760

ae1

MATHEMATICAL WORKS

PUBLISHED BY

MACMILLAN AND CO.

A Treatise on Solid Geometry. By the Rev. PERCIVAL FROST, M.A., and the Rev. J. WOLSTENHOLME, M.A. 8vo. 18*s*.

A Collection of Mathematical Problems and Examples. By H. A. MORGAN, M.A. Crown 8vo. 6*s*. 6*d*.

Cambridge Course of Elementary Natural Philosophy, for the Degree of B.A. Originally compiled by J. C. SNOWBALL, M.A. Fifth Edition, revised and enlarged, and adapted for the Middle Class Examinations, by THOMAS LUND, B.D. Crown 8vo. 5*s*.

A Collection of Elementary Test-Questions in Pure and Mixed Mathematics, with Answers and Appendices on Synthetic Division, and on the Solution of Numerical Equations by Horner's Method. By JAMES R. CHRISTIE, F.R.S. Crown 8vo. 8*s*. 6*d*.

The First Three Sections of Newton's Principia. With Notes and Problems. By PERCIVAL FROST, M.A. Second Edition. 8vo. 10*s*. 6*d*.

Dynamics of a System of Rigid Bodies. With Examples. By EDWARD JOHN ROUTH. Crown 8vo. 10*s*. 6*d*.

Elementary Hydrostatics. By J. B. PHEAR, M.A. Fourth Edition. Crown 8vo. 5*s*. 6*d*.

A BOOK

OF

MATHEMATICAL PROBLEMS.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

Alfred Baker
Toronto Ap. 2[illegible]
187[illegible]

A BOOK

OF

MATHEMATICAL PROBLEMS

ON SUBJECTS INCLUDED IN

THE CAMBRIDGE COURSE

DEVISED AND ARRANGED BY

JOSEPH WOLSTENHOLME,

FELLOW OF CHRIST'S COLLEGE; SOMETIME FELLOW OF ST JOHN'S COLLEGE; AND LATELY LECTURER IN MATHEMATICS AT CHRIST'S COLLEGE.

"Deduct but what is Vanity or Dress,
"Or Learning's Luxury, or Idleness;
"Or tricks to shew the stretch of human brain,
"Mere curious pleasure, or ingenious pain;

. . . .

"Then see how little the remaining sum."

Pope. Essay on Man
Epistle II.

London and Cambridge:
MACMILLAN AND CO.
1867.

PREFACE.

This "Book of Mathematical Problems" consists, mainly, of questions either proposed by myself at various University and College Examinations during the past fourteen years, or communicated to my friends for that purpose. It contains also a certain number, (between three and four hundred), which, as I have been in the habit of devoting considerable time to the manufacture of problems, have accumulated on my hands in that period. In each subject I have followed the order of the Text-books in general use in the University of Cambridge; and I have endeavoured also, to some extent, to arrange the questions in order of difficulty.

I had not sufficient boldness to seek to impose on any of my friends the task of verifying my results, and have had therefore to trust to my own resources. I have however done my best, by solving anew every question from the proof sheets, to ensure that few serious errors shall be discovered. I shall be much obliged to any one who will give me infoımation as to those which still remain.

I have, in some cases, where I thought I had anything serviceable to communicate, prefixed to certain classes of problems fragmentary notes on the mathematical subjects to which they relate. These are few in number, and I hope will be found not altogether superfluous.

This collection will be found to be unusually copious in problems in the earlier subjects, by which I designed to make it useful to mathematical students, not only in the Universities, but in the higher classes of public schools.

I have to express my best thanks to Mr R. Morton, Fellow of Christ's College, for his great kindness in reading over the proof sheets of this work, and correcting such errors as were thereby discoverable.

JOSEPH WOLSTENHOLME.

WASTDALE HEAD, *July* 31.

CONTENTS.

PLANE TRIGONOMETRY.

CONIC SECTIONS, GEOMETRICAL.

CONIC SECTIONS, ANALYTICAL.

SOLID GEOMETRY.

STATICS.

DYNAMICS, ELEMENTARY.

DYNAMICS OF A POINT.

DYNAMICS OF A RIGID BODY.

ERRATA.

Page	Problem	
13	78	(17) *for* $-$ *read* $+$.
15	80	*for* $a_1 + b_2$ *read* $a_1 + a_2$.
26	133	*for* positive *read* real.
31	167	in the last line *for* a_1 *read* a_0, and *for* a_2 *read* a_1.
33	176	(3) *for* $\lfloor\underline{n} - n - 2^n$ *read* $\lfloor\underline{n} + n - 2^n$.
39	188	in the first member of the equation *for* $r^{2^{n+1}}$ *read* $r^{3.2^{n-1}}$, and in the second member *for* $r^{2^{n+2}}$ *read* $r^{3.2^n}$.
—	191	*for* $(m+n+1)(m+n-1)$ *read* $(m+n-1)(m+n-2)$.
40	192	(2) *for* $\lfloor\underline{3}$ *read* $\lfloor\underline{2}$, and *for* $\lfloor\underline{5}$ *read* $\lfloor\underline{4}$.
41	194	(6) *for* integral part of *read* integer next greater than.
44	206	(1) *for* $\frac{1}{c+}\frac{1}{b+}\frac{1}{a+}\frac{1}{c+}\dots$ *read* $\frac{1}{b+}\frac{1}{a+}\frac{1}{c+}\frac{1}{b+}\dots$.
45	211	*for* is a_1 *read* is $\frac{a_1}{1-a_1}$.
46	212	(5) *for* 6 *read* 3.
...	...	(6) *for* $(2n-1)x$ *read* $(2n-1)x^2$, and *for* $(-x)^n$ *read* $(-1)^n x^{2n+1}$.
47	213	(8) *for* $(a^2-1)^2$ *read* $(a^2-1)^n$.
53	245	*for* $(c+d)^2$ *read* $(c+d)^2 - c^2$.
62	284	*for* $\frac{b+c}{a-d}$ *read* $\frac{a+d}{c-b}$, and *for* $\frac{d-a}{c+b}$ *read* $\frac{b-c}{a+d}$.
...	290	*for* $a-x$ *read* $a+x$.
65	301	in the result *for* sin *read* cos *throughout*.
...	304	*for* $\alpha + \frac{\theta}{2}$, $\beta + \frac{\theta}{2}$, *read* $\frac{\alpha+\theta}{2}$, $\frac{\beta+\theta}{2}$.
73	346	in the last result *for* h *read* $2h$.
75	352	*for* $+\alpha'\beta'\gamma'$ *read* $-\alpha'\beta'\gamma'$.
154	line 20	*for* $\sin\beta$ *read* $\sin\theta$.
248	... 2	*for* xz *read* nz.

BOOK OF MATHEMATICAL PROBLEMS.

GEOMETRY.

1. A POINT O is taken within a polygon $ABC...KL$: prove that OA, $OB...OL$ are together greater than half the perimeter of the polygon.

2. Two triangles are on the same base and between the same parallels; through the point of intersection of their sides is drawn a straight line parallel to the base and terminated by the sides which do not intersect: prove that the segments of this straight line are equal.

3. The sides AB, AC of a triangle are bisected in D, E, and CD, BE intersect in F: prove that the triangle BFC is equal to the quadrilateral $ADFE$.

4. AB, CD are two parallel straight lines, E the middle point of CD, and F, G the respective points of intersection of AC, BE and of AE, BD: prove that FG is parallel to AB.

5. In any quadrilateral the squares on the sides together exceed the squares on the diagonals by the square on twice the line joining the middle points of the diagonals.

6. If a straight line be divided in extreme and mean ratio and produced so that the part produced is equal to the smaller of the segments, the rectangle contained by the whole line thus pro-

duced and the part produced together with the square on the given line, will be equal to four times the square on the greater part.

7. AB is the diameter of a circle, P a point on the circle, PM perpendicular on AB, on AM, MB, as diameters are described two circles meeting AP, BP in Q, R respectively: prove that QR touches both circles.

8. Given two straight lines in position and a point equidistant from them: prove that any circle passing through the given point and the point of intersection of the given lines will intercept on the given lines segments whose sum, or whose difference, is equal to a given length.

9. A triangle circumscribes a circle and from each point of contact is drawn a perpendicular to the line joining the other two: prove that the lines joining the feet of these perpendiculars are respectively parallel to the sides of the original triangle.

10. On a straight line AB and on the same side of it are described two segments of circles, AP, AQ are chords of the two segments including an angle equal to that between the straight lines touching the two circles at A : prove that P, Q, B are in one straight line.

11. The centre A of a circle lies on another circle which cuts the former in B, C; AD is a chord of the latter circle meeting BC in E, and from D are drawn DF, DG to touch the former circle: prove that G, E, F lie on one straight line.

12. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet in P, Q, and if about two of the triangles so formed circles be described meeting again in R: P, R, Q will lie on one straight line.

13. Two circles intersect in A, and through A any two straight lines BAC, $B'AC'$ are drawn terminated by the two circles: prove that the chords BB', CC' of the two circles are inclined at a constant angle.

14. Four points are taken on a circle and the three pairs of straight lines which can be drawn through the four points intersect respectively in E, F, G: prove that the three pairs of straight lines which bisect the angles at E, F, G respectively will be in the same directions.

15. Through a point of intersection of two circles is drawn a straight line at right angles to their common chord and terminated by the circles, and through the other point is drawn a straight line equally inclined to the straight lines joining that point to the extremities of the former straight line: prove that the tangents to the two circles at the points on this latter straight line will intersect in a point on the common chord.

16. Two circles cut each other at A, and a straight line BAC is drawn terminated by the circles; with B, C as centres are described two circles each cutting at right angles one of the former circles: prove that these two circles and the circles of which BC is a diameter will have a common chord.

17. Circles are described on two of the sides of a triangle as diameters, and each meets the perpendicular from the opposite angular point on its diameter in two points: prove that these four points lie on a circle.

18. The tangents from a point O to a circle are bisected by a straight line which meets a chord PQ of the circle in R: prove that the angles ROP, OQR are equal.

19. A straight line PQ of given length is intercepted between two straight lines OP, OQ given in position; through P, Q are drawn straight lines in given directions, intersecting in a point R, and the angles POQ, PRQ are equal and on the same side of PQ: prove that R lies on a fixed circle.

20. From the point of intersection of the diagonals of a quadrilateral inscribed in a circle perpendiculars are let fall on the sides: prove that the sum of two opposite angles of the quadrilateral formed by joining the feet of these perpendiculars is double of one of the angles between the diagonals of the former.

21. If a circle touch each of two other circles the indefinite straight line passing through the points of contact will cut off similar segments from the two circles.

22. Two circles have internal contact at A, a straight line touches one circle at P and cuts the other in Q, Q': prove that QP, PQ' subtend equal angles at A.

If the contact be external, PA bisects the external angle between QA, $Q'A$.

23. A straight line touches one of two fixed circles which do not intersect in P, and cuts the other in Q, Q': prove that there are two fixed points at either of which PQ, PQ' subtend angles equal or supplementary.

24. At two fixed points A, B are drawn AC, BD at right angles to AB and on the same side of it, and of such magnitude that the rectangle AC, BD is equal to the square on AB: prove that the circles whose diameters are AC, BD will touch each other. Prove also that the point of contact lies on a fixed circle.

25. A triangle is inscribed in one of two concentric circles, and from any point on the other circle perpendiculars are let fall on the sides of the triangle: prove that the area of the triangle formed by joining the feet of these perpendiculars is independent of the position of the point.

26. ABC is an isosceles triangle right angled at C, and the parallelogram $ABCD$ is completed; with centre D and distance DC a circle is described: prove that if P be any point on this circle, the squares on PA, PC are together equal to the square on PB.

27. A circle is described about a triangle ABC, and the tangents to the circle at B, C meet in A'; through A' is drawn a straight line meeting AC, AB in the points B', C': prove that BB', CC' will intersect on the circle.

28. If D be the middle point of the side BC of a triangle ABC and the tangents at B, C to the circumscribed circle meet in A', the angles BAA', CAD will be equal.

29. The side BC of a triangle ABC is bisected in D, and on DA is taken a point P such that the rectangle DP, DA is equal to the rectangle DB, DC: prove that the angles BPC, BAC are together equal to two right angles.

30. If the circle inscribed in a triangle ABC touch BC in D, the circles inscribed in the triangles ABD, ACD will touch each other.

31. Given the base and the vertical angle of a triangle, prove that the centres of the four circles which touch the sides of the triangle, will lie on two fixed circles passing through the extremities of the base.

32. A circle is drawn through B, C and the centre of perpendiculars of a triangle ABC, D is the middle point of BC and AD is produced to meet the circle in E: prove that AE is bisected in D.

33. The straight lines joining the centres of the four circles which touch the sides of a triangle are bisected by the circumscribed circle: also the middle point of the line joining any two of the centres and that of the lines joining the other two are the extremities of a diameter of the circumscribed circle.

34. With three given points not lying in one straight line as centres, describe three circles which shall have three common tangents.

35. From the angular points of a triangle straight lines are drawn perpendicular to the opposite sides and terminated by the circumscribed circle: prove that the parts of these lines intercepted between their point of intersection and the circle are bisected by the corresponding sides respectively.

36. The radii from the centre of the circumscribed circle of a triangle to the angular points are respectively perpendicular to the lines joining the feet of the perpendiculars.

37. Three circles are described each passing through the centre of perpendiculars of a given triangle and through two of the angular points: prove that their centres are the angular points of a triangle equal in all respects to the given triangle and similarly situated: and that the relation between the two triangles is reciprocal.

38. If the centres of two of the circles which touch the sides of a triangle be joined, and also the centres of the other two, the squares on the joining lines are together equal to the square on the diameter of the circumscribed circle.

39. The centre of perpendiculars of a triangle is joined to the middle point of a side, and the joining line produced to meet the circumscribed circle: prove that it will meet it in the same point as the diameter through the angular point opposite to the bisected side.

40. From any point of a circle two chords are drawn touching another circle whose centre is on the circumference of the former: prove that the line joining the extremities of these chords is fixed in direction.

41. ABC is a triangle and O the centre of its circumscribed circle; $A'B'C'$ another triangle whose sides are respectively parallel to OA, OB, OC. If through A', B', C' be drawn straight lines respectively parallel to the sides of the former triangle, they will intersect in a point which is the centre of one of the circles touching the sides of the triangle $A'B'C'$.

42. A triangle is drawn having its sides parallel to the lines joining the angular points of a given triangle to the middle points of the opposite sides: prove that the relation between the two triangles is reciprocal.

43. Two triangles are so related that straight lines drawn through the angular points of one parallel respectively to the sides

of the other meet in a point: prove that straight lines drawn at the corresponding angular points of the second parallel to the sides of the first will meet in a point; and that each triangle will be divided into three triangles, which are each to each in the same ratio.

44. In any triangle ABC, O, O' are the centres of the inscribed circle, and of the escribed circle opposite A; OO' meets BC in D, any straight line through D meets AB, AC respectively in b, c, Ob, $O'c$ intersect in P, Oc, $O'b$ in Q: prove that P, A, Q lie on one straight line perpendicular to OO'.

45. The centre of the circumscribed circle of a triangle, and the centre of perpendiculars are joined: prove that the joining line is divided into segments in the ratio of 1 : 2 by each of the straight lines joining the angular points to the middle points of the opposite sides.

46. Inscribe a parallelogram in a given triangle so that its diagonals may intersect at a given point within the triangle.

47. The side BC of a triangle ABC is bisected in D, a straight line parallel to BC meeting AB, AC produced in P, P' respectively is divided in Q so that PQ, BD, QP' are in continued proportion, and through Q is drawn a straight line RQR' terminated by AB, AC and bisected in Q: prove that the triangles ABC, ARR' are equal.

48. On AB, AC two sides of a triangle are taken two points D, E; AB, AC are produced to F, G so that BF is equal to AD and CG to AE; BG, CF, FG are joined the two former meeting in H: prove that the triangle FHG is equal to the two triangles BHC, ADE together.

49. If two sides of a triangle be given in position and their sum be also given, and if the third side be divided in a given ratio, the point of division will lie on one of two fixed straight lines.

50. Two circles intersect in A, B, any straight line through A meets the circles in P, Q: prove that BP has to BQ a constant ratio.

51. Through the centre of perpendiculars of a triangle is drawn a straight line at right angles to the plane of the triangle: prove that any tetrahedron of which the triangle is one face, and whose opposite vertex lies on this line will be such, that through any edge can be drawn a plane perpendicular to the opposite edge.

52. $ABCD$ are four points not in one plane, and AB, AC respectively lie in planes perpendicular to CD, BD: prove that AD lies in a plane perpendicular to BC; and that the middle points of the six edges lie on a sphere, which will also pass through the feet of the shortest distances between the opposite edges.

53. Each edge of a tetrahedron is equal to the opposite edge: prove that the straight line joining the middle points of two opposite edges is at right angles to both.

54. If from any point O be let fall perpendiculars Oa, Ob, Oc, Od on the faces of a tetrahedron $ABCD$, the perpendiculars from A, B, C, D on the corresponding faces of the tetrahedron $abcd$ meet in a point O'; and the relation between O and O' is reciprocal.

55. A solid angle is contained by three plane angles: prove that any straight line through the containing point makes with the edges angles whose sum is greater than half the sum of the containing angles.

56. The circles described on the diagonals of a complete quadrilateral as diameters cut orthogonally the circle circumscribing the triangle formed by the diagonals.

57. Four points are taken on the circumference of a circle, and through them are drawn three pairs of straight lines each intersecting in a point: prove that the straight line joining any one

of these points to the centre of the circle will be perpendicular to the line joining the other two.

58. A sphere is described touching three given spheres: prove that the plane passing through the points of contact, contains one of four fixed straight lines.

59. Four straight lines are given in position: prove that an infinite number of systems of three circles can be found, such that the points of intersection of the four straight lines are the centres of similarity of the circles taken two and two.

60. In two fixed circles are drawn two parallel chords PP', QQ'; PQ, $P'Q'$ are joined meeting the circles again in R, S; R', S', respectively: prove that the points of intersection of QQ', RR' and of PP', SS' lie on a fixed straight line.

61. The six radical axes of the four circles (taken two and two) which touch the sides of a triangle are the straight lines bisecting internally and externally the angles of a triangle formed by joining the middle points of the sides of the former triangle.

62. If two circles have four common tangents, the circles described on these common tangents as diameters will have a common radical axis.

63. Four points are taken on a circle, and from the middle point of the chord joining any two a straight line is drawn perpendicular to the chord joining the other two: prove that the six lines so drawn will meet in a point.

64. Given in position two sides of a triangle, including an angle equal to that of an equilateral triangle, and given the length of the third side: prove that the centre of the Nine Points' Circle of the triangle lies on a fixed straight line.

65. Given in position two sides of a triangle, and given the sum of those sides, the centre of the Nine Points' Circle lies on a fixed straight line.

66. The perpendiculars let fall from the centres of the escribed circles of a triangle on the corresponding sides will meet in a point.

67. A point O is taken within a triangle ABC, and through A, B, C are drawn straight lines parallel to those bisecting the angles BOC, COA, AOB: prove that these lines will meet in a point.

68. ABC is a triangle, AA', BB', CC' are drawn through a point to meet the opposite sides: prove that the straight lines drawn through A, B, C to bisect $B'C'$, $C'A'$, $A'B'$ will meet in a point.

69. If two circles lie entirely without each other, and any straight line meet them in P, P'; Q, Q' respectively, there are two points O such that the straight lines bisecting the angles POP', QOQ' shall be always at right angles to each other.

70. Given two circles which do not intersect, a tangent to one at any point P meets the polar of P with respect to the other in P': prove that the circle on PP' as diameter will pass through two fixed points.

71. A point has the same polar with respect to each of two circles, prove that any common tangent will subtend a right angle at that point.

72. Given two points A, B; if any straight line PAQ be drawn through A so that the angle PBQ is equal to a given angle, and that BP has to BQ a given ratio, P, Q will lie on two fixed circles which pass through A and B.

73. If O be a fixed point, P any point on a fixed circle, and the rectangle be constructed of which OP is a side and the tangent at P a diagonal, the angular point opposite O will lie on the polar of O.

ALGEBRA.

I. *Highest Common Divisor.*

74. Reduce to their lowest terms the fractions

(α) $\dfrac{11x^4 + 24x^3 + 125}{x^4 + 24x + 55}$, (β) $\dfrac{2x^5 - 11x^2 - 9}{4x^5 + 11x^4 + 81}$,

(γ) $\dfrac{x^5 + 11x^3 - 54}{x^5 + 11x + 12}$, (δ) $\dfrac{x^5 - 209x + 56}{56x^5 - 209x^4 + 1}$,

(ε) $\dfrac{8x^7 - 377x^3 + 21}{21x^7 - 377x^4 + 8}$, (ζ) $\dfrac{16x^9 - x^5 + 16x^4 + 32}{32x^9 + 16x^5 - x^4 + 16}$,

(η) $\dfrac{x^8 + 2x^6 + 3x^4 - 2x^2 + 1}{6x^8 + x^7 + 17x^5 - 7x^3 - 2}$,

(θ) $$\frac{1 + x^2}{a + 2bx + cx^2} + \frac{(b + cx)^2 + (a + bx)^2}{a(b + cx)^2 - 2b(a + bx)(b + cx) + c(a + bx)^2}.$$

75. Simplify the expressions

(α) $$\frac{x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2) - 4xyz}{x + y + z - xyz},$$

(β) $$\frac{a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)}{a^2(b - c) + b^2(c - a) + c^2(a - b)},$$

(γ) $$\frac{1}{(a - b)(a - c)(a - d)} + \frac{1}{(b - c)(b - d)(b - a)} + \frac{1}{(c - d)(c - a)(c - b)} + \frac{1}{(d - a)(d - b)(d - c)},$$

(δ) $$\frac{bcd}{(a - b)(a - c)(a - d)} + \frac{cda}{(b - c)(b - d)(b - a)} + \frac{dab}{(c - d)(c - a)(c - b)} + \frac{abc}{(d - a)(d - b)(d - c)},$$

(ε) $$a^2\frac{(a + b)(a + c)}{(a - b)(a - c)} + b^2\frac{(b + c)(b + a)}{(b - c)(b - a)} + c^2\frac{(c + a)(c + b)}{(c - a)(c - b)}.$$

76. Prove that

$$\frac{(ab+cd)(a^2+b^2-c^2-d^2)-(ac+bd)(a^2-b^2+c^2-d^2)}{(a^2+b^2-c^2-d^2)(a^2-b^2+c^2-d^2)+4(ab+cd)(ac+bd)}$$

$$\equiv \frac{(b-c)(a+d)}{(a+d)^2-(b-c)^2}.$$

77. Prove that

$$\frac{(y-z)(1+y^2)(1+z^2)+(z-x)(1+z^2)(1+x^2)+(x-y)(1+x^2)(1+y^2)}{x(y-z)(1+y^2)(1+z^2)+y(z-x)(1+z^2)(1+x^2)+z(x-y)(1+x^2)(1+y^2)}$$

$$\equiv \frac{1-yz-zx-xy}{x+y+z-xyz}.$$

II. *Equations.*

78. Solve the equations

(1) $(x+1)(x+2)(x+3)=(x+4)(x+5)(x-3)$,

(2) $(x+1)(x+2)(x+3)=(x-1)(x-2)(x-3)$
$+3(4x-1)(x+1)$,

(3) $(x+a)(x+a+b)=(x+b)(x+3a)$,

(4) $\dfrac{1}{x+1}+\dfrac{7}{x+5}=\dfrac{5}{x+3}+\dfrac{3}{x+7}$,

(5) $\dfrac{1}{x+6a}+\dfrac{2}{x-3a}+\dfrac{3}{x+2a}=\dfrac{6}{x+a}$,

(6) $\dfrac{x}{x+b-a}+\dfrac{b}{x+b-c}=1$,

(7) $\dfrac{a}{x+b+c}+\dfrac{b}{x+a-c}=\dfrac{a-c}{x+b}+\dfrac{b+c}{x+a}$,

(8) $\dfrac{(a-b)^2}{(a-b)x+\alpha-\beta}+\dfrac{(b-c)^2}{(b-c)x+\beta-\gamma}+\dfrac{(c-d)^2}{(c-d)x+\gamma-\delta}$

$+\dfrac{(d-a)^2}{(d-a)x+\delta-\alpha}=0$,

(9) $\dfrac{x^2+3}{x-1}+\dfrac{x^2-x+1}{x-2}=2\dfrac{x^2-2x+1}{x-3}$,

(10) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}=\dfrac{1}{a+b+x}$,

(11) $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2-3x+1}{x-3}=2x-\dfrac{1}{4x-8}$,

(12) $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{11x+18}{11x-18}$,

(13) $a^2\dfrac{x-b}{a-b}+b^2\dfrac{x-a}{b-a}=x^2$,

(14) $(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10)$,

(15) $(a+x)^{\frac{1}{3}}+(b+x)^{\frac{1}{3}}=(a-b)^{\frac{1}{3}}$,

(16) $$\frac{(x+a)(x+b)}{(x-a)(x-b)}+\frac{(x-a)(x-b)}{(x+a)(x+b)}$$
$$=\frac{(x+c)(x+d)}{(x-c)(x-d)}+\frac{(x-c)(x-d)}{(x+c)(x+d)},$$

(17) $\dfrac{5}{x+2}+\dfrac{5}{x+4}=\dfrac{1}{x+1}-\dfrac{8}{x+3}+\dfrac{1}{x+5}$,

(18) $x^6+1+(x+1)^6=2(x^2+x+1)^3$,

(19) $\dfrac{1}{10x-50}-\dfrac{2}{x-6}+\dfrac{9}{x-7}-\dfrac{14}{x-8}+\dfrac{7}{x-9}=0$,

(20) $13x^2=10\dfrac{8x-15}{x^2-4x+5}-12\dfrac{11x-15}{x^2-6x+6}$,

(21) $\dfrac{1}{x-1}-\dfrac{4}{x-2}+\dfrac{4}{x-3}-\dfrac{1}{x-4}=\dfrac{1}{30}$,

(22) $x^8+1+(x+1)^8=2(x^2+x+1)^4$,

(23) $x^{10}+1+(x+1)^{10}=2(x^2+x+1)^5+15x^2(x^2+x+1)^3$,

(24) $16x(x+1)(x+2)(x+3)=9$,

(25) $x^4+2x^3-11x^2+4x+4=0$,

(26) $\dfrac{4}{x^2-2x}=\dfrac{2}{x^2-x}+x^2-x,$

(27) $\dfrac{5}{x^2-7x+10}+\dfrac{5}{x^2-13x+40}=x^2-10x+19,$

(28) $\dfrac{40}{x^2+2x-48}-\dfrac{20}{x^2+9x+8}+\dfrac{8}{x^2+10x}-\dfrac{12}{x^2+5x-50}+1=0,$

(29) $\dfrac{1}{x(x-1)}+\dfrac{2}{(x-1)(x-3)}-\dfrac{6}{(x-1)(x+2)}$
$+\dfrac{8}{(x-2)(x+2)}+1=0,$

(30) $(x^2+1)^2=4(2x-1),$

(31) $\dfrac{2x}{3}=\dfrac{7}{x^2+3x-7}-\dfrac{3}{x^2+x-3},$

(32) $8x^2=\dfrac{4x+69}{x^2+2x+3}-\dfrac{9x+23}{x^2+x+1},$

(33) $x^3=6x+6,$

(34) $4x^3=6x+3,$

(35) $x^3+6x^2=36,$

(36) $xy(x+y)=12x+3y,\quad xy(4x+y-xy)=12(x+y-3),$

(37) $x\sqrt{1-y^2}-y\sqrt{1-x^2}=xy-\sqrt{1-x^2}\sqrt{1-y^2}=\dfrac{1}{2},$

(38) $xyz=a(y^2+z^2)=b(z^2+x^2)=c(x^2+y^2),$

(39) $cy+bz=\dfrac{1}{a}-\dfrac{1}{x},\quad az+cx=\dfrac{1}{b}-\dfrac{1}{y},\quad bx+ay=\dfrac{1}{c}-\dfrac{1}{z},$

(40) $x+\dfrac{b^3-c^3}{yz}=y+\dfrac{c^3-a^3}{zx}=z+\dfrac{a^3-b^3}{xy}=\sqrt[3]{xyz},$

(41) $\left.\begin{array}{l} x+y+z=mxyz, \\ yz+zx+xy=n, \\ (1+x^2)(1+y^2)(1+z^2)=(1-n)^2, \end{array}\right\}$

(42) $2x+\dfrac{y^2}{z}=2y+\dfrac{z^2}{x}=2z+\dfrac{x^2}{y}=a.$

III. *Theory of Quadratic Equations.*

79. In the equation

$$\frac{a}{x-md}+\frac{b}{x-mc}+\frac{c}{x+mb}+\frac{d}{x+ma}=0,$$

prove that, if $a+b+c+d=0$, the only finite value of x will be

$$\frac{m(ac+bd)}{a+b}.$$

80. In the equation

$$\frac{a_1}{x+b_1}+\frac{a_2}{x+b_2}+\frac{a_3}{x+b_3}+\frac{a_4}{x+b_4}=0,$$

prove that, if

$$a_1+b_2+a_3+a_4=0, \text{ and } a_1b_1+a_2b_2+a_3b_3+a_4b_4=0,$$

the only finite value of x will be

$$\frac{a_1b_1^3+a_2b_2^3+a_3b_3^3+a_4b_4^3}{a_1b_1^2+a_2b_2^2+a_3b_3^2+a_4b_4^2}-(b_1+b_2+b_3+b_4).$$

81. Find limits to the real values of x and y which can satisfy the equation

$$x^2+12xy+4y^2+4x+8y+20=0.$$

82. If the roots of the equation

$$ax^2+2bx+c=0$$

be possible and different, the roots of the equation

$$(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$$

will be impossible, and *vice versâ*.

83. Prove that the equations

$$x+y+z=a+b+c,$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

$$\frac{x}{a^3}+\frac{y}{b^3}+\frac{z}{c^3}=0,$$

are equivalent only to two independent equations, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

84. Obtain the several equations for determining α, β, γ so that the equations

$$x^4 + 2px^3 + qx^2 + rx + s = 0, \quad (x^2 + px + \alpha)^2 = (\beta x + \gamma)^2$$

may coincide: and in this manner solve the equation

$$(x^2 + 3x - 6)^2 + 3x^2 = 72.$$

85. The roots of the equation

$$(x + a - c)(x + b + c)(x + a - d)(x + b + d) = e$$

will all be real, if

$$16e < (a - b - 2c)^2 (a - b - 2d)^2 > -4(c - d)^2 (b + c + d - a)^2.$$

86. If x_1, x_2 be the roots of the equation

$$\frac{a}{(1-m)(1-x)} + b + \frac{c}{mx} = 0,$$

then will $$\frac{a}{(1-x_1)(1-x_2)} + b + \frac{c}{x_1 x_2} = 0.$$

87. If y, z be the roots of the equation

$$\frac{(1-m^2)(1-x^2)}{b-c} + \frac{4mx}{c-a} = \frac{(1+m^2)(1+x^2)}{a-b},$$

then will $$\frac{(1-y^2)(1-z^2)}{b-c} + \frac{4yz}{c-a} = \frac{(1+y^2)(1+z^2)}{a-b}.$$

88. Prove that the equation

$$x^3 + ax^2 + bx + \frac{b^2}{3a} = 0$$

can be solved directly, and that the complete cubic

$$x^3 - px^2 + qx - r = 0$$

may be reduced to this form by the substitution $x \equiv y + h$.

Prove that the roots of the auxiliary quadratic are

$$\frac{\alpha(\beta-\gamma)^2 + \beta(\gamma-\alpha)^2 + \gamma(\alpha-\beta)^2 \pm \sqrt{(-3)}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2},$$

α, β, γ being the roots of the cubic.

IV. *Theory of Divisors.*

89. Determine the condition necessary in order that

$$x^2 + px + q, \text{ and } x^2 + p'x + q'$$

may have a common divisor of the form $x + c$, and prove that such a divisor will also be a divisor of $px^2 + (q - p')x - q'$.

90. The expression

$$x^6 + 3ax^5 + 3bx^4 + cx^3 + 3dx^2 + 3ex + f$$

will be a complete cube if

$$\sqrt[3]{f} = \sqrt{\frac{e}{a}} = \frac{d}{b} = \frac{c - a^3}{6a} = b - a^2.$$

91. The expression $x^5 - bx^3 + cx^2 + dx - e$ will be the product of a complete square and a complete cube, if

$$\frac{12b}{5} = \frac{9d}{b} = \frac{5e}{c} = \frac{d^2}{c^2}.$$

92. Prove that $ax^3 + bx + c$, and $a + bx^2 + cx^3$ will have a common quadratic factor, if

$$b^2c^2 = (c^2 - a^2 + b^2)(c^2 - a^2 + ab).$$

93. The expressions

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4, \text{ and } a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

will have a common quadratic factor, if

$$(a_2 - a_4 - a_0)(a_4 - a_0)^2 + (a_3 - a_1)(a_3a_0 - a_1a_4) = 0.$$

94. The expression $x^3 + px^2 + qx + r$ will be divisible by $x^2 + ax + b$, if

$$a^3 - 2pa^2 + (p^2 + q)a + r - pq = 0, \text{ and } b^3 - qb^2 + rpb - r^2 = 0.$$

95. Prove that $x^4 + px + q$ will be divisible by $x^2 + ax + b$, if

$$a^6 - 4qa^2 = p^2, \text{ and } (b^2 + q)(b^2 - q)^2 = p^2b^3.$$

96. The highest common divisor of $p(x^q - 1) - q(x^p - 1)$, and $(q - p)x^q - qx^{q-p} + p$ is $(x - 1)^2$, p, q being numbers whose greatest common measure is 1, and $q > p$.

97. If n be any positive whole number not divisible by 3, the expression $x^{2n} + 1 + (x + 1)^{2n}$ will be divisible by $x^2 + x + 1$; and, if n be of the form $3r + 2$, by $(x^2 + x + 1)^2$.

V. *Identities and Equalities.*

98. Prove that,

(1) $(a + b + c)^3 \equiv a^3 + b^3 + c^3 + 3(b + c)(c + a)(a + b)$,

(2) $$\frac{2}{b-c} + \frac{2}{c-a} + \frac{2}{a-b} + \frac{(b-c)^2 + (c-a)^2 + (a-b)^2}{(b-c)(c-a)(a-b)} \equiv 0,$$

(3) $$(S - b^2)(S - c^2) + (S - c^2)(S - a^2) + (S - a^2)(S - b^2)$$
$$\equiv 4s(s - a)(s - b)(s - c),$$

where $2S \equiv a^2 + b^2 + c^2$, and $2s \equiv a + b + c$,

(4) $$2y^2z^2(z + x)^2(x + y)^2 + 2z^2x^2(x + y)^2(y + z)^2 + 2x^2y^2(y + z)^2(z + x)^2$$
$$\equiv x^4(y + z)^4 + y^4(z + x)^4 + z^4(x + y)^4 + 16x^2y^2z^2(yz + zx + xy),$$

(5) $$(b^2c^2 + a^2d^2)(b - c)(a - d) + (c^2a^2 + b^2d^2)(c - a)(b - d) + (a^2b^2 + c^2d^2)(a - b)(c - d)$$
$$\equiv (b - c)(c - a)(a - b)(d - a)(d - b)(d - c),$$

(6) $$(bcd + cda + dab + abc)^2 - abcd(a + b + c + d)^2$$
$$\equiv (bc - ad)(ca - bd)(ab - cd),$$

(7) $$\frac{(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)}{a^3+b^3+c^3-3abc} \equiv 2,$$

(8) $$(x^2-x+1)(x^4-x^2+1)\ldots\ldots(x^{2^n}-x^{2^{n-1}}+1)$$
$$\equiv \frac{x^{2^{n+1}}+x^{2^n}+1}{x^2+x+1}.$$

99. If $\dfrac{x}{a}+\dfrac{y}{b}=1$, and $\dfrac{x^2}{a}+\dfrac{y^2}{b}=\dfrac{ab}{a+b}$;

then will $$\frac{x^{n+1}}{a}+\frac{y^{n+1}}{b}=\left(\frac{ab}{a+b}\right)^n.$$

100. Having given $\dfrac{x}{y+z}=a$, $\dfrac{y}{z+x}=b$, $\dfrac{z}{x+y}=c$,

find the relation between a, b, c; and prove that

$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

101. Having given the equations

$$\left.\begin{aligned} x+y+z&=1,\\ ax+by+cz&=d,\\ a^2x+b^2y+c^2z&=d^2,\end{aligned}\right\}$$

prove that

$$a^3x+b^3y+c^3z=d^3-(d-a)(d-b)(d-c).$$

102. If $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c},$$

then, for all integral values of n,

$$\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{(a+b+c)^{2n+1}}.$$

103. If $x+y+z=xyz$, then will

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\,\frac{2y}{1-y^2}\,\frac{2z}{1-z^2},$$

and $$\frac{y+z}{1-yz}+\frac{z+x}{1-zx}+\frac{x+y}{1-xy}=\frac{y+z}{1-yz}\,\frac{z+x}{1-zx}\,\frac{x+y}{1-xy}.$$

104. If $x(b-c)+y(c-a)+z(a-b)=0$, then will

$$\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

105. If x, y, z, u be all finite and satisfy the equations

$$\begin{aligned}x&=by+cz+du,\\ y&=ax+cz+du,\\ z&=ax+by+du,\\ u&=ax+by+cz,\end{aligned}$$

then will

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$$

106. If

$$\frac{y^2-z^2}{b-c}=\frac{yz}{x} \text{ and } \frac{z^2-x^2}{c-a}=\frac{zx}{y},$$

then will

$$\frac{x^2-y^2}{a-b}=\frac{xy}{z}:$$

and if

$$a+\frac{y^2-z^2}{b-c}=b+\frac{z^2-x^2}{c-a},$$

then will each member of the equation be equal to $c+\dfrac{x^2-y^2}{a-b}$.

107. If $\dfrac{x-\dfrac{yz}{x}}{1-yz}=\dfrac{y-\dfrac{zx}{y}}{1-zx}$, and x, y be unequal, then will each member of this equation be equal to $\dfrac{z-\dfrac{xy}{z}}{1-xy}$, to $x+y+z$, and to $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$.

108. Having given the equations

$$alx+bmy+cnz=al'x+bm'y+cn'z=ax^2+by^2+cz^2=0,$$

prove that

$$x(mn'-m'n)+y(nl'-n'l)+z(lm'-l'm)=0,$$

and that

$$\frac{1}{a}\{(m-m')z-(n-n')y\}^2+\frac{1}{b}\{(n-n')x-(l-l')z\}^2$$
$$+\frac{1}{c}\{(l-l')y-(m-m')x\}^2=0.$$

109. Having given

$$a+b+c+d=a'-b'-c'+d'=0=aa'+bb'+cc'+dd',$$

prove that

$$\frac{aa'^2+bb'^2+cc'^2+dd'^2}{\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}+\frac{d}{d'}}=b'c'\frac{a+d}{\frac{a}{a'}+\frac{d}{d'}}=a'd'\frac{b+c}{\frac{b}{b'}+\frac{c}{c'}}.$$

110. In the equations

$$\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)},$$

l, m, n are all finite; prove that

$$\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}.$$

111. If a, b, c, x, y, z be any six quantities, and

$$a_1=bc-x^2,\quad b_1=ca-y^2,\quad c_1=ab-z^2,$$
$$x_1=yz-ax,\quad y_1=zx-by,\quad z_1=xy-cz;$$

and a_2, b_2, c_2, x_2, y_2, z_2 be similarly formed from a_1, b_1, c_1, x_1, y_1, z_1 and so on; then will

$$\frac{a_{2n}}{a}=\frac{b_{2n}}{b}=\frac{c_{2n}}{c}=\frac{x_{2n}}{x}=\frac{y_{2n}}{y}=\frac{z_{2n}}{z}$$
$$=(ax^2+by^2+cz^2-abc-2xyz)^{\frac{2^{2n}-1}{3}}.$$

112. Having given

$$\frac{x}{1-x^2}=\frac{y+z}{m+nyz},\quad \frac{y}{1-y^2}=\frac{z+x}{m+nzx};$$

prove that if x, y be unequal,

$$\frac{z}{1-z^2}=\frac{x+y}{m+nxy}.$$

113. Prove the following equalities, having given that $a+b+c=0$,

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\,\frac{a^2+b^2+c^2}{2},$$

$$\frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\,\frac{a^2+b^2+c^2}{2}=\frac{a^3+b^3+c^3}{3}\,\frac{a^4+b^4+c^4}{2}.$$

$$\frac{a^{11}+b^{11}+c^{11}}{11}=\frac{a^3+b^3+c^3}{3}\,\frac{a^8+b^8+c^8}{2}-\frac{(a^3+b^3+c^3)^3}{9}\,\frac{a^2+b^2+c^2}{2}.$$

114. If $a+b+c+d=0$, then will

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\,\frac{a^2+b^2+c^2+d^2}{2}.$$

115. Having given the equations

$$(y+z)^2=4a^2yz,\quad (z+x)^2=4b^2zx,\quad (x+y)^2=4c^2xy;$$

prove that

$$a^2+b^2+c^2\pm 2abc=1.$$

116. Having given

$$\frac{y}{z}-\frac{z}{y}=a,\quad \frac{z}{x}-\frac{x}{z}=b,\quad \frac{x}{y}-\frac{y}{x}=c;$$

prove that

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4+a^2b^2c^2=0.$$

117. Having given the equations

$$\frac{x}{a}(y^2-z^2)+\frac{y}{b}(z^2-x^2)+\frac{z}{c}(x^2-y^2)=0,$$

$$\frac{x}{x'(by'+cz'-ax')}=\frac{y}{y'(cz'+ax'-by')}=\frac{z}{z'(ax'+by'-cz')};$$

prove that, if x, y, z be all finite,

$$\frac{x'}{a}(y'^2-z'^2)+\frac{y'}{b}(z'^2-x'^2)+\frac{z'}{c}(x'^2-y'^2)=0.$$

118. If the quantities x, y, z be all unequal and satisfy the equations

$$\frac{a(y^2z^2+1)+y^2+z^2}{yz}=\frac{a(z^2x^2+1)+z^2+x^2}{zx}=\frac{a(x^2y^2+1)+x^2+y^2}{xy};$$

each member of the equations $=a^2-1$, and

$$yz+zx+xy=\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}.$$

119. If $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=0;$

then will

$$\frac{x}{a}\left(\frac{b'}{b}+\frac{c'}{c}-\frac{a'}{a}\right)^2+\frac{y}{b}\left(\frac{c'}{c}+\frac{a'}{a}-\frac{b'}{b}\right)^2+\frac{z}{c}\left(\frac{a'}{a}+\frac{b'}{b}-\frac{c'}{c}\right)^2=0.$$

VI. *Inequalities.*

The symbols employed in the following questions are always supposed to denote real quantities.

The fundamental proposition on which the solution generally depends is $a^2+b^2>2ab$.

Limiting values of certain expressions involving an unknown quantity in the second degree only may be found from the condition that a quadratic equation shall have real roots:—e. g. "To find the greatest and least values of $\frac{x^2-4x+3}{x^2-2x+4}$." Assuming the expression $\equiv y$, we obtain the quadratic

$$x^2(1-y)-2x(2-y)+3-4y=0,$$

and if x be a real quantity satisfying this equation we must have

$$(2-y)^2>(1-y)(3-4y),$$

or $$3y^2-3y<1,\ \text{ or }\ (2y-1)^2<\frac{7}{3},$$

shewing that y must lie between the values

$$\frac{1}{2}\left(1+\sqrt{\frac{7}{3}}\right),\quad \frac{1}{2}\left(1-\sqrt{\frac{7}{3}}\right).$$

120. If x, y, z be three positive quantities whose sum is unity, then will

$$\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right)>8.$$

121. Prove that

$$4(a^4+b^4+c^4+d^4)>(a+b+c+d)(a^3+b^3+c^3+d^3)>16abcd.$$

122. Prove that

$$\lfloor n > n^{\frac{n}{2}} < \left(\frac{n+1}{2}\right)^n;$$

$$\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\ldots\ldots\left(2-\frac{2n-1}{n}\right)<\frac{1}{\lfloor n}.$$

123. If a, b, c be three positive quantities of which any two are together greater than the third, then will

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>\frac{9}{a+b+c};$$

and, x, y, z being any real quantities,

$$a^2(x-y)(x-z)+b^2(y-z)(y-x)+c^2(z-x)(z-y)$$

cannot be negative.

If $x+y+z=0$, $a^2yz+b^2zx+c^2xy$ cannot be positive.

124. If a, b, c be positive and not all equal, the expressions

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b),$$
$$a^3(a-b)(a-c)+b^3(b-c)(b-a)+b^3(c-a)(c-b),$$

are positive.

125. Prove that

$$\{ax(b+c)+by(c+a)+cz(a+b)\}^2$$
$$>4abc(x+y+z)(ax+by+cz);$$

a, b, c, x, y, z being all positive, and a, b, c unequal.

126. If $xyz=(1-x)(1-y)(1-z)$,

the greatest value of either of these equals is $\frac{1}{8}$, x, y, z being each positive and less than unity.

127. Find the greatest numerical values without regard to sign which the expression

$$(x-8)(x-14)(x-16)(x-22),$$

can have for values of x between 8 and 22.

128. If $a > b$, and c be positive, the greatest value which the expression

$$(x-a)(x-b)(x-a-c)(x-b+c),$$

can have for values of x between a and b is

$$\frac{(a-b)^2(a-b+2c)^2}{16}.$$

129. If $p > m$,

$$\frac{x^2-2mx+p^2}{x^2+2mx+p^2} > \frac{p-m}{p+m} < \frac{p+m}{p-m}.$$

130. The expression

$$\frac{ax^2+bx+c}{cx^2+bx+a},$$

will be capable of all values whatever if

$$b^2 > (a+c)^2;$$

there will be two values between which it cannot lie if

$$b^2 < (a+c)^2 > 4ac;$$

and two values between which it must lie if

$$b^2 < 4ac.$$

131. The expression

$$\frac{(x-\alpha)(x-\beta)}{(x-\gamma)(x-\delta)},$$

can have any real value whatever if both, or neither, of the two α, β, lie between γ and δ: otherwise there will be two values between which it cannot lie.

132. The expression

$$\frac{ax^2 + 2bx + c}{a'x^2 + 2b'x + c'},$$

will be capable of all values, provided that

$$(ac' - a'c)^2 < 4(a'b - ab')(b'c - bc').$$

Prove that this inequality involves the two

$$b^2 > ac, \quad b'^2 > a'c';$$

and investigate the condition (1) that two limiting values exist between which the expression cannot lie, (2) that two limiting values exist between which the expression must lie.

133. If $x_1, x_2, x_3 \ldots x_n$ be positive, and if

$$x_1^2 + x_2^2 + \ldots + x_n^2 - x_1x_2 - x_2x_3 - \ldots - x_{n-1}x_n - x_n + \frac{n-1}{2n} = 0,$$

then will $x_1, x_2, \ldots x_n$, 1 be in ascending order of magnitude.

VII. *Proportion, Variation, Scales of Notation.*

134. If $b+c+d$, $c+d+a$, $d+a+b$, $a+b+c$ be proportionals, then will

$$\frac{a^3-d^3}{a-d} = \frac{b^3-c^3}{b-c}.$$

135. If y vary as the sum of three quantities, of which the first is constant, the second varies as x, and the third as x^2: and if $(a, 0)$, $(2a, a)$, $(3a, 4a)$ be three pairs of simultaneous values of x and y, then when

$$x = na, \; y = (n-1)^2 a.$$

136. A triangle has two sides given in position and a given perimeter $2s$: if c be the length of the side opposite to the given angle, the area of the triangle $\propto s - c$.

137. The radix of a scale in which 49 denotes a square number must be of the form $(r+1)(r+4)$, where r is some whole number.

138. The radix of a scale being $4r+2$, prove that if the digit in the units' place of any number be either $2r+1$, or $2r+2$, the square of the number will have the same digit in the units' place.

139. Find a number of (1) three digits, (2) four digits, in the denary scale such that if the first and last digits be interchanged, the result represents the same number in the nonary scale, and prove that there is only one solution in each case.

140. If the radix of any scale have more than one prime factor, there will exist at least one digit different from unity such that if any number have that digit in the units' place, its square will have the same digit in the units' place.

VIII. *Arithmetical, Geometrical, and Harmonical Progressions.*

141. If the sum of m terms of an A.P. be to the sum of n terms as $m^2 : n^2$; prove that the m^{th} term will be to the n^{th} term as $2m-1 : 2n-1$.

142. The series of natural numbers are divided into groups 1; 2, 3, 4; 5, 6, 7, 8, 9; and so on: prove that the sum of the numbers in the n^{th} group is $n^3+(n-1)^3$.

143. The sum of the products of every two of n terms of an A.P., whose first term is a and last term l, is

$$\frac{n(n-2)(3n-1)(a+l)^2+4n(n+1)al}{24(n-1)}.$$

144. Having given that $\frac{a}{b-c}$, $\frac{b}{c-a}$, $\frac{c}{a-b}$ are in A.P.; prove that

$$\frac{a^3+c^3-2b^3}{a^2+c^2-2b^2}=\frac{a+b+c}{2}.$$

145. If a, b, c; b, c, a; or c, a, b be in A.P., then will

$$\frac{2}{9}(a+b+c)^3=a^2(b+c)+b^2(c+a)+c^2(a+b);$$

and if in G.P.,

$$a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3.$$

146. If a, l be the first and n^{th} terms of an A.P. the continued product of all the n terms is

$$>(al)^{\frac{n}{2}}<\left(\frac{a+l}{2}\right)^n.$$

147. The first term of a G.P. is a, and the n^{th} term l; prove that the r^{th} term is $(a^{n-r}l^{r-1})^{\frac{1}{n-1}}$.

148. If a, b, c be in A.P., α, β, γ in H.P., and $a\alpha, b\beta, c\gamma$ in G.P., then will

$$a:b:c::\frac{1}{\gamma}:\frac{1}{\beta}:\frac{1}{\alpha}.$$

149. The first term of an H.P. is a and the n^{th} term l, prove that the r^{th} term is

$$\frac{(n-1)\,al}{(n-r)\,l+(r-1)\,a}.$$

Prove that the sum of these n terms $<(a+l)\frac{n}{2}$, and their continued product $<(al)^{\frac{n}{2}}$.

150. If a, b, c be in H.P., then will

$$\frac{1}{b-c}+\frac{4}{c-a}+\frac{1}{a-b}=\frac{1}{c}-\frac{1}{a}.$$

151. If a, b, c, d be four positive quantities in H.P.,

$$a+d>b+c.$$

152. Prove that $b+c$, $c+a$, $a+b$ will be in H.P., if a^2, b^2, c^2 be in A.P.

153. If three numbers be in G.P. and the mean be added to each of the three, the three sums will be in H.P.

154. If n harmonic means be inserted between two positive quantities a and b, the difference between the first and last of these means bears to the difference between a and b a ratio less than $n-1 : n+1$.

155. If a_0, a_1, a_2 ... be an A.P., b_0, b_1, b_2 ... a G.P., and A, B, C, D be any four consecutive terms of the series a_0+b_0, a_1+b_1, a_2+b_2, ..., then will

$$Ab_1-B(b_0+2b_1)+C(b_1+2b_0)-Db_0=0.$$

IX. *Permutations and Combinations.*

The number of permutations of n *different* things taken r together is denoted by ${}_nP_r$, and the corresponding number of combinations by ${}_nC_r$.

156. Prove *a priori* that

$$\begin{aligned}
{}_nP_r &\equiv {}_{n-1}P_r + 2r\,{}_{n-2}P_{r-1} + r(r-1)\,{}_{n-2}P_{r-2},\\
&\equiv {}_{n-3}P_r + 3r\,{}_{n-3}P_{r-1} + 3r(r-1)\,{}_{n-3}P_{r-2} + r(r-1)(r-2)\,{}_{n-3}P_{r-3},\\
&\equiv {}_{n-p}P_r + pr\,{}_{n-p}P_{r-1} + \frac{p(p-1)}{2}r(r-1)\,{}_{n-p}P_{r-2} + \dots\\
&\qquad + r(r-1)\dots(r-p+1)\,{}_{n-p}P_{r-p},
\end{aligned}$$

p being a whole number $< r$.

157. In the expansion of $(a_1 + a_2 + \ldots + a_p)^n$, where n is a whole number not greater than p, prove that the coefficient of any term in which none of the quantities $a_1, a_2 \ldots a_p$ appears more than once as a factor is $\lfloor n$.

158. The number of permutations of n different letters taken all together, in which no letter occupies the same place as in a certain given permutation, is

$$\lfloor n \left\{ \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots + \frac{(-1)^n}{\lfloor n} \right\}.$$

159. Prove that

$${}_nC_r \equiv {}_{n+2}C_r - 2\,{}_{n+2}C_{r-1} + 3\,{}_{n+2}C_{r-2} - \ldots + (-1)^r(r+1).$$

160. The number of combinations of $2n$ things taken n together, when n of the things and no more are alike, is 2^n; and the number of combinations of $3n$ things, n together, when n of the things and no more are alike, is

$$2^{2n-1} + \frac{\lfloor 2n}{2(\lfloor n)^2}.$$

161. The number of ways in which mn different things can be distributed among m persons so that each person shall have n of them is $\dfrac{\lfloor mn}{(\lfloor n)^m}$.

162. There are p suits of cards, each suit consisting of q cards numbered from 1 to q; prove that the number of sets of q cards numbered from 1 to q which can be made from all the suits is p^q.

163. The number of ways in which p things may be distributed among q persons, so that everybody may have one at least, is

$$q^p - q(q-1)^p + \frac{q(q-1)}{\lfloor 2}(q-2)^p + \ldots\ldots$$

164. The number of ways in which r things may be distributed among $n+p$ persons, so that certain n of those persons may each have one at least, is (S_r)

$$(n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{\lfloor 2}(n+p-2)^r - \ldots\ldots$$

Hence prove that

$$S_1 = S_2 = \ldots\ldots = S_{n-1} = 0, \quad S_n = \lfloor n, \quad S_{n+1} = \left(\frac{n}{2}+p\right)\lfloor n+1.$$

X. *Binomial Theorem.*

165. Prove that

(1) $$1 - n\frac{1+x}{1+nx} + \frac{n(n-1)}{\lfloor 2}\frac{1+2x}{(1+nx)^2} - \frac{n(n-1)(n-2)}{\lfloor 3}\frac{1+3x}{(1+nx)^3} + \ldots\ldots = 0;$$

(2) $$1 + 3\left(\frac{2n+1}{2n-1}\right) + 5\left(\frac{2n+1}{2n-1}\right)^2 + \ldots + (2n-1)\left(\frac{2n+1}{2n-1}\right)^{n-1} \equiv n(2n-1),$$

n being a whole number.

166. Determine a, b, c, d, e in order that the n^{th} term in the expansion of

$$\frac{a+bx+cx^2+dx^3+ex^4}{(1-x)^5}$$

may be n^4x^{n-1}.

167. Prove that the series

$$1^n + 2^nx + 3^nx^2 + \ldots\ldots + r^nx^{r-1} + \ldots\ldots$$

is the expansion of a function of x of the form

$$\frac{a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n}{(1-x)^{n+1}};$$

also prove that

$$a_n = 0;\ a_{n-1} = a_1 = 1,\ a_{n-2} = a_2 = 2^n - (n+1);\ a_{n-r} = a_{r-1}.$$

168. The sum of the first $r+1$ coefficients of the expansion of $(1-x)^{-m}$ is equal to $\dfrac{\lfloor m+r}{\lfloor m \lfloor r}$.

169. Prove that

$$2^n - (n-1)\,2^{n-2} + \frac{(n-2)(n-3)}{\lfloor 2}\,2^{n-4} - \frac{(n-3)(n-4)(n-5)}{\lfloor 3}\,2^{n-6} + \ldots\ldots \equiv n+1,$$

$$1 - (n-1)\frac{m}{(1+m)^2} + \frac{(n-2)(n-3)}{\lfloor 2}\,\frac{m^2}{(1+m)^4} - \frac{(n-3)(n-4)(n-5)}{\lfloor 3}\,\frac{m^3}{(1+m)^6} + \ldots\ldots \equiv \frac{m^{n+1}-1}{m-1}\,\frac{1}{(m+1)^n},$$

$$(p+q)^n - (n-1)(p+q)^{n-2}pq + \frac{(n-2)(n-3)}{\lfloor 2}(p+q)^{n-4}p^2q^2 - \frac{(n-3)(n-4)(n-5)}{\lfloor 3}(p+q)^{n-6}p^3q^3 + \ldots\ldots \equiv \frac{p^{n+1}-q^{n+1}}{p-q},$$

n being a positive integer.

170. If p be nearly equal to q, then will $\dfrac{q+2p}{p+2q}$ be nearly equal to $\sqrt[3]{\dfrac{p}{q}}$.

171. If a_r denote the coefficient of x^r in the expansion of $\left(\dfrac{1+x}{1-x}\right)^n$ in a series of ascending powers of x; the following relation will hold among any three consecutive coefficients,

$$(r+1)\,a_{r+1} - 2na_r - (r-1)\,a_{r-1} = 0.$$

172. If $\dfrac{(1+x)^n}{(1-x)^2}$ be expanded in ascending powers of x, the coefficient of x^{n+r-1} is $(n+2r)\,2^{n-1}$, n, r being positive integers (including zero).

173. The sum of the first n coefficients of the expansion, in ascending powers of x, of $\frac{(1+x)^n}{(1-x)^3}$ is $2^{n-4}\frac{n(n+2)(n+7)}{3}$.

174. Prove that

$$1+3n+\frac{3.4}{1.2}\frac{n(n-1)}{\lfloor 2}+\frac{4.5}{1.2}.\frac{n(n-1)(n-2)}{\lfloor 3}+\dots$$
$$\dots+\frac{(n+1)(n+2)}{2}$$
$$\equiv 2^{n-3}(n^2+7n+8).$$

175. Prove that

$$\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

is equal to the sum of the homogeneous products of n dimensions of a, b, c.

XI. *Exponential and Logarithmic Series.*

In the questions under this head, n always denotes a positive whole number.

176. Obtain the following equalities;—

(1) $n^n-(n+1)(n-1)^n+\frac{(n+1)n}{\lfloor 2}(n-2)^n-\dots\dots\equiv 1$,

(2) $(n-1)^n-n(n-2)^n+\frac{n(n-1)}{\lfloor 2}(n-3)^n-\dots\equiv \lfloor n-1$,

(3) $(n-2)^n-n(n-3)^n+\frac{n(n-1)}{\lfloor 2}(n-4)^n-\dots\equiv \lfloor n-n-2^n$,

(4) $1^n-n\,2^n+\frac{n(n-1)}{\lfloor 2}3^n-\dots+(-1)^n(n+1)^n\equiv(-1)^n\lfloor n$.

177. The coefficient of x^r in the expansion of $(1+x)^n$ being denoted by a_r, prove that

$$a_0 p^{n-1} - a_1 (p-1)^{n-1} + a_2 (p-2)^{n-1} - \ldots\ldots + (-1)^{p-1} a_{p-1},$$
$$\equiv a_0 (n-p)^{n-1} - a_1 (n-p-1)^{n-1} + a_2 (n-p-2)^{n-1} - \ldots\ldots$$
$$+ (-1)^{n-p-1} a_{n-p-1},$$

p being a whole number $< n$.

178. Denoting by u_n the series

$$1^n + 2^n + \frac{3^n}{\lfloor 2} + \ldots + \frac{(r+1)^n}{\lfloor r} + \ldots \text{ to infinity,}$$

prove that

$$u_{n-1} + u_n \equiv u_{n+1} - nu_n + \frac{n(n-1)}{\lfloor 2} u_{n-1} - \frac{n(n-1)(n-2)}{\lfloor 3} u_{n-2} + \ldots$$
$$+ (-1)^n u_1;$$

and
$$u_{n+1} - u_n \equiv u_n + nu_{n-1}$$
$$+ \frac{n(n-1)}{\lfloor 2} u_{n-2} + \frac{n(n-1)(n-2)}{\lfloor 3} u_{n-3} + \ldots + u_0;$$

and by means of either of these, prove that $u_7 \equiv 4140\epsilon$.

179. If $u_n \equiv 1^{n-1} - 2^{n-1} + \frac{3^{n-1}}{\lfloor 2} - \frac{4^{n-1}}{\lfloor 3} + \ldots$ to infinity,

then will

$$1 = u_{n+1} + u_n + nu_{n-1} + \frac{n(n-1)}{\lfloor 2} u_{n-2} + \ldots + nu_1 + u_0.$$

180. If $v_n \equiv \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots + \frac{(-1)^{n+1}}{\lfloor n+1}$;

then will

$$1 = v_n + v_{n-1} + \frac{v_{n-2}}{\lfloor 2} + \ldots + \frac{v_1}{\lfloor n-1} + \frac{1}{\lfloor n+1}.$$

181. Having given

$$u_n = nv_{n-1},\ v_n = u_n + u_{n-1},$$

prove that the limit of $\frac{u_n}{\lfloor n+1}$ when n is indefinitely increased is

$$u_0 + \frac{u_1 - 2u_0}{\epsilon}.$$

182. If there be a series of terms u_0, u_1, u_2, u_3 ... u_n ..., of which any one is obtained from the preceding by the formula

$$u_n = nu_{n-1} + (-1)^n, \text{ and if } u_0 = 1; \text{ then will}$$

$$\lfloor n \equiv u_n + nu_{n-1} + \frac{n(n-1)}{\lfloor 2} u_{n-2} + \ldots + \frac{n(n-1)}{\lfloor 2} u_2 + nu_1 + u_0.$$

Prove also that $\frac{u_n}{\lfloor n}$ tends to become equal to $\frac{1}{\epsilon}$, as n is indefinitely increased.

183. Prove that

$$\frac{2^{n+2}-2}{n+2} \equiv 2^n - \frac{n-1}{\lfloor 2} 2^{n-2} + \frac{(n-2)(n-3)}{\lfloor 3} 2^{n-4} - \ldots,$$

and that

$$p^n + q^n \equiv (p+q)^n - n(p+q)^{n-2}pq + \frac{n(n-3)}{\lfloor 2}(p+q)^{n-4}p^2q^2$$
$$- \frac{n(n-4)(n-5)}{\lfloor 3}(p+q)^{n-6}p^3q^3 + \ldots$$

184. By means of the identity

$$\log(1-x^3) \equiv \log(1-x) + \log(1+x+x^2),$$

prove that

$$0 \equiv 1 - \frac{3n(3n+1)}{\lfloor 3} + \frac{(3n-1)\,3n\,(3n+1)\,(3n+2)}{\lfloor 5} - \ldots,$$

the series being carried to $3n$ terms.

185. If there be n quantities $a, b, c, \ldots$ and s_n represent their sum, s_{n-1} the sum of *any* $n-1$ of them, and so on, and if

$$S_r \equiv (s_n)^r - \Sigma(s_{n-1})^r + \Sigma(s_{n-2})^r - \ldots + (-1)^{n-1}\Sigma(s_1)^r;$$

prove that

$$S_1 = S_2 = S_3 = \ldots = S_{n-1} = 0,$$

$$S_n = \lfloor n \; abc \ldots, \quad 2S_{n+1} = \lfloor n+1 \; abc \ldots (a+b+c+\ldots),$$

and $$12\, S_{n+2} = \lfloor n+2 \; abc \ldots \{2\,\Sigma(a^2) + 3\,\Sigma(ab)\}.$$

If α be any other quantity, and if S_r now denote

$$(\alpha+s_n)^r - \Sigma(\alpha+s_{n-1})^r + \Sigma(\alpha+s_{n-2})^r - \ldots + (-1)^{n-1}\,\Sigma(\alpha+s_1)^r,$$

then will $$S_1 = S_2 = S_3 = \ldots = S_{n-1} = 0,$$

$$S_n = \lfloor n \; abc \ldots, \quad 2S_{n+1} = \lfloor n+1 \; abc \ldots (2\alpha + a + b + c \ldots),$$

and $$12\, S_{n+2} = \lfloor n+2 \; abc \ldots \{2\,\Sigma(a^2) + 3\,\Sigma(ab) + 6\alpha\,\Sigma(a) + 6\alpha^2\}.$$

XII. *Summation of Series.*

If u_n denote a certain function of n, and

$$S_n \equiv u_1 + u_2 + \ldots\ldots + u_n,$$

the summation of the series means expressing S_n as a function of n involving only a fixed number of terms. The usual artifice by which this is effected consists in expressing u_n as the difference of two quantities, one of which is the same function of n as the other is of $n-1$, $(U_n - U_{n-1})$. This being done, we have at once

$$S_n \equiv (U_1 - U_0) + (U_2 - U_1) + \ldots\ldots + (U_n - U_{n-1}) \equiv U_n - U_0.$$

Thus, if u_n be the product of r consecutive terms of a given A. P., beginning with the n^{th},

$$u_n \equiv \{a+(n-1)\,b\}\,(a+nb) \ldots\ldots \{a+(n+r-2)\,b\}$$

$$\equiv \frac{\{a+(n-1)b\}(a+nb)\ldots\{a+(n+r-1)b\} - \{a+(n-2)b\}\ldots\{a+(n+r-2)b\}}{(r+1)\,b},$$

whence $U_n \equiv \dfrac{\{a+(n-1)b\}(a+nb)\ldots\ldots\{a+(n+r-1)b\}}{(r+1)b}$,

and $$S_n \equiv \frac{1}{(r+1)b}[\{a+(n-1)b\}(a+nb)\ldots\ldots\{a+(n+r-1)b\}$$
$$-(a-b)(a)(a+b)\ldots\ldots\{a+(r-1)b\}].$$

The sums of many series can also be expressed in a finite form by equating the coefficients of x^n in the expansions of the same function of x effected by two different methods, of which examples have been given in the Binomial, Exponential, and Logarithmic Series.

In the following examples, n always means a positive whole number.

186. Sum the series:

(1) $\dfrac{1}{1.2}+\dfrac{3}{2.5}+\dfrac{5}{5.10}+\ldots\ldots+\dfrac{2n-1}{\{1+(n-1)^2\}(1+n^2)}$,

(2) $\dfrac{1}{2.4}+\dfrac{1.3}{2.4.6}+\dfrac{1.3.5}{2.4.6.8}+\ldots$
$$\ldots+\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n(2n+2)},$$

(3) $\dfrac{1.2}{1.2.3}+\dfrac{2.2^2}{\lfloor 4}+\ldots\ldots+\dfrac{n.2^n}{\lfloor n+2}$,

(4) $\dfrac{3}{\lfloor 4}+\dfrac{2.3^2}{\lfloor 5}+\ldots\ldots+\dfrac{n.3^n}{\lfloor n+3}$,

(5) $\dfrac{r}{\lfloor r+1}+\dfrac{2.r^2}{\lfloor r+2}+\ldots\ldots+\dfrac{nr^n}{\lfloor n+r}$,

(6) $\dfrac{1}{(1+x)(1+2x)}+\dfrac{1}{(1+2x)(1+3x)}+\ldots$
$$\ldots+\frac{1}{(1+nx)\{1+(n+1)x\}},$$

(7) $$\frac{1}{1.3}+\frac{2}{1.3.5}+\ldots\ldots+\frac{n}{1.3.5\ldots(2n+1)},$$

(8) $$\frac{1}{3}+\frac{3}{3.7}+\frac{5}{3.7.11}+\ldots\ldots+\frac{2n-1}{3.7.11\ldots(4n-1)},$$

(9) $$\frac{1}{\lfloor\underline{3}}+\frac{5}{\lfloor\underline{4}}+\ldots\ldots+\frac{n^2+n-1}{\lfloor\underline{n+2}},$$

(10) $$\frac{4}{\lfloor\underline{3}}+\frac{9}{\lfloor\underline{4}}+\ldots\ldots+\frac{2n^2+3n-1}{\lfloor\underline{n+2}},$$

(11) $$\frac{5}{5.10}+\frac{11}{10.17}+\ldots\ldots+\frac{n^2+n-1}{(1+n^2)\{1+(n+1)^2\}},$$

(12) $$\frac{1}{(1+x)(1+x^2)}+\frac{x}{(1+x^2)(1+x^3)}+\ldots$$
$$\ldots+\frac{x^{n-1}}{(1+x^n)(1+x^{n+1})},$$

(13) $$\frac{x}{1+x^2}+\frac{x}{1+x^2}\,\frac{x^2}{1+x^4}+\frac{x}{1+x^2}\,\frac{x^2}{1+x^4}\,\frac{x^4}{1+x^8}+\ldots$$
$$\ldots+\frac{x}{1+x^2}\,\frac{x^2}{1+x^4}\ldots\ldots\frac{x^{2^{n-1}}}{1+x^{2^n}},$$

(14) $$\frac{x(1-ax)}{(1+x)(1+ax)(1+a^2x)}$$
$$+\frac{ax(1-a^2x)}{(1+ax)(1+a^2x)(1+a^3x)}+\ldots$$
$$\ldots+\frac{a^{n-1}x(1-a^nx)}{(1+a^{n-1}x)(1+a^nx)(1+a^{n+1}x)},$$

(15) $$\frac{1}{p+r}+\frac{r+1}{(p+r)(p+2r)}+\frac{(r+1)(2r+1)}{(p+r)(p+2r)(p+3r)}+\ldots$$
$$\ldots+\frac{(r+1)(2r+1)\ldots\{(n-1)r+1\}}{(p+r)(p+2r)\ldots\quad(p+nr)}.$$

187. Prove that

$$1 \equiv \frac{n(n-1)}{\underline{\lvert 2}} - 2\,\frac{n(n-1)(n-2)}{\underline{\lvert 3}}$$
$$+ 3\,\frac{n(n-1)(n-2)(n-3)}{\underline{\lvert 4}} - \ldots$$
$$\ldots + (n-1)(-1)^n.$$

188. Prove that

$$(1+r+r^2+r^3)(1+r^2+r^4+r^6)\ldots(1+r^{2^{n-1}}+r^{2^n}+r^{2^{n+1}})$$
$$\equiv \frac{1-r^{2^n}-r^{2^{n+1}}+r^{2^{n+2}}}{1-r-r^2+r^3}.$$

189. Prove that

$$4^n - \frac{n-1}{\underline{\lvert 2}}\,4^{n-1} + \frac{(n-2)(n-3)}{\underline{\lvert 3}}\,4^{n-2} - \ldots\ldots \equiv \frac{4^{n+1}-2^{n+1}}{n+2}.$$

Expand $\log_e\{1-4x(1-x)\}$ & $2\log(1-2x)$ & equate coeff. of x^{n+2}

190. Prove that

$$1 + n + \frac{n(n+1)}{\underline{\lvert 2}} + \frac{n(n+1)(n+2)}{\underline{\lvert 3}} + \ldots$$
$$\ldots + \frac{n(n+1)\ldots(2n-1)}{\underline{\lvert n}}$$
$$\equiv \frac{(n+1)(n+2)\ldots 2n}{\underline{\lvert n}}.$$

191. Prove that

$$\frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \ldots \text{ to } \infty = 1,$$

$$\frac{1}{m+n} + \frac{m}{(m+n)(m+n-1)}$$
$$+ \frac{m(m-1)}{(m+n)(m+n+1)(m+n-1)} + \ldots \text{ to } m+1 \text{ terms} = \frac{1}{n},$$

$$\frac{m}{m+p} + \frac{m(m+1)}{(m+p)(m+p+1)} + \ldots \text{ to } \infty = \frac{m}{p-1},$$

p being any positive quantity > 1.

