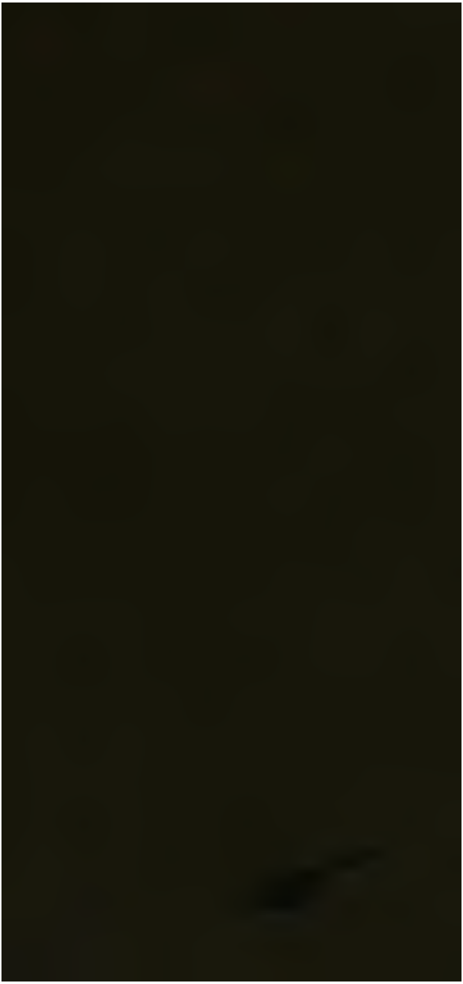

ELEMENTARY ALGEBRA.

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ELEMENTARY ALGEBRA

EDITED FOR THE SYNDICS OF THE UNIVERSITY PRESS

BY

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PREFACE.

THIS work on Elementary Algebra has been written at the request of the Syndics of the Cambridge University Press, and is intended to include those parts of the subject which most Schools and Examination Boards consider as covered by the adjective *elementary*. The discussion, herein contained, of Permutations and Combinations, the Binomial Theorem, and the Exponential Theorem—subjects which are sometimes included in Elementary Algebra, and sometimes excluded from it—should be regarded as introductory to their treatment in larger text-books.

I have in general followed the order of arrangement and method of presenting the subject which are traditional in England. I hope that the Table of Contents will enable the reader to find with ease the articles in which any particular part of the subject is discussed.

It may assist a student who is reading the subject for the first time, without the aid of any one to explain his difficulties, if I add that the propositions here given fall naturally into five groups, and that in the text these groups are divided one from the other by collections of miscellaneous questions or examination papers which have been set recently by various representative Examining Bodies. All articles and examples which are marked with an asterisk (*) may be omitted by any one who is reading the subject for the first time.

I am indebted to the kindness of the Secretaries of the Cambridge Local Examinations Syndicate and of the Oxford and Cambridge Schools Examination Board for permission to use the papers and questions which have been set in the examinations held under their authority. A large number of the examples inserted at the end of each chapter are, except for a few verbal alterations, derived from one or other of these sources, and indicate the tests of a knowledge of the subject which are usually applied: those questions which are marked with an asterisk are intended for the more advanced students only. The numerous examples interspersed in the text of each chapter are in most cases easier than those placed at the end of the chapters, and can be solved by a direct application of the rules given in the text.

I gratefully acknowledge my obligations to Dr Forsyth of Trinity College, Cambridge, Mr Platts of Trinity College, Cambridge, Mr Tucker, the Secretary of the London Mathematical Society, formerly of St John's College, Cambridge, and now of University College School, London, and Mr R. T. Wright of Christ's College, Cambridge, who have generously devoted considerable time to the dreary task of reading the proof-sheets and verifying the results of the examples. Their suggestions and remarks have been of great assistance to me, and have saved the book from many imperfections and obscurities.

I shall be grateful for notices of misprints, corrections, or criticisms on the work which may occur to any of my readers.

W. W. ROUSE BALL.

TRINITY COLLEGE, CAMBRIDGE,
October, 1890.

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CHAPTER I.

DEFINITIONS AND NOTATION.

1. **Algebra** is a science which treats mainly of numbers.

The distinction between arithmetic (which is also concerned with numbers) and algebra may be stated as follows. In arithmetic, every number is distinguished from every other number by the use of a certain figure or figures; but in algebra, we use symbols (such as the letters of the alphabet a, b, x, y, &c.) to represent *any* number whatever. We may, in a special problem, give a particular numerical value to one or more of the letters introduced, but our processes are general, and the letters may usually stand for any numbers whatever. Numbers represented by letters are often called *quantities.*

It is customary to employ not only the letters of the English alphabet, such as $a, b, c, \ldots$, or $A, B, C, \ldots$, and those of the Greek alphabet, such as $\alpha, \beta, \gamma, \ldots$, but also letters with accents, like $a', a'', \ldots$, or with suffixes, like $a_1, a_2, \ldots$, each of which may represent *any* number.

Thus the numbers represented by letters, like a, a', a'', a_1, a_2, will generally be different, and will have no connection one with the other.

2. **Symbols.** Each of the quantities like a, b, a', b', a_1, b_1, is termed a *symbol.*

The numbers denoted by algebraical symbols are abstract numbers.

Every concrete quantity, such as a length, an area, a time, a weight, &c., is measured by the number of units of its own kind which it contains. Thus a length may be expressed as $\frac{1}{2}$ foot, 6 inches, &c., according as a foot or an inch is the unit of length. The numerical measure of the quantity, that is, the number of times the unit is contained in it, is called an abstract number; such as $\frac{1}{2}$ or 6 in the case above given.

3. **Symbols of Operation.** The operations or processes of algebra are denoted by certain signs which are known as *symbols of operation.* With many of these symbols, such as $+$, $-$, $\times$, $\div$, &c. the student has already become acquainted in arithmetic.

The words "of operation" are often omitted, and, for brevity, these symbols of operation are called symbols. They cannot well be confused with the symbols defined in Art. 2, and no difficulty is found to arise from this double use of the word.

4. **Expressions.** Any combination of symbols by algebraical processes is called an *algebraical expression.*

In other words, any combination of letters, which denote numbers, by means of symbols of operation is called an algebraical expression.

5. It follows from the definition given in Art. 1, that algebra may from one point of view—and this is the best way of presenting it to one who is reading it for the first time—be regarded as a generalization of arithmetic. This is historically the origin of the science, which was indeed once known as *universal arithmetic.*

The description of algebra as a universal arithmetic may be illustrated by shewing how its notation enables us to express various arithmetical relations in a concise and general manner.

For example, if a man walk for 4 hours at the rate of 3 miles an hour, he will walk 3×4 miles; if he walk for 2 hours at the rate of 4 miles an hour, he will walk 4×2 miles. Now these and all similar conclusions may be included in a single statement or formula. Let us say that a man walks at the rate of v miles an hour (where the letter v stands for 1, 2, 3, $3\frac{1}{2}$, or any number, whether fractional or not), and let us suppose that he walks for t hours (where the letter t also stands for any number whatever), then the number of miles he will walk will be the product of v and t. If we represent this number of miles by the letter s, then $s=v\times t$. This algebraical relation includes every particular numerical example of the kind in a single statement.

6. We shall begin by describing some of the processes, and defining some of the terms, used in algebra. It will be noticed that in many cases these definitions are the same as those with which the student is familiar in arithmetic.

7. **Addition.** The result of adding two or more numbers together is called their *sum*.

The operation of addition is denoted by the word *plus*, which is represented by the symbol +. This symbol, when written between two numbers, signifies the operation of adding the number placed after the symbol to the one placed before it.

Thus $2+3$ (read as *two plus three*) indicates that we are to add 3 to 2. So $a+b$ (read as *a plus b*) indicates that we are to add the number denoted by the letter b to that denoted by a. Similarly $x+y+z$ (read as *x plus y plus z*) indicates that we are first to add y to x, and then to add z to that sum.

8. **Subtraction.** The result of subtracting a smaller number from a greater number is called the *difference* of the two numbers.

The operation of subtraction is denoted by the word *minus*, which is represented by the symbol −. This symbol, when written between two numbers, signifies the operation of subtracting the number placed after the symbol from that placed before it.

Thus $7-3$ (read as *seven minus three*) indicates that we are to subtract 3 from 7. So $a-b$ (read as *a minus b*) indicates that we are to subtract the number denoted by the letter b from that denoted by the letter a. Similarly $a+b-c$ indicates that we are first to add b to a, and then from the sum to subtract c.

9. **Signs.** Many symbols of operation are employed in algebra, but the word *sign*, when used alone, is generally taken to refer only to the symbols + and −.

10. **Positive and Negative Quantities.** It is evident that addition and subtraction are processes opposed to one another. If $+a$ means increasing a quantity by a, then $-a$ must mean decreasing it by a; and if these operations be performed in succession, no effect will be produced.

If, for instance, a certain length measured along a line from a fixed point (estimated say in feet) be given, then $+a$ will represent a feet in that direction. Hence $-a$ must represent a feet in the opposite direction, since the effect of the two taken in succession is to be nothing.

So, if we are considering a man's income (reckoned say in pounds sterling), then $+a$ will signify an addition of £a, and $-a$ will signify a decrease of £a. If however we are considering his expenditure, then $+a$ will signify an increase of £a, while $-a$ will refer to what decreases his expenditure by £a. Thus, if I earn 20 shillings and then lose 5 shillings, the result may be stated either by saying that I have gained 15 shillings, or that I have lost -15 shillings.

Similarly, if $+a$ represent a distance of a miles to the north, then $-a$ will represent a miles to the south; and vice versa. Thus, if I walk 10 miles to the north, I may be said to have walked -10 miles to the south.

So again, if a man be x years older than a boy, the fact may also be expressed by saying that the boy is $-x$ years older than the man.

11. The quantities $+a$ and $-a$ are in fact always *equal in magnitude but opposite in character*, and we use the signs + and − to signify this difference in their *nature* or *quality*, without any regard to whether the quantities to which they are prefixed are actually added to or subtracted from any other quantity.

12. **Multiplication.** The number obtained by multiplying two or more numbers together is called their *product*. Where more than two numbers are multiplied together their product is sometimes called their *continued product*.

The operation of multiplication is denoted either by a dot (.) or by the symbol ×. Either of these symbols, when placed between two numbers, signifies the operation of multiplying the number placed before the symbol by the number placed after it.

Thus 7×5 or $7 . 5$ (either of which is read as *five times seven* or *seven multiplied by five* or *seven into five*) indicates that we are to multiply 7 by 5. The latter form, namely $7 . 5$, might be mistaken for the decimal fraction 7·5; and it is therefore better to avoid using it, if there be any chance of confusion. So the product of 2 and the number denoted by a can be represented either by $2 \times a$, or by $2 . a$; it is also often written as $2a$, the dot between the 2 and the a being left out. Similarly the result of multiplying the number denoted by a by the number denoted by b is represented either by $a \times b$, or by $a . b$, or by placing the symbols side by side; thus, ab. It is evident that this latter form of representing the result cannot be used where both the quantities are arithmetical numbers: thus the symbol 75 is used to denote seventy-five and therefore cannot be also used to denote five times seven.

Of the methods of denoting multiplication of algebraical quantities which are above described, that of placing the symbols (which represent the quantities) side by side is the most common.

13. **Factors.** Each number in the product of several numbers is called a *factor* of the product.

If the factor be a number expressed in figures, it is called a *numerical factor*: if it be a number denoted by a letter or letters, it is called a *literal factor*.

14. Where several literal factors occur in the same product, it is usual to write them in their alphabetical order and to place the numerical factors first. Thus we generally write the product

of 5, a, b, and c as $5abc$, and not $a \times 5 \times c \times b$ (or any similar order). Similarly it is usual to write the product of a and x as ax, and not as xa, though it is always permissible to use the latter form if it be more convenient. We shall see later that we infer from arithmetic that it is immaterial in what order the numbers which form a product are multiplied together.

15. **Coefficient.** Each expressed factor in a product (or the product of some of the factors) is called the *coefficient* of the product of the remaining factors.

A coefficient may be a product of an arithmetical number and a number denoted by a letter or letters.

If the coefficient be a number expressed in figures, it is called a *numerical coefficient*: if it be a number denoted by a letter or letters, it is called a *literal coefficient*.

Where the coefficient is unity it is usually omitted. For example, we write x and not $1 \times x$.

Thus in the product $6abx$, the coefficient of abx is 6, which is a numerical coefficient; the coefficient of ax is $6b$; the coefficient of x is $6ab$; and so on.

Similarly in the quantity y, the coefficient of y is unity.

16. **Division.** The result of dividing one number by another number is called the *quotient* of the first by the second. The number divided is known as the *dividend*, and the number by which it is divided is called the *divisor*.

If there be no remainder, then the dividend is said to be *exactly divisible* by the divisor.

The operation of division is denoted either by the symbol $\div$, or by the symbol /. Either of these symbols, when placed between two numbers, signifies the operation of dividing the number placed first (the dividend) by the number placed after it (the divisor). The symbol / is called a *solidus*: it is desirable that the student should know its signification, but except in this chapter we shall use it but rarely.

The operation of division may also be represented by a fraction having the dividend for numerator and the divisor for denominator.

Thus either $35 \div 7$ or $35/7$ (read as *thirty-five divided by seven* or *thirty-five by seven*) indicates that we are to divide thirty-five (the dividend) by seven (the divisor). The operation can also be indicated by the use of a fractional form, as $\frac{35}{7}$. So any of the forms $a \div b$, a/b, or $\dfrac{a}{b}$ indicates that we are to divide the number denoted by a by the number denoted by b.

17. **Brackets.** We sometimes want to isolate a particular set of quantities, and treat them for the moment by themselves as if they were a single quantity. This is effected by placing them within a pair of *brackets*. The same result may be otherwise denoted by drawing a line, called a *vinculum*, over the quantity it is desired to isolate.

Brackets of various shapes are used, such as ( ), { }, [ ].

The methods of treating brackets, and of removing or inserting them, will be fully explained in chapters II. and III.; but their use and meaning may be here illustrated by considering an expression such as $a-(b+c)$. Here the part $b+c$ is enclosed in a bracket, it is therefore to be treated as a single quantity: thus the sum of b and c is to be subtracted from a. The same result might also be denoted by drawing a vinculum over the $b+c$: thus, $a-\overline{b+c}$.

Similarly $(a+b)(c-d)$ signifies the product of the sum of a and b and the quantity obtained by subtracting d from c.

So again $\{a-(b+c)\}[b-c]$ indicates the product of the expression in the brackets { } by the expression in the brackets [ ]: the quantity in the brackets { } is found by first adding c to b and then subtracting their sum from a, the quantity in the brackets [ ] is found by subtracting c from b.

18. **Equality.** The symbol = represents equality, and stands for the words *is equal to.*

Thus $a=b$ (read as *a is equal to b* or *a equals b*) indicates that the number denoted by a is equal to the number denoted by b.

19. **Other Symbols of Operation.** The following symbols of operation are used as abbreviations for the words against which they are placed.

The symbol $>$ stands for *is greater than.*

....	$<$	*is less than.*
....	$\neq$	*is not equal to.*
....	$\ngtr$	*is not greater than.*
....	$\nless$	*is not less than.*
....	$\pm$	*plus or minus.*
....	$\therefore$	*therefore.*
....	$\because$	*because.*

.... $\sim$, when placed between two numbers, stands for *the difference between* them.

Thus $7 \times 5 = 35$ signifies that five times seven is equal to thirty-five.

Again $(a-b) > c$ signifies that the result of subtracting the number denoted by b from the number denoted by a is greater than the number denoted by c.

So $a \sim b$ indicates the difference between the numbers denoted by a and b.

20. We shall now give a few examples to illustrate the above notation. The beginner will find it desirable

(i) to write every step in a line by itself,

(ii) to place each fresh line *below* the one last written,

(iii) to keep the symbols for equality in a vertical line.

Should it be necessary to explain how one step is derived from the one immediately preceding it, the explanation should be written between the two lines.

To save space, explanatory statements and successive steps are often *printed* in the same line, but in *writing* his work the student is recommended to follow the above rules.

Ex. 1. *What is the numerical value of $x \div a$ when $x=6, a=2$?*

In this case, $x \div a = 6 \div 2$

$= 3.$

Ex. 2. *If $m=2, n=1, x=3, y=1$, find the numerical values of* (i) $2m+n(x-2y)$; (ii) $(2m+n)(x-2y)$; (iii) $(m-n)(x-3)$.

These are three separate examples. We shall take them in their order.

(i) Here $2m = 2 \times 2 = 4.$

Also, $x-2y = 3-(2\times 1) = 3-2 = 1,$

and $\therefore\ n(x-2y) = 1 \times 1 = 1.$

$\therefore\ 2m+n(x-2y) = 4+1$

$= 5.$

(ii) Here $2m+n = 4+1 = 5,$

and $x-2y = 3-2 = 1.$

$\therefore\ (m+2n)(x-2y) = 5 \times 1 = 5.$

(iii) Here $m-n = 2-1 = 1,$

and $x-3 = 3-3 = 0.$

But the product of two numbers, one of which is zero, is itself zero. Hence $(m-n)(x-3) = 1 \times 0$

$= 0.$

Note. The student should remember that the sum of a number of quantities, each of which is zero, is necessarily equal to zero; and therefore **the product of a number of quantities, of which one is zero, must be equal to zero.**

EXAMPLES ON THE ELEMENTARY PROCESSES. I. A.

1. Write down the continued product of a, x, 3, and b.

2. What is the numerical value of $7ax$ when $a=\frac{1}{21}$ and $x=3$? And what is the numerical value when $a=2$ and $x=\frac{1}{14}$?

Write down the coefficient of x in the following quantities, numbered 3 to 6; and state whether the coefficient is literal or numerical.

3. $7x$. **4.** $23ax$. **5.** xy. **6.** x.

What are the numerical values of the following quantities, numbered **7** to **10**, when $a=1$, $x=2$, $y=3$?

7. $\frac{3ax}{y}$. **8.** $y \div x$. **9.** $\frac{xy}{3a}$. **10.** x/y.

What are the numerical values of the following quantities, numbered **11** to **14**, when $a=2$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=1$?

11. $abcd$. **12.** $a \div b$. **13.** $b \div a$. **14.** $\frac{ab}{cd}$.

15. What is the numerical value of the quotient of $(a-b)$ by c, when $a=11$, $b=2$, and $c=3$?

16. State in words the meaning of the expression

$$\{(a-b)-(c-d)\} \div (a-c).$$

17. Find the numerical value of

$$5a+3b-\{(c+d) \div (c-d)\}+2e,$$

when $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{2}$, $d=\frac{1}{3}$, $e=\frac{1}{4}$.

18. If *A* gained £7 and lost 12*s.* how much did he lose as the result of the whole transaction?

19. If *A* walked 7 miles in a S.W. direction, how far did he walk in a N.E. direction?

21. **Powers.** When a quantity is multiplied by itself a number of times the resulting product is called a *power* of the quantity.

Thus xx is called the *second power of* x, or the *square of* x, or x *squared*; xxx is called the *third power of* x, or the *cube of* x, or x *cubed*; $xxxx$ is called the *fourth power of* x; and so on.

22. **Indices. Exponents.** The square of x is usually denoted by x^2 instead of by xx, the small number placed above and to the right of x shewing the number of times the factor x has been repeated to form the product. The cube of x is similarly denoted by x^3 instead of xxx. And generally, if the factor x be repeated n times, the result is written as x^n. The small number or letter placed above and to the right of the symbol, and which

denotes the number of times the symbol is repeated in the product, is called the *index*, or *exponent*.

When the number x is taken by itself, it might be called the first power of x and denoted by x^1, but if the index be unity it is usual to omit it.

Thus, the fifth power of x, or $xxxxx$, is denoted by x^5; and in this case, 5 is the index, exponent, or power. So x^n denotes the n^{th} power of x, and is usually read as *x to the power n*.

A similar rule applies to more complicated products.

Thus a^2b^3 is written instead of $aabbb$. So $7ax^3y^4$ is written instead of $7axxxyyyy$.

Note. Beginners are sometimes apt to confuse the index (which denotes the power to which a quantity represented by a letter is raised) with a suffix (which is merely used to distinguish the quantity from other quantities as explained in Art. 1). They have no connection. The student will also notice that the number which represents an index is written above and to the right of the symbol to which it refers, while the number which represents a suffix is usually written below the symbol to which it refers.

Example. If $a=2$, $b=3$, $x=4$, $y=1$, find the numerical values of (i) $3by^2$; (ii) x^2-y^2; (iii) x^a-b^y; (iv) $a^y \div y$.

(i)
$$\begin{aligned} 3by^2 &= 3\times b\times y\times y \\ &= 3\times 3\times 1\times 1 \\ &= 9. \end{aligned}$$

(ii) Here $x^2 = xx = 4\times 4 = 16$,

and $y^2 = yy = 1\times 1 = 1$.

$$\begin{aligned} \therefore\ x^2-y^2 &= 16-1 \\ &= 15. \end{aligned}$$

(iii) Here $x^a = 4^2 = 4\times 4 = 16$,

and $b^y = 3^1 = 3$.

$$\therefore\ x^a - b^y = 16-3 = 13.$$

(iv) Here $a^y = 2^1 = 2$, and $y=1$.

$$\therefore\ a^y \div y = 2\div 1 = 2.$$

EXAMPLES ON THE NOTATION OF INDICES. I. B.

If $a=1$, $b=3$, $c=2$, $x=2$, $y=5$, find the numerical values of the following expressions.

1. $27x^2$.
2. $2ay^2$.
3. $\dfrac{x^2}{ab}$.
4. $x^2 \div c$.
5. $x^2 \div y^2$.
6. $\dfrac{a+b}{c}$.
7. 2^c.
8. x^y.
9. $\dfrac{y^x}{b+c}$.

If $\alpha=2$, $\beta=1$, $l=m=3$, find the numerical values of

10. $\alpha^2+\beta^2$.
11. $\alpha^3-3\beta^3$.
12. $\dfrac{\beta^2}{\alpha^2}$.
13. $2m^\alpha$.
14. $\frac{1}{2}\alpha^2 l^3$.
15. $3^{\frac{l+m}{\beta}}$.
16. $\frac{1}{3}\alpha\beta^2 l^3 m^4$.
17. $l^\alpha - \alpha^l$.
18. $\dfrac{l-m}{\alpha}$.

If $a_1=3$, $a_2=2$, $a_3=4$, find the numerical values of

19. $a_1{}^2 - a_2 a_3$.
20. $a_2{}^2 - a_3$.
21. $a_1 a_2{}^3 a_3{}^2$.

If $a=1$, $b=3$, $x=0$, $y=2$, find the numerical values of

22. $7ax^2$.
23. y^2-x^2.
24. $2a^2by^2$.
25. $b^2 \div y$.
26. $3ay^2-2b^2x$
27. b^2-2xy.

If $a=4$, $b=3$, $x=2$, $y=1$, find the numerical values of

28. $(a^2+b^2)(x^2-y^2)$.
29. $3a-(b-x+2)^2$.
30. $2\{a^2-(b^2-x^2)\}$.
31. $x^2 \div (a-b)$.
32. $(x^3+y^3)/(x+y)$.
33. $(x^3-y^3)^2$.

23. **Roots. Surds.** The quantity which when raised to the n^{th} power is equal to any number such as a is called the n^{th} *root of a*.

The n^{th} root of a is represented by the symbol $\sqrt[n]{a}$, which is called a *surd*.

It is usual to call the number which when squared is equal to a the *square root of a* (and not the second root of a). It is denoted by $\sqrt[2]{a}$, or more often by $\sqrt{a}$.

Similarly the number which when cubed is equal to a is generally called the *cube root of a* (and not the third root of a); it is denoted by $\sqrt[3]{a}$.

Where no exact number can be found which is the n^{th} root of a, then the n^{th} root of a is called an *irrational quantity* or an *irrational surd.*

An expression which involves no irrational quantity is said to be *rational.*

The symbol $\surd$ is known as the *radical* or the *radical sign.*

Wherever the radical sign is followed by more than one symbol it is desirable to put all the quantities on which it operates within brackets or under a vinculum.

Thus the square root of the product of a and b would be denoted either by $\surd(ab)$ or by $\sqrt{ab}$. The expression $\surd ab$ signifies the product of b and the square root of a, but it is so likely to be mistaken for $\surd(ab)$ that we should avoid its use, and should express the product of b and the square root of a by $b\surd a$, where the radical sign only affects the quantity immediately before which it stands.

Ex. 1. *If* $a=3$, $x=7$, $c=11$, *find the numerical values of*

(i) $\sqrt{3a}$; (ii) $\surd(x-a)$; (iii) $\sqrt[3]{x^2-2c}$; (iv) $\sqrt[3]{}(ax^2)$.

(i)
$$\begin{aligned}\sqrt{3a}&=\sqrt{3\times 3}\\&=\surd 9\\&=3.\end{aligned}$$

(ii)
$$\begin{aligned}\surd(x-a)&=\surd(7-3)\\&=\surd 4\\&=2.\end{aligned}$$

(iii)
$$\begin{aligned}\sqrt[3]{x^2-2c}&=\sqrt[3]{49-22}\\&=\sqrt[3]{27}\\&=3.\end{aligned}$$

(iv)
$$\begin{aligned}\sqrt[3]{}(ax^2)&=\sqrt[3]{}(3\times 49)\\&=\sqrt[3]{}(147),\end{aligned}$$

and as there is no exact number whose cube is 147, either we must leave the result as an irrational surd, or we can (by arithmetic) find the value to as many places of decimals as we like.

Ex. 2. *If* $a=4$, $b=1$, $x=3$, *find the numerical values of*

(i) $(\sqrt{x^2})\div a$; (ii) $\sqrt{(x^2\div a)}$; (iii) $\sqrt[3]{a^x-2(a+x)(b+x)}$.

(i) $(\sqrt{x^2})\div a=x\div a=3\div 4=\frac{3}{4}$.

(ii) $\sqrt{(x^2\div a)}=\sqrt{9\div 4}=\sqrt{\frac{9}{4}}=\frac{3}{2}$.

(iii) $\sqrt[3]{a^x-2(a+x)(b+x)}=\sqrt[3]{4^3-2(4+3)(1+3)}$

$=\sqrt[3]{64-2\times 7\times 4}$

$=\sqrt[3]{8}$

$=2$.

EXAMPLES ON THE NOTATION OF SURDS. I. C.

Find the numerical values of the following expressions, when $a=1$, $b=2$, $c=1$, $d=2$.

1. $a\sqrt{b^2c}$.
2. $(b-d)\sqrt{a^3}$.
3. $\sqrt{(b^2-d^2)(a+c)}$.
4. $\sqrt[3]{b^2d}+\sqrt{(a+2b-1)}$.
5. $\sqrt{(2b^2+d^3)}$.
6. $\sqrt[4]{\dfrac{4b^3d^3}{ac}}$.

If $a=\frac{1}{2}$, $b=2$, $x=1$, $y=\frac{1}{3}$, find the numerical values of

7. $\sqrt{abx}$.
8. $\sqrt{1-\frac{5}{3}y}$.
9. $\sqrt{(x^2-4by^2)}$.
10. $\sqrt[3]{bx\div a}$.
11. $(x-y)\sqrt{b-x}$.
12. $(x^2-y^2)\sqrt{a^2+b}$.

13. Express in words the meaning of the expressions

(i) $(a+b)\sqrt{a^2-(b^2+c^2)}$; (ii) $a+b\sqrt{a^2-(b^2+c^2)}$.

And find their numerical values, when $a=3$, $b=2$, $c=1$.

24. **Terms.** When an algebraical expression is made up of a number of component parts, connected by the symbols + and −, each part is called a *term.*

Terms which differ only in their numerical coefficients are called *like terms.*

A term preceded by the symbol + is called a *positive term*, or a *positive quantity*; and it is said to *have a positive sign.* A term preceded by the symbol − is called a *negative term*, or a *negative quantity*; and it is said to *have a negative sign.*

When no symbol is prefixed to a term it is considered to be positive.

Thus in the expressions $2ax^2 - 3b^3 - 5xyz$ and $7a^2x + 4b^3 + xyz$ the terms $-3b^3$ and $4b^3$ are like, as are also the terms $-5xyz$ and $+xyz$; but the terms $2ax^2$ and $7a^2x$ (though they contain the same letters) are unlike, since the powers to which a and x are respectively raised are different. The terms $2ax^2$, $7a^2x$, $4b^3$, and xyz are positive terms: the terms $-3b^3$ and $-5xyz$ are negative terms.

25. **Simple and Compound Expressions.** A *simple expression* consists of one term only. A *compound expression* contains more than one term.

A simple expression is sometimes called a *monomial.* If a compound expression consist of two terms, it is called a *binomial*; if of three terms, a *trinomial*; and if of more than three terms, a *multinomial* or *polynomial.*

Thus $5ab^2c^3$ is a simple expression, $a + b$ is a binomial, $x^3 + y^3 + z^3$ is a trinomial, and so on.

Again $2x^3 - 3ax^2 - bx + 4 + a^2/b$ is a compound algebraical expression, made up of five terms. The first term is $2x^3$, the second is $3ax^2$, the third is bx, the fourth is 4, and the fifth is the quotient of a^2 by b. The first, fourth, and fifth terms are positive terms, the second and third terms are negative terms. No two of the terms are like.

As one more illustration, consider the expression

$$[\sqrt{\{x^2 + \sqrt{b^2 + x^2}\}} - a]\,(x + b).$$

Although this looks complicated, it is a simple expression; it consists of the product of two quantities, namely that enclosed in the brackets ( ) and that enclosed in the brackets [ ]. The quantity enclosed in the brackets ( ) is a binomial expression, consisting of the sum of two terms. The quantity enclosed in the brackets [ ] is also a binomial expression, consisting of the result of subtracting a from the square root of the expression enclosed in the brackets { }. This last expression, namely that enclosed in the brackets { }, consists of the sum of x^2 and of the square root of the sum of b^2 and x^2.

26. **Degree or Dimensions of a quantity.** A quantity which is the product of n letters is said to be of *the* n^{th} *degree* or of *n dimensions.*

In reckoning dimensions, numerical factors are not counted.

Thus each of the quantities abc, $2a^2b$, and $-3x^3$ is of the third degree, or of three dimensions.

Sometimes we speak of the dimensions in a particular letter, and we then confine ourselves to that letter.

Thus $2a^2b$ is of two dimensions in a and one dimension in b.

27. **Degree or Dimensions of an expression.** The *degree* of an expression is the degree of the term of highest dimensions in it. In estimating it, it is usual to confine ourselves to only one letter.

An expression which is of the n^{th} degree is said to be of *n dimensions.*

An expression which is of the first degree in a letter is said to be *linear* in that letter.

Thus $x^2y+xy^2+y^3$ is of the second degree in x, but it may be said to be of the third degree if all the letters are taken into account.

28. **Homogeneous Expressions.** A compound expression is said to be *homogeneous* when every term in it is of the same dimensions.

Thus the expression $x^2y+xy^2+y^3$ is homogeneous in x and y, and is of the third degree.

29. **Formula. Identity.** When an algebraical expression can be written in two ways the result obtained by equating one to the other is said to be an algebraical *formula* or *identity.*

Examples will be given later [see *ex. gr.* Art. 87].

30. It is desirable to warn the beginner to be careful to observe both the *order* and the *extent* of the operations indicated in an algebraical expression.

Thus $a+b\times c$ or $a+bc$ signifies that the product of b and c is to be found and then added to a, while either $(a+b)\times c$ or $(a+b)\,c$ or $\overline{a+b}\,.\,c$ signifies that the number which is formed by adding b to a is to be multiplied by c.

So again $\sqrt{}a+b$ signifies that b is to be added to the square root of a, but $\sqrt{}(a+b)$ or $\sqrt{a+b}$ signifies the square root of the sum of a and b.

It is obviously undesirable to use forms of notation which resemble one another so closely as those above written, and which are liable to be mistaken the one for the other; and in practice we shall avoid the use of forms which a careless reader might regard as ambiguous.

EXAMPLES ON ALGEBRAIC NOTATION. I. D.

1. What are the numbers seven and three called respectively in the expression $7a^3$?

2. Define what is meant (i) by x^3, and (ii) by $3x$. Which is the greater when $x=1$, and which is the greater when $x=2$?

3. Write down the continued product of 3, b, b, c, and y.

4. Write down the sum of the five quantities given in Ex. 3.

What are the dimensions of each of the following quantities?

5. xyz. **6.** $3ax^4$. **7.** $2a^2x$. **8.** bx^3. **9.** x. **10.** x^n.

11. What are the dimensions in x of each of the quantities given in Exs. 5, 6, 7, 8, 9, 10?

State the dimensions in x of the expressions numbered 12 to 14.

12. $2-3x$. **13.** $3ax^2-7x^3+2a^4x$. **14.** x^n-a^n.

15. Are any of the terms in the following expression like terms?

$$2ax^3-b^2+3a^2x^2+3b^3-5abx.$$

16. If $a=8$, $b=5$, $c=1$, find the numerical value of

$$c\sqrt{10ab}+b\sqrt{8ac}+a\sqrt{45bc}.$$

If $x=2$, find the numerical values of the expressions 17, 18.

17. $\frac{2x-1}{3}+\frac{3x+4}{2}-\frac{21x-6}{9}$. **18.** $x^2+\sqrt{x^2-42x+89}$.

19. If $a=1$, $b=3$, $c=4$, $d=0$, find the numerical values of

(i) $3ab^2-d[bc^2+2(c-b)]+ac(b^3-cd)+\frac{2}{3}b$;

(ii) $\frac{\sqrt{a+2c}}{a+2b+d}+\frac{3(a+2b)}{2c-a}$.

20. If $a=3$, $b=4$, $c=5$, $d=6$, find the numerical values of

(i) $\frac{2\sqrt{a^2+b^2}+\sqrt[3]{b^3-a^3-d^3}}{d-c+b-a}$; (ii) $\frac{2(a+b)(c+d)-(b+c)(d+a)}{ab+cd-bc-da}$.

CHAPTER II.

ADDITION AND SUBTRACTION.

31. THE first thing that we have now to do is to learn how to add, subtract, multiply, and divide algebraical expressions. These operations will constantly occur in all our subsequent investigations, and it is necessary to know how to effect them with accuracy and facility before we proceed any farther.

We shall deal in this chapter with the rules for adding and subtracting algebraical expressions, and shall consider first the case of simple expressions and next the case of compound expressions.

32. **Addition and Subtraction of Simple Expressions.** The addition of a number of simple quantities is indicated by writing them down in succession, each preceded by the sign of addition, namely + [see Art. 7]. If any one of them has to be subtracted, it must be preceded [Art. 8] by the sign of subtraction, namely −.

Thus to indicate the addition of b to a and then the subtraction of c from the result we write $a+b-c$.

We cannot simplify this until we know what numbers or expressions are represented by a, b, c.

33. **Order of addition is immaterial.** In arithmetic it is shewn that the sum or the difference of several numbers is the same in whatever order the additions or subtractions are made. We assume that

the same will therefore be true of numbers when they are represented by algebraical symbols.

Thus the sum of a and b may be written indifferently as $a+b$ or $b+a$.

It is easy to verify this statement by giving a and b numerical values, or considering some *particular* case (such as one in which a and b stand for a number of shillings, or for lengths measured in some given direction); and all similar references to arithmetic can be tested in like manner.

By similar reasoning the expression

$$a-b+c-d$$

may also be written (among other ways) as

$$a+c-b-d, \quad \text{or } a-d-b+c, \quad \text{or } c+a-d-b.$$

Any difficulty that may arise from negative quantities will be explained later.

34. **Combination of like terms.** *Where like terms occur, they can be combined into a single term.*

(i) Where the like terms are of the same sign they can be replaced by a single term of the same sign, like either of them, and having a numerical coefficient equal to the sum of the numerical coefficients of the separate terms.

For example, to add $2a$ to any quantity and then to add $3a$ to it is equivalent to adding $5a$ to it. That is,

$$2a+3a=5a.$$

Similarly, to subtract $2a$ from any quantity and then to subtract $3a$ from it is equivalent to subtracting $5a$ from it. That is,

$$-2a-3a=-5a.$$

(ii) Where two like terms are of opposite signs we take the difference of the numerical coefficients and affix the sign of the greater.

For example,

$$3a-2a=2a+\ a-2a=+a,$$
$$2a-3a=2a-2a-\ a=-a.$$

The terms may cancel one another. For example,

$$2x^2-x^2-x^2=0.$$

(iii) Where there are several like terms, some positive and some negative, first, as in case (i), we combine all the positive terms into one term, and all the negative terms into another term; and then, as in case (ii), we combine these two terms into one term.

For example,

$$\begin{aligned} 4a+2a-3a+a-5a &= 4a+2a+a-3a-5a \\ &= 7a-8a \\ &= -a. \end{aligned}$$

35. **Simplification of Expressions by Collection of like terms.** There may be different sets of like terms in the quantities to be added. In such a case all the like terms of each kind can be collected together, and then combined into one term. The final result will be the sum or difference of the terms so formed.

For example, the expression

$$\begin{aligned} 3a+b+2a-3b-4a+b &= 3a+2a-4a+b+b-3b \\ &= 5a-4a+2b-3b \\ &= a-b. \end{aligned}$$

EXAMPLES. II. A.

Find the values of the following expressions by combining like terms.

1. $-a+2b+3c-2a-2c+2b+3a.$
2. $-\frac{1}{2}a+c-2b+n-\frac{3}{2}a+4n-\frac{1}{2}c.$
3. $x^2+2y^2-z^2-3y^2+2x^2-7z^2+y^2.$
4. $\frac{3}{4}a-b+\frac{1}{3}b-\frac{1}{4}a+\frac{2}{3}b-\frac{1}{2}a.$
5. $2p^2+3q^2-\frac{1}{2}r+\frac{1}{4}q^2-\frac{3}{2}p^2+2r^2-\frac{1}{2}r.$
6. $3\frac{a}{b}-\frac{a}{b}+2\frac{a}{b}-4\frac{a}{b}.$
7. $\sqrt{a}+2\sqrt{b}-\sqrt{a}-\sqrt{b}.$
8. $2\sqrt{a}+c-3\sqrt{b}+\sqrt{a}+2\sqrt{b}-3c.$

36. **Addition of Multinomials.** *To add to any quantity a multinomial expression write down every term of the multinomial with its own sign prefixed.*

To prove this rule, let us consider the addition to any quantity of a binomial expression like $(a+b)$ or $(a-b)$.

In the case of $a+b$ both the terms are positive. Now we know from arithmetic that to add a to any quantity and then to add b to it is equivalent to adding their sum $(a+b)$ to it. That is,

$$+(a+b)=+a+b \ldots\ldots\ldots\ldots\ldots (A).$$

In the case of $a-b$ the terms are of opposite signs, and we will suppose for the present that $a>b$. Then, if we add a to the given quantity we shall have added b too much, and therefore must subtract b from the result. That is,

$$+(a-b)=+a-b \ldots\ldots\ldots\ldots\ldots (B).$$

A similar proof evidently applies to the case of any multinomial, and hence the rule given at the head of this article follows.

Thus, for example,

$$+(a+b+c)=+a+b+c,$$

and $$+(a-b-c)=+a-b-c.$$

Ex. Find the sum of $(a+b-c)$, $(2a+4c)$, *and* $(3a-2b-3c)$.

The sum $=(a+b-c)+(2a+4c)+(3a-2b-3c)$

$=a+b-c+2a+4c+3a-2b-3c.$

Collect like terms,

$\therefore$ the sum $=a+2a+3a+b-2b-c+4c-3c$

$=6a-b.$

37. **Process of addition.** It is often convenient to write the expressions so that the like terms come in vertical columns, and then add them as in arithmetic.

Thus the above example would be written

$$\begin{array}{rrrr} & a+ & b- & c \\ & 2a & & +4c \\ & 3a- & 2b- & 3c \\ \hline \text{Add,} & 6a- & b & \\ \hline\hline \end{array}$$

38. The numerical coefficients may be fractional as in the following example.

Ex. Add together $\frac{1}{2}x+y-\frac{1}{3}z$, $\frac{2}{3}x-\frac{1}{2}y$, *and* $\frac{1}{3}y+z$.

Here we have

$$\begin{array}{rrr} \frac{1}{2}x+ & y- & \frac{1}{3}z \\ \frac{2}{3}x- & \frac{1}{2}y & \\ & \frac{1}{3}y+ & z \\ \hline \frac{7}{6}x+ & \frac{5}{6}y+ & \frac{2}{3}z \\ \hline\hline \end{array}$$

39. **Subtraction of Multinomials.** *To subtract from any quantity a multinomial expression write down every term of the multinomial with its sign changed; that is, change every plus into a minus, and change every minus into a plus.*

To prove this rule, let us consider the subtraction from any quantity of a binomial expression like $(a+b)$ or $(a-b)$.

In the case of $a+b$ both the terms are positive. Now we know from arithmetic that to subtract a from any quantity and then to subtract b from it is equivalent to subtracting their sum $(a+b)$. That is,

$$-(a+b)=-a-b \ldots\ldots\ldots\ldots\ldots (\text{C}).$$

In the case of $a-b$ the terms are of opposite signs, and we will suppose for the present that $a>b$. Then, if we subtract a from the given quantity we shall have subtracted b too much, and we must therefore add b to the result. That is,

$$-(a-b)=-a+b \ldots\ldots\ldots\ldots\ldots (\text{D}).$$

A similar proof applies to the case of a multinomial, and hence the rule given at the head of this article follows.

Thus $\qquad -(a+b+c)=-a-b-c,$

$\qquad -(a-b-c)=-a+b+c,$

and $\qquad -(a+b-c)=-a-b+c.$

40. **Extension of results.** The results (B) and (D) of Arts. 36 and 39 were proved true on the hypothesis that a was greater than b. We shall now see under what conditions we may regard them as true for all values of a and b.

If they be true for all values of a and b, we may put $a=0$ in them. We shall then get

$$+(-b)=-b,$$
$$-(-b)=+b.$$

The first of these results shews that *the addition of a negative quantity must be taken as being equivalent to the subtraction of a positive quantity of the same magnitude.* The second shews that *the subtraction of a negative quantity must be taken as being equivalent to the addition of a positive quantity of the same magnitude.*

Neither of these operations is discussed in elementary arithmetic, but if we take them to have the meanings above given, then it will be found on trial that the equations (B) and (D) are true whatever be the numerical values that we give to the symbols. Therefore, on this hypothesis, we may consider those equations to be true for all values of the quantities involved.

For example, to subtract $x-y+z$ from $2x+3y$, we have

$$(2x+3y)-(x-y+z)=2x+3y-x+y-z$$
$$=x+4y-z.$$

Hitherto we have supposed that our algebraical symbols denoted positive numbers only, for if a symbol had denoted a negative number we could not have subtracted it from any other number. We now know the meaning to be assigned to the addition or subtraction of a negative number, and henceforth we shall consider that our symbols may stand for negative as well as positive numbers.

*41. The meanings obtained, in the last article, for the addition and the subtraction of a negative quantity will be found on consideration to be a natural extension of the results of arithmetic, and to be consistent with the description given in Arts. 10, 11 of negative quantities. But the method by which we have found a meaning for these operations is worthy of close attention, since it is one of which we shall make frequent use, and on it large parts of algebra are founded. The following account of the method may help the student to understand it better.

We have two relations, (B) and (D), which we have proved in Arts. 36, 39 to be true in every case in which they are arithmetically intelligible, namely, in every case in which the number denoted by a is greater than the number denoted by b. But if certain numbers (*ex. gr.* whenever a is less than b) are substituted in these relations they involve operations (such as the subtraction of a negative number) which are arithmetically unintelligible. Such operations have no meaning, and we can make them stand for any thing we please, or define them in any manner we like, provided that our subsequent use of them is always consistent with the meaning so selected, and leads to results which are not inconsistent with the meaning of operations already employed. Now the use of algebra depends largely on employing relations which are true whatever be the numbers for which the symbols stand. We therefore try to find a meaning for these operations, not in an arbitrary way, but by extending the arithmetical results so that our new meanings shall be consistent with all the results already obtained, and shall thus allow of our symbols having *any* numerical values.

42. **Process of subtraction.** When one expression is to be subtracted from another, it is often convenient to write the expressions so that the like terms come in vertical columns, and then to subtract them as in arithmetic—changing mentally the sign of every term in the quantity subtracted, but not altering the signs on the paper.

Thus the example given at the end of Art. 40, where $x-y+z$ is subtracted from $2x+3y$, would be written thus:

$$\begin{array}{lr} & 2x+3y \\ & x-y+z \\ \hline \text{Subtract,} & x+4y-z \end{array}$$

43. Removal of Brackets. A plus sign before a bracket enclosing a quantity merely signifies that the quantity is to be added; while a minus sign before a bracket enclosing a quantity signifies that the quantity is to be subtracted.

Hence the results of Arts. 36, 39, 40 may be stated thus. *If, in a given expression, we have a quantity in a bracket preceded by the plus sign, we can remove the bracket provided we write down the terms inside it each with its own sign prefixed: if the bracket be preceded by the minus sign, we can remove the bracket provided we write down the terms inside it each with its sign changed.*

For example,

$$a+(b-c)+(c-d)=a+b-c+c-d$$
$$=a+b-d.$$
$$a+(b-c-d)-(-a-b+d)=a+b-c-d+a+b-d$$
$$=2a+2b-c-2d.$$

44. Sometimes the quantities contained within brackets are themselves compound expressions involving other brackets. In this case the brackets are made of different shapes so as to enable us to pick out each pair.

Thus $a-\{b-[c-(d-e)]\}$ means that we are to subtract from a the quantity enclosed within the brackets $\{\ \}$. This quantity is itself formed by subtracting from b the quantity within the brackets $[\ ]$; and so on.

To find the value of such an expression, the beginner will find it best to remove only one pair of brackets at a time, and to begin with the innermost brackets.

Thus $\quad a-\{b-[c-(d-e)]\}=a-\{b-[c-d+e]\}$

$$=a-\{b-c+d-e\}$$
$$=a-b+c-d+e.$$

45. Introduction of Brackets. Conversely, we can introduce brackets. Any of the terms of an expression can be placed within a pair of brackets, preceded by the plus sign, provided no alteration is made in the signs of the terms inside the brackets. Similarly, any of the terms of an expression can be placed within a pair of brackets, preceded by the minus sign, provided the sign of every term within the brackets is changed.

For example, we may write $a+b-c$ in the form

$$a+(b-c) \quad \text{or} \quad a-(-b+c).$$

Other ways of writing it are the following:

$$b+(a-c), \quad b-(c-a), \quad (a+b)-c, \quad -(-a-b)-c.$$

Each of these forms reduces to $a+b-c$ when the brackets are removed.

46. **Algebraical Sum.** We have shewn [Art. 40] that the subtraction of a positive quantity is equivalent to the addition of a negative one of the same numerical magnitude; and also that the subtraction of a negative quantity is equivalent to the addition of a positive one of the same numerical magnitude. Hence, subtraction may be regarded as equivalent to an algebraical summation; and the result of subtracting one quantity from another is often described as their *algebraical sum.*

Thus the algebraical sum of $5a$, $-2a$, and $-4a$ is equal to

$$5a-2a-4a=5a-6a=-a.$$

This extension of the word *sum* to cover subtraction as well as addition is convenient and saves much circumlocution.

NOTE. When hereafter we speak of the sum of two or more quantities, or of adding certain quantities together, we shall in all cases mean the algebraical sum.

47. **Inequalities.** A similar extension of arithmetical language has been given to the phrases *greater than* and *less than.*

In arithmetic, we say that the excess of a number a over a number b is $a-b$, or that a number a is greater than a number b by $a-b$, where, in both cases, a is supposed to be greater than b. So long as a is greater than b this is intelligible, but if a be less than b it is arithmetically unintelligible. We therefore extend the meaning of the phrase "greater than," and say that (whatever numbers a and b may denote) *a is greater than b by $a-b$.*

Thus 5 is greater than 3 by $5-3$, that is, by 2; similarly 3 is greater than 4 by $3-4$, that is, by -1. So -2 is greater than 7 by $-2-7$, that is, by -9. Again -3 is greater than -1 by $-3-(-1)$, that is, by -2; and so on.

48. We say also that a is greater than b, if $a-b$ be positive; and a is less than b, if $a-b$ be negative.

This enables us to compare the magnitudes of any two numbers, whether they be positive or negative.

For example, -2 is greater than -3, because $-2-(-3)$ is equal to $+1$, which is positive.

MISCELLANEOUS EXAMPLES. II. B.

1. Add 3 to -2.

2. Add -3 to 2.

3. Add $-x$ to $-y$.

4. Find the sum of x^2, y^2, $-3x^2$, $7y^2$, z^2, and $-5y^2$.

5. Find the sum of a^3, $3x^3$, $-2a^2x$, $-2a^3$, $-3x^3$, and $2a^2x$.

6. Subtract 7 from 2.

7. Subtract 7 from -2.

8. Subtract b from a.

9. Subtract $-x$ from $-y$.

10. If A gained 10*s.* and then lost £2, how much did he lose? How much did he gain?

11. To $a+b-x$ add $-2a-b+y$.

12. Add together $x+y$ and $x-y$.

13. Find the sum of $\frac{1}{2}a+x$, $-\frac{1}{3}a-2y$, and $x+3y$.

14. From $x+y$ subtract $x-y$.

15. From the sum of $2a+3b$ and $2b-3a$ subtract $x+a$.

16. What is the difference between a^2-b^2 and a^2+x^2?

17. From $bc+2ab-3ca$ subtract $2bc+ab-3ca$.

18. Add together $3a-(b-c)$, $3b-(c-a)$, $3c-(a-b)$; and find the numerical value of the result when $a=6$, $b=3$, $c=2$.

19. Add together
$$4b^3+(3a^2c-6b^2c),\ 5b^2c-(a^3+3a^2c),\ a^3-(2b^3+2b^2c).$$

20. Subtract $(x+2y+z)$ from $(x-2y+z)+(2y-z)$.

21. Simplify $[a+b-\{a+b+c-(a+b+c+d)\}]-a$.

22. Simplify $(a-b)-\{3a-(a+b)\}+\{(a-2b)-(5a-2b)\}$.

23. Add together $\frac{1}{3}a+\frac{1}{4}b+\frac{1}{5}c$, $\frac{1}{4}a+\frac{1}{5}b-\frac{1}{3}c$, $-\frac{1}{5}a-\frac{1}{3}b+\frac{1}{4}c$.

24. From $\frac{2}{3}x^2+\frac{1}{4}y^2$ subtract x^2-y^2+2x.

25. From the sum of $\frac{1}{2}l+\frac{1}{3}m$ and $-\frac{1}{2}m+\frac{1}{3}n$ subtract the sum of $\frac{3}{2}l-\frac{4}{3}n$ and $\frac{1}{4}m-\frac{2}{3}n$.

26. A man is now a years old, how long will it be before he is b years old? What would be the meaning of the result if b were less than a?

27. A walks 10 miles due north, then 15 miles due south, and then 7 miles due north. What is the total distance towards the south that he has gone?

28. What quantity must be added to a in order that the sum may be b?

29. What quantity must be subtracted from a in order that the result may be $a+b$?

30. What quantity must be added to a^3+b^3 in order that the sum may be x^3?

31. What quantity must be subtracted from x^2-y^2 in order that the result may be $2xy$?

32. By how much is x greater than y?

33. By how much is $-b$ greater than $-a$?

34. By how much is 11 greater than -11?

35. Shew that, if $x=2a+3b$, $y=3a-2b$, $z=b-4a$, then $x+y+z=a+2b$.

CHAPTER III.

MULTIPLICATION.

49. We proceed now to the consideration of the multiplication of algebraical quantities, and shall discuss successively (i) the product of two or more simple expressions, (ii) the product of a compound expression and a simple expression, and (iii) the product of two or more compound expressions.

50. **Order of multiplication is immaterial.** In arithmetic it is shewn that the product of one number a by another b is the same as the product of b by a, that is, $ab = ba$. Similarly,

$$abc = acb = bca = bac = cab = cba.$$

Thus it is immaterial in what order we multiply the different numbers together. We assume that the same is true of all algebraical expressions (whether they represent positive or negative quantities).

51. **Product of Simple Expressions.** The product of two or more simple quantities is indicated by writing them down in a line separated, each from the one next to it, by the symbol ×, or by a dot (.), or, in the case of literal factors, side by side without any intermediate symbol [see Art. 12].

Where the result contains the same quantity repeated more than once, the result can be somewhat simplified by using the index notation [Art. 22].

Ex. 1. *Find the product of* $2abx$ *and* $3axy$.

$$(2abx)(3axy) = 2 \times a \times b \times x \times 3 \times a \times x \times y$$
$$= 2 \times 3 \times a \times a \times b \times x \times x \times y$$
$$= 6a^2bx^2y.$$

With a little practice, the result can be written down by inspection: and as soon as possible the student should solve such questions by inspection only.

Ex. 2. *Find the product of* $3a^3xy^3$ *and* $\frac{1}{3}ax^3$.

$$(3a^3xy^3)(\tfrac{1}{3}ax^3) = 3 \times \tfrac{1}{3} \times a^3a \times xx^3 \times y^3$$
$$= a^4x^4y^3.$$

52. **Product of a positive and a negative quantity.** The explanation of the multiplication of two numbers which is given in arithmetic is only intelligible if both the numbers be positive. If one of them be negative, we have to find the meaning of an expression like $(-a)\,b$ or like $a\,(-b)$.

The expression $(-a)\,b$ signifies that $-a$ has to be taken b times.

$$\therefore\ (-a)\,b = (-a) + (-a) + (-a) + \ldots\ldots [b \text{ terms}]$$
$$= -a - a - a - \ldots\ldots\ldots\ldots\ldots\ldots [b \text{ terms}]$$
$$= -(a + a + a + \ldots\ldots\ldots) \qquad [b \text{ terms}]$$
$$= -(ab)$$
$$= -ab.$$

The above proof applies only to cases where b is a whole number, but, by the principle explained in Art. 41, the rule can be extended to cases where b is a positive fraction.

The expression $a\,(-b)$ may be written as $(-b)\,a$, since the order in which the multiplication is made is immaterial. But, by the above reasoning, $(-b)\,a = -ba$, which is the same as $-ab$. Therefore $a\,(-b) = -ab$.

We shall defer for the present the consideration of the meaning to be attached to the product of two negative quantities [see Art. 55].

For example, the product of $3ab$ and $-4cd$ is $-12abcd$. Here all the letters are different, and no further simplification is possible.

Similarly, the product of $-2xy$ and $3x$ is $-6x^2y$.

Similarly, the continued product of ab, bc, and $-ac$ is

$$(ab)(bc)(-ac)=(ab^2c)(-ac)=-a^2b^2c^2.$$

EXAMPLES. III. A.

Write down the product of the following expressions.

1. $2ax$ and $3by$.
2. $3ab$ and cd.
3. $7a^2x^3$ and $7b^3x^2$.
4. $\frac{1}{2}abc$ and $2ab^2c^3$.
5. $4ab$, $7bc$, $11cd$, and $8da$.
6. ax, bx, cx, dx^2, ex^3, and fx^4.
7. ayz, bzx, and cxy.
8. a^2yz, $abzx$, and b^2xy.
9. $\frac{1}{7}a^2x^3$, $\frac{1}{9}b^2x$, and $63abxy^2$.
10. $\frac{1}{3}l^2m$, $\frac{1}{4}m^2n$, and $\frac{1}{5}ln^2$.
11. $7ax$ and $-bx$.
12. $-7ax$ and bx.
13. abx, bcx, and $-acx$.
14. l^2mx, $-m^2ny$, and $3x^2y^2$.
15. $-8a_1a_2a_3$, $3a_2a_3a_4$, $2a_3a_4a_5$.
16. $-\frac{1}{2}x^2y^3$, $\frac{1}{3}y^4z$, and bx^3z^4.

53. **Product of a Simple Expression and a Multinomial.** *The product of a multinomial and a number denoted by a letter is found by multiplying every term of the multinomial by the number, and taking the algebraic sum of these products.*

Consider first the case of the product of a number and a binomial. We know from arithmetic that to add to any quantity n times the sum (that is, the algebraic sum) of two numbers, a and b, is equivalent to first adding na and then adding nb. That is,

$$n(a+b)=na+nb,$$

and

$$n(a-b)=na-nb.$$

These results may also be proved directly by a method similar to that given in Art. 52. For

$$\begin{aligned} n(a+b) &= (a+b)+(a+b)+ \ldots\ldots[n \text{ terms}] \\ &= \left.\begin{aligned} &a+a+\ldots\ldots\ldots\ldots\ldots\ldots[n \text{ terms}] \\ &+b+b+\ldots\ldots\ldots\ldots\ldots\ldots[n \text{ terms}] \end{aligned}\right\} \\ &= na+nb. \end{aligned}$$

Similarly,

$$\begin{aligned} n(a-b) &= (a-b)+(a-b)+ \ldots\ldots[n \text{ terms}] \\ &= \left.\begin{aligned} &a+a+\ldots\ldots\ldots\ldots\ldots\ldots[n \text{ terms}] \\ &-b-b-\ldots\ldots\ldots\ldots\ldots\ldots[n \text{ terms}] \end{aligned}\right\} \\ &= na-nb. \end{aligned}$$

Moreover, since the order of multiplication is immaterial, we shall also have

$$(a+b)n = an+bn, \quad \text{and} \quad (a-b)n = an-bn.$$

The same method is applicable to multinomials, and hence the rule given at the head of this article follows.

Ex. 1. *Multiply* $7a-2b+3c$ *by* x.

$$\begin{aligned} x(7a-2b+3c) &= x \cdot 7a - x \cdot 2b + x \cdot 3c \\ &= 7ax-2bx+3cx. \end{aligned}$$

Ex. 2. *Multiply* $7x+3y+11z$ *by* $-3x$.

$$\begin{aligned} (7x+3y+11z)(-3x) &= (7x)(-3x)+(3y)(-3x)+(11z)(-3x) \\ &= -21x^2-9xy-33xz. \end{aligned}$$

Ex. 3. *Multiply* $ax-by-cz$ *by* xyz.

$$\begin{aligned} (ax-by-cz)(xyz) &= (ax)(xyz)-by(xyz)-cz(xyz) \\ &= ax^2yz-bxy^2z-cxyz^2. \end{aligned}$$

EXAMPLES. III. B.

Find the product of the following expressions.

1. $2x+3y+z$ and x.
2. $al+bm+cn$ and $-7an$.
3. $lyz-mxz-nxy$ and lny.
4. Find the coefficient of x in $\frac{1}{2}(x+a)-\{2a-b(c-x)\}$.

Reduce the following expressions (numbered **5** to **10**) to their simplest forms; and state in each case what is the coefficient of x, and whether the form is a binomial, trinomial, or multinomial.

5. $3a-2(b-x)-\{2(a-b)-3(x+a)\}-\{9x-4(x-a)\}$.

6. $12\left(\frac{x+y}{2}-\frac{y-2x}{6}\right)-8\left(\frac{3y-x}{2}-\frac{3x+2y}{4}\right)$.

7. $2[x-a-3\{x-4(x-5\overline{x-a}-a)-a\}]$.

8. $6a-2[b-4(3c-2x)+3\{a-(4c+x)\}]$.

9. $2(3x-y)-4\{2x-(x-y)\}-3\{(x-4y)+2(3x-y)\}$.

10. $4\left(x-\frac{y}{2}\right)-6\left[\left(2x-\frac{y}{3}\right)-12\left(\frac{x}{3}-\frac{3x+2y}{24}\right)\right]$.

11. Express $a^2(x^2+x-1)-ab(1+x^2-x)+b^2(1+x-x^2)$ with numerical coefficients, when $a=4$ and $b=8$; collect like terms, and arrange the result according to powers of x.

12. Simplify $b^2+[a(a-b)-\{ab-b(a+b)\}]$.

13. Simplify $xy-\{y\overline{3a-2x}-a(4y-x)-x(a-2y)\}$. What is the coefficient of y in the result?

54. **Product of Two Binomials.** *The product of two binomials is found by multiplying every term in the one by every term in the other, prefixing the proper sign according to the rule of signs enunciated in Article* 56, *and taking the algebraic sum of these products.*

To prove this rule, we have to find the product of two binomial expressions, one like $a+b$ or $a-b$, and the other like $x+y$ or $x-y$.

(i) To find the value of $(a+b)(x+y)$.

Denote $a+b$ by n,

$\therefore\ (a+b)(x+y)=n(x+y)$

$=nx+ny$ [Art. 53

$=(a+b)x+(a+b)y$

$=ax+bx+ay+by$. [Art. 53

(ii) To find the value of $(a+b)(x-y)$.

Denote $a+b$ by n,

$$\begin{aligned} \therefore\ (a+b)(x-y) &= n(x-y) \\ &= nx - ny & \text{[Art. 53} \\ &= (a+b)x - (a+b)y \\ &= (ax+bx) - (ay+by) & \text{[Art. 53} \\ &= ax + bx - ay - by. & \text{[Art. 39} \end{aligned}$$

(iii) To find the value of $(a-b)(x-y)$.

Denote $a-b$ by n, and let us assume that a is greater than b,

$$\begin{aligned} \therefore\ (a-b)(x-y) &= n(x-y) \\ &= nx - ny & \text{[Art. 53} \\ &= (a-b)x - (a-b)y \\ &= (ax-bx) - (ay-by) & \text{[Art. 53} \\ &= ax - bx - ay + by. & \text{[Art. 39} \end{aligned}$$

All these results are included in the rule enunciated at the head of this article.

55. **Extension of results.** The results of the last article have been proved on the assumption that we are finding the product either of two positive quantities, or of a positive and a negative quantity. We now proceed to find the meaning which must be given to the product of two negative quantities so that those results may be true for all values of the symbols a, b, x, y. To do this, we use a method analogous to that employed in Arts. 40, 41.

If, in the result of (iii), we put $b=0$ and $y=0$, we obtain $a \times x = ax$, which is of course true.

If, in the same result, we put $a=0$ and $y=0$, we obtain

$$(-b)\,x = -bx.$$

Similarly, if we put $b=0$ and $x=0$, we obtain

$$a(-y)=-ay.$$

These give us the rule for forming the product of a positive and a negative quantity, which we had previously obtained in Art. 52.

If, in the same result, we put $a=0$ and $x=0$, we get

$$(-b)(-y)=+by,$$

which gives us the meaning to be attached to the product of two negative quantities.

If we agree that the product of two negative numbers shall be taken as a positive number equal to the product of the numbers, then it will be found on trial that the results (i) (ii), (iii) of Art. 54 are true whatever be the numerical values (positive or negative) that we give to the symbols, and any one of these results may be deduced from either of the other two of them.

For example, if we take the first result, namely,

$$(a+b)(x+y)=ax+bx+ay+by,$$

and put $-c$ for b, and $-z$ for y, we obtain

$$(a-c)(x-z)=ax+(-c)x+a(-z)+(-c)(-z)$$
$$=ax-cx-az+cz,$$

which is equivalent to the result (iii).

Similarly, from any one of the three results of Art. 54 the other two can be deduced.

56. **Rule of Signs.** The results of the last article are known as the **rule of signs**, which may be stated in the following form. *The product of two quantities of the same sign (either both positive or both negative) is a positive quantity; while the product of two quantities of opposite signs (one positive and the other negative) is a negative quantity.*

The rule is sometimes enunciated in the following form. *Like signs produce plus, and unlike signs produce minus.*

57. The application of the rule gives the results

$$(-1)(b) = -b, \text{ and } (-1)(-b) = +b.$$

Also, by Art. 40, we have

$$-(b) = -b, \text{ and } -(-b) = +b.$$

Hence to multiply a quantity by -1 is equivalent to subtracting it.

This result is sometimes assumed; but it is desirable that the student should notice the different meanings which are attached to expressions like $(-1)(b)$ and $-(b)$, though it happens that the results are equal to one another.

58. As particular instances of the rule of signs, we have $(-a)(-a) = +a^2$, and also $(+a)(+a) = +a^2$. Thus the square of any quantity, whether positive or negative, is itself positive.

The special case that $(-1)(-1) = +1$ is worthy of notice.

59. The continued application of the rule of signs enables us to write down the product of several factors of different signs.

Thus

$$(-a)(-b)(-c) = (-a)(+bc) = -abc,$$
$$(-a)^3 = (-a)(-a)(-a) = -a^3,$$
$$(-a)^4 = (-a)(-a)^3 = (-a)(-a^3) = a^4,$$
$$(-a)^5 = (-a)(-a)^4 = (-a)(a^4) = -a^5.$$

The method is general, and the product of a number of factors will be positive or negative according as the number of factors which are negative is even or odd.

60. **Process of multiplication.** It is usual to perform the process of multiplication in the manner shewn in the following example, where the product of $x+3$ and $x-2$ is determined.

$$\begin{array}{l} x+3 \\ x-2 \\ \hline x^2+3x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}, \\ \qquad -2x-6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}, \\ \hline x^2+x-6 \\ \hline\hline \end{array}$$

The multiplier, $x-2$, is written in a line under the multiplicand, $x+3$, and a horizontal line is then drawn. Each term of the multiplicand is then multiplied by the first term (in this case, x) of the multiplier, and the results with their proper signs are written down in a line, which is denoted above by (i).

Next, each term of the multiplicand is multiplied by the next term in the multiplier (in this case, -2), and the results with their proper signs are written in a second line, which is denoted above by (ii); any term in this product which is "like" one of the terms in row (i) is put immediately under it.

The final result is obtained by adding these rows together. The fact that the like terms are in vertical columns facilitates the addition.

This method of multiplying is sometimes called "cross multiplication."

Note. The lines of dots and the numbers (i) and (ii), in the work given above, are inserted only to facilitate the explanation, and they form no part of the process itself.

61. The numerical coefficients of the terms in the expressions which are multiplied together may be fractions. In such a case, we may either multiply the expressions together in a manner similar to that given above, or we may reduce each expression to a fraction with a numerical denominator (which is the arithmetical L.C.M. of the denominators of the fractional coefficients) and in which all the coefficients in the numerator are integers. The beginner will find the latter course the easier.

For example, to find the product of $\frac{1}{2}x-\frac{1}{3}y$ and $\frac{1}{3}x+\frac{1}{4}y$ we may proceed in a manner similar to that given in Art. 60; thus

$$\begin{array}{lll} \frac{1}{2}x - \frac{1}{3}y & & \\ \frac{1}{3}x + \frac{1}{4}y & & \\ \hline \frac{1}{6}x^2 - \frac{1}{9}xy & & \\ \quad\quad + \frac{1}{8}xy & - \frac{1}{12}y^2 & \\ \hline \frac{1}{6}x^2 + \frac{1}{72}xy & - \frac{1}{12}y^2 & \\ \hline\hline \end{array}$$

Or we may put

$$\tfrac{1}{2}x-\tfrac{1}{3}y=\tfrac{1}{6}(3x-2y), \text{ and } \tfrac{1}{3}x+\tfrac{1}{4}y=\tfrac{1}{12}(4x+3y).$$

Hence $(\frac{1}{2}x-\frac{1}{3}y)(\frac{1}{3}x+\frac{1}{4}y)=\frac{1}{6}(3x-2y)\frac{1}{12}(4x+3y)$
$$=\frac{1}{72}(3x-2y)(4x+3y)$$
$$=\frac{1}{72}(12x^2+xy-6y^2),$$
the product of $3x-2y$ and $4x+3y$ being found in the manner explained in Art. 60.

62. The following examples are important, and will serve to illustrate the above remarks.

Ex. 1. *Find the square of the binomial* $x+a$.

$$\begin{array}{l} x+a \\ x+a \\ \hline x^2+ax \\ \quad\quad ax+a^2 \\ \hline x^2+2ax+a^2 \end{array}$$

Ex. 2. *Find the square of* $x-a$.

$$\begin{array}{l} x-a \\ x-a \\ \hline x^2-ax \\ \quad\quad -ax+a^2 \\ \hline x^2-2ax+a^2 \end{array}$$

Ex. 3. *Find the product of* $x+a$ *and* $x-a$.

$$\begin{array}{l} x+a \\ x-a \\ \hline x^2+ax \\ \quad\quad -ax-a^2 \\ \hline x^2 \quad\quad -a^2 \end{array}$$

We see, from this example, that **the product of the sum and the difference of two quantities is equal to the difference of their squares.**

Thus, $98\times102=(100-2)(100+2)=10000-4=9996$.

Ex. 4. *Find the product of* $x+a$ *and* $x+b$.

$$\begin{array}{l} x+a \\ x+b \\ \hline x^2+ax \\ \quad\quad +bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

EXAMPLES. III. C.

1. Write down the product of
   (i) $-ax$ and $-by$; (ii) ax and $-by$; (iii) $-x$ and $-x$.
2. Multiply $x+2$ by $y-3$.
3. Multiply $x-1$ by $x-2$.
4. Find the product of $x-a$ and $x+b$.
5. Multiply $7x-1$ by $x-7$.
6. Multiply $7a-m$ by $m+7a$.
7. Multiply $3x-\frac{1}{3}a$ by $\frac{1}{3}x-3a$.
8. Multiply $ax-b^2$ by $bx-a^2$.
9. Find the product of x^2-a^2 and x^2+a^2.
10. Multiply $ax+by$ by $xa-2yb$.
11. Multiply $2al^2-3bm^2$ by $3bl-2am$.
12. Find the product of $4ax^2-3byz$ and $3by^2-2axz$.

63. **Product of Two Multinomials.** *The product of any two expressions is found by multiplying every term of the one by every term of the other, prefixing the proper sign according to the rule of signs, and taking the algebraic sum of these products.*

To prove this rule, we have only to apply the same method as that given in Art. 54. Consider the product of two multinomials, such as $a+b+c+\ldots$ and $x+y+z+\ldots$, where the $+\ldots$ in these expressions is intended to shew that there may be a number of other terms similar to these written down.

Denote $(a+b+c+\ldots)$ by n,

$\therefore$ the product $(a+b+c+\ldots)(x+y+z+\ldots)$

$=n(x+y+z+\ldots)$

$=nx+ny+nz+\ldots$

$=(a+b+c+\ldots)x+(a+b+c+\ldots)y+(a+b+c+\ldots)z+\ldots$

$=ax+bx+cx+\ldots+ay+by+cy+\ldots+az+bz+cz+\ldots$

If some of the terms be negative, we can still write each expression as an algebraical sum, and then, by the rule of signs, find the corresponding terms in the product.

Thus the product $(a-b)(x-y-z)$

$$=\{a+(-b)\}\{x+(-y)+(-z)\}$$
$$=ax+a(-y)+a(-z)+(-b)x+(-b)(-y)+(-b)(-z)$$
$$=ax-ay-az-bx+by+bz.$$

64. It is usual to perform the process of multiplication in a manner similar to that given in Art. 60.

Thus to multiply x^2+x+1 by x^2-x+1 we proceed as follows

$$\begin{array}{l} x^2+x+1 \\ x^2-x+1 \\ \hline x^4+x^3+x^2 \\ \quad\;\; -x^3-x^2-x \\ \qquad\qquad +x^2+x+1 \\ \hline x^4 \qquad\; +x^2 \qquad +1 \\ \hline\hline \end{array}$$

As another example, let us multiply $2x^2-3x+1$ by $3x-4$.

$$\begin{array}{l} 2x^2-\;\;3x+1 \\ 3x\;-\;\;4 \\ \hline 6x^3-\;\;9x^2+\;\;3x \\ \qquad\;-\;\;8x^2+12x-4 \\ \hline 6x^3-17x^2+15x-4 \\ \hline\hline \end{array}$$

65. If an expression consist of several terms, and these contain different powers of the same letter, it is usual and convenient to arrange the terms so that the term containing the highest power of the letter shall be the one to the extreme left, the term containing the next highest power of the letter shall be next to it, and so on. When so written the expression is said to be arranged in *descending powers of that letter.*

Thus each of the expressions

$$x^2+(a+b)x+c^2, \quad x^3-4a^2x^2-3a^3x+a^4$$

is arranged in descending powers of x. The latter of them is also arranged in ascending powers of a.

The process of multiplication is facilitated if the expressions are arranged in descending powers of the same letter.

EXAMPLES. III. D.

1. Find the product of $3x^2+7x+2$ and $2x^2-4x+3$.
2. Multiply $3x^2-5x-7$ by $7x^2-5x-3$.
3. Multiply x^2-ax+a^2 by $x+a$.
4. Multiply x^2+ax+a^2 by $x-a$.
5. Find the product of x^2+ax+a^2 and x^2-ax+a^2.
6. Multiply $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.

66. A judicious use of brackets will often enable us to write down the product of two expressions by inspection.

Thus the three following examples are included in Art. 62, Ex. 3, where it is proved that $(x-a)(x+a)=x^2-a^2$.

Ex. 1.
$$\begin{aligned}(a^2+b^2)(a^2-b^2)&=(a^2)^2-(b^2)^2\\&=a^4-b^4.\end{aligned}$$

Ex. 2.
$$\begin{aligned}(a-b+c)(a+b-c)&=\{a-(b-c)\}\,\{a+(b-c)\}\\&=a^2-(b-c)^2\\&=a^2-(b^2-2bc+c)\\&=a^2-b^2+2bc-c^2.\end{aligned}$$

Ex. 3.
$$\begin{aligned}\{x^2+x+1\}\,\{x^2-x+1\}&=\{(x^2+1)+x\}\,\{(x^2+1)-x\}\\&=(x^2+1)^2-x^2\\&=(x^4+2x^2+1)-x^2\\&=x^4+x^2+1.\end{aligned}$$

67. Again, if we want to find the square of any multinomial, for example of $a+b+c+d$, we may multiply $a+b+c+d$ by $a+b+c+d$ in the manner indicated in Art. 64, or we may find the product by repeated applications of Art. 62, Ex. 1. Using the latter method, we have

$$\begin{aligned}\{a+b+c+d\}^2&=\{a+(b+c+d)\}^2\\&=a^2+2a(b+c+d)+(b+c+d)^2\\&=a^2+2ab+2ac+2ad+(b+[c+d])^2\\&=a^2+2ab+2ac+2ad+b^2+2b[c+d]+[c+d]^2\\&=a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2.\end{aligned}$$

The required square consists therefore of the following groups of terms, (i) the square of the first term, namely a^2; (ii) twice the product of the first term and the second term, twice the product of the first term and the third term, &c.—every term that *follows* a being multiplied by $2a$ to give one in this group of terms; (iii) the square of the second term, namely b^2; (iv) a group of terms, every term that *follows* b in the original expression being multiplied by $2b$ to give one in this group of terms; and so on.

The method is general, and enables us to write down the square of any multinomial by inspection. For example, the square of $(a^2+b^2-c^2-d^2+e^2)$

$$=(a^2)^2+2a^2(b^2-c^2-d^2+e^2)+(b^2)^2+2b^2(-c^2-d^2+e^2)+(-c^2)^2$$
$$+2(-c^2)(-d^2+e^2)+(-d^2)^2+2(-d^2)(e^2)+(e^2)^2$$
$$=a^4+2a^2b^2-2a^2c^2-2a^2d^2+2a^2e^2+b^4-2b^2c^2-2b^2d^2+2b^2e^2+c^4$$
$$+2c^2d^2-2c^2e^2+d^4-2d^2e^2+e^4.$$

EXAMPLES. III. E.

[*The results of the following examples can be reduced to one of the three following forms:*

$$(x+a)(x-a)=x^2-a^2,$$
$$(x+a)(x^2-ax+a^2)=x^3+a^3,$$
$$(x-a)(x^2+ax+a^2)=x^3-a^3;$$

and the student ought to be able to write down the answers by inspection, in a manner analogous to the examples given in Art. 66.]

Write down the product of the following expressions:

1. x^2+7 and x^2-7.
2. 77 and 83.
3. $x+3$ and x^2-3x+9.
4. $2x+11$ and $2x-11$.
5. $a-b+c$ and $-a-b+c$.
6. y^2-b^2 and $y^4+b^2y^2+b^4$.
7. $y-(a+1)$ and $y^2+(a+1)y+a^2+2a+1$.

68. **Product of a number of expressions.** The product of more than two expressions can be determined by first finding the product of two of them, then multiplying that by another of the factors, and so on in succession.

69. For example, the cube of $x+a$ is denoted by $(x+a)^3$, and its value is obtained by multiplying $(x+a)^2$ by $x+a$. The value of the former of these quantities, namely of $(x+a)^2$, has been determined in Art. 62, Ex. 1, and is $x^2+2ax+a^2$. Hence to find the value of $(x+a)^3$ we have the following work.

$$\begin{array}{l} x^2+2ax+a^2 \\ \underline{x+a} \\ x^3+2ax^2+a^2x \\ \quad\;\; +ax^2+2a^2x+a^3 \\ \hline x^3+3ax^2+3a^2x+a^3 \end{array}$$

Similarly, the cube of $x-a$ will be found by multiplying $(x-a)^2$ by $(x-a)$. The value of $(x-a)^2$ is determined in Art. 62, Ex. 2; and the product of this by $(x-a)$ will be found to be

$$x^3 - 3ax^2 + 3a^2x - a^3.$$

This result can however be found at once from the result of the last example. For, if in it we put $(-a)$ for $+a$ we obtain

$$\{x+(-a)\}^3 = x^3 + 3(-a)x^2 + 3(-a)^2x + (-a)^3,$$

that is, $$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3.$$

These results can be combined in a single formula, thus:

$$(x \pm a)^3 = x^3 \pm 3ax^2 + 3a^2x \pm a^3,$$

where either the upper sign is taken throughout, or the lower sign is taken throughout.

70. Similarly, it can be shewn that

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,$$

and $$(x-a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4.$$

Both results are contained in the formula

$$(x \pm a)^4 = x^4 \pm 4ax^3 + 6a^2x^2 \pm 4a^3x + a^4.$$

*71. We have hitherto confined ourselves to cases where the indices were definite numerical numbers; and we have assumed that results, such as $x \times x^2 = x^3$ and $(x^2)^2 = x^4$, were obvious from the definition of indices given in Art. 22. The indices may however be themselves denoted by algebraical symbols, which may stand for any integral number: thus x^m will mean the product of m factors, each equal to x.

We now proceed to prove two rules of which the above instances are special cases. We shall for the present assume that the indices are positive integers.

*72. **Index Law I.** *To shew that* $x^m \times x^n = x^{m+n}$, *where* x *is any quantity, and* m *and* n *are positive integers.*

We have, by the definition of indices [Art. 22],

$$\begin{aligned} x^m \times x^n &= (xxx \ldots m \text{ factors})(xxx \ldots n \text{ factors}) \\ &= xxx \ldots (m+n) \text{ factors} \\ &= x^{m+n}. \end{aligned}$$

Thus the index of the product of two factors is the sum of their indices, provided the factors are powers of the same quantity.

For example, $x^{13} \times x^{25} = x^{38}$.

*73. A similar rule holds if we have the product of several factors, provided each factor is some power of the same quantity. Thus

$$x^m \times x^n \times x^p = x^{m+n} \times x^p$$
$$= x^{m+n+p}.$$

*74. It follows that the product of two homogeneous expressions is itself homogeneous. For, if the sum of the indices of every term in the first expression be m, and the sum of the indices of every term in the second expression be n, then the product of any term of the one expression and any term of the other will be of $m+n$ dimensions.

Should the student make a mistake in one term of such a product, this test may enable him to detect his blunder.

For example, if it were asserted that the result of multiplying $x^3+2ax^2+3a^2x-a^3$ by $x^2-2ax+5a^2$ contained a term $3a^3x$, we know there must be a mistake, since every term in the product must be of $(3+2)$ dimensions, and $3a^3x$ is of only 4 dimensions [Art. 26]. As a matter of fact $3a^3x^2$ is one term in the product.

It follows similarly that the product of a number of homogeneous expressions is itself homogeneous, and therefore that any power of a homogeneous expression is itself homogeneous.

*75. **Theorem.** *To shew that* $(ab)^n = a^n b^n$.

It is obvious from the definition that $(ab)^2 = a^2b^2$. This is a particular instance of the general theorem that $(ab)^n = a^n b^n$.

We have, by the definition of indices [Art. 22],

$$(ab)^n = ab \times ab \times ab \times \dots n \text{ factors}$$
$$= (aaa \dots n \text{ factors}) \times (bbb \dots n \text{ factors})$$
$$= a^n \times b^n$$
$$= a^n b^n.$$

Similarly, $\{abc\}^n = \{a\,(bc)\}^n = a^n\,(bc)^n = a^n b^n c^n$.

And similarly, the n^{th} power of a product is the product of the n^{th} powers of its factors.

*76. **Index Law II.** *To shew that* $(x^m)^n = x^{mn}$, *where* x *is any quantity, and* m *and* n *are positive integers.*

We have, by the definition of indices,

$$(x^m)^n = x^m \times x^m \times x^m \times \dots n \text{ factors}$$
$$= x^{m+m+m+\dots n \text{ terms}} \qquad \text{[Art. 72.}$$
$$= x^{mn}.$$

For example, $(x^3)^4 = x^{12}$.

*EXAMPLES ON THE INDEX LAWS. III. F.

Write down the product of the following expressions numbered 1 to 9.

*1. x^{17} and x^{33}. *2. $7a^{11}x^{51}$ and $-3a^{25}x^{49}$.

*3. $-4a^{51}y^{19}$ and $-3a^{101}b^{206}y^{43}$. *4. x^n+1 and x^n-1.

*5. x^n+a and x^n+b. *6. x^n+x-1 and x^n-x-1.

*7. $2x^m+3x^n+1$ and x^m-2x^n+3.

*8. a^n-b^n, b^n-c^n, and a^n-c^n.

*9. $(ab)^n-(x^2)^n$, a^n-x^n, and b^n-x^n.

*10. Write down the continued product of x to the power of a million, of x to the power of a thousand, and of x.

MISCELLANEOUS EXAMPLES. III. G.

[*The beginner should take care that all the expressions in each example are arranged in descending powers of some letter before he begins to multiply them together.*]

1. If $x=0$, and $y=-5$, find the value of
$$7(x-y)(2x-y)-3(2x-y)(x+y)+5(x+y)^2.$$

2. Find the value of
$(a-b)^2+(b-c)^2+(a-b)(b-c)+5c^2$, when $a=1$, $b=-2$, $c=\frac{1}{2}$.

3. Subtract $(2a-3b)(c-d)$ from $(2a+3b)(c+d)$.

4. Simplify $(x-a)^2+(x-b)^2+2\{(x-a)(b-x)+ab\}$.

5. Prove that
$$(a-b)(a+b-c)+(b-c)(b+c-a)=(a-c)(a+c-b).$$

6. Simplify

(i) $(a-3b)^2+6(a-b)(b-c)+(3b-c)^2+6(ac-2b^2)$;

(ii) $(a+b+c)^2-(a-b+c)^2+(a+b-c)^2-(-a+b+c)^2$.

7. Reduce to its simplest form
$(a+b+c-d)(a+b-c+d)+(a-b+c+d)(-a+b+c+d)$.

8. From $3(x^2-3x+2)\{3(x+5)-5(5+x)\}$
subtract $2(x+5)\{5(x^2-2x)-2(x^2-2x-6)\}$.

9. Multiply $x^2-5xy+6y^2$ by $x-4y$.

10. Multiply $a^5+3a^4x+9a^3x^2+27a^2x^3+81ax^4+243x^5$ by $a-3x$.

11. Find the product of $1-2x+4x^2$ and $1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}$.

12. Multiply $x^3-ax^2-2a^2x+a^3$ by x^2+ax-a^2.

13. Multiply $x^3-2ax^2+2a^2x-3a^3$ by $x^2-3ax+2a^2$.

14. Multiply $x^3+6x^2y+12xy^2+8y^3$ by $x^3-3x^2y+3xy^2-y^3$.

15. From $(2a-b)^2+(a-2b)^2$ subtract the square of $2(a-b)$.

16. Multiply $x^2+y^2+z^2-xy+xz+yz$ by $x+y-z$.

17. Multiply together $3a+b+2c$, $2a+2b+c$, and $a-2b-3c$.

18. Multiply together $3a-b+2c$, $2a-3b+c$, and $a+2b-3c$.

19. Find the continued product of
$$x+y+z,\ x+y-z,\ x-y+z, \text{ and } -x+y+z.$$

20. Multiply $\frac{1}{6}a^2+\frac{1}{3}ab-\frac{1}{2}b^2$ by $\frac{1}{10}a^2-\frac{1}{2}ab+\frac{2}{3}b^2$.

21. Find the continued product of $2x-1$, $2x^2+\frac{1}{2}$, and $\frac{1}{2}x+\frac{1}{4}$.

22. Find the continued product of

$$\tfrac{1}{2}(x^2+3x+2),\ \tfrac{1}{3}(x^2-5x+6),\ \text{and}\ \tfrac{1}{4}(x^2+2x-3).$$

23. Prove that $(x+3)^3-(x+2)^3=3(x+2)(x+3)+1$.

24. Simplify

$$(a-b)(x+a)(x+b)+(b-c)(x+b)(x+c)+(c-a)(x+c)(x+a).$$

25. Prove that $(a-b)^2+(c-a)(c-b)=(b-c)^2+(a-b)(a-c)$.

26. Shew that $y(y-1)(y-2)(y-3)+1=(y^2-3y+1)^2$.

27. In the expression x^3-2x^2+3x-4, substitute $a-2$ for x, and arrange the result according to descending powers of a.

28. Simplify

(i) $(a+b+c)^2-(b+c)^2-(c+a)^2-(a+b)^2+a^2+b^2+c^2$;

(ii) $(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3+a^3+b^3+c^3$.

29. If $x=a^2-bc$, $y=b^2-ca$, and $z=c^2-ab$; prove that

(i) $ax+by+cz=(x+y+z)(a+b+c)$;

(ii) $bc(x^2-yz)=ca(y^2-zx)=ab(z^2-xy)$.

***30.** Simplify the expression $ax^2+by^2+cz^2+xyz$, where

$$x=b+c-a,\ y=c+a-b,\ \text{and}\ z=a+b-c.$$

***31.** Prove the following identities, by multiplying out each side of the equality.

(i) $(1+a)^2(1+b^2)-(1+a^2)(1+b)^2=2(a-b)(1-ab)$;

(ii) $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$;

(iii) $(y+z-2x)^3+(z+x-2y)^3+(x+y-2z)^3$
$=3(y+z-2x)(z+x-2y)(x+y-2z)$;

(iv) $(x-y)(x-2y)(x-3y)+9y(x-y)(x-2y)+18y^2(x-y)+6y^3$
$=x(x+y)(x+2y)$;

(v) $(x+y+z)^3=(x+y-z)^3+(x-y+z)^3+(-x+y+z)^3+24xyz$;

(vi) $(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3$
$=2yz(y^2-z^2)+2zx(z^2-x^2)+2xy(x^2-y^2)$.

***32.** Find the coefficient of x^5 in the product of $(1-x)^2$ and $1+2x+3x^2+4x^3+5x^4$.

CHAPTER IV.

DIVISION.

77. WE proceed next to the consideration of the division of one algebraical expression by another algebraical expression; and shall discuss successively division by simple expressions and division by multinomials.

We may add that we can always test the correctness of the result of a division by seeing whether the product of the quotient and the divisor when added to the remainder is equal to the dividend. For this reason, division is sometimes said to be the inverse of multiplication.

78. **Division by Simple Expressions.** The quotient of an expression (the dividend) when divided by a simple quantity (or by a product of simple quantities) is indicated by writing the divisor after the dividend and preceded either by the symbol ÷ or by a solidus / [see Art. 16]. The operation may also be represented by a fraction having the dividend for numerator and the divisor for denominator.

For example, we may represent the operation of dividing abc by x either by the form $abc \div x$, or by abc/x, or by $\frac{abc}{x}$.

In using either of the first two of these forms it must be noticed that all the quantities that come immediately after the symbol ÷ or / are to be regarded as a divisor. Thus either ab/xc or $ab \div xc$ would mean that ab was to be divided by xc, and not that ab was to be divided by x and then the result multiplied by c: the latter could be represented either by $(ab \div x)c$, or by $(ab/x)c$, or by $\frac{ab}{x} \times c$.

Similarly we can express the operation of dividing x^2+y^2 by a by any of the following forms, $(x^2+y^2) \div a$, or $(x^2+y^2)/a$, or $\frac{x^2+y^2}{a}$. But an expression like $x^2+y^2 \div a$ would signify that y^2 only was to be divided by a, and the quotient added to x^2; while $x^2 \div a + y^2$ would signify that x^2 was to be divided by a, and that y^2 was to be added to the quotient: we shall however avoid the use of forms such as these which a careless reader might regard as ambiguous.

Where the same quantity appears as a factor of both divisor and dividend it may be cancelled, since to multiply any expression by a certain quantity and then to divide it by the same quantity cannot alter it.

Ex. 1. *Divide* $21ax^2$ *by* $3x$.

Here
$$\frac{21ax^2}{3x} = \frac{21a\cancel{x}x}{3\cancel{x}}$$
$$= 7ax.$$

Ex. 2. *Divide* $6a^4b^3c^2$ *by* $5ab^2c^3$.

Here
$$\frac{6a^4b^3c^2}{5ab^2c^3} = \frac{6\cancel{a}a^3\cancel{b^2}b\cancel{c^2}}{5\cancel{a}\cancel{b^2}\cancel{c^2}c}$$
$$= \frac{6a^3b}{5c}.$$

Ex. 3. *Divide* $9a^2b(c^2+d^2)$ *by* $3ax^2$.

Here
$$\frac{9a^2b(c^2+d^2)}{3ax^2} = 3\frac{ab(c^2+d^2)}{x^2}.$$

Note. Examples such as those given above can generally be solved by inspection, and it is only to explain the reason of the method used that any steps are inserted in the work here printed.

79. **Rule of Signs.** One or more of the factors may be negative. We may, by Arts. 55, 56, regard a negative factor as the product of -1 and an equal positive factor. The following results, where A and B

stand for any algebraical expressions, follow at once:

(i) $$\frac{-A}{B}=\frac{(-1)A}{B}=(-1)\frac{A}{B}=-\frac{A}{B};$$

(ii) $$\frac{A}{-B}=\frac{(-1)(-1)A}{(-1)B}=(-1)\frac{A}{B}=-\frac{A}{B};$$

(iii) $$\frac{-A}{-B}=\frac{(-1)A}{(-1)B}=\frac{A}{B}.$$

Thus the **rule of signs in division** is similar to that in multiplication, as given in Art. 56, and may be stated in the following form. *The quotient of any quantity by another quantity of the same sign (that is either when both are positive or both are negative) is positive, but its quotient by a quantity of the opposite sign is negative.*

This is sometimes enunciated in the form, *Like signs produce plus, and unlike signs produce minus.*

Ex. 1. *Divide* $11a^2bc$ *by* $-2ac^2$.

Here $$\frac{11a^2bc}{-2ac^2}=-\frac{11}{2}\frac{ab}{c}.$$

Ex. 2. *Divide* $-3xy$ *by* $-2ab$.

Here $$\frac{-3xy}{-2ab}=\frac{3}{2}\frac{xy}{ab}.$$

EXAMPLES. IV. A.

1. Divide x^7 by x^2.
2. Divide $18ax^5$ by $3x$.
3. Divide $64ab^2c^3$ by $4bc^2$.
4. Divide $51lm^2z$ by $17lz$.
5. Divide $98a_1{}^7a_2{}^5a_3{}^3$ by $7a_1{}^3a_2{}^5a_3{}^7$.
6. Divide $56a^2x^3$ by $-7ax^2$.
7. Divide $-52l^3x$ by $13l^2x$.
8. Divide $-81x^7$ by $-9x^4$.
9. Divide $51ab^2cx^2$ by $3aby$.
10. Divide $-4p^3x^3$ by $3pqx^2$.

80. **Quotient of a Multinomial by a Simple Expression.** *The quotient of a multinomial quantity by a simple quantity is the algebraic sum of the quotients obtained by dividing each term of the multinomial by the simple quantity.*

We know from arithmetic that the quotient obtained by dividing the sum of two or more numbers by a divisor is the sum of the quotients obtained by dividing the separate numbers by that divisor. The same proposition will therefore be true of numbers represented by algebraical symbols: the principle explained in Art. 41 allowing us to extend its application to negative quantities. This is equivalent to the rule printed at the head of this article.

Thus, if $a+b$ be divided by x, the result is the sum of a/x and b/x, which we may express in any of the forms

$$(a+b)\div x=(a\div x)+(b\div x);$$

or

$$\frac{a+b}{x}=\frac{a}{x}+\frac{b}{x};$$

or

$$(a+b)/x=a/x+b/x.$$

We have a similar result if there are more than two terms. Thus

$$\frac{a+b-c}{x}=\frac{a}{x}+\frac{b}{x}-\frac{c}{x},$$

which may also be written $(a+b-c)/x=a/x+b/x-c/x$.

Ex. 1. *Divide* $7ab^2+3a^2b$ *by* ab.

Here
$$\frac{7ab^2+3a^2b}{ab}=\frac{7ab^2}{ab}+\frac{3a^2b}{ab}$$
$$=7b+3a.$$

We might also have proceeded thus:

$$\frac{7ab^2+3a^2b}{ab}=\frac{ab(7b+3a)}{ab}=7b+3a.$$

Ex. 2. *Divide* $4lm^2n^3-3l^5m$ *by* $-2lm^2$.

Here
$$\frac{4lm^2n^3-3l^5m}{-2lm^2}=-\left(\frac{4lm^2n^3}{2lm^2}-\frac{3l^5m}{2lm^2}\right)$$
$$=-\left(2n^3-\frac{3}{2}\frac{l^4}{m}\right)$$
$$=-2n^3+\frac{3}{2}\frac{l^4}{m}.$$

EXAMPLES. IV. B.

1. Divide $7a^2b+14ab^2$ by $7ab$.
2. Divide $2a^2x-3ab^2$ by $4a$.
3. Divide $12l^3y-15lmy^2$ by $-3ly$.
4. Divide $\frac{3}{2}ax^4+\frac{2}{3}bx^3-\frac{7}{4}cx^2+19x$ by $-\frac{1}{2}x$.
5. Divide $-\frac{7}{3}x^3-\frac{6}{5}x^2-\frac{3}{2}x$ by $-\frac{14}{15}x$.

81. **Quotient of a Multinomial by a Multinomial.** We proceed now to consider the division of one compound expression by another compound expression.

We want to obtain an expression such that the product of it and the divisor shall be equal to the dividend, and our method of finding the expression is equivalent to gradually building up an expression which when multiplied by the divisor will give the dividend. To do this, we arrange both dividend and divisor in descending powers of some common letter, which let us suppose is x. Then, if we divide the first term in the dividend by the first term in the divisor we shall get one term of the quotient. Multiply the whole of the divisor by this, and subtract the result from the quotient; we shall then get a remainder which is at least one degree lower in x than the original dividend. This remainder must then be treated as the dividend, and we shall get another term in the required quotient and a new expression to divide which is of still lower dimensions in x. Finally, we shall get a number of terms in the quotient, and either no remainder, or a remainder which is of lower dimensions than the original divisor.

82. Thus to divide x^4+x^2+1 by $1+x+x^2$ we write the expressions in descending powers of x as shewn herewith.

Divisor.	*Dividend.*	*Quotient.*
x^2+x+1)	x^4+x^2+1 (	

The first term of the dividend, viz. x^4, when divided by the first term of the divisor, viz. x^2, gives x^2; hence the first term in the quotient is x^2. Write x^2 in the space reserved for the quotient; then multiply the divisor by it, writing the result below the dividend, and subtract. It is often convenient to place all like terms (if any) in the same vertical column; thus

$$\begin{array}{l}
x^2+x+1\;)\;x^4 \qquad +x^2+1\;(\;x^2 \\
\qquad\qquad\quad \underline{x^4+x^3+x^2\qquad} \\
\qquad\qquad\qquad -x^3 \qquad +1
\end{array}$$

Next, we have to divide this remainder by x^2+x+1. The highest powers of x in the remainder and the divisor are respectively $-x^3$ and x^2, and the result of dividing $-x^3$ by x^2 is $-x$. This therefore is the next term of the quotient. We write this as another term of the quotient, and then multiply the divisor by it, writing the result below the quantity from which we have now to subtract it. The operation so far will therefore be represented as follows.

$$\begin{array}{l}
x^2+x+1\;)\;x^4 \qquad +x^2 \qquad +1\;(\;x^2-x \\
\qquad\qquad\quad \underline{x^4+x^3+x^2\qquad\qquad} \\
\qquad\qquad\qquad -x^3 \qquad\qquad +1 \\
\qquad\qquad\qquad \underline{-x^3-x^2-x\qquad} \\
\qquad\qquad\qquad\qquad\quad x^2+x+1
\end{array}$$

Next, we have to divide this remainder by x^2+x+1. The first term of this new remainder, x^2, when divided by the first term of the divisor, viz. x^2, gives 1; hence the next term in the quotient is 1. Proceeding as above, we get

$$\begin{array}{l}
x^2+x+1\;)\;x^4 \qquad +x^2 \qquad +1\;(\;x^2-x+1 \\
\qquad\qquad\quad \underline{x^4+x^3+x^2\qquad\qquad} \\
\qquad\qquad\qquad -x^3 \qquad\qquad +1 \\
\qquad\qquad\qquad \underline{-x^3-x^2-x\qquad} \\
\qquad\qquad\qquad\qquad\quad x^2+x+1 \\
\qquad\qquad\qquad\qquad\quad \underline{x^2+x+1}
\end{array}$$

Therefore the quotient is x^2-x+1, and there is no remainder. Hence (x^4+x^2+1) is *exactly divisible* by (x^2+x+1) [Art. 16].

The student should verify this result by forming the product of x^2+x+1 and x^2-x+1; and he will find [see Art. 66, Ex. 3] that

$$(x^2+x+1)(x^2-x+1)=x^4+x^2+1.$$

83. The above process is really equivalent to writing the dividend in the successive forms

$$x^4+x^2+1,$$
$$x^2(x^2+x+1)-x^3+1,$$
$$x^2(x^2+x+1)-x(x^2+x+1)+(x^2+x+1),$$

which are only different ways of writing the same expression. The last form shews that the dividend is exactly divisible by x^2+x+1, and that x^2-x+1 is the quotient of the division.

84. *Note.* The successive terms of the quotient are sometimes written by the side of the steps from which they arise, thus:

$$\begin{array}{r}
x^2+x+1\ )\ x^4 \qquad +x^2 \qquad +1\ (\ x^2 \\
x^4+x^3+x^2 \qquad\qquad\quad \\
\hline
-x^3 \qquad\quad +1\ (\ -x \\
-x^3-x^2-x \qquad\quad \\
\hline
x^2+x+1\ (\ +1 \\
x^2+x+1 \qquad \\
\hline
\end{array}$$

85. The following are additional examples.

Ex. 1. *Divide* x^2-a^2 *by* $x-a$.

$$\begin{array}{r}
x-a\ )\ x^2 \qquad -a^2\ (\ x+a \\
x^2-ax \qquad\qquad \\
\hline
ax-a^2 \qquad \\
ax-a^2 \qquad \\
\hline
\end{array}$$

Hence the quotient is $x+a$, and there is no remainder.

The result of this division ought to have been obvious, since we have already shewn that $(x+a)(x-a)=x^2-a^2$; and hence the quotient of x^2-a^2 by $x-a$ must be $x+a$. It should be noticed that, if we can resolve the dividend into factors, we can write down by inspection the result of dividing it by any one of those factors.

Ex. 2. *Divide* $2x^2-3ax+a^2$ *by* $x+2a$.

We have

$$\begin{array}{r}
x+2a\ )\ 2x^2-3ax+a^2\ (\ 2x-7a \\
2x^2+4ax \qquad\qquad\qquad \\
\hline
-7ax+\ \ a^2 \qquad\qquad \\
-7ax-14a^2 \qquad\qquad \\
\hline
15a^2 \qquad\qquad \\
\end{array}$$

The quotient therefore is $2x-7a$, and there is a remainder $15a^2$.

This result is equivalent to the formula

$$2x^2-3ax+a^2=(2x-7a)(x+2a)+15a^2,$$

which the student can verify by multiplication.

Note. Here we wanted the remainder to be independent of x. If however we had arranged the expressions in descending powers of a, we should have had a quotient $\frac{1}{2}a-\frac{7}{4}x$, and a remainder $\frac{15}{4}x^2$ which is independent of a. This is shewn by the accompanying work.

$$\begin{array}{r}
2a+x\ )\ a^2-3ax+2x^2\ (\ \tfrac{1}{2}a-\tfrac{7}{4}x \\
a^2+\tfrac{1}{2}ax \qquad\qquad\qquad\quad \\
\hline
-\tfrac{7}{2}ax+\ 2x^2 \qquad\qquad\quad \\
-\tfrac{7}{2}ax-\ \tfrac{7}{4}x^2 \qquad\qquad\quad \\
\hline
\tfrac{15}{4}x^2 \qquad\qquad\quad \\
\end{array}$$

This result is equivalent to the formula

$$a^2 - 3ax + 2x^2 = (\tfrac{1}{2}a - \tfrac{7}{4}x)(2a + x) + \tfrac{15}{4}x^2.$$

Ex. 3. *Divide* $\frac{1}{4}x^2 + \frac{11}{12}x - \frac{1}{3}$ *by* $\frac{3}{4}x - \frac{1}{4}$.

We shall treat this by a process analogous to the second method given at the end of Art. 61. We have

$$\tfrac{1}{4}x^2 + \tfrac{11}{12}x - \tfrac{1}{3} = \tfrac{1}{12}(3x^2 + 11x - 4), \text{ and } \tfrac{3}{4}x - \tfrac{1}{4} = \tfrac{1}{4}(3x - 1).$$

Hence $$\frac{\frac{1}{4}x^2 + \frac{11}{12}x - \frac{1}{3}}{\frac{3}{4}x - \frac{1}{4}} = \frac{\frac{1}{12}(3x^2 + 11x - 4)}{\frac{1}{4}(3x - 1)} = \frac{1}{3}\,\frac{3x^2 + 11x - 4}{3x - 1}.$$

Now dividing $3x^2 + 11x - 4$ by $3x - 1$ we have a quotient $x + 4$, as shewn herewith.

$$\begin{array}{r}
3x - 1\ )\ 3x^2 + 11x - 4\ (\ x + 4 \\
3x^2 - x \qquad\quad \\
\hline
12x - 4 \qquad\quad \\
12x - 4 \qquad\quad \\
\hline
\end{array}$$

Therefore the required quotient is $\frac{1}{3}(x + 4)$.

Ex. 4. *Divide* $a^3 + b^3 + c^3 - 3abc$ *by* $a + b + c$.

Here it will be convenient not only to arrange each expression in descending powers of a, but also to arrange the coefficients of every power of a in powers of b: in short, to arrange primarily in powers of a, next, as far as is consistent with this, in powers of b, and so on.

To save space, we shall not concern ourselves with arranging the expressions so that only like terms are in the same vertical column.

$$\begin{array}{l}
a + b + c\ )\ a^3 - 3abc + b^3 + c^3\ (\ a^2 - ab - ac + b^2 - bc + c^2 \\
\qquad\qquad\ \ a^3 + a^2b + a^2c \\
\hline
\qquad\qquad\quad -a^2b - a^2c - 3abc + b^3 + c^3 \\
\qquad\qquad\quad -a^2b - ab^2 - abc \\
\hline
\qquad\qquad\qquad\qquad -a^2c + ab^2 - 2abc + b^3 + c^3 \\
\qquad\qquad\qquad\qquad -a^2c \qquad\ - abc - ac^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\ \ ab^2 - abc + ac^2 + b^3 + c^3 \\
\qquad\qquad\qquad\qquad\qquad\ \ ab^2 \qquad\qquad\quad + b^3 + b^2c \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\ - abc + ac^2 - b^2c + c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\ - abc \qquad\ - b^2c - bc^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ac^2 + bc^2 + c^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ac^2 + bc^2 + c^3 \\
\hline
\end{array}$$

Hence the quotient is $a^2 - ab - ac + b^2 - bc + c^2$, and there is no remainder.

We might also have collected the coefficient of each power of a in a bracket, thus:

$$\begin{array}{l|l}
a+(b+c) & a^3 \qquad\qquad -3abc+b^3+c^3 \;\big(a^2-a(b+c)+(b^2-bc+c^2) \\
 & \underline{a^3+a^2(b+c)} \\
 & \quad -a^2(b+c)-3abc+b^3+c^3 \\
 & \quad \underline{-a^2(b+c)-a(b+c)^2} \\
 & \qquad\qquad a(b^2-bc+c^2)+b^3+c^3 \\
 & \qquad\qquad \underline{a(b^2-bc+c^2)+b^3+c^3}
\end{array}$$

Here we write down the difference of $a(b+c)^2$ and $3abc$ in its simplest form, either performing the work mentally, or doing it as a subsidiary example. Similarly we write down the product of $(b+c)$ and (b^2-bc+c^2) as b^3+c^3, without inserting the details of the multiplication.

**Ex.* 5. *By what expression must x^6+x^4-2 be divided to give x^4+2x^2+2 as quotient, with no remainder?*

We always have the relation

$$\text{dividend}=(\text{divisor}).(\text{quotient})+\text{remainder}.$$

Here there is no remainder.

$$\therefore\; x^6+x^4-2=(\text{divisor}).(x^4+2x^2+2).$$

If we divide both sides of this equality by x^4+2x^2+2, we see that the required divisor is equal to the quotient of x^6+x^4-2 by x^4+2x^2+2. This will be found to be x^2-1, which is therefore the required expression.

EXAMPLES. IV. C.

1. Divide $x^2-17x+16$ by $x-1$.
2. Divide $x^3+24x^2-17x-8$ by $x-1$.
3. Divide $4x^4-5x^3+6x-\frac{99}{8}$ by $2x-3$.
4. Divide x^3+a^3 by $x+a$.
5. Divide x^3-a^3 by $x-a$.
6. Divide $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ by $x+y+z$.
7. Divide $x^7-13x-30$ by x^2-2x+3.
8. Divide $2x^3-\frac{3}{2}x^2y-10y^3$ by $\frac{1}{2}x-y$.
9. Divide $2x^6-3x^4+1$ by x^2+2x+1.
10. Divide $x^4+3ax^3+a^3x+3a^4$ by $x^2+4ax+3a^2$.

11. Divide $a^2x(x+y)+3a(x^2-2xy-y^2)+9y(y-x)$ by $ax-3y$.

12. Divide by $(a-c)$ the sum of the four quantities

$$3a^3-a^2(7b+2c)+3ab^2,\ b^3+ab(4a-5b)-7bc^2,$$
$$2a^2c-b^3+a^2(3b-2a), \text{ and } 6a^2b+bc^2+2ab^2-a^3.$$

13. Multiply $\frac{1}{16}x^4-\frac{1}{2}x^3+\frac{3}{2}x^2-2x+1$ by $\frac{1}{4}x^2-x+1$, and divide the product by $\frac{1}{8}x^3-\frac{3}{4}x^2+\frac{3}{2}x-1$.

14. Divide $abx^3+(ac-bd)x^2-(af+cd)x+df$ by $ax-d$.

*86. **Index Law 1.** We proceed now to consider the case of the quotient of a quantity raised to a literal power divided by the same quantity raised to some other literal power. The method of treating this question is analogous to that given in Art. 72, and we shall for the present assume that the indices are positive integers.

We want to find the value of $x^m \div x^n$ when m and n are positive integers, and we shall consider (i) the case when $m > n$, and (ii) the case when $m < n$.

First. If $m > n$, then $\dfrac{x^m}{x^n} = x^{m-n}$.

For
$$\frac{x^m}{x^n} = \frac{x\ x\ x\ldots\ldots(m \text{ factors})}{x\ x\ x\ldots\ldots(n \text{ factors})}.$$

Now, if $m > n$, then each factor in the denominator will cancel with a factor in the numerator, and therefore all the n factors in the denominator will cancel with n of the factors in the numerator. This will leave $(m-n)$ factors in the numerator, each equal to x, and the product of these is denoted by x^{m-n}. Thus we have

$$\frac{x^m}{x^n} = x^{m-n}.$$

Second. If $n > m$, then $\dfrac{x^m}{x^n} = \dfrac{1}{x^{n-m}}$.

For
$$\frac{x^m}{x^n} = \frac{x\ x\ x\ldots\ldots(m \text{ factors})}{x\ x\ x\ldots\ldots(n \text{ factors})}.$$

Now, if $n > m$, then each factor in the numerator will cancel with a factor in the denominator, and therefore all the m factors in the numerator will cancel with m of the factors in the denominator. This will leave $(n-m)$ factors in the denominator, each equal to x, and the product of these is denoted by x^{n-m}. Thus we have

$$\frac{x^m}{x^n} = \frac{1}{x^{n-m}}.$$

*EXAMPLES ON THE INDEX LAWS. IV. D.

*1. Write down the quotient of $121a^{21}x^7$ by $11a^{11}x^4$.

*2. Write down the quotient of $17a^{15}x^{22}$ by $-3a^{14}x^{24}$.

*3. Write down the quotient of $27a^{2n}$ by $3a^n$.

*4. Divide $x^{2n}-1$ by x^n+1.

*5. Divide $x^{2n}+x^{n+1}+x-1$ by x^n+1.

MISCELLANEOUS EXAMPLES ON DIVISION. IV. E.

1. Divide $1-5x^4+4x^5$ by $1-2x+x^2$.

2. Divide $x^3+3xy+y^3-1$ by $x+y-1$.

3. Divide $a^3+3a^2b+3ab^2+b^3+c^3$ by $a+b+c$.

4. Divide $3x^5+2x^4+13x-10$ by x^3+2x^2+x-2.

5. What is the remainder when $x^4-2x^3+2x^2-x-1$ is divided by x^2+3x-1?

6. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4-c^4$ by $a+b+c$.

7. Divide x^6+27y^6 by $x^2+3xy+3y^2$.

8. Divide $2x^4+5x^3y-16x^2y^2+35xy^3-12y^4$ by $x^2-2xy+3y^2$.

9. Divide $2x^4-10x^3y+25x^2y^2-31xy^3+20y^4$ by $x^2-3xy+4y^2$.

10. Divide $12b^4-3a^4-4a^2b^2-8ab(a^2+b^2)$ by $3a^2+6b^2+2ab$.

11. Divide $x^7-2y^{14}-7x^5y^4-7xy^{12}+14x^3y^8$ by $x-2y^2$.

12. Divide $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$ by $ax+by-1$.

13. Divide $3x^5-10x^4y+16x^3y^2-12x^2y^3+xy^4+2y^5$ by $(x-y)^2$.

14. Divide $x^4-(b-2)x^3-(2b-1)x^2-(b^2+2b-8)x+3b+3$ by $x^2+3x+b+1$.

15. Divide $1+3x$ by $1-2x$ to 5 terms in the quotient.

16. Divide $14a^4+15a^3b+33a^2b^2+36ab^3+28b^4$ by $7a^2-3ab+14b^2$.

17. Divide $9x^4-12x^3y+13x^2y^2-4xy^3+y^4$ by $9x^2-3xy+y^2$.

18. What is the remainder when $a^4-3a^3b+2a^2b^2-b^4$ is divided by $a^2-ab+2b^2$?

19. Divide $y^4+y^3-5y^2-2y$ by y^2+3y+1.

20. Divide $y^8-2b^4y^4+b^8$ by $y^3+by^2+b^2y+b^3$.

21. Divide $x^4+(a-1)x^3-(2a+1)x^2+(a^2+4a-5)x+3a+6$ by $x^2-3x+(a+2)$.

22. By what expression must $x+3$ be multiplied to give x^7+2187?

23. By what expression must $3a^4-8a^3b+4a^2b^2-8ab^3-12b^4$ be divided to give the quotient $3a^2-2ab+6b^2$?

*24. By what expression must x^3+6x^2-4x-1 be divided to give x^2+5x-9 as quotient, with 8 as remainder?

25. The quotient and divisor in a certain example are $a-b-c$ and $a^2+b^2+c^2+ab+ac-bc$. What was the dividend?

26. The product of two factors is $(x+2y)^3+(3x+z)^3$, and one of them is $4x+2y+z$: find the other.

27. The quantity $x^5+x^4+2x^3+3x^2-x+4$ is the product of two algebraical expressions. One of them is x^2-x+1. Find the other.

28. If the product of two expressions be $x^8+x^4y^4+y^8$, and one of them be x^2-xy+y^2; find the other.

*29. Prove that if $n^2=n+1$, then $x^4+ax^3+a^2x^2+a^3x+a^4$ is exactly divisible by $x^2+nax+a^2$.

*30. Find the quotient by $a+b+c+d$ of
$(a+b)(b+c)(c+d)(d+a)(a+c)(b+d)+(abc+bcd+cda+dab)^2$.

*31. If x^2+7x+c be exactly divisible by $x+4$, what is the value of c?

CHAPTER V.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

87. **Identity.** When two algebraical expressions are equal for all values of the letters involved, the statement of their equality is called an *identity* or a *formula* or an *identical equation* [see Art. 29].

In an identity, the expressions which are separated by the symbol of equality are merely different ways of expressing the same quantity.

Thus $(a+b)^2=a^2+2ab+b^2$ is an identity, for it is true for all values of a and b.

88. Some writers use the symbol $\equiv$ to represent the words *is identically equal to*, and would write the above identity as $(a+b)^2\equiv a^2+2ab+b^2$. The symbol $\equiv$ has however another and different meaning, which is in common use in the higher parts of mathematics; and as it is important that every mathematical symbol should be unambiguous and definite it is undesirable to use $\equiv$ as a symbol of identity.

89. **Equation.** When two algebraical expressions are equal to one another only for certain values of some of the letters involved, the statement of the equality is called an *algebraical equation*.

The quantities separated by the sign of equality are called the *sides of the equation*.

Thus $x+7=11$ is an algebraical equation, for it is true only when x is put equal to 4.

The expression $x+7$ is the left-hand side of the equation; the number 11 is the right-hand side of the equation.

90. **Root.** The values of the letters which make the two sides of an equation equal to one another are called the *roots of the equation.*

The roots of an equation are said to *satisfy* it.

To *solve* an equation is the process of finding the roots. The word *solution* is often used to denote the roots, which are the results of this process.

91. **Unknowns.** The letters which represent the unknown quantities are often called the *unknowns*,—an abbreviation for unknown quantities; and the equations are said to be equations in (*i.e.* involving) the unknown quantities.

92. **Notation.** It is usual to represent the value of known quantities (other than mere numbers) by the letters near the beginning of the alphabet, such as a, b, c, &c.; and to represent the unknown quantities, whose values we want to determine, by the letters near the end of the alphabet, such as x, y, z, &c.

93. **Classification of Equations.** Equations which involve (beside the unknown quantities) only numbers are called *numerical equations.* Equations in which some or all of the known quantities are represented by letters are called *algebraical equations.*

Equations are further classified according to their dimensions in the symbols which represent the unknown quantities.

Simple Equations. An equation which contains only one unknown quantity, say x, is said to be a *simple equation*, or an *equation of the first degree*, or a *linear equation*, when only first powers of x occur in it.

Quadratic Equations. An equation which contains only one unknown quantity, say x, is said to be a *quadratic equation*, or an *equation of the second degree*, when the only powers of x which occur in it are the first and second, namely, x and x^2.

*Similarly, an equation in x is called a *cubic equation*, or an *equation of the third degree*, when the only powers of x involved are x, x^2 and x^3. It is called a *biquadratic equation*, or an *equation of the fourth degree*, when the only powers of x involved are x, x^2, x^3, and x^4.

94. **Axioms.** We assume the following axioms.

(i) If equal quantities be added to equal quantities the sums will be equal.

(ii) If equal quantities be taken from equal quantities the differences will be equal.

(iii) If equal quantities be multiplied by the same number (or by equal numbers) the products will be equal.

(iv) If equal quantities be divided by the same number (or by equal numbers) the quotients will be equal.

In other words, we assume that we may multiply or divide both sides of an equation by the same quantity; and also that we may add to or subtract from each side of an equation the same quantity.

95. **Transposition of terms.** *Any term may be moved from one side of an equality to the other provided its sign is changed.*

Suppose that we have an equality such as

$$a + b - c = d.$$

Add c to each side. This, by axiom (i), is permissible.

$$\therefore\ a + b - c + c = d + c,$$

that is, $$a + b = d + c.$$

Thus c has been moved from the left-hand side of the equality to the right-hand side, but it appears on the right-hand side with the opposite sign to that which it had when on the left-hand side.

Next, returning to an equality like

$$a + b - c = d,$$

let us subtract b from each side, or (in other words) add $-b$. This, by axiom (ii), is permissible.

$$\therefore\ a + b - c - b = d - b,$$

that is, $$a - c = d - b.$$

Thus b has been moved from the left-hand side of the equality to the right-hand side, but it appears on the right-hand side with the opposite sign to that which it had when on the left-hand side.

Terms which are thus removed from one side of an equality to the other with a change of sign are said to be *transposed.*

If we transpose all the terms from each side of an equation to the other side, the sign of every term will be changed. This is equivalent to multiplying both sides by -1, which we already know, by axiom (iii), is permissible. We can thus change the sign of every term of an equation.

96. We shall confine ourselves in the rest of this chapter to some of the easier examples of simple equations which involve only one unknown quantity.

97. **Method of Solving Simple Equations.** The method of solving a simple equation will be readily understood from the following examples. The usual process is as follows.

(i) First, we clear the equation of fractions, remove brackets, and perform any other algebraical operations which are indicated.

(ii) Next, we transpose all the terms involving the unknown quantity to one side of the equation (usually to the left-hand side), and all the other terms to the other side.

(iii) Then, we simplify each side of the equation as far as is possible, combine like terms, and in particular collect all the terms which are multiplied by the unknown quantity into one term with it as a factor.

(iv) Lastly, we divide both sides of the equation by this coefficient of the unknown quantity: and we then have the required root, which must involve nothing but numbers and known quantities.

It is desirable to write each successive step in a separate line, and also to indicate how it is deduced from the preceding step. The beginner should also state his answer at the end of the work, so as to make it quite clear what is the result arrived at.

Ex. 1. *Solve the equation* $3x-8=x-2$.

(i) Transpose the term on the right-hand side which involves x to the left-hand side; and transpose the term on the left-hand side which does not involve x to the right-hand side.

$$\therefore\ 3x-x=8-2.$$

(ii) Collect like terms, $\therefore\ 2x=6$.

(iii) Divide by 2, $\therefore\ x=3$.

Thus the required root is 3. This value of x will therefore "satisfy" the given equation, that is, make it identically true. We can verify this result by seeing whether this root does so satisfy the given equation. If we put $x=3$ in the given equation it becomes

$$3\times3-8=3-2,$$

that is, $9-8=3-2$,

which is true. In the following examples we shall not in general verify our results, but a student who is doubtful as to the correctness of his work can always test the final result in the manner above described.

Ex. 2. *Solve the equation* $\frac{1}{3}y-\frac{1}{4}=y+\frac{1}{2}$.

Clear of fractions. The L.C.M. of the denominators is 12; multiplying both sides of the equation by 12, we have

$$4y-3=12y+6.$$

Transpose terms, $\therefore\ 4y-12y=6+3$.

Collect like terms, $\therefore\ -8y=9$.

Change the sign of every term, $\therefore\ 8y=-9$.

Divide by 8, $\therefore\ y=-\frac{9}{8}$.

Therefore the root of the given equation is $-\frac{9}{8}$.

Ex. 3. *Solve the equation* $7x+4(2x-1)-2(x+3)+5=5(x-1)$.

Remove the brackets, $\therefore\ 7x+8x-4-2x-6+5=5x-5$.

Transposing, $\therefore\ 7x+8x-2x-5x=-5+4+6-5$.

Collect like terms, $\therefore\ 8x=0$.

$$\therefore\ x=0.$$

Therefore the root of the given equation is 0.

Ex. 4. *Solve the equation* $\cdot 2x + 3 = 1 - \cdot 3x$.

Transposing, $\therefore \; \cdot 2x + \cdot 3x = 1 - 3$.

Collect like terms, $\therefore \; \cdot 5x = -2$.

Divide by $\cdot 5$, $\therefore \; x = -\frac{2}{\cdot 5} = -\frac{2}{\frac{1}{2}} = -4$.

Therefore the root of the given equation is -4.

[We might also have at once expressed the decimals as vulgar fractions, or have multiplied both sides of the equation by 10, and so got rid of the decimal fractions; and then solved.]

Ex. 5. *Solve the equation* $ax + b = cx + d$.

Transposing, $\therefore \; ax - cx = d - b$.

Collect like terms, $\therefore \; (a - c)\, x = d - b$.

Divide by $a - c$, $\therefore \; x = \frac{d - b}{a - c}$.

Therefore the root of the given equation is $(d - b)/(a - c)$.

The following examples are rather harder. We shall follow the above procedure, but only indicate the more important steps.

Ex. 6. *Solve the equation*

$$7(x - 1)(x + 2) = 11x^2 - 4(x + 1)(x - 2).$$

Multiplying out the products, we have

$$7(x^2 + x - 2) = 11x^2 - 4(x^2 - x - 2),$$

that is, $7x^2 + 7x - 14 = 11x^2 - 4x^2 + 4x + 8$.

