A COURSE

OF

PURE GEOMETRY

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CONTAINING A COMPLETE GEOMETRICAL TREATMENT OF THE PROPERTIES OF THE CONIC SECTIONS

BY

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PREFACE

THIS work is a revision and enlargement of my *Course of Pure Geometry* published in 1903. It differs from the former edition in that it does not assume any previous knowledge of the Conic Sections, which are here treated *ab initio*, on the basis of the definition of them as the curves of projection of a circle. This is not the starting point of the subject generally in works on Geometrical Conic Sections. The curves are usually defined by means of their focus and directrix property, and their other properties are evolved therefrom. Here the focus and directrix property is established as one belonging to the projections of a circle and it is freely used, but the fact that the conics are derived by projection from a circle and therefore possess all its projective properties is kept constantly in the mind of the student.

Many of the properties of the Conic Sections which can only be established with great labour from their focus and directrix property are proved quite simply when the curves are derived directly from the circle.

Nor is the method employed here any more difficult than the prevalent one, though it is true that certain ground has to be covered first. But this is not very extensive and I have indicated (p. xii) the few articles which a Student should master before he proceeds to Chapter IX. Without a certain knowledge of cross ratios, harmonic section, involution and the elementary principles of conical projection no one can follow the argument here adopted. But these things are quite easy, and the advantage gained by the student who from the beginning sees the Conic Sections whole, as he does when they are presented to his mind as the projections of a circle, more than compensates for any delay there may be through the short study of the necessary preliminaries.

I hope that, thanks to the efficiency of the Readers of the Cambridge University Press, there are not many misprints to be found in this book. But if any are found I shall be grateful if I may be informed of them at the address given below.

E. H. A.

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September 1917.

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SIMILARITY OF FIGURES

The student who may be using this work as a first text book on Geometrical Conic Sections will be able to proceed to Chapter IX after reading the following paragraphs of the first eight chapters:

13 to 16*a*, 29 to 35, 40 to 45, 48 to 53, 58, 67, 68, 69 to 7[illegible], 77 to 87.

CHAPTER I

SOME PROPERTIES OF THE TRIANGLE

1 Definition of terms.

}y *lines*, unless otherwise stated, will be meant *straight*

4.

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The lines joining the vertices of a triangle to the middle the opposite sides are called its *medians*.

(*c*) y the *circumcircle* of a triangle is meant the circle passing through its vertices.

The centre of this circle will be called the *circumcentre* of the triangle.

The reader already knows that the circumcentre is the point of intersection of the perpendiculars to the sides of the triangle drawn through their middle points.

(*d*) The *incircle* of a triangle is the circle touching the sides of the triangle and lying within the triangle.

The centre of this circle is the *incentre* of the triangle.

The incentre is the point of intersection of the lines bisecting the angles of the triangle.

(*e*) An *ec..cle* of a triangle is a circle touching one side of ...d the other two sides produced. There are three

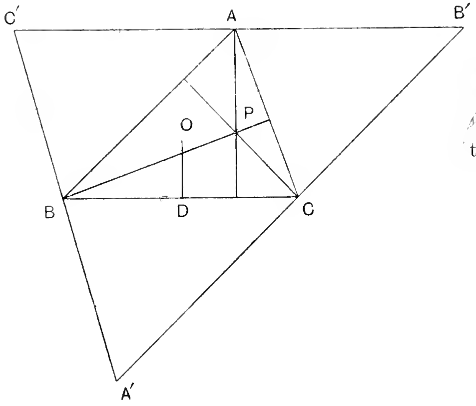
...tre of an ecircle is called an *ecentre*.

An ecentre is the point of intersection of the bisector of one of the angles and of the bisectors of the other two external angles.

(*f*) Two triangles which are such that the sides and angles of the one are equal respectively to the sides and angles of the other will be called *congruent*.

If ABC be congruent with $A'B'C'$, we shall express the fact by the notation: $\triangle ABC \equiv \triangle A'B'C'$.

2. Proposition. *The perpendiculars from the vertices of a triangle on to the opposite sides meet in a point (called the* orthocentre); *and the distance of each vertex from the orthocentre is twice the perpendicular distance of the circumcentre from the side opposite to that vertex.*



Through the vertices of the triangle ABC draw lines parallel to the opposite sides. The triangle $A'B'C'$ thus formed will be similar to the triangle ABC, and of double its linear dimensions.

Moreover, A, B, C being the middle points of the sides of $A'B'C'$, the perpendiculars from these points to the sides on which they lie will meet in the circumcentre of $A'B'C'$.

But these perpendiculars are also the perpendiculars from A, B, C to the opposite sides of the triangle ABC.

Hence the first part of our proposition is proved.

Now let P be the orthocentre and O the circumcentre of ABC.

Draw OD perpendicular to BC.

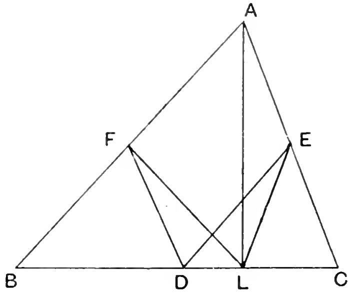
Then since P is also the circumcentre of the triangle $A'B'C'$, PA and OD are corresponding lines in the two similar triangles $A'B'C'$, ABC.

Hence AP is twice OD.

3. **Definition.** It will be convenient to speak of the perpendiculars from the vertices on to the opposite sides of a triangle as the *perpendiculars of the triangle;* and of the perpendiculars from the circumcentre on to the sides as the *perpendiculars from the circumcentre.*

4. **Prop.** *The circle through the middle points of the sides of a triangle passes also through the feet of the perpendiculars of the triangle and through the middle points of the three lines joining the orthocentre to the vertices of the triangle.*

Let D, E, F be the middle points of the sides of the triangle ABC, L, M, N the feet of its perpendiculars, O the circumcentre, P the orthocentre.



Join FD, DE, FL, LE.

Then since E is the circumcentre of ALC,

$$\angle ELA = \angle EAL.$$

And for a like reason

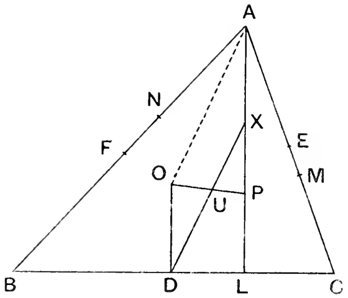
$$\angle FLA = \angle FAL.$$

$\therefore \angle FLE = \angle FAE$

$= \angle FDE$ since $AFDE$ is a parallelogram.

$\therefore L$ is on the circumcircle of DEF.

Similarly M and N lie on this circle.



Further the centre of this circle lies on each of the three lines bisecting DL, EM, FN at right angles.

Therefore the centre of the circle is at U the middle point of OP.

Now join DU and produce it to meet AP in X.

The two triangles OUD, PUX are easily seen to be congruent, so that $UD = UX$ and $XP = OD$.

Hence X lies on the circle through D, E, F, L, M, N.

And since $XP = OD = \frac{1}{2}AP$, X is the middle point of AP.

Similarly the circle goes through Y and Z, the middle points of BP and CP.

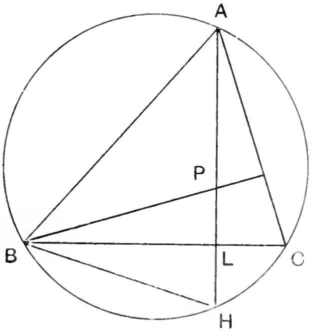
Thus our proposition is proved.

5. The circle thus defined is known as the *nine-points circle* of the triangle. Its radius is half that of the circumcircle, as is obvious from the fact that the nine-points circle is the circumcircle of DEF, which is similar to ABC and of half its linear

dimensions. Or the same may be seen from our figure wherein $DX = OA$, for $ODXA$ is a parallelogram.

It will be proved in the chapter on Inversion that the nine-points circle touches the incircle and the three ecircles of the triangle.

6. **Prop.** *If the perpendicular AL of a triangle ABC be produced to meet the circumcircle in H, then $PL = LH$, P being the orthocentre.*



Join BH.

Then $\angle HBL = \angle HAC$ in the same segment
$= \angle LBP$ since each is the complement of $\angle ACB$.

Thus the triangles PBL, HBL have their angles at B equal, also their right angles at L equal, and the side BL common.

$\therefore PL = LH.$

7. **Prop.** *The feet of the perpendiculars from any point Q on the circumcircle of a triangle ABC on to the sides of the triangle are collinear.*

Let R, S, T be the feet of the perpendiculars as in the figure. Join QA, QB.

$QTAS$ is a cyclic quadrilateral since T and S are right angles.

$\therefore \angle ATS = \angle AQS$

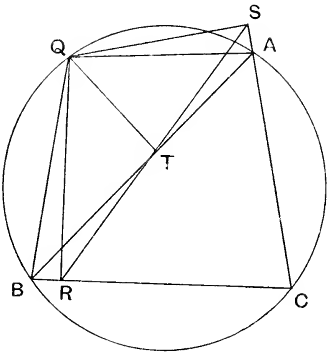
$= \text{complement of } \angle QAS$

$= \text{complement of } \angle QBC$ (since QAC, QBC are supplementary)

$= \angle BQR$

$= \angle BTR$ (since $QBRT$ is cyclic).

$\therefore$ RTS is a straight line.



This line RTS is called the *pedal line* of the point Q. It is known also as the *Simson* line.

The converse of this proposition also holds good, viz.:

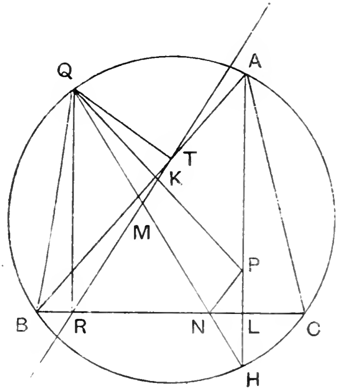
If the feet of the perpendiculars from a point Q on to the sides of a triangle are collinear, Q lies on the circumcircle of the triangle.

For since $QBRT$ and $QTAS$ are cyclic,

$$\angle BQR = \angle BTR = \angle ATS = \angle AQS.$$

$\therefore \angle QBR = \angle QAS$, so that $QBCA$ is cyclic.

8. Prop. *The pedal line of Q bisects the line joining Q to P, the orthocentre of the triangle.*



Join QP cutting the pedal line of Q in K.

Let the perpendicular AL meet the circumcircle in H.

Join QH cutting the pedal line in M and BC in N.

Join PN and QB.

Then since $QBRT$ is cyclic,

$$\begin{aligned} \angle QRT &= \angle QBT \\ &= \angle QHA \text{ in same segment} \\ &= \angle HQR \text{ since } QR \text{ is parallel to } AH. \end{aligned}$$

$$\therefore\ QM = MR.$$

$\therefore$ M is the middle point of QN.

But $\angle PNL = \angle LNH$ since $\triangle PNL \equiv \triangle HNL$

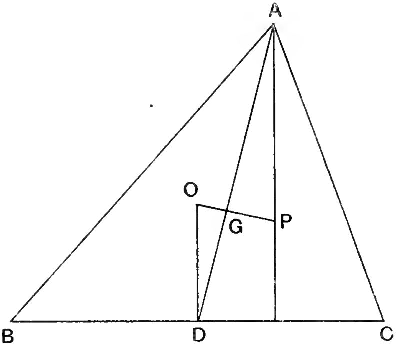
$$\begin{aligned} &= \angle RNM \\ &= \angle MRN. \end{aligned}$$

$\therefore$ PN is parallel to RT.

$\therefore$ $QK : KP = QM : MN$.

$$\therefore\ QK = KP.$$

9. Prop. *The three medians of a triangle meet in a point, and this point is a point of trisection of each median, and also of the line joining the circumcentre O and the orthocentre P.*



Let the median AD of the triangle ABC cut OP in G.

Then from the similarity of the triangles GAP, GDO, we deduce, since $AP = 2OD$, that $AG = 2GD$ and $PG = 2GO$.

Thus the median AD cuts OP in G which is a point of trisection of both lines.

Similarly the other medians cut OP in the same point G, which will be a point of trisection of them also.

This point G is called the *median point* of the triangle. The reader is probably already familiar with this point as the centroid of the triangle.

10. Prop. *If AD be a median of the triangle ABC, then*

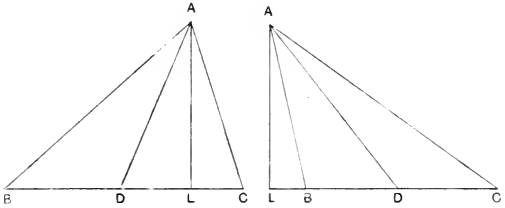
$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

Draw AL perpendicular to BC.

Then $$AC^2 = AB^2 + BC^2 - 2BC \, . \, BL$$

and $$AD^2 = AB^2 + BD^2 - 2BD \, . \, BL.$$

These equalities include the cases where both the angles B and C are acute, and where one of them, B, is obtuse, provided that BC and BL be considered to have the same or opposite signs according as they are in the same or opposite directions.



Multiply the second equation by 2 and subtract from the first, then

$$AC^2 - 2AD^2 = BC^2 - AB^2 - 2BD^2.$$

$$\therefore\ AB^2 + AC^2 = 2AD^2 + BC^2 - 2BD^2$$

$$= 2AD^2 + 2BD^2, \text{ since } BC = 2BD.$$

11. The proposition proved in the last article is only a special case of the following general one:

If D be a point in the side BC of a triangle ABC such that $BD = \frac{1}{n} BC$, then

$$(n-1)\,AB^2 + AC^2 = n\,.\,AD^2 + \left(1 - \frac{1}{n}\right) BC^2.$$

For proceeding as before, if we now multiply the second of the equations by n and subtract from the first we get

$$AC^2 - n\,.\,AD^2 = (1-n)\,AB^2 + BC^2 - n\,.\,BD^2.$$

$$\therefore\ (n-1)\,AB^2 + AC^2 = n\,.\,AD^2 + BC^2 - n\left(\frac{1}{n}\,BC\right)^2$$

$$= n\,.\,AD^2 + \left(1 - \frac{1}{n}\right) BC^2.$$

12. Prop. *The distances of the points of contact of the incircle of a triangle ABC with the sides from the vertices A, B, C are $s-a$, $s-b$, $s-c$ respectively; and the distances of the points of contact of the ecircle opposite to A are s, $s-c$,*

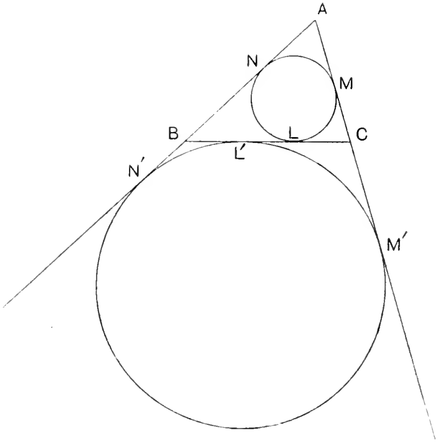
$s-b$ respectively; a, b, c being the lengths of the sides opposite to A, B, C and s half the sum of them.

Let the points of contact of the incircle be L, M, N.

Then since $AM = AN$, $CL = CM$ and $BL = BN$,

$$\therefore\ AM + BC = \text{half the sum of the sides} = s.$$

$$\therefore\ AM = s - a.$$



Similarly $BL = BN = s - b$, and $CL = CM = s - c$.

Next let L', M', N' be the points of contact of the ecircle opposite to A.

Then $$AN' = AB + BN' = AB + BL'$$
and $$AM' = AC + CM' = AC + CL'.$$

$$\therefore \text{ since } AM' = AN',$$

$$2AN' = AB + AC + BC = 2s.$$

$$\therefore\ AN' = s,$$

and $$BL' = BN' = s - c, \text{ and } CL' = CM' = s - b.$$

Cor. $BL' = CL$, and thus LL' and BC have the same middle point.

EXERCISES

1. Defining the pedal triangle as that formed by joining the feet of the perpendiculars of a triangle, shew that the pedal triangle has for its incentre the orthocentre of the original triangle, and that its angles are the supplements of twice the angles of the triangle.

2. A straight line PQ is drawn parallel to AB to meet the circumcircle of the triangle ABC in the points P and Q, shew that the pedal lines of P and Q intersect on the perpendicular from C on AB.

3. Shew that the pedal lines of three points on the circumcircle of a triangle form a triangle similar to that formed by the three points.

4. The pedal lines of the extremities of a chord of the circumcircle of a triangle intersect at a constant angle. Find the locus of the middle point of the chord.

5. Given the circumcircle of a triangle and two of its vertices, prove that the loci of its orthocentre, centroid and nine-points centre are circles.

6. The locus of a point which is such that the sum of the squares of its distances from two given points is constant is a sphere.

7. A', B', C' are three points on the sides BC, CA, AB of a triangle ABC. Prove that the circumcentres of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are the angular points of a triangle which is similar to ABC.

8. A circle is described concentric with the circumcircle of the triangle ABC, and it intercepts chords A_1A_2, B_1B_2, C_1C_2 on BC, CA, AB respectively; from A_1 perpendiculars A_1b_1, A_1c_1 are drawn to CA, AB respectively, and from A_2, B_1, B_2, C_1, C_2 similar perpendiculars are drawn. Shew that the circumcentres of the six triangles, of which Ab_1c_1 is a typical one, lie on a circle concentric with the nine-points circle, and of radius one-half that of the original circle.

9. A plane quadrilateral is divided into four triangles by its internal diagonals; shew that the quadrilaterals having for angular points (i) the orthocentres and (ii) the circumcentres of the four

triangles are similar parallelograms; and if their areas be Δ_1 and Δ_2, and Δ be that of the quadrilateral, then $2\Delta + \Delta_1 = 4\Delta_2$.

10. Prove that the line joining the vertex of a triangle to that point of the inscribed circle which is farthest from the base passes through the point of contact of the escribed circle with the base.

11. Given in magnitude and position the lines joining the vertex of a triangle to the points in which the inscribed circle and the circle escribed to the base touch the base, construct the triangle.

12. Prove that when four points A, B, C, D lie on a circle, the orthocentres of the triangles BCD, CDA, DAB, ABC lie on an equal circle.

13. Prove that the pedal lines of the extremities of a diameter of the circumcircle of a triangle intersect at right angles on the nine-points circle.

14. ABC is a triangle, O its circumcentre; OD perpendicular to BC meets the circumcircle in K. Prove that the line through D perpendicular to AK will bisect KP, P being the orthocentre.

15. Having given the circumcircle and one angular point of a triangle and also the lengths of the lines joining this point to the orthocentre and centre of gravity, construct the triangle.

16. If AB be divided at O in such a manner that

$$l \,.\, AO = m \,.\, OB,$$

and if P be any point, prove

$$l \,.\, AP^2 + m \,.\, BP^2 = (l+m)\, OP^2 + l \,.\, AO^2 + m \,.\, BO^2.$$

If a, b, c be the lengths of the sides of a triangle ABC, find the locus of a point P such that $a \,.\, PA^2 + b \,.\, PB^2 + c \,.\, PC^2$ is constant.

CHAPTER II

SOME PROPERTIES OF CIRCLES

13. Definition. When two points P and P' lie on the same radius of a circle whose centre is O and are on the same side of O and their distances from O are such that $OP \,.\, OP' =$ square of the radius, they are called *inverse points* with respect to the circle.

The reader can already prove for himself that if a pair of tangents be drawn from an external point P to a circle, centre O, the chord joining the points of contact of these tangents is at right angles to OP, and cuts OP in a point which is the inverse of P.

14. The following proposition will give the definition of the *polar* of a point with respect to a circle :

Prop. *The locus of the points of intersection of pairs of tangents drawn at the extremities of chords of a circle, which pass through a fixed point, is a straight line, called the* polar *of that point, and the point is called the* pole *of the line.*

Let A be a fixed point in the plane of a circle, centre O.

Draw any chord QR of the circle to pass through A.

Let the tangents at Q and R meet in P.

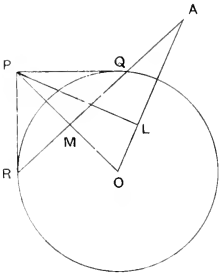
Draw PL perpendicular to OA.

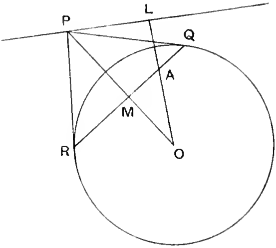
Let OP cut QR at right angles in M.

Then $PMLA$ is cyclic.

$\therefore OL \,.\, OA = OM \,.\, OP =$ square of radius.

$\therefore L$ is a fixed point, viz. the inverse of A.





Thus the locus of P is a straight line perpendicular to OA, and cutting it in the inverse point of A.

15. It is clear from the above that the polar of an external point coincides with the chord of contact of the tangents from that point. And if we introduce the notion of imaginary lines, with which Analytical Geometry has furnished us, we may say that the polar of a point coincides with the chord of contact of tangents real or imaginary from that point.

We may remark here that the polar of a point on the circle is the tangent at that point.

Some writers *define* the polar of a point as the chord of contact of the tangents drawn from that point; others again define it by means of its harmonic property, which will be given in a later chapter. It is unfortunate that this difference of treatment prevails. The present writer is of opinion that the method he has here adopted is the best.

16. Prop. *If the polar of A goes through B, then the polar of B goes through A.*

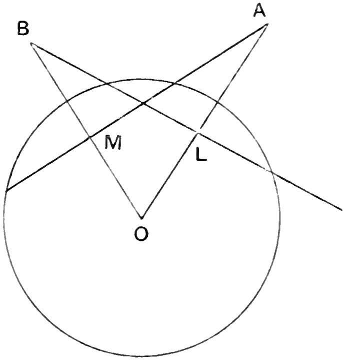
Let BL be the polar of A cutting OA at right angles in L.

Draw AM at right angles to OB.

Then $OM \,.\, OB = OL \,.\, OA =$ sq. of radius,

$\therefore AM$ is the polar of B,

that is, A lies on the polar of B.



Two points such that the polar of each goes through the other are called *conjugate points.*

The reader will see for himself that inverse points with respect to a circle are a special case of conjugate points.

We leave it as an exercise for the student to prove that if l, m be two lines such that the pole of l lies on m, then the pole of m will lie on l.

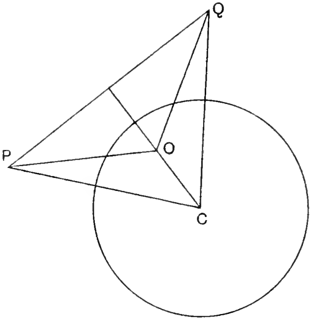
Two such lines are called *conjugate lines.*

From the above property for conjugate points we see that the polars of a number of collinear points all pass through a common point, viz. the pole of the line on which they lie. For if A, B, C, D, &c., be points on a line p whose pole is P: since the polar of P goes through A, B, C, &c., $\therefore$ the polars of A, B, C, &c., go through P.

We observe that the intersection of the polars of two points is the pole of the line joining them.

16a. Prop. *If OP and OQ be a pair of conjugate lines of a circle which meet the polar of O in P and Q, then the triangle OPQ is such that each vertex is the pole of the opposite side, and the centre of the circle is the orthocentre of the triangle.*

For the pole of OQ must lie on the polar of O, and it also lies on OP, since OP and OQ are conjugate lines. Thus OQ is the polar of P. Similarly OP is the polar of Q.



Also the lines joining C the centre to O, P, Q are perpendicular respectively to the polars of those points, and therefore C is the orthocentre of the triangle.

17. Prop. *If P and Q be any two points in the plane of a circle whose centre is O, then*

$$OP : OQ = \textit{perp. from } P \textit{ on polar of } Q : \textit{perp. from } Q \textit{ on polar of } P.$$

Let P' and Q' be the inverse points of P and Q, through which the polars of P and Q pass.

Let the perpendiculars on the polars be PM and QN; draw PT and QR perp. to OQ and OP respectively.

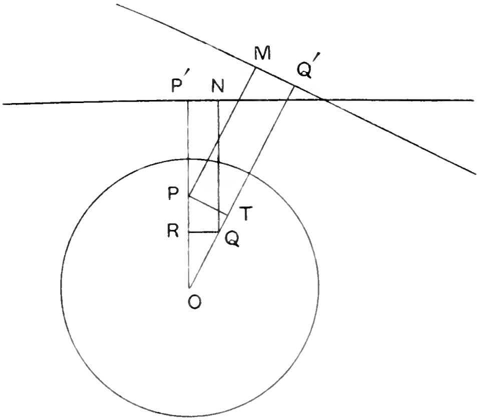
Then we have $\quad OP . OP' = OQ . OQ',$

since each is the square of the radius, and

$$OR . OP = OT . OQ \text{ since } PRQT \text{ is cyclic},$$

$$\therefore \frac{OQ'}{OP'} = \frac{OP}{OQ} = \frac{OT}{OR} = \frac{OQ' - OT}{OP' - OR} = \frac{PM}{QN}.$$

Thus the proposition is proved.



This is known as *Salmon's theorem*.

18. Prop. *The locus of points from which the tangents to two given coplanar circles are equal is a line (called the* radical axis *of the circles) perpendicular to the line of centres.*

Let PK, PF be equal tangents to two circles, centres A and B.

Draw PL perp. to AB. Join PA, PB, AK and BF.

Then $\quad PK^2 = AP^2 - AK^2 = PL^2 + AL^2 - AK^2,$

and $\quad PF^2 = PB^2 - BF^2 = PL^2 + LB^2 - BF^2,$

$$\therefore AL^2 - AK^2 = LB^2 - BF^2,$$

$$\therefore AL^2 - LB^2 = AK^2 - BF^2,$$

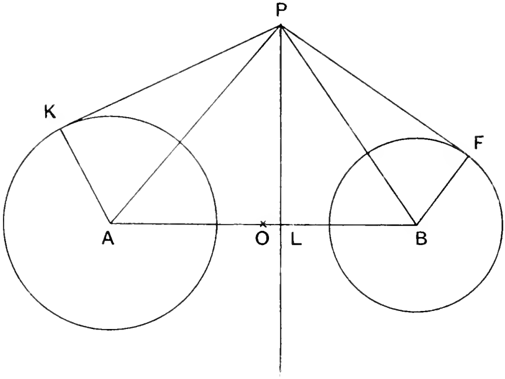
$$\therefore (AL - LB)(AL + LB) = AK^2 - BF^2.$$

Thus if O be the middle point of AB, we have

$$2OL \,.\, AB = \text{difference of sqq. of the radii},$$

$\therefore$ L is a fixed point, and the locus of P is a line perp. to AB.

Since points on the common chord produced of two intersecting circles are such that tangents from them to the two circles are equal, we see that the radical axis of two intersecting circles goes through their common points And introducing



the notion of imaginary points, we may say that the radical axis of two circles goes through their common points, real or imaginary.

19. *The difference of the squares of the tangents to two coplanar circles, from any point P in their plane, varies as the perpendicular from P on their radical axis.*

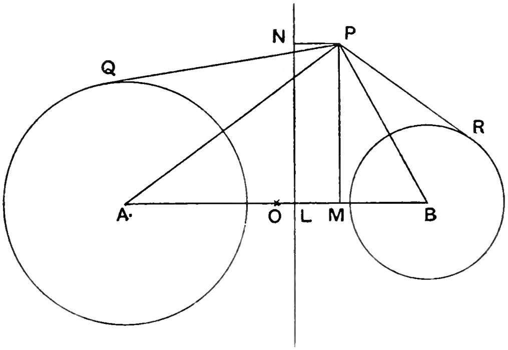
Let PQ and PR be the tangents from P to the circles, centres A and B.

Let PN be perp. to radical axis NL, and PM to AB; let O be the middle point of AB. Join PA, PB.

Then

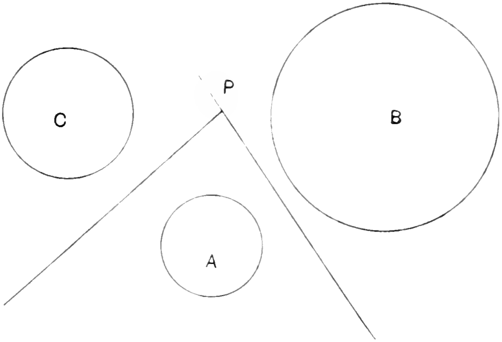
$$\begin{aligned} PQ^2 - PR^2 &= PA^2 - AQ^2 - (PB^2 - BR^2) \\ &= PA^2 - PB^2 - AQ^2 + BR^2 \\ &= AM^2 - MB^2 - AQ^2 + BR^2 \\ &= 2OM \,.\, AB - 2OL \,.\, AB \quad \text{(see § 18)} \\ &= 2AB \,.\, LM = 2AB \,.\, NP. \end{aligned}$$

This proves the proposition.



We may mention here that some writers use the term "power of a point" with respect to a circle to mean the square of the tangent from the point to the circle.

20. Prop. *The radical axes of three coplanar circles taken in pairs meet in a point.*



Let the radical axis of the circles A and B meet that of the circles A and C in P.

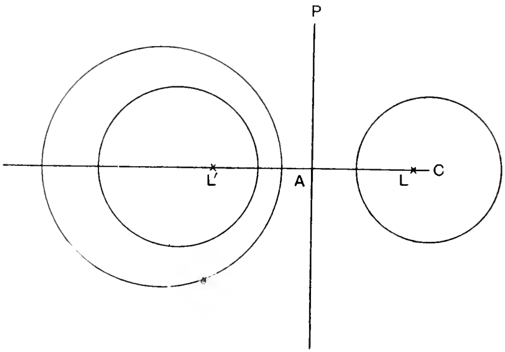
Then the tangent from P to circle C

$$= \text{tangent from } P \text{ to circle } A$$
$$= \text{tangent from } P \text{ to circle } B.$$

$\therefore$ P is on the radical axis of B and C.

21. Coaxal circles. A system of coplanar circles such that the radical axis for any pair of them is the same is called *coaxal*.

Clearly such circles will all have their centres along the same straight line.



Let the common radical axis of a system of coaxal circles cut their line of centres in A.

Then the tangents from A to all the circles will be equal.

Let L, L' be two points on the line of centres on opposite sides of A, such that AL, AL' are equal in length to the tangents from A to the circles; L and L' are called the *limiting points* of the system.

They are such that the distance of any point P on the radical axis from either of them is equal to the length of the tangent from P to the system of circles.

For if C be the centre of one of the circles which is of radius r,

$$PL^2 = PA^2 + AL^2 = PA^2 + AC^2 - r^2 = PC^2 - r^2$$
$$= \text{square of tangent, from } P \text{ to circle } C.$$

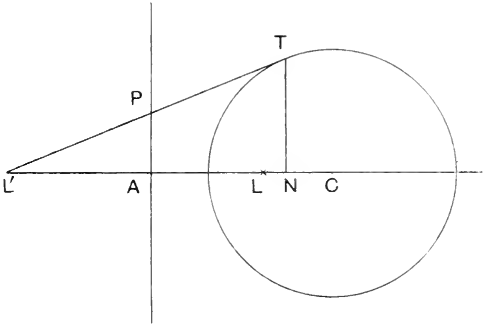
The two points L and L' may be regarded as the centres of circles of infinitely small radius, which belong to the coaxal system. They are sometimes called the *point circles* of the system.

The student will have no difficulty in satisfying himself that of the two limiting points one is within and the other without each circle of the system.

It must be observed that the limiting points are real only in the case where the system of coaxal circles do not intersect in real points. For if the circles intersect, A will lie within them all and thus the tangents from A will be imaginary.

Let it be noticed that if two circles of a coaxal system intersect in points P and Q, then all the circles of the system pass through P and Q.

22. Prop. *The limiting points of a system of coaxal circles are inverse points with respect to every circle of the system.*



Let C be the centre of one of the circles of the system. Let L and L' be the limiting points of which L' is without the circle C.

Draw tangent $L'T$ to circle C; this will be bisected by the radical axis in P.

Draw TN perpendicular to line of centres.

Then $$L'A : AN = L'P : PT,$$
$$\therefore\ L'A = AN,$$
$$\therefore\ N \text{ coincides with } L.$$

Thus the chord of contact of tangents from L' cuts the line of centres at right angles in L.

Therefore L and L' are inverse points.

23. The student will find it quite easy to establish the two following propositions :

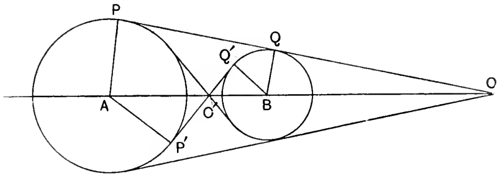
Every circle passing through the limiting points cuts all the circles of the system orthogonally.

A common tangent to two circles of a coaxal system subtends a right angle at either limiting point.

24. Common tangents to two circles.

In general four common tangents can be drawn to two coplanar circles.

Of these two will cut the line joining their centres externally; these are called *direct* common tangents. And two will cut the line joining the centres internally; these are called *transverse* common tangents.



We shall now prove that *the common tangents of two circles cut the line joining their centres in two points which divide that line internally and externally in the ratio of the radii.*

Let a direct common tangent PQ cut the line joining the centres A and B in O. Join AP, BQ.

Then since P and Q are right angles, the triangles APO, BQO are similar,

$$\therefore AO : BO = AP : BQ.$$

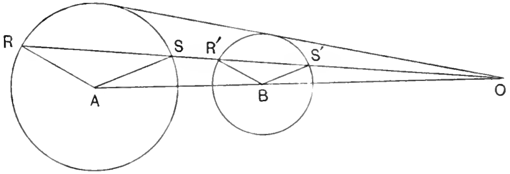
Similarly, if $P'Q'$ be a transverse common tangent cutting AB in O', we can prove $AO' : O'B =$ ratio of the radii.

We have thus a simple construction for drawing the common tangents, viz. to divide AB internally and externally at O' and O in the ratio of the radii, and then from O and O' to draw a tangent to either circle; this will be also a tangent to the other circle.

If the circles intersect in real points, the tangents from O' will be imaginary.

If one circle lie wholly within the other, the tangents from both O and O' will be imaginary.

25. Through the point O, as defined at the end of the last paragraph, let a line be drawn cutting the circles in RS and $R'S'$ as in the figure.



Consider the triangles OAR, OBR'.

We have $$OA : OB = AR : BR',$$

also the angle at O is common to both, and each of the remaining angles at R and R' is less than a right angle.

Thus the triangles are similar, and

$$OR : OR' = AR : BR',$$

the ratio of the radii.

In like manner, by considering the triangles OAS, OBS', in which each of the angles S and S' is greater than a right angle, we can prove that $OS : OS' =$ ratio of radii.

We thus see that the circle B could be constructed from the circle A by means of the point O by taking the radii vectores from O of all the points on the circle A and dividing these in the ratio of the radii.

On account of this property O is called a *centre of similitude* of the two circles, and the point R' is said to correspond to the point R.

The student can prove for himself in like manner that O' is a centre of similitude.

26. In order to prove that the locus of a point obeying some given law is a circle, it is often convenient to make use of the ideas of the last paragraph.

If we can prove that our point P is such as to divide the line joining a fixed point O to a varying point Q, which describes a circle, in a given ratio, then we know that the locus of P must be a circle, which with the circle on which Q lies has O for a centre of similitude.

For example, suppose we have given the circumcircle of a triangle and two of its vertices, and we require the locus of the nine-points centre. It is quite easy to prove that the locus of the orthocentre is a circle, and from this it follows that the locus of the nine-points centre is a circle, since, if O be the circumcentre (which is given) and P the orthocentre (which describes a circle) and U the nine-points centre, U lies on OP and $OU = \frac{1}{2}OP$; therefore the locus of U is a circle, having its centre in the line joining O to the centre of the circle on which P lies.

27. Prop. *The locus of a point which moves in a plane so that its distances from two fixed points in that plane are in a constant ratio is a circle.*

Let A and B be the two given points. Divide AB internally and externally at C and D in the given ratio, so that C and D are two points on the locus.

Let P be any other point on the locus.

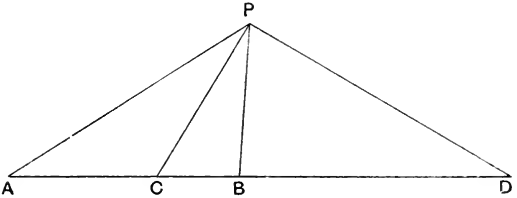
Then since

$$AP : PB = AC : CB = AD : BD,$$

$\therefore$ PC and PD are the internal and external bisectors of the $\angle APB$.

$\therefore$ CPD is a right angle.

$\therefore$ the locus of P is a circle on CD as diameter.



Cor. 1. If the point P be not confined to a plane, its locus is the sphere on CD as diameter.

Cor. 2. If the line AB be divided internally and externally at C and D in the same ratio, and P be any point at which CD subtends a right angle, then PC and PD are the internal and external bisectors of $\angle APB$.

28. If on the line OO' joining the two centres of similitude of circles, centres A and B, as defined in § 25, a circle be described, it follows from § 27 that if C be any point on this circle,

$$CA : CB = \text{radius of } A \text{ circle} : \text{radius of } B \text{ circle}.$$

The circle on OO' as diameter is called the *circle of similitude.* Its use will be explained in the last chapter, when we treat of the similarity of figures.

EXERCISES

1. If P be any point on a given circle A, the square of the tangent from P to another given circle B varies as the perpendicular distance of P from the radical axis of A and B.

2. If A, B, C be three coaxal circles, the tangents drawn from any point of C to A and B are in a constant ratio.

3. If tangents drawn from a point P to two given circles A and B are in a given ratio, the locus of P is a circle coaxal with A and B.

4. If A, B, C &c. be a system of coaxal circles and X be any other circle, then the radical axes of A, X; B, X; C, X &c. meet in a point.

5. The square of the line joining one of the limiting points of a coaxal system of circles to a point P on any one of the circles varies as the distance of P from the radical axis.

6. If two circles cut two others orthogonally, the radical axis of either pair is the line joining the centres of the other pair, and passes through their limiting points.

7. If from any point on the circle of similitude (§ 28) of two given circles, pairs of tangents be drawn to both circles, the angle between one pair is equal to the angle between the other pair.

8. The three circles of similitude of three given circles taken in pairs are coaxal.

9. Find a pair of points on a given circle concyclic with each of two given pairs of points.

10. If any line cut two given circles in P, Q and P', Q' respectively, prove that the four points in which the tangents at P and Q cut the tangents at P' and Q' lie on a circle coaxal with the given circles.

11. A line PQ is drawn touching at P a circle of a coaxal system of which the limiting points are K, K', and Q is a point on the line on the opposite side of the radical axis to P. Shew that if T, T' be the lengths of the tangents drawn from P to the two concentric circles of which the common centre is Q, and whose radii are respectively QK, QK', then

$$T : T' = PK : PK'.$$

12. O is a fixed point on the circumference of a circle C, P any other point on C; the inverse point Q of P is taken with respect to a fixed circle whose centre is at O, prove that the locus of Q is a straight line.

13. Three circles C_1, C_2, C_3 are such that the chord of intersection of C_2 and C_3 passes through the centre of C_1, and the chord of intersection of C_3 and C_1 through the centre of C_2; shew that the chord of intersection of C_1 and C_2 passes through the centre of C_3.

14. Three circles A, B, C are touched externally by a circle whose centre is P and internally by a circle whose centre is Q. Shew that PQ passes through the point of concurrence of the radical axes of A, B, C taken in pairs.

15. AB is a diameter of a circle S, O any point on AB or AB produced, C a circle whose centre is at O. A' and B' are the inverse points of A and B with respect to C. Prove that the pole with respect to C of the polar with respect to S of the point O is the middle point of $A'B'$.

16. A system of spheres touch a plane at the same point O, prove that any plane, not through O, will cut them in a system of coaxal circles.

17. A point and its polar with respect to a variable circle being given, prove that the polar of any other point A passes through a fixed point B.

18. A is a given point in the plane of a system of coaxal circles; prove that the polars of A with respect to the circles of the system all pass through a fixed point.

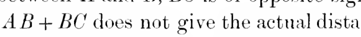
CHAPTER III

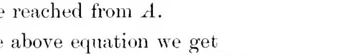
THE USE OF SIGNS. CONCURRENCE AND COLLINEARITY

29. The reader is already familiar with the convention of signs adopted in Trigonometry and Analytical Geometry in the measurement of straight lines. According to this convention lengths measured along a line from a point are counted positive or negative according as they proceed in the one or the other direction.

With this convention we see that, if A, B, C be three points in a line, then, *in whatever order the points occur in the line,*

$$AB + BC = AC.$$





If C lie between A and B, BC is of opposite sign to AB, and in this case $AB + BC$ does not give the actual distance travelled in passing from A to B, and then from B to C, but gives the final distance reached from A.

From the above equation we get

$$BC = AC - AB.$$

This is an important identity. By means of it we can reduce all our lengths to depend on lengths measured from a fixed point in the line. This process it will be convenient to speak of as *inserting an origin.* Thus, if we insert the origin O,

$$AB = OB - OA.$$

30. Prop. *If M be the middle point of the line AB, and O be any other point in the line, then*

$$2OM = OA + OB.$$

For since $$AM = MB,$$
by inserting the origin O we have
$$OM - OA = OB - OM,$$
$$\therefore 2OM = OA + OB.$$

31. A number of collinear points are said to form a *range*.

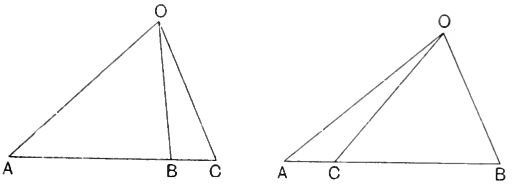
Prop. *If A, B, C, D be a range of four points; then*

$$AB \cdot CD + BC \cdot AD + CA \cdot BD = 0.$$

For, inserting the origin A, we see that the above
$$= AB(AD - AC) + (AC - AB)AD - AC(AD - AB),$$
and this is zero.

This is an important identity, which we shall use later on.

32. If A, B, C be a range of points, and O any point outside their line, we know that the area of the triangle OAB is to the area of the triangle OBC in the ratio of the lengths of the bases AB, BC.



Now if we are taking account of the signs of our lengths AB, BC and the ratio $AB : BC$ occurs, we cannot substitute for this ratio $\triangle OAB : \triangle OBC$ unless we have some convention respecting the signs of our areas, whereby the proper sign of $AB : BC$ will be retained when the ratio of the areas is substituted for it.

The obvious convention is that the area of a triangle PQR shall be accounted positive or negative according as the triangle is to the one or the other side as the contour PQR is described.

Thus if the triangle is to our left hand as we describe the contour PQR, we shall consider $\triangle PQR$ to be a positive magnitude, while $\triangle PRQ$ will be a negative magnitude, for in describing the contour PRQ the area is on our right hand.

With this convention we see that in whatever order the points A, B, C occur in the line on which they lie,

$$AB : BC = \triangle OAB : \triangle OBC,$$

or

$$= \triangle AOB : \triangle BOC.$$

It is further clear that with our convention we may say

$$\triangle OAB + \triangle OBC = \triangle OAC,$$

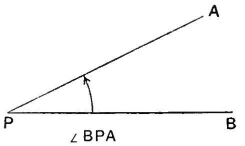
and

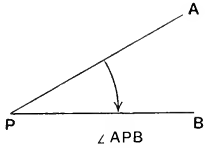
$$\triangle OAB - \triangle OAC = \triangle OCB,$$

remembering always that A, B, C are collinear.

33. Again, we know that the magnitude of the area of a triangle OAB is $\frac{1}{2} OA \,.\, OB \sin AOB$, and it is sometimes convenient to make use of this value. But if we are comparing the areas OAB, OBC by means of a ratio we cannot substitute

$$\tfrac{1}{2} OA \,.\, OB \sin AOB \quad \text{and} \quad \tfrac{1}{2} OB \,.\, OC \sin BOC$$





for them unless we have a further convention of signs whereby the sign and not merely the magnitude of our ratio will be retained.

The obvious convention here again will be to consider angles positive if described in one sense and negative in the opposite sense; this being effective for our purpose, since $\sin(-x) = -\sin x$.

In this case $\angle APB = -\angle BPA$. The angle APB is to be regarded as obtained by the revolution of PB round P from the position PA, and the angle BPA as the revolution of PA round P from the position PB; these are in opposite senses and so of opposite signs.

With this convention as to the signs of our angles we may argue from the figures of § 32,

$$\frac{AB}{BC} = \frac{\triangle AOB}{\triangle BOC} = \frac{\frac{1}{2}\, OA\,.\,OB \sin \angle AOB}{\frac{1}{2}\, OB\,.\,OC \sin \angle BOC}$$

(the lines OA, OB, OC being all regarded as positive)

$$= \frac{OA}{OC} \cdot \frac{\sin \angle AOB}{\sin \angle BOC}.$$

In this way the sign of the ratio $\dfrac{AB}{BC}$, is retained in the process of transformation, since

$$\sin \angle AOB \text{ and } \sin \angle BOC$$

are of the same or opposite sign according as AB and BC are of the same or opposite sign.

The student will see that our convention would have been useless had the area depended directly on the cosine of the angle instead of on the sine, since

$$\cos(-A) = +\cos(A).$$

34. Test for collinearity of three points on the sides of a triangle.

The following proposition, known as Menelaus' theorem, is of great importance.

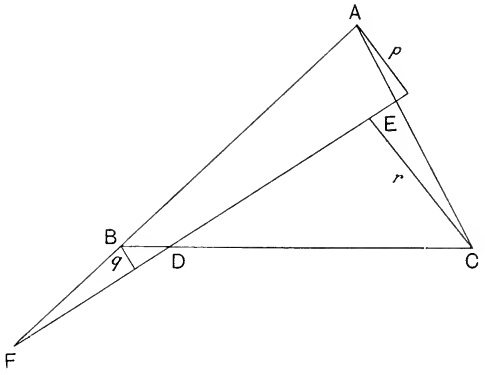
The necessary and sufficient condition that the points D, E, F on the sides of a triangle ABC opposite to the vertices A, B, C respectively should be collinear is

$$AF\,.\,BD\,.\,CE = AE\,.\,CD\,.\,BF,$$

regard being had to the signs of these lines.

All these lines are along the sides of the triangle. We shall consider any one of them to be positive or negative according as the triangle is to our left or right respectively as we travel along it.

We will first prove that the above condition is necessary, if D, E, F are collinear.



Let p, q, r be the perpendiculars from A, B, C on to the line DEF, and let these be accounted positive or negative according as they are on the one or the other side of the line DEF.

With this convention we have

$$\frac{AF}{BF}=\frac{p}{q}, \quad \frac{BD}{CD}=\frac{q}{r}, \quad \frac{CE}{AE}=\frac{r}{p}.$$

Hence $$\frac{AF \,.\, BD \,.\, CE}{BF \,.\, CD \,.\, AE}=1,$$

that is, $$AF \,.\, BD \,.\, CE = AE \,.\, CD \,.\, BF.$$

Next let D, E, F be three points on the sides such that

$$AF \,.\, BD \,.\, CE = AE \,.\, CD \,.\, BF,$$

then shall D, E, F be collinear.

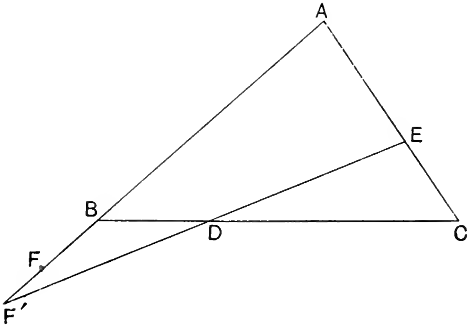
Let the line DE cut AB in F',

$$\therefore AF' \,.\, BD \,.\, CE = AE \,.\, CD \,.\, BF',$$

$$\therefore \frac{AF'}{BF'}=\frac{AF}{BF},$$

$$\therefore (AF + FF')\,BF = AF\,(BF + FF'),$$
$$\therefore FF'\,(BF - AF) = 0,$$
$$\therefore FF' = 0,$$
$\therefore$ F coincides with F'.

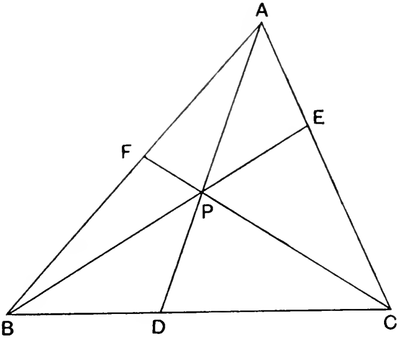
Thus our proposition is completely proved.



35. Test for concurrency of lines through the vertices of a triangle.

The following proposition, known as Ceva's theorem, is fundamental.

The necessary and sufficient condition that the lines AD, BE,



CF drawn through the vertices of a triangle ABC to meet the opposite sides in D, E, F should be concurrent is

$$AF\,.\,BD\,.\,CE = -AE\,.\,CD\,.\,BF,$$

the same convention of signs being adopted as in the last proposition.

First let the lines AD, BE, CF meet in P.

Then, regard being had to the signs of the areas,

$$\frac{AF}{BF} = \frac{\triangle AFC}{\triangle BFC} = \frac{\triangle AFP}{\triangle BFP} = \frac{\triangle AFC - \triangle AFP}{\triangle BFC - \triangle BFP} = \frac{\triangle APC}{\triangle BPC},$$

$$\frac{BD}{CD} = \frac{\triangle BDA}{\triangle CDA} = \frac{\triangle BDP}{\triangle CDP} = \frac{\triangle BDA - \triangle BDP}{\triangle CDA - \triangle CDP} = \frac{\triangle BPA}{\triangle CPA},$$

$$\frac{CE}{AE} = \frac{\triangle CEB}{\triangle AEB} = \frac{\triangle CEP}{\triangle AEP} = \frac{\triangle CEB - \triangle CEP}{\triangle AEB - \triangle AEP} = \frac{\triangle CPB}{\triangle APB}.$$

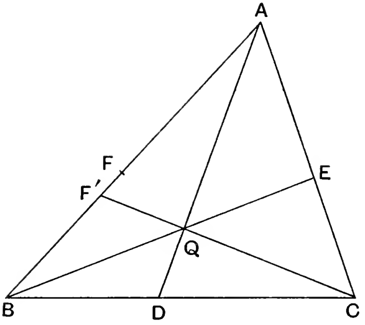
$$\therefore \frac{AF\,.\,BD\,.\,CE}{AE\,.\,CD\,.\,BF} = \frac{\triangle APC}{\triangle CPA}\cdot\frac{\triangle BPA}{\triangle APB}\cdot\frac{\triangle CPB}{\triangle BPC}$$

$$= (-1)(-1)(-1) = -1.$$

Next let D, E, F be points on the sides of a triangle ABC such that

$$AF\,.\,BD\,.\,CE = -AE\,.\,CD\,.\,BF,$$

then will AD, BE, CF be concurrent.



Let AD, BE meet in Q, and let CQ meet AB in F'.

$$\therefore AF' . BD . CE = - AE . CD . BF'.$$

$$\therefore \frac{AF'}{BF'} = \frac{AF}{BF}.$$

$$\therefore (AF + FF') BF = (BF + FF') AF.$$

$$\therefore FF' (BF - AF) = 0.$$

$$\therefore FF' = 0.$$

$\therefore$ F' and F coincide.

Hence our proposition is completely proved.

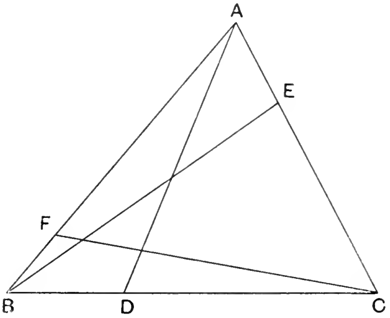
36. Prop. *If D, E, F be three points on the sides of a triangle ABC opposite to A, B, C respectively.*

$$\frac{AF . BD . CE}{AE . CD . BF} = \frac{\sin ACF \sin BAD \sin CBE}{\sin ABE \sin CAD \sin BCF}.$$

For

$$\frac{BD}{CD} = \frac{\triangle BAD}{\triangle CAD} = \frac{\frac{1}{2} AB . AD \sin BAD}{\frac{1}{2} AC . AD \sin CAD} = \frac{AB}{AC} \cdot \frac{\sin BAD}{\sin CAD},$$

with our convention as to sign, and AB, AC being counted positive.



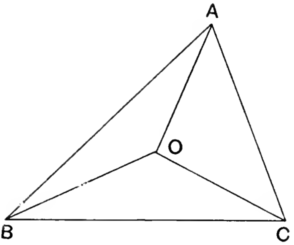
Similarly $$\frac{AF}{BF} = \frac{AC}{BC} \cdot \frac{\sin ACF}{\sin BCF}$$

and $$\frac{CE}{AE} = \frac{BC}{AB} \cdot \frac{\sin CBE}{\sin ABE}.$$

$$\therefore \frac{AF . BD . CE}{AE . CD . BF} = \frac{\sin ACF \sin BAD \sin CBE}{\sin ABE \sin CAD \sin BCF}.$$

COR. The necessary and sufficient condition that AD, BE, CF should be concurrent is

$$\frac{\sin ACF \sin BAD \sin CBE}{\sin ABE \sin CAD \sin BCF} = -1.$$



If O be the point of concurrence this relation can be written in the form

$$\frac{\sin ABO \sin BCO \sin CAO}{\sin ACO \sin CBO \sin BAO} = -1,$$

this being easy to remember.