192. From the equality

$$\log(1-x^3) \equiv \log(1-x) + \log(1+x+x^2),$$

prove that

$$\frac{1}{6n+1} \equiv (-1)^n \left\{1 - \frac{3n(3n+1)}{\lfloor 3} + \frac{(3n-1)\,3n\,(3n+1)(3n+2)}{\lfloor 5} + \dots\right\},$$

$$\frac{1}{6n+2} \equiv (-1)^n \left\{\frac{1}{3n+1} - \frac{3n+1}{\lfloor 3} + \frac{3n(3n+1)(3n+2)}{\lfloor 5} - \dots\right\}.$$

XIII. *Recurring Series.*

The series $u_0, u_1, u_2 \dots\dots u_n$ is a recurring series if any fixed number (r) of consecutive terms are connected by a relation of the form

$$a_n + p_1 a_{n-1} + p_2 a_{n-2} + \dots + p_{r-1} a_{n-r+1} = 0, \quad (A)$$

in which n may have any integral value, but $p_1, p_2 \dots p_{r-1}$ are constant. It follows that the series $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ is the expansion in ascending powers of x of a function of x of the form $\dfrac{A_0 + A_1 x + \dots + A_{r-2} x^{r-2}}{1 + p_1 x + p_2 x^2 + \dots + p_{r-1} x^{r-1}}$ (the *generating function* of the series); and if the *scale of relation* (A) and the first $r-1$ terms of the series be given this function will be completely determined; when, by separating this function into its partial fractions $\dfrac{B_1}{1-a_1 x} + \dfrac{B_2}{1-a_2 x} + \dots\dots$, and expanding each, we obtain the n^{th} term of the series and the sum of n terms.

If the scale of relation is not given, we shall require $2(r-1)$ terms of the series to be known to determine all the constants; thus, if four terms are given we can determine a recurring series with a scale of relation between any three consecutive terms, and whose first four terms are the given quantities.

193. Prove that every A. P. is a recurring series and that its generating function is $\frac{a+(b-a)x}{(1-x)^2}$, a being the first term and b the common difference.

194. Find the generating functions of the following series

(1) $1+3x+5x^2+7x^3+\dots\dots$

(2) $2+5x+13x^2+35x^3+\dots\dots$

(3) $2+4x+14x^2+52x^3+\dots\dots$

(4) $4+5x+7x^2+11x^3+\dots\dots$

(5) $2+2x+8x^2+20x^3+\dots\dots$

(6) $1+3x+12x^2+54x^3+\dots\dots$

and employ the last to prove that the integral part of $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}, n being any integer.

195. The generating function of the recurring series whose first four terms are a, b, c, d, is

$$\frac{ab^2-ca^2+x(a^2d-2abc+b^3)}{b^2-ac+x(ad-bc)+x^2(c^2-bd)}.$$

196. If the scale of relation of a recurring series be

$$a_n-7a_{n-1}+12a_{n-2}=0,$$

and if $u_0=2$, $u_1=7$, find u_n and the sum of the series

$$u_0+u_1+\dots u_{n-1}.$$

197. Prove that, if

$a_0, a_1, a_2 \dots a_n$ be an A.P. and $b_0, b_1 \dots, b_n$ a G. P.,

the series

$$a_0+b_0,\ a_1+b_1,\ \dots\ a_n+b_n,\ \dots.$$

$$a_0b_0,\ a_1b_1,\ \dots\ a_nb_n,\ \dots.$$

are recurring series.

198. Prove that the series

$$\left.\begin{array}{l} 1^2+2^2+3^2+\ldots+n^2 \\ 1^3+2^3+3^3+\ldots+n^3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1^r+2^r+3^r+\ldots+n^r \end{array}\right\}$$

are recurring series, the scales of relation being between 4, 5, ... $r+2$ terms respectively.

199. Find the generating functions of the recurring series

(1) $1+2x+5x^2+10x^3+17x^4+26x^5+\ldots,$

(2) $1+3x+4x^2+\ 8x^3+12x^4+20x^5+\ldots,$

(3) $3+6x+14x^2+36x^3+98x^4+276x^5+\ldots,$

(4) $3-x+13x^2-9x^3+41x^4-53x^5+\ldots,$

and the n^{th} term of each.

200. If the terms of the series a_0, a_1, a_2 ... be derived, each from the preceding, by the formula

$$a_{n+1}=\frac{pq}{p+q-a_n},$$

prove that

$$a_n=pq\,\frac{(a_0-p)p^{n-1}-(a_0-q)q^{n-1}}{(a_0-p)p^n-(a_0-q)q^n}.$$

XIV. *Convergent Fractions.*

201. If $\dfrac{p_n}{q_n}$ be the n^{th} convergent to the continued fraction

$$\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\ldots,$$

we have the equations

$$p_n=b_np_{n-1}+a_np_{n-2},$$

and the like law for q: and for the fraction

$$\frac{a_1}{b_1-}\,\frac{a_2}{b_2-}\,\frac{a_3}{b_3-}\ldots,$$

$$p_n=b_np_{n-1}-a_np_{n-2},$$

and the like law for q. The solution of this equation, a_n, b_n being functions of n, must involve two constants, since it is necessary that two terms be known in order to determine the remaining terms by this formula. These constants may conveniently be taken p_1, p_2; q_1, q_2 respectively. The fraction $\frac{p_n}{q_n}$ thus determined will not usually be in its lowest terms. We will take as an example the question "To find the n^{th} convergent to the continued fraction

$$\frac{1}{1-}\ \frac{1}{3-}\ \frac{4}{6-}\ \frac{12}{9-}\cdots-\frac{2n(n-1)}{3n}-\cdots.$$

Take u_n to represent *either* p_n or q_n (since the same law holds for both),

then $$u_{n+1}=3nu_n-2n(n-1)u_{n-1};$$

or $$u_{n+1}-2nu_n=n\{u_n-2(n-1)u_{n-1}\}.$$

So $$u_n-2(n-1)u_n=(n-1)\{u_{n-1}-2(n-2)u_{n-2}\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_3-4u_2=2(u_2-2u_1),\ (=2 \text{ or } 0 \text{ as } u=p \text{ or } q).$$

Hence $$u_{n+1}-2nu_n=\lfloor n \text{ or } 0,$$

or $$\frac{u_{n+1}}{\lfloor n}-2\frac{u_n}{\lfloor n-1}=1 \text{ or } 0.$$

So $$\frac{2u_n}{\lfloor n-1}-\frac{2^2u_{n-1}}{\lfloor n-2}=2 \text{ or } 0,$$

$$\cdots\cdots\cdots\cdots=\cdots\cdots\cdots$$

$$\frac{2^{n-1}u_2}{\lfloor 1}-2^nu_1=2^{n-1} \text{ or } 0,$$

$$2^nu_1 \qquad =2^n.$$

$$\therefore\ \frac{u_{n+1}}{\lfloor n}=2^{n+1}-1 \text{ or } 2^n,$$

and the n^{th} convergent is

$$\frac{2^n-1}{2^{n-1}}.$$

202. The n^{th} convergent to $\frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\dots$

is
$$\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}.$$

203. If $\frac{p_n}{q_n}$ be the n^{th} convergent to the infinite continued fraction $\frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\dots$; p_n, q_n will be coefficients of x^{n-1} and x^n respectively in the expansion of $\frac{1}{1-ax-x^2}$.

204. Prove that, $\frac{p_n}{q_n}$ being the n^{th} convergent to the infinite continued fraction $\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\dots$,

$$p_{n+2}-(2+ab)p_n+p_{n-2}=0, \quad q_{n+2}-(2+ab)\,q_n+q_{n-2}=0.$$

205. Prove that the products of the infinite continued fractions

$$(1) \quad \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\dots, \quad c+\frac{1}{b+}\frac{1}{a+}\frac{1}{c+}\dots,$$

$$(2) \quad \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}\frac{1}{a+}\dots, \quad d+\frac{1}{c+}\frac{1}{b+}\frac{1}{a+}\frac{1}{d+}\dots,$$

are, (1) $\frac{1+bc}{1+ab}$, (2) $\frac{b+d+bcd}{a+c+abc}$.

206. Prove that the differences of the infinite continued fractions

$$(1) \quad \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\dots, \quad \frac{1}{c+}\frac{1}{b+}\frac{1}{a+}\frac{1}{c+}\dots,$$

$$(2) \quad \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}\frac{1}{a+}\dots, \quad \frac{1}{c+}\frac{1}{b+}\frac{1}{a+}\frac{1}{d+}\frac{1}{c+}\dots,$$

are, (1) $\frac{a-b}{1+ab}$, (2) $\frac{b(a-c)}{a+c+abc}$.

207. The continued fractions

$$4 + \frac{4}{8+}\,\frac{4}{8+}\,\frac{4}{8+}\dots, \quad 2 + \frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+}\dots,$$

each to n quotients, are in the ratio 2 : 1.

208. Having expressed $\sqrt{(n^2+a)}$ as a continued fraction in the form $n + \frac{a}{2n+}\,\frac{a}{2n+}\dots$, $\frac{p_r}{q_r}$ is the r^{th} convergent;

prove that

$$p_r^2 - (n^2+a)\,q_r^2 = (-a)^r, \qquad q_{r+1} + aq_{r-1} = 2p_r,$$

$$p_r p_{r+1} - (n^2+a)\,q_r q_{r+1} = n\,(-a)^r, \qquad p_{r+1} + ap_{r-1} = 2\,(n^2+a)\,q_r.$$

209. Prove that

$$\frac{n}{n+}\,\frac{n-1}{n-1+}\,\frac{n-2}{n-2+}\dots+\frac{2}{2+}\,\frac{1}{1+}\,\frac{1}{2} \equiv \frac{n+1}{n+2}.$$

210. Prove that the value of the infinite continued fraction

$$\frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\dots \text{ is } \frac{1}{\epsilon-1}.$$

211. Any two consecutive terms of the series

$$a_1,\ a_2,\ \dots\ a_n,\ \dots$$

satisfy the equation

$$a_{n+1} = \frac{n\,(n+1)}{2n-a_n},$$

find a_n in terms of a_1; and prove that, when n is indefinitely increased, the limit of $\frac{a_1 a_2 \dots a_{n+1}}{\lfloor n}$ is a_1.

212. Prove that

(1) $$\frac{r\,(r+1)}{1+}\,\frac{r\,(r+1)}{1+}\dots \text{ to } n \text{ terms}$$

$$\equiv \frac{r\,(r+1)^{n+1} + (r+1)\,(-r)^{n+1}}{(r+1)^{n+1} - (-r)^{n+1}},$$

(2) $$\frac{1}{1}+\frac{1}{2}+\frac{3^2}{2}+\frac{5^2}{2}+\ldots\ldots+\frac{(2n-1)^2}{2}$$
$$\equiv 1-\frac{1}{3}+\frac{1}{5}-\ldots\ldots+\frac{(-1)^{n-1}}{2n-1},$$

(3) $$\frac{1}{2}+\frac{2}{2}+\frac{6}{2}+\frac{12}{2}+\ldots\ldots+\frac{n(n+1)}{2}$$
$$\equiv\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\ldots\ldots+\frac{(-1)^n}{(n+1)(n+2)},$$

(4) $$\frac{1}{1}+\frac{1}{r}+\frac{r+1}{r+1}+\frac{r+2}{r+2}+\ldots \text{ to } n \text{ quotients}$$
$$\equiv 1-\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}-\frac{1}{(r+1)(r+2)(r+3)}$$
$$+\ldots \text{ to } n \text{ terms},$$

(5) $$\frac{x}{1}+\frac{x}{2-x}+\frac{4x}{3-2x}+\ldots\ldots+\frac{n^2x}{n+1-nx}$$
$$\equiv x-\frac{x^2}{2}+\frac{x^3}{6}-\ldots\ldots+\frac{(-1)^n x^{n+1}}{n+1},$$

(6) $$\frac{x}{1}+\frac{x^2}{3-x^2}+\frac{(3x)^2}{5-3x^2}+\ldots\ldots+\frac{\{(2n-1)x\}^2}{2n+1-(2n-1)x}$$
$$\equiv x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots+\frac{(-x)^n}{2n+1},$$

(7) $$\frac{1}{1}+\frac{r}{1}+\frac{r(r+1)}{2}+\ldots\ldots+\frac{r(r+n-1)}{n}$$
$$\equiv 1-\frac{r}{r+1}+\frac{r^2}{(r+1)(r+2)}-\ldots$$
$$\ldots+\frac{(-r)^n}{(r+1)(r+2)\ldots(r+n)},$$

(8) $$\frac{1}{1}+\frac{1}{3}+\frac{16}{5}+\frac{81}{7}+\ldots\ldots+\frac{n^4}{2n+1}+\ldots \text{ to } \infty=\frac{\pi^2}{12},$$

(9) $$\frac{1}{1}+\frac{1}{1}+\frac{4}{1}+\frac{9}{1}+\ldots\ldots+\frac{n^2}{1}+\ldots \text{ to } \infty=\log 2,$$

(10) $$\frac{4}{1}+\frac{6}{2}+\frac{8}{3}+\ldots\ldots+\frac{2n}{n-1}+\ldots \text{ to } \infty=2\frac{\epsilon^2-1}{\epsilon^2+1},$$

(11) $$\frac{3^2}{1}+\frac{3.4}{2}+\frac{3.5}{3}+\ldots\ldots+\frac{3n}{n-2}+\ldots \text{ to } \infty=6\frac{2\epsilon^3+1}{5\epsilon^3-2}.$$

213. Prove that

(1) $$\frac{1}{1-}\frac{1}{4-}\frac{1}{1-}\frac{1}{4-}\ldots \text{ to } n \text{ quotients} \equiv \frac{2n}{n+1},$$

(2) $$\frac{1}{4-}\frac{1}{1-}\frac{1}{4-}\frac{1}{1-}\ldots \text{ to } n \text{ quotients} \equiv \frac{n}{2(n+1)},$$

(3) $$\frac{1}{1-}\frac{1}{3-}\frac{4}{5-}\frac{9}{7-}\ldots-\frac{n^2}{2n+1}\equiv 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1},$$

(4) $$\frac{1}{1-}\frac{1}{4-}\frac{9}{8-}\frac{25}{12-}\ldots-\frac{(2n-1)^2}{4n}\equiv 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n+1},$$

(5) $$\frac{1}{r-}\frac{r^2}{2r+1-}\frac{(r+1)^2}{2r+3-}\ldots-\frac{(r+n-1)^2}{2(r+n)-1}$$

$$\equiv\frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{r+n},$$

(6) $$\frac{1}{a-}\frac{a^2}{2a+b-}\frac{(a+b)^2}{2a+3b-}\ldots-\frac{\{a+(n-1)b\}^2}{2a+(2n-1)b}$$

$$\equiv\frac{1}{a}+\frac{1}{a+b}+\ldots+\frac{1}{a+nb},$$

(7) $$\frac{a^2}{1-}\frac{(a-1)^2}{1}-\frac{a^2}{1-}\frac{(a-1)^2}{1}-\ldots \text{ to } n \text{ terms}$$

$$\equiv\frac{a(n+2a-1)}{n+1} \text{ or } \frac{na}{n-2a+2}, \text{ as } n \text{ is odd or even,}$$

(8) $$\frac{a(a+1)}{1}-\frac{a(a-1)}{1}-\frac{a(a+1)}{1}-\ldots \text{ to } 2n \text{ terms}$$

$$\equiv\frac{a(a+1)\{(a^2-1)^n-a^{2n}\}}{(a+1)(a^2-1)^n-a^{2n+1}},$$

(9) $$\frac{x}{1-}\frac{x}{2+x-}\frac{4x}{3+2x-}\dots\frac{n^2x}{-n+1+nx}\equiv x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^{n+1}}{n+1},$$

(10) $$\frac{2}{2-}\frac{3}{3-}\frac{4}{4-}\dots-\frac{n+1}{n}\equiv 1+1+\lfloor 2+\lfloor 3+\dots+\lfloor n,$$

(11) $$\frac{1}{1-}\frac{4}{5-}\frac{9}{13-}\dots-\frac{(n^2-1)^2}{n^2+(n+1)^2}\equiv\frac{n(n+1)(2n+1)}{6},$$

(12) $$\frac{2}{1-}\frac{3}{5-}\frac{8}{7-}\dots-\frac{n^2-1}{2n+1}\equiv\frac{n(n+3)}{2},$$

(13) $$\frac{1}{1-}\frac{1^4}{1^2+2^2-}\frac{2^4}{2^2+3^2-}\dots-\frac{n^4}{n^2+(n+1)^2}\equiv\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{(n+1)^2},$$

(14) $$\frac{1}{1-}\frac{1}{3-}\frac{2}{4-}\frac{3}{5-}\dots-\frac{n}{n+2-}\dots\text{ to }\infty=\epsilon-1,$$

(15) $$1+\frac{r}{1-}\frac{r}{r+2-}\frac{2r}{r+3-}\frac{3r}{r+4-}\dots\text{ to }\infty=\epsilon^r,$$

(16) $$\frac{r^2}{2r+1-}\frac{(r+1)^2}{2r+3-}\frac{(r+2)^2}{2r+5-}\dots\text{ to }\infty=r,$$

(17) $$\frac{r}{r-}\frac{r+1}{r+1-}\frac{r+2}{r+2-}\dots\text{ to }\infty=\frac{r-1}{r-2}.$$

214. The numerator and denominator of any convergent to the fraction $\frac{r}{r-1+}\frac{r}{r-1+}\frac{r}{r-1+}\dots$ differ from each other by unity.

215. If $\frac{p_n}{q_n}$ be the n^{th} convergent to $\frac{r}{r-1+}\frac{r}{r+1+}\frac{r}{r-1+}\dots$ then will

$$(r-1)p_{2n+1}=r(q_{2n+1}-rq_{2n-1}),$$
$$(r-1)p_{2n}=r(r+1)q_{2n-1},$$
$$r(r+1)q_{2n}=(r^2+r-1)p_{2n}-r^2p_{2n-2}.$$

216. If $\dfrac{1+x}{1-2x-x^2} \equiv 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$

and $$\frac{1}{1-2x-x^2} \equiv 1 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots,$$

prove that $$a_n^2 - 2b_n^2 = (-1)^{n-1}.$$

217. If $\dfrac{2-x}{1-4x+x^2} \equiv 2 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$

and $$\frac{1}{1-4x+x^2} \equiv 1 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots,$$

prove that $$a_n^2 - 3b_n^2 = 1.$$

218. If $\dfrac{r-x}{1-2rx+x^2} \equiv r + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

and $$\frac{1}{1-2rx+x^2} \equiv 1 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots,$$

prove that $$a_n^2 - (r^2-1)\, b_n^2 = 1.$$

219. The n^{th} convergent to $1 - \dfrac{1}{4-}\,\dfrac{1}{4-}\ldots$ is equal to the $(2n-1)^{\text{th}}$ convergent to $\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\ldots$.

220. In the equation
$$x^4 - 2nx^2 - x + n(n-1) = 0,$$
prove that $$x = \pm\sqrt{n \pm \sqrt{n+x}},$$
and find all the roots of the equation. Prove that
$$\sqrt{7 - \sqrt{7 + \sqrt{7 - \ldots}}} \text{ to } \infty = 2,$$
and express the other roots of the biquadratic in the same form.

221. Prove that

$$\sqrt{p+\sqrt{p+\sqrt{p+\dots}}} \equiv \frac{m^2+mn+n^2}{2mn},$$

$$\sqrt{p-\sqrt{p-\sqrt{p-\dots}}} \equiv \frac{m^2-mn+n^2}{2mn},$$

$$\sqrt{p-\sqrt{p+\sqrt{p-\dots}}} \equiv \frac{m^2-mn-n^2}{2mn}, \ (m>n),$$

$$-\sqrt{p+\sqrt{p-\sqrt{p+\dots}}} \equiv \frac{n^2-mn-m^2}{2mn};$$

where $p \equiv \dfrac{m^4+m^2n^2+n^4}{4m^2n^2}$.

XV. *Properties of Numbers.*

222. If n be a positive whole number,

$$\begin{aligned} 2^{2n}+15n-1 &\text{ is divisible by } 9, \\ (2n+1)^5-2n-1 &\ \dots\dots\dots\dots\dots \ 120, \\ 3^{2n+2}-8n-9 &\ \dots\dots\dots\dots\dots \ 64, \\ 3^{2n+3}+40n-27 &\ \dots\dots\dots\dots\dots \ 64, \\ 3^{2n+5}+160n^2-56n-243 &\ \dots\dots\dots\dots\dots \ 512. \end{aligned}$$

223. If $2p+1$ be a prime number $(\lfloor p)^2+(-1)^p$ is divisible by $2p+1$.

224. If p be a prime number, ${}_{p-1}C_n+(-1)^{n-1}$ is divisible by p.

225. If $n-1$, $n+1$ be both prime numbers >5, n must be of one of the forms $30t$, or $30t \pm 12$; and $n^2(n^2+16)$ will be divisible by 720.

226. If $n-2$, $n+2$ be both prime numbers >5, n must be of one of the forms $30t+15$ or $30t \pm 9$.

227. If n be a whole number, $n+1$ and n^2-n+1 cannot both be square numbers.

228. The whole number next greater than $(3+\sqrt{5})^n$ is divisible by 2^n.

229. The integral part of $\frac{1}{\sqrt{3}}(\sqrt{3}+\sqrt{5})^{2n-1}$ is divisible by 2^n.

230. The equation $x^3-2y^3=\pm 1$ cannot be satisfied by any integral values of x and y different from unity.

231. The sum of the squares of all the numbers less than a given number N and prime to it is

$$\frac{N^3}{3}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots+\frac{N}{6}(1-a)(1-b)(1-c)\ldots$$

and the sum of the cubes is

$$\frac{N^4}{4}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots+\frac{N^2}{4}(1-a)(1-b)(1-c)\ldots,$$

a, b, c being the different prime factors of N.

XVI. *Probabilities.*

232. A and B throw for a certain stake, each one throw with one die. A's die is marked 2, 3, 4, 5, 6, 7 and B's 1, 2, 3, 4, 5, 6; and equal throws divide the stake. Prove that A's expectation is $\frac{47}{72}$ of the stake. What will A's expectation be if equal throws go for nothing?

233. A certain sum of money is to be given to the one of three persons A, B, C who first throws 10 with three dice; supposing them to throw in the order named until the event happen, prove that A's chance of winning is $\left(\frac{8}{13}\right)^2$, and C's $\left(\frac{7}{13}\right)^2$.

234. Ten persons each write down one of the digits 0, 1, 2...9 at random; find the probability of all ten digits being written.

235. A throws a pair of dice each of which is a cube; B throws a pair one of which is a regular octahedron and the other a regular tetrahedron whose faces are marked from 1 to 8, and from 1 to 4 respectively; which throw is likely to be the higher, the number on the lowest face being taken in the case of the tetrahedron? If A throws 6, what is the chance that B will throw higher?

236. The sum of two positive quantities is known, prove that it is an even chance that their product will be not less than three fourths of their greatest possible product.

237. Two points are taken at random on a given straight line of length a: prove that the probability of their distance exceeding a given length $c\,(<a)$ is $\left(\frac{a-c}{a}\right)^2$.

238. If three points be taken at random on the circumference of a circle the probability of their lying on the same semicircle is $\frac{3}{4}$.

239. If q things be distributed among p persons, the chance that every one of the persons will have at least one is the coefficient of x^q in the expansion of $\underline{|q}\,(\epsilon^{\frac{x}{p}}-1)^p$.

240. If a rod be marked at random in n points and divided at those points the chance that none of the parts shall be greater than $\frac{1}{n}$th of the rod is $\frac{1}{n^n}$.

241. There are $2m$ black balls and m white balls from which 6 balls are drawn at random; prove that when m is very large

the chance of drawing 4 black and 2 white $= \frac{80}{243}$, and the chance of drawing 2 black and 4 white $= \frac{20}{243}$.

242. If n whole numbers taken at random be multiplied together, the chance of the last digit in the product being 1, 3, 7, or 9 is $\left(\frac{2}{5}\right)^n$; of its being 2, 4, 6, or 8 is $\frac{4^n - 2^n}{5^n}$; of its being 5 is $\frac{5^n - 4^n}{10^n}$; and of its being 0 is $\frac{10^n - 8^n - 5^n + 4^n}{10^n}$.

243. If ten things be distributed among three persons, the chance of a particular person having more than 5 of them is $\frac{1507}{19683}$.

244. If on a straight line of length $a + b$ be measured at random two lengths a, b, the probability that the common part of these lengths shall not exceed c is $\frac{c^2}{ab}$, ($c < a$ or b): and the probability of the smaller b lying entirely within the larger a is $\frac{a - b}{a}$.

245. If on a straight line of length $a + b + c$ be measured at random lengths a, b, the chance of their having a common part which shall not be greater than d is $\frac{(c + d)^2}{(c + a)(c + b)}$. ($d <$ either a or b).

246. There are $m + p + q$ coins in a bag each of which is equally likely to be a shilling or a sovereign; $p + q$ being drawn, p are shillings, and q sovereigns: prove that the value of the expectation of the remaining sovereigns in the bag is $\frac{m(q + 1)}{p + q + 2}$ £. If $m = 5$, $p = 2$, $q = 1$, find the chance that if two more coins be drawn they will be a shilling and a sovereign, (1) when the coins previously drawn are not replaced, (2) when they are replaced.

247. A bag contains ten balls each equally likely to be white or black: three balls being drawn turn out two white and one black; these are replaced and five are then drawn, two white and three black: prove that the chance of a draw from the remaining five giving a white ball is $\frac{71}{128}$.

248. From a very large number of balls, each of which is equally likely to be white or black, a ball is drawn and replaced p times and each drawing gives a white ball: prove that the chance of drawing a white ball at the next draw is $\frac{p+1}{p+2}$.

249. A bag contains four white and four black balls; from these four are drawn at random and placed in another bag; three draws are made from the latter the ball being replaced after each draw, and each gives a white ball: prove that the chance of the next draw giving a black ball is ·33.

250. A bag contains m white balls and n black balls, and from it balls are drawn one by one until a white ball is drawn. A bets B at each draw $x : y$ that a black ball is drawn: prove that the value of A's expectation at the beginning of the drawing is $\frac{ny}{m+1} - x$.

251. From an unknown number of balls, each equally likely to be white or black, a ball is drawn and turns out to be white; this is not replaced and $2n$ more draws are made, the balls being not replaced. Prove that the probability that in the $2n+1$ draws more white balls are drawn than black is $\frac{3n+2}{4n+2}$.

XVII. *Miscellaneous Questions.*

252. By performing the operation for the extraction of the square root find a value of x which will make

$$x^4 + 6x^3 + 11x^2 + 3x + 31$$

a perfect square.

253. If $l_2 l_3 + m_2 m_3 = l_3 l_1 + m_3 m_1 = l_1 l_2 + m_1 m_2 = 1$,

then will $$\frac{(m_2 - m_3)(m_3 - m_1)(m_1 - m_2)}{(l_2 - l_3)(l_3 - l_1)(l_1 - l_2)} + \frac{l_1 l_2 l_3}{m_1 m_2 m_3} = 0.$$

254. Prove that

$$(ac - b^2)\{x(x-h) + y(y-k)\}^2 - c(x-h)^2 + 2b(x-h)(y-k) - a(y-k)^2$$

will be divisible by $(x-h)^2 + (y-k)^2$ if

$$\frac{h^2 - k^2}{a - c} = \frac{hk}{b} = \frac{1}{b^2 - ac}.$$

255. Having given the equations

$$lx + my + nz = 0, \quad (b-c)\frac{x}{l} + (c-a)\frac{y}{m} + (a-b)\frac{z}{n} = 0,$$

$$x^2 + y^2 + z^2 = \frac{(b-c)^2}{l^2} + \frac{(c-a)^2}{m^2} + \frac{(a-b)^2}{n^2};$$

prove that

$$l^2 yz(mz - ny) + m^2 zx(nx - lz) + n^2 xy(ly - mx)$$

$$= \frac{(l^2 + m^2 + n^2)^{\frac{3}{2}}}{lmn}(b-c)(c-a)(a-b).$$

256. The product of any r consecutive terms of the series $1 - c$, $1 - c^2$, $1 - c^3$, ... is completely divisible by the product of the first r terms.

257. The equation $\frac{a_1}{x+b_1}+\frac{a_2}{x+b_2}+\ldots+\frac{a_n}{x+b_n}=0$ will reduce to a simple equation if

$$a_1+a_2+\ldots+a_n=0,$$
$$a_1b_1+a_2b_2+\ldots+a_nb_n=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0,$$
$$a_1b_1^{n-3}+a_2b_2^{n-3}+\ldots+a_nb_n^{n-3}=0,$$

and the value of x will in that case be

$$b_1b_2\ldots b_n\frac{\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots+\frac{a_n}{b_n}}{a_1b_1^{n-2}+\ldots+a_nb_n^{n-2}}.$$

258. Prove that

$$(a+b+c+d)^3-4(a+b+c+d)(bc+ca+ab+ad+bd+cd)$$
$$+8(bcd+cda+dab+abc)$$
$$\equiv(b+c-a-d)(c+a-b-d)(a+b-c-d).$$

259. Having given

$$a+b+c+a'+b'+c'=0,\quad a^3+b^3+c^3+a'^3+b'^3+c'^3=0,$$

prove that $$\frac{a^7+\ldots+c'^7}{7}=\frac{a^2+\ldots+c'^2}{2}\,\frac{a^5+\ldots+c'^5}{5}.$$

260. If a, b, c be three positive quantities, of which any two are together greater than the third,

$$(b+c-a)^2(c+a-b)^2(a+b-c)^2$$
$$>(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2),$$

unless a, b, c are all equal.

261. Prove that

$${}_nC_r\equiv{}_{n+p}C_r-p\,{}_{n+p}C_{r-1}$$
$$+\frac{p(p+1)}{\lfloor 2}\,{}_{n+p}C_{r-2}+\ldots+(-1)^r\frac{p(p+1)\ldots(p+r-1)}{\lfloor r}.$$

262. Prove that

$$\frac{1}{1-}\,\frac{1}{2-}\,\frac{3}{4-}\,\frac{5}{6-}\dots-\frac{2n-1}{2n} \equiv 2+1.3+1.3.5+\dots+1.3.5\dots(2n-1),$$

and $$\frac{2n}{2n-1-}\,\frac{2n-2}{2n-3-}\dots-\frac{4}{3-}\,\frac{2}{1} \equiv 2n,$$

263. If $$x \equiv \frac{n}{n+}\,\frac{n+1}{n+1+}\,\frac{n+2}{n+2+}\dots \text{ to } \infty,$$

prove that $$\frac{1}{n+x} = \frac{1}{n} - \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} - \dots \text{ to } \infty.$$

264. The $2n+1^{\text{th}}$ convergent to $\dfrac{1}{1+}\,\dfrac{1}{2p+}\,\dfrac{1}{1+}\,\dfrac{1}{2p+}\dots$ is equal to the n^{th} convergent to $1-\dfrac{1}{2(p+1)-}\,\dfrac{1}{2(p+1)-}\dots$.

265. If $$u_n \equiv \frac{1-n(n-1)}{\lfloor 2} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} - \dots,$$

and $$v_n \equiv n - \frac{n(n-1)(n-2)}{\lfloor 3} + \frac{n(n-1)\dots(n-4)}{\lfloor 5} - \dots,$$

then will $u_n^2 + v_n^2 \equiv 2(u_n u_{n-1} + v_n v_{n-1})$; n being a positive whole number.

266. If $$u_n \equiv 1 + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}x^4 + \dots$$

and $$v_n \equiv nx + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \dots,$$

then will $u_n u_{n-1} - v_n v_{n-1} \equiv (1-x^2)^{n-1}$; n being a positive whole number.

PLANE TRIGONOMETRY.

I. *Equations.*

In the solution of Trigonometrical Equations, it must be remembered that when an equation has been reduced to the forms (1) $\sin x = \sin \alpha$, (2) $\cos x = \cos \alpha$, (3) $\tan x = \tan \alpha$, the solutions are (1) $x = n\pi + (-1)^n \alpha$, (2) $x = 2n\pi \pm \alpha$, (3) $x = n\pi + \alpha$, n denoting an integer positive or negative.

The formulæ most useful in Trigonometrical reductions are

$$\left.\begin{aligned} 2\sin A\cos B &\equiv \sin(A+B)+\sin(A-B),\\ 2\cos A\cos B &\equiv \cos(A+B)+\cos(A-B),\\ 2\sin A\sin B &\equiv \cos(A-B)-\cos(A+B);\end{aligned}\right\}$$

and

$$\left.\begin{aligned} \sin A+\sin B &\equiv 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},\\ \cos A+\cos B &\equiv 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},\\ \cos A-\cos B &\equiv 2\sin\frac{B-A}{2}\sin\frac{A+B}{2};\end{aligned}\right\}$$

which enables us to transform products of Trigonometrical functions, (sines or cosines) into sums of such functions and conversely. Thus, to transform

$$\sin 2\{(\beta-\gamma)\}+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta);$$

$$\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta) \equiv 2\sin(\gamma-\beta)\cos(\beta+\gamma-2\alpha),$$

$$\sin 2(\beta-\gamma) \equiv 2\sin(\beta-\gamma)\cos(\beta-\gamma);$$

$$\begin{aligned}\therefore\ \sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta) &\\ \equiv 2\sin(\beta-\gamma)\{\cos(\beta-\gamma)-\cos(\beta+\gamma-2\alpha)\} &\\ \equiv 2\sin(\beta-\gamma)\{2\sin(\beta-\alpha)\sin(\gamma-\alpha)\} &\\ \equiv -4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta). &\end{aligned}$$

So to transform $\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$,

$$\cos(\gamma-\alpha)\cos(\alpha-\beta) \equiv \tfrac{1}{2}\{\cos(\gamma-\beta)+\cos(\beta+\gamma-2\alpha)\};$$

$$\therefore \cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$$
$$\equiv \tfrac{1}{4}\{1+\cos 2(\gamma-\beta)+\cos 2(\gamma-\alpha)+\cos 2(\alpha-\beta)\}.$$

267. Solve the equations;

(1) $2\sin x \sin 3x = 1$,

(2) $\cos x \cos 3x = \cos 2x \cos 6x$,

(3) $\sin 5x \cos 3x = \sin 9x \cos 7x$,

(4) $\sin 9x + \sin 5x + 2\sin^2 x = 1$,

(5) $\cos ax \cos bx = \cos(a+c)x \cos(b+c)x$,

(6) $4\sin\alpha \sin\beta \sin x = \sin 2\alpha + \sin 2\beta + \sin 2x$,

(7) $\cos x + \cos(x-\alpha) = \cos(x-\beta) + \cos(x+\beta-\alpha)$,

(8) $2\sin^2 2x \cos 2x = \sin^2 3x$,

(9) $2\cot 2x - \tan 2x = 3\cot 3x$,

(10) $8\cos x = \dfrac{\sqrt{3}}{\sin x} + \dfrac{1}{\cos x}$,

(11) $\sin 2x + \cos 2x + \sin x - \cos x = 0$,

(12) $(1+\sin x)(1-2\sin x)^2 = (1-\cos\alpha)(1+2\cos\alpha)^2$,

(13) $\dfrac{\sin\alpha\cos(\beta+x)}{\sin\beta\cos(\alpha+x)} = \dfrac{\tan\beta}{\tan\alpha}$,

(14) $\cos 2x + 2\cos x\cos\alpha - 2\cos 2\alpha = 1$,

(15) $\sin\alpha\cos 3x - 3\sin 3\alpha\cos x + \sin 4\alpha + 2\sin 2\alpha = 0$,

(16) $\dfrac{\cos^3\alpha}{\cos x} + \dfrac{\sin^3\alpha}{\sin x} = 1$.

268. If $\qquad \cos(x+3y) = \sin(2x+2y)$,

and $\qquad \sin(3x+y) = \cos(2x+2y)$;

then will
$$\left.\begin{aligned} x &= (5m-3n)\frac{\pi}{8}+\frac{\pi}{16} \\ y &= (5n-3m)\frac{\pi}{8}+\frac{\pi}{16} \end{aligned}\right\}; \text{ or } x-y = 2r\pi+\frac{\pi}{2},$$

m, n, r being integers.

269. The real roots of the equation $\tan^3 x \tan \frac{x}{2} = 1$ satisfy the equation $\cos 2x = 2 - \sqrt{5}$.

270. Given $\cos 3x = -\frac{3\sqrt{3}}{4\sqrt{2}}$; prove that the three values of

$$\cos x \text{ are } \sqrt{\frac{3}{2}} \sin \frac{\pi}{10}, \quad \sqrt{\frac{3}{2}} \sin \frac{\pi}{6}, \quad -\sqrt{\frac{3}{2}} \sin \frac{3\pi}{10}.$$

271. If the equation $\tan \frac{x}{2} = \frac{\tan x + a - 1}{\tan x + a + 1}$ have real roots, $a^2 > 1$.

272. Find the limits of the value of $\frac{\tan (x + a)}{\tan (x - a)}$ for possible values of x.

273. If β, γ be different values of x given by the equation $\sin (a + x) = m \sin 2a$,

$$\cos \frac{\beta - \gamma}{2} \pm m \sin (\beta + \gamma) = 0.$$

274. The real values of x which satisfy by the equation

$$\sin \left(\frac{\pi}{2} \cos x\right) = \cos \left(\frac{\pi}{2} \sin x\right) \text{ are } 2n\pi \text{ or } 2n\pi \pm \frac{\pi}{2},$$

n being an integer.

275. If x, y be real, and if

$$\sin^2 x \sin^2 y + \sin^2 (x + y) = (\sin x + \sin y)^2,$$

x, or y, must be a multiple of π.

276. If a, β, γ be three angles, unequal and less than 2π, which satisfy the equation

$$\frac{a}{\cos x} + \frac{b}{\sin x} + c = 0,$$

then will $\quad \sin (\beta + \gamma) + \sin (\gamma + a) + \sin (a + \beta) = 0.$

277. If β, γ be angles, unequal and less than π, which satisfy the equation

$$\frac{\cos a \cos x}{a} + \frac{\sin a \sin x}{b} = \frac{1}{c},$$

then will $(b^2+c^2-a^2)\cos\beta\cos\gamma + (c^2+a^2-b^2)\sin\beta\sin\gamma = a^2+b^2-c^2$.

278. If α, β be angles, unequal and less than π, which satisfy the equation

$$a \cos 2x + b \sin 2x = 1,$$

and if $\qquad (l\cos^2 2\alpha + m\sin^2 2\alpha)(l\cos^2 2\beta + m\sin^2 2\beta)$

$$= \{l\cos^2(\alpha+\beta) + m\sin^2(\alpha+\beta)\}^2,$$

then will either $l = m$, or $a^2 - b^2 = \dfrac{m-l}{m+l}$.

II. *Identities and Equalities.*

279. If $\tan^2 A = 1 + 2\tan^2 B$, then will $\cos 2B = 1 + 2\cos 2A$.

280. Having given that $\sin(B+C-A)$, $\sin(C+A-B)$, $\sin(A+B-C)$ are in A. P.; prove that $\tan A$, $\tan B$, $\tan C$ are in A. P.

281. Having given that

$$1 + \cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = 0;$$

prove that $(\beta-\gamma)$, $(\gamma-\alpha)$, or $(\alpha-\beta)$ is an odd multiple of π.

282. If $\cot\alpha$, $\cot\beta$, $\cot\gamma$ be in A. P., so also will $\cot(\beta-\alpha)$, $\cot\beta$, $\cot(\beta-\gamma)$.

283. Having given

$$\cos\theta = \cos\alpha\ \cos\beta,\quad \cos\theta' = \cos\alpha'\cos\beta,\quad \tan\frac{\theta}{2}\tan\frac{\theta'}{2} = \tan\frac{\beta}{2};$$

prove that $\sin^2\beta = (\sec\alpha - 1)(\sec\alpha' - 1)$.

284. If $\tan 2\alpha = 2\dfrac{ab+cd}{a^2-b^2+c^2-d^2}$, $\tan 2\beta = 2\dfrac{ac+bd}{a^2-c^2+b^2-d^2}$, then will $\tan(\alpha-\beta)$ be equal to $\dfrac{b+c}{a-d}$ or to $\dfrac{d-a}{b+c}$.

285. If $\tan(\phi+\alpha)$, $\tan\phi$, $\tan(\phi+\beta)$ be in A. P. then will $\cot\alpha$, $\tan\phi$, $\cot\beta$ be in A. P.

286. If α, β, γ be all unequal and less than 2π, and if

$$\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0;$$

then will $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \sin^2\alpha+\sin^2\beta+\sin^2\gamma = \dfrac{3}{2}$.

287. Having given $\dfrac{\tan 2\beta}{\tan(\gamma+\alpha)} = \dfrac{\tan 2\gamma}{\tan(\alpha+\beta)}$; prove that each member of the equation is equal to $\dfrac{\tan 2\alpha}{\tan(\beta+\gamma)}$,

and that $\quad \sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta) = 0,$

unless $\tan(\beta-\gamma) = 0$.

288. Prove that

$$\left(1-\tan^2\frac{a}{2}\right)\left(1-\tan^2\frac{a}{2^2}\right)\left(1\ \tan^2\frac{a}{2^3}\right)\ldots\ldots \text{ to } \infty = \frac{a}{\tan a}.$$

289. If

$$C \equiv 2\cos\theta - 5\cos^3\theta + 4\cos^5\theta,\quad S \equiv 2\sin\theta - 5\sin^3\theta + 4\sin^5\theta;$$

then will

$$C\cos 3\theta + S\sin 3\theta \equiv \cos 2\theta, \text{ and } S\cos 3\theta - C\sin 3\theta \equiv -\frac{1}{2}\sin 2\theta.$$

290. Having given

$$x\cos\phi + y\sin\phi = x\cos\phi' + y\sin\phi' = 2a, \text{ and } 2\cos\frac{\phi}{2}\cos\frac{\phi'}{2} = 1;$$

prove that $y^2 = 4a(a-x)$; ϕ, ϕ' being unequal and less than 2π.

291. Having given

$$(x-a)\cos\theta + y\sin\theta = (x-a)\cos\theta' + y\sin\theta' = a,$$

$$\text{and } \tan\frac{\theta}{2} - \tan\frac{\theta'}{2} = \pm 2e;$$

prove that $y^2 = 2ax - (1-e^2)x^2$, θ, θ' being unequal and less than 2π.

292. If $(1+\sin\theta)(1+\sin\phi)(1+\sin\psi) = \cos\theta\cos\phi\cos\psi$,

then will $\sec^2\theta + \sec^2\phi + \sec^2\psi - 2\sec\theta\sec\phi\sec\psi = 1.$

293. Having given

$$\cos A + \cos B + \cos C + \cos A\cos B\cos C = 0;$$

prove that

$$\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C \pm 2\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C = 1.$$

294. If

$$\tan^2 A\tan A' = \tan^2 B\tan B' = \tan^2 C\tan C' = \tan A\tan B\tan C,$$

and $\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0;$

then will

$$\tan(A-A') = \tan(B-B') = \tan(C-C') = \tan A + \tan B + \tan C.$$

295. Having given

$$x\sin 3(\beta-\gamma) + y\sin 3(\gamma-\alpha) + z\sin 3(\alpha-\beta) = 0;$$

prove that

$$\frac{x\sin(\beta-\gamma) + y\sin(\gamma-\alpha) + z\sin(\alpha-\beta)}{x\cos(\beta-\gamma) + y\cos(\gamma-\alpha) + z\cos(\alpha-\beta)} + \frac{\sin 2(\beta-\gamma) + \sin 2(\gamma-\alpha) + \sin 2(\alpha-\beta)}{\cos 2(\beta-\gamma) + \cos 2(\gamma-\alpha) + \cos 2(\alpha-\beta)} = 0.$$

296. Having given the equations

$$\left.\begin{aligned} x^2 &= \beta^2 + \gamma^2 - 2\beta\gamma\cos\theta \\ y^2 &= \gamma^2 + \alpha^2 - 2\gamma\alpha\cos\phi \\ z^2 &= \alpha^2 + \beta^2 - 2\alpha\beta\cos\psi \end{aligned}\right\}, \qquad \left.\begin{aligned} x+y+z&=0 \\ \theta+\phi+\psi&=0 \end{aligned}\right\},$$

prove that $\beta\gamma\sin\theta + \gamma\alpha\sin\phi + \alpha\beta\sin\psi = 0.$

297. If $\cos A + \cos B \cos C + \sin B \sin C \cos A = 0$,

then will $\tan\left(45° \pm \frac{A}{2}\right) \tan\left(45° - \frac{B}{2}\right) \tan\left(45° - \frac{C}{2}\right) = -1$;

and if $\cos A = \cos B \cos C \pm \sin B \sin C \cos A$,

then will $\cos B = \cos C \cos A \pm \sin C \sin A \cos B$.

298. Reduce to its simplest form the equation

$$\{x \cos(\alpha+\beta) + y \sin(\alpha+\beta) - \cos(\alpha-\beta)\}\{x \cos(\gamma+\delta) + y \sin(\gamma+\delta) - \cos(\gamma-\delta)\}$$
$$= \{x \cos(\alpha+\gamma) + y \sin(\alpha+\gamma) - \cos(\alpha-\gamma)\}\{x \cos(\beta+\delta) + y \sin(\beta+\delta) - \cos(\beta-\delta)\}.$$

299. Given the equations

$$yz' - y'z + zx' - z'x + xy' - x'y = 0, \quad A + B + C = 180°,$$
$$xx' \sin^2 A + yy' \sin^2 B + zz' \sin^2 C = (yz' + y'z) \sin B \sin C \cos A$$
$$+ (zx' + z'x) \sin C \sin A \cos B + (xy' + x'y) \sin A \sin B \cos C;$$

prove that, for real values, either $x = y = z$; or $x' = y' = z'$.

300. If $A + B + C = 180°$, and if

$$\frac{y^2 + z^2 + 2yz \cos A}{\sin^2 A} = \frac{z^2 + x^2 + 2zx \cos B}{\sin^2 B} = \frac{x^2 + y^2 + 2xy \cos C}{\sin^2 C};$$

then will either

$$x \sin A + y \sin B + z \sin C = 0, \quad \text{or} \quad \frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{\cos C}.$$

301. Having given the equations

$$\frac{y^2 + z^2 - 2yz \cos \alpha}{\sin^2 \alpha} = \frac{z^2 + x^2 - 2zx \cos \beta}{\sin^2 \beta} = \frac{x^2 + y^2 - 2xy \cos \gamma}{\sin^2 \gamma},$$

then will

$$\left.\begin{aligned}
&\frac{x}{\sin(s-\alpha)}=\frac{y}{\sin(s-\beta)}=\frac{z}{\sin(s-\gamma)},\\
&\frac{x}{\sin s}=\frac{y}{\sin(s-\gamma)}=\frac{z}{\sin(s-\beta)},\\
&\frac{x}{\sin(s-\gamma)}=\frac{y}{\sin s}=\frac{z}{\sin(s-\alpha)},\\
\text{or}\quad&\frac{x}{\sin(s-\beta)}=\frac{y}{\sin(s-\alpha)}=\frac{z}{\sin s};
\end{aligned}\right\}\ 2s\equiv\alpha+\beta+\gamma.$$

302. Having given

$$\frac{\cos\alpha}{\cos\theta}+\frac{\sin\alpha}{\sin\theta}=-1;\quad\text{prove that}\quad\frac{\cos^3\theta}{\cos\alpha}+\frac{\sin^3\theta}{\sin\alpha}=1.$$

303. If α, β, γ be unequal and less than π, and if

$$e=\frac{\sin(\alpha-\beta)+\sin(\alpha-\gamma)}{\sin\beta+\sin\gamma-2\sin\alpha}=\frac{\sin(\beta-\gamma)+\sin(\beta-\alpha)}{\sin\alpha+\sin\gamma-2\sin\beta};$$

then will

$$\sin\alpha+\sin\beta+\sin\gamma=0,\quad\cos\alpha+\cos\beta+\cos\gamma=-3e.$$

304. If

$$\tan\frac{\alpha+\theta}{2}\tan\beta=\tan\frac{\beta+\theta}{2}\tan\alpha,\ \text{ and }\ \sin\frac{\alpha-\beta}{2}\ \text{ be finite,}$$

then will $$\sin(\alpha+\beta)=\sin\left(\alpha+\frac{\theta}{2}\right)+\sin\left(\beta+\frac{\theta}{2}\right).$$

305. Eliminate θ, ϕ from the equations

$$\left.\begin{aligned}
\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1,\\
4\cos\frac{\theta-\phi}{2}\cos\frac{\alpha-\theta}{2}\cos\frac{\alpha-\phi}{2}=1.
\end{aligned}\right\}$$

306. Having given the equations

$$a^2+b^2-2ab\cos\alpha=c^2+d^2-2cd\cos\gamma,$$
$$b^2+c^2-2bc\cos\beta=d^2+a^2-2ad\cos\delta,$$
$$ab\sin\alpha+cd\sin\gamma=bc\sin\alpha+ad\sin\delta;$$

prove that $$\cos(\alpha+\gamma)=\cos(\beta+\delta).$$

307. Having given $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$; prove that

$$(1)\quad \tan\frac{\theta}{2} + \tan\frac{\phi}{2} = \frac{4a}{a^2 + b^2 + 2b},$$

$$(2)\quad \tan\theta + \tan\phi = \frac{4ab}{(a^2 + b^2)^2 - 4a^2},$$

$$(3)\quad \cos\theta\cos\phi = \frac{(a^2 + b^2)^2 - 4a^2}{4(a^2 + b^2)},$$

$$(4)\quad \sin\theta\sin\phi = \frac{(a^2 + b^2)^2 - 4b^2}{4(a^2 + b^2)},$$

$$(5)\quad \cos 2\theta + \cos 2\phi = \frac{(b^2 - a^2)(a^2 + b^2 - 2)}{a^2 + b^2},$$

$$(6)\quad \cos 3\theta + \cos 3\phi = b\left\{4b^2 - 3 - 3\,\frac{(a^2 + b^2)^2 - 4a^2}{a^2 + b^2}\right\}.$$

308. Having given

$$e\cos(\beta + \gamma) + \cos(\beta - \gamma) = e\cos(\gamma + \alpha) + \cos(\gamma - \alpha)$$
$$= e\cos(\alpha + \beta) + \cos(\alpha - \beta),$$

α, β, γ being unequal and less than π; prove that each member of these equations is equal to $\frac{e^2 - 1}{2}$, and that

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0.$$

309. Having given

$$A\cos(\beta - \gamma) + B(\cos\beta + \cos\gamma) + C(\sin\beta + \sin\gamma) + D = 0,$$
$$A\cos(\gamma - \alpha) + B(\cos\gamma + \cos\alpha) + C(\sin\gamma + \sin\alpha) + D = 0,$$
$$A\cos(\alpha - \beta) + B(\cos\alpha + \cos\beta) + C(\sin\alpha + \sin\beta) + D = 0;$$

prove that

$$A^2 + B^2 + C^2 = 2AD, \text{ unless } \sin\frac{\beta - \gamma}{2}\sin\frac{\gamma - \alpha}{2}\sin\frac{\alpha - \beta}{2} = 0.$$

310. Having given the equation

$$A\cos(\beta - \gamma) + B\cos(\beta + \gamma) + C\sin(\beta + \gamma)$$
$$+ A'(\cos\beta + \cos\gamma) + B'(\sin\beta + \sin\gamma) + C' = 0,$$

and the two like equations between γ, α, and α, β respectively; prove that if

$$\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$$

be finite, $\quad B^2+C^2-A^2-A'^2-B'^2+2AC'=0.$

311. Eliminate θ from the equations

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad x\sin\theta-y\cos\theta=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}.$$

312. Having given the equations

$$\tan\frac{\beta+\gamma+\theta}{2}\tan\alpha=\tan\frac{\gamma+\alpha+\theta}{2}\tan\beta=\tan\frac{\alpha+\beta+\theta}{2}\tan\gamma=\frac{m}{n};$$

prove that

$$\left.\begin{array}{c}\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0\\ \dfrac{\cos\beta\cos\gamma}{n}-\dfrac{\sin\beta\sin\gamma}{m}=\dfrac{1}{m-n}\\ \&c.\end{array}\right\};\quad \alpha,\beta,\gamma \text{ being unequal, and } <\pi.$$

313. Prove that

$$\cos^2\theta-2\cos\theta\cos\alpha\cos(\alpha+\theta)+\cos^2(\alpha+\theta)$$

is independent of θ.

314. Prove that

$$\sin 2\alpha\cos\beta\cos\gamma\sin(\beta-\gamma)+\sin 2\beta\cos\gamma\cos\alpha\sin(\gamma-\alpha)$$
$$+\sin 2\gamma\cos\alpha\cos\beta\sin(\alpha-\beta)\equiv 0,$$

$$\cos 2\alpha\cos\beta\cos\gamma\sin(\beta-\gamma)+\ldots\ldots+\ldots\ldots$$
$$\equiv\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

315. Prove that

$$2\sin\frac{3\theta}{2}\left\{\sin^3\frac{5\theta}{2}-\sin^3\frac{3\theta}{2}\right\}$$
$$\equiv\cos^3\theta+\cos^3 2\theta+\cos^3 3\theta-3\cos\theta\cos 2\theta\cos 3\theta,$$

and $$2\sin\frac{3\theta}{2}\left\{\cos^3\frac{3\theta}{2}-\cos^3\frac{5\theta}{2}\right\}$$
$$\equiv\sin^3\theta+\sin^3 2\theta+\sin^3 3\theta-3\sin\theta\sin 2\theta\sin 3\theta.$$

316. Prove that

$$\sin 2\alpha \sin(\beta-\gamma) + \sin 2\beta \sin(\gamma-\alpha) + \sin 2\gamma \sin(\alpha-\beta)$$
$$\equiv \{\sin(\gamma-\beta) + \sin(\alpha-\gamma) + \sin(\beta-\alpha)\}$$
$$\{\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)\},$$

and $\cos 2\alpha \sin(\beta-\gamma) + \cos 2\beta \sin(\gamma-\alpha) + \cos 2\gamma \sin(\alpha-\beta)$

$$\equiv \{\sin(\gamma-\beta) + \sin(\alpha-\gamma) + \sin(\beta-\alpha)\}$$
$$\{\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta)\}.$$

317. Prove that

(1) $$\frac{\sin 3\alpha \sin(\beta-\gamma) + \sin 3\beta \sin(\gamma-\alpha) + \sin 3\gamma \sin(\alpha-\beta)}{\cos 3\alpha \sin(\beta-\gamma) + \cos 3\beta \sin(\gamma-\alpha) + \cos 3\gamma \sin(\alpha-\beta)} \equiv \tan(\alpha+\beta+\gamma),$$

(2) $$\frac{\sin 5\alpha \sin(\beta-\gamma) + \sin 5\beta \sin(\gamma-\alpha) + \sin 5\gamma \sin(\alpha-\beta)}{\cos 5\alpha \sin(\beta-\gamma) + \cos 5\beta \sin(\gamma-\alpha) + \cos 5\gamma \sin(\alpha-\beta)} \equiv \frac{\sin(3\alpha+\beta+\gamma) + \ldots + \ldots}{\cos(3\alpha+\beta+\gamma) + \ldots + \ldots},$$

and that

(3) $$\frac{\sin 7\alpha \sin(\beta-\gamma) + \sin 7\beta \sin(\gamma-\alpha) + \sin 7\gamma \sin(\alpha-\beta)}{\cos 7\alpha \sin(\beta-\gamma) + \cos 7\beta \sin(\gamma-\alpha) + \cos 7\gamma \sin(\alpha-\beta)}$$
$$\equiv \frac{\sin(\alpha+3\beta+3\gamma) + \ldots + \ldots + \sin(5\alpha+\beta+\gamma) + \ldots + \ldots}{\cos(\alpha+3\beta+3\gamma) + \ldots + \ldots + \cos(5\alpha+\beta+\gamma) + \ldots + \ldots}.$$

318. Prove that

$$\frac{\cos^3\alpha \sin(\beta-\gamma) + \cos^3\beta \sin(\gamma-\alpha) + \cos^3\gamma \sin(\alpha-\beta)}{\sin^3\alpha \sin(\beta-\gamma) + \sin^3\beta \sin(\gamma-\alpha) + \sin^3\gamma \sin(\alpha-\beta)} + \cot(\alpha+\beta+\gamma) \equiv 0.$$

319. Prove that

$$\frac{1}{2\cos\alpha -} \, \frac{1}{2\cos\alpha -} \, \frac{1}{2\cos\alpha -} \ldots - \frac{1}{2\cos\alpha + a} \equiv \frac{\sin n\alpha + a\sin(n-1)\alpha}{\sin(n+1)\alpha + a\sin n\alpha};$$

there being n quotients in the left hand member.

320. From the identity

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-b)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}\equiv x^2,$$

deduce the identities

$$\cos 2(\theta+\alpha)\frac{\sin(\theta-\alpha)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\ldots+\ldots\equiv\cos 4\theta,$$

$$\sin 2(\theta+\alpha)\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\ldots+\ldots\equiv\sin 4\theta.$$

321. Prove the identity

$$\cos(\beta+\gamma-\alpha-\delta)\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\delta}{2}$$
$$+\cos(\gamma+\alpha-\beta-\delta)\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta-\delta}{2}$$
$$+\cos(\alpha+\beta-\gamma-\delta)\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\delta}{2}$$
$$\equiv 8\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}\sin\frac{\delta-\alpha}{2}\sin\frac{\delta-\beta}{2}\sin\frac{\delta-\gamma}{2}.$$

III. *Inequalities.*

322. Prove that $\cot\frac{\theta}{2}>1+\cot\theta$ for values of θ between 0 and π.

323. Prove that, for real values of x, $\frac{x^2-2x\cos\alpha+1}{x^2-2x\cos\beta+1}$ lies between $\frac{1-\cos\alpha}{1-\cos\beta}$, and $\frac{1+\cos\alpha}{1+\cos\beta}$.

324. If x, y, z be any real quantities, and A, B, C the angles of a triangle,

$$x^2+y^2+z^2>2yz\cos A+2zx\cos B+2xy\cos C,$$

unless
$$\frac{x}{\sin A}=\frac{y}{\sin B}=\frac{z}{\sin C}.$$

325. Having given the equation

$$\sec\beta\sec\gamma + \tan\beta\tan\gamma = \tan\alpha,$$

prove that, for real values of β and γ, $\cos 2\alpha$ must be negative; and that

$$\frac{\tan\beta + \tan\alpha\tan\gamma}{\tan\gamma + \tan\alpha\tan\beta} \pm \frac{\cos\beta}{\cos\gamma} = 0.$$

326. If $A + B + C = 180^\circ$, prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > (1-\cos A)(1-\cos B)(1-\cos C)$$
$$> \cos A\cos B\cos C,$$

unless $A = B = C$. Also

$$\sin A\sin B\sin C > \sin 2A\sin 2B\sin 2C.$$

327. Prove that

$$\left\{\frac{\cos^2(\alpha-\theta)}{a^2} + \frac{\sin^2(\alpha-\theta)}{b^2}\right\}\left\{\frac{\cos^2(\alpha+\theta)}{a^2} + \frac{\sin^2(\alpha+\theta)}{b^2}\right\} \nless \frac{\sin^2 2\alpha}{a^2b^2};$$

and that the two cannot be equal unless $\tan^2\alpha$ lie between $\frac{b^2}{a^2}$ and $\frac{a^2}{b^2}$.

328. If $\tan\alpha\tan\beta\tan\gamma = 1$, α, β, γ being angles between 0 and $\frac{\pi}{2}$,

$$\sin\alpha\sin\beta\sin\gamma < \frac{1}{2\sqrt{2}},$$

unless $\alpha = \beta = \gamma$.

329. If $\alpha + A$, $\beta + B$, $\gamma + C$ be the angles subtended at any point by the sides of a triangle ABC,

$$\frac{\sin^2\alpha}{\sin A} + \frac{\sin^2\beta}{\sin B} + \frac{\sin^2\gamma}{\sin C} > \frac{\sin\alpha\sin\beta\sin\gamma}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}},$$

unless the point be the centre of the inscribed circle.

IV. *Properties of Triangles.*

In these questions a, b, c denote the sides, A, B, C the respectively opposite angles of any triangle; R the radius of the circumscribed circle, r, r_1, r_2, r_3 the radii of the inscribed circle and of the escribed circles respectively opposite A, B, C.

330. If θ, ϕ, ψ be angles given by the equations

$$\cos\theta = \frac{a}{b+c}, \quad \cos\phi = \frac{b}{c+a}, \quad \cos\psi = \frac{c}{a+b};$$

then will
$$\tan^2\frac{\theta}{2} + \tan^2\frac{\phi}{2} + \tan^2\frac{\psi}{2} = 1;$$

and
$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\psi}{2} = \pm\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

331. If $\sin A$, $\sin B$, $\sin C$ be in harmonical progression, so also are $1-\cos A$, $1-\cos B$, $1-\cos C$.

332. Prove that

$$\begin{aligned}\sin A\sin(A-B)\sin(A-C) &+ \sin B\sin(B-C)\sin(B-A)\\ &+ \sin C\sin(C-A)\sin(C-B) = \sin A\sin B\sin C\\ &\qquad - \sin 2A\sin 2B\sin 2C.\end{aligned}$$

333. From the three relations between the sides and angles given in the form

$$a^2 = b^2 + c^2 - 2bc\cos A, \text{ \&c.},$$

deduce the equations

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \quad A + B + C = 180^\circ;$$

assuming that each angle is $< 180^\circ$.

334. In the side BC, produced if necessary, find a point P such that the square on PA may be equal to the sum of the squares on PB, PC; and prove that this is only possible when A, B, C are all acute and $\tan A < \tan B + \tan C$, or when B or C is obtuse. These conditions being satisfied, prove that there are in general two such points which lie both between B and C, one between and one beyond, or both beyond, according as A is the greatest, the mean, or the least angle of the triangle.

335. P, Q are two points on the circumscribed circle, the distance of either from A being a mean proportional between its distances from B and C; prove that

$$\angle BAP \sim \angle CAQ = \frac{B \sim C}{2}.$$

336. The line joining the middle points of BC and of the perpendicular from A on BC makes with BC an angle

$$\cot^{-1}(\cot B \sim \cot C).$$

337. The line joining the centres of the inscribed and circumscribed circles makes with BC an angle

$$\cot^{-1}\left\{\frac{\sin B \sim \sin C}{\cos B + \cos C - 1}\right\}.$$

338. The line joining the centre of the circumscribed circle and the centre of perpendiculars makes with BC an angle

$$\cot^{-1}\left\{\frac{\tan B \sim \tan C}{\tan B \tan C - 3}\right\}.$$

339. The perpendicular from A on BC is a harmonic mean between r_2 and r_3.

340. If O be the centre of the circumscribed circle and AO meet BC in D,

$$DO : AO :: \cos A : \cos(B - C).$$

341. The perimeter of a triangle : perimeter of the inscribed circle :: the area of the triangle : area of the circle

$$:: \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} : \pi.$$

342. A triangle is formed by joining the feet of the perpendiculars of the triangle ABC; and the circle inscribed in this triangle touches the sides in A', B', C'; prove that

$$\frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{A'B'}{AB} = 2\cos A\cos B\cos C.$$

343. A circle is drawn to touch the circumscribed circle and the sides AB, AC; prove that its radius is $r\sec^2\frac{A}{2}$: and if it touch the circumscribed circle and the sides AB, AC produced, its radius is $r_1\sec^2\frac{A}{2}$. If $B = C$ and the latter radius $= R$, $\cos A = \frac{7}{9}$.

344. Prove the formulæ,

$$b^2\sin 2C - 2bc\sin(B-C) - c^2\sin 2B = 0,$$

$$b^2\cos 2C + 2bc\cos(B-C) + c^2\cos 2B = a^2.$$

345. Having given

$$y\sin^2 C + z\sin^2 B = z\sin^2 A + x\sin^2 C = x\sin^2 B + y\sin^2 A;$$

prove that
$$\frac{x}{\sin 2A} = \frac{y}{\sin 2B} = \frac{z}{\sin 2C}.$$

346. Determine a triangle having a base c, an altitude h, and a given difference α of the base angles; and if θ_1, θ_2 be the two values of the vertical angle, prove that $\cot\theta_1 + \cot\theta_2 = \frac{4h}{c\sin^2\alpha}$. Prove that only one of these values corresponds to a true solution; and, if this be θ_1, that

$$\tan\frac{\theta_1}{2} = \frac{\sqrt{h^2 + c^2\sin^2\alpha} - h}{c(1-\cos\alpha)}.$$

347. Determine a triangle in which are given a side a, the opposite angle A, and the rectangle m^2 under the other two sides: and prove that no such triangle exists if $2m\sin\frac{A}{2} > a$.

348. A triangle $A'B'C'$ has its angles respectively supplementary to the half angles of the triangle ABC; and its side $B'C'$ equal to BC; prove that

$$\frac{\triangle A'B'C'}{\triangle ABC} = \frac{\sin\frac{A}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}.$$

349. If x, y, z be perpendiculars from the angular points on any straight line; prove that

$$a^2(x-y)(x-z) + b^2(y-z)(y-x) + c^2(z-x)(z-y) = (2\triangle ABC)^2,$$

any perpendicular being reckoned negative which is drawn in the opposite direction to the other two.

350. If p_1, p_2, p_3 be the perpendiculars of the triangle,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}, \quad \frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}.$$

351. The distances between the centres of the escribed circles being respectively α, β, γ; prove that

$$4R = \frac{\alpha^2}{r_2+r_3} = \frac{\beta^2}{r_3+r_1} = \frac{\gamma^2}{r_1+r_2}$$

$$= \frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2} = \frac{\alpha\beta\gamma}{2\sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}},$$

where $2\sigma \equiv \alpha+\beta+\gamma$. Prove also that

$$a = \frac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}, \text{ and } \triangle = \frac{r_1r_2r_3}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}.$$

352. The distances between the centre of the inscribed circle and those of the escribed circles being respectively α', β', γ'; prove that

$$4R = \frac{\alpha'^2}{r_1 - r} = \frac{\beta'^2}{r_2 - r} = \frac{\gamma'^2}{r_3 - r},$$

and that $$32R^3 - 2R(\alpha'^2 + \beta'^2 + \gamma'^2) + \alpha'\beta'\gamma' = 0.$$

353. Prove that

$$\frac{8r_1r_2r_3}{\alpha\beta\gamma} = \sin A \sin B \sin C,$$

and $$\frac{8r_1r_2r_3}{\alpha'\beta'\gamma'} = (1 + \cos A)(1 + \cos B)(1 + \cos C).$$

354. Prove that

$$4R = \frac{\alpha}{\cos\frac{A}{2}} = \frac{\beta}{\cos\frac{B}{2}} = \frac{\gamma}{\cos\frac{C}{2}} = \frac{\alpha'}{\sin\frac{A}{2}} = \frac{\beta'}{\sin\frac{B}{2}} = \frac{\gamma'}{\sin\frac{C}{2}}.$$

355. If q, r be the radii of two of the four circles which touch the sides of a triangle, and a the distance between their centres, the area of the triangle will be $qr\sqrt{\frac{a^2}{(q \pm r)^2} - 1}$, the upper sign being taken when either of the circles is the inscribed circle.

356. O, O' are the centres of the circumscribed circle, and of the inscribed or one of the escribed circles, A', B', C' the points of contact of the latter circle with the sides, L the centre of perpendiculars of the triangle $A'B'C'$; prove that O, O', L are in one straight line, and that

$$O'L : OO' :: r : R, \quad \text{or} :: r_1 : R, \text{ \&c.}$$

357. If an isosceles triangle be constructed whose vertical angle is $\cos^{-1}\frac{1}{9}$, the inscribed circle will pass through the centre of perpendiculars.

