Fabella
(Some Marsupials, e.g. *Perameles*, bony kernel to patella in others)

Patella ʂed with) L. Fabella, M. Fabella
(*Ungulata*, *Pinnipedia*) ted of *Halmaturus*)

Patella and Cyamella, L. and M. Fabellae
(Some Rodents, e.g. s, e.g. Rabbit, Squirrel)

Patella, Cyamella, L. and M. Fabellae
(Probable ancestor of *Chiroptera*)

←—*Lapse*

Patella, L. and M. Fa
(Some Carnivores, e.g

Patella, L. and M. Fabellae
(Some *Chiroptera*, e.g. *Cynopterus marginatus*)

Patella, L. and
(Some Carnivores, e.g

Cyamella, L. and M. Fabellae
(Some *Chiroptera*, e.g. certain species *Vespertilio*)

Patella and L.
(Some Carnivores, e.g. *Lu*

L. and M. Fabellae
(Some *Chiroptera*, e.g. *Vespertilio murinus*)

Patella
(Some Carnivores, e.g.

L. Fabella
(Some *Chiroptera*, e.g. *Phyllostoma*)

No Sesamoids
Some *Chiroptera*, e.g. certain species *Pteropus*)

THE EVOLUTION OF THE ORTHOSESAMOIDS OF THE KNEE-JOINT (INCLUDING THE PATELLA).

←—*Acquisition*

←—*Lapse*

No Sesamoids
(Most *Amphibia*)

No Sesamoids
(Some *Reptilia*)

Single Lunula
(Some *Amphibia*? e.g. *Leptodactylus pentadactylus*)

No Sesamoids

Lunulae
(Some *Reptilia*, e.g. *Amphibolurus barbata*)

Tibial Crest
(Some *Reptilia*?)

Fibular Crest
(Some *Reptilia*??)

Tibial Crest

Tibial Crest and Lunulae
(Some *Aves*)

Tibial Crest and Lunulae
(Some *Reptilia*?)

Parafibula and Lunulae
(Some *Reptilia*, e.g. *Varanus*)

Patella and Fibular Crest
(Monotremes)

Tibial Crest
(Some *Aves*, e.g. Albatross)

Tibial Crest, Flake Patella and Lunulae
(Some *Aves*, e.g. *Colymbus*)

Patella, Parafibula and Lunulae
(Some *Reptilia*? e.g. *Corucia zebrata*)

Parafibula
?

Fibular Crest
(Some Marsupials, e.g. occasionally Wombat)

Patella and Lunulae
(Some *Aves*)

Patella, Parafibula and Lunulae
?

Parafibula
(Some Marsupials; usually Wombat)

Patella
(Some Edentates, *Dasypus*, *Chlamydophorus*)

Patella and Lunulae
?

Patella, Parafibula and Lunulae
(Some Edentates, *Megatherium*, *Bradypodidae*, *Manidae*)

L. Fabella and Cyamella
(Some Marsupials, e.g. *Macropus*)

Patella
(Some Insectivores??)

Patella and Lunulae
(Some Insectivores, Myogale, perhaps)

Patella, Lunulae, L. Fabella
(Some Insectivores, e.g. *Talpa*, *Potamogale*)

Patella, and L. Fabella
(Some Edentates? *Orycteropus*)

Patella and Cyamella
(Some Edentates, e.g. *Myrmecophaga*)

Patella, Cyamella and (or fused with) L. Fabella
(Some Marsupials, e.g. *Perameles*, bony kernel to patella in others)

Patella
(*Ungulata*, *Pinnipedia*)

Patella and L. Fabella
(*Hyracoidea*. ? Some *Cervidae*)

Patella, Lunulae, L. and M. Fabella
(Some Insectivores, e.g. *Erinaceus*, *Tupaia*, *Hylomys*)

Patella, Cyamella, Lunulae
(Some Insectivores, e.g. *Ericulus setosus*)

Cyamella and (or fused with) L. Fabella, M. Fabella
(asserted of *Halmaturus*)

Patella and Lunulae
(Some Rodents, e.g. *Castor fiber*)

Patella, Lunulae, L. and M. Fabellae
(Some Rodents, e.g. *Mus musculus*, *Fiber zibethicus*, *Dasyprocta*)

Patella, Lunulae, Cyamella, L. and M. Fabellae
(Some Lemuroids, e.g. Loris and Potto)

Patellae, Lunulae, Cyamella, L. and M. Fabellae
(Some Rodents, e.g. Rabbit, Squirrel)

Patella, Cyamella, L. and M. Fabellae
(Probable ancestor of *Chiroptera*)

Patella, L. and M. Fabellae, Cyamella
(Some Carnivores, e.g. *Felis domestica*)

Patella, Cyamella, L. and M. Fabellae
(Some Lemuroids, e.g. *Chiromys*; *Cebidae*, and *Simiadae*)

Patella, L. and M. Fabellae
(Some *Chiroptera*, e.g. *Cynopterus marginatus*)

Patella, L. and M. Fabellae
(Some Carnivores, e.g. *Felis nebulosa*)

Patella and Cyamella
(Some *Anthropoidea*, Orang, Gorilla)

Patella, L. and M. Fabellae
(Certain *Hylobatinae*)

Cyamella, L. and M. Fabellae
(Some *Chiroptera*, e.g. certain species *Vespertilio*)

Patella and L. Fabella
(Some Carnivores, e.g. *Lutra*, *Gula vittatus*)

Patella
(Some *Anthropoidea*, Chimpanzee, and *Hominidae*, Man)

Patella and L. Fabellae
(Certain *Hylobatinae*

L. and M. Fabellae
(Some *Chiroptera*, e.g. *Vespertilio murinus*)

Patella
(Some Carnivores, e.g. *Ursus*)

L. Fabella
(Some *Chiroptera*, e.g. *Phyllostoma*)

No Sesamoids
Some *Chiroptera*, e.g. certain species *Pteropus*)

(*c*) That of *cyamella* and lateral *fabella*. We are fairly convinced that this lies in the break-up of the parafibula and that the parafibula was the result of the separation of the fibular crest with its attached muscular system. The parafibula exists in certain reptiles. The discovery of a reptilian fibular crest would be of the greatest importance. Of course there may be again a double origin corresponding to the parafibula of the Reptiles and the fibular crest of the Monotremes.

(*d*) That of the appearance at a relatively late stage of evolution (and the early disappearance) of the mesial *fabella*. We have no real light to throw on this point; it needs a very special study. The fact that the mesial *fabella* is never specific without the lateral, while the lateral is in early as well as in late evolutionary stages specific without the mesial, suggests that the explanation may be cytological. At the same time we have to remember that while the lateral head of *gastrocnemius* had originally a fibular or at least a parafibular attachment, the evolutionary history of the inner head seems to point to no such marked change in its attachments*.

The object of this memoir has been to suggest problems to those better equipped for studying them than the present authors, rather than to present solutions. They believe that the least important elements of the skeletal form are each able to contribute important suggestions for the general attack on the enigma of evolution and that all vestiges are of special value in the assault. They might even be content with the outcome of this paper if it led anatomical text-book writers in the future to drop once for all the Galen tradition and assert on the contrary that it is needful that all sesamoids should be discussed.

As final conclusion may we not cite the words used by Pittard in 1849 as appropriate to our own views: "But after all, taking into consideration all the facts related above, and many others that have presented themselves to us in the course of this enquiry, we cannot but believe that some higher law than that of adaptation concurs in determining the presence, if not the size, of even these little bones†."

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We wish further to thank Sir Arthur Keith for the invariable kindness he has shown us in freely placing the immense stores of valuable material in his charge at our disposal. For aid in a great variety of enquiries we have to thank very cordially Dr E. C. Derry of Cairo, Mr W. R. Sherrin of the South London Botanical Institute, Dr W. C. Mackenzie of the Australian Institute of Anatomical Research, Melbourne, and Dr Julia Bell of the Galton Laboratory. We have also to acknowledge the very helpful assistance of Miss I. McLearn, especially in the drawings of the knee-joints of birds.

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* It may have in certain cases an insertion to the tibial crest: see the second footnote on p. 392.

† Article on Sesamoid Bones, Todd's *Cyclopaedia of Anatomy and Physiology*, p. 543.

DESCRIPTION OF PLATES XIX—XXXVII *.

Plate XIX, Fig. 40. Femur of Sea Otter (*Enhydra lutris*), showing the marked development of the lateral *fabella*; the sesamoid appears by its action to have largely eroded the popliteal surface of the femur, and the periosteum of the bone in its neighbourhood has disappeared.

Fig. 41. Fibula, Tibia and Femur of Beaver (*Castor fiber*). The development of the fibular head at A suggests how a second sesamoid might well originate in the tendon of the muscle (*Peroneus longus*?) attached to this projection. The femur exhibits a marked third trochanter.

Plate XX, Fig. 42. Drawing of musculature of knee-joint of *Ornithorhynchus*. We see the lateral head of *M. gastrocnemius* attached solely to the fibular crest; the mesial head of *gastrocnemius* is still attached to the mesial condyle.

Fig. 43 *a*, anterior, and Fig. 43 *b*, lateral aspect of the femur, tibia and fibula of *Ornithorhynchus*. The fibula shows the noteworthy crest, a compressed process with its own centre of ossification. There are no sesamoids except patella. Cf. Fig. 47 for the corresponding bones of *Echidna*.

Fig. 44. Drawing of the posterior aspect of the left knee-joint of *Felis domestica*, showing the positions of the external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and the facet on which the latter rests. Owing to the knee-joint being slightly extended the *cyamella* has been lifted off its articular surface on the head of the fibula †.

Fig. 45. Drawing of postero-lateral aspect of the left knee-joint of *Hyrax* showing the external *fabella* (*e.f.*) above the external condyle, but the tendon of *gastrocnemius* has shrunk in drying, so that the *fabella* has been lifted off the condyle. There is a slight depression on the internal condyle, which might suggest an internal *fabella*, but so far none has been found.

Plate XXI, Fig. 46. *Colymbus glacialis*, a drawing after Owen's figure in his article *Aves* in Todd's *Cyclopaedia*. Note the remarkable cnemial crest of the tibia. Cf. especially all the figures on Plate XXXVI.

Fig. 47. Lateral aspect of pelvic limb of *Echidna* to be compared with the knee-joint of *Ornithorhynchus* in Figs. 43 *a* and 43 *b*. Drawing after a part of the skeleton in a plate of Cuvier's *Recherches sur les ossemens fossiles*, Paris, 1834–6 (Tome II, Atlas Plate 214).

Fig. 48. Drawing of lateral, Fig. 49, drawing of mesial, aspect of knee-joint of left limb of Cereopsis Goose, showing cnemial crest of tibia and also a semi-fibular crest which works in a groove on the articular surface of the lateral condyle. Cf. Plate XXXVI.

Plate XXII, Figs. 50 and 51. Drawings from lateral and mesial aspects of the left knee-joint of the Albatross, showing the remarkable cnemial crest of the tibia.

Figs. 52, 53 and 54. Drawings of anterior, lateral and posterior aspects of knee-joint of *Dendrocygna*, the Tree Duck. There is a cnemial crest (*cn. crest*), internal (*i.p.lu.*) and external (*e.p.lu.*) posterior *lunulae* and also what appear to be two or three very small sesamoids wedged in between fibula and tibia (*sesamoids*). Cf. Plate XXXVI.

Plate XXIII, Fig. 55 *a*. Tamandua Anteater (*Tamandua tetradactyla*); photograph of posterior aspect of right limb showing the large *cyamella* (*c.*) still in the popliteal tendon.

Fig. 55 *b*. Common Squirrel (*Sciurus vulgaris*), photograph of posterior aspect of right limb, showing internal and external *lunulae* (*i.l.* and *e.l.*) and *cyamella* (*c.*). Both *fabellae* also exist in the squirrel.

Figs. 56 *a* and 56 *b*. Common Hedgehog (*Erinaceus europaeus*). The former photograph of posterior aspect of right limb showing internal (*i.f.*) and external (*e.f.*) *fabellae*; the latter photograph of mesial aspect of left limb with the femur doubled right back to show anterior external (*a.e.l.*) and anterior internal (*a.i.l.*) *lunulae*. No *cyamella* was found.

Fig. 57 *a*. Photograph of the right pelvic limb of *Hydromys chrysogaster*, the Golden-bellied Rat, from the posterior aspect showing internal (*i.f.*) and external (*e.f.*) *fabellae*.

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* The drawings are as a rule natural size.

† It must be remembered that in nearly all these drawings the sesamoid bones are still contained in the tendons in which they occur. Consequently no attempt could be made to show the actual size of the sesamoids.

Fig. 57 *b*. Photograph of the posterior aspect of the right pelvic limb of *Fiber zibethicus*, the Musquash, showing internal (*i.f.*) and external (*e.f.*) *fabellae* together with a posterior external *lunula* (*p.e.l.*). Compare Figs. 61 *a* and 61 *b*.

Plate XXIV, Fig. 58. Drawing of posterior aspect of left pelvic limb of *Lepus cuniculus*, the Common Rabbit, showing large external (*e.f.*) and smaller internal (*i.f.*) *fabellae* and *cyamella* (*cy.*). The rabbit has usually (?) two anterior *lunulae* as well, but not in this specimen.

Fig. 59. Drawing of lateral aspect of left pelvic limb of *Potamogale velox* from West Africa, showing the external *fabella* (*e.f.*). No sign of ossification in popliteal tendon. Fibula fused with tibia at distal end.

Fig. 60 *a*. Drawing of posterior aspect of left knee-joint of *Paraxerus jacksoni*, African Squirrel, showing external (*e.f.*) and internal (*i.f.*) *fabellae*, together with *cyamella* (*cy.*).

Fig. 60 *b* is the anterior view of the same knee-joint showing an internal anterior *lunula* (*a.i.lu.*).

Fig. 61 *a* and Fig. 61 *b*. Drawings respectively of the posterior and anterior aspects of the right knee-joint of *Fiber zibethicus*, the Musquash or Musk Rat. The former shows the internal (*i.f.*) and external (*e.f.*) *fabellae* with a posterior external *lunula*; the latter besides the two *fabellae* shows external (*e.a.lu.*) and internal (*i.a.lu.*) anterior *lunulae*, but no *cyamella* was present. Cf. Plate XXIII, Fig. 57 *b*.

Fig. 62 *a* and Fig. 62 *b* provide drawings of the posterior and postero-lateral aspects of right pelvic limb of *Hylomys suillus dorsalis*, a Tree Shrew from Borneo. Only internal (*i.f.*) and external (*e.f.*) *fabellae* were found.

Fig. 63 *a* and Fig. 63 *b*. Drawings of the anterior and posterior aspects of the knee-joint of *Trachysaurus rugosus*, the Australian stump-tailed lizard. The former shows at *a′* the lateral anterior *lunula*, and at *b′* the mesial anterior *lunula*; the latter gives *a* the mesial and *b* the lateral posterior *lunula*, and the *cyamella* as indicated. The *cyamella* is on the *anterior* corner of the fibula, but is seen in both posterior and anterior aspects. The four *lunulae* all lie between tibia and femur, the *cyamella* between fibula and femur.

Fig. 64 *a* and Fig. 64 *b*. Drawings of the posterior and anterior aspects of the right knee-joint of *Varanus niloticus*, the Nile Monitor. The former shows internal (*p.i.lu.*) and external (*p.e.lu.*) posterior *lunulae* and the latter external (*a.e.lu.*) and internal (*a.i.lu.*) anterior *lunulae* as well as a large *cyamella* or better *parafibula* (*paraf.*) on the head of the fibula. This parafibula has tendon in the popliteal groove and is also attached to the head of the fibula; it lies just internal to the attachment of the external lateral ligament as shown in the drawing (Fig. 64 *b*). It will be seen that the *cyamella* is entirely anterior and this position is in partial agreement with that of *Trachysaurus rugosus*, which, however, is also visible from posterior as well as anterior aspect. Cf. Figs. 63 *a* and 63 *b*.

Fig. 65 *a* and Fig. 65 *b*. Drawings of the anterior and posterior aspects of the left knee-joint of *Amphibolurus barbata*. The former shows two anterior *lunulae*, a small internal one (*a.i.lu.*) and a larger external (*a.e.lu.*) or rather mid-trochlear *lunula*. The latter shows an external (*p.e.lu.*) and an internal (*p.i.lu.*) posterior *lunula*; both are on top of the tibia and the external is the larger. There is no trace of a parafibula.

Plate XXV, Fig. 66. Photograph of the right knee-joint of *Megatherium cuvieri* or the Fossil Giant Sloth, in the Royal College of Surgeons' Museum. The photograph shows a sesamoid on the top of the fibula, which has usually been described as a lateral *fabella*. It lies close to the popliteal groove and there is little doubt that it is a *cyamella* (*cy.*) or rather parafibula. On the top of the tibia is a second sesamoid which we have not hitherto seen referred to. It is undoubtedly an external anterior *lunula* (*a.e.lu.*). It lies externally from the base of the large *patella* (*pa.*).

Fig. 67. Photograph of the left knee-joint of *Manis aurita*, the Chinese Pangolin. The knee-joint has been opened out for the purpose of photography, but the *cyamella* (*cy.*), usually described as a *fabella*, appears to have been mounted in the right position. When the knee is flexed the sesamoid does not lie on the top of the external condyle, but on the top of the fibula; it is a parafibula, articulating with both fibula and the *posterior* face of the external condyle.

Plate XXVI, Fig. 68. Drawing of the posterior aspect of left knee-joint of *Simia satyrus*, the Orang-ûtan, showing the *cyamella* (*cy.*) in position in the popliteal tendon. Inset, drawings of three *cyamellae* from three Orang-ûtans' knee-joints dissected in the Biometric Laboratory, posterior and anterior aspects, showing range of sizes from *b* to *c* and *a*. *a* is from Orang-ûtan figured, *b* and *c* are from a second specimen, left and right joints respectively. No other sesamoids were found.

Fig. 69 is a drawing of the same knee-joint as in Fig. 68 with the popliteal tendon containing the *cyamella lateralis* (*cy.*) turned back so as to disclose the facet (*f. for cy.*) on which it articulates.

Plate XXVII, Fig. 70 *a* and Fig. 70 *b*. Drawings from posterior and lateral aspects respectively of the left knee-joint of *Lemur varia*, showing the external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and an anterior external *lunula* (*a.e.lu.*) in position. The *cyamella* is seen in close relation to the head of the *fibula* and one notes how with flexed limb the external *fabella* will come into contact with it.

Fig. 71 *a*. Drawing of posterior aspect of right knee-joint of *Mycetis palliatus*, showing internal *fabella* (*i.f.*), external *fabella* (*e.f.*) and *cyamella* (*cy.*).

Fig. 71 *b*. Sesamoids from the joint in Fig. 71*a*. *a* is posterior aspect, *b* anterior aspect of *cyamella*, *c* and *d* are respectively *fabella mesialis*, and *fabella lateralis* in posterior aspects.

Fig. 72. Drawing of distal extremity of right femur of a Gibbon (*Hylobates lar*) which possessed only an internal sesamoid (*i.f.*) and no ossification in the popliteal tendon (*p.t.*) which was "horny" but not ossified.

Fig. 73. Drawing of postero-lateral aspect of left knee-joint of *Lemur melanocephalus* showing external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) enclosed in popliteal tendon, and an anterior external *lunula*.

Plate XXVIII, Fig. 74 *a*. Drawing of posterior aspect of left knee-joint of *Chiromys*, the Aye-Aye. The two branches of *M. gastrocnemius* (g_1 and g_2) have been deflected to show the external (*e.f.*) and the internal (*i.f.*) *fabellae* lying beneath; below is the popliteal tendon (*p.t.*) with the *cyamella* lying on the head of the fibula (*fib.h.*) and articulating with the tibia. *po.* is the popliteal muscle.

Fig. 74 *b*. Drawing of the lateral aspect of the left knee-joint of *Chiromys* after Owen (*Trans. Zool. Soc. London*, Vol. v, 1866, Plate XIX) to indicate how in flexure of the knee the external *fabella* (*e.f.*) comes in contact with the *cyamella* (*cy.*).

Fig. 75. Drawing in posterior aspect of left knee-joint of *Lemur catta*, showing external (*e.f.*) and internal (*i.f.*) *fabellae*, the *cyamella* (*cy.*) and an anterior external *lunula* (*a.e.lu.*).

Fig. 76. Drawing of the lateral aspect of the left knee-joint of a dry specimen of *Perodicticus potto*, Bosman's Potto, showing the position of the anterior external *lunula* (*a.e.lu.*) and the *cyamella* (*cy.*) on the head of the fibula. The popliteal tendon sends a branch down to the head of the fibula (see our p. 378). The *fabellae* had disappeared from this specimen, but it is clear that, in this flexed position of the limb, the lateral *fabella* must rest on the proximal surface of the *cyamella*.

Fig. 77. Drawing of the lateral aspect of the left knee-joint of *Indris brevicaudata*, the Indri, showing the popliteal tendon (*p.t.*) containing the *cyamella* (*cy.*) and the positions of external (*e.f.*) and internal (*i.f.*) *fabellae*. The intimate relations of external *fabella* and *cyamella* on complete flexure are again indicated.

Plate XXIX, Fig. 78. Right, photograph of right pelvic limb of *Phascolomys wombatus*, the Wombat. This specimen shows the parafibula absolutely fused to the fibula so as to show one continuous bone with very small indication of position of union.

Left, photograph of right pelvic limb of *Sarcophilus ursinus*, the Tasmanian Devil; the parafibula is united to the fibula, but its limits are more clearly defined.

Fig. 79. Photograph of left pelvic limb of Wombat, the two portions of the parafibula, which is really free of the fibula, are shown in position on the head of the fibula, the femur is tied back to show up the parafibula which in the usual flexed position would be in contact or almost in contact with the shaft of the femur.

Plate XXX, Fig. 80. Photograph of the left pelvic limb of *Phalangista vulpina*, the Wolfine Phalanger, showing the free parafibula (*paraf.*) mounted on the head of the fibula and its relation to the femur when the limb is flexed. The two portions of the parafibula, which ultimately become the lateral *fabella* and the *cyamella*, are easily distinguished.

Plate XXXI, Fig. 81. Drawing of the posterior aspect of the right knee-joint of the Wombat with muscles attached. Description chiefly on figure itself. The external *gastrocnemius* has been cut across and the proximal portion deflected to show parafibular sesamoid lying in its fibres, and the surface of the fibula with which the sesamoid articulates. The internal *gastrocnemius* takes origin from nearly the whole of the popliteal surface, while the external *gastrocnemius* originates from the parafibular sesamoid. The drawing is also intended to show the tendinous fibres running from *gastrocnemius externus* to the lateral border of the patella; a few, cut through in drawing in order to expose the patella, run to the lateral face of the external condyle. The popliteal tendon can also be seen taking origin in the popliteal groove of the external condyle. It passes across the head of the fibula, its internal border being connected with the external semilunar cartilage, while the external fibres run down to the muscle fibres of the popliteus. This latter arises from the upper third of the posterior surface of the fibula. The double nature of *gastrocnemius externus* suggests that the deeper portion is *soleus*, although Macalister ("Myology of the Wombat and the Tasmanian Devil," *Annals of Natural History*, 1870, p. 19 (offprint)) suggests that *soleus* has a fibular origin.

Plate XXXII, Fig. 82 *a* and Fig. 82 *b*. Drawings of the posterior aspect of the left pelvic limb of *Phascolarctus cinereus*, the Koala, with the muscles attached. Fig. 82 *a* shows the popliteal muscle and the external *gastrocnemius* with the parafibular sesamoid covered by its fibres. Fig. 82 *b* shows *gastrocnemius* cut and turned back, so that the facet of the sesamoid which articulates with the fibula is exhibited and the position on which it rests on the fibular head is seen. This sketch is also intended to illustrate the following relations. The fleshy portion of *popliteus* is attached to fibula, and popliteal tendon runs from the anterior proximal corner of fibula to popliteal groove on external condyle. Also tendinous fibres from the fleshy portion on fibula cross the posterior surface of the latter and mingle with the tendinous fibres of *gastrocnemius* which run from the sesamoid to the external condyle. (See also Fig. 86.)