Transposing, $\therefore \; 7x^2 - 11x^2 + 4x^2 + 7x - 4x = 8 + 14$.

Collect like terms,

$$\therefore \; (7 + 4 - 11)\, x^2 + (7 - 4)\, x = 22.$$

$$\therefore \; 3x = 22.$$

$$\therefore \; x = \tfrac{22}{3} = 7\tfrac{1}{3}.$$

Ex. 7. *Solve the equation* $\cdot 6x + \frac{1}{3} = \cdot 1\dot{6}x + 2\cdot 1$.

Express the decimals as vulgar fractions,

$$\therefore \; \tfrac{6}{10}x + \tfrac{1}{3} = \tfrac{16 - 1}{90}\, x + \tfrac{21}{10}.$$

$$\therefore \; \tfrac{3}{5}x + \tfrac{1}{3} = \tfrac{1}{6}x + \tfrac{21}{10}.$$

The L.C.M. of the denominators is 30. Multiplying by 30, we have

$$18x+10=5x+63.$$
$$\therefore\ 18x-5x=63-10.$$
$$\therefore\ 13x=53.$$
$$\therefore\ x=\tfrac{53}{13}=4\tfrac{1}{13}.$$

EXAMPLES ON SIMPLE EQUATIONS. V.

1. Is 2 a *root* of any of the following equations; and if so, of which is it a root?

(i) $x-3=2x-5$; (ii) $2x-3+x=7x-4$; (iii) $x^2-2=2$.

2. Shew that 2, $\frac{1}{2}$, and $-\frac{2}{3}$ are roots of $x^2(6x-11)=4(x-1)$.

Solve the following equations, numbered 3 to 45.

3. $10x-11x+1=0.$

4. $-13x=5(x-24).$

5. $4x-7+2x-1=3x+6-x-1.$

6. $4\{x-3[x-2(x-1)]\}=24.$

7. $2x-1-2(3x-2)+3(4x-3)-4(5x-4)+5(6x-5)=0.$

8. $\frac{1}{4}x+\frac{1}{5}x=9.$

9. $4x+\dfrac{7x}{9}+2+\dfrac{x}{5}=34.$

10. $\cdot 8x-\cdot 067=\cdot 473+\cdot 071x.$

11. $\cdot 006x-\cdot 491+\cdot 723x=-\cdot 005.$

12. $\dfrac{2x+1}{3}+\dfrac{4x+2}{5}+\frac{1}{7}=2\frac{12}{35}.$

13. $\frac{2}{3}(x+1)-\frac{1}{2}(x-1)=\frac{3}{2}.$

14. $5x-\frac{1}{7}(4x-1)=4(x+1).$

15. $\frac{4}{5}x-\frac{5}{4}x+17=\frac{4}{9}(x-2).$

16. $\dfrac{3x+1}{11}+\dfrac{2x+1}{5}=\dfrac{2x+1}{3}.$

17. $\dfrac{x-3}{9}+\dfrac{4x-3}{15}-\dfrac{7x-4}{20}=0.$

18. $\dfrac{3x+2}{5}+x-\dfrac{7x-3}{4}=\dfrac{5x+7}{12}.$

19. $\dfrac{471-6x}{2}-\dfrac{402-3x}{12}=9-\dfrac{2x+1}{29}.$

20. $\dfrac{x+3}{2}-\dfrac{11-x}{5}=3\frac{1}{5}+\dfrac{3x-1}{20}.$

21. $\dfrac{2x-1}{5}+\dfrac{5x+3}{17}=3-\dfrac{4x-118}{11}.$

22. $\dfrac{x+3}{4}+\dfrac{x+4}{5}=\dfrac{x+10}{11}+\dfrac{x+12}{13}.$

23. $2\dfrac{x-7}{3}+\dfrac{x+2}{4}=6-\dfrac{x+1}{11}.$

24. $x-\dfrac{x-1}{3}=\dfrac{x+17}{4}-\dfrac{12-x}{5}.$

25. $\frac{1}{12}x-\frac{1}{8}(8-x)-\frac{1}{4}(1+x)+\frac{7}{4}=0.$

26. $\frac{3}{5}(2x-7)-\frac{2}{3}(x-8)=4+\frac{1}{15}(4x+1).$

27. $\dfrac{3x+1}{4}-2(6-x)=\dfrac{5x-4}{7}-\dfrac{x-2}{3}.$

28. $2\dfrac{x-7}{15}+\dfrac{x+3}{2}=8+\dfrac{3x+77}{22}.$

29. $\frac{1}{3}(2x-1)-\frac{3}{4}(x-3)+\frac{4}{5}(3x-8)=25.$

30. $\frac{3}{7}(x+\frac{1}{3})-\frac{2}{5}(x+\frac{1}{2})=\frac{3}{5}(x-\frac{1}{3})-\frac{2}{3}(x-\frac{1}{2}).$

31. $\dfrac{4x+3}{9}+\dfrac{7x-5\frac{1}{2}}{5}=\dfrac{2x+11}{7}-\dfrac{9x-13\frac{1}{2}}{11}.$

32. $\dfrac{x-4}{7}-\dfrac{3-\frac{1}{4}(x-5)}{2}-\dfrac{4\frac{1}{7}-\dfrac{2x}{7}}{3}=5.$

33. $\dfrac{x}{3}-\dfrac{x-1}{2\frac{1}{2}}=\dfrac{3x-4}{15}+\dfrac{x}{12}.$

34. $\cdot 032x-\dfrac{3\cdot 65-7\cdot 24x}{\cdot 3}+\cdot 084=0.$

35. $x^2+(x+1)(x-1)=2(x-2)(x+3).$

36. $(x-1)(x+2)(x-3)=x^2(x-2)+2(x+4).$

37. $(2x-3)(6x-7)=(4x-5)(3x-4).$

38. $(5x+2)(x+7)-(3x-1)(x+10)=(2x-1)(x+14).$

39. $(4x-3)(3x+7)=(7x-11)(3x-4)-(9x+10)(x-3).$

40. $(x+4)(2x+5)-(x+2)(7x+1)=(x-3)(3-5x)+47.$

41. $(x-2)(x+1)+(x-1)(x+4)=(2x-1)(x+3).$

42. $(x-3)(x-4)(x-5)=(x-1)(x-14)(x+3)-24.$

43. $(x-a)(x+b)+c=(x+a)(x-b).$

44. $\frac{x-a}{2}+\frac{x-b}{3}=\frac{a+3x}{3}-\frac{2x-b}{2}.$

45. $(x+a)(x-b)-2a^2b=(x+b)(x-a)-2b^2a.$

46. $\frac{1}{5}\left(x-\frac{1}{5}\left[x-\frac{1}{5}\left(x-\frac{1}{5}\left\{x-\frac{1}{6}x\right\}\right)\right]\right)=1.$

47. Find the value of c which will make $x^4+5x^3+7x^2+cx-2c$ divisible without remainder by x^2+3x+2.

48. Find a value of x (other than zero) which will make

$$x^5-8x^3+11x^2+7x-1780$$

exactly divisible by x^2+7x-1.

CHAPTER VI.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

98. A problem leading to an equation consists of a verbal statement of the relations between certain quantities, from which statement it is desired to determine the values of some of the quantities.

If these quantities can be represented by algebraical symbols, and if this verbal statement can be expressed as an algebraical equality (or algebraical equalities) involving these symbols, we obtain an equation (or equations) whose roots are the required values of the unknown quantities.

The beginner will find that his chief difficulty consists in the translation of the given statement (which is expressed in ordinary language) into algebraical language.

99. We shall confine ourselves in this chapter either to problems where there is only one unknown quantity and the given relation involving it can be expressed algebraically as a simple equation, or to similar problems where all the unknown quantities can be expressed in terms of one of them.

The following examples cover some of the more common cases.

Ex. 1. *What number is that which exceeds* 7 *by as much as its double exceeds* 17?

Let x represent the number. Then its double is $2x$.
By the question, x exceeds 7 by as much as $2x$ exceeds 17,

$$\therefore\ x-7=2x-17.$$

Transposing, $$x-2x=7-17.$$

$$\therefore\ x=10.$$

Thus 10 is the required number; and the reader will see that it fulfils the conditions laid down in the question.

Ex. 2. *Divide* 78 *into two parts such that one part is five times as large as the other.*

Let x be one part. Then the other part must be $78-x$.

One part is five times as large as the other,

$$\therefore\ x=5(78-x)$$

$$=5(78)-5x.$$

Transposing, $$x+5x=5(78),$$

that is, $$6x=5(78).$$

Divide by 6, $$\therefore\ x=5(13)$$

$$=65.$$

Therefore one part is 65, and the other is $78-65=13$.

We might however have supposed that $78-x$ was equal to 5 times the part denoted by x. In this case our equation would have been $$78-x=5x.$$

Transposing, $$-6x=-78.$$

Divide by -6, $$\therefore\ x=13.$$

Therefore one part is 13, and the other is $78-13=65$. This therefore leads to the same result as before.

Ex. 3. *Demochares has lived a fourth of his life as a boy; a fifth as a youth; a third as a man; and has spent* 13 *years in his dotage. How old is he?* [From the Collection of Problems by Metrodorus, circ. 310 A.D.]

Suppose Demochares to be x years old.

Then the sum of $\frac{1}{4}x$ years, $\frac{1}{5}x$ years, $\frac{1}{3}x$ years, and 13 years must amount to his present age,

$$\therefore\ \frac{x}{4}+\frac{x}{5}+\frac{x}{3}+13=x.$$

The L.C.M. of the denominators is 60. Multiply by 60.

$$\therefore\ 15x+12x+20x+780=60x,$$

that is, $$47x+780=60x.$$

Transposing, $47x - 60x = -780.$

Collect like terms, $\therefore\ -13x = -780.$

Change the signs throughout, $\therefore\ 13x = 780.$

Divide by 13, $\therefore\ x = 60.$

Thus Demochares is 60 years old.

Ex. 4. *A's age is four times B's age, and in twenty years time A's age will be double that of B. Find the present ages of A and B.*

Suppose that B is now x years old. Then A is $4x$ years old.

In 20 years time, B will be $(x+20)$ years old, and A will be $(4x+20)$ years old.

But, by the question, A's age will then be double that of B,

$$\therefore\ 4x+20=2(x+20).$$

$$\therefore\ 4x+20=2x+40.$$

$$\therefore\ 4x-2x=40-20.$$

$$\therefore\ x=10.$$

Therefore B is now 10 years old, and A is 40 years old.

Note. It will be observed that in all these examples x is an abstract number, the unit that it multiplies in the two last examples being a year. The beginner should notice that in every problem the symbol for the unknown quantity will similarly stand for an abstract number only [see Art. 2].

Ex. 5. *A train running* 30 *miles an hour starts from a certain station at* 1.0. *Another train running* 40 *miles an hour is ten miles behind the first station at* 1.15. *When and where will it overtake the first train?*

Suppose that the second train overtakes the first x hours after 1.0.

In that time, the first train has gone $30x$ miles.

The second train has been travelling for x hours minus 15 minutes, that is for $(x-\frac{1}{4})$ hours. It is running 40 miles an hour, and has therefore gone $40(x-\frac{1}{4})$ miles.

But when the second train overtakes the first, it must have altogether travelled 10 miles more than the first,

$$\therefore\ 40(x-\tfrac{1}{4})=30x+10.$$

$$\therefore\ 40x-10=30x+10.$$

$$\therefore\ 40x-30x=10+10.$$

$$\therefore\ 10x=20.$$

$$\therefore\ x=2.$$

Therefore the second train overtook the first at 3.0. o'clock. Also the distance covered by the first train in that 2 hours was 30×2 miles $= 60$ miles. Therefore the second train overtook the first at a place 60 miles from the station whence the first train had started.

Ex. 6. *A and B play a game for money stakes. At the beginning of the game A has 38s. and B has 30s. At the end of the game A has twice as much as B. How much did A win?*

Suppose that A won x shillings.

$\therefore$ at the end of the game A has $(38+x)$ shillings,

and " " " " B has $(30-x)$ shillings.

But, by the question, A has then twice as much as B.

$$\therefore\ 38+x=2(30-x)$$
$$=60-2x.$$
$$\therefore\ 3x=60-38=22.$$
$$\therefore\ x=\tfrac{22}{3}=7\tfrac{1}{3}.$$

Therefore A won $7\frac{1}{3}$ shillings, that is, 7 shillings and 4 pence.

Ex. 7. *Four pipes discharge into a cistern: one by itself would fill it in one day; another in two days; the third in three days; the fourth in four days. If all run together, how soon will they fill the cistern?* [From the Collection of Problems by Metrodorus.]

Let x be the number of days required. Let v represent the cubical contents of the cistern.

The first	pipe discharges a volume	v	of water in one day.
The second	" " " "	$\frac{1}{2}v$	" " "
The third	" " " "	$\frac{1}{3}v$	" " "
The fourth	" " " "	$\frac{1}{4}v$	" " "
$\therefore$ all four pipes discharge	"	$(v+\frac{1}{2}v+\frac{1}{3}v+\frac{1}{4}v)$	" " "
but	" " " " "	$\frac{v}{x}$	" " "

$$\therefore\ \frac{v}{x}=\frac{v}{1}+\frac{v}{2}+\frac{v}{3}+\frac{v}{4}.$$
$$\therefore\ \frac{1}{x}=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}$$
$$=\tfrac{25}{12}.$$
$$\therefore\ x=\tfrac{12}{25}.$$

The answer therefore is $\frac{12}{25}$ of a day.

This question might have been solved by arithmetic.

If we had supposed that the third and fourth pipes let the water escape, while the first and second had supplied water to the cistern, we should have had the equation

$$\frac{v}{x}=\frac{v}{1}+\frac{v}{2}-\frac{v}{3}-\frac{v}{4},$$

from which we obtain

$$x=\tfrac{12}{11}.$$

Ex. 8. *One of the sides of a rectangular court is longer than the other by* 10 *yards. If the shorter side were increased by* 4 *yards, and the longer one diminished by* 5 *yards, the area of the court would remain unaltered. What are the lengths of the sides?*

Let the shorter side be x yards long, therefore the longer side is $(x+10)$ yards long, and the area of the court is $x(x+10)$ square yards.

In the second case, the lengths of the sides are respectively $(x+4)$ yards and $(x+10-5)$ yards, and therefore the area is $(x+4)(x+5)$ square yards.

These areas are equal,

$$\therefore\ x(x+10)=(x+4)(x+5).$$
$$\therefore\ x^2+10x=x^2+9x+20.$$
$$\therefore\ x=20.$$

Therefore the shorter side is 20 yards long, and the longer side is 30 yards long.

Ex. 9. *Find four numbers, the sum of every possible arrangement of them taken three at a time being respectively* 20, 22, 24, *and* 27. [From the Arithmetic of Diophantus, circ. 350 A.D.]

Let x be the sum of all four numbers.

$\therefore$ the numbers are $x-20$, $x-22$, $x-24$, and $x-27$.

$$\therefore\ x=(x-20)+(x-22)+(x-24)+(x-27).$$
$$\therefore\ x=31.$$

$\therefore$ the numbers are 11, 9, 7, and 4.

Ex. 10. *At what time between* 1 *and* 2 *o'clock do the hour and minute hands of a watch overlap one another?*

Let the required time be x minutes after 1.0.

The two hands of the watch were together at 12.0. Since that time the minute hand has swept through an angle which is represented on the dial by $(60+x)$ minutes.

Again the minute hand moves round twelve times as quickly as the hour hand, for in 12 hours the former has gone 12 times quite round, while the latter has only been round once. Therefore since 12.0 the hour hand has swept through an angle which is represented on the dial by $\frac{1}{12}(60+x)$ minutes.

Now, by the question, the required time is that time between 1.0 and 2.0 when the hands overlap. The minute hand has therefore gone round a little more than once, and the whole angle it has covered will exceed the whole angle covered by the hour hand by an angle represented on the dial by 60 minutes. Therefore we have

$$(60+x)-\frac{1}{12}(60+x)=60.$$
$$\therefore\ \frac{11}{12}(60+x)=60.$$
$$\therefore\ 11(60+x)=12\times 60.$$
$$\therefore\ 660+11x=720.$$
$$\therefore\ 11x=60.$$
$$\therefore\ x=5\tfrac{5}{11}.$$

The answer therefore is $5\frac{5}{11}$ minutes past 1.0.

100. We now proceed to some problems where in certain cases the results are apparently inconsistent with the given question, and we shall explain in what way such results should be interpreted.

Ex. 11. *A man who is now* a *years old has a son who is* b *years old. How long will it be before the father is three times as old as his son?*

Let the time required be x years. At the end of this time the age of the father will be $(a+x)$ years, and the age of his son will be $(b+x)$ years. But, by the question, the father is then three times as old as his son,

$$\therefore\ a+x=3(b+x).$$
$$\therefore\ a-3b=3x-x=2x.$$
$$\therefore\ x=\tfrac{1}{2}(a-3b).$$

Now a may be greater than, equal to, or less than $3b$.

(i) If $a>3b$, then x is a positive number, and there is no difficulty in the solution. The event takes place x years *after* the present time. For example, $a=36$, $b=10$.

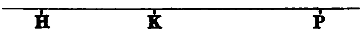
(ii) If $a=3b$, $\therefore x=0$; that is, the father is at the present time three times as old as his son. For example, $a=30$, $b=10$.

(iii) But if $a<3b$, the answer is a negative number of years. This will mean that the event took place *before* the present time [Art. 10]. For example, $a=28$, $b=10$.

Thus our result sums up all the cases in a single statement, which we must interpret according to the usual algebraic rules.

Ex. 12. *Two couriers, A and B, travelling in the same direction along a certain road, start at the same time from two stations, H and K, whose distance apart is* n *miles. A travels at the uniform rate of* a *miles an hour, B travels at the uniform rate of* b *miles an hour. A starts behind B, when will he overtake B?*

A starts from H, B from K. Let A overtake B at the end of t hours after they start, and suppose that A overtakes B at P.



Then the distance travelled by A is equal to $HP=at$,
and „ „ „ „ B „ $KP=bt$;

$$\therefore\ at-bt=HK=n.$$

$$\therefore\ t=\frac{n}{a-b}.$$

(i) If $a>b$, then t is a positive number, and there is no difficulty in the solution.

(ii) If $a<b$, then t is a negative number. In fact, as A travels more slowly than B, he can never overtake B. But, if the couriers be assumed to have been travelling in the same direction and at the same rates before they reached H and K, then at some time before that at which A is at H and B at K, they were together, and it is this time that is given by the answer. Moreover, since $HP=at$ and $KP=bt$ and t is a negative number, it follows that HP and KP are both negative, and therefore are to be taken as measured in the opposite direction to that in which motion is taking place.

(iii) If $a=b$, then $t=\frac{n}{0}$. Now we know from arithmetic that the smaller the denominator of a fraction the bigger does the value of the fraction become (if the numerator remain the same), and when the denominator vanishes the value of the fraction becomes indefinitely large. In this case it will take an infinite time before A overtakes B, which is the same as saying that A never overtakes B. In fact, as they travel at equal rates, they will always be a distance n miles apart.

Ex. 13. As one more example, consider again the problem of filling a cistern by means of four pipes which is given on p. 72, Ex. 7. If we had supposed that the second, third, and fourth pipes had let the water escape, and the first had supplied water we should have had the equation

$$\frac{v}{x}=\frac{v}{1}-\frac{v}{2}-\frac{v}{3}-\frac{v}{4},$$

from which we obtain $x=-12.$

To determine what meaning can be attached to a negative answer, we notice that if we had tried to find the number of days in which the cistern, supposed full, would have been emptied when the last three pipes were emptying it and the first pipe filling it, we should have obtained the equation

$$\frac{v}{x}=-\frac{v}{1}+\frac{v}{2}+\frac{v}{3}+\frac{v}{4},$$

and therefore $x=12$. Hence the negative answer to the problem as stated in the first part of this Example shews us that the problem as it stands is impossible, but that the analogous problem of emptying a cistern which is already full is capable of solution.

101. The following rules may help the beginner to form correctly the equations of the problems hereafter given.

First. In problems concerning distance, time, and (uniform) velocities, we have [see Art. 5] $s=vt$, where s is the distance traversed (i.e. the *number* of units of length in it), t is the time occupied in traversing it (i.e. the *number* of the units of time occupied), and v is the velocity with which it is traversed.

But in all cases the same unit of length should be used throughout the same question; that is, all distances should be expressed either in feet, or in yards, or in miles, and so on—whichever unit is most convenient being chosen. Similarly all durations should be expressed as multiples of the same unit of time; and so for other quantities.

Second. In problems concerning work done by men, taps filling cisterns, &c., we notice,

(i) that if a man do a piece of work in a days, he does $\frac{1}{a}$ of it in each day; if a tap fill a cistern in a hours, it must fill $\frac{1}{a}$ of it in each hour;

(ii) that if x men complete a piece of work in one day, then each man must do $\frac{1}{x}$ of it in 1 day;

(iii) that if x men complete a piece of work in a days, then xa men would do it in 1 day, and therefore 1 man would do $\frac{1}{xa}$ of it in each day.

Should the student feel any difficulty in writing down the foregoing results, a numerical illustration (such as putting $a=2$, $x=3$) will probably guide him aright.

EXAMPLES. VI.

[*Some additional examples on problems leading to simple equations will be found in Chapter* XI.]

1. Divide the number 46 into two parts, such that when the one is divided by 7 and the other by 3, the quotients together may amount to 10.

2. Find a number such that the sum of it and of another number m times as great may be a.

3. Divide the number 237 into two parts, such that one may be contained in the other $1\frac{1}{4}$ times.

4. A person has 264 coins of two kinds, and $4\frac{1}{2}$ times as many of one sort as the other. How many has he of each sort?

5. The difference between the third and eighth parts of a certain number exceeds five times the difference between the eighth and ninth parts by 10; find the number.

6. The difference between the fourth and ninth parts of a certain number exceeds four times the difference between the ninth and tenth parts by 34; find the number.

7. Find a number which when multiplied by 6 exceeds 35 as much as 35 exceeds the original number.

8. Find a number such that if it be subtracted from the sum of its half, third, and fourth parts, the remainder may be 1.

9. An army is defeated, losing $\frac{1}{18}$th of its numbers in killed, and 4,000 prisoners. It is then reinforced by reserves amounting to 5,000 troops; but, in retreating, its rear-guard, which consists of $\frac{1}{4}$th of the numbers now with the colours, is completely cut off. There remain 21,150 men. What was the original force?

10. A man sold a horse for £35, and half as much as he gave for it. He gained £10. 10*s.*; what did he pay for the horse?

11. A fortress has a garrison of 2,600 men; among whom are nine times as many foot soldiers, and three times as many artillery soldiers as cavalry. How many of each corps are there?

12. I think of a certain number. I multiply it by 7, add 3 to the product, divide this by 2, subtract 4 from the quotient, and obtain 15. What number did I think of?

13. Gun metal is composed of 90 per cent. of copper and 10 per cent. of tin. Speculum metal contains 67 per cent. of copper and 33 per cent. of tin. How many cwt. of speculum metal should be melted with 4 cwt. of gun metal in order to make an alloy in which there is four times as much copper as tin?

14. An express train runs 7 miles an hour faster than an ordinary train. The two trains run a certain distance in 4h. 12m. and 5h. 15m. respectively. What is the distance?

15. A man drives to a certain place at the rate of 8 miles an hour. Returning by a road 3 miles longer at the rate of 9 miles an hour he takes $7\frac{1}{2}$ minutes longer than in going. How long is each road?

16. A tourist finds that, if he spends sixteen shillings a day, the money at his disposal will enable him to go on for two days longer than if he spent eighteen shillings a day. How much money has he?

17. Find two consecutive numbers such that the sum of the fifth and eleventh parts of the greater may exceed by 1 the sum of the sixth and ninth parts of the less.

18. The difference of the squares of two consecutive numbers is 17. Find the numbers.

19. In an examination paper one boy A got three marks more than half of the full marks, and another boy B got six marks less than one-third of the full marks. The marks obtained by A were twice as many as those obtained by B. What were the marks that each obtained?

20. I rode one-third of a journey at the rate of 10 miles an hour, one-third more at the rate of 9 miles an hour, and the rest at the rate of 8 miles an hour. If I had ridden half the journey at the rate of 10 miles an hour, and the other half at the rate of 8 miles an hour, I should have been half a minute longer on the way. What distance did I ride?

21. A train travelling at the rate of $37\frac{1}{2}$ miles an hour passes a person walking on a road parallel to the railway in 6 seconds; it also meets another person walking at the same rate as the other, but in the opposite direction, and passes him in 4 seconds. Find the length of the train.

22. A cash box contains three equal sums of money, one in sovereigns, one in shillings, and one in pennies. If the total number of coins is 1305, find how much money is in the box.

23. A person wishes to raffle a gold watch, and for that purpose sells a certain number of tickets. If he sell each ticket for 5*s.* he would lose £5, because the watch cost him more than he would in this case get; but if he sell each ticket for 6*s.*, then he gains £4. How many tickets did he sell?

24. A father's age is equal to the united ages of his five children. In 15 years his age will be only one-half their united ages. How old is the father?

25. A father's age is three times that of his younger son. In seven years' time he will be twice as old as his elder son, who is 5 years older than the younger son. What are their present ages?

26. The prices of the stalls, pit and gallery of a theatre are respectively ten shillings, half-a-crown and one shilling. The pit holds twice as many and the gallery three times as many people as the stalls. If the receipts are £90 when all the seats are full, find the number of people present.

27. A train travels for 35 miles, completing the whole distance in $46\frac{1}{2}$ min. The first 11 miles are accomplished at a uniform rate; the speed is then increased 25 per cent. and remains constant for the next 10 miles, after which it is diminished in the ratio of 14 to 15, at which rate the rest of the journey is performed. Find the original speed of the train.

28. What number is that to which if 13 be added then one-third of the sum will be equal to 13?

29. A man leaves home with a certain sum of money in his pocket; he spends one-eighth of it in travelling expenses, one-half of the remainder in purchases, and the rest, amounting to 21*s.*, he loses. How much did he start with?

30. A bankrupt pays a dividend of 7*s.* 6*d.* in the pound. If his debts had been £1200 more he would have paid only 6*s.* 8*d.* in the pound: what were his debts?

31. In the same time *A* can do twice as much work as *C*, *B* one and a half times as much work as *C*. The three work together for two days, and then *A* works on alone for half a day. In what time could *A* and *C* together do as much as the three will have thus performed?

32. A purse contains 28 coins which together amount to £7. A certain number of the coins are shillings, one-fifth of that number half-sovereigns, and the rest are sovereigns. Find the number of each.

33. At a cricket match the contractor provided dinner for 24 persons, and fixed the price so as to gain $12\frac{1}{2}$ per cent. upon his outlay. Three of the cricketers were absent. The remaining 21 paid the fixed price for their dinner, and the contractor lost 1*s.* What was the charge for dinner?

34. A cattle-dealer spends all his money in buying sheep, and sells them at a profit of 20 per cent. He spends the money which he now has in the same way, and gains 25 per cent. He again spends all his money, and makes a profit of 16 per cent. If the profit which he made upon the last transaction be £300, find how much money he had originally.

35. *A* has 6 more shillings than *B*, but if *A* gives *B* half his money, and then *B* gives back to *A* one quarter of his increased sum, they will each have the same sum: find what each had at first.

36. At what time between 4 and 5 o'clock are the hands of a watch together?

37. At what time between the hours of 10 and 11, will the hands of a clock be equally inclined to the vertical?

EXAMINATION QUESTIONS.

[*The following questions are taken from papers on Elementary Algebra set in recent years to the lower forms of various Public Schools under the authority of the Oxford and Cambridge Schools Examination Board.*]

1. Find the numerical values, when $a=3$, $b=2$, $c=1$, of

(i) a^2+bc-a; (ii) $(a+b)(a-b)-ac+b(c-a)$.

2. Add together $2a+3b-4c$, $3a-2b+3c$, $a-b$, and $4a-c$.

3. Subtract $x-y^2+zx$ from $xy+3x+y^2$. What is the coefficient of x in the result?

4. Solve the equation $2(x-1)-3(x-2)+4(x-3)=4$.

5. A father makes the following agreement with his son. For every day on which the boy is among the first ten boys in his class he is to get three pence; but if he is lower than tenth he is to give his father a penny. At the end of 12 days the boy has 1*s.* 4*d.* During how many days was he among the first ten?

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6. Define *factor*, *term*, *coefficient*, *power*, *index*.

Given $a=3$, $b=2$, $c=-7$, find the value of

(i) $3a^3b-5ab^2c-b^3c^2$; (ii) $\frac{3b^2c}{a}+\frac{5bc^2}{a-c}-\frac{6a^3}{b+2c}$.

7. Remove the brackets from the expressions

(i) $4a+\{a-(3a-2b)+2b\}$; (ii) $7[3a-4\{a-b+3(a+b)\}]$.

8. Multiply $2y^2+4xy-x^2$ by $3x^2(x-y)-2xy(x+y)+y^3$.

9. Divide $x^2-(a-b)x-ab$ by $x-a$.

10. When A and B sit down to play, B has two-thirds as much money as A. After a while A wins 15*s.*, and then he has twice as much money as B. How much had each at first?

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11. Subtract $-2x^3+3x^2y+5y^3-4xy^2$
from $4xy^2-x^3-y^3+4x^2y$.

12. Simplify the following expressions:

(i) $2a-(b-c)-(3a-\overline{2b-c})-(a-\overline{3b-4c})$;

(ii) $-2[x-3\{x-4(x-\overline{5x-1})\}]$;

(iii) $\frac{x-2y+3}{3}-\frac{2x-3y+4}{4}+\frac{2x-y+12}{12}$.

13. Enunciate the rule of signs in multiplication.

Multiply $3a^2-ab+2b^2$ by $3a^2+ab-2b^2$; and verify your result when $a=1$, $b=2$.

14. Solve the equation

$$\frac{2x+1}{3}-\frac{4x-3}{5}-\frac{5x+1}{6}=-6.$$

15. A coach travels between two places in 5 hours; if its speed were increased by 3 miles an hour, it would take $3\frac{1}{2}$ hours for the journey: what is the distance between the places?

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16. If $x=0$, $y=-1$, $z=2$, find the values of

$x^3+y^3+z^3-3xyz$; $(x^2-y^2)\div(y^2-xz)$; $\sqrt{2y^2-3yz+11x^2y+2z^2}$.

17. Simplify (i) $7a-[2a+3(a+1)]$;

(ii) $3x-[2x-3\{x-(x+1)\}]$;

(iii) $(a-b)[(a+c)+\{b-c\}]-a^2+b(b+c)-c\times b$.

18. Form the continued product of $a+b$, $a+2b$, $2a+b$, and $a-b$. What are the dimensions of the product?

19. Distinguish between an equation and an identity.

Solve the equations:

(i) $\frac{x+6a}{3}-\frac{2x-3a}{7}=\frac{5x}{6}-\frac{7x+6a}{14}$;

(ii) $5x(x+3)-7(x+2)(x-11)+2(x^2-3x+7)=0$.

20. In an examination paper one boy, A, gets 8 marks more than the third part of full marks, while another, B, gets 11 marks less than the half of full marks. The marks obtained by A are eight-elevenths of those obtained by B. Find the full marks for the paper.

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21. Multiply $a+b$ by $2a+3b$; $3a^2+2ab+b^2$ by $a^2-2ab-b^2$; and $a+b-c$ by $a-b+c$.

22. Divide $6abc$ by $2a$; a^3-x^3 by $a-x$;
and $3x^2-4y^2-3z^2-4xy+8xz+8yz$ by $x-2y+3z$.

23. Prove that

$$\frac{2(x^3+y^3+z^3-3xyz)}{x+y+z}=(y-z)^2+(z-x)^2+(x-y)^2.$$

24. Solve the equation

$$\frac{3x-8}{5}-\frac{x-1}{4}+\frac{7-x}{3}=\frac{8-2x}{6}-\frac{8-5x}{10}.$$

25. A piece of work could be done by 240 men in 20 days, but when it is half completed 144 workmen strike, and the work has to be finished by the remainder. How many days' delay are thus caused?

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26. Multiply (i) x^3+2x-1 by $x+2$; and (ii) multiply together $a+b-c$, $b+c-a$, and $c+a-b$.

27. Divide the sum of $x+17$, $14x+18$, and x^2+15 by $10+x$; and test your result by putting $x=2$.

28. Divide $4x^5-8x^4+x^3+x^2+4$ by $2x^2-x-2$;
and $(a^2+b^2)^3-(2ab)^3$ by $(a-b)^2$.

29. Shew that a term may be moved from one side of an equality to the other side, provided its sign is changed.

Solve the following equations:

(i) $(x-3)(2x+5)-(x+4)(7x+1)=(x+2)(3-5x)-48$;

(ii) $(5x+3)(7x-4)-(3x+5)(8x-11)=(11x-2)(x+1)$.

30. A man walks up a mountain at the rate of two miles an hour and down again by a way six miles longer at the rate of $3\frac{1}{2}$ miles an hour. He is out eight hours altogether. How far has he walked?

---

31. Simplify

(i) $a(2b+3c)-[c(2a+b)-b(c-2a)]$;

(ii) $2[x-a-3\{x-4(x-5\overline{x-a})-a\}]$.

Explain what is meant by a *coefficient*. Write down the coefficient of x in the second of the examples in this question.

32. Multiply a^2-8b^2 by a^2+8b^2; subtract $17b^4$ from the result, and then divide by $a-3b$.

33. Divide $2a^4+7a^3b-42a^2b^2+47ab^3-14b^4$ by $a^2+6ab-7b^2$.

34. If $x-y=2a$, shew that $x^2-6ax+9a^2=(y-a)^2$.

35. A man starts to walk from A to B at the rate of 4 miles per hour, and on the way is overtaken by a dog-cart; if the cart had started an hour later than it did, he could have walked 8 miles further before he was overtaken: find the rate at which the dog-cart travels.

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36. If $a=0$, $b=1$, $c=2$, $d=3$, find the numerical value of

$$(3abc+2bcd)\sqrt[3]{a^3bc+c^3bd+3}.$$

37. Multiply $x^2+(\sqrt{2}-1)x+1$ by $x^2-(\sqrt{2}+1)x+1$.

38. If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$; prove that

$$ax+by+cz=(a^2-bc)/x.$$

39. If $a+b+c+d=0$, prove that

$$a^3+b^3+c^3+d^3=3(bcd+cda+dab+abc).$$

40. The gross income of a certain man was £40 more in the second of two particular years than in the first, but in consequence of the income-tax rising from 4*d.* in the pound in the first year to 6*d.* in the pound in the second year, his net income (after paying income-tax) was unaltered. Find his income in each year.

CHAPTER VII.

FACTORS.

102. We proceed now to the consideration of some of the simpler algebraical processes. This chapter is devoted to the discussion of factors.

Our investigations will often be greatly facilitated if we can resolve a given expression into factors. This cannot always be effected, and we shall here confine ourselves to a few expressions where it is possible to obtain the factors by inspection.

We shall consider in succession (i) expressions which have a factor common to every term, (ii) certain known forms, (iii) quadratic expressions, and (iv) lastly, a general theorem which enables us to tell by inspection whether any given expression of the first degree is a factor of another expression.

103. **Factor common to every term.** If a certain letter or quantity divide every term in a given expression, it will divide the whole expression; and it will therefore be a factor of the expression.

In general we confine our attention to literal factors. All the numerical factors are usually combined into a single numerical coefficient, which is prefixed to the rest of the expression, but is not reckoned as a factor of it.

Ex. 1. *Find the factors of* $7a^2-3ax$.

Here a is a factor of each term,

$$\therefore\ 7a^2-3ax=a\,(7a-3x).$$

Therefore a and $7a-3x$ are the required factors.

Ex. 2. *Find the factors of* $15ab^2c^3-12a^3bc^2-21ac^4$.

$$15ab^2c^3-12a^3bc^2-21ac^4=3ac^2\,(5b^2c-4a^2b-7c^2).$$

Ex. 3. *Find the factors of* $a\,(2x-3)+b\,(2x-3)$.

$$a\,(2x-3)+b\,(2x-3)=(a+b)\,(2x-3).$$

Ex. 4. *Resolve* $4ax-12bx-6ay+18by$ *into factors.*

$$\begin{aligned}4ax-12bx-6ay+18by&=2\,\{2ax-6bx-3ay+9by\}\\&=2\,\{2x\,(a-3b)-3y\,(a-3b)\}\\&=2\,(2x-3y)\,(a-3b).\end{aligned}$$

This last example is less obvious than the others, but we shall see later how we can obtain the required factors in another way.

EXAMPLES. VII. A.

Resolve the following expressions into factors.

1. $ax+bx$.
2. $5x-20x^2$.
3. a^2-3ab.
4. $3lm^2-9l^2m$.
5. x^2y+xy^2+xy.
6. $11p^3-2p^2q-3pq^3$.
7. $8xy^2z^2+12x^2yz^2-16xyz^4$.
8. $a\,(x-y)-b\,(x-y)$.
9. $ac+ad+bc+bd$.
10. $x^2+lx-3x-3l$.
11. $ax+ay+az+px+py+pz$.
12. $ay^2+(a-1)\,y-1$.

104. **Known Forms.** We proceed next to the consideration of certain known forms, and under this head shall discuss in succession (i) expressions which can be written as the difference of two squares; (ii) trinomial expressions, which are perfect squares; (iii) certain expressions of the third and fourth degree.

105. *DIFFERENCE OF TWO SQUARES.* We have already proved [Art. 61, *Ex.* 3] the formula

$$a^2 - b^2 = (a-b)(a+b).$$

Hence the difference of the squares of two expressions can always be resolved into factors.

Ex. 1. *Resolve $x^2 - 64$ into factors.*

$$x^2 - 64 = x^2 - 8^2$$
$$= (x-8)(x+8).$$

Thus $x-8$ and $x+8$ are the required factors.

Ex. 2. *Resolve $49x^2 - 64y^2$ into factors.*

$$49x^2 - 64y^2 = (7x)^2 - (8y)^2$$
$$= (7x-8y)(7x+8y).$$

Thus $7x-8y$ and $7x+8y$ are the required factors.

Ex. 3. *Resolve $100-9$ into factors.*

$$100 - 9 = 10^2 - 3^2 = (10-3)(10+3) = 7 \times 13.$$

Thus 7 and 13 are the factors of 91.

Ex. 4. *Find the factors of $x^2 - 3$.*

$$x^2 - 3 = x^2 - (\sqrt{3})^2$$
$$= (x-\sqrt{3})(x+\sqrt{3}).$$

That is, $x-\sqrt{3}$ and $x+\sqrt{3}$ are the required factors. A factor involving the square root of x is not a factor such as we want, but $\sqrt{3}$ is a number and may properly form a term in a factor. Similarly, if a be a given quantity or a number, then the factors of $x^2 - a$ are $x - \sqrt{a}$ and $x + \sqrt{a}$.

106. The squares of compound expressions are subject to the same rule, for a and b in Article 105 may stand for any expressions.

Ex. 1. *Find the factors of $(2x+y)^2 - (2y-x)^2$.*

$$(2x+y)^2 - (2y-x)^2 = \{(2x+y)-(2y-x)\}\{(2x+y)+(2y-x)\}$$
$$= (3x-y)(x+3y).$$

Ex. 2. *Find the factors of $(x^2+1)^2 - x^2$.*

$$(x^2+1)^2 - x^2 = \{(x^2+1)-x\}\{(x^2+1)+x\},$$

that is, $$x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1).$$

EXAMPLES. VII. B.

Write down by inspection the factors of the following expressions, each of which is the difference of two squares.

1. a^2-121.
2. $81-144$.
3. x^2y^2-121.
4. $1-64b^2$.
5. x^2-1.
6. x^2-5.
7. $4-x^2$.
8. x^2-4.
9. $9-4a^2$.
10. $121l^2-25m^2$.
11. $81p^2-a^2b^2$.
12. $a^2b^2-a^2x^2$.
13. $a^2b^2-x^2y^2$.
14. $x^2-144l^2m^2$.
15. $x^4-144m^2x^2$.
16. a^4-b^4.
17. $16x^4-81a^4$.
18. $4a^2b^4c^6-9x^4$.
19. $(x+y)^2-121$.
20. $(a-b)^2-9c^2$.
21. $(x+y)^2-(x-y)^2$.
22. $(a^2+b^2)^2-a^2$.
23. $(a^2-b^2)^2-b^2$.
24. $(a+b+c)^2-(a+b-c)^2$.
25. $(3x-4y)^2-(4x-3y)^2$.
26. $a^2-4(a-b)^2$.
27. $(2a-1)^2-(a+1)^2$.
28. $a^2-(x^2+2xy+y^2)$.
29. $(l+m)^4-(m+n)^4$.
30. $(2x+1)^4-(x-2)^4$.

107. *PERFECT SQUARES.* We know [Art. 61, *Exs.* 1, 2] that

$$a^2+2ab+b^2=(a+b)^2,$$

and

$$a^2-2ab+b^2=(a-b)^2.$$

Thus any expression which can be put in either of these forms (where a and b may stand for any compound expressions) can be resolved into factors.

These examples can however generally be solved more simply by the methods given later, Arts. 113, 114, and Chap. XV.

Ex. Find the factors of $4x^2-4x+1$.

The first term is the square of $2x$, and the last term is the square of 1. Hence, if the given expression be of one of the above forms, it can be written

$$(2x)^2-2(2x)+1.$$

This is the case, $\therefore$ the given expression $=(2x-1)^2$.

EXAMPLES. VII. C.

Find the factors of the following expressions, each expression being a perfect square.

1. x^2+2x+1.
2. x^2-6x+9.
3. $4x^2-12xy+9y^2$.
4. x^4-8x^2+16.
5. $a^2-3ab+\frac{9}{4}b^2$.
6. $(a+b)^2-4(a+b)(a-b)+4(a-b)^2$.

108. *EXPRESSIONS OF THE THIRD DEGREE.* The factors of the following expressions of the third degree are known.

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^3+3a^2b+3ab^2+b^3 = (a+b)^3$$
$$a^3-3a^2b+3ab^2-b^3 = (a-b)^3$$
$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

Hence an expression which can be written in any one of these forms can be resolved into factors.

Ex. 1. *Find the factors of* x^3-27.

$$x^3-27=x^3-3^3$$
$$=(x-3)(x^2+3x+3^2)$$
$$=(x-3)(x^2+3x+9).$$

Ex. 2. *Find the factors of* $(2x-y)^3+(x-2y)^3$.

This expression is of the form a^3+b^3, hence, it

$$=\{(2x-y)+(x-2y)\}\{(2x-y)^2-(2x-y)(x-2y)+(x-2y)^2\}.$$

Simplifying, $=\{3x-3y\}\{3x^2-3xy+3y^2\}$

$$=9(x-y)(x^2-xy+y^2).$$

Ex. 3. *Find the factors of* $8a^3+12a^2b+6ab^2+b^3$.

$$8a^3+12a^2b+6ab^2+b^3=(2a)^3+3(2a)^2b+3(2a)b^2+b^3$$
$$=(2a+b)^3.$$

EXAMPLES. VII. D.

Find the factors of the following expressions.

1. x^3-1.
2. a^3+1.
3. $8a^3x^3-1$.
4. $(2x-y)^3-(x-2y)^3$.
5. x^3+3x^2+3x+1.
6. $1-3y+3y^2-y^3$.
7. $x^3+a^3+1-3ax$.

109. *EXPRESSIONS OF THE FOURTH DEGREE.* The factors of the following expressions of the fourth degree are also known.

$$a^4-b^4=(a^2-b^2)(a^2+b^2)=(a-b)(a+b)(a^2+b^2)$$

$$a^4+4a^3b+6a^2b^2+4ab^3+b^4=(a+b)^4$$

$$a^4-4a^3b+6a^2b^2-4ab^3+b^4=(a-b)^4$$

$$a^4+a^2b^2+b^4=(a^2-ab+b^2)(a^2+ab+b^2)$$

Hence an expression which can be written in any one of the above forms can be resolved into factors.

Ex. 1. *Find the factors of* $x^4-(2y)^4$.

$$x^4-(2y)^4=\{x^2-(2y)^2\}\{x^2+(2y)^2\}$$
$$=(x-2y)(x+2y)(x^2+4y^2).$$

Ex. 2. *Find the factors of* $81a^4+144a^2b^2+256b^4$.

$$81a^4+144a^2b^2+256b^4=(3a)^4+(3a)^2(4b)^2+(4b)^4$$
$$=\{(3a)^2-(3a)(4b)+(4b)^2\}\{(3a)^2+(3a)(4b)+(4b)^2\}$$
$$=\{9a^2-12ab+16b^2\}\{9a^2+12ab+16b^2\}.$$

EXAMPLES. VII. E.

Find the factors of the following expressions.

1. $1-a^4$.
2. x^4-16.
3. $(3x+4y)^4-(x+2y)^4$.
4. $1+4a+6a^2+4a^3+a^4$.
5. $y^4-4y^3+6y^2-4y+1$.
6. $(a+1)^4+(a+1)^2(b-1)^2+(b-1)^4$.

110. The beginner may find the following table, containing the factors of some of the forms above considered, useful for reference. A and B may stand for any expressions.

$$A^2-B^2=(A-B)(A+B).$$
$$A^2+B^2\ldots\text{has no real factors,}$$
$$A^3\pm B^3=(A\pm B)(A^2\mp AB+B^2).$$
$$A^4-B^4=(A-B)(A+B)(A^2+B^2).$$
$$A^4+B^4\ldots\text{has no real factors.}$$
$$A^2\pm 2AB+B^2=(A\pm B)^2.$$
$$A^3\pm 3A^2B+3AB^2\pm B^3=(A\pm B)^3.$$

111. It may be noticed that, if we can resolve an expression into factors, the result of dividing it by one of those factors is obvious. This is sometimes useful if the expression be of one of the forms given in the last article.

Ex. Divide $(x-y)^3+(x-z)^3$ *by* $2x-y-z$.

We know that

$$(x-y)^3+(x-z)^3=\{(x-y)+(x-z)\}\{(x-y)^2-(x-y)(x-z)+(x-z)^2\}$$
$$=\{2x-y-z\}\{(x-y)^2-(x-y)(x-z)+(x-z)^2\}.$$

Hence the result of dividing the left-hand side by $2x-y-z$ is

$$(x-y)^2-(x-y)(x-z)+(x-z)^2,$$

which reduces to

$$x^2-xy-xz+y^2-yz+z^2.$$

EXAMPLES. VII. F.

The results of the following examples can be written down by inspection.

1. Divide $(2x)^3+(3y)^3$ by $2x+3y$.
2. Divide $(3a-b)^4-(2a-b)^4$ by $5a-2b$.
3. Divide $(4x+3y-2z)^2-(3x-2y+3z)^2$ by $x+5y-5z$.
4. Divide $(3x+2y+z)^3+(x+2y+3z)^3$ by $x+y+z$.
5. Divide $(2a+3b+4c)^3-(a+b+c)^3$ by $a+2b+3c$.
6. Divide $(ac-bd)^2-(bc-ad)^2$ by $(a-b)(c+d)$.

112. Factors of Quadratic Expressions. We proceed next to the consideration of the factors of quadratic expressions.

A quadratic expression in x is one which is of the second degree in x, and therefore is of the form

$$px^2 + qx + r,$$

where p, q, r are any numbers, positive or negative. The factors of such an expression (if they exist) can always be found. Before proceeding to the general rule, we shall consider a few cases where they are obvious by inspection.

113. We know, by multiplication, that

$$x^2 + (a+b)x + ab = (x+a)(x+b).$$

Hence, if we have to factorize an expression like $x^2 + qx + r$, and we can guess or find two numbers, a and b, such that their sum is q and their product is r, then the required factors will be $x + a$ and $x + b$. For, if $a + b = q$ and $ab = r$, then

$$x^2 + qr + r = x^2 + (a+b)x + ab$$
$$= (x+a)(x+b).$$

Under the same conditions,

$$x^2 - qx + r = x^2 - (a+b)x + ab = (x-a)(x-b).$$

These results can be expressed in one formula, thus:

$$x^2 \pm qx + r = x^2 \pm (a+b)x + ab = (x \pm a)(x \pm b),$$

where, in the ambiguity $\pm$, either the $+$ is to be taken throughout, or the $-$ is to be taken throughout.

Ex. 1. *Find the factors of* $x^2 + 8x + 12$.

We want to find two numbers whose product is 12 and whose sum is 8. Now the only pairs of positive integral factors of 12 are 12 and 1, 6 and 2, 4 and 3; and the only one of these pairs whose sum is 8 is 6 and 2. Hence

$$x^2 + 8x + 12 = (x+6)(x+2).$$

Similarly, $x^2 - 8x + 12 = (x-6)(x-2)$.

Note. An exactly similar rule enables us to determine the factors of $x^2+8xy+12y^2$ and $x^2-8xy+12y^2$, namely,

$$x^2+8xy+12y^2=(x+6y)(x+2y),$$

and $$x^2-8xy+12y^2=(x-6y)(x-2y).$$

Ex. 2. *Find the factors of* $a^2-13a+12$.

We want two numbers whose product is $+12$ and whose sum is -13. These are clearly -1 and -12. Hence we have

$$a^2-13a+12=(a-1)(a-12).$$

Ex. 3. *Find the factors of* $x^2+17xy+60y^2$.

We want two numbers whose product is $+60$ and whose sum is $+17$. These are clearly 5 and 12.

$$\therefore\ x^2+17xy+60y^2=(x+5y)(x+12y).$$

Ex. 4. *Find the factors of* x^2-6x+9.

We want two numbers whose product is $+9$ and whose sum is -6. These are -3 and -3.

$$\therefore\ x^2-6x+9=(x-3)(x-3)=(x-3)^2.$$

This example might have been treated as a known form [Art. 107].

114. In the cases considered in the last article, the term independent of x, which is often called the *absolute* term, (i.e. r), was taken as positive; and thus the numbers a and b were of the same sign. But had r been negative, they would necessarily have been of opposite signs: in such a case, we want to find two numbers, a and b, so that the given expression may be the same as $x^2+(a-b)x-ab$, that is, as $(x+a)(x-b)$. Thus the coefficient of x must be equal to the difference of the two numbers selected.

For example, to find the factors of x^2+x-12, we want to find two numbers whose product is 12 and whose difference is 1. The only pairs of integral factors of 12 are 12 and 1, 6 and 2, 4 and 3, and the only one of these pairs whose difference is 1 is 4 and 3. Hence

$$x^2+x-12=(x+4)(x-3).$$

Similarly, $$x^2-x-12=(x-4)(x+3).$$

If however we regard the rule given in Art. 113 as referring to an algebraical sum and an algebraical product, it will cover all the cases. The only practical points to be observed being that, if the term independent of x be positive, the numbers found must be of the same sign, and the sign of each of them must be the same as the sign of the coefficient of x; but if the term independent of x be negative, the numbers found must be of opposite signs, and the sign of the greater number must be the same as the sign of the coefficient of x.

Ex. 1. *Find the factors of* x^2-x-20.

Here we want to find two numbers whose product is 20 and whose difference is 1; that is, using algebraical notation, whose product is -20 and whose algebraical sum is -1. The fact that the product is negative shews that the numbers must be of opposite signs, and since the sum is negative the one which is arithmetically the greater must be negative. The required numbers are -5 and $+4$.

$$\therefore\ x^2-x-20=(x-5)(x+4).$$

Ex. 2. *Resolve* $1+6a-7a^2$ *into factors.*

Here we want two numbers whose product is -7 and whose sum is $+6$. These are -1 and $+7$.

$$\therefore\ 1+6a-7a^2=(1-a)(1+7a).$$

EXAMPLES. VII. G.

Write down the factors of the following expressions.

1. $x^2+9x+18$.	**2.** $a^2-7a+12$.	**3.** $12+7y+y^2$.
4. $x^2-19x+88$.	**5.** $n^2-14n+49$.	**6.** $y^2-19y+84$.
7. $a^2-5ab+6b^2$.	**8.** $x^2y^2-29xy+54$.	**9.** $1-13l+22l^2$.
10. $22x^4+13x^2+1$.	**11.** $a^2-19ab+88b^2$.	**12.** $a'^2-17a'b'+66b'^2$.
13. $p^2-11p+18$.	**14.** $1-8z-84z^2$.	**15.** x^2+x-2.
16. a^2-a-2.	**17.** $x^2-8x-20$.	**18.** y^2-y-42.
19. $a^2+3ab-28b^2$.	**20.** $1-5m-24m^2$.	**21.** $x^2-9x-36$.
22. b^2+b-42.	**23.** x^4-9x^2-90.	**24.** $a^2+a-110$.
25. x^2-x-42.	**26.** $x^2+6xy-16y^2$.	**27.** $y^2-11y-102$.
28. $x^2-12x-85$.	**29.** $a_1{}^2-16a_1-57$.	**30.** $a^2b^2+11ab-242$.

115. *Method of writing a quadratic expression as the difference of two squares.*

We know that the factors of $a^2 - b^2$ are $(a - b)$ and $(a + b)$. If therefore we can write a quadratic expression as the difference of the squares of two compound expressions we can resolve it into factors.

To write a quadratic expression as the difference of two squares, we try to find some quantity which when added to the terms involving x^2 and x will make a perfect square. Adding that number to the terms involving x^2 and x, and at the same time subtracting it from the rest of the expression, will make no difference to the value of the expression; and if the result of this latter subtraction be a negative quantity, we can apply the rule.

In the following examples the coefficient of x^2 is unity, and in such cases the required number (which we add to the terms involving x^2 and x, and subtract from the other terms) is the square of half the coefficient of x.

Ex. 1.
$$\begin{aligned} x^2+8x+12 &= (x^2+8x+16)+12-16 \\ &= (x+4)^2-4 \\ &= \{(x+4)-2\}\{(x+4)+2\} \\ &= (x+2)(x+6). \end{aligned}$$

Ex. 2.
$$\begin{aligned} x^2+x-20 &= x^2+x+(\tfrac{1}{2})^2-20-\tfrac{1}{4} \\ &= (x+\tfrac{1}{2})^2-(\tfrac{9}{2})^2 \\ &= \{(x+\tfrac{1}{2})-\tfrac{9}{2}\}\{(x+\tfrac{1}{2})+\tfrac{9}{2}\} \\ &= (x-4)(x+5). \end{aligned}$$

**Ex.* 3. Similarly
$$\begin{aligned} x^2+2x-1 &= (x^2+2x+1)-1-1 \\ &= (x+1)^2-2 \\ &= (x+1-\sqrt{2})(x+1+\sqrt{2}). \end{aligned}$$

These factors are rational so far as x is concerned.

*116. If we want the factors of $px^2 + qx + r$, we can write the expression in the form $p\left(x^2+\frac{q}{p}x+\frac{r}{p}\right)$. The factors of the terms in the bracket can then be obtained as described above, and thus the factors of the given expression are known.

For example, to find the factors of $2x^2-5x+2$.

$$\begin{aligned} 2x^2-5x+2 &= 2\{x^2-\tfrac{5}{2}x+1\} \\ &= 2\{x^2-\tfrac{5}{2}x+(\tfrac{5}{4})^2+1-\tfrac{25}{16}\} \\ &= 2\{(x-\tfrac{5}{4})^2-\tfrac{9}{16}\} \\ &= 2\{(x-\tfrac{5}{4})-\tfrac{3}{4}\}\{(x-\tfrac{5}{4})+\tfrac{3}{4}\} \\ &= 2\{x-2\}\{x-\tfrac{1}{2}\} \\ &= (x-2)(2x-1). \end{aligned}$$

*117. Lastly, consider the case of any quadratic expression, such as x^2+qx+r, where q and r stand for any numbers whatever. The form ax^2+bx+c is reducible to this form by the method given in the last article.

The result of adding $(\frac{1}{2}q)^2$ to x^2+qx is $x^2+qx+\frac{1}{4}q^2$, which is the square of $(x+\frac{1}{2}q)^2$.

Hence

$$\begin{aligned} x^2+qx+r &= \{x^2+qx+(\tfrac{1}{2}q)^2\}-\tfrac{1}{4}q^2+r \\ &= (x+\tfrac{1}{2}q)^2-(\tfrac{1}{4}q^2-r) \\ &= \{x+\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}q^2-r}\}\{x+\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}q^2-r}\}. \end{aligned}$$

This is a general formula which includes all the cases given above; and if we write for q and r the values which they have in any particular expression, we obtain its factors.

For example, consider the expression x^2+2x-1 given above; here $q=2$, and $r=1$, and we at once obtain $(x+1-\sqrt{2})$ and $(x+1+\sqrt{2})$ as the factors required.

Similarly, to find the factors of $x^2+\frac{65}{8}x+1$, put $q=\frac{65}{8}$ and $r=1$, and we find that the factors are

$$\left(x+\tfrac{65}{16}-\sqrt{\frac{81\times 49}{16^2}}\right) \text{ and } \left(x+\tfrac{65}{16}+\sqrt{\frac{81\times 49}{16^2}}\right),$$

that is, are $(x+\frac{1}{8})$ and $(x+8)$.

*118. In the analysis given in Art. 117, we took the square root of $\frac{1}{4}q^2-r$. This is always possible (though the result may be a surd whose value cannot be exactly found) provided $\frac{1}{4}q^2 - r$ is a positive quantity. If however $\frac{1}{4}q^2 - r$ be negative, we are landed in a difficulty, since it is impossible, by any rules given in arithmetic, to take the square root of a negative quantity. In fact, a negative quantity, no less than a positive one, when multiplied by itself gives a positive quantity; and hence no real quantity (whether positive or negative) when squared can be a negative quantity. Thus it is only possible to find real factors of $x^2 + qx + r$ when $\frac{1}{4}q^2 > r$, that is, when $q^2 > 4r$.

*119. For example, if we want to find the factors of x^2+2x+2, we put it in the form $(x^2+2x+1)+2-1$, that is, $(x+1)^2+1$; and we can proceed no further.

If we choose to write this result as $(x+1)^2 - a$, where a stands for -1, we can express this as $(x+1-\sqrt{a})(x+1+\sqrt{a})$. The product of these factors is $(x+1)^2 - a$, that is, $(x+1)^2+1$, provided a is supposed to stand for -1.

A quantity like $\sqrt{(-1)}$ is unknown in arithmetic; but if we define it to be a new kind of quantity, which is such that its square is -1, then we can find the factors of x^2+2x+2.

A quantity, such as $\sqrt{(-1)}$, is said to be *imaginary*, and expressions involving such quantities are called imaginary. We shall not concern ourselves in this book with the meaning or the properties of imaginary quantities; but as the student will find them occur constantly in the more advanced parts of mathematics, it is desirable to call his attention to their existence. It may be added that they are not the merely artificial quantities which would be suggested by the arithmetical interpretation of them given above. There are other branches of mathematics, besides the arithmetic with which alone the student is here supposed to be familiar, and a study of those branches enables us to assign a definite intelligible and useful meaning to a quantity like $\sqrt{(-1)}$.

EXAMPLES. VII. H.

Write down the following expressions as the difference of two squares, and thence resolve them into factors.

1. x^2-x-2.	**2.** $y^2-7y+12$.	**3.** $n^2-24n+95$.
4. a^2+a-42.	**5.** $x^2-21x+104$.	**6.** $1-12b-85b^2$.
7. $a^2-11ab-26b^2$.	**8.** x^4+8x^2+7.	**9.** $98a^2-7ab-b^2$.
10. $10x^2+79x-8$.	**11.** $11a^2+75ab-14b^2$.	**12.** $14y^2-25yz+6z^2$.
***13.** x^2+1.	***14.** x^2-4x+5.	***15.** x^2+3x+3.

120. It is always possible to tell whether a quantity, such as $x-a$, is a factor of a given expression by dividing the expression by it; because, if there be no remainder, then $x-a$ is a factor of the expression. The same conclusion can however be obtained at once by the aid of the following theorem.

Theorem. *If an expression involving* x *vanish when* a *is put for* x *wherever* x *occurs, then the expression is exactly divisible by* x − a.

Let us denote the given expression by X. If we divide it by $x-a$, we shall get a certain quotient, which we will denote by Q, and a remainder, which we will denote by R. Hence,

$$X = Q(x-a) + R.$$

Now this equation is true for all values of x, and therefore will be true if we put a for x wherever x occurs in it. This will leave R unaltered; for R does not contain x, since if it did we could continue the division. Also, by hypothesis, the effect of putting a for x in X is to make X vanish. Hence, putting a for x, we obtain

$$0 = 0 + R.$$

Therefore $R = 0$, that is, there is no remainder, and

$$\therefore \quad x-a \text{ is a factor of } X.$$

Note. The reader will remember that we are here confining ourselves to algebraical expressions in which all the numbers denoting powers of x are positive integers, and in which no roots of quantities involving x are involved. Such expressions are said to be *rational* and *integral.* It is only of rational integral algebraical expressions that the above theorem is necessarily true.

121. For example, x^2-7x+6 vanishes if x be put equal to 1. Therefore $x-1$ is a factor of it. Dividing by $x-1$, we find that the other factor is $x-6$.

Again, $x^4-5x^3-7x^2+5x+6$ vanishes if x be put equal to -1, since it then becomes $1+5-7-5+6$, $\therefore$ it is divisible by $x+1$. Dividing by $x+1$, we find that the quotient is x^3-6x^2-x+6. Again, this latter expression vanishes when x is put equal to -1, $\therefore$ $x+1$ is a factor of it. Dividing by $x+1$, we find that the quotient is x^2-7x+6. The factors of this last expression are $x-1$ and $x-6$. Hence the given expression

$$=(x+1)(x+1)(x-1)(x-6)$$
$$=(x+1)^2(x-1)(x-6).$$

122. As another example of this theorem, we will take the following important application to determine whether $x^n \pm a^n$ is divisible by $x+a$ or by $x-a$, where n is any positive integer.

(i) To see if x^n-a^n be divisible by $x-a$ we put $x=a$ in it. It then becomes a^n-a^n which obviously is zero. Therefore x^n-a^n *is always divisible by* $x-a$, whatever integer n may be.

(ii) Again, to see if x^n-a^n be divisible by $x+a$, we put $x=-a$ in it. It then becomes $(-a)^n-a^n$. If n be even, this is equal to a^n-a^n [Art. 59], which is zero. If n be odd, it is equal to $-a^n-a^n$ [Art. 59], which is not zero. Therefore x^n-a^n *is or is not divisible by* $x+a$, *according as* n *is even or odd.*

(iii) Next, let us see whether x^n+a^n is divisible by $x-a$. If it be divisible by $x-a$, it must become equal to zero when x is put equal to a; but if $x=a$ it becomes a^n+a^n, which is not zero. Therefore x^n+a^n *is never divisible by* $x-a$.

(iv) Lastly, to see if x^n+a^n be divisible by $x+a$, we put x equal to $-a$. Now if $x=-a$, x^n+a^n becomes $(-a)^n+a^n$. If n be odd, this is equal to $-a^n+a^n$ [Art. 59], which is zero; but if n be even, it is equal to a^n+a^n, which is not zero. Therefore x^n+a^n *is or is not divisible by* $x+a$, *according as* n *is odd or even.*

EXAMPLES. VII. I.

1. Determine by inspection whether $x-1$ is a factor of each or of any of the following expressions; and if so, find the other factors.

(i) x^3+2x^2-x-2; (ii) x^3-6x+5; (iii) x^3-4x+2.

2. Determine by inspection whether $x+1$ is a factor of each or of any of the following expressions; and if so, find the other factors.

(i) x^3+5x^2+7x+3; (ii) x^3-3x^2+1; (iii) x^3-x^2+4x+6.

3. Each of the following expressions has either $x-1$, $x+1$, or $x-2$ as a factor. Determine all the factors of each expression.

(i) x^3-2x^2-x+2; (ii) x^3-7x^2+6x;

(iii) $x^3+(a-3)x^2+(2-3a)x+2a$; (iv) $x^3+5x^2-4x-20$;

*(v) $x^4+3x^3-3x^2-11x-6$.

MISCELLANEOUS EXAMPLES ON FACTORS. VII. J.

Resolve into the simplest possible factors the following expressions, numbered **1** to **18**.

1. x^3-36x.
2. $(2x+3)^2-(x-3)^2$.
3. $(x-2y)^3+y^3$.
4. $729x^6-y^6$.
5. $(3a^2-b^2)^2-(a^2-3b^2)^2$.
6. $a^2b^2-a^2-b^2+1$.
7. y^3+y^2-y-1.
8. $x^2(x+y)^2-(x^2+y^2)^2$.
9. $9a^2+6ab-4c^2+4cb$.
10. $a^4-ya^3+z^3a-z^3y$.
11. $(a^2+4a)-(b^2-4b)$.
12. $6x^2+7ax-2x-3a^2-3a$.
13. $a^2-b^2+8bc-16c^2$.
14. $a^2+b^2+2(ab+ac+bc)$.
15. $9a^2+6ab-16c^2-8bc$.
16. $72(x^2-1)-17x$.
17. $(a+2b+3c)^2-4(a+b-c)^2$.
18. $(x^2+4x)^2-2(x^2+4x)-15$.

19. Resolve into four factors x^4-6x^2+1.

20. Resolve into factors of the first degree

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$$

21. Prove that $a^4+ma^2b^2+b^4$ is divisible by $a^2+ab\sqrt{2-m}+b^2$, and find the other factor.

22. Find the continued product of

$$x^2+ax+a^2,\ x^2-ax+a^2,\ x^4-a^2x^2+a^4;$$

and deduce (without division) the quotient of $x^8+16a^4x^4+256a^8$ when divided by $x^2+2ax+4a^2$.

23. Find the factors of $\{x^2-(y-z)^2\}\{y^2-(z-x)^2\}\{z^2-(x-y)^2\}$.

***24.** Prove the following identity

$$a^3(c-b)+b^3(a-c)+c^3(b-a)=(a-b)(b-c)(c-a)(a+b+c).$$

Find the factors of the expressions, numbered **25** to **27**.

***25.** $(bc+ca+ab)^2-(b^2c^2+c^2a^2+a^2b^2)$.

***26.** $(a-d)(b^2-c^2)+(b-d)(c^2-a^2)+(c-d)(a^2-b^2)$.

***27.** $(a+b)(a+2b)(a+3b)-9b(a+b)(a+2b)+18b^2(a+b)-6b^3$.

CHAPTER VIII.

HIGHEST COMMON FACTORS.

123. THE **Highest Common Factor** of two or more algebraical expressions is the expression of the highest dimension which will exactly divide each of them. The letters **H. C. F.** are often used as an abbreviation for highest common factor.

Some writers call this factor the *highest common divisor*, and denote it by H.C.D.; others call it the *greatest common measure*, and denote it by G.C.M. The latter name is properly used in arithmetic, but in algebra our symbols stand for any numbers, and we cannot with correctness speak of one symbol being greater or less than another.

124. We shall first consider the rule for finding the H. C. F. of simple expressions and of expressions which can be resolved into factors on inspection, and then the rule for finding the H. C. F. of compound expressions of which the factors are not obvious.

125. **Rule for finding the H. C. F. of Simple Expressions.** The H. C. F. of two or more simple expressions is the product of every factor common to them all, each such factor being raised to the lowest power which it has in any of them.

This is obvious from the definition, and is illustrated in the accompanying examples.

If the expressions have numerical coefficients, it is usual to find the G. C. M. of these coefficients, and prefix it to the H. C. F. as a numerical coefficient.

Ex. 1. *Find the* H.C.F. *of* a^4, a^3, and a^2.

The H.C.F. required must be a power of a, and obviously is a^2.

Ex. 2. *Find the* H.C.F. *of* $4xy^2$ and $2x^2y$.

Here x will divide both quantities, but no power of x higher than the first will divide both. Similarly, y will divide both quantities, but no power of y higher than the first will divide them both. Lastly, the G.C.M. of 4 and 2 is 2. Therefore the H.C.F. of the given expressions is $2xy$.

Ex. 3. *Find the* H.C.F. *of* $6ab^2c^3d^4$, $9a^2b^2cd^2$, and $-3a^6b^3c^2$.

The only letters common to all these expressions are a, b, and c. The first expression involves a to the first power only, and therefore is not divisible by any higher power of a. Similarly, b^2 will divide them all, but any higher power of b will not divide the first or second of them. Similarly, no power of c higher than the first will divide them all. The G.C.M. of 6, 9, and 3 is 3. Therefore the H.C.F. of the given expressions is $3ab^2c$.

EXAMPLES. VIII. A.

Find the H.C.F. of the following quantities.

1. $3ab^2c^3$, $4a^5bc^3$, $6ab^5c^2$, and $5a^4b^2c^7$.
2. $15x^4y^3z^2$, $-20x^3y^2z^4$, $5x^2y^5z^3$, and $-25x^4y^6z^2$.
3. $16p^7q^8r^9$, $12p^4q^7rs^6$, and $-22p^5q^6r^3t^4$.

126. **Rule for finding the H. C. F. of Compound Expressions which can be resolved into factors of the first degree.** This case is covered by the rule just given; since, if we place a factor of the first degree in brackets, we may treat it as if it were a simple quantity.

Hence the required H. C. F. is the product of every factor common to the expressions, each such factor being raised to the lowest power it has in any of them. Numerical coefficients are treated in the manner explained in Art. 125.

Ex. 1. *Find the* H.C.F. *of* $6(a^2-b^2)$ and $9(a-b)^2$.

The given expressions must be resolved into factors of the first degree. They then become respectively

$$6(a-b)(a+b), \quad \text{and } 9(a-b)^2.$$

The H.C.F. required is $3(a-b)$, since $a-b$ is the only factor common to the two expressions, and the G.C.M. of 6 and 9 is 3.

Ex. 2. *Find the* H.C.F. *of*

$$(x^2-a^2)^2,\ (x-a)^3, \text{ and } (x-a)^2(x-b)^2.$$

The first of these expressions can be resolved into factors, $(x-a)^2(x+a)^2$. All the quantities are now expressed as products of factors of the first degree. Their H.C.F. is $(x-a)^2$, since it is clear that the only factor common to them all is a power of $(x-a)$, and no power of it higher than the second will divide them all.

Ex. 3. *Find the* H.C.F. *of*

$$6a^4b+6a^3b^2-36a^2b^3 \text{ and } 9(a^3b^2+4a^2b^3+3ab^4).$$

Here $6a^4b+6a^3b^2-36a^2b^3=6a^2b(a-2b)(a+3b)$,

and $9(a^3b^2+4a^2b^3+3ab^4)=9ab^2(a+b)(a+3b)$.

Hence their H.C.F. is $3ab(a+3b)$.

EXAMPLES. VIII. B.

Find the H.C.F. of the following quantities.

1. x^2-xy and x^2-y^2.
2. a^2-2ab and a^2+2ab.
3. x^4-a^4 and $(x+a)^2$.
4. x^2-4 and x^2+x-6.
5. x^2-3x+2, x^2-1, and x^2+6x-7.
6. $2x^2-3x-2$, and $4x^2+8x+3$.
7. $3a^2b^2(a^3-b^3)$, $6a^4b(a^2-b^2)$, and $9ab^3(a-b)^2$.

127. It is worth noting that if one of the expressions can be resolved into factors of the first degree, we can tell by trial whether each of these factors is also a factor of the other expressions; and we can thus obtain the H.C.F. of the given expressions.

Example. Find the H.C.F. *of*

$$2(a-b)(11a-21b) \text{ and } 209a^3-399a^2b+407ab^2-777b^3.$$

Here we can try whether the factor $a-b$ divides the second expression, and on trial we find that it does not do so; hence it cannot form part of the H.C.F. Next, in the same way trying the factor $11a-21b$, we find that it divides exactly into the second expression. Therefore $11a-21b$ is the required H.C.F.

EXAMPLES. VIII. C.

Find the H.C.F. of the following expressions.

1. $3(a+b)^2$ and $17a^3+19a^2b-5ab^2-7b^3$.
2. $7a(a^2-b^2)$ and $29a^4-60a^3b+12a^2b^2+19ab^3$.
3. $3x^2-3x-6$ and $3x^5-6x^4+3x^3-3x^2-11x+4$.

128. **Rule for finding the H.C.F. of two Compound Expressions.** We confine ourselves, in the first instance, to the case where there are only two compound expressions. To determine their H.C.F. we begin by removing all the simple factors involved. The H.C.F. of the factors so removed must be found, and will be a factor of the final H.C.F. required.

The remaining part of each expression must then be arranged in descending powers of some letter, say x. Next, the expression of higher dimensions (or if both be of the same dimensions, then either of them) must be divided by the other. The remainder, if any, will be of lower dimensions than the expression used as a divisor. This remainder is now used in turn as a divisor and the former divisor as a dividend. Continuing this process until there is no remainder, the last divisor will be the H.C.F. of the two expressions. If there be a final remainder which does not involve x, then there is no factor involving x common to the expressions. The H.C.F. of these expressions (if any), when multiplied by the H.C.F. of the factors removed, will form the H.C.F. required.

129. Before proving the rule, we will illustrate the method by finding the H.C.F. of x^3-2x^2+1 and $2x^3+x^2+4x-7$.

The first step is to divide one expression by the other,

$$\begin{array}{r|l} x^3-2x^2+1 & 2x^3+x^2+4x-7 \; (\,2 \\ & \underline{2x^3-4x^2\qquad +2} \\ & \qquad 5x^2+4x-9 \end{array}$$

We have now to divide x^3-2x^2+1 by $5x^2+4x-9$. To avoid the introduction of fractional coefficients, we will multiply the first of these expressions by 5. This cannot affect the factor we are seeking, since this factor is to be a compound expression common to the two given expressions. *We may therefore at any stage of the process multiply or divide either the divisor or the dividend by any number or simple factor* without altering our result. This will often enable us to simplify the working of an example.

The next step therefore will be as follows,

$$\begin{array}{r|l}
5x^2+4x-9 \;) & 5x^3-10x^2 \qquad +5 \;(\; x \\
 & \underline{5x^3+\;\;4x^2-9x} \\
 & \;\;-14x^2+9x+5
\end{array}$$

Multiply the dividend again by 5 so as to avoid fractions, ∴ we have

$$\begin{array}{r|l}
5x^2+4x-9 \;) & -70x^2+\;\;45x+\;\;25 \;(\; -14 \\
 & \underline{-70x^2-\;\;56x+126} \\
 & \qquad\qquad 101x-101
\end{array}$$

We have now to divide $5x^2+4x-9$ by $101x-101$, that is, by $101(x-1)$. In the same way as before, we may reject the numerical factor 101, since it can form no part of the required H. C. F. The next step will then be,

$$\begin{array}{r|l}
x-1 \;) & 5x^2+4x-9 \;(\; 5x+9 \\
 & \underline{5x^2-5x} \\
 & \qquad 9x-9 \\
 & \qquad \underline{9x-9}
\end{array}$$

There is no remainder, and there is no numerical factor common to the two expressions, therefore $x-1$ (which is the last divisor used) is the H. C. F. required.

130. **Proof of the rule.** The rule enunciated in Art. 128 depends on the principle that any quantity which is a factor of A and B will also be a factor of $mA+nB$, where A and B are two algebraical expressions, and m and n stand for any quantities (except fractions involving the factor in the denominator).

If we suppose the above process to terminate in (say) three steps, the method used will be represented as follows:

$$\begin{array}{l}
A \;)\; B \;(\; Q_1 \\
\qquad \underline{AQ_1} \\
\qquad C \;)\; A \;(\; Q_2 \\
\qquad\qquad \underline{CQ_2} \\
\qquad\qquad D \;)\; C \;(\; Q_3 \\
\qquad\qquad\qquad \underline{DQ_3}
\end{array}$$

That is, Q_1 is the quotient, and C the remainder when B is divided by A. Next, A is to be divided by

C; here, Q_2 represents the quotient, and D the remainder. Next, C is to be divided by D; here, Q_3 represents the quotient, and if the process terminate in three steps, there will be no remainder. Thus we have

$$B = AQ_1 + C, \text{ that is } B - AQ_1 = C \text{......(i)};$$
$$A = CQ_2 + D, \text{ that is } A - CQ_2 = D \text{......(ii)};$$
$$C = DQ_3.$$

From (i), we see that any quantity which divides B and A must divide C. It follows therefore that the required H. C. F. must divide A and C, and therefore from (ii), that it must also divide D. It is therefore a factor of C and D, that is, of DQ_3 and D. Hence it must be D.

131. A direct application of the principle will sometimes enable us to write down at once the H. C. F. of two expressions.

Ex. 1. We might thus have obtained the H. C. F. of the two expressions given in Art. 129 directly by the use of this principle.

The argument in that article was that any quantity which is a factor of $(x^3 - 2x + 1)$ and $(2x^3 + x^2 + 4x - 7)$ is a factor of

$$(2x^3 + x^2 + 4x - 7) - 2(x^3 - 2x + 1),$$

that is, of $5x^2 + 4x - 9$. This was the first step in Art. 129. Now the factors of $5x^2 + 4x - 9$ are readily seen to be $x - 1$ and $5x + 9$. The latter factor does not divide the given expressions, the former factor does; and since there are no other factors of $5x^2 + 4x - 9$, it follows that $x - 1$ is the required H. C. F.

**Ex.* 2. *Find the* H. C. F. *of*

$$ax^{n+1} - (a+1)x^n + 1 \text{ and } x^n - ax + a - 1,$$

n being any positive integer.

Any factor of $ax^{n+1} - (a+1)x^n + 1$ and $x^n - ax + a - 1$ is a factor of $\{ax^{n+1} - (a+1)x^n + 1\} - \{ax - (a+1)\}\{x^n - ax + a - 1\}$. The last expression reduces to $a^2(x-1)^2$. The coefficient a^2 is rejected in the same way as a numerical coefficient, not common to the two expressions, would be rejected. Hence the required H. C. F. is $(x-1)^2$.

132. **H. C. F. of several Compound Expressions.** The rule given in Art. 128 for finding the

H.C.F. of two expressions can be readily extended to include the case of the H. C. F. of several expressions. Let A, B, C, ... stand for the expressions. Find the H. C. F. of two of them, say of A and B: this will include every factor common to A and B. The H. C. F. of this quantity and of C will therefore include every factor common to A, B, and C: and so on.

Example. Find the H.C.F. *of*

$$x^3-7x^2-x+7,\ x^3+3x^2-x-3, \text{ and } x^3-x^2-5x+5.$$

The H.C.F. of the first two expressions can be found either by the method of Art. 126 or by that of Art. 128. The student should perform the analysis, and he will find that x^2-1 is the H.C.F. of these two expressions. He must then determine the H.C.F. of x^2-1 and x^3-x^2-5x+5. This H.C.F. will be found to be $x-1$. Hence $x-1$ is the required H.C.F. of the three given expressions.

Note. The student ought to be able to resolve the above expressions into factors, in which case the answer is at once obvious.

MISCELLANEOUS EXAMPLES ON H.C.F. VIII. D.

[*In working the following examples, the student is recommended to resolve every expression into factors when these are obvious—in which case he will generally be able to write down the answer by inspection.*]

Find the Highest Common Factor of the following expressions numbered **1** to **36.**

1. $4x^2+3x-10$ and $4x^3+7x^2-3x-15$.
2. a^3+a-2 and a^3-3a+2.
3. a^3-2a+4 and a^3+a^2+4.
4. x^3-6x+9 and x^3+4x^2-9.
5. $8x^3-10x^2+7x-2$ and $6x^3-11x^2+8x-2$.
6. $x^3-19x^2+101x-99$ and $x^3-16x^2+72x-81$.
7. $8x^3+18x^2-11x-30$ and $6x^3-11x^2-14x+24$.
8. $9x^3+21x^2-17x+3$ and $3x^3+17x^2+21x-9$.
9. $7x^3-10x^2-7x+10$ and $2x^3-x^2-2x+1$.

10. $7x^3-3x^2-7x+3$ and $2x^3-5x^2-2x+5$.

11. $4x^4-9$ and $2x^4-2x^3+x^2-3x-3$.

12. $x^4+3x^3+12x-16$ and $x^3-13x+12$.

13. x^3-4x^2+2x+3 and $2x^4-x^2-5x-3$.

14. $2x^4-x^3-10x^2-11x+8$ and $2x^3-3x^2-9x+5$.

15. $3x^3-x^2-6x+2$ and x^4+x^3-2x-4.

16. $x^4-15x^2+28x-12$ and $2x^3-15x+14$.

17. $x^4-3x+20$ and $5x^4-3x^3+64$.

18. $x^4-5x^3+5x^2-x-12$ and $x^4-2x^3-12x^2+11x+20$.

19. $3x^4+7x^3-2x^2+x+1$ and $x^4+x^3-2x^2+7x+3$.

20. $2x^4+2x^3-11x^2+13x-3$ and $2x^4-2x^3-5x^2+11x-6$.

21. $2x^4+x^3-4x^2+11x-4$ and $2x^4-7x^3+14x^2-14x+8$.

22. $2x^4-3x^3-2x^2+6x+3$ and $2x^4-7x^3-10x^2+x+2$.

23. $2x^4-7x^3+12x^2-11x+4$ and $3x^4-8x^3+5x^2+2x-2$.

24. $3x^4+5x^3-7x^2+2x+2$ and $2x^4+3x^3-2x^2+12x+5$.

25. x^4+4x^2+16 and $2x^4-x^3+16x-8$.

26. $x^4y-8x^3y^2+17x^2y^3-16xy^4+9y^5$ and
$x^5-9x^4y+26x^3y^2-39x^2y^3+27xy^4$.

27. $x^4y-x^3y^2-15x^2y^3+38xy^4-14y^5$ and
$x^5-7x^4y+21x^3y^2-34x^2y^3+28xy^4$.

28. $6x^3-5x^2+10x-3$ and $6x^3+x^2-10x+3$.

29. $x^3-2ax^2-a^2x+2a^3$ and $x^3+2ax^2-a^2x-2a^3$.

30. $8a^3+16a^2x-40ax^2+16x^3$ and $8a^4-12a^3x-8ax^3+12x^4$.

31. $12a^3-24a^2x-60ax^2-24x^3$ and
$12a^4+18a^3x+12ax^3+18x^4$.

32. $x^3+2x^2y+4xy^2+3y^3$ and $x^4+x^3y+4x^2y^2+xy^3+3y^4$.

33. x^7+1 and $x^{12}+x^3$.

34. $1+x+x^3-x^5$ and $1-x^4-x^6+x^7$.

35. $2x^3+3x^2-11x-6$, $4x^3-16x^2+11x+10$, and
$4x^3+4x^2-29x-15$.

36. $ab(x^2+y^2)+xy(a^2+b^2)$ and $ab(x^3+y^3)+xy(a^2x+b^2y)$.

37. What value of x will make both the quantities in example 4 above vanish?

***38.** What value (other than zero) must be given to a in order that x^3-x-a and x^2+x-a may have a common factor; and what is their highest common factor when a has this value?

39. Find the H.C.F. of the expressions

$$4a^3-47ab^2+7b^3 \text{ and } 6a^3+11a^2b-31ab^2+14b^3;$$

and the G.C.M. of their numerical values when $a=4$, $b=1$.

40. Find the H.C.F. of $2x^3+x^2-x-2$ and $x^5-x^3-2x^2+2x$; and shew that its square is a factor of the latter expression.

***41.** Find a value of x (other than zero) which will make

$$x^5+7x^3-49x^2+8x+2585$$

exactly divisible by x^2-7x+1.

CHAPTER IX.

LOWEST COMMON MULTIPLES.

133. THE **Lowest Common Multiple** of two or more algebraical expressions is the expression of the lowest dimensions which is exactly divisible by each of them. The letters **L.C.M.** are often used as an abbreviation for lowest common multiple.

Some writers call this expression the *least common multiple*. This name is properly used in arithmetic, but in algebra our symbols stand for any numbers, and usually we cannot with correctness speak of one expression being less than another.

134. We shall first consider the rule for finding the L.C.M. of simple expressions, and of expressions which can be resolved into factors of the first degree on inspection, and then the rule for finding the L.C.M. of compound expressions of which the factors are not obvious.

135. **Rule for finding the L.C.M. of Simple Expressions or of Compound Expressions which can be resolved into factors of the first degree.** The L.C.M. of two or more simple expressions (or of expressions which can be resolved into factors of the first degree) is the product of every factor which occurs in them, each such factor being raised to the highest power it has in any of them.

If the expressions have numerical coefficients, it is usual to find the numerical L.C.M. of these coefficients, and prefix it to the required L.C.M. as a numerical coefficient.

Ex. 1. *Find the* L.C.M. *of* x^3, x^2, and x^4.

The required L.C.M. must be some power of x; and, since it is to be exactly divisible by each of the given quantities, it must be x^4.

Ex. 2. *Find the* L.C.M. *of* $3ab^2c^3$, $2a^2b^3c^2$, and $6a^3b^2c$.

The L.C.M. required must contain a^3 as a factor, otherwise a^3b^2c will not divide exactly into it. Similarly, it must contain b^2 and c^3 as factors. The L.C.M. of 3, 2, and 6 is 6. Hence the required L.C.M. is $6a^3b^2c^3$.

Ex. 3. *Find the* L.C.M. *of*
$(x-a)(y-b)^2(z-c)^3$, $(x-a)^2(y-b)^2(z-c)^2$, and $(x-a)^3(y-b)^2(z-c)$.

This, by the same method, will be $(x-a)^3(y-b)^2(z-c)^3$.

Note. The L.C.M. of expressions like the above can be obtained in a manner analogous to that used in arithmetic; but the method described above is so easy of application that it is unnecessary to use a more elaborate process.

EXAMPLES. IX. A.

Find the L.C.M. of the following expressions.

1. $3x^3$, $2x^4$, and x^5.
2. $27abc^2x$, $24a^2bcy$, $60ac^2z$, and $15abcxy$.
3. $5a_1{}^4a_2{}^5a_3{}^6x^2$, $7a_1{}^3a_2{}^6a_4{}^4xy$, and $3a_2{}^4a_3{}^5a_4{}^3y^2$.
4. x^2-y^2, $(x+y)^2$, and $(x-y)^2$.
5. x^2-4y^2, $(x+2y)^2$, and $(x-2y)^3$.
6. $9(x^2-y^2)$, $8(x-y)^2$, and $12(x^3+y^3)$.

136. **Rule for finding the L.C.M. of two Compound Expressions.** The L.C.M. of two compound expressions, whose factors are not obtainable by inspection, is found by dividing one of them by their H.C.F. and multiplying the quotient so obtained by the other of them.

For let A and B stand for the expressions, and let H be their H.C.F. Then A and B are each exactly divisible by H. Suppose the quotients to be respectively a and b. Therefore $A=Ha$, and $B=Hb$. Now a and b have no common factor, because otherwise we could obtain an expression of dimensions higher than H which would divide A and B.

Therefore the L.C.M. of Ha and Hb is Hab.

But $$Hab = (Ha)(b) = Ab,$$

also $$Hab = (Hb)(a) = Ba;$$

and these results are a statement (in algebraical notation) of the rule given at the head of this article.

Note. We can also obtain the L.C.M. of A and B by dividing their product by their H.C.F. For we have

$$\text{the L.C.M. of } A \text{ and } B = Hab = \frac{(Ha)(Hb)}{H} = \frac{AB}{H}.$$

137. **Rule for finding the L.C.M. of several Compound Expressions.** The L.C.M. of three or more algebraical expressions can be obtained by first finding the L.C.M. of two of them; next, finding the L.C.M. of that result and of the third of the given expressions; and so on. The final result is clearly the L.C.M. required.

MISCELLANEOUS EXAMPLES ON L.C.M. IX. B.

[*In working the following examples, the student is recommended to resolve every expression into factors, when possible, and to make use of the method given in Art.* 136 *only when the factors of the expressions cannot be otherwise obtained.*]

Find the L.C.M. of the expressions given in examples 1 to 18.

1. x^2-1, x^2+1, x^3-1, x^3+1, and x^4+x^2+1.
2. (x^3+y^3), $(3x^2+2xy-y^2)$ and $(x^3-x^2y+xy^2)$.
3. $2x^2-5x+3$, $3x^2-x-2$, and $6x^2-5x-6$.
4. $x^{11}+x^4$ and $x^{16}+x^7$.
5. x^3-3x^2+3x-1 and x^3-x^2-x+1.
6. $x^3-15x^2+65x-72$ and $x^3-18x^2+91x-88$.

7. $9x^3 - x - 2$ and $3x^3 - 10x^2 - 7x - 4$.

8. $x^3 - 2x^2 + 2x + 5$ and $x^4 + 5x^3 + 12x^2 + 13x + 5$.

9. $x^4 - x^3 + 8x - 8$ and $x^3 + 4x^2 - 8x + 24$.

10. $x^4 - x^3 + 2x^2 + x + 3$ and $2x^3 - 14x^2 + 26x - 30$.

11. $x^3 - 8x + 3$ and $x^6 + 3x^5 + x + 3$.

12. $x^3 + 2x^2y + 2xy^2 + y^3$ and $x^4 - x^3y - xy^3 + y^4$.

13. $3x^4 + 7x^3 + 13x^2 + 7x + 6$ and $6x^4 + 11x^3 + 10x^2 + 7x + 2$.

14. $x^3 - x^2 - 4x + 4$, $x^3 - 2x^2 - x + 2$, and $x^3 + 2x^2 - x - 2$.

15. $12x^3 + 8x^2 - 27x - 18$, $12x^3 - 8x^2 - 27x + 18$, and $18x^3 + 27x^2 - 8x - 12$.

16. $x(x-y)^3$, $x^2y(x+y)^2$, $xy(x^2-y^2)$, $x^3 - x^2y + xy^2$, $x^2y + xy^2 + y^3$, and $x^6 - y^6$.

17. $x^3(x-y)^2$, $y^2(x+y)^2$, $x^4 - xy^3$, $x^3y + y^4$, $x^2 - xy + y^2$, and $x^2 + xy + y^2$.

18. $x^3y - 3x^2y^2 + 5xy^3 - 6y^4$ and $x^3 - 3x^2y + 2xy^2$.

19. The L.C.M. of two quantities is $x^4 - 5a^2x^2 + 4a^4$, and their H.C.F. is $x^2 - a^2$. One of the quantities is $x^3 - 2ax^2 - a^2x + 2a^3$. Find the other quantity.

20. The H.C.F. of two expressions is $x - 7$, and their L.C.M. is $x^3 - 10x^2 + 11x + 70$. One of the expressions is $x^2 - 5x - 14$. Find the other.

CHAPTER X.

FRACTIONS.

138. **Fraction.** The *fraction*, denoted either by $\frac{a}{b}$ or by a/b, is defined as such a quantity that, if it be multiplied by b, then the resulting product is a.

139. In arithmetic, a fraction is sometimes defined as above. But it is also sometimes otherwise defined by saying that if unity be divided into b equal parts and a of them be taken, the result will be the fraction $\frac{a}{b}$; and it is thence shewn that its value is the quotient of a by b. This latter definition requires that a and b shall be positive integers, but the one given in the last article will allow us to assign any values whatever (simple or compound expressions, positive or negative numbers, integral or fractional numbers, &c.) to our symbols a and b, and is therefore better suited for the purposes of Algebra, since we there suppose the symbols to be unrestricted in value.

140. **Reciprocal Fractions.** Two fractions are said to be *reciprocal* when the numerator of the one is the denominator of the other, and *vice versâ*.

Thus, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocal fractions, and each is said to be the reciprocal of the other.

Similarly, $\frac{a}{1}$ and $\frac{1}{a}$ are reciprocal; thus $\frac{1}{a}$ is the reciprocal of a, and a is the reciprocal of $\frac{1}{a}$.

141. **Properties of Fractions.** We now proceed to consider some of the properties of fractions.

142. *The fraction* $\frac{a}{b}$ *is the quotient of* a *when divided by* b.

The student may perhaps regard this proposition as obvious; but we must deduce it as a consequence of the definition which we have selected.

By definition, $\frac{a}{b} \times b = a.$

Divide each side by b, $\therefore \frac{a}{b} = a \div b.$

Thus $\frac{4x-3}{x-1} = 4 + \frac{1}{x-1}$; for if $4x-3$ be divided by $x-1$, the quotient is 4 and the remainder is 1.

Similarly, $\frac{x^2+ax+b}{x+c} = x+a-c+\frac{c^2-ac+b}{x+c}.$

143. *The value of a fraction is unaltered if both its numerator and its denominator be multiplied by the same quantity.*

Let us denote the fraction $\frac{a}{b}$ by x, and let m be the multiplier. We want to prove that

$$\frac{a}{b} = \frac{ma}{mb}.$$

By definition, [Art. 138], $x \times b = \frac{a}{b} \times b = a.$

Multiply both sides by m, $\therefore xmb = ma.$

Divide both sides by mb, $\therefore x = \frac{ma}{mb}.$

144. Since we may multiply the numerator and denominator of a fraction by the same quantity without altering the value of the fraction, we can (by taking -1 as the multiplier) change the sign of every term in both numerator and denominator.

For example, $\frac{-2a^2+3ab-4b^2}{-3a^2-ab+2b^2} = \frac{2a^2-3ab+4b^2}{3a^2+ab-2b^2}.$

145. Again, if we have a relation like

$$\frac{A}{B}=\frac{C}{D},$$

where A, B, C, D stand for any expressions, we may multiply each side of the equality by BD. We thus obtain

$$AD=CB.$$

This process is known as *multiplying up*.

For example, if $\dfrac{x+1}{x-2}=\dfrac{x-3}{x+4}$, then $(x+1)(x+4)=(x-3)(x-2)$.

146. *The value of a fraction is unaltered if both its numerator and its denominator be divided by the same quantity.*

Let us denote the fraction $\dfrac{a}{b}$ by x, and let m be any quantity. We want to prove that

$$x=\frac{a}{b}=\frac{a/m}{b/m}.$$

By definition [Art. 138], $\qquad xb=a.$

Divide both sides by m, $\qquad \therefore \dfrac{xb}{m}=\dfrac{a}{m}.$

Divide both sides by $\dfrac{b}{m}$, $\qquad \therefore x=\dfrac{\dfrac{a}{m}}{\dfrac{b}{m}}.$

147. *Removal of factors common to the numerator and the denominator of a fraction.* It follows from Art. 146 that if the numerator and the denominator of a fraction have any common factor, and both of them be divided by it, the value of the fraction will be unaltered.

Ex. 1. $$\frac{a^2b}{ab^2}=\frac{(ab)\,a}{(ab)\,b}=\frac{a}{b}.$$

Ex. 2. $$\frac{(x+a)(x-a)}{(x-a)^2}=\frac{x+a}{x-a}.$$

148. Simplification of Fractions. The last article enables us to reduce a fraction to its lowest terms, that is, to express it in such a form that the numerator and the denominator have no common factor.

To do this, we have only to find the H.C.F. of the numerator and the denominator, and divide each of them by it. Wherever any factors common to each are obvious, it will generally be better to divide by them first; but the beginner must remember that **a factor cannot be thus cancelled unless it is a factor of every term in both numerator and denominator.** By thus successively removing factors we can often reduce the fraction to its lowest terms without having to find the H.C.F. of the numerator and the denominator; and even when the complete simplification cannot be thus effected, we can so simplify the fraction that it is comparatively easy to find the H.C.F. of the numerator and the denominator of the fraction thus simplified.

Ex. 1. *Simplify the fraction* $\dfrac{ab^2(a^3-b^3)}{a^2b(a^2-b^2)}$.

Resolving the numerator and denominator into factors [Art. 110], we have

$$\frac{ab^2(a^3-b^3)}{a^2b(a^2-b^2)}=\frac{ab^2(a-b)(a^2+ab+b^2)}{a^2b(a-b)(a+b)}.$$

Cancelling the factors common to both numerator and denominator, the latter form becomes

$$\frac{b(a^2+ab+b^2)}{a(a+b)}.$$

Since $a+b$ and a^2+ab+b^2 have no factor common to them, the fraction is incapable of further reduction.

Ex. 2. *Simplify the fraction* $\dfrac{(x+1)^3-(x-1)^3}{3x^3+x}$.

Resolving the numerator and denominator into factors, we have

$$\frac{(x+1)^3-(x-1)^3}{3x^3+x}=\frac{2(3x^2+1)}{x(3x^2+1)}=\frac{2}{x}.$$

EXAMPLES. X. A.

Reduce the following fractions to their lowest terms.

1. $\dfrac{x^2-14x+13}{x^2-8x-65}$.

2. $\dfrac{x^4-6x^2y^2-16y^4}{x^4-64y^4}$.

3. $\dfrac{x^4-7x^2y^2-18y^4}{x^4-81y^4}$.

4. $\dfrac{(a+1)^4-(a-1)^4}{(a+1)^3-(a-1)^3}$.

5. $\dfrac{10x^4-7x^3+x^2}{4x^4-2x^3-2x+1}$.

6. $\dfrac{x^4-15x^2+28x-12}{2x^3-15x+14}$.

7. $\dfrac{x^3-8x^2y+17xy^2-6y^3}{2x^3-9x^2y+10xy^2-3y^3}$.

8. $\dfrac{a^2b^2-a^2-b^2+1}{ab+a+b+1}$.

9. $\dfrac{12a^4-4a^3b-23a^2b^2+9ab^3-9b^4}{8a^4-14a^2b^2-9b^4}$.

10. $\dfrac{(a^2+b^2-c^2-d^2+2ab+2cd)(a^2+b^2-c^2-d^2-2ab-2cd)}{(a^2+c^2-b^2-d^2+2ac+2bd)(a^2+c^2-b^2-d^2-2ac-2bd)}$.

149. **Addition or Subtraction of Fractions.** If two or more fractions have the same denominator, their sum (or difference) will be equivalent to a fraction having the same denominator, and having for numerator the sum (or difference) of the numerators of the separate fractions.

The method of proof will be sufficiently illustrated by finding the value of $\dfrac{a}{d} \pm \dfrac{b}{d}$.

Multiply by d, $\quad \therefore d\left(\dfrac{a}{d} \pm \dfrac{b}{d}\right) = d\dfrac{a}{d} \pm d\dfrac{b}{d}$

$$= a \pm b.$$

Divide each side by d, $\therefore \dfrac{a}{d} \pm \dfrac{b}{d} = \dfrac{a \pm b}{d}$.

For example, $\dfrac{a^2}{d^2} - \dfrac{b^2}{d^2} + \dfrac{c^2}{d^2} = \dfrac{a^2-b^2+c^2}{d^2}$.

150. If the given fractions have not the same denominator, we must begin by expressing them as equivalent fractions having a common denominator. This is always possible. For the value of a fraction is unaltered if its numerator and its denominator be each multiplied by the same quantity [Art. 143]. If then we find the L.C.M. of the denominators of all the fractions, the denominator of any one of the given fractions will divide exactly into it; and if both the numerator and the denominator of this fraction be multiplied by this quotient, the fraction will be changed to an equivalent fraction having this L.C.M. for its denominator. Thus all the fractions can be reduced to equivalent fractions having a common denominator; and these equivalent fractions can be added by the rule already given in Art. 149.

Note. The converse problem of resolving a given fraction into the sum of a number of simpler fractions (called partial fractions) will not be considered in this book.

Ex. 1. *Find the value of* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$.

Here the L.C.M. of the denominators is abc.

$$\therefore \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{xbc}{abc}+\frac{yca}{abc}+\frac{zab}{abc}$$

$$=\frac{xbc+yca+zab}{abc}.$$

Ex. 2. *Find the value of* $a+\frac{x^2}{a}+\frac{a^2}{x}$.

Here the L.C.M. of the denominators is ax.

$$\therefore a+\frac{x^2}{a}+\frac{a^2}{x}=\frac{a^2x}{ax}+\frac{x^3}{ax}+\frac{a^3}{ax}=\frac{a^2x+x^3+a^3}{ax}.$$

Ex. 3. *Find the value of* $\dfrac{1}{x-a}-\dfrac{1}{x+a}$.

Here the L.C.M. of the denominators is $(x+a)(x-a)$.

$$\therefore \frac{1}{x-a}-\frac{1}{x+a}=\frac{x+a}{(x-a)(x+a)}-\frac{x-a}{(x+a)(x-a)}$$

$$=\frac{(x+a)-(x-a)}{x^2-a^2} \quad\text{.......................}(a)$$

$$=\frac{2a}{x^2-a^2}.$$

Ex. 4. *Simplify*

$$\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}.$$

First, arrange each factor in every term in descending order of a, b, c. The expression then takes the form

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}-\frac{1}{(a-b)(a-c)}.$$

The L.C.M. of the denominators is $(a-b)(b-c)(a-c)$.

$$\therefore \text{ the expression}=\frac{(a-c)-(a-b)-(b-c)}{(a-b)(b-c)(a-c)} \quad\text{.....................}(a)$$

$$=\frac{a-c-a+b-b+c}{(a-b)(b-c)(a-c)}$$

$$=\frac{0}{(a-b)(b-c)(a-c)}$$

$$=0.$$

Ex. 5. *Simplify* $\dfrac{1}{(1-x)^2}+\dfrac{2}{1-x^2}+\dfrac{1}{(1+x)^2}$.

The denominators of the component fractions are $(1-x)^2$, $(1-x)(1+x)$, and $(1+x)^2$. Their L.C.M. is $(1-x)^2(1+x)^2$, which is equal to $(1-x^2)^2$. Hence

$$\text{the given expression}=\frac{(1+x)^2+2(1-x^2)+(1-x)^2}{(1-x^2)^2} \quad\text{...........}(a)$$

$$=\frac{(1+2x+x^2)+(2-2x^2)+(1-2x+x^2)}{(1-x^2)^2}$$

$$=\frac{4}{(1-x^2)^2}.$$

EXAMPLES. X. B.

[*In solving the following examples, the beginner will find it convenient, when he is adding together the numerators of the component fractions, to insert a step in which the separate numerators are enclosed in brackets and added together, before he commences to simplify their sum. This step is shewn in the lines marked* (a) *in the Examples* 3, 4, *and* 5 *above. The lines of dots and the letter* (a) *are only inserted to facilitate this explanation, and form no part of the process of simplification.*]

Simplify the following expressions by reducing them to single fractions in their lowest terms.

1. $\dfrac{a^2+ab+b^2}{a+b}-\dfrac{a^2-ab+b^2}{a-b}$.

2. $\dfrac{a+b}{c}+\dfrac{b+c}{a}+\dfrac{c+a}{b}-\dfrac{(a+b)(b+c)(c+a)}{abc}$.

3. $\dfrac{x-a}{x+a}+\dfrac{a^2+3ax}{a^2-x^2}+\dfrac{x+a}{x-a}$.

4. $\dfrac{x-2a}{x+a}+\dfrac{2(a^2-4ax)}{a^2-x^2}-\dfrac{3a}{x-a}$.

5. $\dfrac{1}{(x-1)(x+1)}-\dfrac{2}{(x+1)(x+3)}+\dfrac{1}{(x+3)(x-1)}$.

6. $\dfrac{3}{x-3}-\dfrac{2}{x-4}-\dfrac{x-6}{(x-2)(x-5)}$.

7. $\dfrac{b+y}{b-y}+\dfrac{4by}{b^2-y^2}+\dfrac{b-y}{b+y}$.

8. $\dfrac{a-3b}{a+2b}-\dfrac{a+3b}{a-2b}-\dfrac{5ab}{a^2+4b^2}$.

9. $\dfrac{1}{x-2y}-\dfrac{1}{2(x+2y)}-\dfrac{x+3y}{2(x^2+4y^2)}$.

10. $\dfrac{1}{x-3}+\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-6x+8}$.

11. $\dfrac{5x+4}{x-2}-\dfrac{3x-2}{x-3}-\dfrac{x^2-2x-17}{x^2-5x+6}$.

12. $\dfrac{a-b}{2(a+b)}+\dfrac{a+b}{2(a-b)}-\dfrac{a^2+b^2}{a^2-b^2}$.

13. $\dfrac{3x^2-8}{x^3-1}-\dfrac{5x+7}{x^2+x+1}+\dfrac{2}{x-1}$

14. $\dfrac{3}{a-b} - \dfrac{ab}{a^3-b^3} + \dfrac{a-b}{a^2+ab+b^2}$.

15. $\dfrac{1}{x-2} + \dfrac{1}{x^2-3x+2} - \dfrac{2}{x^2-4x+3}$.

16. $\dfrac{1}{a-b} - \dfrac{b}{a^2+ab+b^2} - \dfrac{2b^2}{a^3-b^3}$.

17. $\dfrac{1}{x^2-7x+12} + \dfrac{2}{x^2-9x+20} - \dfrac{3}{x^2-8x+15}$.

18. $\dfrac{10x^2}{(1+x^2)(1-4x^2)} + \dfrac{2}{1+x^2} - \dfrac{1}{1-2x}$.

19. $\dfrac{1}{(x-2)(x-1)x(x+1)} + \dfrac{1}{(x-1)x(x+1)(x+2)}$.

20. $\dfrac{1}{(x-2)(x-1)(x+1)} - \dfrac{1}{(x-1)(x+1)(x+2)}$.

21. $\dfrac{a^2-b^2+2bc-c^2}{b^2-c^2+2ca-a^2} - \dfrac{c^2+2ca+a^2-b^2}{b^2+2bc+c^2-a^2}$.

22. $\dfrac{1}{1-x} + \dfrac{1}{1+x} + \dfrac{2}{1-x^2} + \dfrac{2x^2}{(1-x)(1+x^2)} + \dfrac{2x^2}{(1+x)(1+x^2)}$.

23. $\dfrac{x}{(x-y)(x-z)} + \dfrac{y}{(y-z)(y-x)} + \dfrac{z}{(z-x)(z-y)}$.

24. $\dfrac{a+b}{(b-c)(c-a)} + \dfrac{b+c}{(c-a)(a-b)} + \dfrac{a+c}{(a-b)(b-c)}$.

25. $\dfrac{2a-b-c}{(a-b)(a-c)} + \dfrac{2b-c-a}{(b-c)(b-a)} + \dfrac{2c-a-b}{(c-a)(c-b)}$.

26. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-c)(b-a)} + \dfrac{c^2}{(c-a)(c-b)}$.

27. $\dfrac{1}{x(x-y)(x-z)} + \dfrac{1}{y(y-x)(y-z)} + \dfrac{1}{z(z-x)(z-y)}$.

28. $\dfrac{mx^2+nyz}{(x-y)(x-z)} + \dfrac{my^2+nzx}{(y-z)(y-x)} + \dfrac{mz^2+nxy}{(z-x)(z-y)}$.

29. $\dfrac{b-c}{a^2-(b-c)^2} + \dfrac{c+a}{b^2-(c+a)^2} + \dfrac{a+b}{(a+b)^2-c^2}$.

151. **Multiplication by Fractions.** *The product of two or more fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.*

First, consider the case of two fractions such as $\frac{a}{b}$ and $\frac{c}{d}$. To find the value of $\frac{a}{b} \times \frac{c}{d}$.

Let $$x = \frac{a}{b} \times \frac{c}{d}.$$

Multiply by $b \times d$, $$\therefore xbd = \frac{a}{b} \times \frac{c}{d} \times b \times d$$
$$= \left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right)$$
$$= a \times c = ac.$$

Divide each side by bd, $\therefore x = \frac{ac}{bd}$.

Similarly, the product of three fractions may be determined. For

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf}.$$

The method is clearly applicable to any number of fractions.

*152. **Theorem.** *To shew that* $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

As particular cases of the last proposition, we have the relations $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$, and $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. These are instances of the general theorem which we are about to prove.

We have, by the definition of indices [Art. 22],

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \ldots\ldots (n \text{ factors})$$

$$= \frac{a\ a\ a \ldots\ldots (n \text{ factors})}{b\ b\ b \ldots\ldots (n \text{ factors})} \qquad \text{[Art. 151.}$$

$$= \frac{a^n}{b^n}. \qquad \text{[Art. 22.}$$

Ex. 1. *Find the product of* $\frac{7ab}{cd}$ *and* $\frac{3cd}{ab}$.

The product $= \frac{7ab}{cd} \times \frac{3cd}{ab} = 21 \frac{abcd}{abcd} = 21.$

Ex. 2. *Find the product of* $\frac{ax^2+a^3}{x^2-a^2}$ *and* $\frac{x+a}{ax-a^2}$.

The product

$$= \frac{ax^2+a^3}{x^2-a^2} \cdot \frac{x+a}{ax-a^2} = \frac{a(x^2+a^2)(x+a)}{(x-a)(x+a)\,a(x-a)} = \frac{x^2+a^2}{(x-a)^2}.$$

EXAMPLES. X. C.

Reduce the following expressions to their simplest forms.

1. $\frac{x^2-x-6}{x^2+4x+4} \times \frac{x^2-2x-8}{x^2-7x+12}$. 2. $\frac{x^2+y^2}{x^2-xy} \cdot \frac{xy-y^2}{x^4-y^4} \cdot \frac{x}{y}$.

3. $\frac{a^4+a^3b-ab^3-b^4}{a^2-2ab+b^2} \times \frac{a-b}{a^2+ab}$. 4. $\frac{x}{2}\left(\frac{1}{x-y} - \frac{1}{x+y}\right) \cdot \frac{x^2-y^2}{x^2y+xy^2}$.

5. $\left(x - \frac{x^3-2x^2y}{x^2-y^2}\right)\left(y + \frac{xy-y^2}{x+3y}\right) \times \frac{x-y}{2x-y}$.

*6. $\left(\frac{x}{y}\right)^{97} \left(\frac{y}{z}\right)^{99} \left(\frac{z}{x}\right)^{98}$.

153. **Division by Fractions.** *To divide any quantity by a fraction is equivalent to multiplying by the reciprocal of the fraction.*

Let Q be any quantity, integral or fractional, and suppose that we want to divide it by $\frac{a}{b}$. Let $x = Q \div \frac{a}{b}$.

Multiply each side by $\frac{a}{b}$, $\qquad \therefore x\frac{a}{b} = Q.$

Next, multiply each side by $\frac{b}{a}$, $\qquad \therefore x\frac{a}{b}\frac{b}{a} = Q\frac{b}{a}$,

that is, $\qquad x = Q\frac{b}{a}.$

Thus x is equal to the product of Q and the reciprocal of $\frac{a}{b}$.

Ex. 1. *Divide* $\frac{x^3yz^2}{3a^3b^2c}$ *by* $\frac{xy^3z^2}{ab^2c^3}$.

The quotient $= \frac{x^3yz^2}{3a^3b^2c} \times \frac{ab^2c^3}{xy^3z^2}$.

Cancelling the factors common to both numerator and denominator, this

$$= \frac{x^2c^2}{3a^2y^2}.$$

Ex. 2. *Divide* $\frac{ax^3 - a^4}{x^2 - ax + a^2}$ *by* $\frac{x^2 + ax + a^2}{x^3 + a^3}$.

The quotient $= \frac{ax^3 - a^4}{x^2 - ax + a^2} \times \frac{x^3 + a^3}{x^2 + ax + a^2}$.

Resolve into factors $= \frac{a(x-a)(x^2+ax+a^2)}{x^2 - ax + a^2} \cdot \frac{(x+a)(x^2-ax+a^2)}{x^2+ax+a^2}$.

Cancelling the factors common to both numerator and denominator,

$$= a(x-a)(x+a)$$
$$= a(x^2 - a^2).$$

EXAMPLES. X. D.

Reduce to simple fractions in their lowest terms the following expressions which are numbered 1 to 6.

1. $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2} \div \dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$.

2. $\dfrac{x^2+2x-15}{x^2+8x-33} \div \dfrac{x^2+9x+20}{x^2+7x-44}$. 3. $\left(\dfrac{x}{x-a}-\dfrac{a}{x+a}\right) \div \dfrac{x^2+a^2}{x^2-ax}$.

4. $\left(x+\dfrac{xy}{x-y}\right) \times \left(x-\dfrac{xy}{x+y}\right) \div \dfrac{x^2+y^2}{x^2-y^2}$.

5. $\left\{\dfrac{b^2-c^2}{a}+\dfrac{c^2-a^2}{b}+\dfrac{a^2-b^2}{c}\right\} \div \left\{\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c}\right\}$.

*6. $\dfrac{6x^2y^2}{m+n} \div \left[\dfrac{3(m-n)x}{7(r+s)} \div \left\{\dfrac{4(r-s)}{21xy^2} \div \dfrac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$.

*7. Divide $\dfrac{4x^2}{49y^2}-\dfrac{20x}{7y}+\dfrac{178}{7}-\dfrac{15y}{2x}+\dfrac{9y^2}{16x^2}$ by $\dfrac{2x}{7y}-5+\dfrac{3y}{4x}$.

*8. Shew that the first four terms in the quotient of $a+dc$ divided by $a+bc$ are

$$1-(b-d)\frac{c}{a}+(b-d)\frac{bc^2}{a^2}-(b-d)\frac{b^2c^3}{a^3}.$$

154. **Simplification of Expressions involving Fractions.** The rules given above frequently enable us to reduce expressions of considerable complexity to a simpler form. The following are a few examples.

Ex. 1. *Simplify* $\dfrac{2}{\dfrac{1}{x}-\dfrac{1}{y}}+\dfrac{y}{1-\dfrac{y}{x}}-\dfrac{x}{1-\dfrac{x}{y}}$.

This expression $=\dfrac{2}{\dfrac{y-x}{xy}}+\dfrac{y}{\dfrac{x-y}{x}}-\dfrac{x}{\dfrac{y-x}{y}}$

$=2\dfrac{xy}{y-x}+y\dfrac{x}{x-y}-x\dfrac{y}{y-x}$ [Art. 153.

$=-\dfrac{2xy}{x-y}+\dfrac{xy}{x-y}+\dfrac{xy}{x-y}$

$=\dfrac{-2xy+xy+xy}{x-y}$

$=\dfrac{0}{x-y}$

$=0.$

Ex. 2. *Simplify* $\left[\frac{5}{2(x+3)}-\frac{3}{2(x+1)}\right]\times\left[\frac{5}{x-2}-\frac{4}{x-1}\right]$.

This $$=\left[\frac{5(x+1)-3(x+3)}{2(x+1)(x+3)}\right]\left[\frac{5(x-1)-4(x-2)}{(x-2)(x-1)}\right]$$

$$=\frac{2x-4}{2(x+1)(x+3)}\,\frac{x+3}{(x-2)(x-1)}$$

$$=\frac{2(x-2)(x+3)}{2(x+1)(x+3)(x-2)(x-1)}$$

$$=\frac{1}{(x+1)(x-1)}$$

$$=\frac{1}{x^2-1}.$$

Ex. 3. *Simplify* $$\cfrac{a^2}{x+\cfrac{a^2}{x-\cfrac{a^2}{x}}}.$$

A fraction of this form is known as a *Continued Fraction.* The best way of simplifying a fraction of this form is to begin with the last denominator, and work up, as illustrated by the following analysis.

The fraction $$=\cfrac{a^2}{x+\cfrac{a^2}{\cfrac{x^2-a^2}{x}}}$$

$$=\cfrac{a^2}{x+\cfrac{a^2x}{x^2-a^2}}$$

$$=\cfrac{a^2}{\cfrac{x^3-a^2x+a^2x}{x^2-a^2}}$$

$$=\frac{a^2(x^2-a^2)}{x^3}.$$

155. Additional examples on fractions, especially on the comparison of unequal fractions will be found in Arts. 181, 296.

MISCELLANEOUS EXAMPLES ON FRACTIONS. X. E.

1. Find the value of

$$-7ac-\{2c(a-3b)-3a(5c-2b)\}, \text{ when } c=\frac{ab}{a+b}.$$

2. If $x=\dfrac{b-c}{a}$, $y=\dfrac{c-a}{b}$, $z=\dfrac{a-b}{c}$, prove that

$$xyz+x+y+z=0.$$

3. Prove that the sum of two quantities divided by the sum of their reciprocals is equal to the product of the quantities.

4. Shew that the expression $\dfrac{1}{x-a}+\dfrac{1}{x-b}$ has the same value when $x=a+b$ as it has when $x=2ab/(a+b)$.

5. Resolve into their simplest possible factors each of the expressions $x^2-\left(\dfrac{a}{b}-\dfrac{b}{a}\right)x-1$; $x^2-\left(a+\dfrac{1}{a}\right)x+1$.

Reduce to their simplest forms the expressions given in Examples **6** to **48**.

6. $\left[\dfrac{5}{2(x-3)}-\dfrac{3}{2(x-1)}\right]\times\left[\dfrac{5}{x+2}-\dfrac{4}{x+1}\right].$

7. $\left(1-\dfrac{1-x}{1+x}-\dfrac{1-10x^2}{1-x^2}\right)\left(\dfrac{x-1}{4x-1}\right).$

8. $\left(\dfrac{x^2+xy+y^2}{x^2-xy+y^2}-\dfrac{x^2-xy+y^2}{x^2+xy+y^2}\right)\dfrac{x^6-y^6}{x^4-y^4}.$

9. $\left\{\dfrac{a}{a+1}-\dfrac{a-1}{a}\right\}\div\left\{\dfrac{a}{a+1}+\dfrac{a-1}{a}\right\}.$

10. $\left(y-\dfrac{a^2-xy}{y-x}\right)\left(x+\dfrac{a^2-xy}{y-x}\right)+\left(\dfrac{a^2-xy}{y-x}\right)^2.$

11. $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)-\dfrac{(b+c)(c+a)(a+b)}{abc}.$

12. $\dfrac{a}{1+\dfrac{a}{b}}+\dfrac{b}{1+\dfrac{b}{a}}-\dfrac{2}{\dfrac{1}{a}+\dfrac{1}{b}}.$

13. $\left(\dfrac{y}{3x-y}+\dfrac{3x}{3x+y}\right)\times\dfrac{3x-y}{9x^2+y^2}\div\left(\dfrac{1}{3x-y}-\dfrac{1}{3x+y}\right).$

14. $$\frac{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}{\dfrac{x^2}{y^2}-\dfrac{x}{y}+1}\times\frac{\dfrac{x^3}{y}+y^2}{x^2-\dfrac{y^3}{x}}\times\frac{x^2-xy}{y^2+xy}.$$

15. $$\frac{1}{1-x}-\frac{1}{1+x}-\frac{2x}{1+x^2}+\frac{2x^2}{(1-x)(1+x^2)}-\frac{2x^2}{(1+x)(1+x^2)}-\frac{8x^3}{1-x^4}.$$

16. $$\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right).$$

17. $$\frac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}\times\frac{\dfrac{1}{y}-\dfrac{1}{x}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}.$$

18. $$\left(\frac{3a}{3a-2b}-\frac{2b}{3a+2b}\right)\times\frac{3a+2b}{9a^2+4b^2}\div\left(\frac{1}{3a-2b}+\frac{1}{3a+2b}\right).$$

19. $$\frac{(b-c)^2}{(a-b)(a-c)}+\frac{(c-a)^2}{(b-c)(b-a)}+\frac{(a-b)^2}{(c-a)(c-b)}.$$

20. $$\frac{a^2-b^2+2bc-c^2}{b^2-c^2+2ca-a^2}-\frac{c^2+2ca+a^2-b^2}{b^2+2bc+c^2-a^2}.$$

21. $$\frac{(a-b)^2-(b-c)^2}{a^2+ab-bc-c^2}+\frac{(b-c)^2-(c-a)^2}{b^2+bc-ca-a^2}+\frac{(c-a)^2-(a-b)^2}{c^2+ca-ab-b^2}.$$

22. $$\frac{(b+c)(x^2+a^2)}{(c-a)(a-b)}+\frac{(c+a)(x^2+b^2)}{(a-b)(b-c)}+\frac{(a+b)(x^2+c^2)}{(b-c)(c-a)}.$$

23. $$\frac{a-b}{b}+\frac{ab^2}{a^3-b^3}+\frac{a^6}{a^6-b^6}.$$

24. $$\frac{1}{\left(1-\dfrac{b}{a}\right)\left(1-\dfrac{c}{a}\right)}+\frac{1}{\left(1-\dfrac{a}{b}\right)\left(1-\dfrac{c}{b}\right)}+\frac{1}{\left(1-\dfrac{a}{c}\right)\left(1-\dfrac{b}{c}\right)}.$$

25. $$\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a-b)(b-c)(c-a)}.$$

26. $$\frac{\dfrac{a+2}{2a+3}-\dfrac{4a+5}{5a+6}}{\dfrac{2a+3}{3a+4}-\dfrac{3a+4}{4a+5}}.$$

27. $$\frac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}.$$

28. $\dfrac{a+\dfrac{ab}{a-b}}{a^2-\dfrac{2a^2b^2}{a^2+b^2}} \times \dfrac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

29. $\dfrac{\left(\dfrac{a^2}{a^2+b^2}+\dfrac{b^2}{a^2-b^2}\right)(a^2+b^2)^2}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}$.

30. $\dfrac{\left(\dfrac{b^2}{b^2+c^2}+\dfrac{c^2}{b^2-c^2}\right)(b^2+c^2)^2}{\dfrac{b}{b-c}-\dfrac{c}{b+c}}$.

31. $\dfrac{\left(\dfrac{x}{y}+\dfrac{y}{x}+1\right)\left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2}{\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\left(\dfrac{x}{y}+\dfrac{y}{x}\right)}$.

32. $\dfrac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}$.