358. If O, o be the centres of the circumscribed and inscribed circles, and L the centre of perpendiculars

$$OL^2 - 2oL^2 = R^2 - 4r^2;$$

and if o_1 be the centre of the escribed circle opposite A,

$$OL^2 - 2o_1L^2 = R^2 - 4r_1^{\,2}.$$

359. If the centre of the inscribed circle, or of one of the escribed circles, be equidistant from the centre of the circumscribed circle and from the centre of perpendiculars, one angle of the triangle must be equal to 60°.

360. The angle at which the circumscribed circle of a triangle intersects the escribed circle opposite A is

$$\cos^{-1} \frac{1 + \cos A - \cos B - \cos C}{2}.$$

If α, β, γ be the three such angles,

$$(\cos\beta + \cos\gamma)(\cos\gamma + \cos\alpha)(\cos\alpha + \cos\beta) = 2(\cos\alpha + \cos\beta + \cos\gamma - 1)^2.$$

361. If P be any point on the circumscribed circle,

$$PA \sin A + PB \sin B + PC \sin C = 0,$$

a certain convention being made with respect to sign: also

$$PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C = 4 \triangle ABC.$$

362. If P be any point in the plane of the triangle, and O the centre of the circumscribed circle,

$$PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C = 2 \triangle ABC + 4OP^2 \sin A \sin B \sin C.$$

363. If P be any point on the inscribed circle,

$$PA^2 \sin A + PB^2 \sin B + PC^2 \sin C$$

is constant; and if P be any point on the circle with respect to which the triangle is self-conjugate,

$$PA^2 \tan A + PB^2 \tan B + PC^2 \tan C = 2 \triangle ABC.$$

If ρ be the radius of this latter circle, and δ the distance of its centre from the centre of the circumscribed circle,

$$\delta^2 = R^2 + 2\rho^2.$$

264. The line joining the centres of the circumscribed and inscribed circles will subtend a right angle at the centre of perpendiculars, if

$$1 + (1 - 2\cos A)(1 - 2\cos B)(1 - 2\cos C) = 8\cos A\cos B\cos C.$$

365. If P be a point within a triangle at which the sides subtend angles $A + \alpha$, $B + \beta$, $C + \gamma$, respectively,

$$PA\frac{\sin A}{\sin \alpha} = PB\frac{\sin B}{\sin \beta} = PC\frac{\sin C}{\sin \gamma}.$$

366. Any point P is taken within the triangle ABC, and the angles BPC, CPA, APB are A', B', C' respectively; prove that

$$\triangle BPC(\cot A - \cot A') = \triangle CPA(\cot B - \cot B')$$
$$= \triangle APB(\cot C - \cot C').$$

V. *Heights and Distances. Polygons.*

367. At a point A are measured the angle α subtended by two objects P, Q in the same horizontal plane as A and the distances a, b at right angles to AP, AQ respectively to points at which PQ subtends the same angle α; find the distance between P and Q.

368. An object is observed at three points A, B, C lying in a horizontal line which passes directly underneath the object; the angular elevation at B is twice and at C is three times that at A; also $AB = a$, $BC = b$; prove that the height of the object is

$$\frac{a}{2b}\sqrt{(a+b)(3b-a)}.$$

If the angle of elevation at A be $\tan^{-1}\frac{1}{3}$, $a : b :: 13 : 5$.

369. The sides of a rectangle are $2a$, $2b$, and the angles subtended by its diagonals, at a point whose distance from the centre is c, are α, β; prove that

$$\frac{4c^2(a^2+b^2)^2}{(a^2+b^2-c^2)^2} = a^2(\tan\alpha \pm \tan\beta)^2 + b^2(\tan\alpha \mp \tan\beta)^2.$$

370. The diagonals $2a$, $2b$ of a rhombus subtend angles α, β at a point whose distance from the centre is c; prove that

$$(a^2-c^2)^2 b^2 \tan^2\alpha + (b^2-c^2)^2 a^2 \tan^2\beta = 4a^2b^2c^2.$$

371. Three circles A, B, C touch each other two and two, and one common tangent to A and B is parallel to one common tangent to A and C; prove that, if a, b, c be their radii, and p, q the distances of the centres of B and C from that diameter of A which is perpendicular to the two parallel tangents,

$$pq = 2a^2 = 8bc.$$

372. AB is the diameter of a circle, C any point on AB, on AC, BC as diameters are described two other circles: if a circle be described touching the three, its diameter will be equal to the distance of its centre from AB.

373. Four points A, P, Q, B lie in a straight line, circles are described on $AQ\,(\equiv 2a)$, $BP\,(\equiv 2b)$, and $AB\,(\equiv 2c)$ as diameters; prove that the radius of a circle touching the three is

$$\frac{c(c-a)(c-b)}{c^2-ab}.$$

374. A polygon of n sides inscribed in a circle is such that its sides subtend angles α, 2α, ... $n\alpha$ at the centre; prove that its area is to the area of the regular inscribed polygon of n sides in the ratio

$$\sin\frac{n\alpha}{2} : n\sin\frac{\alpha}{2}.$$

375. $ABCD$ is a parallelogram and P any point within it; prove that

$$\triangle APC \cot APC \sim \triangle BPD \cot BPD$$

is independent of the position of P.

376. The distances of any point P on a circle from the angular points of a regular polygon of n sides inscribed in the circle are the positive roots of the equation

$$(d^2-x^2)^n - \frac{2n(2n-1)}{\lfloor 2}(d^2-x^2)^{n-1}x^2$$
$$+\frac{2n(2n-1)(2n-2)(2n-3)}{\lfloor 4}(d^2-x^2)^{n-2}x^4 - \ldots = d^{2n}\cos n\theta\,;$$

d being the diameter of the circle and θ the angle subtended at the centre by any one of the distances.

377. The sides of a convex quadrilateral are a, b, c, d and $2s$ is their sum; prove that

$$\sqrt{s(s-a-d)(s-b-d)(s-c-d)}$$

cannot be greater than the area of the quadrilateral.

378. The equation giving the length x of the diagonal joining the angles (a, d), (b, c) of a quadrilateral, whose sides taken in order are a, b, c, d, is

$$\{x^2(ab+cd)-(ac+bd)(ad+bc)\}^2$$
$$=4abcd\cos^2\alpha\,\{(x^2-a^2-b^2)(x^2-c^2-d^2)+4abcd\sin^2\alpha\};$$

2α being the sum of two opposite angles.

379. In any quadrilateral $ABCD$, BC, AD meet in E; CA, BD in F, and AB, CD in G; prove that

$$\frac{(EB.EC-EA.ED)^2}{EA.EB.EC.ED\sin^2 E}=\frac{(FC.FA-FB.FD)^2}{FA.FB.FC.FD\sin^2 F}$$
$$=\frac{(GA.GB-GC.GD)^2}{GA.GB.GC.GD\sin^2 G}.$$

VI. *Expansions of Trigonometrical Functions.*

380. By means of the equivalence of the expansions of

$$2\epsilon^x \sin x \times \epsilon^x \cos x, \text{ and } \epsilon^{2x} \sin 2x;$$

prove that

$$\Sigma_{r=0}^{r=n-1} \frac{\sin (n-r)\frac{\pi}{4} \cos \frac{r\pi}{4}}{\underline{|n-r}} = \frac{2^{n-1} \sin \frac{n\pi}{4}}{\underline{|n}}.$$

381. Prove by comparing the coefficients of θ^{2n-1} that the expansions of $\sin\theta$ and $\cos\theta$ in terms of θ satisfy the equality

$$2 \sin\theta \cos\theta \equiv \sin 2\theta.$$

382. Prove that

$$2^n - (n-1)\, 2^{n-1} + \frac{(n-2)(n-3)}{\underline{|2}}\, 2^{n-2} - \ldots \equiv \frac{\sin (n+1)\frac{\pi}{4}}{\left(\sin\frac{\pi}{4}\right)^{n+1}}, \quad (n \text{ integral}).$$

383. From the expansion of $(\sin\theta)^{2n+1}$ in terms of sines of multiples of θ, prove that

$$0 \equiv 1 - (2n-1) + \frac{2n\,(2n-3)}{\underline{|2}} - \frac{2n\,(2n-1)(2n-5)}{\underline{|4}} + \ldots$$

to $(n+1)$ terms.

384. If n be an odd integer,

$$1 + \frac{n-1}{n}\cos\theta + \frac{n-1}{n}\,\frac{2n-1}{2n}\cos 2\theta + \ldots \text{ to } \infty$$

$$= (n-1)^{\frac{n-1}{2}} \frac{\cos\frac{n-1}{2n}\theta}{\left(2\sin\frac{\theta}{2}\right)^{\frac{n-1}{n}}}.$$

Within what limits of θ is this true?

385. From the equivalence

$$\log(1+x\epsilon^{-\theta\sqrt{-1}})+\log(1-x\epsilon^{\theta\sqrt{-1}})\equiv\log\{1-x(x+2\sqrt{-1}\sin\theta)\},$$

obtain the expansions of $\cos 2n\theta$, and $\sin(2n+1)\theta$ in terms of $\sin\theta$.

386. From the equivalence

$$\epsilon^{x}-2\cos\theta+\epsilon^{-x}\equiv 4\sin\frac{\theta-x\sqrt{-1}}{2}\sin\frac{\theta+x\sqrt{-1}}{2},$$

resolve the former into its real quadratic factors.

387. If $\tan(\alpha+\beta\sqrt{-1})=\sqrt{-1}$, α, β being real, then will α be indeterminate and β infinite.

388. If $\cos(\alpha+\beta\sqrt{-1})=\cos\phi+\sqrt{-1}\sin\phi$, where α, β, ϕ are real; then will $\sin\phi=\pm\sin^2\alpha$. Find also the relation between α and β.

389. If $\tan(\alpha+\beta\sqrt{-1})=\cos\phi+\sqrt{-1}\sin\phi$, α, β, ϕ being real, then will

$$\alpha=\frac{n\pi}{2}+\frac{\pi}{4},\quad 2\beta=\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right).$$

390. If $\tan(\alpha+\beta\sqrt{-1})=\tan\theta+\sqrt{-1}\sec\theta$, α, β, θ being real, then will

$$2\alpha=n\pi+\frac{\pi}{2}+\theta,\quad 2\beta=\log\cot\frac{\theta}{2}.$$

391. Prove that

$$\frac{1^2}{1^2+1}\,\frac{2^2}{2^2+1}\,\frac{3^2}{3^2+1}\ldots\text{ to }\infty=\frac{2\pi}{\epsilon^{\pi}-\epsilon^{-\pi}}.$$

392. If in a triangle the sides a, b, and the angle $\pi-\theta$ opposite b be given, and θ be small; prove that, approximately

$$\frac{c}{b-a}=1+\frac{a}{b}\frac{\theta^2}{\underline{|2}}-\frac{ab^2-3a^2b-3a^3}{b^3}\frac{\theta^4}{\underline{|4}}.$$

393. Prove that

$$\frac{a}{2\sin a} = -\frac{\pi^2}{a^2-\pi^2} + \frac{(2\pi)^2}{a^2-(2\pi)^2} - \frac{(3\pi)^2}{a^2-(3\pi)^2} + \ldots \text{ to } \infty.$$

394. Prove that

$$\frac{\tan\theta}{8\theta} = \frac{1}{\pi^2-4\theta^2} + \frac{1}{(3\pi)^2-4\theta^2} + \frac{1}{(5\pi)^2-4\theta^2} + \ldots$$

395. Prove that if θ be an angle between $\frac{\pi}{4}$ and $-\frac{\pi}{4}$,

$$\theta^2 = \sin^2\theta + \frac{2}{3}\,\frac{\sin^4\theta}{2} + \frac{2\,.\,4}{3\,.\,5}\,\frac{\sin^6\theta}{3} + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}\,\frac{\sin^8\theta}{4} + \ldots \text{ to } \infty$$

$$= \tan^2\theta - \left(1+\frac{1}{3}\right)\frac{\tan^4\theta}{2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6\theta}{3} - \ldots \text{ to } \infty.$$

396. Prove that

$$\frac{1}{\sin^2\theta} = \frac{1}{\theta^2} + \frac{1}{(\pi+\theta)^2} + \frac{1}{(2\pi+\theta)^2} + \ldots\ \frac{1}{(\pi-\theta)^2} + \frac{1}{(2\pi-\theta)^2} + \ldots$$

397. The expansion of tan tan......tan x is

$$x + 2n\frac{x^3}{\lfloor 3} + 4n(5n-1)\frac{x^5}{\lfloor 5} + \frac{4n}{3}(350n^2-168n+22)\frac{x^7}{\lfloor 7} + \ldots$$

the tangent being taken n times.

VII. *Series.*

In the summation of many Trigonometric series in which the r^{th} term is of the form $a_r\cos r\theta$, or $a_r\sin r\theta$, a_r being a function of r, it is convenient to sum the series in the manner following. "To find the sum of $1 + 2\cos\theta + 3\cos 2\theta + \ldots + n\cos(n-1)\theta$."

$$\left.\begin{aligned} &\text{Let } C \equiv 1 + 2\cos\theta + 3\cos 2\theta + \ldots + n\cos(n-1)\theta \\ &\text{and } \quad S \equiv \quad\ 2\sin\theta + 3\sin 2\theta + \ldots + n\sin(n-1)\theta \end{aligned}\right\},$$

and let $\cos\theta + \sqrt{(-1)}\sin\theta \equiv z$.

Then $C+S\sqrt{-1} \equiv 1+2z+3z^2+\ldots+nz^{n-1} \equiv \dfrac{1-z^n(n+1-nz)}{(1-z)^2}$

$$\equiv \frac{1-(\cos n\theta+\sqrt{-1}\sin n\theta)(n+1)+n\{\cos(n+1)\theta+\sqrt{-1}\sin(n+1)\theta\}}{\left(2\sin\frac{\theta}{2}\right)^2\left(\sin\frac{\theta}{2}-\sqrt{-1}\cos\frac{\theta}{2}\right)^2}$$

$$\equiv \frac{1-(n+1)(\cos n\theta+\sqrt{-1}\sin n\theta)+n\{\cos(n+1)\theta+\sqrt{-1}\sin(n+1)\theta\}}{-\left(2\sin\frac{\theta}{2}\right)^2(\cos\theta+\sqrt{-1}\sin\theta)}$$

$$\equiv \frac{\cos\theta-\sqrt{-1}\sin\theta-(n+1)\{\cos(n-1)\theta+\sqrt{-1}\sin(n-1)\theta\}+n(\cos n\theta+\sqrt{-1}\sin n\theta)}{-\left(2\sin\frac{\theta}{2}\right)^2}.$$

Whence, equating possible and impossible parts,

$$C \equiv \frac{(n+1)\cos(n-1)\theta-\cos\theta-n\cos n\theta}{2(1-\cos\theta)},$$

$$S \equiv \frac{(n+1)\sin(n-1)\theta+\sin\theta-n\sin n\theta}{2(1-\cos\theta)}.$$

Many others may be summed by separating the r^{th} term into a difference of the form $U_{r+1}-U_r$, U_r being some function of r, and U_{r+1} the same function of $r+1$. Thus to sum the series

$$\frac{1}{\sin x}+\frac{1}{\sin 2x}+\frac{1}{\sin 2^2x}+\ldots+\frac{1}{\sin 2^{n-1}x},$$

we have

$$\frac{1}{\sin 2^rx} \equiv \frac{\sin 2^{r-1}x}{\sin 2^{r-1}x\sin 2^rx} \equiv \frac{\sin(2^r-2^{r-1})x}{\sin 2^{r-1}x\sin 2^rx}$$
$$\equiv \cot 2^{r-1}x-\cot 2^rx,$$

whence

$$\frac{1}{\sin x}=\cot\frac{x}{2}-\cot x,$$

$$\frac{1}{\sin 2x}=\cot x-\cot 2x,$$

$$\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{\sin 2^{n-1}x}=\cot 2^{n-2}x-\cot 2^{n-1}x,$$

and the sum of the series is $\cot\frac{x}{2}-\cot 2^{n-1}x$.

398. Sum the series:

(1) $\sin^3\theta + \dfrac{\sin^3 3\theta}{3} + \dfrac{\sin^3 3^2\theta}{3^2} + \dots + \dfrac{\sin^3 3^{n-1}\theta}{3^{n-1}}$,

(2) $\cos^3\theta - \dfrac{\cos^3 3\theta}{3} + \dfrac{\cos^3 3^2\theta}{3^2} - \dots + (-1)^{n-1}\dfrac{\cos^3 3^{n-1}\theta}{3^{n-1}}$,

(3) $\dfrac{\cos\theta}{\sin 3\theta} + \dfrac{\cos 3\theta}{\sin 3^2\theta} + \dots + \dfrac{\cos 3^{n-1}\theta}{\sin 3^n\theta}$,

(4) $\dfrac{\sin 2\theta}{1+2\cos 2\theta} + \dfrac{3\sin 6\theta}{1+2\cos 6\theta} + \dfrac{3^2\sin 2.3^2\theta}{1+2\cos 2.3^2\theta} + \dots$
$$+\frac{3^{n-1}\sin 2.3^{n-1}\theta}{1+2\cos 2.3^{n-1}\theta},$$

(5) $\dfrac{2\cos\theta - \cos 3\theta}{\sin 3\theta} + 2\dfrac{2\cos 3\theta - \cos 3^2\theta}{\sin 3^2\theta} + \dots$
$$+\frac{2^n\cos 3^{n-1}\theta - 2^{n-1}\cos 3^n\theta}{\sin 3^n\theta},$$

(6) $\dfrac{1+2\cos 2\theta}{\sin 4\theta} + \dfrac{1+2\cos 2^3\theta}{\sin 2^4\theta} + \dots + \dfrac{1+2\cos 2^{2n-1}\theta}{\sin 2^{2n}\theta}$,

(7) $\dfrac{1-2\cos 2\theta}{\sin 2\theta} + 3\dfrac{1-2\cos 2^2\theta}{\sin 2^2\theta} + \dots + 3^{n-1}\dfrac{1-\cos 2^n\theta}{\sin 2^n\theta}$,

(8) $\dfrac{5\sin 3\theta - 3\sin 5\theta}{\cos 3\theta - \cos 5\theta} + 4\dfrac{5\sin 12\theta - 3\sin 20\theta}{\cos 12\theta - \cos 20\theta} + \dots$
$$+4^{n-1}\frac{5\sin 3.4^{n-1}\theta - 3\sin 5.4^{n-1}\theta}{\cos 3.4^{n-1}\theta - \cos 5.4^{n-1}\theta},$$

(9) $\dfrac{1+4\sin\theta\sin 3\theta}{\sin 4\theta} + 3\dfrac{1+4\sin 4\theta\sin 12\theta}{\sin 16\theta} + \dots$
$$+3^{n-1}\frac{1+4\sin 4^{n-1}\theta\sin 3.4^{n-1}\theta}{\sin 4^n\theta},$$

(10) $\cos\theta + \dfrac{1}{2}\cos 2\theta + \dfrac{1}{3}\cos 3\theta + \dots$ to ∞,

(11) $\cos\theta - \dfrac{\cos 3\theta}{\lfloor 3} + \dfrac{\cos 5\theta}{\lfloor 5} - \dots$ to ∞,

(12) $\cos\theta - \dfrac{\cos 3\theta}{3} + \dfrac{\cos 5\theta}{5} - \ldots$ to ∞,

(13) $\cos n\theta + n\cos(n-1)\theta + \dfrac{n(n-1)}{\lfloor 2}\cos(n-2)\theta + \ldots + n\cos\theta + 1$,

(14) $\cos\theta - \dfrac{\cos 2\theta}{\lfloor 2} + \dfrac{\cos 3\theta}{\lfloor 3} - \ldots$ to ∞,

(15) $\cos\theta + \dfrac{1}{2}\,\dfrac{\cos 3\theta}{3} + \dfrac{1\,.\,3}{2\,.\,4}\,\dfrac{\cos 5\theta}{5} + \ldots$ to ∞,

(16) $\cos\theta\sin\theta + \cos^2\theta\,\dfrac{\sin 2\theta}{2} + \cos^3\theta\,\dfrac{\sin 3\theta}{3} + \ldots$ to ∞,

(17) $\cos 2\theta\cos\theta + \dfrac{\cos^2 2\theta}{2}\;\dfrac{\cos 3\theta}{3} + \dfrac{1\,.\,3}{2\,.\,4}\cos^2 2\theta\,\dfrac{\cos 5\theta}{5} + \ldots$ to ∞.

399. Prove that

$$\sec\theta + \sec\left(\frac{2\pi}{m} + \theta\right) + \sec\left(\frac{4\pi}{m} + \theta\right) + \ldots + \sec\left\{2(m-1)\frac{\pi}{m} + \theta\right\}$$

is equal to 0, or to $(-1)^{\frac{m-1}{2}}\, m\sec m\theta$, according as m is even or odd: also that

$$\sec^2\theta + \sec^2\left(\frac{2\pi}{m} + \theta\right) + \ldots + \sec^2\left\{2(m-1)\frac{\pi}{m} + \theta\right\}$$

is equal to $\dfrac{m^2}{1-(-1)^{\frac{m}{2}}\cos m\theta}$, or to $m^2\sec^2 m\theta$, according as m is even or odd.

400. Prove that

$$\frac{n\sin n\phi}{\cos n\phi - \cos n\theta} = \frac{\sin\phi}{\cos\phi - \cos\theta} + \frac{\sin\phi}{\cos\phi - \cos\left(\frac{2\pi}{n} + \theta\right)}$$

$$+ \frac{\sin\phi}{\cos\phi - \cos\left(\frac{4\pi}{n} + \theta\right)} + \ldots + \frac{\sin\phi}{\cos\phi - \cos\left\{2(n-1)\frac{\pi}{n} + \theta\right\}}.$$

VIII. *Miscellaneous Questions.*

401. The ambiguities in the equations

$$\cos\frac{A}{2} + \sin\frac{A}{2} = \pm\sqrt{(1+\sin A)},\quad \cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{(1-\sin A)}$$

may be replaced by $(-1)^m$, $(-1)^n$, where m, n denote the greatest integers in $\dfrac{A+90^\circ}{360^\circ}$, and $\dfrac{A+270^\circ}{360^\circ}$ respectively.

402. If $a + b(\tan\theta + \tan\phi) + c\tan\theta\tan\phi = 0$, prove that

$$\frac{1}{a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta} + \frac{1}{a\cos^2\phi + 2b\sin\phi\cos\phi + c\sin^2\phi}$$

is equal to $\dfrac{a+c}{ac-b^2}$.

403. AB is a fixed straight line, C a point in it such that $3AC = AB$, and P a point between B and C: if $CP \equiv CB\sin\theta$, then will $BP\,.\,AP^2 \propto 1 + \sin 3\theta$. Hence prove that $BP\,.\,AP^2$ is greatest when P bisects BC.

404. Find x from the equation

$$\tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{n^2 - x + 1} = \tan^{-1}\frac{1}{n-1};$$

and find the tangent of the angle

$$\tan^{-1}\frac{1}{3} + 3\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{26} - \frac{\pi}{4}.$$

405. Three parallel straight lines are drawn through the angular points of a triangle ABC to meet the opposite sides in D, E, F; prove that

$$\frac{DB\,.\,DC}{DA^2} + \frac{EC\,.\,EA}{EB^2} + \frac{FA\,.\,FB}{FC^2} = 1,$$

the segments of a side being affected with opposite signs when they fall on opposite sides of the point of section.

406. Reduce to its simplest form

$$\frac{(x\cos 2\alpha+y\sin 2\alpha-1)(x\cos 2\beta+y\sin 2\beta-1)-\{x\cos(\alpha+\beta)+y\sin(\alpha+\beta)-\cos(\alpha-\beta)\}^2}{\sin^2(\alpha-\beta)}.$$

407. Prove that

$$(1)\quad \tan^{-1}\frac{1}{70}=\tan^{-1}\frac{1}{83}+\tan^{-1}\frac{1}{447},$$

$$(2)\quad \tan^{-1}\frac{1}{99}=\tan^{-1}\frac{1}{157}+\tan^{-1}\frac{1}{268},$$

$$(3)\quad 0=\tan^{-1}\frac{1}{19}-\tan^{-1}\frac{1}{27}-\tan^{-1}\frac{1}{46}+\tan^{-1}\frac{1}{162},$$

$$(4)\quad \frac{\pi}{4}=\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{21}$$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{8}-\tan^{-1}\frac{1}{43}$$

$$=3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}+\tan^{-1}\frac{1}{1985}.$$

408. Prove that

$$\frac{\sin 4\alpha\sin(\gamma-\beta)+\sin 4\beta\sin(\alpha-\gamma)+\sin 4\gamma\sin(\beta-\alpha)}{\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)}$$

$$=2\Sigma\sin(2\alpha+\beta+\gamma)+\Sigma\sin 2(\beta+\gamma)+\Sigma\sin(3\beta+\gamma).$$

409. Prove that

$$\sin\sin\ldots\sin x=x-\frac{nx^3}{\lfloor 3}+\frac{n(5x-4)}{\lfloor 5}x^5$$

$$-\frac{n}{3}(175n^2-336n+164)\frac{x^7}{\lfloor 7}+\ldots$$

the sine being taken n times.

410. If P be any point on the Nine Points' Circle of the triangle ABC,

$$PA^2\sin A\cos(B-C)+PB^2\sin B\cos(C-A)+PC^2\sin C\cos(C-A)$$

$$=R^2(4\sin A\sin B\sin C+\sin 2A\sin 2B\sin 2C).$$

CONIC SECTIONS, GEOMETRICAL.

I. *Parabola.*

411. Two parabolas having the same focus intersect; prove that the angles between their tangents at the points of intersection are either equal or supplementary.

412. A chord PQ of a parabola is a normal at P and subtends a right angle at the focus S; prove that SQ is twice SP.

413. A chord PQ of a parabola normal at P subtends a right angle at the vertex; prove that SQ is three times SP.

414. Two circles each touch a parabola and touch each other at the focus of the parabola; prove that the angle between the focal distances of the points of contact with the parabola $= 120^\circ$.

415. Two parabolas have a common focus and their axes in opposite directions; prove that if a circle be drawn through the focus touching both the parabolas the line joining the points of contact subtends at the focus an angle of 60°.

416. In a parabola AQ is drawn through the vertex A at right angles to a chord AP to meet the diameter through P in Q; prove that Q lies on a fixed straight line.

417. Through any point P of a parabola a straight line QPQ' is drawn perpendicular to the axis and terminated by the tangents at the extremities of the latus rectum; prove that the distance of P from the latus rectum is a mean proportional between QP, PQ'.

418. The locus of a point dividing in a given ratio a chord of a parabola which is parallel to a given line is a parabola.

419. From any point on the tangent at any point of a parabola perpendiculars are let fall on the focal distance and on the

axis; prove that the sum, or the difference, of the focal distances of the feet of these perpendiculars is equal to half the latus rectum.

420. Two points are taken on a parabola such that the sum of the parts of the normals intercepted between the points and the axis is equal to the part of the axis intercepted between the normals; prove that the difference of the normals is equal to the latus rectum.

421. SY is the perpendicular from the focus of a parabola on any tangent, a straight line is drawn through Y parallel to the axis to meet in Q a straight line through S at right angles to SY; prove that the locus of Q is a parabola.

422. At one extremity of a given finite straight line is drawn any circle touching the line, and from the other extremity is drawn a tangent to the circle; prove that the point of intersection of this tangent with the tangent parallel to the given straight line lies on a fixed parabola.

423. Two parabolas have a common focus; from any point on their common tangent are drawn the other tangents to the two; prove that the distances of these from the common focus are in a constant ratio.

424. Two tangents are drawn to a parabola making equal angles with a given straight line; prove that their point of intersection lies on a fixed straight line passing through the focus.

425. Two parabolas have their axes parallel and two parallel tangents are drawn to them; prove that the straight line joining the points of contact passes through a fixed point.

426. Two parabolas have a common focus S, parallel tangents drawn to them at P, Q meet their common tangent in P', Q'; prove that the angle PSQ is equal to the angle between the axes of the parabolas, and the angle $P'SQ'$ supplementary.

427. If on a tangent to a parabola be taken two points equidistant from the focus, the two other tangents drawn to the parabola from these points will intersect on the axis.

428. A circle is described on the latus rectum of a parabola as diameter, and a straight line drawn through the focus meets the curves in P, Q; prove that the tangents at P, Q intersect either on the latus rectum, or on a straight line parallel to the latus rectum at a distance from it equal to the latus rectum.

429. A chord of a parabola is drawn parallel to a given straight line and on this chord as diameter a circle is described; prove that the distance between the middle points of this chord and of the chord joining the other two points of intersection of the circle and parabola is of constant length.

430. On any chord of a parabola as diameter is described a circle cutting the parabola again in two points; if these points be joined the portion of the axis of the parabola intercepted between the two chords is equal to the latus rectum.

431. A parabola is described having its focus on the arc, its axis parallel to the axis, and touching the directrix, of a given parabola; prove that the two curves will touch each other.

432. Circles are described having for diameters a series of parallel chords of a given parabola; prove that they will all touch another parabola related to the given one in the manner described in the last question.

433. The locus of the centre of the circle circumscribing the triangle formed by two fixed tangents to a parabola and any other tangent is a straight line.

434. Two equal parabolas, A and B, have a common vertex and axes in the same straight line; prove that the locus of the poles with respect to B of tangents to A is A.

435. Three common tangents PP', QQ', RR' are drawn to two parabolas and PQ, $P'Q'$ intersect in L; prove that LR, LR' are parallel to the axes of the two parabolas.

436. Two equal parabolas have a common focus and axes opposite; two circles are described touching each other, each with its centre on one parabola and touching the tangent at the vertex of that parabola: prove that the rectangle under their radii is constant whether the circles touch internally or externally, but in the former case is four times as great as in the latter.

437. Two equal parabolas are placed with their axes in the same straight line and their vertices at a distance equal to the latus rectum; a tangent drawn to one meets the other in two points: prove that the circle on this chord as diameter will touch the parabola of which this is the chord.

438. Two equal parabolas have their axes parallel and opposite, and one passes through the centre of curvature at the vertex of the other; prove that this relation is reciprocal and that the parabolas intersect at right angles.

439. PP' is any chord of a parabola, PM, $P'M'$ are drawn perpendicular to the tangent at the vertex; prove that the circle on MM' as diameter, and the circle of curvature at the vertex will have PP' for their radical axis.

440. A parabola touches the sides of a triangle ABC in A', B', C', $B'C'$ meets BC in P, another parabola is drawn touching the sides and P is its point of contact with BC; prove that its axis is parallel to $B'C'$.

441. The directrix of a parabola and one point of the curve being given; prove that the parabola will touch a fixed parabola to which the given straight line is the tangent at the vertex.

442. If a triangle be self-conjugate to a parabola, the lines joining the middle point of its sides will touch the parabola; and the lines joining any angular point of the triangle to the point of contact of the corresponding tangent will be parallel to the axis.

443. If a complete quadrilateral be formed by four tangents to a parabola, the common radical axis of the three circles on the diagonals as diameters will be the directrix of the parabola.

444. A circle and parabola meet in four points and tangents are drawn to the parabola at these points; prove that the axis of the parabola will bisect the three diagonals of the quadrilateral formed by these tangents.

II. *Central Conics.*

445. If SY be perpendicular on the tangent, SZ on the normal, S being the focus, YZ will pass through the centre.

446. A common tangent is drawn to a conic and to the circle whose diameter is a latus rectum; prove that the latus rectum bisects the angle between the focal distances of the points of contact.

447. A perpendicular from the centre on the tangent meets the focal distances of the point of contact in two points; prove that either of these points is at a constant distance from the feet of the perpendiculars from the foci on the tangent.

448. The tangent at a point P meets the major axis in T; prove that $SP : ST :: AN : AT$, N being the foot of the ordinate and A the nearer vertex.

449. The circle passing through the feet of the perpendiculars from the foci on the tangent and through the foot of the ordinate will pass through the centre; and the angle subtended at either extremity of the major axis by the distance between the feet of the perpendiculars is equal or supplementary to the angle which either focal distance makes with the corresponding perpendicular.

450. A series of conics having a common focus S and major axes equal and in the same straight line will all touch the two parabolas having the same focus S, and latus rectum a line coincident with the major axes in direction and of double the length.

451. A conic is described having the same focus as a parabola and major axis coincident in direction with the latus rectum of

the parabola and equal to half the latus rectum; prove that it will touch the parabola.

452. In a conic PG is the normal at P, S the focus; prove that if $SG = PG$, SP is equal to the latus rectum.

453. CPQ is a common radius to the circles on the minor and major axes of an ellipse, and tangents to the circles at P, Q meet these axes in U, T; prove that TU will touch the ellipse.

454. S, S' are the foci of a conic, SPY, $S'P'Y'$ perpendiculars on a tangent to the auxiliary circle meeting the conic in P, P'; prove that the rectangle SY, $S'P'$ = the rectangle $S'Y'$, $SP = BC^2$.

455. Given the foci and the length of the major axis; obtain by a geometrical construction the points in which the conic meets a given straight line drawn through one of the foci.

456. A tangent to a conic at P meets the minor axis in T, and TQ is drawn perpendicular to SP one of the focal distances; prove that SQ is of constant length: and, PM being drawn perpendicular to the minor axis, that QM will pass through a fixed point.

457. One focus of a conic, a tangent line, and the length of the major axis is given; prove that the locus of the second focus is a circle. Determine the portions of the locus which correspond to an ellipse, and to a hyperbola of which the given point is an interior focus to the branch touched by the given straight line.

458. PCG is a diameter of a conic, QVQ' a parallel chord bisected in V, PV intersects CQ, or CQ' in R; prove that the locus of R is a parabola.

459. If CP, CD be conjugate radii of an ellipse, and if through C a straight line be drawn parallel to either focal distance of P, the distance of D from this straight line will be equal to half the minor axis.

460. If three tangents to a conic be such that their points of intersection are at equal distances from one of the foci, each distance will be equal to the major axis: and the second focus will be the centre of perpendiculars of the triangle formed by the tangents.

461. A straight line is drawn touching the circle on the minor axis of an ellipse, meeting the ellipse in P, and the director circle of the ellipse in Q, Q'; prove that the focal distances of P are equal to QP, $Q'P$.

462. A conic is inscribed in a triangle and is concentric with the Nine Points' Circle; prove that it will have double contact with the Nine Points' Circle.

463. EF is a chord of a circle, S is its middle point; construct a conic of which E is one point, S one focus, and the given circle the circle of curvature at E.

464. If P be a point on an ellipse equidistant from the minor axis and from one of the directrices, the circle of curvature at P will pass through one of the foci.

465. If S be the focus of a conic, K the foot of the directrix, Q a point on the tangent at P, QR, QR' perpendiculars on SP, SK respectively; then will SR bear to KR' a constant ratio.

466. If an equilateral triangle PQR be inscribed in the auxiliary circle of an ellipse, and P', Q', R' be the corresponding points on the ellipse, the circles of curvature at P', Q', R' meet in one point lying on the ellipse and on the circle circumscribing $P'Q'R'$.

467. From a point on an ellipse perpendiculars are drawn to the axes and produced to meet the circles on these axes respectively; prove that the line joining the points of intersection passes through the centre.

468. TP, TQ are tangents to a conic, Qq, Pp chords parallel to TP, TQ respectively; prove that pq is parallel to PQ.

469. QQ' is a chord of an ellipse parallel to one of the equi-conjugate diameters, QN, $Q'N'$ are drawn perpendicular to the major axis; prove that the triangles QCN, $Q'CN'$ are equal; also that the normals at QQ' intersect on the diameter which is perpendicular to the other equi-conjugate.

470. Any ordinate NP of an ellipse is produced to meet the auxiliary circle in Q, and normals to the ellipse and circle at P, Q meet in R; RK, RL are drawn perpendicular to the axes; prove that K, P, L lie on one straight line and that KP, PL are equal respectively to the semi axes.

471. Two conics are described having a common minor axis, and such that the outer touches the directrices of the inner; MPP' is a common ordinate; prove that MP' is equal to the normal at P.

472. QPP' drawn perpendicular to the major axis of an ellipse meets the ellipse in P, P' and the auxiliary circle in Q; prove that the part of the normal to the circle at Q intercepted between the normals to the ellipse at P and P' is equal to the minor axis.

473. The perpendicular from the focus of a conic on any tangent and the central radius to the point of contact will intersect on the directrix.

474. On the normal to an ellipse at P are taken two points Q, Q', such that $QP = Q'P = CD$; prove that the cosine of the angle QCQ' is $\dfrac{CP^2 - CD^2}{AC^2 - BC^2}$.

475. A hyperbola is described through the focus of a parabola and with its foci lying on the parabola; prove that one of its asymptotes is parallel to the axis of the parabola.

476. A parabola passes through two given points and its axis is parallel to a given line; prove that the locus of its focus is a hyperbola.

477. If two tangents of a hyperbola be the asymptotes of another hyperbola and that other touch one of the asymptotes of the former it will touch both.

478. Two similar conics A, B are placed with their major axes in the same straight line, and the focus of A is the centre of B; if a common tangent be drawn the focal distance of its point of contact with A will be equal to the semi major axis of B.

479. A series of similar conics are described having the same focus and direction of major axis, and tangents are drawn to them at points where they meet a fixed circle having its centre at the common focus; prove that these tangents will all touch a similar fixed conic whose major axis is a diameter of the circle.

480. If a chord of a conic subtend a right angle at each of the foci, it must be either parallel to the major axis or a diameter.

481. From the foci S, S' of an ellipse perpendiculars SY, $S'Y'$ are let fall on any tangent; prove that the perimeter of the quadrilateral $SYY'S'$ will be the greatest possible when YY' subtends a right angle at the centre.

482. The angle which a diameter of an ellipse subtends at the extremity of the axis major is supplementary to that which its conjugate subtends at the extremity of the axis minor.

483. From the focus of an ellipse is drawn a straight line perpendicular to the tangent at a point of the auxiliary circle; prove that this perpendicular is equal to the focal distance of the corresponding point of the ellipse.

484. If on any tangent to a conic be taken two points equidistant from one focus and subtending a right angle at the other focus; their distance from the former focus is constant.

485. If a conic be described having one side of a triangle for directrix, the opposite angle for centre, and the centre of perpendiculars for focus; the sides of the triangle which meet in the centre will be conjugate diameters.

486. An ellipse is described touching two confocal ellipses, and having the same centre; prove that the tangents to the two ellipses at two points of contact will be perpendicular to each other.

487. An ellipse is described having double contact with each of two confocal ellipses; prove that the sum of the squares on its axes is constant.

488. If SY, SZ be perpendiculars from the focus S on two tangents drawn from T to a conic, the perpendicular from T on YZ will pass through the other focus.

489. The tangent to a conic at P meets the axes in T, t, and the central radius at right angles to CP in Q; prove that QT bears to Qt a constant ratio.

490. The foot of the perpendicular from the focus of a conic on the tangent at the extremity of the farther latus rectum lies on the minor axis.

491. The tangents and normals drawn to a series of confocal conics at the extremities of their latera recta will touch two parabolas having their foci at the given foci and touching each other at the centre.

492. Through a given point O on a given conic are drawn two chords OP, OQ, equally inclined to a given straight line; prove that PQ passes through a fixed point.

493. A chord PQ of a conic is normal at P, and a diameter LL' is drawn bisecting the chord; prove that PQ makes equal angles with LP, $L'P$, and that $LP \pm L'P$ is constant.

494. A given finite straight line is an equi-conjugate diameter of an ellipse; prove that the locus of its foci is a lemniscate.

495. A parallelogram is inscribed in a conic, and from any point on this conic are drawn two straight lines each parallel to

two sides of the parallelogram; prove that the rectangle under the segments of these lines made by the parallelogram are in a constant ratio.

496. Any two central conics in the same plane have two conjugate diameters of the one parallel respectively to two conjugate diameters of the other; and in general no more.

497. In two similar and similarly situated ellipses are taken two parallel chords PP', QQ'; PQ, $P'Q'$ meet the two conics in R, S; R', S' respectively; prove that RR', SS' are parallel to each other. Also QQ', RR', and PP', SS' intersect in points lying on a fixed straight line.

498. A circle is described touching the focal distances of any point on a given conic, and passing through a given point on the major axis; prove that it will meet the major axis in another fixed point. The given point must be between the foci for a hyperbola, and beyond them for an ellipse.

499. A circle described on the part of the tangent at P intercepted between the tangents at the ends of the major axis meets the conic again in Q; prove that the ordinate of Q is to the ordinate of P as the minor axis to the sum of the minor axis and the diameter conjugate to P.

500. If a conic be inscribed in a triangle ABC and have its focus at O; and if the angles BOC, COA, AOB be denoted by A', B', C',

$$\frac{OA \sin A}{\sin(A'-A)} = \frac{OB \sin B}{\sin(B'-B)} = \frac{OC \sin C}{\sin(C'-C)} = \text{major axis}.$$

With what convention will this be true if O be a point without the triangle?

501. OA, OB are tangents to a conic, a straight line is drawn meeting OA, OB in Q, Q', AB in R, and the conic in P, P'; prove that $QP \,.\, PQ' : QP' \,.\, P'Q' :: RP^2 : RP'^2$; and that, for a

series of parallel straight lines the ratio $QP.PQ' : RP^2$ is constant.

502. S, S' are foci of an ellipse whose minor axis is equal to SS', P any point on the ellipse, O the centre of the circle circumscribed to SPS'; prove that the circle on OP as diameter will touch the major axis at the foot of the normal at P.

503. Through different points of a given straight line are drawn chords of a given conic, bisected respectively at the points; prove that they will touch a fixed parabola.

504. With a fixed point O on a conic as focus is described a parabola touching any pair of conjugate diameters of the conic; prove that this parabola will have a fixed tangent parallel to the tangent at O, and that this tangent divides CO in the ratio $CO^2 : CO'^2$, CO, CO' being conjugate radii.

505. Through a point O are drawn two straight lines, each passing through the pole of the other with respect to a given conic; any tangent to the conic meets them in P, Q; prove that the other tangents drawn from P, Q to the conic intersect on the polar of O.

506. A parabola is described having the focus S of a given conic for its focus and touching the minor axis; prove that a common tangent to the two curves will subtend a right angle at S, and that its point of contact with either conic lies on the directrix of the other.

III. *Rectangular Hyperbola.*

507. A, B, C, D are four points on a rectangular hyperbola and BC is perpendicular to AD; prove that CA is perpendicular to BD and AB to CD.

508. The angle between two diameters of a rectangular hyperbola is equal to the angle between the conjugate diameters.

509. AA' is the transverse axis, P any point on the curve, PK, PL are drawn at right angles to AP, $A'P$ to meet the axis; prove that $PK = A'P$ and $PL = AP$, and that the normal at P bisects KL.

510. The foci of an ellipse are the extremities of a diameter of a rectangular hyperbola; prove that the tangent and normal to the ellipse at any one of the points where it meets the hyperbola are parallel to the asymptotes of the hyperbola.

511. On a series of parallel chords of a rectangular hyperbola as diameters are described a series of circles; prove that they will have a common radical axis.

512. A circle and a rectangular hyperbola intersect in four points, two of which are the extremities of a diameter of the hyperbola; prove that the other two will be the extremities of a diameter of the circle.

513. Any chord of a rectangular hyperbola subtends at the extremities of any diameter angles which are either equal or supplementary: equal if the extremities of the chord be on the same branch and on the same side of the diameter, or on opposite branches and on opposite sides: otherwise supplementary.

514. AB is a chord of a circle and a diameter of a rectangular hyperbola, P any point on the circle; PA, PB meet the hyperbola again in Q, R; prove that BQ, AR will intersect on the circle.

515. Two points are taken on a rectangular hyperbola and its conjugate, the tangents at which are at right angles to each other; prove that the central radii to the point are also at right angles to each other.

516. CP, CQ are radii of a rectangular hyperbola, tangents at P, Q meet in T and intersect CQ, CP respectively in P', Q'; prove that a circle can be described about $CP'TQ'$.

517. A parallelogram has its angular points on the arc of a rectangular hyperbola and from any point on the hyperbola are drawn two straight lines parallel to the sides; prove that the four points in which these straight lines meet the sides of the parallelogram lie on a circle.

518. The tangent at a point P of a rectangular hyperbola meets a diameter QCQ' in T; prove that CQ, TQ' subtend equal angles at P.

519. Through any point on a rectangular hyperbola are drawn two chords at right angles to each other; prove that the circle passing through the point and bisecting the chords will pass through the centre.

520. If PG be a fixed diameter and Q any point on the curve, the angles QPG, QGP will differ by a constant angle.

521. A, B are two fixed points, P a point such that AP, BP make equal angles with a given straight line; prove that the locus of P is a rectangular hyperbola.

522. QR is any chord of a rectangular hyperbola, CP a radius perpendicular to it; prove that the distance of P from either asymptote is a mean proportional between the distances of Q, R from the other.

523. Two circles touch the same branch of a rectangular hyperbola in the points P, Q, and touch each other at the centre C; prove that the angle $PCQ = 60°$.

524. On opposite sides of any chord of a rectangular hyperbola are described equal segments of circles; prove that the four points in which the completed circles again meet the hyperbola are the angular points of a parallelogram.

525. A circle and rectangular hyperbola intersect in four points; prove that the diameter of the hyperbola which is perpendicular to the chord joining any two of the points will bisect the chord joining the other two.

526. PP' is a diameter of a rectangular hyperbola, QQ', RR' two ordinates to it on opposite branches; prove that a common tangent to the circles whose diameters are QQ', RR' will subtend a right angle at P and at P'.

527. Circles are drawn through two given points, and diameters drawn parallel to a given straight line; prove that the locus of the extremities of these diameters is a rectangular hyperbola which asymptotes make equal angles with the line of centres of the circles and with the given straight line.

528. The tangent at a point P of a rectangular hyperbola, and the diameter perpendicular to CP are drawn; prove that the segments of any other tangent, from the point of contact to the points where it meets these two lines, will subtend supplementary angles at P.

529. The normal at a point P of a rectangular hyperbola meets the curve again in Q: two chords PR, PR', are drawn through P at right angles to each other; prove that the points of intersection of QR, PR', and of QR', PR lie on the diameter at right angles to CP.

CONIC SECTIONS, ANALYTICAL.

Cartesian Co-ordinates.

I. *Straight Line, Linear Transformation, Circle.*

In any question relating to the intersections of a curve and two straight lines, it is generally convenient to use one equation representing both straight lines. Thus, to prove the theorem that "Any chord of a given conic subtending a right angle at a given point of the conic passes through a fixed point in the normal at the given point;" we may take the equation of the conic referred to the tangent and normal at a point

$$ax^2 + 2bxy + cy^2 + 2x = 0\ ;$$

the equation of any pair of straight lines through this point, at right angles to each other, is

$$x^2 + 2\lambda xy - y^2 = 0\ ;$$

and at the points of intersection

$$(a + c)\, x^2 + 2\,(b + c\lambda)\, xy + 2x = 0\ ;$$

or, at the points other than the origin,

$$(a + c)\, x + 2\,(b + c\lambda)\, y + 2 = 0,$$

which is therefore the equation of a chord subtending a right angle at the origin. This passes through the point $y = 0$, $(a + c)\, x + 2 = 0$; a fixed point in the normal.

If two points be given as the intersections of a straight line and a conic, the equation of the line joining these points to the origin may be found immediately, since it must be a homogeneous equation in x, y of the second degree. Thus the straight lines joining the origin to the points

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0,$$

$$a'y + b'x + c = 0,$$

are $$c\,(ax^2 + by^2 + 2c'xy) = (a'y + b'x)^2.$$

The results of linear transformations may generally be obtained from the consideration that, the origin being unaltered,

$$x^2 + 2xy \cos \omega + y^2$$

must be transformed into

$$X^2 + 2XY \cos \Omega + Y^2,$$

if (x, y), (X, Y) represent the same point, and ω, Ω be the angles between the axes. Thus if

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy$$

be transformed into $AX^2 + \ldots\ldots$; and if we give h such a value that

$$h\,(x^2 + y^2 + 2xy \cos \omega) - \{ax^2 + \ldots\ldots\}$$

separate into factors, the corresponding transformed expression must also separate into factors. Hence the two equations

$$-c\,(h-a)(h-b) + 2a'b'\,(h \cos \omega - c') - (h-a)\,a'^2$$
$$- (h-b)\,b'^2 + c\,(h \cos \omega - c')^2 = 0,$$

and $-c\,(h-A)\,(h-B) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0,$

must coincide: and thus all the invariants may be deduced.

One form of the equation of the circle may be mentioned: if (x_1, y_1), (x_2, y_2) be the extremities of a diameter, the equation of the circle is

$$(x - x_1)\,(x - x_2) + (y - y_1)\,(y - y_2) = 0.$$

530. The equation of the two straight lines which pass through the origin and make an angle α with the straight line $x + y = 0$ is

$$x^2 + 2xy \sec 2\alpha + y^2 = 0.$$

531. If the straight lines represented by the equation

$$x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$$

make angles α, β with the axis of x,

$$\tan\alpha \sim \tan\beta = 2.$$

532. Form the equation of the straight lines joining the origin to the points given by the equations

$$(x-h)^2 + (y-k)^2 = c^2, \quad kx + hy = 2hk;$$

and prove that they will be at right angles if $h^2 + k^2 = c^2$.

533. The locus of the equation

$$y = 2 + \frac{x^2-1}{2+}\,\frac{x^2-1}{2+}\dots \text{ to } \infty$$

is the part of two straight lines at right angles to each other which include one quadrant.

534. If the formulæ for effecting any transformation of co-ordinates be

$$x = aX + bY + c, \quad y = a'X + b'Y + c',$$

then will $$(ab - a'b')(ab' - a'b) = bb' - aa'.$$

535. The expression

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy$$

is transformed to

$$AX^2 + BY^2 + C + 2A'X + 2B'Y + 2C'XY,$$

the origin being unaltered; prove that

$$\frac{a'^2 + b'^2 - 2a'b'\cos\omega}{\sin^2\omega} = \frac{A'^2 + B'^2 - 2A'B'\cos\Omega}{\sin^2\Omega},$$

$$\frac{2a'b'c' - aa'^2 - bb'^2}{\sin^2\omega} = \frac{2A'B'C' - AA'^2 - BB'^2}{\sin^2\Omega};$$

ω, Ω being the angles between the axes.

536. If ABC be an acute-angled triangle, P any point in its plane; the three circular loci

$$PA^2 = PB^2 + PC^2, \quad PB^2 = PC^2 + PA, \quad PA^2 = PB^2 + PC^2$$

will have their radical centre at the centre of the circle circumscribing the triangle.

537. The radii of two circles are a, b, and the distance between their centres $\sqrt{\{2(a^2+b^2)\}}$; prove that any common tangent subtends a right angle at the point bisecting the distance between their centres.

538. A certain point has the same polar with respect to each of two circles; prove that any common tangent subtends a right angle at that point.

539. AB is a diameter of a circle, O a point in a fixed straight line passing through A, from O two tangents are drawn to the circle meeting the tangent at B in P, Q; prove that $BP+BQ$ is constant.

540. AB is a diameter of a circle, a chord through A meets the tangent at B in P, and from any point in the chord produced are drawn two tangents to the circle; prove that the lines joining A to the points of contact will meet the tangent at B in points equidistant from P.

541. Three circles A, B, C have a common radical axis, and from any point on C two tangents are drawn to A, B respectively; prove that the ratio between the squares on these tangents is equal to the ratio between the distances of the centres of A, B from the centre of C.

542. On two circles are taken two points such that the tangents drawn each from one point to the other circle are equal; prove that the points are equidistant from the radical axis.

543. There are two systems of circles such that any circle of one system cuts any circle of the other system at right angles;

prove that the circles of either system have a common radical axis which is the line of centres of the circles of the other system.

544. Given two circles, a tangent to one at P meets the polar of P with respect to the other in P'; prove that the circle on PP' as diameter will pass through two fixed points, which will be imaginary or real, according as the circles intersect in real or imaginary points.

545. One circle lies entirely within another, a tangent to the inner meets the outer in P, P', and the radical axis in Q: if S be the internal vanishing circle which has the same radical axis, the ratio $\sin\frac{PSP'}{2} : \cos\frac{SQP}{2}$ is constant.

546. Prove that the equation

$$\begin{aligned}&\{x\cos(\alpha+\beta)+y\sin(\alpha+\beta)-a\cos(\alpha-\beta)\}\\&\{x\cos(\gamma+\delta)+y\sin(\gamma+\delta)-a\cos(\gamma-\delta)\}\\=&\{x\cos(\alpha+\gamma)+y\sin(\alpha+\gamma)-a\cos(\alpha-\gamma)\}\\&\{x\cos(\beta+\delta)+y\sin(\beta+\delta)-a\cos(\beta-\delta)\}\end{aligned}$$

is equivalent to the equation $x^2+y^2=a^2$; and state the property of the circle expressed by the equation in this form.

547. The radii of two circles are R, r, the distance between their centres is $\sqrt{(R^2+2r^2)}$, and $r<2R$; prove that an infinite number of triangles can be inscribed in the first, which are self-conjugate with respect to the second: and that an infinite number of triangles can be circumscribed to the second which are self-conjugate with respect to the first.

548. A triangle is inscribed in the circle $x^2+y^2=R^2$ and two of its sides touch the circle $(x-\delta)^2+y^2=r^2$; prove that the third side will touch the circle

$$\left\{x-\frac{4R^2r^2\delta}{(R^2-\delta^2)^2}\right\}^2+y^2=R^2\left\{\frac{2r^2(R^2+\delta^2)}{(R^2-\delta^2)^2}-1\right\}^2.$$

Prove that this circle coincides with the second if $\delta^2=R^2\pm 2Rr$.

II. *Parabola referred to its axis.*

The equation of the parabola being taken $y^2 = 4ax$, the co-ordinates of any point on it may be represented by $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, and with this notation, the equation of the tangent is $y = mx + \frac{a}{m}$; of the normal $my + x = 2a + \frac{a}{m^2}$; and of the chord through two points (m_1, m_2), $2m_1m_2x - y(m_1 + m_2) + 2a = 0$. The equation of the two tangents drawn through a point (X, Y) is

$$(Y^2 - 4aX)(y^2 - 4ax) = \{Yy - 2a(x + X)\}^2.$$

As an example, we may take the following, "To find the locus of the point of intersection of normals to a parabola at right angles to each other."

If (X, Y) be a point on the locus, the points on the parabola to which the normals are drawn from (X, Y) are given by the equation

$$m^3Y + m^2(X - 2a) - a = 0;$$

and if m_1, m_2, m_3 be the three roots of this equation,

$$m_1 + m_2 + m_3 = \frac{2a - X}{Y}, \quad m_2m_3 + m_3m_1 + m_1m_2 = 0, \quad m_1m_2m_3 = \frac{a}{Y}:$$

also since two normals meet at right angles in the point, the product of two of the roots is -1. Let then $m_2m_3 = -1$. Then

$$m_1 = \frac{-a}{Y}, \quad m_2 + m_3 = \frac{3a - X}{Y} = \frac{-Y}{a};$$

or the locus is the parabola

$$Y^2 = a(X - 3a).$$

Again, "The sides of a triangle touch a parabola, and two of its angular points lie on another parabola having the same axis, to find the locus of the third angular point."

Let the equations of the parabolas be $y^2 = 4ax$, $y^2 = 4a'(x + \alpha)$, and let the three tangents be at m_1, m_2, m_3. The point of intersection of (1) and (2) is $\frac{a}{m_1 m_2}$, $a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$, and if this lie on the second parabola

$$a^2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 = 4a'\left(\frac{a}{m_1 m_2} + \alpha\right),$$

and similarly for m_1, m_3. Hence m_2, m_3 are the two roots of the quadratic in z,

$$a^2\left(\frac{1}{z} + \frac{1}{m_1}\right)^2 = \frac{4aa'}{m_1 z} + 4a'\alpha,$$

or
$$\frac{1}{m_2} + \frac{1}{m_3} = \frac{1}{a^2}\left(\frac{4aa'}{m_1} - \frac{2a^2}{m_1}\right), \quad \frac{1}{m_2 m_3} = \frac{1}{m_1^2} - \frac{4a'\alpha}{a^2}.$$

But if (X, Y) be the point of intersection of the tangents (2), (3),

$$X = \frac{a}{m_2 m_3}, \quad Y = a\left(\frac{1}{m_2} + \frac{1}{m_3}\right);$$

$$\therefore X + \frac{4a'\alpha}{a} = \frac{a}{m_1^2}, \quad Y = \frac{4a' - 2a}{m_1};$$

and the equation of the locus is

$$Y^2 = \frac{4(2a' - a)^2}{a}\left(X + \frac{4a'\alpha}{a}\right):$$

a parabola which coincides with the second if $a = 4a'$.

549. Two parabolas have a common vertex A and a common axis, an ordinate NPQ meets them, and a tangent at P meets the outer parabola in R, R'; AR, AR' meet the ordinate in L, M; prove that NP, NQ are respectively harmonic and geometric means between NL, NM.

550. A triangle is inscribed in a parabola and another triangle similar and similarly situated circumscribes it; prove that the sides of the former triangle are respectively four times the corresponding sides of the latter.

551. TP, TQ are two tangents to a parabola; prove that the perpendiculars let fall from P, T, Q on any other tangent are in geometric progression.

552. Four fixed tangents are drawn to a parabola, and from the angular points taken in order of a quadrangle formed by them are let fall perpendiculars p_1, p_2, p_3, p_4 on any other tangent; prove that $p_1p_3 = p_2p_4$.

553. The distance of the middle point of any one of the three diagonals of a quadrilateral from the axis of the parabola which touches the sides is one fourth of the sum of the distances of the four points of contact from the axis.

554. Through the point T, where the tangent to a given parabola at P meets the axis, is drawn a straight line TQQ' meeting the parabola in Q, Q', and dividing the ordinate of P in a given ratio; prove that PQ, PQ' will both touch a fixed parabola having the same vertex and axis as the given one.

555. Two equal parabolas have their axes in the same straight line, and from any point on the outer tangents are drawn to the inner; prove that these tangents will intercept a constant length on the tangent at the vertex of the inner.

556. If p, q, r be the perpendiculars from the angular points of a triangle ABC, whose sides touch a parabola, on the directrix, and x, y, z perpendiculars from the same points on any other tangent,

$$p \tan A + q \tan B + r \tan C = 0,$$

and

$$\frac{p \tan A}{x} + \frac{q \tan B}{y} + \frac{r \tan C}{z} = 0.$$

557. A tangent is drawn to the circle of curvature at the vertex of a parabola, and the ordinates of the points where it meets the parabola are y_1, y_2; prove that $\frac{1}{y_1} \sim \frac{1}{y_2} = \frac{1}{c}$, $2c$ being the latus rectum.

558. OA, OB are two tangents to a parabola meeting the tangent at the vertex in P, Q; prove that

$$PQ = OA \cos QPA = OB \cos PQB.$$

559. Two parabolas have a common focus and direction of axis, QPQ' a chord of the outer is bisected by the inner in P, PP' parallel to the axis meets the outer in P'; prove that QP is a mean proportional between the tangents drawn from P' to the inner.

560. The locus of the centre of the Nine Points' Circle of a triangle, formed by three tangents to a parabola, of which two are fixed, is a straight line.

561. Prove that the parabolas $y^2 = 4ax$, $y^2 + 2cy + 4ax = 8a^2$, cut each other at right angles in two points.

562. Through each point of the straight line $\frac{x}{h} + \frac{y}{k} = 1$ is drawn a chord of the parabola $y^2 = 4ax$ bisected in the point; prove that this chord touches the parabola

$$\left(y - 2a\frac{h}{k}\right)^2 = 8a(x - h).$$

563. Two equal parabolas have a common focus and axes in the same straight line; from any point of either two tangents are drawn to the other: prove that the centres of two of the four circles which touch the sides of the triangle formed by the tangents and their chord of contact lie on the parabola to which the tangents are drawn.

564. The two parabolas $y^2 = ax$, $y^2 = 4a(x + a)$ are so related that if a normal to the latter meet the former in P, P', and A be the vertex of the former, AP or AP' is perpendicular to the normal.

565. The normals at three points of the parabola $y^2 = 4ax$ meet in the point (h, k); prove that the equation of the circle through the three points is

$$2(x^2 + y^2) - 2x(h + 2a) - ky = 0.$$

566. A straight line parallel to the directrix of a parabola meets the axis produced at a distance from the vertex equal to the latus rectum, and a point P on this straight line is joined to the vertex A by a straight line meeting the directrix in Q; with centre Q and radius QA is described a circle meeting the parabola again in R; prove that PR will be normal to the parabola at R.

567. A chord of a parabola passes through the centre of curvature at the vertex; prove that the normals to the parabola at the extremities of the chord intersect on the parabola.

568. From any point of a straight line drawn through the focus of a parabola, and making an angle α with the axis, three normals are drawn; prove that the sum of the angles which they make with the axis exceeds α by a multiple of π.

569. Normals are drawn at the extremities of any chord passing through a fixed point on the axis of a parabola; prove that their point of intersection lies on a fixed parabola.

570. Two normals to a parabola meet at right angles, and from the foot of the perpendicular let fall from their point of intersection on the axis, is measured towards the vertex a distance equal to one fourth of the latus rectum; prove that the straight line joining the end of this distance with the point of intersection of the normals will also be a normal.

571. Two equal parabolas have their axes coincident but their vertices separated by a distance equal to the latus rectum; through the centres of curvature at the vertices are drawn chords PQ, $P'Q'$, equally inclined in opposite directions to the axis, P, P' being on the same side of the axis; prove that (1) PQ', $P'Q$ are normals to the outer parabola; (2) their intersection R' lies on the inner parabola; (3) the normals to the inner parabola at P', Q', R' meet in a point which lies on a third equal parabola.

572. From a point O are drawn three normals OP, OQ, OR, and two tangents OL, OM, to a parabola; prove that the latus rectum $= 4\dfrac{OP\,.\,OQ\,.\,OR}{OL\,.\,OM}$.

573. The normals to the parabola $y^2 = 4ax$ at points P, Q, R, meet in the point (X, Y); prove that the co-ordinates of the centre of perpendiculars of the triangle PQR are

$$X - 6a, \quad -\tfrac{1}{2}Y.$$

574. A circle and parabola intersect in four points A, B, C, D; AB, CD produced meet in P; BC, AD in Q; both points being without the parabola; and from any point on the parabola perpendiculars are let fall on these lines; prove that the rectangle contained by the perpendiculars on the two former : that contained by the perpendiculars on the two latter

$$:: 1 \pm \cos P : 1 \mp \cos Q.$$

575. In the two parabolas $y^2 = 2c\,(x \pm c)$, a tangent drawn to one meets the other in two points, and on the part intercepted is described a circle; prove that this circle will touch the second parabola.

576. On a chord of a given parabola as diameter a circle is described, and the other common chord of the circle and parabola is conjugate to the former chord with respect to the parabola; prove that each chord will touch a fixed parabola equal to the given one.

577. Two parabolas have a common focus S, and axes in the same straight line, and from a point P on the outer are drawn two tangents PQ, PQ' to the inner; prove that the ratio

$$\cos\frac{QPQ'}{2} : \cos\frac{PSA}{2}$$

is constant, A being the vertex of either parabola.

578. If a parabola circumscribe a triangle ABC, and its axis make with BC an angle θ (measured from CB towards CA), its

latus rectum is $2R \sin\theta \sin(C-\theta) \sin(B+\theta)$, R being the radius of the circumscribed circle of the triangle; and if a parabola touch the sides of the triangle, its latus rectum is

$$8R \sin\theta \sin(C-\theta) \sin(B+\theta).$$

579. A triangle ABC is inscribed in a parabola, and the focus is the centre of perpendiculars of the triangle; prove that

$$(1-\cos A)(1-\cos B)(1-\cos C) = 2\cos A \cos B \cos C;$$

and that each side of the triangle will touch a fixed circle which passes through the focus, and whose diameter is equal to the latus rectum.

580. Through a fixed point O within a parabola is drawn any chord PP', and the diameters through P and P' are drawn; prove that there are two fixed straight lines perpendicular to the axis, the part of either of which intercepted between the diameters subtends a right angle at O.

581. A triangle, self-conjugate to a given parabola, has one angular point O given; prove that the circle circumscribing the triangle passes through another fixed point Q, such that OQ is parallel to the axis and bisected by the directrix.

582. A triangle is inscribed in a parabola, its sides are at distances x, y, z from the focus, and subtend angles θ, ϕ, ψ at the focus; prove that

$$\frac{\sin\theta}{x^2} + \frac{\sin\phi}{y^2} + \frac{\sin\psi}{z^2} = \frac{\sin\theta + \sin\phi + \sin\psi + 2\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\psi}{2}}{l^2},$$

l being the latus rectum, and the angles being measured all in the same direction so that their sum is 2π.

583. If a parabola touch the sides of a triangle ABC in A', B', C', and O be the point of intersection of AA', BB', CC', then will

$$\frac{OA}{\sin BOC} + \frac{OB}{\sin COA} + \frac{OC}{\sin AOB} = 0;$$

a certain convention being made as to sign.

584. A parabola touches the sides of a triangle ABC in A', B', C', and O is a point such that OA' bisects the angle BOC, and OB', OC' bisect the external angles between OC, OA, and OA, OB, respectively; prove that $OA = OB + OC$.

585. A triangle circumscribes the circle $x^2 + y^2 = a^2$, and two angular points lie on the circle $(x-2a)^2 + y^2 = 2a^2$; prove that the third angular point lies on the parabola

$$y^2 = a\left(x - \frac{3a}{4}\right).$$

Prove also that the curves have two real common tangents.

586. Two parabolas have a common focus, axes inclined at an angle 2α, and are such that triangles can be inscribed in one whose sides touch the other; prove that $l_2 = 4l_1 \cos^2\alpha$, l_1, l_2 being their latera recta.

587. A circle is described with its centre at a point P of a parabola, and its radius equal to twice the normal at P; prove that triangles can be inscribed in the parabola whose sides touch the circle.

588. Two parabolas A, B have their axes parallel, and the latus rectum of B is four times that of A; prove that an infinite number of triangles can be inscribed in A whose sides are tangents to B.

III. *Ellipse referred to its axes.*

The equation of the ellipse in the following questions is always supposed to be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, unless otherwise stated. The point whose eccentric angle is θ, is called the point θ. The eccentricity is denoted by e. The tangent and normal at the point θ are respectively

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \quad \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2;$$

the chord through α, β is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2};$$

and the intersection of tangents at α, β,

$$a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\quad b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

The polar of (X, Y) is $\frac{xX}{a^2}+\frac{yY}{b^2}=1$; and the equation of the two tangents through (X, Y) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}-1\right)=\left(\frac{xX}{a^2}+\frac{yY}{b^2}-1\right)^2.$$

It follows from the equation of the tangent that, if the equation of any straight line be $lx+my=1$, and l, m satisfy the equation $a^2l^2+b^2m^2=1$, the straight line touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a result often useful.

The equation of the tangent in the form

$$x\cos\theta+y\sin\theta=\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}$$

may be occasionally employed with advantage.

The points α, β are extremities of conjugate diameters if $\alpha\sim\beta=\frac{\pi}{2}$. Any two points are called conjugate, if each lies on the polar of the other; and any two straight lines, if each passes through the pole of the other.

If (X, Y) be the pole of the chord through α, β, it will be found immediately that

$$\frac{\sin\alpha\sin\beta}{1-\frac{X^2}{a^2}}=\frac{\cos\alpha\cos\beta}{1-\frac{Y^2}{b^2}}=\frac{\sin\alpha+\sin\beta}{\frac{2Y}{b}}=\frac{\cos\alpha+\cos\beta}{\frac{2X}{a}}=\frac{1}{\frac{X^2}{a^2}+\frac{Y^2}{b^2}},$$

which are of use in finding the locus of (X, Y) when (α, β) are connected by some fixed equation.

Thus, "If a triangle be circumscribed about an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and two angular points lie on the ellipse,

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

to find the locus of the third angular point."

If α, β, γ be the three points of contact, and (α, β), (α, γ) be the pairs of points whose tangents intersect on the second ellipse, we have

$$\frac{a^2}{a'^2}\cos^2\frac{\alpha+\beta}{2} + \frac{b^2}{b'^2}\sin^2\frac{\alpha+\beta}{2} = \cos^2\frac{\alpha-\beta}{2};$$

and a like equation in α, γ.

Hence β, γ are the two roots of the equation

$$A\cos\alpha\cos\theta + B\sin\alpha\sin\theta = C:$$

where

$$A \equiv -\frac{a^2}{a'^2} + \frac{b^2}{b'^2} + 1, \quad B \equiv \frac{a^2}{a'^2} - \frac{b^2}{b'^2} + 1, \quad C \equiv \frac{a^2}{a'^2} + \frac{b^2}{b'^2} - 1;$$

we have then

$$\frac{\cos\dfrac{\beta+\gamma}{2}}{A\cos\alpha} = \frac{\sin\dfrac{\beta+\gamma}{2}}{B\sin\alpha} = \frac{\cos\dfrac{\beta-\gamma}{2}}{C};$$

and therefore

$$\frac{1}{A^2}\cos^2\frac{\beta+\gamma}{2} + \frac{1}{B^2}\sin^2\frac{\beta+\gamma}{2} = \frac{1}{C^2}\cos^2\frac{\beta-\gamma}{2};$$

the locus of the third point is then

$$\frac{X^2}{A^2a^2} + \frac{Y^2}{B^2b^2} = \frac{1}{C^2}.$$

This will be found to coincide with the second ellipse, if

$$\frac{a}{a'} \pm \frac{b}{b'} \pm 1 = 0.$$

Under this condition, therefore, the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

will be such that an infinite number of triangles can be described, whose sides touch the first, and whose angular points lie on the second. If this condition be not satisfied, the two given conics and the locus will be found to have four common tangents, real or impossible.