Plate XXXIII, Fig. 83 *a* and Fig. 83 *b*. *Macropus* sp. The portion of the knee-joint of a kangaroo (species?) here drawn was most kindly sent to us by Dr W. C. Mackenzie of the Australian Institute of Anatomical Research. It consists only of the lateral branch of *gastrocnemius* with its sesamoid and the *cyamella*, together with the external semilunar cartilage attached. It was thus not possible to draw these in their natural position. It should not be impossible, however, to understand the relations to other structures if the reader will compare these drawings with those of other *Marsupialia*. The three important components are the external *fabella*, the cartilaginous *cyamella*, with surface which articulates with the fibular head, and the external semilunar cartilage. These are drawn in two aspects, the interesting point being that in the kangaroo the lower part of the parafibula, still articulating with the fibular head on the distal face and with the external *fabella* on the proximal has become purely cartilaginous.

Fig. 84. Drawing of the posterior aspect of the left knee-joint of *Didelphys virginiana*, the Virginian Opossum. Here *a* is the popliteal tendon from groove on external condyle, *b* is the *cyamella* in the popliteal tendon as it passes over fibular head, *c* and *d* are branches of popliteal muscle running from tendon attached to *cyamella*, *e* is the external *fabella* in tendon of *gastrocnemius*, *g* and *g'* are the cut ends of *gastrocnemius*. The *gastrocnemius* has been cut just below its insertion on top of the external condyle to enable it to be pulled aside and display the *fabella*. When the leg is fully flexed the external *fabella* lies on the external condyle, and the *cyamella* is in direct contact with the lower surface of the condyle and the *fabella*.

Fig. 85. Drawing of the lateral aspect of the left knee-joint of *Phalangista vulpina*, the Wolfine Phalanger. *a* is the parafibular sesamoid resting on the head of the fibula, *b* is the popliteal tendon. If the reader will compare this drawing with Fig. 78, he will grasp how parafibula and fibula united give a single complete bone. It is not possible to term the sesamoid of Phalangista merely a *fabella*.

Fig. 86. A further drawing of the posterior aspect of the right knee-joint of *Phascolarctus cinereus* after removal of the muscles. *a* is the tendon of *gastrocnemius externus* attached to femur above external condyle, *b* is the cut end of *gastrocnemius externus*, *c* is the fibular "collateral" ligament, *d* is the popliteal tendon running to head of fibula, the lower portion of popliteal tendon has been removed. From a specimen kindly sent by Dr W. Colin Mackenzie.

Plate XXXIV, Fig. 87. Femur of *Macropus bennettii*, the Wallaby, with parafibular sesamoid attached above the external condyle.

Fig. 88. *Perameles lagotis*, the Rabbit-eared Perameles, with parafibular sesamoid attached above the external condyle.

Figs. 87 and 88 are photographs of museum specimens and we hold that they place the sesamoid in an erroneous position, leading it to be confused with a lateral *fabella*.

Fig. 89 *a* and Fig. 89 *b*. Photographs of anterior and lateral aspects of a moist preparation of the right knee-joint of *Sarcophilus ursinus*, the Tasmanian Devil, from the Museum of the Royal College of Surgeons, showing the parafibular sesamoid in what we believe to be the true position, i.e. mounted on the head of the fibula with a ligament connecting its summit with the lateral face of the femur above the external condyle.

Plate XXXV, Fig. 90. Drawing of the lateral aspect of the right knee-joint of *Halmaturus sp.*, the Wallaby, to indicate the position of the parafibular sesamoid on the head of the fibula.

Fig. 91. Drawing of the postero-lateral aspect of the left limb of *Corucia zebrata* to show a noteworthy sesamoid arrangement. *a* is a parafibular sesamoid united by a ligament to *b* a paratibial sesamoid, which might be a *lunula*. Drawing by Mr S. Steward.

Plate XXXVI. Drawings of the knee-joints of Birds. Cf. Plate XXI, Figs. 46, 48, 49, and Plate XXII, Figs. 50, 51, 52, and 53.

Fig. 92. Drawing of lateral aspect of left pelvic limb **A** of *Hesperornis regalis*, after Marsh. Note the giant patella—longer than femur—and the condyles of the femur enclosed by patella; tibia and fibula.

Fig. 93 *a* and Fig. 93 *b* give the mesial (B_1) and lateral (B_2) aspects of the right and left knee-joints of the Cormorant; the patella is of relatively smaller size than that of *Hesperornis regalis*, but clearly built on the same lines; the continuity of the ridges *aa'* and *bb'*,—the pro- and ecto-cnemial ridges of the tibia carried right up the lateral face of the patella,—must be emphasised, as in favour of released cnemial crest rather than a co-ossified patella.

Fig. 94. Drawing of the lateral aspect of the left pelvic limb **C** of *Colymbus torquatus* after Coues. There is a giant cnemial crest. A comparison of **A** and **C** suggests either that (i) the cnemial crest is a co-ossified patella or (ii) the patella a released cnemial crest; independent origin of both is hard to accept. B_2 is certainly an argument in favour of (ii).

Plate XXXVII, Fig. 95 and Fig. 96. Drawings of sections respectively of the fibular crest of a young and an embryo platypus. In the former case ossification is complete but the arrangement of the Haversian canals at the top of the plate, i.e. the proximal end of the fibular crest, seems far from incompatible with a separate centre of ossification. In the latter there appear signs of ossification starting towards the summit of the crest in an area widely removed from the centres of ossification of the shaft or of the fibular head.

Fig. 41. Fibula, Tibia and Femur of Beaver (*Castor fiber*).

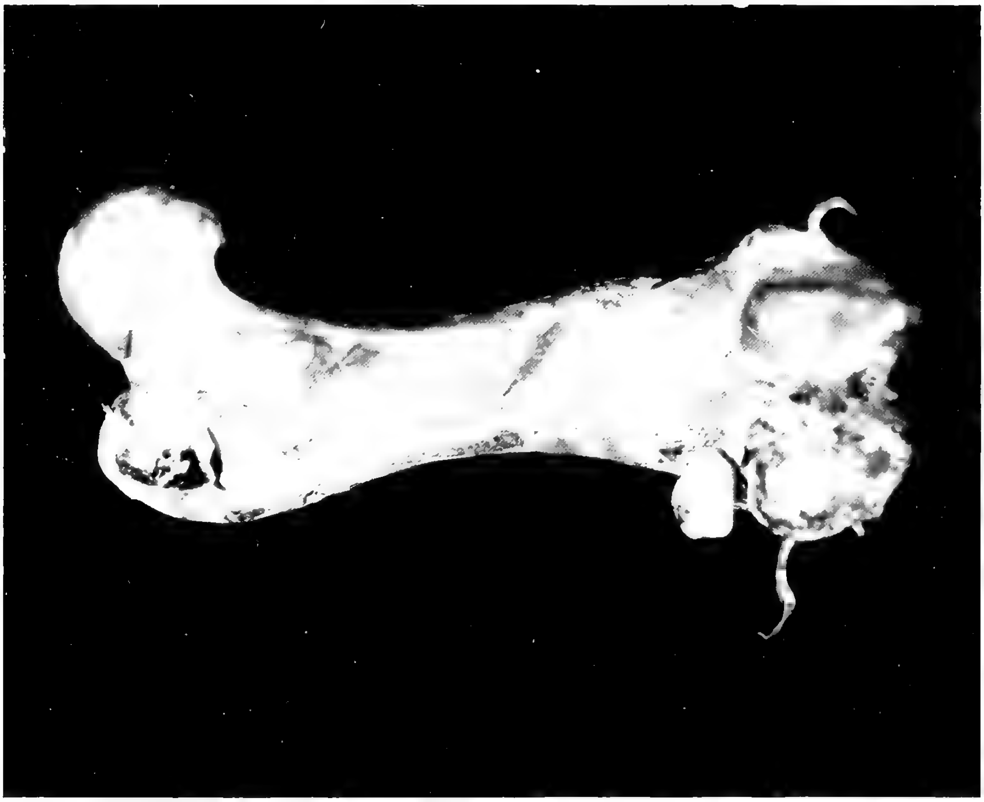


Fig. 40. Femur of Sea Otter (*Enhydra lutris*).

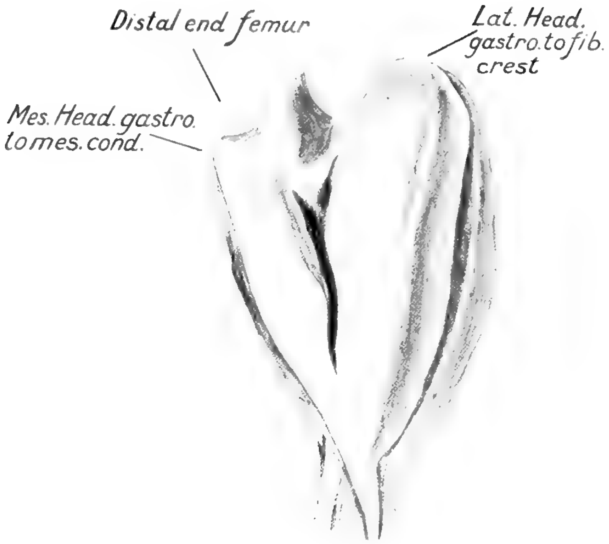


Fig. 42. *Ornithorhynchus* (Musculature).

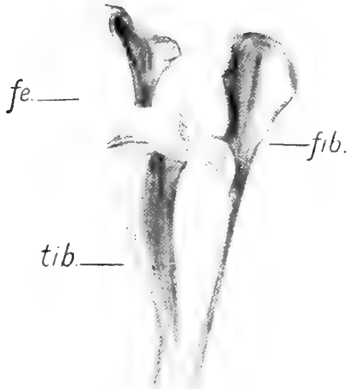


Fig. 43 *a*.

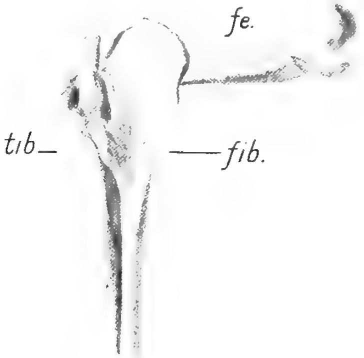


Fig. 43 *b*.

Ornithorhynchus (Bones of Lower Limb).

Fig. 44. Cat (*Felis domestica*).

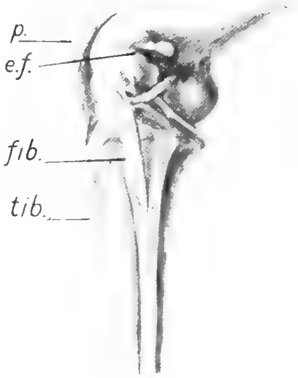


Fig. 45. *Hyrax*.

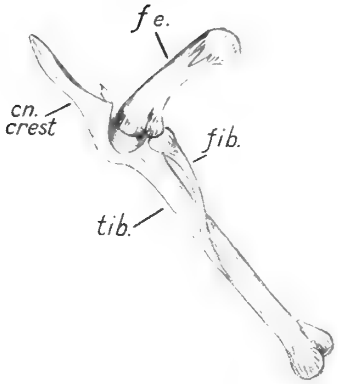


Fig. 46. *Colymbus glacialis* (after Owen).

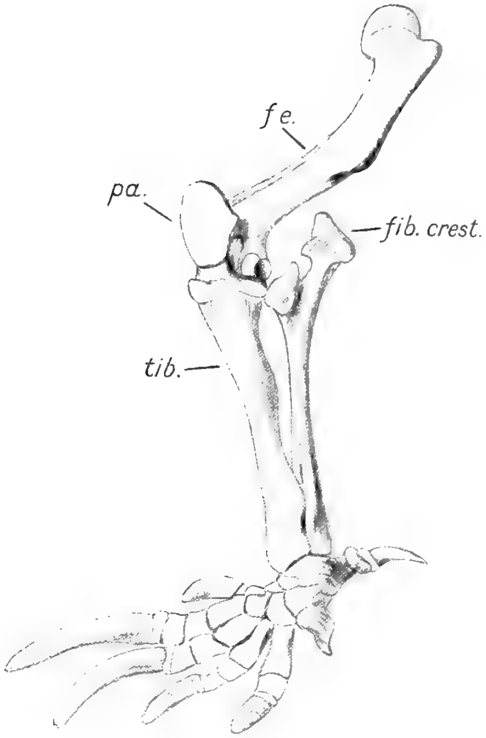


Fig. 47. *Echidna* (after Cuvier).

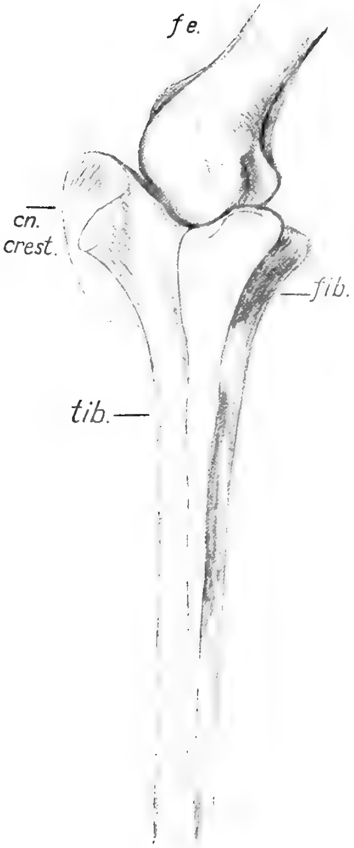


Fig. 48 Cereopsis Goose.

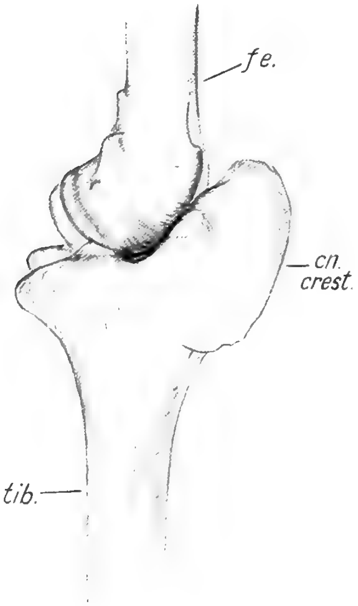
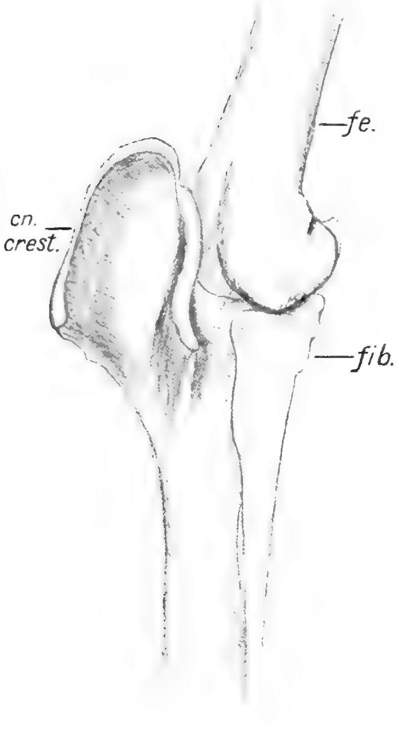


Fig. 49. Cereopsis Goose.



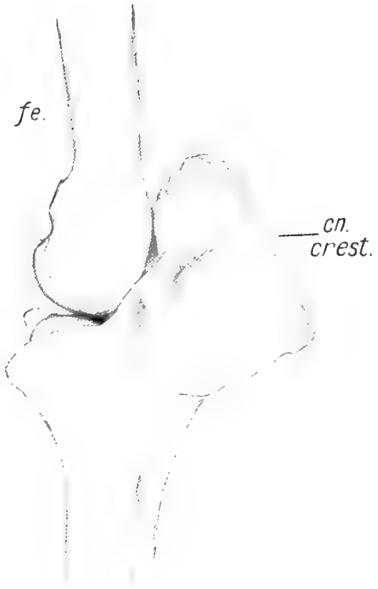
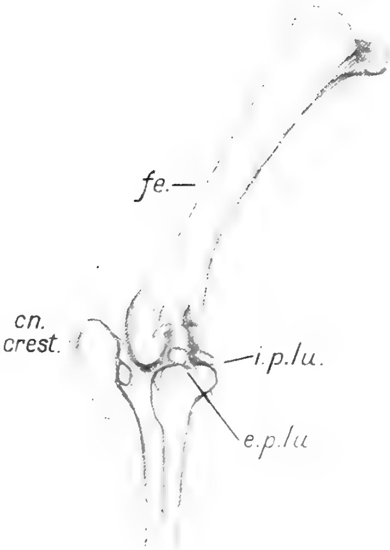


Fig. 50. Albatross (*Diomedea exulans*). Fig. 51.



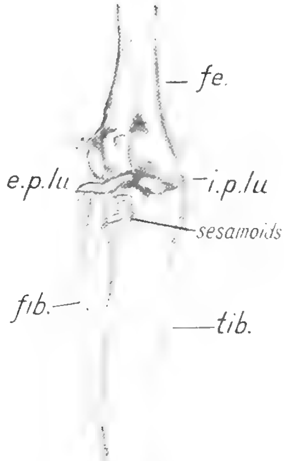


Fig. 52. Fig. 53. Fig. 54.

Dendrocygna.

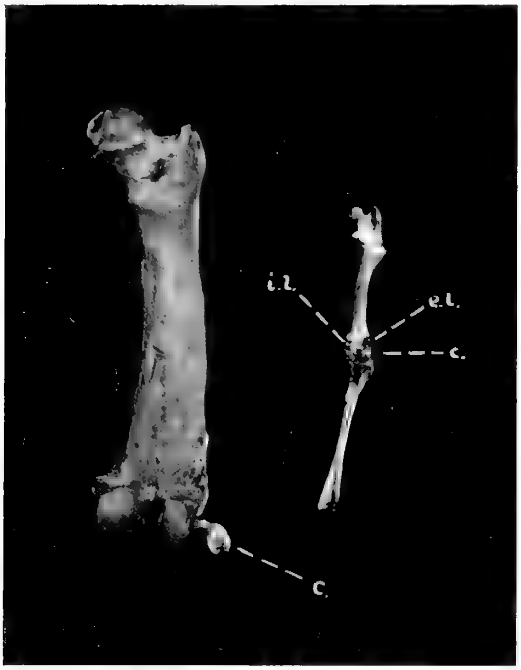


Fig. 55 *a*.
Tamandua Anteater
(*Tamandua tetradactyla*).

Fig. 55 *b*.
Common Squirrel
(*Sciurus vulgaris*).

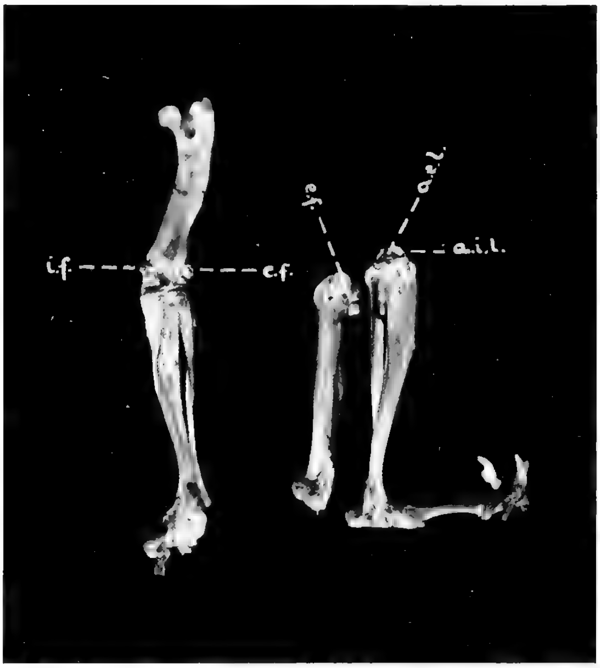


Fig. 56 *a*. Fig. 56 *b*.
Common Hedgehog (*Erinaceus europaeus*)
in two aspects.

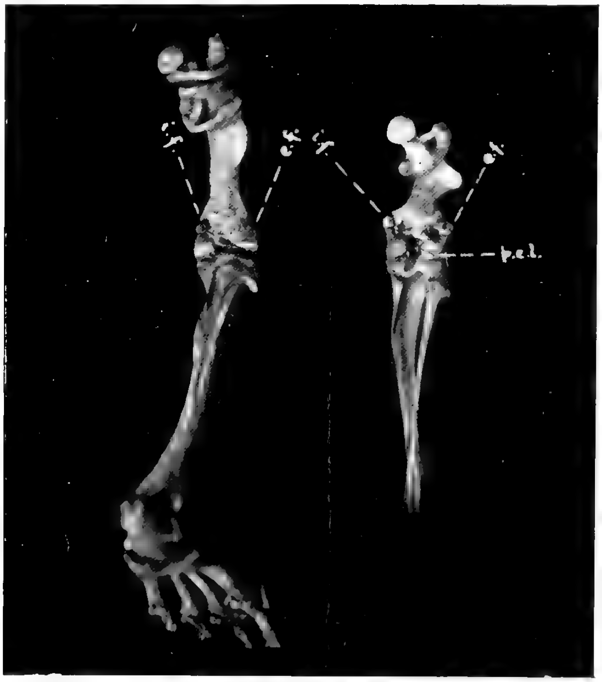


Fig. 57 *a*. Golden-bellied Rat
(*Hydromys chrysogaster*).

Fig. 57 *b*. Musquash
(*Fiber zibethicus*).

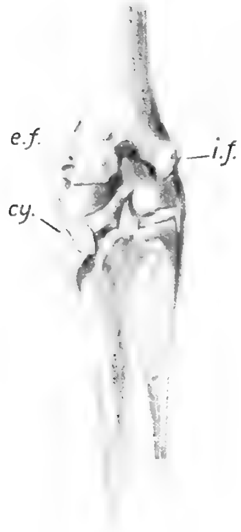


Fig. 58. Common rabbit (*Lepus cuniculus*).

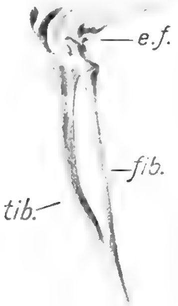


Fig. 59. *Potamogale velox* (West Africa).

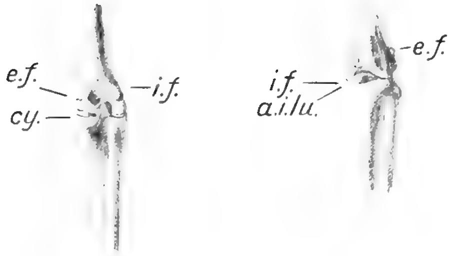


Fig. 60 *a*. Fig. 60 *b*.
Paraxerus jacksoni (African Squirrel).

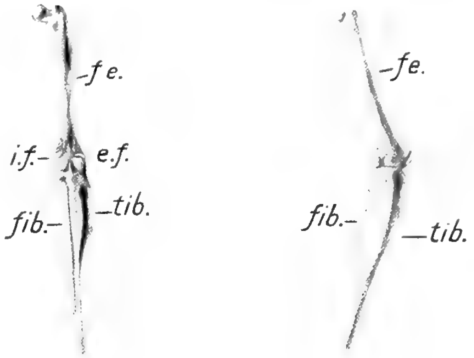


Fig. 62 *a*. Fig. 62 *b*.
Hylomys suillus dorsalis (Borneo).

