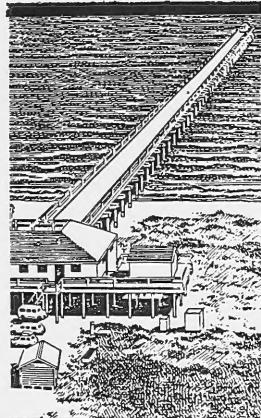




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TECHNICAL REPORT CERC-87-16

# COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

by

H. S. Chen

Coastal Engineering Research Center

DEPARTMENT OF THE ARMY  
Waterways Experiment Station, Corps of Engineers  
PO Box 631, Vicksburg, Mississippi 39180-0631

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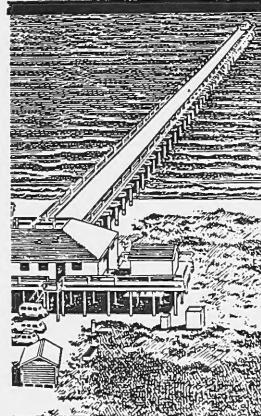
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Under Waves at Entrances Work Unit 31673





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## PREFACE

The work in this report was authorized by the Office, Chief of Engineers (OCE), Coastal Engineering Functional Area of Civil Works Research and Development, under Waves at Entrances Work Unit 31673, Harbor Entrances and Coastal Channels Program, at the Coastal Engineering Research Center (CERC) of the US Army Engineer Waterways Experiment Station (WES). Messrs. John H. Lockhart, Jr., and John G. Housley were OCE Technical Monitors. Dr. Charles L. Vincent is CERC Program Manager.

This report was prepared by Dr. H. S. Chen, Coastal Oceanography Branch (CR-O), Research Division (CR). Work was performed under direct supervision of Dr. Edward F. Thompson, Chief, CR-O, and Mr. H. Lee Butler, Chief, CR; and under general supervision of Dr. James R. Houston and Mr. Charles C. Calhoun, Jr., Chief and Assistant Chief, CERC, respectively.

This study was initiated after a discussion by Dr. Vincent and the author on the possibility of implementing a scheme to redistribute wave energy behind islands in numerical models. The author acknowledges and appreciates the review and comments provided by Drs. Edward F. Thompson and Norman W. Scheffner. This report was edited by Ms. Shirley A. J. Hanshaw, Information Products Division, Information Technology Laboratory, WES.

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COMBINED REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE

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PART I: INTRODUCTION

1. The boundary value problem of linear wave reflection and diffraction by a vertical wedge of arbitrary wedge angle has been well formulated and presented by Stoker (1957) among many other investigators. The technique to obtain an analytical solution for the problem is also depicted in the cited book. However, analytical solutions are not available for the problem, except for the special case of wave diffraction by a thin semi-infinite breakwater, that is, a wedge with wedge angle equal to zero.

2. The solution of the thin semi-infinite breakwater was presented in the dimensionless diffraction diagrams by Wiegel (1962). The diagrams have been especially useful in preliminary engineering design and have been included in the Shore Protection Manual (SPM) (1984). Although equally useful, the combined reflection and diffraction diagrams are not available, perhaps because of the complexity of the diagrams which makes them difficult to create without using modern high-speed computers for computation and graphing.

3. The objectives of the present study are (a) to obtain an analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle subject to excitation of a plane simple harmonic wave train coming from infinity and (b) to provide the combined reflection and diffraction diagrams. The diagrams included in this report have two cases: one for a thin semi-infinite breakwater and the other for a 90-deg vertical wedge. Subroutine WEDGE for computing the combined reflection and diffraction by a vertical wedge of arbitrary wedge angle is also documented in the report (Appendix A).

PART II: BOUNDARY VALUE PROBLEM

Mathematical Formulation

4. In this study our primary interest is the wave reflection and diffraction by a vertical wedge of arbitrary wedge angle in a constant water depth  $h$ \* subject to the excitation of monochromatic incident waves of infinitesimal amplitude coming from infinity. Let  $(r, \theta, z)$  be cylindrical coordinates, with  $z = 0$  representing the undisturbed water free surface and upward direction representing the positive  $z$ -axis. The tip of the wedge is chosen to be the origin of the coordinates and two rigid walls of the wedge to coincide with  $\theta = 0$  and  $\theta = \theta_0$ , respectively, as illustrated in Figure 1. Cartesian coordinates  $(x, y, z)$ , corresponding to the cylindrical coordinates, are also occasionally used and shown in the same figure. Therefore, the wedge

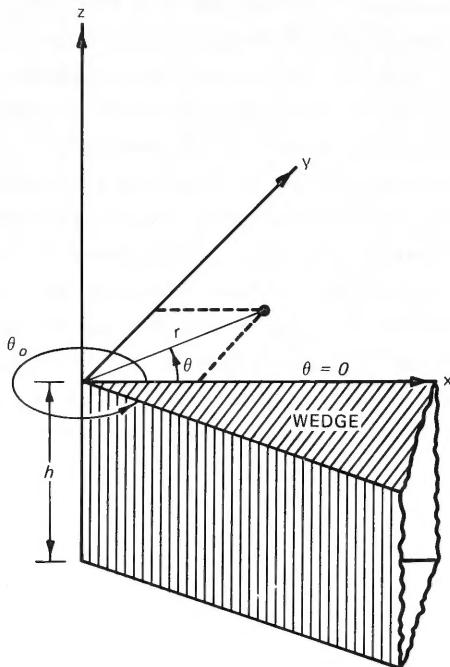


Figure 1. A vertical wedge of arbitrary wedge angle

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\* For convenience, symbols and abbreviations are listed in the notation (Appendix B).

angle is  $2\pi - \theta_0$ , and the water region is defined by  $\theta_0 \geq \theta \geq 0$  and  $0 \geq z \geq -h$ .

5. The velocity field for the wave reflection and diffraction in an ideal fluid can be represented by the velocity potential function  $\Phi(r, \theta, z, t)$  which must satisfy the Laplace equation, where  $t$  is the temporal coordinate. We assume that the waves are sinusoidal in time with radian frequency  $\omega$ . Water depth is constant, and the bottom is rigid and impermeable. Therefore, the vertical and temporal components of the velocity potential function, which follow from separation of variables, can be factored out and the velocity potential written as

$$\Phi(r, \theta, z, t) = A_0 \frac{\cosh k(z + h)}{\cosh kh} \phi(r, \theta) e^{i\omega t} \quad (1)$$

where

$$A_0 = -iga_0/\omega$$

$$i = \sqrt{-1}$$

$g$  = gravitational acceleration

$a_0$  = incident wave amplitude

$k$  = wave number

$\phi$  = horizontal component of the velocity potential function

6. Substituting Equation 1 into the Laplace equation and using both the kinematic and dynamic boundary conditions at the free surface, the Laplace equation is then reduced to the Helmholtz equation which is written in polar coordinates as follows:

$$r^2 \frac{\partial^2 \phi}{\partial r^2} + r \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2} + k^2 r^2 \phi = 0 \quad (2)$$

where  $k$  must be a real number and satisfy the dispersion relationship

$$\omega^2 = gk \tanh kh \quad (3)$$

7. The free surface displacement  $\eta$  from the mean water level  $z = 0$  can be obtained from linear wave theory and is represented as

$$\eta(r, \theta, t) = \frac{1}{g} \frac{\partial \Phi}{\partial t} = a_0 \phi(r, \theta) e^{i\omega t} \quad (4)$$

8. Thus only the horizontal part of the velocity potential function  $\phi$  is needed to be determined as a solution of Equation 2 in the water region  $\theta_0 \geq \theta \geq 0$ , with the following boundary conditions at the rigid and impermeable walls of the wedge:

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{at } \theta = 0 \quad \text{and } \theta_0 \quad (5)$$

9. A condition at infinity is also required to ensure a unique solution. The classic approach is to use the Sommerfeld radiation condition at infinity which states that the scattered wave  $\phi_s$  must behave like a cylindrical outgoing progressing wave at infinity such that

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial \phi_s}{\partial r} + ik\phi_s \right) = 0 \quad (6)$$

The total wave represented by  $\phi$  is the linear superposition of an incident wave  $\phi_i$ , a reflected wave from the the  $\theta = 0$  wall of the wedge  $\phi_r$ , and the scattered wave  $\phi_s$  from the tip of the wedge.

$$\phi = \phi_i + \phi_r + \phi_s \quad (7)$$

Equation 6 can be satisfied if

$$\phi_s \sim \frac{e^{-ikr}}{\sqrt{kr}} \quad \text{at } r \rightarrow \infty \quad (8)$$

10. The incident wave coming from a large distance from the tip of the wedge is assumed to be a plane progressive wave of amplitude  $a_0$  and incident angle  $\alpha$  to the x-axis as given by

$$\phi_i = e^{ikr \cos(\theta - \alpha)} \quad (9)$$

Consequently, the perfectly reflected wave from the  $y = 0$  wall of the wedge is

$$\phi_r = e^{ikr \cos(\theta + \alpha)} \quad (10)$$

Thus the boundary value problem (in which the governing equation is Equation 2, the boundary condition is Equation 5, and the radiation condition is Equation 6) is completely formulated.

#### Analytical Solution

11. Analytical solution to the problem formulated in the preceding section is obtained by following the solution technique by Stoker (1957). To obtain the solution, the water region is divided into three subregions--I, II, and III--by the incident wave ray passing through the tip of the wedge and the reflected wave ray reflected away from the tip of the wedge, as shown in Figure 2. Obviously, the total wave in subregion I is the sum of the incident, reflected, and scattered waves; the total wave in subregion II, where the reflected wave does not exist, is the sum of the incident and scattered waves; and the total wave in subregion III, where the incident and reflected waves have been shaded out, is only the scattered wave. For certain combinations of

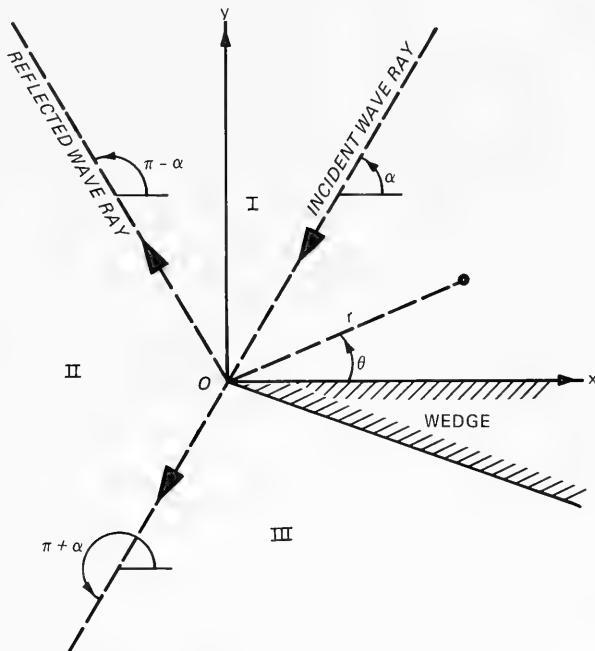


Figure 2. Three subregions and the wedge

the wedge angle and incident wave angle, subregions II and III may not exist at all. In general, the solution function can be written as

$$\phi = \phi_o(r, \theta) + \phi_s(r, \theta) \quad (11)$$

where

$$\phi_o(r, \theta) = \begin{cases} \phi_i + \phi_r & \pi - \alpha > \theta > 0 \\ \phi_i & \pi + \alpha > \theta > \pi - \alpha \\ 0 & \theta_o > \theta > \pi + \alpha \end{cases} \quad (12)$$

The equation reveals that  $\phi_o$  is the sum of the incident and reflected waves  $\phi_i$  and  $\phi_r$  and is a known function. The scattered wave  $\phi_s$  is the only unknown function to be determined in the problem. Nevertheless, the total wave  $\phi$  instead of the scattered wave  $\phi_s$  is the desired solution to be obtained in this study.

12. The solution for the total wave  $\phi$  is pursued. The finite cosine transform of  $\phi$ , denoted by  $\bar{\phi}$ , is introduced by the formula

$$\bar{\phi}(kr, n) = \int_0^{v\pi} \phi(kr, \theta) \cos \frac{n\theta}{v} d\theta \quad (13)$$

where  $n = 0, 1, 2, \dots$  are integers, and  $v$  is related to the wedge angle as defined by

$$\theta_o = v\pi \quad (14)$$

Applying the finite cosine transform and using the boundary condition in Equation 5, Equation 2 becomes

$$r^2 \frac{\partial^2 \bar{\phi}}{\partial r^2} + r \frac{\partial \bar{\phi}}{\partial r} + \left[ (kr)^2 - \left(\frac{n}{v}\right)^2 \right] \bar{\phi} = 0 \quad (15)$$

Equation 15 is a form of the Bessel equation for which general solutions are the Bessel functions of the first and second kinds,  $J_{n/v}(kr)$  and  $Y_{n/v}(kr)$ , respectively. Since  $Y_{n/v}(kr)$  are singular at the origin, the solution is chosen to be

$$\bar{\phi}(kr, n) = a_n J_{n/v}(kr) \quad (16)$$

where  $a_n$  are constants to be determined.

13. Taking the finite cosine transform of Equation 11 and using Equation 16, we have

$$\int_0^{v\pi} \phi_s \cos \frac{n\theta}{v} d\theta = a_n J_{n/v}(kr) - \int_0^{v\pi} \phi_o \cos \frac{n\theta}{v} d\theta \quad (17)$$

or

$$\bar{\phi}_s = a_n J_{n/v}(kr) - \bar{\phi}_o \quad (18)$$

Then applying the operation  $\lim_{r \rightarrow \infty} \sqrt{r}(\partial/\partial r + ik)$  to both sides of Equation 18, and using the Sommerfeld radiation condition (Equation 6) we have

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) \left[ a_n J_{n/v}(kr) - \int_0^{v\pi} \phi_o \cos \frac{n\theta}{v} d\theta \right] = 0 \quad (19)$$

14. Equation 19 can be asymptotically evaluated to determine  $a_n$ .

Firstly, the first term involving the Bessel function is evaluated. The function  $J_{n/v}(kr)$  at  $r \rightarrow \infty$  behaves asymptotically (Abramowitz and Stegun 1964) as follows:

$$J_{n/v}(kr) \sim \sqrt{\frac{2}{\pi kr}} \cos \left( kr - \frac{n\pi}{2v} - \frac{\pi}{4} \right) \quad (20)$$

Hence, we have

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) J_{n/v}(kr) \sim \sqrt{\frac{2k}{\pi}} e^{i(kr - n\pi/2v + \pi/4)} \quad (21)$$

Secondly, the second term involving the integral of  $\phi_o$  is evaluated. The asymptotic behavior of the integral over  $\theta = (0, v\pi)$  and at large distance  $r \rightarrow \infty$  can be found by the method of stationary phase. The integral, after substituting  $\phi_o$  from Equations 9, 10, and 12, can be written as

$$\int_0^{v\pi} \phi_o \cos \frac{n\theta}{v} d\theta = \int_0^{\pi-\alpha} \left[ e^{ikr \cos(\theta-\alpha)} + e^{ikr \cos(\theta+\alpha)} \right] \cos \frac{n\theta}{v} d\theta$$

$$+ \int_{\pi-\alpha}^{\pi+\alpha} e^{ikr \cos(\theta-\alpha)} \cos \frac{n\theta}{v} d\theta \quad (22)$$

In the integrals, there are three points of stationary phase at  $\theta = \alpha$  and  $\theta = \pi \pm \alpha$ . If the same argument as that of Stoker (1957) is followed, of the three contributions only the first one  $\theta = \alpha$  furnishes a nonvanishing contribution for  $r \rightarrow \infty$  when the operator  $\sqrt{r}(\partial/\partial r + ik)$  is applied to it. The physical significance of this statement is that only the incident wave is effective in determining the cosine coefficients of the solution. Therefore,

$$\lim_{r \rightarrow \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) \int_0^{v\pi} \phi_o \cos \frac{n\theta}{v} d\theta \sim 2\sqrt{2\pi k} \cos \frac{n\alpha}{v} e^{i(kr+\pi/4)} \quad (23)$$

Substituting Equations 21 and 23 into Equation 19, we obtain the unknown coefficients  $a_n$ :

$$a_n = 2\pi \cos \frac{n\alpha}{v} e^{in\pi/2v} \quad (24)$$

15. Since the solution  $\phi$  in the cosine series expression is

$$\phi(r, \theta) = \frac{1}{v\pi} \bar{\phi}(r, 0) + \frac{2}{v\pi} \sum_{n=1}^{\infty} \bar{\phi}(r, n) \cos \frac{n\theta}{v} \quad (25)$$

the solution is obtained by substituting Equations 16 and 24 into Equation 25 as follows:

$$\phi(r, \theta) = \frac{2}{v} \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/2v} J_{n/v}(kr) \cos \frac{n\alpha}{v} \cos \frac{n\theta}{v} \right] \quad (26)$$

Equation 26 is the solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle and is considered to be extended from the solution by Stoker (1957) who only solved the problem of a thin semi-infinite breakwater. The solution in Equation 26 and the one by Stoker are not only in nonclosed form but also in terms of Bessel functions. It seems that the calculations of the solutions are very difficult without using a modern high-speed computer. This is probably the reason why Stoker arrived at his solution expressed in the same cosine series but did not use it to calculate the result. Instead, he further transformed the expression into a very complex integral form for further approximation in calculating the result.

16. Notably, the solution at the origin point is obtained by simply substituting  $r = 0$  into Equation 26 to arrive at

$$\phi(0, \theta) = \frac{2}{v} \quad (27)$$

Therefore, wave response at the origin point depends only on the wedge angle and does not depend on the incident wave angle.

#### Two Special Cases

17. The solutions for two special cases are used to verify Equation 26: one for the case of a thin semi-infinite breakwater and the other for the case of an infinite wall extending from  $x = -\infty$  to  $\infty$ .

18. The vertical wedge should reduce to a thin semi-infinite breakwater as the wedge angle reduces to 0 deg. Therefore, solution of the combined wave reflection and diffraction by a thin semi-infinite breakwater is obtained by substituting  $v = 2$  (that is,  $\theta_o = 2\pi$ ) into Equation 26 which then becomes

$$\phi(r, \theta) = J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/4} J_{n/2}(kr) \cos \frac{n\alpha}{2} \cos \frac{n\theta}{2} \quad (28)$$

Equation 28 is precisely the same one obtained by Stoker (1957).

19. The vertical wedge should also become an infinite wall extending from  $x = -\infty$  to  $\infty$  with the water occupying only the half plane of  $y \geq 0$  as the wedge angle increases to 180 deg. In this situation the scattered wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves as follows:

$$\phi(r, \theta) = e^{ikr} \cos(\theta - \alpha) + e^{ikr} \cos(\theta + \alpha) \quad (29)$$

After expansion of the exponential functions in terms of Bessel functions (Abramowitz and Stegun 1964), Equation 29 becomes

$$\phi(r, \theta) = 2 \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos n\alpha \cos n\theta \right] \quad (30)$$

Equation 30 is the same equation reduced from Equation 26 by substituting  $v = 1$  into it.

### PART III: CALCULATION AND RESULTS

20. Results of the combined reflection and diffraction by a wedge of arbitrary wedge angle can be calculated from Equation 26. Since the solution is not only in terms of Bessel functions but also in a nonclosed form, the computer program WEDGE is therefore written to calculate the solution.

21. In the program the subroutine BESJ for calculating Bessel function of fractional or integer order was used. The subroutine was originally written by Amos, Daniel, and Weston in 1975 (Morris 1984) and is collected in the Naval Surface Weapons Center Library of Mathematics Subroutines (Morris 1984).

22. In the calculation the summation of the infinite terms in Equation 26 was carried out to the term which is preceded by eight successive terms of the absolute value of the Bessel function, all equal to or less than  $10^{-8}$ . The solution has a truncation error less than  $10^{-8}$ , and it is of the order of one.

23. In this study, results of the combined wave reflection and diffraction for the wedge are calculated for two cases: one for a vertical wedge of 0-deg wedge angle and the other for a vertical wedge of 90-deg wedge angle.

#### Vertical Wedge of 0-Deg Wedge Angle

24. When the wedge angle is equal to zero, the wedge is actually a thin semi-infinite breakwater extending from  $x = 0$  to  $\infty$ . Figure 3 shows the thin semi-infinite breakwater along with the polar coordinates. In this case the diffraction results for various incident wave angles in the water region from  $\theta = \pi$  to  $2\pi$  and  $r/\lambda \leq 10$ , where  $\lambda$  is the incident wave length, have already been presented by Wiegel (1962) and are shown in the SPM, Volume I (1984). The present results combine reflection and diffraction effects and cover the water region from  $\theta = 0$  to  $2\pi$  and  $r/\lambda \leq 10$ . Therefore, the present results for this particular case can be considered to be a complementary and extended version to the ones in the SPM.