37. Isogonal conjugates. Two lines AD, AD' through the vertex A of a triangle which are such that

$$\angle BAD = \angle D'AC \text{ (not } \angle CAD')$$

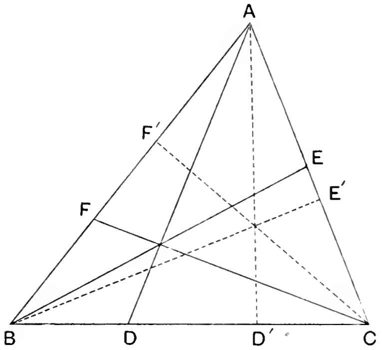
are called *isogonal conjugates.*

Prop. *If AD, BE, CF be three concurrent lines through the vertices of a triangle ABC, their isogonal conjugates AD', BE', CF' will also be concurrent.*

For $$\frac{\sin BAD}{\sin CAD} = \frac{\sin D'AC}{\sin D'AB} = \frac{\sin CAD'}{\sin BAD'}.$$

Similarly $$\frac{\sin CBE}{\sin ABE} = \frac{\sin ABE'}{\sin CBE'}$$

and $$\frac{\sin ACF}{\sin BCF} = \frac{\sin BCF'}{\sin ACF'}.$$



$$\therefore \frac{\sin CAD' \sin ABE' \sin BCF'}{\sin BAD' \sin CBE' \sin ACF'}$$

$$= \frac{\sin BAD \sin CBE \sin ACF}{\sin CAD \sin ABE \sin BCF} = -1.$$

$\therefore AD', BE', CF'$ are concurrent.

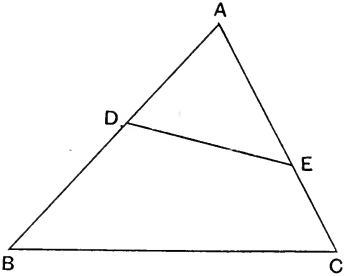
38. The isogonal conjugates of the medians of a triangle are called its *symmedians*. Since the medians are concurrent, the symmedians are concurrent also. The point where the symmedians intersect is called the *symmedian point* of the triangle.

The student will see that the concurrence of the medians and perpendiculars of a triangle follows at once by the tests of this chapter (§§ 35 and 36). It was thought better to prove them by independent methods in the first chapter in order to bring out other properties of the orthocentre and the median point.

39. We will conclude this chapter by introducing the student to certain lines in the plane of a triangle which are called by some writers *antiparallel* to the sides.

Let ABC be a triangle, D and E points in the sides

AB and AC such that $\angle ADE = \angle BCA$ and therefore also $\angle AED = \angle CBA$. The line DE is said to be antiparallel to BC.



It will be seen at once that $DBCE$ is cyclic, and that all lines antiparallel to BC are parallel to one another.

It may be left as an exercise to the student to prove that the symmedian line through A of the triangle ABC bisects all lines antiparallel to BC.

EXERCISES

1. The lines joining the vertices of a triangle to its circumcentre are isogonal conjugates with the perpendiculars of the triangle.

2. The lines joining the vertices of a triangle to the points of contact with the opposite sides of the incircle and ecircles are respectively concurrent.

3. ABC is a triangle; AD, BE, CF the perpendiculars on the opposite sides. If AG, BH and CK be drawn perpendicular to EF, FD, DE respectively, then AG, BH and CK will be concurrent.

4. The midpoints of the sides BC and CA of the triangle ABC are D and E: the trisecting points nearest B of the sides BC and BA respectively are H and K. CK intersects AD in L, and BL intersects AH in M, and CM intersects BE in N. Prove that N is a trisecting point of BE.

5. If perpendiculars are drawn from the orthocentre of a triangle ABC on the bisectors of the angle A, shew that their feet are collinear with the middle point of BC.

6. The points of contact of the ecircles with the sides BC, CA, AB of a triangle are respectively denoted by the letters D, E, F with suffixes 1, 2, 3 according as they belong to the ecircle opposite A, B, or C. BE_2, CF_3 intersect at P; BE_1, CF_1 at Q; E_2F_3 and BC at X; F_3D_1 and CA at Y; D_1E_2 and AB at Z. Prove that the groups of points A, P, D_1, Q; and X, Y, Z are respectively collinear.

7. Parallel tangents to a circle at A and B are cut in the points C and D respectively by a tangent to the circle at E. Prove that AD, BC and the line joining the middle points of AE and BE are concurrent.

8. From the angular points of any triangle ABC lines AD, BE, CF are drawn cutting the opposite sides in D, E, F, and making equal angles with the opposite sides measured round the triangle in the same direction. The lines AD, BE, CF form a triangle $A'B'C'$. Prove that

$$\frac{A'B \cdot B'C \cdot C'A}{AE \cdot BF \cdot CD} = \frac{A'C \cdot B'A \cdot C'B}{AF \cdot BD \cdot CE} = \frac{BC \cdot CA \cdot AB}{AD \cdot BE \cdot CF}.$$

9. Through the symmedian point of a triangle lines are drawn antiparallel to each of the sides, cutting the other two sides. Prove that the six points so obtained are equidistant from the symmedian point.

[The circle through these six points has been called the *cosine circle*, from the property, which the student can verify, that the intercepts it makes on the sides are proportional to the cosines of the opposite angles.]

10. Through the symmedian point of a triangle lines are drawn parallel to each of the sides, cutting the other sides. Prove that the six points so obtained are equidistant from the middle point of the line joining the symmedian point to the circumcentre.

[The circle through these six points is called the *Lemoine circle*. See Lachlan's *Modern Pure Geometry*, § 131.]

11. AD, BE, CF are three concurrent lines through the vertices of a triangle ABC, meeting the opposite sides in D, E, F. The circle circumscribing DEF intersects the sides of ABC again in D', E', F'. Prove that AD', BE', CF' are concurrent.

12. Prove that the tangents to the circumcircle at the vertices of a triangle meet the opposite sides in three points which are collinear.

13. If AD, BE, CF through the vertices of a triangle ABC meeting the opposite sides in D, E, F are concurrent, and points D', E', F' be taken in the sides opposite to A, B, C so that DD' and BC, EE' and CA, FF' and AB have respectively the same middle point, then AD', BE', CF' are concurrent.

14. If from the symmedian point S of a triangle ABC, perpendiculars SD, SE, SF be drawn to the sides of the triangle, then S will be the median point of the triangle DEF.

15. Prove that the triangles formed by joining the symmedian point to the vertices of a triangle are in the duplicate ratio of the sides of the triangle.

16. The sides BC, CA, AB of a triangle ABC are divided internally by points A', B', C' so that

$$BA' : A'C = CB' : B'A = AC' : C'B.$$

Also $B'C'$ produced cuts BC externally in A''. Prove that

$$BA'' : CA'' = CA'^2 : A'B^2.$$

CHAPTER IV

PROJECTION

40. If V be any point in space, and A any other point, then if VA, produced if necessary, meet a given plane π in A', A' is called the projection of A on the plane π by means of the vertex V.

It is clear at once that the projection of a straight line on a plane π is a straight line, namely the intersection of the plane π with the plane containing V and the line.

If the plane through V and a certain line be parallel to the π plane, then that line will be projected to infinity on the π plane. The line thus obtained on the π plane is called *the line at infinity* in that plane.

41. Suppose now we are projecting points in a plane p by means of a vertex V on to another plane π.

Let a plane through V parallel to the plane π cut the plane p in the line AB.

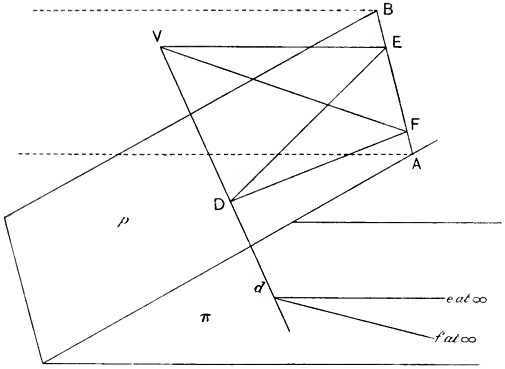
This line AB will project to infinity on the plane π, and for this reason AB is called the *vanishing line* on the plane p.

The vanishing line is clearly parallel to the line of intersection of the planes p and π, which is called *the axis of projection*.

42. Now let EDF be an angle in the plane p and let its lines DE and DF cut the vanishing line AB in E and F, then the angle EDF will project on to the π plane into an angle of magnitude EVF.

For let the plane VDE intersect the plane π in the line de.

Then since the plane VEF is parallel to the plane π, the intersections of these planes with the plane VDE are parallel; that is, de is parallel to VE.



Similarly df is parallel to VF.

Therefore $\angle edf = \angle EVF$.

Hence we see that *any angle in the plane p projects on to the π plane into an angle of magnitude equal to that subtended at V by the portion of the vanishing line intercepted by the lines containing the angle.*

43. Prop. *By a proper choice of the vertex V of projection, any given line on a plane p can be projected to infinity, while two given angles in the plane p are projected into angles of given magnitude on to a plane π properly chosen.*

Let AB be the given line. Through AB draw any plane p'.

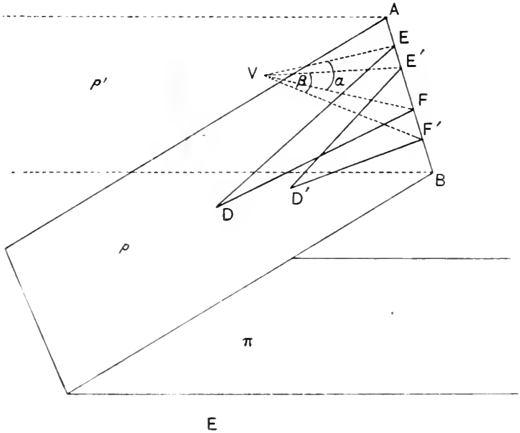
Let the plane π be taken parallel to the plane p'.

Let EDF, $E'D'F'$ be the angles in the plane p which are to be projected into angles of magnitude α and β respectively.

Let E, F, E', F' be on AB.

On EF, $E'F'$ in the plane p' describe segments of circles containing angles equal to α and β respectively. Let these segments intersect in V.

Then if V be taken as the vertex of projection, AB will project to infinity, and EDF, $E'D'F'$ into angles of magnitude α and β respectively (§ 42).



Cor. 1. *Any triangle can be projected into an equilateral triangle.*

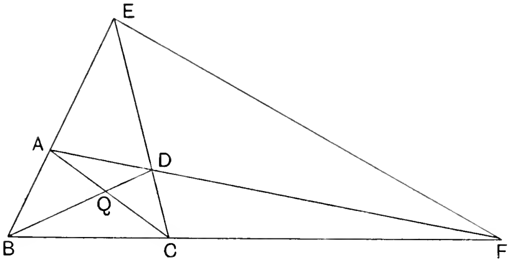
For if we project two of its angles into angles of 60° the third angle will project into 60° also, since the sum of the three angles of the triangle in projection is equal to two right angles.

Cor. 2. *A quadrilateral can be projected into a square.*

Let $ABCD$ be the quadrilateral. Let EF be its third diagonal, that is the line joining the intersections of opposite pairs of sides.

Let AC and BD intersect in Q.

Now if we project EF to infinity and at the same time project ∠s BAD and BQA into right angles, the quadrilateral will be projected into a square.



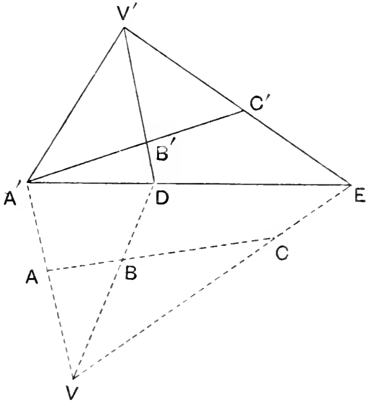
For the projection of EF to infinity, secures that the projection shall be a parallelogram; the projection of $\angle BAD$ into a right angle makes this parallelogram rectangular; and the projection of $\angle AQB$ into a right angle makes the rectangle a square.

44. It may happen that one of the lines DE, $D'E'$ in the preceding paragraph is parallel to the line AB which is to be projected to infinity. Suppose that DE is parallel to AB. In this case we must draw a line FV in the plane p' so that the angle EFV is the supplement of α. The vertex of projection V will be the intersection of the line FV with the segment of the circle on $E'F'$.

If $D'E'$ is also parallel to AB, then the vertex V will be the intersection of the line FV just now obtained and another line $F'V$ so drawn that the angle $E'F'V$ is the supplement of β.

45. Again the segments of circles described on EF, $E'F'$ in the proposition of § 43 may not intersect in any real point. In this case V is an imaginary point, that is to say it is a point algebraically significant, but not capable of being presented to the eye in the figure. The notion of imaginary points and lines which we take over from Analytical Geometry into our present subject will be of considerable use.

46. Prop. *A range of three points is projective with any other range of three points in space.*



Let A, B, C be three collinear points, and A', B', C' three others not necessarily in the same plane with the first three.

Join AA'.

Take any point V in AA'.

Join VB, VC and let them meet a line $A'DE$ drawn through A' in the plane VAC in D and E.

Join DB', EC'. These are in one plane, viz. the plane containing the lines $A'C'$ and $A'E$.

Let DB', EC' meet in V'. Join $V'A'$.

Then by means of the vertex V, A, B, C can be projected into A', D, E; and these by means of the vertex V' can be projected into A', B', C'.

Thus our proposition is proved.

47. The student must understand that when we speak of one range being projective with another, we do not mean necessarily that the one can be projected into the other by a single projection, but that we can pass from one range to the other by successive projections.

A range of *four* points is not *in general* projective with any other range of four points in space. We shall in the next chapter set forth the condition that must be satisfied to render the one projective with the other.

EXERCISES

1. Prove that a system of parallel lines in a plane p will project on to another plane into a system of lines through the same point.

2. Two angles such that the lines containing them meet the vanishing line in the same points are projected into angles which are equal to one another.

3. Shew that in general three angles can be projected into angles of the same magnitude a.

4. Shew that a triangle can be so projected that any line in its plane is projected to infinity while three given concurrent lines through its vertices become the perpendiculars of the triangle in the projection.

5. Explain, illustrating by a figure, how it is that a point R lying on a line PQ, and *outside* the portion PQ of it, can be projected into a point r lying between p and q, which are the projections of P and Q.

6. Any three points A_1, B_1, C_1 are taken respectively in the sides BC, CA, AB of the triangle ABC; B_1C_1 and BC intersect in F; C_1A_1 and CA in G; and A_1B_1 and AB in H. Also FH and BB_1 intersect in M, and FG and CC_1 in N. Prove that MG, NH and BC are concurrent.

7. Prove that a triangle can be so projected that three given concurrent lines through its vertices become the medians of the triangle in the projection.

8. If AA_1, BB_1, CC_1 be three concurrent lines drawn through the vertices of a triangle ABC to meet the opposite sides in $A_1B_1C_1$; and if B_1C_1 meet BC in A_2, C_1A_1 meet CA in B_2, and A_1B_1 meet AB in C_2; then A_2, B_2, C'_2 will be collinear.

[Project the concurrent lines into medians.]

9. If a triangle be projected from one plane on to another the three points of intersection of corresponding sides are collinear.

CHAPTER V

CROSS-RATIOS

48. Definition. If A, B, C, D be a range of points, the ratio $\dfrac{AB \cdot CD}{AD \cdot CB}$ is called a *cross-ratio* of the four points, and is conveniently represented by $(ABCD)$, in which the order of the letters is the same as their order in the numerator of the cross-ratio.

Some writers call cross-ratios 'anharmonic ratios.' This is however not a fortunate term to use, and it will be best to avoid it. For the term 'anharmonic' means not harmonic, so that an anharmonic ratio should be one that is not harmonic, whereas a cross-ratio may be harmonic, that is to say may be the cross-ratio of what is called a harmonic range. The student will better appreciate this point when he comes to Chapter VII.

49. The essentials of a cross-ratio of a range of four points are: (1) that each letter occurs once in both numerator and denominator; (2) that the elements of the denominator are obtained by associating the first and last letters of the numerator together, and the third and second, and *in this particular order*.

$\dfrac{AB \cdot CD}{AD \cdot BC}$ is not a cross-ratio but the negative of one, for it $= -\dfrac{AB \cdot CD}{AD \cdot CB} = -(ABCD)$.

$\dfrac{BA \cdot CD}{AC \cdot DB}$, though not appearing to be a cross-ratio as it stands, becomes one on rearrangement, for it $= \dfrac{BA \cdot CD}{BD \cdot CA}$, that is $(BACD)$.

Since there are twenty-four permutations of four letters taken all together, we see that there are twenty-four cross-ratios which can be formed with a range of four points.

50. Prop. *The twenty-four cross-ratios of a range of four points are equivalent to six, all of which can be expressed in terms of any one of them.*

Let $$(ABCD) = \lambda.$$

First we observe that if the letters of a cross-ratio be interchanged in pairs simultaneously, the cross-ratio is unchanged.

For $$(BADC) = \frac{BA \,.\, DC}{BC \,.\, DA} = \frac{AB \,.\, CD}{AD \,.\, CB} = (ABCD),$$

$$(CDAB) = \frac{CD \,.\, AB}{CB \,.\, AD} = \frac{AB \,.\, CD}{AD \,.\, CB} = (ABCD),$$

$$(DCBA) = \frac{DC \,.\, BA}{DA \,.\, BC} = \frac{AB \,.\, CD}{AD \,.\, CB} = (ABCD).$$

Hence we get

$$(ABCD) = (BADC) = (CDAB) = (DCBA) = \lambda \ldots (1).$$

Secondly we observe that a cross-ratio is inverted if we interchange *either* the first and third letters, *or* the second and fourth.

$$\therefore (ADCB) = (BCDA) = (CBAD) = (DABC) = \frac{1}{\lambda} \ldots (2).$$

These we have obtained from (1) by interchange of second and fourth letters; the same result is obtained by interchanging the first and third.

Thirdly, since by § 31

$$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0,$$

$$\therefore \frac{AB \,.\, CD}{AD \,.\, CB} - 1 + \frac{CA \,.\, BD}{AD \,.\, CB} = 0.$$

$$\therefore 1 - \lambda = \frac{CA \,.\, BD}{AD \,.\, CB} = \frac{AC \,.\, BD}{AD \,.\, BC} = (ACBD).$$

Thus the interchange of the second and third letters changes λ into $1-\lambda$. We may remark that the same result is obtained by interchanging the first and fourth.

Thus from (1)

$$(ACBD)=(BDAC)=(CADB)=(DBCA)=1-\lambda \quad \ldots (3),$$

and from this again by interchange of second and fourth letters,

$$(ADBC)=(BCAD)=(CBDA)=(DACB)=\frac{1}{1-\lambda} \quad \ldots (4).$$

In these we interchange the second and third letters, and get

$$(ABDC)=(BACD)=(CDBA)=(DCAB)$$
$$=1-\frac{1}{1-\lambda}=\frac{\lambda}{\lambda-1} \quad \ldots (5).$$

And now interchanging the second and fourth we get

$$(ACDB)=(BDCA)=(CABD)=(DBAC)=\frac{\lambda-1}{\lambda} \quad \ldots (6).$$

We have thus expressed all the cross-ratios in terms of λ. And we see that if one cross-ratio of four collinear points be equal to one cross-ratio of four other collinear points, then each of the cross-ratios of the first range is equal to the corresponding cross-ratio of the second.

Two such ranges may be called *equi-cross*.

51. Prop. *If A, B, C be three separate collinear points, and D, E other points in their line such that*

$$(ABCD)=(ABCE),$$

then D must coincide with E.

For since $$\frac{AB \cdot CD}{AD \cdot CB}=\frac{AB \cdot CE}{AE \cdot CB},$$

$$\therefore\ AE \cdot CD = AD \cdot CE.$$
$$\therefore\ (AD+DE)\,CD = AD\,(CD+DE).$$
$$\therefore\ DE\,(AD-CD)=0.$$
$$\therefore\ DE \cdot AC = 0.$$
$$\therefore\ DE=0 \text{ for } AC \neq 0,$$

that is, D and E coincide.

52. Prop. *A range of four points is equi-cross with its projection on any plane.*

Let the range $ABCD$ be projected by means of the vertex V into $A'B'C'D'$.

Then

$$\frac{AB \cdot CD}{AD \cdot CB} = \frac{\triangle AVB}{\triangle AVD} \cdot \frac{\triangle CVD}{\triangle CVB},$$ regard being had to the signs of the areas,

$$= \frac{\frac{1}{2} VA \cdot VB \sin AVB}{\frac{1}{2} VA \cdot VD \sin AVD} \cdot \frac{\frac{1}{2} VC \cdot VD \sin CVD}{\frac{1}{2} VC \cdot VB \sin CVB},$$

regard being had to the signs of the angles,

$$= \frac{\sin AVB \sin CVD}{\sin AVD \sin CVB}.$$

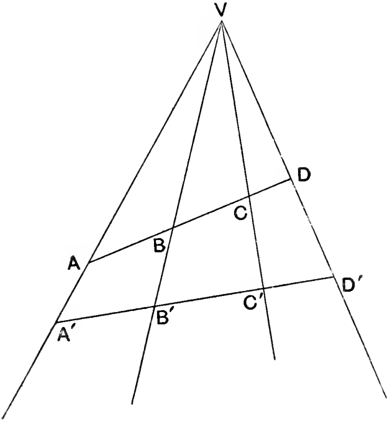


Fig. 1.

Similarly

$$\frac{A'B' \cdot C'D'}{A'D' \cdot C'B'} = \frac{\sin A'VB' \sin C'VD'}{\sin A'VD' \sin C'VB'}.$$

Now in all the cases that arise

$$\frac{\sin A'VB' \sin C'VD'}{\sin A'VD' \sin C'VB'} = \frac{\sin AVB \sin CVD}{\sin AVD \sin CVB}.$$

This is obvious in fig. 1.

In fig. 2

$$\sin A'VB' = \sin B'VA, \text{ these angles being supplementary,}$$
$$= -\sin AVB,$$

and $\sin A'VD' = \sin D'VA = -\sin AVD.$

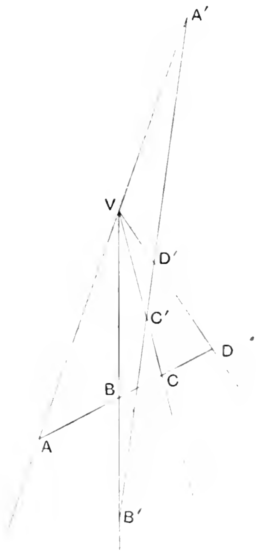


Fig. 2.

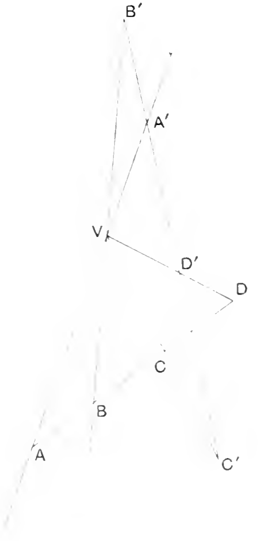


Fig. 3.

Further $\sin C'VD' = \sin CVD,$

and $\sin C'VB' = \sin CVB.$

In fig. 3

$$\sin A'VB' = \sin AVB,$$
$$\sin C'VD' = \sin CVD,$$
$$\sin A'VD' = \sin D'VA = -\sin AVD,$$
$$\sin C'VB' = \sin BVC = -\sin CVB.$$

Thus in each case

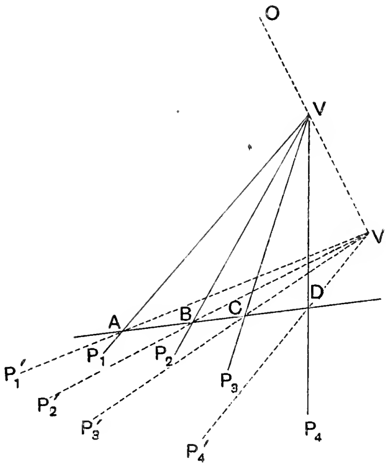
$$(A'B'C'D') = (ABCD).$$

53. A number of lines in a plane which meet in a point V are said to form a *pencil*, and each constituent line of the pencil is called a *ray*. V is called the *vertex* of the pencil.

Any straight line in the plane cutting the rays of the pencil is called a *transversal* of the pencil.

From the last article we see that if VP_1, VP_2, VP_3, VP_4 form a pencil and any transversal cut the rays of the pencil in A, B, C, D, then $(ABCD)$ is constant for that particular pencil; that is to say it is independent of the particular transversal.

It will be convenient to express this constant cross-ratio by the notation $V(P_1P_2P_3P_4)$.



We easily see that a cross-ratio of the projection of a pencil on to another plane is equal to the cross-ratio of the original pencil.

For let $V(P_1, P_2, P_3, P_4)$ be the pencil, O the vertex of projection.

Let the line of intersection of the p and π planes cut the rays of the pencil in A, B, C, D, and let V' be the projection of V, $V'P_1'$, of VP_1, and so on.

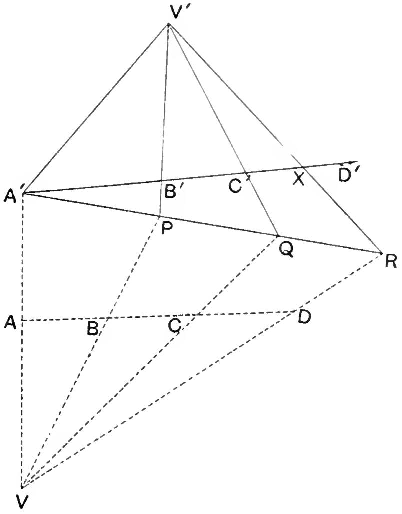
Then $ABCD$ is a transversal also of

$$V'(P_1', P_2', P_3', P_4').$$

$$\therefore V(P_1P_2P_3P_4) = (ABCD) = V'(P_1'P_2'P_3'P_4').$$

54. We are now in a position to set forth the condition that two ranges of four points should be mutually projective.

Prop. *If $ABCD$ be a range, and $A'B'C'D'$ another range such that $(A'B'C'D') = (ABCD)$, then the two ranges are projective.*



Join AA' and take any point V upon it.

Join VB, VC, VD and let these lines meet a line through A' in the plane VAD in P, Q, R respectively.

Join PB', QC' and let these meet in V'. Join $V'A'$, and $V'R$, the latter cutting $A'D'$ in X.

Then $(ABCD) = (A'PQR) = (A'B'C'X)$.

But $(ABCD) = (A'B'C'D')$ by hypothesis.

$\therefore (A'B'C'X) = (A'B'C'D')$.

$\therefore X$ coincides with D' (§ 51).

Thus, by means of the vertex V, $ABCD$ can be projected into $A'PQR$, and these again by the vertex V' into $A'B'C'D'$.

Thus our proposition is proved.

55. Def. Two ranges $ABCDE\ldots$ and $A'B'C'D'E'\ldots$ are said to be *homographic* when a cross-ratio of *any* four points of the one is equal to the corresponding cross-ratio of the four corresponding points of the other. This is conveniently expressed by the notation

$$(ABCDE\ldots) = (A'B'C'D'E'\ldots).$$

The student will have no difficulty in proving by means of § 54 that two homographic ranges are mutually projective.

Two pencils

$$V(P, Q, R, S, T\ldots) \text{ and } V'(P', Q', R', S', T'\ldots)$$

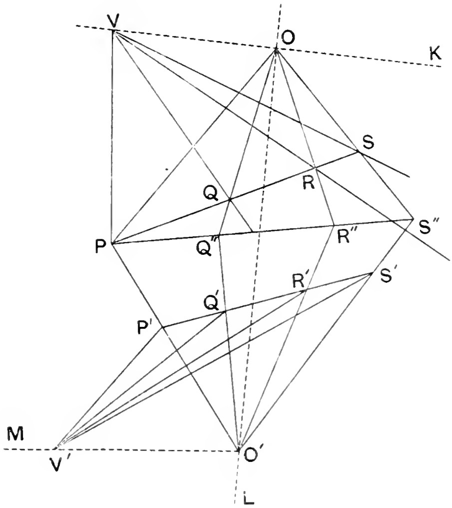
are said to be homographic when a cross-ratio of the pencil formed by any four lines of rays of the one is equal to the corresponding cross-ratio of the pencil formed by the four corresponding lines or rays of the other.

56. Prop. *Two homographic pencils are mutually projective.*

For let $PQRS\ldots$, $P'Q'R'S'\ldots$ be any two transversals of the two pencils, V and V' the vertices of the pencils.

Let $PQ''R''S''\ldots$ be the common range into which these can be projected by vertices O and O'.

Then by means of a vertex K on OV the pencil $V(P, Q, R, S\ldots)$ can be projected into $O(P, Q'', R'', S''\ldots)$; and this last pencil can, by a vertex L on OO', be projected into $O'(P, Q'', R'', S''\ldots)$, that is, $O'(P', Q', R', S'\ldots)$; and this again by means of a vertex M on $O'V'$ can be projected into $V'(P', Q', R', S' \ldots)$.



57. We will conclude this chapter with a construction for drawing through a given point in the plane of two given parallel lines a line parallel to them, *the construction being effected by means of the ruler only.*

Let $A\omega$, $A_1\omega'$ be the two given lines, ω and ω' being the point at infinity upon them, at which they meet.

Let P be the given point in the plane of these lines.

Draw any line AC to cut the given lines in A and C, and take any point B upon it.

Join PA cutting $A_1\omega'$ in A_1.

Join PB cutting $A_1\omega'$ in B_1 and $A\omega$ in B_2.

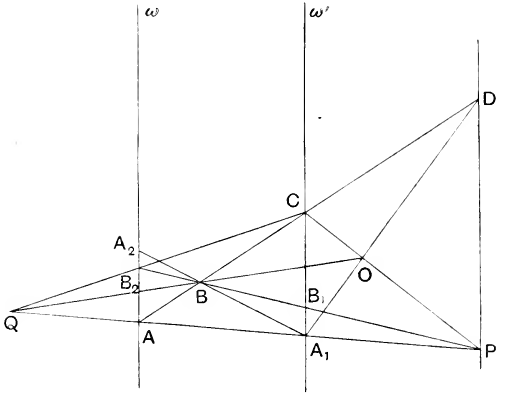
Join PC.

Let A_1A and B_2C meet in Q.

Let QB meet CP in O.

Let A_1O and AC meet in D.

PD shall be the line required.



For

$$(A_1B_1C\omega') = B(A_1B_1C\omega') = (A_2B_2A\omega) = A_1(A_2B_2A\omega)$$
$$= (BB_2PB_1) = C(BB_2PB_1) = (AQPA_1) = O(AQPA_1)$$
$$= (ABCD).$$

$$\therefore\ P(A_1B_1C\omega') = P(ABCD).$$

$\therefore$ PD and $P\omega'$ are in the same line,

that is, PD is parallel to the given lines.

EXERCISES

1. If $(ABCD) = -\frac{1}{3}$ and B be the point of trisection of AD towards A, then C is the other point of trisection of AD.

2. Given a range of three points A, B, C, find a fourth point D on their line such that $(ABCD)$ shall have a given value.

3. If the transversal ABC be parallel to OD, one of the rays of a pencil O (A, B, C, D), then

$$O(ABCD) = \frac{AB}{CB}.$$

4. If $(ABCD) = (ABC'D')$, then $(ABCC') = (ABDD')$.

5 If A, B, C, D be a range of four separate points and

$$(ABCD) = (ADCB),$$

then each of these ratios $= -1$.

6. Of the cross-ratios of the range formed by the circumcentre, median point, nine-points centre and orthocentre of a triangle, eight are equal to -1, eight to 2, and eight to $\frac{1}{2}$.

7. Any plane will cut four given planes all of which meet in a common line in four lines which are concurrent, and the cross-ratio of the pencil formed by these lines is constant.

8. Taking a, b, c, d to be the distances from O to the points A, B, C, D all in a line with O, and

$$\lambda \equiv (a-d)(b-c), \quad \mu \equiv (b-d)(c-a), \quad \nu \equiv (c-d)(a-b),$$

shew that the six possible cross-ratios of the ranges that can be made up of the points A, B, C, D are

$$-\frac{\mu}{\nu}, \quad -\frac{\nu}{\mu}, \quad -\frac{\nu}{\lambda}, \quad -\frac{\lambda}{\nu}, \quad -\frac{\lambda}{\mu}, \quad -\frac{\mu}{\lambda}.$$

CHAPTER VI

PERSPECTIVE

58. Def. A figure consisting of an assemblage of points P, Q, R, S, &c. is said to be in *perspective* with another figure consisting of an assemblage of points P', Q', R', S', &c., if the lines joining corresponding points, viz. PP', QQ', RR', &c. are concurrent in a point O. The point O is called the *centre of perspective.*

It is clear from this definition that a figure when projected on to a plane or surface is in perspective with its projection, the vertex of projection being the centre of perspective.

It seems perhaps at first sight that in introducing the notion of perspective we have arrived at nothing further than what we already had in projection. So it may be well to compare the two things, with a view to making this point clear.

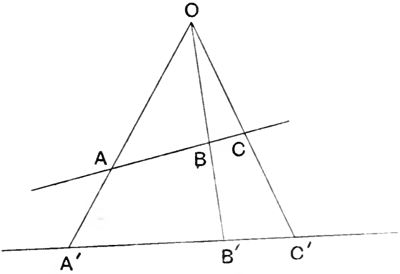
Let it then be noticed that two figures *which are in the same plane* may be in perspective, whereas we should not in this case speak of one figure as the projection of the other.

In projection we have a figure on one plane or surface and project it by means of a vertex of projection on to another plane or surface, whereas in perspective the thought of the planes or surfaces on which the two figures lie is absent, and all that is necessary is that the lines joining corresponding points should be concurrent.

So then while two figures each of which is the projection of the other are in perspective, it is not necessarily the case that of two figures in perspective each is the projection of the other.

59. It is clear from our definition of perspective that if two ranges of points be in perspective, then the two lines of the ranges must be coplanar.

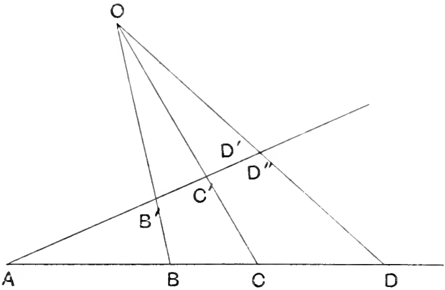
For if A, B, C, &c. are in perspective with A', B', C', &c., and O be the centre of perspective, $A'B'$ and AB are in the same plane, viz. the plane containing the lines OA, OB.



It is also clear that ranges in perspective are homographic.

But it is not necessarily the case that two homographic ranges in the same plane are in perspective. The following proposition will shew under what condition this is the case.

60. Prop. *If two homographic ranges in the same plane be such that the point of intersection of their lines is a point corresponding to itself in the two ranges, then the ranges are in perspective.*



For let $(ABCDE\ldots) = (AB'C'D'E'\ldots)$.

Let BB', CC' meet in O.

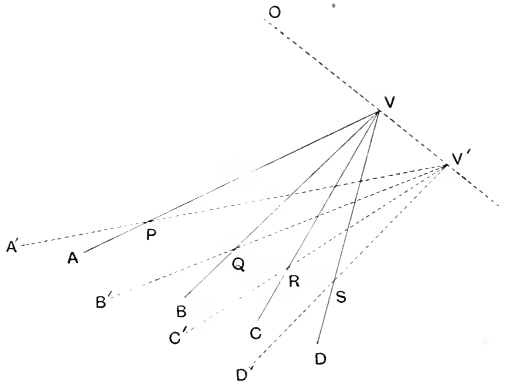
Join OD to cut AB' in D''.

Then $$(AB'C'D') = (ABCD)$$
$$= (AB'C'D'').$$

$\therefore$ D' and D'' coincide. (§ 51.)

Thus the line joining any two corresponding points in the two homographic ranges passes through O; therefore they are in perspective.

61. Two pencils $V(A, B, C, D\ldots)$ and $V'(A', B', C', D'\ldots)$ will according to our definition be in perspective when V and V' are in perspective, points in VA in perspective with points in $V'A'$, points in VB in perspective with points in $V'B'$ and so on.



We can at once prove the following proposition:

If two pencils in different planes be in perspective they have a common transversal and are homographic.

Let the pencils be $V(A, B, C, D\ldots)$ and $V'(A', B', C', D'\ldots)$.

Let the point of intersection of VA and $V'A'$, which are coplanar (§ 59), be P; let that of VB, $V'B'$ be Q; and so on.

The points P, Q, R, S, &c. each lie in both of the planes of the pencils, that is, they lie in the line of intersection of these planes.

Thus the points are collinear, and since

$$V(ABCD\ldots) = (PQRS\ldots) = V'(A'B'C'D'\ldots),$$

the two pencils are homographic.

The line $PQRS\ldots$ containing the points of intersection of corresponding rays is called *the axis of perspective.*

62. According to the definition of perspective given at the beginning of this chapter, two pencils *in the same plane* are always in perspective, with any point on the line joining their vertices as centre.

Let the points of intersection of corresponding rays be, as in the last paragraph, P, Q, R, S, &c.

We cannot now prove P, Q, R, $S\ldots$ to be collinear, for indeed they are not so necessarily.

But if the points P, Q, &c. are collinear, then we say that the pencils are *coaxal.*

If the pencils are coaxal they are at once seen to be homographic.

63. It is usual with writers on this subject to define two pencils as in perspective if their corresponding rays intersect in collinear points.

The objection to this method is that you have a different definition of perspective for different purposes.

We shall find it conducive to clearness to keep rigidly to the definition we have already given, and we shall speak of two pencils as *coaxally in perspective* if the intersections of their corresponding rays are collinear.

As we have seen, two non-coplanar pencils in perspective are always coaxal; but not so two coplanar pencils.

Writers, when they speak of two pencils as in perspective, mean what we here call 'coaxally in perspective.'

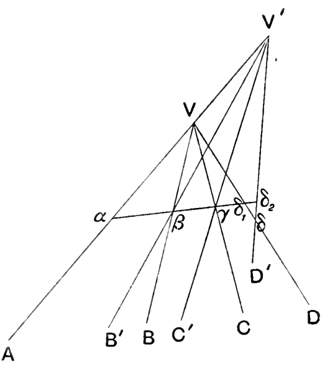
64. Prop. *If two homographic pencils in the same plane have a corresponding ray the same in both, they are coaxally in perspective.*

Let the pencils be

$$V(A, B, C, D, \&c.) \text{ and } V'(A, B', C', D', \&c.)$$

with the common ray $V'VA$.

Let VB and $V'B'$ intersect in β, VC and $V'C'$ in γ, VD and $V'D'$ in δ, and so on.



Let $\gamma\beta$ meet $V'VA$ in α, and let it cut the rays VD and $V'D'$ in δ_1 and δ_2 respectively.

Then since the pencils are homographic,

$$V(ABCD) = V'(AB'C'D').$$
$$\therefore (\alpha\beta\gamma\delta_1) = (\alpha\beta\gamma\delta_2).$$

Therefore δ_1 and δ_2 coincide with δ.

Thus the intersection of the corresponding rays VD and $V'D'$ lies on the line $\beta\gamma$.

Similarly the intersection of any two other corresponding rays lies on this same line.

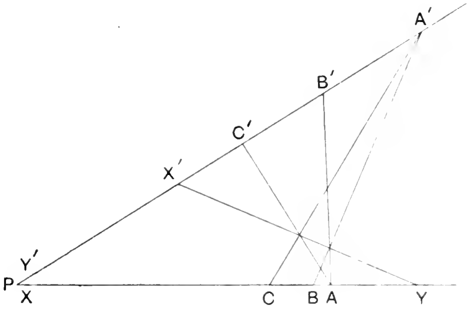
Therefore the pencils are coaxally in perspective.

65. Prop. *If $ABC...$, $A'B'C'...$, be two coplanar homographic ranges not having a common corresponding point, then if two pairs of corresponding points be cross-joined (e.g. AB' and $A'B$) all the points of intersection so obtained are collinear.*

Let the lines of the ranges intersect in P.

Now according to our hypothesis P is not a corresponding point in the two ranges.

It will be convenient to denote P by two different letters, X and Y', according as we consider it to belong to the $ABC...$ or to the $A'B'C'...$ range.



Let X' be the point of the $A'B'C'...$ range corresponding to X in the other, and let Y be the point of the $ABC...$ range corresponding to Y' in the other.

Then $$(ABCXY...) = (A'B'C'X'Y'...).$$

$$\therefore A'(ABCXY...) = A(A'B'C'X'Y'...).$$

These two pencils have a common ray, viz. AA', therefore by the last proposition the intersections of their corresponding rays are collinear, viz.

$$A'B, AB'; \; A'C, AC'; \; A'X, AX'; \; A'Y, AY';$$

and so on.

From this it will be seen that the locus of the intersections of the cross-joins of A and A' with B and B', C and C' and so on is the line $X'Y$.

Similarly the cross-joins of any two pairs of corresponding points will lie on $X'Y$.

This line $X'Y$ is called the *homographic axis* of the two ranges.

This proposition is also true if the two ranges have a common corresponding point. The proof of this may be left to the student.

66. The student may obtain practice in the methods of this chapter by proving that if

$$V(A, B, C\ldots) \text{ and } V'(A', B', C'\ldots)$$

be two homographic coplanar pencils not having a common corresponding ray, then if we take the intersections of VP and $V'Q'$, and of VQ and $V'P'$ (VP, $V'P'$; and VQ, $V'Q'$ being any two pairs of corresponding lines) and join these, all the lines thus obtained are concurrent.

It will be seen when we come to Reciprocation that this proposition follows at once from that of § 65.

TRIANGLES IN PERSPECTIVE

67. Prop. *If the vertices of two triangles are in perspective, the intersections of their corresponding sides are collinear, and conversely.*

(1) Let the triangles be in different planes.

Let O be the centre of perspective of the triangles ABC, $A'B'C'$.

Since BC, $B'C'$ are in a plane, viz. the plane containing OB and OC, they will meet. Let X be their point of intersection.

Similarly CA and $C'A'$ will meet (in Y) and AB and $A'B'$ (in Z).

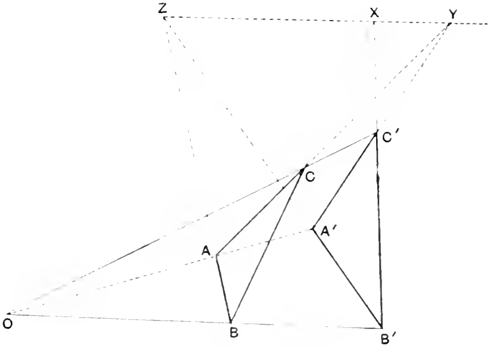
Now X, Y, Z are in the planes of both the triangles ABC, $A'B'C'$.

Therefore they lie on the line of intersection of these planes.

Thus the first part of our proposition is proved.

Next let the triangles ABC, $A'B'C'$ be such that the intersections of corresponding sides (X, Y, Z) are collinear.

Since BC and $B'C'$ meet they are coplanar, and similarly for the other pairs of sides.



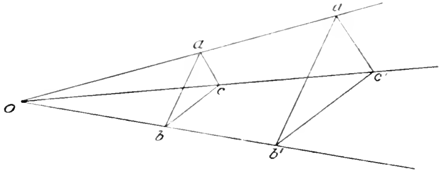
Thus we have three planes $BCC'B'$, $CAA'C'$, $ABB'A'$, of which AA', BB', CC' are the lines of intersection.

But three planes meet in a point.

Therefore AA', BB', CC' are concurrent, that is, the triangles are in perspective.

(2) Let the triangles be in the same plane.

First let them be in perspective, centre O.



Let X, Y, Z be the intersections of the corresponding sides as before.

Project the figure so that XY is projected to infinity.

Denote the projections of the different points by corresponding small letters.

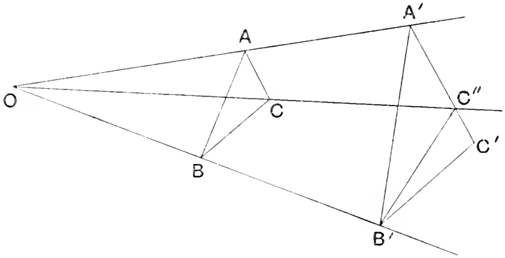
We have now

$$ob : ob' = oc : oc' \text{ since } bc \text{ is parallel to } b'c'$$
$$= oa : oa' \text{ since } ca \text{ is parallel to } c'a'.$$
$$\therefore\ ab \text{ is parallel to } a'b'.$$
$$\therefore\ z \text{ is at infinity also,}$$

that is, x, y, z are collinear.

$$\therefore\ X,\ Y,\ Z \text{ are collinear.}$$

Next let X, Y, Z be collinear; we will prove that the triangles are in perspective.



Let AA' and BB' meet in O.

Join OC and let it meet $A'C'$ in C''.

Then ABC and $A'B'C''$ are in perspective.

$\therefore$ the intersection of BC and $B'C''$ lies on the line YZ.

But BC and $B'C'$ meet the line YZ in X by hypothesis.

$$\therefore\ B'C'' \text{ and } B'C' \text{ are in the same line,}$$

i.e. C'' coincides with C'.

Thus ABC and $A'B'C'$ are in perspective.

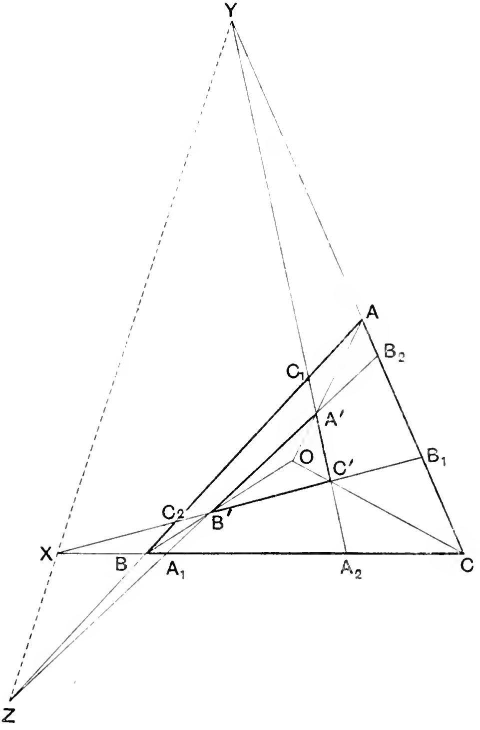
68. **Prop.** *The necessary and sufficient condition that the coplanar triangles ABC, $A'B'C'$ should be in perspective is*

$$AB_1 \,.\, AB_2 \,.\, CA_1 \,.\, CA_2 \,.\, BC_1 \,.\, BC_2$$
$$= AC_1 \,.\, AC_2 \,.\, BA_1 \,.\, BA_2 \,.\, CB_1 \,.\, CB_2,$$

A_1, A_2 being the points in which $A'B'$ and $A'C'$ meet the non-corresponding side BC,

B_1, B_2 being the points in which $B'C'$ and $B'A'$ meet the non-corresponding side CA,

C_1, C_2 being the points in which $C'A'$ and $C'B'$ meet the non-corresponding side AB.



First let the triangles be in perspective; let XYZ be the axis of perspective.

Then since X, B_1, C_2 are collinear,

$$\therefore \frac{AB_1 \,.\, CX \,.\, BC_2}{AC_2 \,.\, BX \,.\, CB_1} = 1.$$

Since Y, C_1, A_2 are collinear,

$$\therefore \frac{AY \,.\, CA_2 \,.\, BC_1}{AC_1 \,.\, BA_2 \,.\, CY} = 1.$$

Since Z, A_1, B_2 are collinear,

$$\therefore \frac{AB_2 \,.\, CA_1 \,.\, BZ}{AZ \,.\, BA_1 \,.\, CB_2} = 1.$$

Taking the product of these we have

$$\frac{AB_1 \,.\, AB_2 \,.\, CA_1 \,.\, CA_2 \,.\, BC_1 \,.\, BC_2 \,.\, AY \,.\, CX \,.\, BZ}{AC_1 \,.\, AC_2 \,.\, BA_1 \,.\, BA_2 \,.\, CB_1 \,.\, CB_2 \,.\, AZ \,.\, BX \,.\, CY} = 1.$$

But X, Y, Z are collinear,

$$\therefore \frac{AY \,.\, CX \,.\, BZ}{AZ \,.\, BX \,.\, CY} = 1.$$

$$\therefore AB_1 \,.\, AB_2 \,.\, CA_1 \,.\, CA_2 \,.\, BC_1 \,.\, BC_2 = AC_1 \,.\, AC_2 \,.\, BA_1 \,.\, BA_2 \,.\, CB_1 \,.\, CB_2.$$

Next we can shew that this condition is sufficient.

For it renders necessary that

$$\frac{AY \,.\, CX \,.\, BZ}{AZ \,.\, BX \,.\, CY} = 1.$$

$\therefore$ X, Y, Z are collinear and the triangles are in perspective.

Cor. If the triangle ABC be in perspective with $A'B'C'$, and the points A_1, A_2, B_1, B_2, C_1, C_2 be as defined in the above proposition, it is clear that the three following triangles must also be in perspective with ABC, *viz.*

(1) the triangle formed by the lines A_1B_2, B_1C_1, C_2A_2,

(2) „ „ „ „ A_1B_1, B_2C_2, C_1A_2,

(3) „ „ „ „ A_1B_1, B_2C_1, C_2A_2.

EXERCISES

1. ABC, $A'B'C'$ are two ranges of three points in the same plane; BC' and $B'C$ intersect in A_1, CA' and $C'A$ in B_1, and AB' and $A'B$ in C_1; prove that A_1, B_1, C_1 are collinear.

2. ABC and $A'B'C'$ are two coplanar triangles in perspective, centre O, through O any line is drawn not in the plane of the triangle; S and S' are any two points on this line. Prove that the triangle ABC by means of the centre S, and the triangle $A'B'C'$ by means of the centre S', are in perspective with a common triangle.

3. Assuming that two non-coplanar triangles in perspective are coaxal, prove by means of Ex. 2 that two coplanar triangles in perspective are coaxal also.

4. If ABC, $A'B'C'$ be two triangles in perspective, and if BC' and $B'C$ intersect in A_1, CA' and $C'A$ in B_1, AB' and $A'B$ in C_1, then the triangle $A_1B_1C_1$ will be in perspective with each of the given triangles, and the three triangles will have a common axis of perspective.

5. When three triangles are in perspective two by two and have the same axis of perspective, their three centres of perspective are collinear.

6. The points Q and R lie on the straight line AC, and the point V on the straight line AD; VQ meets the straight line AB in Z, and VR meets AB in Y; X is another point on AB; XQ meets AD in U, and XR meets AD in W, prove that YU, ZW, AC are concurrent.

7. The necessary and sufficient condition that the coplanar triangles ABC, $A'B'C'$ should be in perspective is

$$Ab' . Bc' . Ca' = Ac' . Ba' . Cb',$$

where a', b', c' denote the sides of the triangle $A'B'C'$ opposite to A', B', C' respectively, and Ab' denotes the perpendicular from A on to b'.

[Let $B'C'$ and BC meet in X; $C'A'$ and CA in Y; $A'B'$ and AB in Z. The condition given ensures that X, Y, Z are collinear.]

8. Prove that the necessary and sufficient condition that the coplanar triangles ABC, $A'B'C'$ should be in perspective is

$$\frac{\sin ABC' \sin ABA' \sin BCA' \sin BCB' \sin CAB' \sin CAC'}{\sin ACB' \sin ACA' \sin CBA' \sin CBC' \sin BAC' \sin BAB'} = 1.$$

[This is proved in Lachlan's *Modern Pure Geometry*. The student has enough resources at his command to establish the test for himself. Let him turn to § 36 Cor., and take in turn O at A', B', C' and at the centre of perspective. The result is easily obtained. Nor is it difficult to remember if the student grasps the principle, by which all these formulae relating to points on the sides of a triangle are best kept in mind—the principle, that is, of travelling round the triangle in the two opposite directions, (1) AB, BC, CA, (2) AC, CB, BA.]

9. Two triangles in plane perspective can be projected into equilateral triangles.

10. ABC is a triangle, I_1, I_2, I_3 its ecentres opposite to A, B, C respectively. I_2I_3 meets BC in A_1, I_3I_1 meets CA in B_1 and I_1I_2 meets AB in C_1, prove that A_1, B_1, C_1 are collinear.

11. If AD, BE, CF and AD', BE', CF' be two sets of concurrent lines drawn through the vertices of a triangle ABC and meeting the opposite sides in D, E, F and D', E', F', and if EF and $E'F'$ intersect in X, FD and $F'D'$ in Y, and DE and $D'E'$ in Z, then the triangle XYZ is in perspective with each of the triangles ABC, DEF, $D'E'F'$.

[Project the triangle so that AD, BE, CF become the perpendiculars in the projection and AD', BE', CF' the medians, and then use Ex. 7.]

CHAPTER VII

HARMONIC SECTION

69. Def. Four collinear points A, B, C, D are said to form a *harmonic range* if

$$(ABCD) = -1.$$

We have in this case

$$\frac{AB \cdot CD}{AD \cdot CB} = -1.$$

$$\therefore \frac{AB}{AD} = -\frac{AB - AC}{AD - AC} = \frac{AB - AC}{AC - AD},$$

thus AC is a harmonic mean between AB and AD.