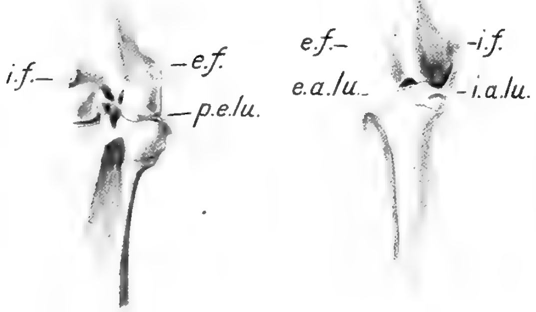


Fig. 61 *a*. Fig. 61 *b*.
Musquash (*Fiber zibethicus*).

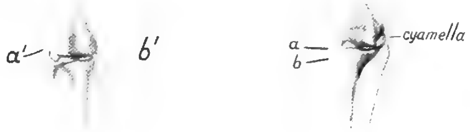


Fig. 63 *a*. Fig. 63 *b*.
Trachysaurus rugosus (Australian Lizard).

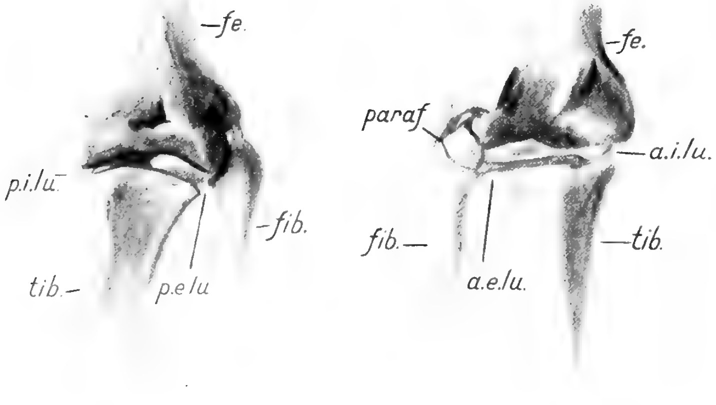


Fig. 64 *a*. Fig. 64 *b*.
Nile Monitor (*Varanus niloticus*).

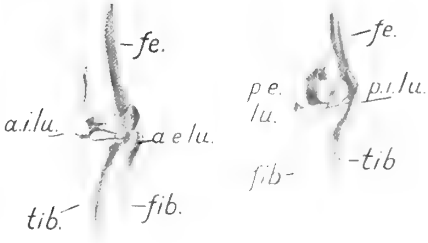


Fig. 65 *a*. Fig. 65 *b*.
Amphibolurus barbata.

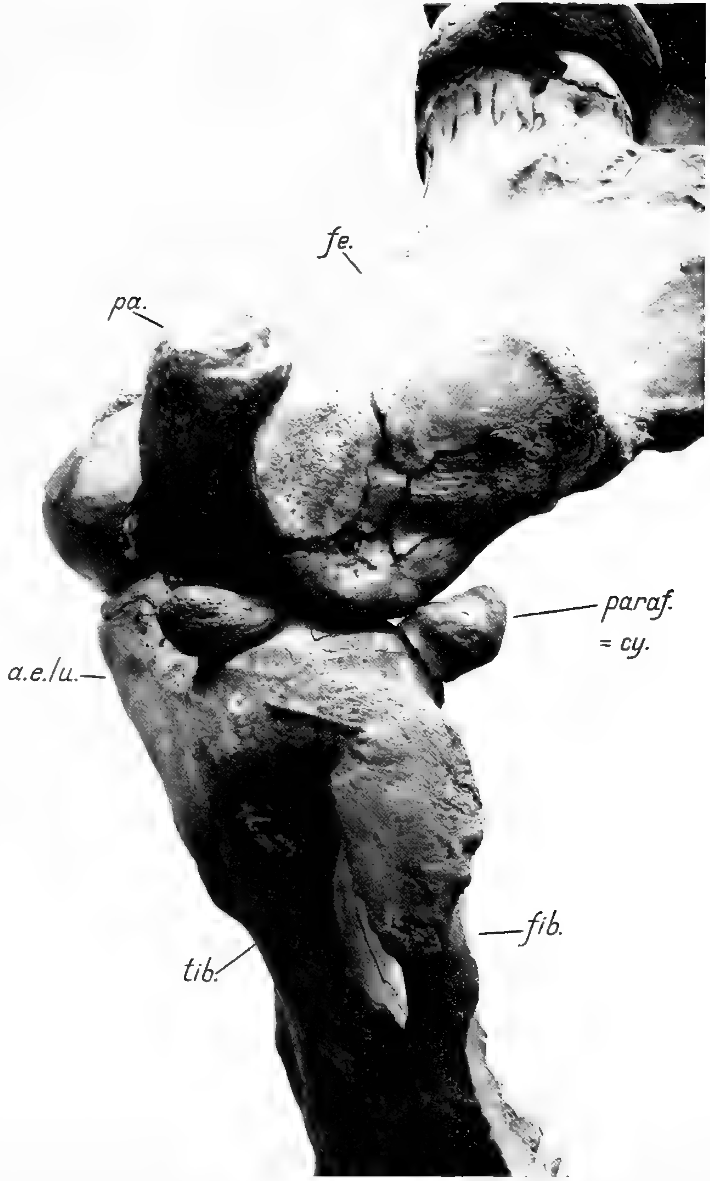


Fig. 66. Fossil Giant Sloth in R. C. of S.'s Museum (*Megatherium cuvieri*).

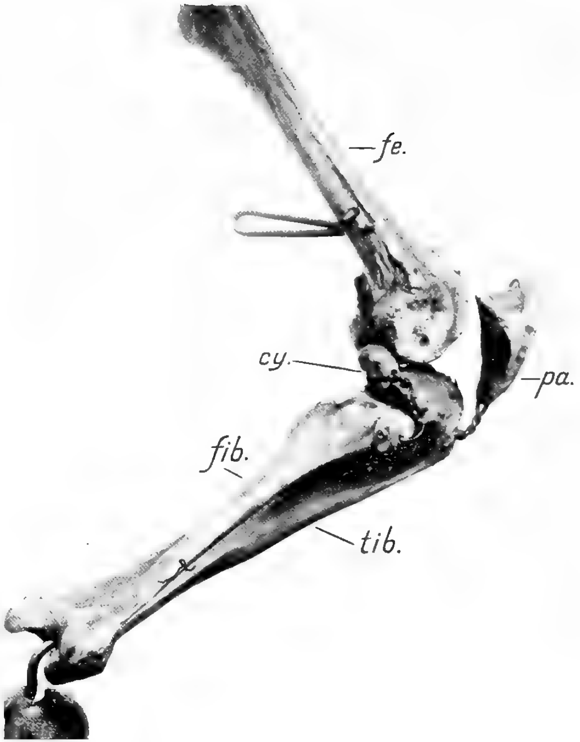
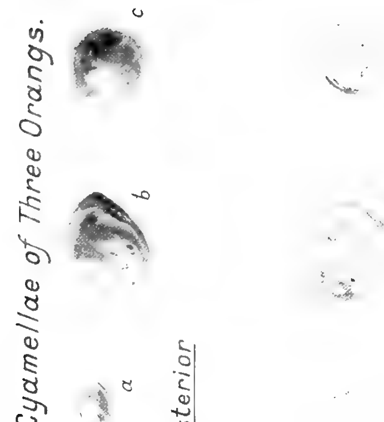


Fig. 67. Chinese Pangolin (*Manis aurita*).

Fig. 69. Orang. Cyamella raised from its facet.



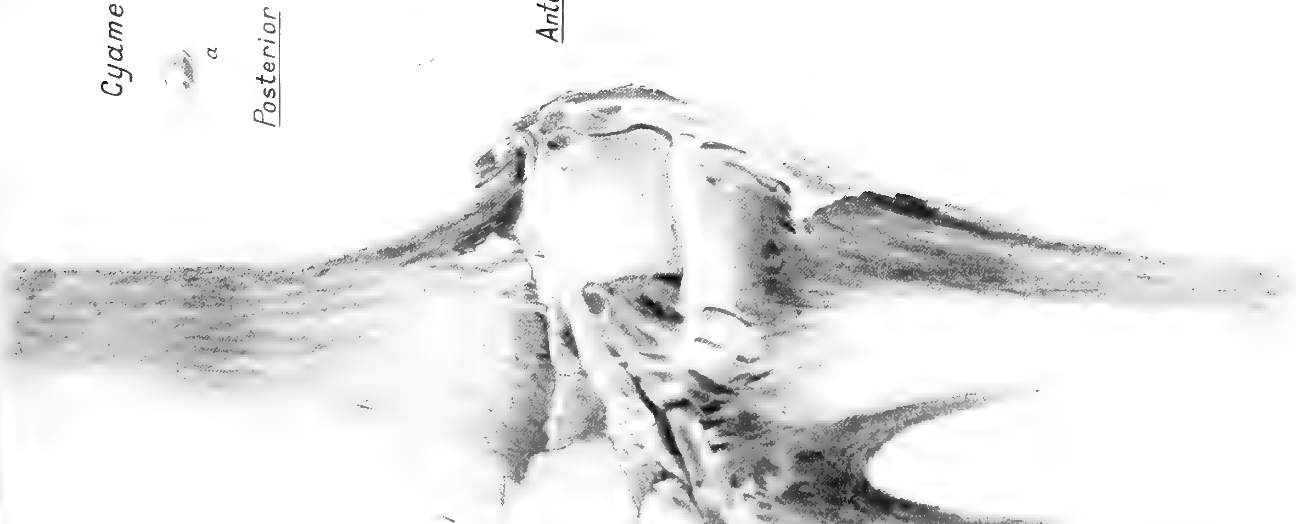


Fig. 68. Orang. Cyamella shown in position.

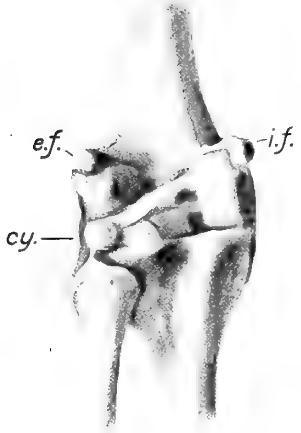


Fig. 70 *a*.

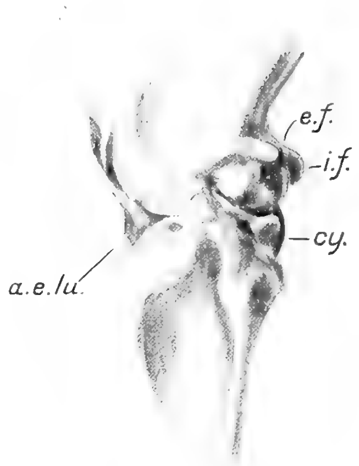


Fig. 70 *b*.

Lemur varia in two aspects.

Fig. 72. Gibbon.

Fig. 73. *Lemur melanocephalus.*

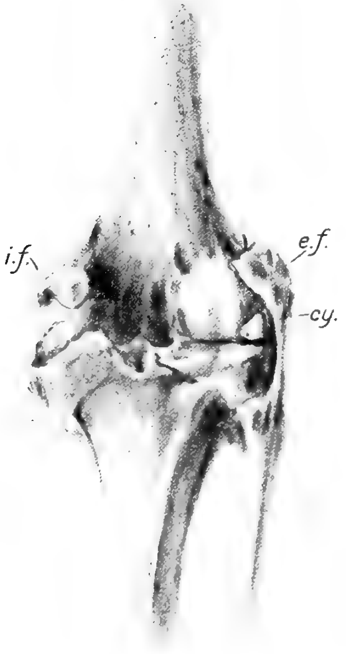


Fig. 71 *a*. *Mycetis palliatus.*

Fig. 71 *b*. *Mycetis palliatus.*

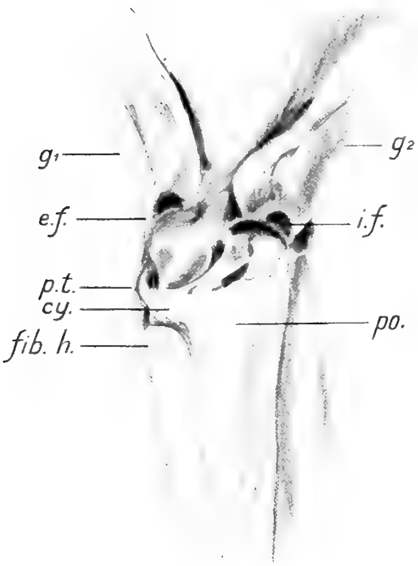


Fig. 74 *a*. Aye-Aye (*Chiromys*).

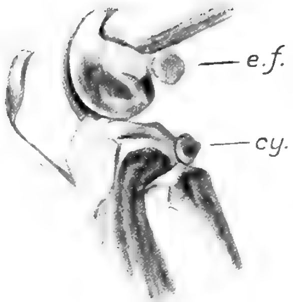


Fig. 74 *b*. *Chiromys* (after Owen).

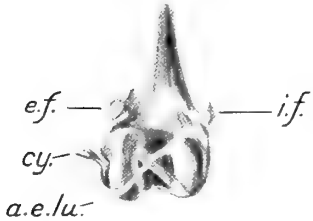
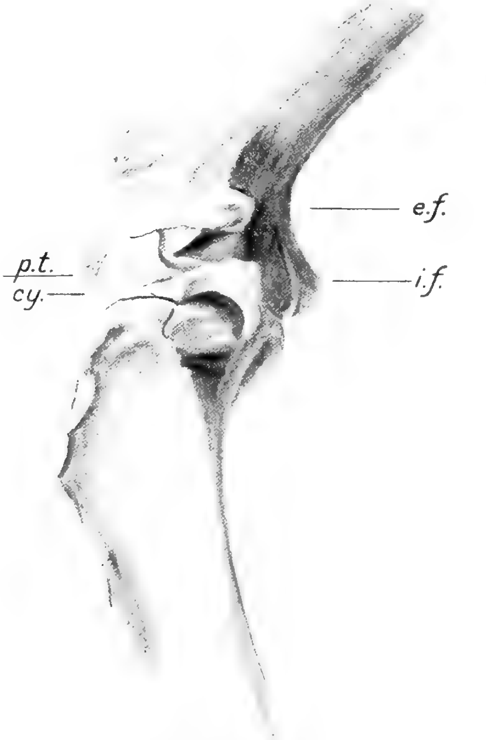


Fig. 75. *Lemur catta*.



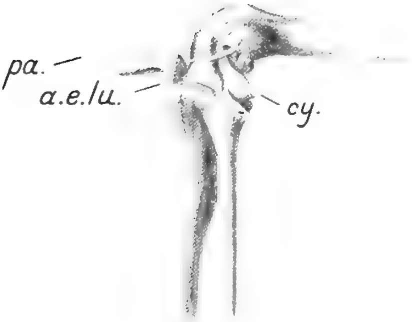


Fig. 76. *Perodicticus potto*.

Fig. 77. *Indris brevicaudata*.

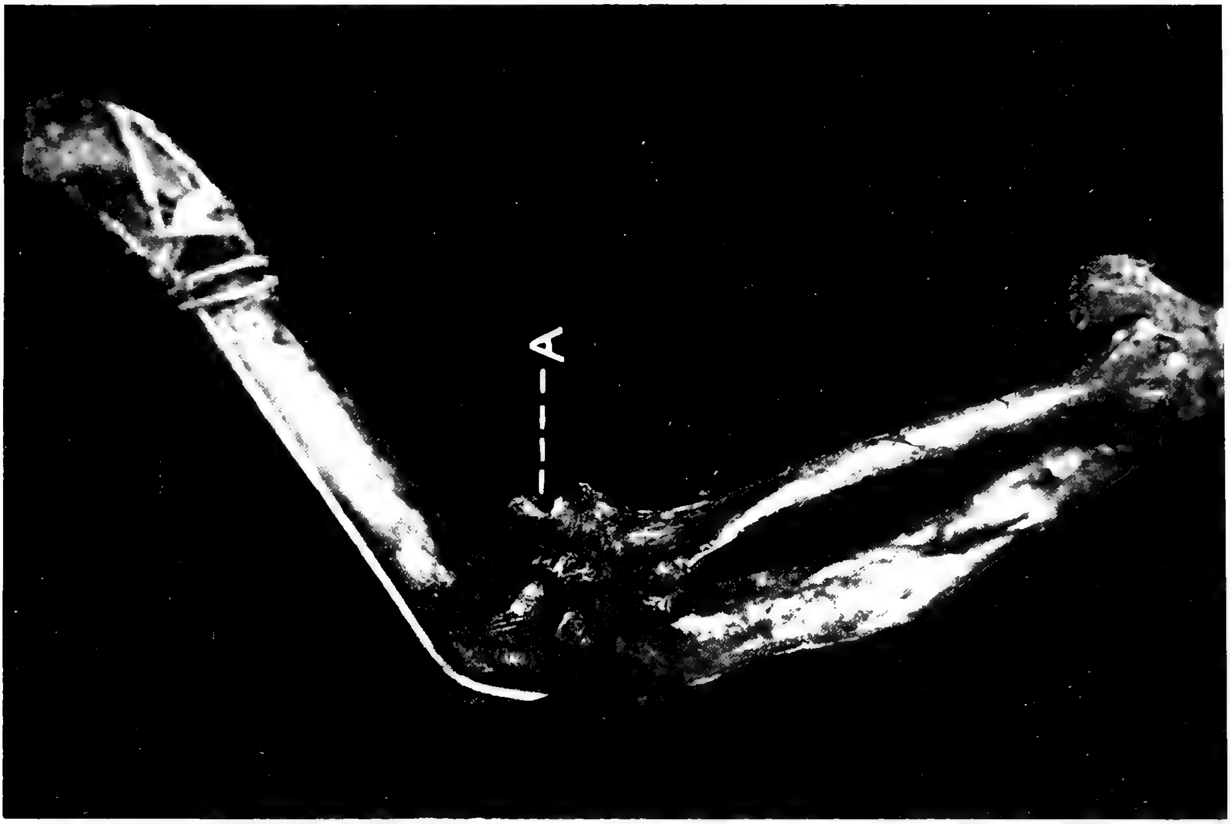


Fig. 79. Wombat. Parafibula free.

Fig. 78.

Tasmanian Devil (*Sarcophilus ursinus*).
Parafibula fused.

Wombat (*Phascolomys wombat*).
Parafibula fused.

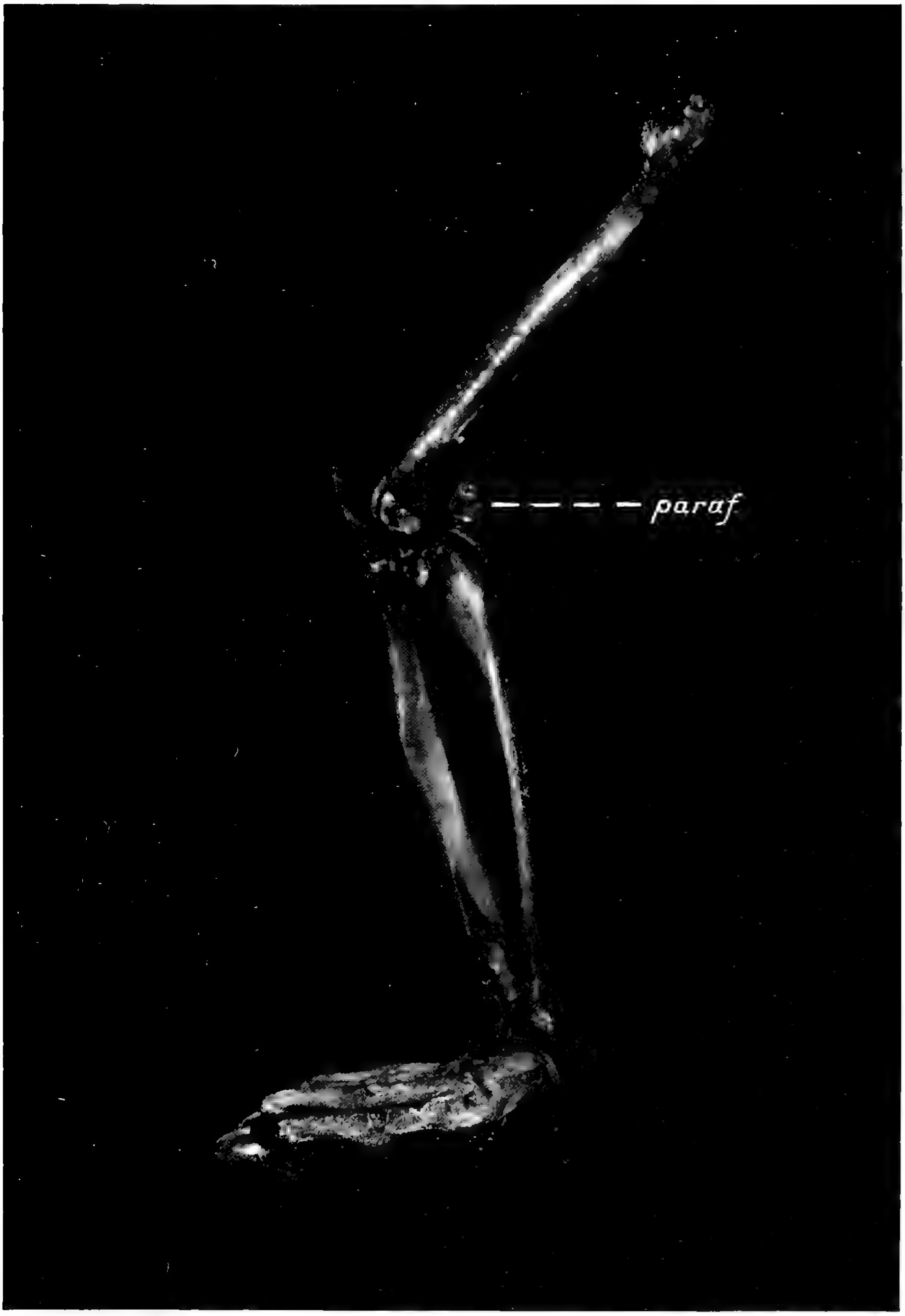


Fig. 80. *Phalangista vulpina.* Free parafibula.

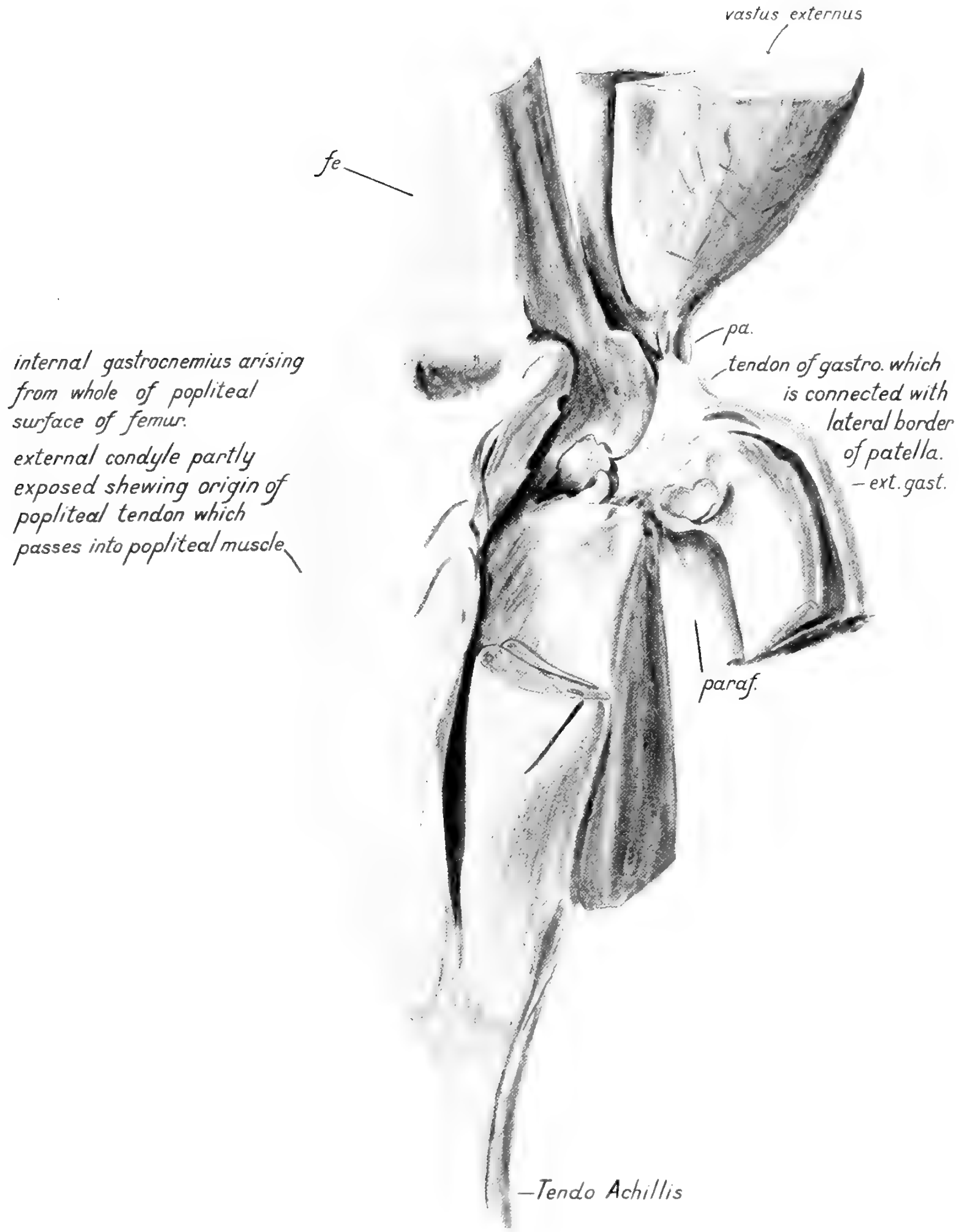


Fig. 81. Knee-joint of *Phascolomys* (Wombat) with attached muscles.