33. $\dfrac{a^2}{\dfrac{a^2+b^2}{a+b}-b}+\dfrac{b^2}{\dfrac{a^2+b^2}{a+b}-a}$.

34. $\dfrac{x+y+\dfrac{y^2}{x}}{x-y+\dfrac{y^2}{x}} \times \dfrac{x+\dfrac{y^3}{x^2}}{x-\dfrac{y^3}{x^2}} \div \left(\dfrac{x+y}{x-y}\right)^2$.

35. $\dfrac{\dfrac{1+x^2}{1+x}-\dfrac{1+x^3}{1+x^2}}{\dfrac{1+x^3}{1+x^2}-\dfrac{1+x^4}{1+x^3}}$.

36. $\dfrac{a-\dfrac{ab}{a+b}}{a^2+\dfrac{a^2b^2}{a^2-b^2}} \times \dfrac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

37. $\dfrac{\left(2+\dfrac{x}{y}\right)\left(1+\dfrac{y}{x}\right)}{1+\dfrac{x}{y}+\dfrac{y}{x}}+\dfrac{3\left(1+\dfrac{x}{y}\right)}{\dfrac{x^3}{y^3}-1}$.

38. $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$.

39. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}$.

40. $\dfrac{\left(\dfrac{p}{q}+1\right)^2}{\dfrac{p}{q}-\dfrac{q}{p}} \times \dfrac{\dfrac{p^3}{q^3}-1}{\dfrac{p^2}{q^2}+\dfrac{p}{q}+1} \div \dfrac{\dfrac{p^3}{q^3}+1}{\dfrac{p}{q}+\dfrac{q}{p}-1}$.

41. $\dfrac{\dfrac{x}{y}-1+\dfrac{y}{x}}{\dfrac{x}{y}+1+\dfrac{y}{x}} \times \dfrac{\left(\dfrac{x}{y}\right)^3-1}{1+\left(\dfrac{y}{x}\right)^3} \times \dfrac{y^2+xy}{x^2-xy}$.

42. $\dfrac{x^2-(2y-3z)^2}{(x+2y)^2-9z^2}+\dfrac{4y^2-(3z-x)^2}{(2y+3z)^2-x^2}+\dfrac{9z^2-(x-2y)^2}{(3z+x)^2-4y^2}$.

43. $\left\{\dfrac{x}{(x+y)^2}-\dfrac{y}{x^2-y^2}+\dfrac{2y^2}{(x+y)^2(x-y)}\right\} \times \dfrac{x+y}{x-y}$.

44. $\dfrac{x}{(x+y)(x+2y)}+\dfrac{2y}{(x+y)(x+3y)}+\dfrac{x}{(x+2y)(x+3y)}-\dfrac{2}{x+3y}$.

45. $\left\{1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right\}\left\{1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right\}$.

46. $\dfrac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1-\dfrac{c^3}{(a+b)^3}\right\}\left\{1+\dfrac{c}{a+b}\right\}}$.

47. $\dfrac{1-\dfrac{y}{x}+\dfrac{y^2}{x^2}}{1+\dfrac{y}{x}+\dfrac{y^2}{x^2}}\times\dfrac{\dfrac{x^3}{y^3}-1}{\dfrac{x^3}{y^3}+1}\div\dfrac{\left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2}{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2}$.

48. $\left(x-y-\dfrac{1}{x-y+\dfrac{xy}{x-y}}\right)\times\dfrac{x^3+y^3}{x^2-y^2}$.

*49. If $x=\dfrac{b^2+c^2-a^2}{2bc}$ and $y=\dfrac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}$, prove that

$$(x+1)(y+1)=2.$$

*50. If $\dfrac{x-y}{x+y}=a,\ \dfrac{y-z}{y+z}=b,\ \dfrac{z-x}{z+x}=c$, shew that

$$(1-a)(1-b)(1-c)=(1+a)(1+b)(1+c).$$

CHAPTER XI.

SIMPLE EQUATIONS CONTINUED.

156. WE return now to the further discussion of simple equations. We shall give in this chapter some more examples of simple equations—particularly of those involving fractions—and of problems leading to simple equations. We shall follow the procedure explained in Chapter V. without specifically indicating every step.

Ex. 1. *Solve the equation* $\frac{1}{x-1}+\frac{2}{x+1}=\frac{3}{x+2}$.

The L.C.M. of the denominators is $(x-1)(x+1)(x+2)$. Multiplying each side of the equation by this L.C.M., we have

$$(x+1)(x+2)+2(x-1)(x+2)=3(x-1)(x+1).$$

$$\therefore\ x^2+3x+2+2x^2+2x-4=3x^2-3.$$

$$\therefore\ x^2+2x^2-3x^2+3x+2x=-3-2+4.$$

$$\therefore\ 5x=-1.$$

$$\therefore\ x=-\tfrac{1}{5}.$$

Ex. 2. *Solve the equation* $\frac{x+2}{x-1}=\frac{3x+1}{3x-2}$.

The L.C.M. of the denominators is $(x-1)(3x-2)$. Multiplying each side of the equation by this L.C.M., we have

$$(x+2)(3x-2)=(3x+1)(x-1).$$

$$\therefore\ 3x^2+4x-4=3x^2-2x-1.$$

$$\therefore\ 3x^2-3x^2+4x+2x=4-1.$$

$$\therefore\ 6x=3.$$

$$\therefore\ x=\tfrac{1}{2}.$$

Ex. 3. *Solve the equation* $\frac{x-2}{x-4}-\frac{x-1}{x-3}=\frac{x-10}{x-8}-\frac{x-11}{x-9}$.

We will begin by dividing the numerator of each fraction by the denominator [Art. 142]. The equation may therefore be written

$$\left(1+\frac{2}{x-4}\right)-\left(1+\frac{2}{x-3}\right)=\left(1-\frac{2}{x-8}\right)-\left(1-\frac{2}{x-9}\right).$$

$$\therefore \frac{2}{x-4}-\frac{2}{x-3}=-\frac{2}{x-8}+\frac{2}{x-9}.$$

Divide each side by 2; and then, instead of at once bringing all the fractions to a common denominator, simplify each side of the equation by itself,

$$\therefore \frac{(x-3)-(x-4)}{(x-4)(x-3)}=\frac{-(x-9)+(x-8)}{(x-8)(x-9)}.$$

$$\therefore \frac{1}{(x-4)(x-3)}=\frac{1}{(x-8)(x-9)}.$$

Multiply up [Art. 145],

$$\therefore (x-8)(x-9)=(x-4)(x-3).$$

$$\therefore x^2-17x+72=x^2-7x+12.$$

$$\therefore -10x=-60.$$

$$\therefore x=6.$$

Ex. 4. *Solve the equation*

$$\frac{x^4+2x^3+x^2-7x-3}{x^2+3x+5}=\frac{x^4+6x^3+2x^2-16x-4}{x^2+7x+10}.$$

Here the numerators are of higher dimensions than the denominators, and it will be convenient to begin by simplifying each fraction by dividing the numerator by the denominator [Art. 142]. The equation then becomes

$$x^2-x-1+\frac{x+2}{x^2+3x+5}=x^2-x-1+\frac{x+6}{x^2+7x+10}.$$

Simplify, $\therefore \frac{x+2}{x^2+3x+5}=\frac{x+6}{x^2+7x+10}$.

Multiply up, $\therefore (x+2)(x^2+7x+10)=(x+6)(x^2+3x+5)$.

Simplify, $\therefore x^3+9x^2+24x+20=x^3+9x^2+23x+30$.

$$\therefore x=10.$$

Ex. 5. *Solve the equation* $x-\frac{a-b}{a+b}=4\frac{ax}{a+b}+\frac{4a^2(b-ax)}{(a+b)(a^2+b^2)}$.

The L.C.M. of the denominators is $(a+b)(a^2+b^2)$. Multiplying every term by this L.C.M., we have

$$x(a+b)(a^2+b^2)-(a-b)(a^2+b^2)=4ax(a^2+b^2)+4a^2(b-ax).$$

$$\therefore\ x(a+b)(a^2+b^2)-4ax(a^2+b^2)+4a^3x=4a^2b+(a-b)(a^2+b^2).$$

$$\therefore\ x(a^3+a^2b-3ab^2+b^3)=a^3+3a^2b+ab^2-b^3.$$

$$\therefore\ x(a-b)(a^2+2ab-b^2)=(a+b)(a^2+2ab-b^2).$$

$$\therefore\ x(a-b)=a+b.$$

$$\therefore\ x=\frac{a+b}{a-b}.$$

Ex. 6. *A certain number is added both to the numerator and to the denominator of the fraction* $\frac{2}{3}$. *The value of the resulting fraction is* $\frac{8}{13}$. *What was the number added?*

Let the required number be x,

$$\therefore\ \frac{2+x}{3+x}=\frac{8}{13}.$$

Multiply up, $\therefore\ 13(2+x)=8(3+x).$

$$\therefore\ 5x=-2.$$

$$\therefore\ x=-\tfrac{2}{5}.$$

This is negative. Hence the answer is that $\frac{2}{5}$ must be *subtracted* from both the numerator and the denominator of $\frac{2}{3}$ to make it equal to $\frac{8}{13}$. It is easy to verify that this answer satisfies the conditions of the question.

Ex. 7. *Of the candidates in a certain examination* 36 *per cent. failed. If there had been* 11 *more candidates, and if of these* 11 *candidates* 2 *had passed, the total number of failures would have been* 37·5 *per cent. How many candidates were there?*

Let x be the number of candidates. The number of those who failed was 36 per cent. of x, that is, was $\frac{36}{100}x$.

If there had been $(x+11)$ candidates, there would have been 9 more failures, that is, the number of failures would have been $(\frac{36}{100}x+9)$.

This latter number would, by hypothesis, have been 37·5 per cent. of a total of $(x+11)$ candidates. Hence, by the question,

$$\frac{36}{100}x+9=\frac{37\cdot 5}{100}(x+11).$$

$$\therefore\ 36x+900=37\cdot 5x+412\cdot 5.$$

$$\therefore\ 1\cdot 5x=487\cdot 5.$$

$$\therefore\ x=\tfrac{487\cdot 5}{1\cdot 5}=325.$$

Therefore the number of candidates was 325.

Ex. 8. *A can row a mile in* $\frac{3}{4}$ *of a minute less time than B. In a mile race, B gets* 250 *yards start, and loses by* 14 *yards. Find the time A and B take to row a mile, on the assumption that they row at the same pace throughout.*

Suppose that A can row a mile in x minutes.

$\therefore$ B can row a mile in $(x+\frac{3}{4})$ minutes.

Now B rows $(1760-250-14)$ yards, that is, 1496 yards, in the same time that A rows 1760 yards.

But B rows 1760 yards in $(x+\frac{3}{4})$ minutes.

$\therefore$ „ 1 „ $\frac{1}{1760}(x+\frac{3}{4})$ „

$\therefore$ „ 1496 „ $\frac{1496}{1760}(x+\frac{3}{4})$ „

Hence, by the question, $\frac{1496}{1760}(x+\frac{3}{4})=x.$

$$\therefore\ \tfrac{17}{20}(x+\tfrac{3}{4})=x.$$

$$\therefore\ 68x+51=80x.$$

$$\therefore\ x=\tfrac{51}{12}=4\tfrac{1}{4}.$$

Hence A can row a mile in $4\frac{1}{4}$ minutes, that is, in 4 min. 15 secs.

EXAMPLES. XI.

Solve the following equations numbered 1 to 22.

1. $\dfrac{1}{x-1}+\dfrac{1}{x}=\dfrac{2}{x+1}.$

2. $\dfrac{1}{x+2}+\dfrac{2}{x-2}=\dfrac{3}{x-3}.$

3. $\dfrac{5}{x-5}-\dfrac{3}{x+5}=\dfrac{2}{x}.$

4. $\dfrac{4}{2x-5}-\dfrac{3}{3x-7}=\dfrac{1}{x}.$

5. $\dfrac{x+3}{x-2}+\dfrac{3x-3}{x+2}=4.$

6. $\dfrac{x}{x+1}-\dfrac{3x}{x+2}=-2.$

7. $\dfrac{x+3}{x-1}+\dfrac{x+1}{x-3}=2.$

8. $\dfrac{2x+7}{x+1}+\dfrac{3x-5}{x+2}=\dfrac{5x+9}{x+3}.$

9. $\dfrac{x^2+7x-6}{x^2+5x-10}=\dfrac{x+1}{x-1}.$

10. $\dfrac{x^2-6x+10}{x^2+8x+17}-\dfrac{(x-3)^2}{(x+4)^2}=0.$

11. $\frac{a}{x-b}=\frac{b}{x-a}$.

12. $\frac{x-b}{a}+\frac{x+a}{b}=\frac{a+b}{b}$.

13. $\frac{2x}{a-2b}=3+\frac{x}{2a-b}$.

14. $\frac{c}{x-d}-\frac{d}{x+c}=\frac{c-d}{x}$.

15. $\frac{x+a}{x+c}+\frac{x-c}{x-a}=2$.

16. $\frac{c^3+x^3}{c+x}+\frac{d^3+x^3}{d+x}=2(x^2-cd)$.

17. $\frac{x-\frac{1}{a}}{c}+\frac{x-\frac{1}{b}}{a}+\frac{x-\frac{1}{c}}{b}=0$.

18. $\frac{1+\frac{4bx}{a}}{bx}=\frac{\frac{4a}{bx}+1}{a}$.

19. $\frac{x-a}{bc}+\frac{x-b}{ac}+\frac{x-c}{ab}=\frac{2}{a}+\frac{2}{b}+\frac{2}{c}$.

20. $\frac{x^4-2x^3+x^2-7x+4}{x^2+x+3}=\frac{x^4+2x^3-10x^2-6x+9}{x^2+5x+4}$.

21. $\frac{x^3+6x^2+13x+10}{5x^2+26x+38}=\frac{x^2+3x+3}{5x+11}$.

*22. $\frac{b+c-a}{x^2-(b+c)x+bc}+\frac{c+a-b}{x^2-(c+a)x+ac}+\frac{a+b-c}{x^2-(a+b)x+ab}=0$.

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23. A person sells 100 acres more than the third part of his estate, and there remain 2 acres less than the half. What was the extent of his estate?

24. Find a number such that if we divide it by 12 and then divide 12 by the number and add the quotients, we obtain a result which is equal to the quotient of the number increased by 9 when divided by 12.

25. A man's income rises £10 a year. But, owing to a change in the income-tax from 4*d.* to 8*d.* in the pound, he finds his net income the same in two successive years. What was it?

26. A man can walk from *A* to *B* and back in a certain time at the rate of 4 miles an hour. If he walk at the rate of $3\frac{1}{2}$ miles an hour from *A* to *B*, and at the rate of $4\frac{1}{2}$ miles an hour from *B* to *A*, he requires $3\frac{1}{3}$ minutes longer for the double journey. What is the distance *AB*?

27. The time required to walk from *A* to *B*, at a uniform rate of $3\frac{1}{3}$ miles an hour, is 5 minutes less than that required to walk half the distance at 3 miles an hour and the other half at 4 miles an hour: what is the distance?

28. If you have an hour and a half between school and calling over, how far can you go out on a bicycle at 10 miles an hour, and walk back at 4 miles an hour, so as to be just in time?

29. *A* and *B* start together to run a mile race. *A* runs at a uniform speed till he gets within 110 yards of the winning-post, when he increases his speed in the ratio of 88 to 65. *B* runs level with *A* till he gets within 135 yards of the winning-post, when he increases his speed in the ratio of 36 to 25, and wins the race by two seconds. Find the time in which the mile was run by each.

30. A man can row from *A* to *B* up stream in 30 minutes, and can row from *B* to *A* down stream in 25 minutes: find the rate of the stream as compared with the rate at which the boat is rowed in still water.

***31.** If two boats *A* and *B* row in a race at their usual speed, *A* will win by 80 yards; but the day proving unfavourable, *A* only rows at $\frac{5}{8}$ths of its usual speed, while *B* rows at $\frac{9}{10}$ths of its usual speed. *A* wins by 26 yards. Find the length of the course.

32. A tradesman marks his goods at a certain rate per cent. above the cost price, and, deducting 10 per cent. on this marked price for ready money, finds that he makes $21\frac{1}{2}$ per cent. on his outlay. How does he mark his goods?

33. A farmer buys sheep and oxen, paying for an ox 4 times as many shillings as the number of sheep he bought, and for a sheep 20 times as many shillings as the number of oxen he bought. He sells them, gaining as much per cent. on the sheep as he loses per cent. on the oxen, and gains on the whole $\frac{1}{25}$ of his outlay. Determine the gain per cent. on the sheep.

34. The expenses of a tram-car company are fixed. When it only sells threepenny tickets for the whole journey, it loses 10 per cent. It then divides the route into two parts, selling two-penny tickets for each part, thereby gaining 4 per cent. and selling 3300 more tickets every week. How many persons used the cars weekly under the old system?

***35.** A wine-merchant buys spirit; and after mixing water with it, sells the mixture at two shillings per gallon more than he paid for the spirit, making $23\frac{3}{4}$ per cent. on his outlay: if he had used double the quantity of water he would have made $37\frac{1}{2}$ per cent. profit. What proportion of water was there in the mixture?

36. In a certain examination, the number of those who passed was three times the number of those who were rejected. If there had been 14 more candidates and if 4 fewer had been rejected, the number of those who passed would have been four times the number of those rejected. Find the number of candidates.

37. One quarter of the candidates in a certain examination failed. The number of marks required for passing was less by 2 than the average marks obtained by all the candidates, was less by 11 than the average marks of those who passed, and was equal to double the average marks of those who failed. How many marks were required for passing?

38. Find the two times between 5 o'clock and 6 o'clock, when the hands of a watch are separated by 14 minute spaces. Find also the interval between each of these times and the time when the hands are together.

39. A watch has a seconds-hand on the same axis as the other two hands. All three hands are together at 12 o'clock. Find at what time the hour-hand and seconds-hand are next together.

40. A man started for a walk when the hands of his watch were coincident between three and four o'clock. When he finished his walk, the hands were coincident between five and six o'clock. What was the time when he started, and for how long did he walk?

***41.** A clock gains 4 minutes per day, what time should it indicate at noon that it may give the true time at 7·15 in the evening?

***42.** Two clocks are both set right at noon on a certain day; one gains as many minutes in a day as the other loses seconds in an hour; they are first again together after 600 days. What time do they then shew?

***43.** Two anchorites lived at the top of a perpendicular cliff of height h, whose base was distant mh from a neighbouring village. One descended the cliff, and walked to the village; the other flew up a height x, and then flew in a straight line to the village. The distance traversed by each was the same. Find x. (Brahmagupta, circ. 640 A.D.)

***44.** If a men or b boys can just mow m acres of grass in n days, how many boys will be required to assist $a-p$ men so as to enable them to mow $m+p$ acres in $n-p$ days?

CHAPTER XII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

157. **Simultaneous Equations.** Equations, all of which are to be satisfied by the same values of the unknown quantities, are said to be *simultaneous.*

158. **Degree of an Equation.** The *degree* of an equation which contains more than one unknown quantity is the degree of that term in it which is of the highest dimensions in the unknown quantities. (See Art. 26.)

Thus $ax+by=c$ is an equation of the first degree in x and y. But $ax^2+bx+cy^2+d=0$ and $ax+by=cxy$ are each of the second degree in x and y.

159. We shall confine ourselves in this chapter to simultaneous equations of the first degree, where *the number of equations which are given is the same as the number of unknown quantities* contained in them.

Thus, if there be two unknown quantities, there will be two equations; if there be three unknowns, there will be three equations; and so on. Similarly, in the last chapter, only one equation was given when there was only one unknown quantity.

160. **Principle of the Method of Solution.** The principle on which the method of solution depends may be illustrated by the following example, where we are required to solve the two simultaneous equations

$$\left.\begin{aligned} 3x + 2y &= 21, \\ 5x - 7y &= \ \ 4, \end{aligned}\right\}$$

involving two unknown quantities: and it is required to find values of x and y which will satisfy both equations.

Before describing the usual procedure, we may observe that the first equation gives

$$x = \tfrac{1}{3}(21 - 2y) \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Therefore to *every* value of y there is a value of x which satisfies this equation; and we can thus get an infinite number of pairs of solutions of this equation.

Similarly, the second equation can be written

$$x = \tfrac{1}{5}(4 + 7y) \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)};$$

and we can get an infinite number of pairs of solutions of this equation.

Now we want a pair of solutions which shall be the same for both equations, and therefore shall satisfy both equations.

Hence the values of x are to be the same, and therefore we must have

$$\tfrac{1}{3}(21 - 2y) = \tfrac{1}{5}(4 + 7y).$$

The solution of this equation is $y = 3$; and if in either (i) or (ii) y be put equal to 3, we obtain $x = 5$. Hence this pair of solutions is common to both equations.

The above method of solution consists in forming from the given equations another equation in which only one of the unknown quantities enters. This equation can then be solved, and thus the value of one of the unknowns is determined. The other unknown quantity can be obtained in a similar manner, but its value may generally be found more simply by making use of the value of the unknown quantity already determined.

161. The principle on which the method of solution depends is explained in the last Article. We now proceed to describe the usual process for effecting this solution in the most conve-

nient way. We will take the pair of equations given above, namely,

$$3x+2y=21,$$
$$5x-7y=4.$$

162. *First method of solution.* Multiply every term in the first equation by the coefficient of x in the second equation, that is, by 5. Next, multiply every term in the second equation by the coefficient of x in the first equation, that is, by 3. The resulting equations are

$$15x+10y=105,$$
$$15x-21y=\ 12.$$

The coefficients of x in the two equations are now equal. Hence, if we subtract the left-hand side of the second equation from the left-hand side of the first equation, and also the right-hand side of the second equation from the right-hand side of the first equation, we shall eliminate x, that is, shall get rid of the terms involving it. This process gives

$$10y-(-21y)=105-12,$$

that is, $\qquad 31y=\ 93.$

Divide by 31, $\qquad \therefore\ y=\ \ 3.$

To obtain the corresponding value of x, we now substitute this value of y in one of the given equations (say, the first). This gives

$$3x+2\,.\,3=21.$$
$$\therefore\ 3x=21-6=15.$$
$$\therefore\ x=5.$$

Hence $x=5$ and $y=3$ is the required solution. The beginner should verify for himself that these values satisfy *both* the given equations.

163. The object of the process above described is to multiply the equations by such numbers as will make the coefficients of one of the unknowns numerically equal in the two equations. Then, by addition or subtraction, we can eliminate that unknown. In this way, we get a simple equation involving the other unknown which can be solved by the methods given in Chapter V.

Note. We might have obtained the value of x by a method similar to that by which we obtained the value of y. We might also have commenced by finding the value of x, and then deduced the value of y. To do this, we should have multiplied the first equation by 7 and the second by 2. This would have made the coefficients of y equal but of opposite signs. We should then have added (instead of subtracting) corresponding sides of the resulting equations.

The analysis would be as follows.

$$\left.\begin{aligned}21x+14y&=147\\10x-14y&=8\end{aligned}\right\}$$

Add, $\therefore\ 31x=155.$

Divide by 31, $\therefore\ x=5.$

Note. We add or subtract the two equations, according as the two equal coefficients have opposite or the same signs.

164. *Second method of solution.* From the first equation we have

$$3x=21-2y.$$

$$\therefore\ x=\tfrac{1}{3}(21-2y).$$

Substitute this value of x in the second equation,

$$\therefore\ \tfrac{5}{3}(21-2y)-7y=4.$$

Multiply by 3, and simplify,

$$\therefore\ 105-10y-21y=12.$$

Transposing, $\therefore\ -31y=-93.$

$$\therefore\ y=3.$$

The corresponding value of x can be obtained as in Art. 162.

165. The following are additional examples. We may remark that if we can simplify our given equations by adding, subtracting, or in any way combining them, before we eliminate one of the unknowns we shall be at liberty to do so, since our methods of solution depend only on combining our two equations so as to give us a new equation.

Ex. 1. *Solve the equations* $2x-3y+13=7x+6y-235=0.$

The equations are

$2x-3y+13=0$ ....(i),

and $7x+6y-235=0$ ....(ii).

Multiply (i) by 2, $\therefore\ 4x-6y+26=0.$

The other equation is $7x+6y-235=0.$

The coefficients of y in the two equations are now the same; hence, adding, we obtain

$$11x-209=0.$$

Transpose, and divide by 11, $\therefore\ x=19$.
Substitute this value of x in (i), $\therefore\ 38-3y+13=0$,
the solution of which is $y=17$.
Thus the required solution is $x=19$, $y=17$.

Or we might proceed thus.
From (i) $2x=3y-13$.

$$\therefore\ x=\tfrac{1}{2}(3y-13) \quad\text{.....................(iii)}.$$

Substitute this value of x in (ii),

$$\therefore\ \tfrac{7}{2}(3y-13)+6y-235=0.$$

The solution of which is $y=17$.
Hence by (iii), $x=19$.

Ex. 2. *Solve the equations*

$$\left.\begin{aligned} 10x-\ \ 9y&=1,\\ -12x+11y&=1.\end{aligned}\right\}$$

Multiply the first equation by 12 and the second by 10,

$$\therefore\ \left.\begin{aligned} 120x-108y&=12\\ -120x+110y&=10\end{aligned}\right\}.$$

Add, $\therefore\ 2y=22$.

$$\therefore\ y=11.$$

Substitute this value of y in the first of the given equations,

$$\therefore\ 10x-99=1,$$

$$\therefore\ x=10.$$

Hence $x=10$ and $y=11$ are the required roots.

Ex. 3. *Solve the equations*

$$3x+2y-1=2x+5y-18=x+4y-11.$$

The equations are

$$\left.\begin{aligned} 3x+2y-\ \ 1&=x+4y-11\\ \text{and}\quad 2x+5y-18&=x+4y-11\end{aligned}\right\}.$$

These reduce to $\left.\begin{aligned} 2x-2y&=-10\\ x+\ y&=\ \ \ 7\end{aligned}\right\}.$

That is, $\left.\begin{aligned} x-y&=-5\\ x+y&=\ \ 7\end{aligned}\right\}.$

Adding, we get, $2x=2$
Subtracting, we get, $2y=12$

Hence $x=1$, $y=6$ are the required roots.

Ex. 4. *Solve the equations*

$$\left.\begin{aligned} ax + by &= c, \\ a'x + b'y &= c'. \end{aligned}\right\}$$

Multiply the first equation by a' and the second by a,

$$\therefore \left.\begin{aligned} a'ax + a'by &= a'c \\ aa'x + ab'y &= ac' \end{aligned}\right\}.$$

Subtract,
$$\therefore (a'b - ab')y = a'c - ac'.$$

$$\therefore y = \frac{a'c - ac'}{a'b - ab'} \quad \text{.......................} \text{(i)}.$$

Similarly, if we multiply the first equation by b' and the second by b and then subtract, we find that

$$x = \frac{bc' - b'c}{a'b - ab'} \quad \text{.......................} \text{(ii)}.$$

The solution consists of the values given in (i) and (ii).

By giving the proper numerical values to a, b, c, a', b', and c', this example can be made to include the results of all the examples hitherto treated in this chapter. For instance, in the equations worked out in Art. 160, we have $a=3$, $b=2$, $c=21$, $a'=5$, $b'=-7$, and $c'=4$.

Ex. 5. *Solve the equations*

$$\left.\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 10, \\ \frac{3}{x} + \frac{2}{y} &= 10. \end{aligned}\right\}$$

Here we proceed to find $\frac{1}{x}$ and $\frac{1}{y}$, from which x and y can be found.

First method. Eliminate (that is, get rid of) the absolute terms. If these terms had been unequal we should have multiplied the equations by such numbers as would have made these terms equal, and then have subtracted the equations; but in the given equations these terms are equal. Subtract the equations [Art. 165],

$$\therefore -\frac{1}{x} + \frac{1}{y} = 0.$$

$$\therefore y - x = 0.$$

$$\therefore y = x.$$

Substitute this value of y in the first equation,

$$\therefore \frac{2}{x} + \frac{3}{x} = 10.$$

Multiply by x, $\therefore\ 2+3=10x.$

$\therefore\ x=\frac{1}{2}.$

Also we have proved that $y=x$, and $\therefore\ y=\frac{1}{2}.$

Hence the required solution is $x=\frac{1}{2}$, $y=\frac{1}{2}$.

Second method. Or we may proceed as in Art. 162. Multiply each term in the first equation by 3, and each term in the second equation by 2,

$$\therefore\ \left.\begin{aligned}\frac{6}{x}+\frac{9}{y}&=30\\ \frac{6}{x}+\frac{4}{y}&=20\end{aligned}\right\}$$

and

Subtract, $\therefore\ \frac{5}{y}=10.$

Hence $y=\frac{1}{2}.$

Substituting this value of y in one of the given equations, we obtain an equation of which the solution is $x=\frac{1}{2}$.

Third method. Or we may proceed as in Art. 164. The first equation gives

$$\frac{2}{x}=10-\frac{3}{y}.$$

$$\therefore\ \frac{1}{x}=5-\frac{3}{2y}.$$

Substitute this value of $\frac{1}{x}$ in the second equation,

$$\therefore\ 3\left(5-\frac{3}{2y}\right)+\frac{2}{y}=10,$$

the solution of which is $y=\frac{1}{2}$.

The corresponding value of x can then be obtained.

Ex. 6. *Solve the equations*

$$\left.\begin{aligned}\frac{2}{x-1}+\frac{3}{y+1}&=10,\\ \frac{3}{x-1}+\frac{2}{y+1}&=10.\end{aligned}\right\}$$

If we treat $(x-1)$ and $(y+1)$ as the unknown quantities, we have, by Ex. 5,

$$x-1=\frac{1}{2},\ \text{and}\ y+1=\frac{1}{2}.$$

$$\therefore\ x=\frac{3}{2},\ \text{and}\ y=-\frac{1}{2}.$$

Ex. 7. *Solve the equations*

$$\left.\begin{aligned}(x-a)(y+b)-(x+a)(y-b)=2a^2,\\(x-b)(y+a)-(x+b)(y-a)=2b^2.\end{aligned}\right\}$$

Perform the multiplications indicated, and simplify. Hence we have

$$\therefore\ bx-ay=a^2 \quad \text{......................(i),}$$

and

$$ax-by=b^2 \quad \text{......................(ii).}$$

Multiply (i) by a, and (ii) by b,

$$\therefore\ \left.\begin{aligned}abx-a^2y=a^3\\abx-b^2y=b^3\end{aligned}\right\}.$$

Subtract,

$$\therefore\ -(a^2-b^2)\,y=a^3-b^3.$$

$$\therefore\ y=-\frac{a^3-b^3}{a^2-b^2}$$

$$=-\frac{a^2+ab+b^2}{a+b}.$$

Similarly,

$$x=-\frac{ab}{a+b}.$$

166. The method for solving *three simultaneous equations of the first degree involving three unknown quantities* is as follows.

Suppose x, y, z to be the unknown quantities. Take one pair of the equations (say, for example, the first and second), and eliminate z between them; this can be effected in either of the ways described in Arts. 162, 164. The result will be an equation involving x and y only.

Next, take another pair of the equations (say for example the first and third), and eliminate z between them also. The result will be an equation involving x and y only.

We now have two simultaneous equations involving only x and y; and these can be solved in the manner already described.

For example, to solve

$$x - y - 2z = 0 \quad \text{.......................(i)},$$
$$-x + 2y - z = -1 \quad \text{.......................(ii)},$$
$$2x - y - z = 5 \quad \text{.....................(iii)}.$$

First, eliminate z between (i) and (ii). [To effect this, multiply (ii) by 2, and subtract from (i); or find z from (ii), and substitute the value in (i).] The result is

$$3x - 5y = 2 \quad \text{..........................(iv)}.$$

Next, eliminate z between (ii) and (iii). The result is

$$\therefore \; -3x + 3y = -6 \quad \text{.......................(v)}.$$

We have now to solve (iv) and (v).

Add,
$$\therefore \; -2y = -4.$$
$$\therefore \; y = 2.$$

Substituting this value of y in (iv), we obtain

$$3x - 10 = 2.$$
$$\therefore \; x = 4.$$

Lastly, substituting $x=4$ and $y=2$ in (i), we obtain

$$4 - 2 - 2z = 0.$$
$$\therefore \; -2z = -2.$$
$$\therefore \; z = 1.$$

Hence the required solution is $x=4$, $y=2$, $z=1$.

167. It is however clear that, if the elimination of z between the second and third of the given equations had led to the same equation as the elimination of z between the first and second equations, we should have only had one equation between x and y, and this could have been satisfied by an infinite number of pairs of roots. In such a case, the equations are said to be *indeterminate* or *not independent*.

A system of three equations is *indeterminate whenever from two of the equations the third can be deduced.*

Thus the equations

$$\left.\begin{aligned} 3x - 2y + 2z &= -1 \\ 2x - y + z &= 0 \\ x - y + z &= -1 \end{aligned}\right\}$$

are not independent. In fact, if the second be subtracted from the first, we obtain the third. The given equations are thus equivalent to only two independent equations.

*168. The above remarks will enable us to complete the discussion of the solution of the equations $ax+by=c$, $a'x+b'y=c'$ which is given above in Ex. 4 on p. 144, and which includes all possible cases.

If $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, then one of the equations is a multiple of the other, and is therefore deducible from it; that is, the equations are *indeterminate* because they are not independent. In this case, the solution found in Ex. 4 on p. 144 becomes $x=\frac{0}{0}$, $y=\frac{0}{0}$.

Again, if $\frac{a}{a'}=\frac{b}{b'}$, and if the value of each of them be not equal to $\frac{c}{c'}$, the equations cannot be solved, because they are *inconsistent*: in fact, the second equation can $\left(\text{since } \frac{a}{a'}=\frac{b}{b'}\right)$ be written in the form $ax+by=c'\frac{a}{a'}$, and since $c'\frac{a}{a'}$ is (by hypothesis) not equal to c, this is inconsistent with the first equation. In this case, the solution found on p. 144, Ex. 4, becomes $x=\frac{bc'-b'c}{0}$, $y=\frac{a'c-ac'}{0}$.

EXAMPLES. XII.

[*In solving the following questions, the beginner should remember that he must find the values of* each *of the unknown quantities, and also that each root must contain nothing but numbers and known constants.*

In using the method of solution given in Art. 162, *it is generally better to eliminate that unknown which has the simpler coefficients. When the value of one of the unknowns is determined, it is better to select the simpler of the original equations as the one in which it is to be substituted.*]

Solve the following systems of simultaneous equations, numbered 1 to 74.

1. $\left.\begin{aligned} 11y-x&=10,\\ 11x-101y&=110.\end{aligned}\right\}$

2. $\left.\begin{aligned} 3x+2y&=43,\\ x+5y&=49.\end{aligned}\right\}$

3. $\left.\begin{array}{l}9x-41y=40,\\4x-19y=3.\end{array}\right\}$

4. $\left.\begin{array}{l}7x-y=3,\\5x+4y=10.\end{array}\right\}$

5. $\left.\begin{array}{l}7x-2y=1,\\3x+5y=59.\end{array}\right\}$

6. $\left.\begin{array}{l}7x-3y=3,\\5x+7y=25.\end{array}\right\}$

7. $\left.\begin{array}{l}x+17y=53,\\8x+y=19.\end{array}\right\}$

8. $\left.\begin{array}{l}x+5y=49,\\3x-11y=95.\end{array}\right\}$

9. $\left.\begin{array}{l}33x+35y=4,\\55x-55y=-16.\end{array}\right\}$

10. $\left.\begin{array}{l}7x+11y=68,\\9x-4y=33.\end{array}\right\}$

11. $\left.\begin{array}{l}7x-9y=17,\\9x-7y=71.\end{array}\right\}$

12. $\left.\begin{array}{l}14x-9y=5,\\35x+6y=3.\end{array}\right\}$

13. $\left.\begin{array}{l}2bx-ay=ab,\\bx+2ay=3ab.\end{array}\right\}$

14. $\left.\begin{array}{l}3abx+y=8b,\\4abx-3y=15b.\end{array}\right\}$

15. $\left.\begin{array}{l}ax+by=2,\\a^2x+b^2y=a+b.\end{array}\right\}$

16. $\left.\begin{array}{l}ax+by=2,\\ab(x+y)=a+b.\end{array}\right\}$

17. $5x-4y=3x+2y=1.$

18. $x+19y=79,\ 9x+y=31.$

19. $4x-6y-3=7x+2y-4=-2x+3y+24.$

20. $5x+2y-1=3x-y+14=x+19y+6.$

21. $12x+13y=19,\ 13x+12y=31.$

22. $77x-15y=8,\ 14x+21y=6{\cdot}2.$

23. $91x+8y=15,\ 21x-35y=-5{\cdot}75.$

24. $\dfrac{x}{{\cdot}6}-{\cdot}8y=1,\ {\cdot}2\dot{3}x+{\cdot}26y=2.$

25. $\dfrac{x-y}{3}-\dfrac{2y-3x}{6}=8,\ \dfrac{x}{6}-\dfrac{y}{3}=1.$

26. $x+\dfrac{y-11}{3}=15,\ \dfrac{x-5}{4}+2y=36.$

27. $7x+\dfrac{5y+9x}{11}=17,\ 9x+\dfrac{11y+9x}{17}=21.$

28. $\dfrac{x+4}{7}+3y=13,\ 4x+\dfrac{y+3}{7}=13.$

29. $\dfrac{x+3}{7}+4y=13,\ 3x+\dfrac{y+4}{7}=13.$

30. $\frac{1}{2}(x-y)+\frac{1}{3}(x+y)=\frac{25}{6},\ x+y-5=\frac{2}{3}(y-x).$

31. $\frac{x}{7}+\frac{y}{8}=37, \quad \frac{x}{8}+\frac{y}{7}=38.$

32. $\frac{x}{6}+\frac{y}{5}=14, \quad \frac{x}{9}+\frac{y}{2}=24.$

33. $\frac{2x+3y}{9}=\frac{x+y}{3\frac{1}{2}}=2.$

34. $\frac{3x-1}{4}+\frac{7y+2}{6}=2x-y=0.$

35. $3x+\frac{y}{5}=\frac{5x}{2}+\frac{2y}{3}=\frac{3}{4}.$

36. $\frac{3x}{5}+\frac{y}{4}=13, \ \frac{x}{3}-\frac{y}{8}=3.$

37. $\frac{2x}{3}-\frac{3y}{4}=3, \quad \frac{5x}{6}-\frac{9y}{12}=1.$

38. $\frac{3x}{4}-\frac{5y}{8}=-1, \quad \frac{5x}{6}+\frac{y}{4}=14.$

39. $\frac{3x}{4}-\frac{5y}{7}=-1, \ \frac{2x}{3}-\frac{y}{2}=1.$

40. $\frac{x}{3}+\frac{y}{2}=8, \quad \frac{x}{5}+\frac{y}{3}=5.$

41. $\frac{x}{9}+\frac{y}{8}=43, \ \frac{x}{8}+\frac{y}{9}=42.$

42. $\frac{x}{8}+\frac{y}{3}=15, \ \frac{x}{4}-\frac{y}{5}=4.$

43. $\frac{x}{77}+\frac{y}{7}=7, \quad \frac{x}{28}-\frac{y}{3}=4.$

44. $\frac{3}{5}x-\frac{1}{3}y=3, \quad \frac{2}{3}x-\frac{4}{9}y=2.$

45. $\frac{3}{4}x-\frac{1}{3}y=1, \ \frac{2}{3}x+\frac{3}{4}y=26.$

46. $\frac{6}{7}x+\frac{5}{9}y=22, \ \frac{1}{7}x-\frac{1}{2}y=-7.$

47. $\frac{4x}{5}+\frac{5y}{6}=18, \ \frac{x}{2}-\frac{y}{3}=1.$

48. $\frac{2x+1}{5}-\frac{3y+2}{7}=2y-x=0.$

49. $3x+\frac{y}{2}=\frac{15x}{2}+\frac{y}{3}=\frac{11}{6}.$

50. $\frac{x}{3}+\frac{y}{4}=6, \quad y-\frac{x-y}{4}=7.$

51. $\frac{4}{x}-\frac{9}{y}=2, \quad \frac{3}{y}-\frac{1}{x}+\frac{1}{6}=0.$

52. $\frac{9}{x}-\frac{4}{y}=8, \quad \frac{13}{x}+\frac{7}{y}=101.$

53. $\frac{4}{x}-\frac{3}{y}=3, \ \frac{3}{x}+\frac{2}{y}=32.$

54. $\frac{3}{x}-\frac{5}{y}=1, \ \frac{2}{x}+\frac{3}{y}=26.$

55. $\frac{3}{x}+\frac{5}{y}=19, \quad \frac{7}{x}-\frac{9}{y}=3.$

56. $\frac{2}{x}+\frac{7}{y}=29, \ \frac{5}{x}-\frac{6}{y}=2.$

57. $ax-by=0, \ c(x-y)=a-b.$

58. $bx-ay=ax-by=ab.$

59. $ax+by=c^2, \quad a(a+x)=b(b+y).$

60. $x+y=a+b, \quad a(x+a)=b(y+b).$

61. $y=\frac{1}{2}(x+a)+\frac{1}{3}b, \quad x=\frac{1}{2}(y+b)+\frac{1}{3}a.$

62. $(a-b)x-(b-c)y=c-a, \ (c-a)x-(a-b)y=b-c.$

63. $\left.\begin{array}{l}(a+c)x-(b+c)y=(a-b)(a+b+2c),\\(a-c)x-(b-c)y=a^2-b^2.\end{array}\right\}$

64. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2,\ ax-by=a^2+b^2.$

65. $\frac{1}{5}(x+y)=\frac{1}{7}(x-y),\ 3x+17y=2.$

66. $\dfrac{4x+y}{5}+\dfrac{2x+y}{3}=6,\quad \dfrac{5x-1}{7}+\dfrac{x-y}{5}=2.$

67. $\frac{1}{3}(x+y)=\frac{1}{5}(x-y),\quad 3x+11y=4.$

68. $\frac{1}{3}(x-11)+y=18,\quad 2x+\frac{1}{4}(y-13)=29.$

69. $2y+\dfrac{3}{x}-4=5y+\dfrac{12}{x}+2=y-\dfrac{2}{x}+4.$

70. $9\left(\dfrac{1}{3x}+\dfrac{1}{5y}\right)=8\left(\dfrac{1}{5x}+\dfrac{1}{3y}\right)=2.$

71. $\dfrac{a}{x}-\dfrac{b}{y}+1=0,\quad \dfrac{b^2}{x}+\dfrac{a^2}{y}=a-b.$

72. $x+2y+3z=32,\quad 4x-5y+6z=27,\quad 7x+8y-9z=14.$

73. $x=3(y-z),\quad z=4(y-x),\quad x+z=2y-5.$

*74. $a(y+z)=b(z+x)=c(x+y)=d^2.$

*75. Shew that the following system of equations is indeterminate. $2x+y=10,\quad x+z-3y+4=0,\quad 2y+6=3x+z.$

CHAPTER XIII.

PROBLEMS LEADING TO SIMULTANEOUS SIMPLE EQUATIONS.

169. We discussed in Chapters VI. and XI. the solution of problems which could be expressed algebraically by simple equations. We shall now treat of problems involving more than one unknown quantity, and such that the given relations between the unknown quantities can be expressed by two or more algebraical equations of the first degree.

The chief difficulty is the translation of the expression of certain relations from ordinary language into algebraical language. As soon as the given relations are expressed by algebraical equations, the methods given in the last chapter will enable us to solve the equations.

It is however worth remembering that we can only solve a system of equations when we have as many equations as there are unknown quantities. To ensure this, the beginner will generally find it advisable to begin by writing down all the equations without attempting to simplify them, and not to commence the actual solution of the equations until he has seen that he has as many independent equations as unknown quantities. To facilitate this, it is convenient to number the equations as they are written down.

170. The following examples are typical of some of the more common problems.

Ex. 1. *A number of two digits is equal to four times the sum of the digits, and it exceeds the sum of the digits by twenty-seven. What is the number?*

Let x be the digit in the tens' place and y the digit in the units' place. Then the number is $10x+y$.

Then, by the question, $(10x+y)$ is equal to 4 times $(x+y)$;

$$\therefore\ 10x+y=4(x+y)\ \ldots\ldots\ldots\ldots\ldots\ldots\ \text{(i)}.$$

Again, by the question, $(10x+y)$ is greater than $(x+y)$ by 27;

$$\therefore\ 10x+y=x+y+27\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

We have now two equations involving two unknown quantities.

Simplifying these equations, and collecting like terms,

(i) becomes $6x=3y$,

and (ii) becomes $9x=27$.

Hence $x=3$,

and $y=2x=6$.

$\therefore$ the required number is $(3\times10)+6=36$.

Ex. 2. *When unity is added to the numerator and denominator of a certain fraction, the result is* $\frac{3}{2}$, *and when unity is subtracted from its numerator and denominator, the result is* 2. *Find the fraction.*

Let x be the numerator of the fraction and y be its denominator. Then we have, by the question,

$$\frac{x+1}{y+1}=\frac{3}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and

$$\frac{x-1}{y-1}=2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{(ii)}.$$

We have therefore two equations involving two unknown quantities. Multiplying up, they become respectively

$$2x-3y=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

and

$$x-2y=-1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

The equations (iii) and (iv) can be solved by any of the methods given in the last chapter. For example, multiply (iv) by 2, and subtract from (iii),

$$\therefore\ y=3.$$

Substituting $y=3$ in (iv), we obtain $x=5$.

$\therefore$ the required fraction is $\frac{5}{3}$.

Ex. 3. *A and B together can do a piece of work in fifteen days. After working together for six days, A went away, and B finished it by himself twenty-four days after. In what time would A alone do the whole?*

Suppose that A working alone would take x days to do the whole: and that B working alone would take y days.

Then in one day A does $\frac{1}{x}$ of the whole.

$\therefore$ in 15 days A ... $\frac{15}{x}$.................

Similarly, in 15 days B ... $\frac{15}{y}$.................

But in 15 days A and B would finish it;

$$\therefore \frac{15}{x}+\frac{15}{y}=1 \quad\text{.......................(i).}$$

Again in 6 days they had done $\frac{6}{x}+\frac{6}{y}$ of the whole.

In the next 24 days B did $\frac{24}{y}$ of the whole.

This served to finish the whole,

$$\therefore \frac{6}{x}+\frac{6}{y}+\frac{24}{y}=1 \quad\text{.......................(ii).}$$

We have thus two equations involving two unknown quantities. Simplifying, they become respectively

$$\left.\begin{aligned}\frac{15}{x}+\frac{15}{y}&=1\\ \text{and}\qquad \frac{6}{x}+\frac{30}{y}&=1\end{aligned}\right\}.$$

Multiply the first of these by 2, and subtract the second,

$$\therefore \frac{30}{x}-\frac{6}{x}=2-1.$$

$$\therefore \frac{24}{x}=1.$$

$$\therefore x=24.$$

Hence A working alone would take 24 days to do the piece of work.

We are not asked to find y; but if we substitute this value of x in either of the equations, we shall find $y=40$.

Ex. 4. *A mule and a donkey were going to market laden with wheat. The mule said "if you gave me one measure, I should carry twice as much as you; but if I gave you one, we should bear equal burdens." Tell me what were their burdens.*

[This problem is said, by tradition, to have been given by Euclid in his lectures at Alexandria, circ. 280 B.C., and is perhaps one of the earliest problems of this kind ever asked.]

Let x be the number of measures carried by the mule; and let y be the number of measures carried by the donkey.

Then if the mule had received one measure from the donkey, the mule would have carried $x+1$ measures and the donkey would have carried $y-1$ measures; hence, by the question,

$$x+1=2(y-1) \qquad \text{(i)}.$$

But if the mule had given one measure to the donkey, the mule would have carried $x-1$ measures and the donkey would have carried $y+1$ measures; hence, by the question,

$$x-1=y+1 \qquad \text{(ii)}.$$

Thus we have two equations involving two unknown quantities.

From (ii), $x=y+2$ ....(iii).

Substitute this value of x in (i),

$$\therefore\ (y+2)+1=2(y-1).$$

$$\therefore\ y=5.$$

Therefore from (iii), $x=7$.

Therefore the mule carried 7 measures, and the donkey carried 5 measures.

Ex. 5. *A man starts to walk a certain distance in a certain time, but after a time being obliged to diminish his pace by one-fifth, he is four minutes late in reaching his destination. If he had walked another mile before diminishing his pace, he would have been only one minute late. What was his original pace, and how far from the end of his journey did he slacken his speed?*

Suppose that he starts at the rate of v miles an hour; and let the distance from the end of his journey at which he begins to slacken speed be x miles.

Now, if he had not slackened his pace, he would have walked x miles in $\frac{x}{v}$ hours [Art. 101]. But when he diminished his pace by one-fifth, he walked at the rate of $\frac{4}{5}v$ miles an hour, and he therefore took $\frac{x}{\frac{4}{5}v}$ hours to cover x miles. The difference between these times is stated in the question to be 4 minutes, that is, $\frac{1}{15}$ hour;

$$\therefore \frac{x}{\frac{4}{5}v} - \frac{x}{v} = \frac{1}{15} \text{..........................(i)}.$$

Again, if he had walked a distance of $(x-1)$ miles at these rates, the difference of times would by the question have been 1 minute, or $\frac{1}{60}$ of an hour. Hence

$$\frac{x-1}{\frac{4}{5}v} - \frac{x-1}{v} = \frac{1}{60} \text{.......................(ii)}.$$

We have therefore two equations involving two unknown quantities. Simplifying them, they become respectively

$$\left.\begin{aligned} 15x &= 4v \\ \text{and} \qquad 15(x-1) &= v \end{aligned}\right\}.$$

Subtracting, we obtain $\quad 15 = 3v, \therefore v = 5.$

Hence $\quad 15x = 20, \therefore x = \frac{4}{3} = 1\frac{1}{3}.$

Therefore the man started at the rate of 5 miles an hour, and slackened pace at a distance of a mile and a third from the end of his journey.

Ex. 6. *The value of* 112 *coins, consisting of half-crowns, shillings, and sixpences, amounts to* £5. 16*s.* 6*d.* *If there were twice as many sixpences, half as many half-crowns, and three times as many shillings, the total value of the coins would be* £16. 3*s.* *How many coins are there of each kind?*

Let x be the number of half-crowns, y the number of shillings, and z the number of sixpences.

The number of coins was 112;

$$\therefore x+y+z=112 \text{ .... (i)}.$$

The value of the x half-crowns was $\frac{5}{2}x$ shillings, since 2*s.* 6*d.* $=\frac{5}{2}$ shillings.

The value of the y shillings was y shillings.

The value of the z sixpences was $\frac{1}{2}z$ shillings.

The value of the whole was £5. 16*s.* 6*d.*, which is equal to $116\frac{1}{2}$ shillings;

$$\therefore \frac{5}{2}x+y+\frac{1}{2}z=116\frac{1}{2} \text{ .... (ii)}.$$

If there had been $\frac{1}{2}x$ half-crowns, $3y$ shillings, and $2z$ sixpences, the value of the whole would have been £16. 3*s.*, which is equal to 323 shillings;

$$\therefore\ \tfrac{5}{2}\left(\tfrac{1}{2}x\right)+3y+\tfrac{1}{2}(2z)=323 \ \ldots\ldots\ldots\ldots \text{(iii)}.$$

We have therefore three equations involving three unknown quantities.

Getting rid of fractions, these reduce to

$$\left.\begin{aligned} x+\ \ y+\ z&=112 \\ 5x+\ 2y+\ z&=233 \\ 5x+12y+4z&=1292 \end{aligned}\right\}.$$

We shall proceed to eliminate z.

Subtracting the first of these equations from the second, we get

$$4x+y=121 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Multiplying the first by 4, and subtracting from the last, we get

$$x+8y=844 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

We have now two equations involving two unknown quantities. To solve these, multiply (b) by 4, and subtract (a),

$$\therefore\ 31y=3255.$$

$$\therefore\ y=105.$$

Substitute this value of y in (b),

$$\therefore\ x+840=844.$$

$$\therefore\ x=4.$$

Substitute these values of x and y in (i),

$$\therefore\ 4+105+z=112.$$

$$\therefore\ z=3.$$

Therefore there were 4 half-crowns, 105 shillings, and 3 sixpences.

Note. Several of the examples given in Chapters VI. and XI. might have been treated as simultaneous equations. For example, in Art. 99, Ex. 4 (p. 71), we might have supposed that B was x years old, and that A was y years old. We should then have had the simultaneous equations $y=4x$, and $y+20=2(x+20)$.

But in this instance, and in all other similar cases in those chapters, it was not necessary to introduce more than one unknown quantity.

Similarly, many of the problems given in this chapter can be solved by the introduction of only one unknown quantity.

EXAMPLES. XIII.

1. Find two numbers, whose difference is 1, such that the sum of the fifth and the seventh parts of the less is less by 1 than the sum of the fourth and the ninth parts of the greater.

2. A number of two digits is equal to four times the sum of its digits: shew that one digit is double the other.

3. A number of two digits is added to another consisting of the same digits reversed, and the sum is 55. The difference of the numbers is 27. Find the numbers.

4. A number, consisting of two digits, is such that when divided by the sum of its digits, the quotient is 7 and the remainder is 3. The number formed from the given number by reversing the digits is less than the given number by 36. Find the number.

5. A number consists of two digits, one of which is treble the other. Another number is formed from the first by reversing the digits. The difference between the numbers is equal to 18. Find the numbers.

6. In a certain proper fraction, the difference between the numerator and the denominator is 12, and if each be increased by 5 the fraction becomes equal to $\frac{3}{4}$. Find the fraction.

7. What is that fraction which becomes $\frac{2}{3}$ when its numerator is doubled and its denominator is increased by 1, and becomes $\frac{3}{5}$ when its denominator is doubled and its numerator increased by 4?

8. If 1 be added to the numerator of a fraction it becomes equal to $\frac{1}{2}$, if 1 be added to the denominator it becomes equal to $\frac{1}{3}$: find the fraction.

9. The sum of three numbers is 21. The greatest exceeds the least by 4, and the other number is half the sum of the greatest and least: find the numbers.

10. Two workmen save one-third and one-fourth of their daily earnings respectively. At the end of a year their united savings amount to £20, and the total amount of their earnings was £67. 10*s.*? What did each earn during the year?

11. *A* and *B* have £70 between them; but if *A* were to lose half his money, and *B* were to lose one-quarter of his, they would then have only £43. How much has each?

12. *A* has 6 more shillings than *B*, but if *A* give *B* half his money, and then *B* give back to *A* one quarter of his increased sum, they will each have the same sum; find what each had at first.

13. Three rabbits cost 6*s*. 4*d*. The second was worth 10*d*. more than the first, and the third 4*d*. less than the second. Required the cost of each.

14. When *A* and *B* sit down to play, *B* has two-thirds as much money as *A*. After a time *A* has won 15*s*., and then he has twice as much money as *B*. How much had each at first?

15. A person bought 40 yards of cloth for £18, some at 10*s*. a yard, and the rest at 7*s*. 6*d*. a yard; how many yards of each kind did he buy?

16. A grocer buys a quantity of tea at 3*s*. a lb. and also an equal weight of tea at 2*s*. 6*d*. a lb. If he had divided his money equally between the two kinds he would altogether have bought one lb. more of tea. What amount did he buy, and how much did he spend?

17. A father's age is four times that of his elder son, and five times that of his younger son: when the elder son has lived to three times his present age, the father's age will exceed twice that of his younger son by three years. Find their present ages.

18. A boy is one-third the age of his father, and has a brother one-sixth his own age; the ages of all three amount to 50 years. What is the boy's age?

19. Seven years ago, the eldest of three sisters, who is three years older than the next, was twice as old as the youngest, and their united ages were 22. What are their present ages, and how long is it since the age of the eldest sister was equal to the sum of the ages of the two younger ones?

20. A boy is a years old; two years after his birth, his mother was 25 years old; his father is now half as old again as his mother was when the boy was b years old. Find the present ages of his father and mother.

21. In a certain community, consisting of p persons, a per cent. can read and write. Of the males alone, b per cent. can read and write; and of the females alone, c per cent. can read and write. Find the number of males and females in the community.

22. An income of £120 a year is derived from a sum of money invested, partly in a $3\frac{1}{2}$ per cent. stock, and partly in a 4 per cent. stock. If the stock be sold out when the $3\frac{1}{2}$ per cents. are at 108 and the 4 per cents. at 120, the capital realised £3672. How much stock of each kind was there?

23. £1000 is divided between A, B, C, and D: B receives half as much as A; the excess of C's share over D's share is equal to one-third of A's share; if B's share were increased by £100, B would have as much as C and D have between them; find how much each receives.

24. Two trains, 92 feet and 84 feet long respectively, moving with uniform velocities, on parallel rails, in opposite directions, pass each other in $1\frac{1}{2}$ seconds. If they move in the same direction, their respective velocities being the same as before, the faster train passes the other in 6 seconds. Find the rate at which each train moves.

***25.** In a contested election 728 votes were polled; the first of three candidates obtained only 10 less than the second and third obtained together, and he could have given enough votes to the third candidate to have brought him in above the second. How many votes were recorded for each candidate?

26. A person buys 9 oxen and 20 sheep for £230; he sells the oxen at a gain of 25 per cent., and the sheep at a loss of 20 per cent., gaining by the transaction £35. Find the price he gave for each.

27. A man spent £200 in buying heifers and lambs, purchasing in all 20 animals. If the animals that he bought had all been heifers, he would have paid £160 more than he did; if they had all been lambs, he would have paid £160 less than he did. How many were there of each kind?

28. The rent of a farm consists of a fixed money payment, together with the value of 325 quarters of corn, partly wheat and partly barley. When wheat is at 56*s.* per quarter and barley at 40*s.*, the rent is £900; but when wheat is at 48*s.* and barley at 36*s.*, the rent falls to £810. Find the amounts of money, wheat, and barley payable as rent.

29. By investing a certain sum in railway shares paying 3 per cent. per annum, at a certain discount per cent., an income of £315 is obtained. If the same sum be invested in the shares of another railway, paying 4 per cent. per annum at a premium equal to the former discount, the income is increased by £65. Find the amount invested, and the prices of the shares.

30. Divide the number 28 into 4 parts such that, if the first part be increased by 2, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results shall all be equal.

31. Find the number of three digits which is equal to 6 times the number formed by its first and third digits, and also is equal to 12 times the sum of its digits, and also is equal to half the number formed by adding 1 to each of its first and second digits and subtracting 2 from its third digit.

32. Divide 48 into three parts such that the first part shall exceed the second part by 2, and be less than the third part by 3.

33. A certain number of 4 digits is unchanged when the digits are reversed: the sum of the digits is equal to the number formed by the first two digits. What is the number?

34. The number of candidates who entered for a certain examination was a number of four digits. The sum of the digits was 20, the two middle digits were alike, and when the whole number was divided by 100 the remainder was 70. Find the number of candidates.

35. A train travelling from A to C, at a uniform rate of 54 miles an hour, accomplishes the distance in the same time as a train which travels from A to a station B (between A and C) at the uniform rate of 60 miles an hour, and without stopping at B, proceeds from B to C at the uniform rate of 50 miles an hour. If the distance between B and C be 3 miles greater than that between A and B, find the number of miles between each pair of stations.

36. A and B run a long-distance race round a $\frac{1}{4}$-mile course; they start together, but when A has completed his first lap, B is 40 yards behind; in 15 minutes after starting A overtakes B by overlapping him. How long does B take to run a mile?

37. A and B run a mile race. If B receive 12 seconds' start, he is beaten by 44 yards. If B receive 165 yards' start, he arrives at the winning post 10 seconds before A. Find the time in which each can run a mile.

38. A and B work together upon a piece of work for six days, A then leaves off work, and after B has worked alone for two days more, it is found that the work is half done. B then leaves off work, A resumes work, and is joined by a third workman who can do in one day twice the excess of the work done by A in one day over that done by B in one day; and the work is completed in eight days. Find the time in which each workman could do the whole work, and the proportions in which they should be paid.

39. *A* and *B* start simultaneously from two towns to meet one another: *A* travels 2 miles per hour faster than *B*, and they meet in 3 hours: if *B* had travelled one mile per hour slower, and *A* had travelled at two-thirds his previous pace, they would have met in 4 hours. Find the distance between the towns.

40. A man has £100 in sovereigns, half-crowns, and shillings; the weight of the coins is 235 ounces, and the number of coins is 852. How many coins of each kind has he? [Assume, for the purpose of this question, that a sovereign weighs $\frac{1}{4}$ ounce, a half-crown $\frac{1}{2}$ ounce, and a shilling $\frac{1}{5}$ ounce.]

41. *A* has a certain number of coins, all being sixpences; *B* has eight coins less, all being half-crowns; *C* has the same number of coins as *A* and *B* together have, all of them being shillings. The value of *C*'s coins is the same as the sum of the values of *A*'s coins and *B*'s coins. What sums have they respectively?

42. A sum of £2000 is divided into two unequal portions, and these are lent out at rates of interest which differ by one per cent. per annum. It is observed that the income arising from the portion lent at the higher rate of interest, is twice that arising from the other portion; also that the whole income arising from the £2000 is twice that which would be obtained by lending out at the lower rate the portion which is lent at the higher rate. Find the rates of interest, and how much is lent at each rate.

43. The change for a shilling consisted of a certain number of pence together with twice as many half-pence and some three-penny pieces, making 11 coins in all; how many coins were there of each sort?

44. Nineteen shillings' worth of silver consisted of a certain number of florins, twice as many sixpences, and the rest half-crowns, making 13 coins in all; how many of each sort were there?

45. A boy spent his week's pocket money in oranges: if he had got 5 more for his money, each orange would have cost a halfpenny less; if 3 fewer, a halfpenny more. How much was his week's pocket money?

***46.** A gardener took to market two baskets of the same size, one filled with currants and the other with raspberries. He sold all the fruit, except 6 quarts, for 8 shillings, obtaining 4*d.* per quart for the currants and 7*d.* per quart for the raspberries; and then found that the value of the raspberries sold was seven times that of the currants unsold. Find how many quarts each basket contained.

47. A quantity of land, partly arable and partly pasture, is sold at the rate of £60 an acre for the pasture, and £40 an acre for the arable; and the whole sum obtained is £10,000. If the average price per acre had been £50, the sum obtained would have been 10 per cent. greater. How much of the land is arable, and how much is pasture?

48. A and B enter into partnership with unequal sums of money. They agree that each shall receive interest on his capital at the rate of 4 per cent. per annum, and that all remaining profits shall be equally divided. At the end of the first year they receive £428 and £508 respectively; and it is found that A thus receives £77 more than he would have got if they had shared in proportion to the capital invested by each. Find each man's capital.

49. An income of £196 is derived from two sums invested, one at 4 per cent., the other at 7 per cent.: if the interest on the former had been 5 per cent., and on the latter 6 per cent., the income derived would have been £212. Find the sums invested.

50. The gross income of a certain man was £30 more in the second of two particular years than in the first, but in consequence of the income-tax rising from 5*d.* in the pound in the first year to 8*d.* in the pound in the second year, his net income after paying income-tax was unaltered. Find his income in each year.

51. A ship, provisioned for a certain voyage, encounters a storm 16 days after starting, which it is calculated will delay it for 8 days: the daily rations are therefore reduced to $\frac{7}{8}$ of the original quantity: a boat is subsequently picked up containing 9 men without provisions, in consequence of which the daily rations are reduced to $\frac{5}{7}$ of the original quantity. What was the number of men at starting, and how long was the voyage expected to last?

*52. A newspaper proprietor finds that his receipts are reduced by a shilling in the pound through his town customers paying for their penny papers in foreign bronze and his country subscribers in postage stamps. The bronze pence are purchased from him by the Post Office at the rate of 13 for a shilling, and the stamps at a charge of $2\frac{1}{2}$ per cent. The number of his country subscribers exceeds the number of his town customers by a thousand. Find the number of each.

53. A cask contains a certain number of gallons of water, and another contains twice as many gallons of wine: six gallons are drawn from each, and what is drawn from the one cask is then put into the other. If the mixture in each cask be now of the same strength, find the amounts of water and wine which they originally contained.

54. A, B, and C walk from P to Q, a distance of 30 miles; A starts $2\frac{1}{2}$ hours before B, and B $1\frac{1}{2}$ hours before C, and they arrive at Q together. If B had started $\frac{1}{2}$ an hour earlier, he would have passed A 2 hours before A reached Q. Find the rates at which A, B, C walk.

55. A man takes five times as long to run a quarter-mile as he does to run a hundred yards; but if he could run the quarter-mile at the same pace as the hundred yards he would do it in $6\frac{2}{5}$ seconds less time than he does. How long does he take to run each?

56. A journey is performed in a certain time. By travelling 2 miles an hour faster, it would be performed in half-an-hour less time; by travelling 2 miles an hour slower, it would take one hour longer. Find the length of the journey.

57. A person with a sum of £2,596 to invest finds that he can obtain £750 more nominal stock in the 3 per cents. than in the 4 per cents., and 10*s*. greater income: what is the price of each stock?

58. If each of the two greater sides of a rectangle be increased by 3 yards, and each of the two smaller sides be increased by 2 yards its area is doubled: if each of the greater sides be diminished by 3 yards and each of the smaller sides be increased by 2 yards the area is unaltered. Find the sides.

59. A passenger is anxious to reach his destination 21 miles distant at the earliest possible time. Two steamers go there, a slow one which starts at 6 A.M. and a more rapid one which starts at 8 A.M. Sixteen hours after the latter passes the former they are 80 miles apart; but if the slow steamer after being passed had increased its pace by one-fourth, and at the same time the quick one its pace by one-fifth, they would have then been 92 miles apart. Which is the better steamer for him to choose?

CHAPTER XIV.

MISCELLANEOUS PROPOSITIONS AND EXAMPLES.

171. WE here add a few miscellaneous examples which require the application of algebraic processes, but do not come exactly within the range of the preceding chapters.

EXPRESSION OF THEOREMS IN ALGEBRAIC NOTATION.

172. We can often render theorems involving properties of numbers immediately obvious by the use of the notation of algebra. The following are a few simple examples.

Ex. 1. *Shew that if the number* 4 *be divided into any two parts, their product is less than* 4 *by the square of half their difference.*

Let x be one part, $\therefore$ $4-x$ is the other part.

We want to shew that the product $x(4-x)$ is less than 4 by the square of half the difference between x and $4-x$. We want therefore to shew that

$$x(4-x)=4-\left\{\frac{x-(4-x)}{2}\right\}^2,$$

that is,

$$\begin{aligned}4x-x^2&=4-\{x-2\}^2\\&=4-(x^2-4x+4)\\&=-x^2+4x,\end{aligned}$$

a result which is obviously true.

Ex. 2. *Shew that the difference between the squares of two consecutive integers is equal to the sum of the integers.*

Let x be any integer, $\therefore$ the next higher integer is $x+1$. We want to shew that

$$(x+1)^2 - x^2 = (x+1) + x,$$

that is, $$(x^2+2x+1) - x^2 = 2x+1,$$

a result which is obviously true.

EXAMPLES. XIV. A.

[*In this set of examples, the word* number *refers only to* integers.]

1. Prove that, if the sum of two numbers be equal to 3, then their difference is equal to one-third of the difference of their squares.

2. A boy is told to think of a number; to subtract 1 from it; to multiply the result by any number, n; then to subtract 1; and finally to add the original number. Shew that the number he originally thought of is one more than the $(n+1)$th part of the final result.

3. Prove that the sum of the squares of two quantities is equal to twice the sum of the squares of half their sum and half their difference.

4. Express the following statement in algebraical symbols. "The difference of the cubes of any two numbers divided by the difference of the numbers is equal to the sum of the squares of the two numbers together with their product." Prove that the statement is true.

5. From the difference of the squares of two numbers subtract the square of their difference, and shew that the result is a multiple of the smaller of the given numbers.

6. From the difference of the cubes of two quantities subtract the cube of their difference, and shew that the result is a multiple of both the given quantities.

SUBSTITUTIONS.

173. The following examples may be treated as illustrations of substitution, though many of them can be proved (often more elegantly) by other processes.

Ex. 1. *Shew that, if* $x-y=1$, *then* $(x^2-y^2)^2=x^3-y^3+xy$.

Since $x-y=1$, $\quad\therefore\ x=y+1$.

Hence $(x^2-y^2)^2=(x-y)^2(x+y)^2$

$$=1^2\times(y+1+y)^2=(2y+1)^2=4y^2+4y+1.$$

Also $\quad x^3-y^3+xy=(y+1)^3-y^3+y(y+1)=4y^2+4y+1.$

These results are the same, $\therefore\ (x^2-y^2)^2=x^3-y^3+xy$.

Ex. 2. *Find the value of* $a^2-b^2-(a-b)^2$, *when* $a+2b=13$ *and* $2a+b=32$.

The equations $a+2b=13$ and $2a+b=32$ are simultaneous equations. They can be solved by the methods given in the last chapter, and it will be found that $a=17$, $b=-2$.

Hence $\quad a^2-b^2-(a-b)^2=17^2-(-2)^2-(19)^2$

$$=-76.$$

Ex. 3. *Shew that, if* $x=pq$, $y=qr$, $z=rs$, *and* $x+y+z=ps$, then $(x+y)^2+(y+z)^2=(x+z)^2$.

The relation $\quad (x+y)^2+(y+z)^2=(x+z)^2$

is true, if $\quad x^2+2xy+y^2+y^2+2yz+z^2=x^2+2xz+z^2$.

Transpose all the terms to the left-hand side, $\therefore$ it is true if

$$2y^2+2xy+2yz-2xz=0,$$

that is, if $\quad y(y+x+z)-xz=0,$

that is, if $\quad (qr)(ps)-(pq)(rs)=0,$

which is the case.

Ex. 4. *Shew that, if* $x^2=x-2$, then $x^5=-x+6$.

We have $\quad x^2=x-2.$

$$\therefore\ x^4=(x-2)^2$$

$$=x^2-4x+4.$$

But $x^2 = x - 2$, $\quad\therefore\; x^4 = (x-2) - 4x + 4$

$$= -3x + 2.$$

$$\therefore\; x^5 = x \times x^4 = x(-3x+2)$$

$$= -3x^2 + 2x.$$

But $x^2 = x - 2$, $\quad\therefore\; x^5 = -3(x-2) + 2x = -x + 6$,

which was to be proved.

Ex. 5. *If* $x + y + z = 0$, *prove that*

$$x(x^2 - yz) + y(y^2 - zx) + z(z^2 - xy) = 0.$$

Multiplying out, we have

$$x(x^2 - yz) + y(y^2 - zx) + z(z^2 - xy)$$

$$= x^3 + y^3 + z^3 - 3xyz$$

$$= (x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz) \qquad \text{[Art. 108.]}$$

$$\therefore\; = 0 \times (x^2 + y^2 + z^2 - xy - xz - yz)$$

$$= 0.$$

EXAMPLES. XIV. B.

1. Find the value of $(a-c)(a+c) - (a+c)^2$, when $3a + 2c = 45$ and $3c + 2a = 15$.

2. If $a + b = 1$, prove that $(a^2 - b^2)^2 = a^3 + b^3 - ab$.

3. Shew that, if $x^2 = x + 1$, then $x^5 = 5x + 3$.

4. Shew that the expression $x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2)$ is not changed by adding the same quantity to x, to y, and to z.

5. If $a + \frac{1}{b} = 1$, and if $abc + 1 = 0$, then will $b + \frac{1}{c} = 1$.

6. If $a + b + c = 0$, prove that $a^3(b-c) + b^3(c-a) + c^3(a-b) = 0$.

7. Shew that, if $a + b + c = 0$, then

$$ab(a+b) + bc(b+c) + ca(c+a) + 3abc = 0.$$

8. If $x + y + z = 0$, prove that $(x^3 + y^3 + z^3)^3 = 27x^3y^3z^3$.

9. Prove that, if $s = a + b + c$, then

$$(as + bc)(bs + ca)(cs + ab) = (b+c)^2(c+a)^2(a+b)^2.$$

10. Shew that, if $\frac{1}{a-b} + \frac{1}{a-c} = \frac{2}{a}$, then $\frac{1}{b} + \frac{1}{c} = \frac{2}{a}$.

ELIMINATION.

174. We solved two simultaneous equations of the first degree between two unknown quantities by eliminating one of them—that is, we combined our equations in such a way as to get rid of one of the quantities involved [see Art. 160]. In general we can eliminate one quantity between two equations, two quantities between three equations, three quantities between four equations, and so on; in each case obtaining one equation as the result.

The following are a few simple examples.

Ex. 1. *Eliminate* y *between the equations* $y-x+1=0$ *and* $y^2+2x^2=3$.

The first equation gives $y=x-1$. Substituting this value for y in the second equation wherever y occurs there, we have

$$(x-1)^2+2x^2=3.$$

$$\therefore\ 3x^2-2x-2=0,$$

an equation in which y does not occur, and which is therefore the required result.

Note. Wherever we are given two equations involving a certain quantity (such as y in the above example), and one of these equations is of the first degree in it, we can always eliminate the quantity by finding its value from the equation which is of the first degree in it, and substituting this value in the other equation.

Ex. 2. *Eliminate* x *between the equations* $ax^2+by^2+c=0$ *and* $bx^2-ay^2+d=0$.

The second equation gives $bx^2=ay^2-d$.

$$\therefore\ x^2=\frac{ay^2-d}{b}.$$

Substitute this value of x^2 in the first equation,

$$\therefore\ a\left(\frac{ay^2-d}{b}\right)+by^2+c=0.$$

Simplifying, this reduces to $a^2y^2-ad+b^2y^2+bc=0$.

$$\therefore\ (a^2+b^2)\,y^2-ad+bc=0,$$

an equation which is independent of x, and is therefore the required result.

EXAMPLES. XIV. C.

Eliminate x between the equations given in examples **1, 2, 3.**

1. $2x+3y-1=0$ and $3x-2y+1=0$.

2. $ax-by=0$ and $x^2+xy+y^2=ab$.

3. $x+a-1=0$ and $x^2-a^2+1=0$.

Eliminate y between the equations given in examples **4, 5, 6.**

4. $y-x=0$ and $2x^2+3y^2+4=0$.

5. $y=mx+c$ and $y^2=4ax$.

6. $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

7. Eliminate y and z between the equations

$$2x+3y-z=3x-2y+2z=4x+y-5z=3.$$

SYMMETRY.

175. **Symmetrical Expressions.** If an expression involving certain letters be such that its value is unaltered when two of the letters are interchanged, it is said to be *symmetrical* with respect to them.

Thus each of the expressions $a+b$ and $2a^2+3ab+2b^2$ is symmetrical with respect to a and b; for, if a and b be interchanged, the values of the expressions are unaltered.

If an expression involving certain letters be such that its value is unaltered when *any* two of those letters are interchanged, it is said to be symmetrical with respect to all of them.

Thus each of the expressions abc and $a^3+b^3+c^3+d^3+3abc$ is symmetrical with respect to a, b, and c; but the second of them is not symmetrical with respect to a, b, c, and d; since if a and d be interchanged, its value is altered.

Similarly, the expression

$$\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}$$

is symmetrical with respect to a, b, and c.

176. **Cyclical interchanges.** The value of the expression at the end of the last paragraph was found in Art. 150, Ex. 4 (p. 120). As written in Art. 175, the three terms which compose it are so arranged that

the second term is obtainable from the first by writing b for a, c for b and a for c; the third term is obtainable from the second by the same rule; and if we apply the same rule to the third term, we get back to the first term. Letters so arranged are said to be taken in *cyclical order*.

One advantage of writing the above expression in this way is that from one term each of the other terms can be written down by symmetry.

And always, if the letters of an expression, such as $a, b, c, \ldots$, be arranged round the circumference of a circle, then to make a *cyclical* change in the expression we replace every letter by the letter immediately in front of it.

177. The *form* in which an expression can be written is a matter of great importance, and the student will find that his power of successfully applying analysis to the problems he has to solve will often depend to a large extent on his power of arranging his symbols in a symmetrical form.

178. Attention to the symmetry of expressions will moreover frequently save the student from mistakes in his work.

For example, if in the product

$$(x+a)(x+b)(x+c)$$

the coefficient of x^2 were said to be $a+b+2c$, it is obvious (by inspection) that there must be an error; for the given expression is symmetrical with respect to a, b, and c, and therefore the coefficient of every power of x in the product must be so.

179. Examples of the following kind are not uncommon in the applications of algebra.

A rational integral algebraic expression of two dimensions in x and y is symmetrical and homogeneous. Its value is 4, when $x=1$ and $y=1$; and its value is 1, when $x=0$ and $y=1$. Find the expression.

Since the expression is homogeneous and of two dimensions in x and y, every term in it must be of one of the forms ax^2, bxy, and cy^2. Let u stand for the expression,

$$\therefore\ u=ax^2+bxy+cy^2,$$

where a, b, c are numerical coefficients which we have to find.

Again, the expression is symmetrical in x and y; that is, if x and y be interchanged, no change is made in the expression. This requires that a shall be equal to c. Hence we must have

$$u=ax^2+bxy+ay^2.$$

Now we are also given that if $x=1$ and $y=1$, then $u=4$,

$$\therefore\ 4=a+b+a \text{..........................(i).}$$

Also, if $x=0$ and $y=1$, then $u=1$,

$$\therefore\ 1=0+0+a \text{..........................(ii).}$$

The equations (i) and (ii) give, when solved, $a=1$ and $b=2$.

Substituting these values, we obtain

$$u=x^2+2xy+y^2,$$

which is the required expression.

It will be observed that we first wrote down a homogeneous expression, next we made it symmetrical, and lastly we determined the coefficients that were then unknown by means of the given relations.

EXAMPLES. XIV. D.

1. Which of the following expressions are symmetrical, and with respect to which letters are they symmetrical?

(i) $abx+aby^2$;
(ii) $a^2+b^2+c^2$;
(iii) $a^2+b^3+2c^2$;
(iv) $(a-b)c^2+(b-c)a^2+(c-a)b^2$.

2. Can $a+2b+c$ be the coefficient of x^2 in the product $(x+b-c)(x+c-a)(x+a-b)$? Can $a^2+b^2+c^2$ be the coefficient of x^2 in this product?

3. Write down by cyclical interchanges of a, b, and c the quantities corresponding to

(i) $b-c$; (ii) b^2-c^2; (iii) $a(b-c)$; (iv) $(a-b)(b+c)$.

4. Find a rational integral homogeneous expression of the first degree in x and y, which is equal to 3 if $x=1$ and $y=1$, and is equal to 4 if $x=2$ and $y=1$.

5. Find a rational integral homogeneous and symmetrical expression of the first degree in x and y, which is equal to 6 if $x=1$ and $y=1$.

COMPARISON OF UNEQUAL QUANTITIES.

180. **Comparison of unequal quantities.** The comparison of the magnitudes of unequal quantities depends on the following propositions.

An inequality is unaltered in character (i) if equal quantities be added to each side of it, or (ii) if equal quantities be subtracted from each side of it, or (iii) if each side be multiplied or divided by a positive number; but (iv) an inequality is reversed in character if each side be multiplied or divided by any negative number.

To prove these propositions, consider the definition of an inequality. A quantity a is said to be greater than a quantity b, that is, $a > b$, if $a - b$ be positive [Art. 48]. But, if $a - b$ be positive, then $(a \pm x) - (b \pm x)$ is positive, and therefore $a \pm x > b \pm x$, that is $a \pm x$ is greater than $b \pm x$. Similarly, if $a < b$, then $a \pm x < b \pm x$. This proves (i) and (ii).

It follows from (i) and (ii) that we may transpose a term from one side of an inequality to the other, provided its sign is at the same time changed [see Arts. 94, 95].

Again, if $a > b$, then $a - b$ is positive. Therefore, if m be a positive number, $m(a - b)$ is positive, and therefore $ma > mb$: but, if m be a negative number, then $m(a - b)$ is negative, and therefore $ma < mb$. This proves (iii) and (iv).

As a particular case of (iv), we have the result that *an inequality is reversed in character if the sign of each side be changed* (that is, if each side be multiplied by -1).

181. To compare two or more unequal fractions, we must express them in an equivalent form having a common positive denominator. To do this, we must find the L.C.M. of their denominators and make this L.C.M. the denominator of each fraction.

The following examples will illustrate the use of Arts. 180, 181. Other examples will be found in Art. 296, which the student may here consult.

Ex. 1. *Arrange the fractions* $\frac{2}{3}$, $\frac{3}{4}$, $\frac{17}{24}$ *in order of magnitude.*

The L.C.M. of their denominators is 24;

$$\therefore \text{ we have } \tfrac{2}{3}=\tfrac{16}{24}, \text{ and } \tfrac{3}{4}=\tfrac{18}{24}.$$

Hence $$\tfrac{2}{3}<\tfrac{17}{24}<\tfrac{3}{4}.$$

Ex. 2. *Determine which of the fractions* $\dfrac{a+3b}{a+2b}$ *and* $\dfrac{a+2b}{a+b}$ *is the greater, a and b being positive quantities.*

The fraction $\dfrac{a+3b}{a+2b}$ is $>$ or $<\dfrac{a+2b}{a+b}$,

according as $\dfrac{(a+3b)(a+b)}{(a+2b)(a+b)}$ is $>$ or $<\dfrac{(a+2b)^2}{(a+b)(a+2b)}$,

that is, as $(a+3b)(a+b)$ is $>$ or $<(a+2b)^2$,

that is, as $a^2+4ab+3b^2$ is $>$ or $<a^2+4ab+4b^2$,

that is, as $3b^2$ is $>$ or $<4b^2$.

But $3b^2$ is $<4b^2$.

$$\therefore \frac{a+3b}{a+2b} \text{ is } < \frac{a+2b}{a+b}.$$

Ex. 3. *Determine which is the greater of the fractions* $\dfrac{6+a}{7+a}$ *and* $\dfrac{6-b}{7-b}$, *where a is any positive number and b is a positive number less than* 7.

The fraction $\dfrac{6+a}{7+a}$ is $>$ or $<\dfrac{6-b}{7-b}$,

according as $\dfrac{(6+a)(7-b)}{(7+a)(7-b)}$ is $>$ or $<\dfrac{(6-b)(7+a)}{(7-b)(7+a)}$,

that is, as $(6+a)(7-b)$ is $>$ or $<(6-b)(7+a)$,
[if $7>b$, Art. 180 (iv).

that is, as $42+7a-6b-ab$ is $>$ or $<42-7b+6a-ab$,

or, transposing, as $7a-6a+7b-6b$ is $>$ or <0,

that is, as $a+b$ is $>$ or <0.

But, if both a and b be positive, $a+b$ is >0.

$$\therefore \frac{6+a}{7+a} \text{ is } > \frac{6-b}{7-b}.$$

Ex. 4. *Shew that, if a and b be unequal quantities, then the sum of a^2 and b^2 will be greater than 2ab.*

We see that $$a^2+b^2 > 2ab,$$
if $$a^2-2ab+b^2 > 0,$$
that is, if $$(a-b)^2 > 0.$$

Now a is not equal to b, and the square of any quantity (whether positive or negative) is a positive quantity and therefore greater than zero;

$$\therefore (a-b)^2 > 0.$$

Hence $$a^2+b^2 > 2ab.$$

Note. If $a=b$, then $a-b=0$, $\therefore (a-b)^2=0$. $\therefore a^2+b^2=2ab$. We may therefore say that, in all cases, $a^2+b^2 \nless 2ab$.

Ex. 5. *Shew that, if $x^2=a^2+b^2$, and $y^2=c^2+d^2$, then xy will be greater than ad+bc, provided that ac is not equal to bd, and x and y are of the same sign.*

We see that $$xy > ad+bc,$$
if $$x^2y^2 > (ad+bc)^2,$$
that is, if $$(a^2+b^2)(c^2+d^2) > (ad+bc)^2,$$
that is, if $$a^2c^2+a^2d^2+b^2c^2+b^2d^2 > a^2d^2+2abcd+b^2c^2,$$
or transposing, if $a^2c^2-2abcd+b^2d^2 > 0$,
that is, if $$(ac-bd)^2 > 0.$$

Now ac is not equal to bd, and therefore $(ac-bd)$ is not equal to 0. Also the square of any quantity (whether positive or negative) is a positive quantity and therefore greater than zero,

$$\therefore (ac-bd)^2 > 0.$$
$$\therefore xy > ad+bc.$$

EXAMPLES. XIV. E.

1. Which is the greater of the fractions $\dfrac{x+3}{x+6}$ and $\dfrac{x+4}{x+7}$?

2. For what values of x is $\dfrac{x+5}{x+7}$ greater than $\dfrac{5}{7}$?

3. For what values of x is $\dfrac{x+3}{x+4}$ greater than $\dfrac{x+2}{x+5}$?

4. Shew that $\dfrac{1}{ab}$ is never greater than $\dfrac{1}{2}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$.

MISCELLANEOUS EXAMPLES. XIV. F.

1. Shew that half the difference between the cubes of two consecutive odd integers is greater by unity than three times the square of the intermediate even integer.

2. A number is equal to the product of two factors; if one of these factors be increased by 1 and the other diminished by 2, the product is increased by 1. Prove that, if the factors be again increased by 1 and diminished by 2 respectively, the product will be less by 2 than the original number.

3. Prove the truth of the following statement. Take any two proper fractions such that their sum is unity, subtract the square of the smaller from the square of the greater, and add unity to the remainder; the result will always be equal to twice the greater of the two fractions.

4. At an election there are two candidates, A and B, of whom one is to be chosen. A has a majority of those who vote by proxy and also of those who vote in person, and it is observed that if a certain number of those who voted for him in person had voted by proxy and thus trebled his majority by proxies only, then his majority by voters in person would be $\frac{1}{3}$ of what it was before. Shew that his whole majority is four times his majority by proxies only.

5. A common conjuring trick is to ask a boy among the audience to throw two dice, or to select at random from a box a domino on each half of which is a number. The boy is then told to recollect the two numbers thus obtained, to choose either of them, to multiply it by 5, to add 7 to the result, to double this result, and lastly to add to this the other number. On mentioning the final number thus obtained, the conjurer knows the two numbers originally chosen. How is this done?

6. Shew that, if $x-3y=2x+y-15=1$, then $x^2+y^2=4xy-3$.

7. Find the value of $25a^2+(a+4b)^2$, when $3a+2b=7$ and $a-b=4$.

***8.** If $x=1+\sqrt{2}$, prove that $x(x-1)=x+1$.

9. If $(x+1)^2=x$, find the value of $11x^3+8x^2+8x-2$.

10. If $\dfrac{1}{a}+\dfrac{1}{a-c}=\dfrac{2}{a-b}$, shew that $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{2}{c}$.

11. Shew that the expression $x^2+y^2+z^2-yz-zx-xy$ is not changed by adding the same quantity to x, to y, and to z.

12. If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$.

13. If $a+b+c=0$, prove that

(i) $a^3+b^3+c^3+3(a+b)(b+c)(c+a)=0$;

(ii) $\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0$.

***14.** Write down the most general possible rational integral algebraical expression of the second degree in x and y, (i) which is homogeneous in x and y, (ii) which is both homogeneous and symmetrical in x and y. Also, write down a rational integral algebraical expression which is of a degree not higher than the second and which is symmetrical.

***15.** A homogeneous expression of two dimensions is symmetrical in x, y, and z. Its value is 9 when $x=y=z=1$, and its value is 36 when $x=1$, $y=2$, $z=3$. Find it.

16. Eliminate x between the equations $2x^2-y^2=3y^2-x^2=2$.

***17.** Eliminate x, y, and z between the equations

$$ax+by+cz=bx+cy+az=cx+ay+bz=0.$$

18. Of the fractions $\frac{a-x}{a+x}$ and $\frac{a^2-x^2}{a^2+x^2}$, which is the greater?

***19.** If $x^2+y^2=z^2$, then will $x^3+y^3<z^3$.

20. If $y+\frac{1}{z}=1$ and $z+\frac{1}{x}=1$, prove that $x+\frac{1}{y}=1$ and $xyz=-1$.

***21.** Prove that if $x+y+z=0$, then

(i) $x^2+xy+y^2=y^2+yz+z^2=z^2+zx+x^2$;

(ii) $(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+24xyz=0$;

(iii) $2(x^5+y^5+z^5)-5xyz(x^2+y^2+z^2)=0$.

***22.** If $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$, and if $a+b+c=0$, shew that $x+y+z=0$.

***23.** Prove that if $a+b+c+d=0$,

then $a^3+b^3+c^3+d^3+3(a+b)(b+c)(c+a)=0$,

and $(a+b)(a+c)(a+d)=(b+c)(b+d)(b+a)=(c+d)(c+a)(c+b)$
$=(d+a)(d+b)(d+c)=-(a+b)(b+c)(c+a)$.

***24.** Shew that, if a, b, c, d be positive quantities of which the sum of any three is greater than the fourth, and if

$$\frac{(c+d)^2-(a-b)^2}{(a+c+d)^2-b^2}+\frac{(d+a)^2-(b-c)^2}{(a+b+d)^2-c^2}=1,$$

then $$a+b=c+d.$$

EXAMINATION PAPERS AND QUESTIONS.

[*The first two of the following papers (A and B) contain those questions on Elementary Algebra on which Junior Students in two of the recent Cambridge Local Examinations were expected to satisfy the Examiners—but the two questions in each paper which are marked with a * are concerned with subjects treated in Chapters XV, XVI. The next two papers (C and D) contain all those questions on Elementary Algebra on which Candidates in two of the recent examinations for Higher Certificates of the Oxford and Cambridge Schools Examination Board were required to pass. These papers are followed by groups of questions taken from papers set in recent years under the authority of the same Examination Board to boys in the lower forms of various Public Schools.*]

Paper A.

1. Simplify $(b-c)(c+a)-(c-a)(a+b)-a(a+b-c)$.

2. Simplify $\dfrac{x+y}{x^3-y^3}+\dfrac{x-y}{x^3+y^3}-\dfrac{2(x^2-y^2)}{x^4+x^2y^2+y^4}$.

3. Divide $(x^2-y^2)^2-2(x^2+y^2)+1$ by $x^2-(y+1)^2$.

4. Resolve $8x^2+13x-6$ into factors.

5. Resolve $ab(c^2+d^2)+cd(a^2+b^2)$ into factors.

*6. Extract the square root of
$$x^8+2x^7+x^6-4x^5-12x^4-8x^3+4x^2+16x+16.$$

7. Solve the equation
$$(x-3a)(3x-a)-(x-2a)(2x-a)=(x-a)^2.$$

*8. Solve the equation
$$\frac{30}{x+2}-\frac{3-x}{5}=x-1.$$

9. Prove that, if $\alpha^2+\alpha+1=0$, and if $\alpha^3=1$, then
$$x^3-1=(x-1)(x-\alpha)(x-\alpha^2).$$

10. The Eiffel Tower is 580 ft. higher than the spire of Salisbury Cathedral, while the number of inches in the height of the spire is 4520 greater than the number of yards in the height of the Tower. Find the height of each building.

Paper B.

1. If $a=3$, $b=4$, $c=5$, $d=6$, find the numerical values of

$$\frac{2\sqrt{a^2+b^2}+\sqrt[3]{a^3+b^3+c^3}}{d-c+b-a}; \text{ and } \frac{(a+b)(c+d)-(b+c)(d+a)}{ab-bc+cd-da}.$$

2. Divide $x^7-13x-30$ by x^2-2x+3.

3. If x^2+7x+c be exactly divisible by $x+4$, what is the value of c?

4. State and prove the rule for finding the Least Common Multiple of two algebraical expressions.

Find the L.C.M. of $9x^3-x-2$ and $3x^3-10x^2-7x-4$.

*5. Find the relation between a, b, c in order that ax^2+bx+c may be a perfect square.

Extract the square root of

$$9x^6-12x^5+22x^4+x^2+12x+4.$$

6. Simplify $\dfrac{\left(2+\frac{x}{y}\right)\left(1+\frac{y}{x}\right)}{1+\frac{x}{y}+\frac{y}{x}}+\dfrac{3\left(1+\frac{x}{y}\right)}{\frac{x^3}{y^3}-1}$.

7. Simplify $\left\{1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right\}\left\{1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right\}$.

8. Solve the equations

(i) $\frac{1}{2}\left[x-\frac{1}{3}\left\{x-\frac{1}{4}\left(x-\dfrac{x-\frac{1}{6}x}{5}\right)\right\}\right]=53$;

*(ii) $\dfrac{7x-11}{4x-7}+\dfrac{3x-2}{12x-1}=\dfrac{2x+5}{x+2}$;

(iii) $11y-x=10$, $11x-101y=110$.

9. A boy spent one-third of his money in cakes, one-fourth in apples, one-fifth in oranges and one-sixth in nuts, and has $1\frac{1}{2}d.$ left: how much had he?

10. In reducing a quantity of ore, it is passed through three processes which remove respectively $\dfrac{1}{m}$th, $\dfrac{1}{n}$th and $\dfrac{1}{p}$th of whatever is subjected to them. If the weight left be 120 lbs., and the weight lost in the third process be 30 lbs., 40 lbs., or 60 lbs. according to the different orders in which the processes can be performed, what was the original weight?

Paper C.

1. Remove the brackets from $a^2-[(b-c)^2-\{c^2-(a-b)^2\}]$.

2. Find the value of $x^3+(p-3)x^2+(q-3p)x-3q$, when $x=3$.

3. Divide $(y^6-b^2y^4+b^4y^2-b^6)(y^6+b^2y^4+b^4y^2+b^6)$ by y^4-b^4.

4. In finding the H.C.F. of two given expressions, can we reject either altogether or temporarily a factor occurring in both expressions? Justify your answer.

Find the H.C.F. of

$$x^7-3x^6+x^5-4x^2+12x-4 \text{ and } 2x^4-6x^3+3x^2-3x+1.$$

5. Simplify $\frac{a+b}{b}-\frac{2a}{a+b}+\frac{a^2b-a^3}{a^2b-b^3}$.

6. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x-2a}{x+2b}-\frac{x+2a}{x-2b}-\frac{16ab}{4b^2-x^2}$.

7. Solve the equations:

(i) $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$;

(ii) $x-\frac{3x-2y}{7}=y-\frac{4x-5y}{19}$, $\quad y-\frac{2x-7y}{6}=2(x-1)-\frac{3x-7}{8}$.

8. Two persons, a certain distance apart, setting out at the same time, are together in 25 minutes if they walk in the same direction, but they meet in 15 minutes if they walk in opposite directions. Compare their rates of walking.

Paper D.

1. If $a=4$, $b=5$, and $c=3$, find the value of

$$\sqrt[3]{\{5(b^2-c^2)-a^2\}}+\sqrt[4]{3\{a(a^2-c^2)-1\}}.$$

2. Express $x^2-8x-84$ in factors.

3. Find the factors of $(2x+3)^2-(x-3)^2$.

4. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$, and find the factors of the quotient.

5. Shew that if a quantity x divide A and B exactly, it will also divide $mA \pm nB$.

Find the H.C.F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$. What value of x will make both these expressions vanish?

6. Define a fraction, and hence prove that $\frac{a}{b}=\frac{ma}{mb}$.

Reduce the following fractions to their simplest forms:

(i) $\frac{x^4-5x^2+4}{x^3-3x+2}$;

(ii) $\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b)}$.

7. Solve the equations:

(i) $\frac{x+2}{x-3}+\frac{x-2}{x-6}=2$;

(ii) $\frac{2x}{x-1}+\frac{3x-1}{x+2}-\frac{5x-11}{x-2}=0$;

(iii) $\left.\begin{array}{l}(a+b)x-(a-b)y=3ab,\\(a+b)y-(a-b)x=\ \ ab.\end{array}\right\}$

8. Find the difference of the squares of the highest and lowest of any three consecutive numbers in terms of the middle number.

Examination Questions.

1. Remove the brackets from the expression

$-3\left[(a+b)-\{(2a-3b)-(5a+\overline{7b-16c})-(-13a+2b-3c-5d)\}\right]$, and find its value when $a=1$, $b=2$, $c=3$, $d=4$.

2. Find the highest common factor of x^6+1 and x^4-x^2-2.

3. Find the lowest common multiple of

$$x^2-5x+6,\ x^2-4x+3,\ \text{and}\ x^2-3x+2.$$

4. Reduce the following fractions to their simplest forms

(i) $\frac{5x^3-4x-1}{2x^3-3x^2+1}$; (ii) $\frac{x^2+x-6}{x^4-13x^2+36}$.

5. At an election where there are three candidates A, B, C, and two persons to be chosen, A obtained 486 votes, B 461, C 457. Every elector voted for two candidates. How many voted for B and C; how many for C and A; and how many for A and B?

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6. Prove that, if $a^3+pa^2+qa+r=0$, then x^3+px^2+qx+r is a multiple of $x-a$.

Find the factors of $x^3+8x^2-79x+70$.

7. Prove the rule for finding the highest common factor of two algebraical expressions: and state the circumstances under which a factor may be introduced or omitted at any stage of the process.

Find the H.C.F. of a^2-b^2 and $27a^3+16a^2b+30ab^2+41b^3$.

8. Add together the fractions

$$\frac{a^2+2bc}{(c-a)(a-b)},\ \frac{b^2+2ca}{(a-b)(b-c)},\ \frac{c^2+2ab}{(b-c)(c-a)}.$$

9. Solve the equations:

$$\text{(i)}\ \frac{x+8}{x-4}=\frac{x-14}{x-2};\quad \text{(ii)}\ \frac{x-2}{x+1}=\frac{x+2}{x-1}-\frac{18}{x^2-1}.$$

10. A vessel can be filled by three pipes, L, M, N. If M and N run together, it is filled in 35 minutes; if N and L, in 28 minutes; if L and M, in 20 minutes. In what time will it be filled if all run together?

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11. A boy is a years old, his mother was b years old when he was born, his father is half as old again as his mother was c years ago. Find the present ages of his father and mother.

12. Prove that every common multiple of any number of algebraical expressions is also a multiple of their lowest common multiple.

Find the highest common factor and lowest common multiple of $a^2-3ab-10b^2$, $a^2+2ab-35b^2$, and $a^2-8ab+15b^2$.

13. Simplify the expressions

$$\frac{2+x}{x+5}-\frac{3-x}{x-4}+\frac{6-x}{x+2};\ \text{and}\ \frac{\dfrac{2a-3b}{2a-6b}-\dfrac{3b}{2a}}{\dfrac{2a-3b}{2a}+\dfrac{3b}{2a-6b}}.$$

14. A merchant added to a cask of wine ten per cent. of water. If he were now to add nine gallons more water, the mixture would contain fifty per cent. of water. How many gallons of wine had he at first?

15. If $\dfrac{y-z}{1+yz}+\dfrac{z-x}{1+zx}+\dfrac{x-y}{1+xy}=0$, prove that two of the quantities x, y, z are equal to one another.

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16. Find the continued product of

$$\left(a+\frac{1}{a}\right)\times\left(a^2+\frac{1}{a^2}\right)\times\left(\frac{1}{a^4}+a^4\right)\times\left(\frac{1}{a}-a\right).$$

17. Find the H.C.F. of $x^7 - x$ and $x^4 - 2x^3 + 3x - 2$; and also the G.C.M. of their numerical values when $x = 2$. Account for the fact that this G.C.M. is not the same as the numerical value of their H.C.F. when $x = 2$.

18. Subtract $\dfrac{1}{x^2+11x+30}$ from $\dfrac{1}{x^2+8x+15}$.

19. Solve the following systems of equations:

(i) $11x - 17y = 17x - 28y + 2 = 28$;

(ii) $(a+b)x - ay = a^2$, $(a^2+b^2)x - aby = a^3$.

20. A farmer bought a certain number of sheep for £30. If he made a profit of 20 per cent. (on those sold) by selling all but five of them for £27, find how many he bought.

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21. Simplify $\left(\dfrac{x-y}{x+y} - \dfrac{x+y}{x-y}\right) \div \left(\dfrac{x^2-y^2}{x^2+y^2} - \dfrac{x^2+y^2}{x^2-y^2}\right)$.

22. What must be the value of x in order that $\dfrac{(a+2x)^2}{a^2+70ax+3x^2}$ may be equal to $1\frac{1}{3}$ when a is equal to 67?

23. Solve the equations:

(i) $\dfrac{42-x}{9} + \dfrac{512-20x}{8} = \dfrac{5x-24}{5} - \dfrac{3}{4}x$;

(ii) $\cdot 4x - \cdot 01x + \cdot 002x = 11\cdot 7 - \cdot 0001x$.

24. Solve the simultaneous equations

$ax + cy - bz = a^2$, $cx + by - az = b^2$, $bx - ay + cz = c^2$.

25. *A* can walk forwards four times as fast as he can backwards, and undertakes to walk a certain distance ($\frac{1}{4}$ of it backwards) in a certain time. But, the ground being bad, he finds that his rate per hour backwards is $\frac{1}{5}$ mile less than he had reckoned, and that to win his wager he must walk forwards two miles an hour faster. What is his usual rate per hour forwards?

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26. Find the factors of $20a^2 + 21ab - 27b^2$.

27. Shew that the following system of equations is indeterminate:

$7x + y - 5z = 24$, $5x - 3y + 2z = 11$, $4y - 7z + 2x = 13$.

28. An egg-dealer bought a certain number of eggs at 1*s.* 4*d.* per score, and five times the number at 6*s.* 3*d.* per hundred. He sold the whole at 10*d.* per dozen, gaining £1. 7*s.* by the transaction. How many eggs did he buy?

CHAPTER XV.

EVOLUTION (SQUARE ROOTS AND CUBE ROOTS).

182. WE have already explained [Art. 23] that the n^{th} *root* of an expression is a quantity such that its n^{th} power is equal to the given expression; and that if the given expression be denoted by X, its n^{th} root is represented by $\sqrt[n]{X}$.

A quantity which has an exact n^{th} root is called a *perfect* n^{th} *power*. In particular, a quantity which has an exact square root is called a *perfect square*, and a quantity which has an exact cube root is called a *perfect cube.*

When no exact quantity can be found which when raised to the n^{th} power is equal to a, then the n^{th} root of a is called an *irrational quantity.*

An expression which involves no irrational quantity is said to be *rational.*

183. The process of finding a root of a given expression is called *evolution.*

The most direct test whether a certain quantity (which we may denote by y) is the n^{th} root of a given expression (which we may denote by X) is that the n^{th} power of y is equal to X.

That is, y will $=\sqrt[n]{X}$, provided $y^n=X$; and if $y^n=X$, then $y=\sqrt[n]{X}$.

Of course $(\sqrt[n]{X})^n=X$.

*184. Since $a^2=(\pm a)^2$, it follows that $\sqrt{a^2}=\pm a$; that is, either $+a$ or $-a$ is the square root of a. Thus, strictly speaking, there are two square roots of a quantity denoted by a^2. Again, we shall see presently that there are three roots of the equation $x^3=a^3$, and strictly speaking, there are therefore three cube roots of a quantity denoted by a^3. The student will find later that similarly the number of n^{th} roots of a given quantity is n, but some of these are imaginary, and since we here confine ourselves [see Art. 119] to real values of the quantities considered, we shall make no attempt to find more than one root out of all the n^{th} roots of the given quantity. We shall moreover adopt the usual notation and speak of any n^{th} root as *the* n^{th} root, though it would be more accurate to say *an* n^{th} root.

185. **Roots of Simple Quantities.** If we want the n^{th} root of a power of a simple quantity, and if the index of the power to which that quantity is raised happen to be a multiple of n (say, p times n), then the root required will be the quantity raised to the p^{th} power: in other words, we must divide the index of the power to which the quantity is raised by n.

For, if y be the n^{th} root of a quantity like a^{np}, we have

$$y=\sqrt[n]{a^{np}}.$$

$$\therefore\quad y^n=a^{np}=(a^p)^n. \qquad \text{[Art. 76]}$$

$$\therefore\quad y=a^p.$$

For example, the cube root of a^6 is a raised to the power of 2, (that is, 6 divided by 3), which is expressed in symbols thus,

$$\sqrt[3]{a^6}=a^2.$$

Similarly, $\sqrt[4]{a^{20}}=a^5$; $\sqrt[5]{x^{10}}=x^2$; and $\sqrt[3]{x^{21}}=x^7$.

186. **Roots of Products.** *The n^{th} root of a product of any factors is the product of the n^{th} roots of the different factors.*

Suppose there are two factors, a and b. We want then to shew that

$$\sqrt[n]{(ab)} \text{ is equal to } \sqrt[n]{a}\,.\,\sqrt[n]{b}.$$

Now $(\sqrt[n]{a} \,.\, \sqrt[n]{b})^n = (\sqrt[n]{a})^n \,.\, (\sqrt[n]{b})^n$ [Art. 75]

$= a \,.\, b.$ [Art. 183]

$= ab.$

$\therefore \quad \sqrt[n]{a} \,.\, \sqrt[n]{b} = \sqrt[n]{(ab)}.$

Similarly, $\sqrt[n]{(abc)} = \sqrt[n]{a} \,.\, \sqrt[n]{b} \,.\, \sqrt[n]{c}.$

Ex. 1. $\sqrt{(a^4b^2)} = \sqrt{a^4} \,.\, \sqrt{b^2} = a^2b.$

Ex. 2. $\sqrt{\dfrac{a^2}{b^2}} = \dfrac{a}{b}.$

Ex. 3. $\sqrt[3]{(a^2b^3)} = \sqrt[3]{a^2} \times \sqrt[3]{b^3} = \sqrt[3]{a^2} \times b.$

Ex. 4. $\sqrt[n]{(a^{pn}b^{qn})} = (\sqrt[n]{a^{pn}})(\sqrt[n]{b^{qn}}) = a^p b^q.$

EXAMPLES. XV. A.

Find the values of the following quantities.

1. $\sqrt{x^{10}}.$ 2. $\sqrt{(a^2b^4)}.$ 3. $\sqrt{\frac{1}{9}a^4}.$ 4. $\sqrt{\dfrac{a^4}{b^2}}.$

5. $\sqrt{\dfrac{144}{x^2}}.$ 6. $\sqrt{49x^4y^6}.$ 7. $\sqrt{\dfrac{81x^8y^4}{100a^4b^8}}.$ 8. $\sqrt[3]{x^3}.$

9. $\sqrt[3]{-x^3}.$ 10. $\sqrt[3]{27a^6b^6}.$ 11. $\sqrt[3]{-8\dfrac{x^6}{y^9}}.$ 12. $\sqrt[3]{\dfrac{64a^9c^4}{27cx^6}}.$

13. $\sqrt[4]{x^8y^{12}}.$ 14. $\sqrt[5]{32a^5b^{10}}.$ 15. $\sqrt[5]{-243\dfrac{x^5}{y^{10}}}.$ 16. $\sqrt[2n]{a^{2n}b^{4n}}.$

187. **Square Root of a Compound Expression.** The general method for obtaining the square root of a compound expression is somewhat more complicated. We shall first confine ourselves to the comparatively simple cases of expressions containing only three terms, and of expressions which are of the second degree in some quantity; and after discussing them shall consider the general case of any compound expression.

We shall assume that the expressions considered are perfect squares. We shall also suppose that every expression has been reduced to its simplest form, and arranged in descending powers of some letter.

188. **Square Root of a Trinomial.** First, consider the case of a trinomial expression, that is, an expression consisting of only three terms. Suppose it to be arranged in descending powers of (say) x.

Let us suppose that its square root is an expression like $A \pm B$, where the quantities, A, B, are arranged in descending powers of x.

Then the given expression must be the same as $(A \pm B)^2$, that is, as $A^2 \pm 2AB + B^2$, the terms of which are arranged in descending powers of x.

If then the trinomial be a perfect square of a quantity like $A \pm B$, and if it be arranged in descending powers of x, then its first term will be A^2, and its last term will be B^2. Hence the square root of the first term will be equal to A, and the square root of the last term will be equal to B; and their sum or difference will be the square root required—the sum or difference being taken according as to whether the sign of the middle term is positive or negative.

The rule applies only to those cases where the given trinomial is a perfect square. Hence (unless we know this to be the case) the correctness of the result must be tested by forming its square and comparing it with the given trinomial.

Example. Find the square root of $9x^4 - 6x^2a^2 + a^4$.

Here we take the square root of the first term, which is $3x^2$; and then the square root of the last term, which is a^2; also, since the middle term is negative, we take their difference. Hence the required square root is $3x^2 - a^2$.

The correctness of the result can be tested by forming its square, which will be found to be equal to the given quantity.

The reader will remember that either $+(3x^2 - a^2)$ or $-(3x^2 - a^2)$ may be taken as the square root required [Art. 184].

EXAMPLES. XV. B.

Find the square roots of the following expressions.

1. $a^2x^2 - 2abx + b^2$.
2. $121x^2 - 374x + 289$.
3. $\frac{1}{4}y^2 + \frac{1}{3}yz + \frac{1}{9}z^2$.
4. $9a_1^2a_2^2 + 16a_3^2a_4^2 - 24a_1a_2a_3a_4$.
5. $64 - 8p + \frac{1}{4}p^2$.
6. $570l^2mn + 361l^2m^2 + 225l^2n^2$.
7. $\dfrac{1}{9x^2} - \dfrac{1}{3y} + \dfrac{x^2}{4y^2}$.
8. $\dfrac{1}{16x^2} + \dfrac{x^2}{9y^2} - \dfrac{1}{6y}$.

189. **Square Root of a Quadratic Expression.** A quadratic expression is one which contains only first and second powers of some quantity, such as x. We can therefore arrange the terms in three groups, namely, those that involve x^2, those that involve x, and those that are independent of x. When thus written as a trinomial, the preceding rule applies; but since it is sometimes difficult to write down by inspection the square root of one of these groups of terms, we can (if we prefer) proceed by the following method.

Let us compare the given expression with the square of an expression like $A+B$, where we will suppose that A involves x but that B is independent of x.

If $A+B$ be the required square root, then the given expression must be identically equal to $(A+B)^2$, that is, to $A^2+2AB+B^2$, the terms of which are arranged in descending powers of x. Hence, if A be taken equal to the square root of that part of the given expression which involves x^2, then the terms in the given expression which involve x to the first power must be $2AB$. Thus, the remaining part of the square root (namely, B) will be obtained by dividing by $2A$ those terms in the given expression which involve the first power of x.

[It may be noticed in passing that the square root of a^2+b^2 is not $a+b$; since the square of $a+b$ is a^2+b^2+2ab.]

Ex. 1. *Find the square root of*

$$x^2+a^2+b^2+2ax-2bx-2ab.$$

Arranging the expression in descending powers of x, we have

$$x^2+2(a-b)x+a^2-2ab+b^2.$$

Hence the first term in the square root is the square root of x^2, that is, x.

If the expression be a perfect square, then the rest of the square root is the quotient of $2(a-b)x$ by $2x$, that is, $(a-b)$.

The remaining terms in the given expression, namely $a^2-2ab+b^2$, are the square of $a-b$.

Hence the required square root is $x+a-b$.

Note. If the square of $a-b$ had not been equal to the group of terms in the given expression which were independent of x, the given quantity would not have had an exact square root.

Ex. 2. *Find the square root of*

$$a^4+a^2x^2-2a^3x-2ax^2+2x+2a^3+2a+x^2+3a^2+1.$$

Arranging the given expression in powers of x and a, we have

$$(a^2-2a+1)\,x^2-2\,(a^3-1)\,x+a^4+2a^3+3a^2+2a+1.$$

The square root of the terms involving x^2 is $(a-1)\,x$. This then is the first term of the required square root.

If the given expression be a perfect cube, then the quotient of $-2\,(a^3-1)\,x$ by $2\,(a-1)\,x$ will give the remaining terms of the root: this quotient is $-(a^2+a+1)$.

The square of this last quantity is equal to the terms in the given expression which are independent of x, namely,

$$a^4+2a^3+3a^2+2a+1.$$

Hence the required square root is

$$(a-1)\,x-(a^2+a+1).$$

Note. If the square of $-(a^2+a+1)$ had not been equal to the group of terms in the original expression which were independent of x, the given quantity would not have had an exact square root.

EXAMPLES. XV. C.

Find the square roots of the following expressions.

1. $x^2+2ax+2bx+a^2+2ab+b^2$.
2. $a^2x^2+2abx-2ax+b^2-2b+1$.
3. $a^2x^2+2abx+b^2-2bx+x^2-2ax^2$.
4. $a^2x^2-4ax^2+4x^2-2abx+4bx+2ax-4x+b^2-2b+1$.
5. $a^2+9b^2+25c^2-30bc+10ca-6ab$.
6. $x^2+\dfrac{a^2}{9}+\dfrac{b^2}{4}+\dfrac{2ax}{3}-\dfrac{ab}{3}-bx$.

190. **Square Root of a Multinomial.** The preceding rule can be extended so as to enable us to find the square root of any multinomial expression.

We shall, as before, assume that the expression is a perfect square, and that it is arranged in descending powers of some letter, such as x. Let us suppose that its square root is an expression like $A+B+C+\ldots$, where the line of dots after C signifies that there may be a number of similar terms, and where the successive terms, A, B, $C\ldots$, are arranged in descending powers of x.

The given expression is equal to $(A+B+C+\ldots)^2$. Denote $B+C+\ldots$ by P. Then

$$(A+B+C+\ldots)^2=(A+P)^2$$
$$=A^2+2AP+P^2$$
$$=A^2+2A(B+C+\ldots)+(B+C+\ldots)^2.$$

The term of the highest dimensions in this last expression is A^2, and since this last expression is the same as the given expression, the term of the highest dimensions in the given expression must be equal to A^2. Hence the first term of the required square root, that is, A, is the square root of the term of highest dimensions in the given expression.

Now, since A, B, C ... are arranged in descending powers of x, therefore $2AB$ is of higher dimensions than any of the other terms in $2A(B+C+\ldots)$. Again, B^2 is the term of highest dimensions in the expansion of $(B+C+\ldots)^2$, but $2AB$ is of higher dimensions than B^2, because A is of higher dimensions than B. Therefore (except for A^2) the term $2AB$ is of higher dimensions than any other term in the square of $(A+B+C+\ldots)$. Hence, if we subtract A^2 from the given expression, the terms of the highest dimensions in the difference must be equal to $2AB$: if therefore these terms be divided by $2A$, we obtain B, which is the second term in the required square root.

We now know A and B. By similar reasoning to that given above, we see that if we subtract $(A+B)^2$ from the given expression, the terms of the highest dimensions left will be equal to $2AC$: if therefore these terms be divided by $2A$, we obtain C, which is the third term in the required square root.

Proceeding in this way, every term in the required square root can be successively obtained.

191. Process of extracting a square root. The process of extracting the square root of a given expression is usually arranged in three columns, as shewn in the following example, where the square root of

$$x^6+4x^5+2x^4-2x^3+5x^2-2x+1$$

is determined.

The first term of the square root is the square root of x^6, that is, x^3. This is written to the right of the given expression in the column III. Subtracting the square of this from the given expression, we obtain $4x^5+2x^4-2x^3+5x^2-2x+1$. Twice the part of the root already determined is then written in the column I. to the left of the expression. By the above rule, we have now to divide the first term of the remainder, which is written in column II., by $2x^3$. The quotient, namely $+2x^2$, is then written both in the column I. and the column III. Thus

I.	II.	III.
	$x^6+4x^5+2x^4-2x^3+5x^2-2x+1$ (	x^3+2x^2
	x^6	
$2x^3+2x^2$	$4x^5+2x^4-2x^3+5x^2-2x+1$	

We now have to subtract $(x^3+2x^2)^2$ from the given expression, that is, we have to subtract $x^6+2x^2(2x^3+2x^2)$. We have already subtracted x^6, and we have thus only to subtract from the remainder the product of $2x^2$ and the expression in the column I. Writing this product below the expression in the column II., and subtracting, we obtain $-2x^4-2x^3+5x^2-2x+1$. Twice the part of the root already determined is then written in the same line as this and in the column I. Dividing the expression in the column II. by $2x^3$, we obtain $-x$ as the next term of the square root. This is then written both in the column I. and the column III. Thus

I.	II.	III.
	$x^6+4x^5+2x^4-2x^3+5x^2-2x+1$ (	x^3+2x^2-x
	x^6	
$2x^3+2x^2$	$4x^5+2x^4-2x^3+5x^2-2x+1$	
	$4x^5+4x^4$	
$2x^3+4x^2-x$	$-2x^4-2x^3+5x^2-2x+1$	

We have now to subtract $(x^3+2x^2-x)^2$ from the given expression. This is equivalent to subtracting from the expression in the last line of the column II. the product of the last term in the column III. (namely, $-x$) and the expression in the last line of I. Writing this below the expression in II., and subtracting, we obtain $2x^3+4x^2-2x+1$. Twice the part of the root already determined is then written in the same line as this and in the column I. Dividing the expression in the column II. by $2x^3$, we obtain $+1$ as the next term in the square root. This is then written both in the column I. and the column III. Proceeding as above, we have now to multiply the quantity in the last line of the column I. by the last term in the column III., and subtract it from the expression in the last line of II. There is no remainder. Hence the original quantity is a perfect square, and its square root is x^3+2x^2-x+1. The whole process is exhibited as follows:

	$x^6+4x^5+2x^4-2x^3+5x^2-2x+1\,(x^3+2x^2-x+1$
	x^6
$2x^3+2x$	$4x^5+2x^4-2x^3+5x^2-2x+1$
	$4x^5+4x^4$
$2x^3+4x^2-x$	$-2x^4-2x^3+5x^2-2x+1$
	$-2x^4-4x^3+x^2$
$2x^3+4x^2-2x+1$	$2x^3+4x^2-2x+1$
	$2x^3+4x^2-2x+1$

192. The following is another example, and exhibits the process for finding the square root of $4x^4-4ax^3-a^2x^2+a^3x+\frac{1}{4}a^4$.

	$4x^4-4ax^3-a^2x^2+a^3x+\frac{1}{4}a^4\,(2x^2-ax-\frac{1}{2}a^2$
	$4x^4$
$4x^2-ax$	$-4ax^3-a^2x^2+a^3x+\frac{1}{4}a^4$
	$-4ax^3+a^2x^2$
$4x^2-2ax-\frac{1}{2}a^2$	$-2a^2x^2+a^3x+\frac{1}{4}a^4$
	$-2a^2x^2+a^3x+\frac{1}{4}a^4$

The process exemplified in this and the last article is analogous to the process used in arithmetic for finding the square root of a number.

193. **Square Roots by inspection.** If however the given expression be of a degree not higher than the sixth, and *if we know that it is a perfect square,* then we can generally write down its square root by inspection, as illustrated by the following method of obtaining the two square roots which are given in Arts. 190, 191.

194. The first term in the square root of

$$x^6+4x^5+2x^4-2x^3+5x^2-2x+1$$

is the square root of x^6, that is, x^3. The next term is obtained by dividing $4x^5$ by twice the term already obtained, that is, by $2x^3$; hence it is $2x^2$.

Similarly, if we write the given expression in ascending powers of x, that is, write it backwards, as

$$1-2x+5x^2-2x^3+2x^4+4x^5+x^6,$$

the first term of its square root, as now written, will be the square root of 1, which is 1. This therefore is the last term of the square root of the expression as originally written. Since however the square root of a^2 is either a or $-a$, we cannot tell whether, if we begin the square root with the terms x^3+2x^2, we shall end with $+1$ or -1. It is better therefore to leave it ambiguous, and take the last term as ± 1. The term before this is got by dividing $-2x$ by twice the term just obtained, that is, by ± 2; hence it is $\mp x$.

Thus four of the terms of the required square root are in order x^3, $2x^2$, $\mp x$, and ± 1. Moreover, since the required square root is of the third degree, there can be no other terms. Thus, the required square root is one of the forms

$$x^3+2x^2\pm(-x+1)\text{......................}(a).$$

Inspection, or at the longest a trial by squaring, will determine which sign in the ambiguity must be taken. In this case, if we form the square of (a) we find that the coefficient of x^4, is 4 ∓ 2, and since the coefficient of x^4 in the given expression is 2, we must take the upper sign in order to make these coefficients the same. Thus the required square root is

$$x^3+2x^2-x+1.$$

195. Similarly, the first term in the square root of

$$4x^4-4ax^3-a^2x^2+a^3x+\tfrac{1}{4}a^4$$

is the square root of $4x^4$, that is, $2x^2$. The next term is the quotient of $-4ax^3$ by $2(2x^2)$, hence it is $-ax$. The last term is similarly the square root of $\frac{1}{4}a^4$, that is, $\pm\frac{1}{2}a^2$. The term before that is the quotient of a^3x by $2(\pm\frac{1}{2}a^2)$; hence it is $\pm ax$. But we have already found that it is $-ax$, hence we must take the lower sign in the ambiguities. Thus the required square root is

$$2x^2-ax-\tfrac{1}{2}a^2.$$

***196. Square Roots of Expressions which are not perfect squares.** If we apply the method of Art. 191 to a multinomial which is not a perfect square, we obtain a series of terms whose square is approximately equal to the given expression.