Now reverting to the table of the twenty-four cross-ratios of a range of four points (§ 50), we see that if $(ABCD) = -1$, then all the following cross-ratios $= -1$:

$$(ABCD),\ (BADC),\ (CDAB),\ (DCBA),$$
$$(ADCB),\ (BCDA),\ (CBAD),\ (DABC).$$

Hence not only is AC a harmonic mean between AB and AD, but also

BD is a harmonic mean between BA and BC,
DB „ „ „ DC and DA,
CA „ „ „ CB and CD.

We shall then speak of A and C as harmonic conjugates to B and D, and express the fact symbolically thus:

$$(AC, BD) = -1.$$

By this we mean that all the eight cross-ratios given above, and in which, it will be observed, A and C are alternate members, and B and D alternate, are equal to -1.

When $(AC, BD) = -1$ we sometimes speak of D as the fourth harmonic of A, B and C; or again we say that AC is divided harmonically at B and D, and that BD is so divided at A and C. Or again we may say that C is harmonically conjugate with A with respect to B and D.

A pencil $P(A, B, C, D)$ of four rays is called harmonic when the points of intersection of its rays with a transversal form a harmonic range.

The student can easily prove for himself that the internal and external bisectors of any angle form with the lines containing it a harmonic pencil.

70. Prop. *If $(AC, BD) = -1$, and O be the middle point of AC, then*

$$OB \cdot OD = OC^2 = OA^2.$$

For since $(ABCD) = -1$,

$$\therefore AB \cdot CD = -AD \cdot CB.$$

Insert the origin O.

$$\therefore (OB - OA)(OD - OC) = -(OD - OA)(OB - OC).$$

But $OA = -OC$.

$$\therefore (OB + OC)(OD - OC) = -(OD + OC)(OB - OC).$$

$$\therefore OB \cdot OD + OC \cdot OD - OB \cdot OC - OC^2$$

$$= -OD \cdot OB + OC \cdot OD - OB \cdot OC + OC^2.$$

$$\therefore 2OB \cdot OD = 2OC^2.$$

$$\therefore OB \cdot OD = OC^2 = AO^2 = OA^2.$$

Similarly if O' be the middle point of BD,

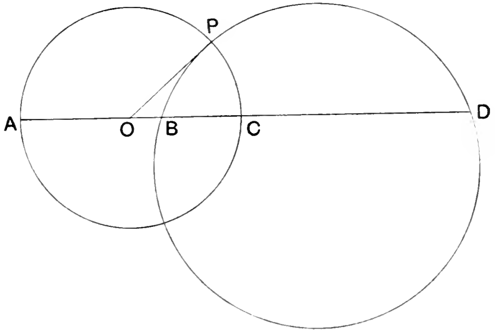
$$O'C \cdot O'A = O'B^2 = O'D^2.$$

Cor. 1. The converse of the above proposition is true, *viz.* that if $ABCD$ be a range and O the middle point of AC and $OC^2 = OB \cdot OD$, then $(AC, BD) = -1$.

This follows by working the algebra backwards.

COR. 2. Given three points A, B, C in a line, to find a point D in the line such that $(AB, CD) = -1$ we describe a circle on AB as diameter, then D is the inverse point of C.

71. Prop. *If $(AC, BD) = -1$, the circle on AC as diameter will cut orthogonally every circle through B and D.*



Let O be the middle point of AC and therefore the centre of the circle on AC.

Let this circle cut any circle through B and D in P; then

$$OB \cdot OD = OC^2 = OP^2.$$

Therefore OP is a tangent to the circle BPD; thus the circles cut orthogonally.

Similarly, of course, the circle on BD will cut orthogonally every circle through A and C.

COR. 1. *If $ABCD$ be a range, and if the circle on AC as diameter cut orthogonally some one circle passing through B and D, then $(AC, BD) = -1$.*

For using the same figure as before, we have

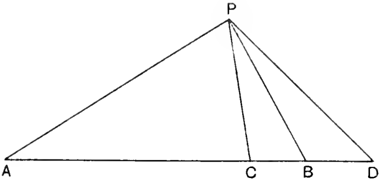
$$OB \cdot OD = OP^2 = OC^2.$$

$$\therefore (AC, BD) = -1.$$

COR. 2. *If two circles cut orthogonally, any diameter of one is divided harmonically by the other.*

72. Prop. *If $P(AB, CD) = -1$ and APB be a right angle, then PA and PB are the bisectors of the angles between PC and PD.*

Let any transversal cut the rays PA, PB, PC, PD of the harmonic pencil in A, B, C, D.



Then
$$(AB, CD) = -1.$$
$$\therefore \frac{AC \cdot BD}{AD \cdot BC} = -1.$$
$$\therefore AC : AD = CB : BD.$$
$\therefore$ as P lies on the circle on AB as diameter we have by § 27
$$PC : PD = CB : BD = AC : AD.$$
$\therefore$ PA and PB are the bisectors of the angle CPD.

73. Prop. *If on a chord PQ of a circle two conjugate points A, A' with respect to the circle be taken, then*
$$(PQ, AA') = -1.$$

Draw the diameter CD through A to cut the polar of A, on which A' lies, in L.

Let O be the centre.

Then by the property of the polar,
$$OL \cdot OA = OC^2.$$
$$\therefore (CD, LA) = -1.$$

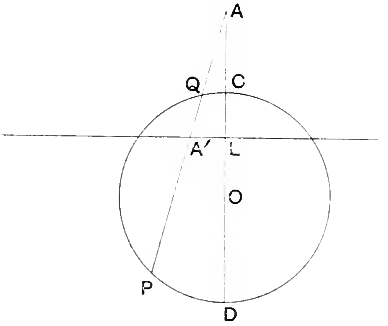
Therefore the circle on CD as diameter (i.e. the given circle) will cut orthogonally every circle through A and L (§ 71).

But the circle on AA' as diameter passes through A and L.

Therefore the given circle cuts orthogonally the circle on AA' as diameter.

But the given circle passes through P and Q.

$$\therefore\ (PQ, AA') = -1.$$



This harmonic property of the circle is of great importance and usefulness. It may be otherwise stated thus:

Chords of a circle through a point A are harmonically divided at A and at the point of intersection of the chord with the polar of A.

74. Prop. *Each of the three diagonals of a plane quadrilateral is divided harmonically by the other two.*

Let AB, BC, CD, DA be the four lines of the quadrilateral: A, B, C, D, E, F its six vertices, that is, the intersections of its lines taken in pairs.

Then AC, BD, EF are its diagonals.

Let PQR be the triangle formed by its diagonals.

Project EF to infinity. Denote the points in the projection by corresponding small letters.

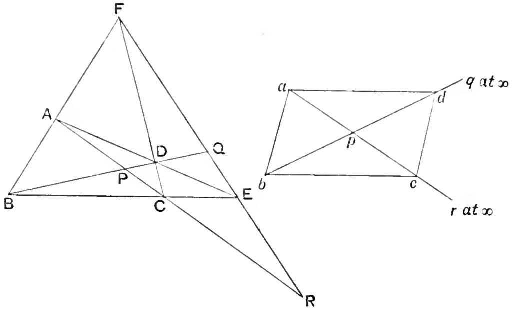
Then $$(BPDQ) = (bpdq) = \frac{bp \cdot dq}{bq \cdot dp} = \frac{bp}{dp}$$

$$\left(\text{since } \frac{dq}{bq} = 1,\ q \text{ being at } \infty\right)$$

$$= -1.$$

Similarly $(APCR) = -1$.

Also $(FQER) = B(FQER) = (APCR) = -1$.



Thus we have proved

$(AC, PR) = -1$, $(BD, PQ) = -1$, $(EF, QR) = -1$.

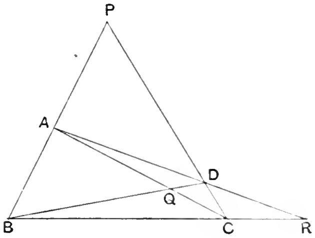
Cor. The circumcircle of PQR will cut orthogonally the three circles described on the three diagonals as diameters.

Note. It has been incidentally shewn in the above proof that if M be the middle point of AB, ω the point at infinity on the line, $(AB, M\omega) = -1$.

75. The harmonic property of the quadrilateral, proved in the last article, is of very great importance. It is important too that at this stage of the subject the student should learn to take the 'descriptive' view of the quadrilateral; for in 'descriptive geometry,' the quadrilateral is not thought of as a closed figure containing an area; but as an assemblage of four lines in a plane, which meet in pairs in six points called the *vertices*; and the three lines joining such of the vertices as are not already joined by the lines of the quadrilateral are called diagonals. By *opposite vertices* we mean two that are not joined by a line of the quadrilateral.

76. A quadrilateral is to be distinguished from a *quadrangle.* A quadrangle is to be thought of as an assemblage of four points in a plane which can be joined in pairs by six straight lines, called its sides or lines; two of these sides which do not meet in a point of the quadrangle are called *opposite sides.* And the intersection of two opposite sides is called a *diagonal point.* This name is not altogether a good one, but it is suggested by the analogy of the quadrilateral.

Let us illustrate the leading features of a quadrangle by the accompanying figure.



$ABCD$ is the quadrangle. Its sides are AB, BC, CD, DA, AC and BD.

AB and CD, AC and BD, AD and BC are pairs of opposite sides and the points P, Q, R where these intersect are the diagonal points.

The triangle PQR may be called the diagonal triangle.

The harmonic property of the quadrangle is *that the two sides of the diagonal triangle at each diagonal point are harmonic conjugates with respect to the two sides of the quadrangle meeting in that point.*

The student will have no difficulty in seeing that this can be deduced from the harmonic property of the quadrilateral proved in § 74.

On account of the harmonic property, the diagonal triangle associated with a quadrangle has been called the *harmonic triangle.*

EXERCISES

1. If M and N be points in two coplanar lines AB, CD, shew that it is possible to project so that M and N project into the middle points of the projections of AB and CD.

2. AA_1, BB_1, CC_1 are concurrent lines through the vertices of a triangle meeting the opposite sides in A_1, B_1, C_1. B_1C_1 meets BC in A_2; C_1A_1 meets CA in B_2; A_1B_1 meets AB in C_2; prove that

$$(BC,\ A_1A_2) = -1, \quad (CA,\ B_1B_2) = -1, \quad (AB,\ C_1C_2) = -1.$$

3. Prove that the circles described on the lines A_1A_2, B_1B_2, C_1C_2 (as defined in Ex. 2) as diameters are coaxal.

[Take P a point of intersection of circles on A_1A_2, B_1B_2, and shew that C_1C_2 subtends a right angle at P. Use Ex. 2 and § 27.]

4. The collinear points A, D, C are given: CE is any other fixed line through C, E is a fixed point, and B is any moving point on CE. The lines AE and BD intersect in Q, the lines CQ and DE in R, and the lines BR and AC in P. Prove that P is a fixed point as B moves along CE.

5. From any point M in the side BC of a triangle ABC lines MB' and MC' are drawn parallel to AC and AB respectively, and meeting AB and AC in B' and C'. The lines BC' and CB' intersect in P, and AP intersects $B'C'$ in M'. Prove that $M'B' : M'C' = MB : MC$.

6. Pairs of harmonic conjugates (DD'), (EE'), (FF') are respectively taken on the sides BC, CA, AB of a triangle ABC with respect to the pairs of points (BC), (CA), (AB). Prove that the corresponding sides of the triangles DEF and $D'E'F'$ intersect on the sides of the triangle ABC, namely EF and $E'F'$ on BC, and so on.

7. The lines VA', VB', VC' bisect the internal angles formed by the lines joining any point V to the angular points of the triangle ABC; and A' lies on BC, B' on CA, C' on AB. Also A'', B'', C'' are harmonic conjugates of A', B', C' with respect to B and C, C and A, A and B. Prove that A'', B'', C'' are collinear.

8. AA_1, BB_1, CC_1 are the perpendiculars of a triangle ABC; A_1B_1 meets AB in C_2; X is the middle point of line joining A to the orthocentre; C_1X and BB_1 meet in T. Prove that C_2T is perpendicular to BC.

9. A is a fixed point without a given circle and P a variable point on the circumference. The line AF at right angles to AP meets in F the tangent at P. If the rectangle $FAPQ$ be completed the locus of Q is a straight line.

10. A line is drawn cutting two non-intersecting circles; find a construction determining two points on this line such that each is the point of intersection of the polars of the other point with respect to the two circles.

11. A_1, B_1, C_1 are points on the sides of a triangle ABC opposite to A, B, C. A_2, B_2, C_2 are points on the sides such that A_1, A_2 are harmonic conjugates with B and C; B_1, B_2 with C and A; C_1, C_2 with A and B. If A_2, B_2, C_2 are collinear, then must AA_1, BB_1, CC_1 be concurrent.

12. AA_1, BB_1, CC_1 are concurrent lines through the vertices of a triangle ABC. B_1C_1 meets BC in A_2, C_1A_1 meets CA in B_2, A_1B_1 meets AB in C_2. Prove that the circles on A_1A_2, B_1B_2, C_1C_2 as diameters all cut the circumcircle of ABC orthogonally, and have their centres in the same straight line.

[Compare Ex. 3.]

13. If A and B be conjugate points of a circle and M the middle point of AB, the tangents from M to the circle are of length MA.

14. If a system of circles have the same pair of points conjugate for each circle of the system, then the radical axes of the circles, taken in pairs, are concurrent.

15. If a system of circles have a common pair of inverse points the system must be a coaxal one.

16. O and O' are the limiting points of a system of coaxal circles, and A is any point in their plane; shew that the chord of contact of tangents drawn from A to any one of the circles will pass through the other extremity of the diameter through A of the circle AOO'.

CHAPTER VIII

INVOLUTION

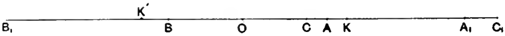
77. Definition.

If O be a point on a line on which lie pairs of points A, A_1; B, B_1; C, C_1; &c. such that

$$OA \cdot OA_1 = OB \cdot OB_1 = OC \cdot OC_1 = \dots\dots\dots\dots = k,$$

the pairs of points are said to be in *Involution*. Two associated points, such as A and A_1, are called *conjugates*; and sometimes each of two conjugates is called the 'mate' of the other.

The point O is called the *Centre* of the involution.



If k, the constant of the involution, be positive, then two conjugate points lie on the same side of O, and there will be two real points K, K' on the line on opposite sides of O such that each is its own mate in the involution; that is $OK^2 = OK'^2 = k$. These points K and K' are called the *double points* of the involution.

It is important to observe that K is not the mate of K'; that is why we write K' and not K_1.

It is clear that $(AA_1, KK') = -1$, and so for all the pairs of points.

If k be negative, two conjugate points will lie on opposite sides of O, and the double points are now imaginary.

If circles be described on AA_1, BB_1, CC_1, &c. as diameters they will form a coaxal system, whose axis cuts the line on which the points lie in O.

K and K' are the limiting points of this coaxal system.

Note also that for every pair of points, each point is inverse to the other with respect to the circle on KK' as diameter.

It is clear that an involution is completely determined when two pairs of points are known, or, what is equivalent, one pair of points and one double point, or the two double points.

We must now proceed to establish the criterion that three pairs of points on the same line may belong to the same involution.

78. Prop. *The necessary and sufficient condition that a pair of points C, C_1 should belong to the involution determined by A, A_1; B, B_1 is*

$$(ABCA_1) = (A_1B_1C_1A).$$

First we will shew that this condition is necessary.

Suppose C and C_1 do belong to the involution. Let O be its centre and k its constant.

$$\therefore\ OA \,.\, OA_1 = OB \,.\, OB_1 = OC \,.\, OC_1 = k.$$

$$\therefore\ (ABCA_1) = \frac{AB \,.\, CA_1}{AA_1 \,.\, CB} = \frac{(OB - OA)(OA_1 - OC)}{(OA_1 - OA)(OB - OC)}$$

$$= \frac{\left(\dfrac{k}{OB_1} - \dfrac{k}{OA_1}\right)\left(\dfrac{k}{OA} - \dfrac{k}{OC_1}\right)}{\left(\dfrac{k}{OA} - \dfrac{k}{OA_1}\right)\left(\dfrac{k}{OB_1} - \dfrac{k}{OC_1}\right)}$$

$$= \frac{(OB_1 - OA_1)(OA - OC_1)}{(OA - OA_1)(OB_1 - OC_1)} = \frac{A_1B_1 \,.\, C_1A}{A_1A \,.\, C_1B_1} = (A_1B_1C_1A).$$

Thus the condition is necessary.

[A more purely geometrical proof of this theorem will be given in the next paragraph.]

Next the above condition is sufficient.

For let $$(ABCA_1) = (A_1B_1C_1A)$$

and let C' be the mate of C in the involution determined by A, A_1; B, B_1.

$$\therefore (ABCA_1) = (A_1 B_1 C'' A).$$
$$\therefore (A_1 B_1 C_1 A) = (A_1 B_1 C'' A).$$
$$\therefore C_1 \text{ and } C'' \text{ coincide.}$$

Hence the proposition is established.

COR. 1. If A, A_1; B, B_1; C, C_1; D, D_1 belong to the same involution

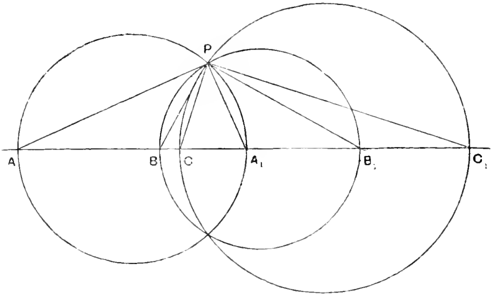
$$(ABCD) = (A_1 B_1 C_1 D_1).$$

COR. 2. If K, K' be the double points of the involution

$$(AA_1 KK') = (A_1 A KK') \text{ and } (ABKA_1) = (A_1 B_1 KA).$$

79. We may prove the first part of the above theorem as follows.

If the three pairs of points belong to the same involution, the circles on AA_1, BB_1, CC_1 as diameters will be coaxal (§ 77).



Let P be a point of intersection of these circles.

Then the angles APA_1, BPB_1, CPC_1 are right angles and therefore

$$P(ABCA_1) = P(A_1 B_1 C_1 A).$$
$$\therefore (ABCA_1) = (A_1 B_1 C_1 A).$$

The circles may not cut in real points. But the proposition still holds on the principle of continuity adopted from Analysis.

80. The proposition we have just proved is of the very greatest importance.

The criterion that three pairs of points belong to the same involution is that a cross-ratio formed with three of the points, one taken from each pair, and the mate of any one of the three should be equal to the corresponding cross-ratio formed by the mates of these four points.

It does not of course matter in what order we write the letters provided that they correspond in the cross-ratios. We could have had

$$(AA_1CB) = (A_1AC_1B_1)$$

or $$(AA_1C_1B) = (A_1ACB_1).$$

All that is essential is that of the four letters used in the cross-ratio, three should form one letter of each pair.

81. Prop. *A range of points in involution projects into a range in involution.*

For let A, A_1; B, B_1; C, C_1 be an involution and let the projections be denoted by corresponding small letters.

Then $$(ABCA_1) = (A_1B_1C_1A).$$

But $$(ABCA_1) = (abca_1)$$

and $$(A_1B_1C_1A) = (a_1b_1c_1a).$$

$$\therefore (abca_1) = (a_1b_1c_1a).$$

$$\therefore a, a_1 : b, b_1 : c, c_1 \text{ form an involution.}$$

NOTE. The centre of an involution does not project into the centre of the involution obtained by projection; but the double points do project into double points.

82. Involution Pencil.

We now see that if we have a pencil consisting of pairs of rays

$$VP, VP'; VQ, VQ'; VR, VR' \text{ \&c.}$$

such that any transversal cuts these in pairs of points

$$A, A_1; B, B_1; C, C_1 \text{ \&c.}$$

forming an involution, then *every* transversal will cut the pencil so.

Such a pencil will be called a *Pencil in Involution* or simply an *Involution Pencil.*

The *double lines* of the involution pencil are the lines through V on which the double points of the involutions formed by different transversals lie.

Note that the double lines are harmonic conjugates with any pair of conjugate rays.

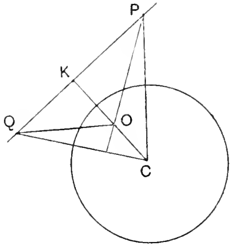
From this fact it results that *if VD and VD' be the double lines of an involution to which VA, VA_1 belong, then VD and VD' are a pair of conjugate lines for the involution whose double lines are VA, VA_1.*

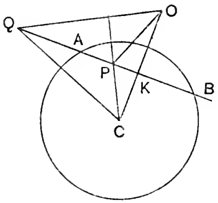
83. We shall postpone until a later chapter, when we come to deal with Reciprocation, the involution properties of the quadrangle and quadrilateral, and pass now to those of the circle which are of great importance. We shall make considerable use of them when we come to treat of the Conic Sections.

84. Involution properties of the circle.

Prop. *Pairs of points conjugate for a circle which lie along a line form a range in involution of which the double points are the points of intersection of the line with the circle.*

Let P and Q be a pair of conjugate points on the line l. Let O be the pole of l.





Thus OPQ is a self-conjugate triangle, and its orthocentre is at C, the centre of the circle (§ 16 a).

Let CK be the perpendicular from C on l.

Then $$PK \,.\, KQ = KO \,.\, KC.$$

$$\therefore\ KP \,.\, KQ = OK \,.\, KC.$$

Thus P and Q belong to an involution whose centre is K, and whose constant is $OK \,.\, KC$. Its double points are thus real or imaginary according as PQ does or does not cut the circle.

If PQ cut the circle in A and B, then $OK \,.\, KC = KA^2 = KB^2$, thus we see that A and B are the double points of the involution.

It is obvious too that A and B must be the double points since each is its own conjugate.

The following proposition, which is the reciprocal of the foregoing, is easily deduced from it.

Prop. *Pairs of conjugate lines for a circle, which pass through a point form an involution pencil of which the double lines are the tangents from the point.*

For pairs of conjugate lines through a point O will meet the polar of O in pairs of conjugate points, which form an involution range, the double points of which are the points in which the polar of O cuts the circle.

Hence the pairs of conjugate lines through O form an involution pencil, the double lines of which are the lines joining O to the points in which its polar cuts the circle, that is the tangents from O.

If O be within the circle the double lines are not real.

85. Orthogonal pencil in Involution.

A special case of an involution pencil is that in which each of the pairs of lines contains a right angle.

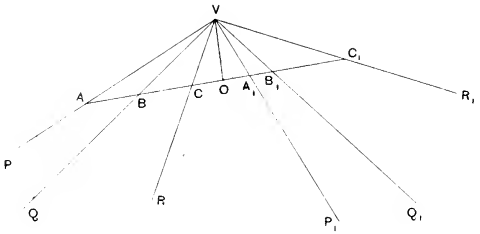
That such a pencil is in involution is clear from the second theorem of § 84, for pairs of lines at right angles at a point are conjugate diameters for any circle having its centre at that point.

But we can also see that pairs of orthogonal lines VP, VP_1; VQ, VQ_1 &c. are in involution, by taking any transversal t to cut these in A, A_1; B, B_1 &c. and drawing the perpendicular VO on to t; then

$$OA \,.\, OA_1 = -OV^2 = OB \,.\, OB_1.$$

Thus the pairs of points belong to an involution with imaginary double points.

Hence pairs of orthogonal lines at a point form a pencil in involution with imaginary double lines.



Such an involution is called an *orthogonal involution.*

Note that this property may give us a test whether three pairs of lines through a point form an involution. If they can be projected so that the angles contained by each pair become right angles, they must be in involution.

86. Prop. *In every involution pencil there is one pair of rays mutually at right angles, nor can there be more than one such pair unless the involution pencil be an orthogonal one.*

Let P be the vertex of the pencil in involution.

Take any transversal l, and let O be the centre of the involution range which the pencil makes on it, and let k be the constant of this involution.

Join OP, and take a point P' in OP such that $OP \cdot OP' = k$.

Thus P and P' will be on the same or opposite sides of O according as k is positive or negative.

Bisect PP' in M and draw MC at right angles to PP' to meet the line l in C.

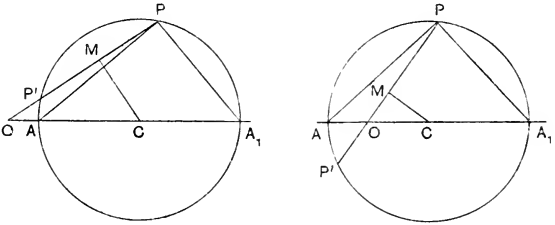
Describe a circle with centre C and radius CP or CP' to cut l in A and A_1.

The points A and A_1 are mates in the involution on l for

$$OA \cdot OA_1 = OP \cdot OP' = k.$$

Also the angle APA_1 being in a semicircle is a right angle.

Hence the involution pencil has the pair of rays PA and PA_1 mutually at right angles.



In the special case where O is the middle point of PP', C and O coincide.

In the case where PO is perpendicular to l, the line through M, the middle point of PP', perpendicular to PP' is parallel to L, and the point C is at infinity. In this case it is PO and the line through P parallel to l which are the pair of orthogonal rays, for O and the point at infinity along l are mates of the involution range on l.

Thus every involution pencil has one pair of orthogonal rays. If the pencil have more than one pair of rays at right angles, then all the pairs must be at right angles, since two pairs of rays completely determine an involution pencil.

87. Prop. *An involution pencil projects into a pencil in involution, and any involution can be projected into an orthogonal involution.*

For let the pairs of rays of an involution pencil at O in the p plane meet the line of intersection of the p and π planes in A. A_1; B, B_1; C, C_1 &c. and let O' be the projection of O.

Then A, A_1; B, B_1; C, C_1 &c. is an involution range.

$\therefore$ O' (A, A_1; B, B_1; C, C_1 &c.) is an involution.

Again, as we can project two angles in the p plane into right angles, we may choose two angles between two pairs of rays of an involution to be so projected. Then the pencil in the projection must be an orthogonal one.

It may be remarked too that at the same time that we project the involution pencil into an orthogonal one we can project any line to infinity (§ 43).

Note. As an orthogonal involution has no real double lines, it is clear that if an involution pencil is to be projected into an orthogonal one, then the pencil thus projected should not have real double lines, if the projection is to be a real one.

An involution pencil with real double lines can only be projected into an orthogonal one by means of an imaginary vertex of projection.

The reader will understand by comparing what is here stated with § 43 that the two circles determining V in that article do not intersect in real points, if the double points of the involution range which the involution pencil in the p plane intercepts on the vanishing line be real, as they are if the involution pencil have real double lines.

EXERCISES

1. Any transversal is cut by a system of coaxal circles in pairs of points which are in involution, and the double points of the involution are concyclic with the limiting points of the system of circles.

2. If K, K' be the double points of an involution to which A, A_1; B, B_1 belong; then A, B_1; A_1, B; K, K' are in involution.

3. If the double lines of a pencil in involution be at right angles, they must be the bisectors of the angles between each pair of conjugate rays.

4. The corresponding sides BC, $B'C'$ &c. of two triangles ABC, $A'B'C'$ in plane perspective intersect in P, Q, R respectively; and AA', BB', CC' respectively intersect the line PQR in P', Q', R'. Prove that the range (PP', QQ', RR') forms an involution.

5. The centre of the circumcircle of the triangle formed by the three diagonals of a quadrilateral lies on the radical axis of the system of circles on the three diagonals.

6. Shew that if each of two pairs of opposite vertices of a quadrilateral is conjugate with regard to a circle, the third pair is also; and that the circle is one of a coaxal system of which the line of collinearity of the middle points of the diagonals is the radical axis.

7. The two pairs of tangents drawn from a point to two circles, and the two lines joining the point to their centres of similitude, form an involution.

8. Prove that there are two points in the plane of a given triangle such that the distances of each from the vertices of the triangle are in a given ratio. Prove also that the line joining these points passes through the circumcentre of the triangle.

CHAPTER IX

THE CONIC SECTIONS

88. Definitions. The *Conic Sections* (or *Conics*, as they are frequently called) are the curves of conical, or vertical, projection of a circle on to a plane other than its own. They are then the plane sections of a cone having a circular base. It is not necessary that the cone should be a right circular one, that is, that its vertex should lie on the line through the centre of the circular base and at right angles to it. So long as the cone has a circular base (and consequently too all its sections parallel to the base are circles), the sections of it are called *conic sections*.

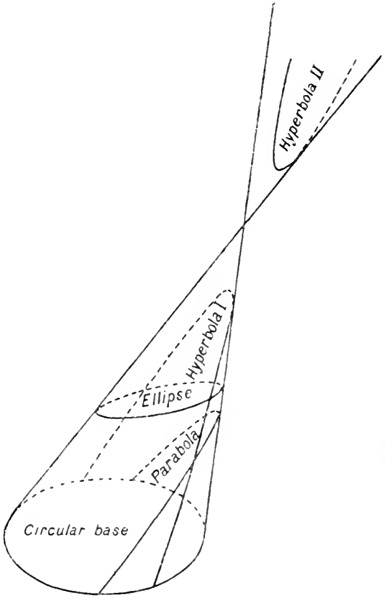
89. The conic sections are classified according to the relation of the vanishing line to the projected circle. If the vanishing line touches the circle, the curve of projection is called a *parabola*; if the vanishing line does not meet the circle, the curve is called an *ellipse*; and if the vanishing line cuts the circle the curve of projection is called a *hyperbola*.

In other words a *parabola* is the section of a cone, having a circular base, by a plane parallel to a generating line of the cone. By a 'generating line' is meant a line joining the vertex of the cone to a point on the circumference of the circle which forms its base.

An *ellipse* is a section of the cone by a plane such that the plane parallel to it through the vertex cuts the plane of the base in a line external to it.

A *hyperbola* is a section of the cone by a plane such that the parallel plane through the vertex cuts the base of the cone.

The curves are illustrated by the following figure and it should be observed that the hyperbola consists of two branches, and that to obtain both these branches the cone must be prolonged on both sides of its vertex.



90. Focus and directrix property.

Every conic section, or projection of a circle, possesses, as we shall presently shew, this property, namely that it is the locus of a point in a plane such that its distance from a fixed point in the plane bears to its distance from a fixed line, also in the plane, a constant ratio. The fixed point is called the *focus* of the conic, the fixed line is called the *directrix*, and the constant ratio the *eccentricity*. It will be proved later that the eccentricity is unity, less than unity, or greater than unity, according as the conic is a parabola, an ellipse, or a hyperbola.

91. Text books on Geometrical Conic Sections usually take the focus and directrix property of the curves as the *definition* of them, and develop their properties therefrom, ignoring for the purpose of this development the fact that every conic section, even when defined by its focus and directrix property, is all the while the projection of some circle. This is to be regretted. For many of the properties which can only be evolved with great labour from the focus and directrix property are proved with great ease when the conics are regarded as the projections of a circle. We shall in the next chapter shew how easy it is to prove that plane curves having the focus and directrix property are the projections of a circle.

92. Projective properties.

The conic sections, being the projections of a circle must possess all the projective properties of the circle.

(1) They will be such that no straight line in their plane can meet them in more than two points, and from points which are the projections of such points in the plane of the circle as lie without the circle, two and only two tangents can be drawn, which will be the projection of the tangents to the circle.

(2) The conic sections will clearly have the 'pole and polar property' of the circle. That is, the locus of the intersections of tangents at the extremities of chords through a given point will be a line, the point and line being called in relation to one another *pole* and *polar*. The polar of a point from which tangents can be drawn to the curve will be the same as the line through the points of contact of the tangents. This line is often called 'the chord of contact,' but strictly speaking the chord is only that portion of the line intercepted by the curve. The polar is unlimited in length.

(3) If the polar of a point A for a conic goes through B, then the polar of B must go through A. Two such points are called *conjugate points*.

(4) Also if the pole of a line l lie on another line l', the pole of l' will lie on l, two such lines being called *conjugate lines*.

(5) The harmonic property of the pole and polar which obtains for the circle must hold also for the conic sections since cross-ratios are unaltered by projection.

(6) As an involution range projects into a range also in involution, pairs of conjugate points for a conic which lie along a line will form an involution range whose double points will be the points (if any) in which the line cuts the curve.

(7) Similarly pairs of conjugate lines through a point will form an involution pencil whose double lines are the tangents (if any) from the point.

93. Circle projected into another circle.

The curve of projection of a circle is under certain conditions another circle.

Prop. *If in the curve of projection of a circle the pairs of conjugate lines through the point P which is the projection of the pole of the vanishing line form an orthogonal involution, then the curve is a circle having its centre at P.*

For since the polar of P is the line at infinity, the tangents at the extremities of any chord through P meet at infinity. But since the involution pencil formed by the pairs of conjugate lines through P is an orthogonal one, these tangents must meet on a line through P perpendicular to the chord. Hence the tangents at the extremities of the chord are at right angles to it.

Thus the curve has the property that the tangent at every point of it is perpendicular to the radius joining the point to P. That is, the curve is a circle with P as centre.

Cor. *A circle can be projected into another circle with any point within it projected into the centre.*

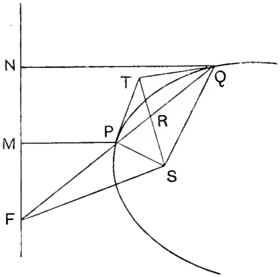
For we have only to project the polar of the point to infinity and the involution pencil formed by the conjugate lines through it into an orthogonal involution.

Note. The point to be projected into the centre needs to be within the circle if the projection is to be a real one (see Note to § 87).

94. Focus and directrix as pole and polar.

Prop. *If in the plane of the curve of projection of a circle there exist a point S such that the involution pencil formed by the conjugate lines through S is an orthogonal one, then S and its polar are focus and directrix for the curve.*

Let P and Q be any two points on the curve, and let the tangents at them meet in T.



Join ST cutting PQ in R, and let PQ meet the polar of S in F. Join SF, and draw PM, QN perpendicular to the polar of S.

Then ST is the polar of F, for the polar of F goes through S since that of S goes through F, and it also goes through T since the polar of T goes through F.

$$\therefore SF \text{ and } ST \text{ are conjugate lines.}$$

But by hypothesis the conjugate lines at S form an orthogonal involution.

$$\therefore TSF \text{ is a right angle.}$$

And by the harmonic property of the pole and polar

$$(FR,\ PQ) = -1.$$

$\therefore$ ST and SF are the bisectors of the angle PSQ (§ 72).

$\therefore SP : SQ = FP : FQ$

$= PM : QN$ (by similar triangles).

$\therefore SP : PM = SQ : QN.$

Thus the ratio of the distance of points on the curve from S to their distance from the polar of S is constant, that is S and its polar are focus and directrix for the curve.

NOTE. If the polar of S should happen to be the line at infinity then the curve of projection is a circle (§ 93).

We may here remark that the circle may be considered to have the focus and directrix property, the focus being at the centre, and the directrix the line at infinity. The eccentricity is the ratio of the radius of the circle to the infinite distance of the points on the circle from the line at infinity.

95. Parallel chords.

Before we go on to establish the focus and directrix property of the curves of projection of a circle, which we shall do by shewing that for all of them there exists at least one point S, the pairs of conjugate lines through which form an orthogonal involution, we will establish a very important general proposition about parallel chords.

Prop. *In any conic section* (or *curve of projection of a circle*) *the locus of the middle points of a system of parallel chords is a straight line, and the tangents at the points where this line meets the curve are parallel to the chords.*

Moreover the tangents at the extremities of each of the parallel chords will intersect on the line which is the locus of the middle points of the chords, and every line parallel to the chords and in the plane of the curve is conjugate with this line containing the middle points.

Let QQ' be one of the chords of the system and M its middle point.

The chords may be considered as concurrent in a point R at infinity which is the projection of a point r on the vanishing line; and we have

$$(QQ', MR) = -1.$$

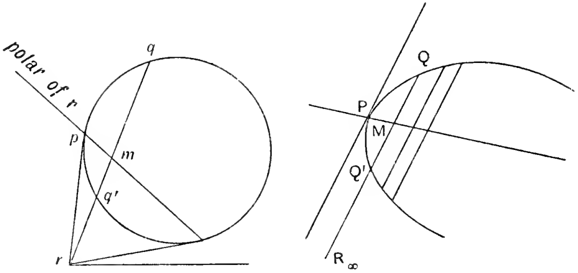
$$\therefore (qq', mr) = -1$$

that is m is on the polar of r.

Thus as the locus of the points m is a line, that of the points M is so too.

Let P be a point in which this locus meets the curve, and let P be the projection of p, then as the tangent at p goes through r that at P must go through R; that is the tangent at P is parallel to QQ'.

Further as the tangents at q and q' meet on the polar of r, those at Q and Q' will meet on the projection of the polar of r, that is on the line which is the locus of the middle points of the chords.



Also every line through r will have its pole on the polar of r, and therefore every line through R in the plane of the conic will have its pole on the line PM, that is, every line parallel to the chords is conjugate with the locus of their middle points.

In other words, the polar of every point on the line which is the locus of the middle points of a system of parallel chords is a line parallel to the chords.

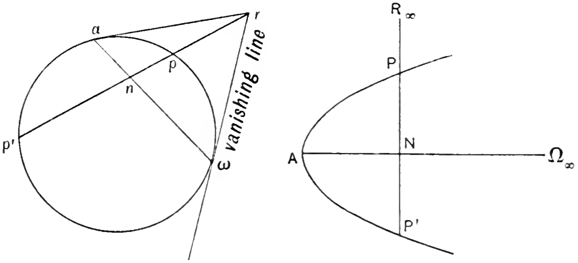
96. Focus and directrix property established.

We are now in a position to establish the focus and directrix property of the conic sections, defined as the projections of a circle. We shall take the parabola, ellipse and hyperbola separately, and in each case prove a preliminary proposition respecting their axes of symmetry.

Prop. *A parabola* (or *the projection of a circle touched by the vanishing line in its plane*) *has an axis of symmetry which meets the curve in two points one of which is at infinity.*

Let the vanishing line touch the circle in ω.

In the plane through V, the vertex of projection, and the vanishing line draw Vr at right angles to $V\omega$, meeting the vanishing line in r. Draw the other tangent ra to the circle.



Now r is the pole of $a\omega$, and therefore if pp' be any chord which produced passes through r and which cuts $a\omega$ in n then

$$(pp', nr) = -1.$$

Thus, using corresponding capital letters in the projection, and remembering that $rn\omega$ will project into a right angle since $r\omega$ subtends a right angle at V, we shall have a chord PP' at right angles to $A\Omega$ and cutting it at N so that

$$(PP', NR) = -1.$$

But R is at infinity. $\therefore$ $PN = NP'$.

Thus all the chords perpendicular to $A\Omega$ are bisected by it, and the curve is therefore symmetrical about this line, which is called the *axis* of the parabola.

The axis meets the curve in the point A, called the *vertex*, and in the point Ω which is at infinity.

As $ra\omega$ projects into a right angle (for $r\omega$ subtends a right angle at V) the tangent at A is at right angles to the axis.

Finally the curve touches the line at infinity at Ω.

97. Prop. *A parabola* (or *projection of a circle touched by the vanishing line*) *has the focus and directrix property, and the eccentricity of the curve is unity.*

Let P be any point on the curve of projection. Let PNP' be the chord through P, perpendicular to the axis, and cutting it in N.

The tangents at P and P' will intersect in a point T on the line of the axis (§ 95).

Then as T is the pole of PP'

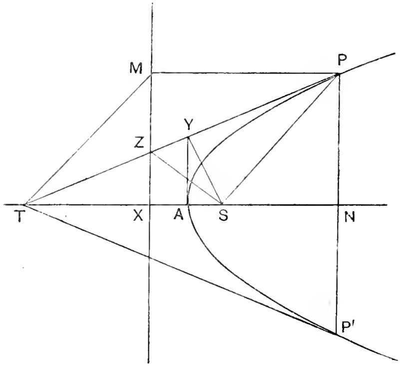
$$(TN, A\Omega) = -1.$$

$\therefore$ as Ω is at infinity, $TA = AN$.

Now let the tangent at P meet that at A in Y and draw YS at right angles to PY to meet the axis in S.

The polar of S will be at right angles to the axis (§ 95). Let it be XM cutting the axis in X, the tangent at P in Z, and the line through P parallel to the axis in M.

Join SP, SZ.



Now as S is the pole of XM

$$(XS, A\Omega) = -1.$$

$\therefore$ $XA = AS$, and as $TA = AN$ we have $TS = XN = MP$.

But $TY : YP = TA : AN = 1$, so that $\triangle TYS \equiv \triangle PYS$, and $TS = PS$.

Thus $PS = PM$, that is P is equidistant from S and the polar of S.

Further, since ST is equal and parallel to PM and $SP = PM$, $SPMT$ is a rhombus and PT bisects the angle SPM.

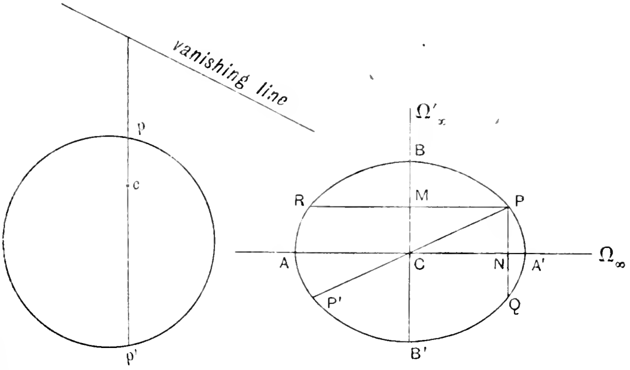
Thus $\triangle SPZ \equiv \triangle MPZ$, and $\angle ZSP = \angle ZMP =$ a right $\angle$.

Now Z is the pole of SP, for the polar of Z must go through S (since that of S goes through Z) and through P since ZP is a tangent at P.

Hence SZ and SP are conjugate lines for the curve, and they are at right angles. So also are ST and the line through S at right angles to it (§ 95).

Therefore the involution pencil formed by the pairs of conjugate lines through S is an orthogonal one. Thus S and its polar XM are focus and directrix for the curve (§ 94), and the eccentricity is $SP : PM$ which is unity.

98. Prop. *The projection of a circle not met by the vanishing line in its plane is either a circle or a closed curve having two axes of symmetry, mutually perpendicular, on which are intercepted by the curve chords of unequal length.*



Let c be the pole of the vanishing line.

Let pp' be any chord through c of the circle.

Then pp' is divided harmonically at c and its intersection with the polar of c.

Using corresponding capital letters in the projection, we shall have that the chord PP' through C is divided harmonically at C and its intersection with the polar of C, which is the line at infinity. $\therefore\ PC = CP'$.

Thus every chord through C is bisected at C. For this reason the point C is called the *centre* of the curve, and the chords through it are called *diameters*.

The tangents at the extremities of any diameter are parallel, for the tangents at the extremities of chords of the circle through c meet in the polar of c, which is the vanishing line.

First suppose that the involution pencil formed by the pairs of conjugate lines through C is an orthogonal one; then the curve is a circle (§ 93).

Next suppose that the involution pencil is not an orthogonal one; then there must be one and only one pair of conjugate lines through C mutually at right angles (§ 86).

Let the curve intercept on these lines chords AA' and BB'.

Draw the chords PQ and PR perpendicular to AA' and BB' respectively cutting them in N and M, and let Ω and Ω' be the points at infinity on the lines of AA' and BB'.

Then as Ω' is the pole of AA', and PQ passes through Ω',

$$(PQ, N\Omega') = -1.$$
$$\therefore\ PN = NQ.$$

Similarly $PM = MR$.

Thus the curve is symmetrical about each of the two lines ACA', BCB', which are called the *axes*.

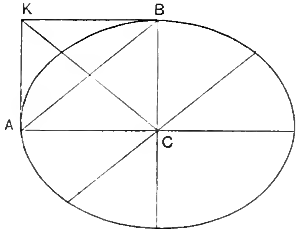
We shall now shew that AA' and BB' cannot be equal.

Let the tangents at A and B meet in K, then $CAKB$ is a rectangle, and CK bisects AB.

But CK bisects the chord through C parallel to AB, for every chord through C is bisected at C.

Hence CK and the line through C parallel to AB are conjugate lines (§ 95).

But these lines would be at right angles if $CA = CB$. And thus if CA and CB were equal the involution pencil formed by the pairs of conjugate lines through C would be an orthogonal one; which is contrary to hypothesis.



Hence AA' and BB' cannot be equal.

We shall suppose AA' to be the greater of the two. Then AA' is called the *major axis* and BB' the *minor axis*.

99. Prop. *An ellipse (or curve of projection, other than a circle, of a circle not met by the vanishing line in its plane) has the focus and directrix property and the eccentricity of the curve is less than unity.*

Assuming that the projection is not a circle we have as shewn in § 98 two axes of symmetry AA', BB' of which AA' is the greater.

With centre B and radius equal to CA describe a circle cutting the major axis in S and S'.

The polars of S and S' are perpendicular to AA' (§ 95).

Let these be XF and $X'F'$, cutting AA' in X and X' and the tangent at B in F and F'.

Now since the polar of S goes through F, that of F goes through S; but the polar of F goes through B, since FB is a tangent.

$\therefore$ SB is the polar of F, and SF, SB are conjugate lines.

We will shew that they are mutually at right angles.

Since S is the pole of XF

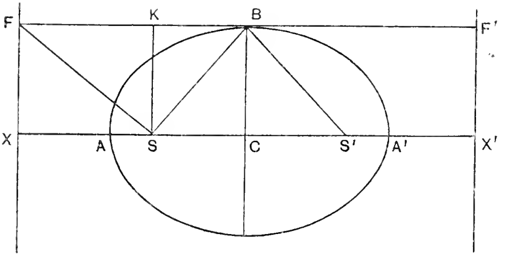
$$(AA', SX) = -1.$$

$$\therefore\ CS \cdot CX = CA^2.$$

Now draw SK parallel to CB to meet BF in K.

Then $$BK \cdot KF = CS \cdot SX = CS(CX - CS)$$
$$= CA^2 - CS^2$$
$$= SB^2 - CS^2 = SK^2.$$

$$\therefore\ FSB \text{ is a right angle.}$$



Thus we have two pairs of conjugate lines through S mutually at right angles, namely SF and SB, as also SX and SK (§ 95) so that the involution pencil formed by the conjugate lines at S is an orthogonal one.

$\therefore$ S and its polar XF are focus and directrix for the curve (§ 94).

Similarly S' and its polar $X'F'$ are focus and directrix.

The eccentricity $= SB : BF = CA : CX$ which is less than unity.

Note too that as $CS \cdot CX = CA^2$, the eccentricity $= CS : CA$.

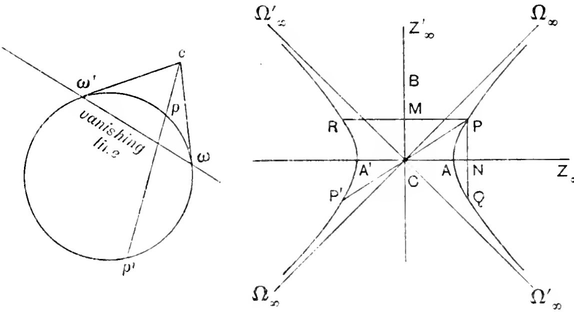
NOTE. We see now how a circle may be regarded as the limiting case of an ellipse whose two foci S and S' coincide with its centre, and its directrix is the line at infinity (see Note to § 94).

100. Prop. *A hyperbola* (or *projection of a circle which is cut by the vanishing line*) *has two axes of symmetry mutually at right angles, only one of which cuts the curve.*

Let the vanishing line cut the circle in ω and ω', and let c be the pole of the line.

Let pp' be any chord the line of which passes through c. Then pp' is divided harmonically at c and its intersection with $\omega\omega'$.

Using corresponding capital letters in the projection, we shall have that the chord PP' through C is divided harmonically at C and its intersection with the polar of C which is the line at infinity. $\therefore PC = CP'$.



Thus every chord through C in the curve of projection is bisected at C, which is therefore called the *centre* of the curve, and the chords through C are called diameters.

Let it be observed that not every line through C meets the curve, since in the plane of the circle there are lines through c which do not meet it.

Of each pair of conjugate lines through c only one will meet the circle, for $c\omega$ and $c\omega'$ are the double lines of the involution formed by these conjugate lines.

Further the involution pencil formed by the conjugate lines through C cannot be an orthogonal one, since it has real double lines, namely the projection of $c\omega$ and $c\omega'$.

Thus there will be one and only one pair of conjugate lines through C mutually at right angles (§ 86).

Let this pair be CA and CB, of which the former is the one that meets the curve, namely in A and A'.

Note that the curve of projection will have two tangents, from C whose points of contact Ω and Ω' the projections of ω and ω' are at infinity. These tangents are called *asymptotes*.

Since $C\Omega$ and $C\Omega'$ are the double lines of the involution pencil formed by the conjugate lines through C

$$C(\Omega\Omega', AB) = -1 \quad (\S\ 82).$$

$\therefore$ Since CA and CB are at right angles, they are the bisectors of the angles between $C\Omega$ and $C\Omega'$ (§ 72).

To prove that the curve is symmetrical about CA and CB we draw chords PQ, PR perpendicular to them and cutting them in N and M. Let Z and Z' be the points at infinity along the lines CA and CB. Then since Z' is the pole of AA' and PQ passes through Z',

$$(PQ, NZ') = -1.$$

$$\therefore PN = NQ.$$

Similarly $PM = MR$.

Thus the curve has two axes of symmetry mutually at right angles, one of which meets the curve, and the other not. AA' which meets the curve is called the *transverse axis* and CB is called the *conjugate axis*.

NOTE. At present B is not a definite point on the line CB. We shall find it convenient later on to make it definite; the point to be emphasised is that the transverse axis does not cut the curve, and we cannot determine points B and B' on it as these points are determined in the case of the ellipse.

101. Prop. *A hyperbola* (or *curve of projection of a circle cut by the vanishing line*) *has the focus and directrix property, and the eccentricity is greater than unity.*

Using the notation of the preceding article, we describe a circle with centre C, and radius CA cutting $C\Omega$ in K and L', and $C\Omega'$ in K' and L, as in the figure.

The lines KL and $K'L'$ will be perpendicular to the transverse axis, since, as we have seen, CA bisects the angle $\Omega C\Omega'$.

$\therefore$ the poles of these lines, which we will denote by S and S', will lie on the line of the transverse axis (§ 95).

We will now shew that S and its polar KL are focus and directrix, as are also S' and $K'L'$.

Let KL and $K'L'$ cut AA' in X and X'.

Then by the harmonic property of the pole and polar

$$(AA', SX) = -1.$$

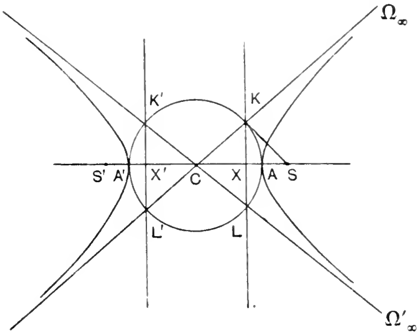
$$\therefore\ CS \, . \, CX = CA^2 = CK^2.$$

$$\therefore\ CKS \text{ is a right angle.}$$

Now the polar of K must go through S, since that of S goes through K. Moreover the polar of K goes through Ω, since $K\Omega$ is a tangent at Ω.

$\therefore$ $S\Omega$ is the polar of K, that is SK and $S\Omega$ are conjugate lines.

But Ω being at infinity $S\Omega$ is parallel to $K\Omega$, that is $S\Omega$ is perpendicular to SK.



Thus we have two pairs of conjugate lines through S, mutually at right angles, namely $S\Omega$ and SK, and SC and the line through S at right angles to SC (§ 95).

Hence the pencil formed by the pairs of conjugate lines through S is an orthogonal one, and therefore S and its polar KX are focus and directrix for the curve.

Similarly S' and $K'L'$ are focus and directrix.

The eccentricity is the ratio

$$\begin{aligned} & S\Omega : \text{perpendicular from } \Omega \text{ on } KL \\ &= K\Omega : \text{the same} \\ &= CK : CX = CA : CX \text{ which is greater than unity.} \end{aligned}$$

Note too that as $CS.CX = CA^2$, the eccentricity also $= CS : CA$.

We might have obtained the eccentricity thus:

It is the ratio $SA : AX$

$$\begin{aligned} &= CS - CA : CA - CX \\ &= CS \,.\, CX - CA \,.\, CX : CX\,(CA - CX) \\ &= CA\,(CA - CX) : CX\,(CA - CX) \\ &= CA : CX. \end{aligned}$$

102. Central and non-central conics. Diameters.

We have seen that the ellipse and hyperbola have each a centre, that is a point such that every chord passing through the same is bisected by it. Ellipses and hyperbolas then are classified together as *central conics.* The parabola has no centre and is called *non-central.*

We have proved in § 95 that the locus of the middle points of a system of parallel chords is a straight line. Clearly in the case of the central conics this line must go through the centre, for the diameter parallel to the chords is bisected at that point.

In the case of the parabola, the line which is the locus of the middle points of a system of parallel chords is parallel to the axis. For such a system is the projection of chords of the circle concurrent at a point r on the vanishing line; and the polar of r, which projects into the locus of the middle points of the system of chords, passes through ω the point of contact of the circle with the vanishing line. Thus the locus of the middle points of the system of parallel chords of the parabola passes through Ω, that is, the line is parallel to the axis.

All lines then in the plane of a parabola and parallel to its axis will bisect each a system of parallel chords. These lines are conveniently called diameters of the parabola. They are not diameters in the same sense in which the diameters of a central conic are, for they are not limited in length and bisected at a definite point.

103. Ordinates of diameters.

Def. The parallel chords of a conic bisected by a particular diameter are called *double ordinates* of that diameter, and the half chord is called an *ordinate* of the diameter.