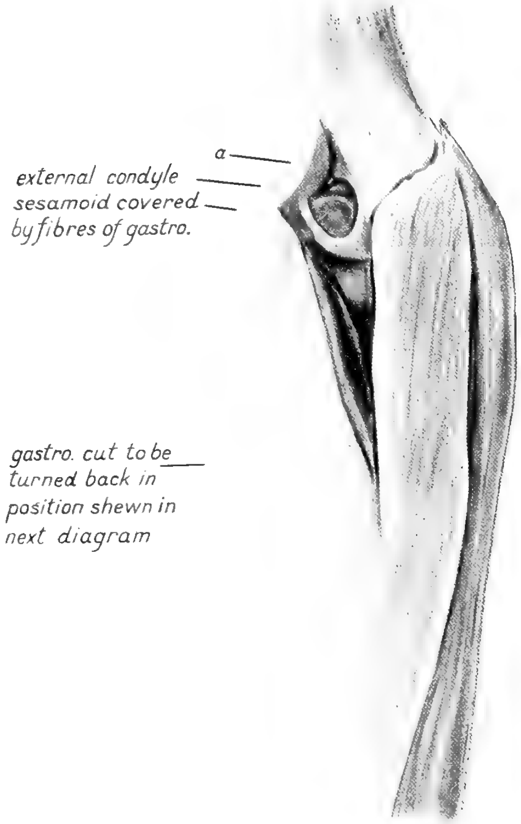


Fig. 82*a*.

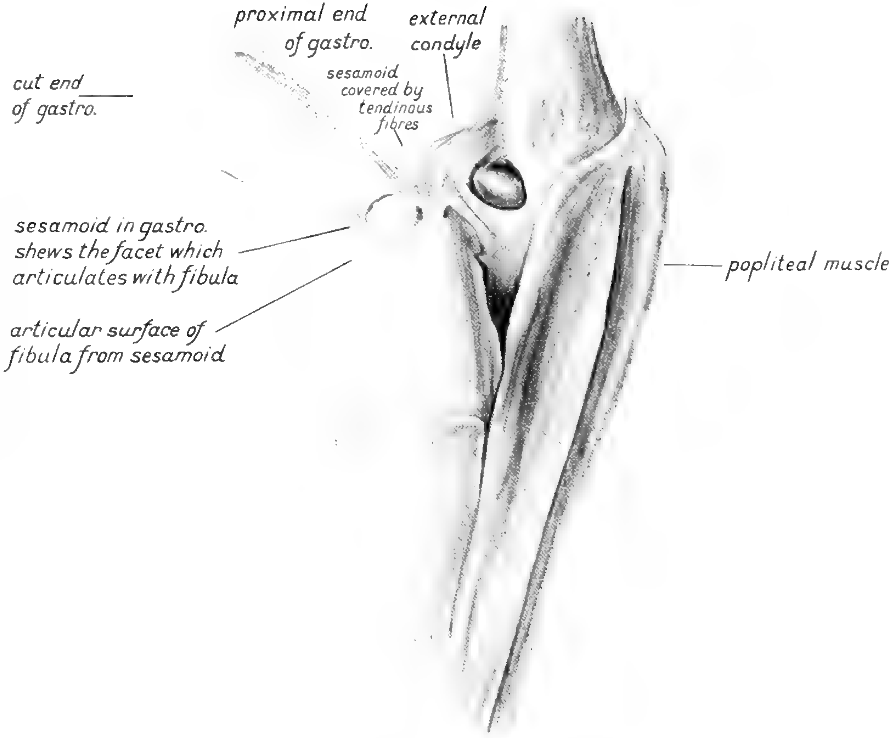


Fig. 82 *b*.

Koala (*Phascolarctus cinereus*). Sesamoid articulating with fibula.

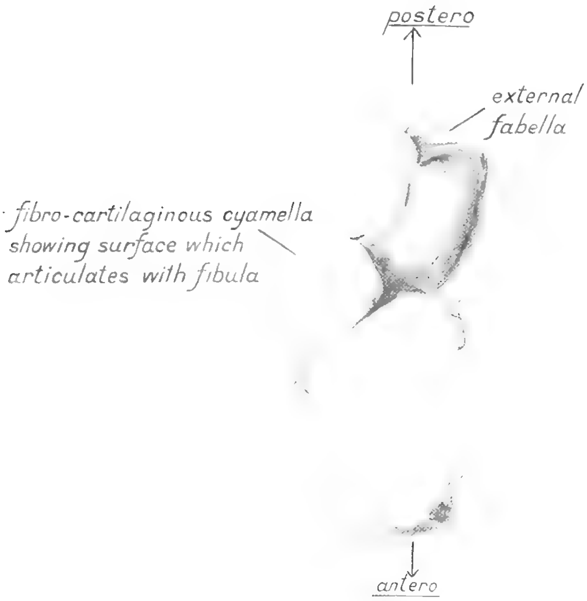


Fig. 83 *a*. *Macropus* sp.?

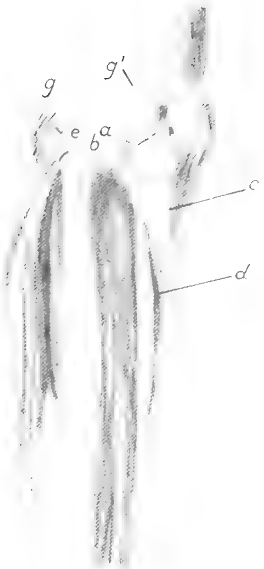


Fig. 84. Virginian Opossum (*Didelphys virginiana*).

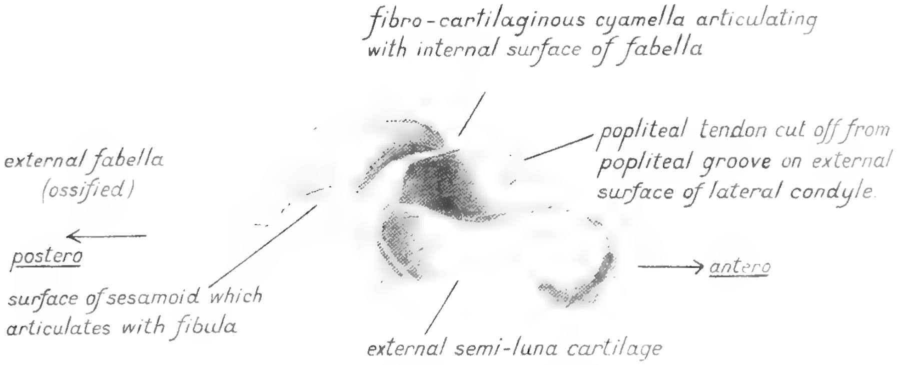


Fig. 83 *b*. *Macropus* sp.?

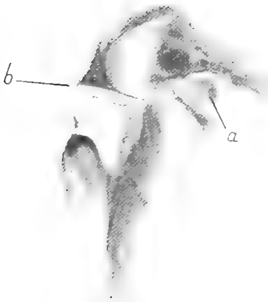


Fig. 85. *Phalangista vulpina*.

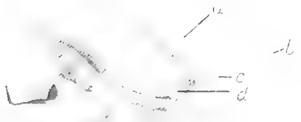


Fig. 86. Koala (*Phascolarctus cinereus*).

Fig. 87. Wallaby (*Macropus bennettii*).

Fig. 88. Rabbit-eared Perameles.

Doubtful Position of Parafibula.

Fig. 89 *a*. Tasmanian Devil (*Sarcophilus ursinus*).
Anterior Aspect.

Fig. 89 *b*. Tasmanian Devil (*Sarcophilus ursinus*).
Lateral Aspect.

Correct Position of Parafibula.

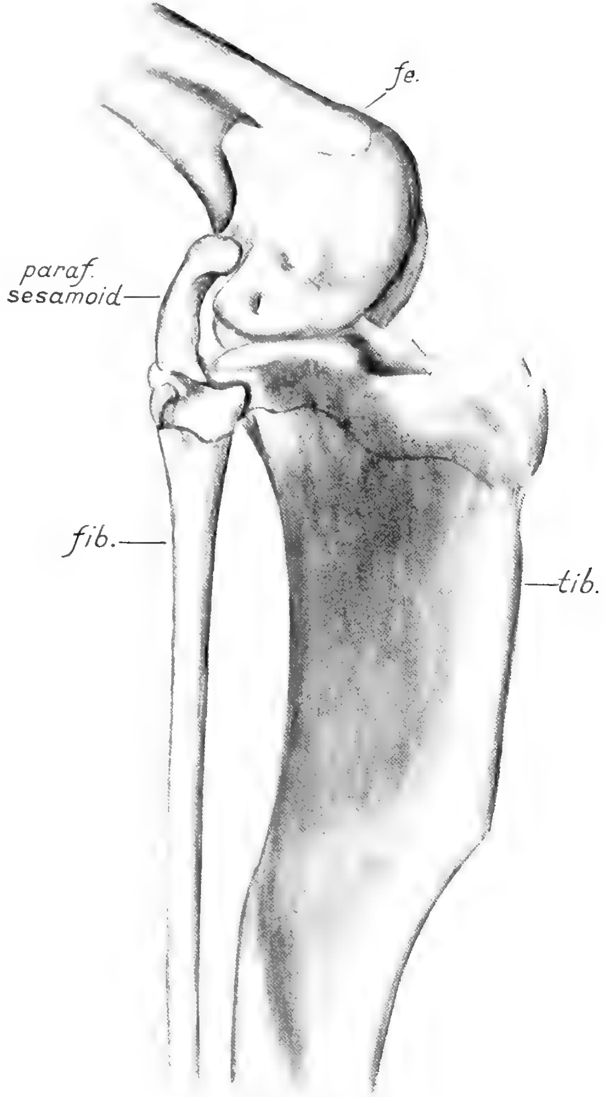


Fig. 90. Wallaby (*Halmaturus* sp.?).

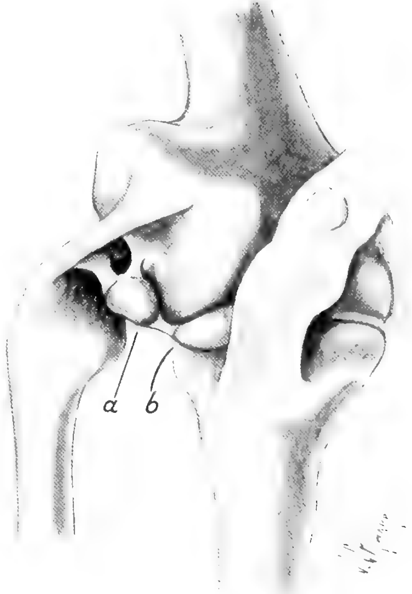


Fig. 91. *Corucia zebrata.* *a*, parafibular sesamoid; *b*, paratibial sesamoid united by ligament.

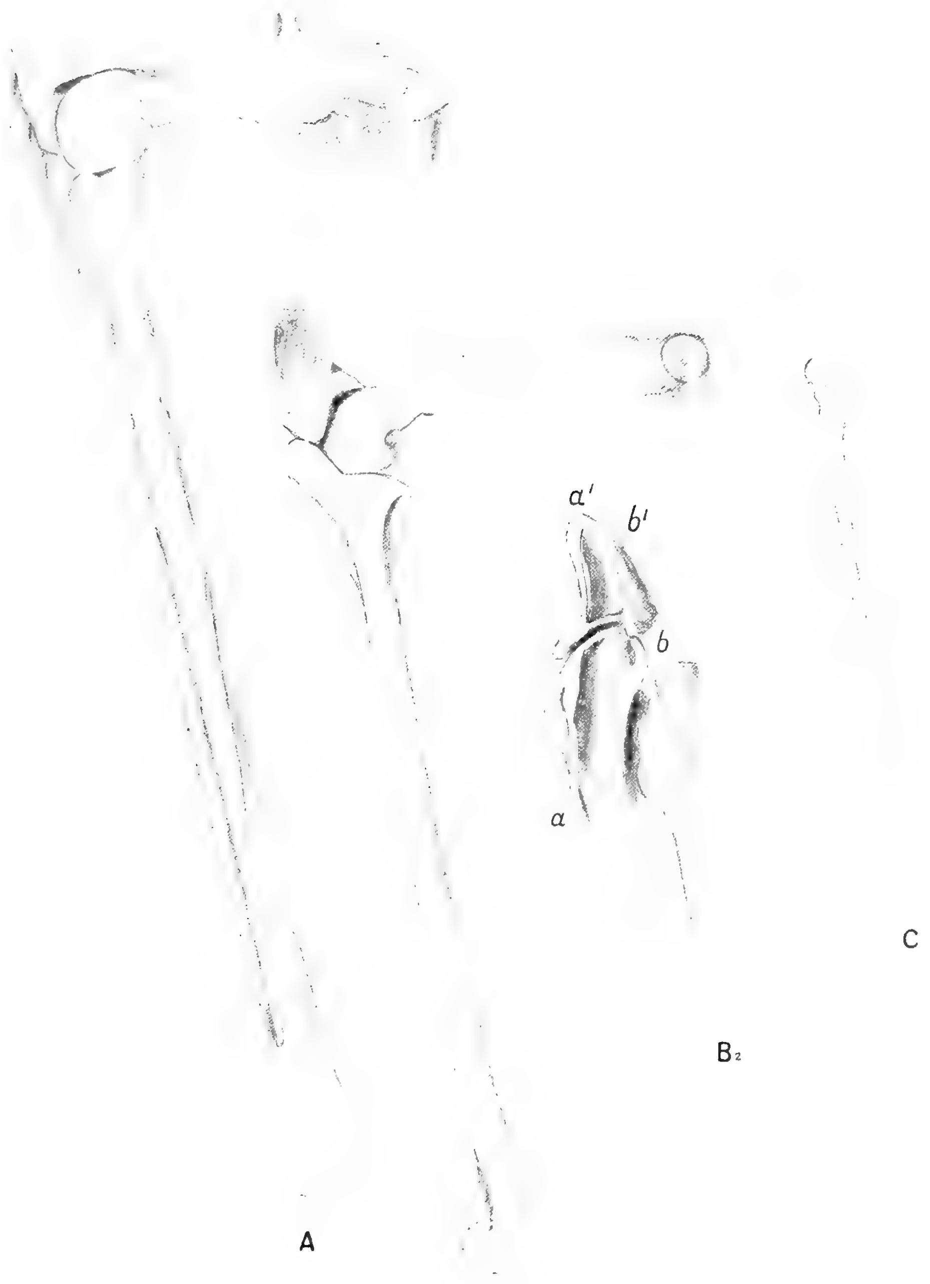


Fig. 92. Fig. 93 *a*. Fig. 93 *b*. Fig. 94.

A. *Hesperornis regalis*, after Marsh ($\frac{1}{2}$ natural size). **B₁**. R. Limb, mesial aspect from actual limb of Cormorant (natural size). **B₂**. L. Limb, lateral aspect from actual limb of Cormorant (natural size).

C. *Colymbus torquatus*, after Coues ($\frac{1}{2}$ natural size).

Fig. 95. Section of Fibular Crest of Young Platypus. × c. 60.

Fig. 96. Section of Fibular Crest of Embryo Platypus. × c. 60.

THE INCOMPLETE MOMENTS OF A NORMAL SOLID.

By A. RITCHIE-SCOTT, D.Sc.

1. *Introductory.*

One of the chief problems of mathematical statistics is the representation as closely as possible of a series of observations by a mathematical formula, or speaking graphically, by fitting a geometrically determinate curve or surface to the observations, these being represented by some kind of space unit. The formula, curve or surface constitutes a résumé of the known facts and gives that conception of them which is the scientific law underlying their relations. The very conception, however, of the statistical method involves a classification of the phenomena into groups, which assumes identity for the purposes of classification and ignores the infinitely fine gradation within the group. We have therefore to deal, not with the actual measurements, but with groups of measurements labelled with some representative value or class mark and to these groups we must fit our formula, curve or surface.

In the process of fitting, the method of most universal application is the method of moments which consists of expressing "the area and moments of the curve or surface for the given range of observations in terms [of the real constants of the theoretical curve] and equating these to the like quantities for the observations" (K. Pearson, "On the systematic fitting of curves," *Biometrika*, Vol. I, 1902, p. 270).

In the early history of mathematical statistics Gauss fitted the normal curve to observations by means of zero, first and second moments, i.e. by means of the sum, mean and standard deviation of the observations. The method of least squares is for any method of polynomial fitting a method of moments in which high moments may have to be used.

It was soon discovered, however, that the incomplete moments of the normal curve are important particularly in regard to plural partial correlation and the fitting of incomplete curves, while the development of the ideas of multiple correlation and variation brought into view the need of multiple moments and multiple product moments which are still further required in the evaluation of the probable error of multiple correlation coefficients.

With multiple variates we have the same problem as with the single variate, viz. the reconstruction of a population from a portion of it, and for this purpose incomplete moments and product moments are essential. Further, the theory of plural partial multiple correlation of observations classed in broad categories depends entirely on a knowledge of these incomplete product moments.

It is therefore from several points of view very desirable to obtain algebraical expression for these incomplete moments, and the present paper is an attempt to deal with the problem.

In the course of the analysis it was found necessary to employ functional operators which lead to the development of functions of the form of

$$x^n - \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4}\,\ldots.$$

These in a somewhat different form have been termed tetrachoric functions and their values tabled (*Tables for Statisticians and Biometricians,* Cambridge, 1914). In order to avoid confusion with these tabled functions and also to suggest their connection with multiple celled tables I have tentatively called the functions developed in the present paper, polychoric functions. The incomplete multiple product moment is found to be expressible in a form of multiple polychoric function which is itself reducible to a function of single polychoric functions.

2. *Notation.*

As the integrals in the following paper are very complicated unless an abbreviated notation is used, it will be necessary to resume some of the well-known formulae connected with the normal equation in order to be intelligible and to avoid confusion.

Consider the surface represented by the normal equation

$$Nz'(x', y') = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}\,e^{-\frac{x'^2/\sigma_x{}^2 + y'^2/\sigma_y{}^2 - 2rx'y'/\sigma_x\sigma_y}{2(1-r^2)}} \quad \ldots\ldots\ldots\ldots(1).$$

If we write $\dfrac{x'}{\sigma_x} = x$ and $\dfrac{y'}{\sigma_y} = y$ and

$$Nz(x, y) = \frac{N}{2\pi\sqrt{1-r^2}}\,e^{-\frac{x^2+y^2-2rxy}{2(1-r^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

then

$$z'(x', y') = \frac{z(x, y)}{\sigma_x\sigma_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If now the surface be cut by the plane $x' = h'$ then the area of the face of the section will be

$$N\int_{-\infty}^{+\infty} z'(h', y')\,dy' = \frac{Ne^{-h'^2/2\sigma_x{}^2}}{\sqrt{2\pi}\,\sigma_x} = \frac{NH}{\sigma_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If another plane be drawn at $y' = k'$ it will in a similar manner expose a surface whose area is

$$N\int_{-\infty}^{+\infty} z'(x', k')\,dx' = \frac{Ne^{-k'^2/2\sigma_y{}^2}}{\sqrt{2\pi}\,\sigma_y} = \frac{NK}{\sigma_y} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The two planes will divide the volume enclosed by the normal surface into four quadrants *a*, *b*, *c*, and *d* as in the usual tetrachoric scheme

a	*b*
c	*d*

in which the x's go from $-\infty$ to $+\infty$ in the direction of a to b and y's similarly from a to c. The a quadrant will be considered as the leading or standard quadrant and where it is necessary to specify the limits of integration they will be those of this quadrant. Its volume will be denoted by m with, where necessary, a distinguishing suffix.

Hence we may write

$$m = N\int_{-\infty}^{h'}\int_{-\infty}^{k'} z'(x', y')\,dx'dy' = N\int_{-\infty}^{h}\int_{-\infty}^{k} z(x_1, y_1)\,dx\,dy \quad \text{......(6).}$$

The area of the face bounding quadrant a parallel to the y axis is

$$N\int_{-\infty}^{k'} z'(h', y')\,dy' = \frac{Ne^{-h'^2/2\sigma_x^2}}{\sqrt{2\pi}\,\sigma_x}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{k-rh}{\sqrt{1-r^2}}} e^{-y^2/2}\,dy = \frac{NH}{\sigma_x}A \quad \text{...... (7).}$$

Since the whole area of the section is $\frac{NH}{\sigma_x}$, A is the fraction which the portion bounding the quadrant a is of the whole section. It has no dimensions and is independent of σ_x and σ_y.

Similarly the face bounding the a quadrant parallel to the x axis is

$$N\int_{-\infty}^{h'} z'(x', k')\,dx' = \frac{Ne^{-k'^2/2\sigma_y^2}}{\sqrt{2\pi}\,\sigma_y}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{h-rk}{\sqrt{1-r^2}}} e^{-x^2/2}\,dx = \frac{NK}{\sigma_y}B \quad \text{...(8).}$$

A and B may be taken from Sheppard's Tables when h, k and r are known.