Again, for the reciprocal problem, "If a triangle be inscribed in the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, and two of its sides touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, to find the envelope of the third side."

If α, β, γ be the angular points, and (α, β), (α, γ) the sides which touch the second ellipse, we have

$$\frac{a^2}{a'^2} \cos^2 \frac{\alpha+\beta}{2} + \frac{b^2}{b'^2} \sin^2 \frac{\alpha+\beta}{2} = \cos^2 \frac{\alpha-\beta}{2},$$

and a like equation in α, γ. Hence, as before,

$$\frac{C^2}{A^2} \frac{\cos^2 \frac{\beta+\gamma}{2}}{\cos^2 \frac{\beta-\gamma}{2}} + \frac{C^2}{B^2} \frac{\sin^2 \frac{\beta+\gamma}{2}}{\cos^2 \frac{\beta-\gamma}{2}} = 1,$$

and the third side is

$$\frac{x}{a'} \frac{\cos \frac{\beta+\gamma}{2}}{\cos \frac{\beta-\gamma}{2}} + \frac{y}{b'} \frac{\sin \frac{\beta+\gamma}{2}}{\cos \frac{\beta-\gamma}{2}} = 1;$$

wherefore the envelope is

$$A^2 \frac{x^2}{a'^2} + B^2 \frac{y^2}{b'^2} = C^2,$$

which coincides with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{if} \quad \frac{a}{a'} \pm \frac{b}{b'} \pm 1 = 0.$$

If this condition be not satisfied, the three conics intersect in the same four points, real or impossible.

The relations between the eccentric angles corresponding to normals may be found from the equation

$$\frac{aX}{\cos\theta} - \frac{bY}{\sin\theta} = a^2 - b^2,$$

which if (X, Y) be given is a biquadratic whose roots give the points, the normals at which meet in (X, Y). If $z \equiv \tan\frac{\theta}{2}$, this equation becomes

$$z^4 bY + 2z^3(aX + a^2 - b^2) + 2z(aX - a^2 + b^2) - bY = 0.$$

This equation having four roots, there must be two relations between the roots independent of (X, Y) as is obvious geometrically. These relations are manifest on inspection of the equation; they are

$$z_1 z_2 z_3 z_4 = -1, \quad z_2 z_3 + \ldots\ldots = 0:$$

and the relation between any three is found from these to be

$$z_2 z_3 + z_3 z_1 + z_1 z_2 = \frac{1}{z_2 z_3} + \frac{1}{z_3 z_1} + \frac{1}{z_1 z_2},$$

which is equivalent to

$$\sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) + \sin(\theta_1 + \theta_2) = 0.$$

Since $1 - (z_2 z_3 + \ldots) + z_1 z_2 z_3 z_4 = 0$, it appears that

$$\tan\left(\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2}\right) = \infty;$$

or $\theta_1 + \theta_2 + \theta_3 + \theta_4$ is an odd multiple of π.

The following is another method of investigating the same question. If the normal at (x, y) to the ellipse pass through (X, Y), we have

$$a^2 xX - b^2 yY = (a^2 - b^2)\,xy. \quad \ldots\ldots\ldots\ldots\ldots (A).$$

Now if $\frac{lx}{a}+\frac{my}{b}=1$, and $\frac{l'x}{a}+\frac{m'y}{b}=1$, be the equations of two lines joining the four points, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{lx}{a}+\frac{my}{b}-1\right)\left(\frac{l'x}{a}+\frac{m'y}{b}-1\right)=0$$

can be made to coincide with (A). This leads to

$$\lambda=1,\quad ll'+1=0,\quad mm'+1=0,$$

whence we see that the normals at the points where the ellipse is met by the straight lines

$$\frac{lx}{a}+\frac{my}{b}=1,\qquad \frac{x}{al}+\frac{y}{bm}=-1$$

meet in a point. If α, β be the two points on the former line, and γ one of the points on the other,

$$l=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\quad m=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\quad 1+\frac{\cos\gamma}{l}+\frac{\sin\gamma}{m}=0,$$

whence

$$\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}+\left(\cos\gamma\sin\frac{\alpha+\beta}{2}+\sin\gamma\cos\frac{\alpha+\beta}{2}\right)\cos\frac{\alpha-\beta}{2}=0,$$

or $$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

The same relation must hold for the fourth point (δ), whence the other equation

$$\alpha+\beta+\gamma+\delta=(2n+1)\pi$$

immediately follows.

589. A is the vertex of an ellipse of eccentricity e, along PA any chord is taken a length $PR=\frac{PA}{e^2}$, and RQ is drawn at right angles to the chord to meet the straight line through P parallel to the axis; prove that the locus of Q is a straight line perpendicular to the axis.

590. Two ellipses have the same major axis, and an ordinate NPQ is drawn; the tangent at P meets the other ellipse in two

points, the lines joining which to either extremity of the major axis meet the ordinate in L, M; prove that NP is a harmonic, and NQ a geometric, mean between NL, NM.

591. On the focal distances of any point of an ellipse as diameters are described two circles; prove that the eccentric angle of the point is equal to the angle which a common tangent to the circles makes with the minor axis.

592. The ordinate NP at a point P of an ellipse is produced to Q, so that $NQ : NP :: CA : CN$, and from Q two tangents are drawn to the ellipse; prove that they intercept on the minor axis produced a length equal to the minor axis.

593. From a point P of an ellipse two tangents are drawn to the circle on the minor axis; prove that these tangents will meet the diameter at right angles to CP in points lying on two fixed straight lines parallel to the major axis.

594. The lengths of two tangents drawn to an ellipse from a point in one of the equi-conjugate diameters are p, q; prove that

$$(a^2+b^2)(p^2-q^2)^2(p^2+q^2+a^2+b^2)^2 = 4(p^2+q^2)^2(a^2-b^2)^2.$$

595. ACA' is the major axis, P a point on the auxiliary circle; AP, $A'P$ meet the ellipse in Q, Q'; prove that the equation of QQ' is

$$(a^2+b^2)\,y\sin\theta + 2b^2x\cos\theta = 2ab^2;$$

where θ is the angle ACP. If an ordinate to P meet QQ' in R, the locus of R is an ellipse to which QQ' is a tangent.

596. A circle is described on a chord of the ellipse lying on the straight line

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta;$$

prove that the equation of the line joining the other two points of intersection of the ellipse and circle is

$$\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha = \frac{a^2+b^2}{a^2-b^2}\cos\beta.$$

597. In an ellipse whose axes are in the ratio $\sqrt{2}+1 : 1$, a circle whose diameter joins the extremities of two conjugate diameters of the ellipse will touch the ellipse.

598. Two circles have each double contact with an ellipse and touch each other; prove that

$$4b^2 = \frac{(r \pm r')^2}{e^2} + \frac{(r \mp r')^2}{1-e^2},$$

r, r' being the radii; also that the point of contact of the two circles is equidistant from their two chords of contact with the ellipse.

599. Two ellipses have common foci S, S', and from a point P on the outer are drawn two tangents PQ, PQ' to the inner; prove that $\cos\frac{QPQ'}{2} : \cos\frac{SPS'}{2}$ is a constant ratio.

600. The sides of a parallelogram circumscribing an ellipse are parallel to conjugate diameters; prove that the rectangle under the perpendiculars let fall from two opposite angles on any tangent is equal to the rectangle under those from the other two angles.

601. Prove that the equation

$$\left\{\frac{x}{a}\cos(\alpha-\beta)+\frac{y}{b}\sin(\alpha-\beta)-1\right\}\left\{\frac{x}{a}\cos(\alpha+\beta)+\frac{y}{b}\sin(\alpha+\beta)-1\right\} = \left\{\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-\cos\beta\right\}^2$$

is true at any point of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

and hence that the locus of a point, from which if two tangents be drawn to the ellipse the centre of the circle inscribed in the triangle formed by the two tangents and the chord of contact shall lie on the ellipse, is the curve

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2}.$$

602. The three points whose eccentric angles are α, β, γ are the angular points of a triangle; prove that the co-ordinates of the centre of perpendiculars of this triangle are

$$\frac{a^2+b^2}{2a}(\cos\alpha+\cos\beta+\cos\gamma)-\frac{a^2-b^2}{2a}\cos(\alpha+\beta+\gamma),$$

$$\frac{a^2+b^2}{2b}(\sin\alpha+\sin\beta+\sin\gamma)-\frac{a^2-b^2}{2b}\sin(\alpha+\beta+\gamma);$$

and of the centre of the circle circumscribing the triangle are

$$\frac{a^2-b^2}{4a}\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\},$$

$$\frac{b^2-a^2}{4b}\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}.$$

Find the loci of each of these points when the triangle is a maximum.

603. If a chord of a parabola be drawn through a fixed point on the axis, and an ellipse be described with the extremities of this chord for foci, and passing through the focus of the parabola; the minor axis of this ellipse will be constant.

604. Two tangents to a given circle intersect a constant length on a fixed tangent; prove that the locus of their point of intersection is a conic which the given circle osculates at a vertex.

605. If a tangent be drawn to an ellipse, and with the point of contact as centre another ellipse be described similar and similarly situated to the former, but of three times the area; then if from any point of this latter ellipse two other tangents be drawn to the former, the triangle formed by the three tangents will be double of the triangle formed by joining their points of contact.

606. TP, TQ are tangents to an ellipse at points whose eccentric angles are α, β, another tangent meets them in P', Q'; prove that

$$PP'.QQ'=TP'.TQ'\cos^2\frac{\alpha-\beta}{2}.$$

607. If two sides of a triangle be given in position and the third in magnitude, the locus of the centre of the Nine Points' Circle of the triangle is an ellipse; which reduces to a limited straight line if the acute angle between the given sides be 60°. If c be the given side, and 2α the given angle, the axes of the ellipse are equal to $\dfrac{c \sin 3\alpha}{4 \sin^2 \alpha \cos \alpha}$, $\dfrac{c \cos 3\alpha}{4 \sin \alpha \cos^2 \alpha}$.

608. If PM, PN be perpendiculars from any point of an ellipse on the axes, and the tangent at P meet the equi-conjugates in Q, R; the tangents from Q, R to the ellipse will be parallel to MN.

609. If a right-angled maximum triangle can be inscribed in an ellipse, the eccentric angle of the point at which is the right angle, is

$$\frac{1}{2} \cos^{-1} \left\{ -\frac{1}{2} \frac{a^2 + b^2}{a^2 - b^2} \right\}.$$

610. CP, CD are conjugate radii, and PQ is measured along the normal at P equal to m times CD; prove that the locus of Q is the ellipse

$$\frac{x^2}{(a - mb)^2} + \frac{y^2}{(b - ma)^2} = 1;$$

PQ being measured inwards or outwards as m is positive or negative.

611. If a triangle circumscribe a given ellipse, and its centre of gravity lie in the axis of x, at a distance c from the centre, its angular points will lie on the fixed conic

$$\frac{(x - 3c)^2}{a^2} + \frac{y^2 (a^2 - 9c^2)}{a^2 b^2} = 4.$$

612. If two tangents be drawn to an ellipse from a point P, the cosine of the angle between them is

$$\frac{CP^2 - AC^2 - BC^2}{2SP \,.\, S'P}.$$

613. Two points H, H' are conjugate with respect to an ellipse, P is any point on the ellipse, and PH, PH' meet the ellipse again in QQ'; prove that QQ' passes through the pole of HH'.

614. A triangle circumscribes the circle $x^2+y^2=a^2$, and two of its angular points lie on the circle $(x-c)^2+y^2=b^2$; prove that the locus of the third angular point is a conic touching the common tangents of the two circles; and that this conic becomes a parabola if $(c\pm a)^2=b^2-a^2$.

615. A triangle circumscribes an ellipse, and two of its angular points lie on a confocal ellipse; prove that the third angular point lies on another confocal ellipse, and that the perimeter of the triangle is constant.

616. The lines

$$l_1\frac{x}{a}+m_1\frac{y}{b}=1,\quad l_2\frac{x}{a}+m_2\frac{y}{b}=1,\quad l_3\frac{x}{a}+m_3\frac{y}{b}=1,$$

form a self-conjugate triangle to the ellipse; prove that

$$l_2l_3+m_2m_3=l_3l_1+m_3m_1=l_1l_2+m_1m_2=1,$$

and that the co-ordinates of the centre of perpendiculars of the triangle are

$$\frac{a^2-b^2}{a}l_1l_2l_3,\quad \frac{b^2-a^2}{b}m_1m_2m_3.$$

617. A triangle is self-conjugate to a given ellipse, and one angular point O is fixed; prove that the circle circumscribing the triangle passes through another fixed point Q; that C, Q, O are in one straight line, and that $CQ\,.\,CO=a^2+b^2$.

618. In the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a}+\frac{y^2}{b}=a+b,$$

a tangent to the former meets the latter in P, Q; prove that the tangents at P, Q are at right angles to each other.

619. Two tangents OP, OQ are drawn at the points α, β; prove that the co-ordinates of the centre of the circle circumscribing the triangle OPQ are

$$\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\,\frac{a^2+(a^2-b^2)\cos\alpha\cos\beta}{2a},$$

$$\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\,\frac{b^2+(b^2-a^2)\sin\alpha\sin\beta}{2b}.$$

If this point lie on the axis of x, the locus of O is a circle.

620. Two points P, Q are taken on an ellipse, such that the perpendiculars from Q, P on the tangents at P, Q intersect on the ellipse; prove that the locus of the pole of PQ is the ellipse

$$a^2x^2+b^2y^2=(a^2+b^2)^2.$$

If R be another point similarly related to P, the same relation will hold between Q, R: and the centre of perpendiculars of the triangle PQR will be the centre of the ellipse.

621. Perpendiculars p_1, p_2 are let fall from extremities of two conjugate diameters on any tangent, and p_3 is the perpendicular from the pole of the line joining the two former points: prove that $p_3^2=2p_1p_2$.

622. The normal at a point P of an ellipse meets the curve in Q, and any other chord PP' is drawn; QP' and the line through P at right angles to PP' meet in R: prove that the locus of R is the straight line

$$\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi=\frac{a^2+b^2}{a^2-b^2},$$

ϕ being the eccentric angle of P. The part of any tangent intercepted between this straight line and the tangent at P will be divided by the point of contact into two parts which subtend equal or supplementary angles at P.

623. Two normals are drawn to an ellipse at the extremities of a chord parallel to the tangent at the point α; prove that the locus of their intersection is the curve

$$2(ax \sin\alpha + by \cos\alpha)(ax \cos\alpha + by \sin\alpha) = (a^2 - b^2)^2 \sin 2\alpha \cos^2 2\alpha.$$

624. The normals at three points of an ellipse whose eccentric angles are α, β, γ will meet in a point if

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0.$$

625. If four normals to an ellipse meet in a point, the sum of the corresponding eccentric angles will be an odd multiple of π. Prove also that two tangents drawn to the ellipse parallel to two chords passing through the four points will intersect on one of the equi-conjugate diameters.

626. If (x_1, y_1), (x_2, y_2), (x_3, y_3) be three points on an ellipse, such that $x_1 + x_2 + x_3 = 0$, $y_1 + y_2 + y_3 = 0$, the circles of curvature at these points will pass through a point on the ellipse, whose co-ordinates are $\dfrac{4x_1x_2x_3}{a^2}$, $\dfrac{4y_1y_2y_3}{b^2}$.

627. If four normals be drawn from a given point to any one of a series of confocal ellipses, the sum of the angles made by them with the axis is constant.

628. The normals to an ellipse at the points where it is met by the straight lines

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta, \quad \frac{x}{a\cos\alpha} + \frac{y}{b\sin\alpha} = \frac{-1}{\cos\beta},$$

will all intersect in a point.

629. Pp is a diameter of an ellipse, PM, PN perpendiculars on the axes, MN produced meets the ellipse in Q, q; prove that the normals at Q, q, intersect in the centre of curvature at p.

630. If from a point O be drawn OP, OQ, OR, OS normals to an ellipse, and p, q, r, s be taken such that their co-ordinates are

the intercepts on the axes of the tangents at P, Q, R, S respectively; p, q, r, s will lie on one straight line. Also, if through the centre C be drawn straight lines at right angles to CP, CQ, CR, CS to meet the tangents at P, Q, R, S respectively, the four points so determined will lie on one straight line.

631. If the normals to an ellipse at P, Q, R, S meet in a point, and circles described about QRS, RSP, SPQ, PQR meet the ellipse again in P', Q', R', S', the normals at P', Q', R', S' will meet in a point.

632. PQ is a chord of an ellipse, normal at P, PP' a chord perpendicular to the axis, the tangent at P' meets the axes in T, T', the rectangle $TCT'R$ is completed, and CR meets PQ in U; prove that $CR \,.\, CU = a^2 - b^2$.

633. Normals drawn to an ellipse at the extremity of a chord passing through a given point on the axis will intersect in another ellipse whose axes are

$$\frac{a^2-b^2}{a}\left(c+\frac{1}{c}\right), \quad \frac{a^2-b^2}{b}\left(c\sim\frac{1}{c}\right);$$

the distance of the given point from the centre being c times the semi major axis.

634. A triangle $A'B'C'$ is inscribed in the ellipse $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, and its sides touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ in the points A, B, C; prove that, if the relation which must exist between the axes be $\dfrac{a}{a'}+\dfrac{b}{b'}=1$, the eccentric angles of A and A' will differ by π: but if the relation be $\dfrac{a}{a'}\sim\dfrac{b}{b'}=1$, the sum of these eccentric angles will be either π or 3π.

635. Two ellipses are concentric and similarly situated, and triangles can be inscribed and circumscribed; prove that the normals to one ellipse at the angular points of any such triangle meet in a point; and also the normals to the other at the points of contact.

636. The ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2+b^2)^2},$$

are so related that (1) an infinite number of triangles can be inscribed in the former whose sides shall touch the latter; (2) the central distance of any angular point of such a triangle will be perpendicular to the opposite side; (3) the normals to the first ellipse at the angles of any such triangle, and to the second at the points of contact, will severally meet in a point.

637. The ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2-b^2)^2}, \quad (a^2>2b^2),$$

are such that the normals to the latter at the angular points of any inscribed triangle which circumscribes the former, meet on the latter.

638. Two ellipses are confocal, and are such that triangles can be inscribed in one whose sides touch the other; prove that the perimeters of all such triangles are equal.

639. Triangles are circumscribed to an ellipse, such that the normal at each point of contact passes through the opposite angular point; prove that the angular points lie on the ellipse

$$\frac{a^2-k}{a^4}x^2+\frac{b^2-k}{b^4}y^2=1,$$

k being the positive root of the equation

$$z^2(a^2-b^2)^2+2a^2b^2(a^2+b^2)z=3a^4b^4.$$

Also the perimeter of the triangle formed by joining the points of contact is constant.

640. The two similar and similarly situated conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=m^2,$$

will be capable of having triangles circumscribing the first and inscribed in the second, if

$$m^2 \pm 2m = \frac{h^2}{a^2} + \frac{k^2}{b^2}.$$

641. Two straight lines are drawn parallel to the major axis at a distance $\frac{b}{e}$ from it; prove that the part of any tangent intercepted between them will be divided by the point of contact into two parts subtending equal angles at the centre.

642. A circle has its centre in the major axis of an ellipse, and triangles can be inscribed in the circle whose sides touch the ellipse; prove that the circle must touch the two circles

$$x^2 + (y \pm b)^2 = a^2.$$

643. Two tangents are drawn to an ellipse from a point (X, Y); prove that the rectangle under the perpendiculars from any point of the ellipse on the tangents bears to the square on the perpendicular to the chord of contact the ratio $1 : \lambda$; where

$$(\lambda^2 - 1)\left(\frac{X}{a^4} + \frac{Y^2}{b^4}\right)^2 = \left(\frac{1}{b^4} - \frac{1}{a^4}\right)\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1\right)\left(\frac{X^2}{a^2} - \frac{Y^2}{b^2} - \frac{a^2 - b^2}{a^2 + b^2}\right).$$

644. If a triangle be inscribed in an ellipse, and the centre of perpendiculars of the triangle be one of the foci, the sides of the triangle will touch one of the circles

$$\left\{x \pm \frac{a^2\sqrt{(a^2 - b^2)}}{a^2 + b^2}\right\}^2 + y^2 = \frac{a^2b^4}{(a^2 + b^2)^2}.$$

IV. *Hyperbola, referred to its axes, or asymptotes.*

645. Prove that the four equations

$$b\{x \pm \sqrt{(x^2 - a^2)}\} = a\{y \pm \sqrt{(y^2 + b^2)}\}$$

represent respectively the portions of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

which lie in the four quadrants.

646. Two points (x_1, y_1), (x_2, y_2) are taken on the hyperbola $xy = c^2$; prove that the equation of the line joining them is

$$\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1.$$

647. The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which is bisected at the point (X, Y) is

$$\frac{xX}{a^2} - \frac{yY}{b^2} = \frac{X^2}{a^2} - \frac{Y^2}{b^2}.$$

648. The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is a rectangular hyperbola.

649. A double ordinate PP' is drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the tangent at P meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in Q, Q'; prove that $P'Q$, $P'Q'$ are tangents to the hyperbola; and, if R, R' be the points in which these lines again meet the ellipse, that RR' divides PP' into two parts in the ratio of $2 : 1$.

650. A circle is drawn to touch the asymptotes of a hyperbola; prove that the tangents drawn to it at the points where it meets the hyperbola will also touch the auxiliary circle of the hyperbola.

651. Two hyperbolas have the same asymptotes, and NPQ is drawn parallel to one asymptote meeting the other in N, and the curves in P, Q; a tangent at Q meets the outer hyperbola in two points, and the lines joining these to the centre meet the ordinate NQ in L, M; prove that NQ is a geometric mean between NL, NM, and that NP is a harmonic mean between NQ and the harmonic mean between NL, NM.

652. The locus of a point from which can be drawn two straight lines at right angles to each other, each of which touches

one of the rectangular hyperbolas $xy = \pm c^2$ is also the locus of the feet of the perpendiculars let fall from the origin on tangents to the hyperbolas $x^2 - y^2 = \pm 4c^2$.

653. The axes of an ellipse are the asymptotes of a rectangular hyperbola which does not meet it in real points; prove that if two tangents be drawn to the ellipse from a point on the hyperbola, the difference of the eccentric angles of the points of contact will be least when the point lies on an equi-conjugate of the ellipse.

654. The locus of the equation

$$y = x + \frac{c^2}{x+}\,\frac{c^2}{x+} \ldots \text{ to } \infty,$$

is that part of the hyperbola $y^2 - x^2 = c^2$, which, starting from a vertex, goes to ∞ on the line $y = x$.

655. Normals are drawn to a rectangular hyperbola at the extremities of a chord parallel to a given line; the locus of their intersection is another rectangular hyperbola whose asymptotes make with the asymptotes of the given hyperbola angles equal and opposite to those made by the given line.

656. An ellipse is described confocal with a given hyperbola, and the equi-conjugates of the ellipse lie on the asymptotes of the hyperbola; prove that if from any point of the ellipse two tangents be drawn to the hyperbola, the centres of two of the circles which touch these tangents and their chord of contact will lie on the hyperbola.

657. A triangle circumscribes a given circle, and its centre of perpendiculars is a given point; prove that its angular points lie on a fixed conic which is an ellipse, parabola, or hyperbola, as the given point lies within, upon, or without the given circle.

V. *Polar Co-ordinates.*

658. The equation of the normal drawn to the circle $r=2a\cos\theta$, at the point where $\theta=\alpha$, is

$$a\sin 2\alpha = r\sin(2\alpha-\theta).$$

659. The equation of the straight line which meets the circle $r=2a\cos\theta$, at the points where $\theta=\alpha$, $\theta=\beta$, is

$$2a\cos\alpha\cos\beta = r\cos(\alpha+\beta-\theta).$$

660. PSQ is a focal chord of a conic, and a parallel chord AP' through the vertex A meets the latus-rectum in Q'; prove that the ratio $SP \,.\, SQ : AP' \,.\, AQ'$ is constant.

661. Prove that the equations

$$\frac{c}{r} = e\cos\theta \pm 1$$

represent the same conic. If (r, θ) be the coordinates of a point on the locus when the upper sign is taken, what will be its co-ordinates when the lower sign is taken?

662. A conic is described having a common focus with the conic $\frac{c}{r} = 1 + e\cos\theta$, similar to it, and touching it at the point $\theta=\alpha$; prove that its latus rectum is $\frac{2c(1-e^2)}{1+2e\cos\alpha+e^2}$, and that the angle between the axes of the two conics is

$$2\tan^{-1}\frac{e+\cos\alpha}{\sin\alpha}.$$

663. A triangle ABC has its angular points on a hyperbola, and the centre of its circumscribed circle at a focus of the hyperbola; prove that $e\sin B\sin C = 1$, B, C being angular points on the same branch, and e the eccentricity.

664. If r_1, r_2, r_3 be the focal distances of three points on a conic, and α, β, γ the angles between them, the latus rectum l is given by the equation

$$\frac{8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{l}=\frac{\sin\alpha}{r_1}+\frac{\sin\beta}{r_2}+\frac{\sin\gamma}{r_3};$$

the angles α, β, γ being always so taken that their sum is 2π.

665. SP is a radius vector from the focus of a conic, Q, Q' two points on it, conjugate with respect to the conic; prove that the latus rectum

$$=\frac{2SP\,.\,SQ}{SQ-SP}+\frac{2SP\,.\,SQ'}{SQ'-SP}.$$

666. If PQ, PR be two chords of an ellipse subtending each the same angle at the focus, the tangent at P and the chord QR will intersect on the directrix.

667. If an ellipse circumscribe a triangle ABC, and the centre of perpendiculars of the triangle be a focus of the ellipse, the latus rectum will be

$$\frac{4R\cos A\cos B\cos C}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}};$$

R being the radius of the circumscribed circle.

668. Prove that any chord of the ellipse

$$\frac{c}{r}=1+e\cos\theta$$

which is normal at a point where the conic is met by the lines

$$\frac{c}{r}\left(e+\frac{1}{e}\right)=\pm\sin\theta-(1-e^2)\cos\theta$$

will subtend a right angle at the pole.

669. Two parabolas have a common focus and axes in the same straight line in opposite directions; a circle is drawn through the focus touching both parabolas; prove that

$$3r^{\frac{2}{3}} = a^{\frac{2}{3}} - (aa')^{\frac{1}{3}} + a'^{\frac{2}{3}},$$

a, a' being the latera recta, and r the radius of the circle.

670. Through a fixed point is drawn any straight line, and on it are taken two points, such that their distances from the fixed point are in a constant ratio, and the line joining them subtends a constant angle at another fixed point; prove that their loci are circles.

671. If a triangle ABC circumscribe an ellipse, and its sides subtend angles α, α'; β, β'; γ, γ' at the foci,

$$\frac{\sin(\alpha - A)\sin(\alpha' - A)}{\sin\alpha\sin\alpha'} = \frac{\sin(\beta - B)\sin(\beta' - B)}{\sin\beta\sin\beta'}$$
$$= \frac{\sin(\gamma - C)\sin(\gamma' - C)}{\sin\gamma\sin\gamma'};$$

the angles being taken so that $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 2\pi$.

672. Four tangents to the parabola $\dfrac{2a}{r} = 1 + \cos\theta$ are drawn at the points θ_1, θ_2, θ_3, θ_4; prove that the centres of the circles circumscribing the four triangles formed by them lie on the circle

$$2r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 = a\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4 - \theta).$$

673. Two straight lines bisect each other at right angles; prove that the locus of the points at which they subtend equal angles is

$$\frac{r^2}{ab} = \frac{a\cos\theta - b\sin\theta}{b\cos\theta - a\sin\theta};$$

$2a$, $2b$ being the lengths of the lines, their point of intersection the pole, and $2a$ the direction of the prime radius.

674. Two circles intersect, a straight line is drawn through one of their common points, and tangents are drawn to the circles

at the points in which this line again meets them: prove that the locus of the point of intersection of these tangents is the curve

$$cr = 2ab\{1 + \cos(\theta + \alpha - \beta)\};$$

the second common point of the circles being pole, the common chord (c) the initial line, a, b the radii, and α, β the angles subtended by c in the segments of the two circles which lie each without the other circle.

675. Two ellipses have a common focus, and axes inclined at an angle α, and triangles can be inscribed in one whose sides touch the other; prove that

$$c_1^2 \pm 2c_1c_2 = e_1^2c_2^2 + e_2^2c_1^2 - 2e_1e_2c_1c_2 \cos\alpha,$$

c_1, c_2 being the latera recta, and e_1, e_2 the eccentricities. Also, if θ, ϕ, ψ be the angles subtended at the focus by the sides of any one of the triangles,

$$4\cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} = \frac{c_2}{c_1}.$$

VI. *General Equation of the Second Degree.*

The general equation of a conic being

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0,$$

the equations giving its centre are

$$ax + c'y + b' = 0, \quad c'x + by + a' = 0.$$

The equation determining its eccentricity may be found from the consideration that

$$\frac{a + b - 2c'\cos\omega}{\sin^2\omega}, \quad \frac{ab - c'^2}{\sin^2\omega}$$

are unchanged by transformation of co-ordinates; and therefore that $\dfrac{(a+b-2c'\cos\omega)^2}{(ab-c'^2)\sin^2\omega}$ is equal to $\dfrac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2}$; if 2α, 2β be the axes. We have then the eccentricity e given by the equation

$$\frac{e^4}{1-e^2}=\frac{(a+b-2c'\cos\omega)^2}{(ab-c'^2)\sin^2\omega}-4.$$

The foci may be determined from the condition that the rectangle under the perpendiculars from them on any tangent is constant. Thus taking the simple case when the origin is the centre, and the axes rectangular, if the equation of the conic be

$$ax^2+by^2+2c'xy=f,$$

and (X, Y) be one focus, $(-X, -Y)$ the other, we must have, in order that the straight line $lx+my=1$ may be a tangent,

$$\frac{(1-lX-mY)(1+lX+mY)}{l^2+m^2}=\text{a constant}\equiv\mu,$$

or, $$l^2(\mu+X^2)+m^2(\mu+Y^2)+2lmXY-1=0\ldots\ldots\text{(A)}.$$

But the condition that this line may be a tangent gives also the condition that

$$ax^2+by^2+2c'xy-f(lx+my)^2$$

must be a perfect square, or

$$(a-fl^2)(b-fm^2)=(c'-flm)^2;$$

whence $$l^2bf+m^2af-2lmc'f+c'^2-ab=0\ldots\ldots\ldots\text{(B)}.$$

Now (A) (B) expressing the same condition must give the same relation between l and m; hence

$$\frac{\mu+X^2}{bf}=\frac{\mu+Y^2}{af}=\frac{XY}{-c'f}=\frac{1}{ab-c'^2}\equiv\frac{X^2-Y^2}{(b-a)f}.$$

The equations are then

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{c'}=\frac{f}{c'^2-ab}.$$

It may be noticed also that we obtain the following equation for μ

$$\left(\mu - \frac{bf}{ab-c'^2}\right)\left(\mu - \frac{af}{ab-c'^2}\right) = \frac{c'^2 f^2}{\{ab-c'^2\}^2}$$

equivalent to
$$\left(\frac{f}{\mu} - a\right)\left(\frac{f}{\mu} - b\right) = c'^2;$$

whose roots are the squares of the semi axes.

To each root correspond two foci, which are real for one and unreal for the other.

The same method will apply to all cases; and, the foci being found, the directrices are their polars.

The more useful forms of the equation are

$$(1) \quad \frac{x^2}{aa'} + 2\lambda xy + \frac{y^2}{bb'} - x\left(\frac{1}{a} + \frac{1}{a'}\right) - y\left(\frac{1}{b} + \frac{1}{b'}\right) + 1 = 0,$$

which, for different values of λ, represents a series of conics passing through four given points; two of the joining lines being taken as axes:

$$(2) \qquad \frac{x}{h} + \frac{y}{k} - 1 = 2\lambda\left(\frac{xy}{hk}\right)^{\frac{1}{2}},$$

the equation of a conic touching the axes at distances h, k respectively from the origin. It is sometimes convenient to use this as the equation of the conic touching four given straight lines; h, k, λ being then parameters connected by the equations

$$\left(\frac{1}{h} - \frac{1}{a}\right)\left(\frac{1}{k} - \frac{1}{b}\right) = \frac{\lambda^2}{hk} = \left(\frac{1}{h} - \frac{1}{a'}\right)\left(\frac{1}{k} - \frac{1}{b'}\right),$$

the equations of the other two given straight lines being

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \frac{x}{a'} + \frac{y}{b'} = 1.$$

When (2) represents a parabola, $\lambda = 1$; and the equation may be written

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1.$$

The equation of the polar of (X, Y) to the conic in the most general form is

$$x(aX + c'Y + b') + y(c'X + bY + a') + b'X + a'Y + c = 0,$$

and this may be adapted to all the special cases. The equation of the tangent at a point (X, Y) to the parabola

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1, \text{ is } \frac{x}{(hX)^{\frac{1}{2}}} + \frac{y}{(kY)^{\frac{1}{2}}} = 1,$$

the signs of the radicals in the equation of the tangent being determined by the corresponding signs in the equation of the curve at the point (X, Y). The equation of the polar of course cannot be expressed in this form.

The equation of two tangents from (X, Y) is

$$(aX^2 + bY^2 + c + 2a'Y + 2b'X + 2c'XY)$$
$$(ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy)$$
$$= \{x(aX + c'Y + b') + y(c'X + bY + a') + b'X + a'Y + c\}^2.$$

676. The equation

$$lx(bx + cy) - my(ax + by) = 0$$

represents a pair of conjugate diameters of the conic

$$ax^2 + 2bxy + cy^2 = d.$$

677. If $\dfrac{x^2}{a'^2} + \dfrac{y^2}{b'^2} = 1$ be an ellipse referred to conjugate diameters inclined at an angle ω, the condition that the circle $x^2 + 2xy\cos\omega + y^2 = r^2$ may touch the ellipse is

$$\left(\frac{1}{r^2} - \frac{1}{a'^2}\right)\left(\frac{1}{r^2} - \frac{1}{b'^2}\right) = \frac{\cos^2\omega}{r^4}.$$

Hence determine the relations between any conjugate diameters and the axes.

678. The axes of the conic $ax^2 + 2bxy + cy^2 = d$ make with the lines bisecting the angles between the axes of co-ordinates angles θ; prove that

$$\tan 2\theta = \frac{(a-c)\sin\omega}{(a+c)\cos\omega - 2b},$$

ω being the angle between the axes.

679. The locus of the foot of the perpendicular let fall from a point (X, Y) on any tangent to the ellipse $ax^2+2bxy+cy^2=1$ is

$$c(x-X)^2-2b(x-X)(y-Y)+a(y-Y)^2$$
$$=(ac-b^2)\{x(x-X)+y(y-Y)\}^2,$$

the axes being rectangular. Prove that this reduces to a circle and a point-circle, if

$$\frac{X^2-Y^2}{a-c}=\frac{XY}{b}=\frac{1}{b^2-ac}.$$

680. The equations determining the foci of the conic

$$ax^2+2bxy+cy^2=d$$

are
$$\frac{y(x+y\cos\omega)}{a\cos\omega-b}=\frac{x(y+x\cos\omega)}{c\cos\omega-b}=\frac{d}{ac-b^2},$$

ω being the angle between the axes.

681. The general equation of a conic confocal with the conic $ax^2+2bxy+cy^2=1$ is

$$(x^2+y^2)(ac-b^2)+\lambda(ax^2+2bxy+cy^2)=\frac{(a+\lambda)(c+\lambda)-b^2}{\lambda};$$

and the given conic is cut at right angles by any conic whose equation is

$$(\lambda-a)x^2+\frac{\lambda(c-a)-2ac}{b}xy-(\lambda+c)y^2=1.$$

682. The equation of the equi-conjugates of the conic

$$ax^2+2bxy+cy^2=1$$

is
$$\frac{ax^2+2bxy+cy^2}{ac-b^2}=\frac{2(x^2+y^2)}{a+c}.$$

683. The equation of the equi-conjugates of the conic

$$ax^2+2bxy+cy^2=1$$

is $$(a-c)(ax^2-cy^2)+2x(bx+cy)(b-a\cos\omega)$$
$$+2y(ax+by)(b-c\cos\omega)=0;$$

when ω is the angle between the axes.

684. The tangents to the two conics

$$\frac{x^2}{a}+\frac{y^2}{b}=1,\quad \frac{x^2}{a}+2\lambda xy-\frac{y^2}{b}=\frac{a-b}{a+b},$$

at any common point are at right angles. If both curves be hyperbolas, they will have four real common tangents.

685. If a common tangent be drawn to the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a^2}+2\lambda\frac{xy}{ab}-\frac{y^2}{b^2}+(1+\lambda^2)\frac{a^2+b^2}{a^2-b^2}=0\,;$$

it will subtend a right angle at the centre.

686. Prove that in general two parabolas can be described passing through the points of intersection of the conics

$$ax^2+by^2+c+2a'y+2b'x+2c'xy=0,$$
$$Ax^2+By^2+C+2A'y+2B'x+2C'xy=0\,;$$

and that the axes of these parabolas will be at right angles to each other if

$$\frac{c'}{a-b}=\frac{C'}{A-B}.$$

687. The equation of the director circle of the conic

$$f(x,\,y)\equiv ax^2+2c'xy+by^2+2b'x+2a'y+c=0,$$

is $(x^2+y^2)(c'^2-ab)+2x(c'a'-bb')+2y(b'c'-aa')$
$$+b'^2-ca+a'^2-bc=0.$$

688. The equation of the asymptotes is

$$(c'^2-ab)f(x,\,y)=aa'^2+bb'^2-2a'b'c'.$$

689. The equation of the chord which passes through the point $(X,\,Y)$ and is bisected at that point is

$$(x-X)(aX+c'Y+b')+(y-Y)(c'X+bY+a')=0\,;$$

and the equation of the tangents drawn through the origin is

$$(ac-b'^2)x^2+2(cc'-a'b')xy+(bc-a'^2)y^2=0.$$

690. The foci are given by the equations

$$x(aa'-b'c')+y(bb'-c'a')+a'b'-cc'=xy(c'^2-ab),$$
$$2x(bb'-c'a')-2y(aa'-b'c')+b'^2-ca-a'^2+bc=(x^2-y^2)(c'^2-ab).$$

691. Trace the locus of the equation

$$\frac{x}{x+y-a}+\frac{y}{x+y-b}=m,$$

for values of $m >=$ and < 1.

692. Trace the conics

$$2x^2-2xy-2ay+a^2=0, \quad 2y^2-2xy-2ay+a^2=0,$$
$$2xy-2ay+a^2=0;$$

shewing that they touch each other two and two.

693. Two parabolas are so situated that a circle can be described through their four points of intersection; prove that the distance of the centre of this circle from the axis of either parabola is equal to half the latus rectum of the other.

694. A hyperbola is drawn touching the axes of an ellipse and the asymptotes of the hyperbola touch the ellipse; prove that the centre of the hyperbola lies on one of the equi-conjugates of the ellipse.

695. Five fixed points are taken, no three of which are in one straight line, and five conics are described, each bisecting all the lines joining four of the points, two and two: prove that these conics will have one common point.

696. A, P and B, Q are points taken respectively on two parallel fixed straight lines, A, B being fixed points and P, Q variable points, subject to the condition that $AP \cdot BQ$ is constant; prove that PQ touches a fixed conic which will be an ellipse or hyperbola according as P, Q are on the same or opposite sides of AB.

697. One side AB of a rectangle $ABCD$ slides between two rectangular axes; prove that the loci of C, D are ellipses whose area is independent of the magnitude of AB; and that the angle between their axes is $\cot^{-1}\dfrac{2BC}{AB}$. If in any position AB make an angle θ with the axis of y, and α, β be the angles which the

tangents at C, D to the loci make with the axes of y, x respectively,

$$\cot\alpha + \cot\theta = \cot\beta + \tan\theta = \frac{AB}{BC}.$$

698. A circle being traced on a plane, the locus of the vertex of all cones on that base whose principal elliptic sections have an eccentricity e, is the surface generated by the revolution about its conjugate axis of an hyperbola of eccentricity e^{-1}.

699. A straight line of given length slides between two fixed straight lines, and from its extremities two straight lines are drawn in given directions; prove that the locus of their intersection is an ellipse.

700. Two circles of radii r, r' $(r > r')$ touch each other, and a conic is described having real double contact with both; prove that, when the points of contact are not on different branches of a hyperbola, the eccentricity $> \frac{1}{2}\sqrt{\left(1 + \frac{r}{r'}\right)}$, and the latus rectum is greater than, equal to, or less than $r - r'$, according as the conic is an ellipse, parabola, or hyperbola. If the contacts be on different branches of a hyperbola, $e > \frac{r + r'}{r - r'}$, and the asymptotes always touch a fixed parabola.

701. If ABC be a triangle circumscribing an ellipse, S, S' the foci, and if $SA = SB = SC$, then will

$$\frac{S'A \,.\, S'B \,.\, S'C}{SA \,.\, SB \,.\, SC} = 1 - e^2;$$

and each angle of the triangle ABC will lie between the values $\cos^{-1}\frac{1 \pm e}{2}$.

702. One angular point of a triangle, self conjugate to a given conic, is given; prove that the circles on the opposite sides as diameters will have a common radical axis, which is normal at

the given point to the similar, concentric, and similarly situated conic through that point.

703. If e be the eccentricity of the conic

$$ax^2 + by^2 + c + 2a'y + 2b'x + 2c'xy = 0,$$

and ω the angle between the axes,

$$\frac{e^4}{1-e^2} = \frac{(a-b)^2 \sin^2 \omega + \{(a+b) \cos \omega - 2c'\}^2}{(ab - c'^2) \sin^2 \omega}.$$

704. The co-ordinates of the focus of the parabola

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1,$$

are given by the equations

$$\frac{x}{k} = \frac{y}{h} = \frac{hk}{h^2 + k^2 + 2hk \cos \omega},$$

and the equation of its directrix is

$$x (h + k \cos \omega) + y (k + h \cos \omega) = hk \cos \omega.$$

705. In the parabola

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1,$$

a tangent is drawn meeting the axes in P, Q; and perpendiculars are drawn from P, Q to the opposite axes respectively; prove that the locus of the point of intersection of these perpendiculars is

$$\frac{x + y \cos \omega}{k} + \frac{y + x \cos \omega}{h} = \cos \omega.$$

706. The condition that the straight line $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ should touch the conic

$$\frac{x}{h} + \frac{y}{k} - 1 = 2\lambda \left(\frac{xy}{hk}\right)^{\frac{1}{2}}$$

is

$$\left(\frac{1}{h} - \frac{1}{\alpha}\right)\left(\frac{1}{k} - \frac{1}{\beta}\right) = \frac{\lambda^2}{hk}.$$

707. The asymptotes of the conic

$$\frac{x}{h}+\frac{y}{k}-1=2\lambda\left(\frac{xy}{hk}\right)^{\frac{1}{2}}$$

will touch the parabola

$$\left(\frac{x}{2h}\right)^{\frac{1}{2}}+\left(\frac{y}{2k}\right)^{\frac{1}{2}}=1.$$

708. The equation of the director circle of the conic

$$\frac{x}{h}+\frac{y}{k}-1=2\lambda\left(\frac{xy}{hk}\right)^{\frac{1}{2}}$$

is $$(\lambda^2-1)(x^2+y^2+2xy\cos\omega)+h(x+y\cos\omega)$$
$$+k(y+x\cos\omega)-hk\cos\omega=0.$$

709. A conic is drawn to touch four given straight lines, two of which are parallel; prove that its asymptotes will touch a fixed hyperbola, and that this hyperbola touches the diagonals of the quadrilateral, formed by the four given straight lines, at their middle points.

710. If a rectangular hyperbola have double contact with a parabola, the centre of the hyperbola and the pole of the chord of contact will be equidistant from the directrix of the parabola.

711. The area of the ellipse of minimum eccentricity which can be drawn touching two given straight lines at distances h, k from their point of intersection is

$$\pi hk(h^2+k^2)\sin\omega\,\frac{(h^2+k^2-2hk\cos\omega)^{\frac{1}{2}}}{(h^2+k^2+2hk\cos\omega)^{\frac{3}{2}}};$$

and, if e be the minimum eccentricity,

$$\frac{e^4}{1-e^2}=\frac{(h^2-k^2)^2}{h^2k^2\sin^2\omega}.$$

712. Four points are such that ellipses can be described through them, and e is the least eccentricity of any such ellipse;

e' is the eccentricity of the hyperbola, which is the locus of the centres of all such conics; prove that

$$\frac{e'^4}{e'^2-1}+\frac{e^4}{e^2-1}=4.$$

Prove also that the equi-conjugates of the ellipse are parallel to the asymptotes of the hyperbola.

713. The equation of the conic of minimum eccentricity through four given points is

$$\frac{x^2}{aa'}+\frac{4xy\cos\omega}{aa'+bb'}+\frac{y^2}{bb'}-x\left(\frac{1}{a}+\frac{1}{a'}\right)-y\left(\frac{1}{b}+\frac{1}{b'}\right)+1=0;$$

the points being on the axes of x, y at distances a, a'; b, b' from the origin.

714. If ϕ be the angle which an axis of any conic through four given points makes with the line bisecting the angle θ between the axes of the two parabolas through the four points, the eccentricity e of the conic is given by the equation

$$\frac{e^4}{1-e^2}=\frac{4\cos^2\theta}{\sin^2\theta-\sin^2 2\phi}.$$

715. If θ be the acute angle between the axes of two parabolas, the minimum eccentricity of a conic passing through their points of intersection is $\sqrt{\dfrac{2}{1+\sec\theta}}$; and either axis of this conic makes equal angles with the axes of the two parabolas.

716. Three points A, B, C are taken on an ellipse, the circle about ABC meets the ellipse again in P, and PP' is a diameter; prove that of all ellipses passing through $ABCP'$ the given ellipse is the one of least eccentricity.

717. Of all ellipses circumscribing a parallelogram, the one of least eccentricity has its equi-conjugates parallel to the sides of the parallelogram.

718. The ellipse of least eccentricity which can be inscribed in a given parallelogram is such that any point of contact divides the side on which it lies into parts which are to each other as the squares on the adjacent diagonals.

719. The axes of the conic which is the locus of the centres of all conics passing through four given points are parallel to the asymptotes of the rectangular hyperbola which passes through the four points.

720. The equation of a conic having the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for focus, and osculating the ellipse at the point θ, is

$$(x^2+y^2)(a^2\cos^2\theta+b^2\sin^2\theta)^3$$
$$=\{(a^2-b^2)(ax\cos^3\theta-by\sin^3\theta)+a^2b^2\}^2.$$

721. A parabola has contact of the third order with an ellipse; prove that the axis of the parabola is parallel to the diameter of the ellipse through the point of contact, and that the latus rectum of the parabola is equal to $\frac{2a^2b^2}{r^3}$, where a, b are the semiaxes and r the central distance of the point of contact.

722. The locus of the foci of all conics which have a contact of the third order with a given curve at a given point, is a curve whose equation referred to the tangent and normal at the given point is of the form

$$m(x^2+y^2)(my+x-2a)=ay(x-a).$$

723. An ellipse of constant area πc^2 is described having contact of the third order with a given parabola whose latus rectum is $2m$: prove that the locus of the centre of the ellipse is an equal parabola whose vertex is at a distance $\left(\frac{c^4}{m}\right)^{\frac{1}{3}}$ from the vertex of the given parabola; also that, when $c=m$, the axes of the ellipse make with the axis of the parabola angles

$$\frac{1}{2}\tan^{-1}(2\tan\phi),$$

where ϕ is the angle which the tangent at the point of contact makes with the axis.

724. The locus of the centre of a rectangular hyperbola having contact of the third order with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$x^2+y^2=(a^2+b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{3}{2}}.$$

725. The locus of a point, such that tangents drawn from it to a fixed conic intercept a constant length on a fixed tangent, is a conic having contact of the third order with the given conic at the extremity of the diameter through the fixed point of contact; and that the locus is an ellipse, parabola, or hyperbola, according as the given length is less than, equal to, or greater than the diameter parallel to the fixed tangent.

726. On every straight line can be found two points, conjugate to each other with respect to a given conic, and the distance between which subtends a right angle at a given point not on the straight line.

727. If a parabola meet an ellipse in four points such that the four normals to the ellipse at those points meet in one point, the axis of the parabola will be parallel to one of the equi-conjugate diameters of the ellipse.

728. A conic is drawn through four given points lying on two parallel straight lines; prove that the asymptotes touch the parabola which touches the other four joining lines.

VII. *Envelopes.*

The equation of the tangent to a parabola in the form

$$y = mx + \frac{a}{m},$$

gives as the condition of equal roots in m, $y^2 = 4ax$, and the equation of the tangent to an ellipse

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1,$$

written in the form

$$z^2\left(1 + \frac{x}{a}\right) - 2z\frac{y}{b} + 1 - \frac{x}{a} = 0, \quad \left(z \equiv \tan\frac{\alpha}{2}\right)$$

gives as the condition of equal roots in z,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

So, in general, if the equation of a straight line, or curve, involve a parameter in the second degree, it follows, that through any proposed point can in general be drawn two straight lines, or curves, of the series represented by the equation. These two curves will be the tangents (rectilinear or curvilinear) from the proposed point to the envelope of the system. In order that they may be coincident, the point from which they are drawn must be a point on the envelope.

Thus, "To find the envelope of a system of conics having a given focus, length and direction of major axis."

The equation of any such conic may be taken

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta,$$

where e is the parameter. The envelope is then the curve

$$(r\cos\theta)^2 = 4a(r-a),$$

or,

$$r^2\sin^2\theta = (r-2a)^2,$$

or, $$r = \frac{2a}{1 \pm \sin\theta};$$

a pair of parabolas having the given point for focus, and their common latus rectum in the given direction and of twice the given length. This admits of very easy geometrical proof.

In this case every one of the system of curves has real contact with the envelope; but it frequently happens that this is true for only a portion of the system.

The same principle applies to systems whose equation involves higher powers of the parameter; but in the following examples it will be found that the general equation of the system whose envelope is required can always be expressed in the form $U + 2\lambda V + \lambda^2 W = 0$, where λ is the parameter; and the corresponding envelope is the curve $UW = V^2$.

729. The envelope of the circles

$$x^2 + (y - \beta)(y - \gamma) = 0,$$

where β, γ are connected by the equation

$$\frac{1}{\beta} \sim \frac{1}{\gamma} = \frac{1}{a},$$

is the two circles $$x^2 + y^2 \pm 2ax = 0.$$

730. The envelope of the circles

$$x^2 + (y - \beta)(y - \gamma) = 0,$$

where β, γ are connected by the equation

$$(\beta + a)(\gamma - a) + h^2 = 0,$$

is the two circles $$y^2 + (x \pm h)^2 = a^2.$$

731. The ellipse $$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

has its axes connected by the equation

$$a^2\alpha^6 = \beta^2(\alpha^2 - \beta^2):$$

prove that the envelope is the two circles

$$x^2 + y^2 \pm 2ax = 0;$$

and if the relation between the axes be

$$a^2\alpha^2 = \beta^2 (p\alpha^2 - q\beta^2),$$

the envelope is $$px^2 + qy^2 + 2aq^{\frac{1}{2}}x = 0.$$

732. Through each point of the straight line $\frac{lx}{a} + \frac{my}{b} = 1$ is drawn a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ bisected in the point; prove that the envelope is the parabola

$$\left(\frac{mx}{a} - \frac{ly}{b}\right)^2 + 4\left(\frac{lx}{a} + \frac{my}{b} - 1\right) = 0.$$

733. The envelope of the ellipse

$$x^2 + y^2 - 2(ax \cos\alpha + by \sin\alpha)\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right)$$
$$+ (a^2 + b^2 - c^2)\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right)^2 = a^2\sin^2\alpha + b^2\cos^2\alpha - c^2$$

is the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1.$$

734. A variable tangent to a parabola meets two fixed tangents in two points; prove that the directrix of the parabola which touches the fixed tangents at these points envelopes another parabola.

735. A variable tangent to a parabola meets two fixed tangents, and a circle is described on the part intercepted as diameter; prove that the envelope of these circles is an ellipse which touches the fixed tangents at the points where the directrix meets them.

736. The director circle of a conic and one point of the conic being given, prove that the envelope is a fixed conic whose major axis is a diameter of the given circle.

737. If two points be taken on an ellipse such that the normals at these points intersect a given normal in the same point, the chord joining the two points envelopes a parabola whose directrix passes through the centre and whose focus is the foot of the perpendicular from the centre on a tangent perpendicular to the given normal.

738. In an ellipse given the centre and directrix, prove that the envelope is two parabolas having their common focus at the given centre.

739. In an ellipse given one extremity of the minor axis and a directrix, prove that the envelope is a circle having its centre at the given point and touching the given straight line.

740. If three points be taken on an ellipse, such that their centre of gravity is a fixed point, the straight lines joining them two and two will touch a fixed conic.

741. If three points be taken on an ellipse, such that the centre of perpendiculars of the triangle formed by joining them is a fixed point, the joining lines will touch a fixed conic whose asymptotes are perpendicular respectively to the tangents drawn from the fixed point to the ellipse.

VIII. *Areal Coordinates.*

In this system, the position of a point P with respect to three fixed points A, B, C, not in one straight line, is determined by the values of the ratios of the triangles PBC, PCA, PAB to the triangle ABC, any one of them PBC being positive or negative, as P and A lie on the same or opposite sides of BC. These are usually denoted by α, β, γ, which satisfy the equation $\alpha+\beta+\gamma=1$. A point is therefore completely determined by equations of the form $l\alpha=m\beta=n\gamma$.

The general equation of a straight line is $l\alpha+m\beta+n\gamma=0$, and l, m, n are proportional to the perpendiculars from A, B, C on the straight line, perpendiculars drawn in opposite directions being of course affected with opposite signs.

The condition that the straight lines

$$l\alpha + m\beta + n\gamma = 0, \quad l'\alpha + m'\beta + n'\gamma = 0,$$

shall be parallel is

$$mn' - m'n + nl' - n'l + lm' - l'm = 0;$$

and that they may be perpendicular is

$$ll' \sin^2 A + \ldots + \ldots = (mn' + m'n) \sin B \sin C \cos A + \ldots + \ldots$$

If $l = m = n$, or $l' = m' = n'$, it will be noticed that both these conditions are satisfied. The straight line $\alpha + \beta + \gamma = 0$, *the line at infinity*, may then be regarded as both parallel and perpendicular to every other straight line; the angle which it makes with any straight line being really indeterminate.

In questions relating to four given points, it is convenient to take the four points to be

$$l\alpha = \pm m\beta = \pm n\gamma;$$

and similarly to take the equations of four given straight lines to be

$$l\alpha \pm m\beta \pm n\gamma = 0.$$

The general equation of a conic is

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0;$$

the polar of any point $(\alpha', \beta', \gamma')$ being

$$\alpha'(l\alpha + n'\beta + m'\gamma) + \beta'(n'\alpha + m\beta + l'\gamma) + \gamma'(m'\alpha + l'\beta + n\gamma) = 0.$$

The special forms of this equation most useful are

(1) circumscribing the triangle of reference

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0;$$

(2) inscribed in the triangle

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0;$$

(3) touching two sides at the points where the third meets them

$$\beta\gamma = l\alpha^2;$$

(4) to which the triangle is self-conjugate

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0.$$

The general equation of a conic passing through four given points ($l\alpha = \pm m\beta = \pm n\gamma$) is

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0,$$

with the condition $$\frac{L}{l^2} + \frac{M}{m^2} + \frac{N}{n^2} = 0\,;$$

and the general equation of a conic touching four given straight lines ($l\alpha \pm m\beta \pm n\gamma = 0$) is

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0,$$

with the condition $$\frac{l^2}{L} + \frac{m^2}{M} + \frac{n^2}{N} = 0.$$

The form (4) of the equation of a conic admits of our denoting any point on the conic by a single variable, analogous to the eccentric angle in the case of a conic referred to its axes, which is indeed a particular case of this form.

Thus, the equation of the conic being $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$, any point on it may be represented by the equations

$$\frac{\sqrt{l\alpha}}{\cos\theta} = \frac{\sqrt{m\beta}}{\sin\beta} = \sqrt{(-n)}\gamma\,;$$

which we may call the point θ.

The equation of the tangent at θ is

$$\sqrt{l\alpha}\cos\theta + \sqrt{m\beta}\sin\theta = \sqrt{(-n)}\,\gamma\,;$$

the equation of the chord through θ, ϕ is

$$\sqrt{l\alpha}\cos\frac{\theta+\phi}{2} + \sqrt{m\beta}\sin\frac{\theta+\phi}{2} = \sqrt{(-n)}\gamma\cos\frac{\theta-\phi}{2}\,;$$

and the point of intersection of the tangents at θ, ϕ is

$$\frac{\sqrt{l\alpha}}{\cos\dfrac{\theta+\phi}{2}} = \frac{\sqrt{m\beta}}{\sin\dfrac{\theta+\phi}{2}} = \frac{\sqrt{(-n)}\,\gamma}{\cos\dfrac{\theta-\phi}{2}}.$$

The equations of any two given conics may be assumed in this form, since we have only to take the equations of their four common points to be $l\alpha = \pm m\beta = \pm n\gamma$, or the equations of their four common tangents to be $l\alpha \pm m\beta \pm n\gamma = 0$. The triangle of reference will, however, be imaginary if the conics intersect in two real and two impossible points.

Any point on the conic $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ may in like manner be taken to be

$$\frac{\alpha \cos^2\theta}{l} = \frac{\beta \sin^2\theta}{m} = \frac{\gamma}{-n};$$

and any point on the conic

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$$

to be
$$\frac{l\alpha}{\cos^4\theta} = \frac{m\beta}{\sin^4\theta} = n\gamma;$$

the tangents at these points being respectively

$$\frac{\alpha}{l}\cos^4\theta + \frac{\beta}{m}\sin^4\theta + \frac{\gamma}{n} = 0;$$

$$\frac{l\alpha}{\cos^2\theta} + \frac{m\beta}{\sin^2\theta} - n\gamma = 0;$$

but these equations are not often needed.

The equation of a tangent to $\beta\gamma = l\alpha^2$ in the form

$$\mu^2\frac{\beta}{l} - 2\mu\alpha + \gamma = 0,$$

the point of contact being

$$\alpha = \mu\frac{\beta}{l} = \frac{\gamma}{\mu},$$

is, however, frequently convenient.

742. The sides of the triangle of reference are bisected in points A_1, B_1, C_1; the triangle $A_1B_1C_1$ is treated in the same way, and so on n times; prove that the equation of B_nC_n is

$$\beta + \gamma = \frac{2^{n+1} + (-1)^n}{2^n + (-1)^{n+1}}\alpha.$$

743. The equation of the straight line passing through the centres of the inscribed and circumscribed circles is

$$\frac{\alpha}{\sin A}(\cos B - \cos C) + \frac{\beta}{\sin B}(\cos C - \cos A) + \frac{\gamma}{\sin C}(\cos A - \cos B) = 0.$$

Prove that the point

$$\frac{\alpha}{\sin A\,(m + n\cos A)} = \frac{\beta}{\sin B\,(m + n\cos B)} = \frac{\gamma}{\sin C\,(m + n\cos C)}$$

lies on this straight line.

744. If α, β, γ be perpendiculars from any point on three straight lines which meet in a point and make with each other angles A, B, C; the equation $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$ will represent two straight lines, which will be real, coincident, or imaginary according as $mn\sin^2 A + nl\sin^2 B + lm\sin^2 C$ is negative, zero, or positive.

745. If x, y, z be the perpendiculars from A, B, C on any straight line, and α, β, γ the areal co-ordinates of any point on that line, then $x\alpha + y\beta + z\gamma = 0$; and the perpendicular from any point (α, β, γ) on the straight line is $x\alpha + y\beta + z\gamma$.

746. Within a triangle ABC are taken two points O, O'; AO, BO, CO meet the opposite sides in A', B', C', and the points of intersection of $O'A$, $B'C'$; $O'B$, $C'A'$; $O'C$, $A'B'$ are respectively D, E, F: prove that $A'D$, $B'E$, $C'F$ will meet in a point; and this point remains the same if O, O' be interchanged in the construction.

747. If x, y, z be the perpendiculars from A, B, C on any tangent (1) to the inscribed circle

$$x\sin A + y\sin B + z\sin C = 2R\sin A\sin B\sin C;$$

(2) to the circumscribed circle

$$x\sin 2A + y\sin 2B + z\sin 2C = 4R\sin A\sin B\sin C;$$

(3) to the Nine Points' Circle

$$x\sin A\cos(B - C) + y\sin B\cos(C - A) \\ + z\sin C\cos(A - B) = 2R\sin A\sin B\sin C;$$

and (4) to the circle to which the triangle is self-conjugate

$$x^2 \tan A + y^2 \tan B + z^2 \tan C = 0.$$

748. If A', B', C' be the feet of the perpendiculars let fall from the point $l\alpha = m\beta = n\gamma$ on the sides of the triangle of reference, the straight lines drawn through A, B, C perpendicular to $B'C'$, $C'A'$, $A'B'$ will meet in the point

$$\frac{\alpha}{la^2} = \frac{\beta}{mb^2} = \frac{\gamma}{nc^2}.$$

749. A triangle $A'B'C'$ has its angular points on the sides of the triangle ABC, and AA', BB', CC' meet in a point, any straight line is drawn meeting the sides of the triangle $A'B'C'$ in points which are joined respectively to the corresponding angles of the triangle ABC; prove that the joining lines meet the sides of the triangle ABC in three points lying in one straight line.

750. The two points at which the escribed circles of the triangle of reference subtend equal angles lie on the straight line

$$\alpha (b - c) \cot A + \beta (c - a) \cot B + \gamma (a - b) \cot C = 0.$$

751. Four straight lines form a quadrilateral, and from the middle points of the sides of a triangle formed by three of them perpendiculars are let fall on the line joining the middle points of the diagonals; prove that these perpendiculars are inversely proportional to the perpendiculars from the angular points of the triangle on the fourth straight line.

752. A straight line meets the sides of a triangle in A', B', C', the straight line joining A to the point (BB', CC') meets BC in D, and E, F are similarly determined. If O be any point, the lines joining D, E, F to the respective intersections of OA, OB, OC with $A'B'C'$ will pass through a point O', and OO' will pass through a point whose position is independent of O.

753. The conic $la^2 + m\beta^2 + n\gamma^2 = 0$
will represent a circle if

$$l \tan A = m \tan B = n \tan C.$$

754. The necessary and sufficient conditions that the equation

$$la^2 + m\beta^2 + n\gamma^2 + l'\beta\gamma + m'\gamma a + n'a\beta = 0$$

may represent a circle are

$$\frac{m+n-l'}{a^2} = \frac{n+l-m'}{b^2} = \frac{l+m-n'}{c^2}.$$

755. The lengths of the tangents from A, B, C to a circle are p, q, r; prove that the equation of the circle is

$$a^2\beta\gamma + b^2\gamma a + c^2 a\beta = (a+\beta+\gamma)(p^2 a + q^2\beta + r^2\gamma).$$

756. P, P_1, P_2, P_3 are the points of contact of the Nine Points' Circle with the inscribed and escribed circles; prove that (1) the equations of the tangents at these points are

$$\frac{a}{b-c} + \frac{\beta}{c-a} + \frac{\gamma}{a-b} = 0,$$

and the three equations obtained from this by changing the sign of a, b, or c; (2) PP_1, P_2P_3 meet BC in the same points as the straight lines bisecting the internal and external angles at A; (3) PP_1, P_2P_3 intersect in the point

$$\frac{-a}{b^2-c^2} = \frac{\beta}{c^2-a^2} = \frac{\gamma}{a^2-b^2};$$

(4) the tangents at P, P_1, P_2, P_3 all touch the ellipse which touches the sides of the triangle at their middle points.

757. The straight line $la + m\beta + n\gamma = 0$ meets the sides of the triangle ABC in A', B', C'; prove that the circles on AA', BB', CC' have the common radical axis

$$a \cot A\left(\frac{1}{m} - \frac{1}{n}\right) + \beta \cot B\left(\frac{1}{n} - \frac{1}{l}\right) + \gamma \cot C\left(\frac{1}{l} - \frac{1}{m}\right) = 0:$$

and the circles will touch each other if

$$(mn + nl + lm)^2 (a^4 + \ldots\ldots - 2b^2c^2 - \ldots\ldots)$$
$$= 8lmn\{la^4 + \ldots\ldots - (m+n)\,b^2c^2 - \ldots\ldots\}.$$

758. The equation of the straight line bisecting the diagonals of the quadrilateral whose four sides are

$$l\alpha \pm m\beta \pm n\gamma = 0,$$

is

$$l^2\alpha + m^2\beta + n^2\gamma = 0,$$

and the equation of the radical axis of the three circles on the diagonals is

$$(m^2 - n^2)(b^2\gamma + c^2\beta) + \dots\dots + \dots\dots = 0.$$

759. One of four straight lines passes through the centre of one of the four circles which touch the diagonals of the quadrilateral; prove that the other three straight lines pass each through one of the other three centres. Prove that in this case the circles described on the three diagonals touch each other in a point lying on the circle circumscribing the triangle formed by the diagonals, and that their common tangent is a normal to this circle.

760. The equation of a circle passing through B and C, and whose segment on BC (on the same side as A) contains an angle θ, is

$$\beta\gamma \sin^2 A + \gamma\alpha \sin^2 B + \alpha\beta \sin^2 C = \alpha(\alpha + \beta + \gamma)\frac{\sin B \sin C \sin(\theta - A)}{\sin\theta}.$$

761. The locus of the radical centre of three circular arcs on BC, CA, AB, respectively containing angles $A + \theta$, $B + \theta$, $C + \theta$ for different values of θ, is the straight line

$$\frac{\alpha}{\sin A}\sin(B - C) + \frac{\beta}{\sin B}\sin(C - A) + \frac{\gamma}{\sin C}\sin(A - B) = 0.$$

If $\theta = 90°$, the radical centre is the centre of the circumscribed circle.

762. Prove that, if P be a point such that

$$\tan BPC - \tan A = \tan CPA - \tan B = \tan APB - \tan C,$$

there are two positions of P; and that the equation of the line joining them is

$$\alpha \cot A(\tan B - \tan C) + \beta \cot B(\tan C - \tan A) + \gamma \cot C(\tan A - \tan B) = 0.$$

763. A straight line drawn through the centre of the inscribed circle meets the sides of the triangle ABC in A', B', C', these points are joined to the centres of the corresponding escribed circles; prove that the joining lines meet two and two on the sides of the triangle; and, if A'', B'', C'' be their points of intersection, the circles on $A'A''$, $B'B''$, $C'C''$ as diameters will touch each other in one point lying on the circumscribed circle; and their common tangent will be normal to the circumscribed circle.

764. A circle meets the sides of a triangle ABC in P, P'; Q, Q'; R, R' respectively, and AP, BQ, CR meet in the point $l\alpha = m\beta = n\gamma$; prove that AP', BQ', CR' meet in the point $l'\alpha = m'\beta = n'\gamma$; where

$$\frac{(m+n)(m'+n')}{a^2} = \frac{(n+l)(n'+l')}{b^2} = \frac{(l+m)(l'+m')}{c^2}.$$

765. The equation of the circle which passes through the centres of the escribed circles of the triangle of reference is

$$bc\alpha^2 + ca\beta^2 + ab\gamma^2 + (a+b+c)(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0.$$

766. The lines joining the feet of the perpendiculars meet the corresponding sides of the triangle ABC in A', B', C'; prove that the circles on AA', BB', CC' as diameters will touch each other if

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 7.$$

767. If the side BC subtend angles θ, θ' at the foci of an inscribed conic,

$$\sin(\theta - A)\sin(\theta' - A) = \sin\theta \sin\theta'.$$

768. If P be any point on the minimum ellipse circumscribing a given triangle ABC,

$$\frac{AP}{\sin BPC} + \frac{BP}{\sin CPA} + \frac{CP}{\sin APB} = 0,$$

$$\cot BPC + \cot CPA + \cot APB = \cot A + \cot B + \cot C;$$

the angles BPC, CPA, APB being so taken that their sum is 360°.

769. If a conic $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ be such that the normals to it at the angular points of the triangle of reference meet in a point,

$$\frac{l}{\sin A}(m^2 - n^2) + \frac{m}{\sin B}(n^2 - l^2) + \frac{n}{\sin C}(l^2 - m^2) = 0\,;$$

the point must lie on the curve

$$\alpha(\beta^2 - \gamma^2)(\cos A - \cos B \cos C) + \ldots + \ldots = 0\,;$$

and the centre must lie on the curve

$$\frac{\alpha}{\sin A}(\beta^2 - \gamma^2) + \ldots + \ldots = 0\,;$$

trilinear co-ordinates being used.

770. The two conics circumscribing the triangle of reference, passing through the point (α, β, γ), and touching the line

$$x\alpha + y\beta + z\gamma = 0$$

will be real if

$$\frac{xyz\,(x\alpha + y\beta + z\gamma)}{\alpha\beta\gamma} < 0.$$

Interpret this result geometrically.

771. A conic touches the sides of a triangle ABC in the points D, E, F, and AD meets the conic again in d; prove that the equation of the tangent at d is $2(m\beta + n\gamma) = l\alpha$; AD, BE, CF meeting in the point $l\alpha = m\beta = n\gamma$.