25. In this study wave response was calculated at 1,460 grid points intersected by  $r/\lambda = 0.5, (0.5), 10$ , which means that the values of  $r/\lambda$  are from 0.5 to 10.0 with each value increment being 0.5. Hereafter, all similar expressions are to be interpreted in the same way (e.g.,  $\theta = 0, (\pi/36), 2\pi$  for

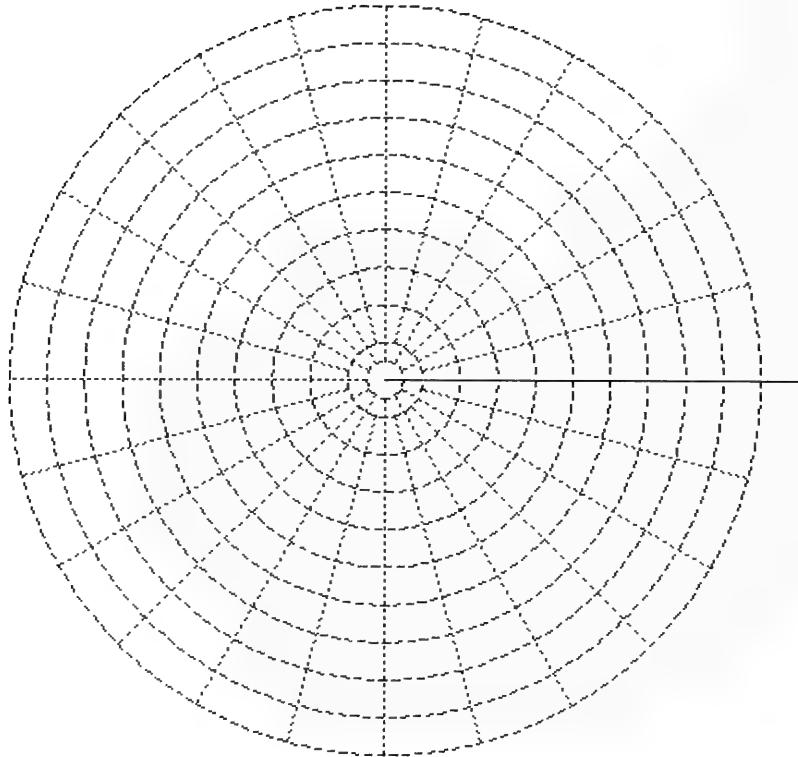


Figure 3. Thin semi-infinite breakwater and polar coordinates

the incident wave angle  $\alpha = 0, (\pi/12), \pi$ . The wave response at the origin point is obtained substituting  $v = 2$  into Equation 27, as follows:

$$\phi(0, \theta) = 1 \quad (31)$$

Those calculated values were used to interpret the value for each non-overlapping pixel of size  $0.1r/\lambda$  by  $0.1r/\lambda$  in the area within the  $10r/\lambda$  radius from the origin. A diagram was then constructed by patching those pixels over the entire area. The wave response diagrams for each incident wave angle are shown in Figures 4 through 15. Notably, the values in the diagrams constitute the amplification factor which is defined as the ratio of the total wave height to the incident wave height. Therefore, in subregions II

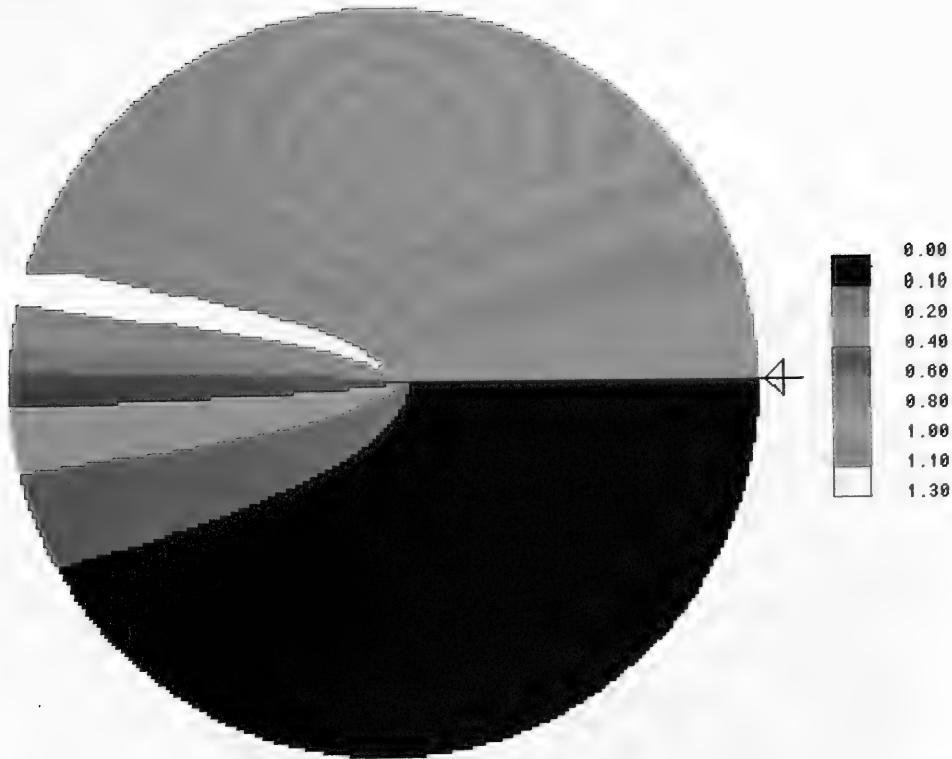


Figure 4. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 0 deg

and III (as defined in Figure 2) where the reflected wave is absent, the amplification factor is essentially the diffraction coefficient as defined in the SPM.

26. Figures 4 through 15 reveal that the amplification factors in subregion I change very rapidly between 0 and 2.35 over the subregion, and the diagram patterns become very complex because of the interesting superposition of the incident, reflected, and scattered waves. (In the legend of Figures 4 through 15, the width of the pixel is one incident wave length, and the values are amplification factors.) Such patterns would be very difficult to construct without using a high-speed computer and computer graphics. In subregions II and III, the amplification factors change smoothly from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly

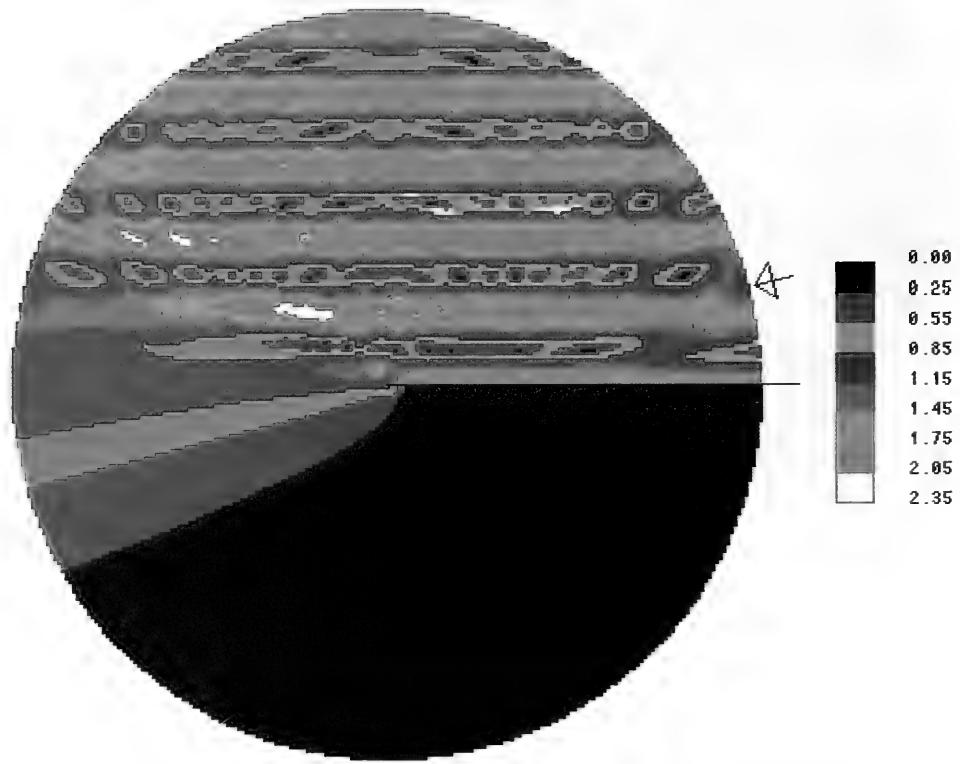


Figure 5. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 15 deg

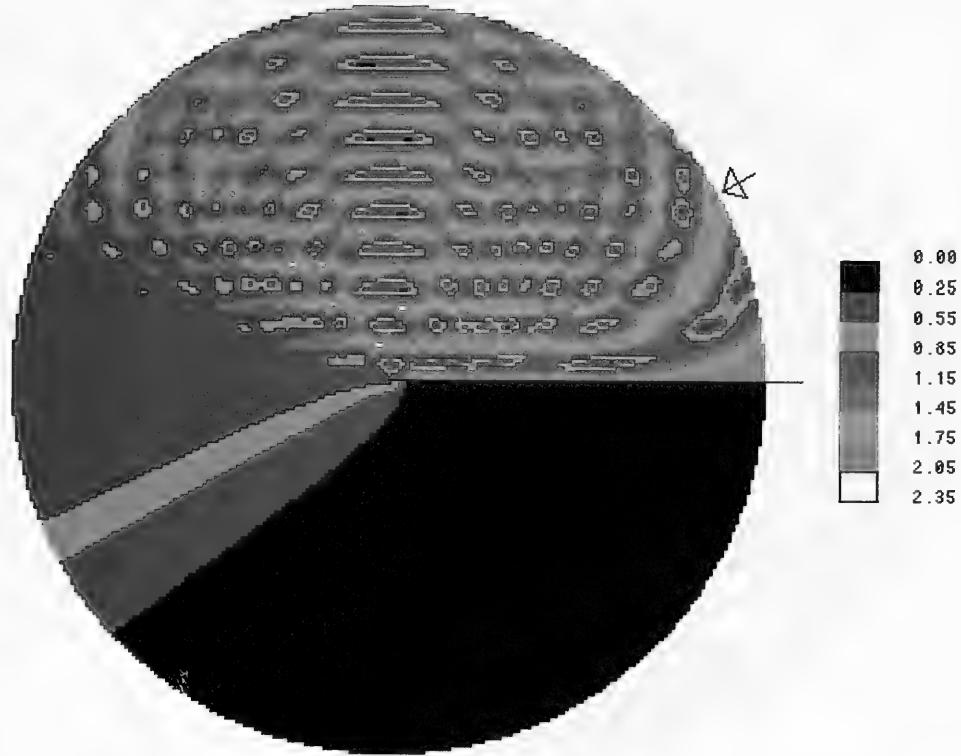


Figure 6. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 30 deg

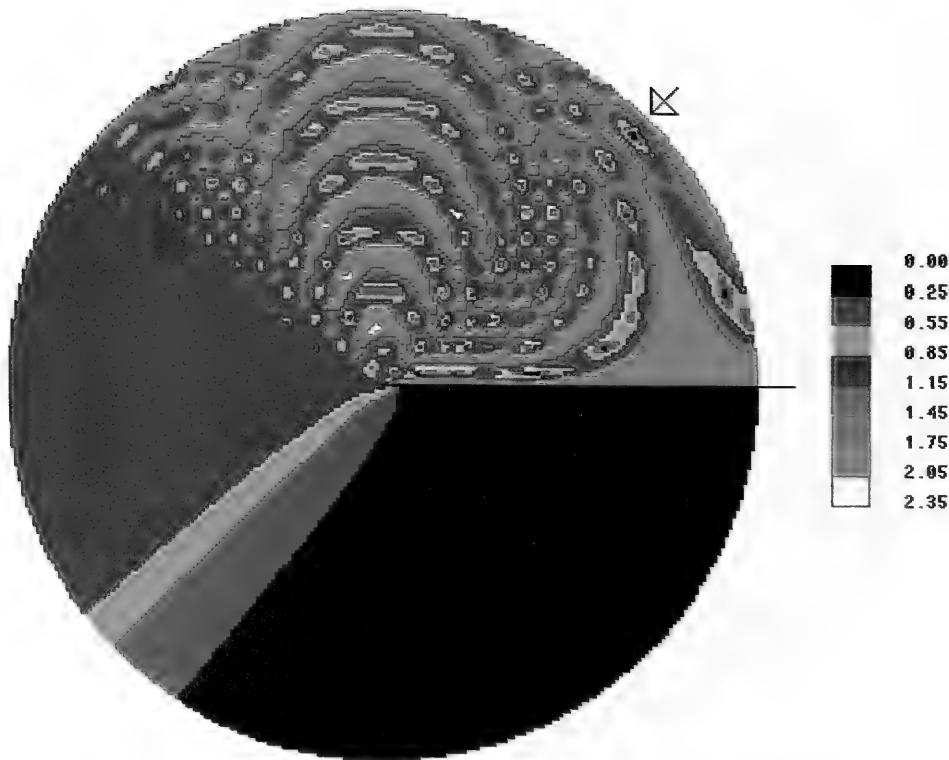


Figure 7. Amplification factor diagram for the thin semi-infinite  
breakwater for incident wave angle = 45 deg