Writing

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx = \mathscr{E}(x),$$

which is the function tabled, we have

$$A = \mathscr{E}\left(\frac{k-rh}{\sqrt{1-r^2}}\right) \text{ and } B = \mathscr{E}\left(\frac{h-rk}{\sqrt{1-r^2}}\right) \quad \text{.....................(9).}$$

Lastly we have the ordinate at the intersection of the planes

$$Nz'(h', k') = \frac{Nz(h, k)}{\sigma_x\sigma_y} = \frac{N\chi}{\sigma_x\sigma_y} \quad \text{..............................(10).}$$

Since

$$\begin{aligned}\chi &= \frac{1}{2\pi\sqrt{1-r^2}}e^{-\frac{h^2+k^2-2rhk}{2(1-r^2)}}\\ &= \frac{1}{\sqrt{1-r^2}}\cdot\frac{e^{-\frac{1}{2}h^2}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(k-rh)^2}{2(1-r^2)}}\\ &= \frac{1}{\sqrt{1-r^2}}\cdot\frac{e^{-\frac{1}{2}k^2}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(h-rk)^2}{2(1-r^2)}} \quad \text{...........(11),}\end{aligned}$$

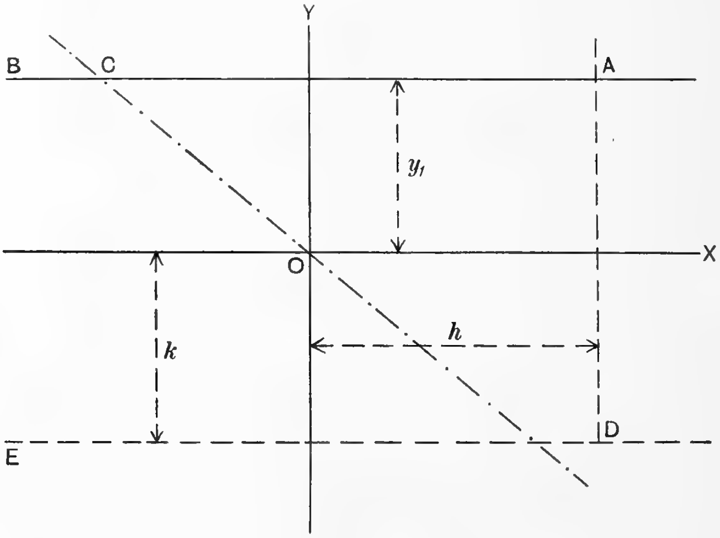
we may write

$$\begin{aligned}\chi &= \frac{1}{\sqrt{1-r^2}}\,.\,E(h)\,.\,E\left(\frac{k-rh}{\sqrt{1-r^2}}\right)\\ &= \frac{1}{\sqrt{1-r^2}}\,.\,E(k)\,.\,E\left(\frac{h-rk}{\sqrt{1-r^2}}\right) \quad \text{....................(12),}\end{aligned}$$

where $E(x)$ is the tabled function $\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. Clearly $E(h) = H$, and $E(k) = K$.

3. *Centroid of a normal quadrant found by geometry.*

In this and the following section I shall show how the lower moments may be found by direct methods and also how these methods lead to results of such complexity with high moments that a symbolic method is necessary.



Let OX, OY be the co-ordinate axes through the mean of a normal solid and let AD, DE be the projections of the bounding planes of a quadrant. AB is the projection of a plane cutting the solid perpendicularly to the Y axis and at a distance $y = y_1$. The equation of the curve in which it cuts the surface will be

$$Nz = N\frac{e^{-\frac{1}{2}y_1^2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-r^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-ry_1}{\sqrt{1-r^2}}\right)^2} \quad \dots\dots\dots\dots\dots(13),$$

i.e. a curve whose mean is at $x = ry_1$, whose standard deviation is $\sqrt{1-r^2}$ and whose area is NK. Considering B to be at an infinite distance the area of BA is

$$N\int_{-\infty}^{h} z(x, y_1)\, dx = \alpha \text{ say,} \quad \dots\dots\dots\dots\dots\dots\dots(14),$$

and since the centroid of a segment of a normal curve

$$= \frac{\text{ordinate}}{\text{area}} \times (\text{stand. dev.})^2 \quad \dots\dots\dots\dots\dots\dots\dots(15),$$

writing the centroid of BA as $\bar{x}_{y_1}$ we have

$$\bar{x}_{y_1} - ry_1 = -\frac{(1-r^2).\, z(h, y_1)}{\alpha} \quad \dots\dots\dots\dots\dots\dots\dots(16).$$

Therefore the x moment of the section AB may be written

$$\bar{x}_{y_1}\alpha = ry_1\alpha - (1-r^2) \,.\, z(h, y_1) \quad \dots\dots\dots\dots\dots\dots(17).$$

Now let the surface BA move from $y = -\infty$ to $y = k_1$ then $\bar{x}_{y_1}\alpha$ will generate the x moment of the quadrant, $y_1\alpha$ will generate the y moment and $z(h, y_1)$ will generate the area of the surface exposed by the plane AD. Writing the centroids of the quadrant $\bar{\bar{x}}$, $\bar{\bar{y}}$

$$\alpha\bar{\bar{x}} = r\alpha\bar{\bar{y}} - (1-r^2)\int_{-\infty}^{k} z(h, y)\,dy \quad \text{.....................(18).}$$

A similar discussion of the y moment will give the equation

$$\alpha\bar{\bar{y}} = r\alpha\bar{\bar{x}} - (1-r^2)\int_{-\infty}^{h} z(x, k)\,dx \quad \text{.....................(19).}$$

Both the integrals are known and the centroids may be found as the solution of a pair of simultaneous equations.

4. *Another method of finding the moments.*

The method of § 2 may be expressed analytically and extended to higher moments. For example second moments may be found as follows:

$$\left.\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= -\frac{1}{(1-r^2)^2}\{x^2 - 2rxy + r^2y^2 - (1-r^2)\}\, z(x, y)\\
\frac{\partial^2 z}{\partial x\partial y} &= +\frac{1}{(1-r^2)^2}\{r(1-r^2) - [rx^2 - (1+r^2)xy + ry^2]\}\, z(x, y)\\
\frac{\partial^2 z}{\partial y^2} &= -\frac{1}{(1-r^2)^2}\{r^2x^2 - 2rxy + y^2 - (1-r^2)\}\, z(x, y)
\end{aligned}\right\} \quad \text{...(20).}$$

Hence

$$\left.\begin{aligned}
(x^2 - 2rxy + r^2y^2)\, z(x, y) &= (1-r^2)^2\frac{\partial^2 z}{\partial x^2} + (1-r^2)\, z(x, y)\\
\{rx^2 - (1+r^2)xy + ry^2\}\, z(x, y) &= -(1-r^2)^2\frac{\partial^2 z}{\partial x\partial y} + r(1-r^2)\, z(x, y)\\
(r^2x^2 - 2rxy + y^2)\, z(x, y) &= (1-r^2)^2\frac{\partial^2 z}{\partial y^2} + (1-r^2)\, z(x, y)
\end{aligned}\right\} \quad \text{...(21).}$$

Multiplying both sides by N and integrating, and writing

$$\left.\begin{aligned}
m\,.\,\bar{m}_{20} &= N\iint x^2\,.\,z(x, y)\,dx\,dy\\
m\,.\,\bar{m}_{11} &= N\iint xy\,.\,z(x, y)\,dx\,dy\\
&\text{etc.}
\end{aligned}\right\} \quad \text{.....................(22),}$$

we get

$$\left.\begin{aligned}
\bar{m}_{20} - 2r\bar{m}_{11} + r^2\bar{m}_{02} &= \frac{N}{m}(1-r^2)^2\iint\frac{\partial^2 z}{\partial x^2}\,dx\,dy + (1-r^2)\\
r\bar{m}_{20} - (1+r^2)\,\bar{m}_{11} + r\bar{m}_{02} &= -\frac{N}{m}(1-r^2)^2\iint\frac{\partial^2 z}{\partial x\partial y}\,dx\,dy + r(1-r^2)\\
r^2\,\bar{m}_{20} - 2r\bar{m}_{11} + \bar{m}_{02} &= +\frac{N}{m}(1-r^2)^2\iint\frac{\partial^2 z}{\partial y^2}\,dx\,dy + (1-r^2)
\end{aligned}\right\} \quad \text{...(23).}$$

The integrals are known and the $\bar{m}$'s can be found by solving the equations. With high moments the equations become cumbrous and it is difficult to follow the relations between the moments. A more powerful method has therefore been adopted.

5. *Moments of a* $\left(\frac{1}{2^n}\right)^{ant}$ *of an n-fold normal correlation surface by means of functional operators.*

The rationale of the process to be developed in this paper will be best understood by considering at once the general case. The details for a normal surface in two variables will be worked out in a later section.

Consider the equation to a normal correlation surface

$$Nz(x_1, x_2, \dots x_n) = \frac{N}{(2\pi)^{\frac{n}{2}}\sqrt{\Delta}}\, e^{-\frac{\Sigma\Delta_{11}x_1^2 + 2\Sigma\Delta_{12}x_1x_2}{2\Delta}} \quad \text{...........(24)},$$

where as usual

$$\Delta = \begin{vmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{vmatrix} \quad \text{..........................(25)},$$

and Δ_{11}, Δ_{12} are the co-factors of r_{11}, r_{12} ... and $r_{11} = r_{22} = \dots = 1$, and $r_{12} = r_{21}$ etc.

The general problem is to find the $l_1, l_2, l_3 \dots l_n$th product moment of the $\left(\frac{1}{2^n}\right)^{\text{ant}}$ of the form, that is to reduce the integral

$$m\,.\,\bar{m}_{l_1 l_2 \dots l_n} = N\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\dots\int_{-\infty}^{h_n} x_1^{l_1}x_2^{l_2}\dots x_n^{l_n}\, z\,(x_1, x_2 \dots x_n)\, dx_1\, dx_2 \dots dx_n \quad \text{......(26)}.$$

Differentiating partially by x_1, x_2, ...

$$\left.\begin{aligned} \frac{\partial z}{\partial x_1} &= -\frac{1}{\Delta}(\Delta_{11}x_1 + \Delta_{12}x_2 + \dots)\,z \\ \frac{\partial z}{\partial x_2} &= -\frac{1}{\Delta}(\Delta_{12}x_1 + \Delta_{22}x_2 + \dots)\,z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial z}{\partial x_s} &= -\frac{1}{\Delta}(\Delta_{1s}x_1 + \Delta_{2s}x_2 + \dots)\,z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \text{.....................(27)}.$$

Hence remembering that Δ_{st} is the co-factor of r_{st} we have

$$\left.\begin{aligned} \left(r_{11}\frac{\partial z}{\partial x_1} + r_{12}\frac{\partial z}{\partial x_2} + \dots r_{1n}\frac{\partial z}{\partial x_n}\right) &= -\frac{1}{\Delta}(\Delta x_1)\,z = -x_1 z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left(r_{s1}\frac{\partial z}{\partial x_1} + r_{s2}\frac{\partial z}{\partial x_2} + \dots r_{sn}\frac{\partial z}{\partial x_n}\right) &= -\frac{1}{\Delta}(\Delta x_s)\,z = -x_s z \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \text{.........(28)}.$$

Now representing the left side of the above equations by means of operators such as

$$-\left(r_{s1}\frac{\partial}{\partial x_1}+r_{s2}\frac{\partial}{\partial x_2}+\dots\right)=\phi_s \dots\dots\dots\dots(29),$$

we have

$$\phi_1(z)=x_1\,.\,z;\quad \phi_2(z)=x_2\,.\,z,\ \dots\ \phi_s(z)=x_s\,.\,z \quad \dots\dots\dots(30).$$

Further

$$\phi_1(x_1)=-r_{11}=-1\,;\quad \phi_1(x_2)=-r_{12}\,;\quad \phi_2(x_1)=-r_{21}\,;\quad \phi_2(x_2)=-r_{22}=-1$$

and generally

$$\phi_s(x_{s'})=\phi_{s'}(x_s)=-r_{ss'} \quad \dots\dots\dots\dots(31).$$

Since these operators are weighted sums of differential coefficients they are distributive, commutative and iterative.

Hence

$$\left.\begin{aligned}
\phi_1^2(z)&=\phi_1(\phi_1(z))\\
&=\phi_1(x_1z)\\
&=\phi_1(x)\,.\,z-x_1\,.\,\phi_1(z)\\
&=-r_{11}z+x_1^2z\\
\phi_1^3(z)&=\phi_1(\phi_1^2(z))\\
&=\phi_1(-r_{11}z+x_1^2z)\\
&=-r_{11}\phi_1(z)+2x_1\,.\,\phi_1(x)\,.\,z+x_1^2\,\phi(z)\\
&=-r_{11}x_1z-2r_{11}x_1z+x_1^3z\\
&=-3r_{11}x_1z+x_1^3z\\
&\text{etc.,}\\
\phi_1\phi_2(z)&=\phi_1(x_2z)\\
&=\phi_1(x_2)\,.\,z+x_2\phi_1(z)\\
&=-r_{12}z+x_2x_1z
\end{aligned}\right\}\dots\dots\dots(32).$$

In this manner any function $\phi_1^{l_1}\phi_2^{l_2}\dots(z)$ may be expanded into a series of terms each of which involves the r's, the x's and z. That is we may write

$$\Sigma r_{ss'}r_{s''s'''}\dots x_1^{p_1}x_2^{p_2}\dots z=\phi_1^{l_1}\phi_2^{l_2}\dots(z) \quad \dots\dots\dots\dots(33).$$

Now if both sides be integrated with respect to all the variables we shall have a sum of mixed moments on the left side and on the right side the integral

$$\int_{-\infty}^{h_1}\int_{-\infty}^{h_2}\dots\int_{-\infty}^{h_n}\phi_1^{l_1}\phi_2^{l_2}\dots\phi_n^{l_n}(z)\,dx_1dx_2\dots dx_n \quad \dots\dots\dots(34),$$

on the reduction of which the solution depends.

Consider the numerator of the index of e

$$\Sigma\Delta_{ss}x_s^2+2\Sigma\Delta_{st}x_sx_t=f(x_1x_2\dots x_n)\dots\dots\dots\dots(35).$$

Write

$$x_1=y_1+r_{11}h_1,$$

$$x_2=y_2+r_{12}h_1,$$

$$\dots\dots\dots\dots$$

$$x_s=y_s+r_{1s}h_1,$$

then

$$\begin{aligned} f(x_1 x_2 \dots x_n) &= f(y_1 + r_{11}h_1,\ y_2 + r_{12}h_1,\ \dots\ y_n + r_{1n}h_1) \\ &= f(y_1,\ y_2, \dots y_n) \\ &\quad + \frac{h_1}{1\,!}\left(r_{11}\frac{\partial}{\partial y_1} + r_{12}\frac{\partial}{\partial y_2} + \dots\right) f(y_1,\ y_2 \dots y_n) \\ &\quad + \frac{h_1^2}{2\,!}\left(r_{11}\frac{\partial}{\partial y_1} + r_{12}\frac{\partial}{\partial y_2} + \dots\right)^2 f(y_1,\ y_2 \dots y_n) \\ &\quad + \dots . \end{aligned}$$

But

$$\begin{aligned} &\left(r_{11}\frac{\partial}{\partial y_1} + r_{12}\frac{\partial}{\partial y_2} + \dots\right) f(y_1,\ y_2 \dots y_n) \\ &= 2(\Delta_{11}y_1 + \Delta_{12}y_2 + \dots) r_{11} + 2(\Delta_{12}y_1 + \Delta_{22}y_2 + \dots) r_{12} \\ &\quad + \dots \\ &= 2(r_{11}\Delta_{11} + r_{12}\Delta_{12} + \dots) y_1 + 2(r_{11}\Delta_{12} + r_{12}\Delta_{22} + \dots) y_2 \\ &\quad + \dots \\ &= 2\Delta y_1, \end{aligned}$$

and

$$\left(r_{11}\frac{\partial}{\partial y_1} + r_{12}\frac{\partial}{\partial y_2} + \dots\right)^2 f = \left(r_{11}\frac{\partial}{\partial y_1} + r_{12}\frac{\partial}{\partial y_2} + \dots\right) . 2\Delta y_1 = 2\Delta$$

and the remaining terms vanish.

Hence

$$f(y_1 + r_{11}h_1,\ y_2 + r_{12}h_1 + \dots) = f(y_1,\ y_2 \dots y_n) + 2h\Delta y_1 + \Delta h_1^2 \quad \dots\dots(36).$$

Put $y_1 = 0$ and

$$\begin{aligned} f(h_1,\ x_2,\ x_3 \dots x_n) &= f(0,\ x_2 - r_{12}h_1,\ x_3 - r_{13}h_1 + \dots) + \Delta h_1^2 \\ &\text{say} = f(x_2',\ x_3' \dots x_n') + \Delta h_1^2 \quad \dots\dots\dots\dots\dots\dots\dots\dots(37) \end{aligned}$$

Hence we may write

$$Nz(h_1 x_2 x_3 \dots x_n) = \frac{N}{(2\pi)^{\frac{n}{2}}\sqrt{\Delta}}\, e^{-\frac{\Sigma \Delta_{ss'} x_s'^2 + 2\Delta_{st} x_s' x_t' + \Delta h_1^2}{2\Delta}} \quad \dots\dots\dots(38)$$

Omitting the first variate from the Σ summation,

$$= \frac{e^{-h_1^2/2}}{\sqrt{2\pi}} \cdot \frac{N}{(2\pi)^{\frac{n-1}{2}}\sqrt{\Delta}} \cdot e^{-\frac{\Sigma \Delta_{ss'} x_s'^2 + 2\Delta_{st} x_s' x_t'}{2\Delta}} .$$

Hence

$$\begin{aligned} &\int_{-\infty}^{h_1}\int_{-\infty}^{h_2} \dots \int_{-\infty}^{h_n} \phi_1^{l_1}\phi_2^{l_2} \dots \phi_n^{l_n} z\, dx_1 dx_2 \dots dx_n \\ &= \int_{-\infty}^{h_1}\int_{-\infty}^{h_2} \dots \int_{-\infty}^{h_n} \phi_1 \phi_1^{l_1 - 1}\phi_2^{l_2} \dots \phi_n^{l_n} z\, dx_1 dx_2 \dots dx_n \\ &= -\int_{-\infty}^{h_1}\int_{-\infty}^{h_2} \dots \phi_1^{l_1 - 1}\phi_2^{l_2} \dots \left(r_{11}\frac{\partial}{\partial x_1} + r_{12}\frac{\partial}{\partial x_2} + \dots\right) z\, dx_1 dx_2 \dots dx_n \\ &= -\sum_1^s r_{1s}\int_{-\infty}^{h_1} \dots \left[\int_{-\infty}^{h_s}\right] \dots \int_{-\infty}^{h_n} \phi_1^{l_1 - 1}\phi_2^{l_2} \dots z(x_1,\ x_2 \dots h_s \dots x_n)\, dx_1 \dots [dx_s] \dots dx_n \\ &\qquad \dots\dots\dots\dots\dots(39), \end{aligned}$$

in which the expressions in brackets disappear,

$$= -\frac{e^{-h_s^2/2}}{\sqrt{2\pi}} \times \text{an integral of order } n - 1.$$

Any ϕ integral may therefore be reduced one degree and since we can evaluate $\int_{-\infty}^{h} z(x, h)\, dx$ the formal solution of the problem is complete.

6. *Polychoric Functions.*

I shall now consider in detail the case for two variables, but before doing so it will be necessary to establish certain ancillary formulae which will greatly simplify both the process of integration and the computation when the formulae are used in numerical work. The first of these formulae are closely related to the well-known tetrachoric functions which have been tabulated in *Tables for Statisticians and Biometricians* (Cambridge, 1914), and to emphasise this relation I have ventured to call them polychoric functions. They are however already known as parabolic cylinder functions* (see Whittaker and Watson, *Modern Analysis*, p. 341, 2nd edition, Cambridge, 1915). Only the properties germane to the present issue will be dealt with here.

From the definition of $z'(x', y')$ we have

$$\left.\begin{aligned} \frac{\partial}{\partial x'} z'(x', y') &= -\frac{1}{1-r^2}\frac{1}{\sigma_x}\left(\frac{x'}{\sigma_x} - r\frac{y'}{\sigma_y}\right) z' \\ \frac{\partial}{\partial y'} z'(x', y') &= -\frac{1}{1-r^2}\frac{1}{\sigma_y}\left(\frac{y'}{\sigma_y} - r\frac{x'}{\sigma_x}\right) z' \end{aligned}\right\} \quad \text{..................(40)},$$

and we may write

$$\left.\begin{aligned} \phi'(z') &= -\left(\sigma_x \frac{\partial}{\partial x'} + r\sigma_y \frac{\partial}{\partial y'}\right)(z') = \frac{x'}{\sigma_x} z' \\ \psi'(z') &= -\left(r\sigma_x \frac{\partial}{\partial x'} + \sigma_y \frac{\partial}{\partial y'}\right)(z') = \frac{y'}{\sigma_y} z' \end{aligned}\right\} \quad \text{..............(41)},$$

where ϕ' and ψ' are functional operators.

In most cases there will be gain in clearness and no loss in generality in writing x for $\frac{x'}{\sigma_x}$ and y for $\frac{y'}{\sigma_y}$ and when desirable the following contracted forms will be used:

$$\left.\begin{aligned} \phi(z) &= -\left(\frac{\partial}{\partial x} + r\frac{\partial}{\partial y}\right)(z) = x \,.\, z \\ \psi(z) &= -\left(r\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(z) = y \,.\, z \end{aligned}\right\} \quad \text{......................(42)}.$$

Since

$$\left.\begin{aligned} \phi(z) &= x \,.\, z, & \psi(z) &= y \,.\, z \\ \phi(x) &= -1, & \psi(x) &= -r \\ \phi(y) &= -r, & \psi(y) &= -1 \end{aligned}\right\} \quad \text{........................(43)},$$

[* For the history of the subject and applications of the functions: see Pearson, K., *Phil. Trans.* Vol. 195, A, pp. 1–47, 1900; Whittaker, E. T., *Proc. Lond. Math. Soc.* Vol. xxxv. pp. 417–427, 1903; Pearson, K., "A Mathematical Theory of Random Migration," *Drapers' Research Memoirs*, Biometric Series, III., Camb. Univ. Press, 1906; and Cunningham, E., *Proc. R. Soc.* Vol. 81, A, pp. 310–331, 1908. ED.]

then
$$\begin{aligned}\phi^2(z) &= \phi\{\phi(z)\}\\ &= \phi(x\,.\,z)\\ &= \phi(x)\,.\,z + x\,.\,\phi(z)\\ &= -z + x\,.\,xz\\ &= (x^2-1)\,z\end{aligned}$$

and any combination of operators may be developed in the same way.