Example. Find three terms (in descending powers of x) of the square root of x^2+ax+b.

$$\begin{array}{l|l}
 & x^2+ax+b \quad \left(x+\dfrac{a}{2}+\dfrac{b-\frac{1}{4}a^2}{2x}+\ldots\right. \\
 & x^2 \\
\hline
2x+\dfrac{a}{2} & ax+b \\
 & ax+\dfrac{a^2}{4} \\
\hline
2x+\dfrac{a}{2}+\dfrac{b-\frac{1}{4}a^2}{2x} & b-\dfrac{a^2}{4}
\end{array}$$

Hence the required terms are $x+\dfrac{a}{2}+\dfrac{4b-a^2}{8x}$. The student will find that the next term in the square root is $-\dfrac{4ab-a^3}{16x^2}$.

EXAMPLES ON SQUARE ROOTS. XV. D.

Find the square roots of the following expressions numbered 1 to 15.

1. x^4+4x^3-8x+4.
2. $x^4-8x^3+10x^2+24x+9$.
3. $4x^4-12x^3+45x^2-54x+81$.
4. $4x^4-4x^3+3x^2-x+\frac{1}{4}$.
5. $36x^4-36x^3+17x^2-4x+\frac{4}{9}$.
6. $4a^4+9(1-2a)+3a^2(7-4a)$.
7. $4a^4+9(1+2a)+3a^2(7+4a)$.
8. $4(x-1)(x^3-1)+9x^2$.
9. $(2x+1)(2x+3)(2x+5)(2x+7)+16$.
10. $9x^6-12x^4+30x^3+4x^2-20x+25$.
11. $9x^6-12x^5+22x^4+x^2+12x+4$.
12. $9x^4-12x^3y+10x^2y^2-4xy^3+y^4$.
13. $9x^4-24x^3y+40x^2y^2-32xy^3+16y^4$.
14. $4x^4+2bx^3-(4a-\frac{1}{4}b)bx^2-ab^2x+a^2b^2$.
15. $(a^2+b^2)^2+(c^2+d^2)^2+(a+b)^2(c-d)^2+(a-b)^2(c+d)^2+8abcd$.

16. Shew that, if $x=\frac{a+b}{c-d}$, then $(a-cx)^2+(x^2-1)(b^2-d^2)$ is a perfect square.

17. Prove that the product of any four consecutive even integers increased by 16 is a perfect square.

***18.** The first two terms of a certain perfect square are $49x^4-28x^3$, and the last two terms are $6x+\frac{9}{4}$. Find the expression, and its square root.

***19.** Extract the square root of $a^{2m}x^{2n}+10ca^{2m-2}x^{2n+1}$
$6a^{m+1}x^{n-1}+25c^2a^{2m-4}x^{2n+2}-30ca^{m-1}x^n+9\frac{a^2}{x^2}$.

*197. **Cube Roots of Multinomials.** The method for finding the cube root of a multinomial expression is analogous to that given above for finding the square root. We shall not here discuss the general method, which is strictly analogous to that explained in Arts. 190, 191; and shall only briefly indicate some methods for finding the cube root by inspection in a few simple cases.

The following articles apply only to expressions which are perfect cubes.

*198. *First, consider the case of an expression which is a perfect cube and consists of four terms.* Suppose it arranged in descending powers of some letter. We know that

$$(A\pm B)^3=A^3\pm 3A^2B+3AB^2\pm B^3.$$

If then the given expression be a perfect cube of a quantity like $A\pm B$, its first term will be A^3, and its last term will be $\pm B^3$. Hence the cube root of its first term will be A, and the cube root of its last term will be $\pm B$. The algebraic sum of these quantities is the required cube root. There is only one real cube root of any given real quantity, and thus there is no ambiguity.

Ex. 1. *Find the cube root of*

$$8x^3-36ax^2+54a^2x-27a^3.$$

The first term is the cube root of $8x^3$, that is, $2x$.

The last term is the cube root of $-27a^3$, that is, $-3a$.

Hence the required cube root is

$$2x-3a.$$

The cube of this is equal to the given expression. If this had not been the case, the given quantity would not have had an exact cube root.

Ex. 2. *Find the cube root of* $125x^6 - \frac{75}{2}ax^4 + \frac{15}{4}a^2x^2 - \frac{1}{8}a^3$.

The cube root is $\sqrt[3]{(125x^6)} - \sqrt[3]{\tfrac{1}{8}a^3}$,

that is, $5x^2 - \frac{1}{2}a$.

The cube of this is equal to the given expression.

*199. *Next, consider a multinomial of a degree not higher than nine.* In this case, we may proceed as in Art. 194. Arrange the expression whose cube root is required in descending powers of some letter, such as x. Let us compare it with the expansion of $(A+B+C+\ldots)^3$, where A, B, $C\ldots$ are arranged in descending powers of x. Then the two terms of highest dimensions in $(A+B+C+\ldots)^3$ are A^3 and $3A^2B$. Thus, if the required cube root be $A+B+C+\ldots$, the cube root of the first term in the given expression will be A, and the quotient of the second term by $3A^2$ will be B.

Similarly, the cube root of the last term in the given expression will be the last term of the required cube root, which we will denote for the moment by H; and the quotient of the last term but one by $3H^2$ will be the last term but one of the required cube root. Thus the two first terms and the two last terms of the required cube root can be written down by inspection.

This rule will enable us to write down the cube root of any compound expression whose degree, say in x, is not higher than nine, provided it is a perfect cube, since the only possible terms in the cube root are terms involving x^3, x^2, x, and an absolute term.

Ex. Find the cube root of

$$8x^9 - 36x^8 + 66x^7 - 87x^6 + 105x^5 - 87x^4 + 61x^3 - 42x^2 + 12x - 8.$$

The first term is the cube root of $8x^9$, *i.e.* is $2x^3$. The next term is the quotient of $-36x^8$ by $3(2x^3)^2$, *i.e.* is $-3x^2$. The last term is the cube root of -8, *i.e.* is -2. The term before the last is the quotient of $12x$ by $3(-2)^2$, *i.e.* is x. We thus have the terms involving x^3, x^2, x, and the term independent of x. There can be no other terms. Therefore the cube root is

$$2x^3 - 3x^2 + x - 2.$$

*200. *Lastly, consider the case of any multinomial.* To find the cube root of any multinomial, we can either use a method analogous to that given in Arts. 190, 191, or we can write down the two first and the two last terms of the cube root, determined as in Art. 199, and insert between them terms involving the intermediate powers of x with unknown coefficients. By cubing this expression, and comparing it with the given one, we can at once obtain these unknown coefficients.

Ex. *Find the cube root of*

$$x^{12}-3x^{11}+3x^{10}+2x^{9}-9x^{8}+9x^{7}-9x^{5}+9x^{4}-2x^{3}-3x^{2}+3x-1.$$

The two first terms are x^4 and $-x^3$ [Art. 199]. The two last terms are similarly x and -1. The only other term which the cube root can contain is one involving x^2: suppose that its coefficient is h. Therefore the required cube root is

$$x^4-x^3+hx^2+x-1.$$

The cube of this is

$$x^{12}-3x^{11}+x^{10}(3+2h)+\ldots$$

This must be the same as the given expression. Comparing the coefficients of x^{10}, we see that

$$3+2h=3.$$

$$\therefore\ h=0.$$

Hence the required cube root is

$$x^4-x^3+x-1.$$

*EXAMPLES ON CUBE ROOTS. XV. E.

Find the cube roots of the following expressions.

*1. $a^3+6a^2+12a+8.$

*2. $8y^6+60y^4+150y^2+125.$

*3. $27-135x+225x^2-125x^3.$

*4. $a^6-6a^5+15a^4-20a^3+15a^2-6a+1.$

*5. $8x^6-12x^5+18x^4-13x^3+9x^2-3x+1.$

*6. $8x^6-36x^5+114x^4-207x^3+285x^2-225x+125.$

*7. $27y^6-54ay^5+63a^2y^4-44a^3y^3+21a^4y^2-6a^5y+a^6.$

*8. $(a+2b)^3+(b-2c)^3+3(a+2b)(b-2c)(a+3b-2c).$

*9. $a^{3m}-6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n}.$

CHAPTER XVI.

QUADRATIC EQUATIONS.

201. We have already defined [Art. 93] a *quadratic equation* involving only one variable, say x, as an equation in which no power of the symbol representing the unknown quantity is involved except the first and second, namely x and x^2.

Thus, $2x^2=3$, $3x^2+2x=0$, and $ax^2+bx+c=0$ are quadratic equations.

A quadratic equation in which the term involving the first power of the unknown quantity is absent is sometimes called a *pure quadratic*. Other quadratic equations are called *adfected quadratics*.

We proceed now to consider the solution of a quadratic equation involving only one variable.

202. Any term of an equation can be transposed from one side of the equation to the other side [Art. 95]. It is therefore possible to move all the terms of an equation to one side, and the equation will then take the form that the algebraical sum of those terms is equal to zero. Thus

$$ax^2+bx+c=0 \quad\text{.................(i)},$$

is the general form of a quadratic equation, all the terms having been brought to the left-hand side.

If we divide by a so that the equation takes the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad\text{.................(ii)}$$

[or, what comes to the same thing, if $a=1$ in (i)] it is said to be expressed in its *simplest form*.

When written in the above form, the term which does not involve x (or the sum of such terms) is called the *absolute term* of the equation.

Thus the absolute term in (i) is c, and in (ii) is $\frac{c}{a}$.

203. **First method of solution. Resolution into Factors.** This method of solution depends on resolving the left-hand side of (i) or (ii) into factors. This can always be effected by Arts. 115—117, but if the factors be not obvious by inspection, it will be less trouble to find the roots by the process of *completing the square*, as hereafter explained, than to use the results of the articles above mentioned. We shall therefore here confine ourselves to cases where the factors are obvious.

This method depends on the following self-evident proposition.

204. *If the product of two quantities be zero, one of them must be zero.*

Let P and Q be the two quantities, then by hypothesis $PQ=0$.

Now, if neither P nor Q be zero, their product cannot be zero, which is contrary to the hypothesis.

Hence either P or Q must be zero.

Conversely, if either P or Q be zero, their product is zero (unless the other be infinite).

205. Thus, if the product $(x-1)(x-2)$ be zero, then either $x-1=0$, or $x-2=0$;

that is, either $x=1$, or $x=2$.

If therefore we have the quadratic equation $(x-1)(x-2)=0$, either $(x-1)$ must be zero, or $(x-2)$ must be zero.

That is, either $x-1=0$, or $x-2=0$.

That is, $x=1$, or $x=2$.

That is, the roots are 1 and 2.

Hence, if we had to solve the equation $x^2-3x+2=0$, then since we can express it in the form $(x-1)(x-2)=0$, we are able to write down the roots at once.

206. Conversely, if we want to form an equation whose roots are 1 and 2, we have merely to reverse the process. We have

$$x=1 \quad \text{or } x=2.$$
$$\therefore x-1=0 \quad \text{or } x-2=0.$$
$$\therefore (x-1)(x-2)=0.$$
$$\therefore x^2-3x+2=0.$$

207. In the following examples the resolution into factors can be performed by inspection. In every case we

(i) simplify the equation,
(ii) take all the terms to one side of the equation,
(iii) resolve the resulting expression into factors.

It follows from Art. 204 that, if we remove a factor involving x, or if we divide each side of the equation by such a factor, we shall obtain one root of the equation by equating that factor to zero. The student must carefully bear this in mind when he is simplifying any equation which he is trying to solve.

Ex. 1. *Solve the equation* $x^2=a^2$.

This is equivalent to $\quad x^2-a^2=0.$

$$\therefore (x-a)(x+a)=0.$$
$$\therefore x-a=0, \text{ or } x+a=0.$$
$$\therefore x=a, \text{ or } x=-a.$$

Hence the roots are $\pm a$, that is, are a and $-a$.

We can therefore solve an equation of this form by taking the square root of each side of the equation, and prefixing to one square root the sign $\pm$.

Ex. 2. *Solve the equation* $x^2=ax$.

This equation is $\quad x^2-ax=0.$

$$\therefore x(x-a)=0.$$
$$\therefore x=0, \text{ or } x=a.$$

Hence the roots are 0 and a.

Ex. 3. *Solve the equation* $x^2+5x+6=0$.

The equation is $\quad (x+2)(x+3)=0.$

$$\therefore x+2=0, \text{ or } x+3=0.$$
$$\therefore x=-2, \text{ or } x=-3.$$

Hence the roots are -2 and -3.

Ex. 4. *Solve the equation* $x^2-x-6=0$.

The equation is $(x-3)(x+2)=0$.

$$\therefore\ x-3=0, \text{ or } x+2=0.$$

$$\therefore\ x=3, \text{ or } x=-2.$$

Hence the roots are -2 and 3.

Ex. 5. *Solve the equation* $abx^2-x(a^2+b^2)+ab=0$.

The equation is $(ax-b)(bx-a)=0$.

$$\therefore\ ax-b=0, \quad \text{or } bx-a=0.$$

$$\therefore\ x=b/a, \text{ or } x=a/b.$$

Hence the roots are b/a and a/b.

EXAMPLES. XVI. A.

Solve the following equations by resolution into factors.

1. $x^2-3x=0$. **2.** $ax^2+bx=0$. **3.** $x^2=9$.

4. $x^2=a^2+2ab+b^2$. **5.** $x^2+3x+2=0$.

6. $x^2+3ax+2a^2=0$. **7.** $x^2-11x=60$.

8. $x^2+9x+20=0$. **9.** $x^2-7x=30$.

10. $x^2-3x-4=0$. **11.** $x^2+7x=60$.

12. $7x^2+6x-1=0$. **13.** $2y^2+3y+1=0$.

14. $x^2+10=13(x+6)$. **15.** $(x-1)(x-2)=5(x-3)+2$.

16. $x^2+(a+b)x+ab=0$. **17.** $y^2-y=2$.

18. $2(x^2+1)=-5x$. **19.** $y^2-9ay+20a^2=0$.

20. $6(2x^2-1)=-x$. **21.** $(2x-3)(3x+4)=39$.

22. $3x^2+2ax-65a^2=0$. **23.** $x^2-(a-b)x=(c-a)(c-b)$.

24. $\left(\dfrac{x+3}{x+2}\right)^2-\left(\dfrac{x+3}{x+2}\right)=2$.

25. $(x+5)^2-9(x+5)(2x+1)+20(2x+1)^2=0$.

26. $(b+c)x^2-(c+a)x+a-b=0$.

208. **Second method of solution. Completing the square.** Where the factors are not obvious, we might find them by the method given in Arts. 115—117; but it is generally better to proceed directly by *completing the square*, as is illustrated by the following solution of the equation

$$x^2-138=-17x.$$

First. Transpose the absolute term to the right-hand side and all the terms involving x or x^2 to the left-hand side,

$$\therefore\ x^2+17x=138.$$

Second. Divide by the coefficient of x^2. In this case it is 1, and the equation is not thereby altered.

Third. Add to each side the square of half the coefficient of x, that is, $(\frac{17}{2})^2$,

$$\therefore\ x^2+17x+(\tfrac{17}{2})^2=138+(\tfrac{17}{2})^2.$$

Fourth. Collect the numbers on the right-hand side into a single term,

$$\therefore\ x^2+17x+(\tfrac{17}{2})^2=138+\tfrac{289}{4}$$

$$=\frac{552+289}{4}$$

$$=\tfrac{841}{4}.$$

Fifth. Express each side as a perfect square. The left-hand side is written in such a form that it is a perfect square, viz. $(x+\frac{17}{2})^2$, hence its square root is $x+\frac{17}{2}$. The square root of the right-hand side must also be found. Where it is a number, we can do this by arithmetic; if letters or algebraical symbols be involved, the result will generally be obvious, but if not, we must use the method described in the last chapter. In this case the square root of $\frac{841}{4}$ is $\frac{29}{2}$.

Hence the equation is equivalent to

$$(x+\tfrac{17}{2})^2=(\tfrac{29}{2})^2.$$

Sixth. Take the square root of each side of the equation. [Art. 207, Ex. 1, p. 200]

$$\therefore\ x+\tfrac{17}{2}=\pm\tfrac{29}{2}.$$

If we take the upper sign in the ambiguity $\pm$, then

$$x+\tfrac{17}{2}=+\tfrac{29}{2}.$$

$$\therefore\ x=\tfrac{29}{2}-\tfrac{17}{2}$$

$$=6.$$

If we take the lower sign, then

$$x+\tfrac{17}{2}=-\tfrac{29}{2}.$$

$$\therefore\ x=-\tfrac{29}{2}-\tfrac{17}{2}$$

$$=-23.$$

Hence the roots of the equation are 6 and -23.

Note. If we had seen that the factors of $x^2+17x-138$ were $x+23$ and $x-6$, we should have obtained the answer at once; and generally, we only make use of the above method of completing the square when we cannot see the factors.

209. The following are additional examples.

Ex. 1. *Solve the equation* $x^2 - 6x + 3 = 7x - 2x^2 - 1$.

(i) Transposing, we have $3x^2 - 13x = -4$.

(ii) Divide by 3, $\therefore x^2 - \frac{13}{3}x = -\frac{4}{3}$.

(iii) Complete the square: that is, add $(\frac{1}{2}\,\frac{13}{3})^2$ to each side,

$$\therefore x^2 - \tfrac{13}{3}x + (\tfrac{13}{6})^2 = -\tfrac{4}{3} + (\tfrac{13}{6})^2.$$

(iv) Simplify,

$$= -\tfrac{4}{3} + \tfrac{169}{36}$$
$$= \tfrac{121}{36}.$$

(v) Express each side as a perfect square,

$$\therefore (x - \tfrac{13}{6})^2 = (\tfrac{11}{6})^2.$$

(vi) Take the square root of each side of the equation,

$$\therefore x - \tfrac{13}{6} = \pm \tfrac{11}{6}.$$

Hence the roots are $x = \frac{13}{6} + \frac{11}{6} = \frac{24}{6} = 4$,

and $x = \frac{13}{6} - \frac{11}{6} = \frac{2}{6} = \frac{1}{3}$.

Ex. 2. *Solve the equation* $x^2 + 6x - 100 = 0$.

Transpose, $\therefore x^2 + 6x = 100$.

Complete the square, $\therefore x^2 + 6x + (3)^2 = 100 + (3)^2$

$$= 109.$$

Take the square root of each side, $\therefore x + 3 = \pm\sqrt{109}$.

Hence the roots are $x = -3 + \sqrt{109}$, and $x = -3 - \sqrt{109}$.

We cannot find the exact square root of 109; and in such cases it is usual to leave surds in the result. But we can, if it be deemed desirable, take the square root of 109 to as many places of decimals as we like, and thus approximate to the roots. In this case

$$\sqrt{109} = 10{\cdot}4403\ldots.$$

Hence the roots are

$$x = -3 + 10{\cdot}4403\ldots = 7{\cdot}4403\ldots,$$

and $x' = -3 - 10{\cdot}4403\ldots = -13{\cdot}4403\ldots$.

**Ex.* 3. *Solve the equation* $x^2 + 2x + 2 = 0$.

Following the above order of procedure, we have

$$x^2 + 2x = -2.$$
$$\therefore x^2 + 2x + 1 = -2 + 1 = -1.$$
$$\therefore (x + 1) = \pm\sqrt{-1}.$$

That is, the roots are $-1 + \sqrt{-1}$ and $-1 - \sqrt{-1}$, both of which are imaginary [Art. 119].

Ex. 4. *Solve the equation* $ax^2+bx+c=0$.

This is the general form of a quadratic equation, and therefore includes all the preceding examples.

Transpose the absolute term $\therefore\ ax^2+bx=-c$,

Divide by a, $$\therefore\ x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Complete the square, $$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$
$$=\frac{b^2-4ac}{4a^2}.$$

Take the square root of each side, $$\therefore\ x+\frac{b}{2a}=\pm\frac{\sqrt{(b^2-4ac)}}{2a}.$$

Hence the roots are $x=\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$, that is, are

$$x=-\frac{b}{2a}+\frac{\sqrt{(b^2-4ac)}}{2a},\quad\text{and}\quad -\frac{b}{2a}-\frac{\sqrt{(b^2-4ac)}}{2a}.$$

210. It is convenient to recollect the result of the last example, namely, that the roots of the equation

$$ax^2+bx+c=0,$$

are $$\frac{-b+\sqrt{b^2-4ac}}{2a},\quad\text{and}\quad\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

These roots are real if $b^2>4ac$.

They are equal if $b^2=4ac$.

They are imaginary if $b^2<4ac$.

The student will notice that one root exceeds $-\dfrac{b}{2a}$ by as much as the other falls short of it.

***211. Third method of solution. Method by Substitution.** The following is another method of solving a quadratic equation, but the student will not find it so convenient in practice as the methods already given. It is known as the *method by substitution.*

Suppose the equation to be

$$ax^2+bx+c=0.$$

Let us put $x=y+h$, where h is some quantity which we will ultimately fix as may be most convenient to us,

$$\therefore\ a(y+h)^2+b(y+h)+c=0,$$

that is, $$ay^2+2ahy+ah^2+by+bh+c=0,$$

that is, $$ay^2+y(2ah+b)+ah^2+bh+c=0.$$

Now we may give any value that we like to h. Let us choose it so that $2ah+b=0$; that is, take $h=-\frac{1}{2}\frac{b}{a}$.

The equation last written will then become

$$ay^2+\frac{1}{4}\frac{b^2}{a}-\frac{1}{2}\frac{b^2}{a}+c=0,$$

that is, $$y^2=\frac{b^2-4ac}{4a^2},$$

the roots of which are $$y=\pm\frac{\sqrt{(b^2-4ac)}}{2a}.$$

But $$x=y+h=y-\frac{1}{2}\frac{b}{a}.$$

Hence the values of x, which are the required roots, are

$$\pm\frac{\sqrt{(b^2-4ac)}}{2a}-\frac{b}{2a},$$

which agree with the results given in Art. 210.

EXAMPLES. XVI. B.

[*Additional examples will be found in the collection at the end of the chapter, and numbered XVI. C.*]

Solve the following equations.

1. $x^2+6x=55.$
2. $2x^2+3x=2.$
3. $2x^2-3x=2.$
4. $5x^2-17x+14=0.$
5. $4x^2+4x-3=0.$
6. $4x^2-13x+3=0.$
7. $15x^2+34x+15=0.$
8. $15x^2+x-6=0.$
9. $13x^2-90x-7=0.$
10. $4x^2-11x-3=0.$
11. $14x^2-13x+3=0.$
12. $5x^2-12x=9.$

13. $3x^2+7x-76=0$.

14. $17x^2+19x-1848=0$.

15. $21x^2-2x=3$.

16. $4x^2-65x+126=0$.

17. $4x^2-8x+3=0$.

18. $8(x^2-1)=3(2x-3)$.

19. $3x^2+1=\frac{28}{5}x$.

20. $2x^2+697x=349$.

21. $(3x-4)(4x-3)=10$.

22. $(x-b)(x+b)=a^2-b^2$.

23. $(x-19)(x-21)=8$.

24. $(x-2)^2+(x-3)^2=(x+6)^2$.

25. $2ax^2+bx=2a-b$.

26. $x^2+2(a-b)x+b^2=2ab$.

27. $a(x^2+1)=x(a^2+1)$.

28. $2ax^2-4x=2a+a^2x$.

29. $abx^2+(a+b)x=-1$.

30. $(a^2-b^2)(x^2-1)=2x(a^2+b^2)$.

31. $5-x\{x-3(3x-5)\}+2(2x-1)^2=0$.

32. $x^2-(a-b)x+(a-b+c)c=2cx+ab$.

33. $(p-q)x^2-(p+q)x+2q=0$.

34. $3x^2-6ax-12b=4x-9bx-8a$.

***35.** $8(x+a)(x+b)(x+c)$

$$=(2x-a+b+c)(2x+a-b+c)(2x+a+b-c).$$

212. The equations may contain fractions, involving x in the denominators. In such cases we must multiply throughout by the L.C.M. of the denominators, and simplify the resulting equations, before we can apply the above rules.

We must however be careful to multiply only by the L.C.M., as we may otherwise introduce a factor which will apparently give us a root of the given equation, but which will not really satisfy it. This is illustrated in Example 2 which is worked out on the next page.

Should the numerator of any fraction be of equal or of higher dimensions than the denominator, it may be convenient to begin by dividing it by the denominator [Art. 142]. This is illustrated in Example 3 on page 208. But although this may simplify the work, it is not necessary.

We may add that the simplification of equations involving fractions, and the combination of the fractions involved, frequently afford opportunity for ingenuity in arranging the work so as to enable the solution to be obtained in a few lines; but no precise rules can be laid down, and nothing but practice combined with a certain natural aptitude will give the requisite skill. All the examples at the end of this chapter are soluble by direct analysis, and do not necessarily require the use of special devices.

213. The following examples illustrate these remarks.

Ex. 1. *Solve the equation* $\dfrac{5}{x-1}-\dfrac{4}{x+1}=\dfrac{3}{x+7}$.

The L.C.M. of the denominators is $(x-1)(x+1)(x+7)$. Multiplying throughout by this, we have

$$5(x+1)(x+7)-4(x-1)(x+7)=3(x-1)(x+1).$$

$$\therefore\ 5(x^2+8x+7)-4(x^2+6x-7)=3(x^2-1).$$

$$\therefore\ 2x^2-16x-66=0.$$

$$\therefore\ x^2-8x-33=0.$$

$$\therefore\ (x-11)(x+3)=0.$$

$$\therefore\ x=11, \text{ or } x=-3.$$

Hence the roots are 11 and -3.

Ex. 2. *Solve the equation* $\dfrac{x+1}{x^2+x-2}+\dfrac{x-1}{x^2+3x+2}-\dfrac{1}{x^2-1}=0$.

The L.C.M. of the denominators is $(x+2)(x-1)(x+1)$. Multiplying throughout by this, we have

$$(x+1)^2+(x-1)^2-(x+2)=0.$$

$$\therefore\ 2x^2-x=0.$$

$$\therefore\ x(2x-1)=0.$$

$$\therefore\ x=0, \text{ or } 2x-1=0.$$

$$\therefore\ x=0, \text{ or } x=\tfrac{1}{2}.$$

Hence the roots are 0 and $\frac{1}{2}$.

[Now if we had multiplied the given equation by the product of the first two denominators, instead of by the L.C.M. of the three denominators, we should have obtained as the resulting equation

$$(x+1)(x^2+3x+2)+(x-1)(x^2+x-2)-(x^2+4x+4)=0,$$

which reduces to $2x^3+3x^2-2x=0$. If we divide by x, which is a factor of the left-hand side, we get $x=0$ and $2x^2+3x-2=0$. The latter equation may be written $(2x-1)(x+2)=0$, and its roots are therefore $x=\frac{1}{2}$ and $x=-2$; and we might think that 0, $\frac{1}{2}$, and -2 were all roots of the given equation. But the root

$x=-2$ arises from the factor $x+2$ in $2x^2+3x-2$; and this factor was introduced by our having multiplied every term in the original equation by $x+2$ in addition to the L.C.M. of the denominators. The resulting expression of course had $x+2$ as a factor, but $x+2$ was no factor of the expression which was given as equal to zero.

We may say generally that no factor of an expression by which the given equation is multiplied will be a root of the equation. Hence we must be careful to multiply the given equation only by the L.C.M. of the denominators, and we shall then avoid the introduction of any unnecessary factor.]

Ex. 3. *Solve the equation* $\dfrac{x^2+1}{x-1}+\dfrac{x^2-2}{x-2}=2x.$

Since the numerators of the two fractions on the left-hand side are of a higher order than their respective denominators, it will be better to begin by dividing each numerator by the corresponding denominator. The equation then becomes

$$\left(x+1+\frac{2}{x-1}\right)+\left(x+2+\frac{2}{x-2}\right)=2x.$$

$$\therefore\ 2x+3+\frac{2}{x-1}+\frac{2}{x-2}=2x.$$

$$\therefore\ 3+\frac{2}{x-1}+\frac{2}{x-2}=0.$$

The L.C.M. of the denominators is $(x-1)(x-2)$. Multiplying throughout by this, we obtain

$$3(x-1)(x-2)+2(x-2)+2(x-1)=0.$$

$$\therefore\ 3x^2-5x=0.$$

$$\therefore\ x(3x-5)=0.$$

$$\therefore\ x=0, \text{ or } 3x-5=0.$$

Hence the roots are 0 and $\frac{5}{3}$.

**Ex.* 4. *Solve the equation* $\dfrac{a+c}{x+a+c}-\dfrac{a}{x+a}=\dfrac{b}{x+b}-\dfrac{b-c}{x+b-c}.$

Since the numerator of each of these fractions is the same as the part of the denominator which is independent of x, we had better begin by writing the equation in the form

$$\left(1-\frac{x}{x+a+c}\right)-\left(1-\frac{x}{x+a}\right)=\left(1-\frac{x}{x+b}\right)-\left(1-\frac{x}{x+b-c}\right).$$

$$\therefore\ -\frac{x}{x+a+c}+\frac{x}{x+a}=-\frac{x}{x+b}+\frac{x}{x+b-c}.$$

x is a factor of every term, and therefore [Art. 206] $x=0$ is one root. The other root is determined by the equation

$$-\frac{1}{x+a+c}+\frac{1}{x+a}=-\frac{1}{x+b}+\frac{1}{x+b-c}.$$

This equation is equivalent to

$$\frac{-(x+a)+(x+a+c)}{(x+a)(x+a+c)}=\frac{-(x+b-c)+(x+b)}{(x+b)(x+b-c)}.$$

$$\therefore\ \frac{c}{(x+a)(x+a+c)}=\frac{c}{(x+b)(x+b-c)}.$$

Divide by c; and multiply up,

$$\therefore\ (x+b)(x+b-c)=(x+a)(x+a+c).$$

$$\therefore\ x^2+x(2b-c)+b^2-bc=x^2+x(2a+c)+a^2+ac.$$

$$\therefore\ x(2b-2c-2a)=a^2+ac-b^2+bc.$$

$$\therefore\ -2x(a-b+c)=(a+b)(a-b+c).$$

$$\therefore\ -2x=a+b.$$

$$\therefore\ x=-\tfrac{1}{2}(a+b).$$

Hence the roots are 0 and $-\frac{1}{2}(a+b)$.

MISCELLANEOUS EXAMPLES. XVI. C.

1. Shew that the definition of the roots of an equation is satisfied by the statement that 1 and 2 are roots of the equation $x^2-3x+2=0$.

2. Determine whether 1, $\frac{1}{3}$, $-\frac{3}{4}$, or any of them, are roots of the equation $3(4x^3+1)=x(7x+8)$.

3. Determine whether $+1$, -1, or either of them, is a root of the equation $(2x^2+1)\sqrt{2x^2+3x+1}=x^2-99x-100$.

4. Form equations of which the roots are respectively (i) 1 and 2; (ii) -3 and -5; (iii) 0 and 4; (iv) a and b.

Solve the following equations.

5. $x^2-4x=5$.
6. $6x^2-x-1=0$.
7. $5x^2-26x+5=0$.
8. $13x^2-17x=66$.
9. $17x^2-14x=40$.
10. $27x^2-24x-16=0$.

11. $15x^2+7x=2.$ **12.** $77(x^2-1)=72x.$

13. $5x^2+6x-104=0.$ **14.** $20x^2-x-1=0.$

15. $15x^2+7x=74.$ **16.** $12x^2-95x-8=0.$

17. $12x^2-x-6=0.$ **18.** $6x^2-11x-35=0.$

19. $15x^2+2x-77=0.$ **20.** $5x^2+4x-1=0.$

21. $4x^2-17x+4=0.$ **22.** $18x^2-11x-24=0.$

23. $28x^2-13x-6=0.$ **24.** $(3x-11)(2x-10)=x^2-9.$

25. $5(x-9)^2-(2x-11)^2+(3x-21)^2=53.$

26. $(2x+1)(3x-7)+(2x-5)^2=(x+2)(2x-5).$

27. $(x-2)^2+(x+5)^2=(x+7)^2.$

28. $6(2x-7)^2-(x-13)^2+5x-15=0.$

29. $x\{2x-4(5x-3)\}=2(3x-2)^2.$

30. $(3x+1)^2-3(3x+1)(x+5)+2(x+5)^2=0.$

31. $(x-1)(x-2)(x-3)=(x^2-2)(x+25).$

32. $ax^2+bx=cx.$ **33.** $ab(x+1)x+a^2=(a^2+2b^2)x+4b^2.$

34. $x^2-2ax-2bx+4ab=0.$ **35.** $ax^2+2bx=a-2b.$

36. $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2.$ **37.** $\dfrac{a^2}{a+x}+\dfrac{b^2}{b+x}=a+b.$

38. $12x-\dfrac{6}{x}=21.$ **39.** $5\,\dfrac{6-x}{4}+\dfrac{4}{x-3}=1.$

40. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=2\tfrac{1}{6}.$ **41.** $\dfrac{x+2}{x+7}+\dfrac{1}{x}=\tfrac{5}{6}.$

42. $\tfrac{1}{7}(x+\tfrac{5}{6})=\dfrac{13}{6x}.$ **43.** $\dfrac{1}{3x-2}-\dfrac{3}{5(x-1)}=-\tfrac{7}{20}.$

44. $\dfrac{4}{3x-5}-\dfrac{5}{2x+1}=3.$ **45.** $\dfrac{x+1}{x+2}+\dfrac{x+2}{x+3}=1\tfrac{5}{12}.$

46. $\dfrac{7x-11}{4x-7}+\dfrac{3x-2}{12x-1}=\dfrac{2x+5}{x+2}.$ **47.** $\dfrac{x-5}{4}+\dfrac{4}{x-5}=\dfrac{3x-1}{4}.$

48. $\dfrac{1}{x-2}+\dfrac{2}{x-1}=\dfrac{6}{x}.$ **49.** $7-\dfrac{2}{x+4}+\dfrac{4}{(x+6)(x+4)}=0.$

50. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-1}=\dfrac{x-2}{x-3}-\dfrac{x-3}{x-2}.$ **51.** $\dfrac{2x+3}{2x-3}+\dfrac{3x+2}{3x-2}=2\,\dfrac{x-1}{x+1}.$

52. $\dfrac{3x+2}{x-3}+\dfrac{3x-2}{x+3}=\dfrac{4x^2+12x+2}{x^2-9}.$ ***53.** $\dfrac{1}{x^2-3x}+\dfrac{1}{x^2+4x}=\dfrac{9}{8x}.$

54. $1+\dfrac{2}{(x-7)(x-3)}=\dfrac{7}{x-3}$.

55. $1-\dfrac{4}{x-3}=\dfrac{3}{(x-3)(x-5)}$.

56. $2\dfrac{x-3}{x+1}+\dfrac{24}{x^2-1}=\dfrac{x+2}{x-1}$.

57. $\dfrac{2x}{x-1}+\dfrac{3x-1}{x+2}-\dfrac{5x-11}{x-2}=0$.

58. $3-\dfrac{1}{x+2}=\dfrac{5}{(x+2)(2x-3)}+\dfrac{3}{4x-6}$.

59. $\dfrac{2}{x-1}-\dfrac{x-1}{2}=\dfrac{2}{x-6}-\dfrac{x-6}{2}$.

60. $\dfrac{x-1}{x^2-5x+6}+\dfrac{x-2}{x^2-4x+3}-\dfrac{x-3}{x^2-3x+2}=0$.

61. $\dfrac{1}{16(x-2)}+\dfrac{1}{16(x+2)}-\dfrac{x}{8(x^2+4)}=\dfrac{1}{2(x-1)}+\dfrac{1}{2(x+1)}-\dfrac{1}{x}$.

62. $\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}-\dfrac{x}{2(x^2+1)}=\dfrac{1}{8(x-2)}+\dfrac{1}{8(x+2)}-\dfrac{1}{4x}$.

63. $\dfrac{1}{7(x-3)(x-2)}+\dfrac{x-4}{(x-1)(x-3)}-\dfrac{x-3}{(x-1)(x-2)}=0$.

64. $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$.

65. $\dfrac{x}{x-\dfrac{1}{2+\dfrac{x}{x-3}}}=1$.

66. $\dfrac{a+1}{x}-\dfrac{x+1}{a}+a+1=0$.

67. $\dfrac{a-x}{x-b}+\dfrac{x-b}{a-x}=\dfrac{13}{6}$.

68. $\dfrac{a^3+b^3}{x^2+ab}-\dfrac{ab}{x}=0$.

69. $\dfrac{1}{a+x}+\dfrac{1}{b+x}=\dfrac{a+b}{ab}$.

70. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=1$.

71. $\dfrac{1}{2x-5a}+\dfrac{5}{2x-a}=\dfrac{2}{a}$.

72. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{a+b}{ab}$.

73. $\dfrac{x+b}{a+b}+\dfrac{a}{b}=\dfrac{x^2+ab}{bx}$.

74. $\dfrac{x}{x+a}+\dfrac{x+b}{x}=\dfrac{c}{a+c}+\dfrac{b+c}{c}$.

75. $\dfrac{1}{x+a+b}=\dfrac{1}{x}+\dfrac{1}{a}+\dfrac{1}{b}$.

76. $\dfrac{ax+b}{cx+b}+\dfrac{bx+a}{cx+a}=\dfrac{(a+b)(x+2)}{cx+a+b}$.

CHAPTER XVII.

SIMULTANEOUS EQUATIONS, OF WHICH AT LEAST ONE IS OF A DEGREE HIGHER THAN THE FIRST.

214. WE now proceed to consider the solution of two or more simultaneous equations, one or more of them being quadratics, or at least of a degree higher than the first. We shall begin by treating the case where there are only two equations involving two unknown quantities, and one of the equations is of the first degree. We shall next discuss the solution of two simultaneous equations involving two unknown quantities, where both the equations are quadratics. The theory of simultaneous equations of a higher order and not reducible to one of these two cases lies outside the limits of this work.

215. **Solution of Two Simultaneous Equations, one being a Simple Equation and the other a Quadratic Equation.** We commence with the case of two equations involving only two unknown quantities, where one of the equations is of the first degree. We can always solve such a system by the following rule.

First. From the simple equation, find the value of one of the unknown quantities (say x) in terms of the other unknown quantity (say y) and the known quantities.

Second. Substitute this value of x in the quadratic equation. The result will be a quadratic equation involving only y and known quantities.

Third. The solution of this quadratic will give two values of y.

Fourth. To each of the values of y so determined will correspond a certain value of x, which can be determined from the value of x in terms of y which was originally obtained from the simple equation.

216. The following examples illustrate the method.

Ex. 1. *Solve the simultaneous equations*

$$\begin{cases} 2x+3y=1 \quad \dots\dots\dots\dots\dots\dots\dots\dots (a), \\ 3x^2-xy+2y^2=16 \quad \dots\dots\dots\dots\dots\dots\dots (b). \end{cases}$$

(i) It is usually best to solve the simple equation for that variable whose numerical coefficient is the smaller. In this case, therefore, we solve it for x. From (a), we have

$$2x=1-3y,$$

$$\therefore\ x=\tfrac{1}{2}(1-3y).$$

(ii) Substitute this value of x in (b),

$$\therefore\ \tfrac{3}{4}(1-3y)^2-\tfrac{1}{2}y(1-3y)+2y^2=16.$$

Multiply by 4, collect like terms, and simplify,

$$\therefore\ 41y^2-20y-61=0.$$

(iii) This quadratic must now be solved. Resolving into factors, we have

$$(y+1)(41y-61)=0.$$

[These factors ought to be obvious by Art. 114, since the only positive integral factors of 41 are 41 and 1, and of 61 are 61 and 1. If however the factors be not obvious, then the quadratic in y must be solved by the method of Art. 208.]

$$\therefore\ y+1=0, \text{ or } 41y-61=0.$$

$$\therefore\ y=-1, \text{ or } y=\tfrac{61}{41}.$$

(iv) We have now to find the values of x which correspond to these values of y. We have $x=\frac{1}{2}(1-3y)$.

If $y=-1$, $\quad\therefore\ x=\frac{1}{2}(1-3y)=\frac{1}{2}(1+3)=2.$

If $y=\frac{61}{41}$, $\quad\therefore\ x=\frac{1}{2}(1-3y)=\frac{1}{2}(1-3\cdot\frac{61}{41})=-\frac{71}{41}.$

Hence the roots of the original system are

$$x=2,\ y=-1;\ \text{and}\ x=-\frac{71}{41},\ y=\frac{61}{41}.$$

Ex. 2. *Solve the simultaneous equations*

$$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=2 \text{ .... (i)}, \\ \dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=2 \text{ .... (ii)}. \end{cases}$$

Here, we treat $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities.

From (i), $$\frac{a}{x}=2-\frac{b}{y}.$$

Substitute this value of $\frac{1}{x}$ in (ii), $\quad\therefore\ \left(2-\frac{b}{y}\right)^2+\frac{b^2}{y^2}=2.$

Simplifying, $$\therefore\ y^2-2by+b^2=0.$$

$$\therefore\ (y-b)^2=0.$$

Hence the two roots of y are equal to one another, and each is equal to b.

We have now to find the values of x which correspond to these values of y. We have $$\frac{a}{x}=2-\frac{b}{y}.$$

$\therefore$ if $y=b$, $$\frac{a}{x}=2-\frac{b}{y}=2-\frac{b}{b}=2-1=1.$$

$$\therefore\ x=a.$$

Hence the roots of the original equations are

$$\left.\begin{matrix} x=a \\ y=b \end{matrix}\right\},\quad \text{and}\quad \left.\begin{matrix} x=a \\ y=b \end{matrix}\right\}.$$

Ex. 3. *Solve the simultaneous equations*

$$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=2 \text{ .... (i)}, \\ ax-by=a^2-b^2 \text{ ....(ii)}. \end{cases}$$

From (ii), we have $$x=\frac{a^2-b^2+by}{a} \text{ .... (iii)}.$$

Now (i) may be written $\quad ay+bx=2xy.$

Substitute in this latter equation, the value of x given in (iii),

$$\therefore\ ay+b\,\frac{a^2-b^2+by}{a}=2\,\frac{a^2-b^2+by}{a}\,y.$$

This, on simplification, reduces to

$$2by^2+y\,(a^2-3b^2)-b\,(a^2-b^2)=0\ \ldots\ldots\ldots\ldots\text{(iv)}.$$

$$\therefore\ (y-b)\,(2by+a^2-b^2)=0.$$

[If these factors be not obvious to the student, he must solve equation (iv) by one of the methods given in the last chapter, when he will obtain the roots given in the next line.]

$$\therefore\ y=b,\quad \text{or } y=-\frac{a^2-b^2}{2b}.$$

These values must now be successively substituted in (iii), and we shall thus obtain the corresponding values of x.

First, if $y=b$, $\therefore$ by (iii), $x=\dfrac{a^2-b^2+by}{a}=\dfrac{a^2-b^2+b^2}{a}=a.$

Second, if $y=-\dfrac{a^2-b^2}{2b}$, $\therefore$ by (iii), $x=\dfrac{a^2-b^2+by}{a}=\dfrac{a^2-b^2}{2a}.$

Hence the roots of the original equations are

$$x=a,\ y=b;\ \text{and}\ x=\frac{a^2-b^2}{2a},\quad y=-\frac{a^2-b^2}{2b}.$$

Ex. 4. *Solve the equations* $x-y=3$, $x^2-y^2=6$.

Although the method by which the three preceding examples were solved is always applicable, yet the work may sometimes be slightly facilitated by noticing whether the terms involving x and y in the given simple equation form a factor of part of the quadratic equation. In this case, the second equation may be written

$$(x-y)\,(x+y)=6.$$

Dividing each side by the corresponding side of the first equation, we obtain

$$\frac{(x-y)\,(x+y)}{x-y}=\frac{6}{3}.$$

$$\therefore\ x+y=2.$$

We have now the simultaneous equations

$$x-y=3,\ x+y=2,$$

the roots of which are $x=\frac{5}{2}$, $y=-\frac{1}{2}$.

The equations given in this example might have been solved in the same way as the equations in Examples 1, 2, and 3 above.

EXAMPLES. XVII. A.

Solve the following systems of equations.

1. $\begin{cases} xy=5, \\ x+y=6. \end{cases}$

2. $\begin{cases} x^2+y^2=41, \\ x+y=9. \end{cases}$

3. $\begin{cases} 9x^2-4y^2=576, \\ 2y-3x=-12. \end{cases}$

4. $\begin{cases} 25x^2-9y^2=675, \\ 3y+5x=45. \end{cases}$

5. $\begin{cases} x+y=8, \\ x^2+y^2=50. \end{cases}$

6. $\begin{cases} x^2+2xy=5, \\ x+y=3. \end{cases}$

7. $\begin{cases} x-y=3, \\ x^2+xy+y^2=93. \end{cases}$

8. $\begin{cases} x+y=7, \\ x^2+y^2=37. \end{cases}$

9. $\begin{cases} 3x+6y=21, \\ 3y+6x=4xy. \end{cases}$

10. $\begin{cases} x^2+y^2=10a^2+10b^2-12ab, \\ x+y=2(a+b). \end{cases}$

11. $\begin{cases} x(y+7)+y(x+5)=5, \\ 7y+4x=1. \end{cases}$

12. $\begin{cases} x^3+y^3=351, \\ x+y=9. \end{cases}$

13. $\begin{cases} x+y=\frac{29}{48}, \\ \frac{1}{x}+\frac{1}{y}=7\frac{11}{15}. \end{cases}$

14. $\begin{cases} \frac{x}{y}-\frac{y}{x}=\frac{3}{2}, \\ x+y=6. \end{cases}$

15. $\begin{cases} bx-ay=a^2-b^2, \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2}{b^2}-\frac{b^2}{a^2}. \end{cases}$

16. $\begin{cases} b^2(x-a)+a^2(y-b)=0, \\ \frac{1}{x-b}-\frac{1}{y-a}-\frac{1}{a-b}=0. \end{cases}$

217. **Solution of Two Simultaneous Quadratic Equations.** The solution of any two simultaneous quadratic equations involving two unknown quantities is not always possible by the methods which are explained in this book. But any simultaneous equations of the form

$$\left.\begin{aligned} ax^2+bxy+cy^2&=d \\ Ax^2+Bxy+Cy^2&=D \end{aligned}\right\},$$

(where all the terms involving the unknown quantities are of the second degree) can always be solved by the following method.

Dividing each side of the first equation by the corresponding side of the second equation, we have

$$\frac{ax^2+bxy+cy^2}{Ax^2+Bxy+Cy^2}=\frac{d}{D}.$$

Now put $y=vx$, and substitute vx for y wherever y occurs,

$$\therefore \quad \frac{x^2(a+bv+cv^2)}{x^2(A+Bv+Dv^2)}=\frac{d}{D}.$$

This equation reduces to a quadratic in v. Solving this, we obtain two values of v; let us denote them by α and β.

(i) If $v=\alpha$, $\therefore$ $y=\alpha x$. Put $y=\alpha x$ in the first of the given equations; we can from the resulting equation obtain two roots of x, to each of which will correspond a value of y, which may be obtained from the equation $y=\alpha x$.

(ii) If $v=\beta$, we can similarly obtain two values of x; and, thence, the two corresponding values of y.

Hence there are altogether four pairs of roots.

218. The following examples illustrate the method.

Ex. 1. *Solve the simultaneous equations*

$$\begin{cases} x^2+3xy=10, \\ xy+4y^2=\ 6. \end{cases}$$

Dividing, we have

$$\frac{x^2+3xy}{xy+4y^2}=\frac{10}{6}=\frac{5}{3}.$$

Let $y=vx$, $\qquad \therefore \quad \dfrac{x^2(1+3v)}{x^2(v+4v^2)}=\dfrac{5}{3}.$

This reduces to $\qquad 3(1+3v)=5(v+4v^2).$

$$\therefore \quad 20v^2-4v-3=0.$$

$$\therefore \quad v=\tfrac{1}{2}, \text{ or } v=-\tfrac{3}{10}.$$

(i) If $v=\frac{1}{2}$, $\therefore\ y=\frac{1}{2}x$. Substitute this value in the first of the given equations,

$$\therefore\ x^2+\tfrac{3}{2}x^2=10.$$
$$\therefore\ 2x^2+3x^2=20.$$
$$\therefore\ x^2=4.$$
$$\therefore\ x=\pm 2.$$

If $x=+2$, $\qquad \therefore\ y=\frac{1}{2}x=1$ .... (i).

If $x=-2$, $\qquad \therefore\ y=\frac{1}{2}x=-1$ .... (ii).

(ii) If $v=-\frac{3}{10}$, $\therefore\ y=-\frac{3}{10}x$. Substitute this value in the first of the given equations,

$$\therefore\ x^2-\tfrac{9}{10}x^2=10.$$
$$\therefore\ x=\pm 10.$$

If $x=+10$, $\qquad \therefore\ y=-\frac{3}{10}x=-3$ ....(iii).

If $x=-10$, $\qquad \therefore\ y=-\frac{3}{10}x=3$ ....(iv).

Hence the four pairs of roots are

$$\left.\begin{matrix}x=2\\y=1\end{matrix}\right\},\quad \left.\begin{matrix}x=-2\\y=-1\end{matrix}\right\},\quad \left.\begin{matrix}x=10\\y=-3\end{matrix}\right\},\quad \text{and}\ \left.\begin{matrix}x=-10\\y=3\end{matrix}\right\}.$$

[The student, in writing down the roots of simultaneous equations, must be careful to arrange them in their proper pairs: thus, in this example, $x=10$, $y=-3$, satisfy both the given equations. Values of x and y taken from different brackets will not in general satisfy the given equations: thus $x=10$, $y=3$, satisfy neither of the given equations.]

Alternative method. We can sometimes arrive at the result more simply by first forming from the two given equations a new one, of which both sides are perfect squares, but this method is only applicable in particular cases. Thus, in the above case where the given equations are

$$x^2+3xy=10,$$
$$xy+4y^2=6,$$

we obtain by addition

$$x^2+4xy+4y^2=16,$$

that is,

$$(x+2y)^2=16.$$
$$\therefore\ x+2y=\pm 4.$$

(i) If $x+2y=4$, we have the simultaneous equations

$$x+2y=4,$$
$$x^2+3xy=10.$$

The roots of these can be found by the method of Art. 215, and are

$$\left.\begin{array}{l} x=2 \\ y=1 \end{array}\right\} \text{ and } \left.\begin{array}{l} x=10 \\ y=-3 \end{array}\right\} \text{......................(i).}$$

(ii) If $x+2y=-4$, we have the simultaneous equations

$$x+2y=-4,$$
$$x^2+3xy=10.$$

The roots of these can be found by the method of Art. 215, and are

$$\left.\begin{array}{l} x=-2 \\ y=-1 \end{array}\right\} \text{ and } \left.\begin{array}{l} x=-10 \\ y=3 \end{array}\right\} \text{.................. (ii).}$$

Ex. 2. *Solve the simultaneous equations*

$$x^2+xy=21, \quad y^2+xy=28.$$

If we adopt the first method of solution, [Art. 217], we have

$$\frac{x^2+xy}{y^2+xy}=\frac{21}{28}=\frac{3}{4}.$$

Let $y=vx$, $\qquad \therefore \dfrac{x^2(1+v)}{x^2(v^2+v)}=\dfrac{3}{4}.$

[The quantity $1+v$ is a factor of both the numerator and the denominator of the fraction on the left-hand side. For the reason given in Art. 213, Ex. 2, it does not lead to a root of the quadratic.]

Dividing the numerator and the denominator of the fraction on the left-hand side by $1+v$, we have

$$\frac{1}{v}=\frac{3}{4}.$$
$$\therefore v=\tfrac{4}{3}.$$

But $y=vx$, hence $\qquad y=\tfrac{4}{3}x.$

Substitute this value of y in the first of the given equations,

$$\therefore x^2+\tfrac{4}{3}x^2=21.$$
$$\therefore 3x^2+4x^2=63.$$
$$\therefore x^2=9.$$
$$\therefore x=\pm 3.$$

But $y=\tfrac{4}{3}x$, $\qquad \therefore y=\pm 4.$

Hence the roots are $\qquad x=\pm 3,\ y=\pm 4.$

The alternative method of solution explained in *Ex.* 1, is applicable to the equations given in this example. If we adopt it, we obtain, by the addition of the given equations,

$$x^2+2xy+y^2=49.$$

$$\therefore\ x+y=\pm 7.$$

If $x+y=7$, this, combined with the first of the given equations, gives, by the method of Art. 215, $x=3$, $y=4$.
If $x+y=-7$, this, similarly, gives $x=-3$, $y=-4$.

**Ex.* 3. *Solve the equations* $x^2=y+12$, $y^2=x+12$.

These equations are of such a form that they can be solved by a process analogous to the second method used in the last two examples.

Subtracting, we obtain

$$x^2-y^2=y-x.$$

$$\therefore\ (x-y)(x+y)=-(x-y).$$

$$\therefore\ (x-y)(x+y+1)=0.$$

$$\therefore\ x-y=0,\ \text{or}\ x+y+1=0.$$

(i) If $x-y=0$, $\quad\therefore\ y=x$.
Substitute this value of y in the first of the given equations,

$$\therefore\ x^2=x+12.$$

$$\therefore\ (x-4)(x+3)=0.$$

$$\therefore\ x=4,\ \text{or}\ x=-3.$$

But $y=x$, $\therefore$ the corresponding values of y are 4 and -3.

Hence $x=4$, $y=4$; and $x=-3$, $y=-3$ are two sets of roots.

(ii) If $x+y+1=0$, $\quad\therefore\ y=-(1+x)$.
Substitute this value of y in the first of the given equations,

$$\therefore\ x^2=-(1+x)+12.$$

$$\therefore\ x^2+x-11=0,$$

the roots of which are $\quad x=\dfrac{-1\pm\sqrt{45}}{2}$.

The corresponding values of y are $y=\dfrac{-1\mp\sqrt{45}}{2}$.

These form two more sets of roots.

Thus altogether there are 4 sets of roots, namely,

$$\left.\begin{matrix}x=4\\y=4\end{matrix}\right\},\quad \left.\begin{matrix}x=-3\\y=-3\end{matrix}\right\},\quad \left.\begin{matrix}x=\frac{1}{2}(-1+\sqrt{45})\\y=\frac{1}{2}(-1-\sqrt{45})\end{matrix}\right\},\ \text{and}\ \left.\begin{matrix}x=\frac{1}{2}(-1-\sqrt{45})\\y=\frac{1}{2}(-1+\sqrt{45})\end{matrix}\right\}.$$

219. We may sometimes combine the equations so as to get rid of all the terms of the second degree and thus obtain an equation of the first degree. The case will then be reduced to one similar to that discussed in Arts. 215, 216.

Ex. 1. *Solve the simultaneous equations*

$$\left.\begin{aligned}(x+1)(y+2)&=10\\ xy&=3\end{aligned}\right\}.$$

Subtracting, we obtain $2x+y+2=7$.

$$\therefore\ y=5-2x.$$

Substitute this value of y in the second of the given equations,

$$\therefore\ x(5-2x)=3.$$
$$\therefore\ 2x^2-5x+3=0.$$
$$\therefore\ (x-1)(2x-3)=0.$$
$$\therefore\ x=1, \text{ or } x=\tfrac{3}{2}.$$

But $y=5-2x$, $\therefore$ (i) if $x=1$, $y=3$; and (ii) if $x=\frac{3}{2}$, $y=2$.

Hence the roots of the given equation are

$$x=1,\ y=3;\ \text{and}\ x=\tfrac{3}{2},\ y=2.$$

Ex. 2. *Solve the equations*

$$axy=c(bx+ay),\quad bxy=c(ax-by).$$

We have, by division, $\dfrac{a}{b}=\dfrac{bx+ay}{ax-by}$.

Multiply up, $\therefore\ a(ax-by)=b(bx+ay)$.

$$\therefore\ (a^2-b^2)x=2aby.$$

$$\therefore\ x=2\frac{ab}{a^2-b^2}y \text{ .... (i).}$$

Substitute this value of x in the first equation,

$$\therefore\ 2\frac{a^2b}{a^2-b^2}y^2=c\left(\frac{2ab^2}{a^2-b^2}y+ay\right),$$

which on reduction gives

$$y=0, \text{ or } y=\frac{(a^2+b^2)c}{2ab}.$$

The corresponding values of x will, by substitution in (i), be found to be

$$x=0, \text{ and } x=\frac{(a^2+b^2)c}{a^2-b^2}.$$

EXAMPLES. XVII. B.

Solve the following systems of equations.

1. $\left.\begin{aligned} x+y&=9 \\ x^2-xy+y^2&=27 \end{aligned}\right\}.$

2. $\left.\begin{aligned} 23x^2-y^2&=22 \\ 7y-23x&=200 \end{aligned}\right\}.$

3. $\left.\begin{aligned} \frac{m}{x}+\frac{n}{y}&=2 \\ ny-mx&=n^2-m^2 \end{aligned}\right\}.$

4. $\left.\begin{aligned} bx+ay&=a^2+b^2 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}&=\frac{b^2}{a^2}+\frac{a^2}{b^2} \end{aligned}\right\}.$

5. $\left.\begin{aligned} x^2+3xy&=2 \\ 3y^2+xy&=1 \end{aligned}\right\}.$

6. $\left.\begin{aligned} x^2+2xy&=39 \\ xy+2y^2&=65 \end{aligned}\right\}.$

7. $\left.\begin{aligned} x^2-5xy&=11 \\ y^2+3xy&=-2 \end{aligned}\right\}.$

8. $\left.\begin{aligned} x^2+2xy&=32 \\ 2y^2+xy&=16 \end{aligned}\right\}.$

9. $\left.\begin{aligned} x^2-xy+y^2&=3 \\ x^2+xy+y^2&=7 \end{aligned}\right\}.$

10. $\left.\begin{aligned} x^2-3xy&=-5 \\ y^2+5xy&=14 \end{aligned}\right\}.$

11. $\left.\begin{aligned} 3x^2-7xy+4y^2&=5 \\ 4x^2-7xy+3y^2&=2 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 2x^2-9xy+9y^2&=5 \\ 4x^2-10xy+11y^2&=35 \end{aligned}\right\}.$

13. $\left.\begin{aligned} x^2+xy-y^2&=29 \\ xy+2y^2&=7 \end{aligned}\right\}.$

14. $\left.\begin{aligned} 3x^2+5xy&=22 \\ 11xy-3y^2&=19 \end{aligned}\right\}$

15. $\left.\begin{aligned} 2x^2-3xy&=14 \\ 3y^2-x^2+1&=0 \end{aligned}\right\}.$

16. $\left.\begin{aligned} 4x^2+xy&=7 \\ 3xy+y^2&=18 \end{aligned}\right\}.$

17. $\left.\begin{aligned} x^2+4xy&=35 \\ 2xy-16y^2&=1 \end{aligned}\right\}.$

18. $\left.\begin{aligned} x^2+xy&=10 \\ y^2-xy&=3 \end{aligned}\right\}.$

19. $\left.\begin{aligned} x^2-y^2&=8x-16 \\ 3xy&=6x+7y \end{aligned}\right\}.$

20. $\left.\begin{aligned} x^2+2xy&=39 \\ 2y^2-3xy&=5 \end{aligned}\right\}.$

21. $\left.\begin{aligned} x^2+xy&=104 \\ xy-y^2&=15 \end{aligned}\right\}.$

22. $\left.\begin{aligned} 3x^2-5xy+2y^2&=14 \\ 2x^2-5xy+3y^2&=6 \end{aligned}\right\}.$

23. $\left.\begin{aligned} x^2+y^2&=50 \\ xy&=7 \end{aligned}\right\}.$

24. $\left.\begin{aligned} 4x^2+3xy&=10 \\ 3y^2+4xy&=20 \end{aligned}\right\}.$

25. $\left.\begin{aligned} 2x^2-xy&=4 \\ 8y^2-7xy&=4 \end{aligned}\right\}.$

26. $\left.\begin{aligned} x^2-xy+y^2&=7 \\ x^4+x^2y^2+y^4&=133 \end{aligned}\right\}.$

27. $\left.\begin{aligned} 32y^2-2xy&=11 \\ x^2+4y^2&=10 \end{aligned}\right\}.$

28. $\left.\begin{aligned} x^2+3xy&=7 \\ xy+3y^2&=14 \end{aligned}\right\}.$

29. $\left.\begin{aligned} x^2-3xy&=-2 \\ y^2+5xy&=11 \end{aligned}\right\}.$

30. $\left.\begin{aligned} x^2+q^2y^2&=p^2+2pq+2q^2 \\ xy&=p+q \end{aligned}\right\}.$

31. $\left.\begin{array}{r} x^2+q^2y^2=p^2-2pq+2q^2 \\ xy=p-q \end{array}\right\}.$ 32. $\left.\begin{array}{r} x^2+xy=a^2 \\ y^2+xy=b^2 \end{array}\right\}.$

33. $\left.\begin{array}{r} x^2+y^2=5a^2+5b^2+8ab \\ xy=2a^2+2b^2+5ab \end{array}\right\}.$ 34. $\left.\begin{array}{r} xy+x=15 \\ xy-y=8 \end{array}\right\}.$

35. $2x^2+xy=y^2-2xy-4=12.$ 36. $x^2-2xy=2y^2+xy+4=16.$

37. $\left.\begin{array}{r} (x+2)(y+3)=15 \\ xy=2 \end{array}\right\}.$ 38. $\left.\begin{array}{r} xy-2x=21 \\ xy+3y=50 \end{array}\right\}.$

39. $\left.\begin{array}{r} 9x+8y+7xy=0 \\ 7x+4y+6xy=0 \end{array}\right\}.$ 40. $\left.\begin{array}{r} 5x+3y=y^2 \\ x-y=x^2 \end{array}\right\}.$

41. $\left.\begin{array}{r} \frac{x}{y}-\frac{y}{x}=\frac{15}{4} \\ x-y=3 \end{array}\right\}.$ 42. $\left.\begin{array}{r} x+\frac{1}{y}=\frac{18}{7} \\ y+\frac{1}{x}=\frac{7}{4} \end{array}\right\}.$ 43. $\left.\begin{array}{r} x-\frac{1}{y}=3\frac{4}{5} \\ y-\frac{1}{x}=4\frac{3}{4} \end{array}\right\}.$

44. $\left.\begin{array}{r} (x-3)^2+(y-3)^2=34 \\ xy-3(x+y)=6 \end{array}\right\}.$ 45. $\left.\begin{array}{r} (3x-y)(3y-x)=21 \\ 3x(3x-2y)=49-y^2 \end{array}\right\}.$

*46. $\left.\begin{array}{r} (2x-y)(2y-x)=10 \\ (x+y)(x-y)=7 \end{array}\right\}.$ 47. $\left.\begin{array}{r} (x+2y)(2x+y)=20 \\ 4x(x+y)=16-y^2 \end{array}\right\}.$

48. $\left.\begin{array}{r} \frac{x}{y}+\frac{y}{x}=\frac{5}{2} \\ x^2+3y^2=28 \end{array}\right\}.$ *49. $\left.\begin{array}{r} \frac{x}{b+y}=\frac{y}{a+x} \\ ax+by=(x+y)^2 \end{array}\right\}.$

*50. $\left.\begin{array}{r} 2x^2+3xy=27 \\ xy+y^2=4 \end{array}\right\}.$ 51. $\left.\begin{array}{r} (x+5)^2+(y+5)^2=85 \\ xy+5(x+y)=17 \end{array}\right\}.$

52. $\frac{1}{4x^2}-\frac{1}{y^2}=3,\quad \frac{1}{4x^2}-\frac{1}{xy}+\frac{1}{y^2}=9.$ 53. $x-\frac{b^2}{y}=a-b=\frac{a^2}{x}-y.$

*54. $\frac{7(x^2+1)}{10(xy+1)}=\frac{5(xy+1)}{7(y^2+1)}=\frac{x+1}{y+1}.$ *55. $\left.\begin{array}{r} x^3+y^3=468 \\ xy=\ \ 35 \end{array}\right\}.$

*56. $\left.\begin{array}{r} (x+y)(x^3+y^3)=1216 \\ x^3-y^3=49(x-y) \end{array}\right\}.$ *57. $\left.\begin{array}{r} xy=a(x+y) \\ x^2y^2=b^2(x^2+y)^2 \end{array}\right\}.$

*58. $\left.\begin{array}{r} x^2y+xy^2=180 \\ x^3+y^3=189 \end{array}\right\}.$ *59. $\left.\begin{array}{r} x^3+y^3=407 \\ x^2y+xy^2=308 \end{array}\right\}.$

*60. $ax^2+bxy+cy^2=bx^2+cxy+ay^2=a+b+c.$

*61. $x^2+yz=7,\quad y^2+zx=7,\quad z^2+xy=11.$

*62. $xy+zu=444,\quad yz+ux=156,\quad zx+yu=180,\quad xyzu=5184$;
where $x>y>z>u$. (Porson's Problem.)

CHAPTER XVIII.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

220. WE proceed now to the consideration of problems in which the given relations can be expressed by means of quadratic equations. We shall first confine ourselves to problems whose solution depends on that of a single quadratic equation, and shall then proceed to problems which require the solution of simultaneous equations.

Here, as in the problems leading to simple equations which have been previously considered, the formation of the equations is a mere translation from ordinary language into algebraical language. The solution of the resulting equation or equations can be effected by the processes explained in the two preceding chapters.

221. The following examples are typical of some of the more common problems in which the relations can be expressed by means of quadratic equations.

We may add that though there are always two roots of a quadratic equation, yet the student will sometimes find that it *is only one of these roots that will satisfy the problem* under discussion. The other root will generally be found to satisfy some analogous problem, the algebraical expression of which leads to the same quadratic equation as that formed from the given problem.

Ex. 1. *Divide sixteen into two parts such that their product shall be six times their difference.*

Let x be one part.

Therefore the other part is $16-x$.

Now the product of the two parts is, by the question, six times their difference,

$$\therefore\ x(16-x)=6\{x-(16-x)\}\ \ldots\ldots\ldots\ldots\ldots\ (\text{i}),$$

that is, $$16x-x^2=6x-96+6x.$$

$$\therefore\ x^2-4x-96=0.$$

$$\therefore\ (x-12)(x+8)=0.$$

$$\therefore\ x=12,\ \text{or}\ x=-8.$$

The latter root is not applicable to this problem; therefore one part is 12, and the other is 16 − 12, that is, is 4.

Note. In forming the equation (i), we subtracted $(16-x)$ from x to get their difference. Had we subtracted x from $16-x$ to get their difference, the equation corresponding to (i) would have been

$$x(16-x)=6\{(16-x)-x\},$$

which would have led to the equation

$$x^2-28x+96=0,$$

of which the roots are 4 and 24. The latter root is inapplicable to this problem, since it is greater than 16. Hence the answer would have been, that one part was 4 and the other part was 16 − 4, that is, 12. This is the same answer as before.

Ex. 2. *The price of photographs is raised* 3*s. per dozen, and customers consequently receive* 7 *less than before for a guinea. What were the prices charged?*

Suppose that before the price was raised, the charge was x shillings a dozen;

$$\therefore\ \text{for}\ \ 1s.\ \text{a man got}\ \frac{12}{x}\ \text{photographs},$$

$$\therefore\ \text{for}\ 21s.\ \ ,,\ \ ,,\ \ ,,\ \ \frac{12\times 21}{x}\ \ \ ,,$$

After the price was raised to $(x+3)$ shillings a dozen,

$$\text{then for}\ 21s.\ \text{a man got}\ \frac{12\times 21}{x+3}\ \text{photographs}.$$

This, by the question, was 7 less than before,

$$\therefore\ \frac{12\times 21}{x}=\frac{12\times 21}{x+3}+7.$$

Divide each term by 7, and multiply up,

$$\therefore\ 36(x+3)=36x+x(x+3).$$

$$\therefore\ x^2+3x-108=0.$$

$$\therefore\ (x-9)(x+12)=0.$$

$$\therefore\ x=9, \text{ or } x=-12.$$

The latter solution is obviously inapplicable to this problem, hence the answer is 9 shillings a dozen.

The student can verify that at this price the given relation would be true.

[*Note.* We can find what is indicated by the negative root of the resulting quadratic equation by observing that if the given relation had been that when the price was *lowered* 3*s.* a dozen the number received was 7 *more* than before, we should have obtained the equation

$$\frac{12\times 21}{x}=\frac{12\times 21}{x-3}-7,$$

which reduces to the equation $x^2-3x-108=0$, of which the roots are 12 and -9; hence in this case the price would have been 12*s.* a dozen. The root -12 in the original problem corresponds to this solution of this problem.]

Ex. 3. *A boat's crew can row at the rate of* 9 *miles an hour in still water. What is the speed of the current of a river, if it take them* $2\frac{1}{4}$ *hours to row* 18 *miles, of which* 9 *miles are up stream and* 9 *miles down stream?*

Suppose that the river is flowing at the rate of x miles an hour.

The crew can row at the rate of 9 miles an hour in still water.

$\therefore$ the crew row " " $(9-x)$ " " up stream,

and " " " " " $(9+x)$ " " down stream.

Therefore to row 9 miles up stream takes $\frac{9}{9-x}$ hours [Art. 101],

and " " down " $\frac{9}{9+x}$ "

The sum of these is stated in the question to be $2\frac{1}{4}$ hours,

$$\therefore\ \frac{9}{9-x}+\frac{9}{9+x}=2\tfrac{1}{4}=\frac{9}{4}.$$

Divide by 9, and multiply up,

$$\therefore\ 4(9+x)+4(9-x)=(9-x)(9+x).$$

$$\therefore\ 36+4x+36-4x=81-x^2.$$

$$\therefore\ x^2=9.$$

$$\therefore\ x=\pm 3.$$

The negative root is obviously inadmissible.

Hence the answer is 3 miles an hour.

**Ex.* 4. *The perimeter of a rectangular field is $\frac{34}{13}$ times its diagonal, and the length exceeds the breadth by* 70 *yards. What is its area?*

Suppose that the breadth of the field is x yards,

$\therefore$ the length of the field is $x+70$ yards;

$\therefore$ the perimeter is $2x+2(x+70)$ yards,

and the diagonal is $\sqrt{\{x^2+(x+70)^2\}}$ yards,

and the required area is $x(x+70)$ sq. yds.

By the relation given in the question, we have

$$2x+2(x+70)=\tfrac{34}{13}\sqrt{\{x^2+(x+70)^2\}}.$$

$$\therefore\ 13(2x+70)=17\sqrt{\{2x^2+140x+4900\}}.$$

Squaring both sides, we obtain

$$169(4x^2+280x+4900)=289(2x^2+140x+4900).$$

Simplifying, this reduces to

$$x^2+70x=6000.$$

Completing the square,

$$\therefore\ x^2+70x+(35)^2=6000+(35)^2$$
$$=7225.$$

$$\therefore\ x+35=\pm 85.$$

$$\therefore\ x=50, \text{ or } -120.$$

The latter root, being negative, is inadmissible,

$\therefore$ the breadth is 50 yards, and the length is 120 yards.

Hence the area is 50×120 square yards, that is, 6000 square yards.

222. We proceed now to give a few examples of problems which lead to simultaneous equations, of which one or more is of a degree higher than the first.

Ex. 1. *Find two numbers whose sum is* 34 *and product is* 145.

Let x and y be the numbers,

$$\therefore\ x+y=34 \ \ldots\ldots\ldots\ldots \text{(i)},$$

and

$$xy=145 \ \ldots\ldots\ldots\ldots \text{(ii)}.$$

We have therefore two equations involving two unknown quantities.

From (i) we have $\qquad x=34-y.$

Substitute this value of x in (ii),

$$\therefore\ (34-y)\,y=145.$$

$$\therefore\ y^2-34y+145=0.$$

$$\therefore\ (y-5)(y-29)=0.$$

[If these factors be not obvious, the equation must be solved by completing the square, and it will be found that the roots are the same as those given in the next line.]

$$\therefore\ y=5, \text{ or } y=29.$$

If $y=5$, $\therefore\ x=34-y=34-5=29.$

If $y=29$, $\therefore\ x=34-y=34-29=5.$

Hence the required numbers are 5 and 29.

Ex. 2. *Find a number of two digits such that, if it be divided by the product of its digits the quotient is* 2, *and if* 27 *be added to the number the order of the digits is reversed.*

Let x be the tens' digit, and y be the units' digit,

$$\therefore\ \text{the number is } 10x+y.$$

We are given that

$$\frac{10x+y}{xy}=2 \ \ldots\ldots\ldots\ldots \text{(i)}.$$

The number formed by reversing the order of the digits is $10y+x$.

Hence, by the relation given in the question,

$$10x+y+27=10y+x \ \ldots\ldots\ldots\ldots \text{(ii)}.$$

We have therefore two equations involving two unknown quantities.

The equation (i) may be written $10x+y=2xy$ ....(iii).
The equation (ii) reduces to $x-y+3=0$.

$$\therefore \; y=x+3.$$

Substitute this value of y in (iii),

$$\therefore \; 10x+(x+3)=2x(x+3).$$

Simplify, $\therefore \; 2x^2-5x-3=0.$

$$\therefore \; (2x+1)(x-3)=0.$$

$$\therefore \; x=-\tfrac{1}{2}, \text{ or } x=3.$$

To satisfy our problem x must be an integer, hence the only root suitable for our purpose is

$$x=3.$$

But $y=x+3$, and therefore if $x=3$, $y=x+3=3+3=6$.

Hence the required number, which is $10x+y$, is 36.

The student can satisfy himself that this number satisfies the given conditions.

Ex. 3. *The number of apples which can be obtained for a shilling is two more than the number of pears which can be obtained for a shilling, and the price of seven pears exceeds that of seven apples by a penny; find the prices charged for apples and pears respectively.*

Suppose that each apple costs x pence,

and that each pear costs y pence.

The number of apples obtained for a shilling is $\dfrac{12}{x}$,

and „ of pears „ „ „ $\dfrac{12}{y}$.

Hence, by the question, $\dfrac{12}{x}=\dfrac{12}{y}+2$,

which reduces to $6y=6x+xy$ .... (i).

Also the price of 7 apples is $7x$ pence,
and „ „ 7 pears is $7y$ „
but seven pears cost one penny more than six apples.

$$\therefore \; 7y=7x+1 \text{ ....(ii).}$$

We have therefore two equations involving two unknown quantities.

From (ii), we obtain $y=x+\tfrac{1}{7}$.

Substitute this value of y in (i),

$$\therefore\ 6\left(x+\tfrac{1}{7}\right)=6x+x\left(x+\tfrac{1}{7}\right).$$

$$\therefore\ 7x^2+x-6=0.$$

$$\therefore\ (7x-6)(x+1)=0.$$

$$\therefore\ x=\tfrac{6}{7},\ \text{or}\ x=-1.$$

Now x cannot be negative, $\therefore x=\frac{6}{7}$;
also, if $x=\frac{6}{7}$, $\therefore y=x+\frac{1}{7}=\frac{6}{7}+\frac{1}{7}=1$.

Therefore the pears cost a penny each, and the apples cost $\frac{6}{7}$ of a penny each, that is, the apples were sold at the rate of seven for sixpence.

Ex. 4. *Find the sides of a rectangle, whose area is unaltered if its length be increased by* 3 *inches while its breadth is diminished by* 2 *inches, and whose area is diminished by one quarter if its length be increased by* 9 *inches while its breadth is diminished by* 5 *inches.*

Suppose that one side is x inches long, and that the next side is y inches long,

$\therefore$ the area is xy square inches.

Had the length been $(x+3)$ in. and the breadth $(y-2)$ in., the area would have been $(x+3)(y-2)$ sq. in.
Hence, by the question, $xy=(x+3)(y-2)$,
which reduces to $3y-2x-6=0$ .... (i).

Had the length been $(x+9)$ in. and the breadth $(y-5)$ in. the area would have been $(x+9)(y-5)$ sq. in.
Hence, by the question, $\frac{3}{4}xy=(x+9)(y-5)$,
which reduces to $xy+36y-20x-180=0$ .... (ii).

We have therefore two equations involving two unknown quantities.

From (i) we obtain $x=\frac{3}{2}y-3$.
Substitute this value of x in (ii),

$$\therefore\ \left(\tfrac{3}{2}y-3\right)y+36y-20\left(\tfrac{3}{2}y-3\right)-180=0,$$

which reduces to

$$y^2+2y-80=0.$$

$$\therefore\ (y+10)(y-8)=0.$$

$$\therefore\ y=-10,\ \text{or}\ y=8.$$

Now y cannot be negative, $\therefore y=8$;
also, if $y=8$, $\therefore x=\frac{3}{2}y-3=\frac{3}{2}8-3=9$.

Hence one side is 9 inches long, and the other is 8 inches long

Ex. 5. *Two places, A and B, are* 168 *miles apart, and trains leave A for B and B for A simultaneously. They pass each other at the end of* 1 *hr.* 52 *min., and the first reaches B half-an-hour before the second reaches A. Find the speed of each train.*

Suppose that the train from A to B travels at the rate of x miles an hour; and that the train from B to A travels at the rate of y miles an hour.

At the end of 1 hr. 52 min. the first train has gone $1\frac{52}{60}\times x$ miles,
and „ „ „ „ second „ „ $1\frac{52}{60}\times y$ „ .
The sum of these is the whole distance of 168 miles,

$$\therefore\ 1\tfrac{52}{60}\times x+1\tfrac{52}{60}\times y=168,$$

which reduces to $x+y=90$ .... (i).

The train from A goes 168 miles in $\dfrac{168}{x}$ hours,

and „ „ B „ „ „ $\dfrac{168}{y}$ „

but the train from A travels 168 miles in $\frac{1}{2}$ hour less than that from B,

$$\therefore\ \frac{168}{x}=\frac{168}{y}-\frac{1}{2},$$

which reduces to $336y=336x-xy$ .... (ii).

We have therefore two equations involving two unknown quantities.

From (i) we have $x=90-y$.

Substitute this value of x in (ii),

$$\therefore\ 336y=336(90-y)-(90-y)y.$$

$$\therefore\ y^2-762y=-30240.$$

Completing the square, $\therefore\ y^2-762y+(381)^2=(381)^2-30240$
$=114921.$

$$\therefore\ y-381=\pm 339.$$

$$\therefore\ y=720,\ \text{or}\ y=42.$$

If $y=720$, $\therefore\ x=90-y=90-720=-630.$

If $y=42$, $\therefore\ x=90-y=90-42=48.$

Both x and y must be positive, hence the only solution suitable for the problem is $x=48$, $y=42$. Therefore the train from A to B travels at the rate of 48 miles an hour, and the train from B to A travels at the rate of 42 miles an hour.

EXAMPLES. XVIII.

1. Divide 18 into two parts whose product may be 12 times their difference.

2. The product of two consecutive odd numbers is greater than the sum of the numbers by 79. Find the numbers.

3. A distributes £180 in equal sums amongst a certain number of people. B distributes the same sum, but gives to each person £6 more than A, and gives to 40 persons fewer than A does. How much does A give to each person?

4. On a certain road the number of telegraph posts per mile is such that, if there were 3 less in each mile, the interval between the posts would be increased by $9\frac{3}{5}$ yards. Find the interval between two consecutive posts.

5. A rectangular room is 14 feet longer than it is broad, and 7 feet broader than it is high. If 1 foot were added to its height, 2 feet added to its breadth, and 6 feet taken from the length of the room, its cubical content would be unaltered. Find its dimensions.

6. The area of the floor of a certain room is 224 square feet; it is found that, by leaving a margin of 18 inches all round, a saving of 12 yards of carpet, which is three-quarters of a yard wide, is effected: what are the dimensions of the room?

7. A colonel draws up his whole regiment in two solid squares; he then increases the side of the smaller square by 6 men, and finds that he must diminish that of the larger by 4; he next increases the side of the (original) large square by 6 men, and finds that he has one man over. How many men are there in the regiment?

8. The men in a regiment can be formed into a solid square, and also into a hollow square 4 deep, the number of men in the front in the latter formation exceeding the number of men in the front in the former formation by 25. Find the number of men in the regiment.

9. The men in a regiment can be arranged in a hollow square 4 deep; if the number of men be increased by 129 they can be arranged in a solid square having on each side 10 men less than were on each outer side of the hollow square. Find how many men were in the regiment at first.

10. The men in a regiment can be arranged in a column twice as deep as its breadth; if the number be diminished by 206, the men can be arranged in a hollow square three deep having the same number of men in each outer side of the square as there were in the depth of the column; how many men were there at first in the regiment?

11. The area of an oblong room is 221 square feet; its perimeter is 60 feet. Find its length and breadth.

12. Two vessels, one of which sails faster than the other by 2 miles an hour, start together upon voyages of 1152 and 720 miles respectively; the slower vessel reaches its destination one day before the other: how many miles per hour did the faster vessel sail?

13. A man travels 39 miles on a bicycle; if his wheel had made 13 more revolutions per minute he would have covered the distance in 10 minutes less time, and if the circumference had been 1 foot greater, and had made the same number of revolutions as it did, he would have gone $3\frac{1}{4}$ miles farther. What time did he take?

14. A man travels 24 miles on a bicycle; if his wheel had made 8 more revolutions per minute he would have covered the distance in 10 minutes less time, and if the circumference had been 1 foot greater and had made the same number of revolutions as it did he would have gone 2 miles farther. What time did he take?

15. Two steamers ply between the same two ports a distance of 420 miles. One travels half-a-mile per hour faster than the other, and is two hours less on the journey. At what rates do they go?

16. Two bricklayers can together build a wall in 18 days. Find how long it would take each of them to build it by himself if it be known that one takes 15 days longer than the other.

17. A farmer bought a number of oxen for £240, and after losing 3, sold the remainder for £8 more a head than they had cost him, and gained £59 on the transaction: what number did he buy?

18. A merchant lays out £16. 3*s*. in buying silk. If it had cost 6*d*. less per yard, he would have got 8 yards more for his money. What was the cost per yard?

19. A man bought a number of articles for £20, if each had cost 2*s*. less, he would have got 10 more for the same money: how many did he buy?

20. The number of pence which a dozen apples cost is less by 2 than twice the number of apples which can be bought for 1*s*. How many apples can be bought for 8*s*.?

21. The number of eggs which can be bought for one shilling is equal to the number of pence which 27 eggs cost. How many eggs can be bought for one shilling?

22. What are eggs a dozen, when if two more were given for a shilling, the price would be lowered a shilling a dozen?

23. What is the price of oranges per dozen, when if two less were given in a shilling's worth the price would be raised one penny per dozen?

24. A woman spends five shillings in eggs; if she had paid threepence a dozen less for them, she would have got a dozen more for the same money. How many did she buy?

25. The number of fives-balls which can be bought for a pound is equal to the number of shillings in the cost of 125 of them. How many can be bought for a pound?

26. A man bought a number of articles for £21. If he had got 6 more for the same money they would have cost 1*s*. 3*d*. less apiece: how many did he buy?

27. A number of articles were sold for £6. 5*s*., and the seller by charging for each 6*d*. more than the cost price received for all but 10 as much as all had cost: how many were there?

28. A number of apples were bought for 5*s*. 10*d*. Half of the apples cost 1*d*. a dozen more than the others, and 12 more of the cheaper than of the dearer were obtained for 1*s*. How many apples were there?

29. A man bought a certain number of shares in a company for £375: if he had waited a few days until each share had fallen £6. 5*s*. in value, he might have bought five more for the same money. How many shares did he buy?

30. What two numbers are those whose sum is 47 and product 312?

31. Find a fraction such that the denominator exceeds the square of half the numerator by unity, and the product of the sum and difference of the numerator and denominator is 225.

32. A number consists of 2 digits of which that in the units' place is the greater; the difference between their squares is equal to the number, and if they be inverted the number thus formed is 7 times their sum. Find the number.

33. A number of two digits is equal to double the product of its digits and also equal to four times their sum. Find it.

34. A certain number exceeds the product of its digits by 52 and exceeds twice the sum of its digits by 53: find the number.

35. Find two numbers expressed by the same two digits in different orders, whose sum is equal to the square of the sum of the digits and whose difference is equal to the cube of the difference of the digits.

36. Find a number of three digits in which the digits in the hundreds' and tens' places each exceed that in the units' place by 1, and which is such that if it be divided by the digit in the units' place the quotient has 1 in the hundreds' place, and the other two digits are equal to half the digit in the hundreds' place in the original number.

*37. Find two numbers such that their product is 48, and the difference of their squares is to the sum of their cubes as 13 to 217.

*38. Divide the number 26 into three parts such that the sum of their squares may equal 300, and the square of the middle part may be half the sum of the squares of the greatest and least parts.

39. Find the sides of a rectangle whose area is unaltered, if its length be increased by 4 feet while its breadth is diminished by 3 feet, and which loses one-third of its area, if its length be increased by 16 feet while its breadth is diminished by 10 feet.

40. The area of a certain rectangle is equal to the area of a square whose side is three inches longer than one of the sides of the rectangle. If the breadth of the rectangle be decreased by one inch and its length increased by two inches the area is unaltered. Find the lengths of the sides of the rectangle.

*41. The diagonal of a rectangular field is 182 yards, and its perimeter is 476 yards. What is its area?

42. It costs twice as much per square yard to stain the floor of a room as to paper the walls, but it is found that the total sum spent on the floor is equal to that spent on the walls; moreover, if the room had been 2 yards shorter and 1 yard broader, the total sum spent on the floor would also have been the same as that spent on the walls. Given that 8 sq. yards of the walls are occupied by doors and windows and that the room is 12 feet high, find its other dimensions.

43. A rectangular grass-plot, the lengths of whose sides are as 3 to 4, is surrounded by a gravel walk of uniform width, the outer boundary of which is also rectangular. The area of the walk is to that of the grass-plot as 2 to 3, and the length of a diagonal of the grass plot is 100 feet: find the width of the walk, and the dimensions of the grass-plot.

44. Two rectangular fields were supposed each to contain 4 acres, but on measurement it was found that one contained 140 sq. yards more, the other 160 sq. yards less than that area, and that one was 10 yards longer and also 10 yards narrower than the other: find the dimensions of the fields.

45. Sixty-four square yards of the wall surface of a room which is 12 feet high are occupied by doors and windows, and the rest of the wall is papered at one shilling a sq. yd. The whole of the floor is also stained at a shilling a sq. yd. It is found that equal amounts of money are spent on paper and on staining, and that this would also be the case if the room were one yard shorter and two yards broader. Find the length and breadth of the room.

46. Two elevens *A* and *B* play a cricket-match. The total of *A*'s first innings is the square of the difference of the totals of *B*'s two innings, and the total of *A*'s second innings is one-fourth the sum of the totals of *B*'s two innings; *A* scored 215 more in their first innings than in their second, and lost the match by three runs. What were the respective scores?

47. *A*, *B*, *C* are candidates at an election. *A* polls a number of votes equal to the square of *B*'s majority over *C*, and *C* polls a number equal to the square of *A*'s majority over *B*: *A* polls 27 more than *C*. Find what each polls.

48. Two trains run, without stopping, over the same 36 miles of rail. One of them travels 15 miles an hour faster than the other and accomplishes the distance in 12 minutes less. Find the speeds of the two trains.

49. *A* starts to walk from *P* to *Q*: *B* starts 1 h. 40 m. later to drive from *P* to *Q*. *B* overtakes *A* at 10 miles from *P*, and picking him up brings him to *Q* in 5 hours from the time at which *A* started, which is 2 h. 30 m. less than *A* would have occupied in walking the whole distance. Find the rate at which *A* walks and *B* drives.

50. A man bought a certain number of articles of equal value for £75. By selling them at £1. 16*s*. each, he gained as much as ten of them cost him. How many did he buy?

51. A boy intends to spend a certain sum of money in lawn-tennis balls. There are two prices, each of the cheaper ones being 3*d.* less than each of the better ones. He finds he can buy as many of the better kind as he gives pence for each; but he can get 4 more of the cheaper ones. How much has he to spend, and what are the prices of the balls?

52. A person has ten guineas to spend in books of equal value; if, however, he has any of them bound, they will cost six shillings a volume extra. He reckons that if he gets half of them bound he will possess nine more volumes than he would were they all bound: what is the cost apiece of the books unbound?

53. A girl worked two square pieces of worsted work of the same kind, one of which was an inch longer than the other; the first took $12\frac{1}{2}$ skeins, the second 18 skeins. What was the length of the side of each square?

54. A sum of £19,950, if invested in the 4 per cents., would give an annual income of £8. 8*s.* more than if it were invested in the 3 per cents.: if however each stock were 1 per cent. higher in price, the former would give £9. 17*s.* 11*d.* more. What is the price of each stock?

***55.** John and Hodge met together at market, John had bought sheep, and Hodge pigs and geese. Whilst taking a friendly pot together they agreed to an exchange of goods, viz. John to give his sheep for Hodge's pigs and geese. The value of each sheep was the same as that of a pig and goose, and that of each goose was two shillings. Now the number of the pigs and geese together exceeded the number of sheep by 16; and the number of geese exceeded that of the pigs by 10. What was the number of each, the pigs and geese together being worth five pounds? (*The Ladies' Diary*, 1756.)

CHAPTER XIX.

EQUATIONS REDUCIBLE TO A QUADRATIC FORM.

223. It is customary to exclude from elementary algebra any discussion as to the general solution of equations of a degree higher than the second, but to include a few equations which are immediately reducible to a quadratic form.

We may remark that the solution of an algebraical equation can always be reduced to the solution of one which is (i) rational [Art. 182] and (ii) integral [p. 98]. Rational integral algebraical equations are classified according to the number of variables involved and their degree, and a rational integral algebraical equation involving one variable can always be solved, if it be of a degree not higher than four.

224. **Quadratics.** The following examples are typical of the more common equations which are quadratics in a power of x, and are at once reducible to a quadratic form.

Ex. 1. *Solve the equation* $x^4-4x^2+3=0$.

This is a quadratic in x^2.

Resolving into factors, $(x^2-3)(x^2-1)=0$.

$$\therefore\ x^2-3=0, \text{ or } x^2-1=0.$$

$$\therefore\ x=\pm\sqrt{3}, \text{ or } x=\pm 1.$$

Hence there are four roots, namely, $\pm\sqrt{3}$, ± 1.

We may also begin the solution by putting $x^2 = y$. The given equation then becomes

$$y^2 - 4y + 3 = 0,$$

the roots of which are $y = 3$ and $y = 1$.

$$\therefore x^2 = 3, \text{ or } x^2 = 1.$$

$$\therefore x = \pm\sqrt{3}, \text{ or } x = \pm 1.$$

Ex. 2. *Solve the equation* $x^6 + 8x^3 + 12 = 0$.

This is a quadratic in x^3,

$$\therefore (x^3 + 6)(x^3 + 2) = 0.$$

$$\therefore x^3 + 6 = 0 \text{ or } x^3 + 2 = 0.$$

$$\therefore x^3 = -6, \text{ or } x^3 = -2.$$

$$\therefore x = \sqrt[3]{(-6)} \text{ or } x = \sqrt[3]{(-2)}.$$

As a matter of fact there are three cube roots of -6, and also three cube roots of -2, but we shall not here concern ourselves with finding them. Thus there are six roots of the given equation. Similarly, if the given equation be of the degree n, there will always be n roots.

We might also have begun by putting $x^3 = y$, and then have proceeded as in the last example.

Note. Any equation of the form

$$ax^{2n} + bx^n + c = 0$$

is a quadratic in x^n, or is reducible to a quadratic by putting $x^n = y$. The solution of this quadratic gives two values of x^n. The n^{th} roots of these values give the roots of the given equation.

EXAMPLES. XIX. A.

Solve the following equations.

1. $x^4 - 13x^2 + 36 = 0$.

2. $3x^4 - 44x^2 + 121 = 0$.

3. $16x^4 - 40a^2x^2 + 9a^4 = 0$.

4. $x^6 - 7x^3 - 8 = 0$.

5. $x^{2n} + 2ax^n + a^2 - b^2 = 0$.

6. $\dfrac{4}{2x^2 - 5} - \dfrac{6}{3x^2 - 7} = 1$.

7. $\dfrac{x^2 + 16}{25} + \dfrac{25}{x^2 + 16} = 2$.

8. $\dfrac{x^2 - b^2}{a^2} + \dfrac{a^2}{x^2 - b^2} = 2$.

225. The method of solving another similar class of equations is illustrated by the following example.

Ex. *Solve the equation* $(x^2+2x-3)^2+3(x^2+2x+2)-13=0.$

Put $x^2+2x=y$,

$$\therefore\ (y-3)^2+3(y+2)-13=0.$$
$$\therefore\ y^2-3y+2=0.$$
$$\therefore\ (y-2)(y-1)=0.$$
$$\therefore\ y=2, \text{ or } y=1.$$

(i) If $y=2$, $\quad \therefore\ x^2+2x=2$,
the roots of which are $\quad -1\pm\sqrt{3}$.

(ii) If $y=1$, $\quad \therefore\ x^2+2x=1$,
the roots of which are $\quad -1\pm\sqrt{2}$.

Hence the four roots of the given equation are

$$-1\pm\sqrt{3},\quad -1\pm\sqrt{2}.$$

If we put x^2+2x in a bracket, and treat it like a separate symbol, we need not introduce y; but the beginner will probably find it easier to use a subsidiary symbol as in the above example.

Note. Any equation like

$$p(ax^2+bx+c)^2+q(ax^2+bx+d)+r=0,$$

can be solved in a similar way. Because, if we put $ax^2+bx=y$, we obtain a quadratic in y. Suppose the roots are h and k.

Then, (i) if $y=h$, $\therefore\ ax^2+bx=h$, which gives two roots of x.
And, (ii) if $y=k$, $\therefore\ ax^2+bx=k$, which gives two more roots of x.
Hence there are altogether four roots of x.

EXAMPLES. XIX. B.

Solve the following equations.

1. $(x^2-4x+5)(x^2-4x+2)+2=0.$
2. $(x^2+x+1)^2-4(x^2+x-1)-5=0.$
3. $(x^2+3x-1)^2-12(x^2+3x-2)+15=0.$
4. $12x^2-9x+3+\dfrac{616}{4x^2-3x+4}=77.$

226. Equations involving Surds. Equations involving surds may frequently be reduced to quadratic equations.

Examples of these are given later in Chapter XXIV. (pp. 298—302).

227. One root known. If one root of an equation be known or be obvious, we can reduce the degree of the equation by unity. If the given equation be of the third degree, we can thus reduce it to a quadratic.

Ex. 1. *One root of the equation* $x^3+10x^2-x=10$ *is* 1. *Find all the roots.*

Since $x=1$ satisfies the equation, therefore x^3+10x^2-x-10 vanishes when x is put equal to 1 [Art. 90], and therefore is divisible by $x-1$ [Art. 120]. Dividing x^3+10x^2-x-10 by $x-1$, we obtain

$$x^2+11x+10=0.$$

$$\therefore\ (x+10)(x+1)=0.$$

$$\therefore\ x=-10, \text{ and } x=-1.$$

Hence the other roots are -10 and -1; and therefore all the roots are $1, -10, -1$.

Note. Before dividing by $x-1$, we have to take all the terms to one side of the equation.

Ex. 2. *One root of the equation* $x^3=a^3$ *is* a. *Find the other roots.*

Since x^3-a^3 vanishes when $x=a$ [Art. 90], $\therefore$ x^3-a^3 is exactly divisible by $x-a$,

$$\therefore\ x^3-a^3=0$$

can be written $\quad (x-a)(x^2+ax+a^2)=0.$

One root is given by $\quad x-a=0.$

The other roots are given by $x^2+ax+a^2=0$, and $\therefore$ by Art. 210, are $\quad x=\frac{1}{2}(-a\pm\sqrt{-3a^2}).$

Hence there are three cube roots of a^3, namely

$$a,\ \tfrac{1}{2}(-a+\sqrt{-3a^2}), \text{ and } \tfrac{1}{2}(-a-\sqrt{-3a^2});$$

for each of these when cubed is equal to a^3. This was what was asserted in Art. 184. Of these three cube roots, two are imaginary, and only one is real.

EXAMPLES. XIX. C.

One root of each of the following equations is printed in square brackets by its side. Find the other roots.

1. $x^3+4x^2+x-6=0$, $[-3]$.

2. $5x^3-13x^2-13x+21=0$, $[1]$.

3. $6x^3-61ax^2-127a^2x-60a^3=0$, $[-a]$.

*4. $x^3+k^3=0$, $[-k]$.

228. **Reciprocal Equations.** An equation in x is said to be *reciprocal* if, when $\frac{1}{x}$ is substituted for x, and the resulting equation simplified, the form of the equation is unaltered.

An equation of the form

$$ax^4 \pm bx^3+cx^2 \pm bx+a=0$$

is reciprocal, since, if $\frac{1}{x}$ be substituted for x and the resulting equation be simplified, its form is unaltered. An equation of this form can be solved by dividing by x^2, and putting

$$x+\frac{1}{x}=y.$$

An equation of the form

$$ax^4 \pm bx^3+cx^2 \mp bx+a=0$$

can be solved by dividing by x^2, and putting

$$x-\frac{1}{x}=y.$$

If $-\frac{1}{x}$ be substituted for x in this equation and the resulting equation be simplified, its form is unaltered. Such equations are analogous to reciprocal equations.

229. The following are examples of reciprocal equations.

Ex. 1. *Solve the equation* $x^4-3x^3+4x^2-3x+1=0$.

Divide by x^2, $\therefore\ x^2-3x+4-3\dfrac{1}{x}+\dfrac{1}{x^2}=0$.

$$\therefore\ \left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)+4=0.$$

Let $x+\dfrac{1}{x}=y$, $\therefore\ x^2+2+\dfrac{1}{x^2}=y^2$, $\therefore\ x^2+\dfrac{1}{x^2}=y^2-2$,

$\therefore$ the given equation becomes

$$(y^2-2)-3y+4=0.$$
$$\therefore\ y^2-3y+2=0.$$
$$\therefore\ (y-2)(y-1)=0.$$
$$\therefore\ y=2, \text{ or } y=1.$$

(i) If $y=2$, $\therefore\ x+\dfrac{1}{x}=2$.

$$\therefore\ x^2+1=2x,$$

the roots of which are 1 and 1.

(ii) If $y=1$, $\therefore\ x+\dfrac{1}{x}=1$.

$$\therefore\ x^2+1=x,$$

the roots of which are $\frac{1}{2}(1\pm\sqrt{-3})$.

Hence the four roots of the given equation are $1, 1, \frac{1}{2}(1\pm\sqrt{-3})$.

Ex. 2. *Solve the equation* $\dfrac{x^2+1}{x+1}+2\dfrac{x+1}{x^2+1}=3$.

Let $$\frac{x^2+1}{x+1}=y,$$

$\therefore$ the given equation becomes

$$y+2\frac{1}{y}=3.$$
$$\therefore\ y^2-3y+2=0,$$

the roots of which are $y=2$ and $y=1$.

(i) If $y=2$, $\therefore \frac{x^2+1}{x+1}=2.$

$$\therefore x^2-2x-1=0.$$

$$\therefore x=1\pm\sqrt{2}.$$

(ii) If $y=1$, $\therefore \frac{x^2+1}{x+1}=1.$

$$\therefore x^2-x=0.$$

$$\therefore x(x-1)=0.$$

$$\therefore x=0, \text{ or } x=1.$$

Hence the four roots of the given equation are $1\pm\sqrt{2}$, 0, and 1.

EXAMPLES. XIX. D.

Solve the following reciprocal equations.

1. $2x^4+x^3-6x^2+x+2=0.$
2. $6x^4+35x^3+62x^2+35x+6=0.$
3. $2x^4-9x^3+14x^2-9x+2=0.$
4. $6x^4-25x^3+12x^2+25x+6=0.$

*5. $2x^4-5x^3+6x^2-5x+2=0.$

CHAPTER XX.

THE THEORY OF QUADRATIC EQUATIONS.

230. We discussed, in Chapter XVI., the means of solving a given quadratic equation, and we also there explained [Art. 206] how we could form a quadratic equation which has given numbers for its roots. We shall now proceed to investigate some of the general properties of the roots of any quadratic equation, and shall conclude the chapter by shewing how the form of the solution of a quadratic equation sometimes enables us to determine the greatest and least values which a given expression can assume.

231. We will begin by reminding the student [see Art. 210] that the roots of the equation

$$ax^2 + bx + c = 0,$$

are
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence the roots are *real* if $b^2 > 4ac$,

" " " *equal* if $b^2 = 4ac$,

" " " *imaginary* if $b^2 < 4ac$.

Moreover, if $b^2 - 4ac$ be a perfect square, then we can take the square root of it, and the roots of the given equation will be *rational* [Art. 182]; but if $b^2 - 4ac$ be not a perfect square, then the roots of the given equation will involve an irrational quantity.

232. **Sum and Product of the Roots of a Quadratic Equation.** The following is an important proposition. *The sum of the roots of the quadratic equation* $ax^2 + bx + c = 0$ *is equal to* $-\frac{b}{a}$, *and the product of the roots is equal to* $\frac{c}{a}$.

If we solve the equation, we find that the roots are

$$x = -\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a},$$

and

$$x = -\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}.$$

The sum of these is $-\frac{b}{2a} - \frac{b}{2a}$, that is $-\frac{b}{a}$.

The product of the roots

$$= \left\{-\frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}\right\}\left\{-\frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}\right\}$$

$$= \frac{b^2}{4a^2} - \left[\frac{\sqrt{(b^2 - 4ac)}}{2a}\right]^2 \qquad \text{[Art. 62, Ex. 3.}$$

$$= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}.$$

Note. The sum of the roots of the equation $x^2 + px + q = 0$ is $-p$, and their product is $+q$.

Hence, if we are given that the sum of two quantities is h and that their product is k, we know that the two quantities must be the roots of the equation

$$x^2 - hx + k = 0.$$

Ex. 1. *Find the sum and the product of the roots of the equation* $x^2+7x+6=0.$

The sum is -7, and the product is 6.

If the equation be solved, the roots will be found to be -6 and -1, which agrees with the above statement.

Ex. 2. *If* α *and* β *be the roots of the equation* $2x^2-5x+3=0$, *find the value of* $\alpha^3+\beta^3$.

We have $\alpha+\beta=\frac{5}{2}$, and $\alpha\beta=\frac{3}{2}$;

$$\therefore\ (\alpha+\beta)^3=(\tfrac{5}{2})^3,$$

that is,
$$\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)=\tfrac{125}{8}.$$

$$\therefore\ \alpha^3+\beta^3+3\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}=\tfrac{125}{8}.$$

$$\therefore\ \alpha^3+\beta^3=\tfrac{125}{8}-\tfrac{45}{4}=\tfrac{35}{8}.$$

EXAMPLES. XX. A.

Write down the sum and the product of the roots of the following equations, numbered 1 to 6.

1. $x^2-3x+4=0.$
2. $2x^2-4x+5=0.$
3. $ax^2-bcx+a^3=0.$
4. $2x^2-3=6(x-1).$
5. $3x(x-2)=2(1-3x).$
6. $(px-q)(qx-p)=(x-p)(x-q).$

7. If α and β be the roots of the equation $3x^2-7x+4=0$, find the values of $\alpha^2+\beta^2$ and $(\alpha-\beta)^2$.

8. If α and β be the roots of the equation $px^2+qx+p=0$, what is the value (i) of $\alpha-\beta$, and (ii) of $\alpha^3-\beta^3$?

9. If α and β be the roots of the equation $x^2-px+q=0$, find the values of $\alpha^2+\beta^2$, and $\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)$.

233. Resolution into Factors. *If h and k be the roots of the equation $ax^2+bx+c=0$, then will*

$$ax^2+bx+c=a(x-h)(x-k).$$

We have $h+k=-\dfrac{b}{a}$, and $hk=\dfrac{c}{a}$; [Art. 232]

$$\therefore\quad b=-a(h+k),\ \text{and}\ c=ahk.$$

Substitute these values of b and c in ax^2+bx+c,

$$\begin{aligned}\therefore\quad ax^2+bx+c&=ax^2-a(h+k)x+ahk\\&=a\{x^2-(h+k)x+hk\}\\&=a(x-h)(x-k).\end{aligned}$$

234. Hence, if we want to resolve the expression ax^2+bx+c into factors, we can proceed thus. Put the given expression equal to zero, and solve the resulting equation. If the roots be h and k, then the required factors will be $a(x-h)(x-k)$.

Ex. 1. *Find the factors of* $42x^2-5x-63$.

If we equate this to zero, we obtain the equation

$$42x^2-5x-63=0.$$

The student, on solving this, will find that the roots are $\frac{9}{7}$ and $-\frac{7}{6}$.

$$\begin{aligned}\therefore\ 42x^2-5x-63&=42\left(x-\tfrac{9}{7}\right)\left(x+\tfrac{7}{6}\right)\\&=(7x-9)(6x+7).\end{aligned}$$

Ex. 2. *Find the factors of the expression*

$$(ab+1)(x^2+1)+(a+b)(x^2-1)-(a^2+b^2-2)x.$$

If we equate this to zero, and arrange it in descending powers of x, we obtain the equation

$$(ab+a+b+1)x^2-(a^2+b^2-2)x+ab-a-b+1=0.$$

The student, on solving this by one of the methods given in Chapter XVI., will find that the roots are $\dfrac{b-1}{a+1}$ and $\dfrac{a-1}{b+1}$.

Hence the given expression is equal to

$$(ab+a+b+1)\left(x-\frac{b-1}{a+1}\right)\left(x-\frac{a-1}{b+1}\right),$$

which reduces to $\{(a+1)\,x-(b-1)\}\,\{(b+1)\,x-(a-1)\}$.

EXAMPLES. XX. B.

Find the factors of the following expressions.

1. $9x^2-24x+16$.
2. $16x^2+16x+3$.
3. $(x-1)(x-2)-6$.
4. $x^2-2ax+a^2-b^2$.
5. $x^2-a^2-2cx-b^2+2ab+c^2$.
6. $x^2-2(a+b)\,x-ab(a-2)(b+2)$.
7. $(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)$.

*8. $yz(y-z)+zx(z-x)+xy(x-y)$.

235. **If one root of a quadratic equation be obvious, the other root can be obtained after division.** For if h be a root of the equation, obtained by equating a quadratic expression to zero, the expression vanishes when x is put equal to h [Art. 90], therefore the expression is exactly divisible by $x-h$ [Art. 120]. The quotient will be an expression of the first degree in x, which gives a simple equation for the other root.

This is the same principle which we used in Art. 227 to reduce the degree of an equation by unity when one root was known.

Example. Solve the equation $\frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}$.

It is obvious that this equation is satisfied by $x=b$, $\therefore$ b is one root of it.

On reduction and simplification, the equation becomes

$$bx^2-x(a^2+b^2)+a^2b=0.$$

Since the left-hand side is known to vanish when $x=b$, it must be exactly divisible by $x-b$. Dividing by $x-b$, we obtain $bx-a^2=0$. Hence the other root is $\frac{a^2}{b}$.

EXAMPLES. XX. C.

Solve the following quadratic equations, numbered 1 to 3.

1. $\frac{a+b}{x}+ax=\frac{a+b}{c}+ac.$ 2. $x-a-\frac{1}{x-a}=b-a-\frac{1}{b-a}.$

3. $(x-1)(x-2)=(a-1)(a-2).$

4. Find for what other value of x, the expression $\frac{x}{c}-\frac{c}{x}$ has the same value as it has when $x=a$.

236. There cannot be more than two unequal roots of a Quadratic Equation.

First proof. For suppose, if possible, that there are three roots of the equation $ax^2+bx+c=0$, no two of them being equal. Denote them by h, k, l.

Since h and k are roots we have [Art. 233]

$$ax^2+bx+c=a(x-h)(x-k).$$

But since l is a root, it satisfies the equation [Art. 90]

$$\therefore \quad a(l-h)(l-k)=0.$$

Therefore one of the factors of this last expression must vanish. [Art. 204.] But a cannot be zero, and if either $l-h$ or $l-k$ vanish, then l is not different from h and k. That is, two of the assumed roots must be equal, which is contrary to the hypothesis.

Thus there cannot be three unequal roots.

Second proof. This proposition may also be proved thus. Since h and k are roots of the equation $ax^2+bx+c=0$,

we have $\quad ah^2+bh+c=0$ and $ak^2+bk+c=0$.

Subtract, $\quad \therefore a(h^2-k^2)+b(h-k)=0.$

$$\therefore a(h-k)(h+k)+b(h-k)=0.$$

$$\therefore (h-k)\{a(h+k)+b\}=0.$$

Now $h-k$ is not equal to 0, therefore the other factor of the product must vanish [Art. 204];

$$\therefore a(h+k)+b=0\text{......................(i).}$$

Similarly, since h and l are roots of $ax^2+bx+c=0$, we have the equations

$$ah^2+bh+c=0 \text{ and } al^2-bl+c=0\,;$$

and from these, in the same way as above, we obtain

$$a(h+l)+b=0 \quad \text{.......................(ii).}$$

Subtracting (ii) from (i), we have

$$a(k-l)=0.$$

And since a is not equal to 0, therefore $k-l=0$, that is, two of the roots must be equal, contrary to the hypothesis.

Note. The proposition that there cannot be more than one root of an equation of the first degree can be proved in a similar way.

237. The use of the propositions in Art. 232 enables us to form **equations whose roots are connected in a given manner with the roots of a given quadratic equation.**

Ex. 1. *If α and β be the roots of $x^2+px+q=0$, form the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.*

The required equation is

$$\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)=0,$$

that is, $$(\alpha x-1)(\beta x-1)=0,$$

that is, $$\alpha\beta x^2-(\alpha+\beta)x+1=0.$$

Now by Art. 232, $\alpha+\beta=-p$ and $\alpha\beta=q$.

Hence the required equation is

$$qx^2+px+1=0.$$

Ex. 2. *If α and β be the roots of the equation $x^2+4x+2=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.*

The required equation is $\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0,$

that is, $$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0;$$

that is, $$x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1=0.$$

Now, in this case, $\alpha+\beta=-4$ and $\alpha\beta=2$, [Art. 232.

$$\therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=(-4)^2-2\times 2$$
$$=12.$$

Hence the required equation is

$$x^2-\tfrac{12}{2}x+1=0,$$

that is, $$x^2-6x+1=0.$$

MISCELLANEOUS EXAMPLES. XX. D.

[*The following miscellaneous examples on the subject of this chapter are to be solved by the use of the foregoing articles, and not by finding the roots of the quadratic equations which are given.*]

1. If α and β be the roots of $4x^2=3x-1$, find the value of $\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}$.

2. Shew that, if α and β be roots of the equation $x^2-px+q=0$, then

$$\frac{\alpha^3}{\alpha^2-q}+\frac{\beta^3}{\beta^2-q}=p.$$

3. Shew that, if α and β be the roots of $x^2-cx+d=0$, then $$(\alpha^2+d)^2-(\beta^2+d)^2=c^3(\alpha-\beta).$$

4. Prove that the difference of the roots of $x^2+px+q=0$ is equal to the difference of the roots of $x^2+3px+2p^2+q=0$.

5. If α and β be the roots of the equation $x^2-2px+q=0$, prove that $\alpha^3+\beta^3=8p^3-6pq$.

6. If the roots of $x^2+px+q=0$, and those of $x^2+qx+p=0$ differ by the same quantity, then $p+q+4=0$.

7. Find the value of c in terms of a and b, in order that the sum of the roots of the equation $x^2+ax+b=0$ may be equal to the difference of the roots of the equation $x^2+cx+(a+c)b=0$.

8. Shew that one root of the equation $ax^2+bx+c=0$ will be the reciprocal of the other root, if $a=c$.

9. If α and β be the roots of the equation $2x^2-5x+3=0$, find the equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

10. If α and β be the roots of the equation $px^2+qx+r=0$, form the equation whose roots are $-\frac{1}{\alpha}$, $-\frac{1}{\beta}$.

11. Prove that the roots of $x^2+2mx+4n=0$ are 2α and 2β, where α and β are the roots of $x^2+mx+n=0$.

12. If α and β are the roots of the equation $7x^2=2x+1$, find the values of $\alpha^2+\beta^2$ and of $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$.

13. If the roots of the equation $ax^2+x+b=0$ be equal to a and b, what are their numerical values?

14. If the roots of the equation $px^2+qx+1=0$ be equal to $\frac{1}{p}$ and $\frac{1}{q}$, what are their numerical values?

15. If α and β be the roots of the quadratic $x^2+rx+s=0$, shew that the roots of $4x^2+2rx+s=0$ are $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$.

16. Shew that, if α and β be the roots of $x^2-bx+c=0$, then

$$(\alpha^2-c)^2-(\beta^2-c)^2=b(\alpha-\beta)^3.$$

17. Find the quadratic which has equal roots, each being equal to the sum of the roots of the equation $3x^2+5x+1=0$.

18. If α and β be the roots of $px^2+qx+r=0$, shew that the equation $pqx^2+(pr+q^2)x+qr=0$ has roots $\frac{\alpha\beta}{\alpha+\beta}$ and $\alpha+\beta$.

19. Prove that, if α and β be the roots of $px^2+qx+r=0$, then $qrx^2+(pr+q^2)x+pq=0$ has roots $\frac{1}{\alpha+\beta}$ and $\frac{1}{\alpha}+\frac{1}{\beta}$.

20. If α and β be the roots of the equation $x^2-px+q=0$, then will

$$\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}=\frac{p^4}{q^2}-\frac{4p^2}{q}+2.$$

21. Find the relation that must exist between the quantities a, b, c, so that the equation $ax^2+bx+c=0$ may have one of its roots double of the other.

22. Prove that, if $a+b+c=0$, then each pair of the equations

$$x^2+ax+bc=0,\quad x^2+bx+ca=0,\quad \text{and}\quad x^2+cx+ab=0$$

will have a common root.

23. If the roots of $ax^2-bx+c=0$ be the reciprocals of the roots of $a_1x^2-b_1x+c_1=0$, prove that $ab_1=bc_1$, and $aa_1=cc_1$.

24. Prove that, if $ax^2+bx+c=0$ and $ax^2+b'x-c=0$ have a common root, then their other roots will be equal in magnitude but opposite in sign.

***25.** If $ax^2+bx+c=0$, $cx^2+kx+a=0$ have a common root, prove that k must be a root of the equation

$$ac(x^2+b^2)-b(a^2+c^2)x+(a^2-c^2)^2=0.$$

***26.** The expressions x^2+6x+b and $x^2+12x+3b$ have a common factor. What numerical values can b have?

***27.** How can you tell, without solving, whether the roots, supposed real, of a quadratic equation are positive or negative?

Prove that the positive root of $x^2-8x-8=0$ is greater than 8.

***28.** The numerical term in a quadratic equation of the form $x^2+px+q=0$ is misprinted 18 instead of 8, and a student in consequence finds the roots to be 3 and 6. What were they meant to be?

***29.** Two boys attempt to solve a quadratic equation. After reducing it to the form $x^2+px+q=0$, one of them has a mistake only in the absolute term, and finds the roots to be 1 and 7; the other has a mistake only in the coefficient of x, and finds the roots to be -1 and -12. Find the roots of the correct equation.

*238. **Application to Maxima and Minima.** The maximum or minimum values of expressions involving a variable quantity (say x) to a power not higher than the second can often be found by equating the expression to y, and solving for x. The method is explained and illustrated in the following examples.

Ex. 1. *Find the least value which the expression* $x^2-6x+10$ *can have for any real value of* x.

Let $$x^2-6x+10=y.$$
$$\therefore\ x^2-6x+10-y=0.$$
$$\therefore\ x=3\pm\sqrt{y-1}.$$

Now, if x be real, $y-1$ cannot be negative, $\therefore$ y is not less than 1. Hence the smallest possible value of y is 1.

Ex. 2. *Divide a line into two parts so that the rectangle contained by them shall be a maximum.*

Let the length of the line be a; that is, suppose that it contains a units of length. Let x be the length of one part into which it is divided, $\therefore a-x$ is the length of the other part. Thus $x(a-x)$ is to be a maximum.

Let $$x(a-x)=y.$$

$$\therefore x^2-ax+y=0.$$

$$\therefore x=\frac{a\pm\sqrt{a^2-4y}}{2}.$$

Now, if x be real, a^2-4y cannot be negative. Therefore the greatest possible value of $4y$ is a^2; that is, $\frac{1}{4}a^2$ is the greatest possible value of y.

If $y=\frac{1}{4}a^2$, then $x=\frac{1}{2}a$; that is, the line must be bisected.

Ex. 3. *Find the greatest and the least values which the expression* $\frac{x^2+x+1}{x^2-x+1}$ *can have, where x is any real quantity.*

Let $$\frac{x^2+x+1}{x^2-x+1}=y.$$

$$\therefore x^2+x+1=y(x^2-x+1).$$

$$\therefore x^2(1-y)+x(1+y)+1-y=0.$$

Solve for x, $$\therefore x=\frac{-(1+y)\pm\sqrt{(1+y)^2-4(1-y)^2}}{2(1-y)}.$$

Now x is real, $\therefore (1+y)^2-4(1-y)^2$ must be positive,

that is, $\{(1+y)+2(1-y)\}\{(1+y)-2(1-y)\}$ is positive,

that is, $\{3-y\}\{3y-1\}$ is positive.

Hence the factors $3-y$ and $3y-1$ must be of the same sign. Now, if $y>3$, the first of these factors is negative and the second positive; also, if $3y<1$, that is, if $y<\frac{1}{3}$, the first is positive and the second negative. But if y lie between 3 and $\frac{1}{3}$, both factors are positive.

Therefore y is not greater than 3 and is not less than $\frac{1}{3}$.

That is, the greatest value y can have is 3 and the least value y can have is $\frac{1}{3}$.

EXAMPLES. XX. E.

1. Shew that $x^2-8x+19$ can never be less than 3.

2. Shew that $4x^2+3-4x$ can never be less than 2.

3. Find the least value of the expression $\frac{1}{2}(x+1)(x+2)$.

4. Find the greatest possible value of

$$(4y^2-8y+9)\div(4y^2+8y+9).$$

5. What is the least value which the expression

$$(x^2-6x+5)\div(x+1)^2$$

can have?

6. What is the greatest possible value of $5+4x-x^2$?

7. If the sum of two numbers be always equal to a, what are the numbers when the sum of their squares is as small as possible?

*8. Four men, A, B, C, D, went to market to buy sheep. A bought 8 sheep more than B, and B bought 16 sheep more than C. The number of sheep bought by C and D was 72. The sum of the squares of the number of sheep that each person bought was a minimum. Find what number each person bought.

CHAPTER XXI.

INDETERMINATE EQUATIONS.

[*The student who is reading the subject for the first time may omit this chapter.*]

***239.** A single equation between two or more unknown quantities, such as $ax + by = c$, can be satisfied by an infinite number of values of the unknown quantities. For, if there be two unknown quantities, *any* number or value may be given to one of the unknown quantities, and we shall then get an equation to determine the corresponding value of the other unknown quantity.

Such equations are called *indeterminate.*

More generally we may say that a system of simultaneous equations where the number of equations is the same as the number of variables is usually *determinate*, that is, the equations enable us to determine a finite number of values of the unknown quantities which satisfy all the equations. But if the number of equations given be less than the number of unknown quantities contained in them, the system is *indeterminate.*

***240.** It may however happen that in a numerical indeterminate equation the number of roots which are whole numbers or integers is determinate. The general method of obtaining such roots lies beyond the scope of this work, but it may be illustrated by one or two easy examples.

**Ex.* 1. *Find all the positive integral solutions of the equation* $5x+3y=22$.

We begin by dividing every term by the coefficient of x or the coefficient of y, taking the smaller of the two. In this case, therefore, we shall divide by 3,

$$\therefore\ x+\tfrac{2}{3}x+y=7+\tfrac{1}{3}.$$

We now take all the integral terms to one side of the equation, and the fractional terms to the other,

$$\therefore\ x+y-7=\tfrac{1}{3}-\tfrac{2}{3}x \text{......................(i).}$$

Now x and y are integers, $\therefore\ x+y-7$ is an integer, $\therefore\ \frac{1}{3}-\frac{2}{3}x$ must also be an integer. Let us denote it by p,

$$\therefore\ \tfrac{1}{3}-\tfrac{2}{3}x=p.$$
$$\therefore\ 1-2x=3p.$$

This is another indeterminate equation between x and p, but of a simpler form than that from which we started. Continuing the above process, we must now divide by 2, and then transpose the integral terms to one side of the equation and the fractional terms to the other side. We thus obtain

$$x+p=\tfrac{1}{2}-\tfrac{1}{2}p \text{ ....(ii).}$$

Now x and p are integers, $\therefore\ \frac{1}{2}-\frac{1}{2}p$ must be an integer. Let us denote it by q,

$$\therefore\ \tfrac{1}{2}-\tfrac{1}{2}p=q.$$
$$\therefore\ 1-p=2q \text{ ....(iii).}$$

This is another indeterminate equation between p and q, but as the coefficient of one of the variables, namely p, is now unity we need not continue the process any further.

From (iii), we have $\qquad p=1-2q.$

Substitute this value of p in (ii), and we obtain

$$x=3q-1.$$

Substitute this value of x in (i), and we obtain

$$y=9-5q.$$

The integral solutions of the equation are all included in the values

$$x=3q-1, \text{ and } y=9-5q.$$

where q may be any integer whatever. It is obvious that, if q be an integer, the corresponding values of x and y will also be integers; and it is easy to verify that whatever value q may have, these values of x and y satisfy the given equation.