The ordinates of a diameter are as we have seen parallel to the tangents at the point or points in which the diameter meets the curve.

The ordinates of an axis of a conic are perpendicular to that axis.

The ordinates of the axis of a parabola, of the major axis of an ellipse, and of the transverse axis of a hyperbola are often called simply 'ordinates' without specifying that whereto they are ordinates. Thus the ordinate of a point P on a parabola, ellipse or hyperbola must be understood to mean the perpendicular PN on the axis, the major axis, or the transverse axis, as the case may be.

Note. When we speak of the axis of a conic there can be no ambiguity in the case of a parabola, but in the case of the ellipse and hyperbola, which have two axes of symmetry, there would be ambiguity unless we determined beforehand which axis was meant. Let it then be understood that by the axis of a conic will be meant that one on which the foci lie.

104. The contents of the present chapter are of great importance for a right understanding of the conic sections. The student should now have a good general idea of the form of the curves, and, as it were, see them whole, realising that they have been obtained by projecting a circle from one plane on to another. We shall in the next chapter set forth properties which all conics have in common, and in subsequent chapters treat of the parabola, ellipse and hyperbola separately, shewing the special properties which each curve has.

CHAPTER X

PROPERTIES COMMON TO ALL CONICS

105. Proposition. *If the line (produced if necessary) joining two points P and Q of a conic meet a directrix in F, and S be the corresponding focus, SF will bisect one of the angles between SP and SQ.*

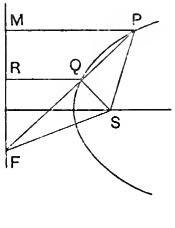


Fig. 1.

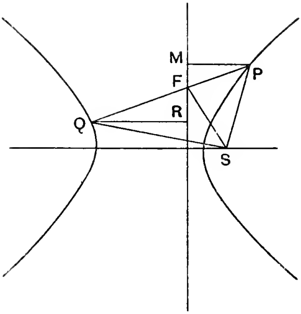


Fig. 2.

For, drawing PM and QR perpendicular to the directrix we have, if e be the eccentricity,

$$SP : PM = e = SQ : QR,$$

$$\therefore\ SP : SQ = PM : QR$$

$$= FP : FQ \text{ (by similar } \triangle\text{s } FQR,\ FPM).$$

∴ in figs 1 and 3, SF bisects the exterior angle of PSQ, and in fig. 2 it bisects the angle PSQ itself.

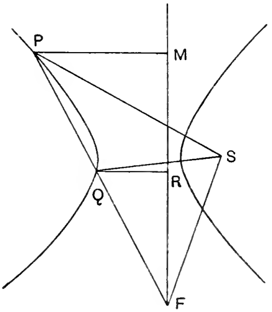


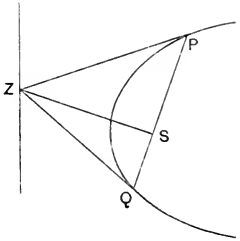
Fig. 3.

We see then that SF bisects the exterior angle of PSQ, if P and Q be on the same branch of the curve, and the angle PSQ itself if P and Q be on opposite branches.

106. Prop. *If the tangent to a conic at a point P meet a directrix in Z and S be the corresponding focus, ZSP is a right angle.*

This is easily seen from the following considerations:

The focus and directrix are 'pole and polar' for the conic, therefore the tangents at the extremities of a focal chord PSQ will meet at Z in the directrix, and Z will be the pole of PQ.



Thus SZ and SP are conjugate lines, since the pole of SP lies on SZ.

But the pairs of conjugate lines through a focus are at right angles. Therefore ZSP is a right angle.

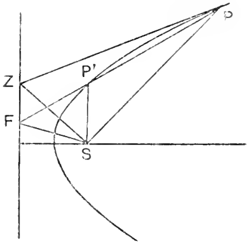
But as we are going to prove in the next article that every plane curve having the focus and directrix property is the projection of some circle, we will give another proof of the proposition dependent only on this property.

Regard the tangent at P as the limiting case of the line of the chord PP' when P' is very close to P.

Now if PP' meet the directrix in F, SF bisects the exterior angle of PSP' (§ 105) for P and P' are on the same branch.

And the nearer P' approaches P, the more does this exterior angle approximate to two right angles.

$$\text{Thus } \angle ZSP = \text{the limit of } FSP' \text{ when } P' \text{ approaches } P$$
$$= \text{a right angle.}$$



It should be observed that this second proof yields also the result that tangents at the extremities of a focal chord intersect in the directrix, since the tangent at either end of the chord PSQ is determined by drawing SZ at right angles to the chord to meet the directrix in Z; then ZP, ZQ are the tangents.

107. In the preceding chapter we defined the conic sections as the curves of projection of a circle and showed that they have the focus and directrix property. We shall now establish the converse proposition.

Prop. *Every plane curve having the focus and directrix property is the projection of some circle.*

For we have shown in the second part of § 106 that a curve having the focus and directrix property is such that tangents at the extremities of any chord through S, the focus, intersect on the directrix on a line through S perpendicular to the chord.

Now project so that the directrix is the vanishing line; and so that the orthogonal involution at S projects into another orthogonal involution (§ 87).

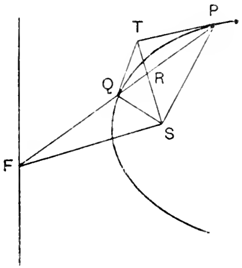
Then the curve of projection has the property that the tangents at the extremities of every chord through s, the projection of S, meet at infinity on a line through s perpendicular to the chord.

Hence the tangent at each point of the curve is at right angles to the radius joining the point to s, and therefore the curve is a circle with s as centre.

108. It follows of course from what we have established in the preceding chapter that if the curve having the focus and directrix property had its eccentricity unity then the circle into which it has been projected must touch the vanishing line in the plane of the circle; if the eccentricity be less than unity then the circle does not meet the vanishing line; if the eccentricity be greater than unity the circle is cut by the vanishing line.

109. Pair of tangents.

Prop. *If a pair of tangents TP, TQ be drawn to a conic from a point P and S be a focus, then SP and SQ make equal angles with ST, and if PQ meet the corresponding directrix in F $\angle TSF$ is a right angle.*



Let TS meet PQ in R.

Since PQ is the polar of T, and this goes through F, the polar of F must go through T.

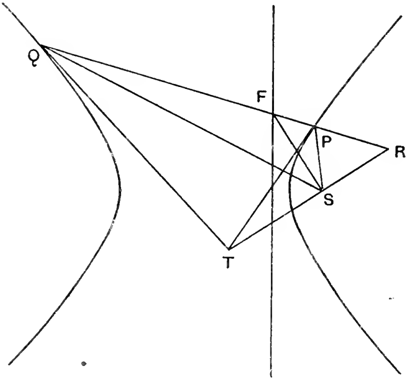
But since F is on the directrix the polar of F must go through S.

Thus ST is the polar of F.

Hence SF and ST are conjugate lines, and as they are through a focus, they must be at right angles.

Further $$(PQ, FR) = -1$$
$$\therefore\ S(PQ, FR) = -1.$$

Thus SR and SF are the bisectors of the angles between SP and SQ (§ 72).



NOTE. It will be seen that if the points of contact of the tangents from T lie on the same branch of the curve ST bisects the angle PSQ, but if they are on different branches then ST bisects the exterior angle of PSQ.

The figures given do not shew the case where TP and TQ both touch the branch remote from S. The student can easily represent this in a figure of his own.

110. The above proposition gives a simple construction for drawing two tangents to a conic from an external point T.

Join ST and let it meet the conic in K and K'. The figure of the preceding article can be utilised. Take R in KK' such that $(TR, KK') = -1$ (§ 70, Cor. 2).

Draw SF at right angles to ST to meet the directrix in F. Draw the line FR and let it cut the conic in P and Q.

Then TP and TQ are the tangents.

For as ST and SF are at right angles and through a focus, they are conjugate lines.

$\therefore$ the pole of ST lies on SF.

But the pole of ST is on the directrix.

$\therefore$ F is the pole of ST, that is, the polar of F goes through T.

$\therefore$ the polar of T goes through F.

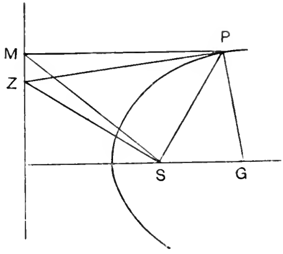
But the polar of T goes through R since $(TR, KK') = -1$. Thus the line FR is the polar of T, that is, PQ is the chord of contact of the tangents from T.

111. The Normal.

Def. The line through the point of contact of a tangent and at right angles to it is called the *normal* at that point.

Prop. *If the normal at P to any conic meet the axis in G, and S be a focus of the conic then $SG = e \cdot SP$, where e is the eccentricity.*

Let the tangent at P meet the directrix corresponding to S in Z.



Draw PM perpendicular to the directrix.

Then since PM is parallel to the axis, $\angle MPS = \angle PSG$.

Also since PMZ and PSZ are right angles, $PSZM$ is cyclic and $\angle SMP = \angle SZP =$ complement of $\angle SPZ = \angle SPG$.

Thus the $\triangle$s SPG, PMS are similar and

$$SG : SP = PS : PM = e\,; \quad \therefore\ SG = e\,.\,SP.$$

The student can make for himself a figure shewing the case where P is on the branch of a hyperbola remote from S. In this case it will be found

$$\begin{aligned}\angle SMP &= 180^\circ - \angle SZP\\ &= 90^\circ + \angle SPZ = \angle SPG.\end{aligned}$$

So that the $\triangle$s SPG and PMS are still similar.

112. The latus rectum.

Def. The focal chord perpendicular to the axis on which the focus lies is called the *latus rectum* of the conic.

Thus the latus rectum is the double ordinate through the focus to the axis (§ 103).

Prop. *The semi-latus rectum of a conic is a harmonic mean between the segments of any focal chord.*

Let SL be the semi-latus rectum, and PSQ any focal chord.

Draw PM and QR perpendicular to the directrix, and PN and QK perpendicular to the axis.

Then $$\begin{aligned}SP : PM = e &= SL : SX\\ &= SQ : QR.\end{aligned}$$

And by similar triangles

$$\begin{aligned}SP : SQ &= SN : KS\\ &= XN - XS : XS - XK \quad \text{(Fig. 1)}\\ &= MP - XS : XS - RQ\\ &= e\,(MP - XS) : e\,(XS - RQ)\\ &= SP - SL : SL - SQ.\end{aligned}$$

$\therefore$ SP, SL and SQ are in harmonic progression and

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{SL}.$$

This proposition requires some modification if P and Q are on opposite branches. We now have

$$SP : SQ = SN : KS = XS - XN : KX + XS \quad \text{(Fig. 2)}$$
$$= e\,(XS - MP) : e\,(QR + XS)$$
$$= SL - SP : SQ + SL$$

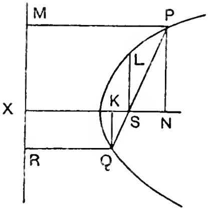


Fig. 1.

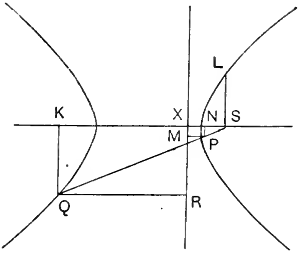


Fig. 2.

$$\therefore\ SP\,(SQ + SL) = SQ\,(SL - SP)$$
$$\therefore\ SQ\,.\,SL - SP\,.\,SL = 2SP\,.\,SQ$$
$$\therefore\ \frac{1}{SP} - \frac{1}{SQ} = \frac{2}{SL}.$$

Thus in this case it is SP, SL and $-SQ$ that are in H.P.

COR. *The rectangle contained by the segments of any focal chord varies as the length of the chord.*

For
$$\frac{1}{SP} \pm \frac{1}{SQ} = \frac{2}{SL},$$
according as P and Q be on the same or opposite branches (P being on the branch adjacent to S), and in both cases we have
$$\frac{PQ}{SP\,.\,SQ} = \frac{2}{SL}$$
$$\therefore\ SP\,.\,SQ = \frac{SL}{2} \times PQ$$
that is
$$SP\,.\,SQ \propto PQ.$$

If SP and SQ are in opposite directions P and Q lie on the same branch of the curve, and if they are in the same direction, P and Q lie on opposite branches.

113. Prop. *Any conic can be projected into a circle with any point in the plane of the conic projected into the centre of the circle.*

For let P be any point in the plane of the conic.

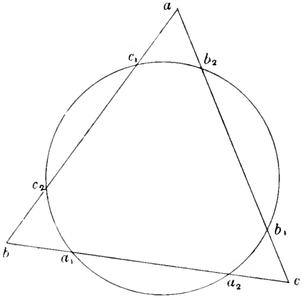
Take the polar of P for the vanishing line and project so that the involution pencil formed by the pairs of conjugate lines through P projects into an orthogonal involution (§ 87); then exactly as in § 93 we can prove that the curve of projection is a circle.

NOTE. In order that the projection may be a real one, the tangents from P to the conic must not be real (Note to § 87), that is P must lie within the conic.

Carnot's theorem.

114. Prop. *If a conic cut the sides of a triangle ABC in A_1, A_2; B_1, B_2; C_1, C_2; then*

$$AB_1 . AB_2 . CA_1 . CA_2 . BC_1 . BC_2$$
$$= AC_1 . AC_2 . BA_1 . BA_2 . CB_1 . CB_2.$$



Project the conic into a circle; and denote the points in the projection by corresponding small letters.

Then since $ab_1 . ab_2 = ac_1 . ac_2,$
$ca_1 . ca_2 = cb_1 . cb_2,$
$bc_1 . bc_2 = ba_1 . ba_2,$

$\therefore\ ab_1 . ab_2 . ca_1 . ca_2 . bc_1 . bc_2 = ac_1 . ac_2 . ba_1 . ba_2 . cb_1 . cb_2.$

$\therefore$ the triangle formed by the lines a_1b_2, b_1c_2, c_1a_2 is in perspective with the triangle abc (§ 68).

∴ the triangle formed by the lines A_1B_2, B_1C_2, C_1A_2 is in perspective with the triangle ABC.

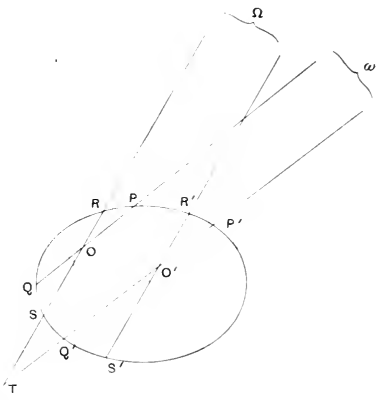
∴ by § 68

$$AB_1 . AB_2 . CA_1 . CA_2 . BC_1 . BC_2$$
$$= AC_1 . AC_2 . BA_1 . BA_2 . CB_1 . CB_2.$$

Newton's theorem.

115. Prop. *If O be a variable point in the plane of a conic, and PQ, RS be chords in fixed directions through O, then*

$$\frac{OP . OQ}{OR . OS} \textit{ is constant.}$$



Let O' be any other point and through O' draw the chords $P'Q'$, $R'S'$ parallel respectively to PQ and RS.

Let QP, $Q'P'$ meet in ω at infinity and SR, $S'R'$ in Ω.

Let $P'Q'$ and RS meet in T.

Now apply Carnot's theorem to the triangle ωOT and get

$$\frac{\omega P . \omega Q . OR . OS . TP' . TQ'}{\omega P' . \omega Q' . TR . TS . OP . OQ} = 1,$$

but $$\frac{\omega P}{\omega P'} = 1 \text{ and } \frac{\omega Q}{\omega Q'} = 1.$$

$$\therefore \frac{TP'.TQ'}{TR.TS} = \frac{OP.OQ}{OR.OS}.$$

Next apply Carnot's theorem to the triangle $\Omega TO'$ and get

$$\frac{\Omega R.\Omega S.TP'.TQ'.O'R'.O'S'}{\Omega R'.\Omega S'.O'P'.O'Q'.TR.TS} = 1.$$

$$\therefore \frac{TP'.TQ'}{TR.TS} = \frac{O'P'.O'Q'}{O'R'.O'S'}.$$

Hence $$\frac{OP.OQ}{OR.OS} = \frac{O'P'.O'Q'}{O'R'.O'S'},$$

that is, $\frac{OP.OQ}{OR.OS}$ is constant.

This proposition is known as Newton's theorem.

NOTE. In applying Newton's theorem it must be remembered that the lines OP, OQ, &c. have sign as well as magnitude. If OP and OQ are opposite in direction, they have opposite sign, and so for OR and OS.

116. Newton's theorem is of great importance, as we shall see in later chapters, where considerable use will be made of it. We give some propositions illustrating its use.

Prop. *If two chords of a conic PP' and QQ', intersect in O the ratio $OP.OP' : OQ.OQ'$ is equal to that of the lengths of the focal chords parallel to PP' and QQ'.*

Let the focal chords parallel to PP' and QQ' be pSp' and qSq'.

Then by Newton's theorem

$$OP.OP' : OQ.OQ' = Sp.Sp' : Sq.Sq'$$
$$= pp' : qq' \quad (\S\ 112 \text{ Cor.}).$$

In the special case where O is the centre of the conic we have $OP' = -OP$ and $OQ' = -OQ$.

$$\therefore OP^2 : OQ^2 = pp' : qq'.$$

NOTE. We have already explained that in using Newton's theorem, the signs of the segments of the line are to be considered.

If $OP.OP'$ and $OQ.OQ'$ have opposite signs so also will $Sp.Sp'$ and $Sq.Sq'$ have opposite signs. This only happens in the case of the hyperbola, and when one of the four points p, p' and q, q' lies on the opposite branch to the other three. Then one of the focal chords pp', qq' will join two points on opposite branches and the other will join two points on the same branch.

If we make the convention that a negative value be attached to the length of a focal chord if it joins two points on opposite branches otherwise it is to count positive the relation

$$OP \,.\, OP' : OQ \,.\, OQ' = pp' : qq'$$

is algebraically as well as numerically correct.

So also it is true, with the same convention as to sign, that if CP, CQ be the semidiameters parallel to the focal chords, pp', qq'

$$pp' : qq' = CP^2 : CQ^2.$$

And from this we see that of the two diameters parallel to two focal chords, one of which joins two points on the same branch and the other two points on opposite branches, only one can meet the curve in real points for the ratio $CP^2 : CQ^2$ has now a negative value.

117. Prop. *If OP and OQ be two tangents to a conic then $OP^2 : OQ^2$ is equal to the ratio of the focal chords parallel respectively to OP and OQ.*

Let the focal chords be pSp', qSq'. Then regarding OP as meeting the curve in two coincident points P, and OQ similarly, we have by Newton's theorem

$$OP \,.\, OP = OQ \,.\, OQ = Sp \,.\, Sp' : Sq \,.\, Sq',$$

$$\therefore\; OP^2 : OQ^2 = pp' : qq'.$$

Whence we see that the focal chords have the same sign.

It is clear too that the ratio of the tangents from a point to a central conic is equal to that of the diameters parallel to them.

118. Prop. *If a circle cut a conic in four points the chords joining their points of intersection in pairs are equally inclined to the axis.*

Let the conic and circle intersect in the four points P, Q, P', Q'.

Let PP' and QQ' intersect in O.

Draw focal chords pSp', qSq' parallel to PP' and QQ'.

Then, by Newton's theorem

$$OP \,.\, OP' : OQ \,.\, OQ' = Sp \,.\, Sp' : Sq \,.\, Sq'$$
$$= pp' : qq'.$$

But from the circle

$$OP \,.\, OP' = OQ \,.\, OQ',$$
$$\therefore \ pp' = qq' \text{ and } Sp \,.\, Sp' = Sq \,.\, Sq'.$$

Thus the parallel focal chords have the same sign, their lengths are equal and the rectangles contained by their segments are equal.

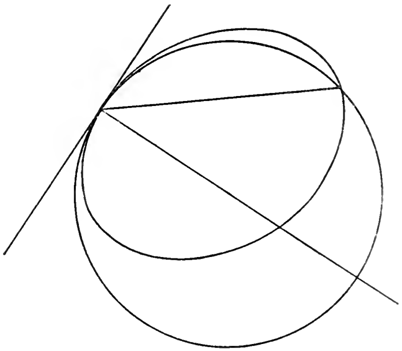
These chords must then be symmetrically placed and make equal angles with the axis. Thus PP' and QQ', parallel to them, make equal angles with the axis.

COR. If a circle touch a conic at one point and cut it at two others, the tangent at the point of contact and the chord joining the two points of intersection make equal angles with the axis.

Circle of curvature.

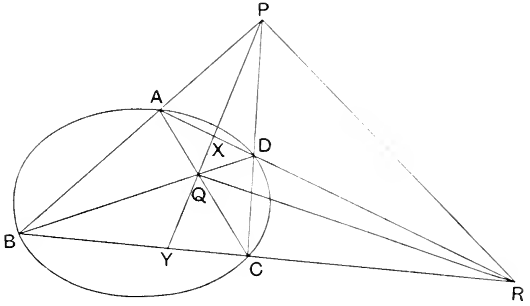
119. An infinite number of circles can be drawn to touch a conic at a given point P, such circles having their centres along the normal at the point. These circles will in general cut the conic in two other points, but in the special case where one of these other two points coincides with the point of contact P the circle is called the circle of curvature at P. This circle may be regarded as the limiting case of the circle passing through P and through two points on the conic consecutive to P, so that the conic and the circle have two consecutive tangents in common. They have then the same rate of curvature at that point. The subject of curvature properly belongs to the Differential Calculus, but it seems desirable to give here the principal properties of the circles of curvature of conics. Accordingly we shall at the end of the chapters on the parabola, ellipse, and hyperbola add a proposition relating to the circles of curvature for these curves. It is clear from § 118 that if the circle of curvature at a point P of a conic cut the conic again in Q then PQ and the tangent at P are equally inclined to the axis. For the tangent at P and the chord PQ are the common chords of the circle and the conic.

The following figure illustrates the circle of curvature at a point of a conic.



Self-Polar Triangle.

119a. Prop. *If a conic pass through the four points of a quadrangle, the diagonal or harmonic triangle is self-polar with regard to the conic—that is, each vertex is the pole of the opposite side.*



Let $ABCD$ be the quadrangle; PQR the diagonal or harmonic triangle.

Let PQ cut AD and BC in X and Y.

Then $(AD, XR) = -1$,

$\therefore$ the polar of R goes through X (§ 92 (5)),

and $(BC, YR) = -1$,

$\therefore$ the polar of R goes through Y.

$\therefore$ PQ is the polar of R.

Similarly QR is the polar of P, and PR of Q.

Thus the proposition is proved.

Another way of stating this proposition would be to say that the diagonal points when taken in pairs are conjugate for the conic.

The triangle PQR is also called *self-conjugate* with regard to the conic.

EXERCISES

1. Given a conic and a focus and corresponding directrix of it, shew how to draw the tangent at any point.

2. Given two points on a conic and a directrix, shew that the locus of the corresponding focus is a circle.

3. POP' and QOQ' are two chords of a conic intersecting in O, prove that PQ and $P'Q'$ meet on the polar of O.

[Project the conic into a circle and O into the centre.]

4. If the tangent at the end of a latus rectum LSL' meet the tangent at the nearer vertex A in T then $TA = AS$.

5. If the tangent at any point P of a conic meet a directrix in F, and the latus rectum through the corresponding focus in D then $SD : SF =$ the eccentricity.

6. If the normal at P to a conic meet the axis in G, and GL be perpendicular to the focal radius SP, then $PL =$ the semi-latus rectum.

7. If PSP' be a focal chord and Q any point on the conic and if PQ and $P'Q$ meet the directrix corresponding to the focus S in F and F', FSF' is a right angle.

8. If a conic touch the sides opposite to A, B, C of a triangle ABC in D, E, F respectively then AD, BE, CF are concurrent. [Use § 114.]

9. By means of Newton's theorem prove that if PN be the ordinate of a point P on a parabola whose vertex is A, then $PN^2 : AN$ is independent of the position of P on the curve.

10. Conics are drawn through two fixed points D and E, and are such that DE subtends a constant angle at a focus of them; shew that the line joining this focus to the pole of DE passes through a fixed point.

11. The polar of any point with respect to a conic meets a directrix on the diameter which bisects the focal chord drawn through the point and the corresponding focus.

12. Prove that the line joining a focus of a conic to that point in the corresponding directrix at which a diameter bisecting a system of parallel chords meets it is perpendicular to the chords. [Use §§ 95 and 106.]

13. P and Q are two points on a conic, and the diameters bisecting the chords parallel respectively to the tangents at P and Q meet a directrix in M and N; shew that MN subtends at the corresponding focus an angle equal to that between the tangents at P and Q.

14. Given a focus and the corresponding directrix of a variable conic, shew that the polar of a given point passes through a fixed point.

15. Given a focus and two points of a variable conic, prove that the corresponding directrix must pass through one or other of two fixed points.

16. If two conics have a common focus, a chord common to the two conics will pass through the point of intersection of the corresponding directrices.

17. If T be any point on the tangent at a point P of a conic of which S is a focus, and if TM be the perpendicular to SP, and TN the perpendicular on the directrix corresponding to S, then $SM : TN = e$. (Adams' theorem.)

18. Given a focus of a conic and a chord through that focus, prove that the locus of the extremities of the corresponding latus rectum is a circle.

19. If TP and TQ be two tangents to a conic prove that the portion of a tangent parallel to PQ intercepted between TP and TQ is bisected at the point of contact.

20. A diameter of a conic meets the curve in P and bisects the chord QR which is a normal at Q, shew that the diameter through Q bisects the chord through P which is a normal at P.

21. PQ is a chord of a conic cutting the axis in K, and T is the pole of PQ; the diameter bisecting PQ meets a directrix in Z and S is the corresponding focus; prove that TS is parallel to ZK.

22. If AA', BB', CC' be chords of a conic concurrent at O, and P any point on the conic, then the points of intersection of the straight lines BC, PA', of CA, PB', and of AB, PC' lie on a straight line through O.

[Project to infinity the line joining O to the point of intersection of AB, PC' and the conic into a circle.]

23. A, B, C, D are four points on a conic; AB, CD meet in E; AC and BD in F; and the tangents at A and D in G; prove that E, F, G are collinear.

[Project AD and BC into parallel lines and the conic into a circle.]

24. If a conic be inscribed in a quadrilateral, the line joining two of the points of contact will pass through one of the angular points of the triangle formed by the diagonals of the quadrilateral.

25. Prove Pascal's theorem, that if a hexagon be inscribed in a conic the pairs of opposite sides meet in three collinear points.

[Project the conic into a circle so that the line joining the points of intersection of two pairs of opposite sides is projected to infinity.]

26. A is a fixed point in the plane of a conic, and P any point on the polar of A. The tangents from P to the conic meet a given line in Q and R. Shew that AR, PQ, and AQ, PR intersect on a fixed line.

[Project the conic into a circle having the projection of A for centre.]

27. A system of conics touch AB and AC at B and C. D is a fixed point, and BD, CD meet one of the conics in P, Q. Shew that PQ meets BC in a fixed point.

28. If a conic pass through the points A, B, C, D, the points of intersection of AC and BD, of AB and CD, of the tangents at B and C, and of the tangents at A and D are collinear.

29. Through a fixed point A on a conic two fixed straight lines AI, AI' are drawn, S and S' are two fixed points and P a variable point on the conic; PS, PS' meet AI, AI' in Q, Q' respectively, shew that QQ' passes through a fixed point.

30. If a conic cut the sides BC, CA, AB of a triangle ABC in A_1A_2, B_1B_2, C_1C_2, and AA_1, BB_1, CC_1 are concurrent, then will AA_2, BB_2, CC_2 be concurrent.

31. When a triangle is self-conjugate for a conic, two and only two of its sides cut the curve in real points.

32. Prove that of two conjugate diameters of a hyperbola, one and only one can cut the curve in real points.

[Two conjugate diameters and the line at infinity form a self-conjugate triangle.]

33. Given four points S, A, B, C, shew that in general four conics can be drawn through A, B, C having S as focus; and that three of the conics are hyperbolas with A, B, C not on the same branch, while the remaining conic may be an ellipse, a parabola, or a hyperbola having A, B, C on the same branch.

34. Prove that a circle can be projected into a parabola with any given point within the circle projected into the focus.

35. Prove that a circle can be projected into an ellipse with two given points within the circle projected into the centre and a focus of the ellipse.

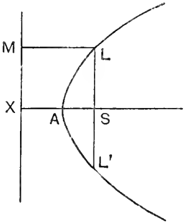
36. Prove that a circle can be projected into a hyperbola with a given point P within the circle and another given point Q without it projected respectively into a focus and the centre of the hyperbola.

CHAPTER XI

THE PARABOLA

120. The form of the parabola has already been indicated in §§ 96 and 97. In this chapter we shall develop the special properties of the curve. Throughout A will stand for the vertex, S for the focus, X for the intersection of the directrix with the axis, and Ω for the point at infinity along the axis, at which point, as we have seen, the parabola touches the line at infinity.

Prop. *The latus rectum* $= 4AS$.



Let LSL' be the latus rectum. Draw LM perpendicular to the directrix. Then $LL' = 2LS = 2LM = 2SX = 4AS$.

121. Prop. *If PN be the ordinate of the point P, then* $PN^2 = 4AS \cdot AN$.

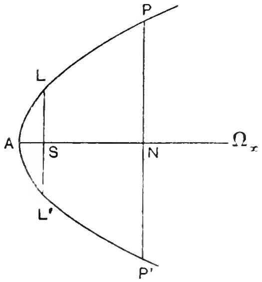
Let PN meet the parabola again in P', and let LSL' be the latus rectum.

Then by Newton's theorem (§ 115),

$$NP \cdot NP' : SL \cdot SL' = NA \cdot N\Omega : SA \cdot S\Omega$$
$$= NA : SA$$

since Ω is at infinity.

$$\therefore\ PN^2 : SL^2 = AN : AS;$$
$$\therefore\ PN^2 : 4AS^2 = AN : AS;$$
$$\therefore\ PN^2 = 4AS \cdot AN.$$



This proposition will later on be seen to be only a special case of a more general theorem.

122. The preceding proposition shews that a parabola is the locus of a point in a plane such that the square of its distance from a line l varies as its distance from a perpendicular line l'. The line l is the axis, l' the tangent at the vertex, and the constant of variation is the length of the latus rectum

To determine the parabola we ought to know on which side of the line l' the point lies. If it may lie on either side then the locus is *two* parabolas, each of which is got from the other by rotating the figure about the tangent at the vertex through two right angles.

123. Tangent and Normal.

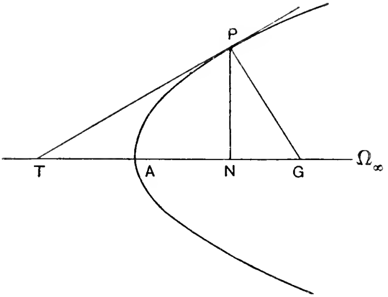
Prop. If the tangent and normal at P meet the axis in T and G respectively, and PN be the ordinate of P,

(1) $TA = AN$

(2) $NG = 2AS$.

The first of these properties has been already proved in § 97. We have seen that if PN meet the curve again in P', the tangents at P and P' meet on the line of the axis, that is, they intersect in T. Then by the harmonic property of the pole and polar

$$(TN, A\Omega) = -1;$$
$$\therefore\ TA = AN.$$



Also since TPG is a right angle

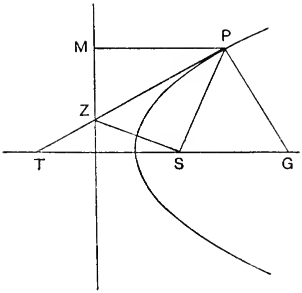
$$PN^2 = TN \,.\, NG = 2AN \,.\, NG.$$

But $$PN^2 = 4AS \,.\, AN \quad (\S\, 121),$$

$$\therefore\ NG = 2AS.$$

Def. NG is called the *subnormal* of the point P. Thus in a parabola the subnormal is constant.

124. Prop. *The tangent at any point of a parabola makes equal angles with the axis and the focal distance of the point.*



Let the tangent at P meet the directrix in Z, and the axis in T. Draw PM perpendicular to the directrix.

Now since $SP = PM$, and PZ is common to the $\triangle$s SPZ, MPZ which have the angles at M and S right angles,

$$\therefore \triangle SPZ \equiv \triangle MPZ$$

$$\text{and } \angle SPT = \angle TPM = \angle STP.$$

COR. If the normal at P meet the axis G then

$$SG = SP = ST.$$

That $ST = SP$ follows from the equality of the angles SPT and STP.

Further the complements of these angles must be equal;

$$\therefore \angle SPG = \angle SGP;$$

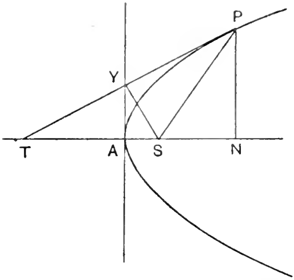
$$\therefore SG = SP.$$

We note that the equality of SG and SP in a parabola follows from the fact that for any conic $SG = e \,.\, SP$ (§ 111).

125. Prop. *The foot of the perpendicular Y from the focus on to the tangent at any point P of a parabola lies on the tangent at the vertex and $SY^2 = SA \,.\, SP$.*

The first part of this proposition is implicitly proved in § 97. We can also prove it thus:

Let the tangent at P meet the axis in T. Since $ST = SP$, SY will bisect TP.



But if PN be the ordinate of P, $TA = AN$.

$\therefore$ AY is parallel to NP, that is AY is the tangent at the vertex.

Further as SYT is a right angle and YA perpendicular to ST, $SY^2 = SA \cdot ST = SA \cdot SP$.

Cor. 1. $\angle SPY = \angle SYA$.

Cor. 2. If the locus of the foot of the perpendicular from a fixed point on a variable line be a straight line, then the variable line touches a parabola having its focus at the fixed point.

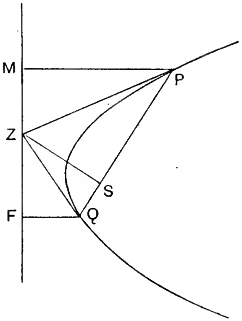
Def. When a line moves in a plane so as always to touch a certain curve, the curve is called the *envelope* of the line.

Pair of Tangents.

126. Prop. *Tangents to a parabola at the extremities of a focal chord intersect at right angles in the directrix.*

That they intersect in the directrix we know already, since the focus and directrix are pole and polar.

Let PSQ be a focal chord, and let the tangents meet in the directrix in Z. Draw PM and QF perpendicular to the directrix.



Then as we have seen in § 124

$$\triangle SPZ \equiv \triangle MPZ,$$

$$\therefore \angle SZP = \angle MZP.$$

Similarly $$\angle SZQ = \angle FZQ.$$

Thus $$\angle QZP = \tfrac{1}{2} \text{ of two right } \angle\text{s}$$
$$= \text{a right angle.}$$

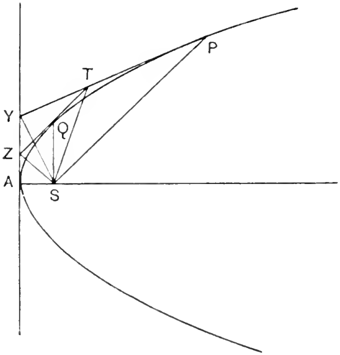
127. Prop. *If TP and TQ be two tangents to a parabola, the triangles SPT, STQ are similar.*

We know already that the angles PST, TSQ are equal (§ 109).

Let the tangents at P and Q meet the tangent at the vertex in Y and Z, then SYT and SZT are right angles (§ 125).

Then $\angle SPY = \angle SYA$ (§ 125)

$= \angle STZ$ since $SZYT$ is cyclic.



Thus $\angle SPT = \angle STQ$;

$\therefore$ the remaining angle $STP = \angle SQT$ and the triangles SPT and STQ are similar.

Cor. 1. $ST^2 = SP \,.\, SQ$

for $SP : ST = ST : SQ$.

Cor. 2. $TP^2 : TQ^2 = SP : SQ$

for $TP^2 : TQ^2 = \triangle SPT : \triangle STQ$

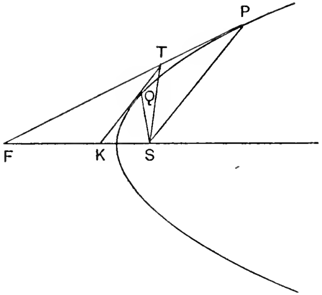
$= SP \,.\, ST : ST \,.\, SQ$

since $\angle PST = \angle TSQ$

$= SP : SQ$.

128. Prop. *The exterior angle between two tangents to a parabola is equal to half the angle which their chord of contact subtends at the focus.*

Let the tangents at P and Q meet in T, and let them meet the axis in F and K respectively.



Then
$$\begin{aligned}\angle FTK &= \angle SKQ - \angle SFP \\ &= \angle SQK - \angle SPT \\ &= 2 \text{ right angles} - \angle SQT - \angle SPT \\ &= 2 \text{ right angles} - \angle STP - \angle SPT \\ &= \angle TSP \\ &= \tfrac{1}{2} \angle PSQ.\end{aligned}$$

129. Parabola escribed to a triangle.

When the sides of a triangle are tangents to a parabola, the triangle is said to *circumscribe* the parabola. But it must be clearly understood that the triangle does not enclose the parabola, for no finite triangle can enclose a parabola, which is infinite in extent. When a triangle circumscribes a parabola, the parabola is really escribed to it, that is, it touches one side of the triangle and the other two sides produced. Only triangles which have the line at infinity for one of their sides can enclose the parabola, and in the strict sense of the word be said to circumscribe it. It is convenient however to extend the meaning of the word 'circumscribe' and to understand by a triangle circumscribing a conic a triangle whose sides touch the conic whether the triangle encloses the conic or not.

130. Prop. *The circumcircle of the triangle formed by three tangents to a parabola passes through the focus.*

Understanding the word 'circumscribe' as explained in § 129 we may state this proposition thus:

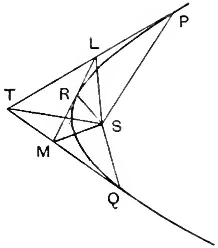
If a triangle circumscribe a parabola its circumcircle goes through the focus.

This can be seen from the fact that the feet of the perpendiculars from the focus on the three sides of the triangle which touch the parabola are collinear, lying as they do on the tangent at A.

$\therefore$ S lies on the circumcircle of the triangle (§ 7).

Or we may prove the proposition in another way:

Let the tangents at P, Q, R form the triangle TLM as in the figure.



Then as $\triangle SPL$ is similar to $\triangle SLR$, $\angle SLR = \angle SPL$.

And as $\triangle SPT$ is similar to $\triangle STQ$, $\angle STQ = \angle SPT$.

$$\therefore \angle SLM = \angle STM,$$

that is, $SLTM$ is cyclic, or S lies on the circle through T, L and M.

COR. *The orthocentre of a triangle circumscribing a parabola lies on the directrix.*

For if TLM be the triangle, the line joining S to the orthocentre is bisected by the tangent at the vertex, which is, as we have seen, the pedal line of S (§ 8).

$\therefore$ the orthocentre must lie on the directrix.

131. Prop. *If the tangents at P and Q to a parabola meet in T, and a third tangent at R cut them in L and M, the triangle SLM is similar to the triangles SPT and STQ, and*

$$PL : LT = TM : MQ = LR : RM.$$

By the preceding proposition S lies on the circumcircle of TLM.

$\therefore \angle SML = \angle STL$

and $\angle SLM = \angle SPT$ from the similar $\triangle$s SPL, SLR.

$\therefore \triangle SLM$ is similar to $\triangle SPT$ and therefore also to $\triangle STQ$.

Further $\triangle SLR$ is similar to $\triangle STM$ for $\angle SLM = \angle STM$,

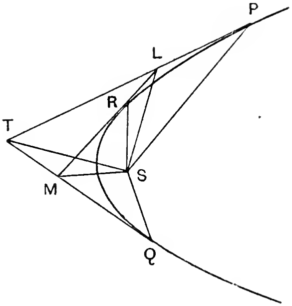
and $\angle SRL = \angle SLP$

$$= 180^\circ - \angle SLT = \angle SMT.$$

$\therefore LR : TM = SR : SM$

$= MR : MQ$ (by similar $\triangle$s SRM, SMQ).

$\therefore LR : RM = TM : MQ.$



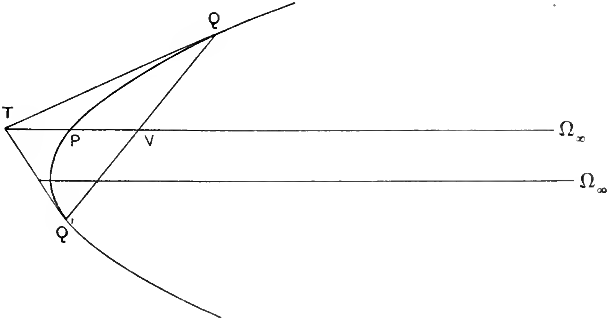
Similarly $MR : RL = TL : LP$.

Hence $PL : LT = TM : MQ = LR : RM$.

132. Diameters.

We have already explained in § 102 what is meant by diameters of a parabola. Every line parallel to the axis is a diameter, and every diameter bisects a system of parallel chords. It must be remembered too that the tangents at the extremities of each of the parallel chords bisected by a diameter intersect on that diameter (§ 95).

Prop. *If TQ and TQ′ be tangents to a parabola, and TV be the diameter bisecting QQ′ in V and cutting the curve in P then* $TP = PV$.



For PV being parallel to the axis goes through Ω.

Thus by the harmonic property of the pole and polar

$$(TV, P\Omega) = -1,$$

$$\therefore\ TP = PV.$$

Note that $TA = AN$ (§ 123) is only a special case of this.

133. Prop. *The length of any focal chord of a parabola is four times the distance of the focus from the point where the diameter bisecting the chord meets the curve.*

Let RSR' be any focal chord, PV the diameter which bisects it in V.

Let the tangents at R and R' meet in Z. Then Z is both on the directrix and on the line of the diameter PV. Also $ZP = PV$ (§ 132).

But ZSV is a right angle (§ 106).

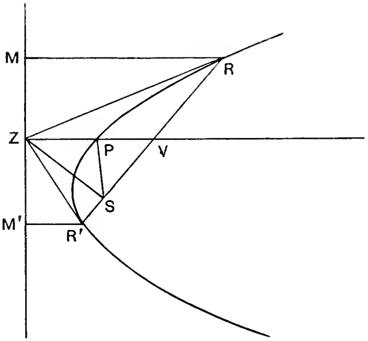
$$\therefore\ SP = PV = ZP.$$

Now draw RM and $R'M'$ perpendicular to the directrix.

Then $2VZ = RM + R'M'$ since V is the middle point of RR'

$$= RS + SR' = RR';$$

$$\therefore\ RR' = 4PV = 4SP.$$



A focal chord bisected by a particular diameter is called the *parameter* of the diameter which bisects the chord. Thus RR' is the parameter of the diameter PV, and we have proved it equal to $4SP$. In particular the latus rectum is the parameter of the axis, and we proved it equal to $4SA$.

134. Prop. *If QV be an ordinate of a diameter PV then*

$$QV^2 = 4SP \,.\, PV.$$

Produce the ordinate QV to meet the curve again in Q'. $\therefore\ QV = VQ'$.

Draw the focal chord RSR' parallel to the chord QQ'. RR' will be bisected by PV in U (say).

The diameter PV meets the curve again in Ω at an infinity, thus by Newton's theorem

$$VQ \,.\, VQ' : VP \,.\, V\Omega = UR \,.\, UR' : UP \,.\, U\Omega;$$

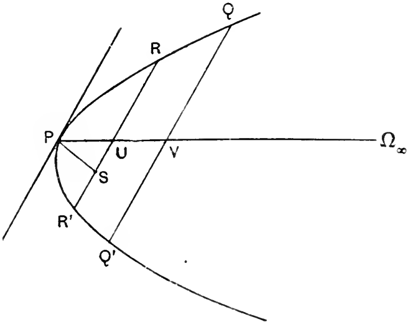
$$\therefore\ VQ \,.\, VQ' : UR \,.\, UR' = VP \,.\, V\Omega : UP \,.\, U\Omega = VP : UP.$$

$$\therefore\ QV^2 : RU^2 = PV : PU;$$

$$\therefore \frac{QV^2}{PV} = \frac{RU^2}{PU} = \frac{4SP^2}{SP} \quad (\S\ 133)$$

$$= 4SP.$$

$$\therefore QV^2 = 4SP \cdot PV.$$



It will be seen that the property $PN^2 = 4AS \cdot AN$ of § 121 is only a special case of the general proposition just proved.

135. The preceding proposition shews that a parabola may be regarded as the locus of a point in a plane such that the square of its distance from a fixed line l varies as its distance from another fixed line l' not necessarily at right angles to l. The line l is a diameter of the parabola and l' is a tangent at the point where l and l' intersect.

If α be the angle which QV makes with the axis in § 134

$$QV = \text{perp. from } Q \text{ on } PV \times \operatorname{cosec} \alpha$$

$$\text{and } PV = \text{perp. from } Q \text{ on tangent at } P \times \operatorname{cosec} \alpha;$$

$$\therefore (\text{Perp. from } Q \text{ on } PV)^2 \times \operatorname{cosec}^2 \alpha$$

$$= 4SP \times \text{perp. from } Q \text{ on tangent at } P \times \operatorname{cosec} \alpha.$$

$$\therefore \frac{(\text{Perp. from } Q \text{ on } PV)^2}{\text{Perp. from } Q \text{ on tangent at } P} = 4SP \sin \alpha = 4SY$$

where SY is the perpendicular from S on the tangent at P.

Thus if a point move in a plane so that the square of its distance from a line l is k times its distance from another line l', k being constant, the locus of the point is a parabola having its axis parallel to l. The focus lies in a line parallel to l' and distant $\frac{k}{4}$ from it, and it also lies in a line through the intersection of l and l' and making with l' the same angle that l makes with it. The line l' is the tangent to the parabola at its point of intersection with l.

As already explained in § 122 if the lines l and l' are at right angles l is the axis itself and l' the tangent at the vertex.

136. Circle of curvature.

We have explained in § 119 what we mean by the circle of curvature at any point of a conic. We shall now shew how the circle of curvature at any point of a parabola can be determined.

The centre of the circle of curvature at P lies on the normal at P, if then we can find the length of the chord through P of this circle in any direction, the length of the diameter of the circle can be found by drawing a line through the other extremity of this chord and perpendicular to it to meet the normal in D, then PD will be the diameter of the circle of curvature.

Prop. *The chords of the circle of curvature parallel to the axis and through the focus at any point P of a parabola* $= 4SP$.

It is clear that these two chords must be equal for they make equal angles with the tangent at P.

Now consider a circle touching the parabola at P and cutting it again at a near point Q. Draw the diameter of the parabola through Q and let it meet the circle again in K and the tangent at P in R. Draw QV the ordinate to the diameter PV.

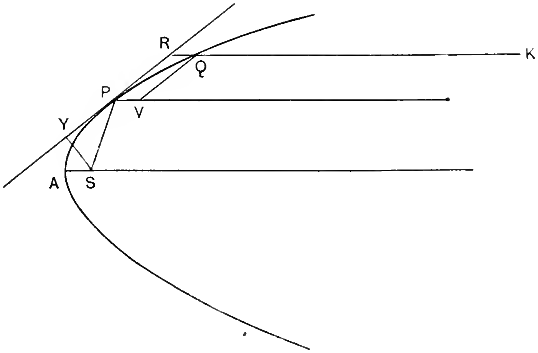
Then from the circle we have

$$RQ \, . \, RK = RP^2;$$

$$\therefore \; RK = \frac{RP^2}{RQ} = \frac{QV^2}{PV} = 4SP. \quad (\S\ 134.)$$

Now in the limit when Q moves up to and ultimately coincides with P, RK becomes the chord of the circle of curvature through P parallel to the axis.

Hence this chord is of length $4SP$.



COR. The diameter of the circle of curvature at $P = \frac{4SP^2}{SY}$, where SY is the perpendicular on the tangent. For

$$\frac{4SP}{\text{Diameter}} = \cos(\angle \text{ between normal and axis})$$
$$= \sin(\angle \text{ between tangent and axis})$$
$$= \sin(\angle \text{ between tangent and } SP)$$
$$= \frac{SY}{SP}.$$

NOTE. The diameter of the circle of curvature is commonly called the diameter of curvature, and the chords of the circle of curvature through P are called simply chords of curvature.

EXERCISES

1. PSP' is a focal chord of a parabola, PN and $P'N'$ ordinates to the axis, prove

$$PN \,.\, P'N' = 4AS^2 = 4AN \,.\, AN'.$$

2. A series of parabolas touch a given line and have a common focus, prove that their vertices lie on a circle.

3. If a straight line rotate about a point in a plane containing the line the directions of motion of each point in it at a given moment are tangents to a parabola.

4. The ordinates of points on a parabola are divided in a given ratio, prove that the locus of the points dividing them is another parabola.

5. The locus of the middle points of focal chords of a parabola is also a parabola, whose latus rectum is half that of the original parabola.

6. If QV be an ordinate of the diameter PV and QD be perpendicular to the diameter $QD^2 = 4AS \,.\, PV$.

7. If the normal at P to a parabola meet the axis in G then $PG^2 = 4AS \,.\, SP$.

8. PQ, PR are two chords of a parabola; PQ meets the diameter through R in the point F and PE meets the diameter through Q in E; prove that EF is parallel to the tangent at P.

[Project the parabola into a circle with E projected into the centre.]

9. If a circle touch a parabola at P and cut it at Q and R, the diameters through Q and R will meet the circle again in points on a line parallel to the tangent at P.

[Use § 116 Cor.]

10. If TP and TQ be two fixed tangents to a parabola, and a variable tangent cut them in L and M respectively the ratios $PL : TM$ and $TL : QM$ are equal and constant.

11. If two tangents TP and TQ to a parabola be cut by a third tangent in L and M respectively then

$$\frac{TL}{TP} + \frac{TM}{TQ} = 1.$$

12. If the normal at P to a parabola meet the curve again in Q and PN be the ordinate of P, and T the pole of PQ,

$$PQ : PT = PN : AN.$$

13. If the tangent at P meet the directrix in Z and the latus rectum produced in D, then $SD = SZ$.

14. TP and TQ are tangents to a parabola, and the diameters through P and Q meet any line drawn through T in M and N; prove

$$TM^2 = TN^2 = TP \,.\, TQ.$$

15. If T be any point on the tangent at P to a parabola and the diameter through T meet the curve in Q, then

$$RP^2 = 4SP \,.\, RQ.$$

16. If the chord PQ which is normal at P to a parabola subtends a right angle at S, then $SQ = 2SP$.

17. If PN and $P'N'$ be two ordinates of a parabola such that the circle on NN' as diameter touches the parabola, then

$$NP + N'P' = NN'.$$

18. TP and TQ are two tangents to a parabola, and PK and QL are drawn perpendicular respectively to TQ and TP, prove that the triangles STK and STL are equal.

19. TP and TP' are tangents to a parabola, and the diameter through T cuts the curve in Q. If PQ, $P'Q$ cut TP', TP respectively in R and R', and the diameters through R and R' cut the curve in V, V' respectively, prove that PV, $P'V'$ intersect on TQ.

[Project the parabola into a circle and the line through T parallel to PP' to infinity.]

20. Prove that no circle described on a chord of a parabola as diameter can meet the directrix unless the chord be a focal chord, and then the circle touches the directrix.

21. If TP and TQ be a pair of tangents to a parabola and the chord PQ be normal at P, then TP is bisected by the directrix.

22. The triangle ABC circumscribes a parabola having S as focus, prove that the lines through A, B, C perpendicular to SA, SB, SC respectively are concurrent.

23. The tangents to a parabola at P and P' intersect in Q; the circles circumscribing the triangles SPQ, SPQ' meet the axis again in R and R'. Prove that PR and QR' are parallel.

24. A given triangle ABC moves in a plane, with one side AB passing through a fixed point, and with the vertex A on a given straight line. Shew that the side AC will envelop a parabola.

25. Shew that the envelope of a line which moves so as to intercept equal chords on two given circles is a parabola having the radical axis of the two given circles as the tangent at its vertex.

26. A line meets a parabola in P and p on the same side of the axis. AQ is drawn parallel to Pp to meet the curve again in Q. Prove that the ordinate of Q is equal to the sum of the ordinates of P and p.

27. The distance of the point of intersection of two tangents to a parabola from the axis is half the sum of the ordinates of their points of contact.

28. If LL' be the latus rectum of a parabola and the tangent at any point P meet that at L in V, then $SL \,.\, LP = VL \,.\, VL'$.

29. A circle touches a parabola at a point P, and passes through the focus S. Shew that the parabola meets the circle again or not according as the latus rectum is or is not less than SP.

30. QQ' is the normal at Q to a parabola meeting the parabola again in Q', QP is equally inclined to the axis with the normal and meets the curve again in P; V is the middle point of QQ', and PV meets the axis in R; shew that $QSPR$ lie on a circle, S being the focus.

31. Two parabolas with equal latus rectum are on the same axis, and are such that the part of any tangent to one which is cut off by the other is equal to the perpendicular upon this tangent from the focus of the first parabola. Shew that the latus rectum of each is sixty-four times the distance between the vertices.