Assume $$\phi^n(z) = x\,.\,\phi^{n-1}(z) - (n-1)\,\phi^{n-2}(z) \ldots\ldots\ldots\ldots(44).$$

Then $$\begin{aligned}\phi^{n+1}(z) &= \phi\,\{x\,.\,\phi^{n-1}(z)\} - (n-1)\,\phi\,\{\phi^{n-2}(z)\}\\ &= \phi(x)\,.\,\phi^{n-1}(z) + x\,.\,\phi^n(z) - (n-1)\,\phi^{n-1}(z)\\ &= x\phi^n(z) - n\phi^{n-1}(n) \quad \ldots\ldots\ldots\ldots(45),\end{aligned}$$

which is of the same form as (48). But the relation is true for $n = 1, 2$, and hence is generally true. The solution of this functional equation is

$$\phi^n(z) = \left(x^n - \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} - \ldots\right) z \quad \ldots(46).$$

Writing the factor

$$x^n - \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} - \ldots = T_n(x) \quad \ldots\ldots(47),$$

we may write $$\phi^n(z) = T_n(x)\,.\,z,$$

and $T_n(x)$ I have called the polychoric function of x of the nth order. A more convenient method of writing it is

$$\frac{T_n(x)}{n\,!} = \frac{x^n}{n\,!} - \frac{x^{n-2}}{2\,(n-2)\,!} + \frac{x^{n-4}}{2^2\,.\,2\,!\,(n-4)\,!} - \ldots = \Sigma\,(-1)^s\,\frac{x^{n-2s}}{2^s\,.\,s\,!\,(n-2s)\,!} \quad \ldots(48).$$

In a similar way it may be easily shown that

$$\bar{\phi}^n(\zeta) = \left\{x^n + \frac{n\,.\,n-1}{2}\,x^{n-2} + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{2\,.\,4}\,x^{n-4} + \ldots\right\}\zeta \quad \ldots(49),$$

where $$\bar{\phi} = \left(\frac{d}{dx} + r\,\frac{d}{dy}\right) \text{ and } \zeta = \frac{1}{2\pi\sqrt{1-r^2}}\,e^{+\frac{x^2+y^2-2rxy}{2\,(1-r^2)}},$$

and it will be found useful to write $x^n + \dfrac{n\,.\,n-1}{2}\,x^{n-2} + \ldots = \mathfrak{U}_n(x)$. The following property of these functions is sufficiently interesting to be noted here. Presuming the argument x throughout

$$\left.\begin{array}{lll} & & \dfrac{T_n}{n\,!} + p\,\dfrac{T_{n-2}}{2\,(n-2)\,!} + p^2\,\dfrac{T_{n-4}}{2^2\,.\,2\,!\,(n-4)\,!} + \ldots \\ (a) & = & \dfrac{x^n}{n!} - \dfrac{x^{n-2}}{2\,(n-2)\,!} + \dfrac{x^{n-4}}{2^2\,.\,2\,!\,(n-4)\,!} - \ldots \\ (b) & & +\dfrac{p}{2}\left(\dfrac{x^{n-2}}{(n-2)\,!} - \dfrac{x^{n-4}}{2\,(n-4)\,!} + \dfrac{x^{n-6}}{2^2\,.\,2\,!\,(n-6)\,!} - \ldots\right) \\ (c) & & +\dfrac{p^2}{2^2\,.\,2\,!}\left(\dfrac{x^{n-4}}{(n-4)\,!} - \dfrac{x^{n-6}}{2\,(n-6)\,!} + \dfrac{x^{n-8}}{2^2\,.\,2\,!\,(n-8)\,!} - \ldots\right) \\ & & + \ldots\ldots \end{array}\right\} \ldots(50).$$

$$\left.\begin{array}{llll} \text{The } s+1\text{th term of the } (a) \text{ line is} & & & (-1)^s \dfrac{x^{n-2s}}{2^s \,.\, s\,!\,(n-2s)\,!} \\ \qquad s\text{th} & ,, & (b) \;\; ,, & (-1)^{s-1} \dfrac{x^{n-2s}}{2^{s-1}\,(s-1)\,!\,(n-2s)\,!} \cdot \dfrac{p}{2} \\ \qquad s-1\text{th} & ,, & (c) \;\; ,, & (-1)^{s-2} \dfrac{x^{n-2s}}{2^{s-2}\,(s-2)\,!\,(n-2s)\,!} \cdot \dfrac{p^2}{2^2 \,.\, 2\,!} \end{array}\right\} \ldots (51).$$

Hence the $s+1$th term of the whole expansion is

$$(-1)^s \frac{x^{n-2s}}{2^s \,.\, s\,!\,(n-2s)\,!} \left\{1 - \frac{s}{1\,!} p + \frac{s \,.\, s-1}{2\,!} p^2 - \frac{s \,.\, s-1 \,.\, s-2}{3\,!} p^3 + \ldots\right\}$$

$$= (-1)^s \frac{x^{n-2s}}{2^s \,.\, s\,!\,(n-2s)\,!} (1-p)^s = \frac{x^{n-2s}}{2^s \,.\, s\,!\,(n-2s)\,!} (p-1)^s \ldots\ldots (52).$$

Hence

$$\frac{T_n}{n!} + p \frac{T_{n-2}}{2\,(n-2)\,!} + p^2 \frac{T_{n-4}}{2^2 \,.\, 2\,!\,(n-4)\,!} + \ldots$$

$$= \frac{x^n}{n!} - (1-p) \frac{x^{n-2}}{2\,(n-2)\,!} + (1-p)^2 \frac{x^{n-4}}{2^2 \,.\, 2\,!\,(n-4)\,!} - \ldots \quad \ldots (53).$$

Putting $p = 1$

$$\frac{x^n}{n!} = \frac{T_n}{n\,!} + \frac{T_{n-2}}{2\,(n-2)\,!} + \frac{T_{n-4}}{2^2 .\, 2\,!\,(n-4)\,!} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

and $$x^n = T_n + \frac{n \,.\, n-1}{2} T_{n-2} + \frac{n \,.\, n-1 \,.\, n-2 \,.\, n-3}{2 \,.\, 4} T_{n-4} + \ldots \quad \ldots\ldots (55),$$

which may be symbolically written

$$x^n = \mathfrak{H}_n \{T(x)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$

if we apply suffixes to the T's on the right side instead of indices to an argument. In a similar manner it may be shown that

$$x^n = \mathfrak{H}_n - \frac{n \,.\, n-1}{2} \mathfrak{H}_{n-2} + \frac{n \,.\, n-1 \,.\, n-2 \,.\, n-3}{2 \,.\, 4} \mathfrak{H}_{n-4} - \ldots \quad \ldots (57),$$

which may be symbolically written

$$x^n = T_n \{\mathfrak{H}(x)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58),$$

and finally $$\mathfrak{H}_n \{T(x)\} = T_n \{\mathfrak{H}(x)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59).$$

From its mode of formation it is obvious that the functional equation of $T_n(x)$ is

$$T_n(x) = x \,.\, T_{n-1}(x) - (n-1)\, T_{n-2}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (60).$$

If we make $n = -1$ and expand we get

$$T_{-1}(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61).$$

But $$\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx = -e^{-\frac{1}{2}x^2} \left(\frac{1}{x} - \frac{1}{x^3} + \ldots\right) = -T_{-1}(x) \,.\, e^{-\frac{1}{2}x^2}.$$

Hence $$T_{-1}(x) = -\frac{\displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx}{e^{-\frac{1}{2}x^2}} = -\frac{\mathscr{E}(x)}{E(x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62).$$

From the operational point of view

$$\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}\,dx = \int_{-\infty}^{x} \phi\phi^{-1}\left(e^{-\frac{1}{2}x^2}\right).\,dx = -\,\phi^{-1}\,.\,e^{-\frac{1}{2}x^2} = -\,T_{-1}(x)\,.\,e^{-\frac{1}{2}x^2}$$

as before.

I have hitherto considered the polychoric function with respect to one variable only. If now we consider two variables the query arises where there is a corresponding function such that

$$\phi^p\psi^q(z) = T_{p,q}(x, y)\,.\,z(x, y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

where ψ stands to y in the same relation that ϕ stands to x. We have

$$\phi\left(\frac{x^n}{n\,!}\right) = -\frac{x^{n-1}}{(n-1)\,!}\,;\quad \psi\left(\frac{x^n}{n\,!}\right) = -\,r\,\frac{x^{n-1}}{(n-1)\,!}\,.$$

Hence

$$\phi^s\left(\frac{x^n}{n\,!}\right) = (-1)^s\,\frac{x^{n-s}}{(n-s)\,!}\,;\quad \psi^s\left(\frac{x^n}{n\,!}\right) = (-\,r)^s\,\frac{x^{n-s}}{(n-s)\,!}\,,$$

and

$$\phi^s\left(\frac{T_n(x)}{n\,!}\right) = \phi^s\left\{\frac{x^n}{n\,!} - \frac{x^{n-2}}{2\,(n-2)\,!} + \frac{x^{n-4}}{2^2\,.\,2\,!\,(n-4)\,!} - \ldots\right\}$$

$$= (-1)^s\left\{\frac{x^{n-s}}{(n-s)\,!} - \frac{x^{n-s-2}}{2\,.\,(n-s-2)\,!} + \frac{x^{n-s-4}}{2^2\,.\,2\,!\,(n-s-4)\,!} - \ldots\right\}$$

$$= (-1)^s\,\frac{T_{n-s}(x)}{(n-s)\,!} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(64),$$

and similarly

$$\psi^s\left(\frac{T_n(x)}{n\,!}\right) = (-\,r)^s\,\frac{T_{n-s}(x)}{(n-s)\,!} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(65),$$

with reciprocal relations between ϕ, ψ and $T_n(y)$. Hence

$$\begin{aligned}
\frac{\psi^q\phi^p}{q\,!\,p\,!}(z) &= \frac{\psi^q}{q\,!}\left(\frac{T_p(x)}{p\,!}\,.\,z\right)\\
&= \frac{\psi^q}{q\,!}\left(\frac{T_p(x)}{p\,!}\right).\,z\\
&\quad + \frac{\psi^{q-1}}{(q-1)\,!}\,.\left(\frac{T_p(x)}{p\,!}\right).\,\frac{\psi}{1\,!}(z)\\
&\quad + \frac{\psi^{q-2}}{(q-2)\,!}\,.\left(\frac{T_p(x)}{p\,!}\right).\,\frac{\psi^2}{2\,!}(z)\\
&\quad + \ldots\ldots\\
&= \left\{\frac{(-\,r)^q}{q\,!}\,.\,\frac{T_{p-q}(x)}{(p-q)\,!} + \frac{(-\,r)^{q-1}}{(q-1)\,!}\,.\,\frac{T_{p-q+1}(x)}{(p-q+1)\,!}\,.\,\frac{T_1(y)}{1\,!}\right.\\
&\quad \left. + \frac{(-\,r)^{q-2}}{(q-2)\,!}\,.\,\frac{T_{p-q+2}(x)}{(p-q+2)\,!}\,.\,\frac{T_2(y)}{2\,!} + \ldots\right\} z \quad \ldots\ldots(66).
\end{aligned}$$

It is more convenient to have the series in the reversed order. The $s+1$th term is

$$\frac{T_{p-q+s}(x)}{(p-q+s)\,!}\,.\,\frac{T_s(z)}{s\,!}\,.\,\frac{(-\,r)^{q-s}}{(q-s)\,!}\,z,$$

and the last term, i.e. when $s=q$, is

$$\frac{T_p(x)}{p!}\cdot\frac{T_q(y)}{q!}\,z.$$

Hence we may write

$$\frac{\phi^p\psi^q}{p!\,q!}(z)=z\left\{\frac{T_p(x)}{p!}\cdot\frac{T_q(y)}{q!}-\frac{T_{p-1}(x)\,.\,T_{q-1}(y)}{(p-1)!\,(q-1)!}\,\frac{r}{1!}+\frac{T_{p-2}(x)\,.\,T_{q-2}(y)}{(p-2)!\,(q-2)!}\,\frac{r^2}{2!}-\dots\right\}\text{ (67).}$$

If now we write $\phi^p\psi^q(z)=T_{p,q}(x,y)\,.\,z,$

so that $T_{p,q}(x, y)$ might be termed a polychoric function in two variables or better a double polychoric function, we shall have

$$\frac{T_{p,q}(x,y)}{p!\,q!}=\frac{T_p(x)}{p!}\cdot\frac{T_q(y)}{q!}-\frac{T_{p-1}(x)}{(p-1)!}\cdot\frac{T_{q-1}(y)}{(q-1)!}\cdot\frac{r}{1!}+\frac{T_{p-2}(x)}{(p-2)!}\cdot\frac{T_{q-2}(y)}{(q-2)!}\cdot\frac{r^2}{2!}-\dots$$

$$=\sum_{s=0}^{p\text{ or }q}\frac{T_{p-s}(x)}{(p-s)!}\cdot\frac{T_{q-s}(y)}{(q-s)!}\,\frac{(-r)^s}{s!}\quad\dots\dots\dots\dots\dots\dots\dots\dots(68),$$

p or q being taken as the upper limit of s, whichever is the greater. When $q=0$

$$\frac{T_{p,0}(x,y)}{p!\,0!}=\frac{T_p(x)}{p!}\cdot\frac{T_0(y)}{0!}=\frac{T_p(x)}{p!}\quad\dots\dots\dots\dots\dots\dots(69),$$

so that the single polychoric function is the particular case of a double polychoric function, viz. when the order with respect to one of the variables is zero.

The above expansion may be reversed in a manner similar to the process in equations (54) to (58) giving the result

$$\frac{T_p(x)}{p!}\,\frac{T_q(y)}{q!}=\frac{T_{p,q}(x,y)}{p!\,q!}+\frac{T_{p-1,q-1}(x,y)}{(p-1)!\,(q-1)!}\,\frac{r}{1!}+\frac{T_{p-2,q-2}(x,y)}{(p-2)!\,(q-2)!}\cdot\frac{r^2}{2!}+\dots\text{ (70).}$$

7. *Some properties of the Polychoric Functions.*

We have already seen (53) that

$$\frac{T_n}{n!}+p\,\frac{T_{n-2}}{2\,(n-2)!}+p^2\,\frac{T_{n-4}}{2^2.2!\,(n-4)!}+\dots$$

$$=\frac{x^n}{n!}-(1-p)\,\frac{x^{n-2}}{2\,(n-2)!}+(1-p)^2\,\frac{x^{n-4}}{2^2.2!\,(n-4)!}-\dots\quad\dots\dots(71).$$

Putting $\sqrt{1-p}=q$ and therefore $p=1-q^2$ we have

$$\frac{T_n}{n!}+(1-q^2)\,\frac{T_{n-2}}{2\,(n-2)!}+(1-q^2)^2\,\frac{T_{n-4}}{2^2.2!\,(n-4)!}+\dots$$

$$=\frac{x^n}{n!}-\frac{x^{n-2}q^2}{2\,(n-2)!}+\frac{x^{n-4}q^4}{2^2.2!\,(n-4)!}-\dots$$

$$=q^n\left\{\frac{\left(\frac{x}{q}\right)^n}{n!}-\frac{\left(\frac{x}{q}\right)^{n-2}}{2\,(n-2)!}+\frac{\left(\frac{x}{q}\right)^{n-4}}{2^2.2!\,(n-4)!}\dots\right\}$$

$$=q^n\,\frac{T_n\left(\frac{x}{q}\right)}{n!}\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(72).$$

Similarly by putting $\frac{1}{\sqrt{1-p}} = q$ we have $p = -\frac{1-q^2}{q^2}$ and

$$\frac{T_n}{n!} - \frac{1-q^2}{q^2}\,\frac{T_{n-2}}{2\,(n-2)!} + \left(\frac{1-q^2}{q^2}\right)^2 \frac{T_{n-4}}{2^2.2!(n-4)!} - \dots = \frac{1}{q^n}\,\frac{T_n(qx)}{n!} \;\dots\dots(73).$$

The following developments may be easily proved by expansion and simple addition:

$$\frac{T_n(x)}{n!} - \frac{x^2\,T_{n-2}(x)}{2\,(n-2)!} + \frac{x^4\,T_{n-4}(x)}{2^2.2!\,(n-4)!} - \dots = \frac{x^n\,T_n(1)}{n!} - \frac{x^{n-2}\,T_{n-2}(1)}{2\,(n-2)!} + \frac{x^{n-4}\,T_{n-4}(1)}{2^2.2!\,(n-4)!} \;\dots\dots(74),$$

$$\frac{T_n(x+a)}{n!} = \frac{x^n}{n!} + \frac{x^{n-1}\,T_1(a)}{(n-1)!} + \frac{x^{n-2}\,T_2(a)}{2!\,(n-2)!} + \dots \;\dots\dots\dots\dots\dots(75).$$

Since we have the binomial coefficients multiplied into terms of the type $x^{n-s}\,T_s(a)$ we may write this symbolically

$$T_n(x+a) = \{x + T(a)\}^n,$$

with the convention that the x's will have indices and the T's suffixes. Clearly we may also write

$$T_n(x+a) = (T(x) + a)^2.$$

Similarly

$$\frac{(x+a)^n}{n!} = \frac{T_n(x)}{n!} + \frac{T_{n-1}(x)\,.\,\mathfrak{A}_1(a)}{(n-1)!\,1!} + \frac{T_{n-2}(x)\,.\,\mathfrak{A}_2(a)}{(n-2)!\,2!} + \dots \;\dots\dots(76).$$

Therefore

$$(x+a)^n = \{T(x) + \mathfrak{A}(a)\}^n$$
$$= \{\mathfrak{A}(x) + T(a)\}^n,$$

with the same convention as before as to indices and suffixes.

The following is of interest. Write $\theta_n = \frac{T_n(x)}{n!}$, then

$$\frac{d\theta_n}{dx} = \frac{d}{dx}\left(\frac{x^n}{n!} - \frac{x^{n-2}}{2\,.\,(n-2)!} + \frac{x^{n-4}}{2^2\,.\,2!\,(n-4)!} - \dots\right)$$
$$= \frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{2\,.\,(n-3)!} + \frac{x^{n-5}}{2^2.2!\,(n-5)!} - \dots$$
$$= \frac{T_{n-1}(x)}{(n-1)!} = \theta_{n-1} \;\dots\dots\dots\dots\dots\dots(77).$$

Since

$$T_n = xT_{n-1} - (n-1)\,T_{n-2},$$
$$\theta_n = \frac{1}{n}(x\theta_{n-1} - \theta_{n-2})$$
$$= \frac{1}{n}\left(x\frac{d\theta_n}{dx} - \frac{d^2\theta_n}{dx^2}\right)$$
$$= \frac{1}{n}\left(x - \frac{d}{dx}\right)\frac{d}{dx}\,\theta_n$$
$$= \frac{1}{n}\left(x - \frac{d}{dx}\right)\theta_{n-1}$$
$$= \frac{1}{n}\left(x - \frac{d}{dx}\right)\left(x - \frac{d}{dx}\right)\frac{\theta_{n-2}}{n-1}$$
$$= \dots\dots\dots\dots\dots\dots\dots\dots$$
$$= \frac{1}{n!}\left(x - \frac{d}{dx}\right)^n.1 \;\dots\dots\dots\dots\dots\dots(78).$$

Various functions may be expanded in terms of T. Thus since

$$T_n \, . \, z = \phi^n(z) = \left(-\frac{d}{dx}\right)^n (e^{-\frac{1}{2}x^2}) = (-)^n \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2},$$

$$e^{-\frac{1}{2}(x-1)^2} = f(x-1) = f_x - \frac{df}{dx} + \frac{1}{2!} \cdot \frac{d^2f}{dx^2} - \dots$$

$$= e^{-\frac{1}{2}x^2} \left(\frac{T_0}{0!} + \frac{T_1}{1!} + \frac{T_2}{2!} + \dots\right)$$

$$= e^{-\frac{1}{2}x^2} (\theta_0 + \theta_1 + \theta_2 + \dots) \quad \dots\dots\dots\dots(79).$$

$$\therefore \; \theta_0 + \theta_1 + \theta_2 + \dots = e^{-\frac{1}{2}(x-1)^2 + \frac{1}{2}x^2} = e^{x - \frac{1}{2}} \quad \dots\dots\dots\dots(80).$$

$$\therefore \; e^x = e^{\frac{1}{2}} (\theta_0 + \theta_1 + \theta_2 + \dots) \quad \dots\dots\dots\dots(81).$$

8. *Expansion of the Polychoric Function.*

If we write T_p, T_q' for $T_p(x)$ and $T_q(y)$ respectively we have seen that (1) T_p may be written T_{p0}, (2) T_q' as T_{0q}. The same symbol T may be used for the single or the double function, as its order is shown by the number of suffixes.

$$\frac{T_{pq}}{p!\,q!} = \frac{T_p T_q'}{p!\,q!} - r\frac{T_{p-1}T'_{q-1}}{(p-1)!\,(q-1)!} + \frac{r^2}{2!} \cdot \frac{T_{p-2}T'_{q-2}}{(p-2)!\,(q-2)!} - \frac{r^3}{3!} \cdot \frac{T_{p-3}T'_{q-3}}{(p-3)!\,(q-3)!}$$

$$= \frac{T_p T_q'}{p!\,q!} - \int \left\{\frac{T_{p-1}T'_{q-1}}{(p-1)!\,(q-1)!} - r\frac{T_{p-2}T'_{q-2}}{(p-2)!\,(q-2)!} + \frac{r^2}{2!} \cdot \frac{T_{p-3}T'_{q-3}}{(p-3)!\,(q-3)!} - \dots\right\} dr$$

$$= \frac{T_p T_q'}{p!\,q!} - \int \frac{T_{p-1,\,q-1}}{(p-1)!\,(q-1)!} \, dr \quad \dots\dots\dots\dots(82),$$

and

$$T_{pq} = T_p T_q' - p \, . \, q \int T_{p-1,\,q-1} \, dr \quad \dots\dots\dots\dots(83),$$

which gives a simple method of expanding T_{pq}.

It may be convenient on occasion to express the above in the θ notation

$$\theta_{pq} = \theta_p \theta_q' - \int \theta_{p-1,\,q-1} \, dr \quad \dots\dots\dots\dots(84).$$

Having expanded T_{pq} we can at once write down the expansion $\phi^p \psi^q$. The two following tables give the expansions required, viz. $\phi^p \psi^q$ in terms of powers of x and y multiplied by z and its converse.

TABLE I.

xy products of z in terms of polychoric functions.

1. $xz \; = \phi(z)$

2. $x^2z \; = (\phi^2 + 1)\,z$
   $xyz \; = (\phi\psi + r)\,z$

3. $x^3z \; = (\phi^3 + 3\phi)\,z$
   $x^2yz \; = (\phi^2\psi + \psi + 2r\phi)\,z$

TABLE I (*continued*).

4.
$$\begin{aligned}
x^4z &= (\phi^4 + 6\phi^2 + 3)\,z\\
x^3yz &= (\phi^3\psi + 3\phi\psi + 3r\phi^2 + 3r)\,z\\
x^2y^2z &= (\phi^2\psi^2 + \phi^2 + 4r\phi\psi + \psi^2 + 2r^2 + 1)\,z
\end{aligned}$$

5.
$$\begin{aligned}
x^5z &= (\phi^5 + 10\phi^3 + 15\phi)\,z\\
x^4yz &= (\phi^4\psi + 6\phi^2\psi + 3\psi + 12r\phi + 4r\phi^3)\,z\\
x^3y^2z &= (\phi^3\psi^2 + 3\phi\psi^2 + 6r\phi^2\psi + 6r\psi + 3\,(1 + 2r^2)\,\phi + \phi^3)\,z
\end{aligned}$$

6.
$$\begin{aligned}
x^6z &= (\phi^6 + 15\phi^4 + 45\phi^2 + 15)\,z\\
x^5yz &= (\phi^5\psi + 10\phi^3\psi + 15\phi\psi + 30r\phi^2 + 5r\phi^4 + 15r)\,z\\
x^4y^2z &= (\phi^4\psi^2 + 6\phi^2\psi^2 + 3\psi^2 + 24r\phi\psi + 8r\phi^3\psi + 6\,(1 + 2r^2)\,\phi^2 + \phi^4 + 3\,(1 + 4r^2))\,z\\
x^3y^3z &= (\phi^3\psi^3 + 3\phi^3\psi + 9r\phi^2\psi^2 + 9r\phi^2 + 3\phi\psi^3 + 9(1 + 2r^2)\,\phi\psi + 9r\psi^2 + 3r\,(3 + 2r^2))\,z
\end{aligned}$$

7.
$$\begin{aligned}
x^7z &= (\phi^7 + 21\phi^5 + 105\phi^3 + 105\phi)\,z\\
x^6yz &= (\phi^6\psi + 15\phi^4\psi + 45\phi^2\psi + 15\psi + 90r\phi + 60r\phi^3 + 6r\phi^5)\,z\\
x^5y^2z &= (\phi^5\psi^2 + 10\phi^3\psi^2 + 15\phi\psi^2 + 10r\phi^4\psi + 60r\phi^2\psi + 30r\psi + 15\,(1 + 4r^2)\,\phi\\
&\qquad + 10\,(1 + 2r^2)\,\phi^3 + \phi^5)\,z\\
x^4y^3z &= (\phi^4\psi^3 + 6\phi^2\psi^3 + 3\psi^3 + 12r\phi^3\psi^2 + 36r\phi\psi^2 + 3\phi^4\psi + 18\,(1 + 2r^2)\,\phi^2\psi\\
&\qquad + 9\,(1 + 4r^2)\,\psi + 12r\,(3 + 2r^2)\,\phi + 12r\phi^3)\,z
\end{aligned}$$

8.
$$\begin{aligned}
x^8z &= (\phi^8 + 28\phi^6 + 210\phi^4 + 420\phi^2 + 105)\,z\\
x^7yz &= (\phi^7\psi + 21\phi^5\psi + 105\phi^3\psi + 105\phi\psi + 315r\phi^2 + 105r\phi^4 + 7r\phi^6 + 105r)\,z\\
x^6y^2z &= (\phi^6\psi^2 + 15\phi^4\psi^2 + 45\phi^2\psi^2 + 15\psi^2 + 180r\phi\psi + 120r\phi^3\psi + 12r\phi^5\psi\\
&\qquad + 45\,(1 + 4r^2)\,\phi^2 + 15\,(1 + 2r^2)\,\phi^4 + \phi^6 + 15\,(1 + 6r^2))\,z\\
x^5y^3z &= (\phi^5\psi^3 + 3\phi^5\psi + 30\,(1 + 2r^2)\,\phi^3\psi + 45\,(1 + 4r^2)\,\phi^4 + 30r\,(3 + 2r^2)\,\phi^2\\
&\qquad + 15r\phi^4 + 10\phi^3\psi^3 + 90r\phi^2\psi^2 + 15\phi\psi^3 + 45r\psi^2 + 15r\phi^4\psi^2 + 15r\,(3 + 4r^2))\,z\\
x^4y^4z &= (\phi^4\psi^4 + 6\phi^4\psi^2 + 6\phi^2\psi^4 + 36\,(1 + 2r^2)\,\phi^2\psi^2 + 3\phi^4 + 3\psi^4\\
&\qquad + 48r\phi^3\psi + 48r\phi\psi^3 + 18\,(1 + 4r^2)\,\phi^2 + 18\,(1 + 4r^2)\,\psi^2\\
&\qquad + 48r\,(3 + 2r^2)\,\phi\psi + 16r\phi^3\psi^3 + 3\,(3 + 24r^2 + 8r^4))\,z.
\end{aligned}$$

TABLE II.