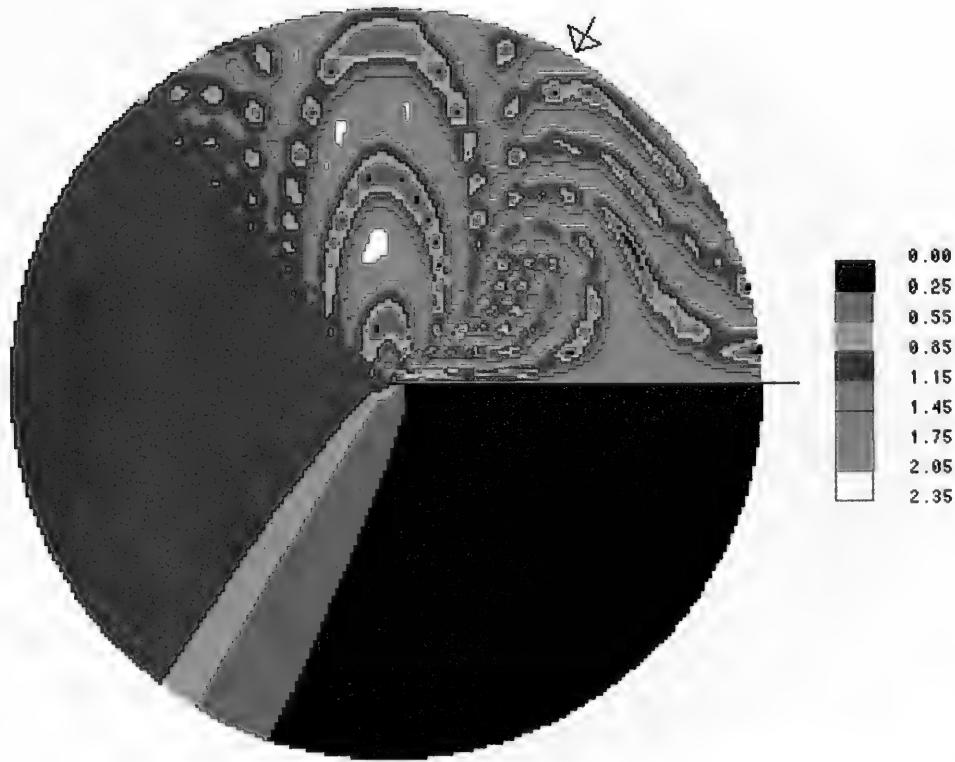


Figure 8. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 60 deg

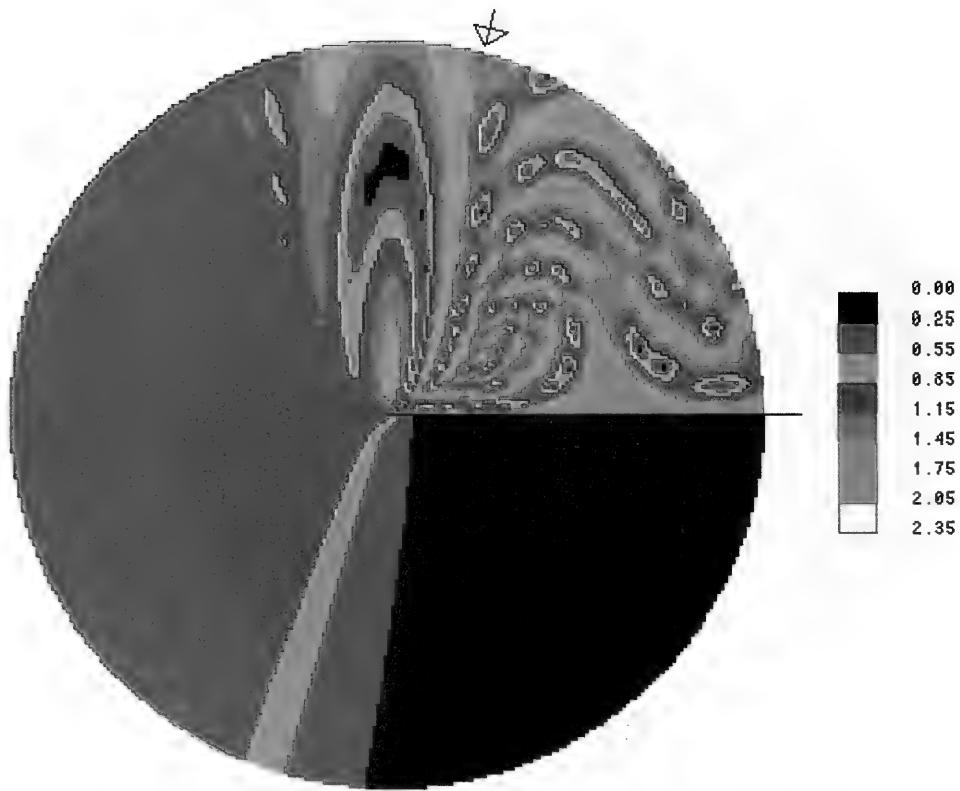


Figure 9. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 75 deg

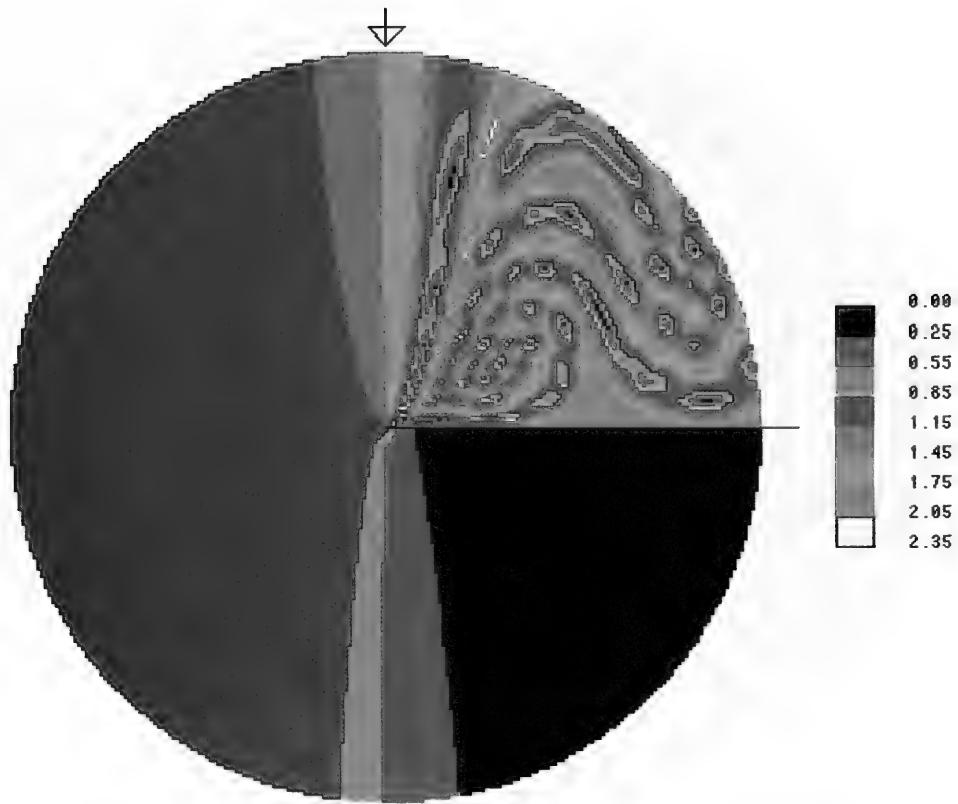


Figure 10. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 90 deg

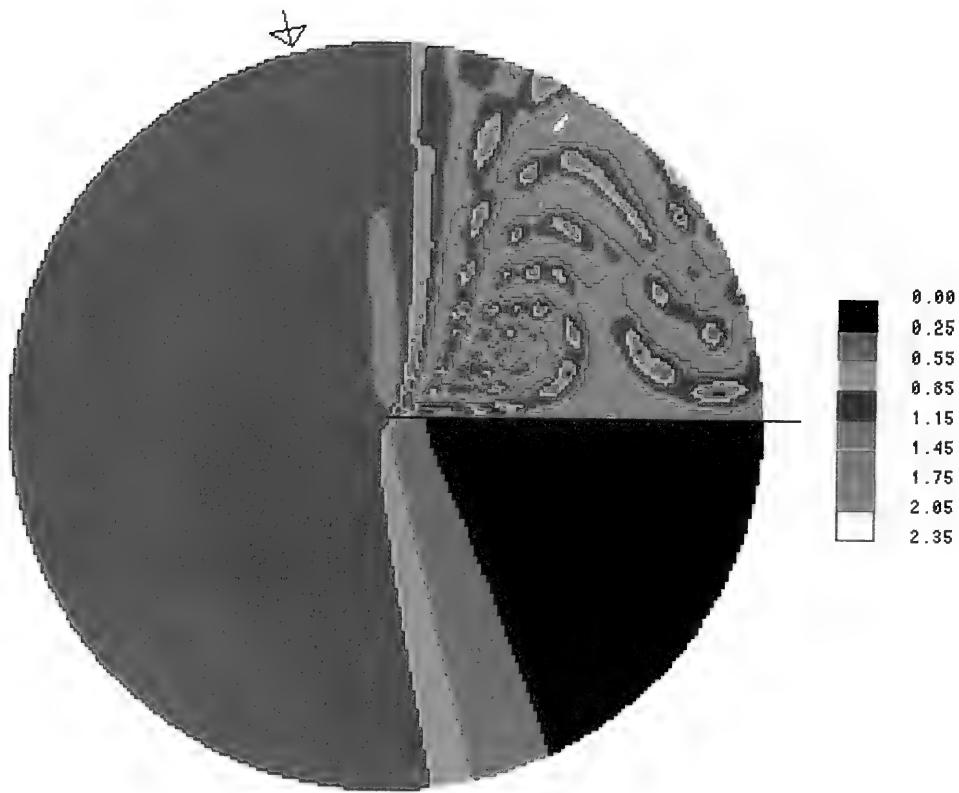


Figure 11. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 105 deg

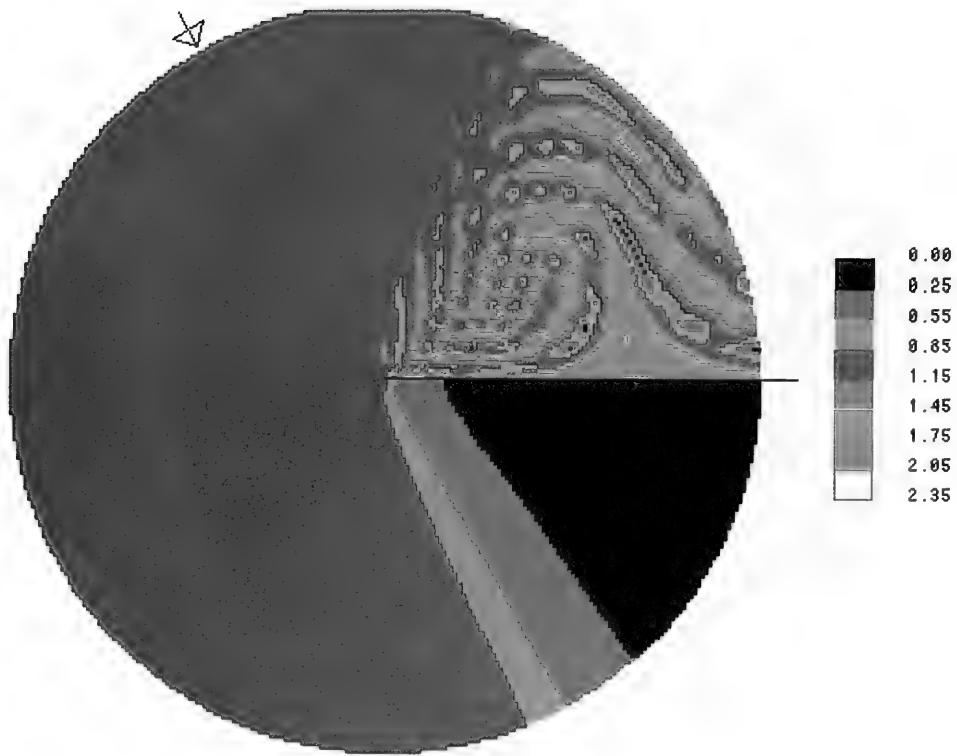


Figure 12. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 120 deg

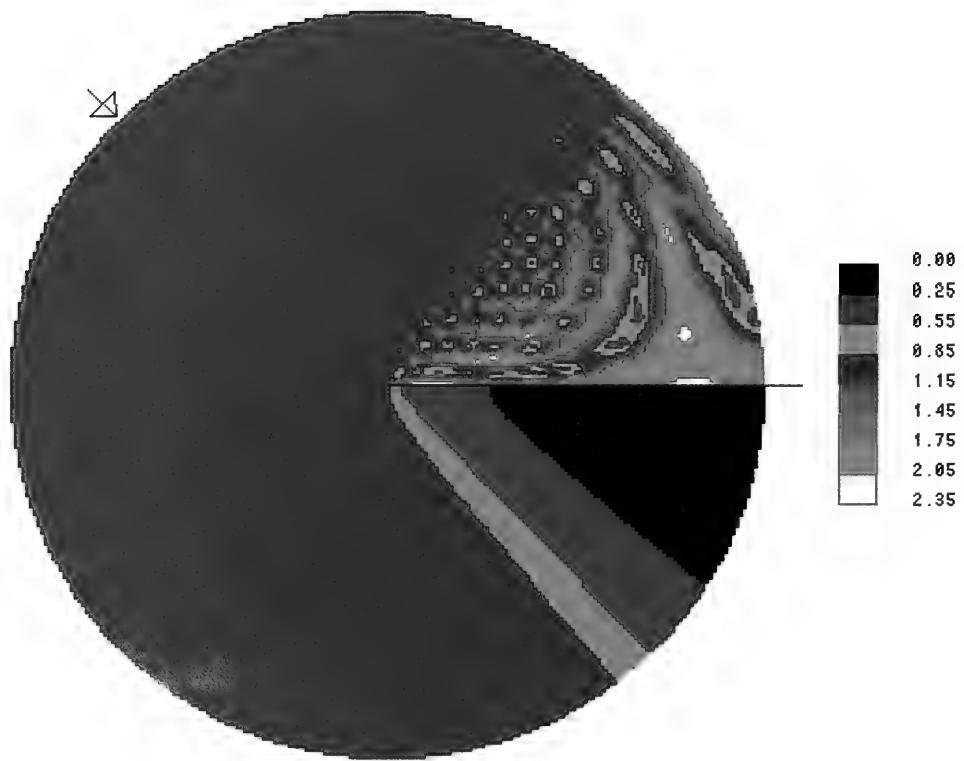


Figure 13. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 135 deg

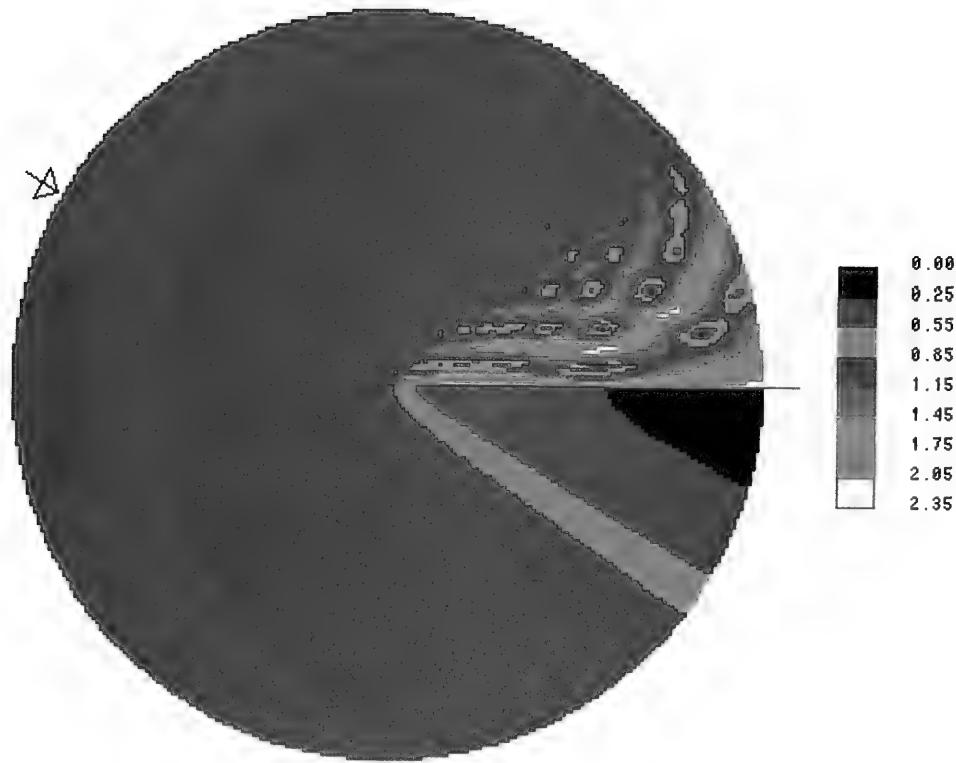


Figure 14. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 150 deg

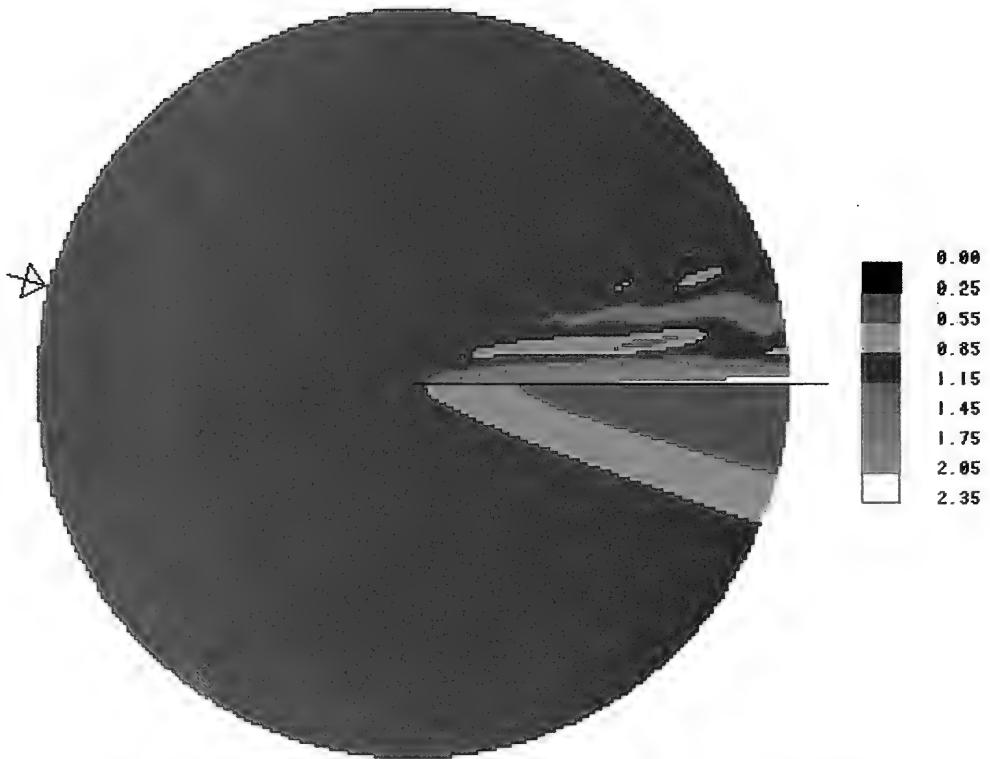


Figure 15. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 165 deg

0.00 at the back wall of the wedge in the shadow zone. The diagram patterns are relatively smooth and simple.

27. Notably, the diagrams do not include phase information of the wave response which is usually unimportant in most engineering practice. Should the phase of the wave response need to be known, one can always use the computer program WEDGE to calculate it.

28. The contour diagrams of the amplification factor, similar to the ones presented by Wiegel (1962) and shown in the SPM (1984), were also plotted as typically shown in Figure 16. Examination of those contour diagrams indicates that, in subregion III, the results are identical to Wiegel's results. But in subregion II the contour patterns for the amplification factor (or the diffraction coefficient  $K'$  used in the SPM (1984)) equal to 1.0; thus the present results are far more complicated than Wiegel's. The author believes that Wiegel's results may lose accuracy because of insufficient resolution of the computational tools during the late fifties and early sixties. Nevertheless, such inaccuracies are usually either tolerable or immaterial in most engineering practice.

29. The contour patterns in subregion I are very complex, and it is difficult to track specific contours. Therefore, for clarity only the patched diagrams are presented, and the contour diagrams are omitted in this report.

#### Vertical Wedge of 90-Deg Wedge Angle

30. When the wedge angle is equal to  $\pi/2(\theta_0 = 3\pi/2)$ , the vertical wedge occupies the entire fourth quadrant of the space as shown in Figure 17. Wave response was calculated at 1,100 grid points intersected at  $r/\lambda = 0.5, (0.5), 10.0$  and  $\theta = 0, (\pi/36), 3\pi/2$  for the incident wave angle  $\alpha = 0, (\pi/12), \pi$ . The wave response at the origin is obtained by substituting  $v = 1.5$  into Equation 27 as follows:

$$\phi(0, \theta) = \frac{4}{3} \quad (32)$$

Those calculated results were used to construct the amplification factor diagrams by following the same procedures described for the case of the thin semi-infinite breakwater. The diagrams are shown in Figures 18 through 27. Because of symmetry of the results, the diagrams for the incident wave angle  $\alpha > 3\pi/4$  can be obtained from those for an incident wave angle of  $\pi - \alpha$ .

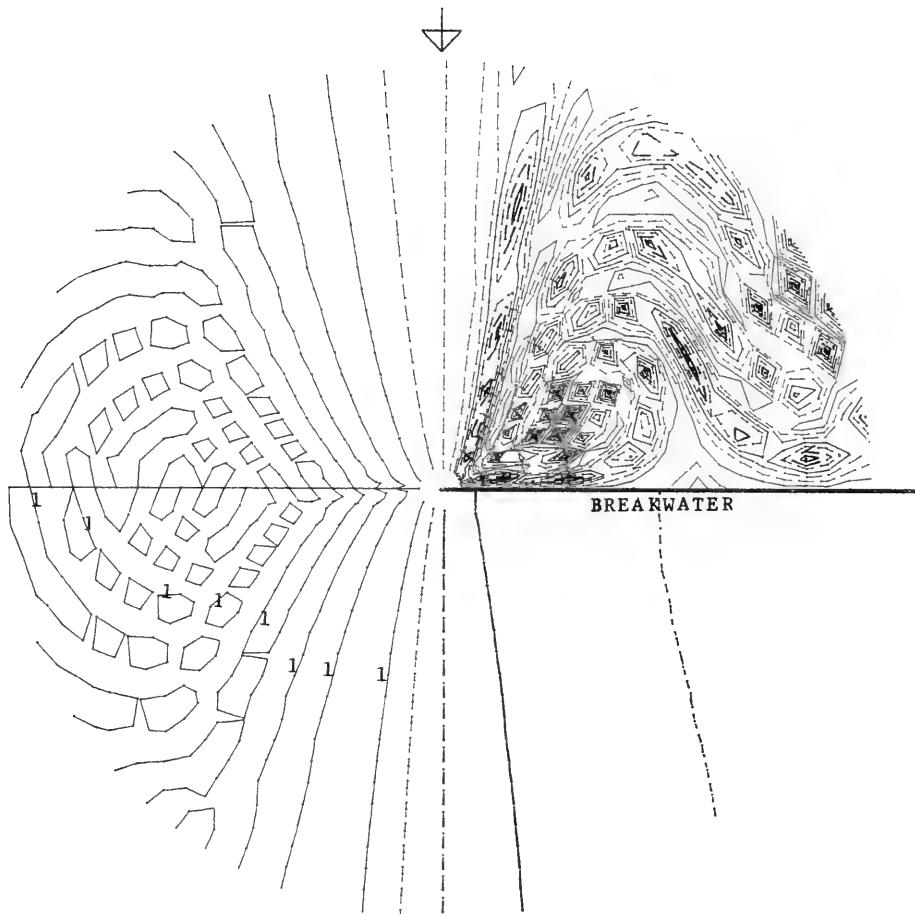


Figure 16. Amplification factor contour diagram for the thin semi-infinite breakwater for incident wave angle = 90 deg

Therefore, the diagrams for the incident wave angles  $\alpha = \pi/6, (\pi/12), \pi$  are omitted in this report.

31. The diagrams indicate that, for each corresponding incident wave angle, the results in subregion I are very similar to those obtained for the vertical wedge of 0-deg wedge angle. However, the results in subregions II and III from both wedges are discernibly different.

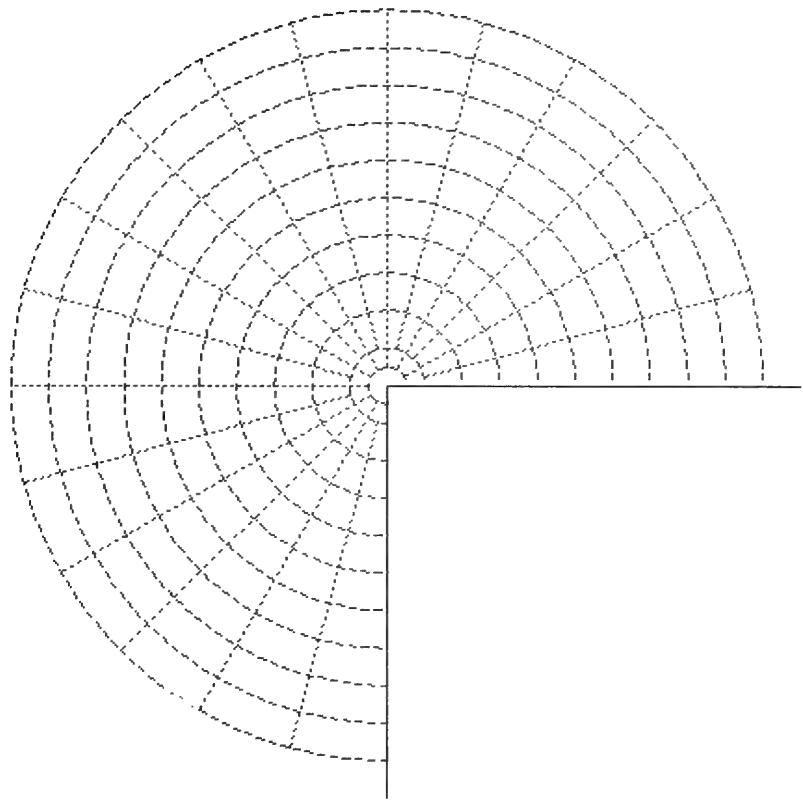


Figure 17. A 90-deg wedge and polar coordinates

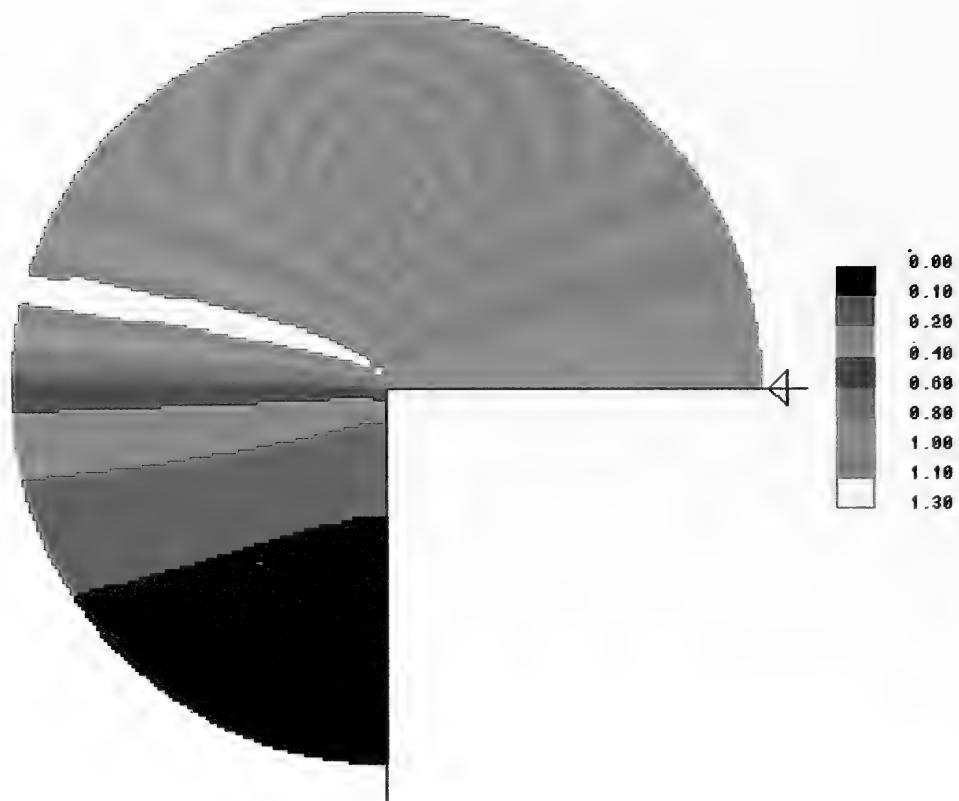


Figure 18. Amplification factor diagram for the  
90-deg wedge for incident wave angle = 0 deg

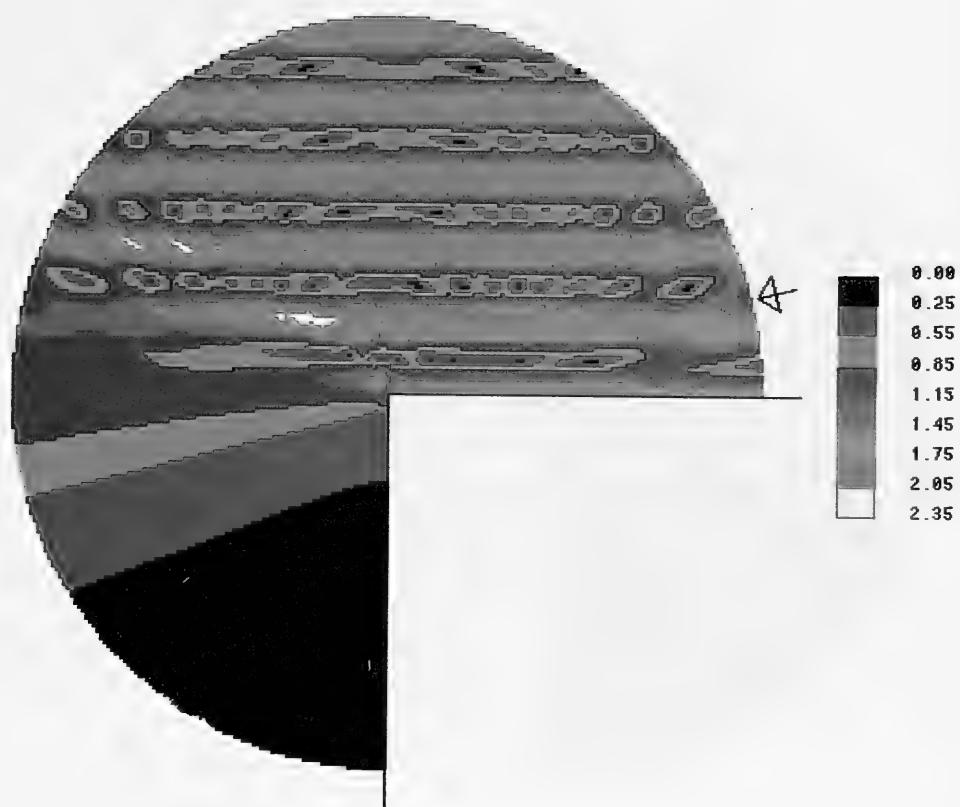


Figure 19. Amplification factor diagram for the  
90-deg wedge for incident wave angle = 15 deg

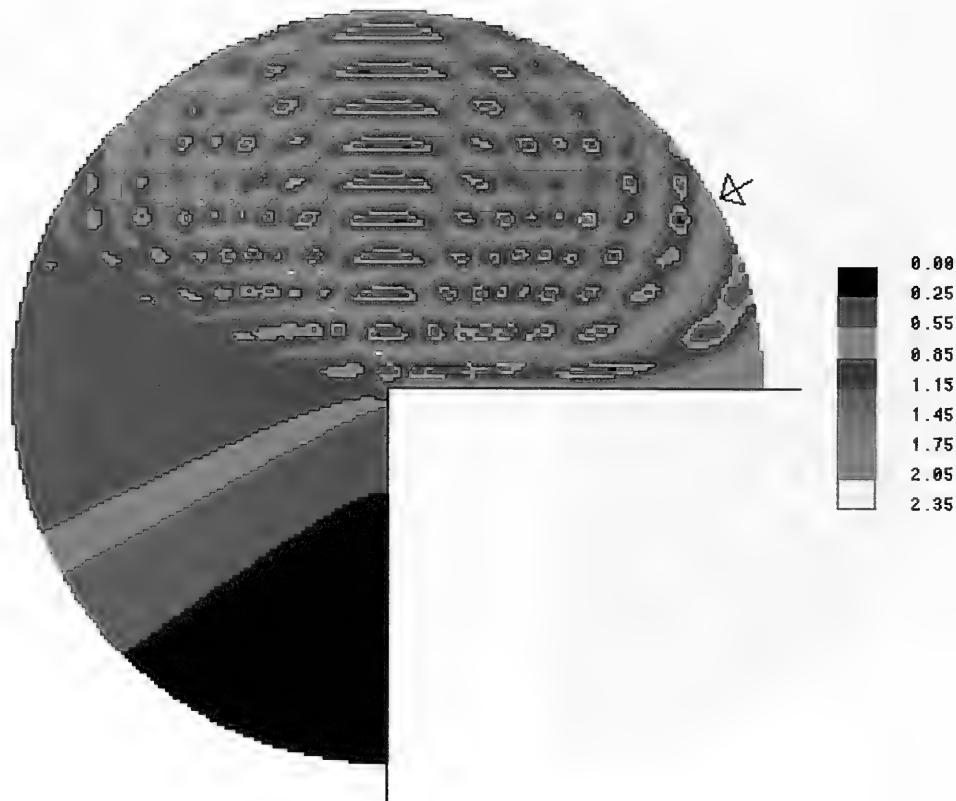


Figure 20. Amplification factor diagram for the 90-deg wedge for incident wave angle = 30 deg