772. Find the two points in which the straight line $\beta = k\gamma$ meets the conic

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0\,;$$

and from the condition that one of these points may be at infinity, determine the directions of the asymptotes. Prove that the conic will be a rectangular hyperbola if

$$a^2l^2 + b^2m^2 + c^2n^2 + 2bcmn\cos A + 2canl\cos B + 2ablm\cos C = 0.$$

773. If $l\alpha + m\beta + n\gamma = 0$ be the equation of the axis of a parabola which touches the sides of the triangle of reference,

$$\frac{la^2}{m-n} + \frac{mb^2}{n-l} + \frac{nc^2}{l-m} = 0.$$

774. A conic touches the sides of the triangle ABC, any point is taken on the straight line which passes through the intersections of the chords of contact with the corresponding sides, and the lines joining this point to the angular points meet the opposite sides in A', B', C'; prove that corresponding sides of the triangles ABC, $A'B'C'$ intersect two and two on one straight line which touches the conic.

775. If a, b be the axes of a conic inscribed in a triangle, and α, β, γ be the *trilinear* co-ordinates of either focus:

$$\frac{b^2}{a^2} = \frac{4\alpha\beta\gamma\,(\alpha \sin A + \ldots + \ldots)\,(\beta\gamma \sin A + \ldots + \ldots)}{(\beta^2 + \gamma^2 + 2\beta\gamma \cos A)\,(\ldots\ldots)\,(\ldots\ldots)}.$$

776. Two parabolas are such that triangles can be inscribed in one whose sides touch the other; prove that

$$\left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{4}} + \left(\frac{\beta}{\beta'}\right)^{\frac{1}{4}} + \left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{4}} = 0,$$

if (α, β, γ), $(\alpha', \beta', \gamma')$ be the foci; the triangle formed by the common tangents being the triangle of reference.

777. A conic touches the sides of the triangle ABC in the points A', B', C'; AA', BB', CC' meet in O; through O is drawn any straight line meeting the sides of the triangle in a, b, c, and from a, b, c are drawn other tangents to the conic; prove that these tangents will intersect two and two in points lying on a fixed conic circumscribing the triangle.

778. The equation of an asymptote of the conic $\beta\gamma = k\alpha^2$, is

$$\mu^2\gamma + k\beta - 2k\mu\alpha = 0,$$

μ being given by the equation $\mu^2 + \mu + k = 0$; also the envelope of the asymptotes for different values of k is the parabola

$$4\alpha\,(\alpha + \beta + \gamma) + (\beta - \gamma)^2 = 0.$$

779. The radius of curvature at the point B of the conic $\beta\gamma = k\alpha^2$ is $\dfrac{2R \sin^2 C}{k \sin A \sin B}$.

780. OA, OB touch a conic at A, B, and the tangent at P meets OA, OB in B', A' respectively; find the locus of the intersection of AA', BB'; and if AP, BP meet OB, OA in A'', B'' respectively, find the envelope of $A''B''$.

781. OA, OB touch a conic at A, B, and C, D are two other fixed points on the conic; a tangent to the conic meets OA, OB in C', D'; prove that the locus of the intersection of CC', DD' is a conic passing through C, D and the intersections of OC, BD; and of OD, AC.

782. The locus of the foci of the conic $\beta\gamma = k\alpha^2$ for different values of k is

$$\frac{\alpha}{\sin A}\left(\frac{\beta^2}{\sin^2 B} - \frac{\gamma^2}{\sin^2 C}\right) + \frac{2\beta\gamma}{\sin B \sin C}(\beta \cot B - \gamma \cot C) = 0.$$

783. Prove that the equation $4\beta\gamma = \alpha^2$ represents a parabola; and that the tangential equation of the same parabola is $yz = x^2$.

784. CA, CB are tangents to a conic, P any point on the conic, and AP, BP meet CB, CA in A', B'; prove that the triangle $A'B'P$ is self-conjugate to another conic touching the former at A, B.

785. The sides of a triangle ABC touch a conic, O, O_1, O_2, O_3 are the centres of the circles which touch the sides; a conic is described through B, C, O, O_1 and one focus, and another through B, C, O_2, O_3 and the same focus; prove that the fourth point of intersection of these conics will be the second focus.

786. A conic passes through four given points; prove that the locus of the points of contact of tangents drawn to it from a given point is in general a curve of the third degree, which reduces to a conic if the point be in the same straight line with two of the former; and in that case the locus passes through the other two given points.

787. Given a point O on a conic, and a triangle ABC self-conjugate to the conic, AO, BO, CO meet the opposite sides in three points, and the lines joining these two and two meet the corresponding sides in A', B', C'; prove that the intersections of BB', CC'; CC', AA'; and AA', BB' also lie on the conic.

788. Any tangent to a conic meets the sides of a self-conjugate triangle in D, E, F; the line joining A to the intersection of BE, CF meets BC in A'; B', C' are similarly determined; prove that $B'C'$, $C'A'$, $A'B'$ are also tangents to the conic.

789. If $l\alpha + m\beta + n\gamma = 0$ be the equation of an asymptote of a rectangular hyperbola self-conjugate to the triangle of reference,

$$\frac{la^2}{m-n} + \frac{mb^2}{n-l} + \frac{nc^2}{l-m} = 0.$$

790. A parabola is described touching the sides of a triangle ABC, S is the focus, and the axis meets the circle circumscribing the triangle again in O; prove that if with centre O a rectangular hyperbola be described, to which the triangle is self-conjugate, one of its asymptotes will coincide with OS.

791. One directrix of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

passes through A; prove that

$$\frac{1}{l} = \frac{\cot^2 B}{m} + \frac{\cot^2 C}{n};$$

and that the second focus lies on the line joining the feet of the perpendiculars from B, C on the opposite sides.

792. The equations determining the foci of the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0$$

are
$$\frac{1}{\sin^2 A}\left\{\frac{(\beta+\gamma)^2}{l} + \frac{\alpha^2}{m} + \frac{\alpha^2}{n}\right\} = \frac{1}{\sin^2 B}\left\{\frac{(\gamma+\alpha)^2}{m} + \frac{\beta^2}{n} + \frac{\beta^2}{l}\right\}$$
$$= \frac{1}{\sin^2 C}\left\{\frac{(\alpha+\beta)^2}{n} + \frac{\gamma^2}{l} + \frac{\gamma^2}{m}\right\}.$$

793. If a triangle be self-conjugate to a parabola, the lines joining the middle points of the sides are tangents to the parabola. Hence prove that the focus lies on the Nine Points' Circle, and that the directrix passes through the centre of the circumscribed circle.

794. A conic is drawn touching the four straight lines

$$l\alpha \pm m\beta \pm n\gamma = 0\,;$$

prove that its equation is

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0,$$

L, M, N satisfying the equation

$$l^2MN + m^2NL + n^2LM = 0.$$

Investigate the species of this conic with reference to the position of its centre on the straight line which is its locus.

795. Any conic through the four points

$$l\alpha = \pm m\beta = \pm n\gamma$$

will divide harmonically the straight line joining the points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, if

$$l^2\alpha_1\alpha_2 = m^2\beta_1\beta_2 = n^2\gamma_1\gamma_2.$$

796. If a triangle be self-conjugate to a rectangular hyperbola, and any conic be inscribed in the triangle, its foci will be conjugate points with respect to the hyperbola.

797. A given triangle is self-conjugate to a conic, and the centre of the conic lies on a given straight line parallel to one of the sides of the triangle; prove that the asymptotes will envelope a fixed conic which touches the other two sides of the triangle.

798. The locus of the foci of all conics touching the four straight lines $l\alpha \pm m\beta \pm n\gamma = 0$ is

$$(\alpha + \beta + \gamma)\,(l^2\alpha^2 \cot A + m^2\beta^2 \cot B + n^2\gamma^2 \cot C) \sin A \sin B \sin C$$
$$= (l^2\alpha + m^2\beta + n^2\gamma)\,(\beta\gamma \sin^2 A + \gamma\alpha \sin^2 B + \alpha\beta \sin^2 C).$$

799. The straight lines

$$x\alpha + y\beta + z\gamma = 0, \quad x'\alpha + y'\beta + z'\gamma = 0,$$

will be conjugate with respect to all parabolas inscribed in the triangle of reference, if

$$yz' + y'z = zx' + z'x = xy' + x'y:$$

and with respect to all conics touching the four straight lines

$$l\alpha \pm m\beta \pm n\gamma = 0,$$

if
$$\frac{xx'}{l^2} = \frac{yy'}{m^2} = \frac{zz'}{n^2}.$$

800. A rectangular hyperbola is inscribed in the triangle ABC; prove that the locus of the pole of the straight line which bisects the sides AB, AC is the circle

$$\alpha^2(a^2 + b^2 + c^2) + (\beta^2 + 2\alpha\beta)(a^2 + b^2 - c^2) + (\gamma^2 + 2\gamma\alpha)(a^2 - b^2 + c^2) = 0.$$

801. Four conics are described with respect to any one of which three of the four straight lines $l\alpha \pm m\beta \pm n\gamma = 0$ form a self-conjugate triangle, and the fourth is the polar of a fixed point $(\alpha', \beta', \gamma')$; prove that the four will have two common tangents meeting in $(\alpha', \beta', \gamma')$, their equation being

$$\frac{l^2\alpha'}{\beta\gamma' - \beta'\gamma} + \frac{m^2\beta'}{\gamma\alpha' - \gamma'\alpha} + \frac{n^2\gamma'}{\alpha\beta' - \alpha'\beta} = 0.$$

802. A triangle circumscribes the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

and two of its angular points lie on the conic

$$l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0;$$

prove that the locus of the third angular point is the conic

$$\frac{l\alpha^2}{\left(\frac{m'}{m} + \frac{n'}{n} - \frac{l'}{l}\right)^2} + \ldots + \ldots = 0;$$

and that this will coincide with the second conic if

$$\left(\frac{l'}{l}\right)^{\frac{1}{3}} + \left(\frac{m'}{m}\right)^{\frac{1}{3}} + \left(\frac{n'}{n}\right)^{\frac{1}{3}} = 0.$$

Also prove that the three conics have four common tangents.

803. The angular points of a triangle lie on the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

and two of its sides touch the conic

$$l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0;$$

prove that the envelope of the third side is the conic

$$l\alpha^2\left(\frac{m}{m'} + \frac{n}{n'} - \frac{l}{l'}\right)^2 + \ldots + \ldots = 0;$$

and that this will coincide with the second conic if

$$\left(\frac{l}{l'}\right)^{\frac{1}{3}} + \left(\frac{m}{m'}\right)^{\frac{1}{3}} + \left(\frac{n}{n'}\right)^{\frac{1}{3}} = 0.$$

Also prove that the three conics have four common points.

804. A triangle is self-conjugate to the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

and two of its angular points lie on the conic

$$l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0;$$

prove that the locus of the third angular point is the conic

$$l\left(\frac{m'}{m} + \frac{n'}{n}\right)\alpha^2 + \ldots + \ldots = 0,$$

and that this will coincide with the second if

$$\frac{l'}{l} + \frac{m'}{m} + \frac{n'}{n} = 0.$$

Prove also that the three conics have four common points, and that the envelope of the line joining the two angular points is the conic

$$\frac{l'\alpha^2}{\frac{m}{m'} + \frac{n}{n'}} + \ldots + \ldots = 0.$$

805. A triangle is self-conjugate to the conic

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

and two of its sides touch the conic

$$l'\alpha^2 + m'\beta^2 + n'\gamma^2 = 0\,;$$

prove that the envelope of the third side is the conic

$$\frac{l\alpha^2}{\frac{m}{m'} + \frac{n}{n'}} + \ldots + \ldots = 0\,;$$

and that this will coincide with the second if

$$\frac{l}{l'} + \frac{m}{m'} + \frac{n}{n'} = 0.$$

Prove also that the three conics have four common tangents, and that the locus of the point of intersection of the two sides is the conic

$$l'\left(\frac{m'}{m} + \frac{n'}{n}\right)\alpha^2 + \ldots + \ldots = 0.$$

IX. *Anharmonic Properties.*

The anharmonic ratio of four points A, B, C, D in one straight line, denoted by $\{ABCD\}$, means the ratio $\frac{AB}{BD} : \frac{AC}{CD}$, or $\frac{AB \,.\, CD}{AC \,.\, BD}$; the order of the letters marking the direction of measurement of any segment, and segments measured in opposite directions being affected with opposite signs. So, if A, B, C, D be any four points in a plane, and P any other point in the same plane, $P\{ABCD\}$ denotes $\frac{\sin APB \,.\, \sin CPD}{\sin APC \,.\, \sin BPD}$, the same rules being observed as to direction of measurement and sign for the angles in this expression as for the segments in the former.

Either of these ratios is said to be harmonic when its value is -1; in which case AD is the harmonic mean between AB and AC, and DA the harmonic mean between DB and DC.

The anharmonic ratio of four points, or four straight lines, can never be equal to 1; as that leads immediately to the condition $AD \cdot BC = 0$, or $\sin APD \cdot \sin BPC = 0$; making two of the points, or two of the lines, coincident.

If A, B, C, D be four fixed points on a conic, and P any other point on the same conic, $P\{ABCD\}$ is constant for all positions of P, and is harmonic when BC, AD are conjugates to the conic. Also, if the tangents at A, B, C, D meet the tangent at P in a, b, c, d, the range $\{abcd\}$ is constant and equal to the former pencil.

The range formed by four points in a straight line is equal to the pencil formed by their polars with respect to any conic.

If the equations of four straight lines can be put in the form $u = \mu_1 v$, $u = \mu_2 v$, $u = \mu_3 v$, $u = \mu_4 v$, the anharmonic ratio of the pencil formed by them, or of the range in which any straight line meets them, is

$$\frac{(\mu_1 - \mu_2)(\mu_3 - \mu_4)}{(\mu_1 - \mu_3)(\mu_2 - \mu_4)}.$$

806. Two fixed straight lines meet in A; B, C, D are three fixed points on another straight line through A; any straight line through D meets the two former in B', C'; BB', CC' meet in P, BC', CB' in Q: prove that the loci of P, Q are straight lines through A which form with the two former a pencil equal to $\{ABCD\}$.

807. ABC is a triangle, two fixed straight lines intersect in a point O on BC, any point is taken on AO, and the straight lines joining it to B, C meet the two fixed lines in B_1, B_2, C_1, C_2, respectively; prove that B_1C_2 and B_2C_1 pass each through a fixed point on BC.

808. Chords are drawn through a fixed point on a conic, making equal angles with a given direction; prove that the straight line joining their extremities passes through a fixed point.

809. Through a given point are drawn chords PP', QQ' to a given conic, so as to touch any the same confocal conic; prove that the points of intersection of PQ, $P'Q'$, and of PQ', $P'Q$ are fixed.

810. If A, B be two fixed points, P, Q any two points on the same straight line, such that the range $\{APQB\}$ is harmonic, and a circle be described on PQ as diameter; all such circles will have a common radical axis, and will cut orthogonally any circle passing through A, B.

811. If two triangles be formed, each by two tangents to a conic and the chord of contact, the six angular points lie on a conic.

812. Four chords of a conic are drawn through a point, and two other conics are drawn through the point, and each through four extremities of the chords, respectively opposite; prove that these conics will have contact at the point.

813. Through a given point O is drawn any straight line meeting a given conic in Q, Q', and P is taken on this line, so that the range $\{OQQ'P\}$ is constant; prove that the locus of P is an arc of a conic having double contact with the former.

814. Given two points A, B of a conic, the envelope of a chord PQ, such that the pencil $\{APQB\}$ on the conic is equal to a given quantity, is a conic touching the former conic at A, B.

815. Through a fixed point is drawn any straight line meeting two fixed straight lines in Q, R respectively; E, F are two other fixed points, QE, RF meet in P; prove that the locus of P is a conic passing through E, F, and the point of intersection of the two fixed straight lines.

816. If A, B, C be three points on a conic, and P, P' any two other points, such that the pencils $\{PABC\}$, $\{P'CBA\}$ at any point on the conic are equal; PP', CA, and the tangent at B, will meet in a point.

817. If A, B, C, A', B', C' be six fixed points on a conic, such that

$$\{B'ABC\}=\{BA'B'C'\},$$

and P, P' be any two points of the conic, such that

$$\{PABC\}=\{P'A'B'C'\},$$

PP' will pass through a fixed point on BB'.

818. A given straight line meets any conic which passes through four given points in two points; prove that these points are conjugate with respect to the conic, which is the locus of the pole of the given straight line with respect to the series of conics through the four points.

819. The anharmonic ratio of the pencil subtended by the four points α_1, α_2, α_3, α_4 on an ellipse at any point on the ellipse is

$$\frac{\sin\dfrac{\alpha_1-\alpha_2}{2}\sin\dfrac{\alpha_3-\alpha_4}{2}}{\sin\dfrac{\alpha_1-\alpha_3}{2}\sin\dfrac{\alpha_2-\alpha_4}{2}}.$$

820. If tangents be drawn to a conic at A, B, C, D, and X_1, X_2, X_3 be the middle points of the diagonals joining the points of intersection of the tangents at (1) A, B; C, D; (2) A, C; B, D; (3) A, D; B, C; and O be the centre; the range $\{OX_1X_2X_3\}$ is equal to the pencil $\{ABCD\}$ at any point of the conic.

821. A conic is drawn through four given points A, B, C, D; AB, CD meet in Y_1, AC, BD in Y_2, AD, BC in Y_3, and O is the centre of the conic; prove that the pencil $\{ABCD\}$ on the conic is equal to the pencil $\{OY_1Y_2Y_3\}$ on the conic which is the locus of O.

822. The anharmonic ratio of the points of intersection of the conics

$$l\alpha^2+m\beta^2+n\gamma^2=0,\ \ l'\alpha^2+m'\beta^2+n'\gamma^2=0,$$

with respect to the former is

$$\frac{\frac{l'}{l}-\frac{m'}{m}}{\frac{l'}{l}-\frac{n'}{n}}, \quad \frac{\frac{m'}{m}-\frac{n'}{n}}{\frac{m'}{m}-\frac{l'}{l}}, \quad \text{or} \quad \frac{\frac{n'}{n}-\frac{l'}{l}}{\frac{n'}{n}-\frac{m'}{m}},$$

or the reciprocal of one of these, according to the order in which the points are taken : and these are also the anharmonic ratios of the range formed by the four common tangents on any tangent to the latter.

X. *Reciprocal Polars, Projections.*

If there be a system of points, and straight lines, lying in the same plane, and we take the polars of the points and the poles of the straight lines with respect to any conic in that plane, we obtain a system reciprocal to the former; so that to a series of points lying on any curve in the first system correspond a series of straight lines touching a certain other curve in the second system, and *vice versâ:* and, in particular, to any number of points lying on a straight line or a conic, correspond a number of straight lines passing through a point or touching a conic. Thus, from any general theorem of position may be deduced a reciprocal theorem. It is in nearly all cases advisable to take a circle for the *auxiliary conic*, with respect to which the system is reciprocated; the point (p) corresponding to any straight line being then found by drawing from O, the centre of the circle, OP perpendicular to the straight line, and taking on it a point p, such that $OP \,.\, Op = k^2$, k being the radius of the circle : and similarly the straight line through p at right angles to OP is the straight line corresponding to the point P.

To draw the figure reciprocal to a triangle ABC, with respect to a circle whose centre is O, or, more shortly, *with respect to the point O*, draw Oa perpendicular to BC, and in it take any point a; through a, O draw straight lines perpendicular to OC, CA, meeting in b; and through b, O draw straight lines perpendicular to OA, AB, meeting in c; then the points a, b, c will be the poles of the sides of the triangle, and the straight lines bc, ca, ab the polars of the points A, B, C, with respect to some circle with centre O. Now, suppose we want to find the point corresponding to the perpendicular from A, it must lie on bc, and on the line through O at right angles to Oa, since Oa is parallel to the straight line whose reciprocal is required; it is then determined. Hence, to the theorem that the three perpendiculars of a triangle meet in a point, corresponds the following: if through any point O in the plane of a triangle abc be drawn straight lines at right angles to Oa, Ob, Oc, to meet the respectively opposite sides, the three points so determined will lie in one straight line.

So, from the theorem, that the bisectors of the angles meet in a point, we get the following: the straight lines drawn through O bisecting the external angles (or one external and two internal angles) between Ob, Oc; Oc, Oa; Oa, Ob, respectively, will meet the opposite sides in three points lying in a straight line.

If a circle, with centre A and radius R, be reciprocated with respect to O, the corresponding curve is a conic whose focus is O, major axis along OA, eccentricity $\frac{OA}{R}$, and latus rectum $\frac{2k^2}{R}$ or $\frac{2}{R}$ if we take the radius of the auxiliary circle unity. The centre A is reciprocated into the directrix. Most of the focal properties of conics may thus be deduced from well known properties of the circle. For instance, if O be a point on the circle, and OP, OQ two chords at right angles, PQ passes through the centre. Reciprocating with respect to O, to the circle corresponds a parabola, and to the points P, Q two tangents to the parabola at right angles to each other, which therefore intersect on the directrix.

Again, to find the condition that two conics which have a common focus should be such that triangles can be inscribed in one whose sides shall touch the other.

Take two circles which have this property, and let R, r be their radii, δ the distance between their centres; then

$$\delta^2 = R^2 \pm 2Rr.$$

Reciprocate the system with respect to a point at distances x, y from the centres, and let a be the angle between these distances. Then a will be the angle between the major axes of the conics, and if $2c_1$, $2c_2$ be the latera recta, e_1, e_2 the eccentricities,

$$c_1 = \frac{1}{r}, \quad c_2 = \frac{1}{R}, \quad e_1 = \frac{y}{r}, \quad e_2 = \frac{x}{R}, \quad \delta^2 = x^2 + y^2 - 2xy \cos a,$$

whence $$R^2 \pm 2Rr = x^2 + y^2 - 2xy \cos a;$$

$$\therefore \frac{1}{c_2^2} \pm \frac{2}{c_1 c_2} = \frac{e_2^2}{c_2^2} + \frac{e_1^2}{c_1^2} - \frac{2e_1 e_2}{c_1 c_2} \cos a,$$

or $$c_1^2 \pm 2c_1 c_2 = e_2^2 c_1^2 + e_1^2 c_2^2 - 2e_1 e_2 c_1 c_2 \cos a;$$

the relation required.

If a system of confocal conics be reciprocated with respect to one of the foci, the reciprocal system will consist of circles having a common radical axis, the radical axis being the reciprocal of the second focus, and the former focus being one of the limiting points of the system.

The reciprocal of a conic with respect to any point in its plane is another conic which is an ellipse, parabola, or hyperbola, according as the point lies within, upon, or without the conic. To the points of contact of the tangents from the point correspond the asymptotes, and to the polar of the point the centre of the reciprocal conic. So also to the asymptotes and centre of the original conic correspond the points of contact and polar with respect to the reciprocal conic.

As an example, we may investigate the elementary property that the tangent at any point of a conic makes equal angles with the focal distances. The reciprocal theorem is, that if we take any point O in the plane of a conic, there exist two fixed straight lines (the reciprocals of the foci), such that if a tangent to the conic at P meet them in Q, Q', OP makes equal angles with OQ and OQ'. If, however, the point O lie on the curve, the original curve was a parabola; and one of the straight lines being the reciprocal of the point at infinity on the parabola will be the tangent at O.

Since the anharmonic ratio of the pencil formed by any four straight lines is equal to that of the range formed by their poles with respect to any conic, it follows that in any reciprocation whatever, a pencil or range is replaced by a range or pencil having the same anharmonic ratio.

The Method of Projections enables us to make the proof of any general theorem of position depend upon that of a more simple particular case of the theorem. Given any figure in a plane, we have five constants disposable to enable us to simplify the projected figure, three depending on the position of the vertex and two on the direction of the Plane of Projection. It is clear that relations of tangency, of pole and polar, and anharmonic ratio, are the same in the original and projected figures.

As an example we will take the following: "To prove that if two triangles be self-conjugate to the same conic, their angular points lie on a conic."

Let the two triangles be ABC, DEF; project the conic into a circle with its centre at D, then E, F will be at infinity, and DE, DF will be at right angles to one another. Draw a conic through $ABCDE$, then since ABC is self-conjugate to a circle whose centre is D, D is the centre of perpendiculars of the triangle ABC, the conic is therefore a rectangular hyperbola, and E being one of its points at infinity, F will be the other. The theorem is therefore true.

Again, retaining the centre at D, take any other conic instead of the circle; DE, DF will still be conjugate diameters, and therefore if any conic pass through A, B, C, D, its asymptotes will be parallel to a pair of conjugate diameters of the conic whose centre is D, and to which ABC is self-conjugate. The same must therefore be the case with respect to the four conics, each having its centre at one of the four points, and the other three self-conjugate. These conics are therefore similar and similarly situated. Moreover, if we draw the two parabolas which can pass through the four points, their axes must be parallel respectively to coincident conjugate diameters of any one of the four conics; i.e. to the asymptotes. But the axes of these parabolas must be parallel to the asymptotes of the conic which is the locus of centres of all conics through the four points; since when the centre is at infinity the conic becomes a parabola. Hence, finally, if we have four points in a plane, the four conics, each of which has its centre at one of the four points, and the other three self-conjugate, and the conic which is the locus of centres of conics through the four points, are all similar and similarly situated.

Let A, B be any two fixed points on a circle, ∞, ∞' the two impossible circular points at infinity, P any other point on the circle; then $P\{A\ \infty\ \infty' B\}$ is constant. Hence PA, PB and the circular points divide the line at infinity in a constant anharmonic ratio. Hence, two straight lines including a constant angle, may be projected into two straight lines, which divide the straight line joining two given points (the projections of the circular points) in a constant anharmonic ratio. In particular, if APB be a right angle, AB passes through the centre (the pole of $\infty\ \infty'$), and the ratio becomes harmonic.

Thus, projecting the property of the director circle of a conic, we obtain the following theorem: "The locus of the intersection of tangents to a conic which divide harmonically the straight line joining two given points is a conic passing through the given points; and the straight line joining the two points has the same pole for both conics."

If tangents be drawn to any conic through the circular points, their four points of intersection are the real and impossible foci of the conic. If the conic be a parabola, the line joining the circular points is a tangent, and one of the real foci is at infinity, while the two impossible foci are the circular points. Many properties of the foci, especially of the parabola, may thus be generalized by projections. Thus, remembering that the directrix is the polar of the focus, we see that if a conic be inscribed in a triangle ABC, and two tangents be drawn dividing BC harmonically, their point of intersection lies on the polar of A. So also, since the locus of the intersection of tangents to a parabola, including a constant angle, is a conic having the same focus and directrix, it follows that, if a conic be inscribed in a triangle ABC, and two tangents be drawn dividing BC in a constant range, the locus of their point of intersection is a conic having double contact with the former at the points where AB, AC touch it.

The circular points at infinity have singular properties in relation to many other curves. All epicycloids and hypocycloids pass through them, the cardioid has cusps at them, and may be projected into a three-cusped hypocycloid.

823. If two conics have a common focus S, and two common tangents PQ, $P'Q'$, the angles PSQ, $P'SQ'$ will be equal or supplementary.

824. If two conics have a common focus and equal minor axes, their common tangents will be parallel.

825. If two conics have a common focus and equal latera recta, one of their common chords will pass through the common focus.

826. If S be the focus of a conic and A any point, the straight line drawn through S at right angles to SA will meet the polar of A on the directrix.

827. If O be a fixed point on a conic, A, B, C any three other points on the conic, and the straight lines through O at right angles to OA, OB, OC meet BC, CA, AB respectively in A', B', C', the straight line $A'B'C'$ will meet the normal at O in a fixed point.

828. Given a conic and a point O; prove that there are two straight lines, such that the distance between any two points on one of them, conjugate to the conic, subtends a right angle at O.

829. O being a fixed point on a conic, OP, OQ any two chords, OR the chord normal at O; prove that there exists a straight line passing through the pole of OR, such that the tangents at P, Q intercept on it a length which subtends at O an angle twice POQ.

830. On any straight line can be found two points, conjugate to a given conic, such that the distance between them subtends a right angle at a given point.

831. ABC is a triangle, O any point, and straight lines through O at right angles to OA, OB, OC meet the respectively opposite sides in A', B', C'; prove that any conic which touches the sides of the triangle and the straight line $A'B'C'$ will subtend a right angle at O.

832. An ellipse is described about an acute-angled triangle ABC, and one focus is the centre of perpendiculars of the triangle; prove that its latus rectum is

$$2R\frac{\cos A\cos B\cos C}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

833. A parabola and hyperbola have a common focus and axis, and the parabola touches the directrix of the hyperbola; prove that any straight line through the focus is harmonically divided by any tangent to the parabola and the two parallel tangents to the hyperbola.

834. A series of conics are described having equal latera recta, the focus of a given parabola their common focus, and tangents to the parabola their directrices; prove that the common tangents of any two intersect on the directrix of the parabola.

835. With a given point as focus four conics can be drawn circumscribing a given triangle, and the latus rectum of one of them will be equal to the sum of those of the other three. Also, if any conic A be drawn touching the directrices of the four

conics, the polar of the given point with respect to it will be a tangent to the conic which has the given point for focus and touches the sides of the triangle, and the conic A will subtend a right angle at the given point.

836. Prove that with the centre of the circumscribed circle as focus, three hyperbolas can be described, passing through the angular points of the triangle ABC; that their eccentricities are $\operatorname{cosec} B \operatorname{cosec} C$, &c.; their directrices the lines joining the middle points of the sides; and that the fourth point of intersection of any two lies on the straight line joining one of the angles to the middle point of the opposite side.

837. From a point P on the circle circumscribing a triangle ABC, are drawn PA', PB', PC' at right angles to PA, PB, PC to meet the corresponding sides; prove that the straight line $A'B'C'$ passes through the centre of the circle.

838. A triangle is inscribed in an ellipse so that the centre of the inscribed circle coincides with one of the foci; prove that the radius of the inscribed circle is $\dfrac{c}{1+\sqrt{(1+e^2)}}$; $2c$ being the latus rectum and e the eccentricity.

839. A triangle is self-conjugate to a hyperbola, and one focus is equidistant from the sides of the triangle; prove that each distance is $\dfrac{c}{\sqrt{(e^2-2)}}$, $2c$ being the latus rectum and e the eccentricity.

840. Two conics have a common focus, and are such that triangles can be inscribed in one which are self-conjugate to the other; prove that

$$2c_1^2 + c_2^2 = e_1^2 c_2^2 + e_2^2 c_1^2 - 2e_1 e_2 c_1 c_2 \cos \alpha;$$

c_1, c_2 being their latera recta; e_1, e_2 their eccentricities; and α the angle between their axes. Prove also that in this case triangles can be circumscribed to the second, which are self-conjugate to the first.

841. Three tangents to a hyperbola are so drawn that the centre of perpendiculars of the triangle formed by them is at one of the foci; prove that the radius of the circle to which the triangle is self-conjugate is constant.

842. If four points lie on a circle, four parabolas can be described having a common focus, and each touching the side of a triangle formed by joining three of the points.

843. BB' is the minor axis of an ellipse, and B is the centre of curvature at B'; on the circle of curvature at B' is taken any point P, and tangents drawn from P to the ellipse meet the tangent at B in Q, Q'; prove that a conic drawn to touch QB', $Q'B'$, with its focus at B and directrix passing through B', will touch the circle at P.

844. If three tangents to a parabola form a triangle ABC, and perpendiculars p_1, p_2, p_3 be let fall on them from the focus S; then will

$$p_2p_3 \sin BSC + p_3p_1 \sin CSA + p_1p_2 \sin ASB = 0,$$

the angles at S being measured in the same direction.

845. A triangle ABC circumscribes a parabola, and x, y, z are the perpendiculars from the focus on the sides; prove that

$$\frac{\sin 2A}{x^2} + \frac{\sin 2B}{y^2} + \frac{\sin 2C}{z^2} = \frac{8 \sin A \sin B \sin C}{l^2},$$

l being the semi latus rectum.

846. A point S is taken within a triangle, such that the sides subtend at it equal angles, and four conics are described with S as focus passing through A, B, C; prove that one of these conics will touch the other three, and that the tangent to this conic at A will meet BC in a point A', such that ASA' is a right angle.

847. With the centre of perpendiculars of a triangle as focus are described two conics, one of which touches the sides of the triangle, and the other passes through the feet of the perpendiculars; prove that these conics will touch each other, and that

the point of contact will lie on the conic touching the sides of the triangle at the feet of the perpendiculars.

848. A conic is inscribed in a triangle, and one directrix passes through the centre of perpendiculars; prove that the corresponding focus lies on the circle to which the triangle is self-conjugate.

849. With the centre of the circumscribing circle of a triangle as focus are described two ellipses, one touching the sides of the triangle, and the other passing through the middle points; prove that these will touch each other.

850. Five points are taken, no three of which lie in one straight line, and with one of the points as focus are described four conics, each of which touches the sides of a triangle formed by joining three of the other points; prove that these conics will have a common tangent.

851. Through a fixed point O are drawn any two straight lines meeting a given conic in P, P'; Q, Q'; and a given straight line in R, R'; and RR' subtends a right angle at another fixed point. Prove that PQ, PQ', $P'Q$, $P'Q'$ all touch a certain fixed conic.

852. Given a conic and a point in its plane O; prove that there exist two points L, such that if any straight line through L meet the polar of L in P, and P' be the pole of this straight line, PP' subtends a right angle at O.

853. The envelope of a straight line which is divided harmonically by two given straight lines and a given conic, is a conic touching the two fixed straight lines at points on the polar of their point of intersection; unless the given lines are conjugate with respect to the conic, when only one such straight line can be drawn.

854. Two equal circles A, B, touch at S, a tangent to B meets A in P, Q, and O is its pole with respect to A; prove that the directrices of two of the conics described with focus S to circumscribe the triangle OPQ will touch the circle A.

855. A conic touches the sides of a triangle ABC in D, E, F, and AD, BE, CF meet in S; three conics are described with S as focus, osculating the former at D, E, F; prove that these three and the former will have one common tangent, which also touches the conic having S as focus and touching the sides of the triangle ABC.

856. Given four straight lines, prove that two conics can be constructed so that an assigned straight line of the four is its directrix and the other three form a self-conjugate triangle: and that, whichever straight line be taken for directrix, the corresponding focus will be one of two fixed points.

857. Through a fixed point O are drawn two chords PP', QQ' to a given conic, such that the two lines bisecting the angles at O are also fixed; prove that the straight lines PQ, $P'Q$, PQ', $P'Q'$ all touch a fixed conic; except when the two fixed straight lines are conjugates to the given conic.

858. OP, OQ; $O'P'$, $O'Q'$ are tangents to a conic, and the conic which touches the sides of the triangles OPQ, $O'P'Q'$ is drawn: prove that any tangent to the latter conic will be divided harmonically by the former conic and the lines PQ, $P'Q'$.

859. Two conics circumscribe the triangle ABC, any straight line through A meets them again in P, Q; and the tangents at P, Q meet BC in $P'Q'$; prove that the range $\{BP'Q'C\}$ is constant.

860. The equation of the polar reciprocal of the evolute of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with respect to the centre is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=\frac{(a^2-b^2)^2}{k^4}.$$

861. If P, Q be two fixed points, and if on the side BC of a triangle ABC be taken a point A', such that the pencil $A'\{APQB\}$ is harmonic; and B', C' be similarly taken on the sides CA, AB; the straight lines AA', BB', CC' will meet in a point; and the

four such points corresponding to the triangles formed by four given straight lines will lie on one straight line.

862. The locus of a point, such that the tangents drawn from it to a given conic are harmonically conjugate to the straight lines joining it to two given points is a conic passing through the two given points, and through the points of contact of the tangents drawn from the two points to the given conic. This locus reduces to a straight line, if the line joining the two points touches the given conic.

863. Two conics touch each other at O, any straight line through O meets them in P, Q; prove that the tangents at P, Q intersect in the straight line joining the two other points of intersection of the conic.

864. Four tangents A, B, C, D are drawn to a conic, the line joining the points of contact of A, B meets C, D respectively in P, Q; prove that if a conic be described touching the four lines, and P be its point of contact with C, Q will be its point of contact with D.

865. Two conics S, S' intersect in A, B, C, D, and the pole of AB with respect to S is the pole of CD with respect to S'; prove that the pole of CD with respect to S is the pole of AB with respect to S'.

866. A quadrilateral can be projected into a rhombus on any plane parallel to one of its diagonals, the vertex being any point on a circle in a certain parallel plane.

867. If ABC be a triangle circumscribing a conic, A' the point of contact of BC, D the point where AA' again meets the conic, and TP be any tangent meeting the tangent at D in T; the pencil $T\{ABCP\}$ will be harmonic.

868. ABC is a triangle circumscribing a conic, TP, TQ two tangents, a conic is described about $TPQBC$, and O is the pole of BC with respect to it; prove that $A\{OBCT\}$ is harmonic.

If T lie on the straight line joining A to the point of contact with BC, O will coincide with A.

869. A conic is described touching the sides of a triangle ABC, one of them BC in a fixed point A'; B', C' are two other fixed points on BC; prove that the point of intersection of tangents drawn from B', C' to the conic lies on a fixed straight line.

870. The sides of a triangle which is self-conjugate to a given rectangular hyperbola touch a parabola, and a diameter of the hyperbola is drawn through the focus of the parabola; prove that the conjugate diameter is parallel to the axis of the parabola.

871. TP, TQ are two tangents to a parabola; a hyperbola described through T, P, Q, and having an asymptote parallel to the axis of the parabola meets the parabola again in R; prove that its other asymptote is parallel to the tangent to the parabola at R.

872. TP, TQ are two tangents to a hyperbola; another hyperbola is described through T, P, Q with asymptotes parallel to those of the former; prove that it will pass through the centre C of the former, and that CT will be a diameter.

873. A triangle is self-conjugate to a conic, and from any other two points conjugate to the conic tangents are drawn to a conic inscribed in the triangle; prove that the other four points of intersection of these tangents will be two pairs of conjugate points to the first conic.

874. If a conic pass through four points, its asymptotes meet the conic which is the locus of centres in two points at the extremities of a diameter.

875. Four points and a straight line being given, four conics are described, such that with respect to any one of them three of the points are the angular points of a self-conjugate triangle,

and the fourth is the pole of the given straight line; prove that these four conics will meet the given straight line in the same two points, and that these points are the points of contact of the two conics through the four points touching the line.

876. Four straight lines and a point being given, four conics are described, with respect to each of which three of the four lines form a self-conjugate triangle, and the fourth is the polar of the given point; prove that all four will have two common tangents through the given point, and these tangents are tangents to the two conics through the point touching the four lines. If two tangents be drawn through the point to any conic touching the four lines, these will form a harmonic pencil with the two common tangents.

877. Four tangents are drawn to a conic, and from a point T on one of the diagonals of the quadrilateral formed by them two other tangents are drawn; prove that the points of contact of these tangents lie on the conic passing through T, and through the points of intersection of the four tangents which do not lie on the diagonal through T.

878. CA, CB are two tangents to a conic, any point P in AB is joined to the point in which its polar meets a fixed straight line; prove that the envelope of the joining line is a conic touching the sides of the triangle ABC and the fixed straight line.

879. ABC is a triangle whose sides are met by a straight line in A', B', C'; the straight line which joins A to the point (BB', CC') meets BC in a; and b, c are similarly determined. Four conics are drawn touching the sides of the triangle ABC, and meeting $A'B'C'$ in the same two points; prove that the other common chord of any two of these conics passes through either a, b, or c; that these six common chords intersect by threes in four points; and that these four points are the poles of $A'B'C'$ with respect to the four conics which intersect $A'B'C'$ in the before-mentioned two points and touch the sides of the triangle abc.

880. The four conics which can pass through three given points and touch two given straight lines are drawn, and their other common tangents drawn to every two; prove that the six points of intersection will lie by threes on four straight lines; and that the diagonals of the quadrilateral formed by these four straight lines pass each through one of the three given points.

881. Two conics intersect in A, B, C, D; through D is drawn a straight line to meet the curves again in two points; prove that the locus of the point of intersection of the tangents at these points is a curve of the fourth degree and third class, having cusps at A, B, C, and touching both conics.

882. Prove that the envelope of the straight line joining the points of contact of parallel tangents to two given parabolas is a curve of the third degree and fourth class; having three points of inflexion the tangents at which are the common tangents of the parabolas.

THEORY OF EQUATIONS.

883. THE product of two unequal roots of the equation

$$ax^3 + bx^2 + cx + d = 0$$

is unity; prove that the third root is $\dfrac{a-c}{b-d}$.

884. The roots of the equation $x^2 - px + q = 0$ may be expressed in the forms

$$\frac{q}{p-}\,\frac{q}{p-}\,\frac{q}{p-}\,\ldots, \text{ and } p - \frac{q}{p-}\,\frac{q}{p-}\,\ldots.$$

Explain these results when $p^2 < 4q$.

885. If the equation $x^3 - px^2 + qx - r = 0$ have two equal roots, the third root must satisfy either of the equations

$$x(x-p)^2 = 4r, \quad (x-p)(3x+p) + 4q = 0.$$

886. The roots of the equation $x^3 - px^2 + qx - r = 0$ are the sines of the angles of a triangle; prove that

$$p^4 - 4p^2q + 8pr + 4r^2 = 0.$$

887. Determine the relation between q and r necessary in order that the equation $x^3 - qx + r = 0$ may be put into the form

$$x^4 = (x^2 + mx + n)^2;$$

and solve in this manner the equation

$$8x^3 - 36x + 27 = 0.$$

888. Find the condition necessary in order that the equation

$$ax^3 + bx^2 + cx + d = 0$$

may be put under the form

$$x^4 = (x^2 + px + q)^2;$$

and solve by this method the equation

$$x^3 + 3x^2 + 4x + 4 = 0.$$

889. The roots of the equation $x^3 - px^2 + qx - r = 0$ are in harmonical progression; prove that those of the equation

$$n(pq - n^2r)x^3 - (p^3 - 2npq + 3n^2r)x^2 + (pq - 3nr)x - r = 0$$

are also in harmonical progression.

890. Reduce the equation $x^3 - px^2 + qx - r = 0$ to the form $y^3 \pm 3y + m = 0$ by assuming $x \equiv ay + b$; and solve this equation by assuming $y \equiv z \mp \frac{1}{z}$. Hence prove that the condition for equal roots is

$$4(p^2 - 3q)^3 = (2p^3 - 9pq + 27r)^2.$$

891. In solving a cubic by Cardan's rule; if α, β, γ be the roots of the complete cubic, the roots of the auxiliary quadratic are

$$\frac{(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) \pm 3\sqrt{(-3)}(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)}{54}.$$

892. Solve the equations

$$(x^2 + 4x - 2)^2 + 3 = 4x(3x^2 + 4),$$

$$x^4 - 6x^3 - 8x - 6 = 0.$$

893. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$x^4 + qx^2 + rx + s = 0,$$

the roots of the equation

$$s^3x^4 + qs(1-s)^2x^2 + r(1-s)^3x + (1-s)^4 = 0$$

will be
$$\beta + \gamma + \delta + \frac{1}{\beta\gamma\delta}, \ \ldots\ldots$$

894. In the equation

$$x^4 - px^3 + qx^2 - rx + s = 0;$$

prove that the sum of two of the roots will be equal to the sum of the other two, if $8r - 4pq + p^3 = 0$; and that the product of two will be equal to the product of the other two, if $p^2s = r^2$.

895. If α, β, γ, δ be the roots of a biquadratic, and the equation be solved by putting it in the form

$$(x^2+ax+b)^2=c(x-d)^2,$$

the values of $2b$ are

$$\beta\gamma+\alpha\delta,\quad \gamma\alpha+\beta\delta,\quad \alpha\beta+\gamma\delta;$$

those of $4c$ are

$$(\beta+\gamma-\alpha-\delta)^2,\quad (\gamma+\alpha-\beta-\delta)^2,\quad (\alpha+\beta-\gamma-\delta)^2;$$

and of d are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\quad \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\quad \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}.$$

896. In the method of solving a biquadratic in x by substituting $x\equiv my+n$, and making the resulting equation in y reciprocal; prove that the three values of n are

$$\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta},\quad \frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta},\quad \text{and}\quad \frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta};$$

and those of m^2 are

$$\frac{(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)}{(\alpha+\beta-\gamma-\delta)^2}, \text{ \&c.};$$

α, β, γ, δ being the roots of the biquadratic.

897. Prove that the equation

$$3x^4+8x^3-6x^2-24x+r=0$$

will have four real roots if $r<-8>-13$; two real roots if $r>-8<19$; and no real roots if $r>19$.

898. Prove that the equation

$$x^n+rx^{n-p}+s=0$$

will have two equal roots if

$$\left\{-\frac{r}{n}(n-p)\right\}^n=\left\{\frac{s}{p}(n-p)\right\}^p.$$

899. If $f(x)\equiv(x-a_1)(x-a_2)\ldots(x-a_n)$; and $a_1, a_2, \ldots a_n$ be all unequal; then will

$$\frac{a_1^{n+r-1}}{f'(a_1)}+\frac{a_2^{n+r-1}}{f'(a_2)}+\ldots\ldots+\frac{a_n^{n+r-1}}{f'(a_n)}$$

be equal to the sum of the homogeneous products of r dimensions of the n quantities $a_1, a_2, \ldots a_n$. Prove also that

$$\frac{f''(a_1)}{a_1 f'(a_1)} + \frac{f''(a_2)}{a_2 f'(a_2)} + \ldots + \frac{f''(a_n)}{a_n f'(a_n)}$$
$$\equiv \frac{1}{a_1^2} + \ldots + \frac{1}{a_n^2} - \left\{\frac{1}{a_1} + \ldots. + \frac{1}{a_n}\right\}^2.$$

900. The equation

$$\frac{a_1}{x+b_1} + \frac{a_2}{x+b_2} + \ldots + \frac{a_n}{x+b_n} = 0$$

will be an identical equation if

$$\Sigma(a) = 0, \quad \Sigma(ab) = 0, \quad \Sigma(ab^2) = 0, \ldots\ldots \quad \Sigma(ab^{n-1}) = 0.$$

901. If n be a prime number, and a an impossible root of the equation $x^n = 1$, then will

$$\frac{1}{m+1} + \frac{a}{m+a} + \frac{a^2}{m+a^2} + \frac{a^3}{m+a^3} + \ldots + \frac{a^{n-1}}{m+a^{n-1}} \equiv \frac{n}{1-(-m)^n}.$$

902. If the system of equations of which the type is

$$a_1^r x_1 + a_2^r x_2 + \ldots + a_n^r x_n = c^r$$

be true for integral values of r from $r = 1$ to $r = n + 1$, they will be true for any value of r.

903. Having given

$$\cos na + p_1 \cos(n-1)a + p_2 \cos(n-2)a + \ldots + p_n = 0,$$
$$\sin na + p_1 \sin(n-1)a + p_2 \sin(n-2)a + \ldots + p_n = 0;$$

prove that

$$1 + p_1 \cos a + p_2 \cos 2a + \ldots + p_n \cos na = 0,$$

and

$$1 + p_1 \sin a + p_2 \sin 2a + \ldots + p_n \sin na = 0.$$

904. Find the commensurable roots of the equations

(1) $x^5 + 5x^4 + 120x^3 - 524x - 24 = 0$,

(2) $6x^4 - x^3 - 2x^2 - 27x + 18 = 0$.

905. One of the roots of the equation

$$(1+x)^4 + x^4 + 1 = 0$$

is the square of another root; explain why, on attempting to solve the equation from the knowledge of this fact, the method fails.

906. If α, β, γ, δ, ... be the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots = 0;$$

then will

$$\Sigma (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta)$$
$$= (n-1)(n-2)p_1^3 - 3n(n-2)p_1 p_2 + 3n^2 p_3;$$

and determine what symmetrical function of the differences of the roots is equal to

$$= -\frac{(n-1)(n-2)(n-3)}{2} p_1^4 + 2n(n-2)(n-3)p_1^2 p_2$$
$$- 4n^2(n-3)p_1 p_3 + 4n^3 p_4.$$

907. The equation $x^5 - 209x + 56 = 0$ has two roots whose product is 1; determine them.

908. Prove that all the roots of the equation

$$(1-x)^n - mnx(1-x)^{n-1} + \frac{m(m-1)}{\underline{|2}} \, \frac{n(n-1)}{\underline{|2}} x^2 (1-x)^{n-2}$$
$$- \ldots \text{ to } m+1 \text{ terms} = 0$$

are real; and that none lie beyond the limits 0 and 1, n being integral and $m < n$.

909. Determine the integers between which lie the roots of the equation

$$f(x) \equiv 12x^3 + 16x^2 - 31x + 10 = 0,$$

having given $f_1(x) \equiv 36x^2 + 32x - 31,$

$f_2(x) \equiv 343x - 197,$

$f_3(x)$ positive.

910. Determine the integers between which lie the real roots of the equation

$$x^6 + 6x^5 - 10x^3 - 15x^2 - 6x - 6 = 0.$$

911. Find the sum of the n^{th} powers of the roots of the equation $x^4 - x^2 + 1 = 0$; and form the equation whose roots are the squares of the differences of the roots of the given equation.

912. If s_r denote the sum of the r^{th} powers of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0,$$

and $S_m \equiv s_1 + s_2 + \dots + s_m$; prove that, if S_m have a finite limit when m is indefinitely increased, that limit is

$$-\frac{p_1 + 2p_2 + 3p_3 + \dots + np_n}{1 + p_1 + p_2 + \dots + p_n}.$$

913. The integer next less than $\frac{1}{\sqrt{3}}(\sqrt{3} + \sqrt{5})^{2n-1}$ is divisible by 2^n, n being integral.

914. The integer next greater than $(\sqrt{3} + \sqrt{7})^{2n}$ is divisible by 2^{2n}.

915. The roots of the equation

$$x^5 - p_1 x^4 + p_2 x^3 - p_3 x^2 + p_4 x - p_5 = 0$$

exceed those of the equation

$$x^5 - q_1 x^4 + q_2 x^3 - q_3 x^2 + q_4 x - q_5 = 0$$

respectively by the same quantity; prove that

$$2p_1^2 - 5p_2 = 2q_1^2 - 5q_2,$$

$$4p_1^3 - 15p_1 p_2 + 25p_3 = 4q_1^3 - 15q_1 q_2 + 25q_3,$$

$$3p_2^2 - 8p_1 p_3 + 20p_4 = 3q_2^2 - 8q_1 q_3 + 20q_4,$$

$$8p_1^2 p_3 - 3p_1 p_2^2 - 50p_1 p_4 + 5p_2 p_3 + 250p_5$$
$$= 8q_1^2 q_3 - 3q_1 q_2^2 - 50q_1 q_4 + 5q_2 q_3 + 250q_5.$$

916. Prove that

$$\begin{vmatrix} 1-n, & 1, & 1, & \ldots\ldots & 1 \\ 1, & 1-n, & 1, & \ldots\ldots & 1 \\ 1, & 1, & 1-n, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & 1, & \ldots & \ldots\ldots & 1-n \end{vmatrix} \equiv 0;$$

n being the order of the determinant.

917. Prove that

$$\begin{vmatrix} x, & 1, & 1, & \ldots\ldots & 1 \\ 1, & x, & 1, & \ldots\ldots & 1 \\ 1, & 1, & x, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & 1, & 1, & \ldots\ldots & x \end{vmatrix} \equiv (x-1)^{n-1}(x+n-1);$$

n being the order of the determinant.

918. Prove that

$$\begin{vmatrix} x_1, & 1, & 1, & \ldots\ldots & 1 \\ 1, & x_2, & 1, & \ldots\ldots & 1 \\ 1, & 1, & x_3, & \ldots\ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & 1, & 1, & \ldots\ldots & x_n \end{vmatrix} \equiv p_n - p_{n-2} + 2p_{n-3} - 3p_{n-4} - \ldots\ldots + (-1)^{n-1}(n-1);$$

$x_1, x_2, \ldots\ldots x_n$ being the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots + (-1)^n p_n = 0.$$

919. Prove that

$$\begin{vmatrix} x, & x^2, & x^3, & \ldots\ldots & x^n \\ x^2, & x^3, & x^4, & \ldots\ldots & x^n, x \\ x^3, & x^4, & \ldots\ldots & x^n, & x, x^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n, & x, & x^2, & \ldots\ldots & x^{n-1} \end{vmatrix} \equiv (-1)^{\frac{n(n-1)}{2}} x^n (x^n-1)^{n-1}$$

920. Prove that

$$\begin{vmatrix} \cos\theta, & \cos 2\theta, & \cos 3\theta, & \dots\dots & \cos n\theta \\ \cos 2\theta, & \cos 3\theta, & \dots\dots & \cos n\theta, & \cos\theta \\ \cos 3\theta, & \dots\dots & & \cos\theta, & \cos 2\theta \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \cos n\theta, & \cos\theta, & \cos 2\theta, & \dots & \cos(n-1)\theta \end{vmatrix} \equiv \frac{\{\cos\theta - \cos(n+1)\theta\}^n - \{1-\cos n\theta\}^n}{2(-1)^{\frac{n(n-1)}{2}}(1-\cos n\theta)},$$

921. The product of the roots of the equation

$$\begin{vmatrix} x-2, & 1, & 0, & 0, & 0, & \dots\dots & 0 \\ 1, & x-2, & 1, & 0, & 0, & \dots\dots & 0 \\ 0, & 1, & x-2, & 1, & 0, & \dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots\dots & & 1, & x-2, & 1 \\ 0, & 0, & \dots\dots & & 0, & 1, & x-2 \end{vmatrix} = 0$$

is $n+1$; and the sum of the roots is $2n$, n being the order of the determinant.

922. Prove that

$$\begin{vmatrix} 1-x_1, & x_1(1-x_2), & x_1x_2(1-x_3), & \dots\dots & x_1x_2\dots x_{n-1}(1-x_n), & x_1x_2\dots x_n \\ -1, & 1-x_2, & x_2(1-x_3), & \dots\dots & x_2\dots x_{n-1}(1-x_n), & x_2\dots x_n \\ 0, & -1, & 1-x_3, & \dots\dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots\dots & 0, & 1 \end{vmatrix}$$

is equal to 1, the second row being formed by differentiating the first with respect to x_1, the third by differentiating the second with respect to x_2, and so on.

DIFFERENTIAL CALCULUS.

923. Having given

$$\sin x \sin\left(\frac{\pi}{m}+x\right)\sin\left(\frac{2\pi}{m}+x\right)\dots\dots$$
$$\sin\left\{(m-1)\frac{\pi}{m}+x\right\}\equiv 2^{-(m-1)}\sin mx\,;$$

prove that

$$\cot x+\cot\left(\frac{\pi}{m}+x\right)+\cot\left(\frac{2\pi}{m}+x\right)+\dots.$$
$$+\cot\left\{(m-1)\frac{\pi}{m}+x\right\}\equiv m\cot mx.$$

924. Prove that the limit of $(\cos x)^{\cot^2 x}$, as x diminishes indefinitely, is $\epsilon^{-\frac{1}{2}}$.

925. If $y=\cot^{-1}x$,

$$\frac{d^n y}{dx^n}=(-1)^n \lfloor n-1 \sin ny \sin^n y.$$

926. Prove that, if n be a positive integer, the expression

$$(x-n)\,\epsilon^x+\frac{x^{n-1}}{\lfloor n-1}+\frac{2x^{n-2}}{\lfloor n-2}+\frac{3x^{n-3}}{\lfloor n-3}+\dots+(n-1)\,x+n$$

will be positive for all positive values of x.

927. If

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0,$$

then will

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^n y}{dx^n}=0.$$

928. Having given

$$y \equiv \{x + \sqrt{(1+x^2)}\}^m + \{x + \sqrt{(1+x^2)}\}^{-m};$$

prove that

$$(1+x^2)\frac{d^{n+2}y}{dx^{n+2}} + (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2 - m^2)\frac{d^n y}{dx^n} = 0.$$

929. If

$$\sin(m \tan^{-1} x) \equiv a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots;$$

prove that

$$(n+1)(n+2)a_{n+2} + (2n^2 + m^2)a_n + (n-1)(n-2)a_{n-2} = 0.$$

930. If

$$\{\log(1+x)\}^3 \equiv a_3 x^3 + a_4 x^4 + \ldots + a_n x^n + \ldots,$$

then will

$$(n+2)a_{n+2} + (n+1)a_{n+1} = \frac{6(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).$$

931. In the equation

$$f(x+h) = f(x) + hf'(x + \theta h),$$

the limiting value of θ, when h is indefinitely diminished, is $\frac{1}{2}$; and, if θ be constant, $f(x) \equiv A + Bx + Cx^2$, where A, B, C are independent of x.

932. In the equation

$$f(x+h) = f(x) + hf'(x + \theta h),$$

prove that the first two terms of the expansion of θ in ascending powers of h are

$$\frac{1}{2} + \frac{h}{24}\frac{f'''(x)}{f''(x)}:$$

and, if $f(x) \equiv \sin x$, the first three terms of the expansion are

$$\frac{1}{2} + \frac{h \cot x}{24} - \frac{h^2}{48 \sin^2 x};$$

provided that $\cot x$ be finite.

933. In the equation

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^n}{\lfloor n} f^n(a+\theta h),$$

the limiting value of θ, when h is indefinitely diminished, is $\frac{1}{n+1}$; and, if $f(x)$ be a rational algebraical expression of $n+1$ dimensions in x, the value of θ is always $\frac{1}{n+1}$.

934. In the equation

$$\frac{F(x+h) - F(x)}{f(x+h) - f(x)} = \frac{F'(x+\theta h)}{f'(x+\theta h)};$$

prove that θ will be constant if $F(x) \equiv \sin x$ and $f(x) \equiv \cos x$. Prove also that, in all cases, the limiting value of θ, when h is indefinitely diminished, is $\frac{1}{2}$.

935. Prove that

$$\frac{\pi}{4} \frac{\epsilon^\pi - 1}{\epsilon^\pi + 1} = \frac{1}{1+1^2} + \frac{1}{1+3^2} + \frac{1}{1+5^2} + \ldots \text{ to } \infty,$$

and that

$$1 + \frac{\pi^4}{\lfloor 4} + \frac{\pi^8}{\lfloor 8} + \ldots \text{ to } \infty = \left(1 + \frac{4}{1^2}\right)\left(1 + \frac{4}{3^2}\right)\left(1 + \frac{4}{5^2}\right) \ldots \text{ to } \infty.$$

936. Prove that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} > \frac{2}{3} n^{\frac{3}{2}} < \frac{2}{3}(n+1)^{\frac{3}{2}};$$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots. + \frac{1}{n} > \log(1+n) < 1 + \log n;$$

$$\cos a + \cos 2a + \ldots + \cos na < \frac{\sin na}{a} > \frac{\sin(n+1)a}{a} - 1;$$

na in the last being $< \frac{\pi}{2}$.

937. If $y = x \cos xy$, the general term of the expansion of $\sin xy$ in terms of x is

$$\frac{(-1)^n}{2} \frac{x^{4n+2}}{\lfloor 2n+1} \left\{(n+1)^{2n} + (2n+2) n^{2n} \right.$$
$$\left. + \frac{(2n+2)(2n+1)}{\lfloor 2} (n-1)^{2n} + \ldots \text{ to } n+1 \text{ terms}\right\}.$$

938. The limiting values of

$$\frac{d^n}{dx^n}\left(\frac{x}{\sin x}\right)^{n+1}, \quad \frac{d^n}{dx^n}(x \cot x)^{n+1}, \quad \frac{d^n}{dx^n}\left\{\frac{x}{\log(1+x)}\right\}^{n+1},$$

are respectively 0, 0, 1, if n be odd; and

$$\{1 . 3 . 5 \dots (n-1)\}^2, \quad (-1)^{\frac{n}{2}} \lfloor n, \text{ and } 1,$$

if n be even.

939. If $f(a) = 0$ and $\phi(a) = 0$, and if the limit, as x approaches a, of $\dfrac{f(x)}{\phi(x)}$ be finite; the limit of $\{f(x)\}^{\phi(x)}$ will be 1.

940. If $f(a) = 1$, $\phi(a) = \infty$, the limiting value of $\{f(x)\}^{\phi(x)}$, as x approaches a, is $\epsilon^{mf'(a)}$, m being the limit of $(x-a)\phi(x)$.

941. If

$$z = \epsilon^x f_1(x+y) + \epsilon^{2x} f_2(x+y) + \dots + \epsilon^{nx} f_n(x+y);$$

then will

$$\left(\frac{d}{dx} - \frac{d}{dy} - 1\right)\left(\frac{d}{dx} - \frac{d}{dy} - 2\right) \dots \left(\frac{d}{dx} - \frac{d}{dy} - n\right) z = 0.$$

942. If

$$z = \epsilon^x f_1\left(\frac{y}{x}\right) + \epsilon^{2x} f_2\left(\frac{y}{x}\right) + \dots + \epsilon^{nx} f_n\left(\frac{y}{x}\right);$$

then will

$$(p+q-1)(p+q-2) \dots (p+q-n) z = 0,$$

where $p^a q^b z$ denotes

$$\left(\frac{y}{x}\right)^b \frac{d^{a+b} z}{dx^a \, dy^b}.$$

943. If x, y be co-ordinates of a point referred to axes inclined at an angle ω, and u any function of the position of the point;

$$\frac{1}{\sin^2 \omega}\left\{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} - 2\cos\omega \frac{d^2u}{dx\,dy}\right\}, \quad \frac{1}{\sin^2\omega}\left\{\frac{d^2u}{dx^2}\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2\right\},$$

will be independent of the particular axes.

944. Having given $x+y=X$, $y=XY$; prove that

$$x\frac{d^2u}{dx^2}+y\frac{d^2u}{dx\,dy}-\frac{du}{dx}=X\frac{d^2u}{dX^2}-Y\frac{d^2u}{dX\,dY}-\frac{du}{dX}.$$

945. If $2x\equiv r(\epsilon^{\theta}+\epsilon^{-\theta})$, $2y\equiv r(\epsilon^{\theta}-\epsilon^{-\theta})$;

$$\frac{d^2u}{dx^2}-\frac{d^2u}{dy^2}\equiv\frac{d^2u}{dr^2}-\frac{1}{r^2}\frac{d^2u}{d\theta^2}+\frac{1}{r}\frac{du}{dr}.$$

946. If $x\equiv\epsilon^{\theta+\phi}+\epsilon^{\theta-\phi}$, $y\equiv\epsilon^{\theta+\phi}-\epsilon^{\theta-\phi}$;

$$4\left(\frac{d^2u}{dx^2}-\frac{d^2u}{dy^2}\right)\equiv\epsilon^{-2\theta}\left(\frac{d^2u}{d\theta^2}-\frac{d^2u}{d\phi^2}\right).$$

947. If $\epsilon^x\equiv r\epsilon^{\cos\theta}$, $\epsilon^y\equiv r\epsilon^{\sin\theta}$;

$$x^2\frac{d^2u}{dy^2}-2xy\frac{d^2u}{dx\,dy}+y^2\frac{d^2u}{dx^2}\equiv\frac{d^2u}{d\theta^2}+r\frac{du}{dr}\log r.$$

948. If u be a function of the four independent variables x_1, x_2, x_3, x_4; and if

$$x_1=r\sin\theta\sin\phi,\quad x_2=r\sin\theta\cos\phi,$$
$$x_3=r\cos\theta\sin\psi,\quad x_4=r\cos\theta\cos\psi;$$

then will

$$\frac{d^2u}{dx_1^2}+\frac{d^2u}{dx_2^2}+\frac{d^2u}{dx_3^2}+\frac{d^2u}{dx_4^2}=\frac{d^2u}{dr^2}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{d^2u}{d\phi^2}$$
$$+\frac{1}{r^2\cos^2\theta}\frac{d^2u}{d\psi^2}+\frac{3}{r}\frac{du}{dr}+\frac{2}{r^2}\cot 2\theta\frac{du}{d\theta}.$$

949. If x, y, z be three variables connected by one equation; prove the formulæ for changing the dependent variable from z to x;

$$\frac{dx}{dz}\equiv\frac{1}{p},\quad\frac{dx}{dy}\equiv-\frac{q}{p},\quad\frac{d^2x}{dz^2}\equiv-\frac{r}{p^3},$$

$$\frac{d^2x}{dy\,dz}\equiv\frac{qr-ps}{p^3},\quad\frac{d^2x}{dy^2}\equiv-\frac{p^2t-2pqs+q^2r}{p^3}.$$

950. The distances of any point from two fixed points are r, r', and a maximum or minimum value of $f(r, r')$ for points lying on a given curve is c; prove that the curve $f(r, r') = c$ will touch the given curve.

951. In the straight line bisecting the angle A of a triangle ABC is taken a point P; prove that the difference of the angles APB, APC will be a maximum when AP is a mean proportional between AB and AC.

952. Find the maximum and minimum values of a normal chord of a given ellipse: proving that if one exist other than the axes, the eccentricity must be $> \frac{1}{\sqrt{2}}$, and that the length of such a chord will be $\frac{3\sqrt{3}\,a^2b^2}{(a^2+b^2)^{\frac{3}{2}}}$, where $2a$, $2b$ are the axes.

953. Normals are drawn to an ellipse at the extremities of two conjugate diameters: prove that a maximum or minimum distance of their point of intersection from the centre is $\frac{(a^2+b^2)^{\frac{3}{2}}}{3\sqrt{3}\,ab}$, provided the eccentricity is $> \frac{2}{\sqrt{5}}$. Examine which of the two it is.

954. If $x + y + z = 3c$, $f(x) \cdot f(y) \cdot f(z)$ will be a maximum or minimum when $x = y = z = c$, according as

$$f''(c) < \text{ or } > \frac{\{f'(c)\}^2}{f(c)}.$$

955. The minimum area which can be included between two parabolas, whose axes are parallel and at a distance c, and which cut each other at right angles in two points, is $c^2 \frac{\sqrt{3}}{2}$.

956. O is the centre of curvature at a point P of a given ellipse, and OP', OQ' are normals drawn from O: prove that, if $a^2 < 2b^2$, $P'Q'$ has its minimum value when the eccentric angle of P is $\tan^{-1}\left\{\frac{2a^2-b^2}{2b^2-a^2}\right\}^{\frac{1}{2}}$.

957. Prove that three parabolas of maximum latus rectum can be drawn circumscribing a given triangle; and that if α, β, γ be the angles which the sides make with the axis of any one of them,

$$\cot\alpha + \cot\beta + \cot\gamma = 0.$$

958. Find the plane sections of greatest and least area which can be drawn through a given point on the surface of a paraboloid of revolution: proving that, if θ_1, θ_2 be the angles which the planes of maximum and minimum section make with the axis,

$$\tan\theta_1 \tan\theta_2 = \frac{3}{2}.$$

959. If x, y, z be the distances of any point in a plane from three given points, and $f(x, y, z)$ be a maximum or minimum;

$$\frac{1}{\sin(y, z)}\frac{df}{dx} = \frac{1}{\sin(z, x)}\frac{df}{dy} = \frac{1}{\sin(x, y)}\frac{df}{dz};$$

(y, z) denoting the angle between the distances y, z.

960. If A, B, C, D be four points not in one plane, and P a point the sum of whose distances from A, B, C, D is a minimum; then will

$$\frac{PA \,.\, PA'}{AA'} = \frac{PB \,.\, PB'}{BB'} = \frac{PC \,.\, PC'}{CC'} = \frac{PD \,.\, PD'}{DD'};$$

PA, PB, PC, PD meeting the opposite faces of the tetrahedron in A', B', C', D'.

961. If A, B, C, D be four points not in one plane, and P a point at which

$$l \,.\, PA^2 + m \,.\, PB^2 + n \,.\, PC^2 + r \,.\, PD^2$$

is a maximum or minimum, then will

$$\frac{\text{vol. } PBCD}{l} = \frac{\text{vol. } PCDA}{m} = \frac{\text{vol. } PDAB}{n} = \frac{\text{vol. } PABC}{r}.$$

962. If u, x, y, z be the distances of any point from four given points not in one plane, and $f(u, x, y, z)$ be a maximum or minimum;

$$\frac{1}{1-a^2-b^2-c^2+2abc}\left(\frac{df}{du}\right)^2 = \frac{1}{1-a^2-b'^2-c'^2+2ab'c'}\left(\frac{df}{dx}\right)^2$$

$$= \frac{1}{1-a'^2-b^2-c'^2+2a'bc'}\left(\frac{df}{dy}\right)^2 = \frac{1}{1-a'^2-b'^2-c^2+2a'b'c}\left(\frac{df}{dz}\right)^2:$$

a, b, c, a', b', c' denoting the cosines of the angles between the distances (y, z), (z, x), (x, y), (u, x), (u, y), (u, z) respectively.

963. P, P' are contiguous points of a curve, PO, $P'O$ are drawn at right angles to the radii vectores SP, SP'; prove that the limiting value of PO, as P' moves up to P, is $\pm \frac{dr}{d\theta}$.

964. S, H are two fixed points, P is a point moving so that the rectangle SP, HP is constant: prove that straight lines drawn from S, H at right angles respectively to SP, HP will meet the tangent at P to the locus of P in points equidistant from P.

965. In the curve $y^3 = 3ax^2 - x^3$, the tangent at P meets the curve again in Q; prove that

$$\tan QOx + 2\tan POx = 0,$$

O being the origin. Prove also that, if the tangent at P be a normal at Q, P lies on the curve

$$4y(3a-x) = (2a-x)(16a-5x).$$

966. In the curve $y^3 = a^2x$, the greatest acute angle between two tangents which intersect on the curve is $\tan^{-1}\frac{3}{4}$.

967. A tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ which makes an angle $\tan^{-1}\frac{\pm 1 \pm \sqrt{5}}{2}$ with the axis of x is also a normal to the

curve. Also if two tangents be drawn to the curve from a point P lying on the curve (P not being the point of contact of either), the acute angle between these tangents cannot exceed 60°.

968. The tangent to the evolute of a parabola at a point where it meets the parabola is also a normal to the evolute.

969. If from a point on the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the two other normals to the ellipse be drawn; the straight line joining the points where they meet the ellipse will be normal to the ellipse

$$(a^2x^2 + b^2y^2)(a^2 - b^2)^2 = a^4b^4.$$

970. Prove that, for any curve of the third degree, there exists one point such that the points of contact of the tangents drawn from it to the curve lie on a circle.