Polychoric functions expanded in terms of x and y.

1.
$$\phi\,(z) = xz$$

2.
$$\begin{aligned}
\phi^2\,(z) &= (x^2 - 1)\,z\\
\phi\psi\,(z) &= (xy - r)\,z
\end{aligned}$$

3.
$$\begin{aligned}
\phi^3\,(z) &= (x^3 - 3x)\,z\\
\phi^2\psi\,(z) &= (x^2y - y - 2rx)\,z
\end{aligned}$$

4.
$$\begin{aligned}
\phi^4\,(z) &= (x^4 - 6x^2 + 3)\,z\\
\phi^3\psi\,(z) &= (x^3y - 3rx^2 - 3xy + 3r)\,z\\
\phi^2\psi^2\,(z) &= (x^2y^2 - x^2 - y^2 - 4rxy + 1 + 2r^2)\,z
\end{aligned}$$

5.
$$\begin{aligned}
\phi^5\,(z) &= (x^5 - 10x^3 + 15x)\,z\\
\phi^4\psi\,(z) &= (x^4y - 6x^2y + 3y - 4rx^3 + 12rx)\,z\\
\phi^3\psi^2\,(z) &= (x^3y^2 - 6rx^2y - 3xy^2 - x^3 + 3\,(1 + 2r^2)\,x + 6ry)\,z
\end{aligned}$$

TABLE II (*continued*).

$$\begin{aligned}
6.\quad & \phi^6(z) = (x^6 - 15x^4 + 45x^2 - 15)\,z \\
& \phi^5\psi(z) = (x^5y - 10x^3y + 15xy - 15r - 5rx^4 + 30rx^2)\,z \\
& \phi^4\psi^2(z) = (x^4y^2 - 6x^2y^2 - 8rx^3y + 6(1+2r^2)\,x^2 + 3y^2 + 24rxy - x^4 - 3(1+4r^2))\,z \\
& \phi^3\psi^3(z) = (x^3y^3 - 3x^3y - 3xy^3 - 9rx^2y^2 + 9(1+2r^2)\,xy + 9rx^2 + 9ry^2 - 3r(3+2r^2))\,z
\end{aligned}$$

$$\begin{aligned}
7.\quad & \phi^7(z) = (x^7 - 21x^5 + 105x^3 - 105x)\,z \\
& \phi^6\psi(z) = (x^6y - 15x^4y + 45x^2y - 15y - 6x^5r + 60x^3r - 90xr)\,z \\
& \phi^5\psi^2(z) = (x^5y^2 - 10x^3y^2 + 15xy^2 - 10rx^4y + 60rx^2y - 30ry \\
& \qquad - 15(1+4r^2)\,x + 10(1+2r^2)\,x^3 - x^5)\,z \\
& \phi^4\psi^3(z) = (x^4y^3 - 6x^2y^3 + 3y^3 - 12rx^3y^2 + 36rxy^2 - 3x^4y \\
& \qquad + 18(1+2r^2)\,x^2y - 9(1+4r^2)\,y - 12r(3+2r^2)\,x + 12rx^3)\,z
\end{aligned}$$

$$\begin{aligned}
8.\quad & \phi^8(z) = (x^8 - 28x^6 + 210x^4 - 420x^2 + 105)\,z \\
& \phi^7\psi(z) = (x^7y - 21x^5y + 105x^3y - 105xy - 315rx^2 + 105rx^4 - 7rx^6 + 105r)\,z \\
& \phi^6\psi^2(z) = (x^6y^2 - 15x^4y^2 + 45x^2y^2 - 15y^2 - 180rxy + 120rx^3y \\
& \qquad - 12rx^5y - 45(1+4r^2)\,x^2 + 15(1+2r^2)\,x^4 - x^6 + 15(1+6r^2))\,z \\
& \phi^5\psi^3(z) = (x^5y^3 - 3x^5y + 30(1+2r^2)\,x^3y - 45(1+4r^2)\,xy - 30r(3+2r^2)\,x^2 \\
& \qquad + 15rx^4 - 10x^3y^3 + 90rx^2y^2 + 15xy^3 - 45ry^2 - 15rx^4y^2 + 15r(3+4r^2))\,z \\
& \phi^4\psi^4(z) = (x^4y^4 - 6x^4y^2 - 6x^2y^4 + 36(1+2r^2)\,x^2y^2 + 3x^4 + 3y^4 \\
& \qquad + 48rx^3y + 48rxy^3 - 18(1+4r^2)\,x^2 - 18(1+4r^2)\,y^2 \\
& \qquad - 48r(3+2r^2)\,xy - 16rx^3y^3 + 3(3+24r^2+8r^4))\,z.
\end{aligned}$$

The above formulae may be immediately expressed in terms of T_{pq} by remembering that $\phi^p\psi^q(z) = T_{pq}\,z$.

Thus from $\qquad x^2yz = (\phi^2\psi + \psi + 2r\phi)\,z$

we get $\qquad x^2y = T_{21} + T_{01} + 2rT_{10}$

and from $\qquad \phi^2\psi(z) = (x^2y - y - 2rx)\,z$

we get $\qquad T_{21} = x^2y - y - 2rx.$

9. *Moments of a quadrant.*

From the preceding tables we may express any moments of z in sums of functions involving ϕ and ψ. Consider one term of such a sum:

$$\begin{aligned}
& \int_{-\infty}^{k}\int_{-\infty}^{h} \phi^p\psi^q z(x,y)\,dx\,dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \phi^{p-1}\psi^q\left(\frac{d}{dx} + r\frac{d}{dy}\right) z(x,y)\,dx\,dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \phi^{p-1}\psi^q \frac{d}{dx} z(x,y)\,dx\,dy - r\int_{-\infty}^{k}\int_{-\infty}^{h} \phi^{p-1}\psi^q \frac{d}{dy} z(x,y)\,dx\,dy \\
&= -\int_{-\infty}^{k} \phi^{p-1}\psi^q z(h,y)\,dy - r\int_{-\infty}^{h} \phi^{p-1}\psi^q z(x,k)\,dx \quad \text{.......................(85).}
\end{aligned}$$

These when expanded by means of the tables will give a series of terms such as

$$(a)\ \int_{-\infty}^{k} h^l y^m z(h, y)\,dy \quad\text{and}\quad (b)\ \int_{-\infty}^{h} x^{l'} k^{m'} z(x, k)\,dx.$$

(a) may be regarded as the (lm)th moment of $\int_{-\infty}^{k} z(h, y)\,dy$, that is of HA (see equation 7), and (b) may be regarded as the ($l'm'$)th moment of $\int_{-\infty}^{h} z(x, k)\,dx$, that is of KB. That is they are moments of the bounding surfaces of the quadrant (equation 8).

If we write the moment coefficients of HA and KB as $\bar{A}_{lm}$ and $\bar{B}_{l'm'}$ respectively we may write

$$\int_{-\infty}^{k} h^l y^m z(h, y)\,dy = HA\,.\,\bar{A}_{lm} = HA\,.\,h^l \bar{A}_{0m} \quad\text{.............(86)},$$

$$\int_{-\infty}^{h} x^{l'} k^{m'} z(x, k)\,dx = KB\,.\,\bar{B}_{l'm'} = KB\,.\,k^{m'} \bar{B}_{l'0} \quad\text{...........(87)}.$$

It remains therefore to find a means of evaluating $\bar{A}_{0m}$ and $\bar{B}_{l'0}$.

An alternative form of the expansion will be got by expanding a ψ instead of a ϕ but the ultimate results will be identical.

To take an actual example

$$\begin{aligned}
&\int_{-\infty}^{k}\int_{-\infty}^{h} x^2 yz\,dx\,dy \\
&= \int_{-\infty}^{k}\int_{-\infty}^{h} (\phi^2\psi + \psi + 2r\phi)\,z\,dx\,dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \left\{(\phi^2 + 1)\left(r\frac{d}{dx} + \frac{d}{dy}\right) + 2r\left(\frac{d}{dx} + r\frac{d}{dy}\right)\right\} z\,dx\,dy \\
&= -\int_{-\infty}^{k}\int_{-\infty}^{h} \left\{(r\phi^2 + r + 2r)\frac{d}{dx} + (\phi^2 + 1 + 2r^2)\frac{d}{dy}\right\} z\,dx\,dy \\
&= -\int_{-\infty}^{k} (r\phi^2 + 3r)\,z(h, y)\,dy - \int_{-\infty}^{h} (\phi^2 + 1 + 2r^2)\,z(x, k)\,dx \\
&= -\int_{-\infty}^{k} (rh^2 - r + 3r)\,z(h, y)\,dy - \int_{-\infty}^{h} (x^2 - 1 + 1 + 2r^2)\,z(x, k)\,dx \\
&= -r\int_{-\infty}^{k} h^2 z(h, y)\,dy - 2r\int_{-\infty}^{k} z(h, y)\,dy - \int_{-\infty}^{h} x^2 z(x, k)\,dx - 2r^2\int_{-\infty}^{h} z(x, k)\,dx \\
&= -rh^2 HA\,. - 2rH\,.\,A - KB\,.\,\bar{B}_{20} - 2r^2 KB \\
&= -HA\,(rh^2 + 2r) - KB\,(\bar{B}_{20} + 2r^2) \quad\text{...(88)}.
\end{aligned}$$

For theoretical purposes the development may be treated more systematically as follows.

Since $$\int_{-\infty}^{k} h^l y^m z(h, y)\,dy = HA\,.\,\bar{A}_{lm} \quad\text{.......................(89)},$$

it is clear that $$\int_{-\infty}^{k} \phi^{p-1} q\, z(h, y)\,dx = \int_{-\infty}^{k} T_{p-1, q}(h, y)\,z(h, y)\,dx$$

$$= HA\,.\,T_{p-1, q}\bar{A} \quad\text{.........................(90)},$$

in which the T will be expanded in A's having the same suffixes as the h's and y's.

Hence we may write immediately

$$\int_{-\infty}^{k}\int_{-\infty}^{h}\phi^p\psi^q z\,dx\,dy = -HA\,.\,T_{p-1,q}\bar{A} - rKB\,.\,T_{p-1,q}\bar{B} = -T_{p-1,q}M \quad \ldots(91),$$

or reducing the ψ

$$= -rHA\,.\,T_{p,q-1}\bar{A} - KB\,.\,T_{p,q-1}\bar{B} = -T_{p,q-1}M',$$

where M or M' may be regarded as a kind of complex moment coefficient.

Applying this to our example we have

$$\begin{aligned}&\int_{-\infty}^{k}\int_{-\infty}^{h}(\phi^2\psi + \psi + 2r\phi)\,z\,dx\,dy\\ &= -\{T_{20}(M') + T_0(M') + 2rT_0M\}\\ &= -\{rHA\,.\,T_{20}(\bar{A}) + KB\,.\,T_{20}\bar{B}\}\\ &\quad -(rHA + KB) - 2r(HA + rKB) \quad \ldots\ldots\ldots\ldots\ldots\ldots(92).\end{aligned}$$

But $$T_{20}(\bar{A}) = \bar{A}_{20} - 1 = h^2 - 1$$

and $$T_{20}(\bar{B}) = \bar{B}_{20} - 1,$$

and the whole reduces as before to

$$-HA\,(rh^2 + 2r) - KB\,(\bar{B}_{20} + 2r^2).$$

10. *Evaluation of the moments of the bounding surfaces of the quadrant.*

In equation (7) the area of the surface bounding the quadrant a and parallel to the y axis was represented by the symbol

$$N\frac{H}{\sigma_x}A = N\int_{-\infty}^{k'} z'(h', y')\,dy',$$

and that parallel to the x axis by

$$N\frac{K}{\sigma_y}B = N\int_{-\infty}^{h'} z'(x', k')\,dx'.$$

Consider now the pqth moment of say the latter, the B face. It will be represented by

$$N\int_{-\infty}^{h'} x'^p k'^q z'(x', k')\,dx' = Nk'^q\int_{-\infty}^{h'} x'^p z'(x', k')\,dx' \quad \ldots\ldots\ldots\ldots(93),$$

since $y' = k'$ over the whole face. Representing this as the product of the area and its pqth moment coefficient we may write

$$N\int_{-\infty}^{h'} x'^p k'^q z'(x', k')\,dx' = N\frac{K}{\sigma_y}B\,.\,\bar{B}'_{pq} = N\frac{K}{\sigma_y}B\,.\,k'^q\,.\,\bar{B}'_{p,0} \quad \ldots\ldots\ldots(94),$$

and it is the value of the coefficient $\bar{B}'_{p,0}$ that we have to determine

$$\begin{aligned}\frac{d}{dx'}\{x'^n\,.\,z'(x', k')\} &= nx'^{n-1}\,.\,z'(x', k') + x'^n\left\{-\frac{x'/\sigma_x - ry'/\sigma_y}{\sigma_x(1-r^2)}\right\}z'(x', k')\\ &= \left(nx'^{n-1} - \frac{x'^{n+1}}{\sigma_x^2(1-r^2)} - r\frac{k'}{\sigma_y}\cdot\frac{x'^n}{\sigma_x(1-r^2)}\right)z'(x', k') \quad \ldots(95).\end{aligned}$$

Transposing and integrating

$$\frac{1}{\sigma_x^2(1-r^2)}\int_{-\infty}^{h'} x'^{n+1}\,.\,z'(x',k')\,dx' = n\int_{-\infty}^{h'} x'^{n-1}\,.\,z'(x',k')\,dx'$$
$$+\frac{rk'}{\sigma_x\sigma_y(1-r^2)}\int_{-\infty}^{h'} x'^{n}\,.\,z'(x',k')\,dx' - \Big|_{-\infty}^{h'} x'^{n}\,.\,z'(x',k') \quad\text{......(96)},$$

that is,

$$\frac{1}{\sigma_x^2(1-r^2)}\cdot\frac{KB}{\sigma_y}\,.\,\bar{B}'_{n+1,0} = n\,\frac{KB}{\sigma_y}\,\bar{B}'_{n-1,0} + \frac{rk'}{\sigma_x\sigma_y(1-r^2)}\cdot\frac{KB}{\sigma_y}\,\bar{B}'_{n,0} - \frac{h'^n\chi}{\sigma_x\sigma_y} \quad\text{...(97)},$$

where $\chi = z(h,k)$, see equation (10).

Reducing by one degree and dividing out by $\dfrac{KB}{\sigma_y\sigma_x^2(1-r^2)}$,

$$\bar{B}'_{n,0} = \frac{rk'\sigma_x}{\sigma_y}\,\bar{B}'_{n-1,0} + (n-1)(1-r^2)\,\sigma_x^2\bar{B}'_{n-2,0} - \frac{h'^{n-1}\sigma_x\chi}{KB}(1-r^2) \quad\text{...(98)}.$$

Hence
$$\bar{B}'_{10} = \frac{rk'\sigma_x}{\sigma_y} - \frac{\sigma_x\chi}{KB}(1-r^2) \quad\text{..........................(99)},$$

and we may write

$$\bar{B}'_{n,0} = \frac{rk'\sigma_x}{\sigma_y}\,\bar{B}'_{n-1,0} + (n-1)(1-r^2)\,\sigma_x^2\bar{B}'_{n-2,0} + h'^{n-1}\left(\bar{B}'_{10} - \frac{rk'\sigma_x}{\sigma_y}\right) \quad\text{...(100)}.$$

Divide throughout by $\sigma_x^n(\sqrt{1-r^2})^n$ and write

$$\sigma_x\sqrt{1-r^2} = S_x,$$
$$\sigma_y\sqrt{1-r^2} = S_y,$$

$$\frac{\bar{B}'_{n,0}}{S_x^n} = \frac{rk'}{S_y}\cdot\frac{\bar{B}'_{n-1,0}}{S_x^{n-1}} + (n-1)\,\frac{\bar{B}'_{n-2,0}}{S_x^{n-2}} + \frac{h'^{n-1}}{S_x^{n-1}}\cdot\frac{\bar{B}'_{10}}{S_x} - \frac{rk'}{S_y}\cdot\frac{h'^{n-1}}{S_x^{n-1}} \quad\text{......(101)}.$$

Let now
$$\frac{\bar{B}'_{n,0}}{S_x^n} = \beta_n;\quad \frac{h'}{S_x} = \iota;\quad \frac{k'}{S_y} = \kappa \quad\text{........................(102)}$$

then
$$\beta_n = r\kappa\beta_{n-1} + (n-1)\,\beta_{n-2} + \iota^{n-1}\beta_1 - r\iota^{n-1}\kappa \quad\text{..............(103)}$$

Putting $n = 1$ in (81),

$$\beta_1 = \frac{\bar{B}_{10}}{\sigma_x\sqrt{1-r^2}} = \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{\chi}{KB}(\sqrt{1-r^2})$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{1}{\sqrt{1-r^2}}\cdot\frac{E(k)\,.\,E\left(\dfrac{h-rk}{\sqrt{1-r^2}}\right)}{K\mathscr{E}\left(\dfrac{h-rk}{\sqrt{1-r^2}}\right)}\sqrt{1-r^2}$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{E\left(\dfrac{h'/\sigma_x - rk'/\sigma_y}{\sqrt{1-r^2}}\right)}{\mathscr{E}\left(\dfrac{h'/\sigma_x - rk'/\sigma_y}{\sqrt{1-r^2}}\right)}$$

$$= \frac{rk'}{\sigma_y\sqrt{1-r^2}} - \frac{E(\iota - r\kappa)}{\mathscr{E}(\iota - r\kappa)}$$

$$= r\kappa + c_\beta \quad\text{..(104)},$$

where c_β is the centroid of the face B. See equation (16).

Substituting in (85),

$$\beta_n = r\kappa\beta_{n-1} + (n-1)\,\beta_{n-2} + \iota^{n-1} c_\beta \quad \text{......................(105).}$$

Similarly for the ***A*** face

$$\int_{-\infty}^{k'} h'^p y'^q z'(h', y')\, dy' = N\frac{H}{\sigma_x} A \,.\, h'^p \,.\, \bar{A}_{0q},$$

$$\frac{\bar{A}'_{0n}}{S_y{}^n} = r\frac{h'}{S_x}\cdot\frac{\bar{A}'_{0,\,n-1}}{S_y{}^{n-1}} + (n-1)\frac{\bar{A}'_{0,\,n-2}}{S_y{}^{n-2}} + \frac{k'^{n-1}}{S_y{}^{n-1}}\cdot\frac{\bar{A}'_{01}}{S_y} - r\frac{h'}{S_x}\cdot\frac{k'^{n-1}}{S_y{}^{n-1}} \quad \text{... (106),}$$

$$\frac{\bar{A}'_{0n}}{S_y{}^n} = \alpha_n\,;$$

$$\alpha_1 = \frac{rh'}{\sigma_x\sqrt{1-r^2}} - \frac{E(\kappa - r\iota)}{\mathscr{E}(\kappa - r\iota)} = r\iota + c_\alpha \quad \text{..............................(107),}$$

$$\alpha_n = r\iota\alpha_{n-1} + (n-1)\,\alpha_{n-2} + \kappa^{n-1} c_\alpha \quad \text{..................................(108).}$$

11. *The moments in terms of the tetrachoric coefficients.*

We have already seen (77) that

$$\frac{d\theta_p(x)}{dx} = \theta_{p-1}(x) \quad\text{and}\quad \frac{d\theta_q(y)}{dy} = \theta_{q-1}(y).$$

Drop the arguments and write θ_p, θ_q' for $\theta_p(x)$, $\theta_q(y)$ respectively,

$$\frac{d\theta_{pq}}{dx} = \frac{d}{dx}\left(\theta_p\theta_q' - r\,\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2\,!}\,.\,\theta_{p-2}\theta'_{q-2}\ldots\right)$$

$$= \theta_{p-1}\theta_q' - r\,\theta_{p-2}\theta'_{q-1} + \frac{r^2}{2\,!}\,.\,\theta_{p-3}\theta'_{q-2}\ldots$$

$$= \theta_{p-1,\,q} \quad \text{..(109).}$$

Similarly

$$\frac{d\theta_{p-1,\,q}}{dy} = \theta_{p-1,\,q-1},$$

$$\therefore\quad \frac{d\theta_{p,\,q}}{dx\,dy} = \theta_{p-1,\,q-1} \quad \text{..............................(110).}$$

But

$$\frac{d\theta_{p,\,q}}{dr} = \frac{d}{dr}\left(\theta_p\theta_q' - r\,\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2\,!}\,\theta_{p-2,\,q-2}\ldots\right)$$

$$= -\,\theta_{p-1}\theta'_{q-1} + r\,\theta_{p-2,\,q-2} - \frac{r^2}{2\,!}\,\theta_{p-3,\,q-3}$$

$$= -\,\theta_{p-1,\,q-1}$$

$$= -\,\frac{d\theta_{p,\,q}}{dx\,dy} \quad \text{..(111).}$$

Hence

$$\left(\frac{d\theta_{pq}}{dr}\right)_{r=0} = -\,\theta_{p-1}\theta'_{q-1} \quad \text{..............................(112).}$$

The result in (111) corresponds to that deduced by Pearson (in *Phil. Trans.* Vol. 195, pp. 1–47), viz.

$$\frac{dz}{dr} = \frac{d^2 z}{dx\,dy} \quad \text{...................................(113).}$$

We have further, since

$$\frac{dz}{dx} = -\frac{(\phi - r\psi)}{1 - r^2}(z), \quad \frac{dz}{dy} = -\frac{(\psi - r\phi)}{1 - r^2}(z) \quad \ldots\ldots\ldots\ldots\ldots(114),$$

$$\frac{d^n z}{dr^n} = \frac{d^{2n} z}{dx^n\, dy^n} = \frac{(\phi - r\psi)^n (\psi - r\phi)^n}{(1 - r^2)^{2n}}(z) \ldots\ldots\ldots\ldots\ldots\ldots(115).$$