Now in this case x and y are not only to be integers, but are also to be positive. To make y positive, q must be less than 2; to make x positive, q must be positive. Hence the only value which q can have is 1. The only positive integral roots are thus given by putting $q=1$, and are

$$x=2, \ y=4.$$

*Ex. 2. *Find all the positive integral solutions of the equation* $x^2-y^2=63$.

We have $$(x+y)(x-y)=63.$$

Since x and y are positive integers, $\therefore x+y$ and $x-y$ are integers. Also, $x+y$ is positive and greater than $x-y$. Now the only positive integral factors of 63 are 63 and 1, 21 and 3, 9 and 7. Hence the only solutions are given by

$$\left.\begin{matrix}x+y=63\\x-y=\ \ 1\end{matrix}\right\}, \quad \left.\begin{matrix}x+y=21\\x-y=\ \ 3\end{matrix}\right\}, \text{ and } \left.\begin{matrix}x+y=9\\x-y=7\end{matrix}\right\}.$$

These give $x=32$ and $y=31$; $x=12$ and $y=9$; $x=8$ and $y=1$.

*Ex. 3. *Find in how many ways a debt of* 3*s.* 8*d. can be paid in sixpences and francs—the value of a franc being taken as* 10*d.*

Let x be the number of francs, and y the number of sixpences, which are used in paying the debt: the value of these is equal to $(10x+6y)$ pence.

This, by the question, is equal to 3*s.* 8*d.* or 44*d.*,

$$\therefore 10x+6y=44.$$

$$\therefore 5x+3y=22.$$

But by Ex. 1, there is only one solution of this equation in positive integers; namely $x=2$ and $y=4$. Therefore there is only one way of paying the debt, namely, by paying two francs and four sixpences.

*EXAMPLES. XXI.

Solve the following equations in positive integers.

1. $2x+3y=9$. 2. $3x+7y=58$. 3. $13x+2y=119$.

Find the general solution of the following equations, and also the least positive integral values of x and y which satisfy each of them.

4. $3x-2y=10$. 5. $7x-9y=29$. 6. $13x-17y=9$.

7. In how many ways can a sum of 15*s.* 5*d.* be paid in threepenny pieces and fourpenny pieces?

8. In how many ways can five pounds be paid in dollars (worth 4*s.* each), and francs (25 of which are worth £1)?

9. Find in how many ways the sum of £4. 15*s.* 6*d.* can be paid in half-guineas and half-crowns.

10. The sum of two numbers is 35, and their highest common factor (or greatest common measure) is 7. Find all the positive integers which these numbers may be.

11. A boy had between 150 and 180 nuts. He divided them into parcels of four, and found there was one over. Then he divided them up into parcels of five, and found that none were left over. How many nuts had he?

12. Three Arab jugglers, travelling to Mecca, with a performing monkey, purchased a basket of dates. The basket contained enough dates to divide equally between the three and one over. During the night one juggler awoke, and, after giving one date to the monkey, secretly ate one-third of the remainder. Afterwards, each of the others in turn awoke, and, after giving one date to the monkey, secretly ate one-third of the remaining dates. In the morning, enough dates remained to divide equally between the three, with again one over. What is the least number of dates which must have been purchased to make this possible?

13. *A* owes *B* 4*s*. 8*d*. *A* has only half-crowns, and *B* has only fourpenny pieces. How can *A* most easily pay *B*?

14. Shew that the equation $4x - 6y = 11$ cannot be satisfied by integral values of x and y.

15. Divide 25 into two parts, such that one of them is divisible by 3 and the other by 2.

16. The sum and the product of two integers are together equal to 41. What are the two integers?

17. A person bought an exact number of shares in a certain company at £24. 10*s*. per share, spending between £200 and £300. Some time after, he bought another exact number at £4. 10*s*. He then sold out the whole for £400, which was the money he gave for them. Find how many shares he bought on each occasion.

18. "There came three Dutchmen of my Acquaintance to see me, being but lately married; they brought their Wives with them. The Men's Names were Hendrick, Claas, and Cornelius; the Women's Geertruij, Catrijn, and Anna: But I forget the Name of each Man's Wife. They told me they had been at Market to buy Hogs; each Person bought as many Hogs as they gave Shillings for each Hog; Hendrick bought twenty-three Hogs more than Catrijn; and Claas bought eleven more than Geertruij; likewise, each Man laid out three Guineas more than his Wife: I desire to know the Name of each Man's Wife." (*The Woman's Almanack*, 1739.)

EXAMINATION QUESTIONS.

1. Find the square root of

$$144x^4 - 600x^3 + 913x^2 - 600x + 144;$$

and prove that the square root is divisible by $3x - 4$.

2. Solve the equations:

(i) $\dfrac{3x-2}{2x-3} - \dfrac{2x-5}{x-1} = 1;$

(ii) $2x - 3y = 1, \ x(x - 2y) + y(y - 2x) + 26 = 0;$

(iii) $x^2 - y^2 + 2x - 4y + 5 = 0, \ xy + 2x + y = 1.$

3. Shew that the sum of the squares of the roots of the equation $x^2 + rx + s = 0$ is equal to $r^2 - 2s$.

If α, β be the roots of the equation $ax^2 + bx + a + b = 0$, prove that $(1 - \alpha^2)(1 - \beta^2) = 4\alpha\beta$.

4. A man buys a number of yards of cloth for £20. He sells three-quarters of them at 3*s.* per yard, and the rest at two-fifths its cost price, when he finds that he has neither gained nor lost: how many yards did he buy?

5. A rectangular plot of grass is surrounded by a gravel walk of equal area 6 feet wide. The diagonal of the rectangle formed by the outer boundary of the walk is 58 feet: find the dimensions of the grass plot.

6. Find the square root of $16a^6-8a^4-16a^3+a^2+4a+4$.

7. Solve the equations:

(i) $3x^2-16x+16=0$;

(ii) $\frac{35}{(x-2)^2}-(x+2)^2=2\frac{x+2}{x-2}$;

(iii) $x+y=\frac{a^2+b^2}{ab}$, $\frac{x}{y}+\frac{y}{x}=\frac{a^4+b^4}{a^2b^2}$.

8. Shew that there cannot be more than two unequal roots of a quadratic equation.

Form the equation whose roots are the reciprocals of those of the equation $(x+5)^2-4x=160$.

9. *A* starts to walk from Wimbledon to London, and *B* from London to Wimbledon at the same time. As they pass one another, *A* diminishes his pace one mile an hour, and *B* increases his one mile an hour. Each arrives at his destination at the same time. Find how much faster one started than the other.

10. The length of a rectangular room is twice its height, and its height is two-thirds of its width. If 5 feet were taken off its length, 5 feet added to its breadth, and 5 feet added to its height, its cubical contents would be increased by 1500 cubic feet. Find its dimensions.

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11. The united ages of a man and his wife are six times the united ages of their children. Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of the children. How many children have they?

12. Find the cube root of

$$64a^6-48a^5-84a^4+47a^3+42a^2-12a-8.$$

13. Find the factors of $4(x^4+1)-20x(x^2+1)+33x^2$.

14. Solve the equations:

(i) $11x^2-13x=18$;

(ii) $x^2+2xy=3$, $x^2-3y^2=6$.

15. A bag contains 180 gold and silver coins of the value altogether of £60. Each gold coin is worth as many pence as there are silver coins, and each silver coin as many pence as there are gold coins. How many are there of each kind?

---

16. In a cricket-match, the score of the captain is three times that of the next best player, and the scores of these two together make up three-fifths of the total of the innings. The average of the rest of the eleven is seven runs, and nine runs are scored for extras. Find the captain's score and the total of the innings.

17. If

$$X = -pqx + (py+qz)(p+q),$$
$$Y = -pqy + (pz+qx)(p+q),$$
$$Z = -pqz + (px+qy)(p+q),$$

prove that

$$X^2 + Y^2 + Z^2 = (p^2+pq+q^2)^2(x^2+y^2+z^2).$$

18. Solve the equations:

(i) $\dfrac{a^2}{a+x} + \dfrac{b^2}{b+x} = a+b;$

(ii) $2x+y = x^2-2x+6y^2+4y = 5.$

19. Shew that the product of the roots of the equation $x^2+px+q=0$ is q.

Find the equation whose roots are the reciprocals of those of $x^2+px+q=0$.

20. From a square field is taken as much ground as is required to form a road of uniform breadth skirting the sides of the field. The ground so taken is $\frac{7}{16}$ of the whole field. The total length of the road measured along the middle is 700 yards. Find the breadth of the road.

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21. Find the square root of

$9a^2+4b^2+c^2+25d^2-12ab-10cd+4bc-6ca+30da-20bd.$

22. Solve the equations:

(i) $\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2} = \dfrac{2x+5\frac{1}{3}}{x};$

(ii) $\dfrac{a-bx^2}{a+bx^2} + \dfrac{ax^2-b}{ax^2+b} = 2\dfrac{a-b}{a+b};$

(iii) $x^2+xy=22, \quad y^2+yx=-18.$

23. If in a number of 3 digits, the sum of the tens' and hundreds' digits be 3 times the sum of the units' and tens' digits, prove that the number is divisible by 7.

24. The sum of the cubes of three consecutive integers is 33 times the middle integer: find the numbers.

25. A man buys two kinds of cloth, black and brown, the brown cloth costing 6*d.* a yard more than the black. He pays £5. 5*s.* for the black cloth, and £4. 10*s.* for the brown, but obtains 3 yards more of the former cloth than of the latter. Find the cost of a yard of each kind of cloth, and the number of yards he bought.

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26. Find the square root of

$$(3x^2-5)^2-4x(3x^2+5)+64x^2.$$

27. Solve the equations:

$$\text{(i)} \quad \frac{x+1}{x-1}+\frac{x-1}{x+1}=\frac{2x+1\frac{1}{2}}{x};$$

$$\text{(ii)} \quad x-y=1, \quad x^3-y^3=19.$$

28. Find the least value which the expression x^2-4x+7 can have for real values of x.

*29. *A* and *B* are two stations on a railway. A fast train leaves *A* for *B* at the same time that a slow train leaves *B* for *A*. They arrive at their destination three-quarters of an hour and an hour and twenty minutes respectively after they passed each other. Find how long each took for the journey.

*30. On two different railways the stations on each are at equal distances. A train on one runs at the rate of 20 miles an hour, and a train on the other runs at the rate of 24 miles an hour, while both trains lose two minutes at every station. The arrival at every station on the first line coincides with the arrival at every fourth station on the second line. Find the distances between the stations, supposing that neither train can run more than 20 miles without stopping, and that on each railway the distances between the stations are an exact number of miles.

CHAPTER XXII.

FRACTIONAL AND NEGATIVE INDICES.

241. We have hitherto supposed that the indices which have been employed are positive integers; and, in fact, the definition of x^n, given in Art. 22 (namely the product of n quantities, each equal to x) is unintelligible unless n is a positive integer.

We shall now proceed to see whether we cannot frame some other definition of x^n, which shall be intelligible whatever n may be; but, of course, the definition must be so framed that, when n is a positive integer, it shall agree with that already given.

242. The method we shall adopt is strictly analogous to that used in Art. 40, when we were seeking to extend our definitions of addition and subtraction to include the case of negative quantities. In that case, we first proved that certain formulae were true so long as a was greater than b, and, next, by assuming those formulae to be true whatever numbers were represented by a and b, we were able to extend our ideas of addition and subtraction to include negative as well as positive quantities.

The process by which, in Art. 55, we obtained the meaning to be assigned to the product of two negative quantities was similar.

243. We have already proved some propositions about indices, and in particular have shewn [Art. 72] that

$$x^m \times x^n = x^{m+n},$$

when m and n are positive integers. We shall now assume this result to be true, whatever m and n may be, and from it shall deduce a meaning to be assigned to x^n, when n is either negative or fractional; and this meaning we shall take as our definition of x^n.

244. The only other propositions about indices which we have proved are [see Arts. 76, 86], that, if m and n be positive integers, then

if $m > n$, $\qquad x^m \div x^n = x^{m-n}$;

if $m < n$, $\qquad x^m \div x^n = 1 \div x^{n-m}$;

and $\qquad (x^m)^n = x^{mn}$

These results may be regarded as true for all values of m and n, provided we can shew that they are satisfied by the values which we are going to assign to x^m and x^n when m and n are fractional or negative. This we shall do in Art. 246.

245. We shall begin then by assuming the formula $x^m x^n = x^{m+n}$, and shall thence deduce in succession the meanings to be assigned to a quantity raised to the power of (i) a positive fractional index, (ii) a zero index, (iii) any negative index (integral or fractional).

Theorem (i). *To shew that $x^{\frac{p}{q}}$ is equal to the q^{th} root of x^p, where p and q are any positive integers.*

Whatever numbers m and n may be, we have

$$x^m \times x^n = x^{m+n};$$

and similarly, $\quad x^l \times x^m \times x^n \times \ldots = x^{l+m+n+\ldots}$ ....(a).

Hence $\quad (x^{\frac{p}{q}})^q = x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times \ldots (q \text{ factors})$ [Art. 22

$$= x^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots(q \text{ terms})} \qquad \text{[by (a)}$$

$$= x^p.$$

Take the q^{th} root of each side of this equality,

$$\therefore \; x^{\frac{p}{q}} \text{ is equal to the } q^{th} \text{ root of } x^p.$$

Theorem (ii). *To shew that x^0 is equal to unity.*

Whatever number m and n may be, we have

$$x^m \times x^n = x^{m+n}.$$

Let $m = 0$, $\qquad \therefore x^m \times x^0 = x^m.$

Divide both sides by x^m, $\therefore x^0 = 1$.

Theorem (iii). *To shew that x^{-n} is equal to the reciprocal of x^n, whatever number n may be.*

Whatever number m and n may be, we have

$$x^m \times x^n = x^{m+n}.$$

Let $m = -n$, $\qquad \therefore x^{-n} \times x^n = x^{-n+n}$

$$= x^0$$

$$= 1.$$

Divide both sides by x^n, $\therefore x^{-n} = \dfrac{1}{x^n}$.

Thus, whatever n may be, positive or negative, integral or fractional, we have found a meaning to be given to x^n.

246. Next, we have to shew that this meaning is consistent with the results of the propositions collected in Art. 244.

Whatever number m and n may be, we have, by Art. 245,

$$\frac{x^m}{x^n} = x^m \times \frac{1}{x^n} = x^m \times x^{-n} = x^{m-n},$$

and $$\frac{x^m}{x^n} = x^m \times \frac{1}{x^n} = \frac{1}{x^{-m}} \times \frac{1}{x^n} = \frac{1}{x^{n-m}}.$$

Thus the first two propositions quoted in Art. 244 are consistent with the extended meaning now given to x^m and x^n.

We have next to shew that, whatever numbers m and n may be, we always have $(x^m)^n = x^{mn}$; and here we shall consider in succession the cases where n is a positive quantity, zero, or a negative quantity.

(i) If m be any number, and $n = p/q$, where p and q are positive integers, then we have to shew that

$$(x^m)^{\frac{p}{q}} = x^{\frac{mp}{q}}.$$

This, by Art. 245 (i), means that $\sqrt[q]{(x^m)^p} = \sqrt[q]{x^{mp}}$, a result which is true. [Art. 76.]

(ii) If m be any number, and $n = 0$, then we have to shew that $(x^m)^0 = x^{m\times 0}$.

This, by Art. 245 (ii), means that $1 = 1$, a result which is obviously true.

(iii) If m be any number, and $n = -s$, where s is a positive quantity, then we have to shew that

$$(x^m)^{-s} = x^{-ms}.$$

This, by Art. 245 (iii), means that $\frac{1}{(x^m)^s} = \frac{1}{x^{ms}}$, a result which by (i) of this section is always true.

247. **Index Laws.** All the above results are included in the statements that, whatever numbers m and n may be,

$$x^m \times x^n = x^{m+n},$$

and $$(x^m)^n = x^{mn}.$$

These results are known as *the index laws.*

248. The following examples illustrate the application of the index laws to the simplification of algebraic expressions.

Ex. 1. *Simplify* $a^{p+3q} \times a^{p+q} \div a^{q-p}$.

The given expression $= a^{p+3q} \times a^{p+q} \times a^{p-q}$

$$= a^{(p+3q)+(p+q)+(p-q)}$$

$$= a^{3p+3q}.$$

Ex. 2. *Simplify the fraction* $\dfrac{(x^{p+2q})^n \,.\, (x^{q+2r})^n \,.\, (x^{r+2p})^n}{(x^p \,.\, x^q \,.\, x^r)^n}$.

This fraction $= \dfrac{x^{np+2nq} \,.\, x^{nq+2nr} \,.\, x^{nr+2np}}{(x^{p+q+r})^n}$

$$= \frac{x^{(np+2nq)+(nq+2nr)+(nr+2np)}}{x^{n(p+q+r)}}$$

$$= \frac{x^{3np+3nq+3nr}}{x^{np+nq+nr}}$$

$$= x^{(3np+3nq+3nr)-(np+nq+nr)}$$

$$= x^{2np+2nq+2nr}.$$

EXAMPLES ON THE INDEX LAWS. XXII. A.

Simplify the following quantities.

1. $\dfrac{a^{2p+q} \times a^{p+4q}}{a^{q-p}}$.

2. $\dfrac{x^{-a+b+c} \,.\, x^{a-b+c} \,.\, x^{a+b-c}}{x^a \,.\, x^b \,.\, x^c}$.

3. $\dfrac{a^{m-n} \,.\, a^{m-3n} \,.\, a^{m-5n}}{a^{n-m} \,.\, a^{n-3m} \,.\, a^{n-5m}}$.

4. $\dfrac{(x^a)^2}{x^{b+c}} \times \dfrac{(x^b)^2}{x^{c+a}} \times \dfrac{(x^c)^2}{x^{a+b}}$.

5. $\dfrac{(x^{a+b})^2 \,.\, (x^{b+c})^2 \,.\, (x^{c+a})^2}{(x^a \,.\, x^b \,.\, x^c)^4}$.

6. $\dfrac{(a^{p-q} \times a^{q-r})^2}{(a^2)^p \times (a^{-q})^2}$.

7. $\dfrac{a^{n-m} \,.\, a^{n-2m} \,.\, a^{n-3m}}{a^{m-n} \,.\, a^{m-2n} \,.\, a^{m-3n}}$.

8. $\dfrac{b^{xy} \,.\, c^{yz+2y}}{b^{yz} \,.\, c^{xy}} \left(\dfrac{b}{c}\right)^{y(z-x+1}$.

9. $a^{\frac{1}{3}} \times a^{-\frac{3}{4}} \times a\sqrt[3]{a^4} \times a^{\frac{1}{12}} \times \sqrt[8]{a^{\frac{25}{3}}} \times (a^{-\frac{7}{4}})^{\frac{7}{6}}$.

249. The following examples illustrate the use of the index laws in questions involving the application of algebraic processes—multiplication, division, evolution, &c.—to quantities raised to fractional or negative indices.

The work will usually be facilitated if all the expressions be arranged in descending powers of some letter (see Art. 65).

Ex. 1. *Multiply* $x-3x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ *by* $2x^{\frac{1}{2}}-5y^{\frac{1}{2}}$.

Following the usual process, we have

$$\begin{array}{rrrr}
x & -\ 3x^{\frac{1}{2}}y^{\frac{1}{2}} & +\ y & \\
2x^{\frac{1}{2}} & -\ 5y^{\frac{1}{2}} & & \\
\hline
2x^{\frac{3}{2}} & -\ 6xy^{\frac{1}{2}} & +\ 2x^{\frac{1}{2}}y & \\
 & -\ 5xy^{\frac{1}{2}} & +\ 15x^{\frac{1}{2}}y & -\ 5y^{\frac{3}{2}} \\
\hline
\therefore\ 2x^{\frac{3}{2}} & -\ 11xy^{\frac{1}{2}} & +\ 17x^{\frac{1}{2}}y & -\ 5y^{\frac{3}{2}} \\
\hline\hline
\end{array}$$

Add

Ex. 2. *Divide* $2x^{\frac{3}{2}}-17xy^{\frac{1}{2}}+17x^{\frac{1}{2}}y-15y^{\frac{3}{2}}$ *by* $2x^{\frac{1}{2}}-15y^{\frac{1}{2}}$.

Following the usual process, we have

$$\begin{array}{r|l}
2x^{\frac{1}{2}}-15y^{\frac{1}{2}} & 2x^{\frac{3}{2}}-17xy^{\frac{1}{2}}+17x^{\frac{1}{2}}y-15y^{\frac{3}{2}}\ (\,x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y \\
 & \underline{2x^{\frac{3}{2}}-15xy^{\frac{1}{2}}} \\
 & \qquad -\ 2xy^{\frac{1}{2}}+17x^{\frac{1}{2}}y \\
 & \qquad \underline{-\ 2xy^{\frac{1}{2}}+15x^{\frac{1}{2}}y} \\
 & \qquad\qquad 2x^{\frac{1}{2}}y-15y^{\frac{3}{2}} \\
 & \qquad\qquad \underline{2x^{\frac{1}{2}}y-15y^{\frac{3}{2}}}
\end{array}$$

Thus the required quotient is $x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

Ex. 3. *Find the square root of*

$$4a^5 - 12a^{\frac{5}{2}}b^{\frac{4}{3}} + 9b^{\frac{8}{3}} + 16a^{\frac{5}{2}}c^{\frac{3}{4}} - 24b^{\frac{4}{3}}c^{\frac{3}{4}} + 16c^{\frac{3}{2}}.$$

Following the usual process, we have

$$\begin{array}{l|l}
 & 4a^5 - 12a^{\frac{5}{2}}b^{\frac{4}{3}} + 16a^{\frac{5}{2}}c^{\frac{3}{4}} + 9b^{\frac{8}{3}} - 24b^{\frac{4}{3}}c^{\frac{3}{4}} + 16c^{\frac{3}{2}} \;(\,2a^{\frac{5}{2}} - 3b^{\frac{4}{3}} + 4c^{\frac{3}{4}} \\
 & 4a^5 \\
\hline
4a^{\frac{5}{2}} - 3b^{\frac{4}{3}} & \quad - 12a^{\frac{5}{2}}b^{\frac{4}{3}} + 16a^{\frac{5}{2}}c^{\frac{3}{4}} + 9b^{\frac{8}{3}} \\
 & \quad - 12a^{\frac{5}{2}}b^{\frac{4}{3}} \qquad\qquad\quad + 9b^{\frac{8}{3}} \\
\hline
4a^{\frac{5}{2}} - 6b^{\frac{4}{3}} + 4c^{\frac{3}{4}} & \qquad\qquad 16a^{\frac{5}{2}}c^{\frac{3}{4}} \qquad - 24b^{\frac{4}{3}}c^{\frac{3}{4}} + 16c^{\frac{3}{2}} \\
 & \qquad\qquad 16a^{\frac{5}{2}}c^{\frac{3}{4}} \qquad - 24b^{\frac{4}{3}}c^{\frac{3}{4}} + 16c^{\frac{3}{2}} \\
\hline
\end{array}$$

Hence the required square root is $2a^{\frac{5}{2}} - 3b^{\frac{4}{3}} + 4c^{\frac{3}{4}}$.

Ex. 4. *Find a root of the equation* $3^{x+1} + 9^x = 6804$.

We have $\quad 3 \times 3^x + (3^2)^x = 6804,$

that is, $\quad 3^{2x} + 3 \, . \, 3^x = 6804.$

This is a quadratic equation in 3^x. Let $3^x = y$,

$$\therefore \; y^2 + 3y = 6804.$$

Completing the square, $\therefore \; y^2 + 3y + (\frac{3}{2})^2 = 6804 + \frac{9}{4} = \frac{27225}{4}$.

Take the square root of each side, $\therefore \; y + \frac{3}{2} = \pm\frac{165}{2}$.

$$\therefore \; y = -\tfrac{3}{2} + \tfrac{165}{2} = 81, \quad \text{or } y = -\tfrac{3}{2} - \tfrac{165}{2} = -84.$$

Now if x be real, 3^x must be positive, hence the negative value of y is inadmissible,

$$\therefore \; 3^x = 81.$$

But $81 = 3^4$, $\therefore \; 3^x = 3^4$, which is satisfied by $x = 4$.

Therefore 4 is a root of the given equation.

Ex. 5. *Find a solution of the simultaneous equations*

$$2^x 4^y = 128, \quad \frac{25^y}{5^x} = 5.$$

These equations can be written

$$2^x 2^{2y} = 2^7, \quad \frac{5^{2y}}{5^x} = 5;$$

that is, $\quad 2^{x+2y} = 2^7, \quad 5^{2y-x} = 5.$

These are obviously satisfied by

$$x+2y=7, \quad 2y-x=1.$$

Adding, we get $4y=8$, $\therefore\ y=2$.

Subtracting, we get $2x=6$, $\therefore\ x=3$.

Hence a solution of the given equations is $x=3$, $y=2$.

*250. The use of the index laws also enables us, in some cases, to extract roots of expressions of an order higher than the second or third.

For example, the *fourth root* of an expression can be found by extracting the square root of the square root of it. For suppose that we desire to find the fourth root of X. Then we have

$$\sqrt[4]{X} = X^{\frac{1}{4}} = (X^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{[\sqrt{X}]}.$$

Example. Find the fourth root of

$6561a^4 - 43740a^3b - 5832a^3c + 109350a^2b^2 + 29160a^2bc + 1944a^2c^2 - 121500ab^3 - 48600ab^2c - 6480abc^2 - 288ac^3 + 50625b^4 + 27000b^3c + 5400b^2c^2 + 480bc^3 + 16c^4$.

Extracting the square root of this expression by any of the methods given in Chapter XV., we find that it is

$$81a^2 - 270ab - 36ac + 225b^2 + 60bc + 4c^2.$$

The square root of this latter expression is $9a - 15b - 2c$, which is therefore the required fourth root.

*251. Similarly, the *sixth root* of an expression can be found by extracting the cube root of its square root. For suppose that we want to find the sixth root of X. Then we have

$$\sqrt[6]{X} = X^{\frac{1}{6}} = (X^{\frac{1}{2}})^{\frac{1}{3}} = \sqrt[3]{[\sqrt{X}]}.$$

The same method shews that we might also obtain the sixth root of an expression by extracting the square root of its cube root.

MISCELLANEOUS EXAMPLES ON INDICES. XXII. B.

1. Simplify $\dfrac{(a^2b^3)^{\frac{1}{6}}\,b^{-2}\,c^{\frac{1}{3}}}{a^{\frac{2}{3}}\,b^{-\frac{5}{4}}\,c^{\frac{1}{4}}}$, and find its value when $a=2$, $b=3$, $c=432$.

Simplify the following expressions, numbered **2** to **9**.

2. $(\sqrt[3]{a^7})\times(\sqrt[5]{a^9})\times a^{-\frac{1}{3}}\div a^{\frac{4}{5}}$.

3. $(\sqrt[7]{a^3})\times(\sqrt[6]{a^7})\times a^{-\frac{1}{2}}\div a^{\frac{2}{21}}$.

4. $\left\{\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}}\right\}^{mn}\cdot\left\{\dfrac{\sqrt[n]{a}}{\sqrt[p]{a}}\right\}^{np}\cdot\left\{\dfrac{\sqrt[p]{a}}{\sqrt[m]{a}}\right\}^{pm}$.

5. $\sqrt[mn]{\dfrac{a^m}{a^n}}\times\sqrt[np]{\dfrac{a^n}{a^p}}\times\sqrt[pm]{\dfrac{a^p}{a^m}}$.

6. $\sqrt[p+q]{\left(\dfrac{a^p}{a^q}\right)^{pq}\div\left\{\dfrac{(a^{p-q})^p}{(a^{p+q})^q}\right\}^q}$.

7. $\dfrac{a^{xy}\,b^{yz}}{a^{yz}\,b^{y(x-1)}}\left(\dfrac{a}{b}\right)^{y(z-x+1)}$.

8. $a^{\frac{3}{4}\left(\frac{4}{5}+\frac{5}{6}\right)}\div a^{\frac{4}{3}\left(\frac{5}{4}+\frac{6}{5}\right)}$.

9. $\left\{\left(\dfrac{x^n}{x^{m-n}}\right)^{m+n}\div[x^{m-n}\,.\,x^{2n}]^m\right\}^{\frac{1}{n-m}}$.

10. Prove that $a-b=(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})$.

11. Prove that $a-b=(a^{\frac{1}{3}}-b^{\frac{1}{3}})(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})$.

12. Prove that $a+b=(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})$.

13. Prove that $a-b=(a^{\frac{1}{4}}-b^{\frac{1}{4}})(a^{\frac{1}{4}}+b^{\frac{1}{4}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})$.

14. Multiply $3x^{\frac{3}{2}}-5x+4$ by $2x^{\frac{3}{2}}-x^{\frac{1}{2}}-4$.

15. Multiply $b^{-1}+2a^{\frac{1}{2}}b^{-\frac{1}{2}}-c+a^{\frac{1}{2}}$ by $c+a^{\frac{1}{2}}-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b^{-1}$.

16. Multiply $(x^{\frac{1}{4}}-y^{\frac{1}{4}})^2+y^{\frac{1}{2}}$ by $(x^{\frac{1}{4}}+y^{\frac{1}{4}})^2+y^{\frac{1}{2}}$.

17. Find the continued product of

$(x+a+a^{\frac{1}{2}}x^{\frac{1}{2}})$, $(a^{\frac{1}{2}}+x^{\frac{1}{2}})$, $(a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x)$, and $(x^{\frac{1}{2}}-a^{\frac{1}{2}})$.

18. Divide by $a^{\frac{1}{2}}-2b^{\frac{1}{2}}-c^{\frac{1}{2}}$ the expression

$a^{\frac{3}{2}}-ab^{\frac{1}{2}}-ac^{\frac{1}{2}}-3a^{\frac{1}{2}}b+a^{\frac{1}{2}}c-2a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}-b^{\frac{1}{2}}c+3bc^{\frac{1}{2}}+2b^{\frac{3}{2}}-c^{\frac{3}{2}}$.

19. Divide $(x-5x^{\frac{1}{2}}+6)(x-5x^{\frac{1}{2}}-14)$ by $x-10x^{\frac{1}{2}}+21$.

20. The product of two quantities is $x+8y-27z+18x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$, and one of them is $x^{\frac{1}{3}}+2y^{\frac{1}{3}}-3z^{\frac{1}{3}}$: find the other.

21. Simplify $\dfrac{x+(xy^2)^{\frac{1}{3}}-(x^2y)^{\frac{1}{3}}}{x+y}$.

22. Simplify $\left\{1-\dfrac{1-x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}}+\dfrac{1+2x}{1-x}\right\}\left\{\dfrac{x^{\frac{1}{2}}+1}{2x^{\frac{1}{2}}+1}\right\}$.

23. Simplify

$$\frac{x^{\frac{1}{4}}}{(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{1}{4}}-z^{\frac{1}{4}})}+\frac{y^{\frac{1}{4}}}{(y^{\frac{1}{4}}-z^{\frac{1}{4}})(y^{\frac{1}{4}}-x^{\frac{1}{4}})}+\frac{z^{\frac{1}{4}}}{(z^{\frac{1}{4}}-x^{\frac{1}{4}})(z^{\frac{1}{4}}-y^{\frac{1}{4}})}.$$

Extract the square root of the expressions numbered **24** to **27**.

24. $x^2+\dfrac{9}{x^2}-4x-\dfrac{12}{x}+10$.

25. $4x^2-12x^{\frac{4}{3}}+28x+9x^{\frac{2}{3}}-42x^{\frac{1}{3}}+49$.

26. $x-4x^{\frac{5}{6}}a^{\frac{1}{6}}+6x^{\frac{1}{2}}c^{\frac{1}{2}}+4x^{\frac{2}{3}}a^{\frac{1}{3}}-12x^{\frac{1}{3}}a^{\frac{1}{6}}c^{\frac{1}{2}}+9c$.

27. $a^6+b^{-6}+4a^5b^{-1}+2b^{-5}a+2a^4b^{-2}-3b^{-4}a^2-6a^3b^{-3}$.

***28.** Extract the fourth root of

$$81x^4-216x^3+216x^2-96x+16.$$

***29.** Extract the sixth root of

$$\frac{x^6}{64}+\frac{x^5y}{16}+\frac{5x^4y^2}{48}+\frac{5x^3y^3}{54}+\frac{5x^2y^4}{108}+\frac{xy^5}{81}+\frac{y^6}{129}.$$

Find solutions of the following equations, numbered **30** to **33**.

30. $4^x=2^{x+h}$. **31.** $9(9^{x-1}+3)=28\times 3^x$.

32. $(a^{\frac{2}{3}}+1)(x^{\frac{2}{3}}-b^{\frac{2}{3}})-(b^{\frac{2}{3}}+1)(x^{\frac{2}{3}}-a^{\frac{2}{3}})=(a^{\frac{2}{3}}b^{\frac{2}{3}}+1)(a^{\frac{2}{3}}-b^{\frac{2}{3}})$.

33. $(x^{m^2+n^2}+x^{-2mn})(a^{m^2+n^2}-a^{-2mn})$
$+(x^{m^2+n^2}-x^{-2mn})(a^{m^2+n^2}+a^{-2mn})=0.$

34. Prove that $x^{x\sqrt{x}}=(x\sqrt{x})^x$ is satisfied by $x=2\frac{1}{4}$.

35. Solve the simultaneous equations $27^x=9^y$, $81^y/3^x=243$.

36. Solve the simultaneous equations $\left.\begin{array}{r}2^x+3^y=17\\ 2^{x+2}-3^{y+1}=5\end{array}\right\}$.

CHAPTER XXIII.

LOGARITHMS.

[*The results of this chapter are not required in the immediately following chapters, and the discussion of logarithms may be deferred for the present if the student desire it.*]

252. **Logarithm.** The *logarithm* of a number to a given *base* is the index of the power to which the base must be raised to be equal to the number.

Thus, if $a^x = n$, then x is called the logarithm of n to the base a. This is written as

$$x = \log_a n.$$

The equations $a^x = n$ and $x = \log_a n$ express the same relation between a, x, and n; and this relation may be written in whichever of these forms is the more convenient.

253. We can approximately calculate the logarithm of a number to any given base in the manner described later [Arts. 414, 416]; but in some cases, the exact value can be at once obtained from the definition. The first two of the following examples are important.

Ex. 1. *The logarithm of* 1 *is zero, whatever the base may be.*

For let a be the base, and let $\log_a 1 = x$.

Therefore, by the definition, $a^x = 1$.

This equation is satisfied by $x = 0$. [Art. 245, (ii).

Ex. 2. *The logarithm of any number to that number as base is unity.*

For let a be the base, and let $\log_a a = x$.

Therefore, by the definition, $\qquad a^x = a$.

This equation is satisfied by $\qquad x = 1$. [Art. 22.

Ex. 3. *Find the value of* $\log_4 \frac{1}{2}$.

Let $\qquad \log_4 \frac{1}{2} = x$.

$$\therefore\ 4^x = \tfrac{1}{2},$$

that is, $\qquad 2^{2x} = 2^{-1}$.

This is satisfied by $2x = -1$, $\therefore\ x = -\frac{1}{2}$.

Ex. 4. *Find the value of* $\log_{\sqrt[3]{9}} (27)^{-1}$.

Let $\qquad x = \log_{\sqrt[3]{9}} (27)^{-1}$.

$$\therefore\ (\sqrt[3]{9})^x = 27^{-1}.$$

$$\therefore\ (9^{\frac{1}{3}})^x = (3^3)^{-1}.$$

$$\therefore\ 3^{\frac{2}{3}x} = 3^{-3}.$$

This is satisfied by $\frac{2}{3}x = -3$, $\therefore\ x = -\frac{9}{2}$.

EXAMPLES. XXIII. A.

Determine the values of the following logarithms.

1. $\log_{10} 100$. 2. $\log_{100} 10$. 3. $\log_4 64$. 4. $\log_{27} 81$.

5. $\log_{\cdot 001} 100$. 6. $\log_{100} \cdot 001$. 7. $\log_{16} 2$. 8. $\log_{\cdot 01} 1000$.

9. $\log_{2\sqrt{7}} 28$. 10. $\log_{\frac{1}{8}} 4$. 11. $\log_{1000} \cdot 0001$.

12. $\log_{\sqrt{2}} 32$. 13. $\log_{\sqrt{5}} \cdot 008$. 14. $\log_5 125$.

15. $\log_{144} \frac{1}{1728}$. 16. $\log_{256} 32$. 17. $\log_{343} \sqrt[5]{49}$.

18. $\log_{\cdot 3} \cdot 027$. 19. $\log_{3\sqrt{3}} (243)^{-\frac{5}{2}}$. *20. $\log_a 0$.

254. We now proceed to discuss some of the elementary properties of logarithms: these are at once deducible from the two fundamental propositions relating to indices, namely, $x^m x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$ [Art. 247].

We shall suppose that the logarithms are calculated to any number, a, as base.

255. Logarithm of a Product. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let m and n be factors of the product.

Let $m = a^x, \quad \therefore \; x = \log_a m;$

and let $n = a^y, \quad \therefore \; y = \log_a n.$

$$\therefore \; mn = a^x . a^y = a^{x+y}.$$

$$\therefore \; \log_a (mn) = x + y$$
$$= \log_a m + \log_a n.$$

Similarly, if $m, n, p, \ldots$ be factors of the product, let

$$m = a^x, \; n = a^y, \; p = a^z, \text{\&c.},$$

then $mnp \ldots = a^x a^y a^z \ldots = a^{x+y+z+\ldots}.$

$$\therefore \; \log_a (mnp \ldots) = x + y + z + \ldots$$
$$= \log_a m + \log_a n + \log_a p + \ldots.$$

256. Logarithm of a Quotient. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

Let m be the dividend, and let n be the divisor.

Let $m = a^x, \quad \therefore \; x = \log_a m;$

and let $n = a^y, \quad \therefore \; y = \log_a n.$

$$\therefore \; \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}.$$

$$\therefore \; \log_a \left(\frac{m}{n}\right) = x - y$$
$$= \log_a m - \log_a n.$$

257. Logarithm of a Power. *The logarithm of a power of a number is equal to the product of the index of the power and the logarithm of the number.*

Let m be the number, and let y be the power to which it is raised; y may be either an integer or a fraction.

Let $m = a^x, \quad \therefore \; x = \log_a m.$

$$\therefore \quad m^y = (a^x)^y = a^{xy}.$$

$$\therefore \quad \log_a (m^y) = xy$$

$$= y \log_a m.$$

258. It follows from the result of the last article that **logarithms can be used to extract the roots of numbers.**

For suppose that N is some given number, and that we want to find the n^{th} root of it.

Let $\sqrt[n]{N} = x.$

$$\therefore \quad N^{\frac{1}{n}} = x.$$

$$\therefore \quad \frac{1}{n} \log_a N = \log_a x. \qquad \text{[Art. 257.}$$

Hence, if the logarithm of N to any base a be known, then we can deduce the logarithm of x to that base; and hence, by means of certain tables which are published, we can find the value of x.

In numerical calculations, the number ten is usually taken for the base [Art. 262].

259. It is on the results of Articles 255, 256, 257, that the use of logarithms largely depends; and in order to apply those propositions, it is necessary to express the number, whose logarithm is required, in the form of a product or a quotient. This is illustrated by the following examples.

Ex. 1. *Having given log* 2 = ·30103 *and log* 3 = ·47712, *find the logarithms of* (i) 5; (ii) 225; *and* (iii) ·003—*the base in all cases being ten.*

We are given the logarithms of 2 and 3, and we know by Art. 253, Ex. 2, that the logarithm of 10 (to the base 10) is unity. Hence, by Art. 257, we know the logarithms of any powers of 2, 3, and 10. Therefore, by Art. 255, we know the logarithms of any product of powers of 2, 3, and 10. We shall therefore in each case begin by expressing the number, whose logarithm is required, in factors which are powers of 2, 3, and 10 only.

(i) We have $5=\frac{10}{2}$.

$$\therefore \log 5=\log 10-\log 2 \qquad \text{[Art. 256.}$$
$$=1-\cdot 30103$$
$$=\cdot 69897.$$

(ii) Dividing 225 by 2, 3, and 5, as often as we can, we find that
$$225=3^2\times 5^2.$$

$$\therefore \log 225=\log(3^2\times 5^2)$$
$$=\log 3^2+\log 5^2$$
$$=2\log 3+2\log 5$$
$$=2\log 3+2\log\tfrac{10}{2}$$
$$=2\log 3+2(\log 10-\log 2)$$
$$=2(\cdot 47712)+2(1-\cdot 30103)$$
$$=2\cdot 35218.$$

(iii) In the same way, $\cdot 00\dot{3}=\frac{3}{900}=\frac{1}{300}=3^{-1}\times 10^{-2}$.

$$\therefore \log(\cdot 00\dot{3})=\log(3^{-1}\times 10^{-2})$$
$$=\log(3^{-1})+\log(10^{-2})$$
$$=-\log 3-2\log 10$$
$$=-\cdot 47712-2$$
$$=-2\cdot 47712.$$

Ex. 2. *Having given log* $4=\cdot 6020600$ *and log* $36=1\cdot 5563025$; *find log* 9 *and log* $\cdot 15$*—the base in all cases being ten.*

We must first resolve 4 and 36 into the simplest possible factors.

We have $4=2^2$, $\quad\therefore\ 2\log 2=\log 4=\cdot 6020600$.

Also, $36=2^2\times 3^2$, $\quad\therefore\ 2\log 2+2\log 3=\log 36=1\cdot 5563025$.

Solving these two equations for log 2 and log 3, we obtain
$$\log 2=\cdot 3010300, \quad \log 3=\cdot 4771213.$$

We now know log 2 and log 3. Hence, if we can express the given numbers as products of powers of 2, 3, and 10 (or 5), we can obtain their logarithms as in the last example.

Now, $9=3^2$.

$$\therefore \log 9=2\log 3=2(\cdot 4771213)=\cdot 9542426.$$

Again, $\cdot 15 = \frac{15}{100} = \frac{3}{20} = \frac{3}{2 \times 10}$.

$$\therefore \log(\cdot 15) = \log 3 - \log 2 - \log 10$$
$$= \cdot 4771213 - \cdot 3010300 - 1$$
$$= -\cdot 8239087.$$

EXAMPLES. XXIII. B.

[In the first four examples, a knowledge of the numerical values of log 2, log 3, log 7, log 11 is assumed. These values are

$$\log 2 = \cdot 3010300,$$
$$\log 3 = \cdot 4771213,$$
$$\log 7 = \cdot 8450980,$$
$$\log 11 = 1 \cdot 0413927.$$

All the logarithms in the following set of examples are calculated to the base ten.]

1. Having given the numerical value of log 2 (see above); find the logarithms of

(i) $\frac{1}{128}$; (ii) 3·125; (iii) $\sqrt[3]{\cdot 025}$; (iv) $\{10 \cdot 24\}^{\frac{5}{2}}$.

2. Given the numerical values of log 2 and log 3 (see above); find the logarithms of the following numbers.

(i) 15; (ii) 1944; (iii) 4·5; (iv) 2400; (v) 75; (vi) ·0045; (vii) ·0036; (viii) ·003; (ix) $1\frac{1}{8}$; (x) ·072; (xi) ·75; (xii) 7·29; (xiii) ·00125.

3. Having given the values of log 2 and log 7 (see above); find the logarithms of $(1 \cdot 75)^{\frac{1}{2}}$; $(24 \cdot 5)^{-\frac{1}{3}}$; $(\cdot 0056)^{\frac{1}{2}}$.

4. Having given the numerical values of log 2, log 7, and log 11 (see above); find the logarithms of 1·4; ·0154; and $\frac{2^{10}}{7^9 11^5}$.

5. Given $\log 2 = \cdot 3010300$ and $\log 6 = \cdot 7781513$; find the logarithms (to the base 10) of 15, and ·0025.

6. Given $\log 21 = 1 \cdot 3222193$ and $\log 49 = 1 \cdot 6901961$; find log 3.

7. Given that $\log 27 = 1 \cdot 4313638$ and $\log 5 = 0 \cdot 6989700$; find log 135, and log 3.

8. If $\log 125 = 2{\cdot}0969100$ and $\log 49 = 1{\cdot}6901961$; find the logarithm of $(35)^{\frac{3}{4}}$.

9. Calculate $\log 14$, it being known that $\log 392 = 2{\cdot}5932861$ and $\log 1715 = 3{\cdot}2342641$.

10. Given $\log 72 = 1{\cdot}857332$ and $\log 45 = 1{\cdot}653212$; find $\log 30$, and $\log {\cdot}0135$.

11. Having given $\log 242 = a$, $\log 80 = b$, $\log 45 = c$; find $\log 66$, $\log 3993$, and $\log 36$ in terms of a, b, c.

260. *To find the relation between the logarithms of the same number to different bases.*

Let m be the number, a and b the bases.

Let $\qquad \log_a m = x$, and $\log_b m = y$.

$$\therefore \quad m = a^x, \qquad m = b^y.$$

$$\therefore \quad a^x = b^y.$$

Therefore $\qquad a = b^{\frac{y}{x}},$

that is, $\qquad \frac{y}{x} = \log_b a.$

$$\therefore \quad y = x \log_b a.$$

$$\therefore \quad \log_b m = \log_a m \times \log_b a \text{..........(i).}$$

Similarly, $\qquad b = a^{\frac{x}{y}},$

that is, $\qquad \frac{x}{y} = \log_a b.$

$$\therefore \quad x = y \log_a b.$$

$$\therefore \quad \log_a m = \log_b m \times \log_a b \text{ .... (ii).}$$

Also, we have $\log_a b \times \log_a b = \frac{x}{y} \times \frac{y}{x} = 1.$

Thus $\log_a b$ and $\log_b a$ are reciprocals each of the other.

Example. Having given $\log_{10} 2 = {\cdot}3010300$, $\log_{10} 3 = {\cdot}4771213$; *find* $\log_5 27$ *correct to four places of decimals.*

It is better to work examples like these from first principles than to quote formulae.

Let $$x = \log_5 27.$$

$$\therefore\ 5^x = 27 = 3^3.$$

$$\therefore\ x \log 5 = 3 \log 3.$$

Now $\log 5 = \log \frac{10}{2} = 1 - \log 2 = 1 - \cdot 30103090 = \cdot 6989700.$

$$\therefore\ x\,(\cdot 6989700) = 3\,(\cdot 4771213).$$

$$\therefore\ x = \frac{3\,(\cdot 4771213)}{\cdot 6989700} = 2\cdot 04782......$$

We are asked to determine x correct to four places of decimals. To obtain the last decimal figure, it is necessary to find x to five places of decimals, in order to see whether x is more nearly equal to 2·0478 or 2·0479. As the fifth decimal figure is less than 5, the answer will be 2·0478.

EXAMPLES. XXIII. C.

1. Having given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 3 = \cdot 4771213$; find, correct to three places of decimals,

(i) $\log_3 6$; (ii) $\log_6 5$; (iii) $\log_5 3$.

2. Having given $\log_{10} 2 = \cdot 3010300$, $\log_{10} 7 = \cdot 8450980$; find $\log_7 4$, and $\log_2 70$.

3. Given $\log_{10} 3 = \cdot 4771213$, $\log_{10} 7 = \cdot 8450980$; find $\log_3 \sqrt{7}$, and $\log_{\sqrt{7}} 3$.

261. **Common Logarithms.** In practical numerical calculations, all logarithms are calculated to the base 10, and the symbol shewing the base to which they are calculated is usually omitted. Logarithms to the base 10 are called *common logarithms.*

The only logarithms considered in the rest of this chapter are common logarithms, and the base 10 need not be inserted.

Tables of the logarithms of the numbers from 1 to 100,000 have been calculated. That is, values of x which satisfy the equations $10^x = 1$, $10^x = 2$,... have been found. The method of finding the values of these logarithms will be explained later, [Art. 416], but it may be stated that the exact roots of these equations cannot in general be found, though the numerical values of the roots can be obtained as accurately as is desired. Most tables give the results to at least five places of decimals.

262. Advantages of taking ten as the base. The advantage of using 10 as the base arises from the consideration that

$$\log 10 \quad = 1 \qquad \text{[Art. 253, Ex. 2,}$$

$$\log 100 \quad = \log (10^2) = 2 \log 10 = 2 \qquad \text{[Art. 257,}$$

$$\log 1000 = \log (10^3) = 3 \log 10 = 3,$$

and, universally, $\log (10^n) = n \log 10 = n$.

Hence $\log (N \times 10^n) = \log N + \log (10^n) = \log N + n$,

and $$\log \frac{N}{10^n} = \log N - \log (10^n) = \log N - n.$$

Thus, if the logarithm of any number, such as N, be known, we can immediately determine the logarithm of the product or quotient of that number by any power of 10.

Thus, if we know that $\log 2 = 0{\cdot}30103$, we have at once

$$\log 20 = \log (2 \times 10) = \log 2 + \log 10 = 1{\cdot}30103.$$

Similarly, $\log {\cdot}02 = \log (2/100) = \log 2 - \log 100 = -2 + {\cdot}30103$.

The latter example illustrates the advantage of keeping the decimal part of the logarithm positive, since then the logarithm of all multiples or quotients of the number by powers of 10 will have the same decimal figures, and will only differ in their integral parts.

263. Mantissa. Characteristic. When a logarithm is written so that it is the algebraic sum of a positive decimal fraction and a certain integer (whether positive or negative), the positive decimal part is called the *mantissa*, and the integral part is called the *characteristic*.

If the characteristic be negative, it is usual to write the negative sign *above* the number.

Thus $\bar{2}\cdot70516$ stands for $-2+0\cdot70516$; while $-2\cdot70516$ would signify $-2-0\cdot70516$. The latter number

$$=-3+(1-0\cdot70516)=-3+\cdot29484=\bar{3}\cdot29484.$$

264. Characteristics can be determined by inspection. *In the common system of logarithms the characteristic of the logarithm of any number can be determined by inspection.*

For suppose the number to be greater than unity, and to lie between 10^n and 10^{n+1}; then its logarithm must be greater than n and less than $n+1$; hence the characteristic of the logarithm is n.

Next, suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$; then its logarithm will be some negative quantity between $-n$ and $-(n+1)$; hence, if we agree that the mantissa shall always be positive, the characteristic will be $-(n+1)$.

265. Since the characteristic of the logarithm of a number can be written down by inspection, it is sufficient to give in the tables the mantissa only.

Example. Find the value of log 2173.

We find in the tables opposite to 2173 the number 3370597. The characteristic is 3. Hence the required logarithm is 3·3370597.

266. Conversely, if we know the characteristic of the logarithm of a number, we know the number of digits in the integral part of the number. If therefore the arrangement of figures in the number be given, we can tell where the decimal point must be.

Example. Find the number whose logarithm is $\bar{2}\cdot4560774$.

Opposite to 4560774 in the tables is the number 28581. Therefore $\log 2\cdot8581=\cdot4560774$.

$$\therefore \log \cdot028581=\bar{2}\cdot4560774.$$

EXAMPLES. XXIII. D.

1. Add together $\bar{8}\cdot2152630$, $6\cdot3212579$, and $\bar{3}\cdot3725700$.

2. Subtract $\bar{3}\cdot1527943$ from $\bar{2}\cdot4984732$.

3. Divide $\bar{2}\cdot5188142$ by 6, and $\bar{8}\cdot5502766$ by 7.

4. Given that $\log_{10} 95\cdot882 = 1\cdot9817371$; write down the logarithms of 9588200, 9588·2, and ·0095882.

5. Given that $\log_{10} 6\cdot4145 = \cdot8071628$; find the numbers whose logarithms are $3\cdot8071628$ and $\bar{3}\cdot8071628$.

6. Find the characteristic of $\log_7 350$, of $\log_{\frac{1}{2}} \cdot065$, and of the logarithm of 500 to the base 3.

7. Find the characteristics of the following logarithms,
$\log_4 21$; $\log_{11} \frac{1}{12}$; $\log_3 95$; $\log_{30} 29$; $\log_{10} \cdot0003$; $\log_2 63$; $\log_7 16829$.

8. Find the logarithms to the base 10 of $\overline{\cdot001}|^{\cdot001}$, and of $\overline{\cdot0001}|^{\cdot0001}$.

267. Uses of Logarithmic Tables. The chief purposes for which we want tables of logarithms are (i) to find the logarithm of a given number, and (ii) from a given logarithm to find the number of which it is the logarithm. If the number or the logarithm be given in the table, this can be done at once. It will be convenient to postpone until Chapter XXXII. the explanation of how a table of logarithms is used to find numbers or logarithms which are not expressly given in the tables [see Arts. 418—422]; but if the student will assume the result of Art. 418, he may read here the examples which are worked out in Arts. 419—422.

268. A few miscellaneous examples on logarithms are here added.

Ex. 1. *Having given* $\log_{10} 3 = \cdot4771213$, *find how many digits there are in* 3^{100}.

Let $$x = 3^{100}.$$

$$\therefore \log x = 100 \log 3 = 47\cdot71213.$$

$$\therefore \text{ the characteristic of } \log x \text{ is } 47.$$

$$\therefore \text{ there are 48 digits in } x.$$

Ex. 2. *Find, correct to three places of decimals, values of x and y which satisfy the equations $2^x 3^y = 1 = 3^{x+1} 2^{y-1}$; having given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$.*

Take the logarithms (to the base ten) of each side of the two given equations.

$$\therefore\ x \log 2 + y \log 3 = \log 1 = 0, \quad \text{[Art. 253, Ex. 1.}$$

and $$(x+1) \log 3 + (y-1) \log 2 = \log 1 = 0.$$

These are two simple equations between x and y. Solving them, we find

$$x = -\frac{\log 3}{\log 2 + \log 3}, \quad y = \frac{\log 2}{\log 2 + \log 3}.$$

Substituting for $\log 2$ and $\log 3$ their numerical values, we obtain

$$x = -\frac{\cdot 4771213}{\cdot 7781513} = -\cdot 6131\ldots$$

$$y = \frac{\cdot 3010300}{\cdot 7781513} = \cdot 3855\ldots$$

The value of x is nearer $-\cdot 613$ than $-\cdot 614$, and that of y is nearer $\cdot 386$ than $\cdot 385$. Hence the answer is $x = -\cdot 613$, $y = \cdot 386$.

Ex. 3. *Find how long it will be before a sum of money put out at compound interest at the rate of* 3 *per cent. per annum, payable annually, has doubled itself: it being given that $\log 2 = \cdot 30103$, and $\log 103 = 2\cdot 0128372$.*

Let n be the number of years required, and P the sum originally put out at interest.

At the end of the first year, the amount is $P + \frac{3}{100} P = P(1\cdot 03)$. Call this amount P_1. Then, at the end of the second year, P_1 has amounted to $P_1(1\cdot 03)$, that is, to $P(1\cdot 03)^2$. Similarly, at the end of the n^{th} year, the total amount will be $P(1\cdot 03)^n$. This, by the question, is $2P$.

$$\therefore\ P(1\cdot 03)^n = 2P.$$

$$\therefore\ (1\cdot 03)^n = 2.$$

$$\therefore\ n \log(1\cdot 03) = \log 2.$$

$$\therefore\ n = \frac{\cdot 30103}{\cdot 0128372} = 23\cdot 45, \text{ very nearly.}$$

It will therefore take a little less than $23\frac{1}{2}$ years before a sum of money doubles itself under these conditions.

MISCELLANEOUS EXAMPLES. XXIII. E.

[*In the first nineteen examples the values of the following logarithms are supposed to be given,*

$$\log_{10} 2 = \cdot 3010300,$$
$$\log_{10} 3 = \cdot 4771213,$$
$$\log_{10} 7 = \cdot 8450980,$$
$$\log_{10} 11 = 1\cdot 0413927.$$

All the logarithms in the following examples are supposed to be taken to the base 10, *unless the contrary is stated.*]

Find the logarithms of the following numbers, numbered **1** to **9**.

1. $\cdot 032$. **2.** $\cdot 36$. **3.** 720. **4.** $\log 72\cdot 9$. **5.** $7\cdot 5$.

6. $\cdot 00045$. **7.** $\sqrt[3]{\cdot 00012}$. **8.** $(29\cdot 7)^{-\frac{3}{5}}$. **9.** $\sqrt{\cdot 002835}$.

10. Find $\log_7 \sqrt{2}$, and $\log_{\sqrt{2}} 7$.

11. Determine which is greater, (i) $\cdot 01$ or $(\frac{2}{3})^9$; (ii) $\cdot 1$ or $(\frac{4}{5})^{10}$.

12. Determine which is the greater $(\frac{21}{20})^{100}$ or 100.

13. Find how many digits there are in 5^{25} and $(54)^{50}$, and in the integral part of $(3\frac{1}{3})^{100}$.

14. How many ciphers are there between the decimal point and the first significant figure in $(\cdot 6)^{100}$?

15. If the number of births in a year be $\frac{1}{30}$ of the population at the beginning of the year, and the number of deaths $\frac{1}{40}$; find in what time the population will be doubled.

16. In what time will £100 amount to £500 at 5 per cent. per annum compound interest?

17. In how many years will a sum of money double itself at compound interest, interest being payable yearly at the rate of 10 per cent. per annum?

18. Solve the equations $3^{1-x-y}\, 4^y = 3^{x-3y}\, 2^{2x-1} = 1$.

19. Given $\log_{12} 36 = a$, find $\log_{12} 48$.

20. Given $\log 24 = 1\cdot 3802112$ and $\log 36 = 1\cdot 5563025$; find $\log 8\cdot 64$, $\log 1\cdot 5$.

21. Given $\log_{10} 864 = 2\cdot 9365137$, $\log_{10} 486 = 2\cdot 6702459$; find $\log_{10} 648$.

22. Find the numbers whose logarithms to the base 64 are 2, $2\frac{1}{2}$, $2\frac{1}{3}$, $2\frac{1}{6}$.

23. What is the smallest number of logarithms (to the base 10) required in order to calculate the logarithms (to the base 10) of each of the following numbers? 64; 125; 50; 30; $\frac{4}{5}$; $\cdot\dot{6}$; $\cdot 375$.

Find the characteristics of the logarithms of these numbers to the base 2.

24. If $\log(1\frac{1}{2}) + \log(6\frac{2}{3}) = 1$, what is the base?

25. If there be 27647 digits in the integral part of $(1\cdot 89)^{100000}$, find log 1890.

26. Given that the integral part of $(3\cdot 981)^{100000}$ contains sixty thousand digits; find $\log_{10} 39810$, correct to five places of decimals.

27. Shew that the square root of $372\cdot 86$ is very nearly ten times its ninth root; having given $\log 37286 = 4\cdot 5714293$.

28. Assuming that the sixth and seventh powers of 7 are 117649 and 793543 respectively; prove that the first digit in the mantissa of $\log_{10} 7$ is 8.

29. How many positive integers are there whose logarithms to the base 2 have 5 for a characteristic?

30. Find x and y, if $(ax)^{\log a} = (cy)^{\log c}$ and $c^{\log x} = a^{\log y}$.

31. Prove that $\log \dfrac{a^2}{bc} + \log \dfrac{b^2}{ca} + \log \dfrac{c^2}{ab} = 0$.

32. Find the value of $7 \log_2 \frac{16}{15} + 5 \log_2 \frac{25}{24} + 3 \log_2 \frac{81}{80}$.

33. Find the sum of the logarithms (to the base 10) of the roots of the equation $x^2 - 14x + 100 = 0$.

34. Express $(a^2+b^2)^2 - (a^2-b^2)^2 - (a^2+b^2-c^2)^2$ in a form fitted for logarithmic calculation.

*35. If $\log_{2a} a = x$, and $\log_{3a} 2a = y$, shew that $2^{1-xy} = 3^{y-xy}$.

*36. Prove that, if the logarithm of y^a to the base x^b be equal to the logarithm of $x^{a'}$ to the base $y^{b'}$, then each of them is equal to $\left(\dfrac{aa'}{bb'}\right)^{\frac{1}{2}}$.

*37. If $\log_a b = \log_b c$, then will each be equal to $\log_{\frac{a}{b}} b + \log_{\frac{b}{a}} c$.

*38. A man borrows £500 from a money-lender. The bill is renewed every half-year with an increase of 12 per cent. What time will elapse before it reaches £5000? [log 112 = 2·049218.]

CHAPTER XXIV.

SURDS.

[*The results of this chapter are not required in the immediately following chapters, and the discussion of surds may be deferred for the present, if the student desire it.*]

269. It may be convenient if we repeat here that the root of a quantity is called a surd. The n^{th} root of a quantity X is denoted by $\sqrt[n]{X}$, and is such a quantity that its n^{th} power is X; it is called a surd of the n^{th} *order*. If there be no exact n^{th} root of X, the surd is said to be irrational [Arts. 23, 182].

270. Surds which have the same irrational factor are called *like* surds.

Thus, $2\sqrt{3}$, $\frac{1}{2}\sqrt{3}$, and $-3\sqrt{3}$ are like surds, because the irrational factor, $\sqrt{3}$, is common to each of them.

The sum of like surds can be combined into a single term in the same way as any other like quantities.

Thus, $2\sqrt{3}+\frac{1}{2}\sqrt{3}-3\sqrt{3}=(2+\frac{1}{2}-3)\sqrt{3}=-\frac{1}{2}\sqrt{3}$.

Surds which have not the same irrational factor are said to be *unlike*.

Thus, $2\sqrt{3}-3\sqrt{2}$ is the algebraical sum of two unlike surds, and cannot be simplified further.

271. We shall consider in this chapter some of the more simple propositions about surds. We shall begin

by discussing surds formed by the roots of simple expressions; next, we shall treat of compound expressions involving surds; and lastly, of the solution of equations which involve surds.

SIMPLE EXPRESSIONS INVOLVING SURDS.

272. The extension of the meaning of indices, which is given in Chapter XXII., enables us to write the root of a quantity either as a surd or as the quantity raised to a fractional power. The latter method of expression enables us to reduce two or more expressions to surds of the same order.

273. *Any rational quantity can be expressed as a surd of any required order.*

For, if a be any quantity, and n any positive integer,

$$a = \sqrt[n]{a^n} = (a^n)^{\frac{1}{n}},$$

which is a surd of the n^{th} order.

Thus, to express 2 as a quadratic surd, we have $2 = \sqrt{4} = 4^{\frac{1}{2}}$. Similarly, 2 can be written as a surd of the third order, $2 = \sqrt[3]{8} = 8^{\frac{1}{3}}$.

274. *The product of a rational quantity and a surd can be expressed as a surd.*

For, let a be the rational quantity, and $\sqrt[n]{b}$ the surd. Then,

$$a\sqrt[n]{b} = \sqrt[n]{a^n} \times \sqrt[n]{b} = \sqrt[n]{a^n b},$$

which is a surd of the n^{th} order.

This proposition may also be proved thus.

$$a \times b^{\frac{1}{n}} = (a^n)^{\frac{1}{n}} \times b^{\frac{1}{n}} = (a^n b)^{\frac{1}{n}}.$$

For example, $7\sqrt{5} = \sqrt{7^2 \times 5} = \sqrt{245}$.

275. *If the quantity under the root-sign of a surd can be resolved into factors, the surd can be expressed as the product of surds.*

For we proved, in Art. 186, that

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}.$$

Hence, if one of these factors, such as a, be the n^{th} power of any quantity, we can resolve the given surd into the product of a rational factor and an irrational factor.

For example, $\sqrt[n]{x^n y} = \sqrt[n]{x^n} \times \sqrt[n]{y} = x\sqrt[n]{y}$,

$$\sqrt[3]{a^3b^2} = \sqrt[3]{a^3} \times \sqrt[3]{b^2} = a\sqrt[3]{b^2},$$

$$\sqrt{49x^2y^3} = \sqrt{49x^2y^2} \times \sqrt{y} = 7xy\sqrt{y},$$

$$\sqrt{x^3} - \sqrt{xy^2} = x\sqrt{x} - y\sqrt{x} = (x-y)\sqrt{x},$$

$$\sqrt{x^3} + \sqrt{x^2y} = x\sqrt{x} + x\sqrt{y} = x(\sqrt{x} + \sqrt{y}).$$

276. Where we have a fraction with a surd (or a product of surds) in the denominator, it is usually convenient to make the *denominator rational.* Thus, if a surd like $\sqrt[n]{a}$ occur in the denominator, we multiply numerator and denominator by $\sqrt[n]{a^{n-1}}$, and thus make the denominator rational and equal to a.

For example,

$$\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2},$$

$$\frac{2\sqrt{3}}{\sqrt{5}} = \frac{2\sqrt{3} \times \sqrt{5}}{(\sqrt{5})^2} = \frac{2\sqrt{15}}{5}.$$

277. *A surd of the* n^{th} *order can be expressed as a surd of the* mn^{th} *order, where* m *and* n *are positive integers.*

This proposition is proved by a method analogous to that given in Art. 273.

For, if $\sqrt[n]{a}$ be the given surd, we have

$$\sqrt[n]{a} = \sqrt[m]{\{(\sqrt[n]{a})^m\}} = \sqrt[m]{(\sqrt[n]{a^m})} = \sqrt[mn]{a^m}.$$

This proposition may also be proved thus.

$$\sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \sqrt[mn]{a^m}.$$

For example,

(i) $\sqrt{a} = \sqrt[6]{a^3}$. (ii) $\sqrt{b} = \sqrt[4]{b^2}$. (iii) $\sqrt{(4x^3)} = 2x\sqrt{x} = 2x\sqrt[4]{x^2}$.

(iv) $\sqrt{(4x^2y^{\frac{3}{2}})} = 2x\sqrt{y^{\frac{3}{2}}} = 2x \times (y^{\frac{3}{2}})^{\frac{1}{2}} = 2xy^{\frac{3}{4}} = 2x\sqrt[4]{y^3}$.

278. *Any two surds can be expressed as surds of the same order.*

For, if $a^{\frac{m}{n}}$ and $b^{\frac{p}{q}}$ be two surds, we can express each of them as a surd whose order is the L. C. M. of n and q.

For, $$a^{\frac{m}{n}} = a^{\frac{mq}{nq}} = \sqrt[nq]{a^{mq}},$$

and $$b^{\frac{p}{q}} = b^{\frac{pn}{qn}} = \sqrt[nq]{b^{pn}}.$$

The above surds are each of the order nq. But the process depends on reducing the fractions $\frac{m}{n}$ and $\frac{p}{q}$ to a common denominator. Therefore if n and q have a common factor, the order of the resulting surds will be the L.C.M. of n and q, and not nq.

Example. Which is the greater $\sqrt[3]{4}$ *or* $\sqrt{3}$?

Reduce the surds to equivalent surds of the same order; in this case, the order will be the L. C. M. of 3 and 2, that is, will be 6.

Then, $$\sqrt[3]{4} = 4^{\frac{1}{3}} = (4^2)^{\frac{1}{6}} = (16)^{\frac{1}{6}},$$

and $$\sqrt{3} = 3^{\frac{1}{2}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}}.$$

But, since $27 > 16$, $\therefore\ (27)^{\frac{1}{6}} > (16)^{\frac{1}{6}}$.

Hence, $$\sqrt{3} > \sqrt[3]{4}.$$

279. *The product or the quotient of any two surds can be written as a surd.*

For, by the last article, we can express the two surds as surds of the same order. Also, we know that the product or quotient of two surds of the same order can be written as a surd, for

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}, \text{ and } \sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{a/b}.$$

Thus,

$$a^{\frac{m}{n}} \times b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \times b^{\frac{pn}{qn}} = (a^{mq} \times b^{pn})^{\frac{1}{nq}} = \sqrt[nq]{(a^{mq} \times b^{pn})}.$$

$$a^{\frac{m}{n}} \div b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \div b^{\frac{pn}{qn}} = (a^{mq} \div b^{pn})^{\frac{1}{nq}} = \sqrt[nq]{(a^{mq} \div b^{pn})}.$$

Ex. 1. $\sqrt{x} \times \sqrt[3]{y} = x^{\frac{1}{2}} \times y^{\frac{1}{3}} = (x^3 y^2)^{\frac{1}{6}} = \sqrt[6]{x^3 y^2}.$

Ex. 2. $\sqrt[4]{a^3} \div \sqrt[3]{b^2} = a^{\frac{3}{4}} \div b^{\frac{2}{3}} = a^{\frac{9}{12}} \div b^{\frac{8}{12}} = (a^9 \div b^8)^{\frac{1}{12}} = \sqrt[12]{a^9/b^8}.$

EXAMPLES. XXIV. A.

1. Express 3 as a surd of the second order, a surd of the third order, and a surd of the fourth order.

Express the following quantities, numbered **2** to **6**, as surds.

2. $a\sqrt[3]{b}$. **3.** $3\sqrt{2}$. **4.** $2\sqrt{3}$. **5.** $x^2\sqrt{y}$. **6.** $x^3y^2\sqrt{xy}$.

Express each of the following fractions, numbered **7** to **11**, as a fraction having a rational denominator.

7. $\frac{1}{\sqrt{2}}$. **8.** $\frac{2}{\sqrt{3}}$. **9.** $\frac{\sqrt{5}}{\sqrt{7}}$. **10.** $\frac{\sqrt{ab}}{\sqrt[3]{bc}}$. **11.** $\frac{x\sqrt[n]{y}}{y\sqrt[n]{x}}$.

12. Prove that (i) $3^{\sqrt{27}} = 27^{\sqrt{3}}$; (ii) $(\sqrt{3})^{3\sqrt{3}} = (3\sqrt{3})^{\sqrt{3}}$.

13. Is $\sqrt[4]{84}$ greater or less than $\sqrt[3]{28}$?

14. Of the following quantities, which is the greatest and which the least?

$$\sqrt{35}, \quad \sqrt[3]{214}, \quad \sqrt[4]{1290}.$$

COMPOUND EXPRESSIONS INVOLVING SURDS.

280. The rules for the multiplication or the division of expressions by quantities like $a + \sqrt{b}$ are the same as those for multiplication or division by rational quantities.

If we have to divide one such expression by another, or if the work involve the multiplication of surds of different orders, it is usually convenient to express all the surds with fractional indices; but simple cases of multiplication may be treated directly.

Ex. 1. *Multiply* $a+\sqrt{x}$ *by* $b+\sqrt{x}$.

We have

$$\begin{array}{r} a+\sqrt{x} \\ b+\sqrt{x} \\ \hline ab+b\sqrt{x} \\ a\sqrt{x}+x \\ \hline ab+(a+b)\sqrt{x}+x \end{array}$$

Ex. 2. *Find the square of* $5-3\sqrt{x}$.

We have

$$\begin{array}{r} 5-3\sqrt{x} \\ 5-3\sqrt{x} \\ \hline 25-15\sqrt{x} \\ -15\sqrt{x}+9x \\ \hline 25-30\sqrt{x}+9x \end{array}$$

Ex. 3. *Find the product of* $\sqrt{x}-\sqrt{y}$ *and* $\sqrt{x}+\sqrt{y}$.

We have $(A-B)(A+B)=A^2-B^2$,

$$\therefore (\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})=(\sqrt{x})^2-(\sqrt{y})^2$$
$$=x-y.$$

EXAMPLES. XXIV. B.

1. Find the product of $2-3\sqrt{x}$ and $4-5\sqrt{x}$.
2. Find the square of $7-2\sqrt{a}$.
3. Divide $6-2a-\sqrt{a}$ by $3-2\sqrt{a}$.
4. Divide $\sqrt{x^3}-11\sqrt{x}+6$ by $\sqrt{x}-3$.
5. Divide $x-81$ by $\sqrt[4]{x}+3$.
6. Prove that, if $x=2+\sqrt{2}$, then $(x-1)(x-2)=x$.

281. Where we are dealing with a compound expression involving surds, it is generally desirable to begin by writing every term and the whole expression so that the denominator is rational. Where the denominator is a surd or a product of surds, this can be effected by the rules given in Art. 276. If the denominator be a compound expression, the process is somewhat more complicated, but it will be sufficient here to remark that a fraction, whose denominator is of the form $a+\sqrt{b}$, will be rationalized, if both its numerator and its denominator be multi-

plied by $a-\sqrt{b}$, since the denominator will then take the form a^2-b. Similarly, a fraction, whose denominator is of the form $a-\sqrt{b}$, will be rationalized, if both its numerator and its denominator be multiplied by $a+\sqrt{b}$.

Thus, $$\frac{X}{a+\sqrt{b}}=\frac{X(a-\sqrt{b})}{(a+\sqrt{b})(a-\sqrt{b})}=\frac{X(a-\sqrt{b})}{a^2-b};$$

and $$\frac{X}{a-\sqrt{b}}=\frac{X(a+\sqrt{b})}{(a-\sqrt{b})(a+\sqrt{b})}=\frac{X(a+\sqrt{b})}{a^2-b}.$$

For example,

$$\frac{5-\sqrt{2}}{3-\sqrt{2}}=\frac{(5-\sqrt{2})(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{15+2\sqrt{2}-2}{9-2}=\frac{13+2\sqrt{2}}{7}.$$

282. The following is an important proposition.

If $x+\sqrt{y}=a+\sqrt{b}$, where x and a are rational quantities, and $\sqrt{y}$ and $\sqrt{b}$ are irrational surds, then will $x=a$, and $y=b$.

We have $$x+\sqrt{y}=a+\sqrt{b}.$$
$$\therefore\ x-a+\sqrt{y}=\sqrt{b}.$$

Squaring both sides of this equation,

$$\therefore\ (x-a)^2+y+2(x-a)\sqrt{y}=b.$$
$$\therefore\ 2(x-a)\sqrt{y}=b-(x-a)^2-y.$$

Therefore, unless $(x-a)$ is equal to zero, we have a multiple of an irrational quantity equal to a rational quantity. This is impossible, and therefore $x-a=0$, that is, $x=a$. Again, if $x=a$, the given relation reduces to $\sqrt{y}=\sqrt{b}$, and therefore $y=b$.

Note. We have assumed that the square root of y is not rational, and our proof requires that $\sqrt{b}$ and $\sqrt{y}$ shall both be irrational surds. Thus, from the relation $2+\sqrt{4}=3+\sqrt{1}$ where $\sqrt{4}$ and $\sqrt{1}$ are not irrational surds, we could not infer that $2=3$ and $4=1$.

283. We have just shewn that, if $x+\sqrt{y}=a+\sqrt{b}$, then $x=a$ and $\sqrt{y}=\sqrt{b}$. It therefore follows that $x-\sqrt{y}=a-\sqrt{b}$.

Expressions like $x+\sqrt{y}$ and $x-\sqrt{y}$ are said to be *conjugate*. Thus if two expressions be equal, their conjugates are also equal.

284. *The square root of an expression like $a+b\sqrt{c}$ can sometimes be found as the sum of two quadratic surds.*

Suppose $\sqrt{(a+b\sqrt{c})}=\sqrt{x}+\sqrt{y}$.

Square both sides, $\therefore\ a+b\sqrt{c}=x+y+2\sqrt{xy}$.

Hence, by Art. 282, $a=x+y$, and $b\sqrt{c}=2\sqrt{xy}$.

These are two equations from which we can obtain x and y: and, since $x+y=a$ and $xy=\frac{1}{4}b^2c$, it follows [Art. 232] that x and y are the two roots of the equation

$$z^2-az+\tfrac{1}{4}b^2c=0.$$

These roots are

$$\frac{a+\sqrt{(a^2-b^2c)}}{2} \text{ and } \frac{a-\sqrt{(a^2-b^2c)}}{2}.$$

Therefore the required square root, namely, $\sqrt{x}+\sqrt{y}$, is

$$\sqrt{\left\{\frac{a+\sqrt{(a^2-b^2c)}}{2}\right\}}+\sqrt{\left\{\frac{a-\sqrt{(a^2-b^2c)}}{2}\right\}}.$$

This expression is however more complicated than the original expression unless a^2-b^2c is a perfect square. In any particular case where a^2-b^2c is a perfect square, the square root reduces to the sum of two quadratic surds.

The square root of $a-b\sqrt{c}$ can be similarly determined.

Example. Find the square root of $27-10\sqrt{2}$.

Let $\sqrt{(27-10\sqrt{2})}=\sqrt{x}-\sqrt{y}$.

[*Note.* It is convenient to assume that $\sqrt{x}$ and $\sqrt{y}$ have opposite signs when the two given numbers (in this case, 27 and $-10\sqrt{2}$) are of opposite signs; and that they have the same sign when the two given numbers are of the same sign.]

Square both sides, $\therefore\ 27-10\sqrt{2}=x+y-2\sqrt{(xy)}$,

$\therefore\ 27=x+y$, and $10\sqrt{2}=2\sqrt{(xy)}$, [Art. 282.

that is, $y=27-x$, and $xy=50$.

$$\therefore\ x(27-x)=50.$$

$$\therefore\ (x-25)(x-2)=0.$$

But $y=27-x$. Hence, if $x=25$, $y=2$; and if $x=2$, $y=25$.

The first solution is alone applicable to this problem.

Therefore $$\sqrt{27-10\sqrt{2}}=5-\sqrt{2}.$$

If we had tried to find the square root of $27+10\sqrt{2}$, we should have assumed it to be equal to $\sqrt{x}+\sqrt{y}$, where $\sqrt{x}$ and $\sqrt{y}$ would have to be taken of the same sign. In this case, we should have obtained the same quadratic equation for x as that written above; and the answer would be $5+\sqrt{2}$.

MISCELLANEOUS EXAMPLES ON SURDS. XXIV. C.

Simplify the following expressions, numbered **1** to **5**.

1. $\left\{\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right\}^{-\frac{1}{2}}$.

2. $\dfrac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y}\div\dfrac{\sqrt{x^2+y^2}-y}{x-\sqrt{x^2-y^2}}$.

3. $\dfrac{a-\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}+b}\div\dfrac{\sqrt{a^2+b^2}-b}{a+\sqrt{a^2-b^2}}$.

4. $\dfrac{1+2x^{\frac{1}{2}}+x^{\frac{3}{4}}+2x}{1+3x^{\frac{1}{2}}+2x^{\frac{3}{4}}+3x}$.

5. $\left(\dfrac{\sqrt{x}}{1+\sqrt{x}}+\dfrac{1-\sqrt{x}}{\sqrt{x}}\right)\div\left(\dfrac{\sqrt{x}}{1+\sqrt{x}}-\dfrac{1-\sqrt{x}}{\sqrt{x}}\right)$.

6. Shew that, if $x=3-\sqrt{3}$, then $x^2+\dfrac{36}{x^2}=24$.

7. If $x=2+\sqrt{3}$, find the value of $\dfrac{1+x^{-\frac{1}{2}}}{1-x^{\frac{1}{2}}}-\dfrac{x^{-\frac{1}{2}}-1}{x^{\frac{1}{2}}+1}$.

8. Multiply $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

9. Divide x^2+y^2 by $x^{\frac{2}{3}}+y^{\frac{2}{3}}$.

10. Divide $2x^3-6x+5$ by $x\sqrt[3]{2}+\sqrt[3]{4}+1$.

11. Divide $(x^4+y^4)\sqrt{3}+6x^2y^2+xy(x^2+y^2)(\sqrt{6}+\sqrt{2})$ by $x^2+xy\sqrt{2}+y^2\sqrt{3}$.

Find the square roots of the expressions, numbered **12** to **15**.

12. $30+12\sqrt{6}$.

13. $7+2\sqrt{6}$.

14. $107-42\sqrt{2}$.

15. $\frac{5}{6}+\frac{\sqrt{2}}{\sqrt{3}}$.

Find the square roots of the expressions, numbered **16** to **20**.

16. $9x^4-6x^3+24x^{\frac{3}{2}}+1-\frac{8}{\sqrt{x}}+\frac{16}{x}$.

17. $\frac{x}{y}+\frac{y}{x}+3-2\sqrt{\frac{x}{y}}-2\sqrt{\frac{y}{x}}$.

18. $\frac{x+\sqrt{x^2-y^2}}{2}$.

19. $a+\sqrt{(a^2+2bc-b^2-c^2)}$.

20. $4+\sqrt{5}+\sqrt{(17-4\sqrt{15})}$.

21. Simplify $\sqrt{30+10\sqrt{5}}$.

22. Simplify $\frac{1}{\sqrt{3}-\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}+1}-\frac{2\sqrt{2}}{\sqrt{3}-1}$.

***23.** Simplify $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

***24.** Simplify $\left(\frac{x+y}{x}+\frac{x+y}{y}+\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\div\frac{\sqrt{x}-\sqrt{y}}{\sqrt{y}}+\frac{1}{4}\right)^{\frac{1}{2}}$.

EQUATIONS INVOLVING SURDS.

285. It not unfrequently happens that an equation proposed for solution contains surds in which the unknown quantity appears under the radical sign, and in such a case we must get rid of the surd before we can solve the equation.

The usual method of effecting this is (after simplifying the equation as far as possible) to transpose one radical to one side of the equation, and to transpose all the other quantities to the other side. By then squaring both sides (or raising them to a suitable power) we get rid of that radical. Repeating the process again, we can get rid of another radical. Continuing the process, we finally obtain an equation which is rational and integral in the unknown quantity; and this equation must then be solved by the methods already described.

We may add that the roots of the equation thus obtained may contain surds which involve nothing but numbers or known quantities [see Art. 105, Ex. 4], though such surds should always be simplified as far as possible.

It may also be well to warn the beginner to be careful that the square of that side of the equation which contains a radical and another term is written down correctly. It is a not uncommon mistake to write the square of $a+\sqrt{b}$ as a^2+b, whereas it is $a^2+2a\sqrt{b}+b$. In a similar way, beginners sometimes think that $(x^2+y^2)^{\frac{1}{2}}$ is equal to $x+y$; this is not the case, for we have already shewn that $(x^2+2xy+y^2)^{\frac{1}{2}}$ is equal to $\pm(x+y)$.

286. The method is illustrated by the following examples.

Ex. 1. *Solve the equation* $\sqrt{x}-4=0$.

Here $\sqrt{x}=4$.

Squaring both sides, $\therefore\ x=16$.

Ex. 2. *Solve the equation* $2\sqrt{x-1}+\sqrt{4x+5}=9$.

Transpose one radical to one side, and all the other terms to the other side,

$$\therefore\ 2\sqrt{x-1}=9-\sqrt{4x+5}.$$

Square both sides,

$$\therefore\ 4(x-1)=81-18\sqrt{4x+5}+4x+5.$$

Transpose the radical to one side of the equation, and all the other terms to the other side; collect like terms, and simplify,

$$\therefore\ 18\sqrt{4x+5}=90.$$

$$\therefore\ \sqrt{4x+5}=5.$$

$$\therefore\ 4x+5=25.$$

$$\therefore\ x=5.$$

The two examples just given led to simple equations. The two following examples lead to quadratic equations.

Ex. 3. *Solve the equation* $2\sqrt{2x+1}-3\sqrt{x-3}=3$

Transposing, $3\sqrt{x-3}=2\sqrt{2x+1}-3.$

Squaring both sides, $\therefore\ 9(x-3)=4(2x+1)-12\sqrt{2x+1}+9.$

Transpose the radical to one side of the equation, and all the other terms to the other side, and simplify,

$$\therefore\ 12\sqrt{2x+1}=40-x.$$

Squaring both sides, $\therefore\ 144(2x+1)=1600-80x+x^2.$

$$\therefore\ x^2-368x+1456=0,$$

the roots of which are 4 and 364.

If now we proceed to verify this solution, we find that $x=4$ satisfies the given equation, since on putting $x=4$ in the equation it becomes $2\sqrt{9}-3\sqrt{1}=3$, which is clearly true.

If however we put $x=364$ in the equation, it becomes

$$2\sqrt{729}-3\sqrt{361}=3,$$

that is, $2\times 27-3\times 19=3$, or $54-57=3$,

which is clearly not true.

Hence, only one of the roots which we have found satisfies the original equation. It is only by trial that we can find which of the two roots is the one we require.

[The explanation of this paradox is that we shall obtain the same quadratic equation for x from another equation involving surds, and the root 364 satisfies this other equation. In fact, if we solve the equation $2\sqrt{2x+1}-3\sqrt{x-3}=-3$, we have

$$3\sqrt{x-3}=2\sqrt{2x+1}+3.$$

$$\therefore\ 9(x-3)=4(2x+1)+12\sqrt{2x+1}+9.$$

$$\therefore\ -12\sqrt{2x+1}=40-x,$$

and squaring both sides of this last equation, we obtain the same quadratic as before, whose roots are 4 and 364. The root 364 satisfies the equation above considered, but the root 4 does not satisfy it.

The equation $2\sqrt{2x+1}+3\sqrt{x-3}=3$ also leads to the same quadratic equation for x; but neither of the roots of the quadratic will satisfy this equation. This equation has no root.]

Ex. 4. *Solve the equation* $\sqrt{x}+\sqrt{x+a}-\sqrt{x+b}=0.$

We shall follow the above procedure without specifically indicating each step.

We have $\sqrt{x}=\sqrt{x+b}-\sqrt{x+a}$.

$$\therefore\ x=(x+b)-2\sqrt{(x+b)(x+a)}+x+a.$$

$$\therefore\ 2\sqrt{(x+b)(x+a)}=x+b+a.$$

$$\therefore\ 4(x+b)(x+a)=(x+a+b)^2.$$

$$\therefore\ 3x^2+2(a+b)x-(a-b)^2=0.$$

$$\therefore\ x=\tfrac{1}{3}\{-(a+b)\pm\sqrt{4(a^2-ab+b^2)}\}.$$

We cannot determine which of these two roots satisfies the given equation unless we know the numerical values of a, or of b, or some relation between them.

If radicals appear in the denominator of a fraction, the equation must be simplified so as to get rid of them.

Ex. 5. *Solve the equation* $\sqrt{x-a}+\sqrt{x}=\dfrac{a}{\sqrt{x-a}}$.

Multiplying up, we have

$$x-a+\sqrt{x}\times\sqrt{x-a}=a.$$

$$\therefore\ \sqrt{x}\times\sqrt{x-a}=2a-x.$$

$$\therefore\ x(x-a)=4a^2-4ax+x^2.$$

$$\therefore\ 3ax=4a^2.$$

$$\therefore\ x=\tfrac{4}{3}a.$$

Ex. 6. *Solve the equation* $\sqrt{x}+3=\dfrac{6}{4-\sqrt{x}}$.

Multiplying up, we have $(\sqrt{x}+3)(4-\sqrt{x})=6$.

$$\therefore\ \sqrt{x}+12-x=6.$$

$$\therefore\ \sqrt{x}=x-6.$$

$$\therefore\ x=(x-6)^2.$$

$$\therefore\ x^2-13x+36=0.$$

$$\therefore\ (x-4)(x-9)=0.$$

Of these roots, $x=9$ alone satisfies the original equation.

In some cases, it is convenient to introduce a subsidiary symbol in the same way as we did in the examples worked out on pp. 239, 240, 242, 243. This is illustrated by the following examples.

Ex. 7. *Solve the equation* $x^2 - \frac{1}{2}x + \sqrt{2x^2 - 3x + 5} = x + 15.$

Simplifying the equation, it becomes

$$2x^2 - 3x + 2\sqrt{2x^2 - 3x + 5} = 30.$$

Let $2x^2 - 3x = y$, then the equation becomes

$$y + 2\sqrt{y+5} = 30.$$

$$\therefore\ 2\sqrt{y+5} = 30 - y \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore\ 4(y+5) = 900 - 60y + y^2.$$

$$\therefore\ y^2 - 64y = -880.$$

$$\therefore\ y^2 - 64y + (32)^2 = (32)^2 - 880$$

$$= 144.$$

$$\therefore\ y - 32 = \pm 12.$$

$$\therefore\ y = 32 + 12 = 44, \quad \text{or } y = 32 - 12 = 20.$$

Of these two roots, the latter alone satisfies the equation (i); therefore $y = 20$.

But $y = 2x^2 - 3x$, $\quad \therefore\ 2x^2 - 3x = 20.$

$$\therefore\ x^2 - \tfrac{3}{2}x = 10.$$

$$\therefore\ x^2 - \tfrac{3}{2}x + (\tfrac{3}{4})^2 = 10 + (\tfrac{3}{4})^2$$

$$= \tfrac{169}{16}.$$

$$\therefore\ x - \tfrac{3}{4} = \pm\tfrac{13}{4}.$$

$$x = \tfrac{3}{4} + \tfrac{13}{4} = 4, \quad \text{or } x = \tfrac{3}{4} - \tfrac{13}{4} = -\tfrac{5}{2}.$$

Hence the required roots of the given equation are 4 and $-\frac{5}{2}$.

Ex. 8. *Of a swarm of bees clustered on a tree, the square root of half their number flew away. Eight-ninths of the original number then departed, leaving but two behind. How many were there at first?* (Bhaskara's *Bija Ganita*, circ. 1160.)

Let x be the number of bees. Then we have, by the question,

$$\sqrt{\tfrac{1}{2}x} + \tfrac{8}{9}x + 2 = x.$$

$$\therefore\ \sqrt{\tfrac{1}{2}x} = x - \tfrac{8}{9}x - 2 = \tfrac{1}{9}x - 2.$$

Square both sides, $\quad \therefore\ \frac{1}{2}x = (\frac{1}{9}x - 2)^2.$

Simplify, $\quad \therefore\ 2x^2 - 153x + 648 = 0,$

of which the roots are 72 and $\frac{9}{2}$. The latter root is not applicable to the problem. Hence there were 72 bees.

EQUATIONS INVOLVING SURDS. XXIV. D.

Solve the following equations, numbered **1** to **46**.

1. $\sqrt{x-4}+x=10.$

2. $\sqrt{a+x}+\sqrt{a-x}=\sqrt{b}.$

3. $\sqrt{x+2}+\sqrt{x+9}=7.$

4. $\sqrt{7x}-\sqrt{3x+1}=1.$

5. $\sqrt{8x+5}-2\sqrt{2x-1}=1.$

6. $\sqrt{x+19}+\sqrt{x+3}=8.$

7. $x+\sqrt{ax-a^2}=3a.$

8. $3x=10-\sqrt{5x+6}.$

9. $\sqrt{x}+\sqrt{a+x}=\dfrac{a}{\sqrt{x}}.$

10. $2\sqrt{x}+2x^{-\frac{1}{2}}=5.$

11. $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=2ax^{-\frac{1}{2}}.$

12. $\sqrt{x+3}+\sqrt{2x-1}=9.$

13. $\sqrt{7x+1}-\sqrt{3x+4}=1.$

14. $\sqrt{10x-1}=3+\sqrt{3x+1}.$

15. $\sqrt{4x+5}-2\sqrt{x-1}=1.$

16. $\sqrt{x}+\sqrt{5x+1}=5.$

17. $\sqrt{3+x}+\sqrt{7-x}=\sqrt{4+2x}.$

18. $\sqrt{4+x}+\sqrt{6-x}=\sqrt{6+2x}.$

19. $\sqrt{x+5}+2\sqrt{x+1}=\sqrt{3x+7}.$

20. $\sqrt{x+28}+\sqrt{9x-28}=4\sqrt{2x-14}.$

21. $\sqrt{7x-6}-\sqrt{4x+1}=\sqrt{3x-17}.$

22. $\sqrt{14x+9}+2\sqrt{x+1}+\sqrt{3x+1}=0.$

23. $\sqrt{12x-3}+\sqrt{x+2}+\sqrt{7x-13}=0.$

24. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}.$

25. $\sqrt{x+3}-2\sqrt{x+1}=\sqrt{5x+4}.$

26. $\sqrt{3x+1}-\sqrt{4x+5}+\sqrt{x-4}=0.$

27. $\sqrt{8x+1}-\sqrt{x+1}=\sqrt{3x}.$

28. $a\sqrt{x^2+a^2}=x^2-a^2.$

29. $\dfrac{c}{\sqrt{x}-d}+\dfrac{d}{\sqrt{x}-c}=2.$

30. $\sqrt{\dfrac{x+7}{2}}+\sqrt{\dfrac{x-7}{2}}=7.$

31. $\dfrac{m\sqrt{c^2-x^2}-n(c+x)}{m\sqrt{c^2-x^2}+n(c+x)}=\dfrac{ma-nb}{ma+nb}.$

32. $x^2+(x-2)(x-3)+\sqrt{2x^2-5x+6}=6.$

33. $2(2x-3)(x-4)-\sqrt{2x^2-11x+15}=60.$

34. $5x^2+11x-12\sqrt{(x+4)(5x-9)}=36.$

35. $4x+4\sqrt{3x^2-7x+3}=3x(x-1)+6.$

36. $x^2+\sqrt{x^2+3x+5}=7-3x.$

37. $y^2+2cy-2a\sqrt{\left\{\frac{y^2+2cy+c-a}{a+c}\right\}}=0.$

38. $4x^2+\sqrt{4x^2-10x+1}=10x+1.$

39. $x^2-3x-3\sqrt{x^2-3x-10}=118.$

40. $x\sqrt{x^2+12}+x\sqrt{x^2+6}=3.$

*41. $2x\sqrt{x^2+a^2}+2x\sqrt{x^2+b^2}=a^2-b^2.$

*42. $x^2+a^2+\sqrt{x^4+a^4}=2a\sqrt{x^2+\sqrt{x^4+a^4}}.$

*43. $\left.\begin{array}{l}\sqrt{x+y}=\sqrt{y}+2\\ x-y=7\end{array}\right\}.$

*44. $\left.\begin{array}{l}\sqrt{x+y}=\sqrt{x}+1\\ x-y=7\end{array}\right\}.$

*45. $\left.\begin{array}{r}x-\sqrt{xy}-26=0\\ 3\sqrt{y}-2\sqrt{x}-7=0\end{array}\right\}.$

*46. $\left.\begin{array}{r}3x-2\sqrt{xy}+9=0\\ 5\sqrt{x}-3\sqrt{y}-3=0\end{array}\right\}.$

***47.** Explain the fact that the value of x obtained by solving the equation $\sqrt{x+8}+\sqrt{x-1}=1$ does not appear to satisfy the equation.

CHAPTER XXV.

RATIO AND PROPORTION.

287. **Ratio.** The relation which two quantities bear to one another in magnitude may be regarded in two ways. We may consider how much one of them is greater than the other: we call this their difference. Or, we may consider *how many times one of them contains the other:* we call this their *ratio.*

288. The quantities in the definition of ratio must be of the same kind. Thus, we can compare a length with a length, or an area with another area, or a sum of money with another sum of money, but we cannot compare inches with shillings, or acres with weeks.

Moreover, the quantities to be compared must be expressed as multiples of the same unit. Every quantity, as we have already remarked, is measured by the number of times it contains a certain unit of its own kind. Thus, if we take a mile as our unit of length, then any length will be measured by the number of miles it contains. If, for example, a certain length be equal to a $\frac{1}{4}$ mile, the numerical measure is $\frac{1}{4}$ when the unit of length is a mile. The same distance might have been expressed as 440 yards, in which case a yard is the unit of length, and 440 is the numerical measure of the magnitude.

To compare quantities, we must express each as a multiple of the *same* unit, and we shall then only have to compare their numerical measures. Thus, whether the quantities in the definition of ratio given in Art. 287 be abstract or concrete, their ratio will be measured by the number of times which one number contains another number.

289. **Notation.** The ratio of a to b is written $a : b$, which is read as *a to b*.

The quantities a and b are called the *terms* of the ratio; of these, a is called the *first term* or *antecedent*, and b is called the *second term* or *consequent*.

A ratio is said to be a *ratio of greater inequality* if the first term be greater than the second term; a ratio is said to be a *ratio of less inequality* if the first term be less than the second term.

290. **Definitions.** The following terms are used:

The *duplicate ratio* of $a : b$ is the ratio $a^2 : b^2$.
The *triplicate ratio* of $a : b$ is the ratio $a^3 : b^3$.
The *subduplicate ratio* of $a : b$ is the ratio $\sqrt{a} : \sqrt{b}$.
The *subtriplicate ratio* of $a : b$ is the ratio $\sqrt[3]{a} : \sqrt[3]{b}$.
The *sesquiplicate ratio* of $a : b$ is the ratio $\sqrt{a^3} : \sqrt{b^3}$.
The ratio *compounded* of the ratios $a : b$ and $c : d$ is the ratio $ac : bd$.

291. **Ratios are measured by Fractions.** The number of times which one number a contains another number b is found by dividing a by b. Hence, the measure of the ratio $a : b$ is the fraction $\frac{a}{b}$.

292. **Incommensurable Quantities.** If two numbers have no common measure (as, for example, if one number be an irrational surd and the other number be an integer), we cannot with accuracy speak of the number of times that one is contained in the other. Such numbers are said to be *incommensurable*, one to the other.

Two numbers are *incommensurable*, one to the other, when their ratio cannot be expressed as the ratio of two integers. Two numbers are *commensurable*, one to the other, when their ratio can be expressed as the ratio of two integers.

A number is said to be incommensurable or commensurable according as it is incommensurable or commensurable to unity.

Thus $\frac{2}{3}$ is commensurable, for the ratio $\frac{2}{3} : 1$ is measured by the fraction $\frac{2}{3}$ [Art. 291], and this fraction is also the measure of the ratio $2 : 3$ [Art. 291]. But $\sqrt{2}$ is incommensurable, for no integers can be found whose ratio, one to the other, is the same as that of $\sqrt{2} : 1$.

293. **Ratios of Incommensurable Quantities.** We have hitherto given no means of comparing the magnitudes of two incommensurable quantities. We shall therefore extend the result of Art. 291 by defining the measure of the ratio of two incommensurable quantities, a and b, as the fraction $\dfrac{a}{b}$.

294. *The value of a ratio is unaltered, if each term be multiplied by the same number, or if each term be divided by the same number.*

The ratio of any two numbers, whether commensurable or incommensurable, is measured by a fraction whose numerator is the first term of the ratio and whose denominator is the second term of the ratio [Arts. 291, 293]. Hence all the properties which in Chapter X. were proved true of fractions are also true of ratios. Hence [Arts. 143, 146] the required result follows.

Thus the ratio $a : b$ is equal either to the ratio $ma : mb$, or to the ratio $a/m : b/m$.

295. **Comparison of Ratios.** Since ratios are measured by fractions we can, by Art. 181, compare the values of two or more ratios, for we can express them as fractions having a common positive denominator.

Thus the ratio $a : b$ is $>$ = or $<$ the ratio $c : d$,

as the fraction $\dfrac{a}{b}$ is $>$ = or $<$ the fraction $\dfrac{c}{d}$,

that is, as $\dfrac{ad}{bd}$ is $>$ = or $<$ $\dfrac{bc}{bd}$,

that is (*provided bd is positive*) as ad is $>$ = or $<$ bc, [Art. 180 (iv)

296. Since ratios are measured by fractions, many of the following examples may be considered as examples of the properties either of fractions or of ratios.

Ex. 1. *What number must be added to each term of the ratio* 7 : 10 *to make it equal to the ratio* 2 : 3?

Let x denote the required number. Therefore, by the question,

$$\frac{7+x}{10+x}=\frac{2}{3}.$$

Multiply up, $\therefore\ 3(7+x)=2(10+x).$

$$\therefore\ 21+3x=20+2x.$$

$$\therefore\ x=-1.$$

Hence the number required is -1. The interpretation of this result is that unity must be *subtracted* from each term of the given ratio to make it equal to the ratio 2 : 3.

Ex. 2. *Of the ratios* $2(x+1)^2+1 : 3x^2+6x+5$ *and* 2 : 3, *which is the greater?*

The quantity $\dfrac{2(x+1)^2+1}{3x^2+6x+5}$ is $>$ or $<\dfrac{2}{3}$

according as $3\{2(x+1)^2+1\}$ is $>$ or $<2(3x^2+6x+5)$, [Art. 295.

that is, as $6x^2+12x+9$ is $>$ or $<6x^2+12x+10$,

that is, as 9 is $>$ or <10.

But 9 is <10, $\therefore\ \dfrac{2(x+1)^2+1}{3x^2+6x+5}$ is $<\dfrac{2}{3}$.

Ex. 3. *Determine whether the ratio* $a+x : b+x$ (*formed from a given ratio* $a : b$ *by adding a positive quantity* x *to each term of it*) *is greater or less than the given ratio* $a : b$.

The quantity $\dfrac{a+x}{b+x}$ is $>$ or $<\dfrac{a}{b}$

as $b(a+x)$ is $>$ or $<a(b+x)$,

that is, as $ab+bx$ is $>$ or $<ab+ax$,

that is, as bx is $>$ or $<ax$,

that is, (since x is positive) as b is $>$ or $<a$.

Hence a ratio of greater inequality is decreased by adding any positive quantity to each term of it. For in this case, $b < a$, and therefore $a+x : b+x < a : b$; that is, the new ratio is less than the given ratio. [For example, $3+1 : 2+1 < 3 : 2$.]

Similarly, a ratio of less inequality is increased by adding any positive quantity to each term of it. For in this case, $b > a$, and therefore $a+x : b+x$ is $> a : b$; that is, the new ratio is greater than the given one. [For example, $1+1 : 2+1 > 1 : 2$.]

Ex. 4. *Determine whether the ratio $a-x : b-x$ (formed from a given ratio $a : b$ by subtracting a positive quantity x from each term of it) is greater or less than the given ratio $a : b$.*

The quantity $\dfrac{a-x}{b-x}$ is $>$ or $< \dfrac{a}{b}$

as $b(a-x)$ is $>$ or $< a(b-x)$,

that is, as $ab - bx$ is $>$ or $< ab - ax$,

that is, as $-bx$ is $>$ or $< -ax$.

Transposing, that is, as ax is $>$ or $< bx$, [Art. 180 (iv).

that is, as a is $>$ or $< b$.

Hence a ratio of greater inequality is increased by subtracting any positive quantity from each term of it. For in this case, $a > b$, and therefore $a-x : b-x > a : b$; that is, the new ratio is greater than the given one. [For example, $3-1 : 2-1 > 3 : 2$.]

Similarly, a ratio of less inequality is decreased by subtracting any positive quantity from each term of it. For in this case, $a < b$, and therefore $a-x : b-x < a : b$; that is, the new ratio is less than the given one. [For example, $3-1 : 4-1 < 3 : 4$.]

EXAMPLES ON RATIO. XXV. A.

1. If $x=2$ and $y=1$, find the ratio of $x^2-y^2 : x^2+y^2$.

2. Express in fractional indices (i) the subduplicate ratio of $x : y$, (ii) the subtriplicate ratio of $x : y$, (iii) the sesquiplicate ratio of $x : y$.

3. Find the ratio compounded of

(i) the ratio $3 : 2$ and the ratio $2 : 3$,

(ii) the subduplicate ratio of $4 : 9$ and the triplicate ratio of $2 : 1$,

(iii) the duplicate ratio of $a : b$ and the sesquiplicate ratio of $b : a$.

4. If $a : b = 4 : 7$, find the ratio of

(i) $ab : b^2 - a^2$; (ii) $\sqrt{(7ab)} : 3b - 2a$.

5. Which is the greater of the ratios $4 - x : 3 - x$ and $4 : 3$, (i) if x be positive, (ii) if x be negative?

6. If $7(x - y) = 3(x + y)$, what is the ratio of x to y?

7. If $8(x + y) = 11(x - y)$, what is the ratio of x to y?

8. If $14x = 35y$, find the duplicate ratio of x to y.

9. Find what number must be added to each term of the ratio $5 : 6$ to make it equal to the ratio $20 : 21$.

10. What number must be added to each term of the ratio $9 : 7$ to make it equal to the ratio $12 : 11$?

11. A certain ratio becomes $2 : 3$, if 2 be added to each of its terms; and becomes $1 : 2$, if 1 be subtracted from each of its terms: find the ratio.

12. If $a : b$ be a ratio of greater inequality, prove that the subduplicate ratio of $a : b$ is less than the ratio of $a : b$.

13. Prove that, if the ratio $a : b$ be compounded with a ratio of less inequality, the ratio thus formed will be less than the ratio $a : b$.

14. Shew that the ratio $a + x : a - x$ is greater or less than the ratio $a^2 + x^2 : a^2 - x^2$, according as the ratio $a : x$ is one of greater or less inequality.

15. If $P : Q$ be the subduplicate ratio of $P - x : Q - x$ (P and Q being each greater than x), prove that $x = PQ/(P + Q)$.

16. Find the quantity which, when subtracted from each term of the ratio $a^2 : b^2$, gives two quantities whose ratio is equal to the triplicate ratio of $a : b$.

17. Two numbers are in the ratio $4 : 11$; the numbers obtained by adding 10 to each of the given numbers are in the duplicate ratio of $3 : 4$. Find the numbers.

18. A is 24 years old, B is 15 years old. What is the least number of years after which the ratio of their ages will be less than $7 : 5$?

19. At present B's age is to A's in the ratio of 3 to 2, but in fifteen years time it will be in the ratio of 4 to 3. Find their ages.

20. Two numbers, each less than 50, and having the same digits, are to one another as $4 : 7$. What are the numbers?

21. Find two numbers such that their product is 91, and the difference of their squares is to the difference of their cubes as 20 to 309.

22. In a certain examination, the number of those who passed was three times the number of those who were rejected. If there had been 16 fewer candidates, and if 6 more had been rejected, the numbers of those who passed and of those who were rejected would have been as 2 : 1. Find the number of candidates.

23. The number of girls in a mixed school increased 7 per cent. during a certain year, while the number of boys diminished 4 per cent.; the total increase in the school during the year was 3 per cent. Compare the numbers of boys and girls.

***24.** Divide £900 between three persons, so that, if their shares be increased by £10, £15 and £20 respectively, the sums shall be in the ratio 4 : 5 : 6.

25. Find two numbers such that their sum, their difference, and the sum of their squares are in the ratio 5 : 3 : 51.

***26.** If n be the ratio of the roots of the equation $x^2-px+q=0$, prove that

$$(n^{\frac{1}{2}}+n^{-\frac{1}{2}})^2=\frac{p^2}{q}.$$

PROPORTION.

297. **Proportion.** Four quantities are said to be *proportional*, or *in proportion*, when the ratio of the first of them to the second is equal to the ratio of the third of them to the fourth.

Thus a, b, c, d are proportional if

$$a : b = c : d,$$

which is read as *a is to b as c is to d*.

The relation is sometimes written in the form

$$a : b :: c : d.$$

298. The quantities a, b, c, d are called the *terms* of the proportion; a is called the first term, b the second term, c the third term, and d the fourth term. The terms a and d are called the *extreme terms* or *extremes*, and the terms b and c are called the *mean terms* or *means*.

299. *The product of the extremes of a proportion is equal to the product of the means.*

For if a, b, c, d be in proportion, then, by definition, the ratio of a to b is equal to the ratio of c to d. That is,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each side by bd,

$$\therefore \quad ad = bc.$$

300. *Conversely, if four quantities a, b, c, d be so related that $ad = bc$, then a, b, c, d will be in proportion.*

We have $$ad = bc.$$

Divide each side by bd, $$\therefore \quad \frac{ad}{bd} = \frac{bc}{bd}.$$

$$\therefore \quad \frac{a}{b} = \frac{c}{d},$$

that is, $$a : b = c : d.$$

301. *Note.* Any one of the four following proportions

$$a : b = c : d, \qquad a : c = b : d, \qquad b : a = d : c, \qquad b : d = a : c$$

leads to the result $$ad = bc.$$

Conversely, from this latter result any one of the four proportions above written can be obtained. Thus, if in the last article, we had divided each side of the relation $ad = bc$ by cd, we should have been led to the proportion $a : c = b : d$.

Hence, if one of the four proportions given above be true, so also are the other three.

302. If we are given a proportion, and if we desire to deduce another proportion, we may proceed in one of two ways, as illustrated by the following examples.

Ex. 1. *Shew that, if* $a:b=c:d$, then $a+b:a-b=c+d:c-d$.

First method. We are given that $\frac{a}{b}=\frac{c}{d}$.

Let $$\frac{a}{b}=x, \quad \therefore \frac{c}{d}=x.$$

$$\therefore a=bx, \qquad c=dx.$$

Now, take successively each ratio in the result; and, wherever an a or a c appears, substitute respectively the values bx and dx. We shall then find that each ratio will reduce to the same expression, and therefore the two ratios will be equal.

Thus, $$\frac{a+b}{a-b}=\frac{bx+b}{bx-b}=\frac{b(x+1)}{b(x-1)}=\frac{x+1}{x-1}.$$

Also, $$\frac{c+d}{c-d}=\frac{dx+d}{dx-d}=\frac{d(x+1)}{d(x-1)}=\frac{x+1}{x-1}.$$

$$\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d},$$

which is the required result.

Second method. Or we might proceed thus.

The relation $$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

is true, if $$(a+b)(c-d)=(a-b)(c+d),$$
that is, if $$ac+bc-ad-bd=ac-bc+ad-bd,$$
that is, if $$2bc=2ad,$$
that is, if $$bc=ad,$$
which, by the given proportion, is true. Hence

$$a+b:a-b=c+d:c-d.$$

Ex. 2. *If* $a:b=c:d=e:f$, *then will each ratio be equal to the ratio* $a+c-e:b+d-f$.

Let $$\frac{a}{b}=x, \quad \therefore \frac{c}{d}=x, \text{ and } \frac{e}{f}=x.$$

$$\therefore a=bx, \qquad c=dx, \qquad e=fx.$$

Hence $$\frac{a+c-e}{b+d-f}=\frac{bx+dx-fx}{b+d-f}=\frac{x(b+d-f)}{b+d-f}=x,$$

which is equivalent to the required result.

Ex. 3. *Shew that, if* $a : b = c : d$, *then will*

$$a^2+c^2 : (a-c)(2a+c) = b^2+d^2 : (b-d)(2b+d).$$

Let $$\frac{a}{b}=x, \quad \therefore \frac{c}{d}=x,$$

$$\therefore a=bx, \qquad c=dx,$$

$$\therefore \frac{a^2+c^2}{(a-c)(2a+c)}=\frac{b^2x^2+d^2x^2}{(bx-dx)(2bx+dx)}=\frac{x^2(b^2+d^2)}{x^2(b-d)(2b+d)}$$
$$=\frac{b^2+d^2}{(b-d)(2b+d)},$$

$$\therefore a^2+c^2 : (a-c)(2a+c)=b^2+d^2 : (b-d)(2b+d).$$

Ex. 4. *If* $(a^2-c^2)(b^2-d^2)=(ab-cd)^2$, *then will* $a : b=c : d$.

Here the given proportion is more complicated than the result we want to prove; and we must therefore use the following method.

We have $$(a^2-c^2)(b^2-d^2)=(ab-cd)^2.$$

$$\therefore a^2b^2-a^2d^2-b^2c^2+c^2d^2=a^2b^2-2abcd+c^2d^2.$$
$$\therefore a^2d^2-2abcd+b^2c^2=0.$$
$$\therefore (ad-bc)^2=0.$$
$$\therefore ad-bc=0.$$
$$\therefore a : b=c : d.$$

303. The following is an important proposition.

If $a : b$, $c : d$, *and* $e : f$ *be unequal positive ratios, then the ratio* $a+c+e : b+d+f$ *is intermediate in magnitude between the greatest and the least of the three given ratios.*

For suppose the three given ratios to be arranged in order of magnitude so that $a : b$ is the greatest and $e : f$ the least of them: thus we have

$$\frac{a}{b} > \frac{c}{d} > \frac{e}{f}.$$

First, let $\frac{a}{b}=x, \quad \therefore \frac{c}{d}<x,$ and $\frac{e}{f}<x.$

$$\therefore a=bx, \qquad c<dx, \qquad e<fx.$$

Add, $\therefore\ a+c+e<bx+dx+fx,$

that is, $a+c+e<x(b+d+f).$

But $x=\dfrac{a}{b}$, $\therefore\ \dfrac{a+c+e}{b+d+f}<\dfrac{a}{b}.$

Next, let $\dfrac{e}{f}=y$, $\therefore\ \dfrac{c}{d}>y$, and $\dfrac{a}{b}>y$.

$\therefore\ e=fy, \quad c>dy, \quad a>by.$

Add, $\therefore\ a+c+e>by+dy+fy,$

that is, $a+c+e>y(b+d+f).$

But $y=\dfrac{e}{f}$, $\therefore\ \dfrac{a+c+e}{b+d+f}>\dfrac{e}{f}.$

Similarly, if we have any number of unequal positive ratios $a:b$, $c:d$, $e:f$, $g:h$, ..., and we form a new ratio

$$a+c+e+g+\ldots : b+d+f+h+\ldots,$$

whose first term is the sum of the antecedents of these ratios and whose second term is the sum of their consequents, then its value will be intermediate in magnitude between the greatest and the least of the given ratios.

EXAMPLES ON PROPORTION. XXV. B.

1. Shew that, if $a:b::c:d$, then
   (i) $ma+b:mc+d::pa+b:pc+d.$
   (ii) $ma+nb:ma-nb::mc+nd:mc-nd.$
   (iii) $a^2+b^2:c^2+d^2::(a+b)^2:(c+d)^2.$

2. If $a:b=c:d$, prove that $(a^2+c^2)(b^2+d^2)=(ab+cd)^2.$

3. If $a:b=c:d$, prove that $a^2+b^2+c^2+d^2$, $(a+b)^2+(c+d)^2$, $(a+c)^2+(b+d)^2$, and $(a+b+c+d)^2$ are in proportion.

4. Shew that, if $c:d::x:y$, then $cd:xy::c^2+d^2:x^2+y^2.$

5. Shew that, if $a:b::c:d$, and a be the greatest of these four quantities, then d will be the least.

6. Prove that, if $x:y::a:b$, then $x^2:a^2::x^2+y^2:a^2+b^2.$

7. If $a : b :: c : d :: e : f$, prove that each of these ratios is equal to $\frac{ma+nc+pe}{mb+nd+pf}$, and also to $\frac{ab+cd+ef}{b^2+d^2+f^2}$.

Shew that, if $a : b = c : d = e : f$, then the results, numbered 8 to 13, will be true.

8. $a+3c+2e : a-e :: b+3d+2f : b-f$.

9. $a+4c+3e : a+c :: b+4d+3f : b+d$.

10. $a^3+c^3-ace : ab^2+cd^2-adf :: (a-e)^2 : (b-f)^2$.

11. $\left(\frac{a+2c+3e}{b+2d+3f}\right)^2 = \frac{ac+ce}{bd+df}$.

12. $(a^2e^2+c^4)(bf+d^2)^2 = (b^2f^2+d^4)(ae+c^2)^2$.

13. $\frac{pa^2+qc^2}{pb^2+qd^2} = \frac{(qc+re)(la+me)}{(qd+rf)(lb+mf)}$.

14. If the ratios $a : x$, $b : y$, $c : z$ be all equal, prove that $pbc+qca+rab$, $pyz+qzx+rxy$, $pa^2+qb^2+rc^2$, and $px^2+qy^2+rz^2$ are in proportion.

15. If $x : 5 :: y : 8$, find the ratio of $x+5$ to $y+8$.

16. If $x-3y : y-2x :: 3 : 2$, find the value of the ratio $x^2-xy : x^2+2y^2$.

17. Shew that, if $x-z : y-z = x^2 : y^2$, then

$$x+z : y+z = \frac{x}{y}+2 : \frac{y}{x}+2.$$

18. The first and fourth terms of a proportion are 5 and 54, and the sum of the mean terms is 39. Find the mean terms.

19. Having given $a+c : b = c : a = a : c-b$, determine the ratios $a : b : c$.

20. Shew that, if $2a+3b$, $2a-3b$, $2c+3d$, and $2c-3d$ be in proportion, so also are a, b, c, d.

21. Shew that, if $a+b-c : c+d+a = a-c : 2d$, then

$$b : a-c = a+c-d : 2d.$$

22. If $a^2+c^2 : ab+cd :: ab+cd : b^2+d^2$, prove that

$$a : b :: c : d.$$

304. **Continued Proportion.** Quantities are said to be in *continued proportion* when the ratio of the first of them to the second, the ratio of the second to the third, the ratio of the third to the fourth, and so on, are equal.

Thus $a, b, c, d, \ldots$ are in continued proportion, if

$$a : b = b : c = c : d = \ldots,$$

that is, if
$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots.$$

If a, b, c be in continued proportion, a is called the *first term* of the proportion, b is called the *mean proportional* between a and c, and c is either called the *third proportional* to a and b or is called the *third term* of the proportion.

305. *The mean proportional between two given numbers is the square root of their product.*

For suppose that a, b, c are in continued proportion. Then

$$\frac{a}{b} = \frac{b}{c}.$$

$$\therefore \quad b^2 = ac.$$

$$\therefore \quad b = \sqrt{ac}.$$

306. The following examples illustrate the methods of treating questions concerning quantities in continued proportion.

Ex. 1. *If three quantities be in continued proportion, the ratio of the first to the third is equal to the duplicate ratio of the first to the second.*

Let a, b, c be the quantities.

We want to prove that $a : c = a^2 : b^2$.

We have
$$\frac{a}{b} = \frac{b}{c}.$$

Let
$$\frac{a}{b} = x, \quad \therefore \frac{b}{c} = x.$$

From the last of these relations, $b = cx$.

From the first of these relations, $a = bx = cx \times x = cx^2$.

Thus each term of the proportion is expressed in terms of c (the last term of the proportion) and of x. In a continued proportion, this is always possible. Now take each of the ratios which it is desired to prove equal, and wherever a or b appears substitute the values just found.

Thus $$\frac{a}{c}=\frac{cx^2}{c}=x^2.$$

Also $$\frac{a^2}{b^2}=x^2.$$

Therefore $$\frac{a}{c}=\frac{a^2}{b^2}.$$

Ex. 2. *Prove that, if a, b, c, d be in continued proportion, then* $a-d:b-c=bd(b+c+d):c^3$.

We have $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}.$$

Let $$\frac{a}{b}=x, \therefore \frac{b}{c}=x, \text{ and } \frac{c}{d}=x.$$

From the last of these relations, $c=dx$.

From the second " " $b=cx=(dx)x=dx^2$.

From the first " " $a=bx=(dx^2)x=dx^3$.

Hence $$\frac{a-d}{b-c}=\frac{dx^3-d}{dx^2-dx}=\frac{d(x^3-1)}{dx(x-1)}=\frac{x^2+x+1}{x}.$$

Also $$\frac{bd(b+c+d)}{c^3}=\frac{dx^2d(dx^2+dx+d)}{d^3x^3}=\frac{x^2+x+1}{x}.$$

$$\therefore \frac{a-d}{b-c}=\frac{bd(b+c+d)}{c^3}.$$

EXAMPLES ON CONTINUED PROPORTION. XXV. C.

1. If $a-b:b-c::b:c$, shew that a, b, c are in continued proportion.

2. If a, b, c, d be in continued proportion, shew that
$$(a-b)^3:(b-c)^3=a:d.$$

3. Find the mean proportional between $3\frac{3}{8}$ and $1\frac{1}{2}$.

4. Find the mean proportional between $x^2-\frac{1}{z^2}$ and $z^2-\frac{1}{x^2}$.

5. Shew that, if $x-y$ be a mean proportional between y and $y+z-2x$, then x will be a mean proportional between y and z.

6. Find a quantity, such that when it is subtracted from each of the quantities a, b, c, the remainders are in continued proportion.

7. Find a quantity, such that when it is added to each of the three quantities $a+b$, $b+c$, $c+a$, the sums are in continued proportion.

8. If b be the mean proportional between a and c, and if the same quantity x be added to a, b, c, determine which is the greater ratio, $a+x : b+x$ or $b+x : c+x$.

9. If the mean proportional between a and d be equal to that between b and c, shew that $a : b :: c : d$.

10. If $a : b :: c : d$, prove that the mean proportional between b and c is a mean proportional between the mean proportional between a and b and that between c and d.

11. If $a+b$, $b+c$, $c+a$ be in continued proportion, prove that $b+c$, $c+a$, $c-a$, $a-b$ are proportionals.

12. The third proportional to two numbers is 48, and the mean proportional between them is 6. Find the numbers.

*13. If $a : b = b : c = c : d = \ldots\ldots$, prove that each of these ratios is equal to

$$\frac{a^2+b^2+c^2+\ldots\ldots}{ab+bc+cd+\ldots\ldots}, \text{ and that } \frac{a}{d}=\frac{a^3+b^3+c^3+\ldots\ldots}{b^3+c^3+d^3+\ldots\ldots}.$$

MISCELLANEOUS EXAMPLES. XXV. D.

1. If $5x-4y : 3x-2y = 4 : 1$, find the ratio of x to y.

2. Find the new ratio formed by subtracting the quantity $\frac{(p-1)ab}{pb-a}$ from each term of the ratio $a : b$.

3. If $A : B$ be the duplicate ratio of $A+x : B+x$, prove that $x^2 = AB$.

4. Find what quantity must be added to each term of the duplicate ratio of $a : b$, in order that the new ratio thus formed may be equal to that of $a : b$.

5. If $x-4y : y-3x = 3 : 2$, find the value of the ratio $x^2-xy+y^2 : x^2+xy+y^2$.

6. If the ratio $a : b$ be compounded with a ratio of greater inequality, prove that the resulting ratio is greater than the ratio of $a : b$.

7. What is the least integer which must be added to the terms of the ratio 9 : 23 so as to make the new ratio so formed greater than the ratio 8 : 13?