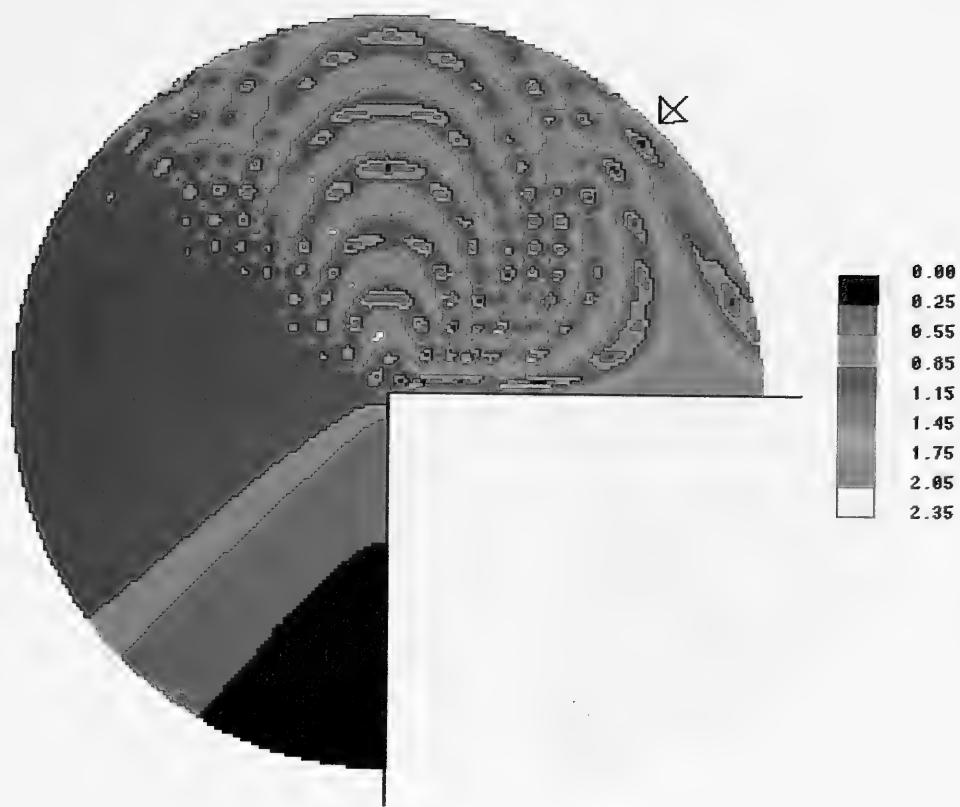


Figure 21. Amplification factor diagram for the 90-deg wedge for incident wave angle = 45 deg

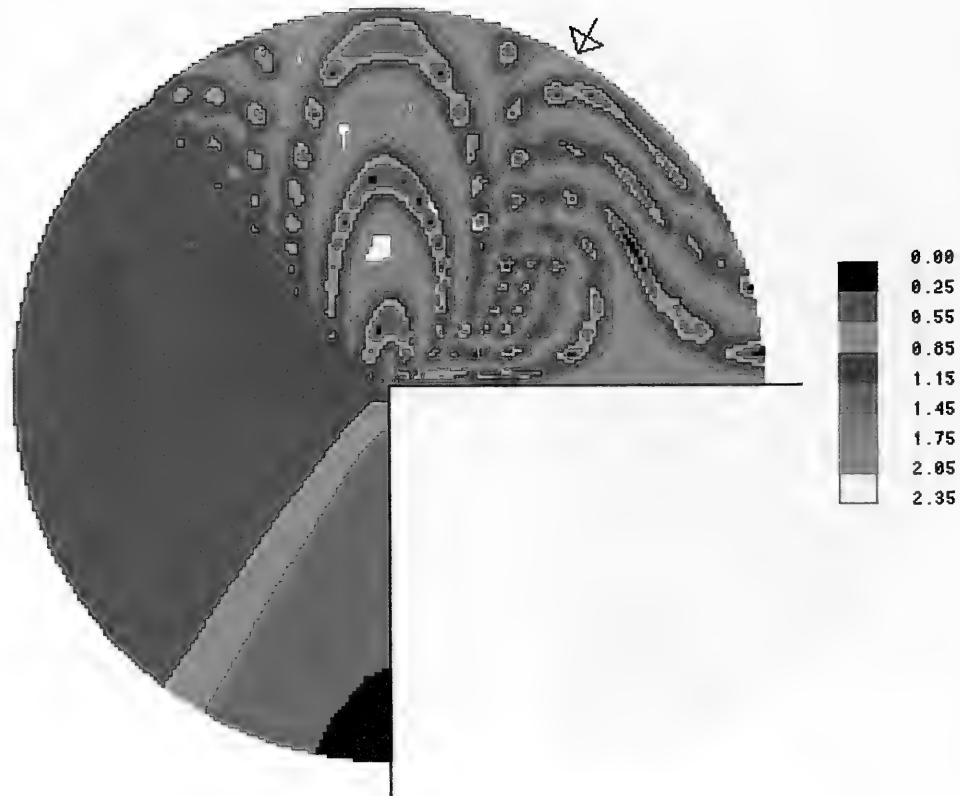


Figure 22. Amplification factor diagram for the 90-deg wedge for incident wave angle = 60 deg

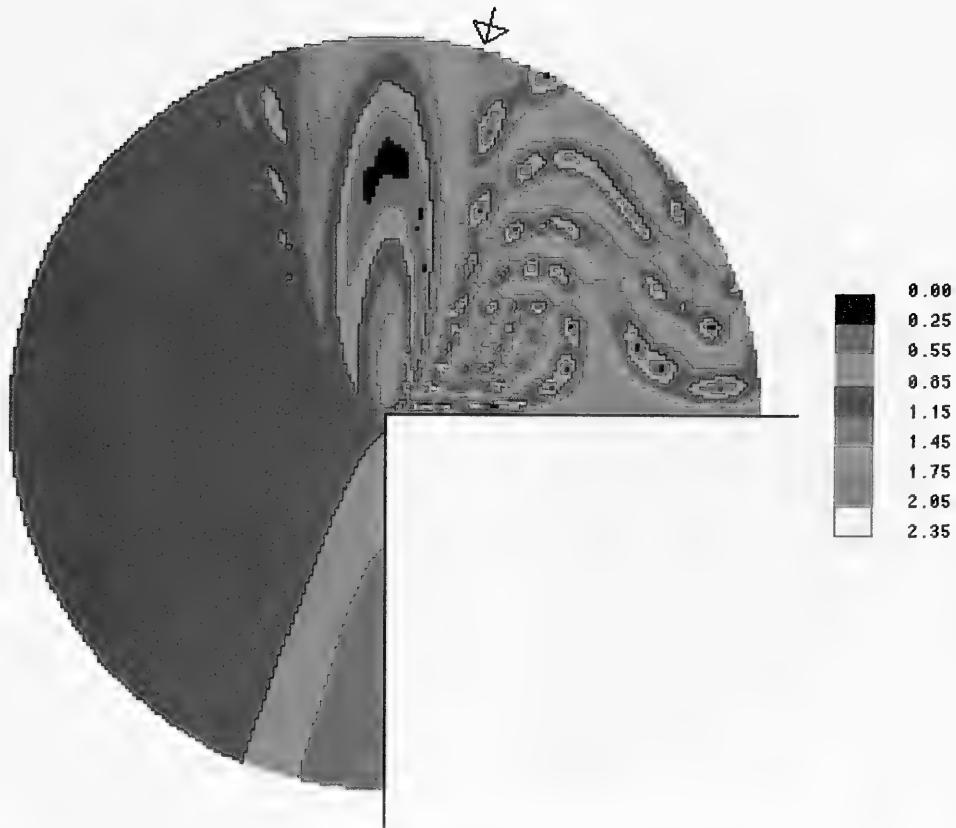


Figure 23. Amplification factor diagram for the  
90-deg wedge for incident wave angle = 75 deg

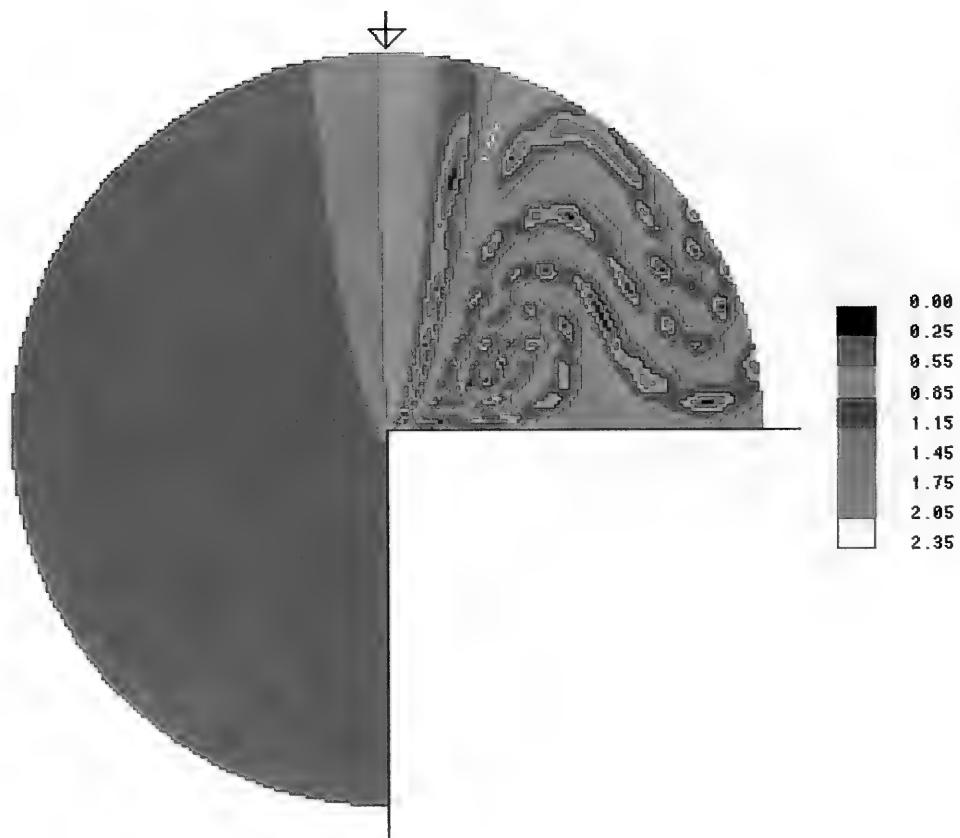


Figure 24. Amplification factor diagram for the  
90-deg wedge for incident wave angle = 90 deg

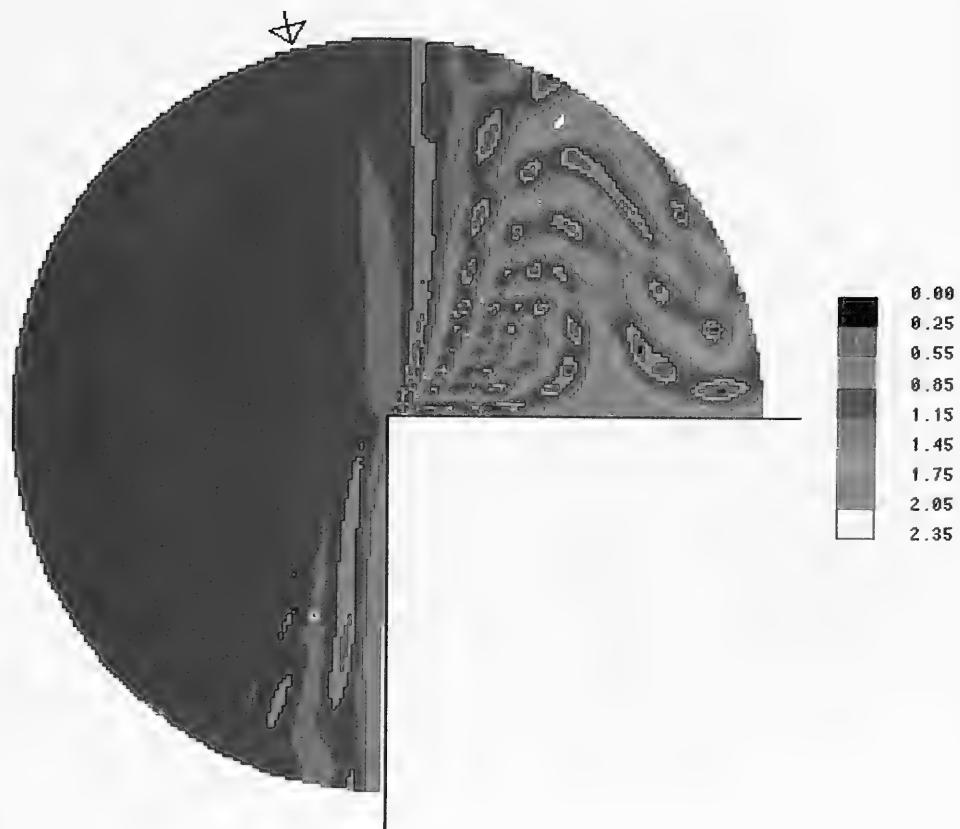


Figure 25. Amplification factor diagram for the 90-deg wedge for incident wave angle = 105 deg

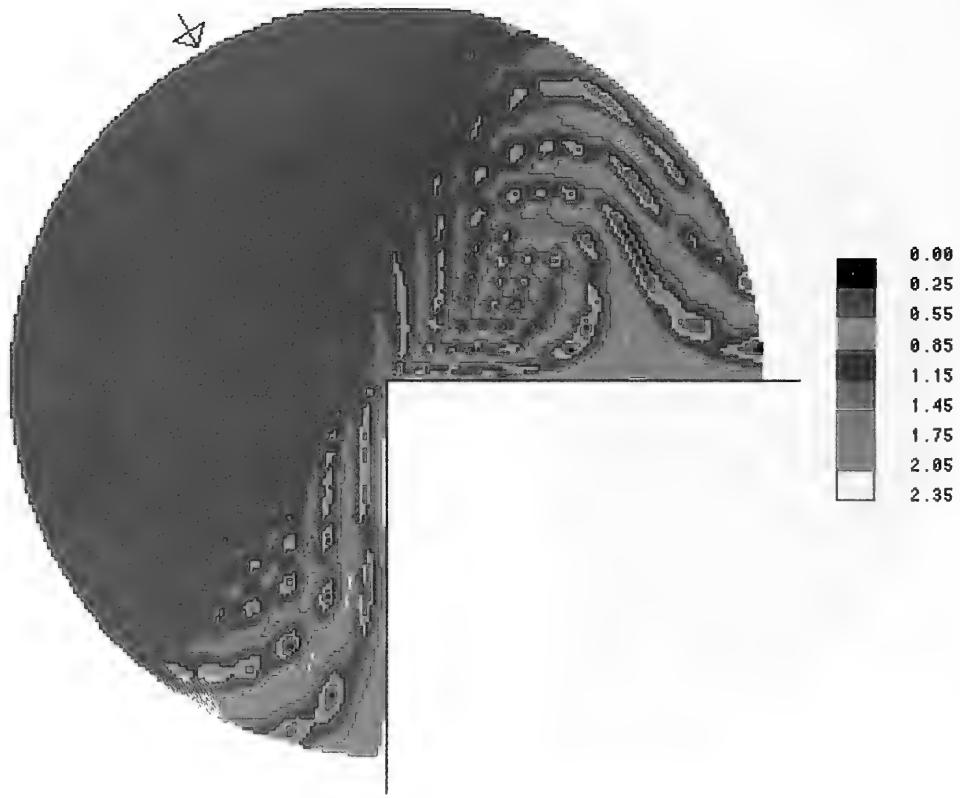


Figure 26. Amplification factor diagram for the  
90-deg wedge for incident wave angle = 120 deg

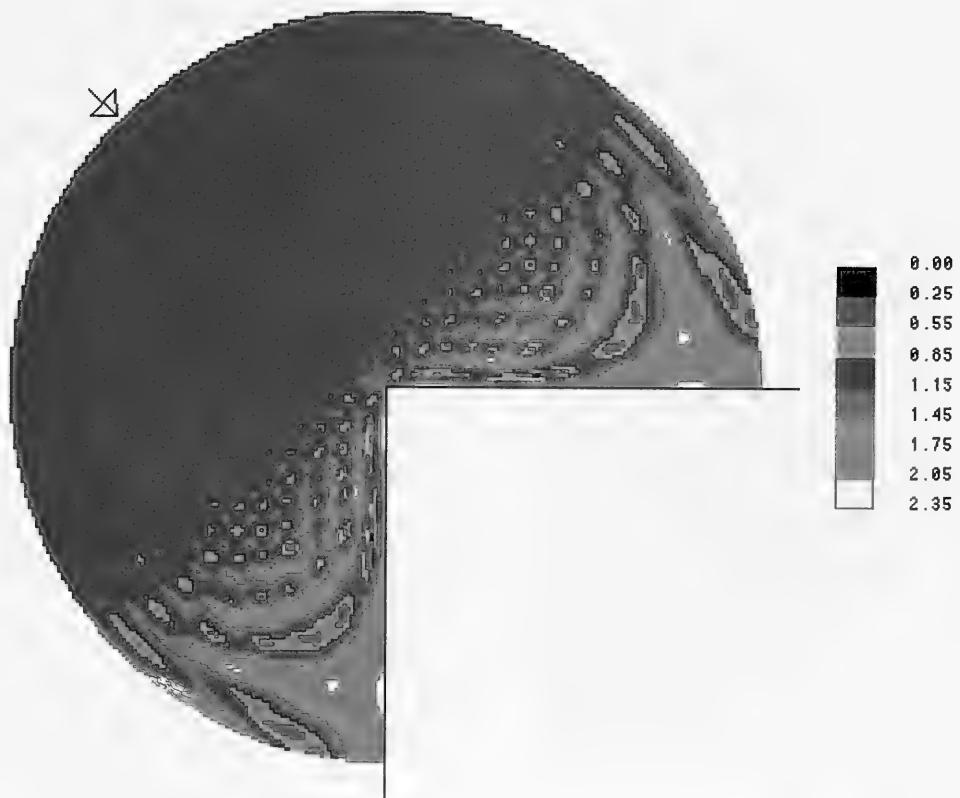


Figure 27. Amplification factor diagram for the 90-deg wedge for incident wave angle = 135 deg

#### PART IV: CONCLUSION

32. An analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle is obtained and expressed in Equation 26. The analytical solution is in terms of Bessel functions and in nonclosed form. The computer subroutine WEDGE, written for calculating the solution, is documented in Appendix A.

33. The amplification factor diagrams for a vertical wedge of 0-deg wedge angle and a vertical wedge of 90-deg wedge angle are calculated and presented. The calculated results indicate that the wave response in subregion I, where the incident, reflected, and scattered waves all exist, is in a very complex pattern with the amplification factor varying from 2.35 to 0.0 over the subregion. The wave response in subregions II and III is a relatively simple pattern with the amplification factor decreasing from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly 0.00 at the back wall of the wedge in the shadow zone.

34. Diagrams of the special case of a vertical wedge of 0-deg wedge angle can be considered complementary and extended versions to the ones presented in the SPM (1984).

#### REFERENCES

- Abramowitz, M., and Stegun, I. A. 1964 (Jun). Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, pp 358-495.
- Morris, A. H., Jr. 1984 (Jun). "NSWC Library of Mathematics Subroutines," NSWC TR 84-143, Strategic Systems Department, Naval Surface Weapons Center, Dahlgren, Va., pp 43-44.
- Shore Protection Manual. 1984. 4th ed., 2 vols, US Army Engineer Waterways Experiment Station, Coastal Engineering Research Center, US Government Printing Office, Washington, DC.
- Stoker, J. J. 1957. Water Waves, Interscience Publishers, Inc., New York, pp 109-133.
- Wiegel, R. L. 1962. (Jan). "Diffraction of Waves by a Semi-infinite Breaker," Journal of the Hydraulics Division, American Society of Civil Engineers, Vol 88, No. HY1, pp 27-44.



## APPENDIX A: SUBROUTINE WEDGE

1. Subroutine WEDGE is used to calculate the value of  $\phi$  in Equation 26 which is generally a complex number. Its absolute value is the amplification factor, and its phase is the phase indicator from the phase of the incident wave. As mentioned in the main text,  $\phi$  is a function of the Bessel function of either fractional or integer order, depending on the wedge angle, and is the summation of a series of infinite terms. The subroutine BESJ, documented in the Naval Surface Weapons Center (NSWC) Library of Mathematics Subroutines (Morris 1984)\* is used in the WEDGE subroutine. The programming of the WEDGE subroutine is very straightforward if a truncation term in the series in Equation 26 is determined. The program is written in FORTRAN language and is listed in this appendix.

### Description

2. The following subroutine is available for computing  $\phi$  in Equation 26:

```
CALL WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)
```

where the arguments are all real values except F which is a complex value.

Input arguments are as follows:

- a. (XRL,XTH)=( $r/\lambda, \theta$ ) where  $(r, \theta)$  are polar coordinates of the location where  $\phi$  is to be computed, and  $\lambda$  is the incident wave length. Therefore, XRL is the radius vector or radius distance normalized by the incident wave length. XTH is the vectorial angle in degree.
- b. WEDGEA = wedge angle in degree.
- c. WAVEA = incident wave angle in degree.
- d. IDX = an index (set to 0 in this subroutine).

Output arguments are as follows:

---

\* References cited in the Appendix can be found in the References at the end of the main text.

- a.  $F = \phi$  in Equation 26 (wave response normalized by incident wave amplitude).
- b. FABS = amplification factor, the absolute value of  $\phi$ .
- c. FPRA = phase difference, the phase of  $\phi$ .