Hence

$$\begin{aligned} \left(\frac{d^n z}{dr^n}\right)_{r=0} &= (\phi^n \psi^n z)_{r=0} \\ &= (T_{nn'} z)_{r=0} \\ &= T_n T_n' z_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(116), \end{aligned}$$

where

$$z_0 = (z)_{r=0}.$$

Hence, developing z as a function of r,

$$\begin{aligned} z &= z_0 + r\left(\frac{dz}{dr}\right)_{r=0} + \frac{r^2}{2\,!}\left(\frac{d^2 z}{dr^2}\right)_{r=0} + \ldots \\ &= z_0\left(1 + rT_1T_1' + \frac{r^2}{2\,!}T_2T_2' + \ldots\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(117). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\phi^p \psi^q z}{p\,!\,q\,!} &= \theta_{pq} z = \theta_{pq}\left(1 + rT_1T_1' + \frac{r^2}{2\,!}T_2T_2' + \ldots\right) z_0 \\ &= \left(\theta_p\theta_q' - r\theta_{p-1}\theta'_{q-1} + \frac{r^2}{2\,!}\theta_{p-2}\theta'_{q-2}\ldots\right)\left(1 + rT_1T_1' + \frac{r^2}{2\,!}T_2T_2' + \ldots\right) z_0 \\ &= \theta_p\theta_q' z_0 \\ &\quad + r(\theta_p\theta_q' T_1T_1' - \theta_{p-1}\theta'_{q-1}) z_0 \\ &\quad + \frac{r^2}{2\,!}(\theta_p\theta_q' T_2T_2' - 2\theta_{p-1}\theta'_{q-1}T_1T_1' + \theta_{p-2}\theta'_{q-2}) z_0 \\ &\quad + \frac{r^3}{3\,!}(\theta_p\theta_q' T_3T_3' - 3\theta_{p-1}\theta'_{q-1}T_2T_2' + 3\theta_{p-2}\theta'_{q-2}T_1T_1' - \theta_{p-3}\theta'_{q-3}) z_0 \\ &\quad + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(118). \end{aligned}$$

The development of the coefficients of the r's is obvious. If we write the coefficient of $\frac{r^n}{n!}$ as $(\theta_{pq}T)_n$, or where there is no possibility of confusion $(\theta T)_n$, we have

$$\frac{\phi^p \psi^q (z)}{p\,!\,q\,!} = \left\{\theta_p\theta_q' + r(\theta T)_1 + \frac{r^2}{2\,!}(\theta T)_2 + \frac{r^3}{3\,!}(\theta T)_3 + \ldots\right\} z_0 \quad \ldots\ldots(119).$$

Hence

$$\begin{aligned} \int_{-\infty}^{k}\int_{-\infty}^{h} \frac{\phi^p \psi^q (z)}{p\,!\,q\,!}\, dx\, dy &= \int_0^r \frac{\phi^p \psi^q (z)}{p\,!\,q\,!}\, dr \\ &= C + \left(\theta_p\theta_q' r + (\theta T)_1\frac{r^2}{2\,!} + (\theta T)_2\frac{r^3}{3\,!} + (\theta T)_3\frac{r^4}{4\,!} + \ldots\right) z_0(h, k) \ldots(120), \end{aligned}$$

where

$$z_0(h, k) = HK.$$

To determine C,

$$\frac{d}{dx}(\theta_n e^{-\frac{1}{2}x^2}) = \frac{d\theta_n}{dx} \cdot e^{-\frac{1}{2}x^2} - \theta_n \cdot x e^{-\frac{1}{2}x^2}$$

$$= -(x\theta_n - \theta_{n-1}) e^{-\frac{1}{2}x^2}$$

$$= -(n+1)\,\theta_{n+1} e^{-\frac{1}{2}x^2} \quad \text{.....................(121).}$$

Therefore $$\int \theta_{n+1} e^{-\frac{1}{2}x^2} dx = -\frac{\theta_n e^{-\frac{1}{2}x^2}}{n+1} \quad \text{.................................(122).}$$

Now put $r=0$ in equation (120),

$$C = \int_{-\infty}^{k}\int_{-\infty}^{h} \frac{\phi^p \psi^q}{p!\,q!} z\,dx\,dy \qquad (r=0)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{h} \theta_p\, e^{-\frac{1}{2}x^2} dx \,.\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} \theta_q'\, e^{-\frac{1}{2}y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{\theta_{p-1}\, e^{-\frac{1}{2}x^2}}{p} \,.\, \frac{1}{\sqrt{2\pi}}\,\frac{\theta'_{q-1}\, e^{-\frac{1}{2}y^2}}{q}$$

$$= \frac{\theta_{p-1}\theta'_{q-1}\, e^{-\frac{1}{2}(x^2+y^2)}}{p\,.\,q\,.\,2\pi} = \frac{\theta_{p-1}\theta'_{q-1}}{pq} HK \quad \text{..................(123).}$$

Hence finally

$$\int_{-\infty}^{k}\int_{-\infty}^{h} \phi^p \psi^q z\,dx\,dy = p!\,q!\,HK\left\{\frac{\theta_{p-1}\theta'_{q-1}}{p\cdot q} + \theta_p\theta_q' r + (\theta T)_1 \frac{r^2}{2!} + (\theta T)_2 \frac{r^3}{3!} + \dots\right\}$$

$$\text{......(124).}$$

When $p = q = 0$ we get the volume of the quadrant. In this case $\frac{\theta_{p-1}}{p}$ becomes indeterminate, since putting $n = 1$ in $\theta_n = \frac{x\theta_{n-1} - \theta_{n-2}}{n}$ we get

$$\theta_1 = x\theta_0 - \theta_{-1} \quad \text{and} \quad \theta_{-1} = x\theta_0 - \theta_1 = x - x = 0.$$

But from (61) we have

$$T_{-1} e^{-\frac{1}{2}x^2} = -\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx \quad \text{........................(125).}$$

Hence we may write in this case

$$C = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} T_0 e^{-\frac{1}{2}x^2} dx \,.\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k} T_0\, e^{-\frac{1}{2}y^2} dy$$

$$= T_{-1}\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \,.\, T'_{-1}\,\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$

$$= T_{-1} T'_{-1} HK \quad \text{..(126).}$$

In this case $\theta_p = \theta_q' = 1$ and all the other θ's vanish; and we have

$$\int_{-\infty}^{k}\int_{-\infty}^{h} z\,dx\,dy = HK\left\{T_{-1}T'_{-1} + rT_0T_0' + \frac{r^2}{2!}T_1T_1' + \frac{r^3}{3!}T_2T_2' + \dots\right\}$$

$$\text{......(127),}$$

which is the ordinary tetrachoric expansion in terms of T.

Applying the above to our former example we have

$$\begin{aligned}
&\int_{-\infty}^{k}\int_{-\infty}^{h} x^2 yz\,dx\,dy \\
&\quad = \int_{-\infty}^{k}\int_{-\infty}^{h} (\phi^2\psi + \psi + 2r\phi)\,z\,dx\,dy \\
&\quad = 2!\,2!\,HK\left\{\frac{\theta_1}{2} + \theta_2\theta_1' r + (\theta_{21}T)_1\frac{r^2}{2!} + (\theta_{21}T)_2\frac{r^3}{3!} + \dots\right\} \\
&\qquad + 0!\,1!\,HK\left\{1 + \theta_1' r + (\theta_{01}T)_1\frac{r^2}{2!} + (\theta_{01}T)_2\frac{r^3}{3!} + \dots\right\} \\
&\qquad + 2r1!\,0!\,HK\left\{1 + \theta_1 r + (\theta_{10}T)_1\frac{r^2}{2!} + (\theta_{10}T)_2\frac{r^3}{3!} + \dots\right\} \\
&\quad = HK\biggl[4\biggl\{\frac{\theta_1}{2} + \theta_2\theta_1' r + (\theta_2\theta_1' T_1 T_1' - 1)\frac{r^2}{2!} \\
&\qquad\qquad\qquad + (\theta_2\theta_1' T_2 T_2' - 2\theta_1 T_1 T_1')\frac{r^3}{3!} \\
&\qquad\qquad\qquad + (\theta_2\theta_1' T_3 T_3' - 3\theta_1 T_2 T_2')\frac{r^4}{4!}\biggr\} \\
&\qquad + \left\{1 + \theta_1' r + (\theta_1' T_1 T_1')\frac{r^2}{2!} + (\theta_1' T_2 T_2')\frac{r^3}{3!} + \dots\right\} \\
&\qquad + 2r\left\{1 + \theta_1 r + (\theta_1 T_1 T_1')\frac{r^2}{2!} + (\theta_1 T_2 T_2')\frac{r^3}{3!} + \dots\right\}\biggr] \quad \dots\dots(128).
\end{aligned}$$

The further simplification of the last two terms is obvious.

12. *The determination of the constants of a whole distribution from those of a quadrant.*

The formulae developed in the preceding pages have many uses, but one of the most useful and interesting is their application to the inverse problem of deducing the constants of a complete normal solid from those of a given quadrant. Thus the records of a body of soldiers might show the age and height distribution, but if recruiting were limited to those over 5′ 8″ in height and over 18 years of age we should have the data of a quadrant only, bounded by a plane at 18 years of age and another at 5′ 8″. The problem is to determine from measurements of this quadrant the constants for the whole population.

If we express the moment coefficients about the general mean, with standard deviations as units, as m_{10}, m_{01}, m_{20}, etc. and write $\frac{N}{m} = \lambda$ we have from the preceding paper

$$\begin{aligned}
m_{10} &= -\lambda(HA + rKB), \\
m_{01} &= -\lambda(rHA + KB), \\
m_{20} &= -\lambda(hHA - r(1-r^2)\chi + r^2 kKB) + 1, \\
m_{02} &= -\lambda(r^2 hHA - r(1-r^2)\chi + kKB) + 1, \\
m_{11} &= -\lambda(rhHA - (1-r^2)\chi + rkKB) + r.
\end{aligned}$$

If now we denote the moments of the quadrant as measured from the bounding planes by q_{10}, q_{01}, q_{20}, etc., with the standard deviations S_x, S_y of the quadrant as units, it can easily be shown that

$$q_{20} S_x^2 = (m_{20} - 2hm_{10} + h^2)\, \sigma_x^2,$$

$$q_{02} S_y^2 = (m_{02} - 2km_{01} + k^2)\, \sigma_y^2,$$

$$q_{11} S_x S_y = (m_{11} - hm_{01} - km_{10} + hk)\, \sigma_x \sigma_y,$$

and

$$Q_{20} = \frac{q_{20}}{q_{10}^2} = \frac{m_{20} - 2hm_{10} + h^2}{(m_{10} - h)^2} = \frac{\lambda \{hHA + r(1 - r^2)\chi + r(2h - rk)KB\} + 1 + h^2}{\{\lambda(HA + rKB) + h\}^2},$$

$$Q_{02} = \frac{q_{02}}{q_{01}^2} = \frac{m_{02} - 2km_{01} + k^2}{(m_{01} - k)^2} = \frac{\lambda \{kKB + r(1 - r^2)\chi + r(2k - rh)HA\} + 1 + k^2}{\{\lambda(rHA + KB) + k\}^2},$$

$$Q_{11} = \frac{q_{11}}{q_{10}\, q_{01}} = \frac{m_{11} - hm_{01} - km_{10} + hk}{(m_{10} - h)(m_{01} - k)} = \frac{\lambda \{hKB + (1 - r^2)\chi + kHA\} + r + hk}{\{\lambda(HA + rKB) + h\}\{\lambda(rHA + KB) + k\}}.$$

Q_{20}, Q_{02}, and Q_{11} are known but so far as I can see at present the solution of the above equations would require the construction of four-entry tables in h, k, r, and λ.

If these were found, then

$$\sigma_x = \frac{q_{10} S_x}{-\lambda(HA + rKB) - h},$$

$$\sigma_y = \frac{q_{01} S_y}{-\lambda(rHA + KB) - k},$$

$$m = \frac{N}{\lambda}.$$

In conclusion my thanks are due to Professor Pearson for suggesting the subject of the above enquiry and the interest he has taken in its course, and again to Miss Alison Robertson for assistance in correcting the proofs.

MISCELLANEA.

Table of Ordinates of the Normal Curve for each Permille of Frequency.

EDITORIAL.

The need of a table such as the present has long been felt. A table of the abscissae of the normal curve for every permille of frequency was calculated by Sheppard at Galton's suggestion[*], but the corresponding table for the ordinates to five significant figures, which is equally needful, has not hitherto been provided.

When we have data classified in broad categories, whether we are dealing with one or two variates it is most desirable to exhibit the results in some graphical form in order that the lazy reader may have an appreciation of the matter under discussion. In the present state of our knowledge, whatever more or less justifiable criticisms may be raised, we do learn something by exhibiting the data with the marginal frequencies represented on the normal scale. The main answer to such criticisms is that such a representation is better than none at all. It will not give an association of the wrong sense, and when the correlation is at all sensible will for many practical purposes indicate its drift. The chief source of error lies, of course, when the number of categories is few, in treating the arrays of a second variable for a broad category of the first as if these arrays were themselves normal distributions. Even if the whole distribution were truly normal this would not be the case, but only the case approximately if the categories ceased to be 'broad,' and were replaced by small subranges. In ordinary practice the chief difficulty of the assumption arises from the fact that the centroid of the whole system for the second variate may not be in adequate numerical agreement with the centroid of the centroids of the arrays obtained on the normal assumption. It is true that if we deal with the whole system as normal and confine ourselves to a tripartite division of the arrays and marginal totals of the second variate we can by aid of the tetrachoric functions obtain better appreciations to the centroids of the arrays; not only does this involve a previous knowledge of the correlation coefficient, but the whole process, especially if it has to be applied to a large number of contingency tables, is very laborious. We are in using it in fact overlooking the main point that all we need is a rough diagrammatic exhibition of the general drift of the association. Those who have had occasion to plot large numbers of contingency tables to normal scales will be the first to admit the weaknesses of the method, but the last to assert that such diagrams are without value. They indicate quite effectively to the casual reader that here the association is of no practical importance, and that there it is an essential feature of the characters under investigation.

In reducing data given in broad categories to a normal scale, all abscissae (x) are measured in terms of the unknown standard deviation, all ordinates (z) in terms of unit frequency and unit standard deviation, i.e. $z=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

Now it is well known that the abscissa of the centroid $\bar{x}_{ss'}$ of a broad category lying between x_s and $x_{s'}$ is numerically given by

$$\bar{x}_{ss'}=(z_{s'}-z_s)/n_{ss'},$$

where $n_{ss'}$ represents the proportional frequency between x_s and $x_{s'}$. In most cases it will be, for diagram purposes, quite adequate to obtain the abscissae and ordinates of the normal scale from the proportional frequencies to three places of decimals. Hence Table I of *Tables for Statisticians* and the present table give the required values in a few minutes, while the old process of determining z from Table II or, where it permitted, Table III was much more lengthy.

Illustration. Find the boundaries and means of the following system of broad categories on the assumption that it corresponds to a normal frequency distribution[†].

---

[*] *Biometrika*, Vol. v. p. 405. It is reissued as Table I of the *Tables for Statisticians and Biometricians.*

[†] The continuous variate may be looked upon as physiological fitness for life.

Health of Yearling Male Babies.

		Frequency	Permille	Differences
I.	Very satisfactory	54	·038	·038
II.	Satisfactory	326	·265	·227
III.	Normal	508	·618	·353
IV.	Indifferent	129	·708	·090
V.	Unsatisfactory	198	·846	·138
VI.	Dead	221	1·000	·154
	Total	1436		

The permilles are obtained by a continuous process with the reciprocal of 1436 on the machine. The differences are obtained from the permilles by subtraction and only differ in one case, that of the Normal Health, which if found directly from the frequency would be ·354, and this is of no importance for our present purposes. Table I and the present Table provide at once:

	Abscissa	Ordinate	Difference of Ordinate	Centroid
	+∞	0		
I.			+·08265	+2·1750
	+1·7744	·08265		
II.			+·24489	+1·0788
	+0·6280	·32754		
III.			+·05382	+ ·1525
	+0·3002	·38136		
IV.			−·03795	− ·4217
	−0·5476	·34341		
V.			−·10614	− ·7691
	−1·0194	·23727		
VI.			−·23727	−1·5407
	−∞	0		

The whole work is very simple, as there is no trouble of interpolation at all.

In actually computing the present table of ordinates to permilles a different process was needful for the first eleven rows from that for the remainder of the rows. From ·110 to ·880 it was found adequate to take the z's from Table III of the *Tables for Statisticians* which are given for α for every ·01 of value—and therefore for every ·005 of frequency—and interpolate by Everett's Central-Difference Formula to every ·001 of frequency. It was also sufficient to use only the δ^2 terms. Such a method was not possible for the upper part of the table. Here up to ·02, the value of x was determined by inverse summation from the values in Table II of the *Tables for Statisticians*. Then z was actually computed from $z=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ for the columns under the rubrics ·000, ·005 and ·010. From these columns the remaining column values were determined using Everett's Central-Difference Formula and including δ^4 terms where requisite. The three rows at the top of the table were found by determining x by inverse summation as before and calculating each individual value of z from its x; the first fifth of the table was thus far more laborious than the last four-fifths. The values of z were ultimately cut down to five figures. A superscript + or − is attached to every value of z ending in 5 for the aid of those who wish to use the table to four figures only, which is often adequate.

We owe a careful revision of the table to Mr H. E. Soper, who used a much simplified process and modified the last figure in a number of cases. If m be the permille, i.e. $\frac{1}{2}(1+\alpha)$, x the abscissa and z the ordinate, he took the formula

$$\delta z = x\,\delta m - \frac{(\delta m)^2}{2z} + \frac{(\delta m)^3}{6z^2} - \dots,$$

which enabled him to start from the ½ of the nearest tabled m in Table II of the *Tables for Statisticians.*

Table of Ordinates of the Normal Curve for each Permille of Frequency.

Per-mille	*·000*	*·001*	*·002*	*·003*	*·004*	*·005*	*·006*	*·007*	*·008*	*·009*	*·010*	
·00	·00000	·00337	·00634	·00915−	·01185−	·01446	·01700	·01949	·02192	·02431	·02665+	*·99*
·01	·02665+	·02896	·03123	·03348	·03568	·03787	·04003	·04216	·04427	·04635+	·04842	*·98*
·02	·04842	·05046	·05249	·05449	·05648	·05845+	·06040	·06233	·06425+	·06615+	·06804	*·97*
·03	·06804	·06992	·07177	·07362	·07545+	·07727	·07908	·08087	·08265−	·08442	·08617	*·96*
·04	·08617	·08792	·08965+	·09137	·09309	·09479	·09648	·09816	·09983	·10149	·10314	*·95*
·05	·10314	·10478	·10641	·10803	·10964	·11124	·11284	·11442	·11600	·11756	·11912	*·94*
·06	·11912	·12067	·12222	·12375−	·12528	·12679	·12830	·12981	·13130	·13279	·13427	*·93*
·07	·13427	·13574	·13720	·13866	·14011	·14156	·14299	·14442	·14584	·14726	·14867	*·92*
·08	·14867	·15007	·15146	·15285+	·15423	·15561	·15698	·15834	·15970	·16105−	·16239	*·91*
·09	·16239	·16373	·16506	·16639	·16770	·16902	·17033	·17163	·17292	·17421	·17550	*·90*
·10	·17550	·17678	·17805−	·17932	·18058	·18184	·18309	·18433	·18557	·18681	·18804	*·89*
·11	·18804	·18926	·19048	·19169	·19290	·19410	·19530	·19649	·19768	·19886	·20004	*·88*
·12	·20004	·20121	·20238	·20354	·20470	·20585+	·20700	·20814	·20928	·21042	·21155−	*·87*
·13	·21155−	·21267	·21379	·21490	·21601	·21712	·21822	·21932	·22041	·22149	·22258	*·86*
·14	·22258	·22365+	·22473	·22580	·22686	·22792	·22898	·23003	·23108	·23212	·23316	*·85*
·15	·23316	·23419	·23522	·23625−	·23727	·23829	·23930	·24031	·24131	·24232	·24331	*·84*
·16	·24331	·24430	·24529	·24627	·24726	·24823	·24921	·25017	·25114	·25210	·25305+	*·83*
·17	·25305+	·25401	·25495+	·25590	·25684	·25778	·25871	·25964	·26056	·26148	·26240	*·82*
·18	·26240	·26331	·26422	·26513	·26603	·26693	·26782	·26871	·26960	·27049	·27137	*·81*
·19	·27137	·27224	·27311	·27398	·27485−	·27571	·27657	·27742	·27827	·27912	·27996	*·80*
·20	·27996	·28080	·28164	·28247	·28330	·28413	·28495−	·28577	·28658	·28739	·28820	*·79*
·21	·28820	·28901	·28981	·29060	·29140	·29219	·29298	·29376	·29454	·29532	·29609	*·78*
·22	·29609	·29686	·29763	·29840	·29916	·29991	·30067	·30142	·30216	·30291	·30365−	*·77*
·23	·30365−	·30439	·30512	·30585−	·30658	·30730	·30802	·30874	·30945+	·31017	·31087	*·76*
·24	·31087	·31158	·31228	·31298	·31367	·31436	·31505+	·31574	·31642	·31710	·31778	*·75*
·25	·31778	·31845−	·31912	·31979	·32045−	·32111	·32177	·32242	·32307	·32372	·32437	*·74*
·26	·32437	·32501	·32565−	·32628	·32691	·32754	·32817	·32879	·32941	·33003	·33065−	*·73*
·27	·33065−	·33126	·33187	·33247	·33307	·33367	·33427	·33486	·33545−	·33604	·33662	*·72*
·28	·33662	·33720	·33778	·33836	·33893	·33950	·34007	·34063	·34119	·34175−	·34230	*·71*
·29	·34230	·34286	·34341	·34395+	·34449	·34503	·34557	·34611	·34664	·34717	·34769	*·70*
·30	·34769	·34822	·34874	·34925+	·34977	·35028	·35079	·35129	·35180	·35230	·35279	*·69*
·31	·35279	·35329	·35378	·35427	·35475+	·35524	·35572	·35620	·35667	·35714	·35761	*·68*
·32	·35761	·35808	·35854	·35900	·35946	·35991	·36037	·36082	·36126	·36171	·36215−	*·67*
·33	·36215−	·36259	·36302	·36346	·36389	·36431	·36474	·36516	·36558	·36600	·36641	*·66*
·34	·36641	·36682	·36723	·36764	·36804	·36844	·36884	·36923	·36962	·37001	·37040	*·65*
·35	·37040	·37078	·37116	·37154	·37192	·37229	·37266	·37303	·37340	·37376	·37412	*·64*
·36	·37412	·37447	·37483	·37518	·37553	·37588	·37622	·37656	·37690	·37724	·37757	*·63*
·37	·37757	·37790	·37823	·37855+	·37888	·37920	·37951	·37983	·38014	·38045−	·38076	*·62*
·38	·38076	·38106	·38136	·38166	·38196	·38225+	·38254	·38283	·38312	·38340	·38368	*·61*
·39	·38368	·38396	·38423	·38451	·38478	·38504	·38531	·38557	·38583	·38609	·38634	*·60*
·40	·38634	·38659	·38684	·38709	·38734	·38758	·38782	·38805+	·38829	·38852	·38875−	*·59*
·41	·38875−	·38897	·38920	·38942	·38964	·38986	·39007	·39028	·39048	·39069	·39089	*·58*
·42	·39089	·39109	·39129	·39149	·39168	·39187	·39206	·39224	·39243	·39261	·39279	*·57*
·43	·39279	·39296	·39313	·39330	·39347	·39364	·39380	·39396	·39411	·39427	·39442	*·56*
·44	·39442	·39457	·39472	·39486	·39501	·39514	·39528	·39542	·39555−	·39568	·39580	*·55*
·45	·39580	·39593	·39605+	·39617	·39629	·39640	·39651	·39662	·39673	·39683	·39694	*·54*
·46	·39694	·39703	·39713	·39723	·39732	·39741	·39749	·39758	·39766	·39774	·39781	*·53*
·47	·39781	·39789	·39796	·39803	·39809	·39816	·39822	·39828	·39834	·39839	·39844	*·52*
·48	·39844	·39849	·39854	·39858	·39862	·39866	·39870	·39873	·39876	·39879	·39882	*·51*
·49	·39882	·39884	·39886	·39888	·39890	·39891	·39892	·39893	·39894	·39894	·39894	*·50*
	·010	*·009*	*·008*	*·007*	*·006*	*·005*	*·004*	*·003*	*·002*	*·001*	*·000*	Per-mille

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