32. A circle touches a parabola at both ends of a double ordinate PP' to the axis. The normal at P meets the circle in R and the parabola in Q. The diameter of the parabola through Q meets PP' in U. Prove that the circle QRU touches PP' at U.

33. EF is a double ordinate of the axis of a parabola, R any point on it, and the diameter through R meets the curve in P; the tangent at P intersects in M and N the diameters through E and F. Prove that PR is a mean proportional between EM and FN.

34. If a parabola roll on another equal parabola, the vertices being originally in contact, its focus will trace out the directrix of the fixed parabola.

35. If P be any point on a parabola whose vertex is A, and PR perpendicular to AP meet the axis R, a circle whose centre is R and radius RP will pass through the ends of the ordinate to the parabola through R. Also if common tangents be drawn to the circle and parabola the ordinates at the points where they touch the parabola will be tangents to the circle.

36. Through any point on a parabola two chords are drawn equally inclined to the tangent there. Shew that their lengths are proportional to the portions of their diameters bisected between them and the curve.

37. TQ, TR, tangents to a parabola, meet the tangent at P in Y and Z, and TU is drawn parallel to the axis, meeting the parabola in U. Prove that the tangent at U passes through the middle point of YZ, and that, if S be the focus,

$$YZ^2 = 4SP \cdot TU.$$

38. PQ is a normal chord of a parabola meeting the axis in G. Prove that the distance of G from the vertex, the ordinates of P and Q, and the latus rectum are four proportionals.

39. Lines are drawn through the focus of a parabola to cut the tangents to it at a constant angle. Prove that the locus of their intersection is a straight line.

40. The radius of curvature at an extremity of the latus rectum of a parabola is equal to twice the normal.

41. The diameter at either extremity of the latus rectum of a parabola passes through the centre of curvature at its other extremity.

42. If the tangent at any point P of a parabola meet the axis in T, and the circle of curvature meet the curve in Q, then $PQ = 4PT$.

43. If R be the middle point of the radius of curvature at P on a parabola, PR subtends a right angle at the focus.

44. The tangent from any point of a parabola to the circle of curvature at its vertex is equal to the abscissa of the point.

45. The chord of curvature through the vertex A at any point P of a parabola is $\dfrac{4PY^2}{AP}$, Y being the foot of the perpendicular from the focus on the tangent at P.

CHAPTER XII

THE ELLIPSE

137. We have already in §§ 98 and 99 indicated the general form of the ellipse, shewing that it has two axes of symmetry at right angles to one another and intersecting in C the centre. On the major axis AA' are two foci S and S', at a distance equal to CA from B and B' the ends of the minor axis, and to these foci correspond directrices at right angles to AA' and cutting it externally in X and X' so that $CS : CA = CA : CX = e$, the eccentricity. It is convenient to call A and A', the ends of the major axis, the *vertices* of the ellipse.

It will be understood that the smaller is the ratio $CS : CA$ the more does the ellipse approximate to circular form, and the greater $CS : CA$ is without reaching unity the more does the ellipse flatten out.

We have always

$$CS^2 = CA^2 - CB^2$$

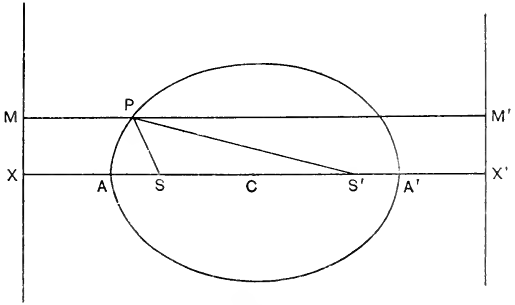
so that, keeping CA constant, CB diminishes as CS increases, and vice versâ. And we have already explained that a circle may be regarded as the limiting case of an ellipse whose two foci coincide with the centre.

We now proceed to establish the chief geometrical properties which all ellipses have in common.

138. Sum of focal distances constant.

Prop. *The sum of the focal distances of any point on an ellipse is constant and equal to AA'.*

Let P be any point on an ellipse, MPM' the perpendicular through P to the directrices as in the figure.



Then $$SP = e \,.\, PM \text{ and } S'P = e \,.\, PM';$$

$$\therefore \quad SP + S'P = e\,(MP + PM') = e \,.\, XX'$$
$$= 2e \,.\, CX = 2CA = AA'.$$

COR. Two confocal ellipses (that is, which have both foci in common) cannot intersect.

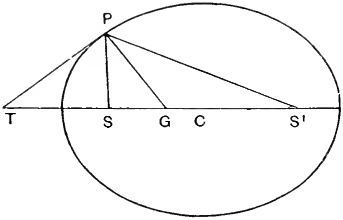
139. The proposition just proved shews that an ellipse may be regarded as the locus in a plane of a point the sum of whose distances from two fixed points in the plane is constant. And we learn that an ellipse can be drawn by tying the ends of a piece of string to two pins stuck in the paper so that the string is not tight, and then holding the string tight by means of the pencil pressed against it, and allowing the point of the pencil to make its mark in all possible positions thus determined.

By keeping the string the same length and changing the distance between the pins we can draw ellipses all having the same major axis but having different eccentricities. It will be seen that the nearer the pins are together the more does the ellipse approximate to circular form.

140. Tangent and Normal.

Prop. *The tangent and normal at any point of an ellipse bisect respectively the exterior and interior angles between the focal distances of the point.*

Let the tangent and normal at P meet the major axis in T and G respectively.



Then by § 111 $SG = e \, . \, SP$, and $S'G = e \, . \, S'P$.

$\therefore \; SG : GS' = SP : PS'$;

$\therefore \; PG$ bisects the angle SPS';

$\therefore \; PT$ which is at right angles to PG must bisect the exterior angle of SPS'.

Cor. $$CG \, . \, CT = CS^2$$

for since PG and PT are the bisectors of the angles between SP and $S'P$,

$$(GT, \; SS') = -1.$$

Prop.

141. *If SY, $S'Y'$ be the perpendiculars from the foci on the tangent at any point P of an ellipse, Y and Y' lie on the circle described on the major axis AA' as diameter, and*

$$SY \, . \, S'Y' = BC^2.$$

Produce SY to meet $S'P$ in K.

Then $$\triangle SPY \equiv \triangle KPY$$

for $$\angle SPY = \angle KPY \quad (\S\ 140)$$

$\angle SYP = \angle KYP$ being right angles and PY is common.

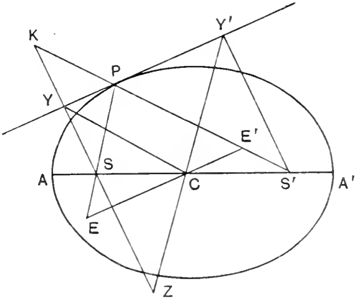
$\therefore \; PK = SP$ and $KY = SY$,

$\therefore \; KS' = KP + PS' = SP + PS' = AA'$ (§ 138).

Now since Y and C are the middle points of SK and SS', CY is parallel to $S'K$ and $CY = \frac{1}{2}S'K = CA$.

Thus Y (and similarly Y') lies on the circle on AA' as diameter.

Moreover as $Y'YS$ is a right angle, YS will meet the circle again in a point Z such that $Y'Z$ will be a diameter, that is, $Y'Z$ goes through C.



And $\triangle CSZ \equiv \triangle CS'Y'$ so that $SZ = S'Y'$.

$\therefore SY . S'Y' = SY . SZ = AS . SA' = CA^2 - CS^2 = BC^2$.

COR. 1. The diameter parallel to the tangent at P will meet SP and $S'P$ in points E and E' such that $PE = PE' = AC$.

For $PYCE'$ is a parallelogram. $\therefore PE' = CY = AC$ and similarly $PE = AC$.

COR. 2. The envelope of a line such that the foot of the perpendicular on it from a fixed point S lies on a fixed circle, which has S within it, is an ellipse having S for a focus.

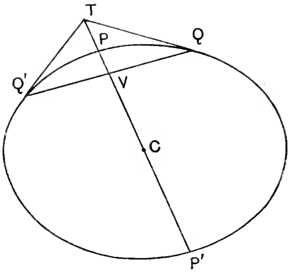
COR. 3. The envelope of a line such that the product of the perpendiculars on it from two fixed points, lying on the same side of it, is constant is an ellipse having the fixed points for its foci.

Def. The circle on the major axis of an ellipse as diameter is called the *auxiliary circle*.

142. Prop. *If TQ and TQ' be a pair of tangents to an ellipse whose centre is C, and CT meet QQ' in V and the curve in P, $CV . CT = CP^2$.*

For let PC meet the ellipse again in P'. Then as T and QQ' are pole and polar, $(TV, PP') = -1$.

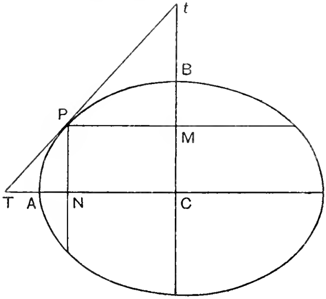
$\therefore$ as C is the middle point of PP', $CV \cdot CT = CP^2$.



143. The preceding proposition is an important one and includes the following as a special case:

If the tangent at P meet the major and minor axes in T and t, and PN, PM be the ordinates to these axes then

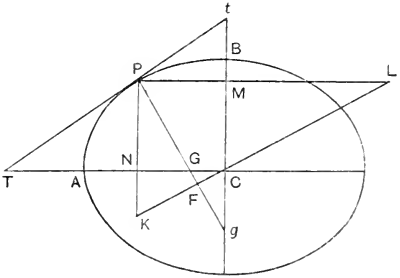
$$CN \cdot CT = CA^2$$
$$CM \cdot Ct = CB^2.$$



For T is the intersection of the tangents at P and at the point where PN again meets the curve, and t is the intersection of the tangents at P and at the point where PM again meets the curve.

144. Prop. *If the normal at any point P on an ellipse meet the major and minor axes in G and g, and the diameter parallel to the tangent at P in F, then $PF.PG = BC^2$ and $PF.Pg = AC^2$.*

Draw the ordinates PN and PM to the axes and let these meet the diameter parallel to the tangent at P in K and L as in the figure.



Let the tangent at P meet the major and minor axes in T and t.

Then $NKFG$ is cyclic since N and F are right angles.

$\therefore\ PF.PG = PN.PK = CM.Ct = BC^2$.

Also $gFML$ is cyclic since F and M are right angles.

$\therefore\ PF.Pg = PM.PL = CN.CT = CA^2$.

144a. Prop. *If the normal at P on an ellipse meet the major axis in G, and PN be the ordinate to this axis, $CG = e^2.CN$.*

Let the tangent at P meet the major axis in T.

We have already proved in § 140 Cor. that $CG.CT = CS^2$.

But $$CN.CT = CA^2;$$

$$\therefore\ CG : CN = CS^2 : CA^2 = e^2;$$

$$\therefore\ CG = e^2.CN.$$

Cor. $NG : NC = BC^2 : AC^2$.

For as $$CG : CN = CS^2 : CA^2,$$

$$\therefore\ CN - CG : CN = CA^2 - CS^2 : CA^2$$
$$= BC^2 : AC^2.$$

144b. Pair of tangents.

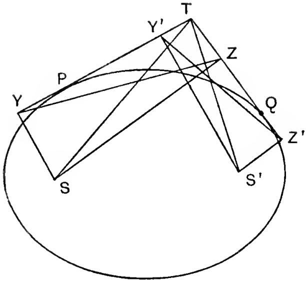
Prop. *The two tangents drawn from an external point to an ellipse make equal angles with the focal distances of the point.*

Let TP and TQ be the tangents; it is required to prove $\angle PTS = \angle S'TQ$.

Draw SY, $S'Y'$ perpendicular to TP, and SZ and $S'Z'$ perpendicular to TQ.

Then $SY \cdot S'Y' = BC^2 = SZ \cdot S'Z'$ (§ 141);

$$\therefore\ SY : SZ = S'Z' : S'Y'.$$



Also $\angle YSZ$ = supplement of $\angle YTZ$ (since $SYTZ$ is cyclic)
$= \angle Y'S'Z'$ (since $Y'TZ'S'$ is cyclic).

$\therefore$ the Δs SYZ and $S'Z'Y'$ are similar, and $\angle SZY = \angle S'Y'Z'$.

But $\angle SZY = \angle STY$ in the same segment and
$\angle S'Y'Z' = \angle S'TZ'$ in the same segment.

$$\therefore\ \angle STY = S'TZ'.$$

145. Director Circle.

Prop. *The locus of points, from which the tangents to an ellipse are at right angles is a circle (called the* **director circle** *of the ellipse).*

Let TP and TQ be two tangents at right angles.

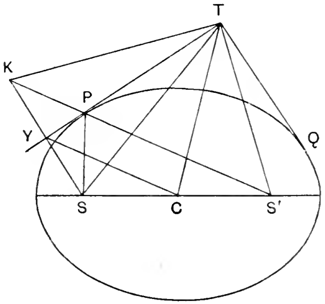
Draw SY perpendicular to TP to meet $S'P$ in K.

Then by § 141, $SY = YK$ and $S'K = AA'$.

Also $\triangle SYT \equiv \triangle KYT$, for $SY = KY$ and YT is common and the angles at Y are right angles.

$\therefore\ ST = KT$ and $\angle KTP = \angle STP = \angle QTS'$ (§ 144 b).

$\therefore\ \angle KTS' = \angle PTQ =$ a right angle.



Now $2CT^2 + 2CS^2 = ST^2 + S'T^2$ (§ 10)
$= KT^2 + S'T^2$
$= KS'^2$
$= 4CA^2.$

$\therefore\ CT^2 = 2CA^2 - CS^2 = 2CA^2 - (CA^2 - CB^2) = CA^2 + CB^2.$

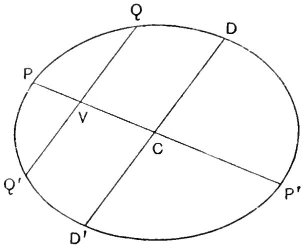
Thus the locus of T is a circle round C, the square of whose radius is $CA^2 + CB^2$.

146. Conjugate Diameters.

The student is already familiar with the idea of conjugate lines of a conic, two lines being called conjugate when each contains the pole of the other. When a pair of conjugate lines meet in the centre of an ellipse, each being a diameter it is convenient to call them *conjugate diameters.* It is clear that they are such that the tangents at the points where either meets the curve are parallel to the other. Moreover all the chords which are parallel to one of two conjugate diameters are bisected by the other (§ 95), and these chords are double ordinates of the diameter which thus bisects them. The axes of the ellipse are that particular pair of conjugate diameters which are mutually at right angles.

147. Prop. *If QV be an ordinate of the diameter PCP' of an ellipse, and DCD' be the diameter conjugate to CP,*

$$QV^2 : PV . VP' = CD^2 : CP^2.$$



For, producing QV to meet the ellipse again in Q', by Newton's theorem we have

$$VQ . VQ' : VP . VP' = CD . CD' : CP . CP'.$$

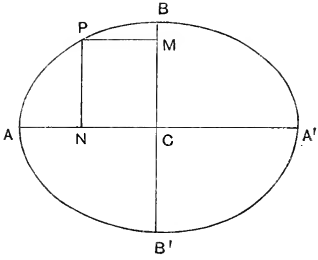
But $$VQ' = -VQ,\ CD' = -CD,\ CP' = -CP;$$

$$\therefore\ QV^2 : PV . VP' = CD^2 : CP^2.$$

148. Special cases of the preceding proposition are these: *If PN and PM be ordinates of the major and minor axes of an ellipse then*

$$PN^2 : AN . NA' = BC^2 : AC^2$$

$$PM^2 : BM . MB' = AC^2 : BC^2.$$



Cor. The latus rectum $= \dfrac{2BC^2}{AC}$.

For if SL be the semi-latus rectum

$$SL^2 : AS \,.\, SA' = BC^2 : AC^2 \text{ and } AS \,.\, SA' = BC^2.$$

149. These properties in § 148 shew that an ellipse may be regarded as the locus of a point in a plane such that the square of its distance from a fixed line l in the plane bears a constant ratio to the product of its distances from two other fixed lines l' and l'', perpendicular to the former and on opposite sides of the point.

The line l is one of the axes of the ellipse, and the lines l' and l'' are the tangents at the ends of the other axis.

The property established in § 147 shews that an ellipse may also be regarded as the locus of a point in a plane such that the square of its distance from a fixed line l in the plane, bears a constant ratio to the product of its distances from two other fixed lines l' and l'' which are parallel to each other (but not necessarily perpendicular to l), and on opposite sides of the point. The line l is a diameter of the ellipse and the lines l' and l'' are the tangents at the points where l meets the curve.

For the student can easily prove for himself that in the notation of § 147

$QV^2 : PV \,.\, VP'$ = square of perpendicular from Q on PP'
: product of perpendiculars from Q on tangents at Q and Q'.

Auxiliary Circle.

150. Prop. *If P be any point on an ellipse and PN the ordinate of the major axis, and if NP meet the auxiliary circle in p, then*

$$NP : Np = BC : AC.$$

For by § 148

$$PN^2 : AN \,.\, NA' = BC^2 : AC^2$$

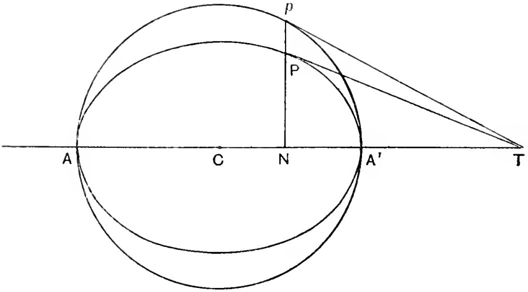
and as $\angle ApA'$ being in a semicircle is a right angle

$$pN^2 = AN \,.\, NA';$$

$$\therefore\ PN : pN = BC : AC.$$

P and p are said to be *corresponding points* on the ellipse and the auxiliary circle. The tangents at two corresponding points will meet the line of the major axis in the same point. For let the tangent at P meet it in T, then $CN \cdot CT = CA^2$ (§ 143).

$\therefore$ T is the pole of pN for the circle, that is, the tangent at p goes through T.



The student can prove for himself by the same method that if an ordinate PM to the minor axis meet the circle on BB' as diameter in p' then

$$PM : p'M = AC : BC.$$

From all this it follows that if the ordinates of a diameter of a circle be all divided in the same ratio, the points of division trace out an ellipse having the diameter of the circle as one of its axes.

151. Prop. *If CP and CD be a pair of conjugate semi-diameters of an ellipse, and p, d the points on the auxiliary circle corresponding to P and D, then pCd is a right angle.*

Let the tangents at P and p meet the major axis in T. Draw the ordinates PN, DM.

Now since CD is parallel to TP, $\triangle PNT$ is similar to $\triangle DMC$.

$$\therefore\ PN : DM = NT : MC.$$

But as $$PN : pN = BC : AC = DM : dM,$$

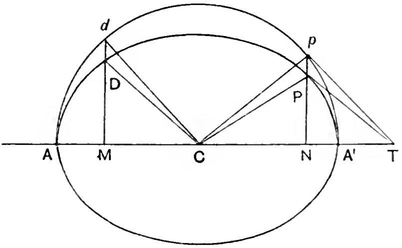
$$\therefore\ PN : DM = pN : dM.$$

$$\therefore\ pN : dM = NT : MC,$$

that is $$pN : NT = dM : MC.$$

$\therefore$ as the $\triangle$s pNT and dMC have the angles at M and N equal, they are similar.

$$\therefore\ \angle MCd = \angle NTp.$$



$$\therefore\ Cd \text{ is parallel to } Tp.$$

$$\therefore\ \angle dCp = \angle CpT = \text{a right angle.}$$

Cor. $pN = CM$ and $dM = CN$ for $\triangle CNp \equiv \triangle dMC$.

Whence also we have

$$PN : CM = BC : AC,$$

$$DM : CN = BC : AC.$$

152. Prop. *If CP and CD be conjugate semidiameters of an ellipse*

$$CP^2 + CD^2 = CA^2 + CB^2.$$

For using the figure of the last proposition

$$CP^2 = CN^2 + PN^2 = CN^2 + \frac{BC^2}{AC^2} \cdot pN^2$$

and $$CD^2 = CM^2 + DM^2 = pN^2 + \frac{BC^2}{AC^2}\, dM^2 = pN^2 + \frac{BC^2}{AC^2}\, CN^2.$$

$$\therefore\ CP^2 + CD^2 = \left(1 + \frac{BC^2}{AC^2}\right)(pN^2 + CN^2)$$

$$= \left(1 + \frac{BC^2}{AC^2}\right) AC^2 = AC^2 + BC^2.$$

Thus the sum of the squares of two conjugate diameters of an ellipse is constant and $= AA'^2 + BB'^2$.

Or we may prove the proposition thus:

In § 151 we proved $pN = CM$,

$$\therefore\ CM^2 + CN^2 = pN^2 + CN^2 = Cp^2 = AC^2.$$

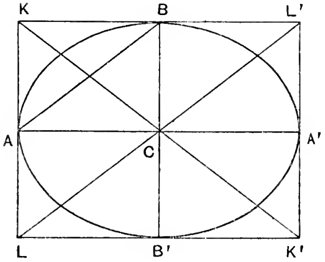
In exactly the same way by drawing ordinates to the minor axis and working with the circle on BB' as diameter we have

$$PN^2 + DM^2 = BC^2.$$

Whence by addition. $CP^2 + CD^2 = AC^2 + BC^2$.

152a. Equiconjugate Diameters.

There is one pair of conjugate diameters of an ellipse which are equal to one another namely those which lie along the diagonals of the rectangle formed by the tangents at the extremities of the major and minor axes.



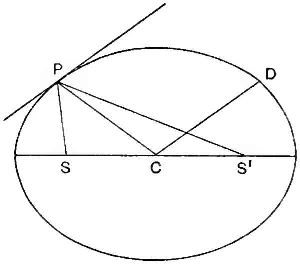
That the diameters along these lines are equal is clear from the symmetry of the curve, and that they are conjugate diameters is seen from the fact that in the figure here presented since KC bisects AB, which is parallel to LL', CK and CL' are conjugate lines.

152b. Prop. *If CP and CD be a pair of conjugate semi-diameters of an ellipse*

$$SP \cdot S'P = CD^2.$$

Since C is the middle point of SS'

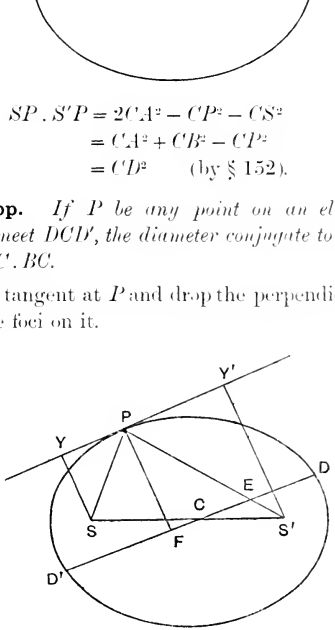
$$\begin{aligned}2CP^2 + 2CS^2 &= SP^2 + S'P^2\\ &= (S'P + SP)^2 - 2SP \,.\, S'P\\ &= 4CA^2 - 2SP \,.\, S'P.\end{aligned}$$



$$\begin{aligned}\therefore\ SP \,.\, S'P &= 2CA^2 - CP^2 - CS^2\\ &= CA^2 + CB^2 - CP^2\\ &= CD^2 \qquad (\text{by § } 152).\end{aligned}$$

153. Prop. *If P be any point on an ellipse, and the normal at P meet DCD', the diameter conjugate to CP, in F, then* $PF \,.\, CD = AC \,.\, BC$.

Draw the tangent at P and drop the perpendiculars SY and $S'Y'$ from the foci on it.



Join SP and $S'P$, and let $S'P$ cut DD' in E.

Then the Δs SPY, $S'PY'$ are similar.

$$\therefore\ SY : SP = S'Y' : S'P$$

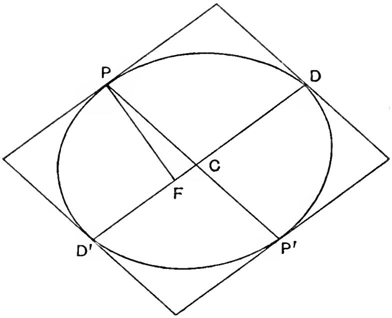
$$\therefore\ SY \,.\, S'Y' : SP \,.\, S'P = SY^2 : SP^2$$

$$= PF^2 : PE^2$$ since the Δs SYP, PFE are similar.

$$\therefore\ BC^2 : CD^2 = PF^2 : AC^2$$

that is, $$PF \,.\, CD = AC \,.\, BC.$$

COR. The area of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters is constant $= 4AC \,.\, BC$.



For the area $= 4$ area of parallelogram PD'
$= 4PF \,.\, CD = 4AC \,.\, BC.$

Circle of Curvature.

154. Prop. *The chord of the circle of curvature at any point P of an ellipse and through the centre of the ellipse is*

$$\frac{2CD^2}{CP}.$$

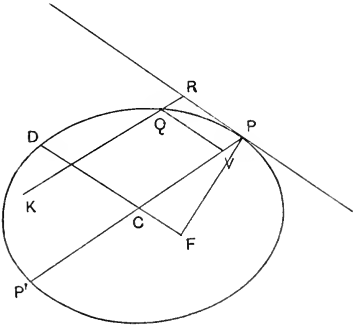
Let Q be a point on the ellipse near to P and QV the ordinate of the diameter PCP'.

Consider the circle touching the ellipse at P and cutting it in Q. Let QK be the chord of this circle parallel to CP. Let QK meet the tangent at P in R.

Then from the circle

$$RQ \,.\, RK = RP^2.$$

$$\therefore\; RK = \frac{RP^2}{RQ} = \frac{QV^2}{PV} = \frac{CD^2}{CP^2}\, VP' \quad (\S\ 147).$$



Thus the chord of the circle of curvature through the centre being the limit of RK when Q approaches P

$$= \frac{CD^2}{CP^2} \times \text{Limit } VP'$$

$$= \frac{CD^2}{CP^2} \times 2CP = \frac{2CD^2}{CP}.$$

Cor. The diameter of the circle of curvature $= \dfrac{2CD^2}{PF}$, F being the point in which the normal meets CD, for

$$\text{Diameter} : \frac{2CD^2}{CP} = \sec(\angle \text{ between normal and } CP)$$

$$= CP : PF.$$

$$\therefore\; \text{Diameter} = \frac{2CD^2}{PF} = \frac{2CD^3}{AC \,.\, BC}.$$

EXERCISES

1. Prove that in the notation of this chapter

$$CS : CX = CS^2 : CA^2.$$

2. If SL be the semi-latus rectum of an ellipse then $SL = e \, . \, SX$; prove from this that

$$SL = \frac{BC^2}{AC}.$$

Obtain also the length of the latus rectum by using the fact (§§ 116, 117) that the lengths of two focal chords are in the ratio of the squares of the diameters parallel to them.

3. If Y, Z be the feet of the perpendiculars from the foci of an ellipse on the tangent at P, of which PN is the ordinate; prove that the circle circumscribing YNZ passes through the centre of the ellipse.

4. If P be any point on an ellipse whose foci are S and S' the circle circumscribing SPS' will cut the minor axis in the points where it is met by the tangent and normal at P.

5. If two circles touch internally the locus of the centres of circles touching them both is an ellipse, whose foci are the centres of the given circles.

6. If the tangent at P to an ellipse meet the major axis in T, and NG be the subnormal,

$$CT \, . \, NG = BC^2.$$

7. If PN be the ordinate of any point P of an ellipse and Y, Y' the feet of the perpendiculars from the foci on the tangent at P, then PN bisects the angle YNY'.

8. If the normal at P to an ellipse meet the minor axis in g, and the tangent at P meet the tangent at the vertex A in V, shew that $Sg : SC = PV : VA$.

9. If the normal at P meet the major axis in G, PG is a harmonic mean between the perpendiculars from the foci on the tangent at P.

10. If an ellipse inscribed in a triangle have one focus at the orthocentre, the other focus will be at the circumcentre.

11. If an ellipse slide between two straight lines at right angles, the locus of its centre is a circle.

12. Lines are drawn through a focus of an ellipse to meet the tangents to the ellipse at a constant angle, prove that the locus of the points in which they meet the tangents is a circle.

13. The locus of the incentre of the triangle whose vertices are the foci of an ellipse and any point on the curve is an ellipse.

14. The opposite sides of a quadrilateral described about an ellipse subtend supplementary angles at either focus.

15. Prove that the foci of an ellipse and the points where any tangent to it meets the tangents at its vertices are concyclic.

16. If CQ be a semidiameter of an ellipse conjugate to a chord which is normal to the curve at P, then CP is conjugate to the normal at Q.

17. If P be any point on an ellipse, foci S and S', and A be a vertex, then the bisectors of the angles PSA, $PS'A$ meet on the tangent at P.

18. In an ellipse whose centre is C and foci S and S', GL is drawn perpendicular to CP, and CM is drawn parallel to $S'P$ meeting PG in M. Prove that the triangles CLM, CMP are similar.

19. A circle is drawn touching an ellipse at two points, and Q is any point on the ellipse. Prove that if QT be a tangent to the circle from Q, and QL perpendicular to the common chord, then

$$QT = e \, . \, QL.$$

20. If a parabola have its focus coincident with one of the foci of an ellipse, and touch its minor axis, a common tangent to the ellipse and parabola will subtend a right angle at the focus.

21. Shew how to determine the magnitude and position of the axes of an ellipse, having given two conjugate diameters in magnitude and position.

22. Construct an ellipse when the position of its centre and a self-conjugate triangle are given.

23. If P be any point on an ellipse whose vertices are A and A', and AP, $A'P$ meet a directrix in E and F, then EF subtends a right angle at the corresponding focus.

24. Deduce from Ex. 23 the property that $PN^2 : AN \, . \, NA'$ is constant.

25. Prove that chords joining any point on an ellipse to the ends of a diameter are parallel to a pair of conjugate diameters.

[Two such chords are called supplemental chords.]

26. If a circle touch a fixed ellipse at P, and intersect it at the ends of a diameter QQ', then PQ and PQ' are fixed in direction.

27. Ellipses have a common fixed focus and touch two fixed straight lines, prove that their director circles are coaxal.

28. SY is the perpendicular from the focus S of an ellipse on a tangent, and K the point in SY produced such that $SY = YK$. Prove that the square of the tangent from K to the director circle is double the square on SY.

29. The circle of curvature at an extremity of one of the equal conjugate diameters of an ellipse meets the ellipse again at the extremity of that diameter.

30. If PN be the ordinate of a point P on an ellipse, the chord of curvature in the direction of $PN = \dfrac{2CD^2}{BC^2} \times PN$.

31. If S and S' be the foci of an ellipse and B an extremity of the minor axis, the circle $SS'B$ will cut the minor axis in the centre of curvature at B.

32. The circle of curvature at a point P of an ellipse passes through the focus S, and SE is drawn parallel to the tangent at P to meet in E the diameter through P; shew that it divides the diameter in the ratio $3 : 1$.

33. The circle of curvature at P on an ellipse cuts the curve again in Q; the tangent at P meets the other common tangent, which touches the ellipse and circle at E and F, in O; prove $(TO, EF) = -1$.

34. The tangent at P to an ellipse meets the equiconjugates in Q and Q'; shew that CP is a symmedian of the triangle QCQ'.

CHAPTER XIII

THE HYPERBOLA

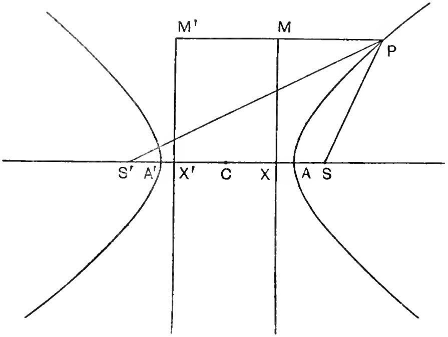
155. We have seen in §§ 100, 101 that a hyperbola, which is the projection of a circle cut by the vanishing line, has two axes of symmetry at right angles, one of which, named the transverse axis, meets the curve in what are called the vertices A and A', while the other called the conjugate axis does not meet the curve. These two axes meet in C the centre of the curve, and there are two tangents from C to the curve having their points of contact at infinity. These tangents are called the asymptotes and they make equal angles with the axes.

The curve has two foci S and S' lying on the line of the transverse axis, and such that the feet of the perpendiculars from them on the asymptotes lie on the circle on AA' as diameter. The directrices which are the polars of the foci are at right angles to the transverse axis, and pass through the feet of these perpendiculars. If X and X' be the points in which the directrices cut the transverse axis and C be the centre, then the eccentricity $(e) = CS : CA = CA : CX$.

156. In this chapter we shall set forth the principal properties that all hyperbolas have in common. Some of these are the same as those of the ellipse and can be established in much the same way. But the fact that the hyperbola has a pair of asymptotes, that is, tangents whose points of contact are at infinity, gives the curve a character and properties of its own.

157. Difference of focal distances constant.

Prop. *The difference of the focal distances of any point on a hyperbola is constant, and equal to the length AA' of the transverse axis.*



Let P be any point on the hyperbola.

Draw PMM' as in the figure perpendicular to the directrices, then

$$S'P = e \,.\, PM' \text{ and } SP = e \,.\, PM.$$

$$\therefore \quad S'P \sim SP = e \,.\, XX' = AA'.$$

For points on the one branch we have $S'P - SP = AA'$ and for points on the other $SP - S'P = AA'$.

Cor. Two confocal hyperbolas cannot intersect.

Tangent and Normal.

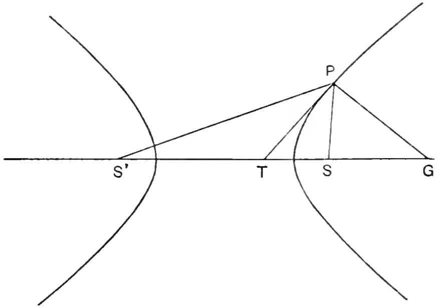
158. Prop. *The tangent and normal at any point of a hyperbola bisect respectively the interior and exterior angles of the focal radii of the point.*

Let the tangent and normal at P meet the transverse axis in T and G.

Then by § 111, $SG = e \,.\, SP$, $S'G = e \,.\, S'P$.

$$\therefore \quad SG : SP = S'G : S'P.$$

$\therefore$ PG is the bisector of the exterior angle of SPS', and PT, perpendicular to PG, must therefore bisect the interior angle.



COR. 1. $CG \cdot CT = CS^2$, for $(SS', TG) = -1$.

COR. 2. If an ellipse and hyperbola are confocal their tangents at the points of intersection of the curves are at right angles, or, in other words, the curves cut at right angles.

159. Prop. *If SY, $S'Y'$ be the perpendiculars from the foci on the tangent to a hyperbola at any point P, Y and Y' will lie on the circle on AA' as diameter (called the* **auxiliary** *circle), and $SY \cdot S'Y'$ will be constant.*

Let SY meet $S'P$ in K.

Then since $\qquad \angle SPY = \angle KPY$ (§ 158)

and $\qquad \angle SYP = \angle KYP$

and PY is common,

$$\therefore \quad \triangle SPY \equiv \triangle KPY$$

and $\qquad SY = YK, \; PK = SP.$

And since Y and C are the middle points of SK and SS', CY is parallel to $S'K$ and

$$CY = \tfrac{1}{2}S'K = \tfrac{1}{2}(S'P - KP) = \tfrac{1}{2}(S'P - SP) = \tfrac{1}{2}AA' = CA.$$

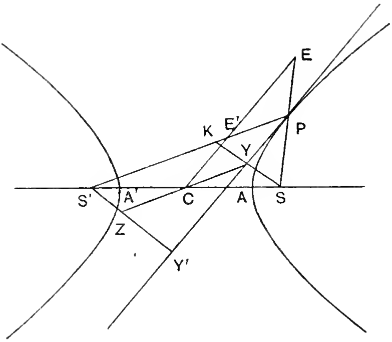
Thus Y (and similarly Y') lies on the circle on AA'.

If $S'Y'$ meet the circle again in Z, then since $YY'Z$ is a right angle, YZ must be a diameter and pass through C.

Also $\triangle SCY \equiv \triangle S'CZ$ and $S'Z = SY$.

$$\therefore \quad SY \,.\, S'Y' = S'Z \,.\, S'Y' = S'A' \,.\, S'A = CS^2 - CA^2$$

which is constant.



Cor. 1. The diameter parallel to the tangent at P will meet SP and $S'P$ in points E and E' such that $PE = PE' = CA$.

Cor. 2. The envelope of a line such that the foot of the perpendicular on it from a fixed point S lies on a fixed circle which has S *outside* it, is a hyperbola having S for a focus.

Cor. 3. The envelope of a line such that the product of the perpendiculars on it from two fixed points, lying on *opposite* sides of it, is constant is a hyperbola having the fixed points for its foci.

Compare § 141, Corr. 2 and 3.

160. On the length of the conjugate axis.

We have seen that the conjugate axis of a hyperbola does not meet the curve, so that we cannot say it has a length in the same way that the minor axis of an ellipse has for its length that portion of it intercepted by the curve.

It is convenient, and this will be understood better as we proceed, to measure off a length BB' on the conjugate axis such that B and B' are equidistant from C, and

$$BC^2 = CS^2 - CA^2 = AS \cdot A'S.$$

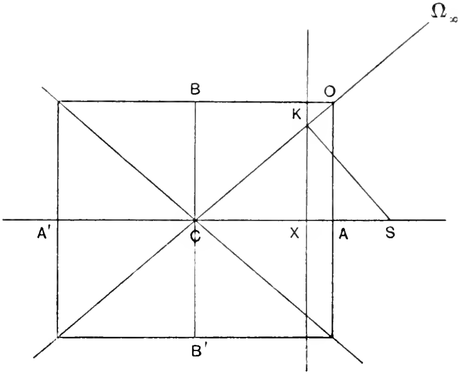
This will make $SY \cdot S'Y'$ in the preceding proposition equal to BC^2. (Compare § 141.)

The length BB' thus defined will for convenience be called the length of the conjugate axis, but it must be clearly understood that BB' is not a diameter length of the hyperbola, for B and B' do not lie on the curve.

161. It will easily be seen that if a rectangle be drawn having a pair of opposite sides along the tangents at A and A', and having its diagonals along the asymptotes, then the portion of the conjugate axis intercepted in this rectangle will be this length BB' which we have marked off as explained above.

For if the tangent at A meet the asymptote $C\Omega$ in O, and the directrix corresponding to S meet $C\Omega$ in K, we have, since CKS is a right angle and $CK = CA$ (§ 101),

$$\triangle CKS \equiv \triangle CAO.$$



Hence

$$AO^2 = SK^2 = CS^2 - CK^2 = CS^2 - CA^2.$$

Pair of tangents.

162. Prop. *The two tangents drawn from a point to a hyperbola make equal or supplementary angles with the focal distances of the point.*

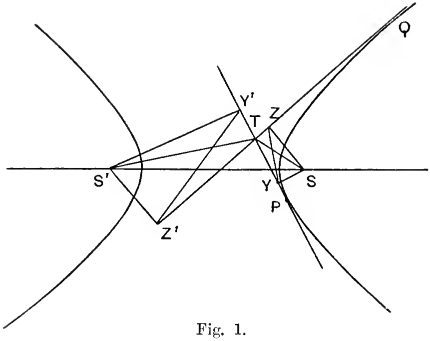


Fig. 1.

Let TP, TQ be the tangents, it is required to prove that the angles STP, $S'TQ$ are equal or supplementary.

Draw SY, $S'Y'$ perp. to TP, and SZ, $S'Z'$ to TQ.

Then $$SY \,.\, S'Y' = BC^2 = SZ \,.\, S'Z'.$$

$$\therefore \quad SY : SZ = S'Z' : S'Y'.$$

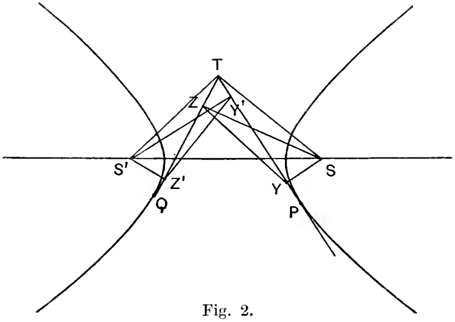


Fig. 2.

Also $\angle YSZ = \angle Z'S'Y'$ these being the supplements of the equal angles ZTY and $Z'TY'$ in fig. 1, and in fig. 2 each being equal to YTZ, since $SZTY$ and $S'Y'TZ'$ are cyclic.

Hence the $\triangle$s SYZ and $S'Z'Y'$ are similar and

$$\angle STP = \angle SZY = \angle S'Y'Z' = \angle S'TZ'$$

$$= \text{supplement of } \angle S'TQ \text{ in fig. 1,}$$

while in fig. 2 it $= \angle S'TQ$.

Thus the two tangents from an external point make equal or supplementary angles with the focal distances of the point according as the tangents belong to opposite branches or the same branch of the curve.

163. Director Circle.

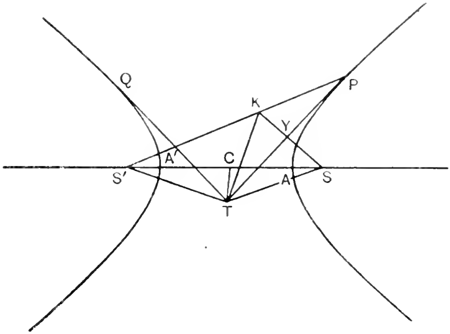
Prop. *The locus of points tangents from which to a hyperbola are at right angles is a circle (called the* **director circle** *of the hyperbola).*

Let TP and TQ be two tangents at right angles.

Draw SY perp. to TP to meet $S'P$ in K.

Then by § 159, $SY = YK$ and $S'K = AA'$.

Also $$\triangle SYT \equiv \triangle KYT.$$



Thus $ST = KT$ and $\angle KTY = \angle STY = \angle QTS'$ (§ 162).

$$\therefore \quad \angle KTS' = \angle PTQ = \text{a right angle.}$$

Now
$$2CT^2 + 2CS^2 = ST^2 + S'T^2 \quad (\S\ 10)$$
$$= KT^2 + S'T^2$$
$$= KS'^2 = AA'^2 = 4CA^2.$$
$$\therefore \quad CT^2 = 2CA^2 - CS^2$$
$$= CA^2 - (CS^2 - CA^2)$$
$$= CA^2 - CB^2.$$

Thus the locus of T is a circle whose centre is C and the square of whose radius is $CA^2 - CB^2$. (Cf. § 145.)

COR. 1. If $CA = CB$, $CT = 0$, that is the tangents from C, or the asymptotes, are at right angles.

COR. 2. If $CA < CB$, there are no points from which tangents at right angles can be drawn.

The Conjugate Hyperbola.

164. A hyperbola is completely determined when we know the length and position of its transverse and conjugate axes; for when AA' and BB' are fixed, S and S' the foci on the line AA' are determined by

$$CS^2 = CS'^2 = CA^2 + CB^2.$$

Also the eccentricity $e = CS : CA$, and the directrices are determined, being the lines perp. to AA' through points X and X' on it such that

$$CA : CX = e = CA' : CX.$$

165. We are going now to take a new hyperbola having for its transverse and conjugate axes respectively the conjugate and transverse axes of the original hyperbola. This new hyperbola will clearly have the same asymptotes as the original hyperbola (§ 161) and it will occupy space in what we may call the exterior angle between those asymptotes, as shown in the figure.

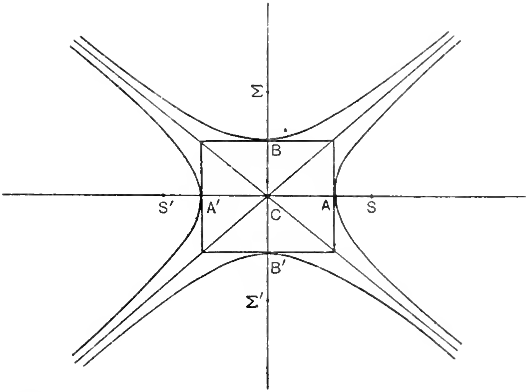
This new hyperbola is called the *conjugate hyperbola* in relation to the original hyperbola, and it follows that the original hyperbola is the conjugate hyperbola of the new one.

The hyperbola and its conjugate are two distinct curves, each having its foci and directrices, nor will they in general have the same eccentricity.

The foci S and S' of the original hyperbola lie on the line of AA' and are such that $CS^2 = CA^2 + CB^2$, and the eccentricity is $CS : CA$. The foci Σ and Σ' of the conjugate hyperbola lie on the line of BB' and are such that $C\Sigma^2 = CA^2 + CB^2$.

Thus $C\Sigma = CS$, but the eccentricity is $C\Sigma : CB$, which is only the same as that of the original hyperbola if $CA = CB$.

In this special case the asymptotes are the diagonals of a square (§ 161) and are therefore at right angles.



When the asymptotes are at right angles the hyperbola is said to be rectangular. In the next chapter we shall investigate the special properties of the rectangular hyperbola.

The conjugate hyperbola is, as we shall see, a very useful adjunct to the hyperbola and considerable use will be made of it in what follows.

Asymptotic properties.

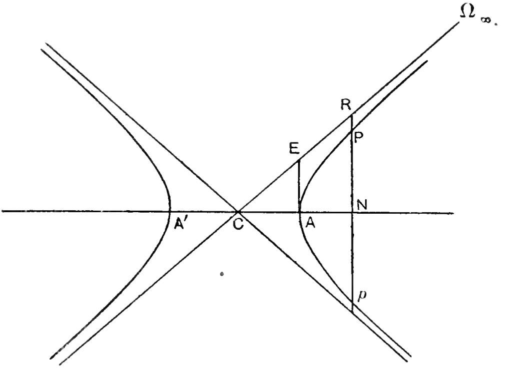
166. Prop. *If R be any point on an asymptote of a hyperbola, and RN perpendicular to the transverse axis meet the hyperbola in P* and p then $RP \,.\, Rp = BC^2$.

Let the tangent at A meet the asymptote on which R lies in E.

Then Ω being the point of contact of the asymptote with the curve at infinity we have by Newton's theorem

$$RP \,.\, Rp : R\Omega^2 = EA^2 : E\Omega^2.$$

$$\therefore\ RP \,.\, Rp = EA^2 = BC^2.$$



This can also be written

$$RN^2 - PN^2 = BC^2.$$

It will presently be seen that this proposition is only a special case of a more general theorem.

167. Prop. *If PN be the ordinate to the transverse axis of any point P of a hyperbola*

$$PN^2 : AN \,.\, A'N = BC^2 : AC^2.$$

Using the figure of § 166 we have

$$RN^2 - PN^2 = BC^2.$$

$$\therefore\ PN^2 = RN^2 - BC^2.$$

But $$RN^2 : BC^2 = RN^2 : EA^2$$

$$= CN^2 : CA^2.$$

$$\therefore\ RN^2 - BC^2 : BC^2 = CN^2 - CA^2 : CA^2.$$

$$\therefore\ PN^2 : BC^2 = AN \,.\, A'N : CA^2.$$

$$\therefore\ PN^2 : AN \,.\, A'N = BC^2 : AC^2,$$

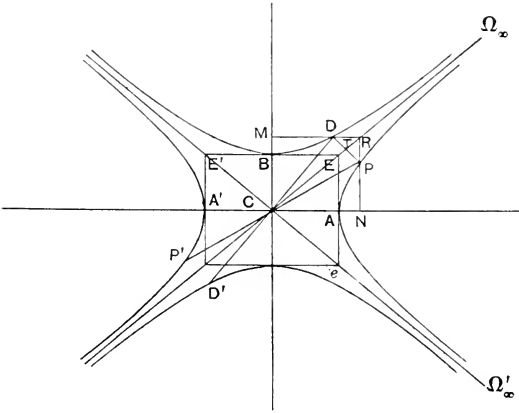
or we may write this

$$PN^2 : CN^2 - CA^2 = BC^2 : AC^2.$$

This too will be found to be but a special case of a more general theorem.

Comparing this property with the corresponding one in the ellipse (§ 148) we see that it was not possible to establish the property for the hyperbola in the same way as for the ellipse, because the conjugate axis does not meet the hyperbola.

168. Prop. *If from any point R in an asymptote of a hyperbola RPN, RDM be drawn perpendicular to the transverse and conjugate axes to cut the hyperbola and its conjugate respectively in P and D, then PD is parallel to the other asymptote, and CP, CD are conjugate lines for both the hyperbola and the conjugate hyperbola.*



Let Ω and Ω′ be the points of contact of the hyperbola and its asymptotes at infinity.

We first observe that AB is parallel to $C\Omega'$.

For drawing the lines through A and B perpendicular to the axes to meet the asymptotes, as indicated in the figure, in E, e, E', we have

$$EB : BE' = EA : Ae.$$

Also MN is parallel to AB, for

$$CA : CB = CA : AE = CN : RN$$
$$= CN : CM.$$

$$\therefore\ CA : CN = CB : CM.$$

Now $\quad RN^2 - PN^2 = BC^2$
and $\quad RM^2 - DM^2 = AC^2$ } § 166.

$$\therefore\ RN^2 - PN^2 : RM^2 - DM^2 = CM^2 : CN^2$$
$$= RN^2 : RM^2.$$

$$\therefore\ RN^2 : RM^2 = PN^2 : DM^2.$$

Thus PD is parallel to MN and therefore to $C\Omega'$.

Thus PD will be bisected by $C\Omega$ in the point T (say).

Now DP will meet $C\Omega'$ at Ω', and we have

$$(DP,\ T\Omega') = -1.$$

$\therefore$ CP and CD will belong to the involution of which $C\Omega$ and $C\Omega'$ are the double lines.

But $C\Omega$ and $C\Omega'$ being tangents from C to both the hyperbola and its conjugate are the double lines of the involution pencil formed by the pairs of conjugate lines through C.

$\therefore$ CP and CD are conjugate lines for both the hyperbola and its conjugate.

For this it follows that the tangent at P to the hyperbola is parallel to CD, and the tangent at D to the conjugate hyperbola is parallel to CP.

169. On the term conjugate diameters.

If the lines PC, DC in the figure of § 168 meet the hyperbola and its conjugate again in the points P' and D' respectively, then PCP' and DCD' are called *conjugate diameters* for each of the hyperbolas. But it must be clearly understood that PCP' is a diameter of the original hyperbola, whereas DCD' is not, but it is a diameter of the conjugate hyperbola.

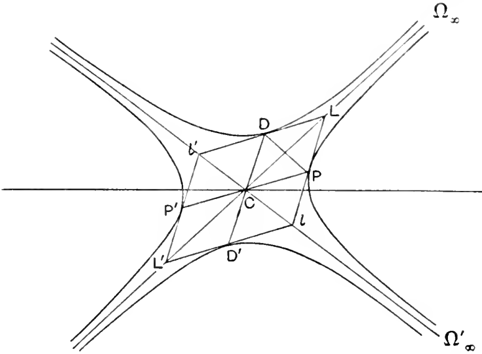
Of two so-called conjugate diameters one is a diameter of the hyperbola and the other of the conjugate hyperbola.

The line DCD' is a diameter even for the original hyperbola in so far as it is a line through the centre and it will bisect a system of parallel chords, but it is not a diameter in the sense that it represents a length intercepted by the curve on the line, for D and D' are not on the hyperbola. DCD' does not meet the hyperbola in real points, though of course as the student acquainted with Analytical Geometry will know it meets the curve in imaginary points, that is, points whose coordinates involve the imaginary quantity $\sqrt{-1}$.

170. Prop. *The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose diagonals lie along the asymptotes.*

Let PCP' and DCD' be the conjugate diameters, as in the figure.

We have already proved (§ 168) that PD is bisected by $C\Omega$.



The tangents at P and D are respectively parallel to CD and CP.

These tangents then form with CP and CD a parallelogram having one diagonal along $C\Omega$.

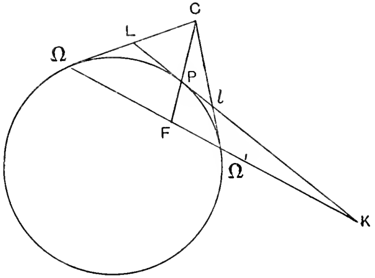
Similarly the tangents at P' and D' meet on $C\Omega$, and those at P', D and P, D' on $C\Omega'$.

COR. The portion of the tangent at any point intercepted between the asymptotes is bisected at the point of contact.

For $$LP = DC = CD' = Pl.$$

171. The property given in the Corollary of § 170 can be independently established by projecting into a circle, and we may use the same letters in the projection without confusion.

Let the tangent at P to the circle meet the vanishing line $\Omega\Omega'$ in K.



The polar of K goes through C, since that at C goes through K, and the polar of K goes through P.

$\therefore$ CP is the polar of K.

Let CP meet $\Omega\Omega'$ in F.

$$\therefore (KF, \Omega\Omega') = -1.$$
$$\therefore C(KF, \Omega\Omega') = -1.$$
$$\therefore (KP, Ll) = -1.$$

Thus in the hyperbola L and l are harmonically conjugate with P and the point at infinity along Ll. $\therefore$ $LP = Pl$.

172. **Prop.** *If through any point R on an asymptote of a hyperbola a line be drawn cutting the same branch of the hyperbola in Q and q, then $RQ \,.\, Rq$ is equal to the square of the semi-diameter of the conjugate hyperbola parallel to RQq.*

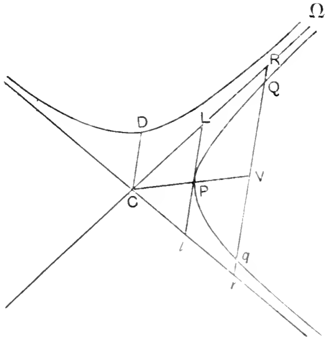
Let V be the middle point of Qq.