Example and Test Run

3. To serve as an example as well as to ensure the subroutine is the correct one, the user should run the test program listed in Figure A1 and make sure the output is the same as that listed in Table A1.

```

PROGRAM WEDGE1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)      TEST1
COMPLEX F
PI=3.141592654                                         TEST2
WRITE(6,4)                                               TEST3
4 FORMAT(//3X,' WEDG-ANG WAV-ANG          LOCATION      WAVE RESPONSE TESTS
1 ABS-VAL   PHASE'/3X,' (DEG)      (DEG)      XRL  XTH(DEG)  (NORMTEST6
2 ALIZED)    AMP-FAC   (RAD)'/3X,35(' -'))           TEST7
WEDGEA=90.                                              TEST8
WAVEA=135.                                              TEST9
XRL=2.0                                                 TEST10
XTH=30.                                                 TEST11
IDX=0                                                   TEST12
CALL WEDGE(F,FABS,FPRA,XRL,XTH,WEDGEA,WAVEA,IDX)      TEST13
WRITE(6,40) WEDGEA,WAVEA,XRL,XTH,F,FABS,FPRA          TEST14
40 FORMAT(1X,BF9.2)                                     TEST15
STOP                                                    TEST16
END                                                     TEST17

```

Figure A1. Computer program list 1

Table A1

Sample Output of the Test Program

---

| WEDG-ANG | WAV-ANG | LOCATION | WAVE RESPONSE | ABS-VAL      | PHASE         |
|----------|---------|----------|---------------|--------------|---------------|
| (DEG)    | (DEG)   | XRL      | XTH(DEG)      | (NORMALIZED) | AMP-FAC (RAD) |
| 90.00    | 135.00  | 2.00     | 30.00         | -.41         | .68 .79 2.11  |

---

4. The input arguments in the test programs are as follows:

- a. WEDGEA = 90; the wedge angle is 90 deg.
- b. WAVEA = 135; the incident wave angle is 135 deg.
- c. XRL = 2.0 and XTH = 30; the location of the wave response  $\phi$  to be computed is at radial vector of two incident wave length distances and vectorial angle of 30 deg in polar coordinates.
- d. IDX = 0; the index IDX is set to 0.

This case is shown in Figure A2. The outputs are given in Table A1 which is self-explanatory.

5. If the location of the wave response  $\phi$  to be computed is a very large distance, for example XRL greater than 18 or so, the output might print the message that the number of terms is insufficient for summation in computing  $\phi$ . In this situation, the user must replace the integer 200 in PARAMETER(NN=200), listed in Card WEDGE15 in the SUBROUTINE WEDGE list, by a larger integer to ensure the accuracy.

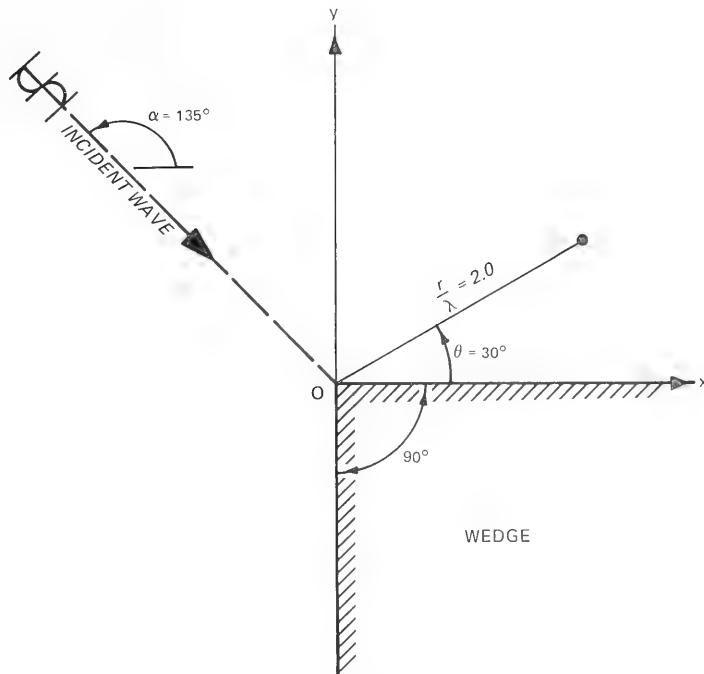


Figure A2. Example problem

Subroutine WEDGE Listing

6. Subroutine WEDGE, which is listed in this section, calls subroutine BESJ which, in turn, calls subroutine JAIRY and function GAMLN. Subroutines BESJ and JAIRY and function GAMLN are borrowed from the NSWC Library of Mathematics Subroutines (Morris 1984). Including these borrowed subroutines here in the list is only for the purpose of allowing subroutine WEDGE to be self-contained and complete. The computer program of subroutine WEDGE is listed in Figure A3.

Figure A3. Computer program list 2 (Sheet 1 of 25)

```

IF(ABS(BJ(N1)).GT.TDLR) ICOUNT=0
10 CONTINUE
14 CONTINUE
F=BJ(1)/2.
DO 20 N=1,NNN
N1=N+1
XMN=XM(N1)
TM=(0.0,1.0)**XMN*BJ(N1)*COS(XMN*WA)*COS(XMN*TH)
F=F+TM
20 CONTINUE
F=4./XNU*F
FR=REAL(F)
FI=AIMAG(F)
FABS=SQRT(FR*FR+FI*FI)
IF(WA.LE.1.E-8) FABS=FABS/2.
IF(FABS.LT.TDLR) GOTO 30
FPHA=ATAN2(FI,FR)
RETURN
30 FPFA=0.0
RETURN
END
SUBROUTINE BESJ(X,ALPHA,N,Y,NZ)
C -----
C WRITTEN BY D.E. AMOS, S.L. DANIEL AND M.K. WESTON, JANUARY, 1975. BESJ102
C REFERENCE SAND-75-0147 BESJ103
C BESJ104
C BESJ105
C ABSTRACT BESJ106
C BESJ COMPUTES AN N MEMBER SEQUENCE OF J BESSEL FUNCTIONS BESJ107
C J/SUB(ALPHA+K-1)/(X), K=1,...,N FOR NON-NEGATIVE ALPHA AND X. BESJ108
C A COMBINATION OF THE POWER SERIES, THE ASYMPTOTIC EXPANSION BESJ109
C FOR X TO INFINITY AND THE UNIFORM ASYMPTOTIC EXPANSION FOR BESJ110
C NU TO INFINITY ARE APPLIED OVER SUBDIVISIONS OF THE (NU,X) BESJ111
C PLANE. FOR VALUES OF (NU,X) NOT COVERED BY ONE OF THESE BESJ112
C FORMULAE, THE ORDER IS INCREMENTED OR DECREMENTED BY INTEGER BESJ113
C VALUES INTO A REGION WHERE ONE OF THE FORMULAE APPLY. BACKWARD BESJ114
C RECURSION IS APPLIED TO REDUCE ORDERS BY INTEGER VALUES EXCEPT BESJ115
C WHERE THE ENTIRE SEQUENCE LIES IN THE OSCILLATORY REGION. IN BESJ116
C THIS CASE FORWARD RECURSION IS STABLE AND VALUES FROM THE BESJ117
C ASYMPTOTIC EXPANSION FOR X TO INFINITY START THE RECURSION BESJ118
C WHEN IT IS EFFICIENT TO DO SO. LEADING TERMS OF THE SERIES AND BESJ119
C UNIFORM EXPANSION ARE TESTED FOR UNDERFLOW. IF A SEQUENCE IS BESJ120
C REQUESTED AND THE LAST MEMBER WOULD UNDERFLOW, THE RESULT IS BESJ121
C SET TO ZERO AND THE NEXT LOWER ORDER TRIED, ETC., UNTIL A BESJ122
C MEMBER COMES ON SCALE OR ALL MEMBERS ARE SET TO ZERO. OVERFLOW BESJ123
C CANNOT OCCUR. BESJ1 CALLS SUBROUTINE JAIRY AND FUNCTION GAMLN. BESJ124
C BESJ125
C DESCRIPTION OF ARGUMENTS BESJ126
C BESJ127
C INPUT BESJ128
C X - X.GE.0 BESJ129
C ALPHA - ORDER OF FIRST MEMBER OF THE SEQUENCE, ALPHA.GE.0 BESJ130
C N - NUMBER OF MEMBERS IN THE SEQUENCE, N.GE.1 BESJ131
C OUTPUT BESJ132
C Y - A VECTOR WHOSE FIRST N COMPONENTS CONTAIN BESJ133
C VALUES FOR J/SUB(ALPHA+K-1)/(X), K=1,...,N BESJ134

```

Figure A3. (Sheet 2 of 25)

```

C      NZ      - ERROR INDICATOR                                BESJ135
C      NZ=0      NORMAL RETURN - COMPUTATION COMPLETED          BESJ136
C      NZ=-1      X IS LESS THAN 0.0                            BESJ137
C      NZ=-2      ALPHA IS LESS THAN 0.0                         BESJ138
C      NZ=-3      N IS LESS THAN 1                           BESJ139
C      NZ.GT.0    LAST NZ COMPONENTS OF Y SET TO 0.0           BESJ140
C                           BECAUSE OF UNDERFLOW                  BESJ141
C
C      ERROR CONDITIONS                                     BESJ142
C
C      IMPROPER INPUT ARGUMENTS - A FATAL ERROR             BESJ143
C      UNDERFLOW - A NON-FATAL ERROR (NZ.GT.0)               BESJ144
C
C-----BESJ147
C-----BESJ148
C-----BESJ149
C      DOUBLE PRECISION DX,TRX,DTM,DFN                      BESJ150
C      DIMENSION Y(N)                                       BESJ151
C      DIMENSION C(11,10),ALFA(26,4),BETA(26,5)            BESJ152
C      DIMENSION C1(88),C2(22)                             BESJ153
C      DIMENSION A1(52),A2(52),B1(52),B2(52),B3(26)       BESJ154
C      DIMENSION GAMA(26),TEMP(3),KMAX(5),AR(8),BR(10),UPOL(10) BESJ155
C      DIMENSION FNULIM(2),PP(4)                           BESJ156
C      DIMENSION CR(10),DR(10)                          BESJ157
C
C      EQUIVALENCE (C(1,1),C1(1))                         BESJ159
C      EQUIVALENCE (C(1,9),C2(1))                         BESJ160
C      EQUIVALENCE (ALFA(1,1),A1(1))                      BESJ161
C      EQUIVALENCE (ALFA(1,3),A2(1))                      BESJ162
C      EQUIVALENCE (BETA(1,1),B1(1))                      BESJ163
C      EQUIVALENCE (BETA(1,3),B2(1))                      BESJ164
C      EQUIVALENCE (BETA(1,5),B3(1))                      BESJ165
C
C      DATA ELIM1,ELIM2,TOL /   667. ,     644. ,     1.E-15 /BESJ167
C
C      DATA PP(1)/8.7290915393555E+00/, PP(2)/2.6569393226503E-01/, BESJ168
C      1    PP(3)/1.2457857686559E-01/, PP(4)/7.7013374743039E-04/ BESJ170
C
C      TOLS=LN(1.E-3)                                    BESJ171
C      DATA TOLS           /-6.9077552789821E+00/        BESJ172
C
C      DATA UPOL(1),CON1,CON2,CON3,CON548           / 1.00000000000000E+00, BESJ175
C      1  6.6666666666667E-01, 3.3333333333333E-01, 1.4142135623731E+00, BESJ176
C      2  1.0416666666667E-01/                         BESJ177
C
C      DATA RTWO,PDF,RTTP,PIDT           / 1.3483997249265E+00, BESJ179
C      1 7.8539816339745E-01, 7.9788456080286E-01, 1.5707963267949E+00/ BESJ180
C
C      DATA FNULIM(1)/100./, FNULIM(2)/60./          BESJ181
C
C      CE=-ALOG(TOL) , TCE=-0.75*ALOG(TOL)          BESJ183
C      DATA CE , TCE           / 3.4538776394911E+01, 2.5904082296183E+01/ BESJ184
C
C      DATA INLIM           /      150           /          BESJ186
C
C      DATA AR(1)/8.355034722222E-02/, AR(2)/1.2822657455633E-01/, BESJ189A