8. Find a number which when subtracted from each term of the ratio 9 : 13 will make the new ratio so formed equal to 13 : 9.

9. A is 32 years old, B is 5 years old; what is the least number of years after which the ratio of their ages will be less than 3 : 1?

10. The terms of a ratio are seven and three; what number must be added to each term in order that the value of the ratio so formed may be half that of the original ratio?

11. If a and x be positive quantities, shew that the ratio $a^2-x^2 : a^2+x^2$ is greater or less than the ratio $a^3-x^3 : a^3+x^3$ according as the ratio $a : x$ is one of less or greater inequality.

12. The age of the eldest of three children is equal to the sum of the ages of the other two, the ages of these two being in the ratio 2 to 3: in ten years time, the age of the eldest will be five years more than half the sum of the ages of the other two. Find their present ages.

13. The ages of a man's three sons are as 1 : 2 : 3. In 12 years time, his age will be equal to the sum of the ages of the three sons; and in 14 years more, his age will be equal to the sum of the ages of the two elder. What are their present ages?

14. If $ax+by : a^3+b^3 = a^3y-b^3x : a^2b^2(a-b)$, find $x : y$.

15. If $a : b :: c : d$, shew that $a^2-b^2 : c^2-d^2 :: ab : cd$.

16. Shew that, if $a : b = c : d$, then

$$\frac{a^2}{m^2}+\frac{b^2}{n^2}+\frac{c^2}{p^2}+\frac{d^2}{q^2}=abcd\left(\frac{1}{a^2q^2}+\frac{1}{b^2p^2}+\frac{1}{c^2n^2}+\frac{1}{d^2m^2}\right).$$

If $a : b = c : d = e : f$, prove the relations numbered **17** to **21**.

17. $pa+qc+re : pb+qd+rf = e : f$.

18. $ae : c^2 = bf : d^2$.

19. $$\frac{a^2x+c^2y+e^2z}{b^2x+d^2y+f^2z}=\frac{cex+eay+acz}{dfx+fby+bdz}.$$

20. $(a+e)^2 : (b+f)^2 :: c\sqrt{a^2-c^2} : d\sqrt{b^2-d^2}$.

21. $a : b :: \sqrt{(m^2a^2+n^2c^2-p^2e^2)} : \sqrt{(m^2b^2+n^2d^2-p^2f^2)}$.

22. If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$, prove that

(i) $\dfrac{c_1a_2a_3 + c_2a_3a_1 + c_3a_1a_2}{c_1b_2b_3 + c_2b_3b_1 + c_3b_1b_2} = \dfrac{a_1^2}{b_1^2}$;

(ii) $\sqrt{a_1b_1} + \sqrt{a_2b_2} + \sqrt{a_3b_3} = \sqrt{(a_1+a_2+a_3)(b_1+b_2+b_3)}$.

23. If $x : a = y : b = z : c$, then will each of these ratios be equal to $\sqrt{\dfrac{lyz + mzx + nz^2}{lbc + mca + nc^2}}$.

24. Prove that, if $l : m = p : q = r : s$, then each of these ratios is equal to $\sqrt{\dfrac{lmpq + l^2r - 17mpqr}{m^2q^2 + m^2r - 17mq^2s}}$.

25. Shew that, if $a : b :: c : d :: e : f :: g : h :: \ldots$, then

$$a : b :: \sqrt{pa^2 + qc^2 + re^2 + sg^2 + \ldots} : \sqrt{pb^2 + qd^2 + rf^2 + sh^2 + \ldots}$$

26. Shew that, if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \ldots = \dfrac{a_n}{b_n}$, then each fraction will be equal to $\left\{\dfrac{\lambda_1a_1^m + \lambda_2a_2^m + \ldots + \lambda_na_n^m}{\lambda_1b_1^m + \lambda_2b_2^m + \ldots + \lambda_nb_n^m}\right\}^{\frac{1}{m}}$.

27. Prove that, if a, b, c, d be in continued proportion, then

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

28. Shew that, if $3x+2u$, $3x-2u$, $3y+2v$, $3y-2v$ be in proportion, so also are x, y, u, v.

29. If $y : x+y = x+z-y : y+z-x = x+y+z : 2x+y+2z$, find the ratios $x : y : z$.

30. If $a(y+z) = b(z+x) = c(x+y)$, prove that

$$x-y : c(a-b) = y-z : a(b-c) = z-x : b(c-a).$$

31. Prove that, if $y-z : a :: z-x : b :: x-y : c$, then

$$a(y-z)^2 + b(z-x)^2 + c(x-y)^2$$
$$= a(z-x)(x-y) + b(x-y)(y-z) + c(y-z)(z-x).$$

32. Shew that, if $x : y : z = a+2b+c : 2a+b-c : 4a-4b+c$, then $a : b : c = x+2y+z : 2x+y-z : 4x-4y+z$.

33. If the fourth proportional to a, b, c be equal to that to a, b', c', shew that $b : b' :: c' : c$.

***34.** If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{a^2-ac}{b^2-bd}$, prove that each of these equal quantities is either zero or 1.

*35. If $ax+by+cz=0$, and $a^2x+b^2y+c^2z=0$; prove that

$$x : y : z :: \frac{b-c}{a} : \frac{c-a}{b} : \frac{a-b}{c}.$$

*36. If a, b, c, d, e, f be in continued proportion, prove that

$$\frac{a^2}{bc}+\frac{c^3}{def}=\frac{a^2}{d^2}+\frac{c}{f}.$$

37. If a, b, c be in continued proportion, then will

$$\frac{\sqrt{(a^2-b^2)}}{a} : \frac{a}{\sqrt{(a^2+b^2)}} :: \frac{\sqrt{(a-c)}}{\sqrt{a}} : \frac{\sqrt{a}}{\sqrt{(a+c)}}.$$

*38. Give the algebraical and geometrical definitions of proportion: and deduce the latter from the former.

39. What number must be added to each of the numbers 3, 4, 13, 16 that the sums may form a proportion?

*40. Find two numbers such that their difference, the difference of their squares, and the difference of their cubes, are in the ratio of 1 : 9 : 61.

41. The third proportional to two numbers is 162, and the mean proportional between them is 6. Find the numbers.

42. Two numbers, each less than 100 and having the same digits, are to one another as 5 : 6; what are the numbers?

43. Divide £1230 among three persons, so that if their shares be diminished by £5, £10, and £15 respectively, the remainders shall be in the ratio 3 : 4 : 5.

44. £3000 is to be divided among A, B, and C: if each received £1000 more than he actually does, the sums received would be proportional to the numbers 4, 3, 2. Find what each receives.

45. A vessel is half full of a mixture of wine and water. If filled up with water the quantity of water bears to that of wine a ratio ten times what it would be were the vessel filled up with wine. Determine the original quantities of wine and water.

46. Two cisterns, connected by a pipe, contain 25 and $24\frac{1}{2}$ gallons of water respectively. Find how much water must be allowed to flow out of one cistern into the other, so that the quantity of water in the first cistern may be to that in the second in the ratio of 5 to 6.

47. A rectangular court has a grass-plot in its centre, surrounded by a gravel walk of uniform width. If the area of the grass-plot be half the area of the court, and the length of the grass-plot be equal to the width of the court, find the ratio of the width of the grass-plot to that of the gravel walk.

CHAPTER XXVI.

VARIATION.

307. **Variation.** When two quantities are so related that the ratio of their numerical measures is always constant, each quantity is said to *vary directly* as the other.

We shall see later [Arts. 312, 314] that there are other kinds of variation besides direct variation; but if one quantity is said to vary as another, it is understood that the former varies directly as the latter.

Thus x *varies directly* as y (or *varies* as y), if $\frac{x}{y}$ be constant.

For example, the distance traversed by a man walking at a uniform pace varies directly as the time during which he walks. If he walk at a pace of three miles an hour for one hour, he will walk three miles; if he walk for two hours, he will walk six miles; and so on: the ratio of the numerical measures of the distance walked and the time occupied being always the same.

308. **Notation.** The symbol $\propto$ is used as an abbreviation for the words *varies directly as.*

Thus $x \propto y$, is read as x *varies directly as* y, or sometimes (for brevity) as x *varies as* y.

309. If $x \propto y$, we have $\frac{x}{y} = m$, where m is some constant quantity, that is, a quantity which contains neither x nor y. Therefore $x = my$.

The beginner, when he is dealing with questions involving variation, will generally find it convenient to introduce a symbol, like m, to denote this constant ratio of the variable quantities. If the particular value of x corresponding to any particular value of y be known, then the value of m can be at once determined.

Ex. 1. *If $x \propto y$, and if $y=3$ when $x=2$, find the relation between x and y.*

We have $x=my$, where m is a constant.

But if $x=2$, $y=3$, $\therefore\ 2=3m$.

$$\therefore\ m=\tfrac{2}{3}.$$

$$\therefore\ x=\tfrac{2}{3}y.$$

Ex. 2. *If $(x+1)^2 \propto y^3$, and if $y=2$ when $x=2$, find the relation between x and y.*

We have $(x+1)^2=my^3$, where m is a constant.

If $x=2$, $y=2$, $\therefore\ (2+1)^2=m2^3$.

$$\therefore\ 9=8m.$$

$$\therefore\ m=\tfrac{9}{8}.$$

$$\therefore\ (x+1)^2=\tfrac{9}{8}y^3.$$

310. *When one quantity varies directly as another, any increase or decrease in the one causes a proportional increase or decrease in the other.*

For let $x \propto y$; and suppose that when x is increased to $x+x'$, then y is increased to $y+y'$.

$$\therefore\ \frac{x}{y}=m, \text{ and also } \frac{x+x'}{y+y'}=m,$$

where m is the constant value of the ratio of the numerical measures of the quantities.

$$\therefore\ x=my, \text{ and } x+x'=m(y+y').$$

Subtract, $\therefore\ x'=my'$.

$$\therefore\ \frac{x'}{y'}=m=\frac{x}{y}.$$

That is, the ratio of x' to y' is the same as that of x to y. Or, in other words, the increase of x bears the same ratio to the corresponding increase of y as x does to y.

EXAMPLES ON DIRECT VARIATION. XXVI. A.

1. If x vary as y, and if $x=7$ when $y=4$, find the value of x when $y=7$.

2. If x vary as the square of y, and if $y=2$ when $x=9$, find the equation between x and y.

3. If $a \propto b$, and if $a=2$ when $b=3$, find the value of ab when $a-b=1$.

4. If $y^2 \propto x^2-1$, and if $y=1$, $x=\sqrt{5}$ be simultaneous values of y and x, find the value of x when $y=2$.

5. Shew that, if a^2-b^2 vary directly as c^2, and if $c=2$ when $a=5$ and $b=3$, then b is a mean proportional between $a-2c$ and $a+2c$.

6. If $x \propto y+z$, and $z \propto x$, and if $x=2$ when $y=4$, find the value of y when $x=1$.

7. If z vary as $(x+a)(y+b)$, and be equal to $(a+b)^2$ when $x=b$ and $y=a$; shew that when $x=a+2b$ and $y=2a+b$, then $z=4(a+b)^2$.

8. If $x \propto a+b$, $a \propto y$, and $b \propto \frac{1}{y}$, and if $x=18$ when $y=1$, and $x=25\frac{1}{2}$ when $y=2$, find x when $y=7$.

9. If $a \propto b$, and $a \propto c$, shew that $bc \propto a^2$.

311. **Inverse Variation.** One quantity is said to *vary inversely* as another quantity when the first varies directly as the reciprocal of the second.

Thus x varies inversely as y if $x \propto \frac{1}{y}$, that is, if $x = m\frac{1}{y}$, where m is some constant quantity.

Also, since if $x = m\frac{1}{y}$, then $y = m\frac{1}{x}$, it follows that if x vary inversely as y, then y varies inversely as x.

Again, if $x = m\frac{1}{y}$, then $xy = m$, hence the product of two quantities, one of which varies inversely as the other, is always the same.

312. As an example of inverse variation, consider the case of a man walking at a uniform rate. Then the time required to walk a given distance varies inversely as the rate at which he walks. In fact, if s be the distance traversed, t be the time occupied in traversing it, and v be the velocity with which it is traversed, we have $s=vt$. Therefore $t=\frac{s}{v}$. Hence, if s be constant, t varies inversely as v.

Again, suppose that the length of the base of a triangle is a feet, and that its altitude is b feet. Then we know that the area of the triangle is $\frac{1}{2}ab$ square feet. Hence, if we have a number of triangles whose bases and altitudes are different but such that the total area of each triangle is the same, then we have $ab=\text{constant}=m$ (say),

$\therefore\ a=m\frac{1}{b}$, that is, a varies inversely as b. Similarly, $b=m\frac{1}{a}$, and therefore b varies inversely as a.

EXAMPLES ON INVERSE VARIATION. XXVI. B.

1. If x vary inversely as y, and if $y=3$ when $x=5$, find the value of x when $y=5$.

2. If x vary inversely as y, and if $y=3$ when $x=10$, find the value of x when $y=8$.

3. If a^3 vary inversely as b^2, and if $b=3$ when $a=2$, find the relation between a and b.

4. A man, walking at a uniform pace, walks from one town to another. He can cover the distance in 4 hours, when he walks at the rate of 3 miles an hour. How long will he take, when he walks at the rate of $3\frac{1}{2}$ miles an hour?

5. If x vary as $p+q$, p vary as y, and q vary inversely as y, and if $x=18$ when $y=1$, and $x=19\frac{1}{2}$ when $y=2$, find x when $y=11$.

6. Shew that, if x vary inversely as $yz/(y-z)$, and be equal to 5 when $y=7$ and $z=2$, then $xyz=14(y-z)$.

7. Prove that, if $\frac{1}{x}+\frac{1}{y}$ vary inversely as $x+y$, then x^2+y^2 varies as xy.

313. **Joint Variation.** One quantity *varies jointly* as two (or more) quantities when it varies as their product.

Thus x varies jointly as y and z when $x \propto yz$. Hence $x = myz$, where m is some constant quantity, that is, a quantity independent of x, y, and z.

314. A quantity x is said to vary directly as y and inversely as z when it varies jointly as y and the reciprocal of z.

In this case, $x \propto y\frac{1}{z}$, that is, $x = m\frac{y}{z}$, where m is some constant quantity.

315. The following is an important proposition.

If x vary as y when z is kept unchanged, and x vary as z when y is kept unchanged, then x will vary jointly as y and z when both y and z are changed.

Let u denote the value of the ratio $x : yz$. Then the proposition will be true, if we prove that u is a constant quantity, that is, one which involves neither x nor y nor z.

Since $$\frac{x}{yz} = u, \quad \therefore \quad x = uyz.$$

Now, by hypothesis, if z be kept unchanged, $x \propto y$. Therefore uz cannot involve x or y. [Art. 309.] Thus, u cannot involve x or y.

Similarly, u cannot involve x or z. Hence u is constant, and therefore the proposition is true.

316. For example, consider the case of a man walking for t hours at the uniform rate of v miles an hour. If v be constant, the space traversed varies as t. If t be constant, the space traversed varies as v. Hence, if both v and t vary, then the space traversed varies as the product of v and t. In fact, we know that $s = vt$, and therefore in this case the constant value of the ratio of the measures of this space and this product is unity.

Again, the area of a triangle varies as the length of its base if its altitude be kept constant (Euc. VI. 1); and a similar proof shews that it varies as its altitude if its base be kept constant. Hence, if both its altitude and its base be changed, the measure of its area varies as the product of the measures of its base and altitude. In fact, if the lengths of the base and altitude of a triangle be respectively a feet and b feet, we know that the area contains $\frac{1}{2}ab$ square feet. In this case, the constant value of the ratio of the measures of this area and this product is $\frac{1}{2}$.

*317. Similarly, physicists have shewn by experiment that the volume of a quantity of gas varies inversely as the pressure to which it is subjected if the temperature be kept constant (Boyle and Mariotte's Law); and that the volume varies as the absolute temperature if the pressure be kept constant (Dalton and Gay-Lussac's Law). Hence, if both the pressure and temperature vary, the volume of the gas varies directly as the pressure and inversely as the absolute temperature.

Again, it has been shewn by experiment that the space described by a body falling from rest varies as the square of the time occupied if the accelerating effect of gravity be constant (which at any given place is the case); and varies as the accelerating effect of gravity (at different places) if the time of fall be constant. Hence, if both the time and the accelerating effect of gravity vary, the space varies as the product of the measures of the accelerating effect of gravity and of the square of the time.

EXAMPLES ON JOINT VARIATION. XXVI. C.

1. If x vary directly as y^2 and inversely as z, and if when $y=2$ and $z=3$ then $x=1$, find the value of x when $y=3$ and $z=2$.

2. If x vary inversely as y^2 and directly as z, and if when $x=4$ and $y=2$ then $z=4$, find the value of x when $z=2$ and $y=\frac{1}{4}$.

3. If A vary directly as the square root of B and inversely as the cube of C, and if $A=3$ when $B=256$ and $C=2$, find B when $A=24$ and $C=\frac{1}{2}$.

4. If x^2 vary jointly as $y-z$ and $y+z$, and if $x=2$ when $y=5$ and $z=3$, shew that z is a mean proportional between $y-2x$ and $y+2x$.

5. Shew that the number of years in which the simple interest on a sum of money will accumulate to a certain fixed amount varies inversely as the sum of money put out to interest and also inversely as the rate per cent. of the interest.

318. A few miscellaneous examples on questions involving variation are here appended.

Ex. 1. *If x vary directly as y and also directly as z, shew that y will vary directly as z.*

Since $x \propto y$, $\therefore\ x = my$, where m is a constant.

Also, since $x \propto z$, $\therefore\ x = nz$, where n is a constant.

$$\therefore\ my = nz.$$

$$\therefore\ y = \frac{n}{m} z.$$

But $\frac{n}{m}$ is a constant. Therefore $y \propto z$.

Ex. 2. *The volume of a right circular cone varies jointly as its height and the square of the radius of its base. If the volume of a cone 7 ft. high with a base whose radius is 3 ft. be 66 cubic feet, find that of a cone half as high, standing on a base whose radius is twice as large as the other one.*

Suppose that the height of a cone is h feet, and that the radius of its base is r feet, and suppose that it contains v cubic feet.

Then, by the question, $v \propto hr^2$,

$\therefore\ v = mhr^2$, where m is a constant.

If $h=7$ and $r=3$, $v=66$, $\therefore\ 66 = m \times 7 \times 3^2$,

$$\therefore\ m = \tfrac{66}{63} = \tfrac{22}{21}.$$

$$\therefore\ v = \tfrac{22}{21} hr^2.$$

The height of the required cone is $3\frac{1}{2}$ ft., and the radius of its base is 6 ft.,

$$\therefore \text{ its volume} = \tfrac{22}{21} \times \tfrac{7}{2} \times 6^2 \text{ cubic feet}$$
$$= 132 \text{ cubic feet.}$$

Ex. 3. *If the quantity of water, which flows through a pipe in a given time, vary as the square of the diameter of the pipe, and if two vessels whose contents are in the ratio of* 8 *to* 3 *be filled by two pipes in* 6 *and* 4 *minutes respectively, compare the diameters of the pipes.*

Suppose that the diameters of the two pipes are x and y (that is, x and y units of length).

The quantity of water which flows through the first pipe in 1 minute $\propto x^2$, and may therefore be taken equal to mx^2, where m is a constant. Hence the quantity of water which flows through the second pipe in 1 minute is equal to my^2.

The quantities of water flowing through the pipes in 6 and 4 minutes respectively are therefore $6mx^2$ and $4my^2$; these (by the question) are in the ratio of 8 to 3,

$$\therefore\ 6mx^2 : 4my^2 = 8 : 3.$$

$$\therefore\ 18mx^2 = 32my^2.$$

$$\therefore\ \frac{x^2}{y^2} = \frac{32}{18} = \frac{16}{9}.$$

$$\therefore\ \frac{x}{y} = \frac{4}{3}.$$

Hence the diameters are in the ratio of 4 to 3.

Ex. 4. *The volume of a sphere varies as the cube of its radius. If three spheres of radii* 6, 8, 10 *inches be melted and formed into a single sphere, find its radius.*

If the radius of a sphere be r inches, its volume varies as r^3, and may therefore be taken equal to mr^3 cubic inches, where m is a constant,

∴ the vol. of a sphere of radius 6 in. $= m \times 6^3 = 216m$ cub. in.,
" " " " 8 in. $= m \times 8^3 = 512m$ cub. in.,
" " " " 10 in. $= m \times 10^3 = 1000m$ cub. in.

∴ the vol. of the single sphere $= (216m + 512m + 1000m)$ cub. in.
$= 1728m$ cub. in.

Let the radius of this sphere be x inches, therefore its volume is mx^3 cubic inches.

Hence, by the question, $mx^3 = 1728m$.

$$\therefore\ x = \sqrt[3]{1728} = 12.$$

Hence the required radius is 12 inches.

Ex. 5. *The price of a passenger's ticket on a certain French railway is proportional to the distance he travels; he is allowed* 25 *kilogrammes of luggage free, but on every kilogramme beyond this amount he is charged a sum proportional to the distance he travels. If a journey of* 200 *miles with* 50 *kilogrammes of luggage cost* 25 *francs, and a journey of* 150 *miles with* 35 *kilogrammes cost* $16\frac{1}{2}$ *francs, what will a journey of* 100 *miles with* 100 *kilogrammes of luggage cost?*

Suppose that the passenger is charged x francs for each mile that he travels, and is charged y francs per mile for each kilogramme of luggage over the 25 kilogrammes which are carried free.

Then, if s miles be the distance travelled and w kilogrammes the total weight of luggage, we are told that the price of the ticket is sx francs, and that (if $w > 25$) the sum charged for the luggage is $sy(w-25)$ francs.

Hence the total sum charged is

$$\{sx + sy(w-25)\} \text{ francs.}$$

If $s=200$ and $w=50$, the charge is 25 francs,

$$\therefore \; 25 = 200\{x + 25y\} \quad \text{.....................(i).}$$

If $s=150$ and $w=35$, the charge is $16\frac{1}{2}$ francs,

$$\therefore \; 16\tfrac{1}{2} = 150\{x + 10y\} \quad \text{.....................(ii).}$$

Solving (i) and (ii), we obtain $x = \frac{1}{10}$, $y = \frac{1}{1000}$.

Therefore, if $s=100$ and $w=100$,

$$\text{the charge} = \{100x + 100y(100-25)\} \text{ francs}$$

$$= 17\tfrac{1}{2} \text{ francs.}$$

EXAMPLES ON VARIATION. XXVI. D.

1. If z vary as x when y is constant, and vary as y^2 when x is constant, how does z vary when neither x nor y is constant?

2. If x vary inversely as yz, and y vary directly as z^2, shew that z varies inversely as $\sqrt[3]{x}$.

3. If x vary directly as y, and y vary inversely as z^2, shew that z varies inversely as $\sqrt{x}$.

4. If $y^2 \propto x^2 - 1$, and if $y = \frac{1}{2}$, $x = \sqrt{2}$ be simultaneous values of y and x, find the value of x when $y=2$.

5. Prove that, if $x \propto y$, and $x \propto z$, then will $x \propto \sqrt{(yz)}$.

6. Given that x varies directly as $\left(y + \frac{1}{y}\right)$, and that $x = 202$ when $y=10$, find x when $y=5$.

7. If $\frac{1}{x} - \frac{1}{y}$ vary inversely as $x-y$, prove that x^2+y^2 varies as xy.

8. If x vary directly as the square of y and inversely as the cube root of z, and if $x=2$ when $y=4$ and $z=8$, find y when $x=3$ and $z=27$.

9. Prove that, if x vary as the sum of the reciprocals of y and z, and be equal to 3 when $y=1$ and $z=2$, then $xyz = 2(y+z)$.

10. If x vary as the sum of the squares of two quantities, y and z, whose product is constant, find the value of x when $y=2$, it being given that $x=3$, when $y=3$ and $z=3$.

11. If x vary as the sum of the cubes of two quantities, y and z, whose sum is constant, find the value of x when $y=2$, it being given that $x=3$, when $y=3$ and $z=3$.

12. If A vary directly as P, inversely as Q, and directly as R; and if when $P=bc$, $Q=ca$, $R=ab$, $A=abc$; find A when $P=a^2$, $Q=b^2$, $R=c^2$.

13. If z vary as $(x+a)(y+b)$, and be equal to $ab(a+b)$ when $x=0$ and $y=0$, prove that $z=(a+b)^3$ when $x=b$ and $y=a$.

14. If $ax+by+1=0$, where a and b are constants and x and y are variable, and if the values of x be 2 and -9 when the values of y are 1 and -4 respectively, what is the value of x when y is zero?

15. Given that w varies as $x+y$, and that y varies as z^2, and that $z=2$, when the values of w, x, and y are 26, 1, and 12 respectively; express w in terms of x and z.

16. Shew that, if x vary directly as $y-z$ and inversely as yz, and be equal to 5 when $y=7$ and $z=2$, then $xyz=14(y-z)$.

17. If a mixture of gold and silver, of which three-quarters of the weight is gold, be worth £49, what will be the value of a mixture of equal weight, of which a half is gold, the value of gold being 16 times that of silver?

18. If a mixture of gold and silver, of whose weight seven-eighths is gold, be worth £15, what will be the value of a mixture of equal weight, of whose weight five-eighths is gold, the value of gold being 16 times that of silver?

19. It being given that the arc of a circle varies as the length of the radius and also as the angle the arc subtends at the centre, find the length of an arc of a circle of radius 20 feet, subtending a certain angle at the centre, when the length of an arc of a circle of radius 4 feet, subtending three times the former angle at the centre, is 9 feet.

20. Given that the area of a sector of a circle varies as the product of the radius and the arc on which the sector stands, find the area of the sector of a circle, of given radius, standing on an arc 15 feet long, when the area of the sector of a circle of 5 times the radius, standing on an arc 12 feet long, is 100 square feet.

21. The volume of a right circular cone varies jointly as its height and the square of the radius of its base. If the volume of a cone 14 ft. high with a base whose radius is 3 ft. be 132 cubic feet, find that of a cone twice as high, standing on a base whose radius is half as large as the other one.

22. The volume of a cylinder varies jointly as its height and the square of the radius of its base. Shew that, if the heights of three cylinders of equal volume be in continued proportion, so also are the radii of their bases.

23. The volume of a sphere varies as the cube of its radius. If three spheres of radii 9, 12, 15 inches be melted and formed into a single sphere, find its radius.

24. If the quantity of water which flows through a pipe in a given time vary as the square of the diameter of the pipe, and if two vessels whose contents are as 16 : 7 be filled by two pipes in 7 and 4 minutes respectively, compare the diameters of the pipes.

25. The value of a diamond varies as the square of its weight, and the value of a ruby varies as the cube of its weight. If the values of a ruby and a diamond, each weighing a carats, be equal, compare the values of a ruby and a diamond weighing respectively b and c carats.

26. It is found that the quantity of work done by a man in an hour varies directly as his pay per hour and inversely as the square root of the number of hours he works per day. He can finish a piece of work in six days, when working 9 hours a day at 1*s.* per hour. How many days will he take to finish the same piece of work, when working 16 hours a day at 1*s.* 6*d.* per hour?

27. The price of a passenger's ticket on a certain French railway varies as the distance he travels; he is allowed 30 kilogrammes of luggage free, but on every kilogramme beyond that amount he is charged a sum proportional to the distance he travels. If a journey of 200 miles with 55 kilogrammes of luggage cost 25 francs, and a journey of 150 miles with 40 kilogrammes cost $16\frac{1}{2}$ francs, what will a journey of 200 miles with 105 kilogrammes of luggage cost?

28. The amount of fuel consumed in a slow-combustion stove varies as the square of the diameter of the stove when the time for which it is kept burning is constant, and varies as the time for which it burns when the diameter of the stove is constant. A stove 10 inches in diameter can be used for 21 days at a cost of 3*s.* 6*d.*, what will it cost to use a stove 12 inches in diameter for 50 days?

29. The value of a silver coin varies directly as the square of the diameter, if the thickness remain the same, it also varies directly as the thickness, if the diameter remain the same. Two silver coins have their diameters in the ratio of 5 : 4, find the ratio of their thicknesses if the value of the first coin be twice that of the second.

30. The pressure of the wind on a plane area varies jointly as the area and the square of the velocity of the wind. The pressure on a square foot is 1 lb. when the wind is blowing at the rate of 16 miles an hour. Find the velocity of the wind when the pressure on two square yards is 50 lbs.

31. The amount of the collection made after a public meeting, held in aid of a certain cause, was found to vary directly as the number of persons present, and inversely as the length of the speeches made. If £40 were collected after a meeting at which 450 persons were present and the speeches lasted for $2\frac{1}{4}$ hours, find how much would be collected at a meeting at which 600 persons were present and the speeches lasted for 4 hours.

***32.** At a certain Regatta, the number of races on each day varied jointly as the number of days from the beginning and end of the Regatta up to and including the day in question. On three successive days, there were respectively 6, 5, and 3 races. Which days were these, and how many days did the Regatta last?

***33.** The expense of running a goods train on a railway varies as the square of the number of trucks in it. The price charged for each truck is the same. If the receipts from a train of 40 trucks just pay the expenses of working it, shew that a railway company will make the same profit on each train by running trains made up of 17 trucks as they would by trains made up of 23 trucks.

Find also how many trucks should be put in each train in order that the profit may be as large as possible.

***34.** If the number of oxen a eat up the meadow b in the time c; and the number of oxen d eat up as good a piece of pasture e in the time f, and the grass grow uniformly; find how many oxen will eat up the like pasture g in the time h. (Sir Isaac Newton. *Universal Arithmetick*, 1707.)

CHAPTER XXVII.

ARITHMETICAL PROGRESSIONS.

319. **Arithmetical Progression.** A series of numbers is said to be in *arithmetical progression* (or to form an arithmetical progression, or to be in *arithmetic progression*) when the difference between any number in the series and the one immediately preceding it is always the same. The letters **A.P.** are often used as an abbreviation for the words Arithmetical Progression.

Each number is called a *term* of the series.

The constant difference, obtained by subtracting from any term the term immediately before it, is called the *common difference* of the progression.

320. **Condition for an A.P.** The condition that the series of numbers denoted by the letters $a, b, c, d \ldots$ shall be in A.P. is that

$$b-a=c-b=d-c=\ldots.$$

321. Each of the following series is an example of an arithmetical progression,

1, 2, 3, 4, 5, ...(common difference = 1),
2, 5, 8, 11, 14, ...(common difference = 3),
4, 2, 0, −2, −4, ...(common difference = −2),
$a, a+d, a+2d, a+3d, a+4d$...(common difference $=d$).

But the series 2, 4, 8, 16,... is not an arithmetical progression, since the difference between the third and second terms $(8-4=4)$

is not the same as the difference between the second and first terms $(4-2=2)$.

Similarly, the series 2, 5, 8, 12,... is not an arithmetical progression, for though the difference between the second and first terms is the same as that between the third and second terms, yet it is not the same as that between the fourth and third terms.

Where all the terms of a series, which are given, form part of an arithmetical progression, it is usual to assume that the whole of the series is in arithmetical progression.

322. Expression for the n^{th} term of an A.P. *The n^{th} term of an arithmetical progression, whose first term is a and common difference is d, is $a+(n-1)d$.*

The second term is $a+d$.

The third term is obtained from the second term by adding d to it, hence it is $a+2d$.

Similarly, each successive term is obtained by adding d to the term before it. Thus, the fourth term is $a+3d$, the fifth term is $a+4d$, and so on; the coefficient of d in any term being 1 less than the number of the term.

Hence the n^{th} term consists of the sum of a and $(n-1)d$; and, if we denote the n^{th} term by l, we have

$$l=a+(n-1)d.$$

Ex. 1. *Find the* 21^{st} *term of the series* 1, 3, 5,...

This is an arithmetical progression. The first term (denoted above by a) is 1, the common difference $d=3-1=2$.

$$\begin{aligned}\text{Hence the } 21^{st} \text{ term} &= a+(n-1)d\\ &= 1+(21-1)\times 2\\ &= 41.\end{aligned}$$

Ex. 2. *Is* 2001 *a term of the series* 2, 7, 12,...?

Suppose it to be the n^{th} term. Then, using the formula $l=a+(n-1)d$, we have $l=2001$, $a=2$, $d=7-2=5$,

$$\therefore\ 2001=2+(n-1)5,$$

that is, $$5n=2004.$$

This would require n to be a fraction, which is impossible. Hence 2001 is not a term of the given series.

[Since $n = 400\frac{4}{5}$, the given number will lie between the 400th and 401st terms. The 400th term of the series will be found to be 1997, and the 401st term to be 2002.]

Ex. 3. *The* 9th *term of an arithmetical progression is zero, and the* 21st *term is* -36. *Find the series.*

We shall know the series if we know the first term and the common difference, since in that case we can write down as many terms of it as we like.

Let a be the first term of the series and d the common difference. Then, using the formula

$$l = a + (n-1)d,$$

we have $l=0$ if $n=9$, and $l=-36$ if $n=21$,

$$\left.\begin{aligned} 0 &= a + 8d \\ -36 &= a + 20d \end{aligned}\right\}.$$

These are two simple equations for a and d.

Subtracting, we have $-36 = 20d - 8d = 12d$.

$$\therefore\ d = -3.$$

But $a + 8d = 0$, $\therefore\ a = -8d = 24$.

Thus the series is 24, 21, 18, 15,....

EXAMPLES. XXVII. A.

1. Which of the following series are in A.P.?

   (i) 1, -2, -4, -7,...;

   (ii) -4, 0, 4, 8,...;

   (iii) 11, -10, -31, -51,....

2. Write down the first five terms of an arithmetical series, whose first term is 6 and whose tenth term is 20.

3. Write down the 17th terms of the following series, each of which is in A.P.

   (i) 4, 7, 10, 13,...;

   (ii) 3, -1, -5, -9,...;

   (iii) $(a+b)^2$, a^2+b^2, $(a-b)^2$,....

4. Is 1000 a term of the series (ii), given in question 1?

5. Find the n^{th} term of an A.P., having given that $a+b$ is the first term and $a-b$ the common difference of the series.

6. Find the arithmetical series whose 10th term is -100 and whose 48th term is 128.

7. Find the 15^{th} term of an arithmetical progression whose 8^{th} and 12^{th} terms are respectively 17 and 25.

8. Find the 18^{th} term of an arithmetical progression whose 6^{th} and 13^{th} terms are respectively 22 and 43.

9. The 5^{th} term of an arithmetical progression is 64, and the 10^{th} term is 729: find the 3^{rd} and the 7^{th} terms.

10. Every term of a series of numbers in A. P. is multiplied by the same quantity. Is the new series so formed in A. P.?

323. **Sum of n terms of an A.P.** *The sum of n terms of an arithmetical progression, whose first term is a and common difference is d, is* $\frac{1}{2}n\{2a+(n-1)d\}$.

Let s be the required sum, and let l be the n^{th} term. Writing down the terms of the series, we have

$$s = a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l.$$

If we write the series in the reverse order, we have

$$s = l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

Adding these series together, we get n terms as follows:

$$2s=(a+l)+(a+l)+(a+l)+\ldots+(a+l)+(a+l)+(a+l),$$

that is, $\quad 2s = n(a+l)$.

$$\therefore \quad s=\frac{n}{2}(a+l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But $l = a+(n-1)d$,

$$\therefore \quad s=\frac{n}{2}\{a+a+(n-1)d\}$$

$$=\frac{n}{2}\{2a+(n-1)d\}\ldots\ldots\ldots\ldots\text{(ii)}.$$

324. *Note.* If the first and last terms of the series had been given, the formula (i) would have given the sum.

If any three of the four quantities a, l, n, s be given, the equation (i) will serve to determine the fourth quantity.

Similarly, if any three of the quantities a, d, n, s be given, the equation (ii) will serve to determine the fourth quantity.

Ex. 1. *Find the sum of* 21 *terms of the series* 17, 11, 5, -1,....

This is an arithmetical progression, whose first term is 17 and whose common difference $=11-17=-6$.

Hence, in the above formula, $a=17$, $d=-6$, $n=21$,

$$\begin{aligned}\therefore\ s&=\frac{n}{2}\{2a+(n-1)d\}\\ &=\tfrac{21}{2}\{34+(20)(-6)\}\\ &=21\{17-60\}\\ &=-903.\end{aligned}$$

Ex. 2. *Find the sum of n terms of the series* $1+3+5+7+\ldots.$

This is an arithmetical progression, whose first term is 1 and whose common difference $=3-1=2$.

Hence, in the above formula, $a=1$, $d=2$,

$$\begin{aligned}\therefore\ s&=\frac{n}{2}\{2a+(n-1)d\}\\ &=\frac{n}{2}\{2+(n-1)2\}\\ &=\frac{n}{2}\{2n\}\\ &=n^2.\end{aligned}$$

Ex. 3. *The* 4[th] *term of an arithmetical progression is* 8, *and the* 11[th] *term is* 22. *Find the sum of* 9 *terms.*

Let a be the first term, and d the common difference.

Then, using the formula $l=a+(n-1)d$, we have,

$$\left.\begin{array}{ll}\text{if } n=4,\ l=8, & \therefore\ 8=a+3d\\ \text{and if } n=11,\ l=22, & \therefore\ 22=a+10d\end{array}\right\}.$$

Solving these two equations, we find $a=2$, $d=2$.

We want the sum of 9 terms of this series, $\therefore\ n=9$.

$$\begin{aligned}\therefore\ s&=\frac{n}{2}\{2a+(n-1)d\}\\ &=\tfrac{9}{2}\{4+8\,.\,2\}\\ &=90.\end{aligned}$$

Ex. 4. *How many terms of the series* $1\frac{1}{2}+3+4\frac{1}{2}+6+\ldots$ *must be taken in order that their sum may be* 99?

We use the equation $s=\frac{n}{2}\{2a+(n-1)d\}$.

Here $s=99$, $a=1\frac{1}{2}$, $d=1\frac{1}{2}$, and n is required,

$$\therefore\ 99=\frac{n}{2}\{3+(n-1)\times\tfrac{3}{2}\}$$

$$=\frac{n}{2}\,\frac{3n+3}{2}.$$

$$\therefore\ n^2+n-132=0.$$

$$\therefore\ (n+12)(n-11)=0.$$

$$\therefore\ n=-12,\ \text{or}\ n=11.$$

To make the solution intelligible, n must be a positive integer. Hence the root -12 is inapplicable to this problem, and the answer is 11.

[*Note.* The interpretation of the answer $n=-12$ is that if, beginning with the first term, we count 12 terms *backwards*, then the sum of those terms will be -99.]

EXAMPLES. XXVII. B.

Sum the series of numbers (each series being in arithmetical progression) given in examples 1 to 17.

1. 8, 12, 16, 20,... to 20 terms.
2. 9, 26, 43,... to 90 terms.
3. 8, 27, 46,... to 100 terms.
4. $32+32\frac{1}{3}+32\frac{2}{3}+\ldots$ to 19 terms.
5. $3+2\frac{1}{3}+1\frac{2}{3}+\ldots$ to 10 terms.
6. $3+3\frac{1}{3}+3\frac{2}{3}+\ldots$ to 15 terms.
7. $2+2\frac{1}{4}+2\frac{1}{2}+\ldots$ to 12 terms.
8. $\frac{2}{3}+1\frac{1}{3}+2+\ldots$ to 10 terms.
9. $-\frac{1}{2}+\frac{1}{4}+1+\ldots$ to 29 terms.
10. $\frac{21}{4}$, $-\frac{9}{2}$, $-\frac{57}{4}$,... to 9 terms.
11. 13·1, 11·4, 9·7,... to 15 terms.
12. 1·6 + 2·4 + 3·2... to 6 terms.

13. $16{\cdot}6+14+11{\cdot}4\ldots$ to 10 terms.

14. $3{\cdot}5+4{\cdot}1+4{\cdot}7+5{\cdot}3+\ldots$ to 12 terms.

15. $2{\cdot}5+3{\cdot}2+3{\cdot}9+4{\cdot}6+\ldots$ to 12 terms.

16. $\frac{x+1}{a}+\frac{x+2}{a}+\frac{x+3}{a}+\ldots$ to a terms.

17. $n-\frac{1}{n},\ 3n-\frac{2}{n},\ 5n-\frac{3}{n},\ldots$ to n terms.

18. The first term of an arithmetical progression is 38, and the fourth term is 86. Find the sum of the first 12 terms.

19. The first term of an arithmetical progression is 36, and the fourth term is 90. Find the sum of the first 10 terms.

20. The 20^{th} term of an A.P. is 15, and the 30^{th} term is 20. Find the sum of the first 25 terms.

21. Find the sum of 24 terms of an arithmetical progression, whose 13^{th} term is 25 and whose 19^{th} term is 37.

22. In an arithmetical series, consisting of 15 terms, the sum of the last five terms exceeds the sum of the first five terms by 100. Find the common difference.

23. If the sum of n terms of an arithmetical progression whose first term is 11 and common difference is 3, be equal to 377, find the value of n.

24. The sum of n terms of the series 3, 7, 11,... is 820. Find n.

25. If n be so chosen that the sum of the first n odd numbers is 900; find n.

26. If the sum of n terms of an arithmetical progression be n^3, and the common difference be 2, find the last term.

27. If the common difference of an A.P. be double the first term, prove that the sum of n terms of the series varies as n^2.

325. **Arithmetic Mean.** When three quantities are in arithmetical progression, the middle one is called the *arithmetic mean* or *average value* of the other two. The letters **A.M.** are often used as an abbreviation for the words Arithmetic Mean.

326. *The arithmetic mean of two quantities is half their sum.*

For let a, x, b be three quantities in arithmetic progression.

$$\therefore\ x - a = b - x.$$
$$\therefore\ 2x = a + b.$$
$$\therefore\ x = \tfrac{1}{2}(a + b).$$

327. **Arithmetic Means.** When any number of quantities are in arithmetical progression, the terms intermediate between the two extreme ones are called the *arithmetic means* (or arithmetical means) of the two extreme terms.

328. *To insert p arithmetic means between two quantities a and b.*

We have two quantities, a and b. We want to insert p quantities between them, so that the $(p+2)$ terms so formed may be in A. P. The first of these $p+2$ terms is a; the last of them is b. Let d be the common difference of this progression. Then the $(p+2)^{\text{th}}$ term must be $a+(p+1)d$.

$$\therefore\ b=a+(p+1)d.$$
$$\therefore\ d=\frac{b-a}{p+1}.$$

Hence the required series is

$$a,\ a+\frac{b-a}{p+1},\ a+2\frac{b-a}{p+1},\ \ldots\ldots,\ a+p\frac{b-a}{p+1},\ b.$$

And the arithmetic means are

$$a+\frac{b-a}{p+1},\ a+2\frac{b-a}{p+1},\ \ldots\ldots,\ a+p\frac{b-a}{p+1},$$

that is, are $\dfrac{pa+b}{p+1},\ \dfrac{(p-1)a+2b}{p+1},\ \ldots\ldots,\ \dfrac{a+pb}{b+1}.$

Ex. 1. *Insert three arithmetic means between* 4 *and* 1.

We want to insert 3 terms between 4 and 1, so that the 5 terms so formed shall be in A. P. Let d be the common difference of the required series. Therefore 1 is to be the 5^{th} term of an A. P. whose first term is 4 and whose common difference is d.

Using the formula $l=a+(n-1)d$, we have

$$1=4+4d,$$

$$\therefore\ d=-\tfrac{3}{4}.$$

$\therefore$ the series is $\quad 4,\ 3\tfrac{1}{4},\ 2\tfrac{1}{2},\ 1\tfrac{3}{4},\ 1;$

and the required means are $3\tfrac{1}{4},\ 2\tfrac{1}{2},\ 1\tfrac{3}{4}$.

Ex. 2. *Find the sum of the p arithmetic means inserted between a and b.*

The sum of an A.P. of n terms, whose first term is a and last term is l, is $\tfrac{1}{2}n(a+l)$ [Art. 323 (i)]. Therefore the sum of the $(p+2)$ terms, whose first term is a and last term is b, is

$$\tfrac{1}{2}(p+2)(a+b).$$

Hence the sum of the arithmetic means is

$$\tfrac{1}{2}(p+2)(a+b)-a-b=\tfrac{1}{2}p(a+b).$$

EXAMPLES. XXVII. C.

1. What is the arithmetic mean between 4 and 8?

2. Insert three arithmetic means between

(i) 4 and 8; (ii) 1 and 9; (iii) 3 and -4.

3. Insert seven arithmetic means between $\tfrac{1}{7}$ and 7.

4. Insert two arithmetic means between a and b. Insert also two arithmetic means between the reciprocals of the means just found.

5. Insert four arithmetic means between x^3 and x^{-2}.

6. Insert five arithmetic means between $(a-b)^2$ and $(a+b)^2$.

7. Prove that the sum of n terms of an arithmetic progression is equal to n times the arithmetic mean between the first and n^{th} term.

8. Find the sum of the arithmetic means inserted between each two consecutive terms of an arithmetical progression of which the first term is a and the last l.

9. A man training for a mile race runs the distance every day for 24 days, his time improving at a uniform rate. On the first day he takes 8 minutes, on the last $4\tfrac{2}{3}$ minutes. What is his average time?

10. The square of the arithmetic mean of three numbers in arithmetical progression is less by 1 than the arithmetic mean of the squares of the extremes, and the cube of the mean is less by 18 than the arithmetic mean of the cubes of the extremes. Find the numbers.

329. The student will notice that all the propositions about arithmetical progressions are deduced from the three formulae,

$$l = a + (n-1)\,d,$$
$$s = \tfrac{1}{2}n\,(a + l),$$
$$s = \tfrac{1}{2}n\,\{2a + (n-1)\,d\},$$

where a is the first term of the series, d the common difference, l the n^{th} term, and s the sum of n terms.

MISCELLANEOUS EXAMPLES. XXVII. D.

[*The following miscellaneous examples are on arithmetical progressions. Other examples on arithmetical progressions will be found on pp.* 377—379.]

1. Sum the following series, each of which is in A. P.

   (i) $7 + 32 + 57 + \ldots$ to 20 terms;

   (ii) $1 + \frac{6}{5} + \frac{7}{5} + \ldots$ to 12 terms;

   (iii) $(a-3b)+(2a-5b)+(3a-7b)+\ldots$ to 40 terms;

   (iv) $(n-1)+(n-2)+(n-3)+\ldots$ to n terms.

2. How many strokes are struck in a day by a clock that tells the hours, but not the quarters or halves?

3. Prove that, if every alternate term of an arithmetical progression be struck out, the remaining terms will form an arithmetical progression.

   If the terms that remain be $\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n} \ldots$, supply the terms that have been removed.

4. Shew that, if any odd number of quantities be in A. P., the first, the middle, and the last of them will be in A. P.

5. The 4^{th} term of an A. P. is 15, and the 20^{th} term is $23\frac{1}{2}$; find the sum of the first 20 terms.

6. The sum of the first three terms of an arithmetical progression is 9, and the sum of the three subsequent terms is 27. Find the series.

7. Shew that the sum of the first 5 terms of the series $11 + 9 + 7 + \ldots$ is equal to the sum of the first 7 terms. Explain this.

8. Find the arithmetical progression whose sum to 5 terms is 45 and whose second term is 7.

9. How many terms of the series $29+27+25+23+\ldots$ must be taken so that their sum may amount to 200?

10. The sum of n terms of the A. P. 2, 5, 8,... is 950; find n.

11. The sum of n terms of the series 3, 6, 9,... is 975; find n.

12. Find the number of terms of the arithmetical progression 18, 15, 12,..., which must be taken so that their sum is 45; and explain the double answer.

13. Find the last term in the series 201, 204, 207,..., when the sum of all the terms is 8217.

14. The sum of a certain number of terms of an A. P. is 36, and the first and last of these terms are 1 and 11 respectively. Find the number of terms and the common difference of the series.

15. If the sum of the first n terms of an A. P. be $32n^2$, where n is any positive integer; find the r^{th} term.

16. If the r^{th} term of an A. P. be $32r-16$, where r is any positive integer; find the sum of the first n terms.

17. Shew that, if unity be added to the sum of any number of terms of the series 8, 16, 24,... the result will be the square of an odd number.

18. An arithmetical progression contains $2n+1$ terms. Shew that the sum of the odd terms : the sum of even terms $=n+1:n$.

19. Find the sum of $2m$ terms of an arithmetical progression, of which the two middle terms are $a-b$ and $a+b$.

20. How many terms of the series $19+17+15+13+\ldots$ must be taken to amount to 100?

21. Shew that the sum of an arithmetical progression, whose first term is a, whose last term is z, and whose common difference is b, is

$$\frac{z^2-a^2}{2b}+\frac{z+a}{2}.$$

22. Shew that, in every arithmetic progression in which the first term is equal to the common difference, the $(p-q)^{\text{th}}$ term is equal to the difference between the p^{th} and q^{th} terms.

23. If each of two arithmetical progressions be composed of $2n+1$ terms, and if their middle terms be equal, shew that their sums are also equal.

24. Find the sum of all the numbers between 200 and 300 which are divisible by 3.

25. A man stands by a heap of 100 stones. How far must he walk, carrying one stone at a time, to place the stones separately, at intervals of 20 feet apart, in a straight line having one end where the heap is?

26. The sum of n terms of an A.P., whose first two terms are 43 and 45, is equal to the sum of $2n$ terms of another A.P., whose first two terms are 45 and 43. Find the value of n.

27. The first term in a series of numbers in A.P. is n, the middle term m, and the sum of the series is $m^2 - mn + m$. Write down the last three terms of the series.

28. Shew that, if $\frac{a}{a-b}, \frac{b}{b-c}, \frac{c}{b-c}$ be in arithmetical progression, then a, b, c will be also in arithmetical progression.

29. The sum of four numbers, which are in arithmetical progression, is 16; and the product of the second and third of these numbers exceeds by 8 the product of the first and fourth. Find the numbers.

30. There are four numbers in arithmetical progression, such that the sum of twice the square of the first and the square of the last is equal to the square of the sum of the first and second, and the square of the last exceeds three times the square of the first by 1. Find them.

31. Divide 15 into 5 parts which are in arithmetical progression, and such that the sum of the squares of the least and greatest of them is less than the sum of the squares of the other three parts by 3.

32. Find three numbers in the ratio of 3 : 6 : 10, such that, if each be increased by 1, the square roots of the sums are in arithmetical progression.

33. The common difference of an A.P. is 2, and the square roots of the first, second, and fourth terms are in A.P. Find the series.

34. Write down the n^{th} term and the last term of the series $1+4+7+\ldots$ continued to $2n+1$ terms; and shew that, if S_1 and S_2 be the sums of the first n and the last n terms respectively, then $S_2 - S_1 = 6S$, where S is the sum of the first n natural numbers 1 2, 3, &c.

35. Prove that any even square, $(2n)^2$, is equal to the sum of n terms of one series of integers in arithmetical progression, and that any odd square, $(2n+1)^2$, is equal to the sum of n terms of another such series increased by 1.

36. The n^{th} terms of the two series

$$2+3\tfrac{7}{8}+5\tfrac{3}{4}+\ldots \text{ and } 187+184\tfrac{1}{4}+181\tfrac{1}{2}+\ldots$$

are the same. Find n.

37. The first terms of two arithmetical series are a and $2a$, and their common differences are respectively d and $\frac{1}{2}d$. Prove that, if $4a$ be a multiple of d, it will be possible to find a number n such that the sum of n terms of each series is the same.

38. Ten balls are placed on the ground in a line at equal distances apart. A boy, standing in a line with them and 12 feet from the nearest ball, places a basket on the ground, fetches each ball in succession, and places it in the basket. When he has returned with the last ball, he finds he has walked a quarter of a mile. How far apart were the balls placed?

39. A man starts to explore an unknown country carrying provisions for 10 days. He can walk 15 miles a day when carrying provisions for 10 days, and he can go an extra mile a day for each day's provisions he gets rid of. What distance will he have walked by the time he has just exhausted his provisions?

40. The cost of boring an Artesian well 500 feet deep is 2*s.* 8*d.* for the first foot, and a halfpenny in addition for each succeeding foot. What is the whole cost?

41. A man, who is training for a race, runs on the first day a number of miles equal to one-sixth part of the number of days he is in training, and each succeeding day runs three miles farther than he did the day before. Altogether he runs 51 miles. How many days is he in training?

42. Three squares of ground, the lengths of whose sides are in A.P., are paved with square tiles of equal size. In the two smaller squares together there are 45 tiles more than in the largest square; and, if there were 9 tiles more in the middle square, the numbers of tiles in the three squares would be in A.P. How many tiles are there in each square?

43. A man walks a distance of 21 miles in 5 hrs. 37 min., starting at the rate of $3\frac{1}{2}$ miles per hour, and increasing his rate by a certain quantity after completing one-third of the distance, and again by the same quantity after completing two-thirds of the distance. What is his final rate?

44. A and B start to walk to a place 96 miles off and back again, B always walking at the rate of two miles a day faster than A. A starts at the rate of 10 miles a day and daily increases his rate by two miles. Determine when and where he will meet B coming back.

CHAPTER XXVIII.

GEOMETRICAL PROGRESSIONS.

330. **Geometrical Progression.** A series of numbers is said to be in *geometrical progression* (or to form a geometrical progression) when the ratio of any term in the series to the one immediately preceding it is always the same. The letters **G.P.** are often used as an abbreviation for the words Geometrical Progression.

Each number is called a *term* of the series.

The constant ratio of any term to the one immediately before it is called the *common ratio* of the progression.

331. **Condition for a G.P.** The condition that the numbers represented by the letters a, b, c, d ... shall be in geometrical progression is that

$$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}=\dots.$$

Thus a, b, c ... are in continued proportion [Art. 304].

332. Each of the following series is an example of a geometrical progresssion,

1, 2, 4, 8, ... (the common ratio being 2),

1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... (the common ratio being $\frac{1}{2}$),

2, $-\frac{1}{3}$, $+\frac{1}{18}$, $-\frac{1}{108}$, ... (the common ratio being $-\frac{1}{6}$),

a, ar, ar^2, ar^3, ... (the common ratio being r).

Where all the terms of a series, which are given, form part of a geometrical progression, it is usual to assume that the whole of the series is in geometrical progression.

333. **Expression for the n^{th} term of a G.P.** *The n^{th} term of a geometrical progression, whose first term is a and common ratio is r, is ar^{n-1}.*

For the second term is ar.

The third term is obtained from the second by multiplying it by r, hence it is ar^2.

Similarly, each successive term is obtained by multiplying the term before it by r. Thus, the fourth term is ar^3, the fifth term is ar^4, and so on; the index of r in any term being 1 less than the number of the term.

Hence the n^{th} term consists of the product of a and r^{n-1}; and, if we denote the n^{th} term by l, we have

$$l = ar^{n-1}.$$

Ex. 1. *Find the* 6^{th} *term of the series* 1, 2, 4, ...

This is a geometrical progression. The first term (denoted above by a) is 1, the common ratio $r = \frac{2}{1} = 2$.

Hence the 6^{th} term $= ar^{n-1}$

$$= 1 \times 2^5$$

$$= 32.$$

Ex. 2. *The fourth term of a geometrical progression is* 4 *and the seventh term is* $-\frac{1}{2}$. *Find the series.*

We shall know the series if we know the first term and the common ratio, since in that case we can write down as many terms of it as we like.

Let a be the first term of the progression, and r the common ratio. Then using the formula $l = ar^{n-1}$, we have

$$\left.\begin{aligned} 4 &= ar^3 \\ -\tfrac{1}{2} &= ar^6 \end{aligned}\right\}.$$

Dividing, we obtain $\quad r^3 = \dfrac{-\frac{1}{2}}{4} = -\frac{1}{8}$,

$$\therefore\ r = -\tfrac{1}{2}.$$

Substitute this value of r in the equation $4 = ar^3$,

$$\therefore\ 4 = a\left(-\tfrac{1}{2}\right)^3,$$

$$\therefore\ a = -32.$$

Hence the series is

$$-32,\ 16,\ -8,\ 4,\ -2,\ 1,\ -\tfrac{1}{2}, \ldots$$

EXAMPLES. XXVIII. A.

1. Which of the following series are in G.P.?

(i) $\frac{3}{2}$, 1, $\frac{2}{3}$, $\frac{4}{9}$,...; (ii) 1, -1, 1, -1,...;

(iii) 2, 4, 8, 12,...; (iv) x^3, x^2y, xy^2, y^3,....

2. Write down the sixth terms of those series given in question 1 which are in G.P.

3. Find the 6th term of the series 3, 6, 12,...

4. What is the eighth term of the geometrical progression whose first and second terms are respectively 2 and -3?

5. What is the sixth term of a geometrical progression whose first and second terms are 3 and -4?

6. The 2nd term of a geometric progression is 21, and the 3rd term is 147. Find the 5th term.

7. The 2nd term of a geometric progression is 15, and the 3rd term is 75. Find the 1st and 6th terms.

8. Find the geometrical progression whose fifth and ninth terms are respectively 1458 and 118098.

9. The sum of the second and fourth terms of a geometrical progression is 20, and the difference of the first and fifth terms is 30. Find the series.

10. Shew that, if the terms of an A.P. be written down and any quantity be raised successively to those powers, the results will be in G.P.

334. **Sum of n terms of a G.P.** *The sum of n terms of a geometrical progression, whose first term is a and common ratio is r, is* $a\dfrac{r^n-1}{r-1}$.

Let s be the required sum. Then

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \ldots\ldots (\text{i}).$$

Multiply by r,

$$\therefore\ sr = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \ldots\ldots (\text{ii}).$$

Subtract (ii) from (i).

$$\therefore\ s - sr = a - ar^n,$$

since all the other terms on the right-hand side cancel.

$$\therefore \quad s(1-r) = a(1-r^n).$$

$$\therefore \quad s = a\frac{1-r^n}{1-r} = a\frac{r^n-1}{r-1}.$$

335. If any three of the four quantities a, r, n, and s be given, this equation will determine the fourth quantity; but, if a, n, and s be given, and if $n > 2$, it will not generally be possible to solve the resulting equation for r.

If l be the last term, we have $l = ar^{n-1}$.

$$\therefore \quad s = \frac{ar^n - a}{r-1} = \frac{rl-a}{r-1},$$

which gives a different form for s.

Ex. 1. *Find the sum of* 8 *terms of the series* 1, 2, 4,....

This is a geometrical progression whose first term is 1 and common ratio $=2$. Hence, in the formula of Art. 334, $a=1$, $r=2$, $n=8$,

$$\therefore \; s = a\frac{r^n-1}{r-1}$$

$$= \frac{2^8-1}{2-1}$$

$$= \frac{256-1}{1} = 255.$$

Ex. 2. *Find the sum of n terms of the series* a^2, ab, b^2,....

This is a geometrical progression whose first term is a^2 and whose common ratio $= \frac{ab}{a^2} = \frac{b^2}{ab} = \frac{b}{a}$. Hence the formula

$$s = a\frac{r^n-1}{r-1}$$

gives

$$s = a^2\frac{\left(\frac{b}{a}\right)^n - 1}{\frac{b}{a}-1}$$

$$= \frac{b^n - a^n}{a^{n-3}(b-a)}.$$

Ex. 3. *It is said that chess was invented by Sessa for the amusement of an Indian rajah, named Sheran, who rewarded the inventor by promising to place* 1 *grain of wheat on the* 1st *square of a chess board,* 2 *grains on the* 2nd, 4 *grains on the* 3rd, *and so on, doubling the number for each successive square on the board. Find the number of grains which Sessa should have received.*

The number of grains on the successive squares form a G.P. whose first term is 1 and common ratio is 2. There are 64 squares,

$\therefore$ in the above notation $a=1$, $r=2$, $n=64$.

Therefore the total number of grains is given by

$$s=a\frac{r^n-1}{r-1}$$
$$=1\times\frac{2^{64}-1}{2-1}$$
$$=2^{64}-1.$$

This number has been calculated, and is equal to

18,446,744,073,709,551,615,

a quantity far greater than all the wheat in the world.

Ex. 4. *How many terms of the series* 3, 9, 27,... *must be taken in order that their sum may be equal to* 120?

Using the formula $s=a\dfrac{r^n-1}{r-1}$, we have $a=3$, $r=3$, $s=120$,

$$\therefore\ 120=3\,\frac{3^n-1}{3-1}=\tfrac{3}{2}(3^n-1),$$
$$\therefore\ 40\times 2=3^n-1,$$
$$\therefore\ 3^n=81.$$

Now $81=9^2=3^4$, $\qquad\therefore\ n=4$.

Hence 4 terms of the series must be taken.

**Ex.* 5. *How many terms of the series* $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8},\ldots$ *must be taken in order that their sum may be equal to* 2?

Using the formula $s=a\dfrac{r^n-1}{r-1}$, we have $a=1$, $r=\frac{1}{2}$, and $s=2$,

$$\therefore\ 2=1\times\frac{(\frac{1}{2})^n-1}{\frac{1}{2}-1}$$
$$=2\{1-(\tfrac{1}{2})^n\},$$

$$\therefore\ 1 = 1 - (\tfrac{1}{2})^n,$$
$$\therefore\ (\tfrac{1}{2})^n = 0.$$

Now the greater we take n, the smaller does $(\frac{1}{2})^n$ become. We can never make $(\frac{1}{2})^n$ quite zero, no matter how great we take n, but by taking n great enough, we can make $(\frac{1}{2})^n$ smaller than any quantity that is mentioned. Therefore an indefinitely large number of the terms of the series can be made to differ from 2 by as small a quantity as we please.

336. **Sum of a G.P. containing an infinite number of terms.** *The sum of an infinite number of quantities in geometrical progression, whose first term is a, and whose common ratio, r, is numerically less than unity, is* $\dfrac{a}{1-r}$.

We proved in Art. 334 that the sum of n terms of the progression is given by the formula $s = a\dfrac{1-r^n}{1-r}$. This can be written $s = \dfrac{a}{1-r} - \dfrac{ar^n}{1-r}$.

In the case we are discussing in this article, r is a proper fraction, and therefore the larger we make n, the smaller will r^n become. Moreover, by taking n sufficiently large, we can make r^n smaller than any assigned quantity. But the difference between the sum of n terms of the series and $\dfrac{a}{1-r}$ is $\dfrac{ar^n}{1-r}$, and therefore this difference may be made as small as we like by sufficiently increasing the number of terms taken, that is, by increasing n.

This is expressed by saying that the limit of the sum of an infinite number of terms of the series

$$a + ar + ar^2 + \dots \text{ is } \frac{a}{1-r}.$$

This however is only true provided that r is a proper fraction, either positive or negative.

Examples, illustrative of the use of this result, are given on pp. 355—357.

*337. **Infinite Series.** Series containing an infinite number of terms are of constant occurrence in mathematics. They are sometimes called infinite series, but by this is only meant series containing an indefinitely large number of terms.

*338. **Convergent and Divergent Series.** Whenever the sum of n terms of a series containing an infinite number of terms can, by sufficiently increasing n, be made to differ from some finite quantity by less than any assignable quantity (no matter how small), such a series is said to be **convergent** (or *converging*): in any other case, it is said to be **divergent** (or *diverging*).

A convergent series may in general be safely employed provided a sufficient number of terms be taken into account; but a divergent series should only be used with extreme caution.

The rules for determining whether a given series is convergent or divergent lie beyond the limits of this book.

*339. As an illustration of convergent and divergent series, let us consider the following example. If we divide 1 by $1-x$, we find that the quotient is $1+x+x^2+x^3+\dots$ and the remainder (if we stop after the n^{th} term in the quotient) is x^n. We might therefore think that

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots\text{(to infinity)}.$$

If x be less than unity, this result is true, for the further we proceed in the division the smaller does the remainder, x^n, become, and thus the series converges. In fact, this is a geometrical progression whose common ratio is x, and if x be a proper fraction, the sum of the series—or more accurately the limit of its sum—is $\frac{1}{1-x}$ [Art. 336].

If however x be greater than unity, then the further we proceed in the division the larger does the remainder x^n become. Thus the difference between $\frac{1}{1-x}$ and the series $1+x+x^2+x^3+\dots$ becomes greater as we take more terms of the series into account.

The series diverges, and the limit of its sum is not represented by $\frac{1}{1-x}$. For example, if we put $x=2$, then $\frac{1}{1-x}$ becomes $\frac{1}{1-2}$, that is, -1; while the series $1+x+x^2+\ldots$ becomes the sum of a number of positive quantities. Thus, if any one had been so careless as to put the series equal to $\frac{1}{1-x}$, he would have been led (in this case) to the absurd result that the sum of a number of positive quantities was equal to a negative quantity. The accurate result, namely,

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^{n-1}+\frac{x^n}{1-x}$$

shews however that the mistake would arise from the neglect of the term $\frac{x^n}{1-x}$. This term can be neglected only when (i) x is numerically less than unity and (ii) n is infinitely large.

Ex. 1. *Find the sum of the series* $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ (*to infinity*).

This is a G.P., whose first term is 1 and whose common ratio is $\frac{1}{2}$. Hence, in the formula $s=\frac{a}{1-r}$, we have $a=1$, $r=\frac{1}{2}$.

$$\therefore\ s=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=2.$$

Ex. 2. *If n be positive, find the sum of*

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots \text{ to infinity.}$$

This is a G.P., whose first term is $\frac{1}{n+1}$ and whose common ratio is $\frac{1}{n+1}$; also, since n is positive, r is less than unity. Hence, in the above formula, $a=\frac{1}{n+1}$, $r=\frac{1}{n+1}$.

$$\therefore\ s=\frac{a}{1-r}$$

$$=\frac{\frac{1}{n+1}}{1-\frac{1}{n+1}}$$

$$=\frac{1}{n}.$$

Ex. 3. **The rule for finding the value of recurring fractions** *in arithmetic affords another illustration of the rule for finding the sum of a geometrical progression containing an infinite number of terms.*

This will be sufficiently illustrated by two examples.

(i) *Find the value of* $0{\cdot}2\dot{7}$.

We have $$0{\cdot}2\dot{7} = 0{\cdot}2777\ldots$$
$$= \frac{2}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots.$$

Now $$\frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots \text{ (to infinity)} = \frac{\frac{7}{10^2}}{1 - \frac{1}{10}} = \frac{7}{90}.$$

$$\therefore\ 0{\cdot}2\dot{7} = \frac{2}{10} + \frac{7}{90} = \frac{(2 \times 9) + 7}{90} = \frac{2(10-1)+7}{90} = \frac{27-2}{90},$$

which is the form in which the result is given by the usual arithmetical rule. It reduces to $\frac{5}{18}$.

(ii) *Find the value of* $0{\cdot}3\dot{1}\dot{2}$.

We have $0{\cdot}3\dot{1}\dot{2} = 0{\cdot}31212\ldots$
$$= \frac{3}{10} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \ldots$$
$$= \frac{3}{10} + \frac{12}{10^3} + \frac{12}{10^5} + \ldots$$

Now $$\frac{12}{10^3} + \frac{12}{10^5} + \ldots \text{ to infinity} = \frac{\frac{12}{10^3}}{1 - \frac{1}{10^2}} = \frac{12}{990},$$

$$\therefore\ 0{\cdot}3\dot{1}\dot{2} = \frac{3}{10} + \frac{12}{990} = \frac{3 \times 99 + 12}{990} = \frac{3(100-1)+12}{990} = \frac{312-3}{990},$$

which is the form in which the result is given by the usual arithmetical rule.

Ex. 4. *Find three numbers in* G.P., *such that their sum is* 7 *and their product* 8.

Let the middle number be a, and let r be the common ratio of the progression.

$$\therefore \text{ the numbers are } \frac{a}{r},\ a,\ ar.$$

The product of the numbers is 8, $\therefore\ \frac{a}{r} \cdot a \cdot ar = 8.$

$$\therefore\ a^3 = 8.$$

$$\therefore\ a = 2.$$

The sum of the numbers is 7, $\therefore\ \frac{a}{r} + a + ar = 7.$

But $a = 2$, $\qquad \therefore\ \frac{2}{r} + 2 + 2r = 7.$

$$\therefore\ 2r^2 - 5r + 2 = 0,$$

$$\therefore\ r = 2, \text{ or } \tfrac{1}{2}.$$

Whichever value of r is taken, we obtain for the required numbers 1, 2, 4. Hence the numbers are 1, 2, 4.

Note. Wherever a problem is concerned with an odd number of quantities in an unknown G.P., it is generally convenient to select the middle quantity of the G.P. and the common ratio of the series as the unknown quantities.

EXAMPLES. XXVIII. B.

Find the sum of the following series, numbered **1** to **16**, each of which is in G.P.

1. $32 + 48 + 72 + \ldots$ to 6 terms.
2. $\frac{7}{45} + \frac{1}{9} + \frac{5}{63} + \ldots$ to 4 terms.
3. $\frac{21}{4},\ -\frac{9}{2},\ \frac{27}{7}, \ldots$ to 6 terms.
4. $\frac{1}{3} - \frac{1}{2} + \frac{3}{4} - \ldots$ to 8 terms.
5. $\frac{1}{4} - \frac{1}{3} + \frac{4}{9} - \ldots$ to 8 terms.
6. $-\frac{3}{21} + \frac{1}{9} - \frac{7}{81} + \ldots$ to 6 terms.
7. $\frac{2}{9} + \frac{4}{9} + \frac{8}{9} + \ldots$ to 12 terms.
8. $1 - 1{\cdot}2 + 1{\cdot}44 - \ldots$ to 9 terms.
9. 250, 100, 40,... to infinity.
10. 108, 72, 48,... to 5 terms.
11. $16 + 12 + 9 + \ldots$ to infinity.
12. $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \ldots$ to infinity.
13. $3 - 1 + \frac{1}{3} - \ldots$ to infinity.
14. $9{\cdot}6,\ 7{\cdot}2,\ 5{\cdot}4, \ldots$ to infinity.
15. $35 + {\cdot}35 + {\cdot}0035 + \ldots$ to inf.
16. $\sqrt{3} - \sqrt{2} + \frac{2}{\sqrt{3}} - \ldots$ to 10 terms.

17. Find the sum of five numbers in geometrical progression, the second term being 5 and the fifth term being 625.

18. The first term of a G.P. is 27, and the third term is 48. Find the sum of the first 6 terms.

19. The first term of a G.P. is 36, and the third term is 81. Find the sum of 6 terms.

20. The sum of the first 6 terms of a geometrical progression is 9 times the sum of the first three terms. Find the common ratio.

21. The sum of the first 8 terms of a G.P. is 17 times the sum of the first four terms. Find the common ratio.

22. Find the geometrical progression whose sum to infinity is 4 and whose second term is $\frac{3}{4}$.

23. The third and fifth terms of a geometrical progression are respectively 12 and 48. Find the sums of eight terms of the two progressions which satisfy the conditions.

24. The first term of a G.P. is 5, and its sum to infinity is 4. Find the sum of the first 5 terms.

25. The sum of a number of terms in G.P. is 20; the last term is $13\frac{1}{2}$, and the first term is $\frac{1}{2}$. Find the common ratio, and the number of terms.

26. The sum of $2n$ terms of a G.P., whose first term is a and whose common ratio is r, is equal to the sum of n terms of a G.P., whose first term is b and whose common ratio is r^2. Prove that b must be equal to the sum of the first two terms of the first series.

***27.** Shew that, if s be the sum of n terms of a geometrical progression, p be the product of all the terms of the given series, and σ be the sum of the reciprocals of the terms composing the given series, then $p^2\sigma^n = s^n$.

340. **Geometric Mean.** When three quantities are in geometrical progression, the middle one is called the *geometric mean* of the other two. The letters **G.M.** are often used as an abbreviation for the words Geometric Mean.

341. *The geometric mean of two quantities is the square root of their product.*

Let a, x, b be three quantities in geometrical progression.

$$\therefore \quad \frac{x}{a} = \frac{b}{x}.$$

$$\therefore \quad x^2 = ab.$$

$$\therefore \quad x = \pm\sqrt{ab}.$$

It is usual to take the positive sign of the square root.

Note. The geometric mean of two quantities is the same as their mean proportional [Art. 305].

342. The arithmetic mean of two unequal quantities is greater than their geometric mean.

Let a and b be two unequal quantities. Their arithmetic mean is $\frac{1}{2}(a+b)$, and their geometric mean is $\sqrt{ab}$. We want to shew that

$$\tfrac{1}{2}(a+b) > \sqrt{ab},$$

that is, that $$a + b - 2\sqrt{ab} > 0,$$

that is, that $$(\sqrt{a} - \sqrt{b})^2 > 0.$$

But the square of any quantity (whether positive or negative) is itself a positive quantity, and therefore greater than zero. Hence $(\sqrt{a} - \sqrt{b})^2$ is > 0.

$$\therefore \quad \tfrac{1}{2}(a+b) > \sqrt{ab}.$$

Note. If we have two equal quantities, a and b, then their arithmetic mean $= \frac{1}{2}(a+b) = \frac{1}{2}(2a) = a$; and their geometric mean $= \sqrt{ab} = \sqrt{a^2} = a$. Thus, in this case, the arithmetic mean is equal to the geometric mean.

We may therefore say that *the arithmetic mean of any two quantities is not less than their geometric mean.*

343. Geometric Means. When any number of quantities are in geometrical progression, the terms intermediate between the two extreme ones are called the *geometric means* (or geometrical means) of the two extreme terms.

344. ***To insert p geometric means between two quantities a and b.***

We have two quantities, a and b. We want to insert p quantities between them so that the $p+2$ terms so formed may be in G. P. The first of these $p+2$ terms is a; the last of them is b. Let r be the common ratio of this progression. Then the $(p+2)^{\text{th}}$ term must be ar^{p+1}.

$$\therefore \quad b = ar^{p+1}.$$

$$\therefore \quad r^{p+1} = \frac{b}{a}.$$

$$\therefore \quad r = \sqrt[p+1]{\frac{b}{a}}.$$

Hence the required series is

$$a, \quad a\sqrt[p+1]{\frac{b}{a}}, \quad a\sqrt[p+1]{\frac{b^2}{a^2}}, \quad \ldots \quad a\sqrt[p+1]{\frac{b^p}{a^p}}, \quad b,$$

of which the terms intermediate between a and b are the required geometric means.

Ex. 1. *Insert two geometric means between* 1 *and* 27.

We want to insert 2 terms between 1 and 27, so that the 4 terms so formed shall be in G. P. Let r be the common ratio of the required geometric progression. Therefore 27 is to be the fourth term of a G. P. whose first term is 1 and whose common ratio is r,

$$\therefore \; 27 = 1 \times r^{4-1}.$$

$$\therefore \; 27 = r^3.$$

$$\therefore \; r = 3,$$

$\therefore$ the series is 1, 3, 9, 27; and the required means are 3 and 9.

Ex. 2. *Find the sum of the p geometric means inserted between a and b.*

Let r be the common ratio of the progression, of which a is the first term and b is the $(p+2)^{\text{th}}$ term.

The sum of the p means is

$$ar + ar^2 + \dots + ar^p = ar\frac{r^p - 1}{r - 1}$$

$$= a\frac{r^{p+1} - r}{r - 1}$$

$$= a\frac{\frac{b}{a} - r}{r - 1}$$

$$= \frac{b - ar}{r - 1}.$$

And we know, by Art. 344, that $r = \sqrt[p+1]{b/a}$, hence the sum can be expressed in terms of p, a, and b.

EXAMPLES. XXVIII. C.

1. Find the geometric mean between

(i) 2 and 8; (ii) $(x+a)^2$ and $(x-a)^2$.

2. Insert three geometric means between

(i) 1 and 16; (ii) $(x+a)^2$ and $(x-a)^2$; (iii) 1 and 2.

3. Insert two geometric means between x^{6n} and y^{6n}.

4. Insert six geometric means between $10\frac{1}{8}$ and $\frac{16}{27}$.

5. Insert eight geometrical means between 512 and 19683.

6. Find the geometrical mean of

$$4x^2 - 12x + 9 \text{ and } 9x^2 + 12x + 4.$$

7. Prove that, in a geometrical series containing an odd number of terms, the middle term is a geometric mean between the first and last terms.

8. Determine the ratio of two numbers, when the ratio of their arithmetic mean to their geometric mean is as 13 to 5.

9. The arithmetic mean between two numbers is 39, and the geometric mean between them is 15. Find the numbers.

10. If one geometrical mean, G, and two arithmetical means, p and q, be inserted between two given quantities, shew that

$$G^2 = (2p - q)(2q - p).$$

11. If one arithmetical mean, A, and two geometrical means, p and q, be inserted between two given quantities, shew that

$$p^3+q^3=2Apq.$$

***12.** Between each consecutive pair of terms of a series of numbers in G.P., m arithmetic means are taken. Find the sum of all the means thus obtained.

345. The student will notice that all the propositions about geometrical progressions are deduced from the formulae

$$l = ar^{n-1},$$

$$s = a\frac{r^n - 1}{r-1},$$

where a is the first term of the series, r the common ratio, l the n^{th} term, and s the sum of n terms.

It has been shewn [Art. 336] that it follows from the latter of these formulae that the limit of the sum of an infinite number of quantities in geometrical progression, whose common ratio, r, is numerically less than unity, is

$$\frac{a}{1-r}.$$

MISCELLANEOUS EXAMPLES. XXVIII. D.

[*The following miscellaneous examples are on geometrical progressions. Other examples on geometrical progressions will be found on pp.* 377—379.]

1. Sum the following series, each of which is in G.P.

(i) $14+42+126+\dots$ to 8 terms;

(ii) $864+1296+1944+\dots$ to 6 terms;

(iii) $8+6+4\frac{1}{2}+\dots$ to infinity;

(iv) $2\sqrt{5}-5\sqrt{6}+15\sqrt{5}-\dots$ to 8 terms.

2. Find, to four places of decimals, the sum to infinity of the series $1+\frac{1}{\sqrt{3}}+\frac{1}{3}+\dots$.

3. Can any of the following geometrical series be summed, the number of terms in each being infinite? If so, find their respective sums.

(i) $1-\frac{3}{2}+\frac{9}{4}-\ldots$; (ii) $1-\frac{2}{3}+\frac{4}{9}-\ldots$;

(iii) $\cdot 9+\cdot 81+\cdot 729+\ldots$; (iv) $\cdot 1+\cdot 5+2\cdot 5+\ldots$;

(v) $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots$.

4. Sum the following series:

(i) $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\ldots$ to n terms;

(ii) $(x+a)+(x^2+2a)+(x^3+3a)+\ldots$ to n terms.

5. Find the sum of n terms of the series whose r^{th} term is $(-a)^r$, r being any positive integer.

6. The first term of a geometrical progression exceeds the second term by 1, and the sum to infinity is 100: find the series.

*7. Find the sum of n terms of a series in which each term is half the sum of all that precede it, and in which the first term is a.

8. The alternate terms of any G. P. form a series in G. P.

9. If an odd number of quantities be in G. P., then will the first, the middle, and the last of them be in G. P.

10. Shew that, if every fourth term of a geometric series be picked out, these terms will themselves form a geometric series.

11. Prove that, if each term of a G. P. be squared, these terms will also form a G. P.

12. Shew that the logarithms of a series of numbers in geometrical progression will themselves be in arithmetical progression.

13. Prove that, if an odd number (n) of consecutive terms of a geometrical progression be multiplied together, the product will be the n^{th} power of the middle term of the progression.

14. Each term in a certain infinite geometric progression is equal to the sum of all that succeed it. Find the common ratio of the progression.

15. Prove that, if S be the sum to infinity of a G. P. whose first term is a, then the common ratio of the series is $1-\frac{a}{S}$.

16. An odd number of consecutive terms in a certain geometrical progression is taken; the middle term is 3, and the continued product of all the terms is 243. How many terms are taken?

17. If the ratio of a G.P. be not less than 2, shew that any term is greater than the sum of all that precede it.

18. Three quantities are in G.P. The first exceeds the sum of the other two by unity; and the excess of the first over the second is greater than the excess of the second over the third by unity. Find the numbers.

19. The sum of four integers in geometrical progression is 255, and the fourth exceeds the second by 180: find the integers.

20. Find three numbers in G.P., such that if they be increased by 4, 7, 1 respectively, the sums form a G.P., whose common ratio is less by 1 than that of the original series.

21. From three numbers, which are in geometrical progression, three others, which are also in G.P., are subtracted. Prove that, if the remainders be in G.P., all these series have the same common ratio.

22. The sum of three quantities in geometrical progression is $\frac{7}{6}$, and the sum of their squares $\frac{133}{36}$. What are the quantities?

23. The continued product of three numbers in geometrical progression is 64, and the sum of the products of them in pairs is 84; find the numbers.

24. If x, y, a be in arithmetical progression and if x, y, b be in geometrical progression, shew that x, $x-y$, $b-a$ are in geometrical progression.

***25.** Shew that, if s_1, s_2, s_3... be an infinite series of sums of infinite geometrical progressions, whose common ratios are the same, and whose first terms are respectively the terms of the series s_1, then $s_2+s_3+\ldots$ is greater or less than s_1 according as the common ratio is greater or less than one-half.

CHAPTER XXIX.

HARMONIC AND OTHER SERIES.

*346. We are constantly concerned in mathematics with series of numbers, of which the successive numbers (or terms) are formed according to various rules. Arithmetical and geometrical progressions are instances of such series. A few series whose terms are formed according to other rules are reducible to arithmetical or geometrical progressions, and we shall deal in this chapter with some of the more simple of such series.

We shall discuss in succession (i) Harmonical Progressions [Arts. 347—356], (ii) Series whose terms are the squares or cubes of numbers which are in arithmetical progression [Arts. 357—361], and (iii) Series whose terms are the products of corresponding terms of a series of quantities in A.P. and a series of quantities in G.P. [Arts. 362—365].

HARMONICAL PROGRESSIONS.

347. **Harmonic Progression.** A series of numbers is said to be in *harmonical progression* (or *harmonic progression*, or to form a harmonical progression), when their reciprocals are in arithmetical progression. The letters **H.P.** are often used as an abbreviation for the words Harmonical Progression.

348. **Condition for an H.P.** The condition that the series of numbers denoted by the letters $a, b, c, d \ldots$ shall be in harmonical progression is that

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \ldots$$

shall be in arithmetical progression; that is, that

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c} = \ldots .$$

For example, 3, 4, and 6 are in harmonic progression because $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ are in arithmetical progression, the common difference being $-\frac{1}{12}$.

349. **Fundamental Property of three numbers in H.P.** *If three quantities be in harmonic progression, the ratio of the difference between the first and second of them to the difference between the second and third of them is equal to the ratio of the first of them to the third of them.*

This property is sometimes taken as the definition of a harmonic progression.

Let a, b, c, be in harmonic progression;

$$\therefore \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}. \qquad \text{[Art. 348}$$

Multiplying throughout by the L.C.M. of the denominators,

$$\therefore \quad ac - bc = ab - ac,$$

that is,

$$c(a-b) = a(b-c).$$

$$\therefore \quad a - b : b - c = a : c.$$

*350. **The n^{th} term of a H.P.** *To find the n^{th} term of a harmonic progression, whose first and second terms are given.*

Let a and b be the two first terms, and let x be the required n^{th} term.

Therefore $\frac{1}{x}$ is the n^{th} term of an arithmetical progression whose two first terms are $\frac{1}{a}$ and $\frac{1}{b}$.

$\therefore$ the common difference of this A.P. $=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$.

Hence, by the formula $l=a+(n-1)d$ [Art. 322], the required term is given by

$$\frac{1}{x}=\frac{1}{a}+(n-1)\frac{a-b}{ab}$$

$$=\frac{b+(n-1)(a-b)}{ab}.$$

$$\therefore\ x=\frac{ab}{b+(n-1)(a-b)}.$$

*351. It is impossible to express the sum of a number of terms of a harmonical progression by an algebraical formula of a concise form similar to the corresponding formulae in A.P. and G.P.

*352. **Harmonic Mean.** When three quantities are in harmonic progression, the middle one is called the *harmonic mean* of the other two. The letters **H.M.** are often used as an abbreviation for the words Harmonic Mean.

*353. *The harmonic mean between a and b is* $2ab/(a+b)$.

Let the harmonic mean between a and b be x.
Therefore a, x, b are in harmonical progression.

$$\therefore\ \frac{1}{a},\ \frac{1}{x},\ \frac{1}{b}\text{ are in A.P.}$$

$$\therefore\ \frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}.$$

$$\therefore\ \frac{2}{x}=\frac{1}{a}+\frac{1}{b}.$$

$$\therefore\ x=\frac{2ab}{a+b}.$$

*354. *The geometric mean of two quantities is also the geometric mean of their arithmetic and harmonic means.*

Let a and b be two quantities; and let A be their arithmetic mean, G their geometric mean, and H their harmonic mean,

$$\therefore\ A=\frac{a+b}{2},\quad G=\sqrt{ab},\quad H=\frac{2ab}{a+b}.$$

$$\therefore\ A\times H=\frac{a+b}{2}\times\frac{2ab}{a+b}=ab=G^2,$$

which is the condition that G may be the G. M. of A and H [Art. 341].

Since $AH=G^2$, we have $A:G=G:H$. But $A \nless G$, [Art. 342], $\therefore\ G \nless H$, that is, the geometric mean of any two quantities is not less than their harmonic mean. This property can be proved directly by a process similar to that given in Art. 342.

*355. **Harmonic Means.** When any number of quantities are in harmonic progression, the terms intermediate between the two extreme terms are called the *harmonic means* of the two extreme terms.

*356. *To insert p harmonic means between a and b.*

We have two quantities, a and b. We want to insert p quantities between them, so that the $p+2$ terms so formed may be in H. P. Therefore the reciprocals of these $p+2$ quantities will be in A. P.

We want, therefore, to form an arithmetical progression, containing $p+2$ terms, of which the first term shall be $\frac{1}{a}$, and of which the last term shall be $\frac{1}{b}$. The series so formed can be constructed by the method given in Art. 328, and will be found to be

$$\frac{1}{a},\quad \frac{(p+1)\,b+(a-b)}{(p+1)\,ab},\quad \frac{(p+1)\,b+2\,(a-b)}{(p+1)\,ab},\ \ldots,\ \frac{1}{b}.$$

The reciprocals of these terms will be in harmonical progression. Hence the required harmonic means are

$$\frac{(p+1)\,ab}{(p+1)\,b+(a-b)},\ \frac{(p+1)\,ab}{(p+1)\,b+2\,(a-b)},\ \ldots,\ \frac{(p+1)\,ab}{(p+1)\,b+p\,(a-b)}.$$

Ex. 1. *Insert two harmonic means between* 2 *and* $\frac{1}{2}$.

We want to find 4 terms in an A. P. of which the first term is $\frac{1}{2}$ and the fourth term is 2.

Let d be the common difference of this A. P. Then using the formula $l=a+(n-1)\,d$ [Art. 322], we have

$$2=\tfrac{1}{2}+3d,$$

$$\therefore\ d=\tfrac{1}{2}.$$

Hence the A.P. is $\frac{1}{2}$, 1, $\frac{3}{2}$, 2. The corresponding H.P. is 2, 1, $\frac{2}{3}$, $\frac{1}{2}$; and the required harmonic means are 1 and $\frac{2}{3}$.

Ex. 2. *Shew that, if a, b, c be in* G.P., *then $a+b$, $2b$, $b+c$ will be in* H.P.

Since a, b, c are in G.P., we have $ac=b^2$.

Now the quantities $a+b$, $2b$, $b+c$ will be in H.P.,

if $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ be in A.P.,

that is, if $$\frac{1}{2b}-\frac{1}{a+b}=\frac{1}{b+c}-\frac{1}{2b},$$

which reduces to $$\frac{a-b}{a+b}=\frac{b-c}{b+c}.$$

This is true, if $(a-b)(b+c)=(b-c)(a+b)$,

that is, if $ac-b^2=b^2-ac$,

that is, if $2(ac-b^2)=0$,

which is true. $\therefore$ $a+b$, $2b$, $b+c$ are in H.P.

Ex. 3. *The sum of three numbers in* H.P. *is* 11, *and their continued product is* 36. *Find the numbers.*

Let the numbers be x, y, z.

Since they are in H.P., $\therefore \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$$\therefore \frac{1}{y}-\frac{1}{x}=\frac{1}{z}-\frac{1}{y} \text{........................ (i).}$$

The sum of the numbers is 11, $\therefore x+y+z=11$...............(ii).

The product of the numbers is 36, $\therefore xyz=36$ ....(iii).

We have therefore three equations to determine three unknown quantities.

Equation (i), on simplification, reduces to

$$2xz=y(x+z) \text{ ....(iv).}$$

Now by (ii), $x+z=11-y$; and by (iii), $xz=\frac{36}{y}$.

Substituting these values of $x+z$ and xz in (iv), we have

$$2\frac{36}{y}=y(11-y),$$

$$\therefore y^3-11y^2+72=0.$$

This is a cubic equation, and we have not discussed the rule which enables us to solve such an equation. This particular equation can however be written in the form

$$y^3 - 3y^2 = 8y^2 - 72,$$

that is, $$y^2(y-3) = 8(y-3)(y+3).$$

Hence one root is given by $y-3=0$. [The other roots are given by $y^2 = 8(y+3)$, hence they are not integers, and therefore they are not applicable to this problem.] Thus the required root is $y=3$.

If in equations (ii) and (iii) we put $y=3$, we obtain $x+z=8$, $xz=12$. From the two latter equations, we obtain $x=6$ and $z=2$, or $x=2$ and $z=6$.

Hence the required numbers are 2, 3, 6.

*EXAMPLES. XXIX. A.

1. What is the fourth term of a H.P. of which the second term is $\frac{1}{7}$ and the fifth term is $\frac{1}{16}$?

2. Find the sum of four terms of a H.P. of which the first term is 1 and the third term is $\frac{1}{5}$.

3. Write down the fourth, fifth, and sixth terms of the H.P. of which the first term is 1 and the second term is 2.

4. Find the harmonic mean of

(i) 1 and 4; (ii) 2 and 5; (iii) a and $-\dfrac{1}{a}$.

5. Insert three harmonic means between

(i) 1 and $\frac{1}{17}$; (ii) 17 and 1; (iii) $\frac{5}{4}$ and $\frac{1}{4}$; (iv) $a+b$ and b.

6. Shew that, if a, $2x$, b be in H.P., then $x-a : x-b = a^2 : b^2$.

7. Shew that, if a, b, c be in geometric progression, and if p, q be the arithmetic means between a, b and b, c respectively, then b will be the harmonic mean between p and q.

SERIES EACH OF WHOSE TERMS IS A POWER OF THE SUCCESSIVE TERMS OF A SERIES OF NUMBERS IN A.P.

*357. We proceed next to the consideration of series each of whose terms is the square (or the cube) of the successive terms of an arithmetical progression.

We shall begin by considering the special case of the determination of the sum of the squares of the numbers 1, 2, 3, 4.... These numbers, when written in this order, are sometimes called the *natural numbers.* To this case, we shall reduce one or two other series.

We shall then determine in a similar way the sum of the cubes of the natural numbers.

*358. *The sum of the squares of the first n natural numbers is equal to* $\frac{1}{6}n(n+1)(2n+1)$.

First proof. Let S_n denote the required sum,

$$\therefore \quad S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2.$$

It is easy to verify that

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

Write $n-1$ for n on both sides of this identity,

$$\therefore \quad n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1.$$

Similarly,

$$(n-1)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1.$$

....

Continuing this process, we finally get the identities

$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1,$$

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1.$$

Now, add all the right-hand sides of these n identities together, and also all the left-hand sides. The sum of the left-hand sides reduces to $(n+1)^3 - 1^3$, since all the other terms cancel one another. Therefore

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + \ldots + n) + n.$$

But $1+2+\ldots+n$ is an A.P., and its sum $=\frac{1}{2}n(n+1)$;

$$\therefore \quad (n+1)^3-1=3S_n+\tfrac{3}{2}n(n+1)+n.$$

$$\therefore \quad S_n=\frac{(n+1)^3-1-\frac{3}{2}n(n+1)-n}{3}.$$

Simplifying the right-hand side, and resolving the result into factors, we obtain

$$S_n=\frac{n(n+1)(2n+1)}{6}.$$

This result is sometimes written $\sum\limits_1^n m^2=\frac{1}{6}n(n+1)(2n+1)$, where $\sum\limits_1^n m^2$ stands for the words "the sum of all quantities like m^2, for integral values of m from $m=1$ to $m=n$."

*359. *Second proof.* This is an important series, and we shall give another proof of the result.

Suppose that we knew or had guessed the value of the sum, and merely wanted to verify the result. We could effect this in the following manner, which is an illustration of what is known as *mathematical induction.*

We assume that we know or suspect that

$$S_n=1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1)\ldots\text{(i)}.$$

Add $(n+1)^2$ to each side,

$$\begin{aligned}\therefore\ 1^2+2^2+\ldots+n^2+(n+1)^2&=\tfrac{1}{6}n(n+1)(2n+1)+(n+1)^2\\&=(n+1)\{\tfrac{1}{6}n(2n+1)+(n+1)^2\},\end{aligned}$$

which on simplification $\quad =\frac{1}{6}(n+1)(n+2)(2n+3)$.

Now this result is exactly what we obtain from (i), if in it we write $n+1$ for n. Hence, if the formula (i) enable us to find the sum of the squares of the first n natural numbers, it will also enable us to find the sum of the squares of the first $n+1$ natural numbers.

But, if $n=1$, the formula (i) is true, since

$$1^2=\tfrac{1}{6}.1.2.3;$$

hence, it is true for the case of $n=2$. But, since it is true if $n = 2$, $\therefore$ it is true if $n = 3$. Again, since it is now known to be true when $n = 3$, $\therefore$ it is true when $n = 4$. Continuing this process, we see that the formula (i) is true for any positive integral value of n.

Ex. 1. *Find the sum of n terms of the series*

$$1 . 2 + 2 . 3 + 3 . 4 + \dots .$$

Let S_n denote the required sum,

$$\begin{aligned} \therefore\ S_n &= 1 . 2 + 2 . 3 + 3 . 4 + \dots + n(n+1) \\ &= 1(1+1) + 2(2+1) + 3(3+1) + \dots + n(n+1) \\ &= (1^2 + 2^2 + 3^2 + \dots + n^2) + (1 + 2 + 3 + \dots n) \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \qquad \text{[Arts. 358, 323.} \\ &= \tfrac{1}{3} n(n+1)(n+2). \end{aligned}$$

Alternative proof. This result is sometimes proved in another way. We have

$$\begin{aligned} 1 . 2 &= \tfrac{1}{3}(1 . 2 . 3) \\ 2 . 3 &= \tfrac{1}{3}(2 . 3 . 4 - 1 . 2 . 3) \\ 3 . 4 &= \tfrac{1}{3}(3 . 4 . 5 - 2 . 3 . 4) \\ &\dots\dots\dots\dots\dots\dots \\ n(n+1) &= \tfrac{1}{3}\{n(n+1)(n+2) - (n-1)n(n+1)\}. \end{aligned}$$

Add,

$\therefore\ 1 . 2 + 2 . 3 + \dots + n(n+1) = \tfrac{1}{3} n(n+1)(n+2)$, since all the other terms on the right-hand sides cancel one another.

Ex. 2. *Find the sum of the squares of n terms of an arithmetical progression.*

Let the series in A.P. be a, $a + d$, $a + 2d$,...

Denote the required sum by S_n. Then

$$\begin{aligned} S_n &= a^2 + (a+d)^2 + (a+2d)^2 + \dots + \{a + (n-1)d\}^2 \\ &= a^2 + \{a^2 + 2ad + d^2\} + \{a^2 + 2 . 2ad + (2d)^2\} + \dots \\ &\qquad + \{a^2 + 2(n-1)ad + (n-1)^2 d^2\}. \end{aligned}$$

Collect like terms,

$$\therefore\ S_n = na^2 + 2ad\{1 + 2 + \dots + (n-1)\} + d^2\{1^2 + 2^2 + \dots + (n-1)^2\}.$$

But $\qquad 1+2+\ldots+(n-1) = \frac{1}{2}(n-1)n,$ [Art. 323

and $\qquad 1^2+2^2+\ldots+(n-1)^2 = \frac{1}{6}(n-1)n(2n-1),$ [Art. 353

$$\therefore\ S_n = na^2 + n(n-1)ad + \tfrac{1}{6}n(n-1)(2n-1)d^2.$$

*360. *The sum of the cubes of the first n natural numbers is equal to* $\frac{1}{4}n^2(n+1)^2$.

Let S_n denote the required sum, $\therefore\ S_n = 1^3+2^3+\ldots+n^3$.

Now $\qquad (n+1)^4 - n^4 = 4n^3+6n^2+4n+1.$

Write $n-1$ for n in this identity,

$$\therefore\ n^4-(n-1)^4 = 4(n-1)^3+6(n-1)^2+4(n-1)+1.$$

Similarly,

$$(n-1)^4-(n-2)^4 = 4(n-2)^3+6(n-2)^2+4(n-2)+1.$$

Continuing this process, we finally obtain the identities

$$3^4-2^4 = 4.2^3+6.2^2+4.2+1,$$

$$2^4-1^4 = 4.1^3+6.1^2+4.1+1.$$

Adding these results, we have

$$(n+1)^4-1^4 = 4S_n+6(1^2+2^2+\ldots+n^2)+4(1+2+\ldots+n)+n,$$

$$= 4S_n + 6.\tfrac{1}{6}n(n+1)(2n+1)+4.\tfrac{1}{2}n(n+1)+n.$$

Transposing S_n to one side of the equality, and all the other terms to the other side, and simplifying, we finally obtain

$$S_n = \tfrac{1}{4}n^2(n+1)^2.$$

This result can also be proved by induction in the same way as the proof given in Art. 359.

The result of this article gives us the theorem that the sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers.

For $\qquad 1^3+2^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2,$

and $\qquad 1+2+\ldots+n = \frac{1}{2}n(n+1),$

$$\therefore\ 1^3+2^3+\ldots+n^3 = (1+2+\ldots+n)^2.$$

Example. Find the sum of n terms of the series

$$1.2.3+2.3.4+3.4.5+\dots.$$

Let S_n denote the required sum.

The n^{th} term is $\qquad n(n+1)(n+2),$

which may be written n^3+3n^2+2n. The other terms can be obtained from this by successively writing for n the numbers 1, 2, 3,....

Hence

$$\begin{aligned}S_n &= (1^3+3.1^2+2.1)+(2^3+3.2^2+2.2)+\dots+(n^3+3.n^2+2n)\\ &= (1^3+2^3+\dots+n^3)+3(1^2+2^2+\dots+n^2)+2(1+2+\dots+n)\\ &= \tfrac{1}{4}n^2(n+1)^2+3.\tfrac{1}{6}n(n+1)(2n+1)+2.\tfrac{1}{2}n(n+1).\end{aligned}$$

[Arts. 360, 358, 323.

Simplifying the right-hand side, we obtain

$$S_n=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

*361. By a process similar to that used in Arts. 358, 360, we can find the sum of the fourth (or higher) powers of quantities in A.P.

*EXAMPLES. XXIX. B.

1. The sum of the squares of the first n natural numbers is equal to $20n$. Find n.

2. Shew that the sum of the squares of the first n odd numbers is equal to $\frac{1}{3}n(4n^2-1)$.

3. Shew that the difference between n and the sum of the squares of any n odd numbers is a multiple of 8.

4. Sum to n terms the series $1.3+2.4+3.5+\dots.$

5. Find the sum of m terms of the series

$$1.2m+2.(2m-1)+3.(2m-2)+\dots.$$

6. Find the sum of 15 terms of a series whose n^{th} term is $(2n-1)(3n+1)$, where n is any positive integer.

7. Shew that the sum of the cubes of the first n odd numbers is equal to $n^2(2n^2-1)$.

8. If n be an even number, prove that the sum of the series $1+2^2+3+4^2+\dots$ to n terms is $\dfrac{n(n+4)(2n+1)}{12}$.

SERIES WHOSE TERMS ARE THE PRODUCTS OF CORRESPONDING TERMS OF AN A.P. AND A G.P.

*362. We proceed next to determine the sum of the terms of a series of which any term, such as the n^{th}, is the product of the n^{th} term of an A.P., and the n^{th} term of a G.P.

*363. *To find the sum of n terms of the series*

$$1 + 2x + 3x^2 + 4x^3 + \dots .$$

Let S denote the required sum,

$$\therefore \quad S = 1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} + nx^{n-1}.$$

$$\therefore \quad Sx = x + 2x^2 + 3x^3 + \dots + (n-1)x^{n-1} + nx^n.$$

Subtract,

$$\therefore \quad S - Sx = 1 + x + x^2 + \dots + x^{n-1} - nx^n,$$

that is,

$$S(1-x) = (1 + x + x^2 + \dots + x^{n-1}) - nx^n$$

$$= \frac{1-x^n}{1-x} - nx^n \qquad \text{[Art. 334}$$

$$\therefore \quad S = \frac{1-x^n}{(1-x)^2} - n\frac{x^n}{1-x}.$$

*364. The following is the general case. *To find the sum of n terms of the series*

$$a,\ \{a+b\}r,\ \{a+2b\}r^2,\ \{a+3b\}r^3,\ \dots .$$

Let S denote the required sum,

$$\therefore \quad S = a + \{a+b\}r + \dots + \{a+(n-2)b\}r^{n-2} + \{a+(n-1)b\}r^{n-1}.$$

$$\therefore \quad Sr = ar + \{a+b\}r^2 + \dots + \{a+(n-2)b\}r^{n-1} + \{a+(n-1)b\}r^n.$$

Subtract, and simplify the result on the right-hand side,

$$\therefore \quad S - Sr = a + br + br^2 + \dots + br^{n-1} - \{a+(n-1)b\}r^n.$$

$$\therefore \quad S(1-r) = a + br\frac{1-r^{n-1}}{1-r} - \{a+(n-1)b\}r^n.$$

$$\therefore \quad S = \frac{a - \{a+(n-1)b\}r^n}{1-r} + br\frac{1-r^{n-1}}{(1-r)^2}.$$

*365. *Note.* In Art. 363, we formed Sx from S, and by subtraction obtained $S(1-x)$; thus we multiplied S by $(1-x)$, and reduced the question to summing a geometrical progression. In a similar way, if

$$S=1^2+2^2x+3^2x^2+\ldots+(n+1)^2x^2,$$

the multiplication of S by $(1-x)^2$ reduces the determination of S to the summation of a geometrical progression.

*MISCELLANEOUS EXAMPLES. XXIX. C.

[*The following examples are on arithmetical and geometrical progression, as well as on the subject-matter of this chapter.*]

1. Determine whether the following series are in arithmetical or in geometrical progression, and sum each of them to six terms.

(i) $18+15+12+\ldots$; (iv) $\frac{2}{3}+1+\frac{3}{2}+\ldots$;

(ii) $3+4\frac{1}{4}+5\frac{1}{2}+\ldots$; (v) $1\cdot48-2\cdot22+3\cdot33-\ldots$;

(iii) $12+4+1\frac{1}{3}+\ldots$; (vi) $1\cdot3-3\cdot1-7\cdot5-\ldots$.

2. Sum the following series to n terms:

(i) $(2n-1)+(2n-3)+(2n-5)+\ldots$;

(ii) $(x+a)+(x^3+3a)+(x^5+5a)+\ldots$;

(iii) $ab+2ab^2+3ab^3+\ldots$.

3. The first and eleventh term of a series are a^5 and a^{-5}. Find the sum of 11 terms of the series, (i) on the supposition that it is in A.P., (ii) on the supposition that it is in G.P.

4. Find the difference between the sums of the series

$$\frac{n}{n}+\frac{n-1}{n}+\frac{n-2}{n}+\ldots \text{ (to } 2n \text{ terms)},$$

and $$\frac{n}{n+1}+\frac{n}{(n+1)^2}+\frac{n}{(n+1)^3}+\ldots \text{ (to infinity)}.$$

5. The sum of 10 terms of an A.P. is 100, and the second term is zero. Find the first term.

6. Prove that, if a, b, c be in arithmetical progression and a, $b-a$, $c-a$ be in geometrical progression, then $a=\frac{1}{3}b=\frac{1}{5}c$.

7. If 1, x, y be in arithmetical progression, and 1, y, x be in geometrical progression, find x and y.

8. The arithmetic mean between two numbers is 25, and the geometric mean between them is 15. Find the numbers.

9. Find three numbers in the ratio of $6 : 11 : 20$, such that if each be increased by 1 they become in G. P.

10. Shew that in every geometric progression in which the common ratio is 5, the arithmetic mean between the 2^{nd} and 4^{th} terms is 13 times the 2^{nd} term.

11. If xy, y^2, z^2 be in arithmetical progression, shew that y, z, $2y - x$ are in geometrical progression.

12. If a, b, c be in G.P., and if x be the A.M. between a and b, and y the A.M. between b and c, prove that $\frac{a}{x} + \frac{c}{y} = 2$.

13. The first term of a geometrical series is a, the sum to infinity is pa. Find the n^{th} term.

14. The sum of an infinite geometrical progression is $4\frac{1}{6}$, and the sum of the first two terms is $2\frac{2}{3}$. Find the series.

15. The first terms of an arithmetical and of a geometrical progression are equal to 8, the second terms are equal, and the third term of the latter series exceeds the third term of the former by 2. Find the two arithmetical and geometrical progressions which satisfy these conditions.

16. Insert between 6 and 16 two numbers such that the first three terms of the series so formed may be in A. P. and the last three terms in G. P.

17. If x, y, a be in A.P., and x, y, b in G. P., shew that, whether the series proceed in ascending or descending order of magnitude (provided only that x and y are real positive quantities) b must be greater than a.

Find the values of x and y, if $a = 21$ and $b = 25$.

18. If a, b, c be in arithmetical progression, and A, G, be the arithmetic and geometric means between a and b, and A', G', be the arithmetic and geometric means between b and c, prove that

$$A^2 - G^2 = A'^2 - G'^2.$$

19. The sum of three numbers in G. P. is 14, and the sum of their reciprocals is $\frac{7}{8}$: find the numbers.

20. The sum of $3n$ terms of a G.P., whose first term is a and common ratio is r, is equal to the sum of n terms of another G.P. series, whose first term is b and common ratio is r^3. Prove that b is equal to the sum of the first three terms of the first G.P.

21. Shew that, if a, b, c be three numbers in G. P., and if p and p' be respectively the A. M. and H. M. between a and b, and q and q' the A. M. and H. M. between b and c, then p, p', q, q' will be proportionals.

22. Three numbers are in H. P.; if 4 be taken from each, they are in G. P.; if 1 be added to the middle one, they are in A. P.: find them.

***23.** An arithmetical, a geometrical, and a harmonical progression have the same first and second terms; and the third terms of the three series are x, y, z respectively. Shew that

$$\left(\frac{x}{z}-3\right)^2+4\left(\frac{x}{y}+\frac{y}{z}\right)=12.$$

24. The m^{th}, n^{th}, and p^{th} terms of an A. P. are in G. P. Shew that the common ratio is $(n-p)/(m-n)$.

25. If x, y, z be in arithmetical progression, and if the harmonic mean between x and z be to their geometric mean as 4 to 5, prove that $\frac{1}{2}x$, $\frac{1}{5}y$, $\frac{1}{8}z$ are in geometrical progression.

26. The sum of four numbers which are in arithmetical progression is 24, and the square of the geometric mean between the second and third of them exceeds by 8 the square of the geometric mean between the first and fourth. Find the numbers.

***27.** The first terms of an endless series of geometrical progressions, having the same common ratio f_1, which is less than 1, themselves form a geometrical progression with a ratio f_2, which is also less than 1. Shew that the sum of all the terms of all the progressions is $a/(1-f_1)(1-f_2)$, where a is the first term of the first progression.

***28.** Shew that, if a, b, c be three numbers in H. P., and n be any positive integer, then a^n+c^n is greater than $2b^n$.

***29.** Find the sum of $x+4x^2+\ldots+n^2x^n+\ldots$ (to infinity), where x is less than unity.

***30.** Find the sum of the first n terms of the series whose r^{th} term is $3r(r+2)(r+3)$, where r is any positive integer.

EXAMINATION PAPERS AND QUESTIONS.

[*The two following papers were set recently in the Previous Examination and the General Examination at Cambridge. These papers are followed by groups of questions on the subject-matter of the last few chapters.*]

Paper A.

1. Multiply $x^2+y^2+z^2+yz+zx-xy$ by $x+y-z$, and divide $(pq+rs)^2-(ps+qr)^2$ by $(p-r)(q-s)$.

2. Reduce to its lowest terms $\dfrac{12x^4+4x^3-23x^2-9x-9}{8x^4-14x^2-9}$.

3. Simplify the expressions:

(i) $\dfrac{3x^2-5}{x^3-1}-\dfrac{4x+5}{x^2+x+1}+\dfrac{1}{x-1}$;

(ii) $\left\{\dfrac{x}{x-1}-\dfrac{x+1}{x}\right\}\div\left\{\dfrac{x}{x-1}+\dfrac{x+1}{x}\right\}$.

4. Solve the equations:

(i) $\dfrac{3(6-5x)}{5}+\dfrac{63x}{50}=\dfrac{3x}{2}-\dfrac{36}{125}$;

(ii) $\dfrac{x^2-4x+5}{x^2+6x+10}-\left(\dfrac{x-2}{x+3}\right)^2=0$;

(iii) $\dfrac{x}{44}+\dfrac{y}{3}=20,\ \dfrac{x}{14}-\dfrac{y}{13}=19$.

5. Extract the square root of $16x^4-24x^3+25x^2-12x+4$.

6. A man can walk from A to B and back in a certain time at the rate of 4 miles an hour. If he walk at the rate of 3 miles an hour from A to B, and at the rate of 5 miles an hour from B to A, he requires 10 minutes longer for the double journey. What is the distance AB?

7. Prove that, when m and n are positive integers,

$$a^m \times a^n = a^{m+n}.$$

How is a meaning given to $a^{\frac{1}{2}}$?

8. Prove that, if $a : b = c : d$,

$$b(a+b-c-d) = (a+b)(b-d).$$

What number must be added to each of the numbers, 3, 5, 7, 10 that the sums may be in proportion?

9. Solve the quadratic equations:

(i) $2x^2 - 21x + 40 = 0$;

(ii) $\dfrac{1}{6x-5a} + \dfrac{5}{6x-a} = \dfrac{2}{a}$.

10. Sum the series:

(i) $1296 + 864 + 576 + \ldots$ to 7 terms, and to infinity;

(ii) $1296 + 1080 + 864 + \ldots$ to 12 terms.

Paper B.

1. Solve the equations:

(i) $(x-1)(x-2)(x-3) + (x-4)(x-5)(x-6)$
$= (x-2)(x-3)(x-4) + (x-3)(x-4)(x-5)$;

(ii) $\dfrac{3x+5}{2x+1} + \dfrac{x+3}{6} + \dfrac{x+1}{4} = \dfrac{x+3}{x+1} + \dfrac{5(x+3)}{12}$;

(iii) $\frac{1}{12}(x+y) = x+1,\ \frac{1}{6}(y-x) = 2x-1$.

2. If the difference between the roots of the equation $x^2 + (a+b)x + c^2 = 0$ is the same as that between the roots of $x^2 + cx + (a+b)c = 0$, prove that $a+b$ is equal either to c or to $-5c$.

3. Solve the equations:

(i) $\dfrac{x+4}{2x+3} + \dfrac{3x+10}{2x} = \dfrac{2x+3}{x-1}$;

(ii) $\sqrt{1-x} + \sqrt{2(1+x)} = \sqrt{6-2x}$;

(iii) $\begin{cases} x^2 + xy + y = 137, \\ y^2 + xy + x = 205. \end{cases}$

4. If $p : q : r :: a+2b+2c : b+2c+2a : c+2a+2b$, prove that $a : b : c :: 2q+2r-3p : 2r+2p-3q : 2p+2q-3r$.

5. If x vary as the sum of two quantities, one of which varies directly as y and the other inversely as y^2, and if $x=37$ when $y=1$, and $x=11$ when $y=2$, find the value of x when $y=3$.

6. Find two numbers whose sum is to their difference as $9 : 2$, and whose product exceeds the difference of their squares by 5.

7. A starts to walk from P to Q at 10 A.M.; B starts to walk from Q to P at 10.24 A.M. They meet 6 miles from Q. B stops 1 hour at P, and A stops 2 hrs. 54 min. at Q; returning they meet midway between P and Q at 6.54 P.M. Find the distance from P to Q.

8. The areas of two rectangles are as $9 : 10$; the greater side of the less : the less side of the greater as $3 : 2$; the diagonal of the less is equal to the greater side of the greater, and the difference of their diagonals is 2 feet. Find their sides.

9. Find the n^{th} term of a geometrical progression whose first two terms are a and b.

If a and b be the first two terms of an arithmetical progression and also of a geometrical progression, and if the ratio of the third term of the former to the third term of the latter be $5 : 9$, find the ratio of their sixth terms.

10. Sum the following series:

(i) $25+20+15+\ldots$ to 8 terms;

(ii) $25+20+16+\ldots$ to 8 terms;

(iii) $1\frac{1}{4} - 1 + \frac{4}{5} - \ldots$ to infinity.

11. Insert four arithmetical means between a and b.

If the square of the arithmetical mean between two quantities be increased by the square of half their difference, the sum is the arithmetical mean between the squares of the two quantities.

12. The difference of the first and second terms of a geometrical progression is 8, and the sum of the second and third terms is 12. Find the series.

Examination Questions.

1. Explain how meanings are assigned to a^0 and a^{-1}.

Divide $x^{\frac{3}{4}}y^{-\frac{3}{4}}+2+x^{-\frac{3}{4}}y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}y^{-\frac{1}{4}}-1+x^{-\frac{1}{4}}y^{\frac{1}{4}}$.

2. Prove that, if a, b, c, d be proportionals, then

$$\frac{a^3}{b}+\frac{b^3}{a}:\frac{c^2}{d}+\frac{d^2}{c}::ab:cd.$$

3. Find the sum of n terms of an arithmetical progression, having given the first term and the common difference.

Find the sums of the series:

(i) $16+24+32+\dots$ to 7 terms;

(ii) $16+24+36+\dots$ to 7 terms;

(iii) $36+24+16+\dots$ to infinity.

4. Find the G.M. of a and b.

The A.M. between two numbers is 1. Shew that their H.M. is is equal to the square of their G.M.

5. The velocity of a train varies directly as the square root of the quantity of coal used per mile and inversely as the number of carriages in the train; and the train is supposed to travel with uniform velocity. In a journey of 25 miles in half an hour with 18 carriages, 10 cwt. of coal are used. How much coal will be consumed in a journey of 15 miles in 20 minutes with 20 carriages?

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6. Prove that, if the equations $x^2+bx+ca=0$, $x^2+cx+ab=0$ have a common root, their other roots will satisfy the equation $x^2+ax+bc=0$.

7. State the index laws. Explain what is meant by $a^{-\frac{3}{2}}$ and $a^{\frac{2}{3}}$.

Simplify the expression $(x^{\frac{2}{3}}y^{\frac{3}{2}})^{-\frac{1}{2}}x^{-3}y^{4}$.

8. If $a : b = c : d$, shew that

$$3a - 2c : 3b - 2d = 2a - 3c : 2b - 3d.$$

If $\frac{x+z}{y+z} = \frac{by - ax}{cx - az} = \frac{az - cy}{ay - bx}$, then will each fraction be equal to y/x unless $b + c = 0$.

9. Find the n^{th} term of a geometrical progression whose first two terms are a and b.

The 5^{th} term of a geometrical progression is 64, and the 11^{th} term is 729. Find the 7^{th} term.

10. Two clerks were awarded pensions, the amount of which was proportional to the square root of the number of years they have served. One had served 9 years longer than the other, and received a pension greater by £50. If the length of service of the first had exceeded that of the second by $4\frac{1}{2}$ years only, their pensions would have been in the ratio $9 : 8$. What were the amounts of their respective pensions?

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11. Define a continued proportion.

If $a : b = b : c = c : d$, shew that $ad^2 = c^3$.

12. If x vary directly as y and inversely as z^2, and if $x = 1$ when $y = 2$ and $z = 3$, find the value of x when $y = 3$ and $z = 2$.

13. Find the sum of

(i) $16 + 19 + 22 + \dots$ to 20 terms;

(ii) $16 + 15 + 14 + \dots$ to 30 terms;

(iii) $16 - 12 + 9 + \dots$ to infinity.

14. Divide 76 into three parts in G. P. such that the sum of the first and third is to the second in the ratio of $13 : 6$.

15. A carrier charges $3d.$ each for all parcels not exceeding a certain weight; and on heavier parcels he makes an additional charge for every pound above that weight. The charge for a parcel of 14 lbs. is $1s.$, and the charge for a parcel of 12 lbs. is twice that for a parcel of 7 lbs. What is the scale of charges?

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16. Simplify $\left\{\frac{\sqrt{(x+a)}}{\sqrt{(x-a)}} - \frac{\sqrt{(x-a)}}{\sqrt{(x+a)}}\right\} \frac{\sqrt{(x^3-a^3)}}{\sqrt{(x^2+ax+a^2)}}$.

17. Shew that $\frac{3}{2}(\sqrt{3}+1)^2 - 2(\sqrt{2}-1)^2 = \sqrt{(59-24\sqrt{6})}$.

18. Find the sum of an infinite number of terms in G. P., the first term and common ratio being given. Determine the limits between which the ratio must lie in order that the formula may be true.

Find the sum of $1 - \cdot 1 + \cdot 01 - \ldots$ to six terms and to infinity.

The first term of a geometrical progression exceeds the second term by 2, and the sum to infinity is 50: find the series.

19. Find x, y, and z where a, x, b are in A. P.; a, y, b are in G. P.; and a, z, b are in H. P.

If $x-a$, $y-a$, $z-b$ be in G. P., find (in terms of a and y) the H. M. between $y-x$ and $y-z$.

20. If G be the G. M. of two quantities A and B, shew that the arithmetic and harmonic means of A and G and the arithmetic and harmonic means of G and B are in proportion.

CHAPTER XXX.

PERMUTATIONS AND COMBINATIONS.

366. **Permutations. Combinations.** The different groups which can be formed, each consisting of a certain definite number of things selected from a given collection of such things, are called *permutations* when the order in which the things are arranged in each group is taken into account; and are called *combinations* when the order in which the things are arranged in each group is not taken into account.

For example, suppose that we have three things, denoted by a, b, c respectively, and we form all possible groups of them taken two at a time. Then there are six permutations, namely, the groups ab, bc, ca, ba, cb, ac; the groups ab and ba being regarded as different, since the order of arrangement is taken into account. On the other hand, there are only three combinations, namely, the groups ab, bc, ca, since arrangements like ab and ba are the same combination.

367. **Notation.** The number of permutations of n things taken r at a time is commonly denoted either by the symbol ${}_nP_r$, or by the symbol ${}^nP_r$, or by the symbol $P(n, r)$. It follows from the definition that r cannot be greater than n.

The number of combinations of n things taken r at a time is commonly denoted either by the symbol ${}_nC_r$, or by the symbol ${}^nC_r$, or by the symbol $C(n, r)$. It follows from the definition that r cannot be greater than n.

Using this notation, the results of the illustration given in the last article would be written ${}_3P_2=6$, and ${}_3C_2=3$.

368. It is obvious from the definitions that ${}_nP_1=n$, and ${}_nC_1=n$, since n things can be taken one at a time in n separate ways.

369. **Number of permutations of n things taken r together.** *The number of permutations of n different things taken r at a time is given by the formula*

$$ {}_nP_r=n(n-1)\ldots\ldots(n-r+1). $$

Let us denote the things by the letters $a, b, c, \ldots$. Suppose that we knew all the different permutations of n things taken $r-1$ at a time; the number of these is denoted by the symbol ${}_nP_{r-1}$. Then, by prefixing to any one of these permutations any one of the $n-r+1$ letters which it does not contain, we obtain one of the permutations of n things taken r at a time. Repeating this process on each of the permutations in ${}_{n-1}P_r$, we obtain all the permutations in ${}_nP_r$. Now every permutation in ${}_nP_{r-1}$ gives rise to $n-r+1$ of the permutations in ${}_nP_r$,

$$ \therefore {}_nP_r=(n-r+1)\,{}_nP_{r-1}. $$

This relation is true for all values of r which are not greater than n. Therefore, writing for r successively $r-1, r-2, \ldots\ldots$, we have

$$ {}_nP_{r-1}=(n-r+2)\,{}_nP_{r-2}, $$
$$ {}_nP_{r-2}=(n-r+3)\,{}_nP_{r-3}, $$
$$ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots $$
$$ {}_nP_2=(n-1)\,{}_nP_1, $$

also $${}_nP_1=n.$$ [Art. 368.

The product of all the right-hand members of these equalities must be equal to the product of all the left-hand members. Cancelling the factors common to both sides, we have

$$ {}_nP_r=(n-r+1)(n-r+2)\ldots\ldots(n-1)n. $$

Reversing the order of the factors on the right-hand side, we have

$$ {}_nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1). $$

370. If in the result of the last article we put $r = n$, we obtain the theorem that the *number of permutations of n things taken all together is*

$$ n(n-1)(n-2)\ldots\ldots 2 \cdot 1. $$

371. **Factorials.** The product of the first n natural numbers (that is, of n consecutive integers beginning with unity and ending with n) is called *factorial n.*

Factorial n is denoted either by the symbol $\underline{|n}$, or by the symbol $n!$ In foreign books, it is sometimes denoted by the symbol $\Gamma(n+1)$.