Let CV cut the hyperbola in P.

Then the tangent at P is parallel to Qq.

Let it meet the asymptotes in L and l.

Let CD be the semi-diameter of the conjugate hyperbola parallel to Qq.



By Newton's theorem we have

$$RQ \,.\, Rq : R\Omega^2 = LP^2 : L\Omega^2.$$

$$\therefore \; RQ \,.\, Rq = LP^2 = CD^2.$$

Thus if the line RQq be always drawn in a fixed direction the rectangle $RQ \,.\, Rq$ is independent of the position of the point R on the asymptote.

We may write the above relation

$$RV^2 - QV^2 = CD^2.$$

And if RQq meet the other asymptote in r we have

$$rV^2 - qV^2 = CD^2.$$

$$\therefore \; rV^2 - qV^2 = RV^2 - QV^2.$$

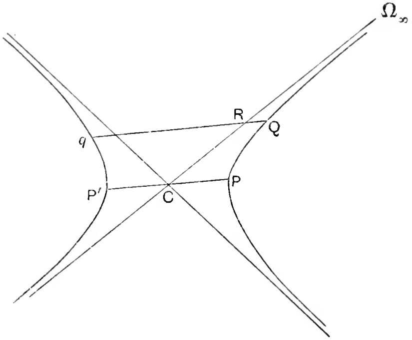
$$\therefore \; RV = Vr$$

and $$\therefore \; RQ = qr.$$

Hence any chord of a hyperbola and the length of its line intercepted between the asymptotes have the same middle point.

We thus have the following remarkable property of the hyperbola: *If Rr joining any two points on the asymptotes of a hyperbola cut the curve in Q and q then $RQ = qr$.*

173. Prop. *If a line be drawn through a point R on an asymptote of a hyperbola to meet opposite branches of the curve in Q and q then $qR \cdot RQ = CP^2$ where CP is the semi-diameter of the hyperbola parallel to Qq.*



For by Newton's theorem

$$RQ \cdot Rq : R\Omega^2 = CP \cdot CP' : C\Omega^2.$$

$$\therefore\ RQ \cdot Rq = -\, CP^2.$$

$$\therefore\ qR \cdot RQ = CP^2.$$

As in the preceding article we can shew that Qq and the portion of it intercepted between the asymptotes have the same middle point.

174. Prop. *If QV be an ordinate of the diameter PCP' and DCD' the diameter conjugate to PP' then*

$$QV^2 : PV \cdot P'V = CD^2 : CP^2.$$

Let QV meet the asymptote $C\Omega$ in R and the curve again in Q'.

Through R draw the chord qq' of the hyperbola parallel to PCP'.

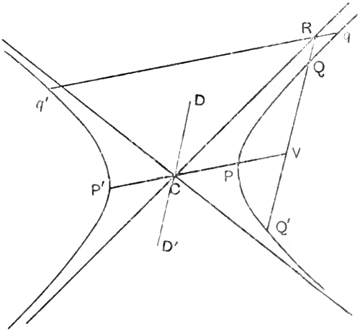
Then by Newton's theorem

$$VQ\,.\,VQ' : VP\,.\,VP' = RQ\,.\,RQ' : Rq\,.\,Rq'.$$

$$\therefore\ -QV^2 : VP\,.\,VP' = CD^2 : -CP^2.$$

that is

$$QV^2 : PV\,.\,P'V = CD^2 : CP^2.$$



This is the general theorem of which that of § 167 is a special case.

We may write the relation as

$$QV^2 : CV^2 - CP^2 = CD^2 : CP^2.$$

175. From §§ 167, 174 we can see that a hyperbola may be regarded as the locus of a point in a plane such that the ratio of the square of its distance from a fixed line l varies as the product of its distances from two other fixed lines l' and l'' parallel to one another and such that the point is on the same side of both of them.

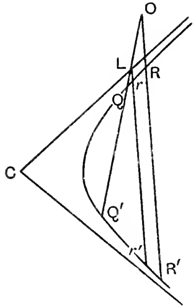
If l' and l'' be perpendicular to l then l is the transverse axis of the hyperbola and l', l'' the tangents at its vertices.

If l' and l'' are not perpendicular to l, then l is a diameter of the hyperbola, and l', l'' are the tangents at the points where it meets the hyperbola.

176. Prop. *If QQ', RR' be chords of a hyperbola intersecting in O then the ratio $OQ \cdot OQ' : OR \cdot OR'$ is equal to that of the squares of the diameters parallel to the respective chords.*

Let OQQ' meet an asymptote in L.

Through L draw Lrr' parallel to ORR' to meet the curve in r and r'.



Then by Newton's theorem

$$OQ \cdot OQ' : OR \cdot OR' = LQ \cdot LQ' : Lr \cdot Lr'$$

$= $ sq. of diameter parallel to QQ'

: sq. of diameter parallel to RR'.

177. This proposition holds equally well if the ends of either or both of the chords lie on opposite branches of the hyperbola, provided that $OQ \cdot OQ'$ and $OR \cdot OR'$ be regarded simply as positive magnitudes.

For suppose that Q, Q' lie on opposite branches and R, R' on the same branch, then as LQ and LQ' are in opposite directions we must for the application of Newton's theorem say

$LQ \cdot LQ' = - QL \cdot LQ' = -$ sq. of diameter parallel to QQ',

so that, as $OQ \cdot OQ' = - QO \cdot OQ'$ we have

$QO \cdot OQ' : OR \cdot OR' = $ sq. of diameter parallel to QQ'

: sq. of diameter parallel to RR'.

178. It may perhaps seem unnecessary to make a separate proof for the hyperbola of the proposition proved generally for the central conics in § 117. But our purpose has been to bring out the fact that in § 117 the diameters must be length diameters of the curve itself, and for these the proposition is true. But as diameters of the hyperbola do not all meet the curve in real points, we wanted to shew how the diameters of the conjugate diameter may be used instead. Whenever the signs of $OQ \cdot OQ'$ and $OR \cdot OR'$ in the notation of §§ 176, 177 are different, this means that the diameters parallel to QQ', RR' are such that only one of them meets the hyperbola. The other meets the conjugate hyperbola.

179. We can see now that if DCD' be a diameter of the conjugate hyperbola, the imaginary points δ, δ' in which it meets the original hyperbola, are given by

$$C\delta^2 = C\delta'^2 = -CD^2,$$

and this to the student acquainted with Analytical Geometry is also clear from the following:

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and of the conjugate hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus corresponding to every point (x, y) on (2) there is a point (ix, iy) on (1) and vice versa.

And if a line through the centre meet the conjugate hyperbola (2) in (x, y) it will meet the original hyperbola in (ix, iy).

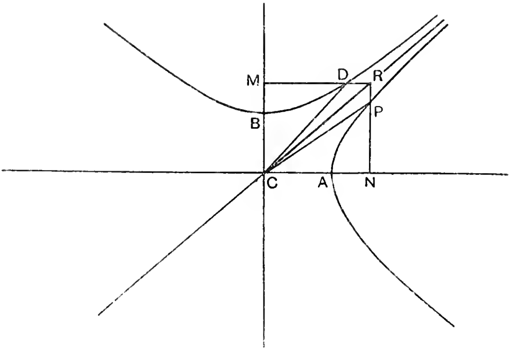
180. Prop. *If CP and CD be conjugate semi-diameters of a hyperbola*

$$CP^2 - CD^2 = CA^2 - CB^2.$$

Draw the ordinates PN and DM to the transverse and conjugate axes.

These intersect in a point R on an asymptote (§ 168), and we have

$$CP^2 = CN^2 + PN^2 = CR^2 - (RN^2 - PN^2)$$
$$= CR^2 - BC^2 \quad (\S\ 166)$$



and

$$CD^2 = CM^2 + DM^2 = CR^2 - (RM^2 - DM^2)$$
$$= CR^2 - CA^2.$$
$$\therefore\ CP^2 - CD^2 = CA^2 - CB^2.$$

Cor. If $CA = CB$, that is if the hyperbola be a rectangular one, then $CP = CD$.

181. Prop. *If P be any point on a hyperbola*

$$SP \,.\, S'P = CD^2$$

where CD is the semi-diameter conjugate to CP.

Since C is the middle point of SS'

$$SP^2 + S'P^2 = 2CS^2 + 2CP^2 \quad (\S\ 10).$$
$$\therefore\ (S'P - SP)^2 + 2SP \,.\, S'P = 2CS^2 + 2CP^2,$$

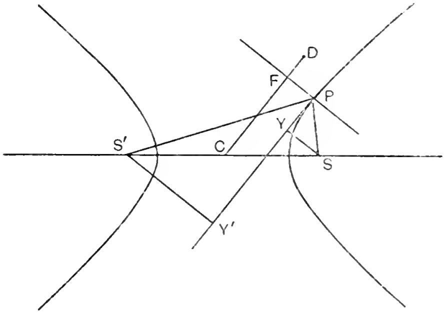
that is

$$4CA^2 + 2SP \,.\, S'P = 2CS^2 + 2CP^2.$$
$$\therefore\ SP \,.\, S'P = CP^2 + CS^2 - 2CA^2$$
$$= CP^2 + CB^2 + CA^2 - 2CA^2$$
$$= CP^2 - (CA^2 - CB^2)$$
$$= CD^2 \quad (\S\ 180).$$

182. Prop. *If CP and CD be conjugate semi-diameters of a hyperbola, and the normal at P meet CD in F, then*

$$PF \, . \, CD = AC \, . \, BC.$$

Draw the perpendiculars SY and $S'Y'$ from the foci on the tangent at P.



Then the Δs SPY, $S'PY$ being similar we have

$$\frac{SY}{SP} = \frac{S'Y'}{S'P} = \frac{S'Y' - SY}{S'P - SP} = \frac{2PF}{2AC} = \frac{PF}{AC}.$$

$$\therefore \frac{SY \, . \, S'Y'}{SP \, . \, S'P} = \frac{PF^2}{AC^2},$$

that is

$$\frac{BC^2}{CD^2} = \frac{PF^2}{AC^2}.$$

$$\therefore PF \, . \, CD = AC \, . \, BC.$$

Cor. The area of the parallelogram formed by the tangents at the ends of a pair of conjugate diameters is constant $= AA' \, . \, BB'$.

183. Prop. *The area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.*

Let the tangent at P meet the asymptotes in L and l. (Use fig. of § 170.)

Let P' be the other end of the diameter through P and let DCD' be the diameter conjugate to PP'. Then L and l are angular points of the parallelogram formed by the tangents at P, P', D, D' (§ 170).

Moreover $\triangle CLl$ is one quarter of the area of the parallelogram formed by these tangents, that is (§ 182),

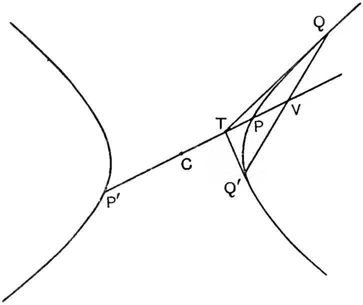
$$\triangle CLl = CA \,.\, CB,$$

which is constant

Cor. The envelope of a line which forms with two fixed lines a triangle of constant area is a hyperbola having the fixed lines for its asymptotes, and the point of contact of the line with its envelope will be the middle point of the portion intercepted between the fixed lines.

184. Prop. *If TQ and TQ' be tangents to the same branch of a hyperbola, and CT meet the curve in P and QQ' in V, then*

$$CV \,.\, CT = CP^2.$$



This follows at once from the harmonic property of the pole and polar, for we have

$$(PP', TV) = -1.$$

$$\therefore\ CV \,.\, CT = CP^2.$$

185. Prop. *If TQ and TQ' be tangents to* **opposite** *branches of a hyperbola, and CT meet QQ' in V and the conjugate hyperbola in P then*

$$VC \,.\, CT = CP^2.$$

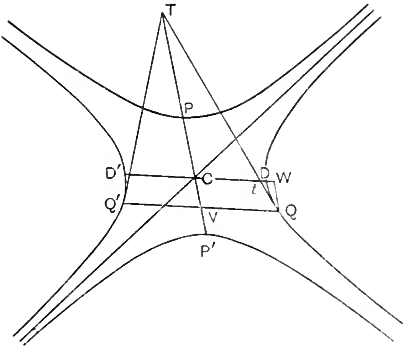
This can be surmised from the preceding proposition, for if CT meet the original hyperbola in the imaginary point p, we have (§ 179)

$$Cp^2 = -CP^2.$$

$$\therefore\ CV.CT = Cp^2 \quad (\S\ 184)$$

$$= -CP^2.$$

$$\therefore\ VC.CT = CP^2.$$



We give however the following purely geometrical proof, which does not introduce imaginary points.

Let DCD' be the diameter conjugate to PP' and meeting the hyperbola in D, D', and TQ in t.

Draw the ordinate QW to the diameter DD', that is, QW is parallel to PP'.

Then by similar $\triangle$s tWQ, tCT

$$TC : WQ = Ct : tW.$$

$$\therefore\ TC.WQ : WQ^2 = Ct.CW : CW.tW$$

$$= Ct.CW : CW^2 - Ct.CW.$$

But by § 184, $Ct.CW = CD^2$.

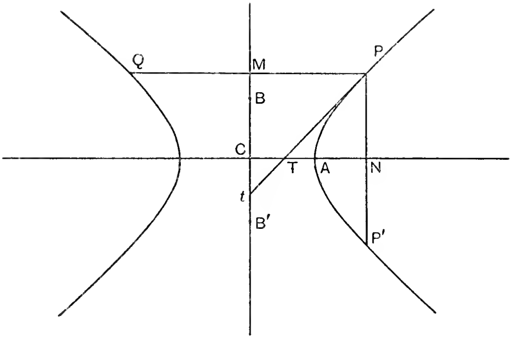
$$\therefore\ TC.WQ : WQ^2 = CD^2 : CW^2 - CD^2.$$

But $$CP^2 : WQ^2 = CD^2 : CW^2 - CD^2 \quad (\S\ 174).$$

$$\therefore\ TC \,.\, WQ = CP^2.$$

$$\therefore\ VC \,.\, CT = CP^2.$$

186. The following are special cases of the two preceding propositions.



If the tangent at P to a hyperbola meet the transverse and conjugate axes in T and t respectively and PN, PM be ordinates to these axes

$$CT \,.\, CN = CA^2$$

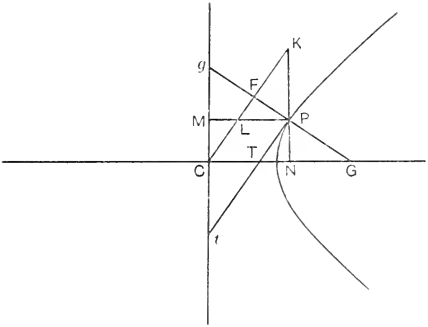
$$MC \,.\, Ct = CB^2.$$

For the tangents from T will be TP and TP' where P' is the point in which PN again meets the hyperbola; and the tangents from t will be tP, tQ where Q is the point in which PM again meets the hyperbola.

187. Prop. *If the normal at P to a hyperbola meet the transverse and conjugate axes in G and g, and the diameter parallel to the tangent at P in F, then*

$$FP \,.\, PG = BC^2, \quad PF \,.\, Pg = AC^2.$$

This proposition can be established exactly like the corresponding one in the ellipse (§ 144).



188. Prop. *If the normal at P to a hyperbola meet the transverse axis in G, and PN be the ordinate*

$$CG = e^2 . CN$$

and $$NG : CN = BC^2 : AC^2.$$

This is proved in the same way as in § 144 *a*.

Circle of curvature.

189. Prop. *The chord of the circle of curvature at any point P of a hyperbola and through the centre of the hyperbola is* $\frac{2CD^2}{CP}$, *and the diameter of the circle is* $\frac{2CD^2}{PF}$.

This is proved in the same way as for the ellipse.

EXERCISES

1. If a line be drawn through a focus S of a hyperbola parallel to one of the asymptotes and a perpendicular $S'K$ be drawn from the other focus S' on to this line $SK = AA'$.

[Use § 157. Take P at Ω.]

2. Find the locus of the centre of a circle touching two fixed circles externally.

3. If the tangent at any point P of a hyperbola cut an asymptote in T and SP cut the same asymptote in Q then $SQ = QT$.

[TP and $T\Omega$ subtend equal angles at S.]

4. Shew that when a pair of conjugate diameters of a hyperbola are given in magnitude and position the asymptotes are completely determined. Hence shew that there are only two hyperbolas having a given pair of conjugate diameters.

5. If two hyperbolas have the same asymptotes a chord of one touching the other is bisected at the point of contact.

6. If PH, PK be drawn parallel to the asymptotes $C\Omega$, $C\Omega'$ of a hyperbola to meet $C\Omega'$ and $C\Omega$ in H and K, then

$$PH \, . \, PK = \tfrac{1}{4}CS^2.$$

[Use § 183.]

7. The tangent to a hyperbola at P meets an asymptote in T and TQ is drawn parallel to the other asymptote to meet the curve in Q. PQ meets the asymptote in L and M. Prove that LM is trisected at P and Q.

8. From any point R on an asymptote of a hyperbola RPN is drawn perpendicular to the transverse axis to cut the curve in P; RK is drawn at right angles to CR to meet the transverse axis in K. Prove that PK is the normal at P.

[Prove that $CN = e^2 \, . \, CK$. § 188.]

9. Prove that in any central conic if the normal at P meet the axes in G and g then $PG \, . \, Pg = CD^2$ where CD is conjugate to CP.

10. If the tangent at a point P of a hyperbola meet the asymptotes in L and l, and the normal at P meet the axes in G and g, then L, l, G, g lie on a circle which passes through the centre of the hyperbola.

11. The intercept of any tangent to a hyperbola between the asymptotes subtends at the further focus an angle equal to half the angle between them.

12. Given a focus of an ellipse and two points on the curve shew that the other focus describes a hyperbola.

13. If P be any point on a central conic whose foci are S and S', the circles on SP, $S'P$ as diameters touch the auxiliary circle and have for their radical axis the ordinate of P.

14. The pole of the tangent at any point P of a central conic with respect to the auxiliary circle lies on the ordinate of P.

15. If PP' and DD' be conjugate diameters of a hyperbola and Q any point on the curve then $QP^2 + QP'^2$ exceeds $QD^2 + QD'^2$ by a constant quantity.

16. Given two points of a parabola and the direction of its axis, prove that the locus of its focus is a hyperbola.

17. If two tangents be drawn to a hyperbola the lines joining their intersections with the asymptotes will be parallel.

18. If from a point P in a hyperbola PK be drawn parallel to an asymptote to meet a directrix in K, and S be the corresponding focus, then $PK = SP$.

19. If the tangent and normal at a point P of a central conic meet the axis in T and G and PN be the ordinate, $NG \,.\, CT = BC^2$.

20. The base of a triangle being given and also the point of contact with the base of the inscribed circle, the locus of the vertex is a hyperbola.

21. If tangents be drawn to a series of confocal hyperbolas the normals at their points of contact will all pass through a fixed point, and the points of contact will lie on a circle.

22. A hyperbola is described touching the principal axes of a hyperbola at one of their extremities; prove that one asymptote is parallel to the axis of the parabola and that the other is parallel to the chords of the parabola bisected by the first.

23. If an ellipse and a hyperbola confocal with it intersect in P, the asymptotes of the hyperbola pass through the points of intersection of the ordinate of P with the auxiliary circle of the ellipse.

24. Prove that the central distance of the point where a tangent to a hyperbola meets one asymptote varies as the distance, parallel to the transverse axis, of the point of contact from the other asymptote.

25. Tangents RPR', TQT' are drawn to a hyperbola, R, T being on one asymptote and R', T' on the other; shew that the circles on RT' and $R'T$ as diameters are coaxal with the director circle.

26. From any point P on a given diameter of a hyperbola, two straight lines are drawn parallel to the asymptotes, and meeting the hyperbola in Q, Q'; prove that PQ, PQ' are to one another in a constant ratio.

27. The asymptotes and one point on a hyperbola being given, determine the points in which a given line meets the curve.

28. If PN be the ordinate and PG the normal of a point P of a hyperbola whose centre is C, and the tangent at P intersect the asymptotes in L and L', then half the sum of CL and CL' is the mean proportional between CN and CG.

29. The tangents to a conic from any point on the director circle are the bisectors of the angles between every pair of conjugate lines through the point.

30. Given a focus, a tangent and the eccentricity of a conic, the locus of the centre is a circle.

31. If P be a point on a central conic such that the lines joining P to the foci are at right angles, $CD^2 = 2BC^2$.

32. Find the position and magnitude of the axes of a hyperbola which has a given line for an asymptote, passes through a given point, and touches a given straight line at a given point.

33. If P be any point on a hyperbola whose foci are S and S' the incentre of the triangle SPS' lies on the tangent at one of the vertices of the hyperbola.

34. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas are constructed; prove that if one hyperbola touch the ellipse the other will do likewise and that the diameters drawn through the points of contact are conjugate to each other.

35. Prove that a circle can be described to touch the four straight lines joining the foci of a hyperbola to any two points on the same branch of the curve.

36. Tangents are drawn to a hyperbola and the portion of each tangent intercepted by the asymptotes is divided in a given ratio; shew that the locus of the point of section is a hyperbola.

37. From a point R on an asymptote of a hyperbola RE is drawn touching the hyperbola in E, and ET, EV are drawn through E parallel to the asymptotes, cutting a diameter in T and V; RV is joined, cutting the hyperbola in P, p; shew that TP and Tp touch the hyperbola.

[Project the hyperbola into a circle and V into the centre.]

38. CP and CD are conjugate semi-diameters of a hyperbola, and the tangent at P meets an asymptote in L; prove that if PD meet the transverse axis in F, LFC is a right angle.

39. From a given point on a hyperbola draw a straight line such that the segment between the other intersection with the hyperbola and a given asymptote shall be equal to a given line. When does the problem become impossible?

40. If P and Q be two points on two circles S_1 and S_2 belonging to a coaxal system of which L is one of the limiting points, such that the angle PLQ is a right angle, prove that the foot of the perpendicular from L on PQ lies on one of the circles of the system, and thus shew that the envelope of PQ is a conic having a focus at L.

41. If a conic touch the sides of a triangle at the feet of the perpendiculars from the vertices on the opposite sides, the centre of the conic must be at the symmedian point of the triangle.

CHAPTER XIV

THE RECTANGULAR HYPERBOLA

190. A rectangular hyperbola as we have already explained is one which has its asymptotes at right angles and its transverse and conjugate axes equal. The eccentricity of a rectangular hyperbola $= \sqrt{2}$, for $e = CS : CA$, and $CS^2 = CA^2 + CB^2 = 2CA^2$.

We will now set forth a series of propositions giving the chief properties of the curve.

191. Prop. *In a rectangular hyperbola conjugate diameters are equal, and if QV be an ordinate of a diameter PCP', $QV^2 = PV \, . \, P'V$.*

For we have $$CP^2 - CD^2 = CA^2 - CB^2 \quad (\S\ 180)$$
$$= 0$$

and $$QV^2 : PV \, . \, P'V = CD^2 : CP^2 \quad (\S\ 174)$$
$$= 1.$$

192. Prop. *Conjugate diameters of a rectangular hyperbola are equally inclined to each of the asymptotes.*

For the asymptotes are the double lines of the involution pencil formed by the pairs of conjugate lines through C, and therefore the asymptotes are harmonically conjugate with any pair of conjugate diameters. Hence as the asymptotes are at right angles they must be the bisectors of the angles between each pair of conjugate diameters (§ 72).

Cor. 1. Any diameter of a rectangular hyperbola and the tangents at its extremities are equally inclined to each of the asymptotes.

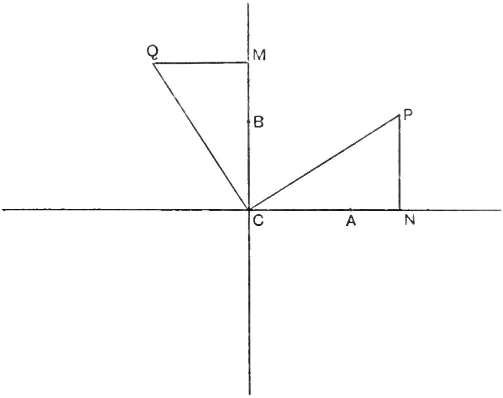
Cor. 2. Any chord of a rectangular hyperbola and the diameter bisecting it are equally inclined to each asymptote.

193. Prop. *Any diameter of a rectangular hyperbola is equal to the diameter perpendicular to it of the conjugate hyperbola.*

This is obvious when we consider that the conjugate hyperbola is in our special case equal to the original hyperbola and can be obtained by rotating the whole figure of the hyperbola through a right angle about an axis through its centre perpendicular to its plane.

194. *If a hyperbola have two perpendicular diameters equal to one another, the one belonging to the hyperbola itself and the other to its conjugate, the hyperbola must be a rectangular one.*

Let CP and CQ be the semi-diameters at right angles to one another and equal, P being on the hyperbola, and Q on the conjugate.



Draw PN and QM perpendicular to the transverse and conjugate axes respectively.

Then $$\triangle CNP \equiv \triangle CMQ.$$

Now $PN^2 : CN^2 - CA^2 = BC^2 : AC^2$ (§ 167)

and $QM^2 : CM^2 - BC^2 = AC^2 : BC^2$,

whence we get $$\frac{CN^2}{AC^2} - \frac{PN^2}{BC^2} = 1$$

and $$\frac{CM^2}{BC^2} - \frac{QM^2}{AC^2} = 1.$$

Subtract and use $CN^2 = CM^2$ and $PN^2 = QM^2$.

$$\therefore \quad CN^2 \left(\frac{1}{AC^2} - \frac{1}{BC^2}\right) - PN^2 \left(\frac{1}{BC^2} - \frac{1}{AC^2}\right).$$

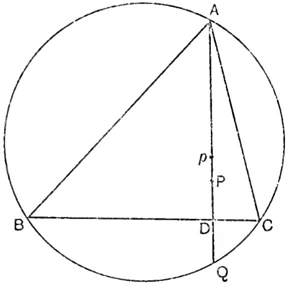
$$\therefore \text{ since } CN^2 + PN^2 \neq 0, \ AC = BC.$$

195. Prop. *If a rectangular hyperbola pass through the vertices of a triangle it passes also through the orthocentre.*

Let ABC be the triangle, P its orthocentre and AD the perpendicular from A on BC.

Let the rectangular hyperbola meet AD again in p.

Since the chords Ap and BC are at right angles the diameters parallel to them will meet one the hyperbola and the other the conjugate hyperbola.



Thus $DB \,.\, DC$ and $Dp \,.\, DA$ will have opposite signs (§ 177), and the ratio of their numerical values will be unity since the diameters parallel to them being at right angles are equal.

$$\therefore \ BD \,.\, DC = Dp \,.\, DA.$$

But $BD \,.\, DC = AD \,.\, DQ$ where Q is the point in which AD produced meets the circumcircle

$$= -AD \,.\, DP \quad (\S\ 6).$$

$$\therefore\ Dp \,.\, DA = DA \,.\, DP.$$

$$\therefore\ Dp = DP$$

that is p coincides with P.

Cor. When a rectangular hyperbola circumscribes a triangle, if the three vertices lie on the same branch of the curve, the orthocentre will lie on the other branch, but if two of the vertices lie on one branch and the third on the other, the orthocentre will lie on that branch on which are the two vertices.

196. Prop. *If a conic circumscribing a triangle pass through the orthocentre it must be a rectangular hyperbola.*

Let ABC be the triangle and AD, BE, CF the perpendiculars meeting in the orthocentre P.

It is clear that the conic must be a hyperbola, since it is impossible for two chords of an ellipse or parabola to intersect at a point external to one of them and not to the other, and the chords AP and BC do so intersect.

Now since $BD \,.\, DC = AD \,.\, PD$, the diameter parallel to BC = the diameter parallel to AP.

And these diameters must belong the one to the hyperbola and the other to its conjugate since $DB \,.\, DC$ and $DP \,.\, DA$ have opposite sign (§ 177).

Therefore the hyperbola is a rectangular one (§ 194).

197. Prop. *If a rectangular hyperbola circumscribe a triangle, its centre lies on the nine points circle of the triangle.*

Let ABC be the triangle, and D, E, F the middle points of the sides.

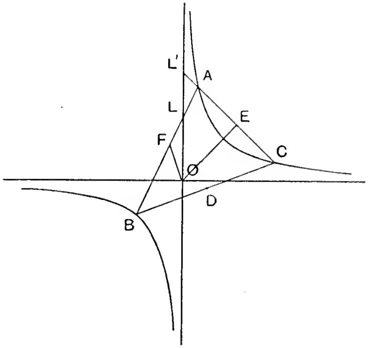
Let O be the centre of the rectangular hyperbola and OLL' an asymptote cutting AB and AC in L and L'.

Since OF bisects the chord AB, OF and AB make equal angles with OLL' (§ 192, Cor. 2).

$$\therefore\ \angle FOL = \angle FLO.$$

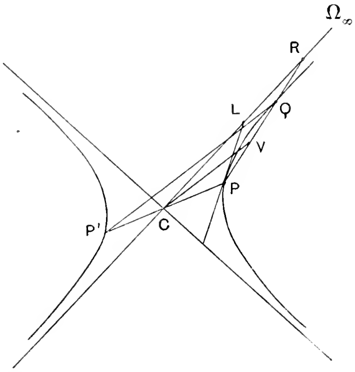
Similarly $\angle EOL' = \angle EL'O$.

$\therefore\ \angle FOE = \angle ALL' + \angle AL'L = \angle BAC = \angle FDE$.



$\therefore$ O lies on the circle round DEF, which circle is the nine points circle of the triangle.

198. Prop. *The angle between any chord PQ of a rectangular hyperbola and the tangent at P is equal to the angle subtended by PQ at P', the other end of the diameter through P.*



Let the chord PQ and the tangent at P meet the asymptote $C\Omega$ in R and L. Let V be the middle point of PQ.

Then $\angle VRC = \angle VCR$ (§ 192, Cor. 2).

and $\angle PLC = \angle PCL$ (§ 192, Cor. 1).

$$\therefore \angle LPR = \angle CLP - \angle CRV$$
$$= \angle PCL - \angle VCR = \angle VCP$$
$$= \angle QP'P \text{ (since } CV \text{ is parallel to } QP'\text{)}.$$

199. Prop. *Any chord of a rectangular hyperbola subtends at the ends of any diameter angles which are equal or supplementary.*

Let QR be a chord, and PCP' a diameter.

Let the tangents at P and P' meet the asymptotes in L, l and L', l'.

In fig. 1, where Q and R lie on the same branch and PP' cuts QR internally,

$$\angle QPL = \angle QP'P \quad (\S\ 198)$$

and $$\angle RPl = \angle RP'P.$$

$$\therefore \angle QPR = \text{supplement of sum of } QPL \text{ and } RPl$$
$$= \text{supplement of } \angle QP'R.$$

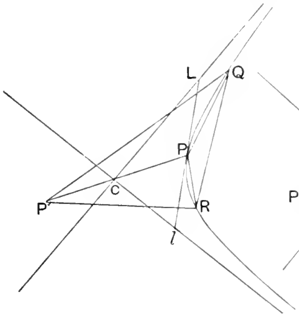


Fig. 1.

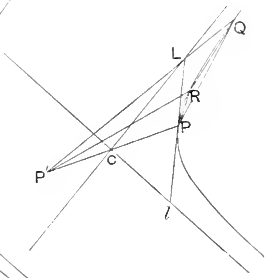


Fig. 2.

In fig. 2, where Q and R lie on the same branch and PP' cuts QR externally,

$$\angle LPR = \angle RP'P$$

and $$\angle LPQ = \angle QP'P.$$

$$\therefore \angle RPQ = \angle QP'P - \angle RP'P = \angle QP'R.$$

In fig. 3, where Q and R lie on opposite branches and PP' cuts QR internally,

$$\begin{aligned}\angle QPR &= \angle QPL + \angle LPP' + \angle P'PR \\ &= \angle QP'P + \angle PP'l' + \angle RP'l' \\ &= \angle QP'R.\end{aligned}$$

In fig. 4, where Q and R lie on opposite branches and PP' cuts QR externally,

$$\angle QPR = \angle QPL + \angle LPR = \angle QP'P + \angle LPR$$

and
$$\angle QP'R = \angle QP'L' + \angle L'P'R = \angle QP'L' + \angle RPP'.$$

$$\therefore \quad \angle QPR + \angle QP'R' = \angle L'P'P + \angle LPP'$$
$$= 2 \text{ right } \angle \text{s}.$$

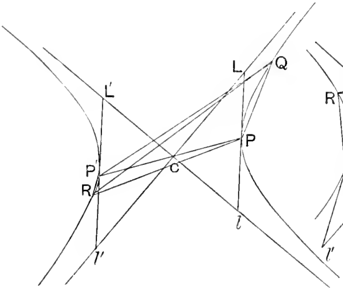


Fig. 3.

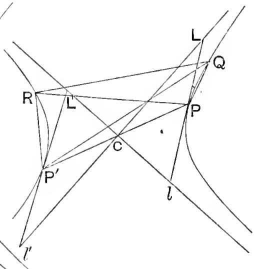


Fig. 4.

EXERCISES

1. The portion of any tangent to a rectangular hyperbola intercepted between its asymptotes is double the distance of its point of contact from the centre.

2. If PN be the ordinate of any point P on a rectangular hyperbola, and PG the normal at P, prove $CN = NG$, and the tangent from N to the auxiliary circle $= PN$.

3. If CK be the perpendicular from the centre on the tangent at any point P of a rectangular hyperbola the triangles PCA, CAK are similar.

4. PQR is a triangle inscribed in a rectangular hyperbola, and the angle at P is a right angle; prove that the tangent at P is perpendicular to QR.

5. If PP' and QQ' be perpendicular chords of a rectangular hyperbola then PQ', QP' will be at right angles, as also PQ and $P'Q'$.

6. PP' is any chord of a rectangular hyperbola, and a diameter perpendicular to it meets the hyperbola in Q; prove that the circle PQP' touches the hyperbola at Q.

7. If from the extremities of any diameter of a rectangular hyperbola lines be drawn to any point on the curve, they will be equally inclined to each asymptote.

8. Focal chords of a rectangular hyperbola which are at right angles to one another are equal.

[See §§ 116, 117.]

9. The distance of any point on a rectangular hyperbola from the centre is the geometric mean between its distances from the foci.

10. If PP' be a double ordinate to the transverse axis of a rectangular hyperbola whose centre is C, then CP' is perpendicular to the tangent at P.

11. The centre of the inscribed circle of a triangle lies on any rectangular hyperbola circumscribing the triangle whose vertices are the e centres.

12. Focal chords parallel to conjugate diameters of a rectangular hyperbola are equal.

13. If the tangent at any point P of a rectangular hyperbola, centre C, meet a pair of conjugate diameters in E and F, PC touches the circle CEF.

14. Two tangents are drawn to the same branch of a rectangular hyperbola. Prove that the angles which these tangents subtend at the centre are respectively equal to the angles which they make with the chord of contact.

15. A circle and a rectangular hyperbola intersect in four points and one of their common chords is a diameter of the hyperbola; shew that the other common chord is a diameter of the circle.

16. Ellipses are described in a given parallelogram; shew that their foci lie on a rectangular hyperbola.

17. If from any point Q in the conjugate axis of a rectangular hyperbola QA be drawn to the vertex, and QR parallel to the transverse to meet the curve, $QR = AQ$.

18. The lines joining the extremities of conjugate diameters of a rectangular hyperbola are perpendicular to the asymptotes.

19. The base of a triangle and the difference of its base angles being given the locus of its vertex is a rectangular hyperbola.

20. The circles described on parallel chords of a rectangular hyperbola are coaxal.

21. If a rectangular hyperbola circumscribe a triangle, the pedal triangle is a self-conjugate one.

22. At any point P of a rectangular hyperbola the radius of curvature varies as CP^3, and the diameter of the curve is equal to the central chord of curvature.

23. At any point of a rectangular hyperbola the normal chord is equal to the diameter of curvature.

24. PN is drawn perpendicular to an asymptote of a rectangular hyperbola from any point P on it, the chord of curvature along PN is equal to $\frac{CP^2}{PN}$.

CHAPTER XV

ORTHOGONAL PROJECTION

200. When the vertex of projection by means of which a figure is projected from one plane p on to another plane π is at a very great distance from these planes, the lines joining corresponding points in the original figure and its projection come near to being parallel. What we may call *cylindrical projection* is the case in which points on the p plane are projected on to the π plane by lines which are all drawn parallel to each other. We regard this as the limiting case of conical projection when the vertex V is at infinity.

In the particular case where the lines joining corresponding points are perpendicular to the π plane on to which the figure on the p plane is projected, the resulting figure on the π plane is said to be the *orthogonal projection* of the original figure.

Points in space which are not necessarily in a plane can be orthogonally projected on to a plane by drawing perpendiculars from them to the plane. The foot of each perpendicular is the projection of the point from which it is drawn. Thus all points in space which lie on the same line perpendicular to the plane on to which the projection is made will have the same projection.

In the present chapter it will be shewn how certain properties of the ellipse can be obtained from those of the circle, for, as we shall see, every ellipse is the orthogonal projection of a circle. It is first necessary to establish certain properties of orthogonal projection.

201. It may be observed at the outset that in orthogonal projection we have no vanishing line as in conical projection.

The line at infinity in the p plane projects into the line at infinity in the plane π. This is clear from the fact that the perpendiculars to the π plane from points in it on the line at infinity meet the p plane at infinity.

It follows that the orthogonal projection of a parabola will be another parabola, and of a hyperbola another hyperbola, while the orthogonal projection of an ellipse will be another ellipse or in particular cases a circle.

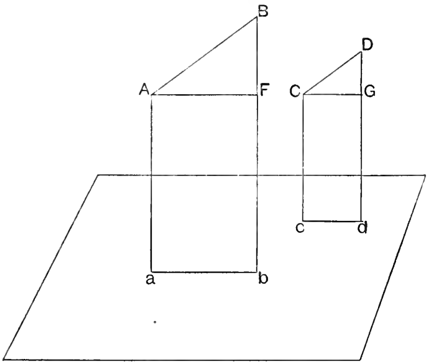
202. The following propositions relating to orthogonal projection are important:

Prop. *The projection of a straight line is another straight line.*

This is obvious from the fact that orthogonal is only a limiting case of conical projection. It is clear that the line in the π plane which will be the projection of a line l will be that in which the plane through l and perpendicular to the plane π cuts this π plane.

203. Prop. *Parallel straight lines project into parallel straight lines, and in the same ratio as regards their length.*

Let AB and CD be two lines in space parallel to one another.



Let ab, cd be their orthogonal projections on to the plane π.

Then ab and cd must be parallel, for if they were to meet in a point p, p would be the projection of a point common to AB and CD.

Now draw AF and CG parallel respectively to ab and cd to meet Bb and Dd in F and G. Then $AabF$ is a parallelogram so that $AF = ab$, and similarly $CG = cd$.

Now since AB is parallel to CD, and AF to CG (for these are respectively parallel to ab and cd which we have proved to be parallel), the angle $FAB =$ the angle GCD; and the angles at F and G are right angles.

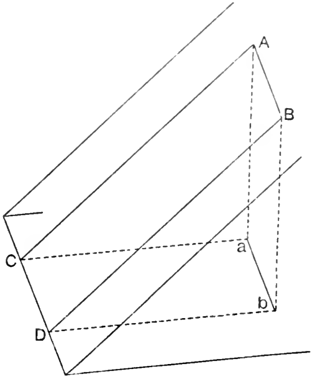
$$\therefore \triangle AFB \text{ is similar to } \triangle CGD.$$

$$\therefore AF : CG = AB : CD.$$

$$\therefore ab : cd = AB : CD.$$

COR. Lengths of line lying along the *same* line are projected in the same ratio.

204. Prop. *If l be a limited line in the p plane parallel to p's intersection with the π plane, the orthogonal projection of l on π will be a line parallel to and of the same length as l.*



Let AB be the limited line l, and ab its orthogonal projection.

Draw AC and BD perpendicular to the line of intersection of p and π.

Then $ACDB$ is a parallelogram.

Also since Aa and Bb are perpendicular to π, Ca and Db are perpendicular to CD and therefore they are parallel to each other.

Further $\qquad \triangle ACa \equiv \triangle BDb$

for $\qquad AC = BD, \ \angle AaC = \angle BbD$

and $\angle ACa = \angle BDb$ for AC and Ca are parallel to BD and Db.

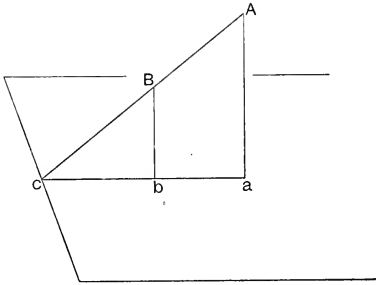
$$\therefore\ Ca = Db.$$

$\therefore$ as Ca and Db are parallel, $CDba$ is a parallelogram.

$$\therefore\ ab = CD = AB.$$

205. Prop. *A limited line in the p plane perpendicular to the line of intersection of p and π will project into a line also perpendicular to this line of intersection and whose length will bear to the original line a ratio equal to the cosine of the angle between the planes.*

Let AB be perpendicular to the intersection of p and π, and let its line meet it in C.



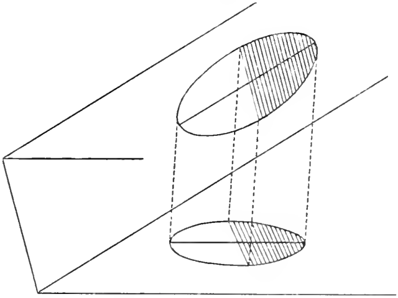
Let ab be the orthogonal projection of AB.

Then ab and AB meet in C, and $ab : AB = ac : AC$

$$= \cos aCA$$

$$= \cos (\angle \text{ between } p \text{ and } \pi).$$

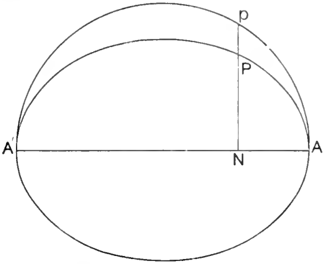
206. Prop. *A closed figure on the p plane will project into a closed figure whose area will bear to that of the original figure a ratio equal to the cosine of the angle between the planes.*



For we may suppose the figure to be made up of an infinite number of narrow rectangular strips the length of which runs parallel to the intersection of p and π. The lengths of the slips are unaltered by projections, and the breadths are diminished in the ratio of the cosine of the angle between the planes.

207. *The ellipse as the orthogonal projection of a circle.*

We have seen (§ 150) that corresponding ordinates of an ellipse and its auxiliary circle bear a constant ratio to one another, viz., $BC : AC$.



Now let the auxiliary circle be turned about its major axis AA' until it comes into a plane making with that of the ellipse an angle whose cosine is $BC : AC$.

It is clear that the lines joining each point on the ellipse to the new position of the point corresponding to it on the auxiliary circle will be perpendicular to the plane of the ellipse.

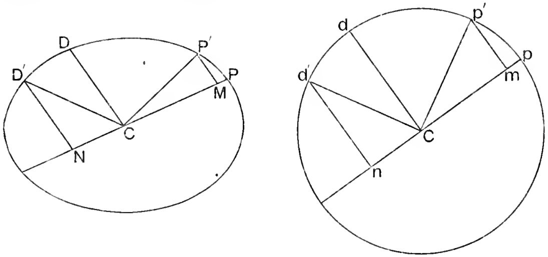
Thus the ellipse is the orthogonal projection of the circle in its new position.

Certain properties of the ellipse then can be deduced from those of the circle by orthogonal projection. We proceed to some illustrations.

208. Prop. *If CP and CD be a pair of conjugate semi-diameters of an ellipse, and CP', CD' another such pair, and the ordinates $P'M$, $D'N$ be drawn to CP, then*

$$P'M : CN = D'N : CM = CD : CP.$$

For let the corresponding points in the auxiliary circle adjusted to make an angle $\cos^{-1}\dfrac{BC}{AC}$ with the plane of the ellipse, be denoted by small letters.



Then Cp and Cd are perpendicular radii as are also Cp' and Cd', and $p'm$, $d'n$ being parallel to Cd will be perpendicular to Cp, and we have

$$\triangle Cmp' \equiv \triangle d'nC.$$

$$\therefore\ p'm : Cd = Cn : Cp$$

and $$d'n : Cd = Cm : Cp.$$

$\therefore$ by § 203

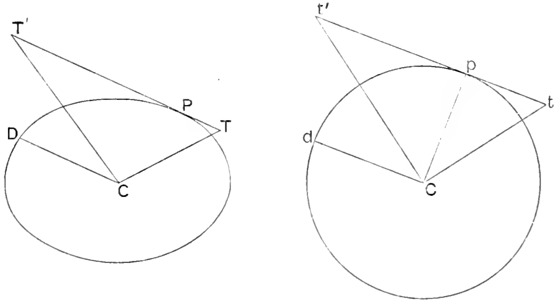
$$P'M : CD = CN : CP$$

$$D'N : CD = CM : CP.$$

$$\therefore\ P'M : CN = CD : CP = D'N : CM.$$

209. Prop. *If the tangent at a point P of an ellipse meet any pair of conjugate diameters in T and T′ and CD be conjugate to CP, then* $TP \cdot PT' = CD^2$.

For in the corresponding figure of the circle Ct and Ct' are at right angles, and Cp is perpendicular to tt'.



$$\therefore\ tp \cdot pt' = Cp^2 = Cd^2.$$
$$\therefore\ tp : Cd = Cd : pt'.$$
$$\therefore\ TP : CD = CD : PT'.$$
$$\therefore\ TP \cdot PT' = CD^2.$$

210. Prop. *The area of an ellipse whose semi-axes are CA and CB is* $\pi \cdot CA \cdot CB$.

For the ellipse is the orthogonal projection of its auxiliary circle tilted to make an angle $\cos^{-1}\dfrac{BC}{AC}$ with that of the ellipse.

$\therefore$ Area of ellipse : Area of auxiliary circle $= BC : AC$ (§ 206).

$\therefore$ Area of ellipse $= \pi \cdot BC \cdot AC$.

211. Prop. *The orthogonal projection of a circle from a plane p on to another plane π is an ellipse whose major axis is parallel to the intersection of p and π, and equal to the diameter of the circle.*

Let AA' be that diameter of the circle which is parallel to the p and π planes.

Let AA' project into aa' equal to it (§ 204).

Let PN be an ordinate to the diameter AA' and let pn be its projection.

$\therefore\ pn = PN \cos\alpha$ where α is the $\angle$ between p and π, and pn is perpendicular to aa' (§ 205).

$$\text{Now } pn^2 : an\,.\,na' = PN^2\cos^2\alpha : AN\,.\,NA'$$
$$= \cos^2\alpha : 1.$$

Hence the locus of p is an ellipse having aa' for its major axis, and its minor axis $= aa' \times \cos\alpha$.

The eccentricity is easily seen to be $\sin\alpha$.

COR. 1. Two circles in the same plane project orthogonally into similar and similarly situated ellipses.

For their eccentricities will be equal and the major axis of the one is parallel to the major axis of the other, each being parallel to the line of intersection of the planes.

COR. 2. Two similar and similarly situated ellipses are the simultaneous orthogonal projections of two circles.

EXERCISES

1. The locus of the middle points of chords of an ellipse which pass through a fixed point is a similar and similarly situated ellipse.

2. If a parallelogram be inscribed in an ellipse its sides are parallel to conjugate diameters, and the greatest area of such a parallelogram is $BC\,.\,AC$.

3. If PQ be any chord of an ellipse meeting the diameter conjugate to CP in T, then $PQ\,.\,PT = 2CR^2$ where CR is the semi-diameter parallel to PQ.

4. If a variable chord of an ellipse bear a constant ratio to the diameter parallel to it, it will touch another similar ellipse having its axes along those of the original ellipse.

5. The greatest triangle which can be inscribed in an ellipse has one of its sides bisected by a diameter of the ellipse and the others cut in points of trisection by the conjugate diameter.

6. If a straight line meet two concentric, similar and similarly situated ellipses, the portions intercepted between the curves are equal.

7. The locus of the points of intersection of the tangents at the extremities of pairs of conjugate diameter is a concentric, similar, and similarly situated ellipse.

8. If CP, CD be conjugate semi-diameters of an ellipse, and BP, BD be joined, and AD, $A'P$ intersect in O, the figure $BDOP$ will be a parallelogram.

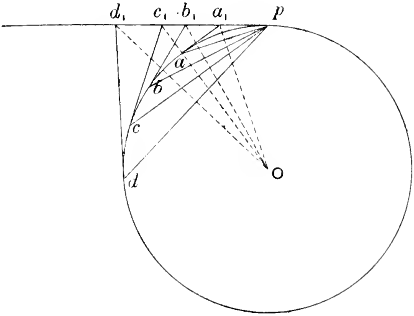
9. Two ellipses whose axes are at right angles to one another intersect in four points. Shew that any pair of common chords will make equal angles with an axis.

10. Shew that a circle of curvature for an ellipse and the ellipse itself can be projected orthogonally into an ellipse and one of its circles of curvature.

CHAPTER XVI

CROSS-RATIO PROPERTIES OF CONICS

212. Prop. *If A, B, C, D be four fixed points on a conic, and P a variable point on the conic, $P\,(ABCD)$ is constant and equal to the corresponding cross-ratio of the four points in which the tangents at A, B, C, D meet that at P.*



Project the conic into a circle and use corresponding small letters in the projection.

Then $$P\,(ABCD) = p\,(abcd).$$

But $p\,(abcd)$ is constant since the angles apb, bpc, cpd are constant or change to their supplements as p moves on the circle; therefore $P\,(ABCD)$ is constant.

Let the tangents at a, b, c, d cut that in p in a_1, b_1, c_1, d_1 and let O be the centre of the circle.

Then Oa_1, Ob_1, Oc_1, Od_1 are perpendicular to pa, pb, pc, pd.

$$\therefore\ p\,(abcd) = O\,(a_1b_1c_1d_1) = (a_1b_1c_1d_1).$$

$$\therefore\ P\,(ABCD) = (A_1B_1C_1D_1).$$

Cor. If A' be a point on the conic near to A, we have

$$A'\,(ABCD) = P\,(ABCD).$$

$\therefore$ if AT be the tangent at A,

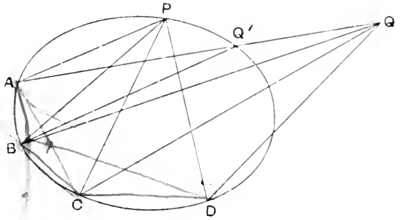
$$A\ (TBCD) = P\,(ABCD).$$

Note. In the special case where the pencil formed by joining any point P on the conic to the four fixed points A, B, C, D is harmonic, we speak of the points on the conic as harmonic. Thus if $P\,(ABCD) = -1$, we say that A and C are harmonic conjugates to B and D.

213. Prop. *If A, B, C, D be four fixed non-collinear points in a plane and P a point such that $P\,(ABCD)$ is constant, the locus of P is a conic.*

Let Q be a point such that

$$Q\ (ABCD) = P\,(ABCD).$$



Then if the conic through the points A, B, C, D, P does not pass through Q, let it cut QA in Q'.

$$\therefore\ P\ (ABCD) = Q'\ (ABCD) \text{ by § 212.}$$

$$\therefore\ Q'\,(ABCD) = Q\ (ABCD).$$

Thus the pencils Q (A, B, C, D) and Q' (A, B, C, D) are homographic and have a common ray QQ'.

Therefore (§ 64) they are coaxally in perspective; that is, A, B, C, D are collinear.

But this is contrary to hypothesis.

Therefore the conic through A, B, C, D, P goes through Q.

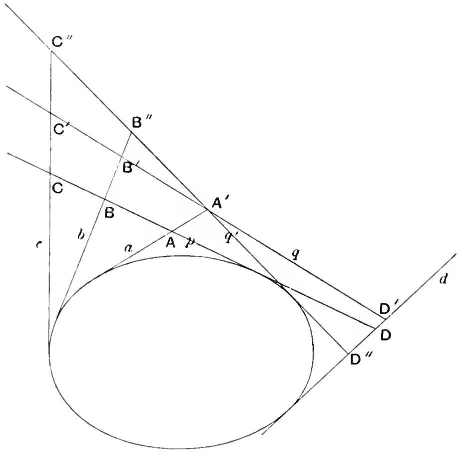
Thus our proposition is proved.

We see from the above that we may regard a conic through the five points A, B, C, D, E as the locus of a point P such that

$$P(ABCD) = E(ABCD).$$

214. Prop. *The envelope of a line which cuts four non-concurrent coplanar fixed straight lines in four points forming a range of constant cross-ratio is a conic touching the four lines.*

This proposition will be seen, when we come to the next chapter, to follow by Reciprocation directly from the proposition of the last paragraph.



The following is an independent proof.

Let the line p cut the four non-concurrent lines a, b, c, d in the points A, B, C, D such that $(ABCD) =$ the given constant.

Let the line q cut the same four lines in A', B', C', D' such that $(A'B'C'D') = (ABCD)$.

Then if q be not a tangent to the conic touching a, b, c, d, p, from A' in q draw q' a tangent to the conic.