```

Figure A3. (Sheet 3 of 25)

```

1      AR(3)/2.9184902646414E-01/, AR(4)/8.8162726744376E-01/, BESJ189B
2      AR(5)/3.3214082818628E+00/, AR(6)/1.4995762986863E+01/, BESJ189C
3      AR(7)/7.8923013011586E+01/, AR(8)/4.7445153886826E+02/ BESJ189D
4      AR(9)/7.8923013011586E+01/, AR(10)/4.7445153886826E+02/ BESJ190
C      DATA BR(1) /-1.45833333333333E-01/, BR(2) /-9.8741319444444E-02/, BSEJ191A
1      BR(3) /-1.4331205391590E-01/, BR(4) /-3.1722720267841E-01/, BSEJ191B
2      BR(5) /-9.4242914795712E-01/, BR(6) /-3.5112030408264E+00/, BSEJ191C
3      BR(7) /-1.5727263620368E+01/, BR(8) /-8.2281439097186E+01/, BSEJ191D
4      BR(9) /-4.9235537052367E+02/, BR(10)/-3.3162185685480E+03/ BSEJ191E
C      DATA C1(1) /-2.08333333333333E-01/, C1(2) / 1.25000000000000E-01/, BESJ193A
1      C1(3) / 0.0/, C1(4) / 0.0/, C1(5) / 0.0/, C1(6) / 0.0/, BESJ193B
2      C1(7) / 0.0/, C1(8) / 0.0/, C1(9) / 0.0/, C1(10) / 0.0/, BESJ193C
3      C1(11) / 0.0/, C1(12) / 3.342013888889E-01/, BESJ193D
4      C1(13) /-4.0104166666667E-01/, C1(14) / 7.03125000000000E-02/, BESJ193E
5      C1(15) / 0.0/, C1(16) / 0.0/, C1(17) / 0.0/, C1(18) / 0.0/, BESJ193F
6      C1(19) / 0.0/, C1(20) / 0.0/, C1(21) / 0.0/, C1(22) / 0.0/, BESJ193G
7      C1(23) /-1.0258125964506E+00/, C1(24) / 1.8464626736111E+00/, BESJ193H
8      C1(25) /-8.9121093750000E-01/, C1(26) / 7.3242187500000E-02/, BESJ193I
9      C1(27) / 0.0/, C1(28) / 0.0/, C1(29) / 0.0/, C1(30) / 0.0/, BESJ193J
1      C1(31) / 0.0/, C1(32) / 0.0/, C1(33) / 0.0/, BESJ193K
2      C1(34) / 4.6695844234262E+00/, C1(35) /-1.1207002616223E+01/, BESJ193L
3      C1(36) / 8.7891235351562E+00/, C1(37) /-2.3640869140625E+00/, BESJ193M
4      C1(38) / 1.1215209960938E-01/, C1(39) / 0.0/, C1(40) / 0.0/, BESJ193N
5      C1(41) / 0.0/, C1(42) / 0.0/, C1(43) / 0.0/, C1(44) / 0.0/, BESJ193O
6      C1(45) /-2.8212072558200E+01/, C1(46) / 8.4636217674601E+01/, BESJ193P
7      C1(47) /-9.1818241543240E+01/, C1(48) / 4.2534998745388E+01/, BESJ193Q
8      C1(49) /-7.3687943594796E+00/, C1(50) / 2.2710800170898E-01/, BESJ193R
9      C1(51) / 0.0/, C1(52) / 0.0/, C1(53) / 0.0/, C1(54) / 0.0/ BESJ193S
C      DATA C1(55) / 0.0/, C1(56) / 2.1257013003922E+02/, BESJ195A
1      C1(57) /-7.6525246814118E+02/, C1(58) / 1.0599904525280E+03/, BESJ195B
2      C1(59) /-6.9957962737613E+02/, C1(60) / 2.1819051174421E+02/, BESJ195C
3      C1(61) /-2.649130486952E+01/, C1(62) / 5.7250142097473E-01/, BESJ195D
4      C1(63) / 0.0/, C1(64) / 0.0/, C1(65) / 0.0/, C1(66) / 0.0/, BESJ195E
5      C1(67) /-1.9194576623184E+03/, C1(68) / 8.0617221817373E+03/, BESJ195F
6      C1(69) /-1.3586550006434E+04/, C1(70) / 1.165539333864E+04/, BESJ195G
7      C1(71) /-5.3056469786134E+03/, C1(72) / 1.2009029132164E+03/, BESJ195H
8      C1(73) /-1.0809091978840E+02/, C1(74) / 1.7277275025845E+00/, BESJ195I
9      C1(75) / 0.0/, C1(76) / 0.0/, C1(77) / 0.0/, BESJ195J
1      C1(78) / 2.0204291330966E+04/, C1(79) /-9.6980598388638E+04/, BESJ195K
2      C1(80) / 1.9254700123253E+05/, C1(81) /-2.0340017728042E+05/, BESJ195L
3      C1(82) / 1.2220046498302E+05/, C1(83) /-4.1192654968898E+04/, BESJ195M
4      C1(84) / 7.1095143024894E+03/, C1(85) /-4.9391530477309E+02/, BESJ195N
5      C1(86) / 6.0740420012735E+00/, C1(87) / 0.0/, C1(88) / 0.0/ BESJ195O
C      DATA C2(1) /-2.4291918790055E+05/, C2(2) / 1.3117636146630E+06/, BESJ197A
1      C2(3) /-2.9980159185381E+06/, C2(4) / 3.7632712976564E+06/, BESJ197B
2      C2(5) /-2.8135632265865E+06/, C2(6) / 1.2683652733216E+06/, BESJ197C
3      C2(7) /-3.3164517248456E+05/, C2(8) / 4.5218768981363E+04/, BESJ197D
4      C2(9) /-2.4998504818112E+03/, C2(10) / 2.4380529699556E+01/, BESJ197E
5      C2(11) / 0.0/, C2(12) / 3.2844698530720E+06/, BESJ197F
6      C2(13) /-1.9706819118432E+07/, C2(14) / 5.0952602492665E+07/, BESJ197G
7      C2(15) /-7.4105148211533E+07/, C2(16) / 6.6344512274729E+07/, BESJ197H
8      C2(17) /-3.7567176660763E+07/, C2(18) / 1.3288767166422E+07/, BESJ197I

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Figure A3. (Sheet 4 of 25)

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9      C2(19)/-2.7856181280864E+06/, C2(20)/ 3.0818640461266E+05/, BESJ197J
1      C2(21)/-1.3886089753717E+04/, C2(22)/ 1.1001714026925E+02/ BESJ197K
C
DATA A1(1) /-4.444444444444E-03/, A1(2) /-9.2207792207792E-04/, BESJ199A
1      A1(3) /-8.8489288489288E-05/, A1(4) / 1.6592768783245E-04/, BESJ199B
2      A1(5) / 2.4669137274179E-04/, A1(6) / 2.6599558934626E-04/, BESJ199C
3      A1(7) / 2.6182429706150E-04/, A1(8) / 2.4873043734466E-04/, BESJ199D
4      A1(9) / 2.3272104008323E-04/, A1(10) / 2.1636248571236E-04/, BESJ199E
5      A1(11) / 2.0073885676275E-04/, A1(12) / 1.8626763663754E-04/, BESJ199F
6      A1(13) / 1.7306077591788E-04/, A1(14) / 1.6109170592902E-04/, BESJ199G
7      A1(15) / 1.5027477416091E-04/, A1(16) / 1.4050349739127E-04/, BESJ199H
8      A1(17) / 1.3166881654592E-04/, A1(18) / 1.2336744559825E-04/, BESJ199I
9      A1(19) / 1.1640527147474E-04/, A1(20) / 1.0979829837271E-04/, BESJ199J
1      A1(21) / 1.0377241042299E-04/, A1(22) / 9.8262607836936E-05/, BESJ199K
2      A1(23) / 9.3212051724950E-05/, A1(24) / 8.8571085247871E-05/, BESJ199L
3      A1(25) / 8.4296310571570E-05/, A1(26) / 8.0349754840779E-05/, BESJ199M
4      A1(27) / 6.9373554135459E-04/, A1(28) / 2.3224174518292E-04/, BESJ199N
5      A1(29) /-1.4198627355669E-05/, A1(30) /-1.1644493167205E-04/, BESJ199O
6      A1(31) /-1.5080355805305E-04/, A1(32) /-1.5512192491810E-04/, BESJ199P
7      A1(33) /-1.4680975664467E-04/, A1(34) /-1.3381550386749E-04/, BESJ199Q
8      A1(35) /-1.1974497568425E-04/, A1(36) /-1.0618431920797E-04/, BESJ199R
9      A1(37) /-9.3769954989119E-05/, A1(38) /-8.2692304558819E-05/ BESJ199S
BESJ200
C
DATA A1(39) /-7.2937434815522E-05/, A1(40) /-6.4404235772102E-05/, BESJ201A
1      A1(41) /-5.6961156600937E-05/, A1(42) /-5.0473104430356E-05/, BESJ201B
2      A1(43) /-4.4813486800888E-05/, A1(44) /-3.9868872771760E-05/, BESJ201C
3      A1(45) /-3.5540053297204E-05/, A1(46) /-3.1741425660902E-05/, BESJ201D
4      A1(47) /-2.8399679390418E-05/, A1(48) /-2.5452272063487E-05/, BESJ201E
5      A1(49) /-2.2845929716472E-05/, A1(50) /-2.0535275310648E-05/, BESJ201F
6      A1(51) /-1.8481621762767E-05/, A1(52) /-1.6651933002139E-05/ BESJ202
C
DATA A2(1) /-3.5421197145774E-04/, A2(2) /-1.5616126394516E-04/, BESJ203A
1      A2(3) / 3.0446550359494E-05/, A2(4) / 1.301986557324E-04/, BESJ203B
2      A2(5) / 1.6747110669971E-04/, A2(6) / 1.7022258768359E-04/, BESJ203C
3      A2(7) / 1.5650142760860E-04/, A2(8) / 1.3633917097744E-04/, BESJ203D
4      A2(9) / 1.1488669202982E-04/, A2(10) / 9.4586909303469E-05/, BESJ203E
5      A2(11) / 7.6449841925090E-05/, A2(12) / 6.0757033496520E-05/, BESJ203F
6      A2(13) / 4.7439429929051E-05/, A2(14) / 3.6275751200534E-05/, BESJ203G
7      A2(15) / 2.66993971497922E-05/, A2(16) / 1.9321093824794E-05/, BESJ203H
8      A2(17) / 1.3005667479396E-05/, A2(18) / 7.8262086574450E-06/, BESJ203I
9      A2(19) / 3.5925748581935E-06/, A2(20) / 1.4404004981425E-07/, BESJ203J
1      A2(21) /-2.6539676969794E-06/, A2(22) /-4.9134686709849E-06/, BESJ203K
2      A2(23) /-6.7273929609125E-06/, A2(24) /-8.1726937967866E-06/, BESJ203L
3      A2(25) /-9.3130471509356E-06/, A2(26) /-1.0201141879802E-05/, BESJ203M
4      A2(27) / 3.7819419920177E-04/, A2(28) / 2.0247195276182E-04/, BESJ203N
5      A2(29) /-6.3793850631886E-05/, A2(30) /-2.3859823060301E-04/, BESJ203O
6      A2(31) /-3.1091625602736E-04/, A2(32) /-3.1368011524758E-04/, BESJ203P
7      A2(33) /-2.7895027379132E-04/, A2(34) /-2.2856408261914E-04/, BESJ203Q
8      A2(35) /-1.7524528034085E-04/, A2(36) /-1.2554406306069E-04/, BESJ203R
9      A2(37) /-8.2298287282021E-05/, A2(38) /-4.6286073058812E-05/ BESJ204
C
DATA A2(39) /-1.7233430236696E-05/, A2(40) / 5.6069048230460E-06/, BESJ205A
1      A2(41) / 2.3139544314829E-05/, A2(42) / 3.6264274585679E-05/, BESJ205B
2      A2(43) / 4.5800612449019E-05/, A2(44) / 5.2459529495911E-05/, BESJ205C
3      A2(45) / 5.6839620854582E-05/, A2(46) / 5.9434982039310E-05/, BESJ205D

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4   A2(47)/ 6.0647852757842E-05/, A2(48)/ 6.0802390778844E-05/, BESJ205E
5   A2(49)/ 6.0157789453946E-05/, A2(50)/ 5.8919965734470E-05/, BESJ205F
6   A2(51)/ 5.7251582377759E-05/, A2(52)/ 5.5280437558585E-05/ BESJ205G
C
    DATA B1(1) / 1.7998872141355E-02/, B1(2) / 5.5996491106439E-03/, BESJ207A
1   B1(3) / 2.8850140223113E-03/, B1(4) / 1.8009660676105E-03/, BESJ207B
2   B1(5) / 1.2475311058920E-03/, B1(6) / 9.2287887657294E-04/, BESJ207C
3   B1(7) / 7.1443042172729E-04/, B1(8) / 5.7178728178970E-04/, BESJ207D
4   B1(9) / 4.6943100760648E-04/, B1(10) / 3.9323283546292E-04/, BESJ207E
5   B1(11) / 3.3481888931830E-04/, B1(12) / 2.8895214849575E-04/, BESJ207F
6   B1(13) / 2.5221161554957E-04/, B1(14) / 2.2228058079888E-04/, BESJ207G
7   B1(15) / 1.9754183803305E-04/, B1(16) / 1.7683685501972E-04/, BESJ207H
8   B1(17) / 1.593168996182E-04/, B1(18) / 1.4434793019733E-04/, BESJ207I
9   B1(19) / 1.3144806811996E-04/, B1(20) / 1.2024544494930E-04/, BESJ207J
1   B1(21) / 1.1044914450460E-04/, B1(22) / 1.0182877074057E-04/, BESJ207K
2   B1(23) / 9.4199822420424E-05/, B1(24) / 8.7413054575383E-05/, BESJ207L
3   B1(25) / 8.1346626216280E-05/, B1(26) / 7.5900226964622E-05/, BESJ207M
4   B1(27) / -4.928295321343E-03/, B1(28) / -8.7820470954639E-04/, BESJ207N
5   B1(29) / -5.0291654957204E-04/, B1(30) / -2.9482213851275E-04/, BESJ207O
6   B1(31) / -1.7546399697078E-04/, B1(32) / -1.0400855046082E-04/, BESJ207P
7   B1(33) / -5.9614195304646E-05/, B1(34) / -3.1203892907610E-05/, BESJ207Q
8   B1(35) / -1.2608973598023E-05/, B1(36) / -2.4289260857573E-07/, BESJ207R
9   B1(37) / 8.0599616541427E-06/, B1(38) / 1.3650700926215E-05/ BESJ207S
C
    DATA B1(39) / 1.7396412547293E-05/, B1(40) / 1.9867297884213E-05/, BESJ209A
1   B1(41) / 2.1446326379082E-05/, B1(42) / 2.2395465923246E-05/, BESJ209B
2   B1(43) / 2.2896778381471E-05/, B1(44) / 2.3078538981118E-05/, BESJ209C
3   B1(45) / 2.3032197608091E-05/, B1(46) / 2.2823607372035E-05/, BESJ209D
4   B1(47) / 2.2500588110529E-05/, B1(48) / 2.2098101536199E-05/, BESJ209E
5   B1(49) / 2.161641842744810E-05/, B1(50) / 2.1150764925622E-05/, BESJ209F
6   B1(51) / 2.0638874978217E-05/, B1(52) / 2.0116524199708E-05/ BESJ209G
C
    DATA B2(1) / 5.5221307672129E-04/, B2(2) / 4.4793258155238E-04/, BESJ211A
1   B2(3) / 2.7952065399202E-04/, B2(4) / 1.5246815619845E-04/, BESJ211B
2   B2(5) / 6.9327110565704E-05/, B2(6) / 1.7625868306999E-05/, BESJ211C
3   B2(7) / -1.3574499634327E-05/, B2(8) / -3.1797241335043E-05/, BESJ211D
4   B2(9) / -4.1886186169669E-05/, B2(10) / -4.6900488937914E-05/, BESJ211E
5   B2(11) / -4.8766544741379E-05/, B2(12) / -4.8701003118674E-05/, BESJ211F
6   B2(13) / -4.7475562089009E-05/, B2(14) / -4.5581305813863E-05/, BESJ211G
7   B2(15) / -4.330964451127E-05/, B2(16) / -4.0923019315775E-05/, BESJ211H
8   B2(17) / -3.8482263860322E-05/, B2(18) / -3.8085716753541E-05/, BESJ211I
9   B2(19) / -3.3779330612337E-05/, B2(20) / -3.1588856077211E-05/, BESJ211J
1   B2(21) / -2.9526956175081E-05/, B2(22) / -2.7597891482834E-05/, BESJ211K
2   B2(23) / -2.5800617466688E-05/, B2(24) / -2.4130935676128E-05/, BESJ211L
3   B2(25) / -2.258235051835E-05/, B2(26) / -2.1147965676891E-05/, BESJ211M
4   B2(27) / -4.7461779655996E-04/, B2(28) / -4.7786456714732E-04/, BESJ211N
5   B2(29) / -3.2039022806704E-04/, B2(30) / -1.6110501611996E-04/, BESJ211O
6   B2(31) / -4.2577810128544E-05/, B2(32) / 3.4457129429497E-05/, BESJ211P
7   B2(33) / 7.9709268407568E-05/, B2(34) / 1.0313823670827E-04/, BESJ211Q
8   B2(35) / 1.124667526220E-04/, B2(36) / 1.1310364210848E-04/, BESJ211R
9   B2(37) / 1.0865163484877E-04/, B2(38) / 1.0143795159766E-04/ BESJ211S
C
    DATA B2(39) / 9.2929839659336E-05/, B2(40) / 8.4029313301609E-05/, BESJ213A
1   B2(41) / 7.5272799134913E-05/, B2(42) / 6.6963252197573E-05/, BESJ213B
2   B2(43) / 5.9256454732320E-05/, B2(44) / 5.2216930882698E-05/, BESJ213C

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Figure A3. (Sheet 6 of 25)

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3   B2(45)/ 4.5853948516536E-05/, B2(46)/ 4.0144551389149E-05/, BESJ213D
4   B2(47)/ 3.5048173003133E-05/, B2(48)/ 3.0515799503435E-05/, BESJ213E
5   B2(49)/ 2.6495611995052E-05/, B2(50)/ 2.2936363369100E-05/, BESJ213F
6   B2(51)/ 1.9789305666402E-05/, B2(52)/ 1.7009198463641E-05/ BESJ213G
C
    DATA B3(1) / 7.3646581057258E-04/, B3(2) / 8.7279080514619E-04/, BESJ215D
1   B3(3) / 6.2261486257314E-04/, B3(4) / 2.8599815419430E-04/, BESJ215A
2   B3(5) / 3.8473767287937E-06/, B3(6) / -1.8790600363697E-04/, BESJ215B
3   B3(7) / -2.9760364659456E-04/, B3(8) / -3.4599812683267E-04/, BESJ215C
4   B3(9) / -3.5338247091604E-04/, B3(10) / -3.3571563577505E-04/, BESJ215D
5   B3(11) / -3.0432112478904E-04/, B3(12) / -2.6672272304761E-04/, BESJ215F
6   B3(13) / -2.2765421412282E-04/, B3(14) / -1.8992261185456E-04/, BESJ215G
7   B3(15) / -1.5505891859909E-04/, B3(16) / -1.2377824076187E-04/, BESJ215H
8   B3(17) / -9.6292614771764E-05/, B3(18) / -7.2517832771442E-05/, BESJ215I
9   B3(19) / -5.2207002889563E-05/, B3(20) / -3.5034775051190E-05/, BESJ215J
1   B3(21) / -2.0648976103555E-05/, B3(22) / -8.7010609684977E-06/, BESJ215K
2   B3(23) / 1.1369868667510E-06/, B3(24) / 9.1642647412278E-06/, BESJ215L
3   B3(25) / 1.5647778542887E-05/, B3(26) / 2.0822362948247E-05/ BESJ215M
C
    DATA GAMMA(1) / 6.2996052494744E-01/, BESJ216
1   GAMMA(2) / 2.5198420997898E-01/, GAMMA(3) / 1.5479030041566E-01/, BESJ217A
2   GAMMA(4) / 1.1071306241616E-01/, GAMMA(5) / 8.5730939552740E-02/, BESJ217B
3   GAMMA(6) / 6.9716131695868E-02/, GAMMA(7) / 5.8608567189371E-02/, BESJ217D
4   GAMMA(8) / 5.0469887353631E-02/, GAMMA(9) / 4.4260058068916E-02/, BESJ217E
5   GAMMA(10) / 3.9372066154351E-02/, GAMMA(11) / 3.5428319592446E-02/, BESJ217F
6   GAMMA(12) / 3.2181885750210E-02/, GAMMA(13) / 2.9464624079116E-02/, BESJ217G
7   GAMMA(14) / 2.7158167711293E-02/, GAMMA(15) / 2.5176827297386E-02/, BESJ217H
8   GAMMA(16) / 2.3457075530608E-02/, GAMMA(17) / 2.1950839013491E-02/, BESJ217I
9   GAMMA(18) / 2.0621082823565E-02/, GAMMA(19) / 1.9438824089788E-02/, BESJ217J
1   GAMMA(20) / 1.8381063380068E-02/, GAMMA(21) / 1.7429321323196E-02/, BESJ217K
2   GAMMA(22) / 1.6568583778661E-02/, GAMMA(23) / 1.5786528598792E-02/, BESJ217L
3   GAMMA(24) / 1.5072950149410E-02/, GAMMA(25) / 1.4419325083996E-02/, BESJ217M
4   GAMMA(26) / 1.3818480573534E-02/ BESJ217N
C
    -----
C
C   TEST INPUT ARGUMENTS
C
    NZ=0
    KT=1
    IF(N-1) 92,108,109
108 KT=2
109 NN=N
    IF(X) 93,110,120
110 IF(ALPHA) 91,114,116
114 Y(I)=1.
    IF(N.EQ.1) RETURN
    I1=2
    GO TO 118
116 I1=1
118 DO 119 I=I1,N
119 Y(I)=0.
    RETURN
120 CONTINUE
    IF(ALPHA.LT.0.) GO TO 91
C

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Figure A3. (Sheet 7 of 25)

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DFN=DBLE(FLOAT(N))+DBLE(ALPHA)-1.D+0          BESJ240
FNU=DFN                                         BESJ241
X02=XX*.5                                       BESJ242
SX02=X02*X02                                     BESJ243
C                                               BESJ244
C DECISION TREE FOR REGION WHERE SERIES, ASYMPTOTIC EXPANSION FOR X BESJ245
C TO INFINITY AND ASYMPTOTIC EXPANSION FOR NU TO INFINITY ARE      BESJ246
C APPLIED.                                              BESJ247
C                                               BESJ248
IF(SX02.LE.(FNU+1.)) GO TO 850                 BESJ249
TA=AMAX1(20.,FNU)                                BESJ250
IF(X.GT.TA) GO TO 880                            BESJ251
IF(X.GT.12.) GO TO 860                           BESJ252
X02L=ALOG(X02)                                    BESJ253
NS=SX02-FNU                                      BESJ254
GO TO 852                                         BESJ255
850 FN=FNU                                         BESJ256
FNP1=FN+1.                                       BESJ257
X02L=ALOG(X02)                                    BESJ258
IS=KT                                            BESJ259
IF(X.LE.0.5) GO TO 134                           BESJ260
NS=0                                              BESJ261
852 DFN=DFN+DBLE(FLOAT(NS))                     BESJ262
FN=DFN                                         BESJ263
FNP1=FN+1.                                       BESJ264
IS=KT                                            BESJ265
IF(N-1+NS.GT.0) IS=3                           BESJ266
GO TO 134                                         BESJ267
860 NS=AMAX1(36.-FNU,0.)                         BESJ268
DFN=DFN+DBLE(FLOAT(NS))                        BESJ269
FN=DFN                                         BESJ270
IS=KT                                            BESJ271
IF(N-1+NS.GT.0) IS=3                           BESJ272
GO TO 130                                         BESJ273
880 CONTINUE                                       BESJ274
RTX=SQRT(X)                                      BESJ275
TAU=RTWO*RTX                                     BESJ276
TA=TAU+FNULIM(KT)                                BESJ277
IF(FNU.LE.TA) GO TO 500                           BESJ278
129 FN=FNU                                         BESJ279
IS=KT                                            BESJ280
C                                               BESJ281
C UNIFORM ASYMPTOTIC EXPANSION FOR NU TO INFINITY      BESJ282
C                                               BESJ283
130 CONTINUE                                       BESJ284
XX=X/FN                                         BESJ285
W2=1.-XX*XX                                     BESJ286
ABW2=ABS(W2)                                     BESJ287
RA=SQRT(ABW2)                                    BESJ288
IF(ABW2.GT.0.2775) GO TO 200                   BESJ289
C                                               BESJ290
C CASES NEAR X=FN, ABS(1.-(X/FN)**2).LE.0.2775      BESJ291
C COEFFICIENTS OF ASYMPTOTIC EXPANSION BY SERIES      BESJ292
C                                               BESJ293
C                                               BESJ294

```

Figure A3. (Sheet 8 of 25)

```

C      ZETA AND TRUNCATION FOR A(ZETA) AND B(ZETA) SERIES          BESJ295
C
C      KMAX IS TRUNCATION INDEX FOR A(ZETA) AND B(ZETA) SERIES=MAX(2,SA) BESJ297
C
C      SA=0.                                                       BESJ299
IF(ABW2.EQ.0.) GO TO 21                                         BESJ300
SA=TOLS ALOG(ABW2)                                               BESJ301
21 SB=SA
DO 22 I=1,5
KMAX(I)=AMAX1(SA,2.)
SA=SA+SB
22 CONTINUE
KB=KMAX(5)
KLAST=KB-1
SA=GAMA(KB)
DO 24 K=1,KLAST
KB=KB-1
SA=SA+W2+GAMA(KB)
24 CONTINUE
Z=W2*SA
AZ=ABS(Z)
RTZ=SQRT(AZ)
FN13=FN**CON2
RTARY=RTZ*FN13
ARY=-RTARY*RTARY
AZ32=AZ*RTZ*CON1
ACZ=FN*AZ32
IF(Z.LE.0.) GO TO 27
C
C      TEST FOR UNDERFLOW, 1.E-280=EXP(-644.), ONE WORD LENGTH        BESJ323
C      UP FROM UNDERFLOW LIMIT OF CDC 6600                           BESJ324
C
C      IF(ACZ.GT.ELIM2) GO TO 180                                     BESJ327
ARY=-ARY
27 PHI=SQRT(SQRT(SA+SA+SA+SA))
C
C      B(ZETA) FOR S=0                                              BESJ330
C
KB=KMAX(5)
KLAST=KB-1
SB=BETA(KB,1)
DO 23 K=1,KLAST
KB=KB-1
SB=SB*W2+BETA(KB,1)
23 CONTINUE
KSP1=1
FN2=FN*FN
RFN2=1./FN2
RDEN=1.
ASUM=1.
RELB=TOL*ABS(SB)
BSUM=SB
DO 25 KS=1,4
KSP1=KSP1+1
RDEN=RDEN*RFN2

```

Figure A3. (Sheet 9 of 25)

```

C          BESJ350
C          A(ZETA) AND B(ZETA) FOR S=1,2,3,4      BESJ351
C                                              BESJ352
C          KB=KMAX(5-KS)                         BESJ353
C          KLAST=KB-1                           BESJ354
C          SA=ALFA(KB,KS)                        BESJ355
C          SB=BETA(KB,KSP1)                      BESJ356
C          DO 26 K=1,KLAST                      BESJ357
C          KB=KB-1                            BESJ358
C          SA=SA*W2+ALFA(KB,KS)                  BESJ359
C          SB=SB*W2+BETA(KB,KSP1)                BESJ360
26 CONTINUE                                BESJ361
      TA=SA*RDEN                          BESJ362
      TB=SB*RDEN                          BESJ363
      ASUM=ASUM+TA                         BESJ364
      BSUM=BSUM+TB                         BESJ365
      IF(ABS(TA).LE.TOL.AND.ABS(TB).LE.RELB) GO TO 152 BESJ366
25 CONTINUE                                BESJ367
152 CONTINUE                                BESJ368
      BSUM=BSUM/(FN*FN13)                  BESJ369
      GO TO 400                           BESJ370
C                                              BESJ371
200 CONTINUE                                BESJ372
      TAU=1./RA                           BESJ373
      T2=1./W2                           BESJ374
      IF(W2.GE.0.) GO TO 30                 BESJ375
C                                              BESJ376
C          CASES FOR (X/FN).GT.SQRT(1.2775)    BESJ377
C                                              BESJ378
      AZ32=ABS(RA-ATAN(RA))                BESJ379
      ACZ=AZ32*FN                         BESJ380
      CZ=-ACZ                           BESJ381
      Z32=1.5*AZ32                        BESJ382
      RTZ=Z32**CON2                      BESJ383
      FN13=FN**CON2                      BESJ384
      RTARY=RTZ*FN13                     BESJ385
      ARY=-RTARY*RTARY                   BESJ386
      GO TO 150                           BESJ387
30 CONTINUE                                BESJ388
C                                              BESJ389
C          CASES FOR (X/FN).LT.SQRT(0.7225)    BESJ390
C                                              BESJ391
      AZ32=ABS ALOG((1.+RA)/XX) -RA        BESJ392
C                                              BESJ393
C          TEST FOR UNDERFLOW, 1.E-280 = EXP(-644.), ONE WORD LENGTH BESJ394
C          UP FROM UNDERFLOW LIMIT OF CDC 6600 BESJ395
C                                              BESJ396
      ACZ=AZ32*FN                         BESJ397
      CZ=ACZ                           BESJ398
      IF(ACZ.GT.ELIM2) GO TO 180           BESJ399
      Z32=1.5*AZ32                        BESJ400
      RTZ=Z32**CON2                      BESJ401
      FN13=FN**CON2                      BESJ402
      RTARY=RTZ*FN13                     BESJ403
      ARY=RTARY*RTARY                   BESJ404

```

Figure A3. (Sheet 10 of 25)

```

150 CONTINUE                                BESJ405
    PHI=SQRT((RTZ+RTZ)*TAU)                 BESJ406
    TB=1.                                     BESJ407
    ASUM=1.                                    BESJ408
    TFN=TAU/FN                                 BESJ409
    UPOL(2)=(C(1,1)*T2+C(2,1))*TFN          BESJ410
    RCZ=CON1/CZ                               BESJ411
    CRZ32=CON54B*RCZ                         BESJ412
    BSUM=UPOL(2)+CRZ32                       BESJ413
    RELB=TOL*ABS(BSUM)                      BESJ414
    AP=TFN                                     BESJ415
    KS=0                                       BESJ416
    KP1=2                                      BESJ417
    RZDEN=RCZ                                  BESJ418
    DO 155 LR=2,8,2                           BESJ419
C
C   COMPUTE TWO U POLYNOMIALS FOR NEXT A(ZETA) AND B(ZETA)      BESJ420
C
    LRP1=LR+1                                 BESJ421
    DO 101 K=LR,LRP1                         BESJ422
    KS=KS+1                                   BESJ423
    KP1=KP1+1                                 BESJ424
    S1=C(1,K)                                 BESJ425
    DO 102 J=2,KP1                           BESJ426
    S1=S1*T2+C(J,K)                         BESJ427
102 CONTINUE                                 BESJ428
    AP=AP*TFN                                BESJ429
    UPOL(KP1)=AP*S1                         BESJ430
    CR(KS)=BR(KS)*RZDEN                     BESJ431
    RZDEN=RZDEN*RCZ                         BESJ432
    DR(KS)=AR(KS)*RZDEN                     BESJ433
101 CONTINUE                                 BESJ434
    SUMA=UPOL(LRP1)                          BESJ435
    SUMB=UPOL(LR+2)+UPOL(LRP1)*CRZ32       BESJ436
    JU=LRP1                                   BESJ437
    DO 151 JR=1,LR                           BESJ438
    JU=JU-1                                   BESJ439
    SUMA=SUMA+CR(JR)*UPOL(JU)                BESJ440
    SUMB=SUMB+DR(JR)*UPOL(JU)                BESJ441
151 CONTINUE                                 BESJ442
    TB=-TB                                    BESJ443
    IF(W2.GT.0.) TB=ABS(TB)                  BESJ444
    ASUM=ASUM+SUMA*TB                        BESJ445
    BSUM=BSUM+SUMB*TB                        BESJ446
    IF(ABS(SUMA).LE.TOL.AND.ABS(SUMB).LE.RELB) GO TO 165
155 CONTINUE                                 BESJ447
165 TB=RTARY                                BESJ448
    IF(W2.GT.0.) TB=-TB                      BESJ449
    BSUM=BSUM/TB                             BESJ450
C
400 CONTINUE                                 BESJ451
    CALL JAIRY(ARY,RTARY,ACZ,AI,DAI)
    TEMP(IS)=PHI*(AI*ASUM+DAI*BSUM)/FN13
    GO TO (401,202,650), IS
402 TEMP(1)=TEMP(3)                         BESJ452