971. A tangent to a given ellipse at P meets the axes in two points, through which are drawn straight lines at right angles to the axes meeting in p: prove that the normal at p to the locus of p and the line joining the centre of the ellipse to the centre of curvature at P make equal angles with the axes.

972. In a curve of the fourth degree, which has four real asymptotes, no two of which are parallel, the asymptotes will meet the curve again in eight points lying on a conic. Determine this conic in the case of the curve

$$xy(x^2 - y^2) + a(x^3 + y^3) = a^2\{(x+y)^2 - a(x+y) + a^2\}:$$

and prove that three of the asymptotes touch the curve in points not at infinity.

973. If the equation of a curve of the n^{th} degree be

$$x^n\phi_1\left(\frac{y}{x}\right) + x^{n-1}\phi_2\left(\frac{y}{x}\right) + \ldots = 0,$$

and $\phi(z) = 0$ have two roots μ_1 and if also $\phi_2(\mu_1) = 0$; the equation of the corresponding rectilinear asymptotes will be

$$(y - \mu_1x)^2\phi_1''(\mu_1) + 2(y - \mu_1x)\phi_2'(\mu_1) + 2\phi_3(\mu_1) = 0.$$

974. Two circles of radii b, $a-b$, respectively roll within a circle of radius a, their points of contact with the fixed circle being originally coincident, and the circles rolling in opposite directions in such a manner that the velocities of points on the circles with respect to their respective centres are equal: prove that they will always intersect in the point which was originally the point of contact.

975. In a hypocycloid the radius of the rolling circle is $\frac{n}{2n+1}$ times the radius of the fixed circle (n integral); prove that the locus of the point of intersection of perpendicular tangents is a circle; and that the line joining the points of contact is also a tangent to a hypocycloid having the same fixed circle.

976. A circle is drawn to touch a cardioid and pass through the cusp: prove that the locus of its centre is a circle. If two such circles be drawn, and through their second point of intersection any straight line be drawn, the tangents to the circles at the ends of this straight line will intersect on the cardioid.

977. If S, H be foci of a lemniscate, PT the tangent at any point P,

$$\sin SPT = \frac{SP \sim 3HP}{2\sqrt{2}\,HP};$$

and if θ, ϕ be the acute angles which the tangent makes with the focal distances,

$$\sqrt{2}\sin\frac{\theta \pm \phi}{2} = \cos\frac{\theta \mp \phi}{2}.$$

978. Two circles touch the curve $r^m = a^m \cos m\theta$ in the points P, Q and touch each other in the pole S: prove that the angle PSQ is equal to $\frac{n\pi}{1-m}$, n being a positive or negative integer.

979. The locus of the centre of a circle touching the curve $r^m = a^m \cos m\theta$ and passing through the pole is the curve

$$(2r)^n = a^n \cos n\theta, \quad \{n(1-m) \equiv m\}.$$

980. In the curve $r = a \sec^n \theta$, prove that, at a point of inflexion, the radius vector makes equal angles with the prime radius and the tangent; and that the distance of the point of inflexion from the pole increases from a to $a\sqrt{\epsilon}$, as n increases from 0 to ∞. If n be negative, there is no real point of inflexion.

981. SY is the perpendicular from the pole S on the tangent to a curve at P; prove that when there is a cusp at P, the circle of curvature at Y to the locus of Y will pass through S: also, that when there is a point of inflexion at Y in the locus of Y, the chord of curvature at P through S will be equal to $4SP$.

982. The general equation of a curve of the fourth degree having cusps at A, B, C is

$$\left(\frac{l}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{m}{\beta}\right)^{\frac{1}{2}} + \left(\frac{n}{\gamma}\right)^{\frac{1}{2}} = 0;$$

ABC being the triangle of reference.

983. The equation

$$y = c\epsilon^{\frac{x}{a}} + b\epsilon^{-\frac{x}{a}}$$

will represent a catenary if $4bc = a^2$.

984. If x, y be rectangular co-ordinates of any point on a curve, ρ the radius of curvature at that point, ϕ the angle which the tangent at the point makes with a fixed straight line,

$$\left(\frac{d^2x}{d\phi^2}\right)^2 + \left(\frac{d^2y}{d\phi^2}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\phi}\right)^2,$$

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 = \frac{1}{\rho^4}\left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\}.$$

985. The centre of curvature at a point P of a parabola is O, OQ is drawn at right angles to OP meeting the focal distance of P in Q; prove that the radius of curvature of the evolute at O is equal to $3OQ$.

986. The reciprocal polar of the evolute of a parabola with respect to the focus is a curve whose equation is of the form

$$r \cos\theta = c \sin^2\theta\,;$$

and that of the evolute of an ellipse is

$$\frac{c}{r\cos\theta} = 1 - \frac{e\sin\theta}{\sqrt{(1-e^2\cos^2\theta)}}\,;$$

the focus being pole and the axis the initial line in each case.

987. A rectangular hyperbola, whose axes are parallel to the co-ordinate axes, has contact of the second order with a given curve at a given point (x, y): prove that the co-ordinates (X, Y) of the centre of the hyperbola are given by the equations

$$\frac{X-x}{\dfrac{dy}{dx}} = Y - y = \frac{\left(\dfrac{dy}{dx}\right)^2 - 1}{\dfrac{d^2y}{dx^2}}:$$

and that the central radius to the point (x, y) is the tangent at (X, Y) to the locus of the centre of the hyperbola. If the given curve be (1) the parabola $y^2 = 4ax$, (2) the circle $x^2 + y^2 = a^2$, the locus of the centre of the hyperbola is

$$(1)\ 4(x+2a)^3 = 27ay^2, \quad (2)\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

988. A series of rectangular hyperbolas have their axes parallel to the axes of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and have with it contact of the second order: prove that the locus of their centres is a curve similar to the evolute of the ellipse, and whose dimensions are to those of the evolute as

$$a^2 + b^2 \;:\; a^2 - b^2.$$

989. A curve is such that any two corresponding points of its evolute and an involute are at a constant distance: prove that the line joining the two points is also constant in direction.

990. Prove that in any epicycloid or hypocycloid, the radius of curvature is proportional to the perpendicular on the tangent from the centre of the fixed circle.

991. The curvature at any point of a lemniscate varies as the difference of the focal distances.

992. If the tangent and normal at a curve be taken as the axes of x, y, the co-ordinates of a neighbouring point are, approximately,

$$x = s - \frac{s^3}{6\rho^2}, \quad y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\left(\frac{d\rho}{ds}\right)_0;$$

ρ being the radius of curvature at the origin, and s the arc measured from the origin.

993. A loop of a lemniscate rolls in contact with the axis of x, prove that the locus of the node is given by the equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{a}{y}\right)^{\frac{4}{3}};$$

and if ρ, ρ' be corresponding radii of curvature of this locus and of the lemniscate, $2\rho\rho' = a^2$.

994. If the curve $r^m = a^m \cos m\theta$ roll along a straight line, the radius of curvature of the path of the pole is

$$r\left(1 + \frac{1}{m}\right).$$

995. A rectangular hyperbola rolls on a straight line: prove that the radius of curvature of the path of the centre is half the distance from the centre to the point of contact; and that the length of any portion of the path of the centre is equal to the corresponding arc of the locus of the feet of the perpendiculars let fall from the centre on the tangent.

996. A plane curve rolls along a straight line; prove that the radius of curvature of the path of any point fixed with respect

to the curve is $\dfrac{r^2}{r-\rho\sin\phi}$, where r is the distance from the fixed point to the point of contact, ϕ the angle between this distance and the straight line, and ρ the radius of curvature at the point of contact.

997. If the curve

$$\frac{a}{r}=1+\sec a\sin(\theta\sin a)$$

roll on a straight line, the locus of the pole is a circle.

998. If (α, β, γ) be areal co-ordinates of any point on a curve, ρ the radius of curvature at that point,

$$\frac{9}{4k^2\rho^2}+\frac{\left\{\dfrac{d^2\alpha}{dt^2}\left(\dfrac{d\beta}{dt}-\dfrac{d\gamma}{dt}\right)+\dfrac{d^2\beta}{dt^2}\left(\dfrac{d\gamma}{dt}-\dfrac{d\alpha}{dt}\right)+\dfrac{d^2\gamma}{dt^2}\left(\dfrac{d\alpha}{dt}-\dfrac{d\beta}{dt}\right)\right\}^2}{\left(a^2\dfrac{d\beta}{dt}\dfrac{d\gamma}{dt}+b^2\dfrac{d\gamma}{dt}\dfrac{d\alpha}{dt}+c^2\dfrac{d\alpha}{dt}\dfrac{d\beta}{dt}\right)^3}=0,$$

a, b, c being the sides, and k the area, of the triangle of reference.

999. If two tangents to an ellipse be drawn intersecting a given length on a fixed straight line, the locus of their intersection is a curve of the fourth degree having contact of the third order with the ellipse at the points where the tangents are parallel to the given line: trace the curve for positions of the fixed straight line in which it intersects the ellipse (1) in real points, (2) in impossible points, the length intercepted being equal to the parallel diameter.

1000. Find the envelopes of

$$(1)\quad x\cos^3\theta+y\sin^3\theta=a,$$

$$(2)\quad \frac{x^3}{a^3\cos\theta}+\frac{y^3}{b^3\sin\theta}=1,$$

θ being the parameter in each case.

1001. A curve is generated by a point of a circle which rolls along a fixed curve: prove that the diameter of the circle through the generating point will envelope a curve similarly generated by a circle of half the dimensions.

1002. OY is a perpendicular let fall from a fixed point O on any one of a series of straight lines drawn according to some fixed law; prove that when OY is a maximum or minimum, Y is in general a point on the envelope: and, if Y be not on the envelope, the line to which OY is the perpendicular is an asymptote to the envelope.

1003. On any radius vector of the curve

$$r = a \sec^n \frac{\theta}{n}$$

as diameter is described a circle: the envelope of such circles is the curve

$$r = c \sec^{n-1} \frac{\theta}{n-1}.$$

Prove this property geometrically in the cases when $n = 2$, and when $n = 3$.

1004. Find the envelope of the system of circles represented by the equation

$$(x - am^2)^2 + (y - 2am)^2 = a^2 (1 + m^2)^2$$

for different values of m.

1005. The envelope of the straight line

$$x \cos\phi + y \sin\phi = a (\cos n\phi)^{\frac{1}{n}}$$

is the curve whose polar equation is

$$r^{\frac{n}{1-n}} = a^{\frac{n}{1-n}} \cos\frac{n\theta}{1-n}.$$

1006. Tangents drawn to a series of confocal conics, at points where they meet a fixed straight line through one of the foci, envelope a parabola, of which the given straight line is directrix and the other given focus the focus.

1007. If a parabola roll along a straight line, the envelope of its directrix is a catenary.

1008. A parabola is described touching a given circle, and having its focus at a given point on the circle: prove that the envelope of its directrix is a cardioid.

1009. A straight line is drawn through each point of the curve $r^m = a^m \cos m\theta$ at right angles to the radius vector: prove that the envelope of such lines is the curve

$$r^n = a^n \cos n\theta, \quad \{n(1-m) \equiv m\}.$$

1010. SY is the perpendicular from the pole on a tangent to the curve $r^m = a^m \cos m\theta$; with S as pole and Y as vertex is described a curve similar to $r^n = a^n \cos n\theta$: prove that the envelope of such curves is the curve $r^p = a^p \cos p\theta$, where

$$\frac{1}{p} \equiv 1 + \frac{1}{m} + \frac{1}{n}.$$

1011. If A be the vertex, P any point of the parabola $y^2 = 4ax$, the straight line through P at right angles to AP will envelope the curve $27ay^2 = (x-4a)^3$.

1012. A circle is described on each radius vector of a given curve: prove that the envelope is the locus of the foot of the perpendicular from the pole on the tangent.

INTEGRAL CALCULUS.

1013. The area common to two ellipses which have the same centre and equal axes inclined at an angle α is

$$2ab \tan^{-1} \frac{2ab}{(a^2-b^2)\sin\alpha}.$$

1014. The arc of the curve

$$y = \sqrt{(a^2-b^2)}\left(1-\cos\frac{x}{b}\right),$$

between the origin and the point where the curve again meets the axis of x, is equal to the perimeter of an ellipse of axes $2a$, $2b$. Determine the ratio of $a : b$ in order that the area included between this part of the curve and the axis of x may be equal to the area of the ellipse.

1015. A series of spheres touch each other at a given point and from each is cut off a segment of given curve surface by a plane perpendicular to the line of centres: prove that the circular sections made by these planes lie on the same sphere.

1016. The sum of the products of each element of an elliptic lamina multiplied by its distance from the focus is equal to $Ma\dfrac{2+e^2}{3}$, M being the mass of the lamina, $2a$ the major axis, and e the eccentricity.

1017. If the areas of the curves

$$a^2y^2(x-b)^2 = (a^2-x^2)(bx-a^2)^2, \quad x^2+y^2=a^2, \quad (b>a)$$

be A, A'; prove that, as b decreases to a, the limiting value of

$$\frac{A'-A}{b-a} \text{ is } 6\pi a.$$

1018. Perpendiculars are let fall upon the tangents to an ellipse from a point within it, whose distance from the centre is c: prove that the area of the curve traced out by the feet of these perpendiculars is

$$\frac{\pi}{2}(a^2 + b^2 + c^2).$$

1019. Find the whole length of the arc enveloped by the directrix of an ellipse rolling along a straight line during a complete revolution; and prove that the curve will have two cusps if the eccentricity of the ellipse exceed $\frac{\sqrt{5}-1}{2}$.

1020. Two catenaries touch each other at the vertex, and the linear dimensions of the outer are twice those of the inner; two common ordinates MPQ, mpq are drawn from the directrix of the outer: prove that the volume generated by the revolution of Pp about the directrix is $2\pi \times$ area $MQqm$.

1021. Find the limiting values of

$$(1)\quad \left\{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\dots\sin(n-1)\frac{\pi}{n}\right\}^{\frac{1}{n}},$$

$$(2)\quad \left\{\sin\frac{\pi}{n}\sin^2\frac{2\pi}{n}\sin^3\frac{3\pi}{n}\dots\sin^{n-1}(n-1)\frac{\pi}{n}\right\}^{\frac{1}{n}},$$

$$(3)\quad \left\{\left(1+\tan\frac{\pi}{4n}\right)\left(1+\tan\frac{2\pi}{4n}\right)\dots\left(1+\tan(n-1)\frac{\pi}{4n}\right)\right\}^{\frac{1}{n}},$$

when n is indefinitely increased.

1022. If $P_m = \int_1^2 x^m\sqrt{(2-3x+x^2)}\,dx$; then will

$$2(m+2)P_m - 3(2m+1)P_{m-1} + 4(m-1)P_{m-2} = 0.$$

1023. An arithmetical, a geometrical, and a harmonical progression have each the same number of terms, and the same first and last terms a and l; the sums of their terms are respectively

s_1, s_2, s_3, and the continued products p_1, p_2, p_3: prove that when the number of terms is indefinitely increased

$$\frac{2s_1}{s_2} = \frac{l+a}{l-a}\log\left(\frac{l}{a}\right), \quad \frac{s_1^2}{s_2 s_3} = \frac{(a+l)^2}{4al}, \quad \frac{p_1 p_3}{p_2^2} = 1.$$

1024. If $2x = r(\epsilon^\theta - \epsilon^{-\theta})$, $2y = r(\epsilon^\theta + \epsilon^{-\theta})$; prove that

$$\int_0^\infty \int_x^\infty v\,dx\,dy = \int_0^\infty \int_0^\infty v'\,rdr\,d\theta\,;$$

v being a function of x, y, which becomes v' when their values are substituted.

1025. If $Xx^2 = Yy^2 = Zz^2 = \ldots = xyz\ldots$, then will

$$\iiint \ldots VdX\,dY\,dZ\ldots = 2^{n-1}(n-2)\iiint \ldots v\,(xyz\ldots)^{n-3}\,dx\,dy\,dz\ldots,$$

V being a function of X, Y, Z ... which becomes v when their values are substituted, and n being the number of integrations.

1026. Prove that

$$\int_0^\infty \int_0^\infty \int_0 Vdx\,dy\,dz = \int_0^\infty \int_0^1 \int_0^1 V'u^2v\,du\,dv\,dw,$$

where $\quad x + y + z = u, \quad y + z = v, \quad z = uvw.$

1027. Prove that

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty Vdx_1\,dx_2\,dx_3\,dx_4 = \int_0^\infty \int_0^1 \int_0^1 \int_0^1 V'u_1^3u_2^2u_3\,du_1\,du_2\,du_3\,du_4,$$

where $\quad x_1 + x_2 + x_3 + x_4 = u_1, \quad x_2 + x_3 + x_4 = u_1u_2,$

$\quad x_3 + x_4 = u_1u_2u_3, \quad x_4 = u_1u_2u_3u_4.$

1028. Prove that

$$\int_0^\infty \int_0^\infty \ldots Vdx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \ldots V'r^5 \sin^4\theta_1 \cos^2\theta_1 \sin\theta_2 \sin\theta_3\,drd\theta_1\ldots d\theta_5,$$

where $\quad x_1 = r\sin\theta_1\cos\theta_2,\qquad x_2 = r\cos\theta_1\cos\theta_3,$

$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_4,\quad x_4 = r\sin\theta_1\sin\theta_2\sin\theta_4,$$

$$x_5 = r\cos\theta_1\sin\theta_3\cos\theta_5,\quad x_6 = r\cos\theta_1\sin\theta_3\sin\theta_5.$$

1029. Prove that

$$\iiint \dots dx_1\, dx_2 \dots dx_n,$$

is equal to

$$\frac{1}{\sqrt{n+1}}\left\{\frac{\pi}{n(n+1)}\right\}^{\frac{n}{2}} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)},$$

the limits being given by the equation,

$$x_1^2 + x_2^2 + \dots + x_n^2 - x_1x_2 - x_2x_3 - \dots - x_{n-1}x_n - x_n + \frac{n-1}{2n} = 0.$$

1030. Prove that the limit of the sum of the series

$$x - \frac{1}{3}\frac{x^3}{\lfloor 3} + \frac{1}{5}\frac{x^5}{\lfloor 5} - \dots \text{ to } \infty,$$

when x is indefinitely increased is $\frac{\pi}{2}$.

1031. Prove that, if n be a positive integer,

$$\int_0^1 (1-x)^{n-1}\log\left(\frac{1}{x}\right) dx = \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right),$$

and that

$$\int_0^1\int_0^x (1-y)^{n-2}\log\frac{1}{y}\, dx\, dy = \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right).$$

1032. Prove that

$$(1)\quad \int_0^{\frac{\pi}{4}} \sin 4x \log\cot x\, dx = \frac{1}{2},$$

$$(2)\quad \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{x\cos x}{1+\sin^2 x}\, dx = -\frac{\pi^2}{2},$$

(3) $\displaystyle\int_0^\infty \frac{x^{2n-1}}{(1+x^2)^{n+1}} \log \frac{1+x^2}{x^2}\, dx = \frac{1}{4n^2},$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{-1}(c \sin x)\, dx = c + \frac{c^3}{3^2} + \frac{c^5}{5^2} + \dots \text{ to } \infty, \ (c < 1),$

(5) $\displaystyle\int_0^{\frac{\pi}{2}} x \cot x\, dx = \frac{\pi}{2} \log 2,$

(6) $\displaystyle\int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin x}\right)^2 dx = \pi \log 2,$

(7) $\displaystyle\int_0^\infty \frac{\log x}{a^2 + x^2}\, dx = \frac{\pi}{2a} \log a,$

(8) $\displaystyle\int_0^\infty \log x \log\left(1 + \frac{a^2}{x^2}\right) dx = \pi a (\log a - 1),$

(9) $\displaystyle\int_0^\infty x \sin x\, \epsilon^{-x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\sqrt{\epsilon}}},$

(10) $\displaystyle\int_0^\infty \frac{x}{1+x^2} \log \frac{1+x^2}{x^2}\, dx = \frac{\pi^2}{24},$

(11) $\displaystyle\int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} + x\right) \log \cot x\, dx = \frac{\pi^2}{8},$

(12) $\displaystyle\int_0^1 \frac{\log\left(\frac{1}{x}\right) dx}{\sqrt{(1-x)}} = 4(1 - \log 2),$

(13) $\displaystyle\int_0^\pi \frac{x \sin x\, dx}{\sqrt{(1 - c^2 \sin^2 x)}} = \frac{\pi}{2c} \log \frac{1+c}{1-c}, \quad (c < 1),$

(14) $\displaystyle\int_0^\pi \frac{x}{1 + \sin x}\, dx = \pi.$

1033. Prove that, c being < 1,

$$\int_0^{\frac{\pi}{2}} \sin^{-1}(c \sin x)\, dx = \frac{1}{2\pi} \int_0^{\pi} \left\{\tan^{-1}\left(\frac{2c \cos x}{1-c^2}\right)\right\}^2 dx.$$

1034. On a straight line of length $a+b+c$ are measured at random two distances $a+c$, $b+c$; prove that the mean value of the part common to the two is

$$b+c-\frac{b^2}{3a}, \quad (a>b).$$

1035. A point is taken at random on a given finite straight line of length a, prove that the mean value of the sum of the squares on the two parts of the line is $\frac{2}{3}a^2$; and that the chance of the sum being less than this mean value is $\frac{1}{\sqrt{3}}$.

1036. A triangle is inscribed in a given circle whose radius is a, prove that, if all positions of the angular points be equally probable, the mean value of the perimeter is $\frac{12a}{\pi}$, and the mean value of the radius of the inscribed circle is

$$a\left(\frac{12}{\pi^2}-1\right).$$

1037. If $2a$ be the given perimeter of a triangle, and all values of the sides for which the triangle is possible be equally probable, the mean value of the radius of the circumscribed circle is five times the mean value of the radius of the inscribed circle.

1038. If $2a$ the perimeter, and c the side of a triangle, be given, find the mean value of its area; and prove that the mean value of these mean values, c being equally likely to have any value from 0 to a, is $\frac{\pi a^2}{30}$.

1039. The mean value of the area of all acute angled triangles inscribed in a given circle of radius a is $\frac{3a^2}{\pi}$.

1040. Of all acute angled triangles inscribed in a given circle of radius a, the mean value of the perimeter is $\frac{48a}{\pi^2}$.

1041. The mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{a(3+e^2)}{4}$, $2a$ being the major axis and e the eccentricity.

1042. If a rod of length a be marked at random in two points and divided at those points, the mean value of the sum of the squares on the parts is $\frac{a^2}{2}$: and if the rod be first divided at random into two parts, and the larger part again divided at random, the mean value of the sum of the squares on the three parts is $\frac{35}{72}a^2$.

1043. If α, β, γ be the areal coordinates of a point, the mean value of $\sqrt{\alpha\beta\gamma}$ for all points within the triangle of reference is $\frac{4\pi}{105}$.

1044. In the equation $x^3 - qx + r = 0$, it is known that q and r both lie between $+1$ and -1; assuming all values between these limits to be equally probable, prove that the chance of all the roots of the equation being real is $\frac{2}{15\sqrt{3}}$.

1045. If a given finite straight line be divided at random in two points, the chance that the three parts can be sides of an acute angled triangle is $3\log 2 - 2$.

1046. If a rod be divided at random in two points, and it is an even chance that n times the sum of the squares on the parts is less than the square on the whole line, prove that

$$n = \frac{12\pi}{4\pi + 3\sqrt{3}}.$$

1047. If n points be taken at random on a given finite straight line, the chance that one of the $n+1$ parts into which the rod will be divided shall be greater than half the line is $\frac{n+1}{2^n}$.

1048. If a rod be divided at random into four parts, the chance that one of the parts shall be greater than half the rod is $\frac{1}{2}$; and the chance that three times the sum of the squares on the parts shall be less than the square on the whole line is $\frac{\pi}{6\sqrt{3}}$.

1049. If a given finite straight line of length a be divided at random in two points, the chance that the product of the three parts shall exceed $\frac{a^3}{108}$ is

$$\frac{4}{9}\int_{\frac{4\pi}{9}}^{\frac{8\pi}{9}} \sin\frac{\theta}{2}\sqrt{(1+2\cos 3\theta)}\,d\theta.$$

1050. If a given finite straight line be divided at random in (1) four points, (2) n points, the chance that (1) four times, (2) n times the sum of the squares on the parts shall be less than the square on the whole line is

$$(1)\ \frac{3\pi^2}{100\sqrt{5}},\quad (2)\ \frac{1}{\sqrt{(n+1)}}\left\{\frac{\pi}{n(n+1)}\right\}^{\frac{n}{2}}\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)}.$$

1051. Find the singular solution of the equation

$$x^2+y^2-a^2=2ay\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\frac{a(x-a)}{2};$$

and trace the locus of the points whose coordinates satisfy the singular solution.

1052. Find the differential equation of a curve, such that the foot of the perpendicular from a fixed point on the tangent lies on a fixed circle; and obtain the general integral and singular solution.

1053. Along the normal to a curve at P is measured a constant length PQ; O is a fixed point and the curve is such that the circle described about OPQ has a fixed tangent at O; find the differential equation of the curve, the general integral, and singular solution.

1054. If PM, PG be the ordinate and normal from a point P of a curve to the axis of x, find a curve (1) in which PM^2 varies as PG; (2) in which the curvature varies as $\frac{PM^4}{PG^3}$: and prove that one species of curve satisfies both conditions.

1055. Find the orthogonal trajectory of the circles

$$x^2 + y^2 - 2\lambda y + a^2 = 0,$$

λ being the parameter.

1056. Integrate the equations:

(1) $x\dfrac{dy}{dx} = 2y + x^{n+1}y - x^n y\sqrt{y}$,

(2) $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} + y\sin^2 x = 0$,

(3) $\dfrac{d^2x}{dt^2} + 2x + 2y = \dfrac{d^2y}{dt^2} + x + 3y = \cos nt$,

(4) $2xyz\dfrac{d^2z}{dx\,dy} + 2xy\dfrac{dz}{dx}\dfrac{dz}{dy} - 2x\dfrac{dz}{dx} - 2y\dfrac{dz}{dy} + z^2 = 0$,

(5) $u_{x+1} - 3u_x + 4u_x^2 = 0$.

1057. The general solution of the equation

$$u_x(2x + 1 - u_{x+1}) = x^2,$$

is

$$u_x = x + \left\{C + \Sigma\left(\frac{1}{x}\right)\right\}^{-1}.$$

1058. Find the general solution of the equations

$$f(x+y)=f(x)\,\phi(y)+\phi(x)\,f(y),$$

$$\phi(x+y)=f(x)\,f(y)+\phi(x)\,\phi(y)\,;$$

and hence prove that $2\cos x$, $2\sqrt{-1}\sin x$, can be expressed in the form k^x+k^{-x}, k^x-k^{-x}.

1059. Prove that

$$\log\frac{\epsilon^x-1}{x}=\frac{x}{2}+\frac{x^2}{\lfloor 2}\cdot\frac{B_1}{2}-\frac{x^4}{\lfloor 4}\,\frac{B_3}{4}+\frac{x^6}{\lfloor 6}\cdot\frac{B_5}{6}+\ldots\ldots$$

$$+(-1)^{n-1}\frac{x^{2n}}{\lfloor 2n}\cdot\frac{B_{2n-1}}{2n}+\ldots\ldots$$

SOLID GEOMETRY.

I. *Straight Line and Plane.*

1060. The co-ordinates of four points are $a-b$, $a-c$, $a-d$; $b-c$, $b-d$, $b-a$; $c-d$, $c-a$, $c-b$; $d-a$, $d-b$, $d-c$; respectively: prove that the straight line, joining the middle points of any two opposite edges of the tetrahedron of which they are the angular points, passes through the origin.

1061. Of the three acute angles which any straight line makes with three rectangular axes, any two are together greater than the third.

1062. The straight line joining the points (a, b, c), $(a', b', c',)$ will pass through the origin if

$$aa' + bb' + cc' = \rho\rho',$$

ρ, ρ' being the distances of the points from the origin, and the axes rectangular. Obtain the corresponding equation when the axes are inclined respectively at angles α, β, γ.

1063. From any point P are drawn PM, PN perpendicular to the planes of zx, zy, O is the origin, and α, β, γ, θ the angles which OP makes with the axes (rectangular) and with the plane OMN: prove that

$$\frac{1}{\sin^2\theta} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

1064. The equations of a straight line being given in the form

$$(1)\quad \frac{a+mz-ny}{l} = \frac{b+nx-lz}{m} = \frac{c+ly-mx}{n},$$

$$(2)\quad \frac{a+mz-ny}{\lambda}=\frac{b+nx-lz}{\mu}=\frac{c+ly-mx}{\nu};$$

obtain them in the form

$$\frac{x-x_0}{L}=\frac{y-y_0}{M}=\frac{z-z_0}{N}.$$

1065. A straight line moves parallel to a fixed plane and intersects two fixed straight lines not in the same plane: prove that the locus of a point which divides the part intercepted in a constant ratio is a straight line.

1066. Determine what straight line is represented by the equations

$$\frac{a+mz-ny}{m-n}=\frac{b+nx-lz}{n-l}=\frac{c+ly-mx}{l-m}.$$

Result. The line at ∞ in the plane

$$x(m-n)+y(n-l)+z(l-m)=0;$$

unless $la+mb+nc=0$, when it is indeterminate.

1067. The equations

$$lx+my+nz=0,\ ax^2+by^2+cz^2=0,$$

represent two straight lines, the cosine of the angle between which is

$$\frac{l^2(b+c)+m^2(c+a)+n^2(a+b)}{\sqrt{l^4(b-c)^2+\ldots+\ldots+2m^2n^2(a-b)(a-c)+\ldots+\ldots}}.$$

1068. A straight line moves parallel to the plane $y=z$, and intersects the curves

$$y=0,\ z^2=mx;\ z=0,\ y^2=-mx;$$

prove that the locus of its trace on the plane of yz is two straight lines at right angles to each other.

1069. A straight line always intersects at right angles the straight line

$$x+y=z=0,$$

and also intersects the curve

$$y=0,\ x^2=az:$$

prove that the equation of its locus is

$$x^2-y^2=az.$$

1070. The equations

$$\frac{ax+c'y+b'z}{x}=\frac{c'x+by+a'z}{y}=\frac{b'x+a'y+cz}{z}$$

represent in general three straight lines mutually at right angles; but, if

$$a-\frac{b'c'}{a'}=b-\frac{c'a'}{b'}=c-\frac{a'b'}{c'},$$

they represent a plane and a straight line perpendicular to that plane.

1071. The two straight lines

$$\frac{x\pm a}{0}=\frac{\pm y}{\cos\alpha}=\frac{z}{\sin\alpha},$$

meet the axis of x in O, O'; and P, P' are points on the two such that

(1) $OP=kO'P'$;

(2) $OP\,.\,O'P'=c^2$;

(3) $OP+O'P'=2c$;

prove that the equation of the locus of PP' is

(1) $(x+a)(y\sin\alpha+z\cos\alpha)=k(x-a)(y\sin\alpha-z\cos\alpha)$;

(2) $\dfrac{x^2}{a^2}-\dfrac{y^2}{c^2\cos^2\alpha}+\dfrac{z^2}{c^2\sin^2\alpha}=1$;

and

(3) $\dfrac{xy}{\cos\alpha}-\dfrac{az}{\sin\alpha}=\dfrac{c}{a}(x^2-a^2)$;

the points being taken on the same side of the plane xy.

1072. A triangle is projected orthogonally on each of three planes mutually at right angles: prove that the algebraical sum of the tetrahedrons which have these projections for bases and a common vertex in the plane of the triangle is equal to the tetrahedron which has the triangle for base and the intersection of the plane for vertex.

1073. A plane is drawn through the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}:$$

prove that the two other straight lines in which it meets the surface

$$(b-c)\,yz\,(mz-ny) + (c-a)\,zx\,(nx-lz) + (a-b)\,xy\,(ly-mx) = 0$$

are at right angles to each other.

1074. If (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction cosines of three straight lines which are, two and two, at right angles, and if

$$\frac{a}{l_1} + \frac{b}{m_1} + \frac{c}{n_1} = \frac{a}{l_2} + \frac{b}{m_2} + \frac{c}{n_2} = 0;$$

then will

$$\frac{a}{l_3} + \frac{b}{m_3} + \frac{c}{n_3} = 0; \text{ and } \frac{a}{l_1 l_2 l_3} = \frac{b}{m_1 m_2 m_3} = \frac{c}{n_1 n_2 n_3}.$$

1075. The equations of the straight lines bisecting the angles between the two straight lines given by the equations

$$lx + my + nz = 0, \quad ax^2 + by^2 + cz^2 = 0,$$

are

$$lx + my + nz = 0, \quad lyz\,(b-c) + mzx\,(c-a) + nxy\,(a-b) = 0.$$

1076. The straight lines bisecting the angles between the two lines given by the equations

$$lx + my + nz = 0, \quad ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

lie on the cone

$$x^2\,(c'n - b'm) + \ldots + \ldots + yz\,\{c'm - b'n + (c-b)\,l\} + \ldots + \ldots = 0.$$

1077. If x, y be the lengths of two of the lines joining the middle points of opposite edges of a tetrahedron, ω the angle between these lines, and a, a' those edges of the tetrahedron which are not met by either of the lines,

$$\cos\omega = \frac{a^2 \sim a'^2}{4xy}.$$

1078. The lengths of the three pairs of opposite edges of a tetrahedron are a, a'; b, b'; c, c': prove that, if θ be the acute angle between the directions of a and a',

$$\cos\theta = \frac{(b^2+b'^2) \sim (c^2+c'^2)}{2aa'}.$$

1079. The line joining the centres of the two spheres which touch the faces of the tetrahedron $ABCD$ opposite to A, B respectively, and the other faces produced, will intersect the edge CD in a point P such that $CP : PD :: \Delta ACB : \Delta ADB$; and the edge AB (produced) in a point Q such that

$$AQ : BQ :: \Delta CAD : \Delta CBD.$$

1080. On three straight lines, meeting in a point, are taken points A, a; B, b; C, c respectively: prove that the intersections of the planes ABC, abc; aBC, Abc; AbC, aBc; and ABc, abC all lie on one plane which divides each of the three straight lines harmonically.

1081. If through any point be drawn three straight lines each meeting two opposite edges of a tetrahedron $ABCD$; and if a, α; b, β; c, γ be the points where these straight lines meet the edges BC, AD; CA, BD; AB, CD; then will

$$Ba \cdot C\gamma \cdot D\beta = B\beta \cdot Ca \cdot D\gamma,$$
$$Cb \cdot A\alpha \cdot D\gamma = C\gamma \cdot Ab \cdot D\alpha,$$
$$Ac \cdot B\beta \cdot D\alpha = A\alpha \cdot Bc \cdot D\beta,$$
$$Ab \cdot Bc \cdot Ca = Ac \cdot Ba \cdot Cb.$$

1082. Any point O is joined to the angular points of a tetrahedron $ABCD$, and the joining lines meet the opposite faces in a, b, c, d: prove that

$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}+\frac{Od}{Dd}=1,$$

regard being had to the signs of the segments. Hence prove that the reciprocals of the radii of the eight spheres which can be drawn to touch the faces of the tetrahedron are the eight positive values of the expression

$$\pm\frac{1}{p_1}\pm\frac{1}{p_2}\pm\frac{1}{p_3}\pm\frac{1}{p_4};$$

p_1, p_2, p_3, p_4 being the perpendiculars from the angular points on the opposite faces.

1083. If A, B, C, D be the areas of the faces of a tetrahedron; $a, b, c, \alpha, \beta, \gamma$, the cosines of the dihedral angles (BC), (CA), (AB), (DA), (DB), (DC), respectively; then will

$$\frac{A^2}{1-a^2-b^2-c^2-2abc}=\frac{B^2}{1-a^2-\beta^2-c^2-2a\beta c}=\frac{C^2}{1-a^2-b^2-\gamma^2-2ab\gamma}$$
$$=\frac{D^2}{1-a^2-b^2-c^2-2abc}.$$

1084. With the same notation as in the last question, prove that for all real values of l, m, n, r,

$$l^2+m^2+n^2+r^2>2mna+2nl\beta+2lm\gamma+2lra+2mrb+2nrc;$$

except when

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{r}{D}.$$

1085. Three straight lines are drawn, two and two at right angles, through a given point, and two of them lie respectively in two fixed planes: the locus of the third is a cone of the second degree, whose sections parallel to the fixed planes are circles.

1086. A point O is taken within a tetrahedron $ABCD$ so as to be the centre of gravity of the feet of the perpendiculars let fall from O on the faces: prove that the distances of O from the several faces are proportional respectively to the faces.

1087. The equation of the cone of revolution which can be drawn touching a system of co-ordinate planes is

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0,$$

the ratios $l : m : n$ being given by the equations

$$\frac{m^2 + n^2 - 2mn\cos\alpha}{\sin^2\alpha} = \frac{n^2 + l^2 - 2nl\cos\beta}{\sin^2\beta} = \frac{l^2 + m^2 - 2lm\cos\gamma}{\sin^2\gamma};$$

where α, β, γ are the angles between the axes. (See question 301).

1088. The inscribed sphere of a tetrahedron $ABCD$ touches the faces in A', B', C', D': prove that AA', BB', CC', DD' will meet in a point, if

$$\cos\frac{a}{2}\cos\frac{\alpha}{2} = \cos\frac{b}{2}\cos\frac{\beta}{2} = \cos\frac{c}{2}\cos\frac{\gamma}{2};$$

where a, α; b, β; c, γ are pairs of dihedral angles at opposite edges.

II. *Linear Transformations. General Equation of the Second Degree.*

The following simple method of obtaining the conditions for a surface of revolution is worthy of notice.

When the expression $ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$ is transformed into $AX^2 + BY^2 + CZ^2$, we obtain the coefficients A, B, C from the equivalence of the conditions that

$$h(x^2 + y^2 + z^2) - ax^2 - by^2 - \ldots\ldots\ldots,$$

and $$h(X^2 + Y^2 + Z^2) - AX^2 - BY^2 - CZ^2$$

may separate into (real or impossible) linear factors: which is the case when $h = A$, B, or C.

But if two of the three coincide as $B = C$; then when $h = B$ the two factors become coincident, or either expression is a complete square. The conditions that this may be simultaneously the case in the former expression give us

$$(B-a)\,a' = -b'c', \text{ \&c.,}$$

or $$B = a - \frac{b'c'}{a'} = b - \frac{c'a'}{b'} = c - \frac{a'b'}{c'} \text{ if } a', b', c' \text{ be all finite.}$$

If $a' = 0$, then $b'c'$ must also vanish; suppose then $a', b' = 0$, therefore $B = c$, and we must have

$$(c-a)\,x^2 + (c-b)\,y^2 - 2c'xy$$

a perfect square, whence

$$c'^2 = (c-a)\,(c-b).$$

In the case of oblique axes, inclined at angles α, β, γ, we must have

$$h\,(x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma) - ax^2 - \ldots - 2a'yz - \ldots$$

a complete square.

It follows that the three equations

$$(h-a)\,(h\cos\alpha - a') = (h\cos\beta - b')\,(h\cos\gamma - c'),$$
$$(h-b)\,(h\cos\beta - b') = (h\cos\gamma - c')\,(h\cos\alpha - a'),$$
$$(h-c)\,(h\cos\gamma - c') = (h\cos\alpha - a')\,(h\cos\beta - b'),$$

must be simultaneously true, and the two necessary conditions may be found by eliminating h.

1089. If there be two systems of rectangular co-ordinates, and θ_1, θ_2, θ_3 be the angles made by the axes of x', y', z' with that of z, and ϕ_1, ϕ_2, ϕ_3 the angles made by the planes of zx', zy', zz' with that of zx; then will

$$\tan^2\theta_1 + \frac{\cos(\phi_2 - \phi_3)}{\cos(\phi_3 - \phi_1)\cos(\phi_1 - \phi_2)} = 0,$$

with two similar equations.

1090. By transformation of co-ordinates, prove that the equation

$$x^2 + y^2 + z^2 + yz + zx + xy = a^2$$

represents an oblate spheroid whose polar axis is to its equatoreal in the ratio 1 : 2; and the equations of whose polar axis are $x = y = z$.

1091. If a cone of the second degree touch one system of three planes, which are two and two at right angles, it will touch an infinite number of such systems: and if one system be co-ordinate planes, and (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction cosines of another; the equation of the cone will be

$$(l_1 l_2 l_3 x)^{\frac{1}{2}} + (m_1 m_2 m_3 y)^{\frac{1}{2}} + (n_1 n_2 n_3 z)^{\frac{1}{2}} = 0.$$

1092. Prove that the surface whose equation, referred to axes inclined each to each at an angle of 60°, is

$$yz + zx + xy + a^2 = 0,$$

is cut by the plane $x + y + z = 0$ in a circle whose radius is a.

1093. In the expression

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x + 2b''y + 2c''z + d;$$

prove that

$$(b + c)\, a''^2 + \ldots + \ldots - a'b''c'' - b'c''a'' - c'a''b'',$$

and $$a''^2(bc - a'^2) + \ldots + \ldots + 2b''c''(b'c' - aa') + \ldots + \ldots$$

are invariants for all systems of rectangular co-ordinates having the same origin.

1094. Prove also that the coefficients in the following equation in h

$$\begin{vmatrix} h + a, & h\cos\gamma + c', & h\cos\beta + b', & a'' \\ h\cos\gamma + c', & h + b, & h\cos\alpha + a', & b'' \\ h\cos\beta + b', & h\cos\alpha + a', & h + c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix} = 0,$$

α, β, γ being the angles between the axes, are invariants for all systems of co-ordinates having the same origin.

1095. Assuming the formulæ for transforming from a system of co-ordinate axes inclined at angles α, β, γ to another inclined at angles α', β', γ' to be

$$x = l_1X + m_1Y + n_1Z, \quad y = l_2X + m_2Y + n_2Z, \quad z = l_3X + m_3Y + n_3Z;$$

prove that

$$1 = l_1^2 + l_2^2 + l_3^2 + 2l_2l_3 \cos\alpha + 2l_3l_1 \cos\beta + 2l_1l_2 \cos\gamma;$$

with similar equations in m and n; and that

$$\cos\alpha' = m_1n_1 + m_2n_2 + m_3n_3$$
$$+ (m_2n_3 + m_3n_2)\cos\alpha + (m_3n_1 + m_1n_3)\cos\beta + (m_1n_2 + m_2n_1)\cos\gamma;$$

with similar equations in n, l; and l, m.

1096. If $ax^2 + by^2 + cz^2$ become $AX^2 + BY^2 + CZ^2$ by any transformation of co-ordinates, the positive and negative coefficients will be in like number in the two expressions.

1097. The equation

$$ax^2 + \dots + 2a'yz + \dots + 2a''x + \dots + d = 0$$

will in general represent a paraboloid of revolution, if

$$\frac{a}{a'} + \frac{b'}{c'} + \frac{c'}{b'} = \frac{b}{b'} + \frac{c'}{a'} + \frac{a'}{c'} = \frac{c}{c'} + \frac{a'}{b'} + \frac{b'}{a'} = 0;$$

and a cylinder of revolution if, in addition to these conditions,

$$\frac{a''}{a'} + \frac{b''}{b'} + \frac{c''}{c'} = 0.$$

1098. The surface whose equation, referred to axes inclined at angles α, β, γ, is $ax^2 + by^2 + cz^2 = 1$, will be one of revolution if

$$\frac{a\cos\alpha}{\cos\alpha - \cos\beta\cos\gamma} = \frac{b\cos\beta}{\cos\beta - \cos\gamma\cos\alpha} = \frac{c\cos\gamma}{\cos\gamma - \cos\alpha\cos\beta}.$$

1099. The surface whose equation, referred to axes inclined at angles α, β, γ, is $ayz + bzx + cxy = 1$, will be one of revolution if

$$\frac{a}{1 \pm \cos\alpha} = \frac{b}{1 \pm \cos\beta} = \frac{c}{1 \pm \cos\gamma};$$

one, or three, of the ambiguities being taken negative.

1100. Prove that the equation of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

can be obtained in the form $x^2 + y^2 - z^2 = d^2$ in an infinite number of ways, provided that either a^2 or b^2 is greater than c^2; and that the new axes of x, y lie on the cone

$$\frac{x^2}{a^2}(b^2 - c^2) + \frac{y^2}{b^2}(a^2 - c^2) - \frac{z^2}{c^2}(a^2 + b^2) = 0.$$

1101. The equation of a given hyperboloid may be obtained in the form

$$ayz + bzx + cxy = 1$$

in an infinite number of ways; and, if α, β, γ be the angles between the axes in any such case, the expression

$$\frac{abc}{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}$$

will be constant.

1102. Prove that the only conoid of the second degree is a hyperbolic paraboloid; and that it will be a right conoid, if the two principal sections be equal parabolas.

1103. The equation

$$ax^2 + \ldots + \ldots + 2a'yz + \ldots + \ldots = 0$$

will represent a cone of revolution, if

$$\frac{b'c'}{a'} + \frac{b'^2 - c'^2}{b - c} = \frac{c'a'}{b'} + \frac{c'^2 - a'^2}{c - a} = 0.$$

1104. The radius r of the central circular sections of the surface $ayz + bzx + cxy = 1$ is given by the equation

$$abcr^6 + (a^2 + b^2 + c^2)\, r^4 = 4;$$

and the direction-cosines of the sections by the equations

$$\frac{l(m^2 + n^2)}{a} = \frac{m(n^2 + l^2)}{b} = \frac{n(l^2 + m^2)}{c} = -lmnr^2.$$

1105. If a cone be described having a plane section of a given sphere for base, and vertex at a point V on the sphere, the subcontrary sections will be parallel to the tangent plane at V.

1106. If a cone whose vertex is the origin and base a plane section of the surface $ax^2 + by^2 + cz^2 = 1$ be a cone of revolution, the plane must touch one of the cylinders

$$(b-a)\,y^2 + (c-a)\,z^2 = 1, \quad (c-b)\,z^2 + (a-b)\,x^2 = 1,$$
$$(a-c)\,x^2 + (b-c)\,y^2 = 1.$$

1107. A cone is described whose base is a given conic and one of whose axes passes through a fixed point in the plane of the conic: prove that the locus of the vertex is a circle.

1108. If the locus of the feet of the perpendiculars let fall from a fixed point on the tangent planes to the cone

$$ax^2 + by^2 + cz^2 = 0$$

be a plane curve, it will be a circle; and in order that this may be the case, the point must lie on one of the systems of straight lines (of which only one is possible)

$$x = 0, \quad \frac{by^2}{b-a} + \frac{cz^2}{c-a} = 0; \ \&c.$$

1109. Prove also that, if the point lie on one of these straight lines, the plane of the circle will be perpendicular to the other: and that a plane section of the cone perpendicular to one of the straight lines will have one of its foci on that straight line and its eccentricity equal to $\sqrt{\left\{\frac{(b-a)(c-a)}{a^2}\right\}}$.

1110. If a plane cut the cone $ayz + bzx + cxy = 0$ in two straight lines at right angles to each other, the normal to the plane through the origin will also lie on the cone.

1111. Prove that, when $bb' = c'a'$, and $cc' = a'b'$, the equation

$$ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + f = 0$$

represents in general a paraboloid whose axis is parallel to the straight line

$$x = 0, \quad c'y + b'z = 0.$$

1112. Prove that the locus of tangent lines, drawn from the origin to the surface

$$u \equiv ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + f = 0,$$

is
$$fu - (a''x + b''y + c''z + f)^2 = 0;$$

and investigate the condition that the surface may be a cone from the consideration that this locus will then become two planes.

1113. The section of the surface $yz + zx + xy = a^2$ by the plane $lx + my + nz = p$ will be a parabola if

$$l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0;$$

and that of the surface

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = a^2$$

will be a parabola if

$$mn + nl + lm = 0.$$

1114. The semiaxes of a central section of the surface

$$ayz + bzx + cxy + abc = 0$$

made by a plane whose direction-cosines are l, m, n, are given by the equation

$$r^4 (2bcmn + \ldots - a^2l^2 - \ldots) - 4abcr^2 (amn + \ldots) + 4a^2b^2c^2 = 0.$$

1115. Prove that the section of the surface

$$ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + d = 0$$

by the plane $lx + my + nz = 0$ will be a rectangular hyperbola, if

$$l^2 (b + c) + m^2 (c + a) + n^2 (a + b) = 2a'mn + 2b'nl + 2c'lm;$$

and a parabola, if

$$l^2 (bc - a'^2) + \ldots + \ldots + 2mn (b'c' - aa') + \ldots + \ldots = 0.$$

Explain why this last equation becomes identical if

$$b'c' = aa', \quad c'a' = bb', \quad \text{and} \quad a'b' = cc'.$$

1116. The generators drawn through the point (x, y, z) of the surface

$$ayz + bzx + cxy + abc = 0$$

will be at right angles, if

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

1117. Normals are drawn to a conicoid at points lying along a generator: prove that they will lie on a hyperbolic paraboloid whose principal sections are equal parabolas.

1118. The two conicoids

$$(A + b^2 + c^2)\,x^2 + (B + c^2 + a^2)\,y^2 + (C + a^2 + b^2)\,z^2 \\ - 2bcyz - 2cazx - 2abxy = \epsilon^4,$$

$$\left(\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C}\right)\left(\frac{a^2}{A} + \frac{b^2}{B} + \frac{c^2}{C} - 1\right) - \left(\frac{ax}{A} + \frac{by}{B} + \frac{cz}{C}\right)^2 = 1,$$

have their axes coincident in direction.

1119. The two conicoids

$$ax^2 + \ldots + 2a'yz + \ldots = 1,$$

$$Ax^2 + By^2 + Cz^2 = 1,$$

have one, and in general only one, system of conjugate diameters coincident in direction; but, if

$$\frac{1}{A}\left(a - \frac{b'c'}{a'}\right) = \frac{1}{B}\left(b - \frac{c'a'}{b'}\right) = \frac{1}{C}\left(c - \frac{a'b'}{c'}\right),$$

there will be an infinite number of such systems, the direction of one of the diameters being the same in all.

1120. Prove that eight conicoids can in general be drawn, containing a given conic and touching four given planes.

1121. A, B is the shortest distance between two generators, of the same system, of a conicoid; and any opposite generator meets them in P, Q respectively: prove that the lengths x, y of AP, BQ are connected by a constant relation of the form

$$axy + bx + cy + d = 0.$$

1122. A, B; P, Q are the points where two fixed generators are met by two of the opposite system; if A, B be fixed, the lengths x, y of AP, BQ will be connected by a constant relation of the form

$$axy + bx + cy = 0.$$

1123. A hyperboloid of revolution is drawn containing two given straight lines which do not intersect: prove that the locus of its axis is a hyperbolic paraboloid, and that its centre lies on one of the generating lines through the vertex of this paraboloid.

III. *Conicoids referred to their axes.*

1124. The curve traced out on the surface $\frac{y^2}{b} + \frac{z^2}{c} = x$ by the extremities of the latera recta of sections made by planes through the axis of x lies on the cone

$$y^2 + z^2 = 4x^2.$$

1125. The locus of the middle points of all straight lines passing through a fixed point and terminated by two fixed planes is a hyperbolic cylinder.

1126. An ellipsoid and hyperboloid are concentric and confocal: prove that a tangent plane to the asymptotic cone of the hyperboloid will cut the ellipsoid in a plane of constant area.

1127. The locus of the centres of all plane sections of a given conicoid drawn through a given point is a similar and similarly situated conicoid, on which the given point and the centre of the given surface are extremities of a diameter.

1128. An ellipse and a circle have a common diameter, and on any chord of the ellipse parallel to this diameter is described a circle in a plane parallel to that of the given circle: prove that the locus of these circles is an ellipsoid.

1129. An ellipsoid is generated by the motion of a point fixed in a certain straight line, which moves so that three other fixed points on it lie in the co-ordinate planes: prove that there are four such systems of points; and that if the corresponding straight lines be drawn through any point on the ellipsoid, the angle between any two is equal to that between the other two.

1130. Of two equal circles, one is fixed and the other moves parallel to a given plane and intersects the former in two points: prove that the locus of the moving circle is two elliptic cylinders.

1131. At each point of a generating line of a conicoid is drawn a straight line touching the conicoid and at right angles to the generating line: prove that the locus of such straight lines is a hyperbolic paraboloid whose principal sections are equal parabolas.

1132. The locus of the axes of sections of the surface

$$ax^2 + by^2 + cz^2 = 1,$$

which contain the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

is the cone

$$(b-c)\,yz\,(mz-ny) + (c-a)\,zx\,(nx-lz) + (a-b)\,xy\,(ly-mx) = 0.$$

1133. The three acute angles made by any system of equal conjugate diameters of an ellipsoid will be together equal to two right angles, if

$$2\,(2a^2-b^2-c^2)(2b^2-c^2-a^2)(2c^2-a^2-b^2) = 27a^2b^2c^2;$$

$2a$, $2b$, $2c$ being the axes.

1134. From different points of the straight line

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{0},$$

asymptotic straight lines are drawn to the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1:$$

prove that they will lie on the two planes

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2=\frac{2z^2}{c^2}.$$

1135. The asymptotes of sections of the conicoid

$$ax^2+by^2+cz^2=1,$$

made by planes parallel to

$$lx+my+nz=0$$

lie on the two planes

$$(l^2bc+m^2ca+n^2ab)(ax^2+by^2+cz^2)=abc(lx+my+nz)^2.$$

1136. The locus of points, from which rectilinear asymptotes can be drawn to the conicoid

$$ax^2+by^2+cz^2=1$$

at right angles to each other, is the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

1137. A sphere is described, having for a great circle a plane section of a given conicoid: prove that the plane of the circle in which it again meets the conicoid intersects the plane of the former circle in a straight line which lies in one of two fixed planes.

1138. In the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2-z^2}{b^2}=1$, $(a>b)$, the spheres, of which one series of circular sections of the hyperboloid are great circles, will have a common radical plane.

1139. The plane containing two parallel generators of a conicoid will pass through the centre. Two generators of the paraboloid

$$\frac{x^2}{a}-\frac{y^2}{b}=4z$$

are drawn through the point $(X, 0, Z)$: prove that the angle between them is

$$\cos^{-1}\left(\frac{a-b+Z}{a+b+Z}\right).$$

1140. The perpendiculars let fall from the vertex of a hyperbolic paraboloid on the generators will lie on two cones of the second degree, whose circular sections are parallel to the principal parabolic sections of the paraboloid.

1141. If A, A' be one of the real axes of a hyperboloid of one sheet, and P, P' the points where any generator meets the generators of the opposite system through A, A' respectively; the rectangle AP, $A'P'$ will be constant.

1142. In a hyperboloid of revolution of one sheet, the shortest distance between two generators of the same system is not greater than the diameter of the principal circular section.

1143. The equation of the cone generated by straight lines, drawn through the origin parallel to normals to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at points where it is met by the confocal surface

$$\frac{x^2}{a^2-d^2}+\ldots=1,$$

is
$$\frac{a^2x^2}{a^2-d^2}+\frac{b^2y^2}{b^2-d^2}+\frac{c^2z^2}{c^2-d^2}=0.$$

1144. The points on a conicoid, the normals at which intersect the normal at a given point, all lie on a cone of the second degree having its vertex at the given point.

1145. Straight lines are drawn in a given direction, and the tangent planes drawn through each straight line to a given conicoid are at right angles to each other: prove that the locus of such straight lines is a cylinder of revolution, or a plane.

1146. A cone is described having for base the section of the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

made by the plane

$$lx + my + nz = 0,$$

and intersects the conicoid in a second plane perpendicular to the given plane: prove that the vertex must lie on the surface

$$(l^2 + m^2 + n^2)(ax^2 + by^2 + cz^2 - 1) = 2(lx + my + nz)(alx + bmy + cnz).$$

1147. The six normals drawn to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

from the point (x_0, y_0, z_0) all lie on the cone

$$(b^2 - c^2)\frac{x_0}{x - x_0} + (c^2 - a^2)\frac{y_0}{y - y_0} + (a^2 - b^2)\frac{z_0}{z - z_0} = 0.$$

1148. The six normals drawn to the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

from any point on one of the lines

$$a(b - c)x = \pm b(c - a)y = \pm c(a - b)z$$

will lie on a cone of revolution.

1149. A section of the conicoid $ax^2 + by^2 + cz^2 = 1$ is made by a plane parallel to the axis of z, and the trace of the plane on xy is normal to the ellipse

$$\frac{ax^2}{(a - c)^2} + \frac{by^2}{(b - c)^2} = \frac{c^2}{(c^2 - ab)^2}:$$

prove that the normals to the ellipsoid at points in this plane will all intersect the same straight line.

1150. If the normals to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at points on the plane

$$l\frac{x}{a} + m\frac{y}{b} + n\frac{z}{c} = 1$$

all intersect the same straight line, the normals at points on the plane

$$\frac{x}{al}+\frac{y}{bm}+\frac{z}{cn}+1=0$$

will also intersect that line. Prove also that the condition for this is

$$(m^2n^2-l^2)(b^2-c^2)^2+(n^2l^2-m^2)(c^2-a^2)^2+(l^2m^2-n^2)(a^2-b^2)^2=0.$$

If $l=m=n=1$, the straight line which the normals intersect is

$$ax(b^2-c^2)=by(c^2-a^2)=cz(a^2-b^2).$$

1151. The normals to the paraboloid

$$\frac{y^2}{b}+\frac{z^2}{c}=2x$$

at points on the plane $px+qy+rz=1$ will all meet one straight line if

$$p^3(b-c)+2p(q^2b-r^2c)=\frac{2(q^2b+r^2c)}{b-c}.$$

1152. PQ is a chord of a conicoid, normal at P; any plane conjugate to PQ meets the conicoid in a curve A: prove that one axis of the cone whose vertex is P, and base the curve A, is the normal at P, and that the other axes are parallel to the axes of any section parallel to the tangent plane at P.

1153. Straight lines are drawn through the point (x_0, y_0, z_0), such that their conjugates with respect to the paraboloid

$$\frac{y^2}{a}+\frac{z^2}{b}=2x$$

are perpendicular to them respectively: prove that these straight lines must lie on the cone

$$\frac{y_0}{y-y_0}-\frac{z_0}{z-z_0}+\frac{a-b}{x-x_0}=0;$$

and that their conjugates will envelope the parabola

$$\frac{yy_0}{a}+\frac{zz_0}{b}=x+x_0; \quad \left(-\frac{yy_0}{a}\right)^{\frac{1}{2}}+\left(\frac{zz_0}{b}\right)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}=0.$$

1154. If a straight line be perpendicular to its conjugate with respect to any conicoid, it will be perpendicular to its conjugate with respect to any conicoid confocal with the former.

1155. Any generator of the surface $y^2+z^2-x^2=m$ will be perpendicular to its conjugate with respect to the surface

$$ax^2+\ldots+2a'yz+\ldots=1,$$

if $$bc-a'^2=ca-b'^2=ab-c'^2; \text{ and } aa'=b'c'.$$

1156. In the two conicoids

$$ax^2+by^2+cz^2=1, \qquad Ax^2+By^2+Cz^2=1,$$

eight generators of the first will be perpendicular respectively to their conjugates with respect to the second.

1157. O is a fixed point, P a point such that the polar planes of O, P with respect to a given conicoid are perpendicular to each other; prove that the locus of P is the plane bisecting chords perpendicular to the polar plane of O.

1158. A hyperbolic paraboloid, whose principal sections are equal, is drawn through two given non-intersecting straight lines; prove that the locus of its vertex is a straight line.

1159. If two conicoids have two common generators of the same system, they will also have two common generators of the opposite system.

1160. If two given straight lines be generators of the same system of a conicoid, the polar plane of any given point, with respect to the conicoid, will pass through a fixed point.

1161. If two conicoids touch each other in three points, they will touch each other in an infinite number; or will have four common generators.

IV. *Tetrahedral Coordinates.*

1162. When the opposite edges of the tetrahedron $ABCD$ are, two and two, at right angles, the three shortest distances between the opposite edges meet in the point

$$\alpha(AB^2 + AC^2 + AD^2 - k) = \beta(BC^2 + BD^2 + BA^2 - k) = \ldots = \ldots,$$

k being the sum of the squares on any pair of opposite edges.

1163. Determine the condition that the straight line

$$\frac{\alpha}{\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{\nu}$$

may touch the conicoid

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta + l'\alpha\delta + m'\beta\delta + n'\gamma\delta = 0;$$

and thence prove that the equation of the tangent plane, at the point $\alpha = \beta = \gamma = 0$, is

$$l'\alpha + m'\beta + n'\gamma = 0.$$

1164. Any conicoid which touches seven of the planes

$$\pm l\alpha \pm m\beta \pm n\gamma \pm r\delta = 0$$

will touch the eighth; and its centre will lie on the plane

$$l^2\alpha + m^2\beta + n^2\gamma + r^2\delta = 0.$$

Prove that this plane bisects the part of each edge of the fundamental tetrahedron which is intercepted by the given planes.

1165. If a hyperbolic paraboloid be drawn containing the sides AB, BC, CD, DA of a quadrilateral which is not plane, and P be any point on this surface,

$$\text{vol. } PBCD \times \text{vol. } PDAB = \text{vol. } PDAC \times \text{vol. } PABC.$$

Also, if any tangent plane meet AB, CD in P, Q respectively,

$$AP : BP :: DQ : CQ.$$

1166. The locus of the centres of all conicoids which have four common generators, two of each system, is a straight line.

1167. Perpendiculars are let fall from the point $(\alpha, \beta, \gamma, \delta)$ on the faces of the fundamental tetrahedron, and the feet of these perpendiculars lie in one plane; prove that

$$\frac{1}{p_1^2\alpha} + \frac{1}{p_2^2\beta} + \frac{1}{p_3^2\gamma} + \frac{1}{p_4^2\delta} = 0,$$

p_1, p_2, p_3, p_4 being the perpendiculars of the tetrahedron.

1168. If a tetrahedron be self-conjugate to a given sphere, any two opposite edges will be at right angles to each other, and all the plane angles at one of the solid angles will be obtuse.

1169. If the opposite edges of a tetrahedron be, two and two, at right angles to each other; the circumscribed sphere, the sphere bisecting the edges, and the sphere to which the tetrahedron is self-conjugate will have a common radical plane.

1170. A tetrahedron is such that a sphere can be described touching its six edges; prove that any two of the four tangent cones drawn to this sphere from the angular points will have a common tangent plane and a common plane section; and the planes of these common sections will all six meet in a point.

1171. A tetrahedron is such that the straight lines joining its angular points to the points of contact of the inscribed sphere with the respectively opposite faces meet in a point; prove that, at any point of contact, the sides of the face on which the point lies subtend equal angles.

1172. If a conicoid circumscribe a tetrahedron $ABCD$, and the tangent planes at A, B, C, D form a tetrahedron $A'B'C'D'$; then if AA', BB' intersect, CC', DD' will also intersect.

1173. Four points are taken on a conicoid; prove that, if the straight line joining one of the points to the pole of the plane through the other three pass through the centre, the tangent plane at that point is parallel to the plane through the three points.

1174. The equation of a conicoid is

$$mn\beta\gamma + nl\gamma\alpha + lm\alpha\beta + lr\alpha\delta + \ldots + \ldots = 0;$$

prove that it can never be a ruled surface, and that it will be a paraboloid if

$$\Sigma\left(\frac{1}{l^2}\right) = \Sigma\left(\frac{1}{lm}\right).$$

1175. The surface

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta + l'\alpha\delta + m'\beta\delta + n'\gamma\delta = 0$$

will be a cylinder, if

$$ll'(m+n-l) + mm'(n+l-m) + nn'(l+m-n) = 2lmn,$$

and

$$ll'(m'+n'-l) + mm'(n'+l-m') + nn'(l+m'-n') = 2lm'n'.$$

1176. If l, m, n, r be respectively the rectangles of segments of chords drawn from four points A, B, C, D (not in one plane) to meet a certain sphere, and ρ be the radius of the sphere; then will

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & AB^2, & AC^2, & AD^2, & l+\rho^2 \\ 1, & BA^2, & 0, & BC^2, & BD^2, & m+\rho^2 \\ 1, & CA^2, & CB^2, & 0, & CD^2, & n+\rho^2 \\ 1, & DA^2, & DB^2, & DC^2, & 0, & r+\rho^2 \\ 1, & l+\rho^2, & m+\rho^2, & n+\rho^2, & r+\rho^2, & 0 \end{vmatrix} = 0.$$

1177. The perpendiculars u, x, y, z let fall from the angular points of a given finite tetrahedron on a plane are connected by the equation

$$pu^2 + qx^2 + ry^2 + sz^2 + 2lxy + 2myu + 2nux + 2l'uz + 2m'xz + 2n'yz = 0;$$

prove that the envelope is a conicoid, which degenerates into a plane curve if

$$\begin{vmatrix} p, & n, & m, & l' \\ n, & q, & l, & m' \\ m, & l, & r, & n' \\ l', & m', & n', & s \end{vmatrix} = 0.$$

V. *Focal Curves, Reciprocal Polars.*

1178. The equations of the focal lines of the cone

$$ayz + bzx + cxy = 0,$$

are $$\frac{(cy+bz)^2}{y^2+z^2} = \frac{(az+cx)^2}{z^2+x^2} = \frac{(bx+ay)^2}{x^2+y^2}.$$

1179. A parallelogram of minimum area is circumscribed about the focal ellipse of a given ellipsoid, and from its angular points taken in order are let fall perpendiculars p_1, p_2, p_3, p_4 on any tangent plane to the ellipsoid; prove that

$$p_1p_3 + p_2p_4 = 2c^2,$$

$2c$ being the length of that axis which is perpendicular to the plane of the focal ellipse.

1180. If ϖ_1, ϖ_2, ϖ_3, ϖ_4 be the perpendiculars on any tangent plane from the extremities of two conjugate diameters of the focal ellipse, and p the perpendicular from the centre

$$\varpi_1\varpi_3 + \varpi_2\varpi_4 = p^2 + c^2.$$

1181. With any two points of the focal ellipse as foci, can be described a prolate spheroid touching an ellipsoid along a plane curve; provided the tangents to the focal at the two points intersect without the ellipse.

1182. The four straight lines drawn in a given direction, and intersecting both focal curves of an ellipsoid, lie upon a cylinder of revolution whose radius is $\sqrt{(a^2-p^2)}$; a being the semi major axis, and p the perpendicular from the centre on the tangent plane normal to the given direction.

1183. If with a given point as vertex, a cone of revolution be described whose base is a plane section of a given conicoid; this base will touch a fixed cone whose vertex lies on one of the axes of the enveloping cone drawn from the given point.

1184. The straight line joining the points of contact of a common tangent plane to the two conicoids,

$$ax^2+by^2+cz^2=1, \quad (a-k)\,x^2+(b-k)\,y^2+(c-k)\,z^2=1$$

subtends a right angle at the centre.

1185. Through a given point can in general be drawn two straight lines, either of which is a focal line of any cone, enveloping a given conicoid, and having its vertex on the straight line. If two enveloping cones be drawn with their vertices one on each of these straight lines, a prolate conicoid of revolution can be inscribed in them, having its focus at the given point.

1186. If any point O be taken on the umbilical focal conic of a conicoid, there exist two fixed points L, such that if any plane A be drawn through L and a be its pole, Oa is at right angles to the plane through O and the line of intersection of A with the polar of L.

1187. With a given point as vertex there can in general be drawn one tetrahedron self-conjugate to a given conicoid, and such that the edges meeting in the point are two and two at right angles; but if the given point lie on a focal curve, an infinite number.

VI. *General Functional and Differential Equations.*

1188. A surface is generated by a straight line which always intersects two fixed straight lines

$$y = mx, \ z = c; \ y = -mx, \ z = -c:$$

prove that the equation of the surface generated is of the form

$$\frac{mcx - yz}{c^2 - z^2} = f\left(\frac{mzx - cy}{c^2 - z^2}\right).$$

1189. The general functional equation of surfaces generated by a straight line, intersecting the axis of z, and meeting the plane of xy in the circle $x^2 + y^2 = a^2$, is

$$x^2 + y^2 = \left\{a + zf\left(\frac{y}{x}\right)\right\}^2;$$

and the general differential equation is

$$(x^2 + y^2)(px + qy - z)^2 = a^2(px + qy)^2.$$

1190. The general functional equation of all surfaces, generated by a straight line which always intersects the axis of z, is

$$z = f\left(\frac{y}{x}\right) + x\phi\left(\frac{y}{x}\right);$$

and the differential equation is

$$rx^2 + 2sxy + ty^2 = 0.$$

1191. The differential equation of a family of surfaces, such that the perpendicular from the origin on the normal always lies in the plane of xy, is

$$z(p^2 + q^2) + (px + qy) = 0.$$

1192. The differential equation of a family of surfaces, generated by a straight line which is always parallel to the plane of xy and whose intercept between the planes of zx, yz is constant and equal to c, is

$$(px + qy)^2(p^2 + q^2) = c^2p^2q^2.$$

1193. The general differential equations of surfaces, generated by a straight line (1) always parallel to the plane $lx + my + xz = 0$; (2) always intersecting the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

are respectively,

(1) $(m + nq)^2 r - 2(m + nq)(l + np)s + (l + np)^2 t = 0$;

(2) $$(ly - mx)^2(q^2 r - 2pqs + p^2 t)$$
$$+ 2(ly - mx)(nx - lz)(qr - ps)$$
$$+ 2(ly - mx)(ny - mz)(qs - pt)$$
$$+ (nx - lz)^2 r + 2(nx - lz)(ny - mz)s + (ny - mz)^2 t = 0.$$

VII. *Envelopes.*

1194. The envelope of the plane $lx + my + nz = a$; l, m, n, being parameters connected by the relations

$$l + m + n = 0, \quad l^2 + m^2 + n^2 = 1$$

is the cylinder

$$(y - z)^2 + (z - x)^2 + (x - y)^2 = 3a^2.$$

1195. Find the envelope of the planes

(1) $$\frac{x}{a}\cos(\theta + \phi) + \frac{y}{b}\cos(\theta - \phi) + \frac{z}{c}\sin(\theta + \phi) = \sin(\theta - \phi),$$

(2) $$\frac{x}{a}\cos(\theta - \phi) + \frac{y}{b}(\cos\theta + \cos\phi) + \frac{z}{c}(\sin\theta + \sin\phi) = 1,$$

both when θ, ϕ are parameters, and when θ only is a parameter.