Let b, c, d cut q' in B'', C'', D''.

$$\therefore (A'B''C''D'') = (ABCD) \text{ by § 212}$$
$$= (A'B'C'D').$$

The ranges $A'B''C''D''$ and $A'B'C'D'$ are therefore homographic and they have a common corresponding point.

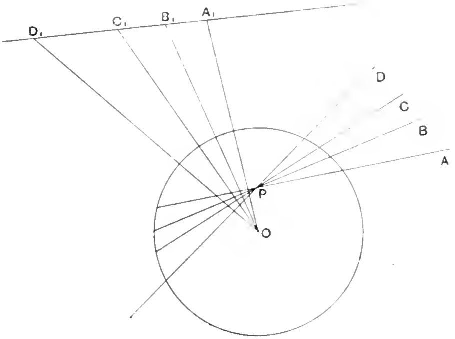
Therefore they are in perspective (§ 60), which is contrary to our hypothesis that a, b, c, d are non-concurrent.

Thus q touches the same conic as that which touches a, b, c, d, p.

And our proposition is established.

215. Prop. *If $P(A, B, C, D)$ be a pencil in the plane of a conic S, and A_1, B_1, C_1, D_1 the poles of PA, PB, PC, PD with respect to S, then*

$$P(ABCD) = (A_1B_1C_1D_1).$$



We need only prove this in the case of a circle, into which as we have seen, a conic can be projected.

Let O be the centre of the circle.

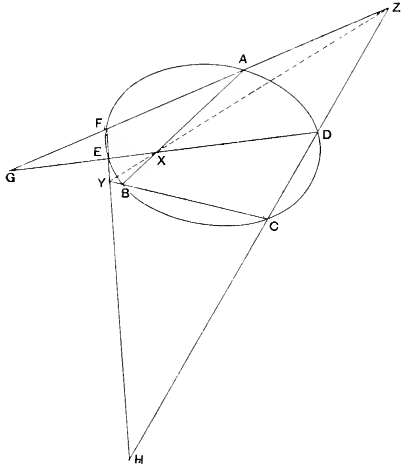
Then OA_1, OB_1, OC_1, OD_1 are perpendicular respectively to PA, PB, PC, PD.

$$\therefore\ P(ABCD) = O(A_1B_1C_1D_1) = (A_1B_1C_1D_1).$$

This proposition is of the greatest importance for the purposes of Reciprocation.

We had already seen that the polars of a range of points form a pencil; we now see that the pencil is homographic with the range.

216. Pascal's theorem. *If a conic pass through six points A, B, C, D, E, F, the opposite pairs of sides of each of the sixty different hexagons (six-*sided *figures) that can be formed with these points intersect in collinear points.*



This theorem may be proved by projection (see Ex. 25, Chap. X). Or we may proceed thus:

Consider the hexagon or six-sided figure formed with the sides AB, BC, CD, DE, EF, FA.

The pairs of sides which are called opposite are AB and DE; BC and EF; CD and FA.

Let these meet in X, Y, Z respectively.

Let CD meet EF in H, and DE meet FA in G.

Then since $\quad A\ (BDEF) = C\ (BDEF),$

$$\therefore\ (XDEG) = (YHEF).$$

These homographic ranges $XDEG$ and $YHEF$ have a common corresponding point E.

$$\therefore\ XY,\ DH \text{ and } FG \text{ are concurrent } (\S\ 60),$$

that is, Z, the intersection of DH and FG, lies on XY.

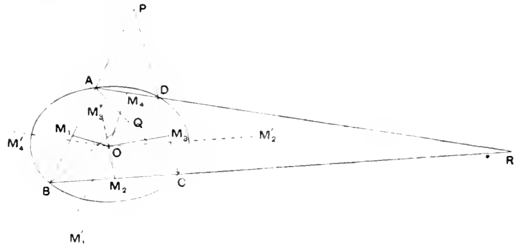
Thus the proposition is proved.

The student should satisfy himself that there are sixty different hexagons that can be formed with the six given vertices.

217. Brianchon's theorem. *If a conic be inscribed in a hexagon the lines joining opposite vertices are concurrent.*

This can be proved after a similar method to that of § 216, and may be left as an exercise to the student. We shall content ourselves with deriving this theorem from Pascal's by Reciprocation. To the principles of this important development of modern Geometry we shall come in the chapter immediately following this.

218. Prop. *The locus of the centres of conics through four fixed points is a conic.*



Let O be the centre of one of the conics passing through the four points A, B, C, D.

Let M_1, M_2, M_3, M_4 be the middle points of AB, BC, CD, DA respectively.

Draw OM_1', OM_2', OM_3', OM_4' parallel to AB, BC, CD, DA respectively.

Then OM_1, OM_1'; OM_2, OM_2'; OM_3, OM_3'; OM_4, OM_4' are pairs of conjugate diameters.

Therefore they form an involution pencil.

$$\therefore\ O(M_1M_2M_3M_4) = O(M_1'M_2'M_3'M_4').$$

But the right-hand side is constant since OM_1', OM_2' &c. are in fixed directions.

$$\therefore\ O(M_1M_2M_3M_4) \text{ is constant.}$$

$\therefore$ the locus of O is a conic through M_1, M_2, M_3, M_4.

Cor. 1. The conic on which O lies passes through M_5, M_6 the middle points of the other two sides of the quadrangle.

For if O_1, O_2, O_3, O_4, O_5 be five positions of O, these five points lie on a conic through M_1, M_2, M_3, M_4 and also on a conic through M_1, M_2, M_5, M_6.

But only one conic can be drawn through five points.

Therefore M_1, M_2, M_3, M_4, M_5, M_6 all lie on one conic, which is the locus of O.

Cor. 2. The locus of O also passes through P, Q, R the diagonal points of the quadrangle.

For one of the conics through the four points is the pair of lines AB, CD; and the centre of this conic is P.

So for Q and R.

219. Prop. *If $O\{AA', BB', CC'\}$ be an involution pencil and if a conic be drawn through O to cut the rays in A, A', B, B', C, C', then the chords AA', BB', CC' are concurrent.*

Let AA' and BB' intersect in P.

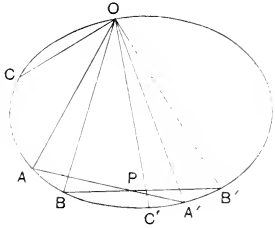
Project the conic into a circle with the projection of P for its centre.

Using small letters in the projection, we see that aoa', bob' are right angles, being in a semicircle.

Hence they determine an orthogonal involution.

$\therefore$ coc' is a right angle; that is, cc' goes through p.

$\therefore$ CC' goes through P.



It will be understood that the points AA', BB', CC' when joined to any other point on the conic give an involution pencil; for this follows at once by the application of § 212.

A system of points such as these on a conic is called *an involution range on the conic.*

The point P where the corresponding chords intersect is called *the pole of the involution.*

EXERCISES

1. If (P, P'), (Q, Q') be pairs of harmonic points on a conic (see Note on § 212), prove that the tangent at P and PP' are harmonic conjugates to PQ and PQ'. Hence shew that if PP' be normal at P, PQ and PQ' make equal angles with PP'.

2. The straight line PP' cuts a conic at P and P' and is normal at P. The straight lines PQ and PQ' are equally inclined to PP' and cut the conic again in Q and Q'. Prove that $P'Q$ and $P'Q'$ are harmonic conjugates to $P'P$ and the tangent at P'.

3. Shew that if the pencil formed by joining any point on a conic to four fixed points on the same be harmonic, two sides of the quadrangle formed by the four fixed points are conjugate to each other with respect to the conic.

4. The tangent at any point P of a hyperbola intersects the asymptotes in M_1 and M_2 and the tangents at the vertices in L_1 and L_2, prove that

$$PM_1^2 = PL_1 \, . \, PL_2.$$

5. Deduce from Pascal's theorem that if a conic pass through the vertices of a triangle the tangents at these points meet the opposite sides in collinear points.

[Take a hexagon $AA'BB'CC'$ in the conic so that A', B', C' are near to A, B, C.]

6. Given three points of a hyperbola and the directions of both asymptotes, find the point of intersection of the curve with a given straight line drawn parallel to one of the asymptotes.

7. Through a fixed point on a conic a line is drawn cutting the conic again in P, and the sides of a given inscribed triangle in A', B', C'. Shew that $(PA'B'C')$ is constant.

8. A, B, C, D are any four points on a hyperbola; CK parallel to one asymptote meets AD in K, and DL parallel to the other asymptote meets CB in L. Prove that KL is parallel to AB.

9. The sixty Pascal lines corresponding to six points on a conic intersect three by three.

10. Any two points D and E are taken on a hyperbola of which the asymptotes are CA and CB; the parallels to CA and CB through D and E respectively meet in Q; the tangent at D meets CB in R, and the tangent at E meets CA in T. Prove that T, Q, R are collinear, lying on a line parallel to DE.

11. The lines CA and CB are tangents to a conic at A and B, and D and E are two other points on the conic. The line CD cuts AB in G, AE in H, and BE in K. Prove that

$$CD^2 : GD^2 = CH \, . \, CK : GH \, . \, GK.$$

12. Through a fixed point A on a conic two fixed straight lines AI, AI' are drawn, S and S' are two fixed points and P a variable point on the conic; PS, PS' meet AI, AI' in Q, Q' respectively, shew that QQ' passes through a fixed point.

13. If two triangles be in perspective, the six points of intersection of their non-corresponding sides lie on a conic, and the axis of perspective is one of the Pascal lines of the six points.

14. If two chords PQ, PQ' of a conic through a fixed point P are equally inclined to the tangent at P, the chord QQ' passes through a fixed point.

15. If the lines AB, BC, CD, DA touch a conic at P, Q, R, S respectively, shew that conics can be inscribed in the hexagons $APQCRS$ and $BQRDSP$.

16. The tangent at P to an ellipse meets the auxiliary circle in Y and Y'. $ASS'A'$ is the major axis and SY, $S'Y'$ the perpendiculars from the foci. Prove that the points A, Y, Y', A' subtend at any point on the circle a pencil whose cross-ratio is independent of the position of P.

17. If A, B be given points on a circle, and CD be a given diameter, shew how to find a point P on the circle such that PA and PB shall cut CD in points equidistant from the centre.

CHAPTER XVII

RECIPROCATION

220. If we have a number of points P, Q, R, &c. in a plane and take the polars p, q, r, &c. of these points with respect to a conic Γ in the plane, then the line joining any two of the points P and Q is, as we have already seen, the polar with respect to Γ of the intersection of the corresponding lines p and q.

It will be convenient to represent the intersection of the lines p and q by the symbol (pq), and the line joining the points P and Q by (PQ).

The point P corresponds with the line p, in the sense that P is the pole of p, and the line (PQ) corresponds with the point (pq) in the sense that (PQ) is the polar of (pq).

Thus if we have a figure F consisting of an aggregate of points and lines, then, corresponding to it, we have a figure F' consisting of lines and points. Two such figures F and F' are called in relation to one another *Reciprocal figures.* The medium of their Reciprocity is the conic Γ.

Using § 215 we see that a range of points in F corresponds to a pencil of lines, homographic with the range, in F'.

221. By means of the principle of correspondence enunciated in the last paragraph we are able from a known property of a figure consisting of points and lines to infer another property of a figure consisting of lines and points.

The one property is called the *Reciprocal* of the other, and the process of passing from the one to the other is known as *Reciprocation*.

We will now give examples.

222. We know that if the vertices of two triangles ABC, $A'B'C'$ be in perspective, the pairs of corresponding sides (BC) $(B'C')$, (CA) $(C'A')$, (AB) $(A'B')$ intersect in collinear points X, Y, Z.

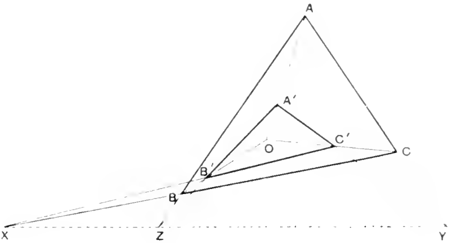


Fig. F.

Now if we draw the reciprocal figure, corresponding to the vertices of the triangle ABC, we have three lines a, b, c forming a triangle whose vertices will be (bc), (ca), (ab). And similarly for $A'B'C'$.

Corresponding to the concurrency of (AA'), (BB'), (CC') in the figure F, we have the collinearity of (aa'), (bb'), (cc') in the figure F'.

Corresponding to the collinearity of the intersections of (BC) $(B'C')$, (CA) $(C'A')$, (AB) $(A'B')$ in figure F, we have the concurrency of the lines formed by joining the pairs of points (bc) $(b'c')$, (ca) $(c'a')$, (ab) $(a'b')$ in the figure F'.

Thus the theorem of the figure F reciprocates into the following:

If two triangles whose sides are abc, $a'b'c'$ respectively be such that the three intersections of the corresponding sides are collinear, then the lines joining corresponding vertices, viz. (ab) and $(a'b')$, (bc) and $(b'c')$, (ca) and $(c'a')$, are concurrent.

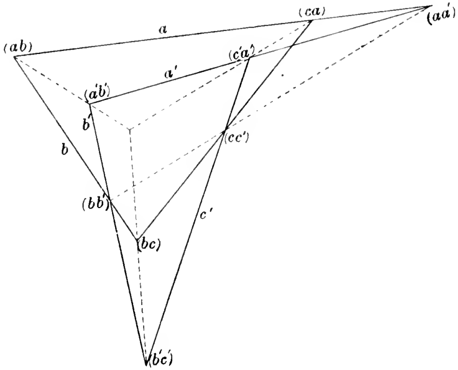


Fig. F'.

The two reciprocal theorems placed side by side may be stated thus:

Triangles in perspective are coaxal.	*Coaxal triangles are in perspective.*

The student will of course have realised that a triangle regarded as three lines does not reciprocate into another triangle regarded as three lines, but into one regarded as three points; and *vice versa*.

223. Let us now connect together by reciprocation the harmonic property of the quadrilateral and that of the quadrangle.

Let a, b, c, d be the lines of the quadrilateral; A, B, C, D the corresponding points of the quadrangle.

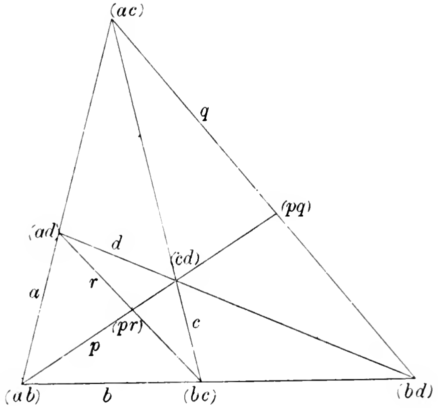


Fig. F.

Let the line joining (ab) and (cd) be p,
" " " (ac) and (bd) be q,
" " " (ad) and (bc) be r.

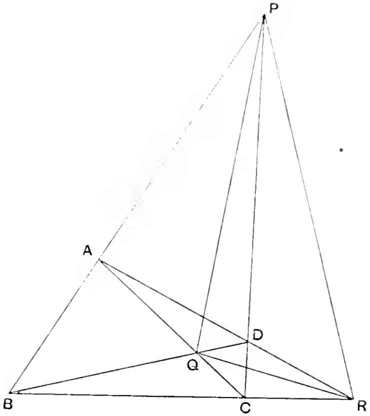


Fig. F′.

The harmonic property of the quadrilateral is expressed symbolically thus:

$$\{(ab)(cd),\ (pr)(pq)\} = -1,$$
$$\{(ad)(bc),\ (pr)(qr)\} = -1,$$
$$\{(ac)(bd),\ (pq)(qr)\} = -1.$$

The reciprocation gives

$$\{(AB)(CD),\ (PR)(PQ)\} = -1,$$
$$\{(AD)(BC),\ (PR)(QR)\} = -1,$$
$$\{(AC)(BD),\ (PQ)(QR)\} = -1.$$

If these be interpreted on the figure we have the harmonic property of the quadrangle, viz. that the two sides of the diagonal triangle at each vertex are harmonic conjugates with the two sides of the quadrangle which pass through that vertex.

The student sees now that the 'diagonal points' of a quadrangle are the reciprocals of the diagonal lines of the quadrilateral from which it is derived. Hence the term 'diagonal points.'

224. Prop. *An involution range reciprocates with respect to a conic into an involution pencil.*

For let the involution range be

$$A,\ A_1;\ B,\ B_1;\ C,\ C_1 \text{ \&c.}$$

on a line p.

The pencil obtained by reciprocation will be a, a_1; b, b_1; c, c_1 &c. through a point P.

Also $\qquad (abca_1) = (ABCA_1)$

and $\qquad (a_1b_1c_1a) = (A_1B_1C_1A)$ by § 215.

But $\qquad (ABCA_1) = (A_1B_1C_1A)$ by § 78.

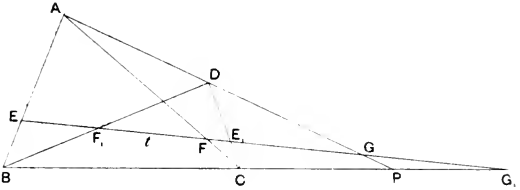
$$\therefore\ (abca_1) = (a_1b_1c_1a).$$

Thus the pencil is in involution.

225. Involution property of the quadrangle and quadrilateral.

Prop. *Any transversal cuts the pairs of opposite sides of a quadrangle in pairs of points which are in involution.*

Let $ABCD$ be the quadrangle (§ 76).

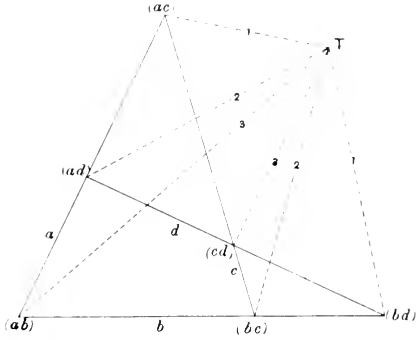


Let a transversal t cut

the opposite pairs of sides AB, CD in E, E_1,
" " " " AC, BD in F, F_1,
" " " " AD, BC in G, G_1.

Let AD and BC meet in P.

Then
$$\begin{aligned}(GEFG_1) &= A\,(GEFG_1)\\ &= (PBCG_1)\\ &= D\,(PBCG_1)\\ &= (GF_1E_1G_1)\\ &= (G_1E_1F_1G)\end{aligned}$$
by interchanging the letters in pairs.



Hence E, E_1: F, F_1: G, G_1 belong to the same involution.

We have only to reciprocate the above theorem to obtain this other:

The lines joining any point to the pairs of opposite vertices of a complete quadrilateral form a pencil in involution.

Thus in our figure T, which corresponds to the transversal t, joined to the opposite pairs of vertices (ac), (bd); (ad), (bc); (ab), (cd) gives an involution pencil.

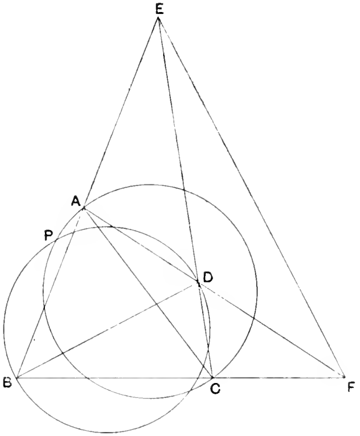
226. Prop. *The circles described on the three diagonals of a complete quadrilateral are coaxal.*

Let AB, BC, CD, DA be the four sides of the quadrilateral.

The diagonals are AC, BD, EF.

Let P be a point of intersection of the circles on AC and BD as diameters.

∴ APC and BPD are right angles.



But PA, PC; PB, PD; PE, PF are in involution (§ 225).

∴ by § 86 $\angle EPF$ is a right angle.

∴ the circle on EF as diameter goes through P.

Similarly the circle on EF goes through the other point of intersection of the circles on BD and AC.

That is, the three circles are coaxal.

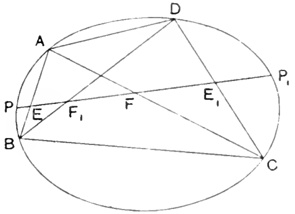
Cor. The middle points of the three diagonals of a quadrilateral are collinear.

This important and well-known property follows at once, since these middle points are the centres of three coaxal circles.

The line containing these middle points is sometimes called the *diameter of the quadrilateral.*

227. Desargues' theorem.

Conics through four given points are cut by any transversal in pairs of points belonging to the same involution.



Let a transversal t cut a conic through the four points A, B, C, D in P and P_1.

Let the same transversal cut the two pairs of opposite sides AB, CD; AC, BD of the quadrangle in E, E_1: F, F_1.

We now have

$$\begin{aligned}(PEFP_1) &= A\,(PEFP_1)\\ &= A\,(PBCP_1)\\ &= D\,(PBCP_1) \text{ by § 212}\\ &= (PF_1E_1P_1)\\ &= (P_1E_1F_1P) \text{ by interchanging the letters in pairs.}\end{aligned}$$

$\therefore$ P, P_1 belong to the involution determined by E, E_1; F, F_1.

Thus all the conics through $ABCD$ will cut the transversal t in pairs of points belonging to the same involution.

Note that the proposition of § 225 is only a special case of Desargues' theorem, if the two lines AD, BC be regarded as one of the conics through the four points.

228. As we shall presently see, the reciprocal of Desargues' theorem is the following:

If conics touch four given lines the pairs of tangents to them from any point in their plane belong to the same involution pencil, namely that determined by the lines joining the point to the pairs of opposite vertices of the quadrilateral formed by the four lines.

Reciprocation applied to conics.

229. We are now going on to explain how the principle of Reciprocation is applied to conics.

Suppose the point P describes a curve S in the plane of the conic Γ, the line p, which is the polar of P with regard to Γ, will envelope some curve which we will denote by S'.

Tangents to S' then correspond to points on S; but we must observe further that tangents to S correspond to points on S'.

For let P and P' be two near points on S, and let p and p' be the corresponding lines.

Then the point (pp') corresponds to the line (PP')

Now as P' moves up to P, (PP') becomes the tangent to S at P and at the same time (pp') becomes the point of contact of p with its envelope.

Hence to tangents of S correspond points on S'.

Each of the curves S and S' is called the polar reciprocal of the other with respect to the conic Γ.

230. Prop. *If S be a conic then S' is another conic.*

Let A, B, C, D be four fixed points on S, and P any other point on S.

Then $P\,(ABCD)$ is constant.

But $P(ABCD) = \{(pa)(pb)(pc)(pd)\}$ by § 215.

∴ $\{(pa)(pb)(pc)(pd)\}$ is constant.

∴ the envelope of p is a conic touching the lines a, b, c, d (§ 214).

Hence S' is a conic.

This important proposition might have been proved as follows.

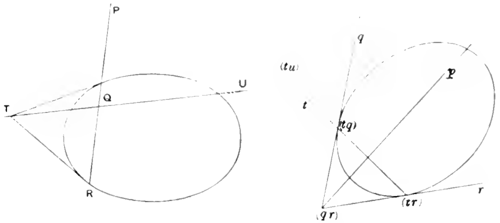
S, being a conic, is a curve of the second *order*, that is, straight lines in its plane cut it in two and only two points, real or imaginary.

Therefore S' must be a curve of the second *class*, that is a curve such that from each point in its plane two and only two tangents can be drawn to it; that is, S' is a conic.

231. Prop. *If S and S' be two conics reciprocal to each other with respect to a conic Γ, then pole and polar of S correspond to polar and pole of S' and vice versa.*

Let P and TU be pole and polar of S.

[It is most important that the student should understand that TU is the polar of P with respect to S, not to Γ. The polar of P with respect to Γ is the line we denote by p.]



Let QR be any chord of S which passes through P; then the tangents at Q and R meet in the line TU, at T say.

Therefore in the reciprocal figure p and (tu) are so related that if any point (qr) be taken on p, the chord of contact t of tangents from it to S' passes through (tu).

∴ p and (tu) are polar and pole with respect to S'.

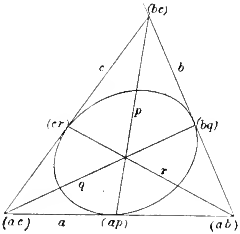
COR. 1. Conjugate points of S reciprocate into conjugate lines of S' and *vice versa.*

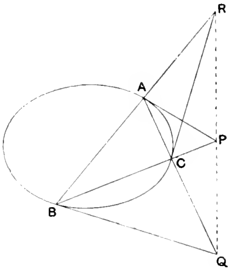
COR. 2. A self-conjugate triangle of S will reciprocate into a self-conjugate triangle of S'.

232. We will now set forth some reciprocal theorems in parallel columns.

1. If a conic be inscribed in a triangle (i.e. a three-side figure), the joining lines of the vertices of the triangle and the points of contact of the conic with the opposite sides are concurrent.

If a conic be circumscribed to a triangle (a three-point figure), the intersections of the sides of the triangle with the tangents at the opposite vertices are collinear.





2. The six points of intersection with the sides of a triangle of the lines joining the opposite vertices to two fixed points lie on a conic.

The six lines joining the vertices of a triangle to the points of intersection of the opposite sides and two fixed lines envelope a conic.

3. The three points of intersection of the opposite sides of each of the six-side figures formed by joining six points on a conic are collinear.—*Pascal's theorem.*

The three lines joining the opposite vertices of each of the six-point figures formed by the intersections of lines touching a conic are concurrent. — *Brianchon's theorem.*

4. If a conic circumscribe a quadrangle, the triangle formed by its diagonal points is self-conjugate for the conic.

If a conic be inscribed in a quadrilateral, the triangle formed by its diagonals is self-conjugate for the conic.

233. Prop. *The conic S' is an ellipse, parabola or hyperbola, according as the centre of Γ is within, on, or without S.*

For the centre of Γ reciprocates into the line at infinity, and lines through the centre of Γ into points on the line at infinity.

Hence tangents to S from the centre of Γ will reciprocate into points at infinity on S', and the points of contact of these tangents to S will reciprocate into the asymptotes of S'.

Hence if the centre of Γ be outside S, S' has two real asymptotes and therefore is a hyperbola.

If the centre of Γ be on S, S' has one asymptote, viz. the line at infinity, that is, S' is a parabola.

If the centre of Γ be within S, S' has no real asymptote and is therefore an ellipse.

234. Case where Γ is a circle.

If the auxiliary or base conic Γ be a circle (in which case we will denote it by C and its centre by O) a further relation exists between the two figures F and F' which does not otherwise obtain.

The polar of a point P with respect to C being perpendicular to OP, we see that all the lines of the figure F or F' are perpendicular to the lines joining O to the corresponding points of the figure F' or F.

And thus the angle between any two lines in the one figure is equal to the angle subtended at O by the line joining the corresponding points in the other.

In particular it may be noticed that if the tangents from O to S are at right angles, then S' is a rectangular hyperbola.

For if OP and OQ are the tangents to S, the asymptotes of S' are the polars of P and Q with respect to C, and these are at right angles since POQ is a right angle.

If then a parabola be reciprocated with respect to a circle whose centre is on the directrix, or a central conic be reciprocated with respect to a circle with its centre on the director circle, a rectangular hyperbola is always obtained.

Further let it be noticed that a triangle whose orthocentre is at O will reciprocate into another triangle also having its orthocentre at O. This the student can easily verify for himself.

235. It can now be seen that the two following propositions are connected by reciprocation:

1. *The orthocentre of a triangle circumscribing a parabola lies on the directrix.*

2. *The orthocentre of a triangle inscribed in a rectangular hyperbola lies on the curve.*

These two propositions have been proved independently (§§ 95, 130).

Let us now see how the second can be derived from the first by reciprocation.

Let the truth of (1) be assumed.

Reciprocate with respect to a circle C having its centre O at the orthocentre of the triangle.

Now the parabola touches the line at infinity, therefore the pole of the line at infinity with respect to C, viz. O, lies on the reciprocal curve.

And the reciprocal curve is a rectangular hyperbola because O is on the directrix of the parabola.

Further O is also the orthocentre of the reciprocal of the triangle circumscribing the parabola.

Thus we see that if a rectangular hyperbola be circumscribed to a triangle, the orthocentre lies on the curve.

It is also clear that *no conics but rectangular hyperbolas can pass through the vertices of a triangle and its orthocentre.*

236. Prop. *If S be a circle and we reciprocate with respect to a circle C whose centre is O, S' will be a conic having O for a focus.*

Let A be the centre of S.

Let p be any tangent to S, Q its point of contact.

Let P be the pole of p, and a the polar of A with respect to C.

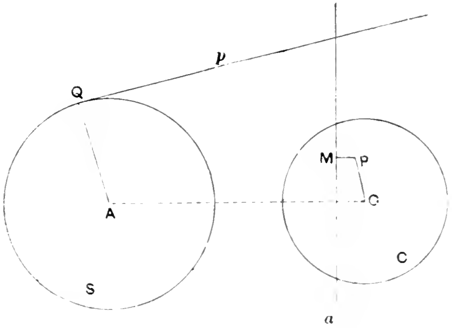
Draw PM perpendicular to a.

Then since AQ is perpendicular to p, we have by Salmon's theorem (§ 17)

$$\frac{OP}{OA} = \frac{PM}{AQ}.$$

$$\therefore \frac{OP}{PM} = \text{the constant } \frac{OA}{AQ}.$$

Thus the locus of P which is a point on the reciprocal curve is a conic whose focus is O, and corresponding directrix the polar of the centre of S.



Since the eccentricity of S' is $\frac{OA}{AQ}$ we see that S' is an ellipse, parabola, or hyperbola according as O is within, on, or without S. This is in agreement with § 233.

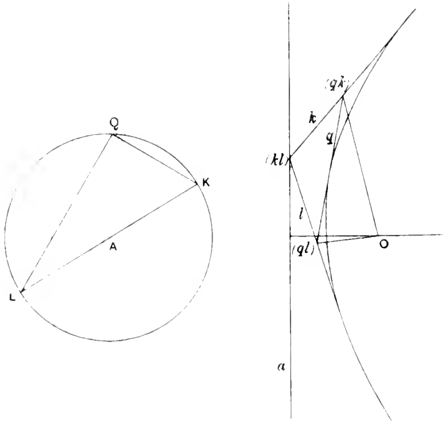
Cor. The polar reciprocal of a conic with respect to a circle having its centre at a focus of the conic is a circle, whose centre is the reciprocal of the corresponding directrix.

237. Let us now reciprocate with respect to a circle the theorem that the angle in a semicircle is a right angle.

Let A be the centre of S, KL any diameter, Q any point on the circumference.

In the reciprocal figure we have corresponding to A the directrix a, and a point (kl) on it corresponds to (KL).

k and l are tangents from (kl) to S' which correspond to K and L, and q is the tangent to S' corresponding to Q.



Now (QK) and (QL) are at right angles.

Therefore the line joining (qk) and (ql) subtends a right angle at O the focus of S'.

Hence the reciprocal theorem is that *the intercept on any tangent to a conic between two tangents which intersect in the directrix subtends a right angle at the focus.*

238. Prop. *A system of non-intersecting coaxal circles can be reciprocated into confocal conics.*

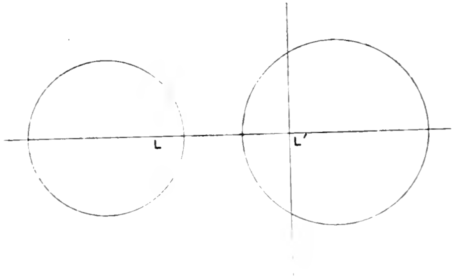
Let L and L' be the limiting points of the system of circles.

Reciprocate with respect to a circle C whose centre is at L.

Then all the circles will reciprocate into conics having L for one focus.

Moreover the centre of the reciprocal conic of any one of the circles is the reciprocal of the polar of L with respect to that circle.

But the polar of L for all the circles is the same, viz. the line through L' perpendicular to the line of centres (§ 22).



Therefore all the reciprocal conics have a common centre as well as a common focus.

Therefore they all have a second common focus, that is, they are confocal.

239. We know that if t be a common tangent to two circles of the coaxal system touching them at P and Q, PQ subtends a right angle at L.

Now reciprocate this with regard to a circle with its centre L. The two circles of the system reciprocate into confocal conics, the common tangent t reciprocates into a common point of the confocals, and the points P and Q into the tangents to the confocals at the common point.

Hence confocal conics cut at right angles.

This fact is of course known and easily proved otherwise. We are here merely illustrating the principles of reciprocation.

240. Again it is known (see Ex. 40 of Chap. XIII) that if S_1 and S_2 be two circles, L one of the limiting points, and P and Q points on S_1 and S_2 respectively such that PLQ is a right angle, the envelope of PQ is a conic having a focus at L.

Now reciprocate this property with respect to a circle having its centre at L. S_1 and S_2 reciprocate into confocals with

L as one focus; the points P and Q reciprocate into tangents to S_1' and S_2', viz. p and q, which will be at right angles; and the line (PQ) reciprocates into the point (pq).

As the envelope of (PQ) is a conic with a focus at L, it follows that the locus of (pq) is a circle.

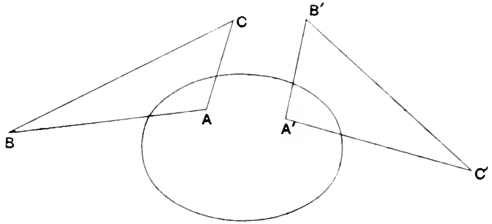
Hence we have the theorem:

If two tangents from a point T, one to each of two confocals, be at right angles, the locus of T is a circle.

This also is a well-known property of confocals.

241. We will conclude this chapter by proving two theorems, the one having reference to two triangles which are self-conjugate for a conic, the other to two triangles reciprocal for a conic.

Prop. *If two triangles be self-conjugate to the same conic their six vertices lie on a conic and their six sides touch a conic.*



Let ABC, $A'B'C'$ be the two triangles self-conjugate with respect to a conic S.

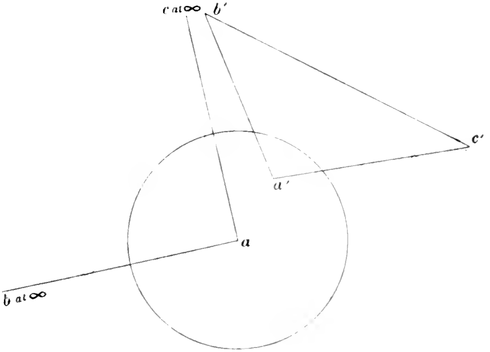
Project S into a circle with A projected into the centre; then (using small letters for the projections) ab, ac are conjugate diameters and are therefore at right angles, and b and c lie on the line at infinity.

Further $a'b'c'$ is a triangle self-conjugate for the circle.

$\therefore$ a the centre of the circle is the orthocentre of this triangle.

Let a conic be placed through the five points a', b', c', a and b.

This must be a rectangular hyperbola, since as we have seen no conics but rectangular hyperbolas can pass through the vertices of a triangle and its orthocentre.



∴ c also lies on the conic through the five points named above, since the line joining the two points at infinity on a rectangular hyperbola must subtend a right angle at any point.

Hence the six points a, b, c, a', b', c' all lie on a conic.

∴ the six points A, B, C, A', B', C' also lie on a conic.

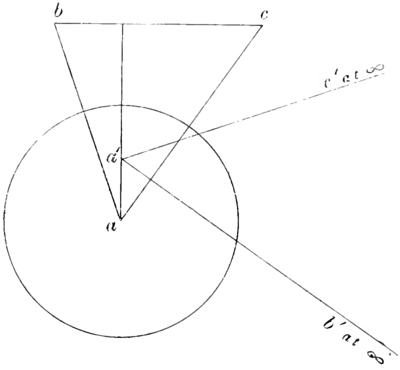
The second part of the proposition follows at once by reciprocating this which we have just proved.

242. Prop. *If two triangles are reciprocal for a conic, they are in perspective.*

Let ABC, $A'B'C'$ be two triangles which are reciprocal for the conic S; that is to say, A is the pole of $B'C'$, B the pole of $C'A'$, C the pole of $A'B'$; and consequently also A' is the pole of BC, B' of CA, and C' of AB.

Project S into a circle with the projection of A for its centre. ∴ B' and C' are projected to infinity.

Using small letters for the projection, we see that, since a' is the pole of bc, aa' is perpendicular to bc.



Also since b' is the pole of ac, ab' is perpendicular to ac; $\therefore$ bb' which is parallel to ab' is perpendicular to ac.

Similarly cc' is perpendicular to ab.

$\therefore$ aa', bb', cc' meet in the orthocentre of the triangle abc.

$\therefore$ AA', BB', CC' are concurrent.

EXERCISES

1. If the conics S and S' be reciprocal polars with respect to the conic Γ, the centre of S' corresponds to the polar of the centre of Γ with respect to S.

2. Parallel lines reciprocate into points collinear with the centre of the base conic Γ.

3. Shew that a quadrangle can be reciprocated into a parallelogram.

4. Reciprocate with respect to any conic the theorem: The locus of the poles of a given line with respect to conics passing through four fixed points is a conic.

Reciprocate with respect to a circle the theorems contained in Exx. 5—12 inclusive.

5. The perpendiculars from the vertices of a triangle on the opposite sides are concurrent.

6. The tangent to a circle is perpendicular to the radius through the point of contact.

7. Angles in the same segment of a circle are equal.

8. The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

9. The angle between the tangent at any point of a circle and a chord through that point is equal to the angle in the alternate segment of the circle.

10. The polar of a point with respect to a circle is perpendicular to the line joining the point to the centre of the circle.

11. The locus of the intersection of tangents to a circle which cut at a given angle is a concentric circle.

12. Chords of a circle which subtend a constant angle at the centre envelope a concentric circle.

13. Two conics having double contact will reciprocate into conics having double contact.

14. A circle S is reciprocated by means of a circle C into a conic S'. Prove that the radius of C is the geometric mean between the radius of S and the semi-latus rectum of S'.

15. Prove that with a given point as focus four conics can be drawn circumscribing a given triangle, and that the sum of the latera recta of three of them will equal the latus rectum of the fourth.

16. Conics have a focus and a pair of tangents common; prove that the corresponding directrices will pass through a fixed point, and all the centres lie on the same straight line.

17. Prove, by reciprocating with respect to a circle with its centre at S, the theorem: If a triangle ABC circumscribe a parabola

whose focus is S, the lines through A, B, C perpendicular respectively to SA, SB, SC are concurrent.

18. Conics are described with one of their foci at a fixed point S, so that each of the four tangents from two fixed points subtends the same angle of given magnitude at S. Prove that the directrices corresponding to the focus S pass through a fixed point.

19. If O be any point on the common tangent to two parabolas with a common focus, prove that the angle between the other tangents from O to the parabolas is equal to the angle between the axes of the parabolas.

20. A conic circumscribes the triangle ABC, and has one focus at O, the orthocentre; shew that the corresponding directrix is perpendicular to IO and meets it in a point X such that $IO \cdot OX = AO \cdot OD$, where I is the centre of the inscribed circle of the triangle, and D is the foot of the perpendicular from A on BC. Shew also how to find the centre of the conic.

21. Prove that chords of a conic which subtend a constant angle at a given point on the conic will envelope a conic.

[Reciprocate into a parabola by means of a circle having its centre at the fixed point.]

22. If a triangle be reciprocated with respect to a circle having its centre O on the circumcircle of the triangle, the point O will also lie on the circumcircle of the reciprocal triangle.

23. Prove the following and obtain from it by reciprocation a theorem applicable to coaxal circles: If from any point pairs of tangents p, p'; q, q', be drawn to two confocals S_1 and S_2, the angle between p and q is equal to the angle between p' and q'.

24. Prove and reciprocate with respect to any conic the following: If ABC be a triangle, and if the polars of A, B, C with respect to any conic meet the opposite sides in P, Q, R, then P, Q, R are collinear.

25. A fixed point O in the plane of a given circle is joined to the extremities A and B of any diameter, and OA, OB meet the circle again in P and Q. Shew that the tangents at P and Q intersect on a fixed line parallel to the polar of O.

26. All conics through four fixed points can be projected into rectangular hyperbolas.

27. If two triangles be reciprocal for a conic (§ 242) their centre of perspective is the pole of the axis of perspective with regard to the conic.

28. Prove that the envelope of chords of an ellipse which subtend a right angle at the centre is a concentric circle.

[Reciprocate with respect to a circle having its centre at the centre of the ellipse.]

29. ABC is a triangle, I its incentre; A_1, B_1, C_1 the points of contact of the incircle with the sides. Prove that the line joining I to the point of concurrency of AA_1, BB_1, CC_1 is perpendicular to the line of collinearity of the intersections of B_1C_1, BC; C_1A_1, CA; A_1B_1, AB.

[Use Ex. 27.]

CHAPTER XVIII

CIRCULAR POINTS. FOCI OF CONICS

243. We have seen that pairs of concurrent lines which are conjugate for a conic form an involution, of which the tangents from the point of concurrency are the double lines.

Thus conjugate diameters of a conic are in involution, and the double lines of the involution are its asymptotes.

Now the conjugate diameters of a circle are orthogonal.

Thus the asymptotes of a circle are the imaginary double lines of the orthogonal involution at its centre.

But clearly the double lines of the orthogonal involution at one point must be parallel to the double lines of the orthogonal involution at another, seeing that we may *by a motion of translation, without rotation*, move one into the position of the other.

Hence the asymptotes of one circle are, each to each, parallel to the asymptotes of any other circle in its plane.

Let a, b be the asymptotes of one circle C, a', b' of another C', then a, a' being parallel meet on the line at infinity, and b, b' being parallel meet on the line at infinity.

But a and a' meet C and C' on the line at infinity,
and b and b' „ C and C' „ „ „

Therefore C and C' go through the same two imaginary points on the line at infinity.

Our conclusion then is that all circles in a plane go through the same two imaginary points on the line at infinity. These two points are called *the circular points at infinity* or, simply, the *circular points.*

The *circular lines* at any point are the lines joining that point to the circular points at infinity; and they are the imaginary double lines of the orthogonal involution at that point.

244. Analytical point of view.

It may help the student to think of the circular lines at any point if we digress for a moment to touch upon the Analytical aspect of them.

The equation of a circle referred to its centre is of the form

$$x^2 + y^2 = a^2.$$

The asymptotes of this circle are

$$x^2 + y^2 = 0,$$

that is the pair of imaginary lines

$$y = ix \text{ and } y = -ix.$$

These two lines are the circular lines at the centre of the circle.

The points where they meet the line at infinity are the circular points.

If we rotate the axes of coordinates at the centre of a circle through any angle, keeping them still rectangular, the equation of the circle does not alter in form, so that the asymptotes will make angles $\tan^{-1}(i)$ and $\tan^{-1}(-i)$ with the new axis of x as well as with the old.

This at first sight is paradoxical. But the paradox is explained by the fact that the line $y = ix$ makes the same angle $\tan^{-1}(i)$ with every line in the plane.

For let $y = mx$ be any other line through the origin.

Then the angle that $y = ix$ makes with this, measured in the positive sense from $y = mx$, is

$$\tan^{-1}\left(\frac{i - m}{1 + im}\right) = \tan^{-1}\left\{\frac{i(1 + im)}{1 + im}\right\} = \tan^{-1} i.$$

245. Prop. *If AOB be an angle of constant magnitude and Ω, Ω' be the circular points, the cross-ratios of the pencil $O(\Omega, \Omega', A, B)$ are constant.*

For $$O(\Omega\Omega' AB) = \frac{\sin \Omega O\Omega' \sin AOB}{\sin \Omega OB \sin AO\Omega'},$$

but the angles $\Omega O\Omega'$, ΩOB, $AO\Omega'$ are all constant since the circular lines make the same angle with every line in the plane, and $\angle AOB$ is constant by hypothesis.

$$\therefore\ O(\Omega\Omega' AB) \text{ is constant.}$$

246. Prop. *All conics passing through the circular points are circles.*

Let C be the centre of a conic S passing through the circular points, which we will denote by Ω and Ω'.

Then $C\Omega$, $C\Omega'$ are the asymptotes of S.

But the asymptotes are the double lines of the involution formed by pairs of conjugate diameters.

And the double lines completely determine an involution, that is to say there can be only one involution with the same double lines.

Thus the conjugate diameters of S are all orthogonal.

Hence S is a circle.

The circular points may be utilised for establishing properties of conics passing through two or more fixed points.

For a system of conics all passing through the same two points can be projected into circles simultaneously.

This is effected by projecting the two points into the circular points on the plane of projection. The projections of the conics will now go through the circular points in the new plane and so they are all circles.

The student of course understands that such a projection is an imaginary one.

247. We will now proceed to an illustration of the use of the circular points.

It can be seen at once that any transversal is cut by a system of coaxal circles in pairs of points in involution (the centre of this involution being the point of intersection of the line with the axis of the system).

From this follows at once Desargues' theorem (§ 227), namely that conics through four points cut any transversal in pairs of points in involution.

For if we project two of the points into the circular points the conics all become circles. Moreover the circles form a coaxal system, for they have two other points in common.

Hence Desargues' theorem is seen to follow from the involution property of coaxal circles.

The involution property of coaxal circles again is a particular case of Desargues' theorem, for coaxal circles have four points in common, two being the circular points, and two the points in which all the circles are cut by the axis of the system.

248. We will now make use of the circular points to prove the theorem: *If a triangle be self-conjugate to a rectangular hyperbola its circumcircle passes through the centre of the hyperbola.*

Let O be the centre of the rectangular hyperbola, ABC the self-conjugate triangle, Ω, Ω' the circular points.

Now observe first that $O\Omega\Omega'$ is a self-conjugate triangle for the rectangular hyperbola. For $O\Omega$, $O\Omega'$ are the double lines of the orthogonal involution at O to which the asymptotes, being at right angles, belong. Therefore $O\Omega$, $O\Omega'$ belong to the involution whose double lines are the asymptotes (§ 82), that is the involution formed by pairs of conjugate lines through O.

$\therefore$ $O\Omega$, $O\Omega'$ are conjugate lines, and O is the pole of $\Omega\Omega'$ which is the line at infinity.

$\therefore$ $O\Omega\Omega'$ is a self-conjugate triangle.

Also ABC is a self-conjugate triangle.

$\therefore$ the six points A, B, C, O, Ω, Ω' all lie on a conic (§ 241); and this conic must be a circle as it passes through Ω and Ω'.

$\therefore$ A, B, C, O are concyclic.

Cor. If a rectangular hyperbola circumscribe a triangle, its centre lies on the nine-points circle.

This well-known theorem is a particular case of the above proposition, for the pedal triangle is self-conjugate for the rectangular hyperbola. (Ex. 21, Chapter XIV.)

249. Prop. *Concentric circles have double contact at infinity.*

For if O be the centre of the circles, Ω, Ω' the circular points at infinity, all the circles touch $O\Omega$ and $O\Omega'$ at the points Ω and Ω'.

That is, all the circles touch one another at the points Ω and Ω'.

250. Foci of Conics.

Prop. *Every conic has four foci, two of which lie on one axis of the conic and are real, and two on the other axis and are imaginary.*

Since conjugate lines at a focus form an orthogonal involution, and since the tangents from any point are the double lines of the involution formed by the conjugate lines there, it follows that the circular lines through a focus are the tangents to the conic from that point.

But the circular lines at any point go through Ω and Ω' the circular points.

Thus the foci of the conic will be obtained by drawing tangents from Ω and Ω' to the conic, and taking their four points of intersection.

Hence there are four foci.

To help the imagination, construct a figure as if Ω and Ω' were real points.

Draw tangents from these points to the conic and let S, S', F, F' be their points of intersection as in the figure; S, S' being opposite vertices as also F and F'.

Let FF' and SS' intersect in O.

Now the triangle formed by the diagonals FF', SS' and $\Omega\Omega'$ is self-conjugate for the conic, because it touches the sides of the quadrilateral. (Reciprocal of § 119 a.)

$\therefore$ O is the pole of $\Omega\Omega'$, i.e. of the line at infinity.

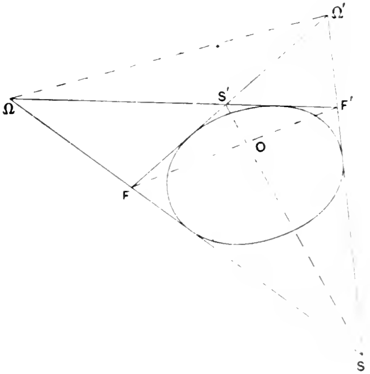
$\therefore$ O is the centre of the conic.

Further $O\Omega\Omega'$ is the diagonal, or harmonic, triangle of the quadrangle $SS'FF'$.

$$\therefore\ O\,(\Omega\Omega',\ FS) = -1. \quad (\S\ 76)$$

$\therefore$ OF and OS are conjugate lines in the involution of which $O\Omega$ and $O\Omega'$ are the double lines.

$\therefore$ OF and OS are at right angles.



And OF and OS are conjugate lines for the conic since the triangle formed by the diagonals FF', SS', $\Omega\Omega'$ is self-conjugate for the conic; and O is, as we have seen, the centre.

$\therefore$ OF and OS, being orthogonal conjugate diameters, are the axes.

Thus we have two pairs of foci, one on one axis and the other on the other axis.

Now we know that two of the foci, say S and S', are real.

It follows that the other two, F and F', are imaginary. For if F were real, the line FS would meet the line at infinity in a real point, which is not the case.

$\therefore$ F and F' must be imaginary.

COR. The lines joining non-corresponding foci are tangents to the conic and the points of contact of these tangents are concyclic.

251. Prop. *A system of conics touching the sides of a quadrilateral can be projected into confocal conics.*

Let $ABCD$ be the quadrilateral, the pairs of opposite vertices being A, C; B, D; E, F.

Project E and F into the circular points at infinity on the plane of projection.

∴ A, C and B, D project into the foci of the conics in the projection, by § 250.

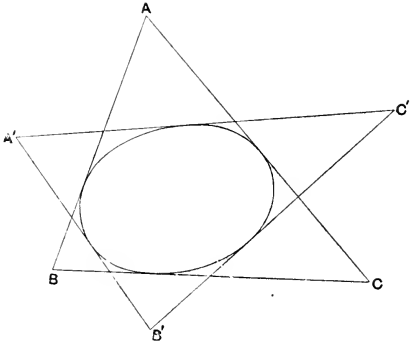
Cor. Confocal conics form a system of conics touching four lines.

252. We will now make use of the notions of this chapter to prove the following theorem, which is not unimportant.

If the sides of two triangles all touch the same conic, the six vertices of the triangles all lie on a conic.

Let ABC, $A'B'C'$ be the two triangles the sides of which all touch the same conic S.

Denote the circular points on the π plane or plane of projection by ω, ω'.



Project B and C into ω and ω'; ∴ S projects into a parabola, since the projection of S touches the line at infinity.

Further A will project into the focus of the parabola, since the tangents from the focus go through the circular points.

Using corresponding small letters in the projection, we see that, since the circumcircle of a triangle whose sides touch a parabola goes through the focus, a, a', b', c' are concyclic.

$$\therefore\ a, a', b', c', \omega, \omega' \text{ lie on a circle.}$$

$$\therefore\ A, B, C, A', B', C' \text{ lie on a conic.}$$

The converse of the above proposition follows at once by reciprocation.

253. We have in the preceding article obtained a proof of the general proposition that if the sides of two triangles touch a conic, their six vertices lie on another conic by the projection of what is a particular case of this proposition, viz. that the circumcircle of a triangle whose sides touch a parabola passes through the focus.

This process is known as *generalising by projection*. We will proceed to give further illustrations of it.

254. Let us denote the circular points in the p plane by Ω, Ω', and their projections on the π plane by ω, ω'. Then of course ω and ω' are not the circular points in the π plane. But by a proper choice of the π plane and the vertex of projection ω and ω' may be any two points we choose, real or imaginary. For if we wish to project Ω and Ω' into the points ω and ω' in space, we have only to take as our vertex of projection the point of intersection of the lines $\omega\Omega$ and $\omega'\Omega'$, and as the plane π some plane passing through ω and ω'.

255. The following are the principal properties connecting figures in the p and π planes when Ω and Ω' are projected into ω and ω':

1. Circles in the p plane project into conics through the points ω and ω' in the π plane.

2. Parabolas in the p plane project into conics touching the line $\omega\omega'$ in the π plane.

3. Rectangular hyperbolas in the p plane, for which, as we have seen, Ω and Ω' are conjugate points, project into conics having ω and ω' for conjugate points.

4. The centre of a conic in the p plane, since it is the pole of $\Omega\Omega'$, projects into the pole of the line $\omega\omega'$.

5. Concentric circles in the p plane project into conics having double contact at ω and ω' in the π plane.

6. A pair of lines OA, OB at right angles in the p plane project into a pair of lines oa, ob harmonically conjugate with $o\omega$, $o\omega'$. This follows from the fact that $O\Omega$, $O\Omega'$ are the double lines of the involution to which OA, OB belong, and therefore $O(AB, \Omega\Omega') = -1$ (§ 82); from which it follows that $o(ab, \omega\omega') = -1$.

7. A conic with S as focus in the p plane will project into a conic touching the lines $s\omega$, $s\omega'$ in the π plane.

And the two foci S and S' of a conic in the p plane will project into the vertices of the quadrilateral formed by drawing tangents from ω and ω' to the projection of the conic in the π plane.

256. It is of importance that the student should realise that ω and ω' are not the circular points in the π plane when they are the projections of Ω and Ω'.

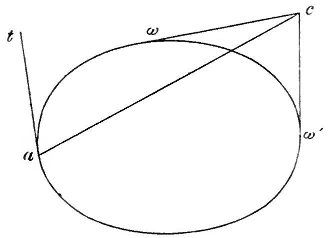
In § 252 we have denoted the circular points in the π plane by ω and ω', but they are not there the projections of the circular points in the p plane.