```

Figure A3. (Sheet 11 of 25)

```

KT=1                                BESJ460
401 IS=2                                BESJ461
DFN=DFN-1.D+0                         BESJ462
FN=DFN                                BESJ463
GO TO 130                             BESJ464
C
C      SERIES FOR (X/2)**2.LE,NU+1      BESJ465
C
134 CONTINUE                           BESJ466
GLN=GAMLN(FNP1)                      BESJ467
ARG=FN*X02L-GLN                      BESJ468
IF(ARG.LT.-ELIM1) GO TO 123          BESJ469
EARG=EXP(ARG)                         BESJ470
BESJ471
300 CONTINUE                           BESJ472
S=1.                                  BESJ473
AK=3.                                 BESJ474
T2=1.                                 BESJ475
T=1.                                  BESJ476
S1=FN                                BESJ477
DO 125 K=1,17                          BESJ478
S2=T2+S1                            BESJ479
T=-T*SX02/S2                         BESJ480
S=S+T                                BESJ481
IF(ABS(T).LT.TOL) GO TO 127          BESJ482
T2=T2+AK                            BESJ483
AK=AK+2.                            BESJ484
S1=S1+FN                            BESJ485
125 CONTINUE                           BESJ486
127 CONTINUE                           BESJ487
TEMP(IS)=S*EARG                      BESJ488
GO TO (301,202,600), IS              BESJ489
301 EARG=EARGL*FN/X02                BESJ490
DFN=DFN-1.D+0                         BESJ491
FN=DFN                                BESJ492
IS=2                                  BESJ493
GO TO 300                             BESJ494
BESJ495
C
C      SET UNDERFLOW VALUE AND UPDATE PARAMETERS BESJ496
C
180 Y(NN)=0.                           BESJ497
NN=NN-1                               BESJ498
DFN=DFN-1.D+0                         BESJ499
FN=DFN                                BESJ500
IF (NN-1) 170,171,130                 BESJ501
171 KT=2                                BESJ502
IS=2                                  BESJ503
GO TO 130                             BESJ504
123 Y(NN)=0.                           BESJ505
NN=NN-1                               BESJ506
FNP1=FN                               BESJ507
DFN=DFN-1.D+0                         BESJ508
FN=DFN                                BESJ509
IF(NN-1) 170,172,173                 BESJ510
172 KT=2                                BESJ511
IS=2                                  BESJ512
BESJ513
BESJ514

```

Figure A3. (Sheet 12 of 25)

```

173 IF(SX02.LE.FNP1) GO TO 133          BESJ515
    GO TO 130                           BESJ516
133 ARG=ARG-X02L+ALOG(FNP1)
    IF(ARG.LT.-ELIM1) GO TO 123          BESJ517
    GO TO 134                           BESJ518
170 NZ=N-NN                           BESJ519
    RETURN                               BESJ520
C
C      BACKWARD RECURSION SECTION      BESJ522
C
202 CONTINUE                           BESJ525
    NZ=N-NN                           BESJ526
    IF(KT.EQ.2) GO TO 250               BESJ527
203 CONTINUE                           BESJ528
C      BACKWARD RECUR FROM INDEX ALPHA+NN-1 TO ALPHA      BESJ529
    Y(NN)=TEMP(1)                      BESJ530
    Y(NN-1)=TEMP(2)                     BESJ531
    IF(NN.EQ.2) RETURN                 BESJ532
    DX=X                               BESJ533
    TRX=2.D+0/DX                      BESJ534
    DTM=DFN*TRX                       BESJ535
    TM=DTM                            BESJ536
    K=NN+1                            BESJ537
    DO 230 I=3,NN                      BESJ538
    K=K-1                            BESJ539
    Y(K-2)=TM*Y(K-1)-Y(K)             BESJ540
    DTM=DTM-TRX                      BESJ541
    TM=DTM                            BESJ542
230 CONTINUE                           BESJ543
    RETURN                             BESJ544
250 Y(1)=TEMP(2)                      BESJ545
    RETURN                             BESJ546
C
C      ASYMPTOTIC EXPANSION FOR X TO INFINITY WITH FORWARD RECURSION IN      BESJ547
C      OSCILLATORY REGION X.GT.MAX(20, NU), PROVIDED THE LAST MEMBER          BESJ548
C      OF THE SEQUENCE IS ALSO IN THE REGION.                                BESJ549
C
500 CONTINUE                           BESJ550
    IN=ALPHA-TAU+2.                    BESJ551
    IF(IN.LE.0) GO TO 502               BESJ552
    INP1=IN+1                          BESJ553
    DALPHA=ALPHA-FLOAT(INP1)           BESJ554
    KT=1                               BESJ555
    GO TO 511                           BESJ556
502 DALPHA=ALPHA                      BESJ557
    IN=0                               BESJ558
511 IS=KT                            BESJ559
512 ARG=X-PIDT*DALPHA-PDF           BESJ560
    SA=SIN(ARG)                      BESJ561
    SB=COS(ARG)                      BESJ562
    RA=RTTP/RTX                      BESJ563
    ETX=B.*X                         BESJ564
503 DX=DALPHA                        BESJ565
    DX=DX+DX                          BESJ566
    DTM=DX*DX                         BESJ567
                                         BESJ568
                                         BESJ569

```

Figure A3. (Sheet 13 of 25)

```

T2=DTM-1.D+0          BESJ570
T2=T2/ETX              BESJ571
S2=T2                  BESJ572
RELB=TOL*ABS(T2)      BESJ573
T1=ETX                BESJ574
S1=1.                  BESJ575
FN=1.                  BESJ576
AK=8.                  BESJ577
DO 504 K=1,13          BESJ578
T1=T1+ETX              BESJ579
FN=FN+AK              BESJ580
DX=FN                 BESJ581
TRX=DTM-DX             BESJ582
AP=TRX                BESJ583
T2=-T2*AP/T1           BESJ584
S1=S1+T2               BESJ585
T1=T1+ETX              BESJ586
AK=AK+8.               BESJ587
FN=FN+AK              BESJ588
DX=FN                 BESJ589
TRX=DTM-DX             BESJ590
AP=TRX                BESJ591
T2= T2*AP/T1           BESJ592
S2=S2+T2               BESJ593
IF(ABS(T2).LE.RELB) GO TO 505
AK=AK+8.               BESJ594
504 CONTINUE            BESJ595
505 TEMP(IS)=RA*(S1*SB-S2*SA)
GO TO (506,507),IS      BESJ596
506 DALPHA=DALPHA+1.
IS=2                   BESJ599
TB=SA                 BESJ600
SA=-SB                BESJ601
SB=TB                 BESJ602
GO TO 503              BESJ603
BESJ604
C FORWARD RECURSION SECTION
C
507 IF(KT.EQ.2) GO TO 250
S1=TEMP(1)              BESJ605
S2=TEMP(2)              BESJ606
TX=2./X                 BESJ607
TM=DALPHA*TX             BESJ608
IF(IN.EQ.0) GO TO 520
C FORWARD RECUR TO INDEX ALPHA
C
DO 510 I=1,IN           BESJ614
S=S2                   BESJ615
S2=TM*S2-S1             BESJ616
TM=TM+TX                BESJ617
S1=S                   BESJ618
510 CONTINUE             BESJ619
IF(NN.EQ.1) GO TO 535
S=S2                   BESJ620
BESJ621
BESJ622
BESJ623
BESJ624

```

Figure A3. (Sheet 14 of 25)

```

S2=TM*S2-S1          BESJ625
TM=TM+TX             BESJ626
S1=S                 BESJ627
520 CONTINUE          BESJ628
C      FORWARD RECUR FROM INDEX ALPHA TO ALPHA+N-1 BESJ629
C
C      Y(1)=S1          BESJ630
C      Y(2)=S2          BESJ631
C      IF(NN.EQ.2) RETURN BESJ632
C      DO 530 I=3,NN    BESJ633
C      Y(I)=TM*Y(I-1)-Y(I-2) BESJ634
C      TM=TM+TX          BESJ635
C
530 CONTINUE          BESJ636
      RETURN             BESJ637
535 Y(1)=S2          BESJ638
      RETURN             BESJ639
C
C      BACKWARD RECURSION WITH NORMALIZATION BY BESJ640
C      ASYMPTOTIC EXPANSION FOR NU TO INFINITY OR POWER SERIES. BESJ641
C
600 CONTINUE          BESJ642
C      COMPUTATION OF LAST ORDER FOR SERIES NORMALIZATION BESJ643
KM=AMAX1(3.-FN,0.)    BESJ644
TFN=FN+FLOAT(KM)      BESJ645
TA=(GLN+TFN-0.9189385332-0.0833333333/TFN)/(TFN+0.5) BESJ646
TA=X02L-TA            BESJ647
TB=-(1.-1.5/TFN)/TFN BESJ648
IN=CE/(-TA+SQRT(TA*TA-CE*TB))+1.5 BESJ649
IN=IN+KM              BESJ650
GO TO 603              BESJ651
650 CONTINUE          BESJ652
C      COMPUTATION OF LAST ORDER FOR ASYMPTOTIC EXPANSION NORMALIZATION BESJ653
GLN=AZ32+RA            BESJ654
IF(ARY.GT.30.) GO TO 675 BESJ655
RDEN=(PP(4)*ARY+PP(3))*ARY+1. BESJ656
RZDEN=PP(1)+PP(2)*ARY     BESJ657
TA=RZDEN/RDEN          BESJ658
IF(W2.LT.0.10) GO TO 651 BESJ659
TB=GLN/RTARY           BESJ660
GO TO 677              BESJ661
651 TB=(1.259921049+0.1679894730*W2)/FN13 BESJ662
GO TO 677              BESJ663
675 CONTINUE          BESJ664
TA=CON1*TCE/ACZ        BESJ665
TA=((0.0493827160*TA-0.1111111111)*TA+0.6666666667)*TA*ARY BESJ666
IF(W2.LT.0.10) GO TO 651 BESJ667
TB=GLN/RTARY           BESJ668
GO TO 677              BESJ669
677 IN=TA/TB+1.5        BESJ670
IF(IN.GT.INLIM) GO TO 402 BESJ671
603 DX=FLOAT(IN)        BESJ672
DTM=DFN+DX             BESJ673
DX=X                  BESJ674
TRX=2.D+0/DX           BESJ675
DTM=DTH*TRX            BESJ676
BESJ677
BESJ678
BESJ679

```

Figure A3. (Sheet 15 of 25)

```

TM=DTM          BESJ680
TA=0.           BESJ681
TB=TOL          BESJ682
KK=1            BESJ683
605 CONTINUE    BESJ684
C               BESJ685
C               BACKWARD RECUR UNINDEXED
C               BESJ686
C               DO 601 I=1,IN      BESJ687
S=TB             BESJ688
TB=TM*TB-TA     BESJ689
TA=S             BESJ690
DTM=DTM-TRX     BESJ691
TM=DTM           BESJ692
601 CONTINUE     BESJ693
C               NORMALIZATION   BESJ694
IF(KK.NE.1) GO TO 604  BESJ695
TA=(TA/TB)*TEMP(3)    BESJ696
TB=TEMP(3)           BESJ697
KK=2               BESJ698
IN=NS              BESJ699
IF(NS.NE.0) GO TO 605 BESJ700
604 Y(NN)=TB        BESJ701
615 NZ=N-NN         BESJ702
IF(NN.EQ.1) RETURN  BESJ703
S=TB              BESJ704
TB=TM*TB-TA       BESJ705
TA=S              BESJ706
DTM=DTM-TRX       BESJ707
TM=DTM           BESJ708
K=NN-1            BESJ709
Y(K)=TB           BESJ710
IF(NN.EQ.2) RETURN BESJ711
KM=K-1            BESJ712
616 CONTINUE     BESJ713
C               BESJ714
C               BACKWARD RECUR INDEXED
C               BESJ715
C               DO 602 I=1,KM      BESJ716
Y(K-1)=TM*Y(K)-Y(K+1) BESJ717
DTM=DTM-TRX       BESJ718
TM=DTM           BESJ719
K=K-1            BESJ720
602 CONTINUE     BESJ721
RETURN           BESJ722
C               BESJ723
C               BESJ724
C               BESJ725
C               BESJ726
91 CONTINUE     BESJ727
NZ=-2            BESJ728
RETURN          BESJ729
92 CONTINUE     BESJ730
NZ=-3            BESJ731
RETURN          BESJ732
93 CONTINUE     BESJ733
NZ=-1            BESJ734