372. The factorial notation enables us to express the results of Articles 369, 370 in a more concise form.

We have ${}_nP_r = n(n-1)\ldots(n-r+1)$.

Multiply and divide the right-hand side by $\underline{|n-r}$.

$$ \therefore\ {}_nP_r = \frac{\{n(n-1)\ldots(n-r+1)\}\{(n-r)(n-r-1)\ldots 2 \cdot 1\}}{(n-r)(n-r-1)\ldots 2 \cdot 1} $$

$$ = \frac{\underline{|n}}{\underline{|n-r}}. $$

Similarly, ${}_nP_n = n(n-1)\ldots 2 \cdot 1 = \underline{|n}$.

373. The product of r consecutive positive integers, of which n is the greatest, is denoted either by the symbol $[n]_r$, or by $(n)_r$, or by n_r. Thus, the result of Art. 369 would be written either as

$$ {}_nP_r = [n]_r, \text{ or as } {}_nP_r = (n)_r, \text{ or as } {}_nP_r = n_r. $$

374. *Note.* The method of proof given in Art. 369 is suggested by the form of the answer. For, if the result be true for all values of r less than n, then we have

$$ {}_nP_r = n(n-1)(n-2)\ldots(n-r+2)(n-r+1), $$

and $\quad {}_nP_{r-1} = n(n-1)(n-2)\ldots(n-r+2).$

Hence $$_nP_r=(n-r+1)\,_nP_{r-1},$$
which is the relation we commenced by proving.

We might, in a similar way, have proved the relation
$$_nP_r=n\,.\,_{n-1}P_{r-1},$$
which would have given us a different form of the proof.

Ex. 1. *How many different numbers can be formed by using* 3 *out of the* 9 *digits* 1, 2, 3, 4, 5, 6, 7, 8 *and* 9?

We have 9 different things, and we are to take them 3 at a time,

$$\therefore \text{ the required number} = {}_9P_3 = 9\,.\,8\,.\,7 = 504.$$

Ex. 2. *How many of the numbers formed as in Ex.* 1 *lie between* 300 *and* 400?

Each number contains three digits. Hence the digit in the hundreds' place must be a 3. There remain 8 different digits, and any pair of these can be put in the tens' and units' places.

$$\therefore \text{ the required number} = {}_8P_2 = 8\,.\,7 = 56.$$

Ex. 3. *Four men hire a four-oared rowing boat. In how many ways can they be arranged as a crew?*

They are all to row at the same time, ∴ we want the number of permutations of 4 men taken all together.

$$\therefore \text{ the required result} = {}_4P_4 = \underline{|4} = 4\,.\,3\,.\,2\,.\,1 = 24.$$

375. *To find the number of permutations of n things taken all together, when the things are not all different.*

Let the n things be denoted by the letters a, b, ... Suppose that p of them are alike, all being "a"s; q of them are alike, all being "b"s; r of them are alike, all being "c"s; and so on.

Let P be the required number of permutations.

Each of the permutations contains all the letters, and therefore contains p "a"s. If, in any permutation, the "a"s were replaced by p new letters (quite distinct both from one another and from the letters already used), then by changing the arrangement of these new letters amongst themselves, and keeping the other letters unaltered in position, we should get $\underline{|p}$ different permutations. If this were done to each of the P permutations, we should get altogether $P \times \underline{|p}$ different permutations.

Similarly, if in any one of these $P \times \underline{|p}$ permutations, all the "b"s were replaced by q new letters, then by permuting these new symbols amongst one another, and keeping the other letters unaltered in position, we should get $\underline{|q}$ different permutations. If this were done to each of the $P \times \underline{|p}$ permutations, we should get altogether $P \times \underline{|p} \times \underline{|q}$ different permutations.

Similarly, if we replaced all the letters of which any one was like any other by new symbols distinct both from one another and from all the others used, then we should get altogether $P \times \underline{|p} \times \underline{|q} \times \underline{|r} \ldots$ different permutations. But the case is now reduced to finding the number of permutations of n things, which are all different, taken all together; and the number of these is $\underline{|n}$. [Art. 370.]

$$\therefore P \, \underline{|p} \, \underline{|q} \, \underline{|r} \ldots = \underline{|n}.$$

$$\therefore P = \frac{\underline{|n}}{\underline{|p} \, \underline{|q} \, \underline{|r} \ldots}.$$

Ex. 1. *Find the number of permutations of the letters in the word* ALGEBRA *taken all together.*

Here there are 7 letters, of which 2 are alike; that is, $n=7$, $p=2$, in the above formula.

$$\therefore \text{ the required number} = \frac{\underline{|7}}{\underline{|2}} = 2520.$$

Ex. 2. *Find the number of permutations of the letters in the word* COMBINATION *taken all together.*

Here there are 11 letters, 2 of them are "o"s, 2 of them are "i"s, and 2 two of them are "n"s.

$$\therefore \text{ the required number} = \frac{\lfloor 11}{\lfloor 2 \lfloor 2 \lfloor 2} = 4989600.$$

EXAMPLES. XXX. A.

1. Write down the values of ${}_3P_2$; ${}_7P_5$; ${}_4P_4$; ${}_6P_1$.

2. If ${}_nP_2 = 110$, what is the numerical value of n?

3. In how many different ways can the letters of the word *woman* be arranged (i) taken all together, and (ii) taken three at a time?

4. How many different arrangements can be made of the letters of the following words, in each case all the letters of the word being used in every arrangement?

(i) *school;* (ii) *number;* (iii) *feeler;* (iv) *cricket.*

5. Find the number of permutations of the letters in the words *fiddle-de-dee* and *Mississippi*, in each case all the letters being used in every permutation.

6. If the number of permutations of 12 things taken r together be 42 times the number taken $r-2$ together, find r.

7. In how many ways can six boys stand in a line to receive an electric shock, two only being willing to stand at the extremities of the line?

8. With three consonants and three vowels, how many words of six letters can be formed, each word beginning with a consonant and ending with a vowel?

9. If the number of permutations which contain a particular thing be equal to the number which do not contain it, prove that n must be even, and r must be equal to $\frac{1}{2}n$.

376. Number of Combinations of n things taken r together. *The number of combinations of n different things taken r at a time is given by the formula*

$$ {}_nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}. $$

First Proof. Each of the combinations of n things taken r at a time contains r different things: and by arranging these r things in every possible order, it would give rise to $\underline{|r}$ permutations. Hence the number of permutations of n things taken r at a time is ${}_nC_r \times \underline{|r}$.

$$ \therefore {}_nC_r \times \underline{|r} = {}_nP_r. $$
$$ = n(n-1)\dots(n-r+1). \quad \text{[Art. 369.} $$
$$ \therefore {}_nC_r = \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}. $$

Second Proof. The above proof reduces the determination of ${}_nC_r$ to that of ${}_nP_r$. We now proceed to give an independent proof.

Let us denote the things by the letters $a, b, c, \dots$. Then those combinations of them (taken r at a time) in which a occurs can be obtained by first writing down all possible combinations of the $n-1$ letters $b, c, \dots$ taken $r-1$ at a time, and then prefixing a to them. Therefore the number of combinations in which a occurs will be ${}_{n-1}C_{r-1}$. Similarly, the number of combinations in which b occurs will be ${}_{n-1}C_{r-1}$: and so also, the number of combinations in which each of the other letters occurs will be ${}_{n-1}C_{r-1}$. Now, if we collect these n combinations of $n-1$ letters taken $r-1$ at a time together, the number of them will be $n \times {}_{n-1}C_{r-1}$; and in this collection we shall get all possible combinations of n things taken r together.

But in this collection, every combination will be repeated r times; because, if the combination happen to contain the r letters $hkl\ldots$, it will occur once with h in the first place, once with k in the first place, and so on; and since the order of arrangement in a combination is immaterial, these will all be the same combination. Hence the number of arrangements in this collection is also $r \times {}_nC_r$.

$$\therefore n \times {}_{n-1}C_{r-1} = r \times {}_nC_r.$$

$$\therefore {}_nC_r = \frac{n}{r} \times {}_{n-1}C_{r-1}.$$

This relation is true for all values of n and r (provided of course that n and r are positive integers, and that r is not greater than n). Therefore, writing $n-1$ for n, and $r-1$ for r, we have

$${}_{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}_{n-2}C_{r-2}.$$

Similarly, $${}_{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}_{n-3}C_{r-3}.$$

....

Finally, $${}_{n-r+2}C_2 = \frac{n-r+2}{2} \times {}_{n-r+1}C_1.$$

Also, $${}_{n-r+1}C_1 = n-r+1.$$ [Art. 368.

The product of all the right-hand members of these equalities must be equal to the product of all the left-hand members. Cancelling the factors common to both sides, we have

$$\begin{aligned} {}_nC_r &= \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \cdots \frac{n-r+2}{2} \cdot \frac{n-r+1}{1} \\ &= \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}. \end{aligned}$$

If we multiply both the numerator and the denominator by $\underline{|n-r}$, we have

$$_nC_r = \frac{n-(n-1)\ldots(n-r+1)}{\lfloor r} \cdot \frac{(n-r)(n-r-1)\ldots 2.1}{\lfloor n-r},$$

that is, $_nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$, which is sometimes a more convenient form for $_nC_r$ than that given in the enunciation of the proposition.

Ex. 1. *Find the number of different whist parties which can be made out of six players.*

We want the number of possible sets of 4 people which can be made out of 6 people.

$$\therefore \text{ the required number} = {_6C_4} = \frac{6.5.4.3}{1.2.3.4} = 15.$$

Ex. 2. *In how many ways may a cricket eleven, of whom two at least must be bowlers, be formed from* 14 *players, it being known that only* 4 *of the players can bowl?*

There are two groups of players, namely, 4 bowlers and 10 other players.

The eleven may contain only 2 bowlers, and 9 others selected out of the remaining 10 players. Now from the 4 bowlers we can select 2 in $_4C_2$ ways, that is, in 6 ways; and from the 10 non-bowlers we can select 9 in $_{10}C_9$ ways, that is, in 10 ways. Any of these 6 pairs of bowlers can be taken with any of the 10 possible sets of 9 non-bowlers which can be selected out of the group of non-bowlers. Thus, altogether there are

$$_4C_2 \times {_{10}C_9},$$

that is, 6×10 possible elevens, each of which would contain two bowlers only.

Similarly, the number of ways of forming an eleven which would contain 3 bowlers and 8 players selected out of the group of non-bowlers is

$$_4C_3 \times {_{10}C_8},$$

since the number of sets of 3 bowlers which can be formed out of the group of 4 bowlers is $_4C_3$, and the number of sets of 8 non-bowlers which can be formed out of the group of 10 non-bowlers is $_{10}C_8$, and any of these sets of 3 bowlers can be taken with any of these sets of 8 non-bowlers to make an eleven.

Lastly, the number of ways of forming an eleven which would contain 4 bowlers and 7 non-bowlers will be

$$_4C_4 \times {_{10}C_7}.$$

Hence, the total number of possible elevens is

$$ {}_4C_2 \times {}_{10}C_9 + {}_4C_3 \times {}_{10}C_8 + {}_4C_4 \times {}_{10}C_7, $$

that is,
$$ \frac{4.3}{1.2} \cdot \frac{10}{1} + \frac{4}{1} \cdot \frac{10.9}{1.2} + 1 \cdot \frac{10.9.8}{1.2.3}, $$

which is equal to $6.10 + 4.45 + 120 = 60 + 180 + 120 = 360$.

Thus, there are 360 possible elevens.

EXAMPLES. XXX. B.

1. Find the values of ${}_7C_6$; ${}_8C_4$; ${}_{47}C_{45}$.

2. If ${}_nC_2 = 21$, find n.

3. If ${}_nC_{n-2} = 10$, find n.

4. If ${}_{10}C_x = 45$, find x.

5. How many parties of 10 men can be formed from a company of 20 men?

6. How many parties of 10 men can be formed from a company of 20 men (i) so as always to include a particular man, (ii) so as always to exclude a particular man?

7. A boat's crew of 4 men has to be selected from 6 men, of whom two can only row on the stroke-side and two can only row on the bow-side. How many crews can be selected, no account being taken of the way in which the crew arrange themselves?

377. *The number of combinations of n things taken r together is equal to the number of combinations of n things taken n − r together.*

For we have

$$ {}_nC_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r}, \text{ and } {}_nC_{n-r} = \frac{\lfloor n}{\lfloor n-r \, \lfloor n-(n-r)} = \frac{\lfloor n}{\lfloor n-r \, \lfloor r}. $$

$$ \therefore {}_nC_r = {}_nC_{n-r}. $$

The result of this proposition is otherwise obvious; since for every combination of r things, which is formed out of the n given things, there must be left a collection of the remaining $n-r$ things, which constitutes a combination of the n things taken $n-r$ together.

378. *To shew that* ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

By Art. 376, we have ${}_nC_r + {}_nC_{r-1}$

$$= \frac{n(n-1)\dots(n-r+2)(n-r+1)}{1.2\dots(r-1)r} + \frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)}$$

$$= \frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)}\left\{\frac{n-r+1}{r}+1\right\}$$

$$= \frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)}\cdot\frac{n+1}{r}$$

$$= \frac{(n+1)n(n-1)\dots(n-r+2)}{\underline{|r}}$$

$$= {}_{n+1}C_r.$$

This result, like that of the last article, can be obtained by considering the meaning of the symbols. For suppose the $n+1$ things to be denoted by the letters a and n other letters $b, c, \dots$ Then the number of combinations of these $n+1$ things taken r at a time will be equal to the number of combinations of the n things $b, c, \dots$ taken r together *plus* the number of combinations of these n things taken $r-1$ together when a is prefixed to each of them; that is, ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

*379. The proposition given in the last article can be extended by repeated applications of it, thus:

$$\begin{aligned} {}_{n+2}C_r &= {}_{n+1}C_r + {}_{n+1}C_{r-1} \\ &= \{{}_nC_r + {}_nC_{r-1}\} + \{{}_nC_{r-1} + {}_nC_{r-2}\} \\ &= {}_nC_r + 2{}_nC_{r-1} + {}_nC_{r-2}. \end{aligned}$$

Similarly, ${}_{n+3}C_r = {}_nC_r + 3{}_nC_{r-1} + 3{}_nC_{r-2} + {}_nC_{r-3}$.

Proceeding in this way, we can prove the following theorem, by mathematical induction,

$${}_{n+m}C_r = {}_nC_r + m\,{}_nC_{r-1} + \frac{m(m-1)}{1.2}\,{}_nC_{r-2} + \dots + {}_nC_{r-m}.$$

The $(k+1)^{\text{th}}$ term is $\dfrac{m(m-1)\dots(m-k+1)}{\underline{|k}}\,{}_nC_{r-k}$; and all the terms (after the first) can be deduced from the $(k+1)^{\text{th}}$ term by putting k successively equal to 1, 2,...m.

This result can also be written in the form

$${}_{n+m}C_r = {}_nC_r + {}_mC_1 \cdot {}_nC_{r-1} + {}_mC_2 \cdot {}_nC_{r-2} + \ldots + {}_mC_k \cdot {}_nC_{r-k} + \ldots + {}_mC_r,$$

since, by Art. 376, $m = {}_mC_1$, $\frac{m(m-1)}{1 \cdot 2} = {}_mC_2$, &c.

It will be noticed that m, n, r are positive integers, and r must be less than m and less than n.

380. *To find what value of r makes ${}_nC_r$ greatest, n being a given number.*

We have $${}_nC_r = \frac{n(n-1)\ldots(n-r+2)(n-r+1)}{1 \cdot 2 \ldots (r-1)r},$$

and $${}_nC_{r-1} = \frac{n(n-1)\ldots(n-r+2)}{1 \cdot 2 \ldots (r-1)}.$$

$$\therefore {}_nC_r = \frac{n-r+1}{r} \times {}_nC_{r-1}.$$

Therefore we can get ${}_nC_r$ from ${}_nC_{r-1}$ by multiplying the latter by the fraction $\frac{n-r+1}{r}$. Hence, if this fraction be greater than unity, ${}_nC_r$ will be greater than ${}_nC_{r-1}$.

$$\therefore {}_nC_r > {}_nC_{r-1},$$

so long as $n - r + 1 > r$,

that is, so long as $n + 1 > 2r$,

that is, so long as $r < \frac{1}{2}(n+1)$.

If n be even, ${}_nC_r$ will be greatest when $r = \frac{1}{2}n$. If n be odd, then $\frac{1}{2}(n+1)$ is an integer, and if $r = \frac{1}{2}(n+1)$ the fraction $\frac{n-r+1}{r}$ is equal to unity, and therefore this value of r makes ${}_nC_r = {}_nC_{r-1}$; hence, if n be odd then when $r = \frac{1}{2}(n+1)$ or $r = \frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1)$, the number of combinations of n things taken r at a time is greater than for any other value of r.

381. *The product of r consecutive positive integers is divisible by* $\lfloor r$.

The number of combinations of n things taken r together is necessarily an integer. Hence, we have the theorem that

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}$$

is an integer, that is, the product of r consecutive positive integers is always divisible by $\lfloor r$.

Another proof will be given later [Art. 437, p. 455, Ex. 5.]

MISCELLANEOUS EXAMPLES. XXX. C.

1. How many of the permutations of the six letters a, b, c, d, e, f taken four at a time contain a, and how many begin with a?

2. How many of the permutations of the six letters a, b, c, d, e, f taken four at a time contain (i) the letters ab in that order, (ii) the letters a and b next to one another?

3. Prove that the number of permutations of $2n$ things taken n together is $2^n \,.\, 1 \,.\, 3 \,.\, 5 \dots (2n-1)$.

4. Out of 5 consonants and 3 vowels, how many words can be formed, each consisting of 3 consonants and 2 vowels?

5. Out of 30 oranges at a halfpenny each, how many selections can be made in buying a shilling's worth?

6. Two landing parties of 50 men each are to be formed. The only men available are four English officers, four English non-commissioned officers, and 100 native troops. Each party is to contain two officers and two non-commissioned officers. How many different possible parties can be made up?

7. How many different numbers can be formed out of the digits 1, 2, 2, 3, 3, 3, 4, all the digits being used in each number?

8. Shew that the whole number of permutations of the letters of the alphabet, when each may occur once, twice, or thrice at a time, is 17576.

9. How many words, each of seven letters, can be formed from three vowels and four consonants, such that no two consonants are next to one another?

10. If the number of permutations of n things 3 together be equal to 6 times the number of combinations of them 4 together, find the value of n.

11. If ${}_{n-1}C_r : {}_nC_r : {}_{n+1}C_r = 6 : 9 : 13$, find n and r.

12. What value should (on the usual conventions as to the meanings of algebraic symbols) be assigned to ${}_nC_0$?

***13.** Shew that ${}_nC_p = \sum_{r=0}^{r=p} \{{}_mC_r \times {}_{n-m}C_{p-r}\}$.

14. There are n railway signal posts at a junction, and the arm of each signal can be moved into three positions: how many different signals can be made?

15. In a railway carriage, holding 5 people on each side, there are 10 people, 4 of whom refuse to travel with their backs to the engine, and 3 of whom insist on doing so. In how many ways can they be arranged?

16. If the ratio of the number of combinations of $2n$ things taken $n-1$ at a time to the number of combinations of $2(n-1)$ things taken n at a time be $132 : 35$, find n.

17. If the number of combinations of n things taken r together be equal to the number of combinations of n things taken $2r$ together, and if the number of combinations of n things taken $r+1$ together be equal to $\frac{11}{3}$ times the number of combinations of n things taken $r-1$ together, find n and r.

18. In how many ways may a man vote at an election where every voter gives six votes which he may distribute as he pleases amongst three candidates?

19. In how many ways can eight counters be arranged in four groups, each group containing two counters?

20. In the list of those who passed a certain examination, 21 candidates were placed in the first class, 27 in the second, and 17 in the third. In how many ways could a staff of 6 masters be selected containing 2 from each class?

21. How many trios can be formed by taking 1 girl and 2 boys from n girls and $2n$ boys; and in how many ways can they be seated at n tables so that no two boys sit next each other?

***22.** In how many ways can two bishops be placed upon a chess-board (i) upon squares of the same colour; (ii) upon squares of different colours?

***23.** Find the sum of all the integral numbers consisting of 5 figures, which can be formed by the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, no digit being used more than once in any number.

24. A boat-club consists of 15 members. Find in how many ways a crew of 9 can be chosen, (i) so as always to include a particular man, (ii) so as always to exclude him.

25. There are 20 stations on a railway, find the number of kinds of tickets required so that a person may travel from any one station to any other.

26. Find how many different sums can be made with the following coins: a penny, a sixpence, a shilling, a half-crown, a crown, and a sovereign.

***27.** In a college of 100 men there are 4 coxswains; shew that there are $\frac{\underline{|4}\ \underline{|96}}{\underline{|64}\,(\underline{|8})^4}$ ways in which four crews, each consisting of a coxswain and 8 men, may be chosen from the members of the college.

28. How many changes can be rung with five out of eight bells, and how many with the whole peal?

29. In how many ways can a whist table be made up out of six married couples and four single persons, so as not to include any husband and wife together?

30. In how many ways can 5 people be arranged at a round table?

***31.** A party of 4 ladies and 4 gentlemen contains two married couples. How many different arrangements of the party can be made at a round table, every lady sitting between two gentlemen, neither of whom is her husband?

***32.** In how many ways can a pack of 52 cards be divided into 4 sets, each containing 13 cards?

33. With 1 red, 1 white, 1 blue, and 3 black balls, in how many ways can 4 of them be arranged in a row?

34. I have four black balls (exactly alike), and also one red, one white, one green, and one blue ball. In how many ways can I make up a row of four balls, no two rows being alike?

***35.** A polygon is formed by joining n points in a plane. Find the number of straight lines, not sides of the polygon, which can be drawn joining any two angular points.

***36.** On each of n given straight lines m points are taken. No other straight line can be drawn through any three of these mn points. How many triangles can be formed by joining the points?

***37.** Find the number of combinations of the letters in the word *annunciation* taken 4 at a time.

CHAPTER XXXI.

THE BINOMIAL THEOREM.

382. WE have found by actual multiplication (see pp. 38, 42), expressions for the square and the cube of a binomial, such as $a+b$. The object of the Binomial Theorem is to find an expression for the n^{th} power of a binomial.

We shall first confine ourselves to the case where n is a positive integer, and shall then discuss the extension to cases where n is either a positive fraction or any negative quantity.

THE BINOMIAL THEOREM FOR A POSITIVE INTEGRAL EXPONENT.

383. *To shew that, if n be a positive integer, then*

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \dots$$

$$+ \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}a^{n-r}b^r + \dots + nab^{n-1} + b^n.$$

It is evident that this result includes such results as we have already proved. If, for example, we put $n=2$, we obtain

$$(a+b)^2 = a^2 + 2ab + b^2.$$

If we put $n=3$, we obtain

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The term $\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}a^{n-r}b^r$ is the $(r+1)^{\text{th}}$ term. It is called the *general term*, because any term may be obtained from it by giving r the proper value—the second term corresponding to $r=1$, the third term to $r=2$, the fourth term to $r=3$, and so on.

The result that we want to prove may be written in the form

$$(a+b)^n = a^n + {}_nC_1 . a^{n-1}b + {}_nC_2 . a^{n-2}b^2 + \dots + {}_nC_r . a^{n-r}b^r + \dots + b^n,$$

which is more convenient for our purpose.

384. *First Proof (Positive Integral Exponent).* Every term in the product of n factors, each equal to $a+b$, must contain one letter taken from every factor, hence it is of n dimensions. The terms of the product consist therefore of certain multiples of a^n, $a^{n-1}b$, ..., b^n.

The term involving a^n arises from the product of an "a" taken from each factor. Hence, a^n is a term in the product.

The terms involving $a^{n-1}b$ arise from the product of a "b" taken from one factor and $n-1$ "a" s, one being taken from each of the remaining $n-1$ factors. But the number of ways in which a "b" can be selected from n factors is the number of combinations of n things taken one at a time, that is, n. Therefore there will be altogether n terms in the product, each equal to $a^{n-1}b$. Hence, $na^{n-1}b$ will be a term in the product.

A similar argument applies to every term in the product. The terms involving $a^{n-r}b^r$ arise from the product of r "b"s (one being taken from each of r of the factors) and of $n-r$ "a" s (one being taken from each of the remaining factors). But the number of ways in which r "b" s can be selected from n factors is ${}_nC_r$. Hence, ${}_nC_r . a^{n-r}b^r$ is a term in the product.

Hence, the expansion of $(a+b)^n$ is

$$(a+b)^n = a^n + {}_nC_1 . a^{n-1}b + \dots + {}_nC_r . a^{n-r}b^2 + \dots + b^n.$$

385. *Second Proof (Positive Integral Exponent).* The proof of the binomial theorem given above is the one which would naturally occur to any one investigating the subject for the first time. But if the result be known or guessed, then the following proof, by *mathematical induction,* is shorter and easier.

Let us assume that the theorem is true for n factors, that is,

$$(a+b)^n = a^n + {}_nC_1 . a^{n-1}b + \ldots + {}_nC_r . a^{n-r}b^r + \ldots + {}_nC_1 . ab^{n-1} + b^n \ldots \text{(i)}.$$

Multiply both sides by $a+b$,

$$\begin{aligned} \therefore\ & (a+b)^{n+1} \\ &= a^{n+1} + {}_nC_1 . a^n b + {}_nC_2 . a^{n-1}b^2 + \ldots + {}_nC_r . a^{n-r}b^r + \ldots + ab^n \\ &\quad + a^n b + {}_nC_1 . a^{n-1}b^2 + {}_nC_2 . a^{n-2}b^3 + \ldots + {}_nC_r . a^{n-r-1}b^{r+1} + \ldots + b^{n+1} \\ &= a^{n+1} + ({}_nC_1 + 1)\, a^n b + ({}_nC_2 + {}_nC_1)\, a^{n-1}b^2 + \ldots \\ &\qquad + ({}_nC_r + {}_nC_{r-1})\, a^{n-r+1}b^r + \ldots + (1 + {}_nC_1)\, ab^n + b^{n+1}. \end{aligned}$$

But $\quad {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r;$ [Art. 378

$$\begin{aligned} \therefore\ (a+b)^{n+1} &= a^{n+1} + {}_{n+1}C_1 . a^n b + {}_{n+2}C_2 . a^{n-1}b^2 + \ldots \\ &\quad + {}_{n+1}C_r . a^{n+1-r}b^r + \ldots + {}_{n+1}C_1 . ab^n + b^{n+1} \quad \ldots\ldots\text{(ii)}. \end{aligned}$$

Now the formula (ii) is exactly the same as the formula (i), except that $n+1$ is written for n wherever it appears in (i). Hence, if the formula (i) be true for the exponent n, then (ii) shews that it is also true when the exponent is $n+1$.

Now (i) is obviously true when $n=1$, therefore it must be true when $n=2$. But since it is true when $n=2$, therefore it must be true when $n=3$. Continuing this line of reasoning, we see that it is true for any positive integral value of n.

386. If, in the formula given in Art. 383, we put $a=1$, $b=x$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 . 2}x^2 + \ldots$$
$$\ldots + \frac{n(n-1)\ldots(n-r+1)}{1 . 2 . \ldots r}x^r + \ldots + x^n.$$

This result includes the expansion of $(a+b)^n$, for

$$(a+b)^n = a^n \left\{1+\frac{b}{a}\right\}^n$$

$$= a^n \left\{1 + n\frac{b}{a} + \frac{n(n-1)}{1 . 2}\frac{b^2}{a^2} + \dots\right\}$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1 . 2}a^{n-2}b^2 + \dots$$

Since the expansion of $(1+x)^n$ given above includes the expansion of $(a+b)^n$, it is often taken as the standard form.

If we change the sign of x in the expansion of $(1+x)^n$, we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1 . 2}x^2 + \dots + (-x)^n,$$

the last term being $+x^n$ or $-x^n$, according as n is even or odd.

Ex. 1. *Write down the expansion of* $(a+b)^5$.

By the formula in Art. 383, we have

$$(a+b)^5 = a^5 + 5a^4b + \frac{5 . 4}{1 . 2}a^3b^2 + \frac{5 . 4 . 3}{1 . 2 . 3}a^2b^3 + \frac{5 . 4 . 3 . 2}{1 . 2 . 3 . 4}ab^4 + b^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Ex. 2. *Write down the expansion of* $(3x-2y)^4$.

In the general formula, Art. 383, put $3x$ for a and $-2y$ for b,

$$\therefore\ (3x-2y)^4 = (3x)^4 + 4(3x)^3(-2y) + \frac{4 . 3}{1 . 2}(3x)^2(-2y)^2 + \frac{4 . 3 . 2}{1 . 2 . 3}(3x)(-2y)^3 + (-2y)^4$$

$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.$$

Ex. 3. *Find the coefficient of* x^7 *in the expansion of* $(1-2x)^9$.

The term involving x^r in the expansion of $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}x^r.$$

$\therefore$ the term involving x^7 in the expansion of $(1-2x)^9$ is

$$\frac{9 . 8 . 7 \dots 3}{1 . 2 . 3 \dots 7}(-2x)^7.$$

Hence, the coefficient of $x^7 = \frac{9 \cdot 8 \cdot 7 \ldots 3}{1 \cdot 2 \cdot 3 \ldots 7}(-2)^7$

$$= -\frac{9 \cdot 8}{1 \cdot 2} 2^7$$

$$= -4608.$$

Ex. 4. *Find the* 12th *term in the expansion of* $(x - \frac{1}{2}y)^{13}$.

The $(r+1)^{\text{th}}$ term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)\ldots(n-r+1)}{\lfloor r} a^{n-r} b^r.$$

$$\therefore \text{ the required term} = \frac{13 \cdot 12 \ldots 3}{1 \cdot 2 \ldots 11} x^2 (-\tfrac{1}{2}y)^{11}$$

$$= -\frac{13 \cdot 12}{1 \cdot 2} (\tfrac{1}{2})^{11} x^2 y^{11}$$

$$= -\tfrac{39}{1024} x^2 y^{11}.$$

387. It is usual to add the following additional propositions connected with the expansion of a binomial when raised to a positive integral power.

388. *To find the greatest coefficient in the expansion of* $(1 + x)^n$.

The coefficient of the $(r+1)^{\text{th}}$ term is ${}_nC_r$.

If n be even, the greatest value of this is when $r = \frac{1}{2}n$ [Art. 380]. Hence, in this case, the coefficient of the $(\frac{1}{2}n+1)^{\text{th}}$ term is greater than that of any others.

If n be odd, the values of ${}_nC_r$ and ${}_nC_{r-1}$ are equal when $r = \frac{1}{2}(n+1)$, and these values are greater than for any other value of r. Hence, in this case, the coefficients of the $\frac{1}{2}(n+1)^{\text{th}}$ and $\frac{1}{2}(n+3)^{\text{th}}$ terms are equal, and either is greater than the coefficient of any other term.

389. *To find the greatest term in the expansion of* $(1 + x)^n$.

The $(r+1)^{\text{th}}$ term is ${}_nC_r \cdot x^r$. The r^{th} term is ${}_nC_{r-1} \cdot x^{r-1}$. Therefore we can obtain the $(r+1)^{\text{th}}$ term from the r^{th} term by multiplying the latter by the fraction

$$\frac{{}_nC_r}{{}_nC_{r-1}}\,x = \frac{n-r+1}{r}\,x = \left(\frac{n+1}{r} - 1\right)x.$$

This multiplier decreases as r increases, but it is greater than unity so long as $\frac{n-r+1}{r}\,x > 1$,

that is, so long as $\qquad (n-r+1)\,x > r,$

that is, so long as $\qquad (n+1)\,x > r + rx,$

that is, so long as $\qquad r < \frac{(n+1)\,x}{1+x}.$

If this be not an integer, and if we take r equal to the integer next below it; then the $(r+1)^{\text{th}}$ term is the greatest. But if $\frac{(n+1)\,x}{1+x}$ be an integer, then for this value of r the fraction above written is equal to unity, and the r^{th} and $(r+1)^{\text{th}}$ terms are equal, and each of them is greater than any other term.

390. The expansion of $(1+x)^n$ is sometimes written as

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + c_nx^n,$$

and we shall use this notation in enunciating the following propositions, Arts. 391—395.

391. *The coefficients of the terms in the expansion of $(1+x)^n$ which are equidistant from the two ends are equal.*

This proposition follows from the fact that

$$c_0 = c_n = 1,$$

$$c_1 = c_{n-1} = n,$$

and $\qquad c_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r} = c_{n-r}.$ [Art. 377.

Note. This is equally true of the expansion of $(a+b)^n$, provided that the numerical coefficients of a and b are equal; but if a and b have unequal numerical coefficients, it may not be true. Thus, in the expansion of $(1-2x)^5$, the coefficients of powers of x which are equidistant from the two ends are not equal.

392. *The sum of the coefficients of all the terms in the expansion of $(1+x)^n$ is 2^n.*

For $\quad (1+x)^n = c_0 + c_1 x + \ldots + c_r x^r + \ldots + c_n x^n.$

Put $x=1$, $\therefore\ 2^n = c_0 + c_1 + \ldots + c_r + \ldots + c_n.$

393. *The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms.*

For $\quad (1+x)^n = c_0 + c_1 x + \ldots + c_r x^r + \ldots + c_n x^n.$

Put $x=-1$ $\therefore (1-1)^n = c_0 - c_1 + c_2 - c_3 + \ldots.$

$$\therefore\ 0 = (c_0 + c_2 + c_4 + \ldots) - (c_1 + c_3 + \ldots).$$

394. The coefficients of x in the expansion of $(1+x)^n$ are the various combinations of n things taken different numbers together. Hence, numerous theorems connected with the combinations of things can be proved by the binomial theorem, and such proofs are often more simple than those suggested by the last chapter. Thus, the theorems just proved may all be read as theorems in combinations. For example, the result of Art. 392 is equivalent to a statement that the sum of all possible combinations of n things taken any number at a time is 2^n-1, for the required sum $= c_1 + c_2 + \ldots + c_n$, and $c_0 = 1$. [See p. 440, Ex. 7.]

395. Numerous theorems concerning the binomial coefficients can be obtained, either directly by making use of their known values, or in some cases by using the binomial expansion.

Example. *Find the value of $c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2$.*

We have $\quad (1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n,$

also $\quad \left(1+\frac{1}{x}\right)^n = c_0 + c_1 \frac{1}{x} + c_2 \frac{1}{x^2} + \ldots + c_n \frac{1}{x^n}.$

Multiplying the two series together, we see that the quantity whose value we want consists of the constant term in the product,

$$\therefore\ c_0^2+c_1^2+c_2^2+\dots+c_n^2 = \text{constant term in } (1+x)^n\left(1+\frac{1}{x}\right)^n$$
$$= \quad " \quad " \quad \frac{(1+x)^n(1+x^n)}{x^n}$$
$$= \text{coefficient of } x^n \text{ in } (1+x)^n\times(1+x)^n$$
$$= \quad " \quad " \quad (1+x)^{2n}$$
$$= \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

396. **Multinomial Theorem.** We can expand the n^{th} power of a multinomial by repeated applications of the binomial theorem.

The process will be sufficiently illustrated by considering the expansion of the n^{th} power of a trinomial, such as $a+b+c$.

$$\{a+b+c\}^n = \{a+(b+c)\}^n$$
$$= a^n + na^{n-1}(b+c) + \dots + \frac{\lfloor n}{\lfloor r \lfloor n-r} a^{n-r}(b+c)^r + \dots + (b+c)^n.$$

Now every power of the binomial $(b+c)$ which occurs on the right-hand side can be expanded by the binomial theorem. If, for example, we want the coefficient of $a^{n-r}b^{r-p}c^p$, we see that it must arise from the term in the original expansion which involves a^{n-r}. The coefficient of $b^{r-p}c^p$ in the expansion of $(b+c)^r$ is $\dfrac{\lfloor r}{\lfloor p \lfloor r-p}$. Hence the coefficient of $a^{n-r}b^{r-p}c^p$ is

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor r}{\lfloor p \lfloor r-p}, \text{ that is, } \frac{\lfloor n}{\lfloor n-r \lfloor r-p \lfloor p}.$$

Hence the coefficient of $a^pb^qc^r$ (where $p+q+r=n$) is

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

397. The expansion of an expression like $(a+b+c+d+\dots)^n$ can be formed in a similar manner; but the process is tedious, and it is preferable (if possible) to express the multinomial $a+b+c+d+\dots$ in some simple form, which frequently enables us to reduce the expansion to that of a binomial.

EXAMPLES. XXXI. A.

Write down the expansion of the following binomials.

1. $(1+x)^8$. 2. $(x-2)^7$. 3. $(x-y)^6$.
4. $(2a+b)^{11}$. 5. $(3x-2y)^5$. 6. $(2a+\frac{1}{3}b)^4$.
7. $(\frac{1}{2}x+\frac{1}{3}y)^6$. 8. $(1-\frac{1}{2}a^2)^7$. 9. $(x^2-\frac{1}{3})^5$.

10. Find the coefficient of x in the expansion of $\left(x^2+\frac{a^3}{x}\right)^5$.

11. Write down the value of $(x+1)^8+(x-1)^8$.

12. Write down the 6th term in the expansion of $(x-x^2)^{25}$.

13. Find the coefficient of x^4 in the expansion of $(x+2)^5$.

14. Find the coefficient of y^3 in the expansion of $(2x-3y)^7$.

15. Find the coefficient of a^8b^3 in the expansion of $(\frac{1}{2}a-\frac{2}{3}b)^{11}$.

16. Write out the expansion of $(2a-x)^7$ in full.

17. Find the middle term in the expansion of $(1+x)^{2n}$.

18. Find the 7th term of $(a-2x)^{12}$.

19. What is the coefficient of x^6 in $(3x-\frac{1}{3})^9$?

20. Find the term involving x^{10} in the expansion of $(ax-3x^2)^7$.

Find the greatest coefficients in the expansions of

21. $(1+x)^4$. 22. $(1-x)^{21}$. 23. $(x+a)^{11}$. 24. $(1+2x)^9$.

25. Find the greatest terms in the expansions of

(i) $(1+x)^9$, when $x=10$;

(ii) $(x+1)^{12}$, when $x=4$;

(iii) $(x+2)^{17}$, when $x=3$.

26. Find the greatest coefficient in the expansion of

$$(1+x)^{10}(1-x).$$

27. Employ the binomial theorem to find the values of $(101)^7$, and of $(99)^5$.

28. The third, fourth, and fifth terms of a binomial series are 20412, 22680, 15120; find the expansion.

*29. If n be a positive integer, find the value of

$$(1+x)^{3n}-3nx(1+x)^{3n-2}+\frac{3n(3n-3)}{1.2}x^2(1+x)^{3n-4}$$
$$-\frac{3n(3n-3)(3n-6)}{1.2.3}x^3(1+x)^{3n-6}+\ldots$$

30. Prove that

$$(1-x)^n=(1+x)^n-2nx(1+x)^{n-1}+\frac{2n(2n-2)}{\lfloor 2}x^2(1+x)^{n-2}-\ldots$$

31. Write down the first four terms in the expansion of $(1+2x+x^2)^7$ in powers of x.

32. Find the expansion of $(1+x-x^2)^4$ in powers of x.

33. Find the coefficient of $a^2b^2c^2$ in $(a+b+c)^6$.

34. Find the middle term of the expansion of $(1-2x+x^2)^n$.

35. What is the sum of all the coefficients in the expansion of $(2x-1)^n$?

36. Find the sum of the numerical coefficients of $(3a-2x)^5$.

37. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$, find the value of

(i) $c_0 - c_1 + c_2 - \ldots + (-1)^n c_n$;

(ii) $c_0c_n + c_1c_{n-1} + \ldots + c_nc_0$.

38. If n be a positive integer, prove that the middle coefficient of $(1+x)^{2n}$ is equal to the sum of the squares of the coefficients of $(1+x)^n$.

***39.** If $c_0, c_1, c_2, \ldots$ be the coefficients of the powers of x in the expansion of $(1+x)^n$; and if $C_0, C_1, C_2, \ldots$ be the coefficients of the powers of x in the expansion of $(1+x)^{2n}$; prove that

$$c_0^2 + c_1^2 + c_2^2 + \ldots = (-1)^n (C_0^2 - C_1^2 + C_2^2 \ldots).$$

***40.** Prove that the sum of the squares of the first, third, &c., coefficients in the expansion of $(a+b)^{2n}$ differs from the sum of the squares of the second, fourth, &c., by $\dfrac{2n(2n-1)\ldots(n+1)}{1 \,.\, 2 \ldots n}$.

***41.** Shew that, if p_k be the coefficient of x^k in the expansion of $(1+x)^n$, then

$$p_1p_2\ldots p_n(n+1)^n = (p_0+p_1)(p_1+p_2)\ldots(p_{n-1}+p_n)\,\lfloor n.$$

***42.** Employ the identity $(1+2x+x^2)^n = (1+x)^{2n}$, to prove that

$$2^n + \frac{n(n-1)}{1^2} 2^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1^2 \,.\, 2^2} 2^{n-4} + \ldots = \frac{\lfloor 2n}{\lfloor n \lfloor n},$$

n being a positive integer.

THE BINOMIAL THEOREM FOR FRACTIONAL AND NEGATIVE EXPONENTS.

398. We have already shewn in Chapter XXII. that a definite meaning can be assigned to $(1+x)^n$ when n is fractional or negative. We now proceed to discuss the possibility of finding an expansion for such an expression in positive powers of x, and we shall find that, *if x be numerically less than unity, then*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \,.\, 2} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \,.\, 2 \ldots r} x^r + \ldots$$

Note. The coefficients of x in the expansion of $(1+x)^n$, namely, the quantities

$$1, \quad n, \quad \frac{n(n-1)}{1.2}, \quad \frac{n(n-1)(n-2)}{1.2.3}, \quad \ldots$$

are known as the **binomial coefficients.**

399. The formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \text{(i)},$$

has been shewn [Art. 386] to be true if n be a positive integer. That, under certain limitations, it is true, whatever be the value of n, is rendered probable by the fact that when we test the result in a few simple cases we find that under certain conditions it is true.

If we put $n=-1$, the right-hand side reduces to $1+x+x^2+\ldots$ while the left-hand side becomes $(1-x)^{-1}$, which is equivalent to $\frac{1}{1-x}$. But, if we divide 1 by $1-x$, we find that the successive terms in the quotient are $1+x+x^2+\ldots$, which agrees with (i).

This illustration is particularly valuable because it leads to a series which we have already discussed at length [Art. 339]; and we have shewn that the series can only be represented by the expression $\frac{1}{1-x}$, when x is numerically less than unity. Thus, the expansion of $(1-x)^{-1}$ in a series in positive integral powers of x by means of the formula (i) is only permissible provided x is numerically less than unity.

400. In a similar manner, if we put $n=\frac{1}{2}$, the left-hand side of (i) is equivalent to $\sqrt{1+x}$. If now we proceed to extract the square root of $1+x$ by the process given in Art. 196, we shall find (however far we go) that the successive terms in the square root are exactly the terms of the right-hand side of (i). But, if x be numerically greater than unity, the remainder (which represents the difference between the root so far as we have extracted it and the given expression $\sqrt{1+x}$) becomes larger and larger the further we proceed, and thus, the further we go, the less is the series on the right-hand side of (i) an accurate representation of the expression on the left-hand side. On the other hand, if x be numerically less than unity, the remainder becomes smaller and smaller, and the further we go, the more accurately does the series on the right-hand side of (i) represent the expression on the left-hand side.

It can be proved that, if n be not a positive integer, then a limitation similar to that given in this and the last article is always necessary; and if n be not a positive integer, then the formula (i) is only true when x is numerically less than unity.

*401. A rigorous proof of the binomial theorem for any value of the exponent cannot be given without introducing considerations beyond the limits of this book, but the following sketch of one method of establishing it, which was suggested by Euler, is worthy of careful study.

Let $f(n)$ stand for the series on the right-hand side of (i),

$$\therefore f(n) = 1 + nx + \frac{n(n-1)}{1 . 2}x^2 + \ldots\ldots$$

Then if n be a positive integer, we know [Art. 386] that $f(n) = (1+x)^n$; also, we have $f(0) = 1$.

Writing m for n, we have

$$f(m) = 1 + mx + \frac{m(m-1)}{1 . 2}x^2 + \ldots\ldots$$

Similarly,

$$f(m+n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 . 2}x^2 + \ldots$$

Now Vandermonde proved that the coefficient of x^r in the product of $f(m)$ and $f(n)$ is equal to the coefficient of x^r in $f(m+n)$; and Cauchy proved that, in that case, the equation $f(m) \times f(n) = f(m+n)$ is true for all values of m and n, provided the series represented by these symbols are convergent. Lastly, Cauchy proved that these series are convergent, provided x is numerically less than unity. We shall not here *prove* these results, but the student will see that if they be granted it follows from them that, if x be less than unity,

$$f(m) \times f(n) = f(m+n).$$

[Euler arrived at this result in another way, which is an illustration of the *principle of the permanence of equivalent forms*, and is an extension of the method used in Arts. 40, 55 to obtain meanings to be assigned to the sum, difference, and product of negative quantities.

First, he shewed that the equation $f(m) \times f(n) = f(m+n)$ was true when m and n are any positive integers, since then it was equivalent to the equation $(1+x)^m \times (1+x)^n = (1+x)^{m+n}$.

Next, he asserted that the product of the series $f(m)$ and the series $f(n)$ must be of the same form, whatever be the values of m and n. (This however is only true of series containing an infinite number of terms if they be convergent; that is, in this case, if $x<1$.)

Hence, he concluded that, since $f(m) \times f(n) = f(m+n)$ is true when m and n are positive integers, therefore this equation is true for all values of m and n. If these series be convergent, that is, if x be less than unity, this argument is valid.]

Thus, if x be less than unity, we have

$$f(m) \times f(n) = f(m+n).$$

$$\therefore f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p).$$

Continuing this process, we obtain the equation

$$f(m) \times f(n) \times f(p) \ldots = f(m+n+p+\ldots).$$

In this result, $m, n, p \ldots$ are any numbers whatever.

Let $m = n = p = \ldots = \frac{r}{s}$, where r and s are positive integers, and take s factors,

$$\therefore \left\{f\left(\frac{r}{s}\right)\right\}^s = f\left(\frac{r}{s} + \frac{r}{s} + \ldots s \text{ terms}\right)$$

$$= f\left(\frac{r}{s} \times s\right)$$

$$= f(r)$$

$$= 1 + rx + \frac{r(r-1)}{1 \cdot 2} x^2 + \ldots$$

$$= (1+x)^r, \quad \because r \text{ is a positive integer.}$$

$$\therefore f\left(\frac{r}{s}\right) = (1+x)^{\frac{r}{s}}.$$

Transposing the sides of this equality, we have

$$\therefore\ (1+x)^{\frac{r}{s}} = f\left(\frac{r}{s}\right)$$

$$= 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1\,.\,2}x^2 + \ldots$$

This proves the binomial theorem to be true when n is a positive fraction (provided $x < 1$). Hence, since we know it to be true when n is a positive integer, it is true for *any positive index* (provided $x < 1$).

Returning to the equation $f(m) \times f(n) = f(m+n)$, let $m = -n$, where n is any positive number (integral or fractional),

$$\therefore f(-n) \times f(n) = f(0) = 1.$$

$$\therefore f(-n) = \frac{1}{f(n)}$$

$$= \frac{1}{(1+x)^n}, \text{ since } n \text{ is a positive number,}$$

$$= (1+x)^{-n}.$$

Hence $(1+x)^{-n} = f(-n)$

$$= 1 + (-n)x + \frac{(-n)(-n-1)}{1\,.\,2}x^2 + \ldots.$$

Therefore the theorem is true for *any negative index* (provided $x < 1$).

402. Hence the following expansion of $(1+x)^n$ in ascending powers of x, namely,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r + \ldots$$

may be used either, *when n is a positive integer* [Art. 386], or *when x is numerically less than unity* [Art. 401].

403. Should x be numerically greater than unity, we can expand $(1+x)^n$ in powers of $1/x$, in the following manner. We have

$$(1+x)^n = x^n\left(\frac{1}{x}+1\right)^n$$

$$= x^n\left(1+\frac{1}{x}\right)^n, \text{ where } \frac{1}{x} \text{ is less than unity,}$$

$$= x^n\left\{1+n\frac{1}{x}+\frac{n(n-1)}{1.2}\frac{1}{x^2}+\dots\right\}.$$

Ex. 1. *Expand* $(1+x)^{-1}$ *in positive powers of* x, *when* x *is less than unity.*

We have

$$(1+x)^{-1} = 1+(-1)x+\frac{(-1)(-2)}{1.2}x^2+\frac{(-1)(-2)(-3)}{1.2.3}x^3+\dots$$

$$= 1-x+x^2-x^3+\dots,$$

the general term, that is the term involving x^n, being $+(-1)^n x^n$.

Ex. 2. *Expand* $(1-x)^{-2}$ *in positive powers of* x, *when* x *is less than unity.*

We have

$$(1-x)^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{1.2}(-x)^2$$

$$+\frac{(-2)(-3)(-4)\dots(-2-n+1)}{1.2.3\dots n}(-x)^n+\dots$$

$$= 1+2x+3x^2+4x^3+\dots+(n+1)x^n+\dots$$

Similarly, the $(r+1)^{\text{th}}$ term in the expansion of $(1-x)^{-n}$ is

$$\frac{n(n+1)\dots(n+r-1)}{\underline{|r}}x^r.$$

Ex. 3. *Find the general term in the expansion of* $(1-x)^{\frac{1}{2}}$.

$$\text{The required term} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n+1)}{1.2.3\dots n}(-x)^n$$

$$= \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\dots\left(-\frac{2n-3}{2}\right)}{1.2.3\dots n}(-1)^n x^n$$

$$= (-1)^{n-1}\frac{1.3\dots(2n-3)}{1.2.3\dots n}\frac{1}{2^n}(-1)^n x^n$$

$$= (-1)^{2n-1}\frac{1.3\dots(2n-3)}{1.2.3\dots n}\left(\frac{x}{2}\right)^n$$

$$= -\frac{1.3\dots(2n-3)}{1.2.3\dots n}\left(\frac{x}{2}\right)^n$$

$$= -\frac{1.3\dots(2n-3)}{2.4.6\dots 2n}x^n.$$

To justify the use of the factor $(2n-3)$ in the numerator, we must suppose that n is greater than 1.

Ex. 4. *Find the general term in the expansion of* $(1-x)^{-\frac{1}{2}}$.

The general term $= \dfrac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\ldots(-\frac{1}{2}-n+1)}{1.2.3\ldots n}(-x)^n$

$$= \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots\left(-\frac{2n-1}{2}\right)}{1.2.3\ldots n}(-1)^n x^n$$

$$=(-1)^n\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}\frac{1}{2^n}(-1)^n x^n$$

$$=(-1)^{2n}\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}\frac{1}{2^n}x^n$$

$$=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}x^n.$$

Ex. 5. *Which is the first term in the expansion of* $(1+\frac{1}{3}x)^{\frac{13}{4}}$, *whose coefficient is negative?*

The $(n+1)^{\text{th}}$ term in the expansion $= \dfrac{\frac{13}{4}.\frac{9}{4}\ldots(\frac{13}{4}-n+1)}{1.2\ldots n}(\frac{1}{3}x)^n$.
This will first be negative, as soon as $\frac{13}{4}-n+1$ is negative,

that is, $\quad 13-4n+4$ is to be negative,

$$\therefore\ 4n>17,$$

$$\therefore\ n=5.$$

Hence the sixth term (that is, the term involving x^5) is the first whose coefficient is negative.

Ex. 6. *Find the greatest term in the expansion of* $(1+x)^{\frac{11}{3}}$, *where* $x=\frac{2}{3}$.

The $(r+1)^{\text{th}}$ term is obtained from the r^{th} by multiplying it by

$$\frac{\frac{11}{3}-r+1}{r}x.$$

Hence, as long as this multiplier is greater than unity, each successive term is greater than the term before it. This multiplier decreases as r increases, but it is greater than unity,

so long as $$\frac{\frac{14}{3}-r}{r}\frac{2}{3}>1,$$

that is, so long as $\qquad 2(\frac{14}{3}-r)>3r,$

that is, so long as $\qquad 5r<\frac{28}{3}.$

$\therefore$ if $r=1$, then this multiplier is >1, but if $r>1$, the multiplier is <1. Therefore the second term is the greatest.

404. We can sometimes sum a given series of numbers by shewing that it consists of the successive terms in the expansion of some binomial. This is sufficiently illustrated by the two following examples.

**Ex.* 1. *Find the sum of the series*

$$1+1+\frac{3.4}{1.2}\cdot\frac{1}{9}+\frac{3.4.5}{1.2.3}\cdot\frac{1}{27}+\ldots \textit{ to infinity.}$$

If the series can be expressed as the expansion of a binomial, we can sum it. Comparing the given series with the expansion of $(1+x)^n$, namely $1+nx+\frac{n(n-1)}{1.2}x^2+\ldots$, we must have $nx=1$, and $\frac{n(n-1)}{1.2}x^2=\frac{3.4}{1.2}\cdot\frac{1}{9}$.

The solution of these equations gives $x=-\frac{1}{3}$, $n=-3$. These values make the fourth terms in the two series the same. Hence we conclude that the given series is the expansion of $(1-\frac{1}{3})^{-3}$. Its value, therefore, is

$$(1-\tfrac{1}{3})^{-3}=(\tfrac{2}{3})^{-3}=(\tfrac{3}{2})^3=\tfrac{27}{8}.$$

With practice, it is generally possible to write down the answer to examples such as this one by inspection.

**Ex.* 2. *Find the sum of the series*

$$\frac{8}{9}+\frac{4.7}{9.18}2^2+\frac{4.7.10}{9.18.27}2^3+\ldots \textit{ to infinity.}$$

Compare the given series with the following expansion

$$(1\pm x)^{-\frac{p}{q}}=1+\left(-\frac{p}{q}\right)(\pm x)+\frac{-\frac{p}{q}\left(-\frac{p}{q}-1\right)}{1.2}(\pm x)^2$$
$$+\frac{-\frac{p}{q}\left(-\frac{p}{q}-1\right)\left(-\frac{p}{q}-2\right)}{1.2.3}(\pm x)^3+\ldots$$
$$=1\mp\frac{p}{1}\frac{x}{q}+\frac{p(p+q)}{1.2}\left(\frac{x}{q}\right)^2\mp\frac{p(p+q)(p+2q)}{1.2.3}\left(\frac{x}{q}\right)^3+\ldots$$

The given series can be written in the form

$$\frac{4}{1}\left(\frac{2}{9}\right)+\frac{4.7}{1.2}\left(\frac{2}{9}\right)^2+\frac{4.7.10}{1.2.3}\left(\frac{2}{9}\right)^3+\ldots$$

These terms are the same as those in the expansion of $(1\pm x)^{-\frac{p}{q}}$ if $p=4$, $q=7-4=3$, and $\frac{x}{q}=\frac{2}{9}$, that is, $x=\frac{2}{3}$. Hence we conclude that the given series is the expansion of $(1-\frac{2}{3})^{-\frac{4}{3}}$, except that the first term in the expansion is missing. Therefore the sum of the given series

$$=(1-\tfrac{2}{3})^{-\frac{4}{3}}-1=(\tfrac{1}{3})^{-\frac{4}{3}}-1=3^{\frac{4}{3}}-1=\sqrt[3]{81}-1.$$

405. **Application to approximations.** The following typical examples illustrate one of the most important applications of the binomial theorem, namely, the determination of the approximate values of given quantities.

Ex. 1. *Find the approximate value of* $\frac{1}{1-x}$, *when* x *is a quantity so small that its square may be neglected.*

We have $$\frac{1}{1-x}=1+x+x^2+\ldots$$

For, since we may neglect the square of x, therefore x must be less than unity, and the series is convergent. Moreover, the cubes and higher powers of x must be smaller than x^2, and therefore may also be neglected. [For example, if $x=\frac{1}{10}$, $x^2=\frac{1}{100}$, $x^3=\frac{1}{1000}$, &c.,]

$$\therefore\ \frac{1}{1-x}=1+x, \text{ approximately.}$$

Ex. 2. *Find the approximate value of* $\frac{(a+x)^n}{(b-x)^n}$, *when* x *is so small compared with* a *or* b *that squares of* x/a *and* x/b *may be neglected.*

$$\frac{(a+x)^n}{(b-x)^n}=\frac{a^n\left(1+\frac{x}{a}\right)^n}{b^n\left(1-\frac{x}{b}\right)^n}$$

$$=\left(\frac{a}{b}\right)^n\left(1+\frac{x}{a}\right)^n\left(1-\frac{x}{b}\right)^{-n}$$

$$=\left(\frac{a}{b}\right)^n\left(1+n\frac{x}{a}+\ldots\right)\left(1+n\frac{x}{b}+\ldots\right),$$

where all terms involving squares and higher powers of $\frac{x}{a}$ and $\frac{x}{b}$ are omitted. Moreover, since $\frac{x^2}{a^2}$ and $\frac{x^2}{b^2}$ are so small that they may be neglected, $\therefore$ $\frac{x^2}{ab}$ will be so small that it also may be neglected.

Hence $$\frac{(a+x)^n}{(b-x)^n}=\left(\frac{a}{b}\right)^n\left(1+n\frac{x}{a}+n\frac{x}{b}\right), \text{ approximately.}$$

Ex. 3. *Find approximately the value of the cube root of* 1001.

Here
$$\begin{aligned}\sqrt[3]{1001}&=(10^3+1)^{\frac{1}{3}}\\&=\left\{10^3\left(1+\frac{1}{10^3}\right)\right\}^{\frac{1}{3}}\\&=10\left(1+\frac{1}{10^3}\right)^{\frac{1}{3}}\\&=10\left\{1+\frac{1}{3}\frac{1}{10^3}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1.2}\frac{1}{10^6}+\dots\right\}\\&=10\{1+0\cdot000\dot{3}-0\cdot000000\dot{1}+\dots\}\\&=10\{1\cdot0003332\dots\}\\&=10\cdot003332\dots,\end{aligned}$$
which gives the result correct to six places of decimals, since the terms neglected cannot affect these figures.

406. The three following examples deal with questions involving the expansion of multinomials.

Ex. 1. *Find the first three terms in the expansion of*
$$(1+2x-x^2)^{\frac{1}{2}}$$
in ascending powers of x.

We have $\{1+(2x-x^2)\}^{\frac{1}{2}}=1+\frac{1}{2}(2x-x^2)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}(2x-x^2)^2+\dots$

We only want the terms involving x and x^2, therefore we need not go further in the expansion, since $(2x-x^2)^3$ must involve x^3: similarly, we may neglect x^3 and higher powers of x in the expansion of $(2x-x^2)^2$.

$$\begin{aligned}\therefore\ \{1+2x-x^2\}^{\frac{1}{2}}&=1+\tfrac{1}{2}(2x-x^2)-\tfrac{1}{8}(4x^2+\dots)\\&=1+x-\tfrac{1}{2}x^2-\tfrac{1}{2}x^2+\dots\\&=1+x-x^2+\dots\end{aligned}$$

Ex. 2. *Find the coefficient of x^{11} in the expansion of*

$$(1-x+x^2-x^3+x^4)^{-2}$$

in ascending powers of x.

The expression in the bracket is a G.P.,

$$\therefore\ \{1-x+x^2-x^3+x^4\}^{-2}=\left\{\frac{1-x^5}{1-x}\right\}^{-2}$$
$$=(1-x^5)^{-2}(1-x)^2$$
$$=(1+2x^5+3x^{10}+\ldots)(1-2x+x^2).$$

The only term in this product which can give x^{11} is the product of $3x^{10}$ and $-2x$. Therefore the coefficient of x^{11} is -6.

Ex. 3. *A candidate is examined in two papers, the full marks of each paper being n. In how many ways can he get a total of n marks?*

In each paper he can get 0, 1, 2,... or n marks. These are the coefficients of the powers of x in $x^0+x^1+x^2+\ldots+x^n$. Now in the product

$$(1+x+x^2+\ldots+x^n)(1+x+x^2+\ldots+x^n),$$

the coefficient of x^n will be the sum of the product of the coefficients of x^0 and x^n, the product of the coefficients of x^1 and x^{n-1}, &c. Each of these products corresponds to one way of obtaining a total of n marks; namely, 0 in the first paper and n in the second, 1 in the first paper and $n-1$ in the second, and so on. Hence, the number of ways in which the candidate can get a total of n marks is the coefficient of x^n in this product. But

$$(1+x+x^2+\ldots+x^n)^2=\left(\frac{1-x^{n+1}}{1-x}\right)^2$$
$$=(1-x^{n+1})^2(1-x)^{-2}$$
$$=\{1-2x^{n+1}+x^{2n+2}\}\{1+2x+3x^2+\ldots+(n+1)x^n+\ldots\}.$$

Hence the coefficient of x^n is $n+1$. Therefore the candidate can get n marks in $n+1$ ways.

This particular example could be solved more simply, but the above work is inserted as an illustration of the method of treating such problems.

Note. Similarly, the number of ways in which the number n can be formed as the sum of k of the numbers a, b, c... is the coefficient of x^n in $(x^a+x^b+x^c+\ldots)^k$.

*407. **Homogeneous Products.** *The number of homogeneous powers and products of r dimensions which can be formed from n letters a, b, c, ..., where each letter may be repeated any number of times, is*

$$\frac{n(n+1)\dots(n+r-1)}{\underline{|r}}.$$

This number is denoted by the symbol ${}_nH_r$.

Consider the continued product

$$\left(1+\frac{a}{x}+\frac{a^2}{x^2}+\dots\right)\left(1+\frac{b}{x}+\frac{b^2}{x^2}+\dots\right)\left(1+\frac{c}{x}+\frac{c^2}{x^2}+\dots\right)\dots,$$

where there are altogether n factors, and each factor is an infinite series. Each factor is homogeneous and of no dimensions. Therefore the product is homogeneous and of no dimensions [Art. 74]. Hence, the coefficient of $\frac{1}{x^r}$ in the product must be a homogeneous function of r dimensions in $a, b, c \dots$, and clearly it will be the sum of every power and product of r dimensions in these letters.

Now we only want the *number* of such products, and if we put $a=b=c=\dots=1$, we make each term in the coefficient of $\frac{1}{x^r}$ equal to unity, and therefore in this case the coefficient of $\frac{1}{x^r}$ will be ${}_nH_r$. Therefore, ${}_nH_r$ is the coefficient of $\frac{1}{x^r}$ in the product

$$\left(1+\frac{1}{x}+\frac{1}{x^2}+\dots\right)\left(1+\frac{1}{x}+\frac{1}{x^2}+\dots\right)\dots n \text{ factors.}$$

$$\therefore\ {}_nH_r = \text{coefficient of } \frac{1}{x^r} \text{ in } \left(1+\frac{1}{x}+\frac{1}{x^2}+\dots\right)^n.$$

Let $\frac{1}{x}=y$, $\therefore\ {}_nH_r=$ coefficient of y^r in $(1+y+y^2+\dots)^n$

$$= \quad ,, \quad ,, \quad \text{in } \left(\frac{1}{1-y}\right)^n$$

$$= \quad ,, \quad ,, \quad \text{in } (1-y)^{-n}$$

$$=\frac{n(n+1)\dots(n+r-1)}{\lfloor r}.$$

[p. 415, Ex. 2.

Note. This result can be proved directly by a method analogous to that given in Art. 369. For, from the given result we have (see Art. 374)

$${}_nH_r=\frac{n+r-1}{r}\cdot{}_nH_{r-1},$$

and this relation can be proved directly, and thence the value of ${}_nH_r$ can be determined.

EXAMPLES. XXXI. B.

[*It may be assumed that the series involved in the following examples are considered only for such values as will make them convergent.*]

Write down the first five terms in the expansions of the following binomials, numbered 1 to 12.

1. $(1-x)^{\frac{1}{2}}$. 2. $(1+x)^{\frac{2}{3}}$. 3. $(1-x)^{\frac{5}{4}}$.

4. $(1+2x)^{-\frac{3}{2}}$. 5. $(1-\frac{1}{2}x)^{\frac{4}{3}}$. 6. $(1+ax)^{-\frac{p}{q}}$.

7. $(a-bx)^{\frac{7}{3}}$. 8. $(\frac{1}{2}-2x)^{-\frac{11}{5}}$. 9. $(x-x^2)^{-\frac{1}{2}}$.

10. $\dfrac{1}{\sqrt[3]{(1-x^2)}}$. 11. $\dfrac{1}{\sqrt{(1-x^3)}}$. 12. $\dfrac{(1+x)^3}{\sqrt{(1+x)}}$.

Express, in their simplest forms, the $(r+1)^{\text{th}}$ terms in the expansions of the following binomials, numbered 13 to 21.

13. $(1-2x)^{-\frac{1}{2}}$. 14. $(1-4x)^{-\frac{3}{2}}$. 15. $(1-x)^{-\frac{1}{n}}$.

16. $(1-nx)^{\frac{1}{n}}$. 17. $(1+2x)^{\frac{3}{2}}$. 18. $(1-\frac{3}{4}x)^{-\frac{11}{3}}$.

19. $\dfrac{1}{\sqrt{(1+3x)^3}}$. 20. $\dfrac{1}{\sqrt[3]{(1-x^2)^4}}$. 21. $\left(x-\dfrac{1}{x}\right)^{-n}$.

22. Determine which is the first term with a negative coefficient in the expansions of (i) $(1+\frac{3}{2}x)^{\frac{9}{4}}$; (ii) $(1+x)^{\frac{p}{q}}$.

23. Prove that, if n be a positive fraction, all the terms in $(1-x)^n$ after a certain term will have the same sign. What is this sign if $n=\frac{17}{3}$? Write down the first two terms which have this sign.

24. Find the greatest term in the expansion of $(1+x)^n$, when $x=\frac{2}{3}$ and $n=-4$.

25. Find the greatest term in the expansion of $(1+x)^{-n}$, when $x=\frac{1}{2}$ and $n=7$.

26. Find the greatest term in the expansion of $(1-x)^{-n}$, when $x=\frac{3}{4}$ and $n=\frac{4}{3}$.

27. The series $1+\frac{2}{1}x+\frac{3\,.\,4}{1\,.\,2}x^2+\frac{4\,.\,5\,.\,6}{1\,.\,2\,.\,3}x^3+\frac{5\,.\,6\,.\,7\,.\,8}{1\,.\,2\,.\,3\,.\,4}x^4+\ldots$ is a binomial expansion, the index being fractional; find the expression which is thus expanded.

28. Find the sum of $1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3+\ldots$

29. Find the sum of $1+\frac{1}{4}+\frac{1\,.\,3}{4\,.\,8}+\frac{1\,.\,3\,.\,5}{4\,.\,8\,.\,12}+\ldots$

***30.** Find the sum of

$$1+\frac{2\,.\,5\,.\,7\,.\,11}{70\,.\,4}+\frac{2\,.\,5\,.\,7\,.\,11\,.\,14}{70\,\underline{|2}\,.\,4^2}+\ldots+\frac{2\,.\,5\ldots(3n+8)}{70\,\underline{|n}\,.\,4^n}+\ldots$$

31. Prove that

$$2\sqrt{2}-1=3\left\{\frac{1}{2}+\frac{1}{2^2\,\underline{|2}}-\frac{1\,.\,1}{2^3\,\underline{|3}}+\frac{1\,.\,1\,.\,3}{2^4\,\underline{|4}}-\frac{1\,.\,1\,.\,3\,.\,5}{2^5\,\underline{|5}}+\ldots\right\}.$$

32. If $p_r=\frac{1\,.\,3\,.\,5\ldots(2r-3)}{2\,.\,4\,.\,6\ldots 2r}$, prove that

$$1-p_1-p_2-\ldots-p_n=2(n+1)\,p_{n+1}.$$

33. Find the square root of 101, correct to five places of decimals.

34. Find $\sqrt[4]{255}$, correct to 3 places of decimals.

35. Find $\sqrt[3]{999}$, correct to 6 places of decimals.

36. Expand $\sqrt{N^2+x}$ in ascending powers of x, as far as the third power of x.

37. If x be so small that its square can be neglected, prove that

$$\frac{\sqrt{9+2x}+\sqrt[3]{8-3x}}{\sqrt[3]{27-6x}+\sqrt[5]{32+10x}}=1+\frac{13}{360}x.$$

38. Find the value of $\dfrac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$, when $x=\frac{1}{50}$, correct to 3 decimal places.

39. If p and q be the n^{th} terms of the expansions of $(1-x)^{-\frac{1}{2}}$ and $(1-x)^{-\frac{3}{2}}$ respectively, shew that $q=(2n-1)p$.

40. Prove that, if p_r and q_r represent the coefficients of x^r in the expansions of $(1+x)^n$ and $(1+x)^{-n}$, then

$$p_n+p_{n-1}q_1+p_{n-2}q_2+\ldots+p_1q_{n-1}+q_n=0.$$

41. Prove that the sum of the coefficients of the first r terms in the expansion of $(1-x)^{-\frac{1}{n}}$ bears to the coefficient of the r^{th} term the ratio of $1+n(r-1)$ to 1.

42. Shew that, if p, q, s be respectively the product, quotient, and sum of two quantities, q being less than unity, then

$$p^2=s^4\left\{q^2-\frac{4}{1}q^3+\frac{4\,.\,5}{1\,.\,2}q^4-\frac{4\,.\,5\,.\,6}{1\,.\,2\,.\,3}q^5+\ldots\right\}.$$

43. If $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n+\ldots$, find the values of

(i) $c_0-2c_1+3c_2-\ldots+(-1)^n(n+1)c_n$.

(ii) $(c_0-c_2+c_4-\ldots)^2+(c_1-c_3+c_5-\ldots)^2$.

44. Find the coefficients of x^{3n} and of x^{3n+2} in the expansion of

$$\sqrt{\frac{1-x}{1+x+x^2}}.$$

45. Find the coefficient of x^{3n+1} in the expansion of

$$(1+x+x^2)^{-1}.$$

46. Find the coefficient of x^{10} in the expansion of

$$(1-2x+4x^2-8x^3+\ldots)^{-\frac{3}{2}}.$$

47. Find the coefficient of x^{10} in the expansion of

$$(1-x+2x^2-2x^3+4x^4-4x^5+8x^6-\ldots)^{-1}.$$

48. Prove that the coefficient of x^n in $\dfrac{(1-3x)^2}{(1-x)^3}$ is $2n^2-6n+1$.

***49.** If n be a positive integer, and

$$\phi(x, n) = \frac{1}{x} - n\frac{1}{x+1} + \frac{n(n-1)}{1 \cdot 2}\frac{1}{x+2} - \ldots,$$

find a relation connecting $\phi(x, n)$ and $\phi(x+1, n-1)$; and thence shew that

$$\phi(x, n) = \frac{|\underline{n}\ |\underline{x-1}}{|\underline{x+n}}.$$

***50.** Prove that the sum of the homogeneous products of n dimensions of three things a, b, c is

$$\frac{a^{n+2}(b-c) + b^{n+2}(c-a) + c^{n+2}(a-b)}{a^2(b-c) + b^2(c-a) + c^2(a-b)}.$$

***51.** Shew that the sum of all homogeneous products of a, b, c of all dimensions from 0 to n is

$$\frac{a^{n+3}}{(a-b)(a-c)(a-1)} + \frac{b^{n+3}}{(b-a)(b-c)(b-1)} + \frac{c^{n+3}}{(c-a)(c-b)(c-1)} - \frac{1}{(a-1)(b-1)(c-1)}.$$

***52.** Prove that the coefficient of x^{n-1} in the expansion of

$$\{(1-x)(1-ax)(1-a^2x)(1-a^3x)\}^{-1}$$

in ascending powers of x is

$$\frac{(1-a^n)(1-a^{n+1})(1-a^{n+2})}{(1-a)(1-a^2)(1-a^3)}.$$

***53.** A man enters for an examination in which there are four papers with a maximum of m marks for each paper. Shew that the number of ways of getting half marks on the whole is

$$\tfrac{1}{3}(m+1)(2m^2+4m+3).$$

***54.** In a certain examination there are 3 papers for each of which 200 marks are obtainable as a maximum; but if less than 50 marks are obtained in a paper, they are not counted towards the aggregate. Shew that there are 1632 ways in which a candidate can just get a total of 200 marks.

CHAPTER XXXII.

THE EXPONENTIAL THEOREM.

408. **Definition of e.** The series

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor n}+\ldots, \text{ to infinity,}$$

is denoted by the letter *e*.

This series contains an infinite number of terms, but it is convergent [Art. 338], since

(i) it is greater than 2 and

(ii) it is less than $1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^p}+\ldots,$

that is, it is less than $1+\dfrac{1}{1-\frac{1}{2}}$, that is, it is less than 3. Its value lies therefore between 2 and 3, and (by actual calculation) is found to be equal to 2·718... We shall now shew that *e* is an incommensurable number, and therefore its numerical value is a non-terminating and non-repeating decimal.

409. *The number denoted by e is incommensurable.*

For, if possible, let *e* be commensurable; and suppose that $e=\frac{m}{n}$, where *m* and *n* are positive integers.

$$\therefore \frac{m}{n}=1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots+\frac{1}{\lfloor n}+\frac{1}{\lfloor n+1}+\frac{1}{\lfloor n+2}+\ldots\text{(to infinity).}$$

Multiplying both sides by $\lfloor n$, we have

an integer = the sum of a number of integers

$$+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

Now $\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\dots$ (to infinity) is obviously greater than $\frac{1}{n+1}$, and it is less than

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\dots$$

This latter series is a geometrical progression of which the common ratio is less than unity, and its sum is $\frac{1}{n}$. [p. 355, Ex. 2]

Thus $\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\dots$ lies between $\frac{1}{n+1}$ and $\frac{1}{n}$, and is therefore a proper fraction. But a proper fraction cannot be equal to the difference between two integers. Therefore e cannot be commensurable.

410. The Exponential Theorem. *To shew that*

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots + \frac{x^r}{\underline{|r}} + \dots \textit{ to infinity.}$$

We know that

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx} \quad \dots\dots\dots\dots (\alpha).$$

We shall now expand each side of this identity by the binomial theorem. We shall suppose n to be greater than unity; and therefore, since $\frac{1}{n}$ is less than unity, this expansion will always be possible. The right-hand side, when expanded, becomes

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{1.2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{1.2.3}\frac{1}{n^3} + \dots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1.2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3} + \dots$$

Putting $x = 1$, we obtain

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1.2.3}+\ldots$$

Hence, the relation (α) may be written

$$\left\{1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1.2.3}+\ldots\right\}^x$$

$$= 1+x+\frac{x\left(x-\frac{1}{n}\right)}{1.2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3}+\ldots$$

This is true for all values of n greater than unity, and we may therefore make n as great as we like. Take n infinitely great, then $\frac{1}{n}$ will be infinitely small and therefore ultimately zero. This relation will then become

$$\left\{1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right\}^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots,$$

that is, $$e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots$$

Note. It can be proved that this series is convergent for all values of x.

411. *If a be any number, then will*

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{\underline{|2}} + \frac{(x \log_e a)^3}{\underline{|3}} + \ldots$$

Let the logarithm of a to the base e be h.

$$\therefore \log_e a = h; \text{ that is, } a = e^h.$$

$$\therefore a^x = e^{hx} = e^{x \log_e a} = 1 + x \log_e a + \frac{(x \log_e a)^2}{\underline{|2}} + \ldots \text{[Art. 410].}$$

412. The Logarithmic Series. *To shew that, if x be numerically less than unity, then*

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^r \frac{x^r}{r} + \dots \text{ (to infinity).}$$

We have $a^n = 1 + n\log_e a + \dfrac{\{n\log_e a\}^2}{\lfloor 2} + \dots$ [Art. 411],

where a and n are any numbers whatever. Let us take $a = 1 + x$,

$$\therefore\ (1+x)^n = 1 + n\log_e(1+x) + \frac{1}{\lfloor 2}\{n\log_e(1+x)\}^2 + \dots$$

$$\therefore\ \frac{(1+x)^n - 1}{n} = \log_e(1+x) + \frac{1}{\lfloor 2} n\{\log_e(1+x)\}^2 + \dots$$

We now proceed to find another expression for the left-hand side of this equality. If x be numerically less than unity, we can expand $(1+x)^n$ by the binomial theorem. Hence we have

$$\frac{(1+x)^n - 1}{n} = \frac{\left[1 + nx + \dfrac{n(n-1)}{1.2}x^2 + \dots\right] - 1}{n}$$

$$= x + \frac{n-1}{1.2}x^2 + \frac{(n-1)(n-2)}{1.2.3}x^3 + \dots$$

Equating the two series to which $\dfrac{(1+x)^n - 1}{n}$ is equal, we have

$$\log_e(1+x) + \dots = x + \frac{n-1}{1.2}x^2 + \frac{(n-1)(n-2)}{1.2.3}x^3 + \dots$$

This is true for all values of n. Put $n = 0$. Every term, after the first, on the left-hand side vanishes.

Therefore $\log_e(1+x) = x + \dfrac{(-1)}{2}x^2 + \dfrac{(-1)(-2)}{1.2.3}x^3 + \dots$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

The student should insert the general term in the expansion of $(1+x)^n$, and satisfy himself that it reduces to the general term in the expansion of $\log_e(1+x)$, which is given in the enunciation of the proposition.

EXAMPLES. XXXII. A.

1. Find the sum of $\frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \ldots$(to infinity).

2. Find the sum of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$(to infinity).

3. Find the sum of

$$1 + \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \ldots \text{(to infinity)}.$$

4. Shew that

$$\left(1 + 1 + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \ldots\right)\left(1 - 1 + \frac{1}{|\underline{2}} - \frac{1}{|\underline{3}} + \ldots\right) = 1.$$

5. Shew that $1 + \frac{3}{|\underline{1}} + \frac{5}{|\underline{2}} + \frac{7}{|\underline{3}} + \ldots = 3e.$

6. Shew, by taking logarithms, that, if $x > y > z$, then

$$\left(\frac{x+z}{x-z}\right)^x < \left(\frac{y+z}{y-z}\right)^y.$$

7. Shew, by taking logarithms of each side of the identity $x^3+1=(x+1)(x^2-x+1)$, and then expanding in powers of x, that, when n is a positive integer,

$$1 - \frac{6n-2}{1.2} + \frac{(6n-3)(6n-4)}{1.2.3} - \frac{(6n-4)(6n-5)(6n-6)}{1.2.3.4} + \ldots = 0.$$

*8. Explain the following paradox. If we put $x=1$ in the logarithmic series [Art. 412], we find

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \ldots$$

$$\therefore\ 2\log 2 = 2 - 1 + \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{2}{5} - \tfrac{1}{3} + \tfrac{2}{7} - \tfrac{1}{4} + \tfrac{2}{9} - \ldots$$

Taking those terms together which have a common denominator, we obtain

$$\begin{aligned} 2\log 2 &= 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} \ldots \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ldots\ldots\ldots \\ &= \log 2. \end{aligned}$$

*9. Shew that $n^{n-1} - n(n-1)^{n-1} + \frac{n(n-1)}{|\underline{2}}(n-2)^{n-1}$

$$- \frac{n(n-1)(n-2)}{|\underline{3}}(n-3)^{n-1} + \ldots = 0.$$

413. Natural or Napierian Logarithms. Logarithms calculated to the base e are called *natural logarithms* or *Napierian logarithms*—the latter name being derived from that of John Napier of Merchistoun, to whom the invention of logarithms is due.

In theoretical investigations, it is usual to assume that all logarithms are calculated to the base e, unless the contrary is expressly stated; and the symbol shewing to what base the logarithms are calculated is frequently omitted.

414. Calculation of Napierian Logarithms. The logarithmic series enables us to calculate the numerical values of the logarithms of the successive natural numbers, namely, of 1, 2, 3, &c.

We have, if x be less than unity,

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\ldots \qquad \text{[Art. 412}$$

and

$$\log_e(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\ldots$$

$$\therefore \log_e\frac{1+x}{1-x}=\log_e(1+x)-\log_e(1-x)$$

$$=2\{x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots\}.$$

Put $\dfrac{1+x}{1-x}=\dfrac{m}{n}$, that is, let $x=\dfrac{m-n}{m+n}$. Since x is numerically less than unity, this is permissible if m and n be positive integers.

$$\therefore \log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\right\}\ldots(a).$$

(i) To find $\log_e 1$. This we know is zero. [Art. 253, Ex. 1.]

(ii) To find $\log_e 2$. In (a), put $m=2$ and $n=1$,

$$\therefore \log_e 2=2\{\tfrac{1}{3}+\tfrac{1}{3}(\tfrac{1}{3})^3+\tfrac{1}{5}(\tfrac{1}{3})^5+\ldots\}$$

$$=0{\cdot}693\ldots$$

(iii) To find $\log_e 3$. In (a), put $m=3$ and $n=2$,

$$\therefore \log_e 3-\log_e 2=2\{\tfrac{1}{5}+\tfrac{1}{3}(\tfrac{1}{5})^3+\tfrac{1}{5}(\tfrac{1}{5})^5+\ldots\}.$$

$$\therefore \log_e 3-0{\cdot}693\ldots=0{\cdot}405\ldots$$

$$\therefore \log_e 3=1{\cdot}098\ldots$$

The method is general. From the logarithm of any number, we can get the logarithm of the next greater number; and thus a table of the Napierian logarithms of all the natural numbers can be calculated.

The logarithm of any fraction or decimal can be deduced by the aid of Art. 256.

415. Common or Briggian Logarithms. Logarithms calculated to the base 10 are called *common logarithms* or *Briggian logarithms* [Art. 261]—the latter name being derived from that of Henry Briggs, to whom their introduction is due.

In numerical calculations—especially where approximations are required—it is usual to assume that all logarithms are calculated to the base 10, unless the contrary is expressly stated; and the symbol shewing the base to which the logarithms are calculated is frequently omitted.

416. Calculation of Common Logarithms. The result of Article 260 enables us to deduce the numerical value of the logarithm of any number calculated to the base ten from the value of its logarithm calculated to the base e.

In the result (i) of Art. 260, put $m=x$; and divide each side by $\log_b a$,

$$\therefore \log_a x = \frac{\log_b x}{\log_b a}.$$

Put $a=10$, and $b=e$,

$$\therefore \log_{10} x = \frac{\log_e x}{\log_e 10}.$$

We have shewn how to calculate the logarithm of any number to the base e; and therefore we know the values of $\log_e x$ and $\log_e 10$. Hence, if we divide the natural logarithm of any number by the value of $\log_e 10$, we shall have the value of the common logarithm of the number.

The fraction $\frac{1}{\log_e 10}$, by which the natural logarithm of any number must be multiplied in order to produce the common logarithm of the same number, is called the *modulus*. Its value is 0·43429...

417. Tables of Logarithms. The numerical values of the common logarithms of the natural numbers from 1 to 100,000 have been calculated, in some cases to fourteen places of decimals; and their values to seven (or five) places are published in a tabular form.

As an illustration, a table of the logarithms of the first 100 natural numbers, calculated to 5 places of decimals, is here added.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	00000	21	32222	41	61278	61	78533	81	90849
2	30103	22	34242	42	62325	62	79239	82	91381
3	47712	23	36173	43	63347	63	79934	83	91908
4	60206	24	38021	44	64345	64	80618	84	92428
5	69897	25	39794	45	65321	65	81291	85	92942
6	77815	26	41497	46	66276	66	81954	86	93450
7	84510	27	43136	47	67210	67	82607	87	93952
8	90309	28	44716	48	68124	68	83251	88	94448
9	95424	29	46240	49	69020	69	83885	89	94939
10	00000	30	47712	50	69897	70	84510	90	95424
11	04139	31	49136	51	70757	71	85126	91	95904
12	07918	32	50515	52	71600	72	85733	92	96379
13	11394	33	51851	53	72428	73	86332	93	96848
14	14613	34	53148	54	73239	74	86923	94	97313
15	17609	35	54407	55	74036	75	87506	95	97772
16	20412	36	55630	56	74819	76	88081	96	98227
17	23045	37	56820	57	75587	77	88649	97	98677
18	25527	38	57978	58	76343	78	89209	98	99123
19	27875	39	59106	59	77085	79	89763	99	99564
20	30103	40	60206	60	77815	80	90309	100	00000

It will be observed that the characteristics of the logarithms are omitted from the table, and only the mantissas are printed [see Art. 265]. The characteristic of the common logarithm of any number can be written down by inspection [Art. 264]. The result of multiplying or dividing the number by any power of ten will change the characteristic of the logarithm, but will not affect the mantissa.

We however constantly meet with numbers which lie between two numbers whose logarithms are known; and to determine the logarithms of such numbers we want the theorem given in the next article, which is known as the rule of proportional parts. Conversely, we frequently meet with a logarithm, opposite to which in the tables no number is placed, though other logarithms very nearly equal to it are the logarithms of known numbers; and to find the number of which it is the logarithm we require the rule of proportional parts.

418. Rule of Proportional Parts. *The numerical value of $\log_a N$ being given, to find approximately the value of $\log_a (N+x)$ where x is small compared with N.*

We have

$$\log_a (N+x) - \log_a N = \log_a \frac{N+x}{N}$$

$$= \log_a \left(1+\frac{x}{N}\right)$$

$$= \log_e \left(1+\frac{x}{N}\right) \times \frac{1}{\log_e a} \qquad \text{[Art. 260.}$$

$$= \mu \left\{\frac{x}{N} - \frac{1}{2}\left(\frac{x}{N}\right)^2 + \dots\right\}, \qquad \text{[Art. 412.}$$

where $\mu = \dfrac{1}{\log_e a}$. Hence, if the squares and higher powers of $\dfrac{x}{N}$ may be neglected in comparison with $\dfrac{x}{N}$, we have, approximately,

$$\log_a (N+x) - \log_a N = \mu \frac{x}{N}.$$

If the tables give us the logarithms of the successive natural numbers, we may suppose N to be an integer and x to be a proper fraction. From the tables, we know the values of $\log_a N$ and $\log_a (N+1)$. Hence, we have

$$\log_a (N+x) - \log_a N = \mu \frac{x}{N},$$

and

$$\log_a (N+1) - \log_a N = \mu \frac{1}{N}.$$

$$\therefore \frac{\log_a (N+x) - \log_a N}{\log_a (N+1) - \log_a N} = x.$$

If $\log_a (N+x)$ be given, the equation last written determines x; if x be given, it determines $\log_a (N+x)$.

419. The rule of proportional parts enables us to find approximately the values of the logarithms of numbers intermediate between numbers whose logarithms are known: and conversely, from a given logarithm to find the number of which it is a logarithm.

To take a simple illustration, we will suppose that we have a table like that printed on page 433, which gives us the logarithms of numbers of only two digits. Suppose then that $\log 92 = 1{\cdot}96379$, and $\log 93 = 1{\cdot}96848$, are given in the tables, and we want to find $\log 92{\cdot}7$.

By Art. 418, we have

$$\frac{\log 92{\cdot}7 - \log 92}{\log 93 - \log 92} = \frac{{\cdot}7}{1} = \frac{7}{10}.$$

$$\therefore\ \log 92{\cdot}7 - \log 92 = \tfrac{7}{10}(\log 93 - \log 92).$$

$$\therefore\ \log 92{\cdot}7 = \log 92 + \tfrac{7}{10}(\log 93 - \log 92)$$

$$= 1{\cdot}96379 + \tfrac{7}{10}({\cdot}00469) = 1{\cdot}96707,$$

if we retain only 5 places of decimals. Thus the value of $\log 92{\cdot}7$ is known approximately.

Conversely, if we are told that $1{\cdot}96707$ is the logarithm of some number, we find from the tables that the number is between 92 and 93. We therefore take it to be $92 + x$. We then have the proportion

$$x = \frac{\log(92 + x) - \log 92}{\log 93 - \log 92},$$

that is, $$x = \frac{1{\cdot}96707 - 1{\cdot}96379}{1{\cdot}96848 - 1{\cdot}96379} = \frac{{\cdot}00328}{{\cdot}00469} = {\cdot}699\ldots$$

Thus the required number is (very approximately) $92{\cdot}7$.

420. We proceed now to take two examples, one of each kind, which will sufficiently illustrate the rule.

421. *To find the logarithm of a number not given in the tables: for example, to find the logarithm of* 575·056.

In the table, opposite the numbers 57505 and 57506 will be found their logarithms, namely, 7597056 and 7597132—the characteristics being omitted.

$$\therefore\ \log 575{\cdot}06 = 2{\cdot}7597132,$$

and $$\log 575{\cdot}05 = 2{\cdot}7597056.$$

The logarithm of the given number lies between these numbers; suppose it to be $2{\cdot}7597056 + x$. The rule of proportional parts shews us that the difference between the numbers is proportional to the difference between the logarithms,

$$\therefore\ 575{\cdot}056 - 575{\cdot}05 : 575{\cdot}06 - 575{\cdot}05 = x : 2{\cdot}7597132 - 2{\cdot}7597056,$$

that is, $$6 : 10 = x : {\cdot}0000076,$$

$$\therefore\ x = \tfrac{6}{10}({\cdot}0000076) = {\cdot}00000456.$$

Since we are only keeping 7 places of decimals, we write this ·0000046, because this is nearer the truth than ·0000045.

Hence $\log 575{\cdot}056 = 2{\cdot}7597056 + {\cdot}0000046 = 2{\cdot}7597102.$

422. *To find the number which has a given logarithm (not printed in the tables): for example, to find the number whose logarithm is* $\bar{1}{\cdot}0211972$.

In the table, opposite the logarithms 0211893 and 0212307 will be found the numbers 10500 and 10501,

$$\therefore\ \log {\cdot}10500 = \bar{1}{\cdot}0211893,$$

$$\therefore\ \log {\cdot}10501 = \bar{1}{\cdot}0212307.$$

The required number is between ·10500 and ·10501. Suppose it to be $\cdot10500 + x$. Then the rule gives us

$$x : {\cdot}10501 - {\cdot}10500 = \bar{1}{\cdot}0211972 - \bar{1}{\cdot}0211893 : \bar{1}{\cdot}0212307 - \bar{1}{\cdot}0211893,$$

that is $$x : {\cdot}00001 = {\cdot}0000079 : {\cdot}0000414,$$

$$\therefore\ x = \tfrac{79}{414}({\cdot}00001) = {\cdot}0000019,$$

if we keep only 6 places of decimals, $= {\cdot}000002$.

Hence $\bar{1}{\cdot}0211972 = \log({\cdot}10500 + {\cdot}000002) = \log {\cdot}105002.$

423. Examples similar to those given in Chapter XXIII. can be worked out to a close degree of approximation by the aid of the rule of proportional parts.

Example. Find approximately the fifth root of 2, having given $\log 2 = {\cdot}30103$, $\log 1{\cdot}1482 = {\cdot}060175$, $\log 1{\cdot}1483 = {\cdot}060554$.

Let $$x = 2^{\frac{1}{5}},$$

$$\therefore\ \log x = \tfrac{1}{5}\log 2 = \tfrac{1}{5}({\cdot}30103) = {\cdot}060206.$$

From the given logarithms, we see that x lies between 1·1482 and 1·1483. Hence, by the rule of proportional parts, we have

$$\therefore\ x = 1{\cdot}1482 + \frac{{\cdot}060206 - {\cdot}060175}{{\cdot}060554 - {\cdot}060175}({\cdot}00001)$$

$$= 1{\cdot}1482 + \tfrac{31}{379}({\cdot}00001) = 1{\cdot}14821, \text{ approximately.}$$

EXAMPLES. XXXII. B.

[*All the logarithms in the following examples are calculated to the base ten.*]

1. Given $\log 11111 = 4{\cdot}0457531$, and $\log 11112 = 4{\cdot}0457922$; find $\log 11{\cdot}1112$, and the number whose logarithm is $2{\cdot}0457777$.

2. Given $\log 8{\cdot}6223 = {\cdot}9356231$, and $\log 8{\cdot}6224 = {\cdot}9356282$; calculate $\log 86{\cdot}22384$.

3. Given $\log 3{\cdot}1156 = {\cdot}4935417$, and $\log 3{\cdot}1157 = {\cdot}4935556$; find $\log {\cdot}03115625$.

4. Given $\log 1{\cdot}3287 = {\cdot}1234269$, and $\log 1{\cdot}3288 = {\cdot}1234596$; find $\log {\cdot}00132874$.

5. Calculate $\log {\cdot}0361356$; having given $\log 3{\cdot}6135 = {\cdot}5579281$, and $\log 3{\cdot}6136 = {\cdot}5579401$.

6. Find $\log {\cdot}172306$; having given $\log 1{\cdot}723 = {\cdot}2362853$, and $\log 1{\cdot}7231 = {\cdot}2363105$.

7. Find $\log 13{\cdot}664357$, and find the number whose logarithm is $2{\cdot}1356053$; having given

$$\log 13664 = 4{\cdot}1355779, \text{ and } \log 13665 = 4{\cdot}1356096.$$

8. Employ logarithms to divide $32{\cdot}0576$ by ${\cdot}69665$, having given

$$\log 3{\cdot}2057 = {\cdot}5059229, \quad \log 4{\cdot}6016 = {\cdot}6629089,$$
$$\log 3{\cdot}2058 = {\cdot}5059364, \quad \log 4{\cdot}6017 = {\cdot}6629183.$$
$$\log 6{\cdot}9665 = {\cdot}8430146,$$

9. Find the value of $(1{\cdot}25)^4$ to six places of decimals, having given $\log 2 = {\cdot}30103$, $\log 2{\cdot}4414 = {\cdot}3876389$, $\log 2{\cdot}4415 = {\cdot}3876567$.

10. Find $(1{\cdot}467799)^7$; having given $\log 14677 = 4{\cdot}1666373$, and $\log 14678 = 4{\cdot}1666669$.

11. Shew that the cube root of 19307 is very nearly ten times its tenth root, having given $\log 19307 = 4{\cdot}2857290$.

12. Given $\log 1{\cdot}7783 = {\cdot}2500050$, and $\log 1{\cdot}7782 = {\cdot}2499806$; find the value of $(17{\cdot}7828)^{\frac{1}{5}}$.

13. Find $(13{\cdot}89492)^{\frac{1}{8}}$; having given $\log 1{\cdot}3894 = {\cdot}1428273$, and $\log 1{\cdot}3895 = {\cdot}1428586$.

14. Find the fifth root of $5{\cdot}4$; having given $\log 3 = {\cdot}4771213$, $\log 5 = {\cdot}6989700$, $\log 14011 = 4{\cdot}1464691$, $\log 14012 = 4{\cdot}1465001$.

15. Find the fifth root of $6{\cdot}4$; having given $\log 2 = {\cdot}30103$, $\log {\cdot}14495 = \bar{1}{\cdot}1612182$, $\log {\cdot}14496 = \bar{1}{\cdot}1612482$.

16. Find the eleventh root of $(39{\cdot}2)^2$; having given

$$\log 2 = {\cdot}3010300, \qquad \log 7 = {\cdot}8450980,$$
$$\log 19484 = 4{\cdot}2896781, \quad \log 19485 = 4{\cdot}2897004.$$

17. Extract the seventh root of 3115455; having given

$$\log 3{\cdot}1154 = {\cdot}4935138, \quad \log 3{\cdot}1155 = {\cdot}4935278,$$
$$\log 8{\cdot}4653 = {\cdot}9276424, \quad \log 8{\cdot}4654 = {\cdot}9276475.$$

18. Find the value of $2^{128}\, 3^{\frac{1}{5}}\, 10^{-35}$; having given

$$\log 2 = {\cdot}3010300, \quad \log 3 = {\cdot}4771213,$$
$$\log 4{\cdot}239 = {\cdot}627263, \quad \log 4{\cdot}24 = {\cdot}627366.$$

19. Prove that $\log 7 + \log 11 + \log 13$ is approximately equal to 3. To how many decimal places is this true?

EXAMINATION PAPERS.

The first of the following papers was set to the Senior Students at one of the recent Cambridge Local Examinations. The next paper (B) was set recently in the Cambridge Higher Local Examinations. The next two papers (C) and (D) are two of those set recently to candidates for admission into the Military Academy, Woolwich.

Paper A.

1. Prove that

$$(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)=1.$$

2. If α, β be the roots of the equation $ax^2+bx+c=0$, find the value of $\alpha^2+\beta^2$; and form the equation whose roots are $\alpha+\dfrac{1}{\beta}$ and $\beta+\dfrac{1}{\alpha}$.

3. If $ab+bc+ca=0$, prove that
   (i) $(a+b+c)^2=a^2+b^2+c^2$;
   (ii) $(a+b+c)^3=a^3+b^3+c^3-3abc$;
   (iii) $(a+b+c)^4=a^4+b^4+c^4-4abc\,(a+b+c)$.

4. Solve the equations:
   (i) $ax+by=mxy=px+qy$;
   (ii) $(b-c)\,x^2+(c-a)\,x+(a-b)=0$;
   (iii) $\sqrt{x+3}-\sqrt{5x-7}=\sqrt{2}$;
   (iv) $\left.\begin{array}{r} x^2+3xy+2y=12 \\ xy+4y^2+x=8 \end{array}\right\}$.

5. The distance from A to B is 20 miles by one road and 24 miles by another. A bicyclist goes from A to B by one road and returns by the other; but in returning, travels 2 miles an hour slower than in going. He finds that, if he goes by the longer and returns by the shorter road, he takes 6 minutes less time to travel than if he went in the reverse order. What is his speed in going?

6. Insert n arithmetic means between a and b, and find the sum of these means.

If the 3rd, the $(p+2)^{\text{th}}$ and the $(3p)^{\text{th}}$ terms of an arithmetical progression be in geometrical progression, prove that the $(p-2)^{\text{th}}$ term of the arithmetical progression is double the first term.

7. Find the number of combinations of n different things taken r together.

Prove that the whole number of ways in which a selection of one or more things can be made out of n different things is $2^n - 1$. If p of the things are alike, prove that the whole number of ways in which such a selection can be made is $(p+1)\,2^{n-p} - 1$.

8. Prove the Binomial Theorem for a positive index.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}.$$

9. Define a logarithm, and prove

(i) $\log_a \frac{m}{n} = \log_a m - \log_a n$, (ii) $\log_a b \,.\, \log_b a = 1$.

Find, correct to two places of decimals, values of x and y which satisfy the equations $(2{\cdot}5)^x = 1000$, $({\cdot}25)^y = 1000$; having given $\log 2 = {\cdot}30103$00; and prove that $\frac{1}{x} - \frac{1}{y} = \frac{1}{3}$.

Paper B.

1. Prove that $ab = ba$, and $(a+b)\,c = ac + bc$, where a, b, and c are positive integers.

Shew that

$$(a+b+c+d)^3 + 3\,(a+b-c-d)(a+c-b-d)(a+d-b-c)$$
$$= 4\,(a^3+b^3+c^3+d^3) + 12abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

2. Solve the simultaneous equations:

(i) $y+z-ax=p$, $z+x-by=q$, $x+y-cz=r$;

(ii) $x^2+ay=y^2+ax=b^2$.

3. Find the relations between the roots and the coefficients of a quadratic equation.

If the length of a field were diminished and its breadth increased by 12 yards, it would be square. If its length were increased and its breadth diminished by 12 yards, its area would be 15049 square yards. Determine the area of the field.

4. Define an arithmetical progression and a geometrical progression; and shew that, if a_1, a_2, a_3 etc. be a G.P., then $\log a_1$, $\log a_2$, $\log a_3$ etc. is an A.P.

If P, Q, R be the p^{th}, q^{th} and r^{th} terms of a G.P., shew that

$$P^{q-r} Q^{r-p} R^{p-q} = 1,$$

and deduce the relation between three given terms of an A.P.

5. Find from first principles the number of combinations of n things taken r together.

In how many different ways can m shillings and $m+n$ florins be given to m' boys and $m'+n'$ girls, one coin to each, where $2m+n=2m'+n'$?

6. Under what limitations does the Binomial Theorem hold when the index is not a positive integer?

Determine the greatest term in $\left(1-\frac{1}{m}\right)^{-n}$, where m and n are integers and m is less than n.

7. Find the cube root of 100 to 3 places of decimals.

Paper C.

1. Find the value of $x^4 - 5x^3 - 12x^2 - 13x - 7$, when

$$x = -\tfrac{1}{2}(1+\sqrt{-3}).$$

2. Find the factors of

$$(a+b+c)^2 - a^2 + b^2 - c^2, \text{ and of } x^5 - y^5 - (x-y)^5.$$

3. Reduce to its lowest terms

$$\frac{2x^4 + 17x^3 + 30x^2 + 8x - 5}{x^4 + 4x^3 - 18x^2 - 29x - 10}.$$

4. Simplify

(i) $$\frac{6x^2 - 5xy - 6y^2}{14x^2 - 23xy + 3y^2} - \frac{15x^2 + 8xy - 12y^2}{35x^2 + 47xy + 6y^2};$$

(ii) $$\frac{c}{ab(b-c)(c-a)} + \frac{b}{ac(a-b)(b-c)} + \frac{a}{bc(a-b)(c-a)}.$$

5. Apply the process for extracting the square root to find m and n, when $x^4 + ax^3 + mx^2 + cx + n$ is a complete square.

6. If the roots of the equation

$$\left(1 - q + \frac{p^2}{2}\right)x^2 + p(1+q)x + q(q-1) + \frac{p^2}{2} = 0$$

be equal, shew that $p^2 = 4q$.

7. Solve the equations:

(i) $\dfrac{5}{x+10}+\dfrac{8}{x+4}=\dfrac{13}{x+7}$;

(ii) $\sqrt{2x+6}-\sqrt{x-1}=2$;

(iii) $\dfrac{a^2(x-b)}{a-b}+\dfrac{b^2(x-a)}{b-a}=x^2$.

8. Find the value of

$$\frac{(4+\sqrt{15})^{\frac{3}{2}}+(4-\sqrt{15})^{\frac{3}{2}}}{(6+\sqrt{35})^{\frac{3}{2}}-(6-\sqrt{35})^{\frac{3}{2}}}.$$

9. The metal of a solid sphere, radius r, is made into a hollow sphere, whose internal radius is r; required its thickness. [The volume of a sphere of radius r may be taken as $\frac{88}{21}r^3$.]

10. Two rectangular lawns have the same area, which is given and is equal to a^2; but the perimeter of the one is one-fourth longer than that of the other, which is a square. Find the dimensions of the lawns.

11. Shew that, if the sum of the first p terms in an A.P. be equal to zero, then the sum of the next q terms is $\dfrac{-a(p+q)q}{p-1}$.

12. Is the coefficient of x^r in the expansion of $(1-x)^{-n}$ equal to the number of combinations of $(n+r)$ things taken r together? If not, amend the proposition, and prove it as amended.

Paper D.

1. Simplify the expression

$$\frac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}\{(a+b+c)\}}.$$

2. Shew that $x^3+y^3+z^3-3xyz$ is divisible exactly by $x+y+z$; and hence, or otherwise, shew that

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

3. Find the highest common factor of $16x^4+36x^2+81$, and $8x^3+27$; and the lowest common multiple of $8x^3+27$, $16x^4+36x^2+81$, and $6x^2-5x-6$.

4. Extract the square root of

$$4x^2a^{-2}+16x^{-2}a^2-12\frac{x}{a}-24\frac{a}{x}+25.$$

5. Determine what relation must hold between a, b, and c in order that the roots of the equation $ax^2+bx+c=0$ may be real and different.

If the roots of the equation $(a^2+b^2)x^2-2b(a+c)x+b^2+c^2=0$ be real, prove that a, b, c are in G.P., and that x is their common ratio.

6. Solve the equations:

(i) $\frac{1}{2}(x-2)-\frac{x-5}{9}+5\frac{x-1}{6}=0$;

(ii) $\frac{b}{x-a}+\frac{a}{x-b}-2=0$;

(iii) $\left.\begin{array}{l} x+y+z=19, \\ x^2+y^2+z^2=133 \end{array}\right\}$, having given that y is a mean proportional to x and z.

7. Two casks, A and B, are filled with two kinds of sherry; in cask A they are mixed in the ratio 2 : 7, and in cask B they are mixed in the ratio 1 : 5. What quantity must be taken from each cask to form a mixture which shall consist of 2 gallons of the first kind of sherry and 9 gallons of the second kind of sherry?

8. Prove the formula for the sum of n terms of an arithmetic progression, whose first and last terms are given.

If s_1, s_2, s_3, &c., be the sums of m arithmetic series, each to n terms, the first terms being 1, 2, 3, &c., respectively, and the differences 1, 3, 5, &c., respectively, shew that

$$s_1+s_2+s_3+\ldots+s_m=\tfrac{1}{2}mn(mn+1).$$

9. The number of combinations of n letters, taken 5 together, in which a, b, c occur is 21. Find the number of combinations of the n letters, taken 6 together, in which a, b, c, d occur.

10. Write down the two middle terms in the expansion of $\left(x+\frac{1}{x}\right)^{2m+1}$; and shew that, if x be a small fraction, then

$\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$ is very nearly equal to $1-\frac{5}{6}x$.

11. Prove that the logarithm of the quotient of two numbers is the difference of the logarithms of the numbers.

If the logarithms of a, b, c be respectively p, q, r, prove that $a^{q-r}\,.\,b^{r-p}\,.\,c^{p-q}=1$.

Prove that $\log\frac{75}{16}-2\log\frac{5}{9}+\log\frac{32}{243}=\log 2$.

12. If the number of persons born in any year be equal to $\frac{1}{44}$th of the whole population at the beginning of the year, and the number who die be equal to $\frac{1}{60}$th of it, find in how many years the population will be doubled. [In this question, the student is supposed to have access to tables of logarithms.]

*CHAPTER XXXIII.

PROPERTIES OF NUMBERS.

*424. The properties of numbers, and the forms of numbers, which are subject to certain conditions, are treated better in works connected with higher arithmetic than in connection with algebra. Some of the more elementary propositions about numbers are however usually inserted in text-books on algebra, and a few of these propositions are given in this chapter, which may be regarded as an appendix to the present work.

SCALES OF NOTATION.

*425. **Denary and other Scales.** In arithmetic, the ordinary method of expressing an integral number by figures is to write down a succession of digits; each digit represents the product of that digit and a power of ten, and the number represents the sum of these products. Thus every digit has a *local value.*

For example, 2017 signifies $(2 \times 10^3) + (0 \times 10^2) + (1 \times 10) + 7$; that is, the 2 represents 2 thousands, *i.e.* the product of 2 and 10^3, the 0 represents 0 hundreds, *i.e.* the product of 0 and 10^2; the 1 represents 1 ten, *i.e.* the product of 1 and 10, and the 7 represents 7 units.

Similarly, a decimal fraction is the sum of a number of fractions whose denominators are powers of ten.

Thus 0·317 stands for $\frac{3}{10} + \frac{1}{10^2} + \frac{7}{10^3}$.

This mode of representing numbers is called the *common scale of notation*, or the *scale to the base ten*, or the *denary scale of notation;* and 10 is said to be the *base* or *radix* of the common scale.

We shall now prove that, in a similar way, a number can be expressed in a scale whose radix (instead of being 10) is any positive integer greater than unity. We shall confine ourselves to scales whose radices are positive integers.

***426. Whole numbers.** *A given whole number can be expressed as a whole number in any other scale of notation.*

By a given whole number we mean a number expressed in words, or else expressed by digits in some assigned scale. If no scale be mentioned, the common scale is supposed to be intended.

Let N denote the given whole number, and let r be the radix of the scale in which it is to be expressed. We have to shew that N can be written in the form

$$N = p_n r^n + p_{n-1} r^{n-1} + \dots + p_2 r^2 + p_1 r + p_0,$$

where each of the numbers $p_0, p_1, \dots, p_n$ is a positive integer less than r, or is zero. Following the analogy of the common system of numeration, the numbers $p_0, p_1, \dots, p_n$ are called the *digits* of the number in the scale r.

Divide N by r, then the quotient, say Q_1, is

$$p_n r^{n-1} + p_{n-1} r^{n-2} + \dots + p_2 r + p_1,$$

and the remainder is p_0. Thus p_0 is determined.

Next, divide Q_1 by r, then the remainder is p_1. Thus p_1 is determined.

Continuing this process (and taking r^n as the highest power of r which is contained in N), we obtain in succession the digits $p_0, p_1, \dots, p_n$ as the remainders of these divisions.

***427.** The following examples illustrate Art. 426.

Ex. 1. *Express* 2176 *in the septenary scale, that is, the scale whose radix is* 7.

The rule given in Art. 426 requires us to divide by 7, and repeat the operation on each successive quotient. Thus

$$\begin{array}{r|l} 7 & 2176 \\ \hline 7 & 310 \text{ with remainder } 6 \\ \hline 7 & 44 \quad\text{''} \quad \text{''} \quad 2 \\ \hline & 6 \quad \text{''} \quad \text{''} \quad 2. \end{array}$$

Hence $2176 = (6 \times 7^3) + (2 \times 7^2) + (2 \times 7) + 6$

$= 6226$, when expressed in the scale 7.

Ex. 2. *Transform* 6226 *from the septenary scale to the common scale.*

This is the converse of Ex. 1. We have to divide 6226 by 10. Our first step in dividing 6226 by 10 is to find how often 10 will divide into 6. Since 6 is less than 10, we cannot divide 6 by 10. Next, we have to divide 62, *i.e.* $(6 \times 7) + 2$, by 10; this gives 4 as quotient with remainder 4. Following the usual process, we divide 42, *i.e.* $(4 \times 7) + 2$, by 10; this gives 3 as quotient, with remainder 0. Next, we divide 6, *i.e.* $(0 \times 7) + 6$, by 10; this gives 0 as quotient, with remainder 6. Hence the quotient obtained by dividing 6226 by 10 is 430 with remainder 6.

Next, we have to divide 430 by 10. Continuing this process, the whole work is as follows:

$$\begin{array}{r|l} 10 & 6226 \\ \hline 10 & 430 \text{ with remainder } 6 \\ \hline 10 & 30 \quad\text{''} \quad \text{''} \quad 7 \\ \hline & 2 \quad \text{''} \quad \text{''} \quad 1. \end{array}$$

Hence 6226 in the septenary scale = 2176 in the common scale.

Ex. 3. *Change* 2418 *from the undenary scale* (*radix* 11) *to the duodenary scale* (*radix* 12).

We shall want symbols for ten and eleven, since they may now be digits. Denote them by t and e respectively.

Following the same process as in Ex. 2, we have

$$\begin{array}{r|l} 12 & 2418 \\ \hline 12 & 21t \text{ with remainder } 9 \\ \hline 12 & 1t \quad\text{''} \quad \text{''} \quad e \\ \hline & 1 \quad \text{''} \quad \text{''} \quad 9 \end{array}$$

Hence 2418 in the undenary scale = $19e9$ in the duodenary scale.

Ex. 4. *Determine which of a series of weights* 1 *lb.*, 2 *lbs.*, 4 *lbs.* *and* 8 *lbs. must be used in a balance to weigh* 13 *lbs.—not more than one weight of each kind being taken.*

This is equivalent to requiring us to express 13 in the binary scale (radix 2).

The above process gives

13 in the denary scale = 1101 in the binary scale,

that is, $13 = (1 \times 2^3) + (1 \times 2^2) + (0 \times 2) + 1 = 8 + 4 + 1.$

Hence the weights required are those marked 8 lbs., 4 lbs., 1 lb.

Ex. 5. *A weight of* 77 *lbs. is placed in one scale-pan of a balance. Shew how to place a series of weights* 1 *lb.*, 3 *lbs.*, 3^2 *lbs.*, 3^3 *lbs.*, 3^4 *lbs. in the scale-pans so as to make the beam of the balance even—not more than one weight of each kind being taken.*

If we express 77 in the ternary scale (radix 3), we find that $77 = (2 \times 3^3) + (2 \times 3^2) + (1 \times 3) + 2$. But since we have only one weight of each kind, each of the digits by which we express it in the scale of 3 must be unity or zero, though as we can put a weight in either scale we can make the digit positive or negative. If therefore we have a 2 as a remainder in our division, we must write it as $3 - 1$. Hence we have

3	77		
3	26	with remainder	−1
3	9	,, ,,	−1
3	3	,, ,,	0
	1	,, ,,	0

$$\therefore\ 77 = (1 \times 3^4) - (1 \times 3) - 1.$$

We must therefore put the 3^4 lbs. weight in one scale-pan, and in the other scale-pan we must place the weights 1 lb. and 3 lbs. in addition to the given weight of 77 lbs.

***428. Vulgar Fractions.** A vulgar fraction in one scale can be expressed as a vulgar fraction in any other scale by expressing the given numerator and the given denominator in the new scale.

***429. Radix Fractions.** The term *radix fraction* is used to denote a fraction, expressed in the scale r, in a manner analogous to that in which a decimal fraction is expressed in the denary scale.

Thus, just as the decimal fraction $0 . abc...$ stands for

$$\frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3} + \dots,$$

so the radix fraction 0 . *abc*... in the scale r stands for

$$\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}+\dots,$$

where a, b, c,... are positive integers, each being less than the radix of the scale.

*430. *A given radix fraction can be expressed as a radix fraction in any other scale of notation.*

Let F denote the given fraction, and let r be the radix of the scale in which it is to be expressed. We have to shew that F can be written in the form

$$F=\frac{q_1}{r}+\frac{q_2}{r^2}+\frac{q_3}{r^3}+\dots,$$

where each of the numbers q_1, q_2, q_3, ... is a positive integer or zero, and is less than r.

Multiply F by r, then q_1 is the integral part of the product, and $\frac{q_2}{r}+\frac{q_3}{r^2}+\dots$ is the fractional part, say F_1. Thus q_1 is determined.

Next, multiply F_1 by r, then the integral part of the product is q_2. Thus q_2 is determined.

Continuing this process, we obtain in succession the numbers q_1, q_2, ... as the integral parts of these products.

If part of a number be integral and part fractional, the parts must be treated separately: that is, the integral part must be treated by the method of Art. 426, and the (radix) fractional part by the method of Art. 430.

Ex. 1. *Express* $\frac{2}{3}$ *as a radix fraction in the binary scale, that is, in the scale whose radix is* 2.

We have

$\frac{2}{3}\times 2=1+\frac{1}{3}$; hence the 1st figure is 1,

$\frac{1}{3}\times 2=0+\frac{2}{3}$; „ „ 2nd „ 0,

and the figures now recur.

Hence $\frac{2}{3}$ in the denary scale $=0{\cdot}\dot{1}\dot{0}$ in the binary scale.

In other words,

$$\frac{2}{3}=\frac{1}{2}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\dots \text{ (to infinity)}.$$

Ex. 2. *Express* 0·261 *as a radix fraction in the scale of* 5.

We have to multiply the given number by 5; take the fractional part of the product, and multiply it by 5; and so on.

The process is as follows

0·261						
5						
1·305	thus the 1st	figure in the	radix	fraction	is	1,
5						
1·525	„	2nd	„	„	„	1,
5						
2·625	„	3rd	„	„	„	2,
5						
3·125	„	4th	„	„	„	3,
5						
0·625	„	5th	„	„	„	0,

and the last two figures recur.

Hence 0·261 in the denary scale $= 0{\cdot}112\dot{3}\dot{0}$ in the scale of 5.

*431. *The difference between any number in a scale of radix r and the sum of its digits is divisible by r* − 1.

Let N be the number, and S the sum of the digits;

$$\therefore \quad N = p_0 + p_1 r + \dots + p_n r^n,$$

and $$S = p_0 + p_1 + \dots + p_n.$$

$$\therefore \quad N - S = p_1(r-1) + p_2(r^2-1) + \dots + p_n(r^n-1).$$

The right-hand side vanishes if $r = 1$, therefore it is divisible by $r - 1$ [Art. 120]. Hence, $N - S$ is divisible by $r - 1$.

For example, in the denary scale, the difference between any number and the sum of its digits is divisible by 9. Now any multiple of 9 is divisible by 9, therefore the sum of its digits is also divisible by 9.

*432. The *rule for casting out the nines*, which is given in some text-books on arithmetic as a check on the accuracy of multiplication, affords another illustration of the use of scales of notation. Suppose that the product of two whole numbers A and B is found to be P. Divide the sum of the digits in A by 9, and let

a be the remainder. Similarly, let b and p be the remainders when the sums of the digits in B and P are respectively divided by 9. Then the rule for casting out the nines is that the difference between ab and p will be either zero or a multiple of 9.

If we use the symbol $M(9)$ to signify any multiple of 9, we have $A=M(9)+a$; $B=M(9)+b$; $P=M(9)+p$.

But $P=AB$, $\therefore M(9)+p=\{M(9)+a\}\{M(9)+b\}$

$$=M(9)+ab.$$

$$\therefore ab-p=M(9).$$

Therefore, if $ab-p$ be neither zero nor a multiple of 9, there must be a mistake in the work. But it does not follow that when the condition is satisfied the result must be right; and in fact, if the number found for the product differ from the correct result by a multiple of 9, the rule will not serve to detect an error.

*433. A number of m digits is necessarily less than 10^m. Hence, its square is necessarily less than 10^{2m}: that is, its square cannot contain more than $2m$ digits. Similarly, its cube cannot contain more than $3m$ digits; and so on.

*434. The last proposition can be used to determine some of the digits of the square root of a number which is a perfect square, by the following rule. *If the square root of a given number (which is a perfect square) contain in all* $(2n+1)$ *digits, and if the first* $(n+1)$ *of these digits have been obtained by the usual arithmetical process, the remaining n digits can be obtained by division.*

Let N represent the given number. Let a be the part of the root already obtained, which consists of $n+1$ digits followed by n ciphers; and let x represent the part of the root which remains to be found. We have

$$\sqrt{N}=a+x.$$

$$\therefore N=a^2+2ax+x^2.$$

$$\therefore \frac{N-a^2}{2a}=x+\frac{x^2}{2a}.$$

Now x contains n digits, $\therefore x^2$ cannot contain more than $2n$ digits [Art. 433]. But a contains $(2n+1)$ digits. Therefore $\frac{x^2}{2a}$ is a proper fraction.

Therefore, if we subtract a^2 from N, and divide the result by $2a$, then the integral part of the quotient will be x.

For example, the first four digits in the square root of 12088868379025 are 3476. Since there are 14 digits in the given number, there will be altogether 7 digits in the square root. Hence the remaining three digits are the integral part of

$$\frac{12088868379025 - (3476000)^2}{2 \times 3476000}$$

which will be found to be 905. Therefore the required square root is

$$3476000 + 905 = 3476905.$$

*EXAMPLES. XXXIII. A.

[*The numbers in the following examples are expressed in the common scale, unless the contrary is stated.*]

1. Express 725 in the scale 6.
2. Express 1171 in the undenary scale.
3. Change 1234 from the scale 5 to the common scale.
4. Change 111000111 from the scale 2 to the scale 12.
5. Express $\frac{13}{41}$ as a vulgar fraction in the scale 7.
6. Change ·42 from the common scale to the scale 9.
7. Change 14·23 from the scale 6 to the scale 5.
8. Express $\frac{1}{7}$ as a radix fraction in the scale 6.
9. A radix fraction in the scale 3 is $\cdot 1\dot{1}020\dot{1}$. Express it as a vulgar fraction in the scale 7.
10. In what scale is 712 expressed as 871; and in what scale as 598?
11. Multiply together 1461 and 6253 in the scale 7.
12. Divide ·1000000 by 10000 in the scale 2.
13. Extract the square root of 25400544 in the senary scale.
14. Extract the square root of 1022121 in the scale 3.
15. Extract the square root of 769 in the scale 12.
16. Shew that the difference between the square of any number and the square of the number obtained by reversing the digits is divisible by $r^2 - 1$.
17. Shew that, if the radix of the scale be odd, the difference between a number and the sum of its digits must be even.

18. Prove that a number is divisible respectively by 7 or 11 or 13, when the number expressed by the last three digits differs from the number expressed by the digits before the last three, by a quantity which is either zero, or a multiple of 7 or 11 or 13.

19. If, in a number, the difference between the sum of the digits in the units', hundreds', ten-thousands', &c. places, and the sum of the digits in the tens', thousands', hundred-thousands', &c. places, be zero or a multiple of 11, prove that the number is divisible by 11.

20. Prove that, if the digits of a number N, expressed in the denary scale, reckoning from the units' place, be $a_0, a_1, \dots a_n$, then $10^n(a_n + 2a_{n-1} + \dots + 2^n a_0) - N$, is divisible by 19.

***21.** Shew that the figure nine cannot occur in the decimal part of a fraction whose denominator, when expressed as a vulgar fraction, is less than ten.

In the scale whose radix is r, what figures, if any, cannot occur in the radix part of a fraction whose denominator, when expressed as a vulgar fraction, is less than r?

22. How many numbers can be formed with 4 digits in the scale of r, no digit being used twice in the same number?

What will be the sum of these numbers?