Our practice has been to use small letters to represent the projections of the corresponding capitals. So then we use ω and ω' for the projection of Ω and Ω' respectively. If Ω and Ω' are the circular points in the p plane, ω and ω' are not the circular points in the π plane; and if ω and ω' are the circular points in the π plane, Ω and Ω' are not the circular points in the p plane. That is to say, only one of the pairs can be circular points at the same time.

257. We will now proceed to some examples of generalisation by projection.

Consider the theorem that the radius of a circle to any point A is perpendicular to the tangent at A.

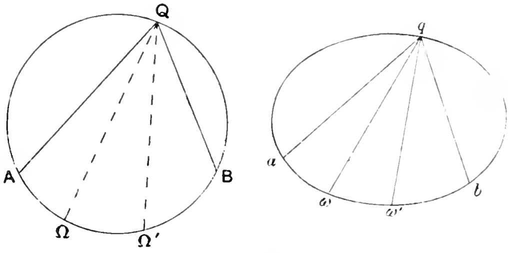
Project the circle into a conic through ω and ω'; the centre C of the circle projects into the pole of $\omega\omega'$.



The generalised theorem is that *if the tangents at two points ω, ω' of a conic meet in c, and a be any point on the conic and a t the tangent there*

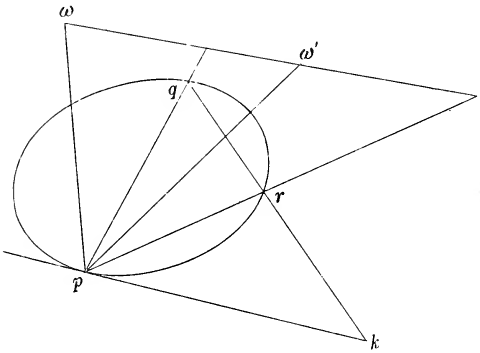
$$a\,(tc,\ \omega\omega') = -1.$$

258. Next consider the theorem that angles in the same segment of a circle are equal. Let AQB be an angle in the segment of which AB is the base. Project the circle into a conic through ω and ω' and we get the theorem that if q be any point on a fixed conic through the four points $a, b, \omega, \omega', q\ (ab\omega\omega')$ is constant (§ 245).



Thus *the property of the equality of angles in the same segment of a circle generalises into the constant cross-ratio property of conics.*

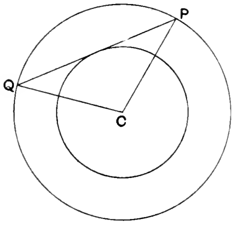
259. Again we have the property of the rectangular hyperbola that if PQR be a triangle inscribed in it and having a right angle at P, the tangent at P is at right angles to QR.

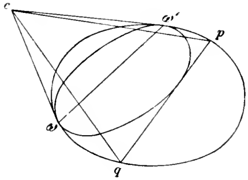


Project the rectangular hyperbola into a conic having ω and ω' for conjugate points and we get the following property.

If p be any point on a conic for which ω and ω' are conjugate points and q, r two other points on the conic such that $p(qr, \omega\omega') = -1$ and if the tangent at p meet qr in k then $k(pq, \omega\omega') = -1$.

260. Lastly we will generalise by projection the theorem that chords of a circle which touch a concentric circle subtend a constant angle at the centre.





Let PQ be a chord of the outer circle touching the inner and subtending a constant angle at C the centre.

The concentric circles have double contact at the circular points Ω and Ω' and so project into two conics having double contact at ω and ω'.

The centre C is the pole of $\Omega\Omega'$ and so c, the projection of C, is the pole of $\omega\omega'$.

The property we obtain by projection is then:

If two conics have double contact at two points ω and ω' and if the tangents at these points meet in c, and if pq be any chord of the outer conic touching the inner conic, then $c(pq\omega\omega')$ is constant.

EXERCISES

1. If O be the centre of a conic, Ω, Ω' the circular points at infinity, and if $O\Omega\Omega'$ be a self-conjugate triangle for the conic, the conic must be a rectangular hyperbola.

2. If a variable conic pass through two given points P and P', and touch two given straight lines, shew that the chord which joins the points of contact of these two straight lines will always meet PP' in a fixed point.

3. If three conics have two points in common, the opposite common chords of the conics taken in pairs are concurrent.

4. Two conics S_1 and S_2 circumscribe the quadrangle $ABCD$. Through A and B lines AEF, BGH are drawn cutting S_2 in E and G, and S_1 in F and H. Prove that CD, EG, FH are concurrent.

5. If a conic pass through two given points, and touch a given conic at a given point, its chord of intersection with the given conic passes through a fixed point.

6. If Ω, Ω' be the circular points at infinity, the two imaginary foci of a parabola coincide with Ω and Ω', and the centre and second real focus of the parabola coincide with the point of contact of $\Omega\Omega'$ with the parabola.

7. If a conic be drawn through the four points of intersection of two given conics, and through the intersection of one pair of common tangents, it also passes through the intersection of the other pair of common tangents.

8. Prove that, if three conics pass through the same four points, a common tangent to any two of the conics is cut harmonically by the third.

9. Reciprocate the theorem of Ex. 8.

10. If from two points P, P' tangents be drawn to a conic, the four points of contact of the tangents with the conic, and the points P and P' all lie on a conic.

[Project P and P' into the circular points.]

11. If out of four pairs of points every combination of three pairs gives six points on a conic, either the four conics thus determined coincide or the four lines determined by the four pairs of points are concurrent.

12. Generalise by projection the theorem that the locus of the centre of a rectangular hyperbola circumscribing a triangle is the nine-points circle of the triangle.

13. Generalise by projection the theorem that the locus of the centre of a rectangular hyperbola with respect to which a given triangle is self-conjugate is the circumcircle.

14. Given that two lines at right angles and the lines to the circular points form a harmonic pencil, find the reciprocals of the circular points with regard to any circle.

Deduce that the polar reciprocal of any circle with regard to any point O has the lines from O to the circular points as tangents, and the reciprocal of the centre of the circle for the corresponding chord of contact.

15. Prove and generalise by projection the following theorem: The centre of the circle circumscribing a triangle which is self-conjugate with regard to a parabola lies on the directrix.

16. P and P' are two points in the plane of a triangle ABC. D is taken in BC such that BC and DA are harmonically conjugate with DP and DP'; E and F are similarly taken in CA and AB respectively. Prove that AD, BE, CF are concurrent.

17. Generalise by projection the following theorem: The lines perpendicular to the sides of a triangle through the middle points of the sides are concurrent in the circumcentre of the triangle.

18. Generalise : The feet of the perpendiculars on to the sides of a triangle from any point on the circumcircle are collinear.

19. If two conics have double contact at A and B, and if PQ a chord of one of them touch the other in R and meet AB in T, then

$$(PQ, RT) = -1.$$

20. Generalise by projection the theorem that confocal conics cut at right angles.

21. Prove and generalise that the envelope of the polar of a given point for a system of confocals is a parabola touching the axes of the confocals and having the given point on its directrix.

22. If a system of conics pass through the four points A, B, C, D, the poles of the line AB with respect to them will lie on a line l. Moreover if this line l meet CD in P, PA and PB are harmonic conjugates of CD and l.

23. A pair of tangents from a fixed point T to a conic meet a third fixed tangent to the conic in L and L'. P is any point on the conic, and on the tangent at P a point X is taken such that $X(PT, LL') = -1$; prove that the locus of X is a straight line.

24. Defining a focus of a conic as a point at which each pair of conjugate lines is orthogonal, prove that the polar reciprocal of a circle with respect to another circle is a conic having the centre of the second circle for a focus.

CHAPTER XIX

INVERSION

261. We have already in § 13 explained what is meant by two 'inverse points' with respect to a circle. O being the centre of a circle, P and P' are inverse points if they lie on the same radius and $OP.OP'$ = the square of the radius. P and P' are on the same side of the centre, unless the circle have an imaginary radius, $= ik$, where k is real.

As P describes a curve S, the point P' will describe another curve S'. S and S' are called *inverse curves.* O is called the *centre of inversion*, and the radius of the circle is called the *radius of inversion.*

If P describe a curve in space, not necessarily a plane curve, then we must consider P' as the inverse of P with respect to a *sphere* round O. That is, whether P be confined to a plane or not, if O be a fixed point in space and P' be taken on OP such that $OP.OP' =$ a constant k^2, P' is called the inverse of P, and the curve or surface described by P is called the inverse of that described by P', and *vice versa.*

It is convenient sometimes to speak of a point P' as inverse to another point P *with respect to a point O.* By this is meant that O is the centre of the circle or sphere with respect to which the points are inverse.

262. Prop. *The inverse of a circle with respect to a point in its plane is a circle or straight line.*

First let O, the centre of inversion, lie on the circle.

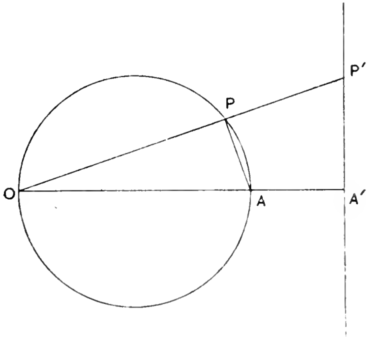
Let k be the radius of inversion.

Draw the diameter OA, let A' be the inverse of A.

Let P be any point on the circle, P' its inverse.

Then $$OP.OP' = k^2 = OA \,.\, OA'.$$

$\therefore$ $PAA'P'$ is cyclic.



$\therefore$ the angle $AA'P'$ is the supplement of APP', which is a right angle.

$\therefore$ $A'P'$ is at right angles to AA'.

$\therefore$ the locus of P' is a straight line perpendicular to the diameter OA, and passing through the inverse of A.

Next let O not be on the circumference of the circle.

Let P be any point on the circle, P' its inverse.

Let OP cut the circle again in Q.

Let A be the centre of the circle.

Then $OP.OP' = k^2$,

and $OP.OQ =$ sq. of tangent from O to the circle $= t^2$ (say).

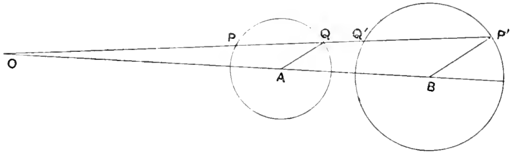
$$\therefore \frac{OP'}{OQ} = \frac{k^2}{t^2}.$$

Take B on OA such that $\frac{OB}{OA} = \frac{k^2}{t^2}$. $\therefore$ B is a fixed point and BP' is parallel to AQ.

And $$\frac{BP'}{AQ} = \frac{OB}{OA} = \frac{k^2}{t^2}, \text{ a constant.}$$

$\therefore$ P' describes a circle round B.

Thus the inverse of the circle is another circle.



Cor. 1. The inverse of a straight line is a circle passing through the centre of inversion.

Cor. 2. If two circles be inverse each to the other, the centre of inversion is a centre of similitude (§ 25); and the radii of the circles are to one another in the ratio of the distances of their centres from O.

The student should observe that, if we call the two circles S and S', and if OPQ meet S' again in Q', Q' will be the inverse of Q.

Note. The part of the circle S which is convex to O corresponds to the part of the circle S' which is concave to O, and *vice versa*.

Two of the common tangents of S and S' go through O, and the points of contact with the circles of each of these tangents will be inverse points.

263. Prop. *The inverse of a sphere with respect to any point is a sphere or a plane.*

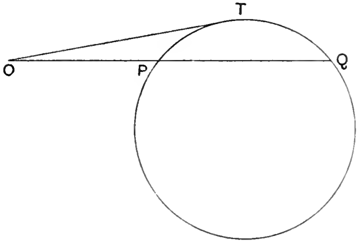
This proposition follows at once from the last by rotating the figures round OA as axis; in the first figure the circle and line will generate a sphere and plane each of which is the inverse of the other; and in the second figure the two circles will generate spheres each of which will be the inverse of the other.

264. Prop. *The inverse of a circle with respect to a point O,* **not in its plane,** *is a circle.*

For the circle may be regarded as the intersection of two spheres, neither of which need pass through O.

These spheres will invert into spheres, and their intersection, which is the inverse of the intersection of the other two spheres, that is of the original circle, will be a circle.

265. Prop. *A circle will invert into itself with respect to a point O in its plane if the radius of inversion be the length of the tangent to the circle from the centre of inversion.*



This is obvious at once, for if OT be the tangent from O and OPQ cut the circle in P and Q, since $OP \cdot OQ = OT^2$ it follows that P and Q are inverse points.

That is, the part of the circle concave to O inverts into the part which is convex and *vice versa*.

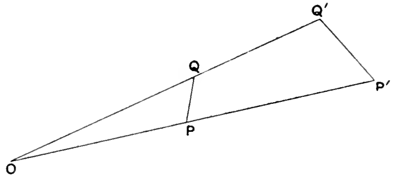
Cor. 1. Any system of coaxal circles can be simultaneously inverted into themselves if the centre of inversion be any point on the axis of the system.

Cor. 2. Any three coplanar circles can be simultaneously inverted into themselves.

For we have only to take the radical centre of the three circles as the centre of inversion, and the tangent from it as the radius.

266. Prop. *Two coplanar curves cut at the same angle as their inverses with respect to any point in their plane.*

Let P and Q be two near points on a curve S, P' and Q' their inverses with respect to O.



Then since $OP \cdot OP' = k^2 = OQ \cdot OQ'$.

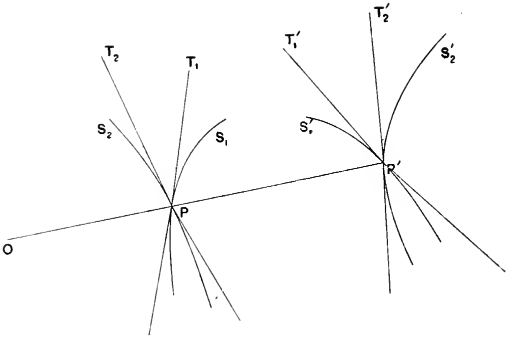
$\therefore$ $QPP'Q'$ is cyclic.

$\therefore$ $\angle OPQ = \angle OQ'P'$.

Now let Q move up to P so that PQ becomes the tangent to S at P; then Q' moves up at the same time to P' and $P'Q'$ becomes the tangent at P' to the inverse curve S'.

$\therefore$ the tangents at P and P' make equal angles with OPP'.

The tangents however are antiparallel, not parallel.



Now if we have two curves S_1 and S_2 intersecting at P, and PT_1, PT_2 be their tangents there, and if the inverse curves be

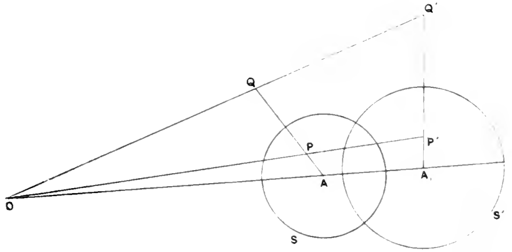
S_1', S_2' intersecting at P', the inverse of P, and $P'T_1'$, $P'T_2'$ be their tangents, it follows at once from the above reasoning that $\angle T_2PT_1 = \angle T_1'P'T_2'$.

Thus S_1 and S_2 intersect at the same angle as their inverses.

Cor. If two curves touch at a point P their inverses touch at the inverse of P.

267. Prop. *If a circle S be inverted into a circle S', and P, Q be inverse points with respect to S, then P' and Q', the inverses of P and Q respectively, will be inverse points with respect to S'.*

Let O be the centre of inversion.



Since P and Q are inverse points for S, therefore S cuts orthogonally every circle through P and Q, and in particular the circle through O, P, Q.

Therefore the inverse of the circle OPQ will cut S' orthogonally.

But the inverse of the circle OPQ is a line; since O, the centre of inversion, lies on the circumference.

Therefore $P'Q'$ is the inverse of the circle OPQ.

Therefore $P'Q'$ cuts S' orthogonally, that is, passes through the centre of S'.

Again, since every circle through P and Q cuts S orthogonally, it follows that every circle through P' and Q' cuts S' orthogonally (§ 266).

Therefore, if A_1 be the centre of S',

$$A_1P' . A_1Q' = \text{square of radius of } S'.$$

Hence P' and Q' are inverse points for the circle S'.

268. Prop. *A system of non-intersecting coaxal circles can be inverted into concentric circles.*

The system being non-intersecting, the limiting points L and L' are real.

Invert the system with respect to L.

Now L and L' being inverse points with respect to each circle of the system, their inverses will be inverse points for each circle in the inversion.

But L being the centre of the circle of inversion, its inverse is at infinity. Therefore L' must invert into the centre of each of the circles.

269. Feuerbach's Theorem.

The principles of inversion may be illustrated by their application to prove Feuerbach's famous theorem, viz. *that the nine-points circle of a triangle touches the inscribed and the three escribed circles.*

Let ABC be a triangle, I its incentre and I_1 its ecentre opposite to A.

Let M and M_1 be the points of contact of the incircle and this ecircle with BC.

Let the line AII_1 which bisects the angle A cut BC in R.

Draw AL perpendicular to BC. Let O, P, U be the circumcentre, orthocentre and nine-points centre respectively.

Draw OD perpendicular to BC and let it meet the circumcircle in K.

Now since BI and BI_1 are the internal and external bisectors of angle B,

$$\therefore (AR, II_1) = -1,$$

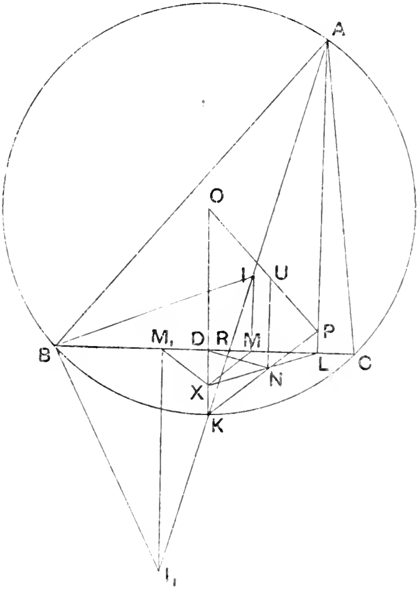
$$\therefore L(AR, II_1) = -1.$$

$\therefore$ since RLA is a right angle, LI and LI_1 are equally inclined to BC (§ 27, Cor. 2).

$\therefore$ the polars of L with regard to the incircle and the ecircle will be equally inclined to BC.

Now the polar of L for the incircle goes through M and that for the ecircle through M_1.

Let MX be the polar of L for the incircle cutting OD in X.



Then since D is the middle point of MM_1 (§ 12, Cor.)

$$\triangle XM_1D \equiv \triangle XMD.$$

$$\therefore \angle XM_1D = \angle XMD.$$

$\therefore$ M_1X is the polar of L for the ecircle, i.e. L and X are conjugate points for both circles.

Let N be the middle point of XL, then the square of the tangent from N to both circles $= NX^2 = ND^2$.

$\therefore$ N is on the radical axis of the two circles; but so also is D since $DM = DM_1$.

$\therefore$ ND is the radical axis, and this is perpendicular to II_1.

Now the pedal line of K goes through D, and clearly also, since K is on the bisector of the angle A, the pedal line must be perpendicular to AK.

$\therefore$ DN is the pedal line of K.

But the pedal line of K bisects KP.

$\therefore$ KNP is a straight line and N its middle point.

And since U is the middle point of OP, $UN = \frac{1}{2}OK$.

$\therefore$ N is a point on the nine-points circle.

Now invert the nine-points circle, the incircle and ecircle with respect to the circle whose centre is N and radius ND or NL.

The two latter circles will invert into themselves; and the nine-points circle will invert into the line BC; for N being on the nine-points circle the inverse of that circle must be a line, and D and L, points on the circle, invert into themselves, $\therefore$ DL is the inverse of the nine-points circle.

But this line touches both the incircle and ecircle.

$\therefore$ the nine-points circle touches both the incircle and ecircle.

Similarly it touches the other two ecircles.

Cor. The point of contact of the nine-points circle with the incircle will be the inverse of M, and with the ecircle the inverse of M_1.

EXERCISES

1. Prove that a system of intersecting coaxal circles can be inverted into concurrent straight lines.

2. A sphere is inverted from a point on its surface; shew that to a system of meridians and parallels on the surface will correspond two systems of coaxal circles in the inverse figure.

[See Ex. 16 of Chap. II.]

3. If A, B, C, D be four collinear points, and A', B', C', D' the four points inverse to them, then

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}.$$

4. If P be a point in the plane of a system of coaxal circles, and P_1, P_2, P_3 &c. be its inverses with respect to the different circles of the system, P_1, P_2, P_3 &c. are concyclic.

5. If P be a fixed point in the plane of a system of coaxal circles, P' the inverse of P with respect to a circle of the system, P'' the inverse of P' with respect to another circle, P''' of P'' with respect to another and so on, then P', P'', P''' &c. are concyclic.

6. POP', QOQ' are two chords of a circle and O is a fixed point. Prove that the locus of the other intersection of the circles POQ, $P'OQ'$ is a second fixed circle.

7. Shew that the result of inverting at any odd number of circles of a coaxal system is equivalent to a single inversion at one circle of the system; and determine the circle which is so equivalent to three given ones in a given order.

8. Shew that if the circles inverse to two given circles ACD, BCD with respect to a given point P be equal, the circle PCD bisects (internally and externally) the angles of intersection of the two given circles.

9. Three circles cut one another orthogonally at the three pairs of points AA', BB', CC'; prove that the circles through ABC, $AB'C'$ touch at A.

10. Prove that if the nine-points circle and one of the angular points of a triangle be given, the locus of the orthocentre is a circle.

11. Prove that the nine-points circle of a triangle touches the inscribed and escribed circles of the three triangles formed by joining the orthocentre to the vertices of the triangle.

12. The figures inverse to a given figure with regard to two circles C_1 and C_2 are denoted by S_1 and S_2 respectively; shew that if C_1 and C_2 cut orthogonally, the inverse of S_1 with regard to C_2 is also the inverse of S_2 with regard to C_1.

13. If A, B, C be three collinear points and O any other point, shew that the centres P, Q, R of the three circles circumscribing the triangles OBC, OCA, OAB are concyclic with O.

Also that if three other circles are drawn through O, A; O, B; O, C to cut the circles OBC, OCA, OAB respectively, at right angles, then these circles will meet in a point which lies on the circumcircle of the quadrilateral $OPQR$.

14. Shew that if the circle PAB cut orthogonally the circle PCD; and the circle PAC cut orthogonally the circle PBD; then the circle PAD must cut the circle PBC orthogonally.

15. Prove the following construction for obtaining the point of contact of the nine-points circle of a triangle ABC with the incircle:

The bisector of the angle A meets BC in H. From H the other tangent HY is drawn to the incircle. The line joining the point of contact Y of this tangent and D the middle point of BC cuts the incircle again in the point required.

16. Given the circumcircle and incircle of a triangle, shew that the locus of the centroid is a circle.

17. A, B, C are three circles and a, b, c their inverses with respect to any other circle. Shew that if A and B are inverses with respect to C, then a and b are inverses with respect to c.

18. A circle S is inverted into a line, prove that this line is the radical axis of S and the circle of inversion.

19. Shew that the angle between a circle and its inverse is bisected by the circle of inversion.

20. The perpendiculars, AL, BM, CN to the sides of a triangle ABC meet in the orthocentre K. Prove that each of the four circles which can be described to touch the three circles about $KMAN$, $KNBL$, $KLCM$ touches the circumcircle of the triangle ABC.

[Invert the three circles into the sides of the triangle by means of centre K, and the circumcircle into the nine-points circle.]

21. Invert two spheres, one of which lies wholly within the other, into concentric spheres.

22. Examine the particular case of the proposition of § 151, where O the centre of inversion lies on S.

23. If A, P, Q be three collinear points, and if P', Q' be the inverses of P, Q with respect to O, and if $P'Q'$ meet OA in A_1, then

$$\frac{AP \cdot AQ}{A_1P' \cdot A_1Q'} = \frac{OA^2}{OA_1^2}.$$

24. A circle is drawn to touch the sides AB, AC of a triangle ABC and to touch the circumcircle internally at E. Shew that AE and the line joining A to the point of contact with BC of the ecircle opposite to A are equally inclined to the bisectors of the angles between AB and AC.

[Invert with A as centre so that C inverts into itself.]

CHAPTER XX

SIMILARITY OF FIGURES

270. Homothetic Figures.

If F be a plane figure, which we may regard as an assemblage of points typified by P, and if O be a fixed point in the plane, and if on each radius vector OP, produced if necessary, a point P' be taken on the same side of O as P such that $OP:OP'$ is constant $(=k)$, then P' will determine another figure F' which is said to be similar and similarly situated to F.

Two such figures are conveniently called, in one word, *homothetic*, and the point O is called their *homothetic centre*.

We see that two homothetic figures are in perspective, the centre of perspective being the homothetic centre.

271. Prop. *The line joining two points in the figure F is parallel to the line joining the corresponding points in the figure F' which is homothetic with it, and these lines are in a constant ratio.*

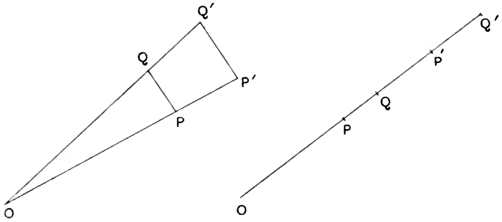
For if P and Q be two points in F, and P', Q' the corresponding points in F', since $OP:OP'=OQ:OQ'$ it follows that PQ and $P'Q'$ are parallel, and that $PQ:P'Q'=OP:OP'$ the constant ratio.

In the case where Q is in the line OP it is still true that $PQ:P'Q'=$ the constant ratio, for since $OP:OQ=OP':OQ'$

$$\therefore\ OP:OQ-OP=OP':OQ'-OP'.$$

$$\therefore\ OP : PQ = OP' : P'Q'.$$

$$\therefore\ PQ : P'Q' = OP : OP'.$$



Cor. If the figures F and F' be curves S and S' the tangents to them at corresponding points P and P' will be parallel. For the tangent at P is the limiting position of the line through P and a near point Q on S, and the tangent at P' the limiting position of the line through the corresponding points P' and Q'.

272. Prop. *The homothetic centre of two homothetic figures is determined by two pairs of corresponding points.*

For if two pairs of corresponding points P, P'; Q, Q' be given O is the intersection of PP' and QQ'.

Or in the case where Q is in the line PP', O is determined in this line by the equation $OP : OP' = PQ : P'Q'$.

The point O is thus uniquely determined, for OP and OP' have to have the same sign, that is, have to be in the same direction.

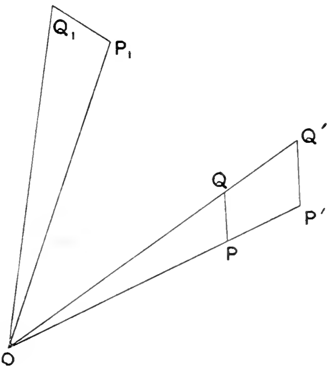
273. Figures directly similar.

If now two figures F and F' be homothetic, centre O, and the figure F' be turned in its plane round O through any angle, we shall have a new figure F_1 which is similar to F but not now similarly situated.

Two such figures F and F_1 are said to be *directly similar* and O is called their *centre of similitude.*

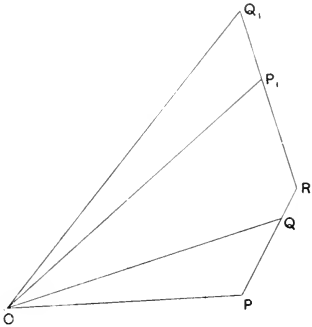
Two directly similar figures possess the property that the

$\angle POP_1$ between the lines joining O to two corresponding points P and P_1 is constant. Also $OP : OP_1$ is constant, and



$PQ : P_1Q_1 =$ the same constant, and the triangles OPQ, OP_1Q_1 are similar.

274. Prop. *If P, P_1; Q, Q_1 be two pairs of corresponding points of two figures directly similar, and if PQ, P_1Q_1 intersect in R, O is the other intersection of the circles PRP_1, QRQ_1.*



For since $\angle OPQ = \angle OP_1Q_1$

∴ $\angle OPR$ and $\angle OP_1R$ are supplementary.

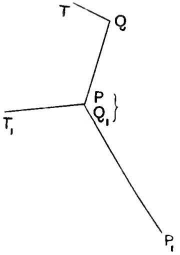
∴ POP_1R is cyclic.

Similarly Q_1OQR is cyclic.

Thus the proposition is proved.

COR. The centre of similitude of two directly similar figures is determined by two pairs of corresponding points.

It has been assumed thus far that P does not coincide with P_1 nor with Q_1.



If P coincide with P_1, then this point is itself the centre of similitude.

If P coincide with Q_1 we can draw QT and Q_1T_1 through Q and Q_1 such that

$$\angle P_1Q_1T_1 = \angle PQT \text{ and } Q_1T_1 : QT = P_1Q_1 : PQ;$$

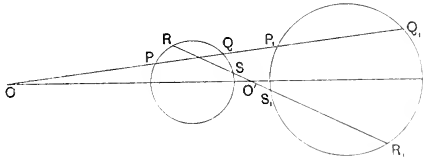
then T and T_1 are corresponding points in the two figures.

275. When two figures are directly similar, and the two members of each pair of corresponding points are on opposite sides of O, and collinear with it, the figures may be called *antihomothetic*, and the centre of similitude is called the *antihomothetic centre.*

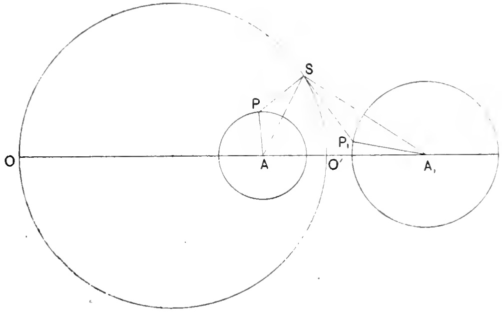
When two figures are antihomothetic the line joining any two points P and Q of the one is parallel to the line joining the corresponding points P' and Q' of the other; *but PQ and $P'Q'$ are in opposite directions.*

276. Case of two coplanar circles.

If we divide the line joining the centres of two given circles externally at O, and internally at O' in the ratio of the radii, it is clear from § 25 that O is the homothetic centre and O' the antihomothetic centre for the two circles.



We spoke of these points as 'centres of similitude' before, but we now see that they are only particular centres of similitude, and it is clear that there are other centres of similitude not lying in the line of these. For taking the centre A of one circle to correspond with the centre A_1 of the other, we may then take any point P of the one to correspond with any point P_1 of the other.



Let S be the centre of similitude for this correspondence.

The triangles PSA, P_1SA_1 are similar, and

$$SA : SA_1 = AP : A_1P_1 = \text{ratio of the radii.}$$

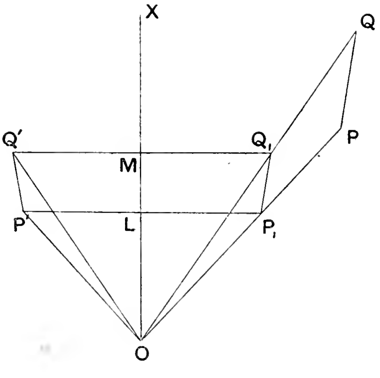
Thus S lies on the circle on OO' as diameter (§ 27).

Thus the locus of centres of similitude for two coplanar circles is the circle on the line joining the homothetic and anti-homothetic centres.

This circle we have already called the *circle of similitude* and the student now understands the reason of the name.

277. Figures inversely similar.

If F be a figure in a plane, O a fixed point in the plane, and if another figure F' be obtained by taking points P' in the plane to correspond with the points P of F in such a way that $OP : OP'$ is constant, and all the angles POP' have the same bisecting line OX, the two figures F and F' are said to be *inversely similar*; O is then called *the centre* and OX *the axis of inverse similitude.*



Draw $P'L$ perpendicular to the axis OX and let it meet OP in P_1.

Then plainly, since OX bisects $\angle POP'$

$$\triangle OLP' \equiv \triangle OLP_1,$$

and $$OP_1 = OP'.$$

$$\therefore\ OP_1 : OP \text{ is constant.}$$

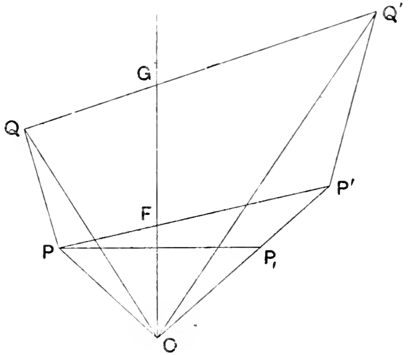
Thus the figure formed by the points P_1 will be homothetic with F.

Indeed the figure F' may be regarded as formed from a figure F_1 homothetic with F by turning F_1 round the axis OX through two right angles.

The student will have no difficulty in proving for himself that if any line OY be taken through O in the plane of F and F', and if $P'K$ be drawn perpendicular to OY and produced to P_2 so that $P'K = KP_2$ then the figure formed with the points typified by P_2 will be similar to F; but the two will not be similarly situated except in the case where OY coincides with OX.

278. If P and Q be two points in the figure F, and P', Q' the corresponding points in the figure F', inversely similar to it, we easily obtain that $P'Q' : PQ =$ the constant ratio of $OP : OP'$, and we see that the angle $POQ =$ angle $Q'OP'$ (*not* $P'OQ'$). In regard to this last point we see the distinction between figures directly similar and figures inversely similar.

279. *Given two pairs of corresponding points in two inversely similar figures, to find the centre and axis of similitude.*



To solve this problem we observe that if PP' cut the axis

OX in F, then $PF:FP' = OP:OP'$ since the axis bisects the angle POP'.

$$\therefore\ PF:FP' = PQ:P'Q'.$$

Hence if P, P'; Q, Q' be given, join PP' and QQ' and divide these lines at F and G in the ratio $PQ:P'Q'$, then the line FG is the axis.

Take the point P_1 symmetrical with P on the other side of the axis, then O is determined by the intersection of $P'P_1$ with the axis.

NOTE. The student who wishes for a fuller discussion on the subject of similar figures than seems necessary or desirable here, should consult Lachlan's *Modern Pure Geometry*, Chapter IX.

EXERCISES

1. Prove that homothetic figures will, if *orthogonally* projected, be projected into homothetic figures.

2. If P, P' ;. Q, Q' ; R, R' be three corresponding pairs of points in two figures either directly or inversely similar, the triangles PQR, $P'Q'R'$ are similar in the Euclidean sense.

3. If S and S' be two curves directly similar, prove that if S be turned in the plane about any point, the locus of the centre of similitude of S and S' in the different positions of S will be a circle.

4. If two triangles, directly similar, be inscribed in the same circle, shew that the centre of the circle is their centre of similitude.

Shéw also that the pairs of corresponding sides of the triangles intersect in points forming a triangle directly similar to them.

5. If two triangles be inscribed in the same circle so as to be inversely similar, shew that they are in perspective, and that the axis of perspective passes through the centre of the circle.

6. If on the sides BC, CA, AB of a triangle ABC points X, Y, Z be taken such that the triangle XYZ is of constant shape, construct the centre of similitude of the system of triangles so formed; and prove that the locus of the orthocentre of the triangle XYZ is a straight line.

7. If three points X, Y, Z be taken on the sides of a triangle ABC opposite to A, B, C respectively, and if three similar and similarly situated ellipses be described round AYZ, BZX and CXY, they will have a common point.

8. The circle of similitude of two given circles belongs to the coaxal system whose limiting points are the centres of the two given circles.

9. If two coplanar circles be regarded as inversely similar, the locus of the centre of similitude is still the 'circle of similitude,' and the axis of similitude passes through a fixed point.

10. P and P' are corresponding points on two coplanar circles regarded as inversely similar and S is the centre of similitude in this case. Q is the other extremity of the diameter through P, and when Q and P' are corresponding points in the two circles for inverse similarity, S' is the centre of similitude. Prove that SS' is a diameter of the circle of similitude.

11. $ABCD$ is a cyclic quadrilateral; AC and BD intersect in E, AD and BC in F; prove that EF is a diameter of the circle of similitude for the circles on AB, CD as diameters.

12. Generalise by projection the theorem that the circle of similitude of two circles is coaxal with them.

MISCELLANEOUS EXAMPLES

1. Prove that when four points A, B, C, D lie on a circle, the orthocentres of the triangles BCD, CDA, DAB, ABC lie on an equal circle, and that the line which joins the centres of these circles is divided in the ratio of three to one by the centre of mean position of the points A, B, C, D.

2. ABC is a triangle, O the centre of its inscribed circle, and A_1, B_1, C_1 the centres of the circles escribed to the sides BC, CA, AB respectively; L, M, N the points where these sides are cut by the bisectors of the angles A, B, C. Shew that the orthocentres of the three triangles LB_1C_1, MC_1A_1, NA_1B_1 form a triangle similar and similarly situated to $A_1B_1C_1$, and having its orthocentre at O.

3. ABC is a triangle, L_1, M_1, N_1 are the points of contact of the incircle with the sides opposite to A, B, C respectively; L_2 is taken as the harmonic conjugate of L_1 with respect to B and C; M_2 and N_2 are similarly taken; P, Q, R are the middle points of L_1L_2, M_1M_2, N_1N_2. Again AA_1 is the bisector of the angle A cutting BC in A_1, and A_2 is the harmonic conjugate of A_1 with respect to B and C; B_2 and C_2 are similarly taken. Prove that the line $A_2B_2C_2$ is parallel to the line PQR.

4. ABC is a triangle the centres of whose inscribed and circumscribed circles are O, O'; O_1, O_2, O_3 are the centres of its escribed circles, and O_1O_2, O_2O_3 meet AB, BC respectively in L and M; shew that OO' is perpendicular to LM.

5. If circles be described on the sides of a given triangle as diameters, and quadrilaterals be inscribed in them having the intersections of their diagonals at the orthocentre, and one side of each passing through the middle point of the upper segment of the corresponding perpendicular, prove that the sides of the quadrilaterals opposite to these form a triangle equiangular with the given one.

6. Two circles are such that a quadrilateral can be inscribed in one so that its sides touch the other. Shew that if the points of contact of the sides be P, Q, R, S, then the diagonals of $PQRS$ are at right angles; and prove that PQ, RS and QR, SP have their points of intersection on the same fixed line.

7. A straight line drawn through the vertex A of the triangle ABC meets the lines DE, DF which join the middle point of the base to the middle points E and F of the sides CA, AB in X, Y; shew that BY is parallel to CX.

8. Four intersecting straight lines are drawn in a plane. Reciprocate with regard to any point in this plane the theorem that the circumcircles of the triangle formed by the four lines are concurrent at a point which is concyclic with their four centres.

9. E and F are two fixed points, P a moving point, on a hyperbola, and PE meets an asymptote in Q. Prove that the line through E parallel to the other asymptote meets in a fixed point the line through Q parallel to PF.

10. Any parabola is described to touch two fixed straight lines and with its directrix passing through a fixed point P. Prove that the envelope of the polar of P with respect to the parabola is a conic.

11. Shew how to construct a triangle of given shape whose sides shall pass through three given points.

12. Construct a hyperbola having two sides of a given triangle as asymptotes and having the base of the triangle as a normal.

13. A tangent is drawn to an ellipse so that the portion intercepted by the equiconjugate diameters is a minimum; shew that it is bisected at the point of contact.

14. A parallelogram, a point and a straight line in the same plane being given, obtain a construction depending on the ruler only for a straight line through the point parallel to the given line.

15. Prove that the problem of constructing a triangle whose sides each pass through one of three fixed points and whose vertices lie one on each of three fixed straight lines is poristic, when the three given points are collinear and the three given lines are concurrent.

16. A, B, C, D are four points in a plane no three of which are collinear and a projective transformation interchanges A and B, and

also C and D. Give a pencil and ruler construction for the point into which any arbitrary point P is changed; and shew that any conic through A, B, C, D is transformed into itself.

17. Three hyperbolas are described with B, C; C, A; and A, B for foci passing respectively through A, B, C. Shew that they have two common points P and Q; and that there is a conic circumscribing ABC with P and Q for foci.

18. Three triangles have their bases on one given line and their vertices on another given line. Six lines are formed by joining the point of intersection of two sides, one from each of a pair of the triangles, with a point of intersection of the other two sides of those triangles, choosing the pairs of triangles and the pairs of sides in every possible way. Prove that the six lines form a complete quadrangle.

19. Shew that in general there are four distinct solutions of the problem: To draw two conics which have a given point as focus and intersect at right angles at two other given points. Determine in each case the tangents at the two given points.

20. An equilateral triangle ABC is inscribed in a circle of which O is the centre: two hyperbolas are drawn, the first has C as a focus, OA as directrix and passes through B; the second has C as a focus, OB as directrix and passes through A. Shew that these hyperbolas meet the circle in eight points, which with C form the angular points of a regular polygon of nine sides.

21. An ellipse, centre O, touches the sides of a triangle ABC, and the diameters conjugate to OA, OB, OC meet any tangent in D, E, F respectively; prove that AD, BE, CF meet in a point.

22. A parabola touches a fixed straight line at a given point, and its axis passes through a second given point. Shew that the envelope of the tangent at the vertex is a parabola and determine its focus and directrix.

23. Three parabolas have a given common tangent and touch one another at P, Q, R. Shew that the points P, Q, R are collinear. Prove also that the parabola which touches the given line and the tangents at P, Q, R has its axis parallel to PQR.

24. Prove that the locus of the middle point of the common chord of a parabola and its circle of curvature is another parabola whose latus rectum is one-fifth that of the given parabola.

25. Three circles pass through a given point O and their other intersections are A, B, C. A point D is taken on the circle OBC, E on the circle OCA, F on the circle OAB. Prove that O, D, E, F are concyclic if $AF \,.\, BD \,.\, CE = -FB \,.\, DC \,.\, EA$, where AF stands for the chord AF, and so on. Also explain the convention of signs which must be taken.

26. Shew that a common tangent to two confocal parabolas subtends an angle at the focus equal to the angle between the axes of the parabolas.

27. The vertices A, B of a triangle ABC are fixed, and the foot of the bisector of the angle A lies on a fixed straight line; determine the locus of C.

28. A straight line $ABCD$ cuts two fixed circles X and Y, so that the chord AB of X is equal to the chord CD of Y. The tangents to X at A and B meet the tangents to Y at C and D in four points P, Q, R, S. Shew that P, Q, R, S lie on a fixed circle.

29. On a fixed straight line AB, two points P and Q are taken such that PQ is of constant length. X and Y are two fixed points and XP, YQ meet in a point R. Shew that as P moves along the line AB, the locus of R is a hyperbola of which AB is an asymptote.

30. A parabola touches the sides BC, CA, AB of a triangle ABC in D, E, F respectively. Prove that the straight lines AD, BE, CF meet in a point which lies on the polar of the centre of gravity of the triangle ABC.

31. If two conics be inscribed in the same quadrilateral, the two tangents at any of their points of intersection cut any diagonal of the quadrilateral harmonically.

32. A circle, centre O, is inscribed in a triangle ABC. The tangent at any point P on the circle meets BC in D. The line through O perpendicular to OD meets PD in D'. The corresponding points E', F' are constructed. Shew that AD', BE', CF' are parallel.

33. Two points are taken on a circle in such a manner that the sum of the squares of their distances from a fixed point is constant. Shew that the envelope of the chord joining them is a parabola.

34. A variable line PQ intersects two fixed lines in points P and Q such that the orthogonal projection of PQ on a third fixed line is of constant length. Shew that the envelope of PQ is a parabola, and find the direction of its axis.

35. With a focus of a given ellipse (A) as focus, and the tangent at any point P as directrix, a second ellipse (B) is described similar to (A). Show that (B) touches the minor axis of (A) at the point where the normal at P meets it.

36. A parabola touches two fixed lines which intersect in T, and its axis passes through a fixed point D. Prove that, if S be the focus, the bisector of the angle TSD is fixed in direction. Shew further that the locus of S is a rectangular hyperbola of which D and T are ends of a diameter. What are the directions of its asymptotes?

37. If an ellipse has a given focus and touches two fixed straight lines, then the director circle passes through two fixed points.

38. O is any point in the plane of a triangle ABC, and X, Y, Z are points in the sides BC, CA, AB respectively, such that AOX, BOY, COZ are right angles. If the points of intersection of CZ and AX, AX and BY be respectively Q and R, shew that OQ and OR are equally inclined to OA.

39. The line of collinearity of the middle points of the diagonals of a quadrilateral is drawn, and the middle point of the intercept on it between any two sides is joined to the point in which they intersect. Shew that the six lines so constructed together with the line of collinearity and the three diagonals themselves touch a parabola.

40. The triangles $A_1B_1C_1$, $A_2B_2C_2$ are reciprocal with respect to a given circle; B_2C_2, C_1A_1 intersect in P_1, and B_1C_1, C_2A_2 in P_2. Shew that the radical axis of the circles which circumscribe the triangles $P_1A_1B_2$, $P_2A_2B_1$ passes through the centre of the given circle.

41. A transversal cuts the three sides BC, CA, AB of a triangle in P, Q, R; and also cuts three concurrent lines through A, B and C respectively in P', Q', R'. Prove that

$$PQ' \cdot QR' \cdot RP' = -P'Q \cdot Q'R \cdot R'P.$$

42. Through any point O in the plane of a triangle ABC is drawn a transversal cutting the sides in P, Q, R. The lines OA, OB, OC are bisected in A', B', C'; and the segments QR, RP, PQ of the transversal are bisected in P', Q', R'. Shew that the three lines $A'P'$, $B'Q'$, $C'R'$ are concurrent.

43. From any point P on a given circle tangents PQ, PQ' are drawn to a given circle whose centre is on the circumference of the

first: shew that the chord joining the points where these tangents cut the first circle is fixed in direction and intersects QQ' on the line of centres.

44. If any parabola be described touching the sides of a fixed triangle, the chords of contact will pass each through a fixed point.

45. From D, the middle point of AB, a tangent DP is drawn to a conic. Shew that if CQ, CR are the semidiameters parallel to AB and DP,

$$AB : CQ = 2DP : CR.$$

46. The side BC of a triangle ABC is trisected at M, N. Circles are described within the triangle, one to touch BC at M and AB at H, the other to touch BC at N and AC at K. If the circles touch one another at L, prove that CH, BK pass through L.

47. ABC is a triangle and the perpendiculars from A, B, C on the opposite sides meet them in L, M, N respectively. Three conics are described; one touching BM, CN at M, N and passing through A; a second touching CN, AL at N, L and passing through B; a third touching AL, BM at L, M and passing through C. Prove that at A, B, C they all touch the same conic.

48. A parabola touches two fixed lines meeting in T and the chord of contact passes through a fixed point A; shew that the directrix passes through a fixed point O, and that the ratio TO to OA is the same for all positions of A. Also that if A move on a circle whose centre is T, then AO is always normal to an ellipse the sum of whose semi-axes is the radius of this circle.

49. Triangles which have a given centroid are inscribed in a given circle, and conics are inscribed in the triangles so as to have the common centroid for centre, prove that they all have the same fixed director circle.

50. A circle is inscribed in a right-angled triangle and another is escribed to one of the sides containing the right angle; prove that the lines joining the points of contact of each circle with the hypothenuse and that side intersect one another at right angles, and being produced pass each through the point of contact of the other circle with the remaining side. Also shew that the polars of any point on either of these lines with respect to the two circles meet on the other, and deduce that the four tangents drawn from any point on either of these lines to the circles form a harmonic pencil.

51. If a triangle PQR circumscribe a conic, centre C, and ordinates be drawn from Q, R to the diameters CR, CQ respectively,

the line joining the feet of the ordinates will pass through the points of contact of PQ, PR.

52. Prove that the common chord of a conic and its circle of curvature at any point and their common tangent at this point divide their own common tangent harmonically.

53. Shew that the point of intersection of the two common tangents of a conic and an osculating circle lies on the confocal conic which passes through the point of osculation.

54. In a triangle ABC, AL, BM, CN are the perpendiculars on the sides and MN, NL, LM when produced meet BC, CA, AB in P, Q, R. Shew that P, Q, R lie on the radical axis of the nine-points circle and the circumcircle of ABC, and that the centres of the circumcircles of ALP, BMQ, CNR lie on one straight line.

55. A circle through the foci of a rectangular hyperbola is reciprocated with respect to the hyperbola; shew that the reciprocal is an ellipse with a focus at the centre of the hyperbola; and its minor axis is equal to the distance between the directrices of the hyperbola.

56. A circle can be drawn to cut three given circles orthogonally. If any point be taken on this circle its polars with regard to the three circles are concurrent.

57. From any point O tangents OP, OP', OQ, OQ' are drawn to two confocal conics; OP, OP' touch one conic, OQ, OQ' the other. Prove that the four lines PQ, $P'Q'$, PQ', $P'Q$ all touch a third confocal.

58. P, P' and Q, Q' are four collinear points on two conics U and V respectively. Prove that the corners of the quadrangle whose pairs of opposite sides are the tangents at P, P' and Q, Q' lie on a conic which passes through the four points of intersection of U and V.

59. If two parabolas have a real common self-conjugate triangle they cannot have a common focus.

60. The tangents to a conic at two points A and B meet in T, those at A', B' in T'; prove that

$$T(A'AB'B) = T'(A'AB'B).$$

61. A circle moving in a plane always touches a fixed circle, and the tangent to the moving circle from a fixed point is always of constant length. Prove that the moving circle always touches another fixed circle.

62. A system of triangles is formed by the radical axis and each pair of tangents from a fixed point P to a coaxal system of circles. Shew that if P lies on the polar of a limiting point with respect to the coaxal system, then the circumcircles of the triangles form another coaxal system.

63. Two given circles S, S' intersect in A, B; through A any straight line is drawn cutting the circles again in P, P' respectively. Shew that the locus of the other point of intersection of the circles, one of which passes through B, P and cuts S orthogonally, and the other of which passes through B, P' and cuts S' orthogonally, is the straight line through B perpendicular to AB.

64. Four points lie on a circle; the pedal line of each of these with respect to the triangle formed by the other three is drawn: shew that the four lines so drawn meet in a point.

65. A, B, C, D are four points on a conic; EF cuts the lines BC, CA, AB in a, b, c respectively and the conic in E and F; a', b' c' are harmonically conjugate to a, b, c with respect to E, F. The lines Da', Db', Dc' meet BC, CA, AB in α, β, γ respectively. Shew that α, β, γ are collinear.

66. Three circles intersect at O so that their respective diameters DO, EO, FO pass through their other points of intersection A, B, C; and the circle passing through D, E, F intersects the circles again in G, H, I respectively. Prove that the circles AOG, BOH, COI are coaxal.

67. A conic passes through four fixed points on a circle, prove that the polar of the centre of the circle with regard to the conic is parallel to a fixed straight line.

68. The triangles PQR, $P'Q'R'$ are such that PQ, PR, $P'Q'$, $P'R'$ are tangents at Q, R, Q', R' respectively to a conic. Prove that

$$P(QR'Q'R) = P'(QR'Q'R)$$

and P, Q, R, P', Q', R' lie on a conic.

69. If A', B', C', D' be the points conjugate to A, B, C, D in an involution, and P, Q, R, S be the middle points of AA', BB', CC', DD',

$$(PQRS) = (ABCD) \,.\, (A'B'C'D').$$

70. ABC is a triangle. If $BDCX$, $CEAY$, $AFBZ$ be three ranges such that $(XBCD) \,.\, (AYCE) \,.\, (ABZF) = 1$, and AD, BE, CF be concurrent, then X, Y, Z will be collinear.

71. If ABC be a triangle and D any point on BC, then (i) the line joining the circumcentres of ABD, ACD touches a parabola:

(ii) the line joining the incentres touches a conic touching the bisectors of the angles ABC, ACB.

Find the envelope of the line joining the centres of the circles escribed to the sides BD, CD respectively.

72. Two variable circles S and S' touch two fixed circles, find the locus of the points which have the same polars with regard to S and S'.

73. QP, QP' are tangents to an ellipse, QM is the perpendicular on the chord of contact PP' and K is the pole of QM. If H is the orthocentre of the triangle PQP', prove that HK is perpendicular to QC.

74. Two circles touch one another at O. Prove that the locus of the points inverse to O with respect to circles which touch the two given circles is another circle touching the given circles in O, and find its radius in terms of the radii of the given circles.

75. Prove that the tangents at A and C to a parabola and the chord AC meet the diameter through B, a third point on the parabola in a, c, b, such that $aB : Bb = Ab : bC = Bb : cB$. Hence draw through a given point a chord of a parabola that shall be divided in a given ratio at that point. How many different solutions are there of this problem?

76. If A, B, C be three points on a hyperbola and the directions of both asymptotes be given, then the tangent at B may be constructed by drawing through B a parallel to the line joining the intersection of BC and the parallel through A to one asymptote with the intersection of AB and the parallel through C to the other.

77. A circle cuts three given circles at right angles; calling these circles A, B, C, Ω, shew that the points where C cuts Ω are the points where circles coaxal with A and B touch Ω.

78. If ABC, DEF be two coplanar triangles, and S be a point such that SD, SE, SF cuts the sides BC, CA, AB respectively in three collinear points, then SA, SB, SC cut the sides EF, FD, DE in three collinear points.

79. ABC is a triangle, D is a point of contact with BC of the circle escribed to BC; E and F are found on CA, AB in the same way. Lines are drawn through the middle points of BC, CA, AB parallel to AD, BE, CF respectively; shew that these lines meet at the incentre.

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