1196. The envelope of the plane

$$\frac{x}{\sin\theta\cos\phi} + \frac{y}{\sin\theta\sin\phi} + \frac{z}{\cos\theta} = a$$

is the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

1197. The envelope of all paraboloids, to which a given tetrahedron is self-conjugate, is the planes each of which bisects three edges of the tetrahedron.

1198. A prolate spheroid can be described having two opposite umbilici of an ellipsoid as foci and touching the ellipsoid along a plane curve: and this spheroid will be the envelope of a series of spheres, having one system of circular sections of the ellipsoid as great circles.

1199. Spheres are described on a series of parallel chords of an ellipsoid as diameters: prove that they will have double contact with another ellipsoid; and that the focal ellipse of the latter will be the diametral section of the former conjugate to the chords. Also, if a, b, c be the axes of the former, and α, β, γ of the latter,

$$a^2 + b^2 + c^2 = \alpha^2 + \beta^2 - \gamma^2,$$

γ being that axis which is perpendicular to the plane bisecting the chords.

1200. The envelope of a sphere, intersecting a given conicoid in two planes and passing through the centre, is a surface of the fourth degree, touching the conicoid along a spherical conic.

VIII. *Curvature.*

1201. If from any point of a curve equal small lengths δs be measured in the same direction along the curve, and along the circle of absolute curvature, respectively; the distance between the extremities of these lengths is ultimately

$$\frac{\delta s^3}{\sigma\rho}\sqrt{\left\{\frac{1}{\rho^2} + \frac{1}{\sigma^2}\left(\frac{d\rho}{ds}\right)^2\right\}},$$

ρ, σ being the radii of curvature and torsion respectively at the point.

1202. Two surfaces have complete contact of the n^{th} order at a point: prove that there are $n+1$ directions of normal sections for which the curves of section will have contact of the $n+1^{\text{th}}$ order: and hence prove that two conicoids which have double contact with each other will intersect in plane curves.

1203. Prove that it is in general possible to determine a paraboloid whose principal sections shall be equal parabolas, and which shall have a complete contact of the second order with a given surface at a given point.

1204. Prove that a paraboloid can in general be drawn having a complete contact of the second order with a given surface at a given point, and such that all normal sections through the point have contact of the third order.

1205. A skew surface is capable of generation in two ways by straight lines; at any point of it the absolute magnitudes of the principal radii of curvature are a, b: prove that the angle between the generators which intersect in the point is

$$\cos^{-1}\frac{a \sim b}{a+b}.$$

1206. The points on the surface

$$xyz = a\,(yz + zx + xy),$$

at which the principal radii of curvature are equal and opposite, lie on the cone

$$x^4\,(y+z) + y^4\,(z+x) + z^4\,(x+y) = 0:$$

and on the surface

$$xyz = a^2\,(x+y+z)$$

all such points lie on the cone

$$x^2(y+z) + y^2(z+x) + z^2(x+y) = 0.$$

1207. A surface is generated by a straight line intersecting the two straight lines

$$\left.\begin{aligned} y &= x\tan a, \\ z &= c, \end{aligned}\right\} \qquad \left.\begin{aligned} y &= -x\tan a, \\ z &= -c, \end{aligned}\right\}$$

and λ, μ are the distances of the points where the generator meets these straight lines from the points where the axis of z meets them: prove that the principal radii of curvature at a point on the first straight line are given by the equation

$$c^2\rho^2\sin^2 2a - 2c\rho\sin 2a\frac{d\lambda}{d\mu}(\lambda - \mu\cos 2a)(4c^2 + \mu^2\sin^2 2a)^{\frac{1}{2}}.$$
$$= \left(\frac{d\lambda}{d\mu}\right)^2 (4c^2 + \mu^2\sin^2 2a)^2.$$

1208. A surface is generated by the motion of a variable circle, which always intersects the axis of x and is parallel to the plane of yz. If, at a point on the axis of x, r be the radius of the circle and θ the angle which the diameter through the point makes with the axis of z, the principal radii of curvature at the point are given by the equation

$$\rho^2 r + \left(\frac{dx}{d\theta}\right)^2 (\rho - r) = 0.$$

1209. A surface is generated by a straight line which always intersects a given circle and the straight line through the centre perpendicular to the plane of the circle; θ is the angle which the generator makes with the fixed straight line, and ϕ the angle which the projection of the generator on the plane of the circle makes with a fixed radius: prove that the principal radii of curvature at the point where the generator meets the fixed straight line are

$$\frac{a\dfrac{d\theta}{d\phi}}{\sin\theta\,(\cos\theta \pm 1)};$$

and that at the point where it meets the circle they are given by the equation

$$\rho^2\left(\frac{d\theta}{d\phi}\right)^2 + a\rho\cos\theta = a^2,$$

a being the radius of the circle.

1210. If l, m, n be the direction cosines of the normal at any point of the conicoid $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$, and ϕ the angle between the geodesic lines through that point and through the umbilici; then will

$$\cos^2\phi=\frac{\{l^2a(c-b)+m^2b(c+a-2b)+n^2c(a-b)\}^2}{l^4a^2(b-c)^2+\ldots+\ldots-2m^2n^2bc(a-b)(a-c)-\ldots-\ldots};$$

the axis of y being parallel to the circular sections.

STATICS.

I. *Composition and Resolution of Forces.*

1211. If O be the centre of the circle circumscribing a triangle ABC, and D, E, F the middle points of the sides, the system of forces OA, OB, OC will be equivalent to the system OD, OE, OF.

1212. Forces P, Q, R act along the sides of a triangle ABC and their resultant passes through the centres of the inscribed and circumscribed circles; prove that

$$\frac{P}{\cos B - \cos C} = \frac{Q}{\cos C - \cos A} = \frac{R}{\cos A - \cos B}.$$

1213. $ABCD$ is a quadrilateral inscribed in a circle, and forces inversely proportional to AB, BC, AD, DC act along the sides in the directions indicated by the order of the letters; prove that their resultant acts along the line joining the intersection of AC, BD to the intersection of the tangents at B, D.

1214. In a triangular lamina ABC, AD, BE, CF are the perpendiculars, and forces BD, CD, CE, AE, AF, BF are applied to the lamina; prove that their resultant passes through the centre of the circumscribed circle.

1215. Three equal forces P act at the angles of a triangle ABC perpendicular respectively to the opposite sides; prove that their resultant is equal to

$$P\sqrt{\left(1 - 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right)}.$$

1216. If O be the centre of the circumscribed circle, and L the centre of perpendiculars of a triangle ABC, three forces represented by LA, LB, LC will have a resultant along LO and equal to twice LO.

1217. Three parallel forces act at the angular points of a triangle ABC, and are to each other as $b+c : c+a : a+b$; prove that their resultant passes through the centre of the inscribed circle of the triangle whose angular points bisect the sides of the former.

1218. The position of a point P such that forces acting along PA, PB, PC, and equal to lPA, mPB, nPC respectively, are in equilibrium, is given by the equations

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n};$$

areal coordinates being used.

1219. Forces act along the sides of a triangle ABC and are proportional to the sides; AA', BB', CC' are the bisectors of the angles; prove that if the forces be turned in the same direction about A', B', C' respectively, through an angle

$$\tan^{-1}\left(-\cot\frac{B-C}{2}\cot\frac{C-A}{2}\cot\frac{A-B}{2}\right),$$

there will be equilibrium.

1220. A system of forces whose components are (X_1, Y_1), (X_2, Y_2) ... act at the points (x_1, y_1), (x_2, y_2) ... and are equivalent to a single couple; prove that if each force be turned about its point of application through an angle θ there will be equilibrium, if

$$\tan\theta = \frac{\Sigma(Yx - Xy)}{\Sigma(Xx + Yy)}.$$

1221. If L, M, N be the sums of the moments of a given system of forces about three rectangular axes, and X, Y, Z the sums of the components along these axes, then will

$$LX + MY + NZ$$

be independent of the particular system of axes.

1222. The necessary and sufficient conditions of equilibrium of a system of forces acting on a rigid body are that the sum of the moments of all the forces about each of the edges of any finite tetrahedron shall be severally equal to zero.

1223. Forces acting on a rigid body are represented by the edges of a tetrahedron, three acting from one angular point and the other three along the sides taken in order of the opposite face; prove that, whichever angular point be taken, the product of the resultant force and minimum couple will be the same.

1224. Three forces act along three non-intersecting straight lines, any other straight line is drawn meeting the three; prove that the shortest distance of this straight line from the central axis of the forces is proportional to the cotangent of the angle its direction makes with the central axis.

1225. A portion of a curve surface of continuous curvature is cut off by a plane, and at a point in each element of the portion a force proportional to the element is applied in direction of the normal; prove that, if all the forces act inwards, or all outwards, they will in the limit have a single resultant.

1226. If a system of forces acting on a rigid body be reducible to a couple, it is always possible by rotation about any proposed point to bring the body into such a position that the forces, acting at the same points of the body in the same directions in space, shall produce equilibrium.

1227. A system of forces are reduced to a force acting through an assumed point and a couple; prove that, if the assumed point be taken on a fixed straight line, and through it the axis of the couple be drawn, the extremity of the axis will lie on another fixed straight line.

1228. Prove that the central axis of two forces P, Q intersects the shortest distance c between their lines of action, and divides it in the ratio

$$Q(Q + P\cos\theta) : P(P + Q\cos\theta),$$

θ being the angle between their directions. Prove that the moment of the principal couple is

$$\frac{cPQ \sin\theta}{\sqrt{(P^2 + Q^2 + 2PQ\cos\theta)}}.$$

1229. If a system of forces be reduced to two, one of which F acts along a fixed straight line; then will

$$\frac{1}{F} = \frac{\cos\theta}{R} + \frac{c\sin\theta}{G},$$

θ being the angle which the given straight line makes with the central axis, c the shortest distance between them, R the resultant force, and G the principal couple.

1230. If a system of forces be reduced to two at right angles to each other, the shortest distance between their lines of action cannot be less than $\frac{2G}{R}$.

1231. If a system of forces be reduced to two P, Q, and the shortest distances of their lines of action from the central axis be x, y respectively; then will

$$G^2(Q^2 - P^2) = R^2(P^2x^2 - Q^2y^2).$$

1232. Two forces act along the straight lines

$$x = a,\ y = z\tan\alpha;\ x = -a,\ y = -z\tan\alpha;$$

prove that their central axis lies on the surface

$$x(y^2 + z^2) = \frac{2ayz}{\sin 2\alpha}.$$

the coordinates being rectangular.

1233. Two forces given in magnitude act along two given non-intersecting straight lines, a third force given in magnitude acts through a given point, and the three have a single resultant; prove that the line of action of the third force must lie on a certain cone of revolution.

II. *Centre of Gravity.*

1234. A rectangular board of weight W is supported in a horizontal position by vertical strings at three of its angular points; a weight $5W$ being placed on the board, the tensions of the strings become W, $2W$, $3W$; prove that the weight must be at one of the angular points of a hexagon whose opposite sides are equal and parallel, and whose area is to that of the board as $3 : 25$.

1235. If particles be placed at the angular points of a tetrahedron, proportional respectively to the areas of the opposite faces, their centre of gravity will be the centre of the sphere inscribed in the tetrahedron.

1236. A uniform wire is bent into the form of three sides of a polygon AB, BC, CD, and $AB = CD = a$, $BC = b$; prove that, if the centre of gravity of the wire be at the intersection of AC, BD, each of the angles B, C is equal to

$$\cos^{-1}\left(-\frac{b^2}{2a^2}\right).$$

1237. A thin uniform wire is bent into the form of a triangle ABC, and particles of weights P, Q, R are placed at the angular points; prove that, if the centre of gravity of the particles coincide with that of the wire,

$$P : Q : R :: b+c : c+a : a+b.$$

1238. A polygon is such that α_1, α_2, α_3 ... being the angles made by its sides with any fixed straight line,

$$\Sigma(\cos 2\alpha) = 0, \quad \Sigma(\sin 2\alpha) = 0;$$

prove that there exists a point O which is the centre of gravity of n equal particles placed at the feet of the perpendiculars from O on the sides; and that the centre of gravity of n equal particles, at the feet of the perpendiculars from any other point P, bisects OP.

1239. The limiting position of the centre of gravity of the area included between the area of a quadrant of an ellipse bounded by the axes and the corresponding quadrant of the auxiliary circle, as the ellipse approaches the circle as its limit, will be a point whose distance from the major axis is twice its distance from the minor axis.

1240. A curve is divided symmetrically by the axis of x, and is such that the centre of gravity of the area included between the ordinates $x=0$, $x=h$ is at a distance mh from the origin; prove that the equation of the curve is

$$y = Cx^{\frac{2m-1}{1-m}}.$$

1241. The circle is the only curve in which the centre of gravity of the area included between any two radii vectores and the curve lies on the straight line bisecting the angle between the radii.

1242. Determine the differential equation of a curve such that the centre of gravity of any arc measured from a fixed point lies on the straight line bisecting the angle between the radii of the extremities. Prove that the curve is a lemniscate, the node being pole.

1243. Two rods AB, BC rigidly united at B and suspended freely from A, rest inclined at angles α, β to the vertical; prove that

$$\frac{AB}{BC} = \sqrt{\left(1 + \frac{\sin\beta}{\sin\alpha}\right)} - 1.$$

1244. AB, BC are two uniform rods freely jointed at B and moveable about A which is fixed; find at what point in BC a prop must be placed so that the rods may be at rest in a horizontal straight line.

1245. Three equal uniform rods, jointed together at their extremities, rest in one horizontal line on three pegs, each rod in contact with one peg; find the positions of equilibrium.

The length of each rod being a, and the pegs at equal distances b; prove that there will be three positions of equilibrium if

$$30b < 36a > 5(4+\sqrt{7})\,a.$$

1246. A rectangular board is supported with its plane vertical by two smooth pegs, and rests with one diagonal parallel to the line joining the pegs; prove that the other diagonal will be vertical.

1247. A rectangular board whose sides are a, b, is supported with its plane vertical on two smooth pegs in the same horizontal line at a distance c; prove that the angle θ made by the side a with the vertical when in equilibrium is given by the equation

$$2c\cos 2\theta = b\cos\theta - a\sin\theta.$$

1248. A portion of a parabola, cut off by a focal chord inclined at an angle α to the axis, rests with its chord horizontal on two smooth pegs in the same horizontal line at a distance c; prove that the latus rectum of the parabola is $c\sqrt{5}\sin^2\alpha$.

1249. A uniform rod AB of length $2a$ is freely moveable about A; a smooth ring of weight P slides on the rod and has attached to it a fine string, which, passing over a pulley at a height b vertically above A, supports a weight Q hanging freely: find the position of equilibrium of the system; and prove that, if in this position the rod and string be equally inclined to the vertical,

$$2Q(Qb - Wa)^2 = P^2Wab.$$

1250. A uniform rod, of length c, rests with one end on a smooth elliptic arc, whose major axis is horizontal, and with the other on a smooth vertical plane at a distance h from the centre of the ellipse; prove that, if θ be the angle made by the rod with the horizon,

$$\tan\theta = \frac{a}{2b}\tan\phi, \quad \text{where } a\cos\phi + h = c\cos\theta.$$

Explain the result when $a = 2b = c$, $h = 0$.

III. *Smooth Bodies under forces in one plane.*

1251. A small smooth heavy ring is capable of sliding on a fine elliptic wire whose major axis is vertical; two strings attached to the ring pass through small smooth rings at the foci and sustain given weights; prove that, if there be equilibrium in any position in which the whole string is not vertical, there will be equilibrium in every position.

Prove also that the pressure on the curve will be a maximum or minimum when the sliding ring is at either extremity of the major axis, and when its focal distances have between them the same ratio as the two sustained weights.

1252. Two spheres of densities ρ, σ and radii a, b, rest in a paraboloid whose axis is vertical, and touch each other at the focus; prove that $\rho^3 a^{10} = \sigma^3 b^{10}$. Also if W, W' be their weights, and R, R' the pressures on the paraboloid at the points of contact,

$$\frac{R}{W} - \frac{R'}{W'} = \frac{1}{2}\left(\frac{R}{W'} - \frac{R'}{W}\right).$$

1253. Four uniform rods freely jointed at their extremities form a parallelogram, and at the middle points of the rods are small smooth rings joined by rigid rods without weight. The parallelogram is suspended freely from one of its angular points; find the tensions of the rods and the reactions of the rings, and prove that (1) if the parallelogram be a rectangle the tensions are equal, (2) if a rhombus the reactions are equal.

1254. An elliptic lamina of axes $2a$, $2b$, rests with its plane vertical on two smooth pegs in the same horizontal line at a distance c; prove that, if $c < b\sqrt{2}$ or $> a\sqrt{2}$, the only positions of equilibrium are when one axis is vertical; and that, if $c > b\sqrt{2} < a\sqrt{2}$, the positions in which an axis is vertical are stable, and there are unstable positions of equilibrium in which the pegs are at the extremities of conjugate diameters.

IV. *Friction.*

1255. Find the least coefficient of friction between a given elliptic cylinder and a particle, in order that, for all positions of the cylinder in which the axis is horizontal, the particle may be capable of resting vertically above the axis.

1256. Two given weights of different material are laid on a given inclined plane, and connected by a string in a state of tension inclined at a given angle to the intersection of the plane with the horizon, and the lower weight is on the point of motion; determine the coefficient of friction of the lower weight, and the magnitude and direction of the force of friction on the upper weight.

1257. A weight w rests on a rough inclined plane ($\mu < 1$) supported by a string, which, passing over a smooth pulley at the highest point of the plane, sustains a weight $> \mu w < w$ hanging vertically; prove that the angle between the two positions of the plane in which w is in a state bordering on motion is

$$2 \tan^{-1} \mu.$$

1258. Two weights of similar material connected by a fine string rest on a rough vertical circular arc on which the string lies; prove that the angle subtended at the centre by the distance between the limiting positions of either weight is $2 \tan^{-1} \mu$.

1259. A uniform rod rests with one extremity against a rough vertical wall, the other being supported by a string of equal length fastened to a point in the wall; prove that the least angle which the string can make with the wall is $\tan^{-1}\left(\frac{3}{\mu}\right)$.

1260. A uniform rod of weight W rests with one end against a rough vertical plane and with the other end attached to a string, which passes over a smooth pulley vertically above the former end and supports a weight P. Find the limiting positions of equilibrium, and prove that equilibrium will be impossible unless P be greater than $W \cos \epsilon$, $\tan \epsilon$ being the coefficient of friction.

1261. A heavy uniform rod of weight W rests inclined at an angle θ to the vertical in contact with a rough cylinder of revolution, whose axis is horizontal and whose diameter is equal in length to the rod. The rod is maintained in its position by a fine string in a state of tension, which passes from one end of the rod to the other round the cylinder; prove that the tension of the string must be not less than

$$\frac{W}{2}\left(\frac{\cos\theta}{\mu} - \sin\theta\right).$$

1262. Two weights support each other on a rough double inclined plane, by means of a fine string passing over the vertex, and both weights are on the point of motion; prove that if the plane be tilted till both weights are again on the point of motion, the angle through which the planes must be turned is

$$2 \tan^{-1} \mu.$$

1263. A square lamina has a string of length equal to that of a side attached at one of the angular points; the string is also attached to a point in a rough vertical wall, and the lamina rests with its plane vertical and perpendicular to the wall; prove that, if the coefficient of friction be 1, the angle which the string makes with the wall lies between $\frac{\pi}{4}$ and $\frac{1}{2}\tan^{-1}\frac{1}{2}$.

1264. Two weights P, Q of similar material resting on a rough double inclined plane are connected by a fine string passing over the common vertex, and Q is on the point of motion down the plane; prove that the weight which may be added to P without producing motion is

$$\frac{P \sin 2\epsilon \sin(\alpha+\beta)}{\sin(\alpha-\epsilon)\sin(\beta-\epsilon)},$$

α, β being the angles of inclination to the horizon, and $\tan\epsilon$ the coefficient of friction.

1265. A uniform rod rests with one extremity against a rough vertical wall $\left(\mu=\frac{7}{3}\right)$, the other extremity being supported by a string three times the length of the rod attached to a point in the wall; prove that the angle which the string makes with the wall in the limiting position of equilibrium is

$$\tan^{-1}\frac{5}{27} \text{ or } \tan^{-1}\frac{1}{3}.$$

1266. A weight W is supported on a rough inclined plane of inclination α by a force P, whose direction makes an angle ι with the plane, and whose component in the plane makes an angle θ with the line of greatest inclination in the plane; prove that, for equilibrium to be possible,

$$\mu^2 > \frac{2\sin^2\alpha \sin^2\theta \cos^2\iota}{1+\cos 2\alpha \cos 2\iota - \sin 2\alpha \sin 2\iota \cos\theta}.$$

1267. A given weight, resting upon a rough inclined plane, is connected with a weight P by means of a string passing over a rough peg, P hanging freely. The coefficients of friction for the peg and plane are $\tan\lambda$, $\tan\lambda'$, $(\lambda>\lambda')$. Prove that the inclination of the string to the plane in limiting equilibrium, when P is a maximum or minimum, is $\lambda-\lambda'$.

V. *Elastic Strings.*

1268. A string whose elasticity varies as the distance from one extremity is stretched by any force; prove that its extension is equal to that of a string of the same length, of uniform elasticity equal to that at the middle point of the former, stretched by the same force.

1269. An elastic string rests on a rough inclined plane, with its upper extremity fixed; prove that its extension will lie between the limits

$$\frac{l^2}{2\lambda}\frac{\sin(\alpha \pm \epsilon)}{\cos\epsilon};$$

α being the inclination of the plane, $\tan\epsilon$ the coefficient of friction, and l, λ natural lengths of the string and of a portion of it whose weight is equal to the coefficient of elasticity.

1270. Two weights P, Q are connected by an elastic string without weight, which passes over two small rough pegs A, B in the same horizontal line at a distance a, Q is just sustained by P, and $AP = b$, $BQ = c$; P, Q are then interchanged, and $AQ = b'$, $BP = c'$: obtain equations for determining the natural length of the string, its elasticity, and the coefficients of friction at A and B.

1271. A weight P just supports another weight Q by means of a fine elastic string passing over a rough circular cylinder whose axis is horizontal, W is the coefficient of elasticity, and a the radius of the cylinder; prove that the extension of the part of the string in contact with the cylinder is

$$\frac{a}{\mu}\log\left(\frac{Q+W}{P+W}\right).$$

1272. An elastic string is laid on a cycloidal arc whose plane is vertical and vertex upwards, and when stretched by its own weight is in contact with the whole of the cycloid, the natural length of the string being equal to the circumference of the generating circle; prove that the coefficient of elasticity is the weight of a portion of the string whose natural length is twice the diameter of the generating circle.

1273. A heavy elastic string, whose natural length is $2l$, is placed symmetrically on the arc of a smooth vertical cycloid, and when in equilibrium a portion of string, whose natural length is x, hangs vertically at each cusp; prove that

$$2\sqrt{(a\lambda)} = (\lambda + x)\tan\frac{l-x}{\sqrt{(4a\lambda)}};$$

$2a$ being the length of the axis of the cycloid, and λ the natural length of a portion of the string whose weight is the coefficient of elasticity.

1274. A heavy elastic string hangs symmetrically over a smooth circular arc, whose plane is vertical, a portion whose natural length is $2a$, and stretched length $3a$, hanging vertically on each side; prove that the natural length of the part in contact is

$$4a\sqrt{2}\log(1+\sqrt{2}),$$

$4a$ being the radius.

1275. A heavy cone resting symmetrically on a rough sphere may be displaced through an angle of $\frac{\pi}{4}$ without upsetting, if the height of the cone be not greater than half a great circle of the sphere.

VI. *Catenaries.*

1276. An endless heavy chain, of length $2l$, is passed over a smooth circular cylinder, whose axis is horizontal; c is the length of a portion of the chain, whose weight is equal to the tension at the lowest point, and 2ϕ the angle between the radii drawn to the points where the chain leaves the cylinder; prove that

$$\tan\phi + \frac{\pi - \phi}{\sin\phi}\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \frac{l}{c}.$$

1277. *ABCD* are four smooth pegs forming a square, *AB*, *CD* being horizontal, and an endless uniform inextensible string passes round the four, hanging in two festoons; prove that

$$\frac{1}{\sin\alpha\log\cot\frac{\alpha}{2}} - \frac{1}{\sin\beta\log\cot\frac{\beta}{2}} = 2,$$

$$\frac{\cot\alpha}{\log\cot\frac{\alpha}{2}} + \frac{\cot\beta}{\log\cot\frac{\beta}{2}} = \frac{l}{a} - 2,$$

α, β being the angles which the tangents at B, C make with the vertical, l the length of the string, and a the length of a side of the square.

1278. A heavy uniform chain rests on a rough circular arc whose plane is vertical, the length of the chain being equal to a quadrant of the circle, and one extremity being at the highest point when the chain is on the point of motion; prove that

$$\epsilon^{\frac{\mu\pi}{2}} = \frac{2\mu}{1-\mu^2}.$$

1279. A heavy uniform chain rests in limiting equilibrium on a rough cycloidal arc, whose axis is vertical and vertex upwards, one extremity being at the vertex and the other at the cusp; prove that

$$\epsilon^{\frac{\mu\pi}{2}} = \frac{3}{1+\mu^2}.$$

1280. A heavy uniform chain fastened at two points rests in the form of a parabola under the action of two forces, one (A) parallel to the axis and constant, and the other (F) tending from the focus; prove that

$$F = 3A + m\cos\phi,$$

ϕ being the angle which the tangent at any point makes with the tangent at the vertex, and m a constant.

1281. Find the law of repulsive force tending from a focus under which an endless uniform chain can be kept in equilibrium in the form of an ellipse; and, if there be two such forces, one in each focus and equal at equal distances, prove that the tension at any point varies inversely as the conjugate diameter.

1282. A uniform chain rests in the form of a cycloid whose axis is vertical under the action of gravity and a certain normal force, and the tension at the vertex vanishes; prove that the tension at any point is proportional to the vertical height above the vertex, and that the normal force at any point is

$$\frac{g}{2\cos\theta}(3\cos^2\theta - 1),$$

where θ is the angle which the normal makes with the vertical.

1283. A heavy chain of variable density, suspended from two points, hangs in the form of a curve whose intrinsic equation is $s = f(\phi)$, the lowest point being origin; prove that the density at any point will vary inversely as $\cos^2\phi f'(\phi)$.

1284. A string is kept in equilibrium in the form of a closed curve by the action of a repulsive force tending from a fixed point, and the density at each point is proportional to the tension;

prove that the repulsive force at any point is inversely proportional to the chord of curvature through the centre of force.

1285. The coordinates of a point P of a rigid body, referred to a system of fixed rectangular axes, are x, y, z; prove that, if the body receive any *small* displacement, the displacements of P parallel to the axes may be represented by

$$\alpha + \theta_2 z - \theta_3 y, \quad \beta + \theta_3 x - \theta_1 z, \quad \gamma + \theta_1 y - \theta_2 x;$$

α, β, γ, θ_1, θ_2, θ_3 being independent of (x, y, z). Hence prove that the sum of the virtual moments of any forces acting on a rigid body is of the form

$$\alpha X + \beta Y + \gamma Z + \theta_1 L + \theta_2 M + \theta_3 N,$$

X, Y, Z being the sums of the resolved parts of the forces along the axes, and L, M, N the sums of the moments about the axes.

DYNAMICS, ELEMENTARY.

I. *Rectilinear motion, Impulses.*

1286. A ball A impinges obliquely on another ball B, and after impact the directions of motion of A and B make equal angles θ with A's previous direction, find θ; and prove that when $A = B$, $\theta = \tan^{-1}\sqrt{e}$, e being the mutual elasticity.

1287. A smooth inelastic ball, mass m, is lying on a horizontal table in contact with a vertical wall, and is struck by another ball, mass m', moving in a direction perpendicular to the wall and inclined at an angle a to the common normal at the point of impact; prove that the angle θ, through which the direction of motion of the striking ball is turned, is given by the equation

$$\cot\theta\cot a = 1 + \frac{m'}{m}.$$

1288. Equal particles $A_1, A_2, \ldots A_n$ are fastened at equal intervals a on a fine string, of length $(n-1)a$, and are then laid on a horizontal table at n consecutive angular points of a regular polygon of p sides $(p > n)$ each equal to a; a blow P is applied to A_1 in direction A_1A_2; prove that the impulsive tension of the string A_rA_{r+1} is

$$P\cos^r a\,\frac{(1+\sin a)^{n-r} - (1-\sin a)^{n-r}}{(1+\sin a)^n - (1-\sin a)^n};$$

a denoting $\dfrac{2\pi}{p}$.

1289. AP, PB are chords of a circle whose diameter AB is vertical; particles falling down AP, PB respectively, start from

A, P simultaneously; prove that the least distance between them during the motion is equal to the distance of P from AB.

1290. A number of heavy particles start at once from the vertex of an oblique circular cone whose base is horizontal, and fall down generating lines of the cone; prove that at any subsequent moment they will lie in a subcontrary section.

1291. The locus of a point P, such that the times of falling down PA, PB to two given points A, B may be equal, is a rectangular hyperbola.

1292. The locus of a point P, such that the time of falling down PA to a given point A is equal to the time of falling vertically from P to a given straight line, is one branch of a hyperbola of which one asymptote is vertical, and the other perpendicular to the given straight line.

1293. A parabola is placed with its axis vertical and vertex downwards; prove that the time of falling down any chord to the vertex is equal to the time of falling vertically through a space equal to the parallel focal chord.

1294. An ellipse is placed with its major axis vertical; prove that the time of descent down any chord to the lower vertex, or from the higher vertex, is proportional to the length of the parallel diameter.

1295. Two weights W, nW move on two inclined planes and are connected by a fine string passing over the common vertex, the whole motion being in one plane; prove that the centre of gravity of the weights describes a straight line with uniform acceleration

$$g\,\frac{n\sin\beta-\sin\alpha}{(n+1)^2}\sqrt{\{n^2+2n\cos(\alpha+\beta)+1\}};$$

α, β being the angles of inclination of the planes.

1296. In the system of pullies in which each hangs by a separate string, P just supports W; prove that, if P be replaced by another weight Q, the centre of gravity of Q and W will descend with uniform acceleration

$$g\frac{W(Q-P)^2}{(P^2+QW)(Q+W)},$$

the weights of the pullies being neglected.

1297. The radii of two circles whose centres are in the same horizontal line are a, b, and the distance between their centres $c\,(>a+b)$; prove that the shortest time of descent from one to the other down a straight line is

$$\sqrt{\left(\frac{2}{g}\,\frac{c^2-(a+b)^2}{a+b}\right)}.$$

1298. A parabola is placed with its axis horizontal; prove that the time of shortest descent down a straight line from the curve to the focus is $\sqrt{\frac{4l}{g}}$, l being the latus rectum.

1299. An ellipse is placed with its major axis vertical; prove that the straight line of quickest descent from the curve to the lower focus is equal in length to the latus rectum, provided the eccentricity be $>\frac{1}{2}$.

1300. A uniform string of length $2a$ is in equilibrium, passing over a small smooth pulley; if it be just displaced, the velocity of the string when the whole has just passed the pulley is $\sqrt{ag}$.

II. *Parabolic Motion.*

1301. A heavy particle is projected in a given direction, determine its velocity in order that it may pass through a fixed point.

1302. If a particle moving under the action of gravity pass through two given points, the locus of the focus of its parabolic path will be a hyperbola.

1303. A heavy imperfectly elastic particle is projected from a point in a horizontal plane in such a manner that at its highest point it impinges directly on a vertical plane, from which it rebounds, and, after another rebound from the horizontal plane, returns to the point of projection; prove that the coefficient of elasticity is $\frac{1}{2}$.

1304. If r_1, r_2, r_3 be three distances of a projectile from the point of projection, and α_1, α_2, α_3 the angular elevations of these points above the point of projection,

$$r_1 \cos^2 \alpha_1 \sin (\alpha_2 - \alpha_3) + r_2 \cos^2 \alpha_2 \sin (\alpha_3 - \alpha_1) + r_3 \cos^2 \alpha_3 \sin (\alpha_1 - \alpha_2) = 0.$$

1305. The axis of a parabola is vertical and the vertex downwards, and a circle has its centre at a point P on the parabola, and passes through the focus S; a perfectly elastic particle sliding down PS is reflected at the circle and then moves freely under the action of gravity; find where it next meets the circle, and, if it be at the lowest point, prove that SP is equal to two thirds of the latus rectum.

1306. A particle is projected up an inclined plane of given inclination so as after leaving the plane to describe a parabola; prove that, for different lengths of the plane, the loci of the focus and vertex of the parabolic path are both straight lines.

1307. A perfectly elastic particle is projected from the middle point of the base of a vertical square towards one of the angles, and after being reflected at the sides containing that angle, falls to the opposite angle; prove that the space due to the velocity of projection is to a side of the square :: 45 : 32.

1308. A perfectly elastic particle is projected with a given velocity from a given point in one of two planes, equally inclined to the horizon and intersecting in a horizontal straight line; determine the angle of projection in order that the particle may after reflection return to the point of projection and be again reflected in the same path. Prove that the inclination of each plane must be 45°.

1309. An imperfectly elastic particle let fall on a fixed inclined plane bounds on to another fixed inclined plane, the line of intersection being horizontal, and the time between the two planes is given; prove that the locus of the point from which the particle is let fall is in general a parabolic cylinder; but that it is a plane if

$$\tan\alpha\tan(\alpha+\beta)=e,$$

α, β being the inclinations of the planes, and e the elasticity.

1310. A particle is projected from a given point, and its resolved velocity parallel to a given straight line is given; prove that the locus of the focus of the parabolic path is a parabola of which the given point is the focus, and whose axis makes with the vertical an angle double that made by the given straight line.

1311. A particle, projected from a point in an inclined plane, after the r^{th} impact begins to move at right angles to the plane, and at the n^{th} impact is at the point of projection; prove that

$$e^n-2e^r+1=0,$$

e being the elasticity.

1312. A particle is projected from a given point in a horizontal plane at an angle α to the horizon, and, after one rebound at a vertical plane, returns to the point of projection; prove that the point of impact must lie on the straight line

$$y(1+e)=x\tan\alpha,$$

x, y being measured horizontally and vertically from the point of projection. If the velocity of projection and not the direction be given, the locus of the point of impact is an ellipse.

1313. An imperfectly elastic particle is projected from a given point with given velocity, so as, after one rebound at an inclined plane passing through the point, to return to the point of projection; prove that the locus of the point where the particle strikes the inclined plane is the ellipse

$$x^2 + (1+e)^2 y^2 = 4ehy,$$

x, y being horizontal and vertical through the given point, e the elasticity, and h the space due to the given velocity.

1314. A particle is projected from a given point so as just to pass over a vertical wall whose height is b, and distance from the point of projection a; prove that when the area of the parabolic path described, before meeting the horizontal plane through the point of projection, is greatest, the range is $\frac{3a}{2}$, and the height of the vertex is $\frac{9b}{8}$.

1315. A heavy particle is projected from a point in a plane whose inclination to the horizon is 30° with given velocity, in a vertical plane perpendicular to the inclined plane; prove that, if all directions of projection in that vertical plane are equally probable, the chance of the range on the inclined plane being at least one third of the greatest possible range is $\frac{1}{2}$.

III. *Motion on a Smooth Curve under the action of Gravity.*

1316. A heavy particle is projected from the vertex of a smooth parabolic arc, whose axis is vertical and vertex downwards, with a velocity due to a height h, and after passing the extremity of the arc proceeds to describe an equal parabola freely; prove that, if c be the vertical height of the extremity of the arc, the latus rectum is $4(h-2c)$.

1317. A particle is projected so as to move on a parabolic arc whose axis is vertical and vertex upwards; prove that the pressure on the curve in any position is proportional to the curvature.

1318. Two heavy particles, connected by a fine string passing through a small fixed ring, describe horizontal circles in equal times; prove that the circles must lie in the same horizontal plane.

1319. A particle P is attached by two strings to fixed points A, B in the same horizontal line, and is projected so as just to describe a vertical circle; PB is cut when the particle is in its lowest position, and P proceeds to describe a horizontal circle; prove that

$$\cos 2PAB = \frac{2}{3},$$

and that, if the tension of PA be unaltered, the angle APB is a right angle.

1320. Two given weights are attached at given points of a fine string, which is attached to a fixed point, and the system revolves with uniform angular velocity about the vertical through the fixed point in a state of relative equilibrium; determine the inclinations of the two parts of the string to the vertical.

1321. A parabola is placed with its axis horizontal and plane vertical, and a heavy smooth particle is projected upwards from the vertex so as to move on the concave side of the curve; prove that the vertical space described before leaving the curve is two thirds of the greatest height attained. If 2θ be the angle described about the focus before leaving the curve

$$h = a(\tan^3\theta + 3\tan\theta),$$

h being the space due to the velocity of projection and $4a$ the latus rectum; also the latus rectum of the subsequent path is

$$4a\tan^2\theta.$$

1322. A heavy particle is projected so as to move on a circular arc whose plane is vertical, and afterwards describe a parabola freely; prove that the locus of the focus of the parabolic path is an epicycloid formed by a circle of radius $\frac{a}{4}$ rolling on a circle of radius $\frac{a}{2}$; a being the radius of the given circle.

1323. A cycloidal arc is placed with its axis vertical and vertex upwards, and a heavy particle is projected from the cusp up the concave side of the curve with velocity due to a height h; prove that the latus rectum of the parabola described after leaving the curve is $\frac{h^2}{2a}$, where a is the length of the axis of the cycloid.

1324. If a cycloidal arc be placed with its axis vertical and vertex upwards, and a heavy particle be projected from the cusp up the concave side of the curve; the focus of the parabola described by the particle after leaving the curve will lie on a fixed cycloid of half the dimensions.

1325. In a certain curve the vertical ordinate of any point bears to the vertical chord of curvature at that point the constant ratio $1 : m$, and a particle is projected, from the point where the tangent is vertical, along the curve with any velocity; prove that the height ascended before leaving the curve : height due to velocity of projection :: $4 : 4 + m$.

1326. A smooth heavy particle is projected from the lowest point of a vertical circular arc with a velocity due to a space equal in length to the diameter $2a$: the length of the arc is such that the range of the particle on the horizontal plane through the point of projection is the greatest possible; prove that this range is equal to $a\sqrt{(9 + 6\sqrt{3})}$.

NEWTON.

1327. Two triangles CAB, cAb, have a common angle A and the sum of the sides containing that angle the same in each; BC, bc intersect in D; prove that ultimately, when b moves up to B,

$$CD : DB :: AB : AC.$$

1328. Two equal parabolas have the same axis, and the focus of the outer is the vertex of the inner one, MPp, NQq are common ordinates; prove that the area of the surface generated by the revolution of the arc PQ about the axis bears to the area $MpqN$ a constant ratio.

1329. Common ordinates from the major axis are drawn to two ellipses, which have a common minor axis, and the outer of which touches the directrices of the inner; prove that the area of the surface generated by the intercepted arc of the inner ellipse revolving about the major axis will bear a constant ratio to the intercepted area of the outer.

1330. AB is a diameter of a circle, P a point on the circle near A, and the tangent at P meets BA produced in T; prove that ultimately the difference of BA, BP bears to AT the ratio $1 : 2$.

1331. If PQ be an arc of continued curvature, and R the point between P and Q at which the tangent is parallel to PQ; then the ultimate ratio $PR : RQ$, when PQ is indefinitely diminished, is one of equality.

1332. If A be a point on a curve, and P, Q two neighbouring points, the ultimate ratio of the triangle formed by the tangents at these points to that formed by the normals is

$$1 : \left(\frac{d\rho}{ds}\right)^2,$$

when P, Q move up to coincidence with A; ρ being the radius of curvature at A, and s the arc to A measured from a fixed point.

1333. A parabola is described about a force in the focus, and along the focal distance SP is measured SQ equal to a constant length; QR is drawn perpendicular to the tangent at P to meet the axis in R; prove that

$$QR : 2SQ :: \text{velocity at } P : \text{velocity at the vertex.}$$

1334. Prove that the equation $2V^2 = F \,.\, PV$ is true when a body moves in a resisting medium, F being the extraneous force, and PV the chord of curvature in direction of F.

1335. Two points P, Q move in the following manner; P describes an ellipse under acceleration tending to the centre, and Q describes relatively to P an ellipse of which P is the centre under acceleration tending to P, and the periodic times of these ellipses are the same; prove that the absolute path of Q is an ellipse concentric with the path of P.

1336. A particle describes a hyperbola under a force tending to one focus; prove that the rate at which areas are described by the central radius vector is inversely proportional to the distance of the particle from the centre of force.

1337. A rectangular hyperbola is described by a point under acceleration parallel to one of the asymptotes; prove that at a point P the acceleration is

$$2U^2 \frac{MP}{CM^2},$$

PM being drawn in direction of the acceleration to meet the other asymptote, C the centre, and U the component velocity perpendicular to the acceleration.

1338. A point describes a cycloid under acceleration tending to the centre of the generating circle; prove that the velocity at any point varies as the radius of curvature.

1339. A particle constrained to move on an equiangular spiral is attracted to the pole by a force proportional to the distance; prove that, in whatever position the particle be placed at starting, the time of describing a given angle about the centre of force will be the same.

1340. An endless string, on which runs a small smooth bead, incloses a fixed elliptic lamina whose perimeter is less than the length of the string; the bead is projected so as to keep the string in a state of tension; prove that it will move with constant velocity, and that the tension of the string will vary inversely as the rectangle under the focal distances.

1341. A parabola is described with constant velocity under the action of two equal forces, one of which tends to the focus; prove that either force varies inversely as the focal distance.

1342. A particle is describing an ellipse about a centre of force μr^{-2}, at a certain point μ receives a small increment $\delta\mu$, and the eccentricity is unaltered; prove that the point is one extremity of the minor axis, and that the major axis $2a$ is diminished by $\dfrac{a\delta\mu}{\mu}$.

1343. A particle is describing an ellipse about a centre of force μr^{-2}, and at a certain point μ is suddenly increased by $\delta\mu$; prove the following equations for determining the corresponding alterations in the major axis $2a$, the eccentricity e, and the longitude of the apse ϖ,

$$\frac{r\delta a}{a(2a-r)} = \frac{e\delta e}{1-e^2}\frac{r}{a-r} = \frac{e\delta\varpi}{\sin(\theta-\varpi)} = -\frac{\delta\mu}{\mu};$$

r, θ being polar co-ordinates of the point at which the change takes place.

1344. In an elliptic orbit about the focus, when the particle is at a distance r from the centre of force, the direction of motion is suddenly turned through a small angle $\delta\beta$; prove that the consequent alteration in the longitude of the apse is

$$\frac{\delta\beta}{e^2}\left(1+e^2-\frac{r}{a}\right);$$

$2a$ being the length of the major axis, and e the eccentricity.

1345. At any point in an elliptic orbit about the focus, the velocity v receives a small increment δv; prove that the alterations in the eccentricity e, and the longitude of the apse ϖ, will be given by the equations

$$\frac{\delta e}{b^2(a-r)} = \frac{e\delta\varpi}{ab\sqrt{\{a^2e^2-(r-a)^2\}}} = \frac{2v\delta v}{\mu ae(2a-r)}.$$

1346. An ellipse is described by a particle under the action of two forces tending to the foci, each varying inversely as the square of the distance; prove that

$$\frac{2a^5}{b^2} = \frac{(\mu\omega^2+\mu'\omega'^2)(\omega+\omega')^4}{\omega^4\omega'^4},$$

a, b being the axes, and ω, ω' the angular velocities at any point about the foci.

1347. Two fixed points of a lamina slide along two straight lines fixed in space, so that the angular velocity of the line joining the points is constant; prove that (1) every fixed point of the lamina describes an ellipse under acceleration tending to the intersection of the two fixed straight lines and proportional to the distance: (2) every fixed straight line of the lamina envelopes during its motion an involute of a four-cusped hypocycloid: (3) the motion of the lamina may be completely represented by supposing a circle fixed in the lamina to roll uniformly with internal contact on a circle of twice its radius fixed in space: (4) for a series of points of the lamina lying in a straight line, the foci of the ellipses described lie on a rectangular hyperbola.

1348. If a lamina move in its plane so that two fixed points in it describe straight lines with acceleration f, f'; the acceleration of the centre of instantaneous rotation is

$$\frac{1}{\sin\theta}\sqrt{(f^2+f'^2-2ff'\cos\theta)};$$

θ being the angle between the lines.

1349. A lamina moves in its own plane, so that two points, fixed in the lamina, describe straight lines with equal accelerations; prove that the acceleration of the centre of instantaneous rotation is constant in direction, and that the acceleration of any point fixed in the lamina is constant in direction.

1350. Two points fixed in a lamina move along two straight lines fixed in space, and the velocity of one of the points is uniform; prove that every point in the lamina moves so that its acceleration is constant in direction and varies inversely as the cube of the distance of the point from a fixed straight line.

1351. Two ellipses are described about a common attractive force in their centre; the axes of the two are coincident in direction, and the sum of the axes of one is equal to the difference of the axes of the other; prove that if the describing particles be at corresponding extremities of the major axes at the same moment and be moving in opposite directions, the line joining them will be of constant length during the motion and will revolve with uniform angular velocity.

1352. A lamina moves in such a manner that two straight lines fixed in the lamina pass through two points fixed in space; prove that the motion of the lamina may be completely represented by supposing a circle fixed in the lamina to roll with internal contact on a circle of half its radius fixed in space.

1353. A triangular lamina ABC moves so that the point A lies on a straight line bc fixed in space, and the side BC passes through a point a fixed in space, and the triangles ABC, abc are equal and similar; prove that the motion of the lamina may be completely represented by supposing a parabola fixed in the lamina to roll on an equal parabola fixed in space.

1354. Two particles describe curves under the action of central attractive forces, and the radius vector of either is always parallel and proportional to the velocity of the other; prove that the curves will be similar ellipses described about their centres.

DYNAMICS OF A POINT.

I. *Rectilinear Motion, Kinematics.*

1355. A heavy particle is attached by an elastic string to a fixed point, from which the particle is allowed to fall freely; when the particle is in its lowest position the length of the string is twice its natural length; prove that the coefficient of elasticity is four times the weight of the particle, and find the time during which the string is extended beyond its natural length.

1356. A particle at B is attached by an elastic string at its natural length to a point A, and attracted by a force varying as the distance to a point C in BA produced, BC being equal to $4BA$, and the particle just reaches the centre of force; prove that the velocity of the particle will be greatest at a point which divides CA in the ratio $8 : 7$.

1357. A particle is attracted to a fixed point by a force $= \mu\,(\text{dist.})^{-2}$, and repelled from the point by a constant force f; the particle is placed at a distance a from the centre, at which point the attractive force is four times as great as the repulsive, and projected directly from the centre with velocity v; prove that, (1) the particle will move to infinity or not according as

$$v >< \sqrt{(2af)}\,;$$

(2) if x, $x+c$ be the distances from the centre of force of two positions of the particle, the time of describing the given distance c between them will be greatest when $x(x+c) = 4a^2$. If $v = \sqrt{(2af)}$ or $3\sqrt{(2af)}$, determine the time of describing any distance.

1358. The accelerations of a point describing a curve are resolved into two, along the radius vector and parallel to the prime radius respectively; prove that these accelerations are

$$\frac{\cot\theta}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)+\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2, \text{ and } -\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

1359. The motion of a point is referred to two axes x, y, of which the axis of x is fixed and the axis of y revolves about the origin; prove that the accelerations in these directions are

$$\frac{d^2x}{dt^2}-\frac{1}{y\sin\theta}\frac{d}{dt}\left(y^2\frac{d\theta}{dt}\right);\quad \frac{d^2y}{dt^2}+\frac{\cot\theta}{y}\frac{d}{dt}\left(y^2\frac{d\theta}{dt}\right)-y\left(\frac{d\theta}{dt}\right)^2;$$

θ being the angle between the axes at a time t.

1360. If PQ be a tangent to a curve at Q, O a fixed point on the curve, $QP=r$, arc $OQ=s$, and ϕ the angle through which the tangent has revolved from O to Q; the accelerations of P in the direction QP, and at right angles to this direction, are respectively

$$\frac{d^2s}{dt^2}+\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2, \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)+\frac{d\phi}{dt}\frac{ds}{dt}.$$

1361. A point describes a curve of double curvature, and its polar co-ordinates at time t are (r, θ, ϕ); prove that its accelerations, (1) along the radius vector, (2) perpendicular to the radius vector in the plane of θ, and (3) perpendicular to the plane of θ, are respectively

$$(1)\quad \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2-r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$(2)\quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)-r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$(3)\quad \frac{1}{r\sin\theta}\frac{d}{dt}\left\{r^2\sin^2\theta\frac{d\phi}{dt}\right\}.$$

1362. A point describes a parabola in such a manner that its velocity, at a distance r from the focus, is

$$\sqrt{\left\{\frac{2f}{r}(r^2-c^2)\right\}},$$

f, c being constant; prove that its acceleration is compounded of f parallel to the axis, and $\frac{fc^2}{r^2}$ from the focus.

1363. A point describes a semi ellipse bounded by the minor axis, and its velocity at a distance r from the focus is

$$a\left\{\frac{f(a-r)}{r(2a-r)}\right\}^{\frac{1}{2}},$$

where $2a$ is the length of the major axis, and f a constant acceleration; prove that the acceleration of the point is compounded of two, each varying universely as the square of the distance, one tending to the nearer focus and the other from the farther focus.

1364. A point is describing a circle, and its velocity at an angular distance θ from a fixed point on the circle varies as

$$\frac{\sqrt{(1+\cos^2\theta)}}{\sin^2\theta};$$

prove that its acceleration is compounded of two tending to fixed points at the extremities of a diameter, each varying inversely as the fifth power of the distance and equal at equal distances.

1365. A point describes a circle, under acceleration constant and not tending to the centre; prove that the point oscillates through a quadrant, and that the direction of the acceleration always touches a certain hypocycloid.

1366. A parabola is described with accelerations F, A, tending to the focus and parallel to the axis respectively; prove that

$$\frac{d}{dr}(F+A)+\frac{2F}{r}=0\,;$$

r being the focal distance.

1367. A point describes an ellipse with accelerations $f(r_1)$, $\phi(r_2)$ tending to the foci; prove that

$$\frac{1}{r_1^2}\frac{d}{dr_1}\{r_1^2 f(r_1)\} = \frac{1}{r_2^2}\frac{d}{dr_2}\{r_2^2 \phi(r_2)\};$$

r_1, r_2 being the focal distances.

1368. The parabola $y^2 = 4ax$ is described by a point under accelerations X, Y parallel to the axes; prove that

$$2x\frac{dY}{dx} - y\frac{dX}{dx} + 3Y = 0.$$

1369. A point describes a parabola under acceleration which makes a constant angle α with the normal, θ is the angle described from the vertex about the focus in a time t; prove that

$$\frac{d\theta}{dt} \propto \cos^3\frac{\theta}{2}\,\epsilon^{-\frac{\theta\tan\alpha}{2}}.$$

Find also the law of acceleration.

1370. A point describes a circle of radius $4a$ with uniform angular velocity ω about the centre, and another point Q describes a circle of radius a with angular velocity 2ω about P; prove that the acceleration of Q varies as the distance of P from a certain fixed point.

1371. An equiangular spiral is described by a point with constant acceleration in a direction making an angle ϕ with the normal; prove that

$$\sin\phi\frac{d\phi}{d\theta} = 2\sin\phi + \cot\alpha\cos\phi;$$

α being the constant angle of the spiral, and θ the angle through which the tangent has turned from a given position.

1372. The only curve which can be described under a constant acceleration in a direction making a constant angle with the normal is an equiangular spiral.

1373. A parabola, $y^2 = 2cx$, is described by a point under acceleration making a constant angle a with the axis, and the velocity when the acceleration is normal is v; prove that the acceleration at any point (x, y) is

$$\frac{v^2c^2}{(c \cos a - y \sin a)^3}.$$

If the point, when the acceleration is normal, be moving towards the vertex, the time in which the direction of motion will turn through a right angle is

$$\frac{2c}{v(\sin 3a + \sin a)}.$$

1374. The intrinsic equation of a curve being $s = f(\phi)$, the curve is described by a point with accelerations X, Y, parallel to the tangent and normal at the point for which $\phi = 0$; prove that

$$\cos\phi\left(\frac{dY}{d\phi} - 3X\right) - \sin\phi\left(\frac{dX}{d\phi} + 3Y\right) + \frac{f''\phi}{f'(\phi)}(Y\cos\phi - X\sin\phi) = 0.$$

1375. The curve $s = f(\phi)$ is described by a point with constant acceleration, which at the origin is in direction of the normal; prove that its inclination θ to this direction at any other point is given by the equation

$$\left(3 - \frac{d\theta}{d\phi}\right) f'(\phi) \tan(\phi - \theta) = f''(\phi).$$

1376. A catenary is described by a point under acceleration, whose vertical component is constant (f); prove that the horizontal component at a point where the tangent makes an angle ϕ with the horizon is

$$f\frac{\cos\phi}{\sin^3\phi}(1 + m\cos\phi + \cos^2\phi).$$

1377. A curve is described under constant acceleration parallel to a straight line revolving uniformly; prove that the curve is a prolate, common, or curtate cycloid; or a circle.

1378. A point describes a certain curve, and initially the acceleration is normal; when the direction of motion has turned through an angle ϕ, that of acceleration has turned through 2ϕ in the same sense; prove that the acceleration $\propto \cos\phi \dfrac{d\phi}{ds}$, the velocity $\propto \cos\phi$, and the angular velocity of the tangent $\propto$ the acceleration.

1379. A point describes an ellipse under accelerations to the foci which are one to another, at any point, inversely as the focal distances of the point; find the law of either force; prove that the velocity of the point varies as the conjugate diameter, and that the periodic time is

$$\frac{\pi}{\omega}\left(\frac{a}{b}+\frac{b}{a}\right),$$

ω being the angular velocity about the centre at the end of either axis.

1380. A parabola is described under constant acceleration; prove that, if ϕ, θ be the angles which the tangent and the direction of the acceleration at any point make with the directrix,

$$3\tan\phi\cos^{\frac{1}{3}}\theta = \int\cos^{\frac{1}{3}}\theta\, d\theta.$$

1381. A point moves with constant acceleration, which is initially normal; and when the direction of motion has turned through an angle ϕ, that of acceleration has turned through $m\phi$ in the same sense; prove that the intrinsic equation of the curve described is

$$\frac{ds}{d\phi} = c\left\{\cos(m-1)\phi\right\}^{\frac{2-m}{m-1}}.$$

Determine the curve when $m = 1$, 2, or 3.

1382. A point describes a curve, which lies on a cone of revolution and crosses all the generating lines at a constant angle, under acceleration whose direction always intersects the axis;

prove that the acceleration makes a constant angle with the axis and varies inversely as the cube of the distance from the vertex.

1383. A cycloid is described under constant acceleration, and θ, ϕ are the angles which the directions of motion and of acceleration at any point make with the tangent at the vertex; prove that

$$\frac{\sin\theta}{\cos\theta\cos(\phi-2\theta)}+\log\left\{n\tan\left(\frac{\phi}{2}-\theta-\frac{\pi}{4}\right)\right\}=0;$$

or that
$$\phi-2\theta=\frac{\pi}{2}.$$

1384. P describes a circle under acceleration tending to S and varying as the distance, S being a point which moves on a fixed diameter initially passing through P; prove that, if θ be the angle described about the centre in a time t,

$$\sin\theta=\sqrt{\left(\frac{m+1}{m}\right)}\sin\sqrt{\mu}t;$$

and the distance of S from the centre is $\frac{a}{m\cos^3\theta}$; a being the radius, and m a constant.

1385. A point describes an arc of a circle, so that its acceleration is always proportional to the n^{th} power of its velocity; prove that the direction of the acceleration touches a certain epicycloid, generated by a circle of radius $\frac{a}{6-2n}$ rolling on one of radius $a\frac{2-n}{3-n}$; a being the radius of the circle described.

II. *Central Forces.*

1386. Prove that the parabola $y^2 = 4ax$ can be described under a constant force parallel to the axis of y, and a force proportional to y parallel to the axis of x. Also under two forces $4\mu(c + x)$, μy, parallel to the axes of x and y.

1387. A cardioid is described with constant angular velocity about the cusp under a constant force to the cusp and another constant force; prove that the magnitude of the latter is twice that of the former, and that its direction always touches an epicycloid generated by a circle of radius a rolling on one of radius $2a$; $8a$ being the length of the axis.

1388. The force to the pole, under which the hyperbola

$$r \cos 2\theta = a \cos \theta$$

can be described, will vary as $\dfrac{\cos^2 2\theta}{\cos^5 \theta}$.

1389. SY is the perpendicular from the pole S of a lemniscate upon the tangent at P, and the locus of Y is described by a particle under a force to S: prove that this force $\propto SP^{-13}$.

1390. The force to the pole under which the pedal of a given curve $r = f(p)$ can be described will vary as

$$\frac{2r^2}{p^5} - \frac{r}{p^4}\frac{dr}{dp}.$$

If the given curve be

$$a^{\frac{5}{3}} = r^{\frac{5}{3}} \sin \frac{3}{5}\theta,$$

the force will be constant.

1391. A parabola is described about a centre of force in C, the centre of curvature at the vertex; prove that the force at any point P of the parabola varies as $\frac{CP}{(AS+SP)^3}$, A being the vertex and S the focus.

1392. The force tending to the pole under which the evolute of the curve $r=f(p)$ can be described will vary as

$$\frac{1}{(r^2-p^2)^2}\frac{\sqrt{r^2\left\{1+\left(\frac{dr}{dp}\right)^2\right\}-2rp\frac{dr}{dp}}}{\left(\frac{dr}{dp}\right)^2+r\frac{d^2r}{dp^2}}.$$

1393. An orbit described under a constant force tending to a fixed point will be the pedal of one of the curves represented by the equation

$$r^2=Ap^5+Bp^4.$$

1394. A particle is projected from a point A at right angles to a straight line SA, and attracted to S by a force varying as cosec PSA; prove that the rate of describing areas about A will be uniformly accelerated.

1395. A particle is projected at a distance a with velocity equal to that in a circle at the same distance and at an angle 45° with the distance, and attracted to a fixed point by a force which at a distance r is equal to $\mu\left(\frac{3}{r^3}+\frac{a^2}{r^5}\right)$: determine the orbit described, and prove that the time to the centre of force is

$$\frac{a^2}{\sqrt{2\mu}}\left(2-\frac{\pi}{2}\right).$$

1396. A particle is attracted to a fixed point by a force which at a distance r is equal to

$$\mu r^{-7}(3a^8+3a^4r^4-r^8),$$

and is projected from a point at a distance a from the centre with velocity equal to that in a circle at the same distance and in a direction making an angle $\cot^{-1} 2$ with the distance; determine the orbit, and prove that the time to the centre of force

$$= \frac{1}{4\sqrt{\mu}} \log 2.$$

1397. A particle is describing a circle under the action of a constant force in the centre, and the force is suddenly increased to ten times its former magnitude; prove that the next apsidal distance will be equal to one fourth the radius of the circle.

1398. A particle is describing a central orbit in such a manner that the velocity at any point is to the velocity in a circle at that distance as $1 : \sqrt{n}$; prove that $p \propto r^n$, p being the perpendicular from the centre of force on the tangent at a point whose distance is r. Prove also that the force varies inversely as r^{2n+1}.

1399. A particle acted on by a central force $\frac{\mu}{r^3}\left(\frac{4}{a} - \frac{3}{r}\right)$ is projected at a distance a, at an angle $45°$, and with velocity $\frac{\sqrt{2\mu}}{a}$; determine the orbit and prove that the time from projection to an apse is $\frac{a^2}{3\sqrt{\mu}}\left(\frac{4\pi}{3\sqrt{3}} - 1\right)$.

1400. In an orbit described under a central force, the velocity of Y, the foot of the perpendicular from the centre of force on the tangent, is constant; prove that the chord of curvature of the orbit through the centre of force is constant.

1401. A particle describing a parabola about a force in the focus comes to an apse, at which point the law of force changes, and the force varies inversely as the distance till the particle next comes to an apse, when the former law is restored. No instantaneous changes being supposed, prove that the major axis of the new orbit will be $\frac{m^2 a}{m-1}$, $4a$ being the latus rectum of the parabola, and m the root of the equation

$$\frac{1}{x^2} = 1 - \log x,$$

which lies between 1 and ϵ.

1402. In an orbit described under a central force a straight line is drawn from a fixed point perpendicular to the tangent and proportional to the force, and this straight line describes equal areas in equal times; prove that the equation of the orbit is of the form

$$\frac{1}{p^{\frac{4}{3}}} = \frac{1}{c^{\frac{4}{3}}} \pm \left(\frac{r}{a^2}\right)^{\frac{4}{3}}.$$

Prove that the rectangular hyperbola is a particular case.

1403. A chain of uniform material rests under normal and tangential forces $-n$, t; prove that the curve in which it rests could be described by a particle, whose mass is equal to that of a unit of length of the chain, under the action of normal and tangential forces $2n$, t.

1404. A centre of force varying inversely as the n^{th} power of the distance moves in the circumference of a circle, and a particle describes an arc of the same circle under the action of the force; prove that the velocity of the centre of force : velocity of the particle :: $5-n$: $1-n$. If $n=3$, the time of describing a semicircle is $4a^2\sqrt{\frac{2}{\mu}}$.

1405. A particle P is repelled from a fixed point S by a force varying as $(\text{dist.})^{-2}$, and attracts another particle Q with a force varying as $(\text{dist.})^{-3}$; initially P and Q are equidistant from S in opposite directions, P is at rest, and the accelerations of the two forces are equal; prove that if Q be projected at right angles to SQ with proper velocity, it will describe a parabola of which S is the focus.

III. *Constrained Motion on Curves or Surfaces: Particles joined by Strings.*

1406. A particle is constrained to move on a curve under the action of forces, such that if projected from a certain point of the curve with velocity v, it would describe the curve freely. Prove that, when projected from that point with velocity V, the pressure on the curve is always $m\dfrac{V^2-v^2}{\rho}$, ρ being the radius of curvature at any point, and m the mass of the particle.

1407. A particle is acted on by two forces, one parallel to a fixed straight line and constant, and the other tending from a fixed point and varying as $(\text{dist.})^{-2}$, and the particle is initially at rest at a point where these forces are equal; prove that it will proceed to describe a parabola whose focus is the fixed point and axis parallel to the fixed line.

1408. Two particles A, B are together in a smooth circular tube; A attracts B with a force whose acceleration is μ . distance, and moves along the tube with uniform angular velocity $2\sqrt{\mu}$; B is initially at rest; prove that the angle ϕ subtended at the centre by AB after a time t is given by the equation

$$\tan\frac{\phi}{4}=\frac{\epsilon^{t\sqrt{\mu}}-1}{\epsilon^{t\sqrt{\mu}}+1}.$$

1409. A particle is placed in a smooth parabolic groove which revolves in its own plane about the focus with uniform angular velocity ω, and the particle describes in space an equal confocal parabola, under an attractive force in the focus; prove that this force at any point is measured by

$$\frac{\omega^2 r}{4c}(3r-4c),$$

r being the focal distance and $2c$ the latus rectum.

1410. A particle is acted on by a repulsive force tending from a fixed point, and by another force in a fixed direction; when at a distance r from the fixed point, the accelerations of these forces are

$$\frac{\mu}{r^2}\left(1-\frac{r}{a}\right), \quad \frac{\mu}{r^2}\left(\frac{r^2}{c^2}+\frac{r}{a}\right)$$

respectively; prove that the particle, abandoned motionless to these forces at any point where they are equal, will proceed to describe a parabola, of which the fixed point is focus.

1411. A bead moves on a smooth elliptic wire, and is attached to the foci by two similar elastic strings, the natural lengths of which are equal, and which remain stretched throughout the motion; prove that, if projected with proper velocity, the velocity will always vary as the conjugate diameter.

1412. A force f resides in the centre of a rough circular arc and from a point of the circle a particle is projected with a velocity $v\,(>\sqrt{af})$ along the interior of the circle; prove that the normal pressure on the curve will be diminished one half after a time

$$\frac{1}{\mu}\sqrt{\frac{a}{f}}\log\left\{\frac{\sqrt{(2af)}+\sqrt{(v^2+af)}}{v+\sqrt{(af)}}\right\};$$

a being the radius and μ the coefficient of friction.

1413. A heavy particle is projected inside a smooth paraboloid of revolution, whose vertex is its lowest point, and its greatest and least vertical heights above the vertex are h, h'; prove that, if v, v' be the velocities at these points,

$$v^2=2gh', \quad v'^2=2gh;$$

and that the pressures on the paraboloid at the corresponding points are inversely as the curvatures of the generating parabolas at these points.

1414. A particle is projected horizontally so as to move on the interior of a smooth hollow sphere of radius a, and the velocity of projection is $2\sqrt{ga}$; prove that when it again moves horizontally, its vertical depth below the highest point of the sphere is a mean proportional between the radius and its initial distance from the lowest point.

1415. A particle in motion on a smooth surface $z = \phi(x, y)$ under the action of gravity describes a curve in a horizontal plane: if u be its velocity, prove that

$$\frac{u^2}{g}\left\{\frac{d^2z}{dx^2}\left(\frac{dz}{dy}\right)^2 - 2\frac{d^2z}{dxdy}\frac{dz}{dx}\frac{dz}{dy} + \frac{d^2z}{dy^2}\left(\frac{dz}{dx}\right)^2\right\} + \left\{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^2 = 0;$$

the axis of z being vertical.

1416. In a smooth surface of revolution whose axis is vertical, a heavy particle is projected so as to move on the surface and describe nearly a horizontal circle; prove that the time of a vertical oscillation is

$$\pi\sqrt{\left\{\frac{kr\sin\alpha}{g(k + 3r\cos\alpha\sin^2\alpha)}\right\}},$$

k being the distance from the axis, r the radius of curvature of the generating curve, and α the inclination of the tangent to the vertical, in the mean position of the particle.

1417. A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical, starting from the cusp and coming to rest at the vertex; prove that the coefficient of friction is given by the equation

$$\mu^2 e^{\mu\pi} = 1.$$

1418. Three equal and similar particles repelling each other with forces varying as the distance are connected by equal inextensible strings and are at rest: if one of the strings be cut, the subsequent angular velocity of either of the other strings will vary as $\sqrt{\left(\frac{1-2\cos\theta}{2+\cos\theta}\right)}$, θ being the angle between them.

1419. A heavy particle is attached to a fixed point by a fine string of length a, and when the string is horizontal and at its full length, the particle is projected horizontally at right angles to the string with a velocity due to a height $2a \cot \alpha$; prove that the greatest depth to which it will fall is $a \tan \frac{\alpha}{2}$.

1420. Two heavy particles are placed on a smooth cycloidal arc whose axis is vertical, and are connected by a fine string passing along the arc: c is the distance of either particle measured along the arc from its position of equilibrium; prove that the time of arriving at a distance s from the position of equilibrium is

$$\sqrt{\frac{4a}{g}} \log \frac{s + \sqrt{(s^2 - c^2)}}{c},$$

a being the radius of the generating circle.

1421. An elliptic wire is placed with its minor axis vertical, and on it slides a smooth ring to which are attached strings, passing through fixed rings at the foci and sustaining each a particle of weight equal to that of the ring: determine the velocity which the particle must have at the highest point that the velocity at the lowest point may be equal to that at the extremity of the major axis.

1422. Two particles of masses p, q are connected by a fine string passing through a small fixed ring; p hangs vertically, and q is held so that the adjacent part of the string is horizontal: if q be let go, the initial tension of the string is $\frac{pqg}{p+q}$, and the initial radius of curvature of q's path is

$$a \frac{\{p^2 + (p+q)^2\}^{\frac{3}{2}}}{p(p+q)(3p+q)};$$

a being the initial distance of q from the ring.

1423. Two particles of masses m, m' lying on a smooth horizontal table are connected by an inextensible string at its full length, and passing through a small fixed ring in the table; the particles are at distances a, a' from the ring, and are projected with velocities v, v' at right angles to the string, so that the parts of the string revolve in the same sense; prove that, if

$$mv^2a' = m'v'^2a,$$

either particle will describe a circle uniformly; and that, if

$$mv^2a^2 = m'v'^2a'^2,$$

their second apsidal distances will be a', a respectively.

1424. Two particles m, m' connected by a string which passes through a small fixed ring are held so that the string is horizontal, their distances from the ring being a, a', and are simultaneously let go; prove that

$$\frac{m}{\rho} = \frac{m'}{\rho'}, \quad \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{a} + \frac{1}{a'};$$

ρ, ρ' being the initial radii of curvature of their paths.

1425. Two particles A and B are connected by a fine string; A rests on a rough horizontal table, and B hangs vertically at a distance a below the edge of the table. A being on the point of motion, B is projected horizontally with a velocity u in the plane perpendicular to the edge of the table; prove that A will begin to move with acceleration $\frac{\mu}{\mu+1}\frac{u^2}{a}$, and that the initial radius of curvature of B's path will be $(\mu+1)a$, μ being the coefficient of friction.