23. If any number be multiplied by a number which exceeds the radix by unity, shew that the result may be obtained by adding each digit to the digit to the left of it, beginning from the right, and carrying unity to the next pair when the sum of any pair is not less than the radix.

Multiply in this way by 11 the number 1243201 in the scale of 5.

24. Enunciate and prove the proposition analogous to that given in Art. 434, about the square root of a number which contains in all $2n$ digits.

25. The highest digits in the square roots of the following numbers, which are perfect squares, are given. Find the remaining digits.

(i) $\sqrt{236144689} = 153\dots$;

(ii) $\sqrt{1420913025} = 376\dots$;

(iii) $\sqrt{285970396644} = 5347\dots$;

(iv) $\sqrt{48303584206084} = 6950\dots$.

26. If the cube root of a given number, which is a perfect cube, contain $2n$ digits, and if the first $(n+1)$ of these digits have been obtained in any way, shew how the remaining $(n-1)$ of them can be obtained by division.

PROPERTIES OF NUMBERS.

*435. We have already seen [Art. 172] that we can make use of algebraical notation to illustrate some of the more obvious properties of numbers. We shall here explain the notation commonly used in the subject, and add a few examples to those given in Art. 172.

*436. **Notation of the theory of numbers.** Either of the following systems of notation is employed.

When a number, denoted by N, is an exact multiple of a number n, the relation is expressed by the symbols $N=M(n)$, which is read as *N is equal to a multiple of n.* Similarly, $N=M(n)+m$ signifies that N is equal to the sum of a multiple of n and of m; that is, that if N be divided by n, the remainder is m.

If two numbers, X and Y, when divided by n, leave the same remainder, they are said to be *congruent* to the modulus n. This relation is expressed by the symbols $X \equiv Y \pmod{n}$, which is called a *congruence*; and is read as *X is congruent to Y to the modulus n.* It is evident that $X-Y$ is exactly divisible by n, and therefore we have $X-Y\equiv 0 \pmod{n}$.

*437. We shall confine ourselves to working out a few examples. We shall, for brevity, use the word number as meaning a positive integer expressed in the common scale of notation.

Ex. 1. *Shew that a number, which is a perfect square, must be of one of the forms* $5n$ *or* $5n\pm 1$, *n being a positive integer.*

Every number, N, can be written in one of the forms $5m$, $5m-1$, $5m-2$, $5m+1$, or $5m+2$.

If $N=5m$, $\therefore\ N^2=25m^2=M(5)=5n$ (say).

If $N=5m\pm 1$, $\therefore\ N^2=M(5)+1=5n+1$ (say).

If $N=5m\pm 2$, $\therefore\ N^2=M(5)+4=M(5)+5-1=M(5)-1$
$=5n-1$ (say).

All of these forms are included in $5n$ or $5n\pm 1$.

Ex. 2. *Shew that a number, which is both a square and a cube, must be of one of the forms* $7n$ or $7n+1$.

Every number, N, can be written in one of the forms $7m \pm a$, where $a=0$, or $a=1$, $a=2$, or $a=3$. Hence, a square is of the form $(7m \pm a)^2$, which can be written $M(7)+a^2$. If $a=0$, then $a^2=0$; if $a=1$, then $a^2=1$; if $a=2$, then $a^2=4$; if $a=3$, then $a^2=9=7+2$. Hence, a square number must be of one of the forms $7n$, or $7n+1$, or $7n+2$, or $7n+4$.

Similarly, a cube must be of the form $(7m \pm a)^3$, which can be written $M(7) \pm a^3$. If $a=0$, then $a^3=0$; if $a=1$, then $a^3=1$; if $a=2$, then $a^3=8=7+1$; if $a=3$, then $a^3=27=28-1$. Hence, a cube number must be of one of the forms $7n$, or $7n+1$, or $7n-1$.

Therefore a number which is both a square and a cube can only be of one of the forms $7n$ or $7n+1$.

Ex. 3. *Shew that a number, which exceeds any odd power of* 7 *by unity, is a multiple of* 8.

The number is of the form $7^{2n+1}+1$, where n is some positive integer.

Let $$\phi(n)=7^{2n+1}+1,$$

$$\therefore \phi(n+1)=7^{2(n+1)+1}+1=7^{2n+3}+1.$$

$$\therefore \phi(n+1)-49\phi(n)=1-49=-48=M(8) \quad \text{......}(a),$$

$$\therefore \text{if } \phi(n) \text{ be a multiple of 8, so also is } \phi(n+1).$$

Now $$\phi(0)=7+1=8.$$

Put $n=0$ in (a), $\quad \therefore \phi(1)=49\phi(0)+M(8)=M(8)$.

Next, put $n=1$ in (a), $\quad \therefore \phi(2)=49\phi(1)+M(8)=M(8)$.

Proceeding in this way, by putting successively in (a) $n=2, 3, \ldots$, we see that if n be any positive integer, $\phi(n)=M(8)$.

Ex. 4. *Shew that, if n be a positive integer,* $n(n+1)(2n+1)$ *is a multiple of* 6.

If we recollect the result of the proposition [Art. 358] that

$$1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1),$$

then, since the left-hand side must necessarily be an integer, we see at once that $\frac{1}{6}n(n+1)(2n+1)$ is a whole number, which proves the proposition.

We can however prove the result directly by induction.

Let $$\phi(n)=n(n+1)(2n+1),$$

$$\therefore \phi(n+1)=(n+1)(n+2)(2n+3),$$

$$\therefore\ \phi(n+1)-\phi(n)=(n+1)(n+2)(2n+3)-n(n+1)(2n+1)$$
$$=6(n+1)^2=M(6)\ldots\ldots\ldots\ldots(a),$$
$$\therefore\ \phi(n+1)=\phi(n)+M(6).$$

$\therefore$ if $\phi(n)$ be a multiple of 6, so also is $\phi(n+1)$.

Now $\qquad \phi(1)=1.2.3=6.$

Put $n=1$ in (a), $\qquad \therefore\ \phi(2)=\phi(1)+M(6)=M(6).$

Next, put $n=2$ in (a), $\therefore\ \phi(3)=\phi(2)+M(6)=M(6).$

Proceeding in this way, we see that if n be any positive integer, $\phi(n)=M(6)$.

Ex. 5. *Shew that the product of any n consecutive integers is divisible by factorial n.*

We deduced this result in Art. 381 from the fact that the number of combinations of n things taken r at a time was necessarily an integer. We now proceed to give a direct proof of it.

We shall first assume that the product of any $n-1$ consecutive integers is divisible by $1.2.3\ldots(n-1)$; and shew that, if this be the case, then the product of any n consecutive integers is divisible by $1.2.3\ldots(n-1)n$. But one integer is always divisible by 1, therefore, the product of 2 consecutive integers is divisible by 1.2; hence, the product of 3 consecutive integers is divisible by $1.2.3$; and continuing the process, the proposition will be proved.

We assume, then, that the product of any $n-1$ consecutive integers is divisible by $1.2.3\ldots(n-1)$. Let $\phi(r)$ stand for the product of n consecutive integers from r upwards, that is,

$$\phi(r)=r(r+1)(r+2)\ldots(r+n-2)(r+n-1),$$
$$\therefore\ \phi(r+1)=(r+1)(r+2)(r+3)\ldots(r+n-1)(r+n),$$
$$\therefore\ \phi(r+1)-\phi(r)=(r+1)(r+2)\ldots(r+n-1)[(r+n)-r]$$
$$=n(r+1)(r+2)\ldots(r+n-1).$$

But $(r+1)(r+2)\ldots(r+n-1)$ is the product of $n-1$ consecutive integers, and is therefore a multiple of $1.2.3\ldots(n-1)$,

$$\therefore\ \phi(r+1)-\phi(r)=\text{multiple of } 1.2.3\ldots n.$$

Therefore, if $\phi(r)$ be a multiple of $1.2.3\ldots n$, so also is $\phi(r+1)$. But $\phi(1)=1.2\ldots n$, therefore the proposition is true of $\phi(2)$; hence, it is true of $\phi(3)$; and continuing the process, it is true of $\phi(r)$. Thus, if it be true for any $n-1$ consecutive integers, it is true for any n consecutive integers, and therefore, as explained above, it is true generally.

*EXAMPLES. XXXIII. B.

1. Shew that a number, which is a perfect cube, must be of the form $7n$ or $7n \pm 1$.

2. Shew that a number, which is a perfect square, cannot be of the form $3n - 1$.

3. Shew that the fourth power of a number is of the form $5n$ or $5n + 1$; also, that it must be of the form $7n$, or $7n + 1$, or $7n + 2$, or $7n + 4$.

4. Shew that the twentieth power of a number is of the form $25n$ or $25n + 1$.

5. Shew that the sixth power of a number is of the form $7n$ or $7n + 1$.

6. Shew that $3^{2n+2} - 8n - 9 = M(64)$.

7. Shew that $2^{2n+1} - 9n^2 + 3n - 2 = M(54)$.

8. Shew that $8^{2n+1} - 14n - 8 = M(49)$.

9. Shew that $n(n^2 - 1) = M(6)$.

ANSWERS TO THE EXAMPLES.

Chapter I. Definitions and Notation.

I. A. Pages 9—10. **1.** $3abx$. **2.** (i) 1; (ii) 1. **3.** 7; numerical. **4.** $23a$. **5.** y; literal. **6.** 1; numerical. **7.** 2. **8.** $\frac{3}{2}$. **9.** 2. **10.** $\frac{2}{3}$. **11.** $\frac{1}{3}$. **12.** 4. **13.** $\frac{1}{4}$. **14.** 3. **15.** 3. **17.** 2. **18.** −£6. 8*s*. **19.** −7 miles.

I. B. Page 12. **1.** 108. **2.** 50. **3.** $\frac{4}{3}$. **4.** 2. **5.** $\frac{4}{25}$. **6.** 2. **7.** 4. **8.** 32. **9.** 5. **10.** 5. **11.** 5. **12.** $\frac{1}{4}$. **13.** 18. **14.** 54. **15.** 18. **16.** 1458. **17.** 1. **18.** 0. **19.** 1. **20.** 0. **21.** 384. **22.** 0. **23.** 4. **24.** 24. **25.** $\frac{9}{4}$. **26.** 12. **27.** 9. **28.** 75. **29.** 3. **30.** 22. **31.** 4. **32.** 3. **33.** 49.

I. C. Page 14. **1.** 2. **2.** 0. **3.** 0. **4.** 4. **5.** 4. **6.** 4. **7.** 1. **8.** $\frac{2}{3}$. **9.** $\frac{1}{3}$. **10.** $\sqrt[3]{4}$. **11.** $\frac{2}{3}$. **12.** $\frac{4}{3}$. **13.** (i) 10; (ii) 7.

I. D. Page 17. **1.** Seven is the *coefficient* of a^3; and three is the *index*, shewing the power to which a is raised. **2.** (i) x^3 stands for $x \times x \times x$; (ii) $3x$ stands for the product of 3 and x. If $x=1$, then $x^3=1$ and $3x=3$, $\therefore 3x > x^3$. If $x=2$, then $x^3=8$ and $3x=6$, $\therefore x^3 > 3x$. **3.** $3b^2cy$. **4.** $3+b+b+c+y$. **5.** 3. **6.** 5. **7.** 3. **8.** 4. **9.** 1. **10.** n. **11.** (i) 1; (ii) 4; (iii) 1; (iv) 3; (v) 1; (vi) n. **12.** 1. **13.** 3. **14.** n. **15.** No. **16.** 180. **17.** 2. **18.** 7. **19.** (i) 137; (ii) $3\frac{3}{7}$. **20.** (i) $\frac{11}{2}$; (ii) 16.

Chapter II. Addition and Subtraction.

II. A. Page 20. **1.** $4b+c$. **2.** $-2a-2b+\frac{1}{2}c+5n$. **3.** $3x^2-8z^2$. **4.** 0. **5.** $\frac{1}{2}p^2+\frac{13}{4}q^2+2r^2-r$. **6.** 0. **7.** $\sqrt{b}$. **8.** $3\sqrt{a}-\sqrt{b}-2c$.

II. B. Pages 27—28. 1. 1. 2. -1. 3. $-x-y$. 4. $-2x^2+3y^2+z^2$. 5. $-a^3$. 6. -5. 7. -9. 8. $a-b$. 9. $x-y$. 10. He lost 30*s*., or he gained $-30s$. 11. $-a-x+y$. 12. $2x$. 13. $\frac{a}{b}+2x+y$. 14. $2y$. 15. $-2a+5b-x$. 16. x^2+b^2 or $-(x^2+b^2)$. 17. $ab-bc$. 18. $3a+3b+3c$; 33. 19. $2b^3-3b^2c$. 20. $-2y-z$. 21. $b+d$. 22. $-5a$. 23. $\frac{83}{60}a+\frac{7}{60}b+\frac{7}{60}c$. 24. $-\frac{1}{3}x^2+\frac{5}{4}y^2-2x$. 25. $-l-\frac{5}{12}m+\frac{1}{3}n$. 26. $b-a$. If a be greater than b, the answer will indicate that it is $(a-b)$ years since the man was b years old. 27. -2 miles. 28. $b-a$. 29. $-b$. 30. $x^3-(a^3+b^3)$. 31. x^2-y^2-2xy. 32. $x-y$. 33. $a-b$. 34. 22.

Chapter III. Multiplication.

III. A. Page 31. 1. $6abxy$. 2. $3abcd$. 3. $49a^2b^3x^5$. 4. $a^2b^3c^4$. 5. $2464a^2b^2c^2d^2$. 6. $abcdefx^{12}$. 7. $abcx^2y^2z^2$. 8. $a^3b^3x^2y^2z^2$. 9. $a^3b^3x^5y^2$. 10. $\frac{1}{60}l^3m^3n^3$. 11. $-7abx^2$. 12. $-7abx^2$. 13. $-a^2b^2c^2x^3$. 14. $-3l^2m^3nx^3y^3$. 15. $-48a_1a_2^2a_3^3a_4^2a_5$. 16. $-\frac{1}{6}bx^5y^7z^5$.

III. B. Pages 32—33. 1. $2x^2+3xy+xz$. 2. $-7a^2ln-7abmn-7acn^2$. 3. $l^2xy^2z-lmnxyz-ln^2xy^2$. 4. $-\frac{1}{2}$. 5. 0. 6. $20x-4y$; a binomial; coefficient of x is 20. 7. $-100x+120a$; a binomial; coefficient of x is -100. 8. $-2b+48c-10x$; a trinomial; coefficient of x is -10. 9. $86x-78y$; a binomial; coefficient of x is 86. 10. $7x-6y$; a binomial; coefficient of x is 7. 11. $-80x^2+112x+16$. 12. $a^2-ab+2b^2$. 13. $xy+ay$; $x+a$.

III. C. Page 39. 1. (i) $abxy$; (ii) $-abxy$; (iii) x^2. 2. $xy-3x+2y-6$. 3. x^2-3x+2. 4. $x^2-ax+bx-ab$. 5. $7x^2-50x+7$. 6. $49a^2-m^2$. 7. $x^2-\frac{82}{9}ax+a^2$. 8. $abx^2-a^3x-b^3x+a^2b^2$. 9. x^4-a^4. 10. $a^2x^2-abxy-2b^2y^2$. 11. $6abl^3-9b^2lm^2-4a^2l^2m+6abm^3$. 12. $12abx^2y^2-8a^2x^3z-9b^2y^3z+6abxyz^2$.

III. D. Page 41. 1. $6x^4+2x^3-15x^2+13x+6$. 2. $21x^4-50x^3-33x^2+50x+21$. 3. x^3+a^3. 4. x^3-a^3. 5. $x^4+a^2x^2+a^4$. 6. $a^3+b^3+c^3-3abc$.

III. E. Page 42. 1. x^4-49. 2. $6460-9=6391$.
3. x^3+27. 4. $4x^2-121$. 5. $(c-b)^2-a^2=-a^2+b^2-2bc+c^2$.
6. y^6-b^6. 7. $y^3-(a+1)^3$.

III. F. Page 45. 1. x^{50}. 2. $-21a^{36}x^{100}$. 3. $12a^{152}b^{206}y^{92}$.
4. $x^{2n}-1$. 5. $x^{2n}+ax^n+bx^n+ab$. 6. $x^{2n}-2x^n-x^2+1$.
7. $2x^{2m}-x^{m+n}-6x^{2n}+7x^m+7x^n+3$.
8. $a^{2n}b^n+b^{2n}c^n+c^{2n}a^n-a^nb^{2n}-b^nc^{2n}-c^na^{2n}$.
9. $a^{3n}b^{2n}-a^nb^n(a^n+b^n)x^n+(a^n+b^n)x^{3n}-x^{4n}$. 10. $x^{1001001}$.

III. G. Pages 46—47. 1. 375. 2. 9. 3. $4ab+6bd$.
4. a^2+b^2. 6. (i) a^2+c^2; (ii) $8ab$. 7. $4ab+4cd$.
8. $-12x^3-30x^2+114x-180$. 9. $x^3-9x^2y+26xy^2-24y^3$.
10. a^6-729x^6. 11. $1-\frac{5}{2}x+\frac{21}{4}x^2-\frac{21}{8}x^3+\frac{5}{4}x^4-\frac{1}{2}x^5$.
12. $x^5-4a^2x^3+3a^4x-a^5$. 13. $x^5-5ax^4+10a^2x^3-13a^3x^2+13a^4x-6a^5$.
14. $x^6+3x^5y-3x^4y^2-11x^3y^3+6x^2y^4+12xy^5-8y^6$.
15. a^2+b^2. 16. $x^3+y^3-z^3+3xyz$.
17. $6a^3-4a^2b-11a^2c-14ab^2-33abc-19ac^2-4b^3-16b^2c-19bc^2-6c^3$.
18. $6a^3+a^2b-11a^2c-19ab^2+40abc-19ac^2+6b^3-23b^2c+25bc^2-6c^3$.
19. $-x^4-y^4-z^4+2y^2z^2+2z^2x^2+2x^2y^2$.
20. $\frac{1}{60}a^4-\frac{1}{20}a^3b-\frac{3}{20}a^2b^2+\frac{23}{60}ab^3-\frac{1}{5}b^4$. 21. $2x^4-\frac{1}{2}$.
22. $\frac{1}{24}(x^6-14x^4+49x^2-36)$. 24. $a^2b-ab^2+b^2c-bc^2+c^2a-ca^2$.
27. $a^3-8a^2+23a-26$. 28. (i) 0; (ii) $6abc$. 30. $4abc$. 32. -6.

Chapter IV. Division.

IV. A. Page 50. 1. x^5. 2. $6ax^4$. 3. $16abc$. 4. $3m^2$.
5. $14a_1^4/a_3^4$. 6. $-80x$. 7. $-4l$. 8. $9x^3$. 9. $17bcx^2/y$.
10. $-4p^2x/39$.

IV. B. Page 52. 1. $a+2b$. 2. $\frac{1}{2}ax-\frac{3}{4}b^2$. 3. $-4l^2+5my$.
4. $-3ax^2-\frac{4}{3}bx^2+\frac{7}{2}cx-38$. 5. $\frac{5}{2}x^2+\frac{9}{7}x+\frac{45}{28}$.

IV. C. Page 56. 1. $x-16$. 2. $x^2+25x+8$.
3. $2x^3+\frac{1}{2}x^2+\frac{3}{4}x+\frac{33}{8}$. 4. x^2-ax+a^2. 5. x^2+ax+a^2.
6. $x^3+y^3+z^3-x^2y-x^2z-xy^2-xz^2-y^2z-yz^2+2xyz$.
7. $x^5+2x^4+x^3-4x^2-11x-10$. 8. $4x^2+5xy+10y^2$.
9. $2x^4-4x^3+3x^2-2x+1$. 10. x^2-ax+a^2. 11. $(a+3)x+(a-3)y$.
12. $6ab+6ac$. 13. $\frac{1}{64}(x^6-12x^5+60x^4-160x^3+240x^2-192x+64)$.
14. bx^2+cx-f.

IV. D. Page 58. **1.** $11a^{10}x^3$. **2.** $-17a/3x^2$. **3.** $9a^n$. **4.** x^n-1. **5.** x^n+x-1.

IV. E. Pages 58—59. **1.** $1+2x+3x^2+4x^3$. **2.** $x^2-x(y-1)+y^2+y+1$. **3.** $(a+b)^2-(a+b)c+c^2$. **4.** $3x^2-4x+5$. **5.** $x^2-5x+18-\dfrac{60x-17}{x^2+3x+1}$. **6.** $(a+b)^3-c(a+b)^2+c^2(a+b)-c^3$. **7.** $x^4-3x^3y+6x^2y^2-9xy^3+9y^4$. **8.** $2x^2+9xy-4y^2$. **9.** $2x^2-4xy+5y^2$. **10.** $-a^2-2ab+2b^2$. **11.** $x^6+2x^5y^2-3x^4y^4-6x^3y^6+2x^2y^8+4xy^{10}+y^{12}$. **12.** $bx+ay+1$. **13.** $3x^3-4x^2y+5xy^2+2y^3$. **14.** $x^2-(b+1)x+3$. **15.** $1+5x+10x^2+20x^3+40x^4$. **16.** $2a^2+3ab+2b^2$. **17.** x^2-xy+y^2. **18.** $2ab^3+3b^4$. **19.** y^2-2y. **20.** $y^5-by^4-b^4y+b^5$. **21.** $x^2+(a+2)x+3$. **22.** $x^6-3x^5+9x^4-27x^3+81x^2-243x+729$. **23.** $a^2-2ab-2b^2$. **24.** $x+1$. **25.** $a^3-b^3-c^3-3abc$. **26.** $4x^2-8xy+4xz+4y^2-4yz+z^2$. **27.** x^3+2x^2+3x+4. **28.** $x^6+x^5y-x^3y^3+xy^5+y^6$. **30.** $a^2(b+c)(c+d)(d+b)+a(bc+cd+db)^2+bcd(bc+cd+db)$. **31.** 12.

Chapter V. Simple Equations.

V. Pages 66—68. **1.** (i) Yes; (ii) No; (iii) Yes [see Art. 90]. **2.** See Art. 90. **3.** 1. **4.** $6\frac{2}{3}$. **5.** $3\frac{1}{4}$. **6.** 3. **7.** $\frac{5}{8}$. **8.** 20. **9.** $6\frac{3}{7}$. **10.** 20/27. **11.** $\frac{2}{3}$. **12.** 1. **13.** 2. **14.** 9. **15.** 20. **16.** 7. **17.** 12. **18.** 1. **19.** 72. **20.** 7. **21.** 13. **22.** 1. **23.** 10. **24.** 7. **25.** 12. **26.** 11. **27.** 5. **28.** 22. **29.** $12\frac{191}{199}$. **30.** 2. **31.** $\frac{3}{2}$. **32.** 25. **33.** $1\frac{19}{21}$. **34.** $\frac{1}{2}$. **35.** $5\frac{1}{2}$. **36.** $-\frac{2}{7}$. **37.** 1. **38.** 2. **39.** $1\frac{32}{63}$. **40.** -1. **41.** -1. **42.** 1. **43.** $\dfrac{c}{2(a-b)}$. **44.** $a+b$. **45.** ab. **46.** 6. **47.** 1. **48.** 2070/2079.

Chapter VI. Problems.

VI. Pages 77—80. **1.** 28, 18. **2.** $a/(m+1)$, $ma/(m+1)$. **3.** $105\frac{1}{3}$, $131\frac{2}{3}$. **4.** 48, 216. **5.** 72. **6.** 360. **7.** 10. **8.** 12. **9.** 28, 800. **10.** £49. **11.** 200 cavalry, 600 artillery, 1800 foot. **12.** 5. **13.** $3\frac{1}{13}$. **14.** 28 miles. **15.** 15 miles and 18 miles. **16.** £14. 8*s*. **17.** 54, 55. **18.** 9, 8. **19.** 48, 24. **20.** 18. **21.** 88 yards. **22.** £15. **23.** 180 tickets; £50. **24.** 45. **25.** 17, 22, 51. **26.** 600; namely, 100 in the gallery, 200 in the pit, and 300 in the stalls. **27.** 40 miles an hour. **28.** 26. **29.** 48*s*. **30.** £9600. **31.** $3\frac{1}{3}$ days. **32.** 20*s*. **33.** 3*s*. **34.** £1250. **35.** 6*s*. each. **36.** $4^h\,21^m\,\frac{9}{11}$. **37.** $9\frac{3}{13}$ past 10, and $21\frac{9}{11}$ minutes past 10.

Examination Questions.

Pages 81—84. **1.** (i) 8; (ii) 2. **2.** $10a-2c$.
3. $xy+2x+2y^2-zx$; $y+2-z$. **4.** 4. **5.** 7. **6.** (i) 190; (ii) $34\frac{1}{2}$.
7. (i) $2a+4b$; (ii) $-91a-56b$. **8.** $-3x^5+17x^4y-12x^3y^2-19x^2y^3+2y^5$.
9. $x+b$. **10.** A had 135*s.*, B had 90*s.* **11.** $8xy^2+x^3-6y^3+x^2y$.
12. (i) $-2a+4b-4c$; (ii) $100x-24$; (iii) 1. **13.** $9a^4-a^2b^2+4ab^3-4b^4$; if $a=1$ and $b=2$, the given expressions become respectively 9 and -3 and their product becomes -27. **14.** 7. **15.** 35 miles. **16.** 7; -1; 4.
17. (i) $2a-3$; (ii) $x-3$; (iii) 0. **18.** $2a^4+5a^3b-5ab^3-2b^4$; four.
19. (i) $10a$; (ii) $-2\frac{1}{3}$. **20.** 528.
21. $2a^2+5ab+3b^2$; $3a^4-4a^3b-6a^2b^2-4ab^3-b^4$; $a^2-b^2+2bc-c^2$.
22. $3bc$; a^2+ax+x^2; $3x+2y-z$. **24.** 3. **25.** 15 days.
26. (i) $x^4+2x^3+2x^2+3x-2$;
(ii) $-a^3-b^3-c^3-2abc+a^2b+ab^2+a^2c+ac^2+b^2c+bc^2$.
27. $(x^2+15x+50)\div(10+x)=x+5$; if $x=2$, $84\div12=2+5$.
28. $2x^3-3x^2+x-2$; $a^4+2a^3b+6a^2b^2+2ab^3+b^4$. **29.** (i) 1; (ii) 3.
30. 22 miles. **31.** (i) ac; (ii) $-100x+124a$; -100.
32. a^4-64b^4; $a^3+3a^2b+9ab^2+27b^3$. **33.** $2a^2-5ab+2b^2$.
35. 12 miles an hour. **36.** 36. **37.** $x^4-2x^3+x^2-2x+1$.
40. £4680; £4720.

Chapter VII. Factors.

VII. A. Page 86. **1.** $x(a+b)$. **2.** $5x(1-4x)$. **3.** $a(a-3b)$.
4. $3lm(m-3l)$. **5.** $xy(x+y+1)$. **6.** $p(11p^2-2pq-3q^3)$.
7. $4xyz^2(2y+3x-4z^2)$. **8.** $(a-b)(x-y)$. **9.** $(a+b)(c+d)$.
10. $(x-3)(x+l)$. **11.** $(a+p)(x+y+z)$. **12.** $(ay-1)(y+1)$.

VII. B. Page 88. **1.** $(a-11)(a+11)$.
2. $(9-12)(9+12)=-3\cdot21=-63$. **3.** $(xy-11)(xy+11)$.
4. $(1-8b)(1+8b)$. **5.** $(x-1)(x+1)$. **6.** $(x-\sqrt{5})(x+\sqrt{5})$.
7. $(2-x)(2+x)$. **8.** $(x-2)(x+2)$. **9.** $(3-2a)(3+2a)$.
10. $(11l-5m)(11l+5m)$. **11.** $(9p-ab)(9p+ab)$. **12.** $a^2(b-x)(b+x)$.
13. $(ab-xy)(ab+xy)$. **14.** $(x-12lm)(x+12lm)$.
15. $x^2(x-12m)(x+12m)$. **16.** $(a-b)(a+b)(a^2+b^2)$.
17. $(2x-3a)(2x+3a)(4x^2+9a^2)$. **18.** $(2ab^2c^3-3x^2)(2ab^2c^3+3x^2)$.

19. $(x+y-11)(x+y+11)$. **20.** $(a-b-3c)(a-b+3c)$. **21.** $4xy$.
22. $(a^2+b^2-a)(a^2+b^2+a)$. **23.** $(a^2-b^2-b)(a^2-b^2+b)$.
24. $4(a+b)c$. **25.** $-7(x-y)(x+y)$. **26.** $(5a-4b)(4b-3a)$.
27. $2a(a-2)$. **28.** $\{\sqrt{a}-(x+y)\}\{\sqrt{a}+(x+y)\}$.
29. $(l-n)(l+2m+n)(l^2+2lm+2m^2+2mn+n^2)$.
30. $5(3x-1)(x+3)(x^2+1)$.

VII. C. Page 89. **1.** $(x+1)^2$. **2.** $(x-3)^2$. **3.** $(2x-3y)^2$.
4. $(x^2-4)^2$. **5.** $(a-\frac{3}{2}b)^2$. **6.** $\{a+b-2(a-b)\}^2=(a-3b)^2$.

VII. D. Page 90. **1.** $(x-1)(x^2+x+1)$. **2.** $(a+1)(a^2-a+1)$.
3. $(2ax-1)(4a^2x^2+2ax+1)$. **4.** $(x+y)(7x^2-13xy+7y^2)$. **5.** $(x+1)^3$.
6. $(1-y)^3$. **7.** $(x+a+1)(x^2+a^2+1-x-a-ax)$.

VII. E. Page 90. **1.** $(1-a)(1+a)(1+a^2)$.
2. $(x-2)(x+2)(x^2+4)$. **3.** $8(x+y)(2x+3y)(5x^2+14xy+10y^2)$.
4. $a(a+2)(a^2+2a+2)$. **5.** $(y-1)^4$.
6. $(a^2+ab+b^2+a-b+1)(a^2-ab+b^2+3a-3b+3)$.

VII. F. Page 91. **1.** $4x^2-6xy+9y^2$. **2.** $a(13a^2-10ab+2b^2)$.
3. $7x+y+z$. **4.** $4(7x^2+4y^2+7z^2+8xy+8yz+2zx)$.
5. $7a^2+13b^2+21c^2+19ab+33bc+24ca$. **6.** $(a+b)(c-d)$.

VII. G. Page 94. **1.** $(x+3)(x+6)$. **2.** $(a-3)(a-4)$.
3. $(3+y)(4+y)$. **4.** $(x-8)(x-11)$. **5.** $(n-7)^2$.
6. $(y-7)(y-12)$. **7.** $(a-2b)(a-3b)$. **8.** $(xy-2)(xy-27)$.
9. $(1-2l)(1-11l)$. **10.** $(11x^2+1)(2x^2+1)$. **11.** $(a-8b)(a-11b)$.
12. $(a'-6b')(a'-11b')$. **13.** $(p-2)(p-9)$. **14.** $(1-14z)(1+6z)$.
15. $(x+2)(x-1)$. **16.** $(a-2)(a+1)$. **17.** $(x-10)(x+2)$.
18. $(y-7)(y+6)$. **19.** $(a+7b)(a-4b)$. **20.** $(a-8m)(a+3m)$.
21. $(x-12)(x+3)$. **22.** $(b+7)(b-6)$. **23.** $(x^2-15)(x^2+6)$.
24. $(a+11)(a-10)$. **25.** $(x-7)(x+6)$. **26.** $(x+8y)(x-2y)$.
27. $(y-17)(y+6)$. **28.** $(x-17)(x+5)$. **29.** $(a_1-19)(a_1+3)$.
30. $(ab-11)(ab+22)$.

VII. H. Page 97. **1.** $(x-\frac{1}{2})^2-(\frac{3}{2})^2=(x-2)(x+1)$.
2. $(y-\frac{7}{2})^2-(\frac{1}{2})^2=(y-4)(y-3)$. **3.** $(n-12)^2-7^2=(n-19)(n-5)$.
4. $(a+\frac{1}{2})^2-(\frac{13}{2})^2=(a-6)(a+7)$. **5.** $(x-\frac{21}{2})^2-(\frac{5}{2})^2=(x-13)(x-8)$.
6. $(1-6b^2)-(7b)^2=(1-17b)(1+5b)$.

7. $(a-\frac{11}{2}b)^2-(\frac{15}{2}b)^2=(a-13b)(a+2b)$. **8.** $(x^2+4)^2-3^2=(x^2+1)(x^2+7)$.
9. $(\frac{21}{2}a)^2-(\frac{7}{2}a+b)^2=(7a-b)\,(14a+b)$.
10. $\{(x+\frac{79}{20})^2-(\frac{81}{20})^2\}=(10x-1)\,(x+8)$.
11. $11\{(a+\frac{75}{22}b)^2-(\frac{79}{22}b)^2\}=(11a-2b)\,(a+7b)$.
12. $14\{(y-\frac{25}{28}z)^2-(\frac{17}{28}z)^2\}=(7y-2z)\,(2y-3z)$.
13. $\{x-\sqrt{(-1)}\}\{x+\sqrt{(-1)}\}$. **14.** $\{x-2-\sqrt{(-1)}\}\{x-2+\sqrt{(-1)}\}$.
15. $\{x+\frac{3}{2}-\frac{1}{2}\sqrt{(-3)}\}\{x+\frac{3}{2}+\frac{1}{2}\sqrt{(-3)}\}$.

VII. I. Pages 99—100. **1.** (i) Yes; $(x+1)\,(x+2)$. (ii) Yes; x^2+x-5. (iii) No. **2.** (i) Yes; $(x+1)\,(x+3)$. (ii) No. (iii) Yes; x^2-2x+6. **3.** (i) $(x-1)(x+1)(x-2)$. (ii) $x(x-1)\,(x-6)$. (iii) $(x-1)\,(x-2)\,(x+a)$. (iv) $(x-2)(x+2)(x+5)$. (v) $(x+1)^2\,(x-2)\,(x+3)$.

VII. J. Page 100. **1.** $x\,(x-6)\,(x+6)$. **2.** $3x\,(x+6)$.
3. $(x-y)\,(x^2-5xy+7y^2)$.
4. $(27x^3-y^3)(27x^3+y^3)=(3x-y)(3x+y)(9x^2+3xy+y^2)(9x^2-3xy+y^2)$.
5. $8\,(a-b)(a+b)(a^2+b^2)$. **6.** $(a-1)(a+1)(b-1)(b+1)$.
7. $(y-1)\,(y+1)^2$. **8.** $y\,(x-y)\,(2x^2+xy+y^2)$.
9. $(3a+2c)\,(3a+2b-2c)$. **10.** $(a-y)(a+z)\,(a^2-ay+z^2)$.
11. $(a-b)\,(a+b+4)$. **12.** $(2x+3a)\,(3x-a-1)$.
13. $(a+b-4c)\,(a-b+4c)$. **14.** $(a+b)\,(a+b+2c)$.
15. $(3a-4c)\,(3a+2b+4c)$. **16.** $(9x+8)\,(8x-9)$.
17. $(-a+5c)\,(3a+4b+c)$. **18.** $(x-1)\,(x+1)\,(x+3)\,(x+5)$.
19. $(x-1-\sqrt{2})\,(x-1+\sqrt{2})\,(x+1-\sqrt{2})\,(x+1+\sqrt{2})$.
20. $(a+b)\,(b+c)\,(c+a)$. **21.** $a^2-ab\sqrt{2-m}+b^2$.
22. $x^8+a^4x^4+a^8$; $x^6-2ax^5+8a^3x^3-32a^5x+64a^6$.
23. $(x+y-z)^2\,(y+z-x)^2(z+x-y)^2$. **25.** $2abc\,(a+b+c)$.
26. $(a-b)\,(b-c)\,(c-a)$. **27.** $a\,(a-b)\,(a-2b)$.

Chapter VIII. Highest Common Factors.

VIII. A. Page 102. **1.** abc^2. **2.** $5x^2y^2z^2$. **3.** $2p^4q^6r$.

VIII. B. Page 103. **1.** $x-y$. **2.** a. **3.** $x+a$. **4.** $x-2$. **5.** $x-1$. **6.** $2x+1$. **7.** $3ab\,(a-b)$.

VIII. C. Page 103. **1.** $a+b$. **2.** $a-b$. **3.** $x+1$.

VIII. D. Pages 107—109. **1.** $4x-5$. **2.** $a-1$. **3.** $a+2$. **4.** $x+3$. **5.** $2x-1$. **6.** $x-9$. **7.** $2x+3$. **8.** $3x^2+8x-3$. **9.** x^2-1. **10.** x^2-1. **11.** $2x^2+3$. **12.** x^2+3x-4. **13.** x^2-x-1. **14.** $2x-1$. **15.** x^2-2. **16.** $x-2$. **17.** x^2-3x+4. **18.** $x+1$. **19.** x^2+3x+1. **20.** $2x^2-4x+3$. **21.** $2x^2-3x+4$. **22.** $2x^2+3x+1$. **23.** x^2-2x+1. **24.** x^2+3x+1. **25.** x^2-2x+4. **26.** $x^2-7xy+9y^2$. **27.** $x^2-5xy+7y^2$. **28.** $3x-1$. **29.** x^2-a^2. **30.** $4(x-a)$. **31.** $6(x+a)$. **32.** $x^2+xy+3y^2$. **33.** $x+1$. **34.** $1+x^3-x^4$. **35.** $2x+1$. **36.** $ay+bx$. **37.** -3. **38.** 6; $x-2$. **39.** $2a+7b$; 75. **40.** $x-1$. **41.** -1.

Chapter IX. Lowest Common Multiples.

IX. A. Page 111. **1.** $6x^5$. **2.** $1080a^2bc^2xyz$. **3.** $105a_1^4a_2^6a_3^6a_4^4x^2y^2$. **4.** $(x^2-y^2)^2$. **5.** $(x+2y)^2(x-2y)^3$. **6.** $72(x-y)^2(x^3+y^3)$.

IX. B. Pages 112—113. **1.** $(x^2+1)(x^6-1)$. **2.** $x(3x-y)(x^3+y^3)$. **3.** $(x-1)(6x^2-5x-6)$. **4.** $x^7(x^9+1)(x^6-x^5+x^4-x^3+x^2-x+1)$. **5.** $(x-1)^3(x+1)$. **6.** $(x-8)(x^2-7x+9)(x^2-10x+11)$. **7.** $(x-4)(3x-2)(3x^2+2x+1)$. **8.** $(x+1)^2(x^2-3x+5)(x^2+3x+5)$. **9.** $(x-1)(x+6)(x^3+8)$. **10.** $(x^2+x+1)(2x^3-14x^3+26x-30)$. **11.** $(x^5+1)(x^3-8x+3)$. **12.** $(x^2-y^2)(x^3-y^3)$. **13.** $(x^2+2x+3)(3x^2+x+2)(2x^2+3x+1)$. **14.** $(x^2-1)(x^2-4)$. **15.** $(9x^2-4)(4x^2-9)$. **16.** $x^2y(x-y)(x^2-y^2)(x^6-y^6)$. **17.** $x^3y^2(x^2-y^2)(x^6-y^6)$. **18.** $xy(x-y)(x^3-3x^2y+5xy^2-6y^3)$. **19.** $x^3+2ax^2-a^2x-2a^3$. **20.** $x^2-12x+35$.

Chapter X. Fractions.

X. A. Page 118. **1.** $\frac{x-1}{x+5}$. **2.** $\frac{x^2+2y^2}{x^2+8y^2}$. **3.** $\frac{x^2+2y^2}{x^2+9y^2}$. **4.** $\frac{4a(a^2+1)}{3a^2+1}$. **5.** $\frac{x^2(5x-1)}{2x^3-1}$. **6.** $\frac{x^3+2x^2-11x+6}{2x^2+4x-7}$. **7.** $\frac{x^2-5xy+2y^2}{2x^2-3xy+y^2}$. **8.** $(a-1)(b-1)=ab-a-b+1$. **9.** $\frac{3a^2-ab+b^2}{2a^2+b^2}$. **10.** 1.

X. B. Pages 121—122. 1. $\frac{-2b^3}{a^2-b^2}$. 2. -2.

3. $\frac{2x+a}{x-a}$. 4. $\frac{x+3a}{x+a}$. 5. $\frac{6}{(x^2-1)(x+3)}$.

6. $-\frac{2(x-6)}{(x-2)(x-3)(x-4)(x-5)}$. 7. $\frac{2(b+y)}{b-y}$. 8. $-\frac{5ab(3a^2+4b^2)}{a^4-16b^4}$.

9. $\frac{y(3x^2+8xy+12y^2)}{2(x^4-16y^4)}$. 10. $\frac{x-5}{(x-3)(x-4)}$. 11. $\frac{x^2-x+1}{x^2-5x+6}$.

12. 0. 13. $\frac{1}{x^3-1}$. 14. $\frac{4(a^2+b^2)}{a^3-b^3}$. 15. $\frac{x-4}{(x-2)(x-3)}$.

16. $\frac{a^2}{a^3-b^3}$. 17. $\frac{1}{(x-3)(x-4)(x-5)}$. 18. $\frac{1}{1+2x}$.

19. $\frac{2}{(x^2-1)(x^2-4)}$. 20. $\frac{4}{(x^2-1)(x^2-4)}$. 21. 0. 22. $\frac{4(1+2x^2)}{1-x^4}$.

23. 0. 24. 0. 25. 0. 26. 1. 27. $\frac{1}{xyz}$. 28. $m+n$. 29. 0.

X. C. Page 124. 1. 1. 2. $\frac{1}{(x^2-y^2)}$. 3. $\frac{a^2+ab+b^2}{a}$.

4. $\frac{1}{x+y}$. 5. $\frac{2xy^2}{x+3y}$. 6. $\frac{y^2}{xz}$.

X. D. Page 126. 1. $\frac{x-3y}{x+3y}$. 2. $\frac{x-4}{x+4}$. 3. $\frac{x}{x+a}$.

4. $\frac{x^4}{x^2+y^2}$. 5. $4(a+b+c)$. 6. 32/3. 7. $\frac{2x}{7y}-5+\frac{3y}{4x}$.

X. E. Pages 128—131. 1. 0. 4. Each $=\frac{1}{a}+\frac{1}{b}$.

5. $\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)$; $(x-a)\left(x-\frac{1}{a}\right)$. 6. $\frac{1}{x^2-1}$.

7. $-\frac{1+2x}{1+x}$. 8. $4xy$. 9. $\frac{1}{2a^2-1}$. 10. a^2. 11. 1. 12. 0.

13. $\frac{3x-y}{2y}$. 14. $\frac{x^2}{y^2}$. 15. 0. 16. $\frac{xy}{x^2+y^2}$. 17. $\frac{5}{3}$. 18. $\frac{3a+2b}{6a}$.

19. -3. 20. 0. 21. 0. 22. 0. 23. $\frac{a^7+a^4b^3+b^7}{b(a^6-b^6)}$.

24. 1. 25. 0. 26. $\frac{3(3a+4)(4a+5)}{(2a+3)(5a+6)}$. 27. $\frac{1+x^4}{x(1+x^2)}$.

28. $\frac{a^2+b^2}{ab(a-b)^2}$. 29. a^4+b^4. 30. b^4+c^4. 31. $\frac{1}{xy}$. 32. $\frac{a-c}{(1+ac)}$.

33. $a+b$. **34.** $\frac{x-y}{x+y}$. **35.** $\frac{1-x+x^2}{x}$. **36.** $\frac{a^2-b^2}{a^2b}$.
37. $\frac{x+y}{x-y}$. **38.** 1. **39.** $\frac{a^2-a+1}{2a-1}$. **40.** 1. **41.** $\frac{x^2}{y^2}$. **42.** 1.
43. $\frac{1}{x+y}$. **44.** 0. **45.** 1. **46.** 1. **47.** $\frac{x+y}{x-y}$.
48. x^2-xy+y^2-1.

Chapter XI. Simple Equations continued.

XI. Pages 135—138. **1.** $\frac{1}{3}$. **2.** $\frac{9}{7}$. **3.** $-\frac{5}{4}$. **4.** 35/16.
5. 7. **6.** $-\frac{4}{5}$. **7.** $2\frac{3}{4}$. **8.** 9. **9.** 2. **10.** $-\frac{1}{2}$. **11.** $a+b$.
12. b. **13.** $\frac{(2a-b)(a-2b)}{a}$. **14.** $\frac{1}{2}(d-c)$. **15.** $\frac{1}{2}(a-c)$.
16. $c+d$. **17.** $\frac{a+b+c}{ab+bc+ca}$. **18.** $\frac{a}{b}$. **19.** $a+b+c$. **20.** 11.
21. 4. **22.** $-\frac{a^2+b^2+c^2-2ab-2ac-2bc}{a+b+c}$. **23.** 588 acres.
24. 16. **25.** £580. **26.** 7 miles. **27.** 14 miles.
28. $4\frac{2}{7}$ miles. **29.** A in $4^m 35^s$; B in $4^m 37^s$. **30.** Rate of boat is 11 times rate of stream. **31.** $2\frac{1}{4}$ miles. **32.** 35 °/₀ above cost price.
33. 6 °/₀. **34.** 4500. **35.** 8 times as much water as spirit.
36. 136. **37.** 50. **38.** $5^h 12^m$ and $5^h 42\frac{6}{11}{}^m$; $15\frac{3}{11}$ minutes.
39. $5\frac{5}{719}$ seconds past 12. **40.** $3\frac{3}{11}$ hours; $2\frac{2}{11}$ hours.
41. $1\frac{5}{24}$ minutes to 12. **42.** $8\frac{4}{7}$ o'clock. **43.** $\frac{mh}{m+2}$.
44. $\frac{pb(mn-mp+am+an)}{ma(n-p)}$.

Chapter XII. Simultaneous Simple Equations.

[*The numbers placed first and second in the answer are respectively the values of* x *and* y *which satisfy the equation. For example, the solution of Ex.* 1 is $x=111$, $y=11$.]

XII. Pages 148—151. **1.** ~~111~~; 11. **2.** 9; 8.
3. 91; 19. **4.** $\frac{2}{3}$; $\frac{5}{3}$. **5.** 3; 10. **6.** $\frac{3}{2}$; $\frac{5}{2}$. **7.** 2; 3. **8.** 39; 2.
9. $-\frac{1}{11}$; $\frac{1}{5}$. **10.** 5; 3. **11.** $16\frac{1}{4}$; $10\frac{3}{4}$. **12.** $\frac{1}{7}$; $-\frac{1}{3}$. **13.** a; b.
14. $\frac{3}{a}$; $-b$. **15.** $\frac{1}{a}$; $\frac{1}{b}$. **16.** $\frac{1}{a}$; $\frac{1}{b}$. **17.** $\frac{3}{11}$; $\frac{1}{11}$. **18.** 3; 4.
19. 3; -1. **20.** 6; 1. **21.** 7; -5. **22.** $\frac{1}{7}$; $\frac{1}{5}$. **23.** $\frac{1}{7}$; $\frac{1}{4}$.

24. 3; 5. **25.** 12; 3. **26.** 13; 17. **27.** 2; 3. **28.** 3; 4.
29. 4; 3. **30.** 4; -5. **31.** 112; 168. **32.** 36; 40. **33.** 3; 4.
34. $-1/37$; $-2/37$. **35.** $\frac{7}{30}$; $\frac{1}{4}$. **36.** 15; 16. **37.** -12; $-14\frac{2}{3}$.
38. 12; 16. **39.** 12; 14. **40.** 15; 6. **41.** 144; 216.
42. 40; 30. **43.** 308; 21. **44.** 15; 18. **45.** 12; 24.
46. 14; 18. **47.** 6; 12. **48.** $\frac{6}{13}$; $\frac{3}{13}$. **49.** $\frac{1}{9}$; 3. **50.** 12; 8.
51. $\frac{2}{3}$; $2\frac{1}{4}$. **52.** $\frac{1}{4}$; $\frac{1}{7}$. **53.** $\frac{1}{6}$; $\frac{1}{7}$. **54.** $\frac{1}{7}$; $\frac{1}{4}$. **55.** $\frac{1}{3}$; $\frac{1}{2}$.
56. $\frac{1}{4}$; $\frac{1}{8}$. **57.** $-\frac{b}{c}$; $-\frac{a}{c}$. **58.** $\frac{ab}{a+b}$; $-\frac{ab}{a+b}$.
59. $\frac{b^2+c^2-a^2}{2a}$; $\frac{c^2+a^2-b^2}{2b}$. **60.** $2b-a$; $2a-b$.
61. $\frac{1}{5}(7a+8b)$; $\frac{1}{5}(8a+7b)$. **62.** -1; 1. **63.** $a+c$; $b+c$.
64. $a+b$; $a-b$. **65.** 12; -2. **66.** 3; 3. **67.** 16; -4.
68. 14; 17. **69.** $\frac{1}{5}$; -17. **70.** 228/65; 228/145.
71. $-(a+b)$; $a+b$. **72.** 4; 5; 6. **73.** 3; $3\frac{2}{3}$; $2\frac{2}{3}$.
74. $\frac{d^2(ab-bc+ca)}{2abc}$; $\frac{d^2(bc-ca+ab)}{2abc}$; $\frac{d^2(ca-ab+bc)}{2abc}$.

Chapter XIII. Problems leading to Simple Equations.

XIII. Pages 158—164. **1.** 36; 35. **3.** 41; 14.
4. 73. **5.** 31; 13. **6.** 31/43. **7.** $\frac{2}{5}$. **8.** $\frac{3}{8}$. **9.** 9; 7; 5.
10. £37. 10*s.*; £30. **11.** £38; £32. **12.** 12*s.*; 6*s.*
13. 1*s.* 8*d.*; 2*s.* 6*d.*; 2*s.* 2*d.* **14.** 135*s.*; 90*s.* **15.** 24 yds.; 16 yds.
16. 120 lbs.; £33. **17.** 30 years; $7\frac{1}{2}$ years; 6 years. **18.** 12 years.
19. 17 years, 14 years, 12 years, 9 years ago. **20.** $23+a$; $\frac{3}{2}(23+b)$.
21. $\frac{a-cp}{b-c}$; $\frac{bp-a}{b-c}$. **22.** £2400; £900.
23. £450; £225; £237. 10*s.*; £87. 10*s.*
24. 50 and 30 miles an hour. **25.** 359; 241; 128.
26. £20 for an ox; £2. 10*s.* for a sheep. **27.** 10 of each.
28. £150, 125 qrs. of wheat, 200 qrs. of barley.
29. £9975; £95, £105. **30.** 4; 10; 2; 12. **31.** 108.
32. $15\frac{2}{3}$; $13\frac{2}{3}$; $18\frac{2}{3}$. **33.** 1881. **34.** 6770.
35. 12 and 15 miles an hour. **36.** 6 minutes.
37. *A* in 5 minutes; *B* in 5 min. 20 secs. **38.** *A* in 24 days, *B* in 32 days, and *C* in 48 days; in the ratio 7 : 3 : 2.
39. 48 miles. **40.** 44 sovereigns, 208 half-crowns, 600 shillings.
41. *A* has 6*s.*, *B* has 10*s.*, *C* has 16*s.*

42. £1200 at 3 %, £800 at 4 %.
43. 3 pence, 6 halfpence, 2 three-penny pieces.
44. 3 florins, 6 sixpences, 4 half-crowns. **45.** 2*s.* 6*d.*
46. 12 quarts. **47.** 60 acres arable, 160 acres pasture.
48. *A* has £3000, *B* has £5000. **49.** £2800 at 4 %, £1200 at 7 %.
50. £2320 and £2350. **51.** 40 men; 72 days.
52. 13000 town subscribers, 14000 country subscribers.
53. 9 gallons; 18 gallons.
54. 3 miles an hour, 4 miles an hour, 5 miles an hour.
55. 55 seconds; 11 seconds. **56.** 12 miles.
57. £88; £118. **58.** 9 yards by 4 yards.
59. The quick, which arrives at $10^h\ 6^m$: the slow steamer arrives at $10^h\ 15^m$.

Chapter XIV. Miscellaneous Propositions.

XIV. A. Page 166. **4.** $\frac{x^3-y^3}{x-y}=x^2+y^2+xy$.

XIV. B. Page 168. **1.** 216.

XIV. C. Page 170. **1.** $13y-5=0$.
2. $y^2(a^2+ab+b^2)=a^2b$. **3.** $a-1=0$. **4.** $5x^2+4=0$.
5. $m^2x^2+2(mc-2a)x+c^2=0$. **6.** $x^2-ax=0$. **7.** $61x-87=0$.

XIV. D. Page 172. **1.** (i) Symmetrical to a and b; (ii) Symmetrical to a, b, c; (iii) Not symmetrical; (iv) Symmetrical to a, b, c. **2.** No (see Art. 178): no (see Art. 74).
3. (i) $c-a$, $a-b$; (ii) c^2-a^2, a^2-b^2; (iii) $b(c-a)$, $c(a-b)$; (iv) $(b-c)(c+a)$, $(c-a)(a+b)$.
4. $x+2y$. **5.** $3(x+y)$.

XIV. E. Page 175. **1.** $\frac{x+4}{x+7}$. **2.** Any positive value of x. **3.** If $x>-\frac{7}{2}$.

XIV. F. Pages 176—177. **1.** $(2n+1)^2-(2n-1)^2=4(2n)$.
5. Subtract 14; the result is a number of two digits, which are the numbers thought of. **7.** 226. **9.** 1. **14.** (i) $ax^2+bxy+cy^2$; (ii) $ax^2+bxy+ay^2$; $a(x^2+y^2)+bxy+c(x+y)+d$.
15. $x^2+y^2+z^2+2(xy+yz+zx)$. **16.** $5y^2-6=0$.
17. $a^3+b^3+c^3-3abc=0$. **18.** The first, if $x>a$; the second, if $x<a$.

Examination Papers and Questions.

Paper A. Page 178. 1. $ab-ac-c^2$. 2. $\frac{4x^2y^2}{x^6-y^6}$.
3. $x^2-(y-1)^2$. 4. $(8x-3)(x+2)$. 5. $(ac+bd)(ad+bc)$.
6. x^4+x^3-2x-4. 7. 0. 8. $5\frac{1}{2}$, -7. 10. 984,404 feet.

Paper B. Page 179. 1. 8; -1.
2. $x^5+2x^4+x^3-4x^2-11x-10$. 3. 12. 4. The H.C.F. is $3x^2+2x+1$; the L.C.M. is $(9x^3-x-2)(x-4)$. 5. $b^2=4ac$; $3x^3-2x^2+3x+2$.
6. $\frac{x+y}{x-y}$. 7. 1. 8. (i) 144; (ii) 3; $-\frac{1}{3}$; (iii) $x=111$, $y=11$.
9. 2*s*. 6*d*. 10. 300 lbs.

Paper C. Page 180. 1. $2ab-2b^2+2bc$. 2. 0.
3. $(y^4+b^4)^2$. 4. x^2-3x+1. 5. $\frac{b}{a+b}$. 6. 0.
7. (i) 72; (ii) $x=5$, $y=4$. 8. 4 : 1.

Paper D. Page 180. 1. 7. 2. $(x-14)(x+6)$.
3. $3x(x+6)$. 4. $-(a-b)(b-c)(c-a)$. 5. $3x-1$; $x=\frac{1}{3}$.
6. (i) x^2+3x+2; (ii) 2. 7. (i) $4\frac{2}{3}$; (ii) $1\frac{1}{2}$; (iii) $x=\frac{1}{2}(2a+b)$, $y=\frac{1}{2}(2a-b)$. 8. 4 times.

Examination Questions. Pages 181—183.
1. $27a-39b+57c+15d$; 180. 2. x^2+1. 3. $(x-1)(x-2)(x-3)$.
4. (i) $\frac{5x^2+x+1}{2x^2-x-1}$; (ii) $\frac{1}{(x+2)(x-3)}$. 5. 216; 241; 245.
6. See Art. 120; $(x-1)(x-5)(x+14)$. 7. $a+b$. 8. -3.
9. (i) 3; (ii) 3. 10. $17\frac{1}{2}$ minutes. 11. $a+b$; $\frac{2}{3}(a+b-c)$.
12. H.C.F. $=a-5b$; L.C.M. $=(a-5b)(a-3b)(a+2b)(a+7b)$.
13. $\frac{x^3+9x^2+3x-166}{(x-4)(x+2)(x+5)}$; 1. 14. 10 gallons. 16. $\frac{1}{a^8}-a^8$.
17. $(x-1)(x^2-x+1)$. 18. $\frac{3}{(x+3)(x+5)(x+6)}$.
19. (i) $x=18$, $y=10$; (ii) $x=a$, $y=b$. 20. 20. 21. $\frac{x^2+y^2}{xy}$.
22. $-\frac{1}{4}$. 23. (i) $-\frac{14}{3}$; (ii) 3/28. 24. $x=a$, $y=b$, $z=c$.
25. 4 miles an hour. 26. $(4a-3b)(5a+9b)$. 28. 4320.

Chapter XV. Evolution.

XV. A. Page 186. 1. x^5. 2. ab^2. 3. $\frac{1}{3}a^2$. 4. $\frac{a^2}{b}$. 5. $\frac{12}{x}$. 6. $7x^2y^3$. 7. $\frac{9x^4y}{10a^2b^4}$. 8. x. 9. $-x$. 10. $3a^2b^2$. 11. $-\frac{2x^2}{y^3}$. 12. $\frac{\frac{4}{3}a^3c}{x^2}$. 13. x^2y^3. 14. $2ab^2$. 15. $-\frac{3x}{y^2}$. 16. ab^2.

XV. B. Page 187. 1. $ax-b$. 2. $11x-17$. 3. $\frac{1}{2}y+\frac{1}{3}z$. 4. $3a_1a_2-4a_3a_4$. 5. $8-\frac{1}{2}p$. 6. $19lm+15ln$. 7. $\frac{1}{3x}-\frac{x}{2y}$. 8. $\frac{1}{4x}-\frac{x}{3y}$.

XV. C. Page 189. 1. $x+a+b$. 2. $ax+b-1$. 3. $(a-1)x+b$. 4. $(a-2)x-(b-1)$. 5. $a-3b+5c$. 6. $x+\frac{1}{3}a-\frac{1}{2}b$.

XV. D. Pages 194—195. 1. x^2+2x-2. 2. x^2-4x-3. 3. $2x^2-3x-9$. 4. $2x^2-x+\frac{1}{2}$. 5. $6x^2-3x+\frac{2}{3}$. 6. $2a^2-3a+3$. 7. $2a^2+3a+3$. 8. $2x^2-x+2$. 9. $4x^2+16x+11$. 10. $3x^3-2x+5$. 11. $3x^3-2x^2+3x+2$. 12. $3x^2-2xy+y^2$. 13. $3x^2-4xy+4y^2$. 14. $2x^2+\frac{1}{2}bx-ab$. 15. $a^2+b^2+c^2+d^2$. 18. $49x^4-28x^3-17x^2+6x+\frac{9}{4}$; $7x^2-2x-\frac{3}{2}$. 19. $a^mx^n+5ca^{m-2}x^{n+1}-3a/x$.

XV. E. Page 197. 1. $a+2$. 2. $2y^2+5$. 3. $3-5x$. 4. a^2-2a+1. 5. $2x^2-x+1$. 6. $2x^2-3x+5$. 7. $3y^2-2ay+a^2$. 8. $a+3b-2c$. 9. a^m-2ax^n.

Chapter XVI. Quadratic Equations.

XVI. A. Page 201. 1. 0, 3. 2. 0, $-\frac{b}{a}$. 3. ± 3. 4. $\pm(a+b)$. 5. -1, -2. 6. $-a$, $-2a$. 7. 0, 11. 8. -4, -5. 9. 10, -3. 10. 4, -1. 11. -12, 5. 12. -1, $\frac{1}{7}$. 13. -1, $-\frac{1}{2}$. 14. 17, -4. 15. 3, 5. 16. $-a$, $-b$. 17. 2, -1. 18. -2, $-\frac{1}{2}$.

19. $5a$, $4a$. **20.** $-\frac{3}{4}$, $\frac{2}{3}$. **21.** $-2\frac{5}{6}$, 3. **22.** -5, $4\frac{1}{3}$.
23. $c-a$, $b-c$. **24.** -1, $-2\frac{1}{2}$. **25.** 0, $\frac{1}{7}$. **26.** 1, $\frac{a-b}{b+c}$.

XVI. B. Pages 205—206.

1. -11, 5. **2.** -2, $\frac{1}{2}$.
3. 2, $-\frac{1}{2}$. **4.** 2, $1\frac{2}{5}$. **5.** $\frac{1}{2}$, $-1\frac{1}{2}$. **6.** 3, $\frac{1}{4}$. **7.** $-\frac{3}{5}$, $-\frac{5}{3}$.
8. $-\frac{2}{3}$, $\frac{3}{5}$. **9.** 7, $-\frac{1}{13}$. **10.** $-\frac{1}{4}$, 3. **11.** $\frac{1}{2}$, $\frac{3}{7}$. **12.** $-\frac{3}{5}$, 3.
13. 4, $-6\frac{1}{3}$. **14.** -11, $9\frac{15}{17}$. **15.** $\frac{3}{7}$, $-\frac{1}{3}$. **16.** 14, $2\frac{1}{4}$. **17.** $\frac{1}{2}$, $\frac{3}{2}$.
18. $\frac{1}{2}$, $\frac{1}{4}$. **19.** $\frac{1}{5}$, $\frac{5}{3}$. **20.** $\frac{1}{2}$, -349. **21.** 2, $\frac{1}{12}$. **22.** $\pm a$.
23. 17, 23. **24.** -1, 23. **25.** -1, $\frac{2a-b}{2a}$. **26.** b, $b-2a$.
27. a, $\frac{1}{a}$. **28.** $\frac{2}{a}$, $\frac{a}{2}$. **29.** $\frac{1}{a}$, $\frac{1}{b}$. **30.** $\frac{a+b}{a-b}$, $-\frac{a-b}{a+b}$.
31. 1, $\frac{10}{7}$. **32.** $a+c$, $c-b$. **33.** 1, $\frac{2q}{p-q}$. **34.** $1\frac{1}{3}$, $2a-3b$.

XVI. C. Pages 209—211.

2. All of them are roots.
3. -1 is a root. **4.** (i) $x^2-3x+2=0$; (ii) $x^2+8x+15=0$; (iii) $x^2-4x=0$; (iv) $x^2-(a+b)x+ab=0$. **5.** 5, -1. **6.** $\frac{1}{2}$, $-\frac{1}{3}$.
7. 5, $\frac{1}{5}$. **8.** 3, $-1\frac{9}{13}$. **9.** 2, $-1\frac{3}{17}$. **10.** $\frac{4}{3}$, $-\frac{4}{9}$. **11.** $-\frac{2}{3}$, $\frac{1}{5}$.
12. $-\frac{7}{11}$, $\frac{11}{7}$. **13.** 4, $-5\frac{1}{5}$. **14.** $-\frac{1}{5}$, $\frac{1}{4}$. **15.** 2, $-37/15$.
16. $-\frac{1}{12}$, 8. **17.** $-\frac{2}{3}$, $\frac{3}{4}$. **18.** $\frac{7}{2}$, $-\frac{5}{3}$. **19.** $\frac{11}{5}$, $-\frac{7}{3}$. **20.** -1, $\frac{1}{5}$.
21. $\frac{1}{4}$, 4. **22.** $-\frac{8}{9}$, $\frac{3}{2}$. **23.** $-\frac{2}{7}$, $\frac{3}{4}$. **24.** $\frac{17}{5}$, 7. **25.** 6, $11\frac{1}{5}$.
26. $\frac{7}{4}$, 2. **27.** -2, 10. **28.** 5, 22/23. **29.** $\frac{1}{3}$, $\frac{2}{3}$. **30.** 2, 9.
31. -1, 44/31. **32.** 0, $\frac{c-b}{a}$. **33.** $\frac{a-2b}{b}$, $\frac{a+2b}{a}$.
34. $2a$, $2b$. **35.** -1, $\frac{a-2b}{a}$. **36.** $a+b$, $\frac{1}{2}(a+b)$. **37.** 0, $\frac{2ab}{a+b}$.
38. $-\frac{1}{4}$, 2. **39.** 2, 31/5. **40.** -3, 2. **41.** 3, 14. **42.** $-\frac{13}{3}$, $\frac{7}{2}$.
43. $\frac{3}{7}$, 2. **44.** $-\frac{11}{9}$, 2. **45.** $-\frac{18}{7}$, 1. **46.** $-\frac{1}{3}$, 3. **47.** -3, 6.
48. $\frac{4}{3}$, 3. **49.** -4, $-\frac{40}{7}$. **50.** 2, 2. **51.** 0, 1. **52.** 1, 5.
53. $-\frac{29}{9}$, 4. **54.** 8, 9. **55.** 4, 8. **56.** 4, 7. **57.** $\frac{3}{2}$, 4.
58. $-23/12$, 2. **59.** 2, 5. **60.** ± 2. **61.** ± 4. **62.** $\pm\frac{1}{2}$.
63. 8. **64.** $\frac{3}{2}$. **65.** 3. **66.** -1, a^2+a. **67.** $\frac{1}{5}(3a+2b)$, $\frac{1}{5}(2a+3b)$.
68. $\frac{a^2}{b}$, $\frac{b^2}{a}$. **69.** 0, $-\frac{a^2+b^2}{a+b}$. **70.** $a+b\pm\sqrt{(a^2-ab+b^2)}$.
71. $\frac{3}{2}a$, $3a$. **72.** $a+b$, $\frac{2ab}{a+b}$. **73.** a, $\frac{b(a+b)}{a}$.
74. c, $\frac{-ab(a+c)}{ab+bc-ca}$. **75.** $-a$, $-b$. **76.** 0, $-\frac{a^3+b^3-c(a^2+b^2)}{c(a^2+b^2-ac-bc)}$.

Chapter XVII. Simultaneous Quadratic Equations.

[*The pairs of roots of each system of equations are separated by a semicolon. For example, the roots of the equations* **1** *in* **XVII. A.** *are* $x=5$, $y=1$; *and* $x=1$, $y=5$.]

XVII. A. Page 216. **1.** 5, 1; 1, 5. **2.** 5, 4; 4, 5. **3.** 10, 9. **4.** 6, 10. **5.** 7, 1; 1, 7. **6.** 1, 2; 5, −2. **7.** 7, 4; −4, −7. **8.** 6, 1; 1, 6. **9.** 3, 2; $\frac{7}{4}$, $\frac{21}{8}$. **10.** $3b-a$, $3a-b$; $3a-b$, $3b-a$. **11.** 2, −1; 15/8, −13/14. **12.** 7, 2; 2, 7. **13.** 5/12, 3/16; 3/16, 5/12. **14.** 4, 2; −6, 12. **15.** $\frac{a^2}{b}$, $\frac{b^2}{a}$. **16.** $\frac{a^2}{b}$, $\frac{b^2}{a}$; $\frac{a(2b-a)}{b}$, $\frac{b(2a-b)}{a}$.

XVII. B. Pages 222—223. **1.** 6, 3; 3, 6. **2.** 19, 91; −47/13, 217/13. **3.** m, n; $\frac{m^2-n^2}{2m}$, $\frac{n^2-m^2}{2n}$. **4.** b, a; $\frac{a^2}{b}$, $\frac{b^2}{a}$. **5.** $\frac{\pm 2}{\sqrt{5}}$, $\frac{\pm 1}{\sqrt{5}}$. **6.** ±3, ±5. **7.** ±1, ∓2; $\pm\frac{11}{4}$, $\mp\frac{1}{4}$. **8.** ±4, ±2. **9.** ±2, ±1; ±1, ±2. **10.** ±1, ±2; $\pm\frac{1}{4}$, $\pm\frac{11}{4}$. **11.** ±1, ±2. **12.** ±4, ±1; $\pm 13\sqrt{(5/68)}$, $\pm 10\sqrt{(5/68)}$. **13.** ±5, ±1; ±13, ∓7. **14.** ±2, ±1; $\pm\frac{11}{8}$, $\pm\frac{19}{8}$. **15.** ±2, ∓1; ±7, ±4. **16.** ±1, ±3; $\pm\frac{7}{12}$, ∓12. **17.** ±5, $\pm\frac{1}{2}$; $\pm 7\sqrt{\frac{2}{3}}$, $\mp\frac{1}{3}\sqrt{\frac{2}{3}}$. **18.** ±2, ±3; $\pm 5\sqrt{\frac{1}{2}}$, $\mp\sqrt{\frac{1}{2}}$. **19.** 7, 3; $\frac{4}{3}$, $-\frac{8}{3}$. **20.** ±3, ±5; $\pm\frac{13}{2}$, $\mp\frac{1}{4}$. **21.** ±8, ±5; $\pm 13\sqrt{\frac{1}{2}}$, $\pm 3\sqrt{\frac{1}{2}}$. **22.** ±3, ±1. **23.** ±1, ±7; ±7, ±1. **24.** ±1, ±2. **25.** ±2, ±2; $\pm\frac{4}{3}$, $\mp\frac{1}{3}$. **26.** 2, 3; 3, 2. **27.** ±3, $\mp\frac{1}{2}$; $\pm 23\sqrt{\frac{1}{65}}$, $\pm\frac{1}{2}\sqrt{\frac{1}{65}}$. **28.** ±1, ±2. **29.** ±2, ±1; $\pm\frac{1}{4}$, $\pm\frac{11}{4}$. **30.** $\pm(p+q)$, ±1; $\pm q$, $\pm\frac{p+q}{q}$. **31.** $\pm(p-q)$, ±1; $\pm q$, $\pm\frac{p-q}{q}$. **32.** $\frac{\pm a^2}{\sqrt{(a^2+b^2)}}$, $\frac{\pm b^2}{\sqrt{(a^2+b^2)}}$. **33.** $\pm(2a+b)$, $\pm(a+2b)$; $\pm(a+2b)$, $\pm(2a+b)$. **34.** 3, 4; 5, 2. **35.** ±3, ∓2; $\pm\sqrt{2}$, $\pm 4\sqrt{2}$. **36.** ±2, ∓3; $\pm 4\sqrt{2}$, $\pm\sqrt{2}$. **37.** 1, 2; $\frac{4}{3}$, $\frac{3}{2}$. **38.** 7, 5; $\frac{9}{2}$, $\frac{20}{3}$. **39.** 0, 0; 4, −1. **40.** 0, 0; 2, −2. **41.** 4, 1; $\frac{3}{5}$, $-\frac{12}{5}$. **42.** $\frac{6}{7}$, $\frac{7}{12}$; $\frac{12}{7}$, $\frac{7}{6}$. **43.** 4, 5; $-\frac{1}{5}$, $-\frac{1}{4}$. **44.** 8, 6; 6, 8. **45.** ±3, ±2. **46.** ±4, ±3; $\pm\frac{1}{3}\sqrt{(-1)}$, $\pm\frac{8}{3}\sqrt{(-1)}$. **47.** ±1, ±2. **48.** ±4, ±2; $\pm 2\sqrt{\frac{7}{13}}$, $\pm 4\sqrt{\frac{7}{13}}$. **49.** $\frac{\pm b\sqrt{a}}{(\sqrt{a}\pm\sqrt{b})}$, $\frac{\pm a\sqrt{b}}{(\sqrt{a}\pm\sqrt{b})}$. **50.** ±3, ±1; $\pm 9\sqrt{(-\frac{1}{2})}$, $\mp 8\sqrt{(-\frac{1}{2})}$. **51.** 1, 2; 2, 1; −11, −12; −12, −11. **52.** $\pm\frac{1}{4}$, ∓1. **53.** a, b; a, b.

54. $-2, -3$. **55.** 5, 7; 7, 5. **56.** 5, 3; 3, 5. **57.** $\frac{ab}{(b \mp a)}, \frac{2ab}{(b \pm a)}$.
58. 5, 4; 4, 5. **59.** 7, 4; 4, 7.
60. $x = y = \pm 1$; $\frac{x}{a-c} = \frac{y}{b-a} = \pm \sqrt{\left\{\frac{a+b+c}{a(a^2+b^2+c^2-bc-ca-ab)}\right\}}$.
61. $x = \pm 2$, $y = \pm 1$, $z = \pm 3$; $x = \pm 1$, $y \pm 2$, $z = \pm 3$; or $x = y$, which gives another (but irrational) solution. **62.** $x = 24$, $y = 18$, $z = 6$, $u = 2$.

Chapter XVIII. Problems leading to Quadratic Equations.

XVIII. Pages 232—237. **1.** 12, 6. **2.** 9, 11. **3.** 60.
4. 25 yards. **5.** 36 feet long, 22 feet broad, 15 feet high.
6. 16 yards by 14 yards. **7.** 2210. **8.** 784. **9.** 496. **10.** 578.
11. 17 feet long, 13 feet broad. **12.** 8 or 12 miles an hour.
13. 2 hours. **14.** 2 *h.* 45 *m.* **15.** 10 miles per hour; $10\frac{1}{2}$ miles per hour.
16. 30 days, 45 days. **17.** 16. **18.** 4*s.* 9*d.* **19.** 40. **20.** 72.
21. 18. **22.** 3*s.* **23.** 8*d.* **24.** 48. **25.** 50. **26.** 42. **27.** 50.
28. 240. **29.** 15. **30.** 39 and 8. **31.** $\frac{8}{17}$. **32.** 48. **33.** 36.
34. 73. **35.** 74 and 47. **36.** 665. **37.** 16 and 3. **38.** 2, 10, 14.
39. 16 feet and 15 feet. **40.** 16 inches and 9 inches. **41.** 11760 sq. yds.
42. 30 feet and 18 feet. **43.** 10 feet; 80 feet by 60 feet.
44. 150 yards by 130 yards. **45.** 9 yds. and 8 yds., or 8 yds. and 6 yds.
46. *A*, 324 and 109; *B*, 227 and 209. **47.** *A*, 196; *B*, 183; *C*, 169.
48. 45 and 60 miles an hour respectively.
49. *A* at $3\frac{1}{2}$ miles per hour, *B* at $7\frac{1}{2}$ miles per hour.
50. 25. **51.** 12*s.*; 9*d.* and 1*s.* **52.** 4*s.* **53.** 5 inches, 6 inches.
54. £125, £95. **55.** 10 sheep worth 10*s.* each, 8 pigs worth 8*s.* each, 18 geese worth 2*s.* each.

Chapter XIX. Equations reducible to Quadratics.

XIX. A. Page 239. **1.** $\pm 3, \pm 2$. **2.** $\pm\sqrt{11}, \pm\sqrt{\frac{1}{3}}$.
3. $\pm\frac{1}{2}a, \pm\frac{1}{3}a$. **4.** $\sqrt[3]{8} = 2$, $\sqrt[3]{(-1)} = -1$. **5.** $\sqrt[n]{(-a \pm b)}$.
6. $\pm\sqrt{3}, \pm\sqrt{\frac{11}{6}}$. **7.** ± 3. **8.** $\pm\sqrt{(a^2+b^2)}$.

XIX. B. Page 240. **1.** 1, 2, 2, 3. **2.** 0, 1, -1, -2.
3. 1, -4, 2, -5. **4.** 2, $-\frac{5}{4}$, $\frac{1}{24}(9 \pm 7\sqrt{33})$.

XIX. C. Page 242. 1. $-2, 1$. 2. $3, -\frac{7}{5}$.
3. $-\frac{5}{8}a, 12a$. 4. $k\{1 \pm \sqrt{(-3)}\}$.

XIX. D. Page 244. 1. $1, 1, -\frac{1}{2}, -2$. 2. $-\frac{1}{2}, -2, -\frac{1}{3}, -3$.
3. $1, 1, 2, \frac{1}{2}$. 4. $-\frac{1}{2}, 2, -\frac{1}{3}, 3$. 5. $1, 1, \frac{1}{4}\{-1 \pm \sqrt{(-15)}\}$.

Chapter XX. The Theory of Quadratic Equations.

XX. A. Page 247. 1. $3; 4$. 2. $2; \frac{5}{2}$. 3. $bc/a;\ a^2$.
4. $3; \frac{4}{3}$. 5. $0; -\frac{2}{3}$. 6. $\dfrac{p^2+q^2-p-q}{pq-1}$; 0. 7. $\frac{25}{6}, \frac{1}{6}$.
8. (i) $\dfrac{\sqrt{q^2-4p^2}}{p}$; (ii) $\dfrac{(q^2-p^2)\sqrt{q^2-4p^2}}{p^3}$ 9. $p^2-2q;\ q+2+\dfrac{1}{q}$.

XX. B. Page 249. 1. $(3x-4)^2$. 2. $(4x+1)(4x+3)$.
3. $(x-4)(x+1)$. 4. $(x-a+b)(x-a-b)$.
5. $(x+a-b-c)(x-a+b-c)$. 6. $(x-2a+ab)(x-2b-ab)$.
7. $\frac{1}{3}\{3x-a-b-c+\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}$
$\{3x-a-b-c-\sqrt{(a^2+b^2+c^2-bc-ca-ab)}\}$.
8. $(x-y)(x-z)(y-z)$.

XX. C. Page 250. 1. $c, \dfrac{a+b}{ac}$. 2. $b, a+\dfrac{1}{a-b}$.
3. $a, 3-a$. 4. $-\dfrac{c^2}{a}$.

XX. D. Pages 252—254. 1. $-\frac{9}{4}$. 7. $c=a+4b$.
9. $6x^2-13x+6=0$. 10. $rx^2-qx+p=0$. 12. $\frac{18}{49}; -\frac{50}{49}$.
13. $a=\pm 1, b=\mp 2$. 14. $p=-2, q=1$. 17. $9x^2+30x+25=0$.
21. $2b^2=9ac$. 26. 0, 9. 28. 1, 8. 29. 2, 6.

XX. E. Page 256. 3. $-\frac{1}{8}$. 4. 5. 5. $-\frac{1}{3}$. 6. 9.
7. Each is $\frac{1}{2}a$. 8. *A* bought 32, *B* bought 24, *C* bought 8, *D* bought 64. [If x be the number bought by *C*, then x^2-16x is a minimum; hence, $x=8$.]

Chapter XXI. Indeterminate Equations.

XXI. Pages 259—260. 1. 3, 1; 0, 3. 2. 17, 1; 10, 4; 3, 7.
3. 1, 53; 3, 40; 5, 27; 7, 14; 9, 1. 4. $x=2t, y=3t-5$; 4, 1.
5. $x=9t-1, y=7t-4$; 8, 3. 6. $x=17t+2, y=13t+1$; 2, 1.

7. 15. **8.** 26 or 24, according as payment in one kind of coin alone is or is not reckoned as permissible. **9.** 2. **10.** 7 and 28; 14 and 21. **11.** 165. **12.** 79. **13.** *A* gives 2 half-crowns and receives 4*d*. **15.** 3 and 22; 9 and 16; 15 and 10; 21 and 4. **16.** 1 and 20; 2 and 13; 5 and 6. **17.** 11, 29. **18.** Hendrick and Anna, Claas and Catrijn, Cornelius and Geertruij, were respectively man and wife.

Examination Questions.

Examination Questions. Pages 261—264.

1. $12x^2-25x+12=(3x-4)(4x-3)$. **2.** (i) 4, $\frac{4}{3}$; (ii) 3, $-\frac{35}{11}$; (iii) 0, 1; -2, -5; $-1\pm3\sqrt{(-1)}$, $-2\mp\sqrt{(-1)}$. **4.** 160. **5.** 28 feet by 30 feet. **6.** $4a^3-a-2$. **7.** (i) $\frac{4}{3}$, 4; (ii) ±3, $\pm\sqrt{(-3)}$; (iii) a/b, b/a; b/a, a/b. **8.** $135x^2-6x-1=0$. **9.** 1 mile an hour. **10.** 20 feet long, 15 feet broad, 10 feet high. **11.** 3. **12.** $4a^2-a-2$. **13.** $(2x-1)^2(x-2)^2$. **14.** (i) $-\frac{9}{11}$, 2; (ii) ±3, ∓1; $\pm\sqrt{(-3)}$, $\mp\sqrt{(-3)}$. **15.** 60 gold coins, 120 silver coins. **16.** 81; 180. **18.** (i) 0, $-2ab/(a+b)$; (ii) 3, -1; $\frac{11}{5}$, $\frac{3}{5}$. **19.** $qx^2+px+1=0$. **20.** 25 yards. **21.** $3a-2b-c+5d$. **22.** (i) -1, 4; (ii) ±1, ±1; (iii) $x=\pm11$, $y=\mp9$. **24.** 2, 3, 4. **25.** 15 yards of black at 7*s*. a yard; 12 yards of brown at 7*s*. 6*d*. a yard. **26.** $3x^2-2x+5$. **27.** (i) $-\frac{1}{3}$, 3; (ii) $x=3$, $y=2$; $x=-2$, $y=-3$. **28.** 3. **29.** 1 hour 45 minutes, 2 hours 20 minutes. **30.** 12 miles, 3 miles.

Chapter XXII. Fractional and Negative Indices.

XXII. A. Page 269. **1.** a^{4p+4q}. **2.** 1. **3.** $a^{12m-12n}$. **4.** 1. **5.** 1. **6.** a^{2q-2r}. **7.** a^{9n-9m}. **8.** $(bc)^y$. **9.** a.

XXII. B. Pages 273—274. **1.** $a^{-\frac{1}{3}}b^{-\frac{1}{4}}c^{\frac{1}{12}}$; 1. **2.** a^3. **3.** a. **4.** 1. **5.** 1. **6.** a^{q^2}. **7.** a^y. **8.** $a^{-49/24}$. **9.** $x^{2(m+n)}$.

14. $6x^3-10x^{\frac{5}{2}}-3x^2+x^{\frac{3}{2}}+20x-4x^{\frac{1}{2}}-16$.

15. $b^{-2}+2a^{\frac{1}{2}}b^{-\frac{1}{2}}c+2a^{\frac{1}{2}}b^{\frac{1}{2}}c+2a^{\frac{1}{2}}b^{-\frac{3}{2}}-c^2+2a^{\frac{1}{2}}b^{-1}+2ab^{-\frac{1}{2}}$
$-2a^{\frac{1}{2}}b^{-\frac{1}{2}}-3a+2ab^{\frac{1}{2}}$.

16. $x+4y$. **17.** x^3-a^3. **18.** $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b-b^{\frac{1}{2}}c^{\frac{1}{2}}+c$. **19.** $x-4$.

20. $x^{\frac{2}{3}}+4y^{\frac{2}{3}}+9z^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+6y^{\frac{1}{3}}z^{\frac{1}{3}}+3x^{\frac{1}{3}}z^{\frac{1}{3}}$. **21.** $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$.

22. $\dfrac{1}{1-x^{\frac{1}{2}}}$. **23.** 0. **24.** $x-2+\dfrac{3}{x}$. **25.** $2x-3x^{\frac{1}{3}}+7$.

26. $x^{\frac{1}{2}}-2a^{\frac{1}{6}}x^{\frac{1}{3}}+3c^{\frac{1}{2}}$. **27.** $a^3+2a^2b^{-1}-ab^{-2}-b^{-3}$. **28.** $3x-2$.
29. $\frac{1}{2}x+\frac{1}{3}y$. **30.** b. **31.** 0, 3. **32.** ab. **33.** a^{-1}. **35.** 1, $\frac{3}{2}$.
36. $x=3$, $y=2$.

Chapter XXIII. Logarithms.

XXIII. A. Page 276. 1. 2. 2. $\frac{1}{2}$. 3. 3. 4. $\frac{4}{3}$.
5. $-\frac{2}{3}$. 6. $-\frac{3}{2}$. 7. $\frac{1}{4}$. 8. $-\frac{3}{2}$. 9. 2. 10. -2. 11. $-\frac{4}{3}$.
12. 10. 13. -6. 14. 3. 15. $-\frac{3}{2}$. 16. $\frac{8}{5}$. 17. $\frac{2}{15}$. 18. 3.
19. $-\frac{25}{3}$. 20. An indefinitely large negative quantity.

XXIII. B. Pages 280—281. 1. (i) $-2·1072100$;
(ii) ·4948500; (iii) $-·53402$; (iv) 2·52575. 2. (i) 1·1760913;
(ii) 3·2886963; (iii) ·6532125; (iv) 3·3802112; (v) 1·8750613;
(vi) $-3+·6532125=-2·3467875$; (vii) $-3+·5563025=-2·4436975$;
(viii) $-3+·4771213=-2·5228787$; (ix) ·2552725;
(x) $-2+·8573325=-1·1426675$; (xi) $-1+8750613=-·1249387$;
(xii) ·8627275; (xiii) $-3+·0969100=-2·9030900$.
3. ·121519; $-·129722$; $-1·125906$.
4. ·1461280; $-4+·1875207=-3·8124793$; $-9·8025455$.
5. 1·1760913; $-4+·3979400=-3·6020600$. 6. ·4671213.
7. 2·1303338; ·4771213. 8. 1·158051. 9. 1·146128.
10. 1·4771213; $-2+·1303338=-1·8696662$.
11. $\frac{1}{6}(3a+2b+3c-5)$; $\frac{1}{6}(9a-2b+3c-1)$; $b+c-2$.

XXIII. C. Page 282. 1. (i) 1·631; (ii) ·898; (iii) ·683.
2. ·712; 6·129. 3. ·886; 1·129.

XXIII. D. Page 285. 1. $\bar{5}·9090909$. 2. 1·3456789.
3. $\bar{2}·9357538$. 4. 6·9817371; 3·9817371; $\bar{3}·9817371$.
5. 6414·5; ·0064145. 6. 3; 4; 5. 7. 2; $\bar{2}$; 4; 0; $\bar{4}$; 5; 5.
8. $-·003=\bar{1}·997$; $-·0004=\bar{1}·9996$.

XXIII. E. Pages 287—288. 1. $\bar{2}·5051500$. 2. $\bar{1}·5563026$.

3. 2·8578325. **4.** 1·8627275. **5.** ·8750613. **6.** $\bar{4}$·6532125. **7.** $\bar{2}$·6930604. **8.** $\bar{1}$·1163460. **9.** $\bar{2}$·7262766. **10.** ·178; 5·615. **11.** $(\frac{3}{5})^9 > \cdot 01$; $(\frac{4}{5})^{10} > \cdot 1$. **12.** $(\frac{21}{20})^{100} > 100$. **13.** 18; 87; 53. **14.** 22. **15.** About 83½ years. **16.** 33 years nearly. **17.** 7 years. **18.** $x = 8293039/7043652$, $y = 4771213/7043652$. **19.** $(\cdot 1239387)\,a$. **20.** ·9365137; ·1760913. **21.** 2·8115750. **22.** 4096; 32768; 16384; 8192. **23.** $\log_{10} 2$ and $\log_{10} 3$. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then the logarithms of the given numbers are $6a$; $3(1-a)$; $2-a$; $1+b$; $3a-1$; $a-b$; $b-3a$. The characteristics of their logarithms (to the base 2) are 6; 6; 5; 4; −1; −1; −2. **24.** 10. **25.** 3·27646. **26.** 4·59999. **29.** 32. **30.** $x = 1/a$, $y = 1/c$. **32.** 6. **33.** 2. **34.** $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$. **38.** 10 years.

Chapter XXIV. Surds.

XXIV. A. Page 293. **1.** $\sqrt{9}$; $\sqrt[3]{27}$; $\sqrt[4]{81}$. **2.** $\sqrt[3]{a^3 b}$. **3.** $\sqrt{18}$. **4.** $\sqrt{12}$. **5.** $\sqrt{x^4 y}$. **6.** $\sqrt{x^7 y^5}$. **7.** $\frac{\sqrt{2}}{2}$. **8.** $\frac{2\sqrt{3}}{3}$. **9.** $\frac{\sqrt{35}}{7}$. **10.** $\frac{\sqrt[6]{a^3 b c^4}}{c}$. **11.** $\frac{\sqrt[n]{x^{n-1} y}}{y}$. **13.** Less. **14.** $\sqrt{35} < \sqrt[3]{214} < \sqrt[4]{1290}$.

XXIV. B. Page 294. **1.** $8 - 22\sqrt{x} + 15x$. **2.** $49 - 28\sqrt{a} + 4a$. **3.** $2 + \sqrt{a}$. **4.** $x + 3\sqrt{x} - 2$. **5.** $x^{\frac{3}{4}} - 3x^{\frac{1}{2}} + 9x^{\frac{1}{4}} - 27$.

XXIV. C. Pages 297—298. **1.** $\frac{\sqrt{2}+\sqrt{6}}{2}$. **2.** $\frac{y^2}{x^2}$. **3.** $\frac{b^2}{a^2}$. **4.** $\frac{1 - x^{\frac{1}{4}} + 2x^{\frac{1}{2}}}{1 - x^{\frac{1}{4}} + 3x^{\frac{1}{2}}}$. **5.** $\frac{1}{2x-1}$. **7.** $2(1-\sqrt{3})$. **8.** $x - y$. **9.** $x^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$. **10.** $2^{\frac{2}{3}}x^2 - (2 + 2^{\frac{1}{3}})\,x + (2^{\frac{4}{3}} - 2^{\frac{2}{3}} + 1)$. **11.** $x^2\sqrt{3} + xy\sqrt{2} + y^2$. **12.** $2\sqrt{3} + 3\sqrt{2}$. **13.** $1 + \sqrt{6}$. **14.** $7\sqrt{2} - 3$. **15.** $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{6}}$. **16.** $3x^2 - 1 + \frac{4}{\sqrt{x}}$. **17.** $\sqrt{\frac{x}{y}} - 1 + \sqrt{\frac{y}{x}}$. **18.** $\frac{\sqrt{x+y} + \sqrt{x-y}}{2}$. **19.** $\frac{\sqrt{a+b-c} + \sqrt{a-b+c}}{\sqrt{2}}$. **20.** $1 + \sqrt{3}$. **21.** $5 + \sqrt{5}$. **22.** 0. **23.** $\sqrt{2}$. **24.** $\sqrt{\frac{x}{y}} + \frac{1}{2} + \sqrt{\frac{y}{x}}$.

XXIV. D. Pages 303—304. **1.** 8. **2.** $\frac{1}{2}\sqrt{(4ab-b^2)}$. **3.** 7. **4.** $\frac{7}{4}$. **5.** $\frac{5}{3}$. **6.** 6. **7.** $2a$. **8.** 2. **9.** $\frac{1}{3}a$. **10.** $\frac{1}{4}$ or 4. **11.** $\frac{4}{3}a$. **12.** 13. **13.** $\frac{11}{4}$. **14.** 5. **15.** 5. **16.** $\frac{9}{4}$. **17.** 6. **18.** 5. **19.** -1. **20.** 17. **21.** 6. **22.** None: the roots of the rationalized equation are 0 and 8, but neither satisfies the given equation. **23.** None: the roots of the rationalized equation are -2 and 7; but neither satisfies the given equation. **24.** 9. **25.** $-\frac{3}{4}$. **26.** 5. **27.** 0, 3. **28.** $a\sqrt{3}$. **29.** $(c+d)^2$. **30.** 25. **31.** $\frac{(b^2-a^2)c}{b^2+a^2}$. **32.** $\frac{1}{2}$, 2. **33.** $-\frac{3}{2}$, 7. **34.** -4, 5, $-\frac{36}{5}$, $\frac{9}{5}$. **35.** 2, 3, $-\frac{2}{3}$, $\frac{1}{3}$. **36.** -4, 1. **37.** $-c \pm \sqrt{2a+c^2}$. **38.** 0, $\frac{5}{2}$. **39.** -11, 14. **40.** $\frac{1}{2}$. **41.** $\frac{a^2-b^2}{2\sqrt{\{2(a^2+b^2)\}}}$. **42.** 0. **43.** $x=16$, $y=9$; $x=8$, $y=1$. **44.** $x=16$, $y=9$. **45.** $x=169$, $y=121$; $x=36$, $y=\frac{25}{9}$. **46.** $\sqrt{x}=-3$, $\sqrt{y}=-6$; $\sqrt{x}=9$, $\sqrt{y}=14$.

Chapter XXV. Ratio and Proportion.

XXV. A. Pages 309—311. **1.** 3 : 5. **2.** (i) $x^{\frac{1}{2}} : y^{\frac{1}{2}}$; (ii) $x^{\frac{1}{3}} : y^{\frac{1}{3}}$; (iii) $x^{\frac{3}{2}} : y^{\frac{3}{2}}$. **3.** (i) 1; (ii) 16 : 3; (iii) $a^{\frac{1}{2}} : b^{\frac{1}{2}}$. **4.** (i) 28 : 33; (ii) 14 : 13. **5.** (i) If x be positive, $4-x : 3-x > 4 : 3$; (ii) If x be negative, $4-x : 3-x < 4 : 3$. **6.** 5 : 2. **7.** 19 : 3. **8.** 25 : 4. **9.** 15. **10.** 15. **11.** 4 : 7. **16.** $\frac{a^2b^2}{a^2+ab+b^2}$. **17.** 8, 22. **18.** 7 years (actually to $7\frac{1}{2}$ years). **19.** 45, 30. **20.** 12, 21; 24, 42. **21.** 7, 13. **22.** 136. **23.** 4 : 7. **24.** £242, £300, £358. **25.** 3, 12.

XXV. B. Pages 315—316. **15.** 5 : 8. **16.** 9 : 209. **18.** 9, 30. **19.** 2 : 3 : 4.

XXV. C. Pages 318—319. **3.** $\frac{9}{4}$. **4.** $xz - \frac{1}{xz}$. **6.** $\frac{ac-b^2}{a+c-2b}$. **7.** $\frac{a^2+ab+ac-b^2-bc-c^2}{b+c-2a}$. **8.** The former. **12.** 3, 12.

XXV. D. Pages 319—322. **1.** 4 : 7. **2.** $a : pb$. **4.** ab. **5.** 1 : 3. **7.** 14. **8.** 22. **9.** 9. **10.** 21. **12.** 10, 6, 4. **13.** Ages of sons are 2, 4, 6; age of man is 36. **14.** $a^2 : b^2$.

29. 2 : 3 : 4. **39.** 2. **40.** 5, 4. **41.** 2, 18. **42.** 45, 54. **43.** £305, £410, £515. **44.** *A* receives £1666. 13*s*. 4*d*., *B* receives £1000, *C* receives £333. 6*d*. 8*d*. **45.** 1 : 2, or 2 : 1. **46.** $2\frac{1}{2}$ gallons. **47.** 4 : 1.

Chapter XXVI. Variation.

XXVI. A. Page 325. **1.** $12\frac{1}{4}$. **2.** $4x = 9y^2$. **3.** 6. **4.** $\sqrt{17}$. **6.** 2. **8.** 78.

XXVI. B. Page 326. **1.** 3. **2.** $3\frac{3}{4}$. **3.** $a^3b^2 = 72$. **4.** $3\frac{3}{7}$ hours. **5.** 78.

XXVI. C. Pages 328. **1.** $3\frac{3}{8}$. **2.** 128. **3.** 4.

XXVI. D. Pages 331—334. **1.** $z \propto xy^2$. **4.** $\sqrt{17}$. **6.** 104. **8.** 6. **10.** $\frac{15}{8}$. **11.** 4. **12.** $\frac{a^3c^3}{b^3}$. **14.** $-\frac{1}{5}$. **15.** $w = 2(x + 3z^2)$. **17.** £34. **18.** £$11\frac{2}{11}$*s*. **19.** 15 feet. **20.** 25 square feet. **21.** 66 cubic feet. **23.** 18 inches. **24.** 8 : 7. **25.** $b^3 : ac^2$. **26.** 3 days. **27.** 40 francs. **28.** 12*s*. **29.** 32 : 25. **30.** $26\frac{2}{3}$ miles. **31.** £30. **32.** The 4th, 5th, and 6th days; 6 days. **33.** 20 trucks. **34.** $\frac{g\{bdf(c-h) - ace(f-h)\}}{beh(c-f)}$.

Chapter XXVII. Arithmetical Progressions.

XXVII. A. Page 337—338. **1.** (ii) Is in A.P. **2.** 6, $7\frac{5}{9}$, $9\frac{1}{9}$, $10\frac{2}{3}$, $12\frac{2}{9}$. **3.** (i) 52; (ii) −61; (iii) $a^2 - 30ab + b^2$. **4.** Yes, the 252nd. **5.** $na - (n-2)b$. **6.** −154, −148, −142, &c. **7.** 81. **8.** 58. **9.** −202, 430. **10.** Yes.

XXVII. B. Pages 340—341. **1.** 920. **2.** 68895. **3.** 94850. **4.** 665. **5.** 0. **6.** 80. **7.** $40\frac{1}{2}$. **8.** $36\frac{2}{3}$. **9.** 290. **10.** $-303\frac{3}{4}$. **11.** 18. **12.** 21·6. **13.** 49. **14.** 163·2. **15.** 152·4. **16.** $x + \frac{1}{2}(a+1)$. **17.** $\frac{1}{2}(2n^3 - n - 1)$. **18.** 1512. **19.** 1170. **20.** $287\frac{1}{2}$. **21.** 576. **22.** 2. **23.** 13. **24.** 20. **25.** 30. **26.** n^2.

XXVII. C. Page 343. **1.** 6. **2.** (i) 5, 6, 7; (ii) 3, 5, 7; (iii) $1\frac{1}{4}$, $-\frac{1}{2}$, $-2\frac{1}{4}$. **3.** 1, $1\frac{6}{7}$, $2\frac{5}{7}$, $3\frac{4}{7}$, $4\frac{3}{7}$, $5\frac{2}{7}$, $6\frac{1}{7}$. **4.** $\frac{2a+b}{3}$, $\frac{a+2b}{3}$; $\frac{4a+5b}{(a+2b)(2a+b)}$, $\frac{5a+4b}{(a+2b)(2a+b)}$.

5. $\frac{1}{5}(4x^3+x^{-2})$, $\frac{1}{5}(3x^3+2x^{-2})$, $\frac{1}{5}(2x^3+3x^{-2})$, $\frac{1}{5}(x^3+4x^{-2})$.
6. $a^2+b^2-\frac{2}{3}ab$, $a^2+b^2-\frac{1}{3}ab$, a^2+b^2, $a^2+b^2+\frac{1}{3}ab$, $a^2+b^2+\frac{2}{3}ab$.

8. $(a+l)^{\frac{n-1}{2}}$. 9. $6\frac{1}{3}$ minutes. 10. 5, 6, 7.

XXVII. D. Pages 344—347. 1. (i) 4890; (ii) $25\frac{1}{3}$; (iii) $820a-1680b$; (iv) $\frac{1}{2}n(n-1)$. 2. 156. 3. $\frac{2n-1}{2n}$, $\frac{2n-3}{2n}$, &c.
5. $369\frac{1}{15}$. 6. 1, 3, 5, 7... 8. 5, 7, 9, 11 ... 9. 10 or 20. 10. 25.
11. 25. 12. 10 or 13. 13. 297. 14. 6; 2. 15. $32(2r-1)$.
16. $16n^2$. 19. ma. 20. 10. 24. 8217 (see Ex. 13).
25. $37\frac{1}{2}$ miles. 26. 10. 27. $2m-n-4$, $2m-n-2$, $2m-n$.
29. 1, 3, 5, 7. 30. 4, 5, 6, 7. 31. 1, 2, 3, 4, 5. 32. 24, 48, 80.
33. $\frac{1}{4}$, $2\frac{1}{4}$, $4\frac{1}{4}$, &c. 34. $3n-2$; $6n+1$. 36. 41. 37. $4a=(n-1)d$.
38. 12 feet. 39. 195 miles. 40. £82. 13*s*. $1\frac{1}{2}d$. 41. 6 days.
42. 12, 15, 18. 43. 4 miles an hour. 44. In 6 days; at 90 miles from the starting place.

Chapter XXVIII. Geometrical Progressions.

XXVIII. A. Page 350. 1. (i), (ii), and (iv) are in G.P.
2. $\frac{16}{81}$; -1; y^5/x^2. 3. 96. 4. $-2187/64$. 5. $-1024/81$.
6. 3; 7203. 7. 9375. 8. 18, ± 54, 162, ± 486, &c.
9. 2, 4, 8, 16, &c.; or -32, 16, -8, 4, &c.

XXVIII. B. Pages 357—358. 1. 665. 2. 296/735.
3. 16383/9604. 4. $-1261/384$. 5. $-8425/8748$. 6. $-25862/343$.
7. 910. 8. 1093711/390625. 9. $416\frac{2}{3}$. 10. $281\frac{1}{3}$. 11. 64.
12. $10\frac{1}{3}$. 13. 4. 14. 38·4. 15. $35\frac{35}{99}$. 16. $211(\sqrt{2}+\sqrt{3})/81$.
17. 781. 18. $374\frac{1}{3}$, or $-53\frac{2}{3}$. 19. $748\frac{1}{3}$, or $-74\frac{13}{18}$. 20. 2.
21. ± 2. 22. 3, $\frac{3}{4}$, $\frac{3}{16}$, &c.; or 1, $\frac{3}{4}$, $\frac{9}{16}$, &c. 23. 765 or -255.
24. $4\frac{1}{256}$. 25. 3; 4.

XXVIII. C. Pages 361—362. 1. (i) ± 4; (ii) $\pm(x^2-a^2)$.
2. (i) 2, 4, 8; (ii) $\pm(x-a)^{\frac{1}{2}}(x+a)^{\frac{3}{2}}$, x^2-a^2, $\pm(x-a)^{\frac{3}{2}}(x+a)^{\frac{1}{2}}$; (iii) $2^{\frac{1}{4}}$, $2^{\frac{1}{2}}$, $2^{\frac{3}{4}}$. 3. $x^{4n}y^{2n}$, $x^{2n}y^{4n}$. 4. $\frac{27}{4}$, $\frac{9}{2}$, 3, 2, $\frac{4}{3}$, $\frac{8}{9}$.
5. 768, 1152, 1728, 2592, 3888, 5832, 8748, 13122. 6. $\pm(6x^2-5x+6)$.
8. 25 : 1. 9. 3, 75. 12. $a\frac{(mr+m+2)(r^{n-1}-1)}{2(r-1)}+ar^{n-1}$, where a is the first term and r is the common ratio of the given G.P.

XXVIII. D. Pages 362—364. **1.** (i) 45920; (ii) 17955; (iii) 32; (iv) $-3893\sqrt{5}(\sqrt{30}-2)/8$. **2.** 2·366. **3.** (i) is not a G.P.; (ii) ·6; (iii) 9; (iv) the common ratio is greater than 1; (v) $\frac{1}{2}(4+3\sqrt{2})$.

4. (i) $\dfrac{x^2(x^{2n}-1)}{x^2-1}+\dfrac{xy(x^ny^n-1)}{xy-1}$; (ii) $\dfrac{x(x^n-1)}{x-1}+\dfrac{n(n+1)}{2}a$.

5. $\dfrac{a\{(-a)^n-1\}}{a+1}$. **6.** 10, 9, 8·1, &c. **7.** $(\frac{3}{2})^{n-1}a$. **14.** $\frac{1}{2}$.

16. 5. **18.** 4, 2, 1. **19.** 3, 12, 48, 192. **20.** 5, 20, 80. **22.** $\frac{3}{2}$, -1, $\frac{2}{3}$. **23.** 1, 4, 16.

Chapter XXIX. Harmonic and other Series.

XXIX. A. Page 370. **1.** $\frac{1}{13}$. **2.** $\frac{812}{585}$. **3.** -2, -1, $-\frac{2}{3}$.

4. (i) $1\frac{3}{5}$; (ii) $2\frac{6}{7}$; (iii) $\dfrac{2a}{1-a^2}$. **5.** (i) $\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{13}$; (ii) $\frac{17}{5}$, $\frac{17}{9}$, $\frac{17}{13}$;

(iii) $\frac{5}{8}$, $\frac{5}{12}$, $\frac{5}{16}$; (iv) $\dfrac{4b(a+b)}{a+4b}$, $\dfrac{2b(a+b)}{a+2b}$, $\dfrac{4b(a+b)}{3a+4b}$.

XXIX. B. Pages 375—376. **1.** 7. **4.** $\frac{1}{6}n(n+1)(2n+7)$. **5.** $\frac{1}{3}m(m+1)(2m+1)$. **6.** 7305.

XXIX. C. Pages 377—379. **1.** (i) in A.P., 63; (ii) in A.P., $36\frac{3}{4}$; (iii) in G.P., $17\frac{79}{81}$; (iv) in G.P., $13\frac{41}{48}$; (v) in G.P., $-6·15125$; (vi) in A.P., $-58·2$. **2.** (i) n^2; (ii) $x\dfrac{1-x^{2n}}{1-x^2}+n^2a$;

(iii) $ab\left\{\dfrac{1-b^n}{(1-b)^2}-n\dfrac{b^n}{1-b}\right\}$. **3.** (i) $\frac{11}{2}(a^5+a^{-5})$; (ii) $\dfrac{a^{11}-1}{a^5(a-1)}$.

4. $1-1=0$. **5.** $-\frac{26}{7}$. **7.** $x=\frac{1}{4}$, $y=-\frac{1}{2}$. **8.** 5, 45.

9. 24, 44, 80. **13.** $\dfrac{a(p-1)^{n-1}}{p^{n-1}}$. **14.** $\frac{5}{3}$, 1, $\frac{3}{5}$,...; or $\frac{20}{3}$, -4, $\frac{12}{5}$,.... **15.** Either the A.P. 8, 12, 16,..., and the G.P. 8, 12, 18...; or the A.P. 8, 4, 0,..., and the G.P. 8, 4, 2,....

16. 9 and 12; or 1 and -4. **17.** $x=9$, $y=15$; or $x=49$, $y=35$.

19. 2, 4, 8. **22.** 6, 8, 12. **26.** 9, 7, 5, 3. **29.** $\dfrac{x(1+x)}{(1-x)^3}$.

30. $\frac{1}{4}n(n+1)(3n^2+23n+46)$.

Examination Papers and Questions.

Paper A. Page 380. **1.** $x^3+y^3-z^3+3xyz$; $(p+r)(q+s)$.

2. $\dfrac{3x^2+x+1}{2x^2+1}$. **3.** (i) $\dfrac{1}{x^3-1}$; (ii) $\dfrac{1}{2x^2-1}$.

4. (i) $1\frac{1}{3}$; (ii) $-2\frac{1}{16}$; (iii) $x=308$ and $y=39$. **5.** $4x^2-3x+2$.

6. 5 miles. **8.** 5. **9.** (i) $2\frac{1}{2}$ or 8; (ii) $\frac{1}{2}a$ or a.

10. (i) $3660\frac{1}{3}$, 3888; (ii) 1296.

Paper B. Pages 381—382.

1. (i) $3\frac{1}{2}$; (ii) 3; (iii) $x=9$, $y=111$.

3. (i) -1 or 6; (ii) $\frac{1}{17}$ or 1; (iii) $x=7$ and $y=11$, or $x=-\frac{39}{5}$ and $y=-\frac{56}{5}$. **5.** 7. **6.** ± 11, ± 7. **7.** 15 miles; *A* and *B* respectively walk at the rates of $3\frac{3}{4}$ and 3 miles an hour. **8.** 12 feet by 9 feet, and 15 feet by 8 feet. **9.** $\dfrac{b^{n-1}}{a^{n-2}}$; 11 : 243, or -3125 : 243.

10. (i) 60; (ii) 1690981/15625; (iii) 19/16. **11.** $\frac{1}{5}(4a+b)$, $\frac{1}{5}(3a+2b)$, $\frac{1}{5}(2a+3b)$, $\frac{1}{5}(a+4b)$. **12.** Either 16, 8, 4,..., or 2, -6, 18,....

Examination Questions. Pages 383—385.

1. $x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{\frac{1}{4}}y^{-\frac{1}{4}}+x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}$. **3.** (i) 280; (ii) $514\frac{2}{7}$; (iii) 108.

5. 6 cwt. **7.** $a^{-\frac{3}{2}}=1\div\sqrt{a^3}$, $a^{\frac{2}{3}}=\sqrt[3]{a^2}$; $x^{-\frac{10}{3}}y^{\frac{13}{4}}$. **9.** 144.

10. £250, £200. **12.** $\frac{27}{8}$. **13.** (i) 890; (ii) 45; (iii) $9\frac{1}{7}$.

14. 16, 24, 36. **15.** 3*d*. up to 5 lbs., and 1*d*. for every additional lb.

16. $\dfrac{2a}{\sqrt{(x+a)}}$. **18.** $\dfrac{999{,}999}{1{,}100{,}000}$; $\dfrac{10}{11}$. **19.** $x=\frac{1}{2}(a+b)$, $y=\sqrt{ab}$, $z=\dfrac{2ab}{a+b}$; $2(y-a)$.

Chapter XXX. Permutations and Combinations.

XXX. A. Page 391. **1.** 6; 2520; 24; 6. **2.** 11.

3. (i) 120; (ii) 60. **4.** (i) 360; (ii) 720; (iii) 120; (iv) 2520.

5. 13860; 6930. **6.** 7. **7.** 48. **8.** 216.

XXX. B. Page 395. **1.** 7; 70; 1081. **2.** 7. **3.** 5.

4. 2 or 8. **5.** 184756. **6.** (i) ${}_{19}C_9$; (ii) ${}_{19}C_{10}$. **7.** 15.

XXX. C. Pages 398—400. **1.** 60; 15. **2.** (i) 6; (ii) 12. **4.** 3600. **5.** 593775. **6.** $36 \times {}_{100}C_{50}$. **7.** 420. **9.** 144. **10.** 7. **11.** $n=12$, $r=4$. **12.** 1. **14.** 3^n. **15.** 43200. **16.** 6. **17.** $n=15$, $r=5$. **18.** 63. **19.** 28. **20.** ${}_{27}C_2 \times {}_{21}C_2 \times {}_{17}C_2 = 10024560$. **21.** $n \cdot {}_{2n}C_2 = (2n-1)\,n^2$; $n \cdot 2n \cdot (2n-1)$. **22.** (i) 992; (ii) 1024. **23.** (11111)(3150)($|\underline{4}$). **24.** (i) ${}_{14}C_8$; (ii) ${}_{14}C_9$. **25.** 380. **26.** 63. **28.** 6720; 40320. **29.** 1071. **30.** 24. **31.** 52. **32.** $\dfrac{52\,!}{(13\,!)^4}$. **33.** 64. **34.** 209. **35.** $\frac{1}{2}n(n-3)$. **36.** $\frac{1}{4}n(n-1)\,m^2(m-1)$. **37.** 90.

Chapter XXXI. The Binomial Theorem.

XXXI. A. Pages 408—410.

1. $1+8x+28x^2+56x^3+70x^4+56x^5+28x^6+8x^7+x^8$.
2. $x^7-14x^6+84x^5-280x^4+560x^3-672x^2+448x-128$.
3. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.
4. $2048a^{11}+11264a^{10}b+28160a^9b^2+42240a^8b^3+42240a^7b^4+29568a^6b^5$ $+14784a^5b^6+5280a^4b^7+1820a^3b^8+220a^2b^9+22ab^{10}+b^{11}$.
5. $243x^5-810x^4y+1080x^3y^2-720x^2y^3+240xy^4-32y^5$.
6. $16a^4+\frac{32}{3}a^3b+\frac{8}{3}a^2b^2+\frac{8}{27}ab^3+\frac{1}{81}b^4$.
7. $x^6/64+x^5y/16+5x^4y^2/48+5x^3y^3/54+5x^2y^4/108+xy^5/81+y^6/729$.
8. $1-7a^2/2+21a^4/4-35a^6/8+35a^8/16-21a^{10}/32+7a^{12}/64-a^{14}/128$.
9. $x^{10}-\frac{5}{3}x^8+\frac{10}{9}x^6-\frac{10}{27}x^4+\frac{5}{81}x^2-\frac{1}{243}$. **10.** $10a^9$.
11. $2(x^8+28x^6+70x^4+28x^2+1)$. **12.** $53130x^{30}$. **13.** 10.
14. $-35 \cdot 2^4 \cdot 3^3x^4 = -30240x^4$. **15.** $-55/288$.
16. $128a^7-448a^6x+672a^5x^2-560a^4x^3+280a^3x^4-84a^2x^5+14ax^6-x^7$.
17. $|\underline{2n}/(|\underline{n})^2$. **18.** $59136a^6x^6$. **19.** -2268. **20.** $-945a^4x^{10}$
21. 3rd term; coefficient $=6$. **22.** 11th and 12th terms.
23. 6th and 7th terms; each coefficient $=462$. **24.** 7th term; coefficient $=5376$. **25.** (i) 10th term; (ii) 11th term; (iii) 8th term.
26. coefficient of $x^5=90$. **27.** $(10+1)^7=107223585210701$; $(10-1)^7=9509900499$. **28.** $(3+2)^7$. **29.** $(1+x^3)^n$.
31. $(1+x)^{14}=1+14x+91x^2+364x^3+\ldots$
32. $1+4x+2x^2-8x^3-x^4+8x^5+2x^6-4x^7+x^8$. **33.** 90.
34. $(-1)^n\dfrac{|\underline{2n}}{(|\underline{n})^2}$. **35.** 1. **36.** $(3a-2)^5$. **37.** (i) 0; (ii) $\dfrac{|\underline{2n}}{(|\underline{n})^2}$.

XXXI. B. Pages 422—425.

1. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4$. 2. $1+2x/3-x^2/9+4x^3/81-7x^4/243$.

3. $1-5x/4+5x^2/32-5x^3/128+35x^4/2048$.

4. $1-3x+\frac{15}{2}x^2-\frac{35}{2}x^3+\frac{315}{8}x^4$. 5. $1-2x/3+x^2/18+x^3/162+5x^4/3888$.

6. $1-\frac{p}{q}ax+\frac{p(p+q)}{2!\,q^2}a^2x^2-\frac{p(p+q)(p+2q)}{3!\,q^3}a^3x^3$
$$+\frac{p(p+q)(p+2q)(p+3q)}{4!\,q^4}a^4x^4.$$

7. $a^{\frac{7}{3}}\left(1-\frac{7}{3}\frac{b}{a}x+\frac{7.4}{3^2.2!}\frac{b^2}{a^2}x^2-\frac{7.4.1}{3^3.3!}\frac{b^3}{a^3}x^3-\frac{7.4.1.2}{3^4.4!}\frac{b^4}{a^4}x^4\right)$.

8. $2^{\frac{11}{5}}\left(1+\frac{11}{5}4x+\frac{88}{5^2}4^2x^2+\frac{616}{5^3}4^3x^3+\frac{16016}{5^4}4^4x^4\right)$.

9. $x^{-\frac{1}{2}}(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4)$. 10. $1+\frac{1}{3}x^2+\frac{2}{9}x^4+\frac{14}{81}x^6+\frac{35}{243}x^8$.

11. $1+\frac{1}{2}x^3+\frac{3}{8}x^6+\frac{5}{16}x^9+\frac{35}{128}x^{12}$. 12. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4$.

13. $\frac{1.3.5\ldots(2r-1)}{r!}x^r$. 14. $\frac{3.5\ldots(2r+1)}{r!}x^r$.

15. $\frac{(n+1)(2n+1)\ldots(rn+1)}{r!\,n^r}x^r$. 16. $-\frac{(n-1)(2n-1)\ldots(rn-1)}{r!}x^r$.

17. $(-1)^r\frac{3.1.1.3.5\ldots(2r-5)}{r!}x^r$. 18. $\frac{11.14\ldots(3r+8)}{4^r.r!}x^r$.

19. $\frac{1.3.5\ldots(2r+1)}{r!}(\frac{2}{3})^rx^r$. 20. $\frac{4.7.10\ldots(3r+1)}{3^r.r!}x^{2r}$.

21. ${}_nH_rx^{-n-2r}$. 22. (i) 5th; (ii) if $\frac{p}{q}$ be negative, then the 2nd; if $\frac{p}{q}$ be positive and be $> n$ and $< n+1$, then the $(n+3)$th.

23. If $n=\frac{17}{3}$, the sign is +;
$$\frac{17.14.11.8.5.2.1}{3^7.7!}x^7,\text{ and }\frac{17.14\ldots2.1}{3^7.7!}x^8.$$

24. The 5th. 25. The 6th and 7th. 26. The 1st and 2nd. 27. $(1-4x)^{-\frac{1}{2}}$.

28. $(1-x)^{-\frac{1}{2}}$. 29. $(1-\frac{1}{2})^{-\frac{1}{2}}=\sqrt{2}$. 30. $(1-\frac{3}{4})^{-\frac{11}{3}}=4^{\frac{11}{3}}$. 33. 10·04987.

34. 3·995. 35. 9·996666. 36. $N+\frac{x}{2N}-\frac{x^2}{8N^3}+\frac{x^3}{16N^5}$. 38. ·983.

43. (i) 0; (ii) 2^n. 44. Coefficient of $x^{3n}=\frac{1.3.5\ldots(2n-1)}{2^r.r!}$; coefficient of $x^{3n+2}=0$. 45. -1. 46. 429/256. 47. -1.

49. $\phi(x,n)=\phi(x,n-1)-\phi(x+1,n-1)$.

Chapter XXXII. The Exponential Theorem.

XXXII. A. Page 430. 1. e^{-1}. 2. log 2. 3. log 4.

XXXII. B. Pages 437—438. 1. 1·0457609; 111·116.
2. 1·9356274. 3. $\bar{2}$·4935452. 4. $\bar{3}$·1234400. 5. $\bar{2}$·5579353.
6. $\bar{1}$·2363004. 7. 1·1356911; 136·6486. 8. 46·0168.
9. 2·441406. 10. 14·67799855. 12. 1·778278. 13. 1·389495.
14. 1·401131. 15. 1·4495593. 16. 1·948446. 17. 8·465369.
18. 4239·33. 19. To 3 places.

Examination Papers.

Paper A. Page 439. 2. $acx^2+b(a+c)x+(a+c)^2=0$.
4. (i) $x=\dfrac{aq-bp}{m(a-p)}$, $y=\dfrac{aq-bp}{m(q-b)}$; (ii) 1, $\dfrac{a-b}{b-c}$; (iii) $\frac{3}{2}$; (iv) $x=8$ and $y=-2$, or $x=2$ and $y=1$. 5. 10 miles an hour.
6. See Art. 328, and p. 343, Ex. 2. 7. See Art. 394.
9. $x=7{\cdot}54$, $y=-{\cdot}498$.

Paper B. Page 440.
2. (i) $x=\dfrac{p(bc-1)+q(c+1)+r(b+1)}{a+b+c-abc+2}$, &c.;
(ii) $x=y=\frac{1}{2}(-a\pm\sqrt{a^2+4b^2})$, or $x=\frac{1}{2}(a\pm\sqrt{4b^2-3a^2})$ and $y=\frac{1}{2}(a\mp\sqrt{4b^2-3a^2})$.
3. 15481 square yards. 4. If P, Q, R be the p^{th}, q^{th}, and r^{th} terms of an A.P., then $P(q-r)+Q(r-p)+R(p-q)=0$. 6. The $(r+1)^{th}$ term, where r is the integer just less than $(n-1)/(m-1)$. 7. 4·642.

Paper C. Page 441. 1. 0. 2. $2(a+b)(b+c)$; $5xy(x-y)(x^2-xy+y^2)$. 3. $\dfrac{2x^2+3x-1}{x^2-3x-2}$. 4. (i) $\dfrac{34xy}{49x^2-y^2}$; (ii) $-\dfrac{1}{abc}$.
5. $m=\frac{1}{4}a^2+2c/a$; $n=c^2/a^2$. 7. (i) -20; (ii) 5; (iii) a, b. 8. $\frac{2}{3}$.
9. $(\sqrt[3]{2}-1)r$. 10. The sides of one are each equal to a; of the other are $2a$ and $\frac{1}{2}a$. 12. The given coefficient $={}_{n+r-1}C_r$.

Paper D. Pages 442—443. 1. 1. 3. $4x^2-6x+9$; $(3x+2)(64x^6-729)$. 4. $2xa^{-1}-3+4ax^{-1}$. 5. $b^2>4ac$. 6. (i) $1\frac{1}{27}$; (ii) $a+b$, $\frac{1}{2}(a+b)$; (iii) $x=9$, $y=6$, $z=4$, or $x=4$, $y=6$, $z=9$.

7. 3 gallons from the first and 8 gallons from the second. **9.** 15.

10. $\dfrac{\lfloor 2m+1}{\lfloor m \, \lfloor m+1} x, \quad \dfrac{\lfloor 2m+1}{\lfloor m \, \lfloor m+1} \dfrac{1}{x}$. **12.** Between 114 and 115 years.

Chapter XXXIII. Properties of Numbers.

XXXIII. A. Pages 451—452. **1.** 3205. **2.** 975. **3.** 194. **4.** 31e. **5.** $\frac{11}{88}$. **6.** $\cdot\dot{5}05915343t0e62t68781\dot{e}$. **7.** $20\cdot\dot{2}\dot{0}$. **8.** $\cdot\dot{0}\dot{5}$. **9.** 111/232. **10.** 9; 11. **11.** 13565523. **12.** ·00001. **13.** 4112. **14.** 1011. **15.** 29. **21.** The digit $(r-1)$. **22.** r^4-r^3; $\frac{1}{2}(r^4+r^3-1)(r^4-r^3)$. **25.** (i) ...67; (ii) ...95; (iii) ...62; (iv) ...078.

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