```

Figure A3. (Sheet 16 of 25)

```

RETURN                                BESJ735
END                                  BESJ736
SUBROUTINE JAIRY(X,RX,C,AI,DAI)      AIRY1
C - - - - - CDC 6600 ROUTINE 1-2-74    AIRY2
C           JAIRY COMPUTES THE AIRY FUNCTION AI(X)    AIRY3
C           AND ITS DERIVATIVE DAI(X) FOR JBESS    AIRY4
C
C   INPUT: X - ARGUMENT, COMPUTED BY JBESSIONT    AIRY5
C          RX - RX=SQRT(ABS(X)), COMPUTED BY JBESSIONT    AIRY6
C          C - C=2.*(ABS(X)*1.5)/3., COMPUTED BY JBESSIONT    AIRY7
C
C   OUTPUT: AI - VALUE OF FUNCTION AI(X)    AIRY8
C          DAI - VALUE OF THE DERIVATIVE DAI(X)    AIRY9
C
C   WRITTEN BY D.E. AMOS, S.L. DANIEL & M.K. WESTON    AIRY10
C - - - - - DIMENSION AK1(14),AK2(23),AK3(14)    AIRY11
C   DIMENSION AJP(19),AJN(19),A(15),B(15)    AIRY12
C   DIMENSION DAK1(14),DAK2(24),DAK3(14)    AIRY13
C   DIMENSION DAJP(19),DAJN(19),DA(15),DB(15)    AIRY14
C   DATA N1,N2,N3,N4/14,23,19,15/    AIRY15
C   DATA M1,M2,M3,M4/12,21,17,13/    AIRY16
C   DATA FPI12,CON1,CON2,CON3,CON4,CON5/    AIRY17
C   1 1.308996389958E+00, 6.6666666666667E-01, 5.0315471619678E+00,    AIRY18
C   2 3.8000458986729E-01, 8.3333333333333E-01, 8.6602540378444E-01/    AIRY19
C   DATA AK1(1) / 2.2042309098779E-01/,    AIRY20
C   1 AK1(2) /-1.2529024278770E-01/, AK1(3) / 1.0388116335919E-02/,    AIRY21
C   2 AK1(4) / 8.2284415200634E-04/, AK1(5) /-2.3461434589123E-04/,    AIRY22
C   3 AK1(6) / 1.6382428017212E-05/, AK1(7) / 3.0690258957319E-07/,    AIRY23
C   4 AK1(8) /-1.2962199935933E-07/, AK1(9) / 8.2290185882367E-09/,    AIRY24
C   5 AK1(10) / 1.5596396862330E-11/, AK1(11) /-3.3916546561568E-11/,    AIRY25
C   6 AK1(12) / 2.0325325742363E-12/, AK1(13) /-1.1067954609788E-14/,    AIRY26
C   7 AK1(14) /-5.1616949778509E-15/    AIRY27
C   DATA AK2(1) / 2.7436615086960E-01/,    AIRY28
C   1 AK2(2) / 5.3979096973690E-03/, AK2(3) /-1.5733922062119E-03/,    AIRY29
C   2 AK2(4) / 4.2742752824875E-04/, AK2(5) /-1.1212491739992E-04/,    AIRY30
C   3 AK2(6) / 2.8876317131890E-05/, AK2(7) /-7.3680422537055E-06/,    AIRY31
C   4 AK2(8) / 1.8729020974102E-06/, AK2(9) /-4.7589279396229E-07/,    AIRY32
C   5 AK2(10) / 1.2113041695591E-07/, AK2(11) /-3.0924537427061E-08/,    AIRY33
C   6 AK2(12) / 7.9245470528265E-09/, AK2(13) /-2.0390244716791E-09/,    AIRY34
C   7 AK2(14) / 5.2686305659574E-10/, AK2(15) /-1.3670476763957E-10/,    AIRY35
C   8 AK2(16) / 3.5614103901371E-11/, AK2(17) /-9.3138829654843E-12/,    AIRY36
C   9 AK2(18) / 2.4446445047364E-12/, AK2(19) /-6.4384026199096E-13/,    AIRY37
C   1 AK2(20) / 1.7010603055935E-13/, AK2(21) /-4.5076010450328E-14/,    AIRY38
C   2 AK2(22) / 1.1977479916481E-14/, AK2(23) /-3.1907704086507E-15/    AIRY39
C   DATA AK3(1) / 2.802714734079E-01/,    AIRY40
C   1 AK3(2) /-1.7812704284438E-03/, AK3(3) / 4.0342257962900E-05/,    AIRY41
C   2 AK3(4) /-1.6324996526900E-06/, AK3(5) / 9.2118148247677E-08/,    AIRY42
C   3 AK3(6) /-6.5229433022916E-09/, AK3(7) / 5.4713840457655E-10/,    AIRY43
C   4 AK3(8) /-5.2440825180026E-11/, AK3(9) / 5.6047790411721E-12/,    AIRY44
C   5 AK3(10) /-6.5637524463931E-13/, AK3(11) / 8.3128576196625E-14/,    AIRY45
C   6 AK3(12) /-1.1270513469106E-14/, AK3(13) / 1.6226797659813E-15/,    AIRY46
C   7 AK3(14) /-2.4648032431243E-16/    AIRY47
C   DATA AJP(1) / 7.7895296643758E-02/,    AIRY48
C   1 AJP(2) /-1.8435636345680E-01/, AJP(3) / 3.0141260521617E-02/,    AIRY49

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2 AJP(4) / 3.0534272427761E-02/, AJP(5) /-4.9542470251308E-03/, AIRY54  
 3 AJP(6) /-1.7274955256395E-03/, AJP(7) / 2.4313763783919E-04/, AIRY55  
 4 AJP(8) / 5.0456477751708E-05/, AJP(9) /-6.1631658269521E-06/, AIRY56  
 5 AJP(10) /-9.0398674551077E-07/, AJP(11) / 9.7024377835588E-08/, AIRY57  
 6 AJP(12) / 1.0963945330520E-08/, AJP(13) /-1.0471633058877E-09/, AIRY58  
 7 AJP(14) /-9.6035944134465E-11/, AJP(15) / 8.2555878945413E-12/, AIRY59  
 8 AJP(16) / 6.3612343901877E-13/, AJP(17) /-4.9662961411602E-14/, AIRY60  
 9 AJP(18) /-3.2981028892962E-15/, AJP(19) / 2.3579825203110E-16/  
 DATA AJN(1) / 3.8049788761724E-02/,  
 1 AJN(2) /-2.4531594184555E-01/, AJN(3) / 1.6582062370270E-01/, AIRY62  
 2 AJN(4) / 7.4933004581879E-02/, AJN(5) /-2.6347628810664E-02/, AIRY63  
 3 AJN(6) /-5.9253559730498E-03/, AJN(7) / 1.4474440958980E-03/, AIRY65  
 4 AJN(8) / 2.1831183132222E-04/, AJN(9) /-4.1066207768030E-05/, AIRY66  
 5 AJN(10) /-4.6687499417177E-06/, AJN(11) / 7.1521880727716E-07/, AIRY67  
 6 AJN(12) / 6.5296477085463E-08/, AJN(13) /-8.4428402756595E-09/, AIRY68  
 7 AJN(14) /-6.4418615897698E-10/, AJN(15) / 7.2080228650528E-11/, AIRY69  
 8 AJN(16) / 4.7246543171785E-12/, AJN(17) /-4.6602263254704E-13/, AIRY70  
 9 AJN(18) /-2.6776271038919E-14/, AJN(19) / 2.3616131657002E-15/  
 DATA A(1) / 4.9027542474279E-01/, A(2) / 1.5764727794620E-03/, AIRY72  
 1 A(3) /-9.661956314031E-05/, A(4) / 1.3591608026882E-07/, AIRY73  
 2 A(5) / 2.9815734265486E-07/, A(6) /-1.8682476755998E-08/, AIRY74  
 3 A(7) /-1.0368573766714E-09/, A(8) / 3.2866081843433E-10/, AIRY75  
 4 A(9) /-2.5709141063278E-11/, A(10) /-2.3235765530068E-12/, AIRY76  
 5 A(11) / 9.5752327904826E-13/, A(12) /-1.2034082804972E-13/, AIRY77  
 6 A(13) /-2.909077167702E-15/, A(14) / 4.5565645458015E-15/, AIRY78  
 7 A(15) /-9.9900387481026E-16/  
 DATA B(1) / 2.7859355280308E-01/, B(2) /-3.5291569188258E-03/, AIRY80  
 1 B(3) /-2.3114967738499E-05/, B(4) / 4.7131784226356E-06/, AIRY81  
 2 B(5) /-1.1241590793133E-07/, B(6) /-2.0010030118434E-08/, AIRY82  
 3 B(7) / 2.6094807530219E-09/, B(8) /-3.5509813610122E-11/, AIRY83  
 4 B(9) /-3.5084997842388E-11/, B(10) / 5.8300718795420E-12/, AIRY84  
 5 B(11) /-2.0464482875333E-13/, B(12) /-1.1052917947674E-13/, AIRY85  
 6 B(13) / 2.8772477803878E-14/, B(14) /-2.8820511100994E-15/, AIRY86  
 7 B(15) /-3.3265631169617E-16/  
 DATA N1D,N2D,N3D,N4D/14,24,19,15/  
 DATA M1D,M2D,M3D,M4D/12,22,17,13/  
 DATA DAK1(1) / 2.0456784230789E-01/, AIRY90  
 1 DAK1(2) /-6.6132273990566E-02/, DAK1(3) /-8.4984580098929E-03/, AIRY91  
 2 DAK1(4) / 3.1218349155629E-03/, DAK1(5) /-2.7001648982943E-04/, AIRY92  
 3 DAK1(6) /-6.3563629867939E-06/, DAK1(7) / 3.0239771240951E-06/, AIRY93  
 4 DAK1(8) /-2.1851119533009E-07/, DAK1(9) /-5.3619428935283E-10/, AIRY94  
 5 DAK1(10) / 1.1309803562231E-09/, DAK1(11) /-7.4302383462907E-11/, AIRY95  
 6 DAK1(12) / 4.2880417082689E-13/, DAK1(13) / 2.2381092575454E-13/, AIRY96  
 7 DAK1(14) /-1.3914013564118E-14/  
 DATA DAK2(1) / 2.9333234388323E-01/, AIRY98  
 1 DAK2(2) /-8.0619678474311E-03/, DAK2(3) / 2.4254017233314E-03/, AIRY99  
 2 DAK2(4) /-6.8229754885024E-04/, DAK2(5) / 1.8578642775118E-04/, AIRY100  
 3 DAK2(6) /-4.9745744768406E-05/, DAK2(7) / 1.3209068123950E-05/, AIRY101  
 4 DAK2(8) /-3.4952824044494E-06/, DAK2(9) / 9.2436245107884E-07/, AIRY102  
 5 DAK2(10) /-2.4473267152187E-07/, DAK2(11) / 6.4930783764891E-08/, AIRY103  
 6 DAK2(12) /-1.7271762150154E-08/, DAK2(13) / 4.6072576360466E-09/, AIRY104  
 7 DAK2(14) /-1.2324905529155E-09/, DAK2(15) / 3.3062040948810E-10/, AIRY105  
 8 DAK2(16) /-8.8925209977240E-11/, DAK2(17) / 2.3977331987830E-11/, AIRY106  
 9 DAK2(18) /-6.4801392115345E-12/, DAK2(19) / 1.7551013202373E-12/, AIRY107  
 1 DAK2(20) /-4.7630382983364E-13/, DAK2(21) / 1.2949824110081E-13/, AIRY108

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2 DAK2(22)/-3.5267962221043E-14/, DAK2(23)/ 9.6200515158592E-15/, AIRY109
3 DAK2(24)/-2.6278691434229E-15/ AIRY110
    DATA DAK3(1) / 2.8467582881135E-01/, AIRY111
1 DAK3(2) / 2.5307307261908E-03/, DAK3(3) /-4.8348113033798E-05/, AIRY112
2 DAK3(4) / 1.8490728394634E-06/, DAK3(5) /-1.0141849117858E-07/, AIRY113
3 DAK3(6) / 7.0592563445715E-09/, DAK3(7) /-5.8532529140038E-10/, AIRY114
4 DAK3(8) / 5.5635768885134E-11/, DAK3(9) /-5.9088909477950E-12/, AIRY115
5 DAK3(10)/ 6.8857435378444E-13/, DAK3(11)/-8.6858825645219E-14/, AIRY116
6 DAK3(12) / 1.1737476261721E-14/, DAK3(13)/-1.6852314651092E-15/, AIRY117
7 DAK3(14) / 2.5537473309706E-16/ AIRY118
    DATA DAJP(1) / 6.5321913131146E-02/, AIRY119
1 DAJP(2) /-1.2026293368882E-01/, DAJP(3) / 9.7801023626382E-03/, AIRY120
2 DAJP(4) / 1.6794842923050E-02/, DAJP(5) /-1.9714614018213E-03/, AIRY121
3 DAJP(6) /-8.4556029509887E-04/, DAJP(7) / 9.4288962070198E-05/, AIRY122
4 DAJP(8) / 2.2582786094548E-05/, DAJP(9) /-2.2906787091599E-06/, AIRY123
5 DAJP(10)/-3.7834399113692E-07/, DAJP(11) / 3.4566393355956E-08/, AIRY124
6 DAJP(12) / 4.2961133200301E-09/, DAJP(13)/-3.5867369121499E-10/, AIRY125
7 DAJP(14)/-3.5724588136190E-11/, DAJP(15) / 2.7269609106634E-12/, AIRY126
8 DAJP(16) / 2.2612065309577E-13/, DAJP(17)/-1.5876320523830E-14/, AIRY127
9 DAJP(18)/-1.1260437448512E-15/, DAJP(19) / 7.3132752951537E-17/ AIRY128
    DATA DAJN(1) / 1.0859453963297E-02/, AIRY129
1 DAJN(2) / 8.5331319485709E-02/, DAJN(3) /-3.1527706811306E-01/, AIRY130
2 DAJN(4) /-8.7842072529426E-02/, DAJN(5) / 5.5325190697605E-02/, AIRY131
3 DAJN(6) / 9.4167406050324E-03/, DAJN(7) /-3.3218702601900E-03/, BEJS132
4 DAJN(8) /-4.1115734315683E-04/, DAJN(9) / 1.0129732689135E-04/, AIRY133
5 DAJN(10) / 9.8763368220840E-06/, DAJN(11)/-1.8731296981239E-06/, AIRY134
6 DAJN(12)/-1.5079850013147E-07/, DAJN(13) / 2.3268766952539E-08/, AIRY135
7 DAJN(14) / 1.5959991741922E-09/, DAJN(15)/-2.0766592266838E-10/, AIRY136
8 DAJN(16)/-1.2410335050030E-11/, DAJN(17) / 1.3963176533104E-12/, AIRY137
9 DAJN(18) / 7.3940097115574E-14/, DAJN(19)/-7.3288747562750E-15/ AIRY137
    DATA DA(1) / 4.9162732110460E-01/, DA(2) / 3.1116493042749E-03/, AIRY139
1 DA(3) / 8.2314076295408E-05/, DA(4) /-4.6176977617214E-06/, AIRY140
2 DA(5) /-6.1315888053463E-08/, DA(6) / 2.8729580465652E-08/, AIRY141
3 DA(7) /-1.8195971537212E-09/, DA(8) /-1.4475282664204E-10/, AIRY142
4 DA(9) / 4.5372404342042E-11/, DA(10)/-3.9965506584722E-12/, AIRY143
5 DA(11)/-3.2408911983032E-13/, DA(12) / 1.6209895256874E-13/, AIRY144
6 DA(13)/-2.4078524797406E-14/, DA(14) / 1.6938481128449E-16/, AIRY145
7 DA(15) / 8.17990748647740E-16/ AIRY146
    DATA DB(1) /-2.7757135694423E-01/, DB(2) / 4.4421283341992E-03/, AIRY147
1 DB(3) /-8.4232852219009E-05/, DB(4) /-2.5804031841871E-06/, AIRY148
2 DB(5) / 3.4238972021762E-07/, DB(6) /-6.2428689470978E-09/, AIRY149
3 DB(7) /-2.3637783684458E-09/, DB(8) / 3.1699104265667E-10/, AIRY150
4 DB(9) /-4.4099569165819E-12/, DB(10)/-5.1867422109358E-12/, AIRY151
5 DB(11) / 9.6487401513702E-13/, DB(12)/-4.9019057660871E-14/, AIRY152
6 DB(13) /-1.7725343067811E-14/, DB(14) / 5.5595061044266E-15/, AIRY153
7 DB(15)/-7.1179333757953E-16/ AIRY154
C -----
    IF(X.LT.0.) GO TO 300 AIRY155
    IF(C.GT.5.) GO TO 200 AIRY156
    IF(X.GT.1.2) GO TO 150 AIRY157
    T=(X+X-1.2)*CON4 AIRY158
    TT = T + T AIRY159
    J=NI AIRY160
    F1=AK1(J) AIRY161
    F2=0. AIRY162

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Figure A3. (Sheet 19 of 25)

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DO 105 I=1,M1          AIRY164
J=J-1                  AIRY165
TEMP1=F1                AIRY166
F1=TT*F1-F2+AK1(J)    AIRY167
F2=TEMP1                AIRY168
105 CONTINUE            AIRY169
AI=T*F1-F2+AK1(1)      AIRY170
C
J=N1D                  AIRY171
F1=DAK1(J)              AIRY172
F2=0.                   AIRY173
DO 106 I=1,M1D          AIRY174
J=J-1                  AIRY175
TEMP1=F1                AIRY176
F1=TT*F1-F2+DAK1(J)   AIRY177
F2=TEMP1                AIRY178
106 CONTINUE            AIRY179
DAI=-(T*F1-F2+DAK1(1)) AIRY180
RETURN                 AIRY181
AIRY182
C
150 CONTINUE             AIRY183
T=(X-X-CON2)*CON3      AIRY184
TT = T + T               AIRY185
J=N2                   AIRY186
F1=AK2(J)               AIRY187
F2=0.                   AIRY188
DO 155 I=1,M2           AIRY189
J=J-1                  AIRY190
TEMP1=F1                AIRY191
F1=TT*F1-F2+AK2(J)    AIRY192
F2=TEMP1                AIRY193
155 CONTINUE             AIRY194
RTRX=SQRT(RX)           AIRY195
EC=EXP(-C)              AIRY196
AI=EC*(T*F1-F2+AK2(1))/RTRX
J=N2D                  AIRY197
F1=DAK2(J)              AIRY198
F2=0.                   AIRY199
DO 156 I=1,M2D          AIRY200
J=J-1                  AIRY201
TEMP1=F1                AIRY202
F1=TT*F1-F2+DAK2(J)   AIRY203
F2=TEMP1                AIRY204
156 CONTINUE             AIRY205
DAI=-EC*(T*F1-F2+DAK2(1))*RTRX
RETURN                 AIRY206
C
200 CONTINUE             AIRY207
T=10./C-1.              AIRY208
TT=T+T                  AIRY209
J=N1                   AIRY210
F1=AK3(J)               AIRY211
F2=0.                   AIRY212
DO 205 I=1,M1           AIRY213
J=J-1                  AIRY214

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        TEMP1=F1          AIRY219
        F1=TT*F1-F2+AK3(J) AIRY220
        F2=TEMP1          AIRY221
205   CONTINUE          AIRY222
        RTRX=SQRT(RX)    AIRY223
        EC=EXP(-C)       AIRY224
        AI=EC*(T*F1-F2+AK3(1))/RTRX AIRY225
        J=N1D            AIRY226
        F1=DAK3(J)       AIRY227
        F2=0.             AIRY228
        DO 206 I=1,M1D   AIRY229
        J=J-1            AIRY230
        TEMP1=F1          AIRY231
        F1=TT*F1-F2+DAK3(J) AIRY232
        F2=TEMP1          AIRY233
206   CONTINUE          AIRY234
        DAI=-RTRX*EC*(T*F1-F2+DAK3(1)) AIRY235
        RETURN           AIRY236
C
300   CONTINUE          AIRY237
        IF(C.GT.5.) GO TO 350
        T=.4*C-1.          AIRY238
        TT=T+T            AIRY239
        J=N3              AIRY240
        F1=AJP(J)         AIRY241
        E1=AJN(J)         AIRY242
        F2=0.             AIRY243
        E2=0.             AIRY244
        DO 305 I=1,M3    AIRY245
        J=J-1            AIRY246
        TEMP1=F1          AIRY247
        TEMP2=E1          AIRY248
        F1=TT*F1-F2+AJP(J) AIRY249
        E1=TT*E1-E2+AJN(J) AIRY250
        F2=TEMP1          AIRY251
        E2=TEMP2          AIRY252
305   CONTINUE          AIRY253
        AI=(T*E1-E2+AJN(1))-X*(T*F1-F2+AJP(1)) AIRY254
        J=N3D            AIRY255
        F1=DAJP(J)       AIRY256
        E1=DAJN(J)       AIRY257
        F2=0.             AIRY258
        E2=0.             AIRY259
        DO 306 I=1,M3D   AIRY260
        J=J-1            AIRY261
        TEMP1=F1          AIRY262
        TEMP2=E1          AIRY263
        F1 = TT*F1-F2+DAJP(J) AIRY264
        E1= TT*E1-E2+DAJN(J) AIRY265
        F2=TEMP1          AIRY266
        E2=TEMP2          AIRY267
306   CONTINUE          AIRY268
        DAI=X*X*(T*F1-F2+DAJP(1))+(T*E1-E2+DAJN(1)) AIRY269
        RETURN           AIRY270
C

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Figure A3. (Sheet 21 of 25)

Figure A3. (Sheet 22 of 25)

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C      GAMLN COMPUTES THE NATURAL LOG OF THE GAMMA FUNCTION FOR      GLN11
C      X.GT.0. A RATIONAL CHEBYSHEV APPROXIMATION IS USED ON      GLN12
C      8.LT.X.LT.1000., THE ASYMPTOTIC EXPANSION FOR X.GE.1000. AND      GLN13
C      A RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3. FOR      GLN14
C      0.LT.X.LT.8. AND X NON-INTEGRAL, FORWARD OR BACKWARD      GLN15
C      RECURSION FILLS IN THE INTERVALS 0.LT.X.LT.2 AND      GLN16
C      3.LT.X.LT.8. FOR X=1.,2.,...,100., GAMLN IS SET TO      GLN17
C      NATURAL LOGS OF FACTORIALS.      GLN18
C
C      DESCRIPTION OF ARGUMENTS      GLN19
C      INPUT      GLN20
C          X      - X.GT.0      GLN21
C      OUTPUT      GLN22
C          GAMLN - NATURAL LOG OF THE GAMMA FUNCTION AT X      GLN23
C -----
C      DIMENSION GLN(100),P(5),Q(2),PCOE(9),QCDE(4)      GLN24
C      DATA XLIM1,XLIM2,RTWPIL/     8. , 1000. , 9.189385332047E-01/      GLN25
C      DATA P(1)/7.663451880000E-04/, P(2)/-5.940956105200E-04/,      GLN26
C      1    P(3)/7.936431104845E-04/, P(4)/-2.777777756577E-03/,      GLN27
C      2    P(5)/8.33333333332E-02/      GLN28
C      DATA Q(1)/-2.777777777778E-03/, Q(2)/8.33333333333E-02/      GLN29
C      DATA PCOE(1)/2.973786644810E-03/,PCOE(2)/9.238194559028E-03/,      GLN30
C      1    PCOE(3)/1.093115956710E-01/,PCOE(4)/3.980671310204E-01/,      GLN31
C      2    PCOE(5)/2.159943128461E+00/,PCOE(6)/6.338067999387E+00/,      GLN32
C      3    PCOE(7)/2.078247253179E+01/,PCOE(8)/3.603677253002E+01/,      GLN33
C      4    PCOE(9)/6.200383800713E+01/      GLN34
C
C      DATA QCDE(1)/1.000000000000E+00/,QCDE(2)/-8.906016659498E+00/,      GLN35
C      1    QCDE(3)/9.822521104714E+00/,QCDE(4)/6.200383800713E+01/      GLN36
C
C      DATA GLN(1) /0.0/, GLN(2) /0.0/, GLN(3) /5.931471805599E-01/,      GLN37
C      1    GLN(4) /1.791759469228E+00/, GLN(5) /3.178053930348E+00/,      GLN38
C      2    GLN(6) /4.787491742782E+00/, GLN(7) /6.579251212010E+00/,      GLN39
C      3    GLN(8) /8.525161361065E+00/, GLN(9) /1.060460290274E+01/,      GLN40
C      4    GLN(10) /1.280182748008E+01/, GLN(11) /1.510441257308E+01/,      GLN41
C      5    GLN(12) /1.750230784587E+01/, GLN(13) /1.998721449566E+01/,      GLN42
C      6    GLN(14) /2.255216385312E+01/, GLN(15) /2.519122118274E+01/,      GLN43
C      7    GLN(16) /2.789927138384E+01/, GLN(17) /3.067186010608E+01/,      GLN44
C      8    GLN(18) /3.350507345014E+01/, GLN(19) /3.639544520803E+01/,      GLN45
C      9    GLN(20) /3.933988418720E+01/, GLN(21) /4.233561646075E+01/,      GLN46
C      1    GLN(22) /4.538013898984E+01/, GLN(23) /4.847118135184E+01/,      GLN47
C      2    GLN(24) /5.160667556776E+01/, GLN(25) /5.478472939811E+01/,      GLN48
C      3    GLN(26) /5.800360522298E+01/, GLN(27) /6.126170176100E+01/,      GLN49
C      4    GLN(28) /6.455753862701E+01/, GLN(29) /6.788974313718E+01/,      GLN50
C      5    GLN(30) /7.125703896717E+01/, GLN(31) /7.465823634883E+01/,      GLN51
C      6    GLN(32) /7.809222355332E+01/, GLN(33) /8.155795945612E+01/,      GLN52
C      7    GLN(34) /8.505446701758E+01/, GLN(35) /8.858082754220E+01/,      GLN53
C      8    GLN(36) /9.213617560369E+01/, GLN(37) /9.571969454214E+01/,      GLN54
C      9    GLN(38) /9.933061245479E+01/, GLN(39) /1.029681986145E+02/      GLN55
C      DATA GLN(40)/1.066317602606E+02/, GLN(41)/1.103206397148E+02/,      GLN56
C      1    GLN(42)/1.140342117815E+02/, GLN(43)/1.177718813997E+02/,      GLN57
C      2    GLN(44)/1.215330815154E+02/, GLN(45)/1.253172711494E+02/,      GLN58
C      3    GLN(46)/1.291239336391E+02/, GLN(47)/1.329525750356E+02/,      GLN59
C      4    GLN(48)/1.368027226373E+02/, GLN(49)/1.406739236482E+02/,      GLN60
C      5    GLN(50)/1.445657439463E+02/, GLN(51)/1.484777669518E+02/,      GLN61

```

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6   GLN(52)/1.524095925845E+02/, GLN(53)/1.553609363031E+02/, GLN66
7   GLN(54)/1.603311282166E+02/, GLN(55)/1.643201122632E+02/, GLN67
8   GLN(56)/1.683274454484E+02/, GLN(57)/1.723527971392E+02/, GLN68
9   GLN(58)/1.763958484070E+02/, GLN(59)/1.804562914175E+02/, GLN69
1   GLN(60)/1.845338288614E+02/ GLN70
DATA GLN(61)/1.886281734237E+02/, GLN(62)/1.927390472878E+02/, GLN71
1   GLN(63)/1.968661816729E+02/, GLN(64)/2.010093163993E+02/, GLN72
2   GLN(65)/2.051681994826E+02/, GLN(66)/2.093425867525E+02/, GLN73
3   GLN(67)/2.135322414946E+02/, GLN(68)/2.177369341140E+02/, GLN74
4   GLN(69)/2.219564418191E+02/, GLN(70)/2.261905483237E+02/, GLN75
5   GLN(71)/2.304390435658E+02/, GLN(72)/2.347017234428E+02/, GLN76
6   GLN(73)/2.389783895618E+02/, GLN(74)/2.432688490030E+02/, GLN77
7   GLN(75)/2.475729140962E+02/, GLN(76)/2.518904022097E+02/, GLN78
8   GLN(77)/2.562211355500E+02/, GLN(78)/2.605649409719E+02/, GLN79
9   GLN(79)/2.649216497986E+02/, GLN(80)/2.692910976510E+02/, GLN80
1   GLN(81)/2.736731242857E+02/, GLN(82)/2.780675734404E+02/, GLN81
2   GLN(83)/2.824742926876E+02/, GLN(84)/2.868931332954E+02/, GLN82
3   GLN(85)/2.913239500943E+02/, GLN(86)/2.957666013508E+02/, GLN83
4   GLN(87)/3.002209486470E+02/, GLN(88)/3.046868567657E+02/, GLN84
5   GLN(89)/3.091641935802E+02/, GLN(90)/3.136528299499E+02/, GLN85
6   GLN(91)/3.181526396202E+02/, GLN(92)/3.226634991267E+02/, GLN86
7   GLN(93)/3.271852877038E+02/, GLN(94)/3.317178871969E+02/, GLN87
8   GLN(95)/3.362611819792E+02/, GLN(96)/3.408150588708E+02/, GLN88
9   GLN(97)/3.453794070623E+02/, GLN(98)/3.499541180408E+02/ GLN89
DATA GLN(99)/3.545390855194E+02/, GLN(100)/3.591342053696E+02/ GLN90
C -----
5 NDX=X
T=X-FLOAT(NDX)
IF(T.EQ.0.0) GO TO 51
DX=XLM1-X
IF(DX.LT.0.0) GO TO 40
C
C   RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3 FOR GAMMA(X) GLN92
C
C   NXM=NDX-2
PX=PCOE(1)
DO 10 I=2,9
10 PX=T*PX+PCOE(I)
QX=QCDE(1)
DO 15 I=2,4
15 QX=T*QX+QCDE(I)
DGAM=PX/QX
IF(NXM.GT.0) GO TO 22
IF(NXM.EQ.0) GO TO 25
C
C   BACKWARD RECURSION FOR 0.LT.X.LT.2 GLN93
C
C   DGAM=DGAM/(1.+T)
IF(NXM.EQ.-1) GO TO 25
DGAM=DGAM/T
GAMLN=ALOG(DGAM)
RETURN
C
C   FORWARD RECURSION FOR 3.LT.X.LT.8 GLN94
C

```

Figure A3. (Sheet 24 of 25)

```

22 XX=2.+T GLN121
   DO 24 I=1,NXM GLN122
      DGAM=DGAM*XX GLN123
24 XX=XX+1. GLN124
25 GAMLN=ALOG(DGAM) GLN125
   RETURN GLN126

C GLN127
C X.GT.XLIM1 GLN128
C GLN129
40 RX=1./X GLN130
   RXX=RXX*RX GLN131
   IF((X-XLIM2).LT.0.) GO TO 41 GLN132
   PX=P(1)*RXX+P(2) GLN133
   GAMLN=P(X)+RXX+(X-.5)*ALOG(X)-X+RTWPIL GLN134
   RETURN GLN135

C GLN136
C X.LT.XLIM2 GLN137
C GLN138
41 PX=P(1) GLN139
   SUM=(X-.5)*ALOG(X)-X GLN140
   DO 42 I=2,5 GLN141
   PX=P(X)+RXX+P(I) GLN142
42 CONTINUE GLN143
   GAMLN=P(X)+RXX+SUM+RTWPIL GLN144
   RETURN GLN145

C GLN146
C TABLE LOOK UP FOR INTEGER ARGUMENTS LESS THAN OR EQUAL 100. GLN147
C GLN148
51 IF(NDX.GT.100) GO TO 40 GLN149
   GAMLN=GLN(NDX) GLN150
   RETURN GLN151
END GLN152

```

Figure A3. (Sheet 25 of 25)



## APPENDIX B: NOTATION

|              |   |
|--------------|---|
| $a_0$        | Incident wave amplitude                                 |
| $A_0$        | $-iga_0/\omega$   |
| $g$          | Gravitational acceleration                              |
| $h$          | Water depth   |
| $i$          | $\sqrt{-1}$   |
| $J$          | Bessel function of the first kind                       |
| $k$          | Wave number   |
| $n$          | Non-negative integers                                   |
| $r$          | Polar coordinate  |
| $t$          | Temporal coordinate                                     |
| $x$          | Horizontal coordinate                                   |
| $y$          | Horizontal coordinate                                   |
| $Y$          | Bessel function of the second kind                      |
| $z$          | Vertical coordinate                                     |
| $\alpha$     | Incident wave angle                                     |
| $\eta$       | Free surface displacement                               |
| $\theta$     | Polar coordinate  |
| $\theta_0$   | Angle related to wedge angle                            |
| $v$          | Value related to wedge angle                            |
| $\phi$       | Velocity potential function                             |
| $\phi_h$     | Horizontal component of the velocity potential function |
| $\phi_o$     | Defined in Equation 12                                  |
| $\phi_i$     | Incident wave velocity potential function               |
| $\phi_r$     | Reflected wave velocity potential function              |
| $\phi_s$     | Scattered wave velocity potential function              |
| $\bar{\phi}$ | Finite cosine transform of $\phi$                       |
| $\omega$     | Wave radian frequency                                   |