1426. A smooth wire in the form of a circle is made to revolve uniformly in a horizontal plane about a point A in its circumference, with angular velocity ω. A small ring P slides on the wire and is initially at rest at its greatest distance c from A; prove that its distance from A at a time t is $\frac{2c}{\epsilon^{\omega t} + \epsilon^{-\omega t}}$, and that the tangent to P's path in space bisects the angle between PA and the radius to P.

1427. Two equal particles are connected by an inextensible string; one lies on a smooth horizontal table, and the string passes through a small fixed ring in the edge of the table to the other which is vertically below the ring: the former particle is projected on the table in a direction at right angles to the string with a velocity $\sqrt{\frac{2cg}{n(n+1)}}$, c being its distance from the ring; prove that its next apsidal distance will be $\frac{c}{n}$, and that its velocity will then be to its initial velocity as $n : 1$.

Prove also that the radius of curvature of the projected particle's initial path is $\frac{4c}{n(n+1)+2}$.

1428. Two particles, whose masses are p, q, are connected by a fine string passing through a small fixed ring; p hangs vertically, and q is projected so as to describe a path which is nearly a horizontal circle; prove that the distance of p at any time from its mean position will be

$$A \sin(mt + B) + A' \sin(nt + B');$$

m, n being the positive roots of the equation

$$\left\{x^2 - \frac{3g}{c}(1 - \cos\alpha)\right\}\left\{x^2 - \frac{g}{c\cos\alpha}(1 + 3\cos^2\alpha)\right\} = \frac{9g^2}{c^2}\cos\alpha\,(1 - \cos\alpha);$$

where c is the mean distance of q from the ring, and $p \equiv q\cos\alpha$.

1429. A particle is placed in a rough tube $\left(\mu = \frac{3}{4}\right)$ which revolves uniformly in one plane about one extremity, and no forces but the pressures of the tube are in operation; prove that the equation of the particle's path is

$$5r = a\left(4\epsilon^{\frac{\theta}{2}} + \epsilon^{-2\theta}\right).$$

1430. A rectilinear tube inclined at an angle α to the vertical revolves with uniform angular velocity ω about a vertical axis which intersects it, and a particle is projected from the stationary point of the tube with a velocity $\frac{g \cos a}{\omega \sqrt{(\sin a)}}$; find its position at any time before it attains relative equilibrium; and prove that the equilibrium is unstable.

1431. A smooth parabolic tube of latus rectum l is made to revolve about its axis, which is vertical, with angular velocity $\sqrt{\frac{g}{l}}$, and a particle is projected from the vertex up the tube; prove that the velocity of the particle is constant, and that the greatest height to which the particle rises in the tube is twice that due to the velocity of projection.

1432. A smooth parabolic tube revolves with uniform angular velocity about its axis, which is vertical, and a particle is placed within the tube very near to the lowest point: find the least angular velocity which the tube can have in order that the particle may ascend; and, if it ascend, prove that its velocity will be proportional to its distance from the axis. Prove also, that if any position of the particle in the tube be one of relative equilibrium, every position will be such.

1433. A curved tube is revolving uniformly about a vertical axis in its plane, and is symmetrical with respect to that axis; the angular velocity is $\sqrt{\frac{g}{a}}$, a being the radius of curvature at the vertex; prove that the equilibrium of a particle placed at the vertex will be stable or unstable according as the conic of closest contact at the vertex is an ellipse or hyperbola.

1434. A circular tube of radius a revolves uniformly about a vertical diameter with angular velocity $n\sqrt{\frac{g}{a}}$, and a particle is projected from its lowest point with such velocity as just to reach the highest point; prove that the time of describing the first quadrant is

$$\sqrt{\frac{a}{(n+1)g}} \log \{\sqrt{(n+2)} + \sqrt{(n+1)}\}.$$

1435. A circular tube containing a smooth particle revolves about a vertical diameter with uniform angular velocity ω, find the position of relative equilibrium of the particle; and prove that it will oscillate about this position in a time $\frac{2\pi}{\omega \sin \alpha}$, α being the angle which the normal at the point makes with the vertical.

1436. A heavy particle is placed in a tube in the form of a plane curve, which revolves with uniform angular velocity ω, about a vertical axis in its plane, and the particle oscillates about its position of relative equilibrium; prove that the time of oscillation is

$$\frac{2\pi}{\omega} \sqrt{\left(\frac{\rho \sin \alpha}{k - \rho \sin \alpha \cos^2 \alpha}\right)};$$

k being the distance from the axis of revolution, α the angle made by the normal with the vertical, and ρ the radius of curvature of the curve, at the point of equilibrium.

1437. A straight tube, inclined to the vertical at an angle α, revolves with uniform angular velocity ω about a vertical axis whose shortest distance from the tube is a, and contains a smooth particle placed initially at its shortest distance from the axis; prove that

$$2s = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \{\epsilon^{\omega t \sin \alpha} + \epsilon^{-\omega t \sin \alpha} - 2\} + c\{\epsilon^{\omega t \sin \alpha} - \epsilon^{-\omega t \sin \alpha}\};$$

s being the space described along the tube in a time t.

1438. A heavy particle is attached to two points in the same horizontal line, at a distance a, by two elastic strings, each of natural length a, and is set free when the strings are at their natural length; prove that the initial radius of curvature of its path is $\frac{2\sqrt{3}a}{m-n}$; the coefficients of elasticity being respectively m and n times the weight of the particle.

1439. A uniform heavy chain is placed on the arc of a smooth vertical circle, its length being equal to a quadrant, and one extremity being at the highest point of the circle; prove that, in the beginning of the motion, the resultant vertical pressure on the circle : the resultant horizontal pressure :: $\pi^2 - 4 : 4$.

IV. *Motion of uniform strings.*

1440. A heavy uniform string PQ of which P is the lower extremity, is in motion on a smooth circular arc in a vertical plane, O being the centre and OA the horizontal radius; prove that the tension at any point R of the string is

$$W\frac{\gamma}{a}\left\{\frac{\sin\gamma}{\gamma}\cos(\gamma+\theta)-\frac{\sin a}{a}\cos(a+\theta)\right\},$$

where θ, $2a$, 2γ are the angles AOP, POQ, POR respectively, and W is the weight of the string.

1441. A portion of a heavy uniform string is placed on the arc of a four-cusped hypocycloid, occupying the space between two adjacent cusps, and runs off the curve at the lower cusp, the tangent at which is vertical; prove that the velocity which the string will have when the whole of it has just left the curve, will be the velocity due to nine-tenths of the length of the string.

1442. A uniform string is placed on the arc of a smooth curve in a vertical plane, and moves under the action of gravity; prove that

$$\frac{d^2s}{dt^2}=\frac{g}{l}(y_2-y_1);$$

l being the length of the string, s the arc described by any point of it at a time t, and y_1, y_2 the depths of its ends below a fixed horizontal straight line.

1443. A uniform heavy string is placed on the arc of a smooth cycloid, whose axis is vertical and vertex upwards; determine the motion, and prove that, so long as the string is wholly in contact with the cycloid, the tension at any given point of the string is constant throughout the motion, and is a maximum at the middle point.

1444. A uniform heavy chain of length l is in motion on the arc of a smooth curve in a vertical plane, and the tangent at the point of greatest tension makes an angle ϕ with the vertical; prove that the difference between the depths of the extremities is $l \cos \phi$.

1445. A uniform inextensible string is at rest in a smooth groove which it just fits, and a tangential impulse P is applied at one extremity; prove that the normal impulse, referred to a unit of length of the string, at a distance s from the other extremity is $\frac{Ps}{a\rho}$, a being the whole length of the string, and ρ the radius of curvature at the point under consideration.

1446. A straight tube of uniform bore is revolving uniformly in a horizontal plane about a point at a distance c from the tube, and within the tube is a smooth uniform chain of length $2a$, which is initially at rest with its middle point at the distance c from the axis of revolution; prove that the chain in a time t will describe a space

$$\frac{c}{2}\left(\epsilon^{\omega t} - \epsilon^{-\omega t}\right)$$

along the tube, and that the tension at any point of the chain is

$$\frac{m}{4a}\left(a^2 - x^2\right)\omega^2;$$

where x is the distance of the point from the middle point, m the mass of the chain, and ω the angular velocity.

1447. A circular tube, of radius a, revolving with uniform angular velocity ω about a vertical diameter contains a heavy uniform chain, which subtends an angle 2α at the centre; prove that the chain will remain in relative equilibrium if the radius through its middle point makes with the vertical an angle

$$\cos^{-1}\left(\frac{g}{a\omega^2 \cos \alpha}\right);$$

and that the tension will be a maximum either at the lowest point of the tube, or at a point whose projection on the axis bisects the distance between the projections of the extremities.

V. *Resisting Medium, Hodograph.*

1448. A heavy particle is projected vertically upwards, and resistance of the air is $m\dfrac{(\text{vel.})^2}{c}$; x, y are the respective heights of two points in the ascent and descent at which the particle has the same velocity; prove that

$$\epsilon^{2\frac{a-x}{c}} + \epsilon^{-2\frac{a-y}{c}} = 2.$$

1449. A heavy particle moves in a medium in which the resistance varies as the square of the velocity; v, v' are its velocities at the two points where its direction of motion makes an angle a with the horizon, and u the velocity at the highest point; prove that

$$\frac{1}{v^2} + \frac{1}{v'^2} = \frac{2\cos^2 a}{u^2}.$$

1450. A particle is moving under the action of gravity in a medium whose resistance varies as the square of the velocity; ρ, ρ' are the radii of curvature of its path at two points at each of which the direction of motion makes an angle a with the horizon, and r the radius of curvature at the highest point; prove that

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2\cos^3 a}{r}.$$

1451. A small smooth bead slides on a fine wire whose plane is vertical, and the height of any point of which is $a\sin\dfrac{2s}{c}$, s being the arc measured from the lowest point, in a medium whose resistance is $m\dfrac{(\text{vel.})^2}{c}$, and starts from the point where $s=\dfrac{\pi c}{8}$; prove that the velocity acquired in falling to the lowest point is $\sqrt{ag}$.

1452. A heavy particle slides on a smooth curve, whose plane is vertical, in a medium whose resistance varies as the square of the velocity, and in any time describes a space which is to the space described in the same time by a particle falling freely in vacuo as $1 : 2n$; prove that the curve is a cycloid, the vertex being the highest point, and that the starting point of the particle divides the arc between two cusps in the ratio $2n-1 : 2n+1$.

1453. A point describes a straight line under acceleration tending to a fixed point and varying as the distance; prove that the corresponding point of the hodograph will move under the same law of acceleration.

1454. The curves

$$r^m = a^m \cos m\theta, \quad r^n = a^n \cos n\theta$$

will be each similar to the hodograph of the other when described about a centre of force in the pole, provided that

$$\frac{1}{m}+\frac{1}{n}+1=0.$$

Prove this property geometrically for both curves when $m=1$.

1455. A point describes a curve in such a manner that its hodograph is described as if under the action of a central force, and T, N are the tangential and normal accelerations of the point; prove that

$$N=3T\frac{ds}{d\rho}=\sqrt[3]{\frac{c^4}{\rho}};$$

ρ being the radius of curvature, s the arc measured from a fixed point, and c a constant.

1456. A point describes half the arc of a cardioid, oscillating symmetrically about the vertex, in such a way that the hodograph is a circle, the origin being in the circumference; prove that the acceleration of the point describing the cardioid will vary as $2r-3a$; r being the distance from the cusp, and $2a$ the length of the axis.

1457. A point P describes a catenary in such a manner that a straight line drawn from a fixed point parallel and proportional to the velocity of P sweeps out equal areas in equal times; prove that the direction of P's acceleration makes with the normal at P an angle

$$\tan^{-1}\left(\frac{2}{3}\tan\phi\right),$$

ϕ being the angle which the normal makes with the axis.

1458. If a circle be described under a constant acceleration not tending to the centre, the hodograph is a lemniscate.

1459. A curve is described under constant acceleration, and its hodograph is a parabola in which the radii vectores are drawn from the focus; prove that the intrinsic equation of the curve is

$$\frac{ds}{d\phi}=c\sec^5\frac{\phi}{2}.$$

1460. A point describes a curve whose hodograph is a circle described with uniform velocity about a point in the circumference; prove that the curve is a cycloid described with constant acceleration.

1461. A point describes a certain curve, its acceleration being initially normal; and when its direction of motion has turned through an angle ϕ, that of acceleration has turned through an angle 2ϕ in the same sense; prove that the hodograph of the path is a circle described about a point in the circumference.

1462. A particle is constrained to move in an elliptic tube under two forces to the foci, each varying inversely as the square of the distance and equal at equal distances, and is just displaced from the position of unstable equilibrium; prove that the hodograph is a circle.

DYNAMICS OF A RIGID BODY.

I. *Moments of Inertia, Principal Axes.*

1463. If m be the mass of an ellipsoid, of which the density at any point is proportional to the product of the distances of the point from the principal planes, the moment of inertia about one of the axes is $\frac{m(b^2+c^2)}{4}$; $2b$, $2c$ being the axes of the corresponding principal section.

1464. If a straight line be at every point of its course a principal axis of a given rigid body, it must pass through the centre of gravity.

1465. If A, B, C be the principal moments of inertia at the centre of gravity of a rigid body, $a^2+b^2+c^2+r^2$ a principal moment at a point whose co-ordinates referred to the principal axes through the centre of gravity are a, b, c; the equation determining r is

$$\frac{a^2}{A-r^2}+\frac{b^2}{B-r^2}+\frac{c^2}{C-r^2}=1,$$

the mass being unity.

1466. The locus of the points at which two of the principal moments of inertia of a given rigid body are equal is the focal curves of the ellipsoid of gyration for the centre of gravity.

1467. The locus of the points at which one of the principal axes of a given rigid body passes through a fixed point, in one of the principal planes through the centre of gravity, is a circle.

1468. The locus of the points at which one of the principal axes of a given rigid body is parallel to a given straight line is a rectangular hyperbola.

1469. In a triangular lamina any one of the sides is a principal axis at the point bisecting the distance between its middle point and the foot of the perpendicular on it from the opposite angle.

1470. If straight lines be drawn in the plane of a given lamina through a given point, the locus of the points at which they are respectively principal axes is a curve of the third degree.

1471. The locus of the straight lines drawn through a given point, each of which is at some point of its course a principal axis of a given rigid body, is the cone

$$a(B-C)yz+b(C-A)zx+c(A-B)xy=0,$$

A, B, C being the principal moments of inertia at the given point, a, b, c the co-ordinates of the centre of gravity; and the principal axes at the given point the axes of reference.

Prove that the locus of the points at which these straight lines are principal axes is the curve

$$x^2+y^2+z^2=\frac{(B-C)yz}{cy-bz}=\frac{(C-A)zx}{az-cx}.$$

1472. If the principal axes at any point be parallel to the principal axes through the centre of gravity, the point must lie on one of the principal axes at the centre of gravity.

II. *Motion about a Fixed Axis.*

1473. A circular disc rolls in one plane on a fixed plane, its centre describing a straight line with uniform acceleration f: find the magnitude and position of the resultant of the impressed forces.

1474. A piece of wire of given length is bent into the form of an isosceles triangle, and revolves about an axis through its vertex perpendicular to its plane; prove that the centre of oscillation will be at the least possible distance from the axis of revolution when the triangle is right-angled.

1475. A heavy sphere of radius [illegible], and a heavy rod of radius $2a$ swing, the one about a horizontal tangent, and the other about a horizontal axis through one extremity, through 90° to their lowest positions, and the pressures on the axes in those positions are equal; prove that their weights are as 35 : 34.

1476. The centre of percussion of a triangular lamina, one of whose sides is a fixed axis, bisects the straight line joining the opposite angle with the middle point of the side.

1477. A lamina $ABCD$ is moveable about AB which is parallel to CD; prove that its centre of percussion will be at the intersection of AC, BD, if $AB^2 = 3CD^2$.

1478. In the motion of a rigid body about a horizontal axis under the action of gravity, prove that the pressure of the axis can only be reduced to a single force, throughout the motion, when the axis of revolution is a principal axis at the point M which is nearest to the centre of gravity. If the axis be a principal axis, but at another point N, and the pressures on the axis be reduced to two, at M and N respectively, the former will be equal and opposite to the weight of the body.

1479. A rough uniform rod, length $2a$, is placed with a length $c\,(> a)$ projecting over the edge of a horizontal table, the rod being perpendicular to the edge; prove that the rod will begin to slide over the edge when it has turned through an angle

$$\tan^{-1} \frac{\mu a^2}{a^2 + 9(c-a)^2}.$$

1480. A uniform beam capable of motion about one extremity is in equilibrium; find at what point a blow must be applied, perpendicular to the rod, in order that the impulse on the fixed extremity may be equal to one-eighth of the blow.

1481. A uniform beam, moveable about its middle point, is in equilibrium in a horizontal position; a perfectly elastic particle whose mass is one fourth that of the beam is dropt upon one extremity, and is afterwards grazed by the other extremity; prove that the height from which the particle falls is to the length of the beam :: $49(2n+1)\pi : 48$, n being integral.

1482. A uniform rod is revolving about its middle point, which is fixed, on a smooth horizontal table, when it strikes a smooth inelastic particle at rest, whose mass is to its own as 1 : 6, and its angular velocity is immediately diminished one ninth; find the point of impact, and prove that when the particle leaves the rod the direction of motion of the ball will make with the rod an angle of 45°.

1483. A smooth uniform rod is moving uniformly on a horizontal table about one extremity, and impinges upon a particle of mass equal to its own, the distance of the point of impact from the fixed extremity being to the length of the rod :: 1 : m; prove that the final velocity of the particle will be to its initial velocity as

$$\sqrt{\{(5m^2-1)(m^2+3)\}} \ : \ 4m.$$

1484. A uniform rod is moving on a horizontal table about one extremity, and driving before it a particle of mass equal to its own, which starts from rest close to the fixed extremity; prove that when the particle has described a distance r along the rod, its direction of motion will make with the rod an angle

$$\tan^{-1}\frac{k}{\sqrt{(r^2+k^2)}};$$

k being the radius of gyration of the rod about the axis of revolution.

1485. A uniform circular disc of mass m is capable of motion about its centre in a vertical plane, and a rough particle of mass p is placed on it close to the highest point; prove that the angle θ, through which the disc will turn before the particle begins to slide, is given by the equation

$$m\sin\theta + 4\mu p = \mu\,(m+6p)\cos\theta.$$

1486. A uniform rod, capable of motion in a vertical plane about its middle point, has attached to its extremities by strings two equal particles which hang freely; when the beam is in equilibrium inclined at an angle a to the vertical, one of the strings is cut; prove that the initial tension of the other string is

$$\frac{mpg}{m + 3p\sin^2 a},$$

and that the radius of curvature of the initial path of the particle is $9l\frac{p}{m}\frac{\sin^3 a}{\cos a}$; m, p being the masses of the rod and of either particle, and l the length of the string.

1487. A uniform rod, moveable about one extremity, is held in a horizontal position, and to a point of the rod is attached a heavy particle by means of a string; prove that the initial tension of the string when the rod is let go is

$$\frac{mpga(4a-3c)}{4ma^2+3pc^2},$$

m, p being the masses of the rod and particle, $2a$ the length of the rod, and c the distance of the string from the fixed extremity.

Prove that the initial path of the particle is the curve whose equation, referred to horizontal and vertical axes, is

$$ma(4a-3c)y^3 + 90c^2l(ma+pc)x = 0;$$

l being the length of the string.

1488. A uniform rod, moveable about one extremity, has attached to the other extremity a heavy particle by means of a string, the rod and string are initially in one horizontal straight line and without motion; prove that the radius of curvature of the initial path of the particle is $\frac{4ab}{a+9b}$, a, b being the lengths of the rod and string.

1489. A uniform rod, of length $2a$ and mass m, capable of motion about one extremity, is held in a horizontal position, and on it slides a small smooth ring of mass p: if the rod be let go, the radius of curvature of the path of the ring is initially

$$\frac{9c^2}{4a-3c}\left(1+\frac{p}{m}\frac{c}{a}\right),$$

c being the initial distance of the ring from the fixed extremity.

1490. A uniform rod, capable of motion about one extremity, has attached to it at the other extremity a particle by means of a string, and the system is abandoned freely to the action of gravity, when the rod makes an angle α with the vertical; prove that the radius of curvature of the particle's initial path is

$$9l\,\frac{m+2p}{m}\,\frac{\sin^3\alpha}{\cos\alpha\,(2-3\sin^2\alpha)},$$

m, p being the masses of the rod and particle, and l the length of the string.

1491. A uniform rod is moveable about one extremity on a smooth horizontal table, and to the other extremity is fastened a particle by means of a string. Initially, the rod and string are in one straight line, the particle is at rest, and the rod has an angular velocity ω; prove that when the rod and string are next in a straight line, the angular velocity of the rod is to that of the string as $b : a$, or as

$$b\,\{3p\,(a-b)^2-ma^2\}\ :\ a\,\{3p\,(a-b)^2+ma\,(a-2b)\};$$

m, p being the masses of the rod and particle, and a, b the lengths of the rod and string.

III. *Motion in Two Dimensions.*

1492. Two equal uniform rods AB, BC, freely jointed at B and moveable about A, start from rest in a horizontal position, BC passing over a smooth peg whose distance from A is

$$4a\sin\alpha\left(>\frac{8a}{3}\right);$$

prove that when BC leaves the peg, the angular velocity of AB is

$$\sqrt{\left(\frac{3g}{4a}\,\frac{\cos\alpha}{1+\cos^2 2\alpha}\right)},$$

$2a$ being the length of either rod.

1493. A lamina, having its centre of gravity fixed, is at rest and is struck by a blow at the point (a, b) perpendicular to its plane; prove that the equation of the instantaneous axis is $Aax + Bby = 0$, the axes of co-ordinates being principal axes at the centre of gravity, and A, B the principal moments of inertia. If (a, b) lie on a certain straight line, there will be no impulse on the fixed point.

1494. A uniform beam revolves uniformly about one extremity in such a manner as to describe uniformly a cone of revolution about a vertical axis; determine the pressure on the fixed extremity and the relation between the angle of the cone and the time of revolution. If θ, ϕ be the angles which the vertical makes with the rod and with the direction of pressure respectively, then will

$$4 \tan \phi = 3 \tan \theta.$$

1495. A fine string, of length $2b$, is attached to two points in the same horizontal line, at a distance $2a$, and carries a particle p at its middle point; a uniform rod, of length $2c$ and mass m, has a ring at each end through which the string passes, and is let fall from a symmetrical position in the same straight line as the two points; prove that in order that the rod may reach the particle

$$(a + b - 2c)(m^2 + 2mp) > 2(2c - a)p^2.$$

1496. A circular disc rolls on a rough cycloidal arc whose axis is vertical and vertex downwards, the length of the arc being such that the curvature at either extremity is equal to that of the circle; prove that, if the contact be initially at one extremity, the point on the auxiliary circle of the cycloid corresponding to the point of contact will move with uniform velocity; and that this velocity will be independent of the radius of the disc.

1497. A sphere rolls from rest down a given length l of a rough inclined plane, and then traverses a smooth portion of the plane of length ml; find the impulse which the sphere sustains when perfect rolling again commences, and prove that the subsequent velocity is less than if the whole plane had been rough.

If $m=120$, the subsequent initial velocity is less than if the whole plane had been rough, in the ratio 67 : 77.

1498. A straight tube AB, of small bore, containing a smooth rod of the same length, is closed at the end B, and is in motion about the fixed end A with angular velocity ω; B being opened, prove that the initial tension of the rod at any point P is equal to $M\omega^2 \dfrac{AP \cdot PB}{2AB}$; M being the mass.

1499. The ends of a uniform heavy rod are fixed by smooth rings to the arc of a circle, which is made to revolve uniformly about a fixed vertical diameter; find the positions of relative equilibrium, and prove that any such position in which the rod is not horizontal is stable.

1500. A smooth semicircular disc rests with its plane vertical and vertex upwards on a smooth horizontal table, and on it rest two equal uniform rods, each of which passes through two fixed rings in a vertical line. The disc is slightly displaced, and in the ensuing motion one rod leaves the disc when the other is at the vertex; prove that

$$\frac{m}{p}=\frac{2(2\sin\alpha-1-\sin\beta)-\sin\beta\cos^2\beta}{\sin^3\beta},$$

m, p being the masses of the disc and of either rod, α the angle which the radius to either point of contact initially makes with the horizon, and $\beta\equiv\cos^{-1}(2\cos\alpha)$.

1501. A uniform rod moves with one extremity on a smooth horizontal plane, and the other attached to a string, which is fixed to a point above the plane; when the rod and string are in one straight line the rod is let go; prove that when the string is vertical its angular velocity is $\sqrt{\dfrac{g}{l}\,\dfrac{a+l-h}{a+l}}$, and the angular acceleration of the rod is $\dfrac{g}{h-l}\,\dfrac{a+l-h}{a+l}$, l, a being the lengths of the string and rod, and h the height of the fixed point above the plane.

1502. A sphere is resting on a rough horizontal plane, half its weight being supported by an elastic string attached to the highest point, the natural length of the string being the radius a, and the stretched length the diameter of the sphere; prove that the time of small oscillations of the sphere parallel to a vertical plane is $\pi\sqrt{\frac{14a}{15g}}$.

1503. A heavy uniform rod, resting in stable equilibrium within a smooth prolate spheroid whose axis is vertical, is slightly displaced in a vertical plane; prove that the length of the simple isochronous pendulum is $a\left(e+\frac{1}{3e}\right)$; $2a$ being the length of the rod, and e the eccentricity of the generating ellipse.

1504. A uniform beam rests with one end on a smooth horizontal table, and has the other attached to a fixed point by means of a string of length l; prove that the time of a small oscillation in a vertical plane is $\pi\sqrt{\frac{2l\tan\alpha}{g}}$, α being the mean inclination of the rod to the vertical.

1505. Two equal uniform rods AB, BC, freely jointed at B, are placed on a smooth horizontal table, at right angles to each other, and a blow is applied at A at right angles to AB; prove that the initial velocities of A, C are in the ratio 8 : 1.

1506. Two equal uniform rods AB, BC, freely jointed at B, are laid on a smooth horizontal table so as to include an angle α, and a blow is applied at A at right angles to AB; determine the initial velocity of C.

1507. Five equal uniform rods, freely jointed at their extremities, are laid in one straight line on a horizontal table, and a blow applied at the middle point at right angles to the line; prove that

$$\frac{v}{14a}=\frac{\omega}{9}=\frac{-\Omega}{3},$$

v being the initial velocity of the central rod, ω, Ω the initial angular velocities of the other pairs of rods, and $2a$ the length of each rod.

1508. Four equal uniform rods AB, BC, CD, DE, freely jointed at B, C, D, are laid on a horizontal table in the form of a square, and a blow is applied at A at right angles to AB from the inside of the square; prove that the initial velocity of A is 79 times that of E.

1509. Two equal uniform rods AB, BC, freely jointed at B and moveable about A, are lying on a smooth horizontal table inclined to each other at an angle α; a blow is applied at C at right angles to BC in a direction tending to decrease the angle ABC; prove that the initial angular velocities of AB, BC are in the ratio

$$\cos\alpha : 8 - 3\cos^2\alpha;$$

that θ, the least value of the angle ABC during the motion, is given by the equation

$$8(5 - 3\cos\theta)(2 - \cos^2\alpha) = (1 - \cos\alpha)^2(16 - 9\cos^2\alpha);$$

and, when $\alpha = \frac{\pi}{2}$, that the angular velocities of the rods when in a straight line are in the ratio

$$-1 : 3, \text{ or } 3 : -5.$$

1510. A uniform rod of length $2a$ rests in a horizontal position with its extremities on a smooth curve, which is symmetrical about a vertical axis; prove that the time of a small oscillation is

$$\pi\sqrt{\left\{\frac{a\rho\cos\alpha\,(1 + 2\cos^2\alpha)}{3g\,(a - \rho\sin^2\alpha\cos\alpha)}\right\}},$$

ρ being the radius of curvature and α the angle which the tangent makes with the horizon at either extremity of the rod in the position of equilibrium.

IV. *Miscellaneous.*

1511. A square is moving freely about a diagonal with angular velocity ω, when one of the angular points not in that diagonal becomes fixed; determine the impulsive pressure on the fixed point, and prove that the instantaneous angular velocity is $\frac{\omega}{7}$.

1512. A uniform rod, of length a, freely moveable about one extremity, is initially projected in a horizontal plane with angular velocity ω; prove that the equations of motion are

$$\sin^2\theta \frac{d\phi}{dt} = \omega, \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{a}\cos\theta - \omega^2 \cot^2\theta;$$

θ, ϕ being respectively the angles which the rod makes with the vertical, and which its projection on a horizontal plane makes with the initial position. If the least value of θ be $\frac{\pi}{3}$, the resolved vertical pressure on the fixed point when $\theta = \frac{\pi}{3}$ is to the weight of the rod as 31 : 16.

1513. A centre of force $= \mu$. distance is at a point O, and from another point A at a distance a, are projected simultaneously an infinite number of equal particles in a direction at right angles to OA, with velocities in arithmetical progression from $\frac{2a\sqrt{\mu}}{7}$ to $\frac{11a\sqrt{\mu}}{7}$; prove that, if after any lapse of time they become suddenly rigidly connected, the system will revolve with angular velocity $\frac{13\sqrt{\mu}}{14}$.

1514. A uniform rod, moveable about one extremity, moves in such a manner as to make always nearly the same angle α with the vertical; prove that the time of its small oscillations is

$$\pi\sqrt{\left(\frac{2a}{3g}\,\frac{\cos\alpha}{1+3\cos^2\alpha}\right)},$$

a being the length of the rod.

1515. A uniform rod is suspended by two strings of equal lengths, attached to its extremities and to two fixed points in the same horizontal line whose distance is equal to the length of the rod; an angular velocity is communicated to the rod, about a vertical axis through its centre, such that it just rises to the horizontal plane in which are the fixed points; find the impulsive couple, and prove that the tension of either string is instantaneously increased in the ratio 7 : 1.

1516. If two equal uniform rods AB, BC, freely jointed at B, rotate uniformly about a vertical axis through A, which is fixed, with angular velocity ω, the equations to determine the angles α, β, which the rods make with the vertical are

$$\frac{(8\sin\alpha+3\sin\beta)\cos\alpha}{3\sin\alpha}=\frac{(3\sin\alpha+2\sin\beta)\cos\beta}{\sin\beta}=\frac{3g}{2a\omega^2},$$

$2a$ being the length of either rod.

1517. A perfectly rough horizontal plane is made to rotate with constant angular velocity about a vertical axis which meets the plane in O; a sphere is projected on it at a point P, so that its centre is initially in the same state of motion as if the sphere had been placed freely on the plane at a point Q; prove that the centre will describe uniformly a circle of radius OQ, and whose centre R is such that OR is parallel and equal to QP.

1518. A rough plane, inclined at an angle α to the horizon, is made to revolve with uniform angular velocity ω about a normal, and a sphere without motion is placed upon it; prove that the path of the centre will be a prolate, a common, or a curtate ling as the initial point of contact is without, upon circle

$$2\omega^2(x^2+y^2)=35gx\sin\alpha,$$

being horizontal, and the axis of x the line of greatest slope. If the initial point of contact be the centre of this circle, the path will be a horizontal straight line.

1519. A rough hollow circular cylinder, whose axis is vertical, is made to rotate with uniform angular velocity ω about a fixed generator, and a heavy uniform sphere is rolling on the concave surface; prove that the equation of motion is

$$\left(\frac{d\phi}{dt}\right)^2=C+\frac{10}{7}\,\frac{a+b}{b}\,\omega^2\cos\phi,$$

where ϕ is the angle which the common normal to the sphere and cylinder makes with the plane containing the axis of revolution and the axis of the cylinder at a time t, and a, $a+b$ are the radii of the sphere and cylinder.

1520. A rough plane is made to revolve uniformly with angular velocity ω about a horizontal line in itself, and a sphere is projected so as to move upon it; determine the motion, and prove that, if when the plane is horizontal the centre of the sphere is vertically above the axis of revolution and moving parallel to it, the contact will cease when the plane has revolved through an angle θ given by the equation

$$\frac{11\cos\theta}{6}=\frac{a\omega^2}{g}+\left(\frac{5}{12}-\frac{a\omega^2}{g}\right)\left(\epsilon^{\theta\sqrt{\frac{5}{7}}}+\epsilon^{-\theta\sqrt{\frac{5}{7}}}\right);$$

a being the radius of the sphere.

1521. A uniform rod is free to move about one extremity in a vertical plane, that plane being constrained to revolve uniformly about a vertical axis through the fixed extremity with angular velocity ω, and the greatest and least angles which the rod makes with the vertical are α, β; prove that

$$2a\omega^2(\cos\alpha+\cos\beta)+3g=0,$$

$2a$ being the length of the rod. If $4a\omega^2 \cos \alpha = 3g$, prove that the time of a small oscillation is

$$\pi \sqrt{\left(\frac{4a \cos \alpha}{3g \sin^2 \alpha}\right)}.$$

1522. A number of concentric spherical shells of equal indefinitely small thickness revolve with uniform angular velocities about a common axis through the centre, each with an angular velocity proportional to the n^{th} power of its radius; the shells being suddenly rigidly united, prove that the subsequent angular velocity is to the previous angular velocity of the outer shell as $5 : n+5$.

1523. An infinite number of concentric spherical shells of equal small thickness are rotating about diameters all in one plane with equal angular velocities, the axis of any one whose radius is r being inclined at an angle $\cos^{-1} \frac{r}{a}$ to the axis of the outer shell; if they become suddenly united into a solid sphere, the new axis of rotation will make an angle $\tan^{-1} \frac{3\pi}{16}$ with the former axis of the outer shell.

1524. Any possible state of motion of a rigid rod may be represented by a single rotation about any one of an infinite number of axes lying in a certain plane.

1525. A free rigid body is in motion about its centre of gravity, when another point of the rigid body is suddenly fixed, and the body assumes a state of permanent rotation about an axis through that point; prove that the point must lie on a certain rectangular hyperbola.

1526. A rigid body is in motion under the action of no forces, and its centre of gravity is at rest; when the instantaneous axis is in a given position in the body, a point rigidly connected with the body is suddenly fixed, and the new instantaneous axis is parallel to one of the principal axes at the centre of gravity; prove that the fixed point must lie on a certain hyperbola, one asymptote of which is the given principal axis.

1527. A free rigid body is at a certain moment in a state of rotation about an axis through its centre of gravity, when another point of the body suddenly becomes fixed; determine the new instantaneous axis, and prove that there are three directions of the former instantaneous axis for which the new axis will be in the same direction, and that these three directions are along conjugate diameters of the central ellipsoid at the centre of gravity.

1528. A rigid body is in motion under the action of no forces, its centre of gravity being at rest, and the instantaneous axis describes a plane in the body; prove that if a point in that diameter of the central ellipsoid which is conjugate to this plane be suddenly fixed, the new instantaneous axis will be parallel to the former.

1529. Two equal uniform rods AB, BC, freely jointed at B and in one straight line, are moving uniformly in a direction at right angles to their length on a smooth horizontal table, when A is suddenly fixed; prove that the initial angular velocities of the rods are in the ratio $3 : -1$, that the least subsequent angle between them is

$$\cos^{-1}\left(-\frac{2}{3}\right),$$

and that when next in a straight line their angular velocities are as $1 : 9$.

1530. Three equal uniform rods AB, BC, CD, freely jointed at B and C, are lying in one straight line on a smooth table, when a blow is applied at the middle point at right angles to the rods; prove that

$$\frac{d\theta}{dt} = \frac{\omega}{\sqrt{(1+\sin^2\theta)}},$$

θ being the angle through which the two outer rods have turned in a time t, and ω their initial angular velocity. Prove also that the velocity of BC is

$$\frac{2a\omega}{3}\left\{1+\frac{\cos\theta}{\sqrt{(1+\sin^2\theta)}}\right\},$$

and that the direction of the strain at B or C makes with BC an angle

$$\tan^{-1}\left(\frac{2}{3}\tan\theta\right).$$

1531. Two equal uniform rods AB, BC, freely jointed at B, are in motion on a smooth horizontal table; when θ is the angle included between them, their angular velocities are ω, Ω; prove that

$$(5-3\cos\theta)(\omega+\Omega)=c_1,$$

$$5\omega^2-6\omega\Omega\cos\theta+5\Omega^2=c_2;$$

c_1, c_2 being constants depending on the initial circumstances.

1532. Three equal uniform inelastic rods, freely jointed at their extremities, are laid in a straight line on a smooth horizontal table, and the outer are set in motion about the ends of the middle one, with equal angular velocities, (1) in the same direction, (2) in opposite directions. Prove that in (1), when the outer rods make the greatest angle with the direction of the middle one produced on each side, the common angular velocity of the three is $\frac{4\omega}{7}$: and in (2) that after the impact of the two outer rods the triangle formed by the three will move with uniform velocity $\frac{a\omega}{3}$, a being the length of each rod.

1533. A uniform rod of length $2a$ has attached to one end a particle by a string of length b; the rod and string are placed in a straight line on a smooth horizontal table, and the particle projected at right angles to the string; prove that the greatest angle which the string can make with the rod produced is

$$2\sin^{-1}\sqrt{\frac{a}{12b}\,\frac{m+p}{p}};$$

where m, p are the masses of the rod and particle.

Prove that if, after any time t, the rod and string make angles θ, ϕ with their initial positions,

$$\{k^2+ab\cos(\phi-\theta)\}\frac{d\theta}{dt}+\{b^2+ab\cos(\phi-\theta)\}\frac{d\phi}{dt}=(a+b)\,v,$$

$$k^2\left(\frac{d\theta}{dt}\right)^2+2ab\cos(\phi-\theta)\frac{d\theta}{dt}\frac{d\phi}{dt}+b^2\left(\frac{d\phi}{dt}\right)^2=v^2;$$

where $k^2\equiv a^2\dfrac{4p+m}{3p}$, and v is the initial velocity of the particle.

1534. A circular disc capable of motion about a vertical axis through its centre, normal to its plane, is set in motion with angular velocity Ω, and at a given point is placed freely a rough uniform sphere: prove the equations of motion

$$\frac{7}{2}r^2\frac{d\theta}{dt}+k^2\omega=k^2\Omega,$$

$$\omega^2(r^2+k^2)(b^2+k^2)=k^4\Omega^2,$$

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2+\frac{2}{7}r\omega\frac{d\theta}{dt}=0,$$

r, θ being the polar co-ordinates of the point of contact measured from the centre of the disc, ω the angular velocity of the disc, b the initial value of r, and $k^2\equiv\dfrac{7pc^2}{4m}$, where m, p are the masses of the sphere and disc, and c the radius of the disc.

1535. A circular disc lies flat on a smooth horizontal table on which it can move freely, and has wound round it a fine string carrying a particle, which is projected with a velocity v from a point of the disc in a direction normal to the disc: prove that

$$\theta=k\sec^{-1}\left(\frac{vt}{ak^2}+1\right),\quad \theta+\phi=k\tan\frac{\theta}{k};$$

where θ, ϕ are the angles through which the string and the disc have turned at a time t, a is the radius, and $k^2 \equiv \frac{3}{2} + \frac{m}{p}$, m, p being the masses of the disc and particle.

1536. Two equal circular discs lying flat on a smooth horizontal table, are connected by a fine string coiled round each which is wound up till the discs are in contact with each other, and are on the same side of the tangent string. One of the discs has its centre fixed and can move freely about it, the other disc is projected with a velocity u at right angles to the tangent string; prove that the angle through which either disc will have turned after a time t is $\sqrt{1 + \frac{u^2t^2}{5a^2}} - 1$, and that the angle through which the string will have turned is $\frac{\sqrt{5}}{2} \tan^{-1} \frac{ut}{a\sqrt{5}}$; a being the radius of either disc.

1537. A smooth tube, mass m, lying on a horizontal table, contains a particle, mass p, which just fits it; the system is set in motion by a blow at right angles to the tube: prove that

$$\left(\frac{dr}{d\theta}\right)^2 (c^2 + A) = (r^2 - c^2)(r^2 + A);$$

r being the distance of the particle from the centre of the tube, when the tube has turned through an angle θ, c the initial value of r, $A \equiv \frac{a^2}{3} \frac{m+p}{p}$, where m, p, are the masses of the tube and particle, and $2a$ the length of the tube.

1538. A circular disc of mass m and diameter d, can move on a smooth horizontal plane about a fixed point A in its circumference, and a fine string is wound round it carrying a particle of mass p, which is initially projected from the disc, at the other end of the diameter through A, with a velocity u normal to the disc, the disc being then at rest. Prove that the angular velocity of the string will vanish when the length of the string unwound is that which initially subtended at A an angle θ, such that

$$8p(\theta \tan\theta + 1)\cos^2\theta + 3m = 0;$$

and that the angular velocity of the disc is then

$$\frac{u}{d}\left(\cos^2\theta + \frac{3m}{8p}\right)^{-\frac{1}{2}}.$$

1539. A rough sphere, radius a, moves on the concave surface of a vertical circular cylinder, radius $a+b$, and the centre of the sphere initially moves horizontally with a velocity v: prove that the depth of the centre below its initial position after a time t is

$$\frac{5gb^2}{2v^2}(1 - \cos nt); \quad \left(n^2 \equiv \frac{2v^2}{7b^2}\right).$$

Prove that, in order that perfect rolling may be maintained, the coefficient of friction must be not less than $\dfrac{12bg}{7v^2}$.

1540. A right circular cylinder is fixed with its axis horizontal and a rough sphere is projected so as to move in contact with the cylinder, being initially at the lowest point and its centre moving in a direction making an angle α with the axis of the cylinder: prove that, in order that the sphere may reach the highest point, its initial velocity must be not less than $\sqrt{\left(\dfrac{27}{7}\,\dfrac{bg}{\sin^2\alpha}\right)}$; a, $a+b$ being the radii of the sphere and cylinder respectively.

Prove the equations of motion

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{u^2\sin^2\alpha}{b^2} - \frac{10g}{7b}(1 - \cos\phi),$$

$$\frac{dz}{dt} = u\cos\alpha\cos\left(\phi\sqrt{\frac{2}{7}}\right),$$

$$a\omega = u\sqrt{\frac{7}{2}}\cos\alpha\sin\left(\phi\sqrt{\frac{2}{7}}\right):$$

z being the distance described by the centre parallel to the axis, ϕ the angle through which the common normal has turned, and ω the angular velocity of the sphere about that normal, at the end of a time t.

1541. A sphere, radius a, is in motion on the surface of a right circular cylinder, radius $a+b$, whose axis makes an angle α with the vertical; and is initially in contact with the lowest generator, its centre moving in a direction perpendicular to the axis with such a velocity that the sphere just makes complete revolutions: prove the equations of motion

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{g \sin \alpha}{7b} (17 + 10 \cos \phi),$$

$$a \frac{d\omega}{dt} = \frac{dz}{dt} \frac{d\phi}{dt},$$

$$\left(\frac{dz}{dt}\right)^2 + \frac{2}{7} a^2\omega^2 = \frac{10}{7} gz \cos \alpha;$$

z being the distance described by the centre parallel to the axis, ϕ the angle through which the common normal has turned, and ω the angular velocity about that normal, at the end of a time t.

1542. A rough sphere, radius a, rolls in a spherical bowl of radius $a+b$; the centre of the sphere is initially of the same height as the centre of the bowl and is moving horizontally with velocity u: prove that if θ be the angle which the common normal makes with the vertical, and ϕ the angle through which the vertical plane containing this normal has turned at the end of a time t,

$$\sin^2\theta \frac{d\phi}{dt} = \frac{u}{b}, \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{10g}{7b} \cos\theta - \frac{u^2}{b^2} \cot^2\theta.$$

Also if R, F, S be the reactions at the point of contact along the common normal, along the tangent which lies in the same vertical plane with the common normal, and at right angles to both these directions,

$$R = m\left(\frac{u^2}{b} + \frac{17g}{7} \cos\theta\right), \quad F = \frac{2mg}{7} \sin\theta, \quad S = 0;$$

and if ω_1, ω_2, ω_3 be the angular velocities of the sphere about these directions,

$$\omega_1 = 0, \quad a\omega_2 \sin\theta = u, \quad a\omega_3 = -b\frac{d\theta}{dt}.$$

1543. A rough sphere, radius a, rolls in a spherical bowl of radius $a+b$ in a state of steady motion, the normal making an angle α with the vertical: prove that the time of small oscillations about this position is

$$\pi\sqrt{\left\{\frac{7b\cos\alpha}{5g(1+3\cos^2\alpha)}\right\}}.$$

HYDROSTATICS.

In these questions a fluid is supposed to be uniform, heavy, and incompressible, unless otherwise stated: and all cones, cylinders, paraboloids, are supposed to be surfaces of revolution, and their bases circles.

1544. A cylinder is filled with equal volumes of n different fluids which do not mix; the density of the uppermost is ρ, of the next 2ρ, and so on, that of the lowest being $n\rho$; prove that the whole pressures on the corresponding portions of the curve surface are in the ratios

$$1^2 : 2^2 : 3^2 : \ldots : n^2.$$

1545. A hollow cylinder containing a weight W of fluid is held so that its axis makes an angle α with the horizon: prove that the resultant pressure on the curve surface is $W \cos \alpha$, and that its direction makes an angle α with the vertical.

1546. Equal volumes of three fluids are mixed and the mixture separated into three parts; to each of these parts is then added its own volume of one of the original fluids and the densities of the mixtures so formed are in the ratios $3 : 4 : 5$, prove that the densities of the fluids are as $1 : 2 : 3$.

1547. A thin tube in the form of an equilateral triangle is filled with equal volumes of three fluids which do not mix, and when held with one side vertical the three points of junction of the fluids bisect the sides: prove that the densities of the fluids are in arithmetical progression.

1548. A circular tube of uniform bore whose plane is vertical is half filled with equal volumes of four fluids which do not mix and whose densities are as 1 : 4 : 8 : 7; prove that the diameter joining the free surfaces will make an angle $\tan^{-1} 2$ with the vertical.

1549. A triangular lamina ABC right-angled at C is attached to a string at A, and rests with the side AC vertical and half its length immersed in fluid: prove that the density of the fluid is to that of the lamina :: 8 : 7.

1550. A lamina in the form of an equilateral triangle suspended freely from an angular point rests with one side vertical and another side bisected by the surface of a heavy uniform fluid: prove that the density of the lamina is to that of the fluid as 15 : 16.

1551. A hollow cone filled with fluid is suspended freely from a point in the rim of the base: prove that the total pressures on the curve surface and on the base in the position of rest are in the ratio

$$1 + 11 \sin^2 \alpha : 12 \sin^3 \alpha,$$

2α being the angle of the cone.

1552. A tube of small bore in the form of an ellipse is half filled with equal volumes of two given fluids which do not mix; find the inclination of its axes to the vertical in order that the free surfaces of the fluids may be at the ends of the minor axis.

1553. A hemisphere is filled with fluid and the surface is divided by horizontal planes into n portions on each of which the whole pressure is the same; prove that the depth of the r^{th} of these planes is to the radius of the hemisphere as $\sqrt{r} : \sqrt{n}$.

1554. A hemisphere is just filled with a uniform heavy fluid and the surface is divided by horizontal planes into n portions the whole pressures on which are in a geometrical progression of ratio k: prove that the depth of the r^{th} plane is to the radius as

$$\sqrt{k^r - 1} : \sqrt{k^n - 1}.$$

1555. An isosceles triangle is immersed with its axis vertical and its base in the surface of a uniform heavy fluid; prove that the resultant pressure on the area intercepted between any two horizontal planes acts through the centre of gravity of that portion of the volume of a sphere, described on the axis as diameter, which is intercepted between the planes.

1556. A conical shell is placed with its vertex upwards on a horizontal table and fluid is poured in through a small hole in the vertex; the cone begins to rise when the weight of the fluid poured in is equal to its own weight: prove that this weight is to the weight of fluid which would fill the cone as $9 - 3\sqrt{3} : 4$.

1557. A parabolic lamina bounded by a double ordinate perpendicular to the axis floats in a heavy uniform fluid with its focus in the surface of the fluid and its axis inclined at an angle $\tan^{-1} 2$ to the vertical; prove that the density of the fluid is to that of the lamina as $8 : 1$; and that the length of the axis is to the latus rectum as $15 : 4$.

1558. A lamina in the form of an isosceles triangle floats in a heavy uniform fluid with its plane vertical and (1) its base out of the fluid, (2) its base totally immersed in the fluid, its axis in these positions making angles θ, ϕ with the vertical; prove that

$$\tan^2\theta + \tan^2\phi = \frac{2\cos^2\alpha - 1}{\sin^2\alpha\cos^2\alpha},$$

and that both positions will not be possible unless 2α the vertical angle $< \cos^{-1}(\sqrt{2} - 1)$.

1559. A right circular cone with its axis vertical and vertex downwards is filled with two fluids which do not mix, and their common surface cuts off one-fourth of the axis from the vertex; prove that, if the whole pressures of the fluids on the curve surface be equal, their densities are as $45 : 1$.

1560. A barometer stands at 29·88 inches and the thermometer is at the dew-point; a barometer and a cup of water are placed under a receiver from which the air is removed and the barometer then stands at ·36 of an inch; find the space which would be occupied by a given volume of the atmosphere if it were deprived of its vapour without changing its pressure or temperature.

1561. In Hawksbee's air-pump, when the n^{th} stroke is half completed the machine is kept at rest; find the difference of the tensions of the two piston rods.

1562. In Smeaton's air-pump, find the position of the piston at that moment of the n^{th} stroke when the upper valve begins to open.

1563. If A, B be the volumes of the receiver and barrel of an air-pump, ρ, σ the densities of atmospheric air and of the air in the receiver respectively, and Π the pressure of the atmosphere; the work done in slowly raising the piston through one stroke is

$$\Pi\left\{B - A\frac{\sigma}{\rho}\log\left(1+\frac{B}{A}\right)\right\},$$

gravity being neglected.

1564. A portion of a right circular cone cut off by a plane through the axis, and two planes perpendicular to the axis is immersed in a uniform heavy fluid in such a manner that the vertex of the cone is in the surface and the axis vertical: prove that the resultant horizontal pressure on the curve surface passes through the centre of gravity of the body immersed.

1565. If it be assumed that the temperature of the atmosphere in ascending from the Earth's surface decreases slowly by an amount proportional to the height ascended, prove that the equation connecting the pressure p and the density ρ at any height will be of the form $p = k\rho^{1+m}$, m being very small.

1566. A right circular cylinder floats in a uniform heavy fluid with its axis inclined at an angle $\tan^{-1}\frac{2}{5}$ to the vertical, its upper circular boundary just out of the fluid, and the lower one completely immersed: prove that the length of the axis : radius :: 7 : 4.

1567. If a circular lamina be in a vertical position with its centre at a depth c below the surface of a heavy uniform fluid, the depth of the centre of pressure below the centre of figure is $\frac{a^2}{4c}$, a being the radius and $c > a$.

1568. A cone of density ρ floats with a generator vertical in a fluid of density σ, the base being out of the fluid; prove that, $2a$ being the vertical angle,

$$\frac{\rho}{\sigma} = (\cos 2a)^{\frac{3}{2}};$$

and that the length of the vertical side immersed is to the length of the axis as $\cos 2a : \cos a$.

1569. A right cone is moveable about its vertex which is fixed at a given distance c below the surface of a heavy fluid, and rests with its axis h inclined at an angle θ to the vertical, its base being completely out of the fluid: prove that

$$\frac{\cos\theta \cos^2 a}{(\cos^2\theta - \sin^2 a)^{\frac{5}{2}}} = \frac{\sigma h^4}{\rho c^4};$$

$2a$ being the vertical angle, ρ, σ the densities of the fluid and cone respectively. Prove that this position is stable; but cannot exist unless $\sigma h^4 \cos^2 a > \rho c^4$.

1570. An elliptic tube half full of heavy incompressible fluid revolves about a fixed vertical axis in its plane with angular velocity ω: prove that the angle which the straight line joining the free surfaces of the fluid makes with the vertical will be

$$\tan^{-1}\left(\frac{g}{p\omega^2}\right),$$

p being the distance of the axis from the centre of the ellipse.

1571. A hollow cone very nearly filled with fluid rotates uniformly about a generating line which is vertical: prove that the pressure on the base is

$$\frac{3W\tan\alpha}{8}\left\{\frac{a\omega^2}{g}(1+5\cos^2\alpha)+8\sin\alpha\right\};$$

W being the weight of the fluid, $2\alpha\left(<\frac{\pi}{2}\right)$ the vertical angle, a the radius of the base, and ω the angular velocity.

1572. A right circular cone, the length of whose axis is h and the radius of its base a, floats in a heavy uniform fluid with $\frac{27}{175}$ of its volume below the surface: prove that, when the fluid revolves about the axis of the cone with angular velocity

$$\sqrt{\frac{525}{592}\frac{gh}{a^2}},$$

the cone will float with a length h or $\frac{3h}{4}$ of its axis immersed. Investigate which of the two positions is stable.

1573. A sphere, radius a, floats in a mass of fluid, which is revolving uniformly about a vertical axis, with its centre at the vertex of the free surface of the fluid: prove that

$$4(p^2+4a^2)(a-pq)=a(p+4aq)^2;$$

where $p\equiv\frac{2g}{\omega^2}$, and $1+q:2$ is the ratio of the densities of the sphere and fluid.

1574. A hollow cone, very nearly filled with uniform heavy fluid, rotates uniformly about a horizontal generating line: prove that the whole pressure on the base in its lowest position is

$$\frac{\pi\rho a^4\omega^2}{8}\left(3+\frac{8g}{a\omega^2}\cos\alpha+5\cos^2\alpha\right).$$

1575. A hollow paraboloid, having a base perpendicular to the axis and at a distance from the vertex equal to the latus rectum, is placed with its axis vertical and vertex upwards, and contains seven-eighths of its volume of heavy uniform fluid. Find the angular velocity with which it must revolve about the axis in order that the surface of the fluid may be confocal with the paraboloid; and prove that in this case the pressure on the base will be greater than it was when the fluid was at rest in the ratio $2\sqrt{2} : 2\sqrt{2}-1$.

1576. If a uniform fluid be acted on by two central forces, each varying as the distance from a fixed point and equal at equal distances from those points, one attractive and the other repulsive, the surfaces of equal pressure will be planes.

1577. In a uniform fluid under the action of two forces tending to fixed points and varying inversely as the square of the distance, one attractive and one repulsive, prove that one surface of equal pressure is a sphere.

1578. A mass of elastic fluid is confined within a hollow sphere and repelled from the centre of the sphere by a force $\frac{\mu}{r}$; prove that the whole pressure on the sphere : the whole pressure which would be exerted if no force acted :: $3k+\mu : 3k$, the pressure being k . density.

1579. A quantity of uniform incompressible fluid, not acted on by gravity, just fills a hollow sphere, and is repelled from a point on the surface of the sphere by a force equal to μ . distance; the fluid revolves about the diameter through the centre of force with uniform angular velocity ω; find the whole pressure on the sphere, and prove that, if when the angular velocity is diminished one half the whole pressure is also diminished one half, $\omega^2 = 6\mu$.

1580. All space being supposed filled with an elastic fluid, the whole mass of which is known, which is attracted to a given point by a force varying as the distance; find the pressure at any point.

1581. Water is contained in a vessel having a horizontal base, and a right cone is supported, partly by the water and partly by the base on which the vertex rests: prove that, for stable equilibrium, the depth of the fluid must be greater than $h\sqrt[4]{(m\cos^2\alpha)}$, m being the number measuring the specific gravity of the cone, h the length of the axis, and 2α the vertical angle.

1582. A solid paraboloid is divided into two parts by a plane through the axis, and the parts are united by a hinge at the vertex; the system is placed in a heavy uniform fluid with its axis vertical and vertex downwards, and floats without separation of the parts: prove that the ratio of the density of the solid to that of the fluid must be greater than x^4, x being given by the equation

$$3hx^3 = 7a(1-x),$$

where h, $4a$ are the lengths of the axis and the latus rectum respectively.

1583. A right cone is floating with its axis vertical and vertex downwards in a fluid whose density varies as the depth: prove that, for stable equilibrium,

$$\cos^2\alpha < \frac{4}{5}\sqrt[4]{\frac{4\rho}{\sigma}};$$

2α being the vertical angle, ρ the density of the cone, and σ the density of the fluid at a depth equal to the height of the cone.

1584. A uniform rod rests in a position inclined to the vertical with half its length immersed in a fluid, and can turn freely about a point in it at a distance equal to one-sixth of its length from the lower end: compare the densities of the rod and fluid, and prove that the equilibrium is stable.

1585. A uniform rod is moveable about one extremity which is fixed below the surface of a fluid, and when slightly displaced from its highest position, it sinks till just immersed before rising: prove that, when at rest in the highest position, the pressure on the point of support was zero.

1586. Two equal uniform rods AB, BC, freely jointed at B, are capable of motion about A which is fixed at a given depth below the surface of a uniform heavy fluid: find the position in which both rods rest partly immersed; and prove that, if such a position be possible, the ratio of the density of the rods to the density of the fluid will be less than 1 : 3.

1587. A hemisphere, a point in whose base is attached to a fixed point by a fine string, rests with its centre in the surface of a fluid and its base inclined at an angle α to the horizon: prove that

$$\frac{\rho}{\sigma} = \frac{16(\pi - \alpha)\cos\alpha - 3\pi\sin\alpha}{2\pi(8\cos\alpha - 3\sin\alpha)};$$

ρ, σ being the densities of the hemisphere and fluid respectively.

1588. A cone is floating, with its axis vertical and vertex downwards, in fluid with $\frac{1}{n}$th of its axis immersed; a weight equal to the weight of the cone is placed upon the base after which the cone sinks till just totally immersed before rising: prove that

$$n^3 + n^2 + n = 7.$$

1589. A hollow cylinder whose axis is vertical contains a quantity of fluid whose density varies as the depth, and a cone, whose axis is coincident with that of the cylinder and which is of equal base, is allowed to sink slowly into the fluid with its vertex downwards. The cone is in equilibrium when just immersed; prove that the density of the cone is equal to the initial density of the fluid at a depth equal to one twelfth of the length of the axis of the cone.

If the cone be allowed to sink freely into the fluid, its vertex being initially at the surface, and sinks till just immersed, the density of the cone is to the density of the fluid at the vertex of the cone in its lowest position as 1 : 30.

1590. A semicircular tube of fine bore whose plane is vertical contains a quantity of fluid which subtends a given angle at the centre; a given heavy particle just fitting the tube is let fall from the extremity of a horizontal radius: find the impulsive pressure at any point of the fluid.

1591. A flexible inextensible envelope when filled with fluid has the form of a paraboloid, whose axis is vertical and vertex downwards and whose height is equal to five-eighths of the latus rectum; determine where the tension of the envelope along the meridian is greatest.

1592. Fluid without weight is contained in a thin flexible envelope in the form of a surface of revolution, and the tensions of the envelope at any point along and perpendicular to the meridian are equal: prove that the surface is a sphere.

1593. A quantity of homogeneous fluid is contained between two parallel planes and is in equilibrium in the form of a cylinder of radius b under a pressure ϖ; that portion of the fluid which lies within a distance a of the axis being suddenly annihilated, prove that the initial pressure at any point at a distance r from the axis is

$$\varpi \frac{\log r - \log a}{\log b - \log a}.$$

1594. A thin hollow cylinder of length h closed at one end, and fitted with an air-tight piston is placed mouth downwards in fluid. The weight of the piston is equal to that of the cylinder, the height of a cylinder of equal weight and radius formed of the fluid is a, the height of fluid which measures the atmospheric pressure is c, and the air enclosed in the cylinder would just fill it at atmospheric density: prove that, for small vertical oscillations, the distances of the piston and of the top of the cylinder from their respective positions of equilibrium, at a time t, are of the form

$$A \sin(\lambda t + \alpha) + B \sin(\mu t + \beta),$$

λ, μ being the positive roots of the equation

$$x^4 - \frac{g}{a} x^2 (2m + 1) + m \frac{g^2}{a^2} = 0,$$

and $m = \dfrac{(a+c)^2}{ch}$. The weight of the air in the cylinder and the resistance of the fluid may be neglected.

GEOMETRICAL OPTICS.

1595. There are three plane mirrors whose lines of intersection are parallel; a ray is incident on one of them in a plane perpendicular to all the mirrors in such a direction that after reflexion at the three mirrors its course is parallel to its original direction: prove that, after another reflexion at each of the three mirrors in the same order, it will return on its original path, and that the whole length of its path between the first and third reflexions at any mirror is independent of the point of incidence.

1596. A ray of light whose direction touches a conicoid is reflected at any confocal conicoid: prove that the reflected ray will also touch the conicoid.

1597. In a hollow ellipsoidal shell, small polished grooves are made coinciding with one series of circular sections, and a bright point is placed at one of the umbilici in which the series terminate: prove that the locus of the bright points seen by an eye in the opposite umbilicus is a central section of the ellipsoid; and that the whole length of the path of any ray by which a bright point is seen is constant.

1598. A ray proceeding from a point on the circumference of a circle is reflected n times at the circle; prove that its point of intersection with the consecutive ray similarly reflected is at a distance from the centre equal to $\dfrac{a}{2n+1}\sqrt{1+4n(n+1)\sin^2\theta}$, θ being the angle of incidence of the ray and a the radius.

1599. If a ray of light be reflected at two plane surfaces, its direction before incidence being parallel to the plane bisecting the angle between the mirrors, and making an angle θ with their line of intersection; its deviation will be $2\sin^{-1}(\sin\theta\sin\alpha)$; α being the angle between the mirrors.

1600. Two prisms of equal refracting angles are placed with one face of each in contact, and their other faces parallel, and a ray passes through the combination in a principal plane: prove that its deviation will be from the edge of the denser prism.

1601. If r, s be the radii of the bounding surfaces of a lens, and its thickness be $\left(1+\frac{1}{\mu}\right)\left(s-r\right)$, all the rays incident on the lens from a certain point will pass through without deviation and without aberration.

1602. What will be the *centre* of a lens whose bounding surfaces are confocal paraboloids having a common axis? Prove that the distance between the focal centres of the lens is

$$\frac{\mu-1}{\mu+1}(a \pm b);$$

$4a$, $4b$ being the latera recta.

1603. If the path of a ray through a medium of variable density be an arc of a circle in the plane of xy, the refractive index at a point (x, y) will be

$$\frac{1}{x-a} f\left(\frac{x-a}{y-b}\right);$$

f being an arbitrary function, and (a, b) the centre of the circle.

1604. A ray of light is propagated through a medium of variable density in a plane which divides the medium symmetrically: prove that the path is such that, if described by a point with velocity always proportional to μ the index of refraction, the accelerations of the point parallel to two rectangular axes x and y will be proportional to $\frac{d(\mu^2)}{dx}$, $\frac{d(\mu^2)}{dy}$ respectively.

1605. A ray is propagated in a medium of variable density in one plane (xy) which divides the medium symmetrically: prove that the projection of the radius of curvature at any point of the path of the ray on the normal to the surface of equal density through the point is equal to

$$\frac{\mu}{\sqrt{\left(\frac{d\mu}{dx}\right)^2 + \left(\frac{d\mu}{dy}\right)^2}}.$$

1606. A small pencil of parallel rays of white light, after transmission in a principal plane through a prism, is received on a screen whose plane is perpendicular to the direction of the pencil: prove that the length of the spectrum will be proportional to

$$\frac{(\mu_v - \mu_r)\sin\iota}{\cos^2 D \cos(D + \iota - \phi)\cos\phi'};$$

where ι is the angle of the prism; ϕ, ϕ' the angle of incidence and reflexion at the first surface, and D the deviation, of the mean ray.

1607. Two prisms of equal refracting angles are placed with one face of each in contact and their other faces parallel: prove that the condition of achromatism for two colours is

$$\frac{\delta\mu_1}{\cos\phi'} = \frac{\delta\mu_2}{\cos\psi'};$$

where ϕ, ϕ' are the angles of incidence and refraction at the first surface,

and ψ', ψ are the angles of incidence and refraction at emergence.

1608. When a ray of white light is refracted through a prism in a principal plane, so that the dispersion of two given colours is a minimum,

$$\frac{\sin(3\phi' - 2\iota)}{\sin\phi'} = 1 - \frac{2}{\mu^2};$$

ϕ' being the angle of refraction at the first surface and ι the refracting angle.

1609. Find the focal length of a lens equivalent to a system of three convex lenses on a common axis, of focal lengths 36 inches, 4 in., and 9 in. respectively, placed at intervals of 24 in. and 13 in., for a pencil proceeding from a point 18 in. in front of the first lens.

1610. Two thin lenses of focal lengths f_1, f_2 are on a common axis and separated by an interval a; the axis of an eccentric pencil before incidence cuts the axis of the lenses at a distance d from the first lens: prove that

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{a}{f_2}\left(\frac{1}{f_1}+\frac{1}{d}\right);$$

F being the focal length of the equivalent single lens.

1611. The focal length F of a single lens, equivalent to a system of three lenses of focal lengths f_1, f_2, f_3 separated by intervals a, b, for an eccentrical pencil parallel to the axis, is given by the equation

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{a}{f_2}\left(\frac{1}{f_1}+\frac{1}{f_3}\right)+\frac{b}{f_2}\left(\frac{1}{f_1}+\frac{1}{f_3}\right)+\frac{ab}{f_1f_2f_3}.$$

1612. Prove that the magnifying power of a combination of three convex lenses of focal lengths f_1, f_2, f_3 on a common axis at intervals a, b, will be independent of the position of the object, if

$$(f_2-a)(f_2-b)+f_2(f_1+f_3-a-b)=0.$$

SPHERICAL TRIGONOMETRY AND PLANE ASTRONOMY.

1613. In a spherical triangle ABC, $a = b = \frac{\pi}{3}$, $c = \frac{\pi}{2}$; prove that the spherical excess is $\cos^{-1}\frac{7}{9}$.

1614. In an equilateral spherical triangle ABC, A', B', C' are the middle points of the sides: prove that

$$2 \sin \frac{B'C'}{2} = \tan \frac{BC}{2}.$$

1615. In an equilateral spherical triangle, whose sides are each a and angles A, $2 \cos \frac{a}{2} \sin \frac{A}{2} = 1$.

1616. ABC is a spherical triangle, each of whose sides is a quadrant, P any point within the triangle: prove that

$$\cos^2 AP + \cos^2 BP + \cos^2 CP = 1\,;$$

$$\cos AP \cos BP \cos CP + \cot BPC \cot CPA \cot APB = 0\,;$$

and that $$\tan BCP \tan CAP \tan ABP = 1.$$

1617. A point P is taken within a spherical triangle ABC whose sides are all quadrants, and another triangle is described whose sides are equal to $2AP$, $2BP$, $2CP$ respectively: prove that the area of the latter triangle is twice that of the former.

1618. A spherical triangle ABC is equal and similar to its polar triangle: prove that

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1.$$

1619. If the sum of the sides of a spherical triangle be given, its area is greatest when the triangle is equilateral.

1620. In a spherical triangle ABC, $a+b+c=\pi$: prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1;$$

$$\cos a=\tan\frac{B}{2}\tan\frac{C}{2},\ \ \&c.;$$

$$\sin\frac{A}{2}=\cos\frac{B}{2}\cos\frac{C}{2}\sin a,\ \ \&c.$$

1621. In a spherical triangle, $A=B+C$: prove that

$$\sin^2\frac{a}{2}=\sin^2\frac{b}{2}+\sin^2\frac{c}{2}.$$

1622. If O be the pole of the small circle circumscribing a spherical triangle ABC,

$$\sin^2\frac{b}{2}+\sin^2\frac{c}{2}-\sin^2\frac{a}{2}=2\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{BOC}{2};$$

and, if P be any point on the circle,

$$\sin\frac{a}{2}\sin\frac{PA}{2}+\sin\frac{b}{2}\sin\frac{PB}{2}+\sin\frac{c}{2}\sin\frac{PC}{2}=0,$$

that arc of the three PA, PB, PC being reckoned negative which crosses one of the sides.

1623. Prove that when the Sun rises in the north east at a place in latitude l, the hour angle at sunrise is $\cot^{-1}(-\sin l)$.

1624. If in latitude 45° the observed time of transit of a star in the equator be unaffected by the combined effect of the errors of level and deviation of the transit instrument, these errors will be very nearly equal to each other.

1625. If m be the ratio of the radius of the Earth's orbit to that of an inferior planet, n the ratio of their motions in longitude considered uniform, the elongation of a planet as seen from the Earth when the planet is stationary is $\tan^{-1}\sqrt{\dfrac{1-m^2n^2}{m^2-1}}$.

1626. The maximum value of the aberration in declination of a given star is

$$20{\cdot}5'' \sqrt{1 - (\cos\delta \cos\omega + \sin\delta \sin\omega \sin a)^2};$$

a, δ being the right ascension and declination of the star, and ω the obliquity of the ecliptic.

1627. If a, a'; δ, δ' be the right ascensions and declinations of two stars whose aberrations in declination vanish simultaneously, and A the sun's right ascension at the time of their vanishing;

$$\tan A = \frac{\tan\delta \sin a - \tan\delta' \sin a'}{\tan\delta \cos a - \tan\delta' \cos a'}.$$

1628. If the latitude of a place has been determined by observation of two zenith distances of the Sun and the time between them, and each observed distance was too great by the same small quantity h; the consequent error in the latitude will be

$$h \frac{\cos \dfrac{a + a'}{2}}{\cos \dfrac{a - a'}{2}};$$

a, a' being the azimuths at the times of observation.

THE END.

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.